SISSA
Scuola Internazionale Superiore di Studi Avanzati

Mathematics Area - PhD course in
Geometry and Mathematical Physics

Pointed Hopf Algebra (Co)actions on Rational Functions

Candidate:
Blessing Bisola Oni

Advisor:
Prof. Ulrich Krähmer

Academic Year 2021-2022

Abstract

Studies have shown that certain singular affine curves $O(X)$ (e.g nodal cubic, lemniscate) admit a quantum homogeneous space structure – a right coideal subalgebra of a Hopf algebra H such that H is faithfully flat as a $O(X)$-module. The starting point of our work is to demand like their classical analogues the extension of the quantum symmetries given by quantum groups H to the field of rational funtions of these singular curves.

Taking this idea futher, we study the construction of Hopf algebras H acting on a given algebra K in terms of algebra morphisms $\sigma\colon K \to \mathrm{M}_n(K)$. This approach is particularly suited for controlling whether these actions restrict to a given subalgebra B of K, whether H is pointed, and whether these actions are compatible with a given $*$-structure on K. In particular, we applied this theory to the field $K = k(t)$ of rational functions containing the coordinate ring $B = k[t^2, t^3]$ of the cusp. We also described an explicit example of this theory and showed how it equips the cusp with a quantum homogeneous space structure.

Acknowledgements

Principally, I will like to appreciate my advisor, mentor and teacher Ulrich Krähmer from whom I have learned a great deal. Thank you Uli for your patience, understanding and encouragement all the way. You are all a PhD student can ask for in an advisor. I sincerely thank Barbara Fantechi who accepted to be my local supervisor when I decided to work with a professor outside of the faculty at SISSA. Thank you Barbara for ensuring that all my logistical needs were met.

Secondly, I will like to thank my family for their love and support throughout this program. Thank you Dad, Mum, Joseph, Dorcas, Abraham, Dickson, and Esther for being my support system. To my number one fan and husband, Oladimeji, I so much appreciate your constant cheer and your faith in me. To my friends who made this journey less lonely and with whom I share my PhD woes, failures and successes. Thank you Mubarak, Youness, Yusuf, Samreena and the Prevailing Faith family. I am forever grateful to the Abdus Salam Interenational Centre (ICTP) for funding my PhD program, because of this funding, I could study and receive an excellent international education without worries.

Finally, I am most grateful to God, my source and sustainer and to the Holy Spirit my greatest teacher.

Contents

Chapter 1

Introduction

The concept of symmetry is ubiquitous in mathematics and in nature. Loosely speaking, an object is called symmetric if it stays the same under certain transformations such as rotations, reflections, and translations among others. Groups were first understood as symmetries of certain objects and thus to study symmetries of spaces, we study actions of groups on these spaces. For example, people study actions of Lie groups on manifolds, and of discrete groups on rings and algebras. Generalizing this theory, we study the actions of quantum groups (non-commutative non-cocommutative Hopf algebras) on algebras. In particular in this thesis we will study the (co)actions of these generalized groups on coordinate rings and function fields of singular plane curves.

Quantum groups

The name quantum groups goes back to the work of Drinfel'd [19] who used the basic notions of states and observables in the theory of classical and quantum mechanics to introduce these generalized groups. He considered in the classical setting elements of a space X (e.g groups, manifolds) as states and functions on X as observables, and in the quantum case, he considered states as 1-dimensional subspaces of a Hilbert space and observables as (self-adjoint) linear operators on the Hilbert space. The collection of these observables form a unital associative algebra which, together with added structures becomes a commutative Hopf algebra in the classical case, and a non-commutative Hopf algebra in the quantum case. Intuitively, this means quantization replaces commutative algebras by noncommutative ones thus, there is a dichotomy between classical groups and quantum groups and their theories. Taking this idea further leads to the notion and theory of quantum homogeneous spaces, the quantum analogue of classical homogeneous spaces which was first introduced by Podleś in [42] when he found that a continuously parametrized family $\{S_q(\nu, \mu) : q \neq 0, (0,0) \neq (\nu, \mu) \in \mathbb{R}_{\geq 0}\}$ of $SU_q(2)$-spaces have "good enough" properties to be called a homogeneous space of the quantum group $SU_q(2)$. He showed that this family is a quantum analog of the classical 2-spheres $SU(2)/SO(2)$ and they are famously called the Podleś' spheres.

Quantum homogeneous space

Classically, we understand that homogenous spaces X of a group G are in bijection with the set of cosets of G by subgroups (stabilizer of a fixed element of X). As one would expect, this carries over to the quantum setting as Podleś showed in [43] that the quotient of a quantum group H by a quantum subgroup corresponds to a homogeneous space of the quantum group H. By a quantum subgroup, we mean a Hopf algebra H_0 together with a surjective morphism of Hopf algebras $\pi : H \longrightarrow H_0$ and the quantum homogeneous space B is the set of elements of H which are invariant under the coaction of H_0 induced by π. Podleś noted that defining quantum homogeneous spaces in this sense is quite restrictive since he showed that quantum groups have fewer quantum subgroups compared to their classical counterpart as shown in the case of $SU_q(2)$ [42, Section 2]. A less restrictive and larger class of quantum homogeneous spaces is the class of embeddable quantum homogeneous spaces – these are right comodule algebras of H together with an injective morphisms of comodule algebras to H [43, 14, 29]. Interestingly, the quotient quantum homogeneous spaces are embeddable. Takeuchi [48] however gave a more rigorous definition of quotient quantum homogeneous spaces using the notion of relative Hopf modules and the equivalence of the categories of right (H, B)-Hopf modules and of left $(\pi(H), H)$-Hopf modules, where π is a surjective morphism of coalgebras and $B \subseteq H$ is a right coideal subalgebra. The equivalence of these categories require the faithful flatness and faithful coflatness of H as a B-module respectively of H as a $\pi(H)$-comodule. Thus following the definition given in [41, 39, 38, 46], we take quantum homogeneous spaces to be right coideal subalgebras over which H is faithfully flat, see Definition 3.4.2.

New examples and more recent works

The study of quantum groups and their actions on commutative algebras goes back to the question raised in Cohen's article [16]: which Hopf algebras can (co)act on a commutative algebra? For semisimple Hopf algebras H over algebraically closed fields k of characteristic 0, it was answered completely by Etingof and Walton [21], see also [20]: if H acts inner faithfully (see Section 2.3.4) on a commutative integral domain K, then H is a group algebra. On the other hand, there are many examples of inner faithful actions of infinite-dimensional pointed Hopf algebras on a commutative algebra even when it admits only few automorphisms and derivations. For example, the coordinate rings B of singular plane curves are conjecturally all quantum homogeneous spaces [34, 33, 12], and their containing pointed Hopf algebras H are dually paired with Hopf algebras $A \subseteq H^\circ$ which are not in general again pointed

Results in this thesis

The starting point of our work is to demand that like classical symmetries (given by actions of group algebras or universal enveloping algebras), these examples of quantum symmetries should extend from the coordinate ring B to the field K of rational functions on the curve. We also examine the example of the quantum homogeneous space structure on the cusp mentioned in [32] and show that in this case, the resulting Hopf algebra H that acts inner faithfully on the coordinate ring is not pointed, see Remark 3.6.10 below. As the classification of Hopf algebra (co)actions on fields is an interesting topic in its own right (see e.g. [22, 23, 25, 49, 50, 18] and the references therein), we felt it worthwhile to begin a systematic study of such actions on function fields that restrict to coordinate rings.

The approach we take was maybe first applied by Manin in his construction of quantum $SL(2)$ as a Hopf algebra (co)acting on the quantum plane [37]. More recently, it was used mostly in the C^*-algebraic quantum group community in the construction and study of compact quantum automorphism groups such as the quantum permutations groups or the liberations of compact Lie groups [4, 6, 52, 7, 5, 9, 3].

In this approach, a bialgebra action on a k-algebra K is constructed from an algebra morphism $K \to \mathrm{M}_n(K)$; the bialgebra is a Hopf algebra if this morphism, viewed as an element of $\mathrm{M}_n(\mathrm{End}_k(K))$ is strongly invertible in the sense of Definition 4.1.2 below. The general approach is well-known, but some aspects of our presentation are to our knowledge novel, such as the connection to the theory of general linear groups over noncommutative rings (see Section 4.1.1), the treatment of $*$-structures in the pointed rather than the semisimple setting (see Section 4.2.8), and the application to the field $K = k(t)$ of rational functions (see Section 4.1.4).

Example: $k[t^2, t^3]$

Our main focus is the construction of pointed Hopf algebra (co)actions on $K = k(t)$ which restrict to the coordinate ring $B = k[t^2, t^3]$ of the cusp. Working over a field

k in which 2 and 3 are invertible, we construct Hopf algebras A_σ and H_σ which are generated as algebras by γ, ψ, φ respectively K, D, Y. These generators satisfy the defining relations in Proposition 4.2.24(1) respectively in Lemmas 4.2.3, 4.2.8. In terms of these generators, the coproduct, counit, and antipode of A_σ and H_σ are given by Proposition 4.2.24(2) respectively by

$$\Delta(K) = K \otimes K, \quad \Delta(D) = 1 \otimes D + D \otimes K,$$
$$\Delta(Y) = 1 \otimes Y - 6D \otimes DK + Y \otimes 1,$$
$$\varepsilon(K) = 1, \quad \varepsilon(D) = \varepsilon(Y) = 0,$$
$$S(K) = K, \quad S(D) = -DK, \quad S(Y) = -Y.$$

The structure of the thesis is as follows. In Chapter 2 we fix notations and give definitions of algebras, coalgebras, bialgebras, and Hopf algebras as well as examples, morphisms of these algebraic objects, their dualizations, (co)representations and results characterizing them. We proceed in Chapter 3 to introduce the definition of quantum homogeneous space in the sense discussed in the paragraphs above, and we also talk about the motivation behind the definition of quantum homogeneous spaces following the notations of [48, 41]. We conclude the chapter with the first contribution of this thesis, where we show in particular that the coaction map by which the coordinate ring $k[t^2, t^3]$ of the cusp admits a quantum homogeneous space structure must be the coaction in [32] see Sections 3.5.2, 3.6 for more details.

Chapter 4 contains the main results of this thesis with the full description of the dually paired Hopf algebras H_σ and A_σ (co)acting on $k(t)$. We show that they are in fact pointed and that the embedding of the Hopf algebra $A_\sigma \subseteq H_\sigma^\circ$ is dense. We also prove that for any point (λ^2, λ^3), $\lambda \in k$ of the cusp, there is an embedding

$$\begin{aligned} \iota : B = k[t^2, t^3] &\longrightarrow A_\sigma \\ t^2 &\mapsto \lambda^2 + \frac{1}{3}\varphi^2, \\ t^3 &\mapsto \gamma + \lambda^2\varphi + \lambda^3\psi \end{aligned}$$

which makes $k[t^2, t^3]$ a quantum homogeneous space of A_σ. Furthermore, the action of H_σ on B and the coaction of A_σ on B both extend to (co)actions on the field $K = k(t)$ of rational functions, which is a step further on previously constructed Hopf algebra (co)actions on $k[t^2, t^3]$ in [34, 33, 12]. In addition, if $k = \mathbb{C}$, then H_σ becomes a Hopf $*$-algebra with $K^* = K$, $D^* = -D$, $Y^* = -Y + 6iD$. The images of A_σ and of B in H_σ° are $*$-subalgebras provided that $\bar{\lambda} = \lambda$; the resulting $*$-structure on $\mathbb{C}[t^2, t^3]$ is given by $t^* = t$.

Chapter 2

Preliminaries

In this chapter we will introduce the main object of our study - Hopf algebras - which is an associative unital algebra with additional structures of a coproduct, a counit and an antipode satisfying some axioms. Commutative Hopf algebras somewhat generalize the notion of algebraic groups as we will show in the next chapter. We denote unless otherwise stated by k a underlying field of characteristic 0, A to be a k-algebra, C to be a k-coalgebra and H as a k-Hopf algebra. The definitions, examples, and results in this chapter were adapted from [30], [31], [40], [44], and [47].

2.1 Algebras and coalgebras

2.1.1 Definitions and examples

Definition 2.1.1. A *unital associative algebra* over a field k is the triple (A, m, η) where A is a k-vector space and

$$\begin{aligned} m : A \otimes A &\longrightarrow A, & \qquad \eta : k &\longrightarrow A \\ a \otimes b &\mapsto ab & 1_k &\mapsto 1 \end{aligned}$$

are k-linear maps called the *multiplication (or product)* respectively the *unit* of A where $m(a \otimes b) := ab$ such that the following diagrams commute

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes id} & A \otimes A \\ {\scriptstyle id \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{\quad m \quad} & A \end{array} \qquad \begin{array}{ccc} k \otimes A \cong A \cong A \otimes k & \xrightarrow{id \otimes \eta} & A \otimes A \\ {\scriptstyle \eta \otimes id}\downarrow & \searrow^{id} & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{\quad m \quad} & A \end{array}$$

The left commutative diagram implies

$$m \circ (m \otimes id)(a \otimes b \otimes c) = m \circ (id \otimes m)(a \otimes b \otimes c)$$
$$(ab)c = abc = a(bc),$$

which means the linear map m is associative and that of the right diagram means

$$m(a \otimes \eta(1_k)) = a\eta(1_k) = a = m(\eta(1_k) \otimes a) = \eta(1_k)a.$$

which is referred to as the unitality of η, which in other words means that A is unital.

Thus from now on, a k-algebra is an associative unital algebra and we write A for (A, m, η) to denote a k-algebra and we denote by **Alg** the category of k-algebras.

Definition 2.1.2. Let A be a k-algebra. The *opposite algebra* of A is the triple (A, m^{op}, η), where $m^{\text{op}}(a \otimes b) := ba$ and we write A^{op} to denote the opposite algebra.

Note that a k-algebra A and its opposite algebra A^{op} have the same underlying k-vector space, the same unit but the multiplication in A^{op} is a flip of the multiplication in A. They are the same only if A is a commutative k-algebra.

Definition 2.1.3. The algebra A is *commutative* if $m^{\text{op}} = m$.

In what follows we give our first example of k-algebras:

Example 2.1.4. Let G be a group. The set

$$k[G] = \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in k \right\}$$

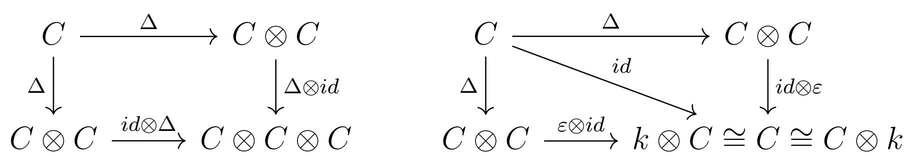
of formal sums of scalar multiples of elements of G together with product defined as

$$\left(\sum_{g \in G} \alpha_g g \right) \left(\sum_{h \in G} \lambda_h h \right) = \sum_{x = gh} \alpha_g \lambda_h x$$

and unit map given by $\eta(1_k) = e$, (e is the unit element of G) is a k-algebra. $k[G]$ is called a *group algebra* and it becomes commutative only when G is an abelian group.

Like k-algebras, k-coalgebras have an underlying vector space that is in addition equipped with some structure maps, but now they go the opposite way round. In the language of category theory, k-algebras are monoids in the category of vector spaces and k-coalgebras are monoids in the opposite category (with the same monoidal structure). Thus, the arrows of the structure maps defining a k-algebra in the commutative diagrams above are now reversed. More precisely

Definition 2.1.5. A *coalgebra* over a field k is a triple (C, Δ, ε) where C is a k-vector space and $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow k$ are k-linear maps called the *coproduct* respectively the *counit* of C such that the diagrams



commute. The commutativity of the first and second diagram imply *coassociativity* of Δ respectively *counitality* of ε.

Note that the tensor product $C \otimes C \otimes C$ in the first commutative diagram is a somewhat sloppy notation for $C \otimes (C \otimes C) \cong (C \otimes C) \otimes C$. The coproduct $\Delta(c) \in C \otimes C$ is denoted by a symbolic sum which in Sweedler's notation is written as $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ but we will drop the summation symbol and write $\Delta(c) = c_{(1)} \otimes c_{(2)}$. The coassociativity condition $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ encoded in the first commutative diagram in Sweedler's notation means

$$c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_2 = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)},$$

which is symbolically written as $c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, and the counitality condition $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$ encoded in the second commutative diagram means

$$c_{(1)}\varepsilon(c_{(2)}) = c = \varepsilon(c_{(1)})c_{(2)}.$$

From now on, we shall denote the k-coalgebra (C, Δ, ε) by C and **Coalg** the category of k-coalgebras. The dual of commutativity is cocommutativity, and to define this we first define the linear map $\tau_{M,N} : M \otimes N \longrightarrow N \otimes M, \quad m \otimes n \mapsto n \otimes m$ for k-vector spaces $M,\ N$ and it is called the *flip* morphism.

Definition 2.1.6. The *opposite coalgebra* of the k- coalgebra (C, Δ, ε) is the triple $(C, \Delta^{\mathrm{cop}}, \varepsilon)$, where $\Delta^{\mathrm{cop}} = \tau_{C,C} \circ \Delta$ that is $\Delta^{\mathrm{cop}}(c) = c_{(2)} \otimes c_{(1)}$ and we write C^{cop} to denote the opposite coalgebra.

Similar to their algebra counterparts, a k-coalgebra C and its opposite coalgebra C^{cop} have the same underlying vector space, counit but the coproduct of C^{cop} is the flip of the coproduct of C. Note that C^{cop} is indeed a coalgebra, since the new coproduct Δ^{cop} satisfies the coassociativity and counitality conditions

$$\begin{aligned}(\Delta^{\text{cop}} \otimes id) \circ \Delta^{\text{cop}} &= ((\tau \circ \Delta) \otimes id) \circ (\tau \circ \Delta) = \tau \circ (\Delta \otimes id) \circ \Delta \\ &= \tau \circ (id \otimes \Delta) \circ \Delta = (id \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta) \\ &= (id \otimes \Delta^{\text{cop}}) \circ \Delta^{\text{cop}}.\end{aligned}$$

$$\begin{aligned}(\varepsilon \otimes \mathrm{id}) \circ \Delta^{\text{cop}}(c) &= \varepsilon(c_{(2)})c_{(1)} = c \\ &= \varepsilon(c_{(1)})c_{(2)} = c_{(2)}\varepsilon(c_{(1)}) \\ &= (\mathrm{id} \otimes \varepsilon) \circ \Delta^{\text{cop}}.\end{aligned}$$

Definition 2.1.7. A coalgebra C is said to be *cocommutative* if $\Delta = \Delta^{\text{cop}}$.

Thus if C is cocommutative then C and C^{cop} are the same.

Example 2.1.8. The group algebra $k[G]$ described in Example 2.1.4 is also a coalgebra: It suffices to define the coproduct on the basis G of $k[G]$ as

$$\Delta(g) = g \otimes g \ \ \forall\, g \in G,$$

and by the counitality condition, $\varepsilon(g) = 1$. This definition extends linearly to the whole of $k[G]$ and it is obvious to see that Δ is coassociative and in fact $\Delta = \Delta^{\text{cop}}$ thus, kG is is a cocommutative coalgebra.

We recall the definitions of subalgebras and ideals of a k-algebra A. A k-subalgebra of A is a k-vector subspace V of A such that $m(V \otimes V) \subseteq V$. A left (resp. right) ideal I of A is a k-vector subspace such that $m(A \otimes I)$(resp. $m(I \otimes A)) \subseteq I$. The dual counterparts of these subobjects in the category of coalgebras are defined as follows

Definition 2.1.9. Let C be a k-coalgebra, then

1. A *subcoalgebra* D of C is a k-vector subspace of C such that $\Delta(D) \subseteq D \otimes D$.
2. A *left (resp. right) coideal* I of C is a subspace with the property that $\Delta(I) \subseteq C \otimes I$ (resp. $I \otimes C$).
3. A subspace I of C is called a *coideal* if $\Delta(I) \subseteq\ C \otimes I + I \otimes C$ and $\varepsilon(I) = 0$.

Just as we can construct a new algebra by taking quotient with a two-sided ideal, so also we can obtain a new coalgebra by taking quotient with a coideal.

Definition 2.1.10. A k-coalgebra C is called *simple* if it has exactly two subcoalgebras - the trivial subcoalgebra $\{0\}$ and itself.

This class of subcoalgebras is very important in the theory of k-coalgebras as they help in the classification of coalgebras as pointed or semisimple coalgebras and also in constructing coradical filtrations of coalgebras. We now state the result which shows that locally, every k-coalgebra is finite dimensional.

Theorem 2.1.11. *[47, Theorem 2.2.1] Let C be a k-coalgebra, and $c \in C$. Then, the subcoalgebra generated by c is finite dimensional.*

The subcoalgebra of C generated by $c \in C$ is the intersection of all subcoalgebras of C containing c. This theorem is called the *fundamental theorem of coalgebras* in [47] and a consequence of this theorem is the following result

Corollary 2.1.12. *[47, Lemma 8.0.1(a)] Every simple subcoalgebra is finite dimensional.*

Recall that a map $f : A \longrightarrow B$ of k-algebras (A, m_A, η_A), (B, m_B, η_B) is a k-linear map such that for all $a \in A$, $b \in B$, $f(ab) = f(a)f(b)$ and $f(1_A) = 1_B$ which is the same as saying

$$f \circ m_A = m_B \circ (f \otimes f), \quad \eta_A \circ f = \eta_B.$$

Passing to the dual, we can also define a coalgebra map between k-coalgebras $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ by reversing the composition in the definition of an algebra map.

Definition 2.1.13. A *coalgebra map* is a linear map of k-vector spaces $f : C \longrightarrow D$ such that $(f \otimes f) \circ \Delta_C = \Delta_D \circ f$ and $\varepsilon_D \circ f = \varepsilon_C$. That is for all $c \in C$,

$$f(c)_{(1)} \otimes f(c)_{(2)} = f(c_{(1)}) \otimes f(c_{(2)}), \text{ and } \varepsilon_D(f(c)) = \varepsilon_C(c). \tag{2.1.1}$$

Recall that if $f : A \longrightarrow B$ is an algebra map, then $\ker f$ is an ideal of A. Similarly given a coalgebra map $g : C \longrightarrow D$, the kernel $\ker g$ of g is a coideal of C: let $x \in \ker g$, then $\varepsilon_C(x) = \varepsilon_D \circ g(x) = 0$ thus, $\varepsilon_C(\ker g) = 0$, and $\Delta(\ker g) \subseteq C \otimes \ker g + \ker g \otimes C$ since $0 = \Delta_D \circ g(x) = (g \otimes g) \circ \Delta_C(x)$, that is $\Delta(x) \subseteq \ker g \otimes g$.

An *isomorphism* of coalgebras is a linear isomorphism f which satisfies equation (2.1.1). The underlying field k is a coalgebra over itself with coproduct $\Delta = \mathrm{id}$ and counit $\varepsilon = \mathrm{id}$, thus a trivial example of a coalgebra map is the counit map of coalgebra C, $\varepsilon : (C, \Delta, \varepsilon) \longrightarrow (k, \mathrm{id}, \mathrm{id})$.

Definition 2.1.14. Let C be a k-coalgebra, an element $0 \neq c \in C$ is called *group-like* if $\Delta(c) = c \otimes c$.

We denote by $G(C)$ the set of group-like elements of C, and from the counitality property of ε, it follows that $\varepsilon(c) = 1$ for all $c \in G(C)$, however, some authors say group-like means $\Delta(c) = c \otimes c$ and $\varepsilon(c) = 1$ to stress that $c \neq 0$. The name group-like is motivated from the fact that these elements have properties like the group elements $g \in kG$ and as we will see later, they also have inverses if C is a Hopf algebra.

Remark 2.1.15. The set $G(C)$ is a linearly independent subset of C since it is the basis of the subcoalgebra (a group algebra) $k[G(C)]$ of C.

Lemma 2.1.16. *[47, Lemma 8.0.1(e)] Let C be a k-coalgebra. Then the elements $g \in G(C)$ are in 1-1 correspondence with 1-dimensional subcoalgebras of C.*

Definition 2.1.17. Let C be a coalgebra over k, an element $x \in C$ is called *primitive* if $\Delta(x) = 1 \otimes x + x \otimes 1$ however, x is called *twisted primitive*(or (g, h)- *primitive*) if for some $g, h \in G(C)$, $\Delta(x) = g \otimes x + x \otimes h$.

The set of (g, h)-primitive elements is denoted by $P_{g,h}$, thus a primitive element belongs to the set $P_{1,1}$. From the counitality condition of ε one easily deduce that for all $x \in P_{g,h}$, $\varepsilon(x) = 0$.

Example 2.1.18. Let $\mathfrak{g}$ be a Lie algebra over k and $(U(\mathfrak{g}), \jmath)$ be its universal enveloping algebra – that is $U(\mathfrak{g})$ is an associative unital k-algebra and $\jmath : \mathfrak{g} \longrightarrow U(\mathfrak{g})_-$ a morphism of Lie algebras, where $U(\mathfrak{g})_-$ is $U(\mathfrak{g})$ together with Lie bracket $[g, h] = gh - hg$ for all $g, h \in \mathfrak{g}$. Suppose $(A, \imath)$ is an enveloping algebra of $\mathfrak{g}$, then by universality of $(U(\mathfrak{g}), \jmath)$, there exist a unique algebra map $\Lambda : U(\mathfrak{g}) \longrightarrow A$ such that $\Lambda \circ \jmath = \imath$. Now take $U(\mathfrak{g}) = \bigoplus_{i=0} \mathfrak{g}^i / I$, where I is the ideal generated by all elements of the form $[x, y] - xy + yx$ for all $x, y \in \mathfrak{g}$, then $\jmath(\mathfrak{g})$ generates $U(\mathfrak{g})$ as an algebra.

The homomorphisms of Lie algebras

$$\delta : \mathfrak{g} \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}), \quad g \mapsto \jmath(g) \otimes 1 + 1 \otimes \jmath(g)$$

and $0 : \mathfrak{g} \longrightarrow k$ yield by the universal property of $U(\mathfrak{g})$ the algebra maps

$$\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}) \quad \text{respectively} \quad \varepsilon : U(\mathfrak{g}) \longrightarrow k$$

such that $\Delta \circ \jmath = \delta$, and $\varepsilon \circ \jmath = 0$. It is straightforward that Δ and ε satisfy the coassociativity respectively counitality conditions on $\jmath(\mathfrak{g})$. Thus, the subalgebra $\jmath(\mathfrak{g}) \subseteq U(\mathfrak{g})$ comprise primitives since for all $x \in \jmath(\mathfrak{g})$ the coproduct and counit are given by

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \varepsilon(x) = 0.$$

The group algebra $k[G]$ and the universal enveloping algebra $U(\mathfrak{g})$ are examples of pointed coalgebra and they are very important in the study of the class of pointed Hopf algebra. One of the important tools used in studying pointed Hopf algebras is the *coradical*.

Definition 2.1.19. Let C be a k-coalgebra. The *coradical* of C is the sum of the simple subcoalgebras of C and it is denoted by C_0.

Definition 2.1.20. A *filtration* of a k-coalgebra C is a family of subspaces $\{V_n\}_{n=0}^{\infty}$ of C satisfying

$$V_0 \subseteq V_1 \subseteq \cdots \bigcup_{n=0}^{\infty} V_n = C$$

and the following property: for all $n \geq 0$,

$$\Delta(V_n) \subseteq \sum_{l=0}^{n} V_{n-l} \otimes V_l.$$

Whenever such a filtration exists, C is called a filtered coalgebra and the existence of a filtration is useful in making inductive proofs of statements about coalgebras.

Proposition 2.1.21. *[44, Proposition 4.1.2] Suppose $\{V_n\}_{n=0}^{\infty}$ is a filtration of a coalgebra C. Then any simple subcoalgebra of C is contained in V_0.*

In view of this result, it is clear that the coradical $C_0 \subseteq V_0$ and this prompts the definition of a very useful type of filtration in the theory of coalgebras – the *coradical filtration* – which we define as follows. Let V, W be two subspaces of C, then their wedge product $V \bigwedge W = \Delta^{-1}(V \otimes C + C \otimes W)$. Note that this is the kernel of the tensor product

$$\pi_V \otimes \pi_W : C \otimes C \longrightarrow C/V \otimes C/W$$

of the projections π_V and π_W of C to the quotient spaces C/V respectively C/W. Replacing W with V in the wedge product formula and setting $\bigwedge^0 V = 0$, $\bigwedge^1 V = V$, then for $n > 1$ define $V_n := \bigwedge^n V$:

Definition 2.1.22. The *coradical filtration* of a k-coalgebra C is $\{C_n\}_{n=0}^{\infty}$ where for all $n \geq 1$, $C_n = \bigwedge^{n+1} C_0$ and $C = \cup_{n=0}^{\infty} C_n$.

Definition 2.1.23. A *pointed coalgebra* is a k-coalgebra C whose simple subcoalgebras are 1-dimensional.

In other words, recalling Lemma 2.1.16, a pointed coalgebra is a coalgebra C whose coradical $C_0 \cong k[G(C)]$ is the grouplike coalgebra.

Remark 2.1.24. In the literature [2, 10] pointed coalgebras are also defined to be coalgebras whose simple comodules are 1-dimensional.

In a pointed coalgebra, we have a better description of the subcoalgebra $C_1 = \bigwedge^2 C_0$ in the coradical filtration: C_1 comprises twisted primitives since $\Delta(C_1) \subseteq C_1 \otimes C_0 + C_0 \otimes C_1$ and for any $x \in P_{g,h}$, we see that $\Delta(x) = g \otimes x + x \otimes h$, $x \in C_1$ thus $C_1 \supset P_{g,h}$, and it is not difficult to see that $P_{g,h} \subseteq C_1$ hence, $C_1 = C_0 \oplus P_{g,h}$.

2.1.2 The dual (co)algebra

Let C be a coalgebra and $C^\star := \mathrm{Hom}_k(C, k)$ be the vector space dual of C, that is, the set of all linear functionals on C. $C^\star$ is a k-algebra, and it is called the *dual algebra* of C. The algebraic structure for the k-vector space $C^\star$ is defined using the coalgebra structure of C: the transpose of the coproduct Δ of C is the linear map $\Delta^\star : (C \otimes C)^\star \longrightarrow C^\star$, and we understand that $C^\star \otimes C^\star \subseteq (C \otimes C)^\star$ via the injection map defined for all $c, d \in C$ as

$$\begin{aligned} j : C^\star \otimes C^\star &\longrightarrow (C \otimes C)^\star, \\ c^\star \otimes d^\star &\mapsto j(c^\star \otimes d^\star)(c \otimes d) = c^\star(c) d^\star(d). \end{aligned}$$

Thus we define the multiplication map m on $C^\star$ as $m = \Delta^\star|_{C^\star \otimes C^\star}$ that is for all $c \in C$

$$\begin{aligned} m : C^\star \otimes C^\star &\longrightarrow C^\star \\ f \otimes g &\mapsto m(f \otimes g)(c) = (f \otimes g) \circ \Delta(c) = f(c_{(1)}) g(c_{(2)}) \end{aligned}$$

and the unit map $\eta = \varepsilon^\star : k^\star (= k) \longrightarrow C^\star$, the dual of the counit ε.

Proposition 2.1.25. *Let C be a k-coalgebra. Then the triple $(C^\star, m, \eta)$ as defined above is a k- algebra.*

Proof. We only need to check that m and η satisfy the associativity and unitality conditions: for all $f, g, h \in C^\star$, $c \in C$,

$$\begin{aligned} m \circ (m \otimes id)(f \otimes g \otimes h)(c) &= m((fg)(c_{(1)})) \otimes h(c_{(2)}) \\ &= f(c_{(1)_{(1)}}) g(c_{(1)_{(2)}}) h(c_{(2)}) = f(c_{(1)}) g(c_{(2)}) h(c_{(3)}) \end{aligned}$$

$$\begin{aligned} m \circ (id \otimes m)(f \otimes g \otimes h)(c) &= f(c_{(1)}) \otimes m(g \otimes h)(c_{(2)}) \\ &= f(c_{(1)}) g(c_{(2)_{(1)}}) h(c_{(2)_{(2)}}) = f(c_{(1)}) g(c_{(2)}) h(c_{(3)}), \end{aligned}$$

the equality of these two equations follows from the coassociativity of Δ on C. The unitality axiom follows from

$$\begin{aligned} m \circ (\eta \otimes \mathrm{id})(1_k \otimes f)(c) &= ((1_k \circ \varepsilon) \otimes f) \circ \Delta(c) \\ &= \varepsilon(c_{(1)}) f(c_{(2)}) = f(\varepsilon(c_{(1)}) c_{(2)}) = f(c) \end{aligned}$$

and a similar straightforward computation as above shows $m \circ (\mathrm{id} \otimes \eta) = \mathrm{id}$. □

Definition 2.1.26. Let C be a coalgebra over k. The algebra $(C^\star, m, \eta)$ defined in Proposition 2.1.25 is called the *dual algebra* of C. We usually write $C^\star$ for $(C^\star, m, \eta)$.

On the other hand, constructing the *dual coalgebra* of an algebra is not as straightforward as its coalgebra counterpart. The challenge is that we do not automatically get the finiteness property that is inherent in coalgebras in the linear dual space $A^\star = Hom_k(A, k)$ of A. To fix this problem, we define a subspace of $A^\star$ and it is this subspace we equip with a coalgebra structure induced by the algebra structure on A.

Definition 2.1.27. Let A be an algebra. A *cofinite ideal* is an ideal $J \subseteq A$ such that A/J is finite dimensional.

Definition 2.1.28. For an algebra A, we define

$$A^\circ = \{g \in A^\star : ker(g) \text{ contains a cofinite ideal}\}.$$

Indeed, A° is a subspace of $A^\star$ since it is closed under scalar multiplication and addition: let $f, g \in A^\circ$, then $f + g \in A^\circ$ since $ker(f+g) \supset ker(f) \cap ker(g)$. However, if A is a finite dimensional algebra over the field k, then $A^\circ = A^\star$.

Lemma 2.1.29. *[47, Lemma 6.0.1], Suppose A and B are algebras and $f : A \longrightarrow B$ is an algebra map. Then*

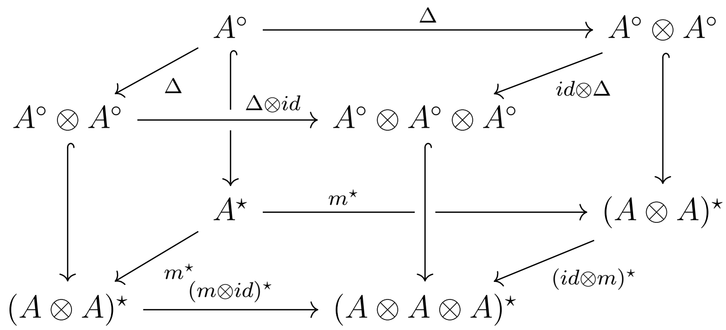
(i) The dual map $f^\star : B^\star \longrightarrow A^\star$ has $f^\star(B^\circ) \subseteq A^\circ$.

(ii) $A^\star \otimes B^\star \subseteq (A \otimes B)^\star$ implies that $A^\circ \otimes B^\circ = (A \otimes B)^\circ$.

Given a k-algebra (A, m, η), with $m^\star : A^\star \longrightarrow (A \otimes A)^\star$, by (i) of Lemma 2.1.29 above, we have $m^\star(A^\circ) \subseteq (A \otimes A)^\circ$ and since $A^\star \otimes A^\star \subseteq (A \otimes A)^\star$, then by (ii) of Lemma 2.1.29, we have $m^\star(A^\circ) \subseteq A^\circ \otimes A^\circ$. The coproduct on A° is defined as $\Delta = m^\star|_{A^\circ}$ and the counit $\varepsilon = \eta^\star|_{A^\circ}$.

Proposition 2.1.30. *Let A be a k-algebra, then the triple (A°, Δ, η) with $\Delta = m^\star|_{A^\circ}$ and $\varepsilon = \eta^\star|_{A^\circ}$ is a coalgebra.*

Proof. We need to show that the coassociativity and counitality axioms are satisfied by Δ and ε as defined above and we show this using commutative diagram



Notice that the bottom diagram is commutative because it is the dual of a commutative diagram encoding the associativity of m, that is

$$(id \otimes m)^\star \circ m^\star = (m \circ (id \otimes m))^\star = (m \circ (m \otimes id))^\star = (m \otimes id)^\star \circ m^\star.$$

Furthermore, since the vertical maps are injections and because of the definition of Δ, we have that the front, rear, right and left square diagrams commute, thus the coassociativity condition follows from the commutativity of the top diagram

$$(\Delta \otimes id) \circ \Delta = (m \otimes id)^\star \circ m^\star = (id \otimes m)^\star \circ m^\star = (id \otimes \Delta) \circ \Delta.$$

Next, we show using the unitality condition of η of A, that ε(as defined above) of A° satisfies the counitality condition

$$\begin{aligned}(\varepsilon \otimes id) \circ \Delta &= (\eta^\star \otimes id) \circ m^\star|_{A^\circ} = (m \circ (\eta \otimes id))^\star|_{A^\circ} = id^\star \\ &= (m \circ (id \otimes \eta))^\star|_{A^\circ} = (id \otimes \eta^\star) \circ m^\star|_{A^\circ} = (id \otimes \varepsilon) \circ \Delta.\end{aligned}$$

□

Definition 2.1.31. Let A be an algebra. The coalgebra $(A^\circ, \Delta, \varepsilon)$ defined in Proposition 2.1.30 is the *dual coalgebra* of A.

Remark 2.1.32. In [44], the definition of A° given is more practical as it is useful for computations involving A°. This alternative definition also ensures that finiteness is built into the coalgebra structure of A° and we give this definition as follows

$$A^\circ = \{a^\star \in A^\star : \ \forall\, a, b \in A,\ a^\star(ab) = \sum_{i=1}^{r} a_i^\star(a) b_i^\star(b)\},$$

that is $\Delta(a^\star) = \sum_i a_i^\star \otimes b_i^\star$, for fixed $a_i^\star, b_i^\star$.

Example 2.1.33. Let $\mathrm{M}_n(k)$ be the algebra of $n \times n$ matrices with entries in the field k. Consider the functional

$$u^i_j : \mathrm{M}_n(k) \longrightarrow k, \quad M \mapsto m_{ij}$$

on $\mathrm{M}_n(k)$, then the linear span $\{u^i_j : i, j \in \mathbb{N}\}$ is a coalgebra with coalgebra structure given by

$$\Delta(u^i_j) = \sum_{k=0}^{n} u^i_k \otimes u^k_j, \qquad \varepsilon(u^i_j) = \delta_{ij}.$$

This coalgebra is the dual coalgebra of the algebra $\mathrm{M}_n(k)$ and it is called the *matrix coalgebra*.

Relationship between (co)algebras and their duals

As one would expect, there is a natural relation between (co)algebras and their duals. Categorically, we can think of these dualizations as contravariant functors:

$$\begin{array}{cc} (-)^\star : \mathbf{Coalg} \longrightarrow \mathbf{Alg}, & (-)^\circ : \mathbf{Alg} \longrightarrow \mathbf{Coalg}. \\ C \mapsto C^\star & A \mapsto A^\circ \end{array}$$

Having shown that $C^\star$ and A° are dual algebra and coalgebra respectively, we only need to show that given a map of coalgebras $f : C \longrightarrow D$, then $(f)^\star = f^\star : D^\star \longrightarrow C^\star$ is an algebra map: let $d_1^\star, d_2^\star \in D^\star,\ c \in C$, then

$$\begin{aligned} (f^\star \circ m_D)(d_1^\star \otimes d_2^\star)(c) &= f^\star(m_D(d_1^\star \otimes d_2^\star))(c) = m_D(d_1^\star \otimes d_2^\star)(f(c)) \\ &= d_1^\star \otimes d_2^\star(\Delta_D \circ f)(c) = (d_1^\star \otimes d_2^\star) \circ ((f \otimes f) \circ \Delta_C)(c) \\ &= d_1^\star(f(c_{(1)}))d_2^\star(f(c_{(2)})) = f^\star(d_1^\star)(c_{(1)})f^\star(d_2^\star)(c_{(2)}), \end{aligned}$$

thus, $f^\star(d_1^\star d_2^\star) = f^\star(d_1^\star)f^\star(d_2^\star)$, in addition, $f^\star$ preserves the unit of the algebra $D^\star$

$$f^\star(1_{D^\star}) = 1_{D^\star} \circ f = \eta_{D^\star}(1_k) \circ f = \varepsilon_D \circ f = \varepsilon_C = \eta_{C^\star} = 1_{C^\star}.$$

Similarly, we will show that given a k-algebra map $g : A \longrightarrow B$, then

$$(g)^\circ = g^\circ : B^\circ \longrightarrow A^\circ,\ \ b^\circ \mapsto g^\circ(b^\circ)(a) = b^\circ(g(a))$$

is a map of k-coalgebras: let $a, a' \in A$ and $b^\circ \in B^\circ$,

$$\begin{aligned} (\Delta_{A^\circ} \circ g^\circ)(b^\circ)(a \otimes a') &= \Delta_{A^\circ}(g^\circ(b^\circ))(a \otimes a') = (g^\circ(b^\circ)) \circ m(a \otimes a') \\ &= b^\circ(g(aa')) = b^\circ(g(a)g(a')) = \Delta_{B^\circ}(b^\circ)(g(a) \otimes g(a')) \\ &= (g^\circ \otimes g^\circ) \circ \Delta_{B^\circ}(b^\circ)(a \otimes a'). \end{aligned}$$

Hence, $\Delta_{A^\circ} \circ g^\circ = (g^\circ \otimes g^\circ) \circ \Delta_{B^\circ}$, in addition $\varepsilon_{A^\circ} \circ g^\circ = \eta_A^\star|_{A^\circ} \circ g^\circ = \eta_B^\star|_{B^\circ} = \varepsilon_{B^\circ}$.

Theorem 2.1.34. *[47, Theorem 6.0.5] The contravariant functors $(-)^\circ$ and $(-)^\star$ are adjoint to each other. In other words, given a k-algebra A and a k-coalgebra C, there is a $1-1$ correspondence between the sets,*

$$\boldsymbol{Alg}(A, C^\star) \longleftrightarrow \boldsymbol{Coalg}(C, A^\circ).$$

The correspondence can be understood by considering the maps

$$\begin{aligned} \Psi : \mathbf{Alg}(A, C^\star) &\longrightarrow \mathbf{Coalg}(C, A^\circ) \\ A \xrightarrow{f} C^\star \quad &\mapsto \quad C \overset{i}{\hookrightarrow} (C^\star)^\circ \xrightarrow{f^\circ} A^\circ \end{aligned}$$

where i is the natural $C \longrightarrow C^{\star\star}$ whose image lies in $C^{\star\circ}$, and

$$\begin{aligned} \Phi : \mathbf{Coalg}(C, A^\circ) &\longrightarrow \mathbf{Alg}(A, C^\star) \\ C \xrightarrow{g} A^\circ \quad &\mapsto \quad A \xrightarrow{\pi} (A^\circ)^\star \xrightarrow{g^\star} C^\star. \end{aligned}$$

From the definition of Ψ and Φ, it can be straightforwardly shown that they are inverses of each other, that is $\Psi \circ \Phi = I_{\mathbf{Coalg}(C,A^\circ)}$ and $\Phi \circ \Psi = I_{\mathbf{Alg}(A,C^\star)}$.

Remark 2.1.35.
(1) The natural map $i : C \longrightarrow C^{\star\star}$ has its image in $C^{\star\circ}$ because the left module action of $C^\star$ on C defined by $c^\star \cdot c = c_{(1)}c^\star(c_{(2)})$ is finite dimensional and generated by the $c_{(1)}$ and by Proposition (6.0.3) of [47], the conclusion follows.

(2) The injection map $A^\circ \hookrightarrow A^\star$ induces the surjective map $A^{\star\star} \longrightarrow A^{\circ\star}$ which in turn yields $\pi : A \subseteq A^{\star\star} \longrightarrow A^{\circ\star}$.

There is also a very nice correspondence between the subobjects of the categories **Alg** and **Coalg** which we state in the following result

Proposition 2.1.36. *[44, Proposition 2.6.4] Let A be a k-algebra. Then,*

(1) If a subspace $I \subseteq A$ is a subalgebra (resp. ideal, left ideal, right ideal) of A, then $I^\perp \cap A^\circ$ is a coideal (resp. subcoalgebra, left coideal, right coideal) of A°.

(2) If a subspace U of A° is a coideal (resp. subcoalgebra, left coideal, right coideal) of A°, then $U^\perp$ is a subalgebra (resp. ideal, left ideal, right ideal) of A.

2.2 Bialgebras and Hopf algebras

2.2.1 Definitions and examples

In the previous sections, we gave definitions and some examples of algebras and coalgebras. Here, we want to discuss objects that have both of these structures - bialgebras and Hopf algebras.

Definition 2.2.1. A *bialgebra* over a field k is a tuple $(B, m, \Delta, \eta, \varepsilon)$ comprising an algebra structure (B, m, η) and a coalgebra structure (B, Δ, ε) such that either of the following is satisfied

(1) m and η are coalgebra maps,

(2) Δ and ε are algebra maps.

The two axioms stated in the above definition are equivalent for instance, if Δ is an algebra map then it follows that $\Delta_B \circ m = (m \otimes m) \circ \Delta_{B\otimes B}$ which means that $m : B \otimes B \longrightarrow B$ is a morphism of k-coalgebras.

In practice, bialgebras are often constructed as quotients of free bialgebras on coalgebras. This process is an extension of the free algebra construction on sets or vector spaces and the universal mapping property of free algebras to bialgebras. Let (C, δ, e) be a k-coalgebra, then define a free algebra $(T(C), \iota)$ on the underlying vector space of C, where $i : C \longrightarrow T(C)$ is an injective linear map, recall that $T(C) = \bigoplus_n C^{\otimes n}$. By the universal property of free algebras, the linear maps $(\iota \otimes \iota) \circ \delta : C \longrightarrow T(C) \otimes T(C)$ and $e \colon C \longrightarrow k$ yield algebra morphisms

$$\Delta : T(C) \longrightarrow T(C) \otimes T(C), \quad \iota(c) \mapsto (\iota \otimes \iota) \circ \delta(c),$$

respectively,

$$\varepsilon : T(C) \longrightarrow k \quad \iota(c) \mapsto e(c).$$

It is easy to check that $(\Delta \otimes \mathrm{id}_{T(C)}) \circ \Delta = (\mathrm{id}_{T(C)} \otimes \Delta) \circ \Delta$ and $(\varepsilon \otimes \mathrm{id}_{T(C)}) \circ \Delta = \mathrm{id}_{T(C)}$, thereby making $(T(C), \Delta, \varepsilon)$ a coalgebra and hence a bialgebra. The analogue of the universal property of free algebras for free bialgebras is given as follows: suppose $f : C \longrightarrow A$ is a coalgebra map from coalgebra C to bialgebra A, then there is a unique map of bialgebras $F : T(C) \longrightarrow A$ such that $F \circ \iota = f$.

Definition 2.2.2. Let C be a k-coalgebra, then the *free bialgebra* on C is the pair $(T(C), \iota)$ described above.

$(T(C), \iota)$ is also called the *tensor bialgebra* on the coalgebra C, and the following result shows that every bialgebra B is a quotient of the free bialgebra $T(C)$ on any coalgebra C that generates B as an algebra.

In particular, taking $C = B$ yields:

Theorem 2.2.3. *[44, Corollary 5.3.3] Let B be a k-bialgebra, then there exists a k-coalgebra C and a surjective bialgebra map $T(C) \longrightarrow B$.*

We shall see the full application of this result in Section 4.2.

Definition 2.2.4. Let $(B, m, \Delta, \eta, \varepsilon)$ be a k-bialgebra, an *antipode* S on B is a k-linear map $S : B \longrightarrow B$ satisfying the relation

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta. \tag{2.2.1}$$

To understand the role of the antipode better, we will describe the convolution algebra $\mathrm{Hom}(C, A)$ of a coalgebra C and an algebra A.

Definition 2.2.5. The *convolution algebra* $\mathrm{Hom}(C, A)$ of a coalgebra C and an algebra A is the algebra of all k-linear maps $f : C \longrightarrow A$ with multiplication defined as

$$f \bullet g = m \circ (f \otimes g) \circ \Delta$$

for all $f, g \in \mathrm{Hom}(C, A)$ and the unit element is $\eta \circ \varepsilon$.

It thus follows that the antipode equation (2.2.1) means that S is the inverse of the identity map id in the convolution algebra $\mathrm{End}(B)$, since the antipode equation can be written as $S \bullet \mathrm{id} = \eta \circ \varepsilon = \mathrm{id} \bullet S$ in the convolution algebra of B.

Remark 2.2.6. The antipode S is a k-linear map but it is anti-multiplicative and is an anti coalgebra map that is for all $a, b \in B$,

$$S(ab) = S(b)S(a), \quad \Delta \circ S = \tau \circ (S \otimes S) \circ \Delta.$$

In other words, the antipode is an algebra map $S : B \longrightarrow B^{\mathrm{op}}$ from B to B^{op}, its opposite bialgebra (has an underlying opposite algebra structure) and is also a coalgebra map $B \longrightarrow B^{\mathrm{cop}}$ from B to its opposite bialgebra(has an underlying opposite coalgebra structure).

Definition 2.2.7. A *Hopf* algebra H is a bialgebra $(H, m, \Delta, \eta, \varepsilon)$ equipped with a linear map S satisfying (2.2.1).

Example 2.2.8. (1) The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ in Example 2.1.18 is a Hopf algebra: recall from Example 2.1.18 that Δ and ε are algebra maps by the universality of $U(\mathfrak{g})$ and they extend from $jmath(\mathfrak{g})$ to the entire algebra $U(\mathfrak{g})$. A direct substitution into the antipode equation yields $S(x) = -x$.

(2) The group algebra kG described in (2.1.4) is a Hopf algebra since for all $g, h \in G$, the coproduct and counit are algebra maps

$$\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta(g)\Delta(h)$$
$$\varepsilon(gh) = 1 = \varepsilon(g)\varepsilon(h),$$

it follows from the antipode equation, $S(g) = g^{-1}$.

(3) Sweedler's Hopf algebra: This is a 4-dimensional Hopf algebra H with basis $\{1, g, x, xg\}$. H is generated as an algebra by x, g satisfying the relations

$$g^2 = 1, \quad x^2 = 0, \quad gx = -xg,$$

with Hopf algebra structure given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g, \quad \varepsilon(g) = 1,$$
$$\varepsilon(x) = 0 \quad S(g) = g^{-1} = g, \quad S(x) = -xg.$$

Moreover, since the simple subcoalgebras are of the form kg, for $g \in G(H)$ hence, they are 1-dimensional and thus H is pointed.

(4) A generalization of the Sweedler's Hopf algebra is the class $H_{n,q}$ of n-dimensional Hopf algebras called Taft's algebras with basis $\{a^i x^j : 0 \leq i, j < n\}$ and generated by grouplike element a and $(1, a)$-twisted primitive element x satisfying the relations

$$a^n = 1, \quad x^n = 0, \quad ax = qxa,$$

where $q^n = 1$ and $q \in k$ and with antipode given by $S(a) = a^{-1},\ S(x) = -xa^{-1}$.

In the following we give definitions of subobjects in the categories of bialgebras and Hopf algebras over a field k.

Definition 2.2.9. (1) A *Hopf subalgebra* $B \subseteq H$ is a subbialgebra of H that is invariant with respect to the antipode, that is $S(B) \subseteq B$.

(2) A *Hopf ideal* is a subspace $I \subseteq H$ which is a *biideal* – an ideal that is also a coideal – such that $S(I) \subseteq I$.

(3) Given a Hopf ideal I of H, we define the *quotient Hopf algebra* as the algebra H/I with Hopf structure inherited from H.

A set X is said to generate an algebra A over a field k if every element $a \in A$ can be expressed as sum of words in the alphabet X. In other words, $A = kX$ is the free algebra on the set X. This construction extends to the construction of free bialgebras $T(C)$ on coalgebras C (where C is viewed as a vector space that generates $T(C)$) as in Definition 2.2.2.

Following [44, section 7.5], we want to briefly discuss the construction of free Hopf algebras on coalgebras. We do this in three steps, first we explain what it means for a coalgebra to generate a Hopf algebra as an algebra, then we equip the free bialgebra constructed above with an antipode and then we conclude.

Suppose C is a coalgebra which generates a Hopf subalgebra $B \subseteq H$ as an algebra, then $\mathcal{C} = C + S(C) + S^2(C) + \cdots$ is a subcoalgebra of B since the sum of coalgebras is again a subcoalgebra and in fact $S(\mathcal{C}) \subseteq \mathcal{C}$. Thus, the subalgebra that $\mathcal{C}$ generates is a sub-bialgebra of B that is invariant under S and contains C. Hence, C generates H as a Hopf algebra if $\mathcal{C}$ generates B as an algebra and note that $S|_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{cop}}$ is a coalgebra map.

Let $(T(C), \iota)$ be the free bialgebra on the coalgebra C, and $\zeta : C \longrightarrow C^{\mathrm{cop}}$ be a coalgebra map. Then, the composite

$$\iota \circ \zeta : C \longrightarrow T(C)^{\mathrm{opcop}}$$

is a coalgebra map (note that we need $^{\mathrm{op}}$ for the composite to be a coalgebra map) and by the universality of $T(C)$, we have a bialgebra map $\mathsf{S} : T(C) \longrightarrow T(C)^{\mathrm{opcop}}$ such that $\mathsf{S} \circ \iota = \iota \circ \zeta$ (that is $\mathsf{S}(\iota(C)) \subseteq \mathrm{im}(\iota)$ meaning $\mathrm{im}(\iota)$ is invariant under S). We now equip $T(C)$ with an antipode by setting $I \subseteq T(C)$ to be the ideal generated by the elements

$$\mathsf{S}(c_{(1)})c_{(2)} - \varepsilon(c)1 \qquad c_{(1)}\mathsf{S}(c_{(2)}) - \varepsilon(c)1,$$

for all $c \in \iota(C)$. I is a coideal: using the fact that S is a bialgebra map, we obtain

$$\varepsilon(\mathsf{S}(c_{(1)})c_{(2)} - \varepsilon(c)1) = \varepsilon \otimes \varepsilon(\iota(c_{(2)}) \otimes c_{(1)}) - \varepsilon(c) = \varepsilon(c_{(2)}\varepsilon(c_{(1)})) - \varepsilon(c) = 0,$$

and setting $u(c) := \mathsf{S}(c_{(1)})c_{(2)} - \varepsilon(c)1$, we have

$$\Delta(u(c)) = u(c) \otimes \mathsf{S}(c_{(1)})c_{(2)} + 1 \otimes u(c).$$

Hence I is a bi-ideal and in fact $\mathsf{S}(I) \subseteq I$ since $\iota(C)$ is S-invariant. Thus $T(C)/I$ is bialgebra and S lifts to the bialgebra map

$$S : T(C)/I \longrightarrow T(C)^{\mathrm{opcop}}/I$$

via the projection $\pi : T(C) \longrightarrow T(C)/I$. By definition, $H_\zeta(C) = T(C)/I$ is a Hopf algebra with antipode S (the elements in I vanish in $H_\zeta(C)$ thus satisfying the antipode equation (2.2.1)).

Theorem 2.2.10. *[44, Theorem 7.5.2] Let C be a k-coalgebra, then there exists the pair $(j, H(C))$ such that*

(a) $H(C)$ is a k-Hopf algebra, and $j : C \longrightarrow H(C)$ is a coalgebra map.

(b) Given any pair (f, A) of k-Hopf algebra A and a coalgebra map $f : C \longrightarrow A$, there exists a Hopf algebra map $F : H(C) \longrightarrow A$ such that $F \circ j = f$.

This theorem applies the above construction of the free Hopf algebras $(H_\zeta(C), \iota)$ to when C generates $H(C)$ as a Hopf algebra, that is construct the free Hopf algebra $(H_\zeta(\mathcal{C}), \iota)$ with $\mathcal{C} = C \oplus C^{\mathrm{cop}} \oplus C \oplus \cdots$, and take ζ to be the injective coalgebra map

$$\zeta : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{cop}} \quad (c_1, c_2, c_3, \cdots) \mapsto (0, c_1, c_2, \cdots),$$

so taking $H(C) = H_\zeta(\mathcal{C})$ and $j = \iota|_C$, gives the *free Hopf algebra* on the coalgebra C. We will see an application of this construction in section 4.1, where we constructed a free Hopf algebra acting on a commutative algebra.

2.2.2 The dual Hopf algebra

Let $(A, m, \eta, \Delta, \varepsilon)$ be a k-bialgebra. Then, the underlying algebra structure (A, m, η) of the bialgebra A defines the dual coalgebra $(A^\circ, m^\star|_{A^\circ}, \eta^\star|_{A^\circ})$, and likewise the underlying coalgebra structure (A, Δ, ε) defines the dual algebra $(A^\star, \Delta^\star, \varepsilon^\star)$. Knowing that both the coalgebra and algebra of a bialgebra have the same underlying vector space, a natural question to ask: is $A^\circ \subseteq A^\star$ a subalgebra?
The answer to this question is in the affirmative as detailed in the following result

Proposition 2.2.11. *[44, Proposition 5.2.1] Let A be a bialgebra over the field k. Then, $A^\circ \subseteq A^\star$ is a subalgebra. In addition, A° is a bialgebra with the dual coalgebra structure and the subalgebra structure of the dual algebra $A^\star$.*

Definition 2.2.12. Let A be a k-bialgebra, the *dual bialgebra* of A is the dual coalgebra A° together with the algebra structure inherited as a subalgebra of $A^\star$.

Let H be a k-Hopf algebra, then the dual Hopf algebra as we expect is the dual bialgebra H° equipped with an antipode which should be the transpose of the antipode S on H, that is $S^\star : H^\star \longrightarrow H^\star$, $f \mapsto S^\star(f)(h) = f(S(h))$ and we want that $S^\star(H^\circ) \subseteq H^\circ$.

Theorem 2.2.13. *[44, Theorem 7.4.1] Let H be a Hopf algebra over the field k with antipode S. Then, $S^\star(H^\circ) \subseteq H^\circ$ and $S^\star|_{H^\circ}$ is an antipode for H°.*

Indeed $S^\star$ is an antipode for H° since it satisfies the antipode equation:

$$\begin{aligned} m_{H^\circ} \circ (S^\star \otimes id) \circ \Delta_{H^\circ} &= \Delta_H^\star \circ (S^\star \otimes id) \circ m_H^\star \\ &= (m_H \circ (S \otimes id) \circ \Delta_H)^\star = (\eta_H \circ \varepsilon_H)^\star = \eta_{H^\circ} \circ \varepsilon_{H^\circ} \\ &= (m_H \circ (id \otimes S) \circ \Delta_H)^\star = m_{H^\circ} \circ (id \otimes S^\star) \circ \Delta_{H^\circ}. \end{aligned}$$

Definition 2.2.14. Let H be a Hopf algebra over the field k with antipode S. Then, the *dual Hopf algebra* is the dual bialgebra H° together with the antipode $S^\star|_{H^\circ}$.

The notion of dualization of Hopf algebra structures can be generalized in the following way

Definition 2.2.15. Let $\mathcal{U}$ and H be two Hopf algebras over the field k. A *dual pairing* between $\mathcal{U}$ and H is a bilinear map

$$\langle \cdot, \cdot \rangle : \mathcal{U} \times H \longrightarrow k$$

such that for all $f, f_1, f_2 \in \mathcal{U}$ and $h, h_1, h_2 \in H$, the following axioms are satisfied

(i) $\langle \Delta_{\mathcal{U}}(f),\ h_1 \otimes h_2 \rangle = \langle f,\ h_1 h_2 \rangle, \quad \langle f_1 f_2,\ h \rangle = \langle f_1 \otimes f_2,\ \Delta_H(h) \rangle.$

(ii) $\langle f,\ 1_H \rangle = \varepsilon_{\mathcal{U}}(f), \quad \langle 1_{\mathcal{U}},\ h \rangle = \varepsilon_H(h).$

(iii) $\langle S_{\mathcal{U}}(f),\ h \rangle = \langle f,\ S_H(h) \rangle.$

Definition 2.2.16. A dual pairing of Hopf algebras $\mathcal{U}$ and H is called *nondegenerate* if $\langle f, h \rangle = 0$ for all $f \in \mathcal{U}$ then $h = 0$, and if $\langle f, h \rangle = 0$ for all $h \in H$ then $f = 0$.

Remark 2.2.17. (1) Indeed the dual Hopf algebra H° discussed above has a nondegenerate dual pairing with H with the pairing $\langle h^\circ, h \rangle = h^\circ(h)$. In addition, there is a canonical nondegenerate dual pairing between any two Hopf algebras $\mathcal{U}$ and H given by $\langle f, h \rangle = \varepsilon_{\mathcal{U}}(f)\varepsilon_H(h)$ for all $f \in \mathcal{U}$ and $h \in H$.

(2) New nondegenerate dual pairing can be obtained from known nondegenerate dual pairings via compositions with Hopf algebra morphisms in the two variables. Suppose $\langle \cdot, \cdot \rangle$ is a dual pairing of $\mathcal{U}$ and H and $\alpha : \mathcal{U}' \longrightarrow \mathcal{U}$ and $\beta : H' \longrightarrow H$ are Hopf algebra morphisms, then the composition $\langle \cdot, \cdot \rangle \circ (\alpha \otimes \beta)$ is a dual pairing of $\mathcal{U}'$ and H'.

(3) The axioms of a dual pairing of Hopf algebras reveal information about the links between the two structures. For example, axiom (i) reveals that $\mathcal{U}$ is cocommutative if and only if H is commutative and vice versa: suppose $\mathcal{U}$ is cocommutative, then

$$\langle f, h_1 h_2 \rangle = \langle \Delta_{\mathcal{U}}(f), h_1 \otimes h_2 \rangle = \langle \Delta_{\mathcal{U}}^{cop}(f), h_1 \otimes h_2 \rangle = \langle f, h_2 h_1 \rangle.$$

On the other hand, if $\mathcal{U}$ is commutative, then

$$\begin{aligned} \langle f_1 \otimes f_2, \Delta_H(h) \rangle &= \langle f_1 f_2, h \rangle = \langle f_2 f_1, h \rangle = \langle f_2 \otimes f_1, \Delta_H(h) \rangle \\ &= f_2(h_{(1)}) f_1(h_{(2)}) = f_1(h_{(2)}) f_2(h_{(1)}) \\ &= \langle f_1 \otimes f_2, \Delta_H^{cop}(h) \rangle. \end{aligned}$$

2.2.3 Hopf *-algebras

Definition 2.2.18. Assume that k is a field with a chosen involutive field automorphism that we denote by $\lambda \mapsto \bar{\lambda}$.

1. For each k-vector space V, we denote by $\bar{V}$ the *conjugate* vector space which is the same abelian group but whose scalar multiplication is twisted by $\bar{\cdot}$ that is, $\lambda \cdot_{\bar{V}} v := \bar{\lambda} v$.

2. A $*$-*structure* on a k-algebra B is an involutive k-algebra isomorphism
$$*\colon B \to \bar{B}^{\mathrm{op}}.$$

3. An *involution* on a k-algebra P is an involutive k-algebra isomorphism
$$\theta\colon P \to \bar{P}.$$

So explicitly, an involution $\theta\colon P \to P$ satisfies for all $\lambda \in k, f, g \in P$

$$\theta(\lambda f + g) = \bar{\lambda}\theta(f) + \theta(g), \quad \theta(fg) = \theta(f)\theta(g), \quad \theta(\theta(f)) = f.$$

Typically, these notions are considered for $k = \mathbb{C}$ with involution given by complex conjugation, see e.g. [31, Section 1.2.7].

On Hopf algebras, one demands the following compatibility between $*$-structures and involutions with the coalgebra structure:

Definition 2.2.19. Let H be a Hopf algebra.

1. A *Hopf $*$-structure* on H is a $*$-structure on the underlying algebra satisfying for all $h \in H$

$$(h^*)_{(1)} \otimes (h^*)_{(2)} = (h_{(1)})^* \otimes (h_{(2)})^*, \quad \varepsilon(h^*) = \overline{\varepsilon(h)}.$$

   In other words, the coalgebra structure maps are $*$-morphisms.

2. A *Cartan involution* on H is an involution θ on the underlying algebra such that for all $h \in H$, we have

$$\theta(h)_{(1)} \otimes \theta(h)_{(2)} = \theta(h_{(2)}) \otimes \theta(h_{(1)}), \quad \varepsilon(\theta(h)) = \overline{\varepsilon(h)}.$$

We remark that in any Hopf $*$-algebra with antipode S, we have

$$S \circ * \circ S \circ * = \mathrm{id}_H,$$

that is S is invertible with inverse $* \circ S \circ *$. One efficient way to show this is to verify that $* \circ S \circ *$ is an antipode for H^{op}, hence as antipodes are unique, $* \circ S \circ * = S^{-1}$. Furthermore, one defines a $*$-structure on the dual Hopf algebra H° as

$$f^*(h) = \overline{f(S(h)^*)},$$

for $f \in H^\circ,\ h \in H$, which in the language of dual pairing, can be re-written as

$$\langle f^*, h \rangle = \overline{\langle f, S(h)^* \rangle}.$$

These structures - $*$-structure and Cartan involution - correspond bijectively to each other:

Lemma 2.2.20. *A map $*\colon H \to H$ is a Hopf $*$-structure if and only if $* \circ S$ is a Cartan involution.*

Proof. Since $*$ is a $*$-structure on H, then the anti-linearity of $*$, the anti-colinearity of the antipode S proves that $* \circ S$ is anti-colinear and anti-linear. The relation $* \circ S \circ * \circ S = \mathrm{id}$ proves that $* \circ S$ is involutive hence, $* \circ S$ is a Cartan involution. Conversely, suppose that $* \circ S$ is a Cartan involution then, for all $h \in H$, we have $h^{**} = S(S(h)^*)^* = h$, that is $*$ is involutive. The compatibility of $*$ with the coalgebra structure on H follows from the properties of the Cartan involution property of $* \circ S$:

$$\begin{aligned}\Delta(h^*) &= \Delta(S(S(h)^*)) = \tau \circ (S \otimes S)(S(h)^*_{(1)} \otimes S(h)^*_{(2)}) \\ &= \tau \circ (S \otimes S)(S(h_{(2)})^* \otimes S(h_{(1)})^*) = S(S(h_{(1)})^*) \otimes S(S(h_{(2)})^*) \\ &= (h_{(1)})^* \otimes (h_{(2)})^*,\end{aligned}$$

and $\varepsilon(h^*) = \varepsilon(S(S(h)^*) = \varepsilon(S(h)^*) = \overline{\varepsilon(h)}$. ∎

2.3 Representations of Hopf algebras

2.3.1 Definitions and examples

Definition 2.3.1. Let A be an algebra and V be a vector space over a field k. A *representation* of A on V is the algebra map

$$\varphi : A \longrightarrow End_k(V),\ a \mapsto \varphi_a.$$

In other words, that φ is an algebra map means for all $a, b \in A$ and $\alpha, \beta \in k$, $\varphi_{\alpha a+\beta b} = \alpha\varphi_a + \beta\varphi_b, \quad \varphi_{ab} = \varphi_a\varphi_b, \quad$ and $\varphi_{1_A} = \mathrm{id}_V$.

Definition 2.3.2. A *left A-module* is a vector space V together with a linear map

$$\varphi : A \otimes V \longrightarrow V,\ a \otimes v \mapsto \varphi(a \otimes v) := a \cdot v$$

such that $\varphi \circ (id \otimes \varphi) = \varphi \circ (m \otimes id)$, and $\varphi \circ (\eta \otimes id) = id$.

The equations in the definition above mean for all $a, b \in A$ $v \in V$ we have $ab \cdot v = a \cdot (b \cdot v)$ and $1_A \cdot v = v$. A *right A-module* is defined similarly only that the action is on the right and the equations are now $\varphi \circ (\varphi \otimes id) = \varphi \circ (id \otimes m)$ and $\varphi \circ (id \otimes \eta) = id$. In practice, we use representations and modules of a k-algebra A interchangeably because there is a correspondence between the set of representations of A and A-modules given by

$$\varphi_a(v) \mapsto \varphi(a \otimes v) = a \cdot v.$$

Let φ be a representation of A on a vector space V, then it defines a left (resp. right) A-module structure on V

$$\varphi : A \otimes V \longrightarrow V, \quad (a, v) \mapsto \varphi(a \otimes v) = \varphi_a(v),$$

the module axioms follow since φ_a is an algebra map. On the other hand, suppose (V, φ) is a left (resp. right) A-module, then the module action φ defines an algebra map

$$\varphi : A \longrightarrow End_k(V), \quad a \mapsto \varphi_a(v) = a \cdot v.$$

Definition 2.3.3. Let A be an algebra over k, then

(1) A k-vector space V is an *A-bimodule* if it is both a left and a right A-module and if in addition $(a \cdot v) \cdot b = a \cdot (v \cdot b)$, $\forall\, a, b \in A,\ v \in V$.

(2) Let V and W be two representations of A, then *a morphism of representations* or *left A-module map* is a linear map $\theta : V \longrightarrow W$ such that $\forall\ a \in A,\ v \in V,\ w \in W \quad \theta(a \cdot v) = a \cdot \theta(v)$, that is θ is A-linear.

If A is a bialgebra, then we can obtain new representations from previously known ones, in particular we can tensor two representations for instance. Let (V, φ_V) and (W, φ_W) be two representations of A then the tensor product $\varphi_V \otimes \varphi_W = \varphi_{V\otimes W}$ is the representation of A on the vector space $V \otimes W$ defined as

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\varphi_V \otimes \varphi_W} End_k(V) \otimes End_k(W)$$
$$a \mapsto a_{(1)} \otimes a_{(2)} \mapsto \varphi_V(a_{(1)})\varphi_W(a_{(2)}),$$

that is for all $a \in A,\ v \in V,\ w \in W$

$$a \cdot (v \otimes w) = \varphi_{V\otimes W}(a) = \varphi_V(a_{(1)})\varphi_W(a_{(2)}) = (a_{(1)} \cdot v)(a_{(2)} \cdot w),$$

the module axioms are satisfied since φ_V and φ_W are module structure maps.

Dually, we define comodules and corepresentations of coalgebras by dualizing the definitions of modules and representations of an algebra.

Definition 2.3.4. Let C be a coalgebra, then a *corepresentation* or *left C-comodule* is a vectorspace V together with a linear map

$$\psi : V \longrightarrow C \otimes V,\ v \mapsto v_{(-1)} \otimes v_{(0)}$$

such that $(id \otimes \psi) \circ \psi = (\Delta \otimes id) \circ \psi$, and $(\varepsilon \otimes id) \circ \psi = id$.

Similarly, the equations in the definition of a corepresentation of coalgebra C mean for all $v \in V$,

$$v_{(-1)} \otimes v_{(0)_{(-1)}} \otimes v_{(0)_{(0)}} = v_{(-1)_{(1)}} \otimes v_{(-1)_{(2)}} \otimes v_{(0)}$$

which is jointly written as

$$v_{(-2)} \otimes v_{(-1)} \otimes v_{(0)}$$

respectively $\varepsilon(v_{(-1)})v_{(0)} = v$. Note that the elements with negative indices lies in C. In similar way, we define a *right C-comodule* as a k-vector space V together with the linear map $\psi : V \longrightarrow V \otimes C,\ v \mapsto v_{(0)} \otimes v_{(1)}$ such that $(\psi \otimes id) \circ \psi = (id \otimes \Delta) \circ \psi$ and $(id \otimes \varepsilon) \circ \psi = id$.

Definition 2.3.5. Let (V, ψ_V) and (W, ψ_W) be two left C-comodules, then a *map of left C-comodules* is a linear map $\alpha : V \longrightarrow W$ such that for all $v \in V$, $\psi_W \circ \alpha = (id \otimes \alpha) \circ \psi_V$, that is $\alpha(v)_{(-1)} \otimes \alpha(v)_{(0)} = v_{(-1)} \otimes \alpha(v_{(0)})$. If α is bijective, then V and W are isomorphic as C-comodules.

Definition 2.3.6. Let A be an algebra, C a comodule and V a vector space over k, then

(1) Suppose V is a representation, then a *left (resp. right) submodule or subrepresentation* of V is a linear subspace $U \subseteq V$ such $a{\cdot}u$ (resp. $u{\cdot}a$) $\in U$ for all $u \in U$. If V is a corepresentation, then a *left (resp. right) C-subcomodule or sub corepresentation* of (V, ψ_V) is a subspace $U \subseteq V$ with $\psi(U) \subseteq C \otimes U$(resp. $U \otimes C$).

(2) A left (resp. right) (co)module (V, ψ_V) is called *simple* if V has no non-trivial sub(co)module, that is the only sub(co)modules of V are $\{0\}$ and V, and ψ_V is called an *irreducible* (co)representation.

Example 2.3.7. (1) The *trivial corepresentation* is defined by the k-linear map

$$\psi : V \longrightarrow C \otimes V \quad v \mapsto 1 \otimes v,$$

for all $v \in V$. The comodule axioms are trivially satisfied thus, any vector space V can be made into a left C-comodule via ψ.

(2) The coproduct $\Delta : C \longrightarrow C \otimes C$ of the coalgebra C defines both a left and right comodule action on C.

In what follows is a characterization of finite dimensional corepresentations in terms of matrices with entries in the coalgebra C. A corepresentation V of C is finite-dimensional if V has a vector space basis.

Proposition 2.3.8. *[31, Proposition 13] Let V be a finite dimensional vector space with basis $e_1, e_2, \cdots, e_d$, $\psi : V \longrightarrow C \otimes V$ be a linear map, and $c = (c_{ij})$ be the $d \times d$ matrix of elements of C such that*

$$\psi(e_i) = \sum_{k=1}^{d} c_{ik} \otimes e_k, \qquad i = 1, 2, \cdots, d$$

then ψ is a corepresentation of C if and only if for all $i, j = 1, 2, \cdots, d$

$$\Delta(c_{ij}) = \sum_{k=1}^{d} c_{ik} \otimes c_{kj}, \quad \varepsilon(c_{ij}) = \delta_{ij}. \tag{2.3.1}$$

Notice that the formula of the coproduct and counit in the above proposition are similar to those of the matrix coalgebra in Example 2.1.33. ψ is a corepresentation of C on V if and only if the comodule axioms

$$\sum_{j,k=1}^{d} c_{ik} \otimes c_{kj} \otimes e_j = \sum_{j=1}^{d} \Delta(c_{ij}) \otimes e_j, \qquad \sum_{i=1}^{d} \varepsilon(c_{ij}) e_j = e_i,$$

are satisfied $\iff$ $\Delta(c_{ij}) = \sum_{j,k=1}^{d} c_{ik} \otimes c_{kj}$ and $\varepsilon(c_{ij}) = \delta_{ij}$. The matrix $(c_{ij}) \in M_d(C)$ is called a *matrix corepresentation* of C.

Definition 2.3.9. Let H be a Hopf algebra. A *left (resp. right) H-(co)module* is a vector space V that is a left (resp. right) (co)module with respect to the (co)algebra structure of H.

It is possible for a vector space V to be both a left and a right H-(co)module. If this is the case, V is called a *bi(co)module* over H: suppose V is a right H-comodule via φ then

$$\psi : V \xrightarrow{\varphi} V \otimes H \xrightarrow{id \otimes S} V \otimes H \xrightarrow{\tau} H \otimes V$$

defines a left H-comodule structure on V.

2.3.2 Hopf (co)module (co)algebras

A k-Hopf algebra H can act (as in modules) and coact (as in comodules) on algebras and coalgebras as they do on vectorspaces. However, it is required that the (co)module (co)action be compatible with the underlying algebra and coalgebra structures. Before we give definitions of (co)modules (co)algebras, we recall from the comment after Definition 2.3.3 that if A is a left H-module then $A \otimes A$ is again a left H-module with module structure map $\varphi_{A\otimes A}$. Similarly, one can describe the tensor products of two corepresentations: Let (C, ψ_C) and (D, ψ_D) be two right H-comodules, then the tensor product corepresentation $\psi_{C\otimes D}$ of H on the vector space $C \otimes D$ is defined as the composite

$$C \otimes D \xrightarrow{\psi_C \otimes \psi_D} (C \otimes H) \otimes (D \otimes H) \xrightarrow{id_C \otimes \tau \otimes id_H} C \otimes D \otimes H \otimes H \xrightarrow{id_C \otimes id_D \otimes m} C \otimes D \otimes H$$

that is

$$\psi_{C\otimes D}(c \otimes d) = c_{(0)} \otimes d_{(0)} \otimes c_{(1)} d_{(1)}.$$

Thus, if C is a coalgebra and also a right H-comodule, then the tensor product $C \otimes C$ is again a right H-comodule and the comodule structure maps $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow k$ are morphisms of right H-comodules.

Definition 2.3.10. Let H be a Hopf algebra, then a *left (resp. right) H-(co)module algebra* is an algebra (A, m, η) which is a left (resp. right) H-(co)module such that the algebra structure maps m, η are maps of H-(co)modules.

Definition 2.3.11. A *left (resp. right) H-(co)module coalgebra* is a coalgebra (C, Δ, ε) which is a left (resp. right) H-(co)module and whose coalgebra structure maps Δ, ε are morphisms of H-(co)modules.

Remark 2.3.12. (1) The underlying field k is a H-module and H-comodule via the trivial action given by $H \otimes k \longrightarrow k,\ (h, \lambda) \mapsto \varepsilon(h)\lambda$ and coaction $k \longrightarrow k \otimes H,\ \lambda \mapsto \lambda \otimes 1$.

(2) The requirement in the definitions above that the algebra structure maps m, η be H-module maps means that for all $h \in H,\ a, b \in A$

$$h \cdot ab = (h_{(1)} \cdot a)(h_{(2)} \cdot b), \quad \text{and} \quad h \cdot 1_k = \varepsilon(h) 1_A,$$

and as H-comodule maps means the coaction ψ_A of H on algebra A satisfies

$$\psi_A(ab) = \psi_A(a)\psi_A(b), \quad \text{and} \quad \psi_A(1) = 1 \otimes 1,$$

that is the comodule structure map ψ_A must be an algebra map. Similarly the requirement that the coalgebra structure maps Δ, ε be H-module maps means for all $c \in C,\ h \in H$

$$\Delta(c \cdot h) = c_{(1)} \cdot h_{(1)} \otimes c_{(2)} \cdot h_{(2)}, \quad \text{and} \quad \varepsilon(c \cdot h) = \varepsilon(c)\varepsilon(h).$$

In [31], comodule algebras A over Hopf algebras H were called quantum spaces and as we will see in Chapter 3, H becomes the coordinate ring of some "quantum group" having satisfied some axioms. If H is a Hopf $*$-algebra, and A has a $*$-algebra structure, then A is a $*$-quantum space – a comodule $*$-algebra over a Hopf $*$-algebra – and the comodule structure map ψ_A is a $*$- algebra morphism, that is $\psi_A(a^*) = \psi_A(a)^*$.

2.3.3 (Co)modules under dual pairing

As one would expect there are relations and correspondences between the representations and corepresentations of two dually paired Hopf algebras $\mathcal{U}$ and H. It is easy to see that a corepresentation of H via the dual pairing $\langle -, - \rangle$ is a representation of $\mathcal{U}$: suppose (V, ψ) is a right H-comodule then the composite

$$\begin{gathered} \mathcal{U} \otimes V \xrightarrow{id_{\mathcal{U}} \otimes \psi} \mathcal{U} \otimes V \otimes H \xrightarrow{\tau \otimes \mathrm{id}_H} V \otimes \mathcal{U} \otimes H \xrightarrow{\mathrm{id}_V \otimes \langle -, - \rangle} V \\ u \otimes v \mapsto u \otimes v_{(0)} \otimes v_{(1)} \otimes u \mapsto v_{(0)} \langle u, v_{(1)} \rangle \end{gathered}$$

defines a left $\mathcal{U}$-module structure $\triangleright$ on V: since for $u, u' \in \mathcal{U},\ v \in V$ we have

$$\begin{aligned} uu' \triangleright v = v_{(0)} \langle uu', v_{(1)} \rangle &= v_{(0)} \langle u \otimes u', \Delta(v_{(1)}) \rangle = v_{(0)} \langle u, v_{(1)} \rangle \langle u', v_{(2)} \rangle \\ &= u \triangleright v_{(0)} \langle u', v_{(1)} \rangle = u \triangleright u' \triangleright v, \end{aligned}$$

and $1_{\mathcal{U}} \triangleright v = v_{(0)}\langle 1_{\mathcal{U}}, v_{(1)}\rangle = \varepsilon_H(v_{(1)})v_{(0)}$. Thus on any right H-comodule, the dual pairing induces a left $\mathcal{U}$-module structure. Similarly, one can show that left H-comodules are again right $\mathcal{U}$-modules and this prompts the question: is the converse true? In other words, do all left (resp. right) $\mathcal{U}$-modules arise as left (resp. right) H-comodules? The answer is no.

In particular when H is a coalgebra C and $\mathcal{U}$ is its dual algebra $C^\star$, there is a class of $C^\star$-modules which have certain properties that make them to arise from C-comodules. These $C^\star$-modules are called *rational modules* and they have the following property:

Let (V, φ) be a left $C^\star$-module, and define for all $c^\star \in C^\star,\ v \in V$ the map

$$\rho : V \longrightarrow Hom(C^\star, V), \quad v \mapsto \varphi_v,$$

where $\varphi_v(c^\star) = c^\star \triangleright v$. Then for elementary tensors $v \otimes c$, the linear map

$$V \otimes C \overset{\phi}{\hookrightarrow} Hom(C^\star, V) \quad v \otimes c \mapsto \langle -, c\rangle v$$

is a well defined embedding: suppose $(v \otimes c) \in ker\ \phi$, then for all $c^\star \in C^\star$ we have $0 = \phi(v \otimes c)(c^\star) = \langle c^\star, c\rangle v$ that is $\langle c^\star, c\rangle = 0$ for all $c^\star$ hence, $\langle\ ,\ \rangle$ is degenerate – a contradiction – thus ϕ is injective.

Definition 2.3.13. A left $C^\star$-module V is called *rational* if $\rho(V) \subseteq V \otimes C$.

In view of this, a rational left $C^\star$-module has a right C-comodule structure induced by ρ. Thus we have the correspondence

$$\{\text{Rational left } C^\star\text{-modules}\} \longleftrightarrow \{\text{Right } C\text{-comodules}\}.$$

For the purpose of this thesis, especially our discussion in Section 3.6 we state and prove the following useful results for which we have no references.

Proposition 2.3.14. *Let (M, φ) be a left finite dimensional module over an algebra A. Then $(M^\star, \varphi^T)$ is a left A°-comodule, where φ^T is the transpose of φ.*

Proof. Denote by $\{m_1, \cdots, m_d\}$ and $\{f^1, \cdots, f^d\}$ the basis respectively the dual basis of M. We claim that $A^\star \otimes M^\star \cong (A \otimes M)^\star$: every element in $A \otimes M$ and $A^\star \otimes M^\star$ can be uniquely written as $\sum_{i=1}^d x_i \otimes m_i$ respectively $\sum_{j=1}^d a_j \otimes f^j$. Then we define

$$\phi : A^\star \otimes M^\star \longrightarrow (A \otimes M)^\star, \quad \sum_{j=1}^d a_j \otimes f^j \mapsto a_j(x_j).$$

ϕ is injective since $a_j(x_j) = 0$ for all x_j implies $a_j = 0$. Given $\psi \in (A \otimes M)^\star$, define $a_j := \psi(-, m_i) \in A^\star$ then take $\psi = \sum_{j=1}^{d} a_j \otimes f^j$ and evaluating at $\sum_{i=1}^{d} x_i \otimes m_i$, we conclude that ϕ is surjective.

Consider the transpose $\varphi^T : M^\star \longrightarrow (A \otimes M)^\star \cong A^\star \otimes M^\star$ of the A-module structure map φ on M. We claim that the image $\varphi^T(M^\star) \subseteq A^\circ \otimes M^\star$. This follows by observing that for $f \in M^\star$, $\xi \in M^{\star\star} \cong M$ we have $(\mathrm{id} \otimes \xi) \circ \varphi^T(f) = f(-, \xi) \in A^\star$. $f(-, \xi)$ is the matrix coefficients of the representation on M and by [28, Corollary 1.4.5], it is in A°. ☐

This means given a finite dimensional representation of an algebra A, its dual gives a corepresentation of the dual coalgebra of A.

Corollary 2.3.15. *Let A be an algebra. Then the following are equivalent*

(1) A° is pointed.

(2) All the finite dimensional irreducible representations of A are 1*-dimensional.*

Proof. By Proposition 2.3.14, all the simple comodules of A° are 1-dimensional, hence by Remark 2.1.24 the conclusion follows. ☐

2.3.4 Inner faithful representations

Banica and Bichon in [4, construction 2.1] introduced the notion of *Hopf image* H_ρ of a representation of a Hopf algebra H on an algebra A. By a representation of a Hopf algebra H on A we mean an algebra map $\rho : H \longrightarrow A$.

Definition 2.3.16. A *factorization* of a representation ρ of H on A is the triple (G, f, s) where G is a Hopf algebra, $s : H \longrightarrow G$ is a surjective morphism of Hopf algebras and $f : G \longrightarrow A$ is a representation such that $\rho = f \circ s$.

In other words, the representation ρ factors through G. We can form a category of factorizations of ρ where the objects are triples (G, f, s) and morphism between two objects (G, f, s) and $(\tilde{G}, \tilde{f}, \tilde{s})$ is a morphism of Hopf algebras $\sigma : G \longrightarrow \tilde{G}$ such that $\tilde{f} = \sigma \circ f$ and $s = \tilde{s} \circ \sigma$.

Definition 2.3.17. The *Hopf image* H_ρ of a representation ρ of Hopf algebra H on algebra A is the factorization $(H_\rho, \theta, \tilde{\rho})$ such that for all objects (G, f, s) in the category of factorizations of ρ there exists a single morphism from G to H_ρ.

In the language of category theory, the Hopf image is the final (or terminal) object in the category of factorizations of ρ.

Theorem 2.3.18. *[4, Theorem 2.1] Let $\rho : H \longrightarrow A$ be a representation of a Hopf algebra H on algebra A. Then ρ has a Hopf image: there exists a triple $(H_\rho, \theta, \tilde{\rho})$ where H_ρ is a Hopf algebra, $\theta : H \longrightarrow H_\rho$ a surjective Hopf algebra map and $\tilde{\rho} : H_\rho \longrightarrow A$ is a representation such that $\rho = \tilde{\rho} \circ \theta$ and if (L, q, φ) is another factorization of ρ, then there exists a unique Hopf algebra map $f : L \longrightarrow H_\rho$ such that $f \circ q = \theta$ and $\tilde{\rho} \circ f = \varphi$.*

In classical representation theory, a representation $\pi : k(G) \longrightarrow A$ of the group algebra $k(G)$ on A is said to be *faithful* if the map π is injective. However, we can ask is the restriction

$$\pi|_G : G \longrightarrow A^\times$$

also faithful as a group homomorphism? So one can talk about the faithfulness of π in two ways depending on the object one is interested in. Observe that

Lemma 2.3.19. *$\pi|_G$ is faithful if and only if $\ker \pi$ does not contain any non-zero Hopf ideal.*

Proof. Suppose $\pi|_G$ is faithful, that is $\ker \pi|_G = \{e_G\}$ and suppose $0 \neq I \subseteq \ker \pi$ is a Hopf ideal. Then, for $s \in I$ we have

$$0 = \pi(s) = \pi\big(\sum_{g_\alpha \in G} \alpha\ g_\alpha\big) = \sum_{g_\alpha \in G} \alpha\ \pi|_G(g_\alpha)$$

this means $s = \alpha e_G$, hence a contradiction. Conversely, suppose $\ker \pi$ contains no non-zero Hopf ideal. Let $s \in \ker \pi|_G \subseteq \ker \pi$, then the two-sided ideal generated by $s - e_G$ is a non-zero Hopf ideal of $k[G]$ contained in $\ker \pi$. Thus by assumption $s = e_G$ and therefore $\ker \pi|_G$ is trivial. $\square$

This result leads to the notion of *inner faithful* representations

Definition 2.3.20. A representation $\pi : H \longrightarrow A$ of a Hopf algebra H on an algebra A is called an *inner faithful* representation if $\ker \pi$ does not contain a non-zero Hopf ideal.

This definition means that if the representation of H on A is inner faithful, then the Hopf image of π is H itself. In other words, aside the Hopf algebra H there is no minimal Hopf algebra representation on algebra A. Arguably, the notion of inner faithfulness is a more useful notion and less restrictive compared to faithfulness since one can pass uniquely to inner faithful actions by taking quotients by the Hopf ideals. We shall heavily apply this notion in Chapter 4.

Chapter 3

Quantum symmetries of coordinate rings

In this chapter, following [27] we give definitions and examples of affine algebraic sets as well as their coordinate rings. We introduce concepts and results such as faithful flatness which aids the study of quantum group actions on commutative algebras, which under some additional assumptions are said to equip these commutative domains with quantum homogeneous space structures. We also give a short literature review of studies conducted on quantum homogeneous spaces starting with the Poodleś' spheres [42] which are examples of quantum homogeneous spaces. Then we briefly discuss more recent studies such as [24, 32, 34, 33, 12] on quantum symmetries of coordinate rings of singular affine curves. We conclude the chapter with Sections 3.5.2 and 3.6 which contain the first contributions of this thesis – a detailed example of the coordinate ring of the cusp being equipped with such a quantum homogeneous space structure.

3.1 Coordinate rings

3.1.1 Algebraic sets

Definition 3.1.1. Let k be an algebraically closed field and $n \in \mathbb{Z}_{\geq 0}$. An *affine space* is the set

$$k^n = \{(a_1, \cdots, a_n) : a_i \in k,\ 1 \leq i \leq n\}$$

of all n-tuples of elements of k.

Elements $p = (a_1, \cdots, a_n)$ of k^n are called *points* and the a_i are called the *coordinates* of p. Let $A = k[x_1, \cdots, x_n]$ be a polynomial ring over k, we can think of polynomials $f \in A$ as functions $k^n \longrightarrow k$ defined by $f(p) = f(a_1, \cdots, a_n)$ for points p in the affine space.

Definition 3.1.2. Let $f \in A$ then a point $p \in k^n$ is called a *zero* of f if $f(p) = 0$. In particular the set $Z(f) = \{p \in k^n : f(p) = 0\}$ is called the *zero set* of f.

Thus in general, given a subset T of A the set $Z(T)$ consists of all points $p \in k^n$ on which every polynomial $f \in T$ vanishes. That is

$$Z(T) = \{p \in k^n : f(p) = 0 \; \forall \; f \in T\}$$

and it is a subspace of the affine space k^n.

Definition 3.1.3. A subspace C of k^n is called an *algebraic set* if $C = Z(T)$ for some subset T of A.

Example 3.1.4. (1) The field k is an algebraic set since it is the zero set of the zero polynomial. It is called the *affine line*.

(2) The set $V := \{(t^2, t^3) : t \in k\}$ is an algebraic set: it is the zero set of the plane curve $x^3 - y^2 \in k[x, y]$.

Note that A is a Noetherian ring thus every ideal is finitely generated. To every ideal I of A we can associate an algebraic set $Z(I)$ – the set of common zeros of the polynomials that generate I. On the other hand, we can associate to a subset of k^n an ideal of A

Definition 3.1.5. For any subset C of k^n define an *ideal of* C to be the set

$$I(C) = \{f \in A : f(p) = 0 \; \forall \; p \in C\}.$$

The ideal $I(C)$ of C is the set of all polynomials that vanish on C. This thus leads to a map between subsets of k^n and ideals of A

$$\{\text{Subsets of } k^n \} \longleftrightarrow \{\text{Ideals of } A\}$$
$$C \rightarrow I(C)$$
$$Z(I) \leftarrow I$$

This map is an inclusion reversing map: suppose $C_1 \subseteq C_2$ are subsets of k^n, then $I(C_2) \subseteq I(C_1)$ since all polynomials $f \in I(C_2)$ vanish on all of $C_2 \supset C_1$. Similarly for ideals $I_1 \subseteq I_2$ we have $Z(I_2) \subseteq Z(I_1)$. This mapping becomes a $1-1$ correspondence between algebraic sets in k^n and *radical* ideals of A which is a consequence of the Nullstellensatz's theorem:

Theorem 3.1.6. *(Hilbert's Nullstellensatz) Let k be an algebraically closed field. Let J be an ideal of A and $f \in A$ be a polynomial that vanish at all points of $Z(T)$. Then $f^r \in J$ for all $r > 0$.*

Definition 3.1.7. Let I be an ideal of A. The *radical* $\sqrt{I}$ of I is the set

$$\{f \in A : f^r \in I \; \forall \; r > 0\}.$$

An ideal is called a radical ideal if it is equal to its radical, that is $\sqrt{I} = I$. Another consequence of the Nullstellensatz's theorem is that the points $(a_1, \cdots, a_n)$ (which are minimal algebraic sets) of k^n correspond to maximal ideals $\langle x_1 - a_1, \cdots, x_n - a_n \rangle$ of A.

Definition 3.1.8. Let $C \in k^n$ be an algebraic set, the *coordinate ring* $O(C)$ of C is the quotient

$$k[x_1, \cdots, x_n]/I(C).$$

In other words, the coordinate ring $O(C)$ of the algebraic set C consists of all polynomials that do not vanish on C. Building on our thoughts about polynomials in A as functions on k^n, the polynomials in $O(C)$ are called *regular* functions on C.

Example 3.1.9. (1) The coordinate ring of the affine space k^n is the polynomial ring $k[x_1, \cdots, x_n]$.

(2) The ideal $I(V)$ of the algebraic set V in Example 3.1.4 is the ideal generated by the irreducible polynomial $x^3 = y^2$. Thus $O(V) = k[x, y]/\langle x^3 - y^2 \rangle$.

Definition 3.1.10. For any $f, g \in O(C)$ the quotient $\frac{f}{g} : C \longrightarrow k$ such that $g(c) \neq 0$ for all $c \in C$ is called a *rational function* on C. We denote by $k(C)$ the field of rational functions on C.

The commutative ring $O(C)$ equipped with pointwise multiplication becomes an algebra. It can also be equipped with a Hopf algebra structure: Let G be a group and also an algebraic set of k^n. Then the group structure maps multiplication and inversion

$$\mu : G \times G \longrightarrow G, \; (g_1, g_2) \mapsto g_1 g_2 \quad \iota : G \longrightarrow G, \; g \mapsto g^{-1}$$

are morphisms of algebraic sets (i.e are polynomial functions). Associated to G is the algebra of regular functions $O(G)$, with a commutative associative product given by pointwise multiplication: for all $g \in G, \; f, h \in O(G)$,

$$m(f \otimes h)(g) = fh(g) = f(g)h(g)$$

and with unit given by $\eta : k \longrightarrow O(G)$, where $1_{O(G)}$ is the regular function on G which assigns 1_k to all $g \in G$.

The product μ on G induces an algebra map

$$\Delta : O(G) \longrightarrow O(G) \otimes O(G), \quad \Delta(f)(g, g') = f \circ \mu(g, g') = f(gg'),$$

similarly the unit and inversion operations on G induce the counit

$$\varepsilon : O(G) \longrightarrow k, \quad f \mapsto \varepsilon(f) = f(e)$$

respectively the antipode $S : O(G) \longrightarrow O(G)$ with $S(f)(g) = f(g^{-1})$. It follows from the group structure of G that the axioms of a Hopf algebra are satisfied by the algebra maps Δ, ε and S.

It is important to mention the $1-1$ correspondence between algebraic groups G and commutative Hopf algebras $O(G)$: from the above discussion, we see how an algebraic group G yields a commutative Hopf algebra $O(G)$. Conversely, given a commutative Hopf algebra A over an algebraically closed field k, associated is the algebraic group $\mathrm{Alg}_k(A, k)$ of algebra maps $A \longrightarrow k$ also known as characters of A. For better insight on this correspondence, we consider the following example

Example 3.1.11. Consider the algebraic group

$$\mathrm{SL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) : ad - bc = 1 \right\},$$

whose group structure is given by matrix multiplication. It is an algebraic set when considered as the locus of the polynomial $ad - bc - 1 \in \mathbb{C}[a, b, c, d]$. Thus its ring of regular function is the quotient polynomial ring

$$O(\mathrm{SL}_2(\mathbb{C})) = \mathbb{C}[a, b, c, d]/\langle ad - bc - 1 \rangle.$$

We shall write $O(\mathrm{SL}_2)$ and SL_2 for $O(\mathrm{SL}_2(\mathbb{C}))$ respectively $\mathrm{SL}_2(\mathbb{C})$.
$O(\mathrm{SL}_2)$ as a commutative algebra, is generated by regular functions $a, b, c,$ and d on SL_2 since for any matrix $M \in \mathrm{SL}_2$ they are defined by

$$a(M) = m_{11}, \quad b(M) = m_{12}, \quad c(M) = m_{21}, \quad d(M) = m_{22}.$$

Furthermore, $O(\mathrm{SL}_2)$ is a Hopf algebra over $\mathbb{C}$ with Hopf structure given by

$$\begin{aligned} \Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d, \quad \Delta(c) = c \otimes a + d \otimes c \\ \Delta(d) = c \otimes b + d \otimes d, \quad \varepsilon(a) = 1 = \varepsilon(d), \quad \varepsilon(b) = 0 = \varepsilon(c) \\ S(a) = d, \quad S(b) = -b, \quad S(c) = -c, \quad S(d) = a. \end{aligned} \tag{3.1.1}$$

Notice that this is a specific example of the matrix coalgebra structure described in Example 2.1.33. In this case, we have $u_1^1 = a,\ u_2^1 = b,\ u_1^2 = c,\ u_2^2 = d$.

We remark that every matrix $M \in \mathrm{SL}_2$ is an element of $\mathrm{Alg}_{\mathbb{C}}(O(\mathrm{SL}_2), \mathbb{C})$ – the group of algebra maps (otherwise known as characters on SL_2). That is as a map, $M : O(\mathrm{SL}_2) \longrightarrow \mathbb{C}$ is

$$a \mapsto M(a) = a(M) = m_{11}, \quad b \mapsto M(b) = b(M) = m_{12}$$
$$c \mapsto M(c) = c(M) = m_{21}, \quad d \mapsto M(d) = d(M) = m_{22}.$$

Conversely, every element $\gamma \in \mathrm{Alg}_{\mathbb{C}}(O(\mathrm{SL}_2), \mathbb{C})$ defines a matrix in SL_2 by specifying γ on the generators a, b, c, d to obtain $\begin{pmatrix} \gamma(a) & \gamma(b) \\ \gamma(c) & \gamma(d) \end{pmatrix}$. Indeed this matrix lies in SL_2, since its determinant $\gamma(a)\gamma(d) - \gamma(b)\gamma(c) = \gamma(ad - bc) = \gamma(1) = 1$. This proves that $\mathrm{SL}_2 \cong \mathrm{Alg}_{\mathbb{C}}(O(\mathrm{SL}_2), \mathbb{C})$.

Thus in this sense, algebraic groups correspond to commutative Hopf algebras and in fact according to Cartier, commutative Hopf algebras arise in this way – regular functions defined on algebraic groups.

Theorem 3.1.12. *[15] Let H be a commutative Hopf algebra over an algebraically closed field k of characteristics 0. Then H is isomorphic to the algebra of regular functions $O(G)$ on an algebraic group G. In particular, if H is finite dimensional, then H is isomorphic to $k^G = \{f : G \longrightarrow k : f(x) = 0 \text{ for almost all } x\}$ where G is the finite group $G = Alg_k(H, k)$.*

3.1.2 Quantum groups

Depending on the source, the term "quantum group" is often reserved for deformations and quantizations of classical objects such as algebras of regular functions on algebraic groups, universal enveloping algebras of semisimple Lie groups among others. As Gastón remarked in [1]: there is no rigorous universally acceptable definition for quantum groups so for us, quantum groups mean non-commutative non-cocommutative Hopf algebras.

We give examples of quantum groups obtained as quantizations of algebras of regular functions and universal enveloping algebras.

Quantum SL_2

The quantum SL_2 otherwise denoted by $O_q(\mathrm{SL}_2)$ is a one-parameter $q \in \mathbb{C}^\times \setminus \{1\}$ deformation of the commutative algebra $O(\mathrm{SL}_2)$. As an algebra, $O_q(\mathrm{SL}_2)$ is generated by a, b, c, d such that the following relations hold

$$ba = qab, \quad db = qbd, \quad ca = qac, \quad dc = qcd,$$
$$bc = cb, \quad ad - da = (q^{-1} - q)bc.$$

Clearly, $O_q(\mathrm{SL}_2)$ is not commutative (unless $q = 1$). The coalgebra structure is given on generators by the same formulas as for $q = 1$ (3.1.1) that is $O(\mathrm{SL}_2)$ but the antipode is slightly modified as follows

$$S(a) = d, \quad S(d) = a, \quad S(b) = -qb, \quad S(c) = -q^{-1}c,$$

because in this quantum setting, the normal determinant is replaced by the *quantum determinant* $det_q = ad - q^{-1}bc = da - qbc$.

Quantum $\mathfrak{sl}_2$

The linear group SL_2, apart from being a subspace of $\mathbb{C}^4$ is also a smooth manifold, hence a Lie group. The Lie algebra associated to SL_2 is the group of traceless matrices $\mathfrak{sl}_2$, and we are interested in $U(\mathfrak{sl}_2)$, the universal enveloping algebra of $\mathfrak{sl}_2$. Let $q \in \mathbb{C}^\times \setminus \{\pm 1\}$. The one-parameter deformation $U_q(\mathfrak{sl}_2)$ of $U(\mathfrak{sl}_2)$ is a non-commutative $\mathbb{C}$-algebra generated by E, F, K, K^{-1} satisfying the relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = q^2EK, \quad KF = q^{-2}FK,$$
$$FE - EF = \frac{K - K^{-1}}{q - q^{-1}}. \tag{3.1.2}$$

There is a unique Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ with coproduct, counit and antipode given by

$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(E) = E \otimes K + 1 \otimes E,$$
$$\varepsilon(K) = 1 = \varepsilon(K^{-1}), \quad \varepsilon(E) = 0 = \varepsilon(F)$$
$$S(K) = K^{-1}, \quad S(K^{-1}) = K, \quad S(E) = -EK^{-1}, \quad S(F) = -KF.$$

These hold true as one first extend Δ to an algebra morphism of the free algebra $\mathbb{C}[E, F, K, K^{-1}]$ to $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ and then finally pass to the quotient $U_q(\mathfrak{sl}_2)$ of $\mathbb{C}[E, F, K, K^{-1}]$ by ensuring that Δ satisfies (3.1.2). In similar way, we extend the counit and antipode to $U_q(\mathfrak{sl}_2)$ and by straightforward computations, one checks that the axioms of Hopf algebra are satisfied.

Remark 3.1.13. The quantum groups $O_q(\mathrm{SL}_2)$ and $U_q(\mathfrak{sl}_2)$ have a non-trivial Hopf dual pairing stated in [31, Theorem 21]). This means that $O_q(\mathrm{SL}_2) \subseteq U_q(\mathfrak{sl}_2)^\circ$ and vice-versa.

3.2 Flatness and faithful flatness

New modules can be obtained from known ones through various operations such as direct sums, direct products among others. In particular the Hom functor is used to characterize important modules such as injective and projective modules.

Our interest is a special class of modules called *flat* modules. The motivation for their study stems from the fact that tensor products of injective linear maps fail to be injective. However, certain modules preserve injections and these are *flat modules*. In this subsection, we will briefly discuss flat and faithfully flat modules as well as their characterizations and we will follow the notations and definitions in [45].

Let ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$ be the categories of left respectively right modules over a ring R. Suppose M is a right R-module and N a left R-module, then their tensor product is an abelian group $M \otimes_R N$ together with a R-biadditive map $f : M \times_R N \longrightarrow M \otimes_R N$, that is for all $r \in R,\ m, m' \in M,\ n, n' \in N$,

$$f(m + m', n) = f(m, n) + f(m', n), \quad f(m, n + n') = f(m, n) + f(m, n'),$$

$$f(rm, n) = f(m, rn).$$

Furthermore, if $g : M \times_R N \longrightarrow P$ is another R-biadditive map to the abelian group P, then there exist a unique group homomorphism $\tilde{f} : M \otimes_R N \longrightarrow P$ such that $\tilde{f} \circ f = g$. Therefore, given a right R-module M and a morphism $\theta : N \longrightarrow N'$ of left R-modules, then

$$\begin{aligned} M \otimes_R - :\ {}_R\mathbf{Mod} &\longrightarrow \mathbf{Ab}, \\ N &\mapsto M \otimes_R N, \\ \theta &\mapsto \mathrm{id}_M \otimes \theta. \end{aligned}$$

is a functor from the category of left R-modules to the category $\mathbf{Ab}$ of abelian groups.

Definition 3.2.1. A category $\mathcal{C}$ is called additive if for any two objects X and Y, the set $\mathrm{Hom}(X, Y)$ of morphisms is an abelian group such that $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(X, Z),\ (f, g) \mapsto g \circ f$ is linear in both arguments and $\mathcal{C}$ contains a zero object as well as the product $X \times Y$.

The categories $\mathbf{Ab}$, ${}_R\mathbf{Mod}$ and $\mathbf{Mod}_R$ are additive categories.

Definition 3.2.2. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between additive categories is called *additive* if the map

$$\mathcal{C}(X, Y) \longrightarrow \mathcal{D}(FX, FY), \quad f \mapsto F(f)$$

is a morphism of abelian groups.

In other words, F is additive if for all $f, g \in \mathcal{C}(X,Y)$ we have $F(f+g) = F(f)+F(g)$ and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$. Clearly, the functor $M \otimes_R -$ is additive since for $\theta, \epsilon \in {}_R\mathbf{Mod}(N, N'')$, we have $\mathrm{id}_M \otimes (\theta+\epsilon) = \mathrm{id}_M \otimes \theta + \mathrm{id}_M \otimes \epsilon$.

Definition 3.2.3. An additive functor F is called

(i) *left exact* if $F(\ker(f)) = \ker F(f)$,

(ii) *right exact* if $F(\mathrm{Im}(f)) = \mathrm{Im}\ F(f))$,

(iii) *exact* if it is both left and right exact.

for all $f \in \mathcal{C}(X,Y)$.

In more explicit terms, F is exact if whenever $0 \longrightarrow N' \xrightarrow{g} N \xrightarrow{f} N'' \longrightarrow 0$ is a short exact sequence, then

$$0 \longrightarrow F(N') \xrightarrow{F(g)} F(N) \xrightarrow{F(f)} F(N'') \longrightarrow 0$$

is also exact in the target category. The additive functor $M \otimes_R -$ is right exact and not left exact because tensor products do not in general preserve kernels of morphisms. Nevertheless as mentioned before, there are modules for which the functor $M \otimes_R -$ is exact and these are the modules we are interested in.

Definition 3.2.4. Let R be a ring. A right R-module M is called *flat* if $M \otimes_R -$ is an exact functor.

The ring R is flat as a right R-module: suppose $f : N' \longrightarrow N$ is an injective morphism of left R-modules, then $R \otimes_R N' \underset{\tau'}{\xrightarrow{\cong}} N'$ and likewise $R \otimes_R N \underset{\tau}{\xrightarrow{\cong}} N$, where τ and τ' are the module structure maps on N respectively N'. Note that $\mathrm{id}_R \otimes f = \tau^{-1} \circ f \circ \tau'$ and $\ker(\mathrm{id}_R \otimes f) = R \otimes \ker(f) + \ker(\mathrm{id}_R) \otimes N' = 0$. Other examples of flat modules include localisations of rings and modules over R (since $S^{-1}R \otimes_R N \cong S^{-1}N$ where $S \subseteq R$ is a multiplicatively closed) among others. The following result gives a characterization of flat modules

Theorem 3.2.5. *[45, Proposition 3.46] Let R be an arbitrary ring. Then, the direct sum $\bigoplus_j M_j$ of right R-modules is flat if and only if each M_j is flat. In particular every projective R-module is flat.*

The proof of the theorem follows from the fact that tensor products commute with direct sums, that is $(\bigoplus_j M_j) \otimes N \cong \bigoplus_j (M_j \otimes N)$ and that $\bigoplus_j(\mathrm{id}_{M_j} \otimes \theta)$ is injective if and only if each summand $\mathrm{id}_{M_j} \otimes \theta$ is injective. A consequence of this theorem is

Corollary 3.2.6. *Every free R-module is flat.*

Proof. Suppose M is a free right R-module then $M \cong \bigoplus_i R^i$. Since R is flat as a R-module, then by the theorem above, M is flat. □

For our purpose, we are most interested in the subcategory of flat modules for which the exactness condition in Definition 3.2.3 is bi-conditional.

Definition 3.2.7. A flat right R-module M is *faithfully flat* if the exact sequence of abelian groups

$$0 \longrightarrow M \otimes_R N' \xrightarrow{1_M \otimes g} M \otimes_R N \xrightarrow{1_M \otimes f} M \otimes_R N'' \longrightarrow 0$$

implies that the sequence

$$0 \longrightarrow N' \xrightarrow{g} N \xrightarrow{f} N'' \longrightarrow 0$$

of left R-modules is exact.

Proposition 3.2.8. *[35, Theorem 4.70] A right R-module M is faithfully flat if M is flat, and if for all left R- module N we have $M \otimes_R N = 0 \implies N = 0$.*

Proposition 3.2.9. *Let N, P be faithfully flat right R-modules, then their direct sum $N \bigoplus P$ is faithfully flat.*

Proof. Suppose for all left R-modules Q, $N \otimes_R Q = 0$ then since N is faithfully flat, $Q = 0$ (similarly $P \otimes Q = 0$ implies $Q = 0$). Now, $(N \oplus P) \otimes Q = (N \otimes Q) \oplus (P \otimes Q) = 0$ implies $Q = 0$ thus $(N \bigoplus P)$ is faithfully flat. □

Corollary 3.2.10. *Free modules are faithfully flat.*

Proof. A free R-module is isomorphic to the direct sum $\bigoplus_i R^i$. The conclusion follows since each of the summand are faithfully flat. □

3.3 Bergman's diamond lemma

Bergman in [8] introduced an approach of constructing vector space basis for associative algebras presented by generators and relations. This construction is particularly useful in proving faithful flatness of these algebras as a module. Since we will use this approach in our work, we give a rather brief introduction of notations and definitions (as given in [8]) to aid our understanding and application of Bergman's diamond Lemma.

Let X be a set, $\langle X\rangle$ the free semigroup with 1 in X, and $k\langle X\rangle$ be the semigroup algebra over a commutative associative ring k with 1 on $\langle X\rangle$. $k\langle X\rangle$ is a free associative k-algebra of linear combinations of monomials formed by letters in X. A set

$$S = \{\epsilon = (W_\epsilon, f_\epsilon) : W_\epsilon \in \langle X\rangle,\ f_\epsilon \in k\langle X\rangle\}$$

is called a *reduction system*. In other words, W_ϵ is a monomial with letters in X, f_ϵ is a linear combination of these monomials, and the relations $W_\epsilon = f_\epsilon$ will be used as instructions to reduce other expressions in $k\langle X\rangle$. A *reduction* for $k\langle X\rangle$ is a k-module endomorphism

$$r_{A\epsilon B} : k\langle X\rangle \longrightarrow k\langle X\rangle, \quad AW_\epsilon B \mapsto Af_\epsilon B,$$

such that $r_{A\epsilon B}(A) = A$ for all $A, B \in \langle X\rangle$.

A reduction $r_{A\epsilon B}$ is said to act *trivially* on $f \in k\langle X\rangle$ (or f is invariant under $r_{A\epsilon B}$) if no term of f contains the monomial $AW_\epsilon B$. An element f is said to be *irreducible*(under S) if it is invariant under every reduction using the reduction system S. We denote by $k\langle X\rangle_{irr}$ the k- submodule of all irreducible elements of $k\langle X\rangle$.

A 5-tuple $(\epsilon, \tau, A, B, C)$ with $\epsilon, \tau \in S$ and $A, B, C \in \langle X\rangle \setminus \{1\}$ is an *overlap ambiguity* of S if $W_\epsilon = AB$ and $W_\tau = BC$. This ambiguity is said to be *resolvable* if there exist compositions (which is also a reduction) of reductions r and r' such that for monomials D, E, G, H, $r_{D\epsilon C}(AB)C = Ar'_{G\tau H}(BC)$ that is $f_\epsilon C = Af_\tau$. This equality is used when reducing expressions involving the monomial ABC. If $\tau \neq \epsilon$ and $A, B, C \in \langle X\rangle$ such that $W_\epsilon = B$ and $W_\tau = ABC$, the 5-tuple $(\epsilon, \tau, A, B, C)$ is called an *inclusion ambiguity*. The inclusion ambiguity is resolvable if $Af_\epsilon C$ and f_τ can be reduced to a common expression.

Finally in building the set up for the diamond lemma, we equip $\langle X\rangle$ with a *semigroup partial ordering* which is defined to be a partial order " $\leq$ " on $\langle X\rangle$ such that for all monomials A, B, B', C, $ABC < AB'C$ only if $B < B'$. This ordering is compatible with S if for all $\epsilon \in S$, f_ϵ is a linear combination of monomials $< W_\epsilon$.

Theorem 3.3.1. *[8, Theorem 1.2] Let S be a reduction system for a free associative algebra $k\langle X\rangle$ and $\leq$ a semigroup ordering on $\langle X\rangle$ which is compatible with S and having a descending chain condition. Suppose all ambiguities of S are resolvable and $I = \langle\{W_\epsilon - f_\epsilon : \epsilon \in S\}\rangle$ be a two sided ideal of $k\langle X\rangle$. Then $k\langle X\rangle_{irr} \cong k\langle X\rangle/I$ by sending $a \mapsto a + I$ as k-vector spaces. In particular, the irreducible words in $\langle X\rangle$ form a vector space basis of $k\langle X\rangle/I$.*

We will see an application of this result in the Sections 3.5.2, 3.6 and Chapter 4.

3.4 Quantum homogeneous space

It is understood that a homogeneous space X of a Lie group G can be realized as a quotient of G by a closed subgroup. Thus there is a $1-1$ correspondence between closed subgroups of G and homogeneous G-spaces. A generalization of this notion to quantum groups suffers a setback since quantum groups have fewer quantum subgroups compared to their classical counterparts as Podleś [42] proved for the quantum group $SU_q(2)$ - the quantization of the classical group $SU(2)$.

In [43], Podleś introduced quantum analogues of homogeneous spaces in two ways: first, *embeddable* homogeneous spaces and second, *quotients* of quantum groups by *quantum subgroups*. Embeddable homogeneous spaces are left comodule algebras B over a Hopf algebra A such that B embeds into A as a subalgebra via $\iota : B \longrightarrow A$ and ι is a morphism of A-comodule algebras. Examples of this class of quantum homogeneous space include the Podleś' spheres $S_q(\nu,\mu)$ – a polynomial algebra generated by $1, x, y, z$ such that

$$xz = q^2 zx, \quad yz = q^{-2} zy, \quad xy = -q(\mu - z)(\nu + z),$$

$$yx = -q(\mu - q^2 z)(\nu + q^{-2} z),$$

where $q \neq 0,\ \nu, \mu \in \mathbb{R}_{\geq 0}$ and $(\nu,\mu) \neq (0,0)$. These embed into the quantum group $SU_q(2)$, see [43, 14, 29] for more details.

The other class of quantum homogeneous spaces comprises those that can be realized as quotients of quantum groups. By *quantum subgroup*, we mean a Hopf algebra A_0 together with a surjective morphism $\pi : A \longrightarrow A_0$ of Hopf algebras. The *quantum quotient* is the subalgebra B of all points $a \in A$ that are invariant under the left coaction $(\pi \otimes \mathrm{id}_A) \circ \Delta_A$ of A_0 on A. Brzezinski in [14] showed that in fact, certain embeddable quantum homogeneous spaces can be understood as quotients spaces, see [14, Proposition 2.5].

As in [48, 32, 41, 51, 11, 38] we take in this thesis quantum homogeneous spaces to be quantum quotient spaces.
Recall from Definition 2.1.9 the definitions of right (resp. left) coideal and coideals of a Hopf algebra then we define

Definition 3.4.1. A right (resp. left) *coideal subalgebra* of a Hopf algebra A is a subalgebra $B \subseteq A$ which is a right (resp. left) coideal.

We now formally define a quantum homogeneous space as

Definition 3.4.2. A right coideal subalgebra B of a Hopf algebra A is called a *quantum homogeneous space* if A is faithfully flat as a right B-module.

From now on, we shall use the abbreviation QHS for quantum homogeneous space.

Remark 3.4.3. We remark that a QHS can be defined using a left coideal subalgebra of A since it becomes a right coideal subalgebra of A^{cop}.

Following the notations and results in [48, 41], we will discuss the construction of QHS as a quotient space and we will see the importance of the requirement of faithful flatness of A as a B-module.

Let $B \subseteq A$ be a right coideal subalgebra, and we denote $\mathcal{M}_B^A$ to be the category whose objects are of the form (M, ρ_M, μ) where (M, ρ_M) is a right A-comodule and (M, μ) is a right B-module structure on M such that

$$\rho_M \circ \mu(m \otimes b) = m_{(0)} \cdot b_{(1)} \otimes m_{(1)} b_{(2)}$$

for all $m \in M,\ b \in B$ and morphisms are right A-comodule maps which are right B-linear. Similarly, we denote by ${}_{A}^{\pi(A)}\mathcal{M}$ the category of left $\pi(A)$-comodules and left A-modules, where $\pi : A \longrightarrow \pi(A)$ is a quotient left A-module coalgebra map and morphisms of this category are left $\pi(A)$-colinear and left A-linear maps. A simpler way to think of π is: the coalgebra $\pi(A)$ is a left A-module and for all $a, b \in A$, we have $\pi(ba) = b\pi(a)$.

It is important to observe that given a right coideal subalgebra B, we can construct a quotient coalgebra as we show in the following:

Proposition 3.4.4. *Let $B \subseteq A$ be a right coideal subalgebra of Hopf algebra A and define $B^+ = B \cap ker\, \varepsilon$. Then the set $AB^+ = span_k\{ab : a \in A,\ b \in B^+\}$ is a coideal and A/AB^+ is a quotient coalgebra.*

Proof. For any $ab \in AB^+$ we have

$$\Delta(ab) = (ab)_{(1)} \otimes (ab)_{(2)} = a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)},$$

which can be rewritten as

$$a_{(1)}(b_{(1)} - \varepsilon(b_{(1)}) \otimes a_{(2)} b_{(2)} + a_{(1)} \varepsilon(b_{(1)}) \otimes a_{(2)} b_{(2)}.$$

thus $\Delta(AB^+) \subseteq AB^+ \otimes A + A \otimes AB^+$. Furthermore, since $B^+ \subseteq \ker \varepsilon$, it follows that $\varepsilon(AB^+) = 0$. Therefore the coalgebra structure of A descends to A/AB^+. □

Thus we have a projection $\pi_B : A \longrightarrow A/AB^+$ which is also a morphism of left A-module coalgebras: the left A-module structure (given by multiplication in A) on A descends to A/AB^+, and A admits a left A/AB^+-comodule structure given by $(\pi_B \otimes \mathrm{id}) \circ \Delta$, hence $A \in {}_{A}^{\pi_B}\mathcal{M}$. Therefore, we conclude that if given a right coideal subalgebra, one can construct a quotient coalgebra map of A-modules.

Proposition 3.4.5. *[48, Proposition 1] If $\pi : A \longrightarrow \pi(A)$ is a quotient left A-module coalgebra map, then the set*

$$B_\pi = \{a \in A : \pi(a_{(1)}) \otimes a_{(2)} = 1 \otimes a\}$$

is a right coideal subalgebra of A.

The set B_π comprises all elements of A which are invariant under the coaction of $(\pi \otimes id) \circ \Delta$ of $\pi(A)$ on A. This thus gives a converse to Proposition 3.4.4. That is if given a quotient coalgebra of a Hopf algebra A, we can construct a right coideal subalgebra B_π. In particular as shown below, if $\pi = \pi_B$ then $B_{\pi_B} \cong B$ as right coideal subalgebras of A and this gives a $1-1$ correspondence between right coideal subalgebras over which A is a faithfully flat module and quotient coalgebras of A. This can be reformulated using the *cotensor product* of comodules:

Definition 3.4.6. Let (P, ρ) and (Q, λ) be a right respectively a left A-comodule. The *cotensor product* $P\square_A Q$ of P and Q is the module over a field k which makes the sequence

$$0 \longrightarrow P\square_A Q \longrightarrow P \otimes Q \xrightarrow{w_{P,Q}} P \otimes A \otimes Q$$

exact.

The map $w_{P,Q} = \rho \otimes 1_Q - 1_P \otimes \lambda$, and exactness of the sequence means $P\square_A Q = \ker w_{P,Q}$ so, in other words,

$$P\square_A Q = \left\{ \sum_i x_i \otimes y_i \in P \otimes Q : \sum_i x_{i_{(0)}} \otimes x_{i_{(1)}} \otimes y_i = \sum_i x_i \otimes y_{i_{(-1)}} \otimes y_{i_{(0)}} \right\}.$$

Observe that the cotensor product can be viewed as a functor from the category ${}^A\mathcal{M}$ of left A-comodules to the category of k-modules.

Definition 3.4.7. A right A-comodule M is called *faithfully coflat* if as a functor, $M\square_A-$ is faithful and preserves exact sequences.

We define the functors

$$\Psi : \mathcal{M}^\pi \longrightarrow \mathcal{M}_B^A, \;\; S \mapsto S\square_\pi A$$

$$\Phi : \mathcal{M}_B^A \longrightarrow \mathcal{M}^\pi, \;\; M \mapsto \bar{M} = M/MB^+$$

where $S \in \mathcal{M}^\pi$, $M \in \mathcal{M}_B^A$ and $S\square_\pi A \subseteq S \otimes A$. It is clear that $S\square_\pi A$ inherits the right B-module of the ambient space $S \otimes A$ and is a right A-comodule via the coproduct Δ of A. In addition, M has a right $\pi(A)$-comodule structure induced by the surjective coalgebra map $\pi : A \longrightarrow \pi(A)$. Suppose $f : M \longrightarrow S$ is a morphism of right $\pi(A)$-comodules, then f induces the map

$$M \xrightarrow{\rho_M} M \otimes A \xrightarrow{f \otimes id_S} S \otimes A.$$

This composite restricts to the morphism $F : M \longrightarrow S\Box_\pi A$ of right A-comodules which is B-linear (if $F(MB^+) = 0$). On the other hand, let $F\colon M \longrightarrow S\Box_\pi A$, be a map of right A-comodules, then it induces the composite

$$M \xrightarrow{F} S\Box_\pi A \subseteq S \otimes A \xrightarrow{id \otimes \varepsilon} S \otimes k \cong S$$

that is $f = (\mathrm{id} \otimes \varepsilon) \circ F$ which is a morphism of right $\pi(A)$-comodules. Hence there is an isomorphism

$$\mathcal{M}^\pi(\bar{M}, S) \cong \mathcal{M}_B^A(M, S\Box_\pi A)$$

that defines the adjunction of Ψ and Φ so we can say Ψ is right adjoint to Φ and vice versa.

Dually, one can show that ${}_B\mathcal{M}(N^\pi, T) \cong {}_A^\pi\mathcal{M}(N, A \otimes_B T)$ where T is a left B-module, $N \in {}_A^\pi\mathcal{M}$, and N^π is the set of elements of N that are coinvariant under the comodule action of A on N. Now we can state the following theorem due to Takeuchi

Theorem 3.4.8. *[48, Theorem 1 & 2]*

(1) Let $B \subseteq A$ be a right coideal subalgebra and $\pi = \pi_B$. Suppose there is a left A-module M which is faithfully flat as a left B-module. Then, the categories $\mathcal{M}_B^A$ and $\mathcal{M}^\pi$ are equivalent via Ψ and Φ. Moreover, $B = B_\pi$ and A is faithfully coflat as a left $\pi(A)$-comodule.

(2) Suppose there is a right A-comodule which is faithfully coflat as a right $\pi(A)$-comodule. Then, the categories ${}_B\mathcal{M}$ and ${}_A^\pi\mathcal{M}$ are equivalent, $\pi = \pi_B$ and A is faithfully flat right B-module.

This theorem establishes a one to one correspondence

$$\left\{\begin{array}{l}\text{Right coideal sub-}\\ \text{algebras } B \text{ of } A\\ \text{over which } A \text{ is}\\ \text{faithfully flat}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Quotient left } A\text{-}\\ \text{module coalgebras}\\ \text{over which } A \text{ is}\\ \text{faithfully coflat}\end{array}\right\}$$

$$B \to \pi_B$$
$$B^\pi \leftarrow \pi$$

A version of the above theorem was given by Masuoka and Wigner [39, Theorem 2.1] but with the requirement that the Hopf algebra A has a bijective antipode. They stated that the equivalence of the functors Φ and Ψ is the same as saying A is faithfully flat as a module over its coideal subalgebra B.

To this end, we can say that quantum homogeneous spaces correspond (not bijective) to quotient coalgebras of their containing Hopf algebras and they are precisely the subalgebras of all elements that are invariant under the comodule action induced by these quotients coalgebras.

However as Takeuchi pointed out, when A is commutative, $\pi(A)$ becomes a quotient bialgebra and the antipode S (which is now bijective) defines a right $\pi(A)$-comodule structure on A. Thus, the correspondence in the above theorem becomes a bijection between the set of right coideal subalgebra over which A is a faithfully flat module and the set of quotient Hopf algebras over which A is faithfully coflat. Recall from Example 3.1.11 the correspondence between algebraic groups and commutative Hopf algebras, we get close to the classical bijection of homogeneous G-spaces and cosets of G by taking G to be the group of characters $A \longrightarrow k$ of A and the set of cosets G/H where H is the group of characters on the quotient Hopf algebra A/AB^+.

To show that a certain coideal subalgebra B of A is a QHS, one must establish that A is a faithfully flat module over B. One of the ways to show this is to use Bergman's diamond Lemma 3.3.1 to construct a vector space basis for A. If the basis elements are of the form $b_i e_j$, where $\{b_i\}_{i\in I}$ is a basis of B, then A is free as a B-module thus faithful flatness follows by Corollary 3.2.10. However, this approach is limited to Hopf algebras presented by generators and relations. In the literature, other methods have been studied for Hopf algebras which are not finitely presented (which are also applicable to finitely presented Hopf algebras), and we will briefly state these results.

Masuoka and Wigner proved that given a non-commutative Hopf algebra A with a bijective antipode such that A is a flat module over a coideal subalgebra $B \subseteq A$, then A is faithfully flat (see [39, Theorem 2.1]). Building on this, they also showed that every commutative Hopf algebra is flat over their coideal subalgebra see [39, Theorem 3.4]. A consequence of this later result is that commutative Hopf algebras are faithfully flat over their right coideal subalgebras.

In addition, Masuoka showed in [38] that Hopf algebras with cocommutative coradical are faithfully flat, more precisely

Theorem 3.4.9. *[38, Theorem 1.3] Let A be a Hopf algebra with bijective antipode S and suppose the coradical A_0 of A is cocommutative. If B is a right coideal subalgebra of A such that $S(B_0) = B_0$ where $B_0 := B \cap A_0$, then A is faithfully flat as a right* *or left B-module.*

The bijectivity of the antipode means that A^{op} is a Hopf algebra with S^{-1} as the antipode, thus the categories ${}_B\mathcal{M}^A$ and $\mathcal{M}_{B^{\text{op}}}^{A^{\text{op}}}$ are equivalent. The condition that $S(B_0) = B_0$ is used to show that every module $M \in \mathcal{M}_{B^{\text{op}}}^{A^{\text{op}}}$ is free as a B-module. Thus whenever A is a pointed Hopf algebra faithful flatness of A as a module over its coideal subalgebras is implied. This result will be applied in our discussion in Chapter 4.

3.5 Quantum symmetry of singular affine curves

Section 3.5.1 is reviewing the results on the containing Hopf algebras which makes certain coordinate rings of singular affine curves admit a QHS. Section 3.5.2 contains the first contribution of this thesis.

3.5.1 The Hopf algebra $A(g, f)$

The earliest work on quantum symmetries of affine curves goes back to the work of Goodearl and Zhang. In [24, Construction 1.2] they constructed a Noetherian Hopf algebra $B(1, 1, n, m, q)$ of GK dimension 2, where $(n, m) = 1$ and q is a primitive mn^{th} root of unity. The Hopf algebra $B(1, 1, n, m, q)$ is the skew group algebra of the infinite cyclic group $B := k[t^n, t^m][x^{\pm 1}, \sigma]$ generated as an algebra by the grouplike element x and twisted primitive elements t^n and t^m such that $xt^m = \sigma(t^m)x$, $xt^n = \sigma(t^n)x$. σ is an automorphism of the ring of coefficients $k[t^n, t^m]$ which is the coordinate ring of the cusp $y^m = x^n$. Trivially $k[t^n, t^m]$ embeds into B as a subalgebra and is a left coideal since $\Delta(t^n) = t^n \otimes 1 + x^{mn} \otimes t^n$ and $\Delta(t^m) = t^m \otimes 1 + x^{mn} \otimes t^m$.

Krähmer and Tabiri showed that the coordinate ring of the nodal cubic $y^2 = x^3 + x^2$ is a coideal subalgebra of the pointed Hopf algebra described in [34, Theorem 1]. This pointed Hopf algebra obtained has GK dimension 3 and as shown in [33], it is not the smallest containing Hopf algebra for which the nodal cubic admits a QHS since Krähmer and Martins [33, Theorem 1] constructed a quotient Hopf algebra of GK dimension 1. Furthermore, Brown and Tabiri in [12] showed that coordinate rings $O(C)$ of plane curves C which decompose as $f(y) = g(x)$ such that $2 \leq \deg f = m, \deg g = n \leq 5$ admit a quantum homogeneous space structure in the containing pointed Hopf algebra

$$A(g, f) = A(a, x, g) \otimes A(b, y, f)/\langle a^n - b^m, f - g\rangle.$$

We will briefly explain the construction of $A(g, f)$ as this Hopf algebra in some sense, encompasses previously discussed containing Hopf algebras for which the coordinate rings of the cusp and the nodal cubic become a QHS.

Suppose g and f are monic polynomials, the pointed Hopf algebras $A(x, a, g)$ and $A(y, b, f)$ where a, b are grouplikes, and x and y are $(1, a)$-twisted primitives respectively $(1, b)$-twisted primitives are constructed as follows:

Let F_0 be the free algebra on the set of generators $\{x, a, c\}$. We define $F := F_0/I$, where I is the ideal generated by the relations $ac - 1$ and $ca - 1$. F has a unique Hopf algebra structure (see [12, Lemma 1.1]) whose coproduct, counit and antipode satisfy

$$\begin{gathered} \Delta(a) = a \otimes a, \quad \Delta(x) = 1 \otimes x + x \otimes a, \quad \Delta(c) = c \otimes c \\ \varepsilon(a) = \varepsilon(c) = 1, \quad \varepsilon(x) = 0 \\ S(a) = c, \quad S(x) = -xc, \quad S(c) = c. \end{gathered} \tag{3.5.1}$$

Define a $(\mathbb{Z}_{\geq 0}, \mathbb{Z}_{\geq})$-grading on F by assigning to a and x the degrees $(1, 0)$ respectively $(0, 1)$. We denote by $P(j, i)_{(a,x)}$ the sum of all monomials in F of degree (j, i) and set $P(0, 0) = 1$. For $\deg g(x) := n \geq 2$, $g(x) = \sum_{i=1}^{s} s_i x^i$ and $s_n \neq 0$ define

$$\sigma_j := \sigma_j(a, x, g(x)) := \sum_{i=j}^{n} s_i P(j, i - j) - s_j a^n.$$

For example, if $g(x) = x^3 + x^2$ then $\sigma_1 = ax + xa + ax^2 + x^2 a + xax$ and $\sigma_2 = a^2 + a^2 x + axa + xa^2 - a^3$.

Definition 3.5.1. $I_{g(x)} := \langle \sigma_j : 1 \leq j \leq n-1 \rangle$ is the ideal generated by the elements σ_j of F.

Following the example given above, $I_{x^3+x^2} = \langle \sigma_1, \sigma_2 \rangle$.

Proposition 3.5.2. *[12, Lemma 1.2] The ideal $I_{g(x)}$ is a Hopf ideal of F.*

Definition 3.5.3. Let F and $I_{g(x)}$ be as described above. Then $A(x, a, g)$ is defined to be the quotient $F/I_{g(x)}$ of F.

Similarly one defines $A(y, b, f)$ to be the quotient $F/I_{f(y)}$ of F. However F here is the Hopf algebra obtained from the quotient of the free algebra on the set $\{y, b, d\}$ by the ideal $J = \langle bd - 1, db - 1 \rangle$. Furthermore, $I_{f(y)}$ is defined similarly as $I_{g(x)}$.

In particular following [12, Lemma 1.2], $A(x, a, g)$ and $A(y, b, f)$ are Hopf algebra with Hopf algebra structure descended from F:

Theorem 3.5.4. *[12, Theorem 1.3] The k-algebra $A(x, a, g)$ is a Hopf algebra with coproduct, counit and antipode given by (3.5.1).*

Consider the tensor product $T = A(a, x, g) \otimes A(b, y, f)$ with Hopf algebra structure inherited from the $A(a, x, g)$ and $A(b, y, f)$, that is tensor coalgebra and tensor algebra structures. In particular, following the definitions of the Hopf algebra $A(a, x, g)$ $A(b, y, f)$, we can write $T = k\langle x, y, a^{\pm 1}, b^{\pm 1}\rangle$ where the elements x, y are primitive elements and a, b group-likes thus T is a pointed Hopf algebra [44, Corollary 5.1.14].

Proposition 3.5.5. *[12, Proposition 1.6] The elements f, g, a^n, b^m are central in T*

Thus we define $\mathcal{I}$ to be the ideal generated by $\{a^n - b^m, f - g\}$.

Definition 3.5.6. Let T and $\mathcal{I}$ be as defined above then $A(g, f) := T/\mathcal{I}$ as a k-algebra.

Theorem 3.5.7. *[12, Theorem 5.1, 5.2]*

1. *$A(g, f)$ is a Hopf algebra.*

2. *The coordinate ring $O(C)$ is a quantum homogeneous space of the Hopf algebra $A(g, f)$.*

The embedding of the coordinate ring $O(C) = k[x, y]/\langle g - f\rangle$ as a right coideal subalgebra of $A(g, f)$ is premised on the fact that the tensor product algebra $k[x, a^{\pm n}] \otimes k[y, b^{\pm m}]$ of the subalgebras $k[x, a^{\pm n}] \subseteq A(x, a, g)$ and $k[y, b^{\pm m}] \subseteq A(y, b, f)$ can be identified with $k[x, y, a^{\pm n}, b^{\pm m}]$ which is generated by twisted primitive elements x, y, and grouplike elements $a^{\pm n}, b^{\pm m}$. Thus, $k[x, y, a^{\pm n}, b^{\pm m}]$ is a right coideal subalgebra of the Hopf algebra $A(a, x, g) \otimes A(b, y, f)$ (see [12, Proposition 4.1] for more details).

An application of this construction to $k[t^2, t^3]$ yields the pointed Hopf algebra $A(x^2, y^3)$ (of GK dimension 3) thus making the cusp admit a quantum homogeneous space structure. It was shown in [12] that the representation of $A(x^2, y^3)$ on $k[t^2, t^3]$ factors through the Hopf algebra $B(1, 1, 2, 3, q)$ found in [24]. In other words $A(x^2, y^3)$ does not act inner faithfully on $k[t^2, t^3]$.

3.5.2 Affine coaction on the cusp

In this section we give a detailed construction of a quantum group A for which the coordinate ring $k[t^2, t^3]$ of the cusp is a quantum homogeneous space. In comparison with previous works discussed above, what is new is that we start off with an ansatz given by affine transformations of the plane as a comodule structure map of a bialgebra A on $k[t^2, t^3]$ and we proved that it is not different from the coaction in [32].

Proposition 3.5.8. *Let $B = k[t^2, t^3]$ be the coordinate ring of the cusp and set $x = t^2$ and $y = t^3$. Suppose A is any bialgebra coacting on B via*

$$\begin{aligned} \rho : B &\to A \otimes B \\ x &\mapsto a \otimes x + b \otimes y + r \otimes 1 \\ y &\mapsto c \otimes x + d \otimes y + s \otimes 1 \end{aligned}$$

and the elements a, b, c, d, r, s of A are linearly independent. Then these elements satisfy the following relations

$$\begin{gathered} [a, c] = 0, \quad [b, d] = 0, \quad [s, r] = 0, \quad [a, d] = [c, b], \quad [a, s] = [c, r], \\ [b, s] = [d, r], \quad c^3 = 0, \quad c^2d + cdc + dc^2 = 0, \quad cd^2 + dcd + d^2c = 0, \\ c^2s + csc + sc^2 + d^3 = a^2, \quad d^2s + dsd + sd^2 = b^2, \quad r^2 = s^3, \\ ab + ba = cds + csd + dcs + dsc + scd + sdc, \\ ar + ra = cs^2 + scs + s^2c, \quad br + rb = s^2d + sds + ds^2. \end{gathered}$$

The coalgebra structure on the elements a, b, c, d, r, s is given by

$$\begin{aligned} &\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d, \quad \Delta(r) = a \otimes r + b \otimes s + r \otimes 1, \\ &\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d, \quad \Delta(s) = c \otimes r + d \otimes s + s \otimes 1, \\ &\varepsilon(a) = 1 = \varepsilon(d), \quad \varepsilon(b) = \varepsilon(c) = \varepsilon(r) = \varepsilon(s) = 0. \end{aligned} \tag{3.5.2}$$

Proof. The relations are obtained using the defining equation of the algebra B and the fact that ρ is an algebra map. That is rename $X := a \otimes x + b \otimes y + r \otimes 1$ and $Y := c \otimes x + d \otimes y + s \otimes 1$, we have

$$\begin{aligned} XY = {} & ac \otimes x^2 + ad \otimes xy + bc \otimes yx + (as + rc) \otimes x \\ & + bd \otimes y^2 + (bs + rd) \otimes y + rs \otimes 1 \end{aligned}$$

$$\begin{aligned} YX = {} & ca \otimes x^2 + da \otimes yx + cb \otimes xy + (sa + cr) \otimes x \\ & + db \otimes y^2 + (sb + dr) \otimes y + sr \otimes 1. \end{aligned}$$

Solving $XY = YX$ and using the relations $xy = yx$ yields

$$[a, c] = [r, s] = [b, d] = 0 \quad [a, d] = [b, c], \quad [a, s] = [c, r], \quad [b, s] = [d, r].$$

In a similar way solving $X^3 = Y^2$ and using the relation $x^3 = y^2$ yields the remaining relations. The coalgebra structure maps stated can be obtained from evaluating the coaction axioms

$$(\Delta \otimes \mathrm{id}_B) \circ \rho = (\mathrm{id}_A \otimes \rho) \circ \rho, \qquad (\varepsilon \otimes \mathrm{id}_B) \circ \rho = \mathrm{id}_B,$$

on the generators x and y of B. That is equating

$$\begin{aligned}(\mathrm{id}_A \otimes \rho) \circ \rho(x) = (a \otimes a + b \otimes c) \otimes x + (a \otimes b + b \otimes d) \otimes y \\ + (a \otimes r + b \otimes s + r \otimes 1) \otimes 1\end{aligned}$$

with $(\Delta \otimes \mathrm{id}_B) \circ \rho(x) = \Delta(a) \otimes x + \Delta(b) \otimes y + \Delta(r) \otimes 1$ yields

$$\Delta(a) = a \otimes a + b \otimes c \quad \Delta(b) = a \otimes b + b \otimes d$$
$$\Delta(r) = a \otimes r + b \otimes s + r \otimes 1.$$

Furthermore, $x = (\varepsilon \otimes \mathrm{id}_B) \circ \rho(x) = \varepsilon(a)x + \varepsilon(b)y + \varepsilon(r)1$ implies

$$\varepsilon(a) = 1, \quad \varepsilon(b) = 0 = \varepsilon(r).$$

The other formulas can be obtained by evaluating the coaction axioms on y. □

Lemma 3.5.9. *The bialgebra A described above is a Hopf algebra if and only if $bc = cb$.*

Proof. Suppose A is a Hopf algebra, then an antipode S is well defined and satisfies

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = \mu \circ (\mathrm{id} \otimes S) \circ \Delta.$$

We set $t := ad - cb = da - bc$ and evaluate the antipode equation at a, b, c, d and simplify to obtain $S(a)t = d, \quad S(b)t = -b, \quad S(c)t = -c, \quad S(d)t = a$. Post multiply the antipode equation evaluated at a, $aS(a) + bS(c) = 1$ by t to obtain $ad - bc = t = ad - cb$ thus $bc = cb$. The converse follows likewise. □

Corollary 3.5.10. *If the bialgebra A is a Hopf algebra, then a, b, c, d all commute with one another.*

Proof. Suppose A is a Hopf algebra then by the above proposition $bc = cb$. Since Δ is an algebra map we have

$$\begin{aligned}0 = \Delta(b)\Delta(c) - \Delta(c)\Delta(b) = ac \otimes (ba - ab) + (ad - da) \otimes bc \\ + bc \otimes (da - ad) + bd \otimes (dc - cd),\end{aligned}$$

but from the relation $[a,d]=[b,c]$ we have $ad=da$ and the above equation becomes

$$ac\otimes(ba-ab)+bd\otimes(dc-cd)=0.$$

Using the assumption that a,b,c,d are linearly independent, the conclusion follows. $\square$

Note that in the Hopf algebra A, the element $t=ad-cb$ is grouplike: $\Delta(t)=t\otimes t$ and $\varepsilon(t)=1$. Hence, t^{-1} exists and both t and t^{-1} commute with a,b,c,d.

Proposition 3.5.11. *The coordinate ring B of the cusp is a left coideal subalgebra of the Hopf algebra A whose elements a,b,c,d,r,s satisfy the relations*

$$\begin{gathered}[a,c]=0,\quad [b,d]=0,\quad [a,d]=0,\quad [b,c]=0,\quad [s,r]=0,\quad [a,s]=[c,r],\\ [b,s]=[d,r],\quad c^3=0,\quad [a,b]=0,\quad [c,d]=0,\quad c^2d=0=cd^2,\\ c^2s+csc+sc^2+d^3=a^2,\quad d^2s+dsd+sd^2=b^2;\quad r^2=s^3,\\ 2ba=2cds+csd+dsc+2scd,\quad ar+ra=cs^2+scs+s^2c,\\ br+rb=s^2d+sds+ds^2,\end{gathered}$$

with coalgebra structure (3.5.2) and antipode given by

$$S(a)=t^{-1}d,\quad S(b)=-t^{-1}b,\quad S(c)=-t^{-1}c,\quad S(d)=t^{-1}a,$$

$$S(r)=t^{-1}(bs-dr),\quad S(s)=t^{-1}(cr-as).$$

Proof. The proof follows from taking $\Delta_A(B)=\rho(B)$ and identifying x and y with r respectively s. $\square$

The Hopf algebra A described above contains commuting elements a,b,c,d,t^{-1} as shown in Lemma 3.5.10 and they generate a commutative Hopf subalgebra with a nilpotent element c. However, we know that

Theorem 3.5.12. *[15] A finitely generated commutative Hopf algebra over a field k of $\operatorname{char}(k)=0$ has no nilpotent element that is, it is reduced.*

Therefore we have that

Lemma 3.5.13. *In the commutative Hopf subalgebra of A generated by a,b,c,d,t^{-1}, $c=0$.*

Proof. Notice that $c^3=0$, so c is nilpotent and by Theorem 3.5.12, $c=0$. $\square$

Implementing this result in A, yields new relations $a^2 = d^3$ and $ab = 0$ precisely when $c = 0$ is substituted into the relations $c^2s + csc + sc^2 + d^3 = a^2$ respectively $2ba = 2cds + csd + dsc + 2scd$. In view of this, we now have

Corollary 3.5.14. $b = 0$.

Proof. By Lemma 3.5.13, we know that $c = 0$, and hence we have that $\Delta(a) = a \otimes a$, $\Delta(d) = d \otimes d$ that is a and d are group-like elements. Hence, their inverses exist and since $ab = 0$, we have that $b = 0$. □

We have shown that B is a left coideal subalgebra of A, hence it is a right coideal subalgebra of A^{cop}. We will now show that A is a faithfully flat right B-module using Bergman's diamond lemma in Theorem 3.3.1 to construct a vector space basis for A

Proposition 3.5.15. *The set* $\mathcal{B} = \{r^j s^l a^i d^k : i, j \in \{0, 1\},\ k, l \in \mathbb{Z}_{\geq 0}\}$ *is a* k*-vector space basis for* A.

Proof. Following the setup of the Bergman's diamond lemma 3.3.1, $X = \{a, d, r, s\}$ and $I = span_k\{ad - da,\ rs - sr,\ as - sa,\ dr - rd,\ a^2 - d^3, d^2s + dsd + sd^2,\ r^2 - s^3,\ ar + ra,\ s^2d + sds + ds^2\}$. Clearly the Hopf algebra $A \cong k\langle X\rangle/I$ and there are no unresolved ambiguities. Notice that the set $\mathcal{B}$ comprises irreducible words in $\langle X\rangle$, thus by Theorem 3.3.1 it is a vector space basis of A. □

Hence, by 3.2.10, we have A is free and thus faithfully flat as a B-module therefore, the coordinate ring B of the cusp is a quantum homogeneous space of the Hopf algebra A whose Hopf structure is described in Theorem 3.5.11 and is presented by generators a, d, r, s and relations:

$$ad = da, \quad rs = sr, \quad as = sa, \quad dr = rd, \quad a^2 = d^3,$$

$$d^2s + dsd + sd^2 = 0, \quad r^2 = s^3, \quad ar = -ra, \quad s^2d + sds + ds^2 = 0.$$

3.6 True quantum symmetry

In the classical setting, symmetries of algebraic sets given by actions of group algebras or universal enveloping algebras do extend to their coordinate rings and fields of rational functions. One naturally ask if this carries over to the quantum setting. The answer is no in general, however in this section we will motivate an example of a quantum symmetry of $k[t^2, t^3]$ which extends to $k(t)$, see Remark 3.6.10. Furthermore, we show that the dual Hopf algebra to the pointed Hopf algebra A generated by this quantum symmetry is not pointed, see Proposition 3.6.8.

Definition 3.6.1. Let G be an algebraic set equipped with a group structure and X be an algebraic set. A G-*action* on X is a map $\alpha : G \times X \longrightarrow X,\ (g, x) \mapsto g \triangleright x$ such that for all $g, h, e_G \in G,\ x \in X$

$$g \triangleright (h \triangleright x) = gh \triangleright x, \quad \text{and} \quad e_G \triangleright x = x.$$

Every G-action α of G on an algebraic set X yields an action of G on the coordinate ring $O(X)$ of X via

$$\hat{\alpha} : G \times O(X) \longrightarrow O(X),\ \ (g, f) \mapsto (g \triangleright f)(x) := f(g^{-1} \triangleright x),$$

and as we have for all $g, h \in G$

$$(g \triangleright h \triangleright f)(x) = (h \triangleright f)(g^{-1} \triangleright x) = f(h^{-1} \triangleright g^{-1} \triangleright x) = f(h^{-1}g^{-1} \triangleright x) = (gh \triangleright f)(x)$$

and

$$(e_G \triangleright f)(x) = f(e_G \triangleright x) = f(x).$$

In particular, $O(X)$ becomes a left G-module algebra since for all $a, b \in O(X)$ and $g \in G$

$$(g \triangleright ab)(x) = ab(g^{-1} \triangleright x) = a(g^{-1} \triangleright x)\, b(g^{-1} \triangleright x) = (g \triangleright a)(g \triangleright b)(x),$$

and

$$(g \triangleright 1)(x) = 1(g^{-1} \triangleright x) = 1.$$

Furthermore, this module action of G on $O(X)$ extends to the rational function field $k(X)$ of X with module action defined by

$$g \triangleright \left(\frac{a}{b}\right) = \frac{g \triangleright a}{g \triangleright b},$$

with $b \neq 0$ on X. This module action is well defined:

$$\begin{aligned} \frac{a}{b} = \frac{a'}{b'} &\iff a'b = ab' \\ &\iff g \triangleright (a'b) = g \triangleright (ab') \\ &\iff (g \triangleright a')(g \triangleright b) = (g \triangleright a)(g \triangleright b') \\ &\iff \frac{g \triangleright a'}{g \triangleright b'} = \frac{g \triangleright a}{g \triangleright b}. \end{aligned}$$

A morphism of algebraic sets induces a contravariant algebra homomorphism between the algebras of their regular functions. Thus, α induces an algebra map

$$\begin{aligned}\alpha^{\#} : O(X) &\longrightarrow O(G \times X) \cong O(G) \otimes O(X)\\ f &\mapsto \alpha^{\#}(f)(g, x) = f \circ \alpha(g, x) = f(g \triangleright x).\end{aligned}$$

Note that $\alpha^{\#}$ is not only an algebra map but also a left coaction of $O(G)$ on $O(X)$. However, this coaction does not in general extend to the function field $k(X)$ of X as shown in what follows.

Example 3.6.2. Recall the algebraic group $G = \mathrm{SL}_2(\mathbb{C})$ described in Example 3.1.11. Define a left action of G on the affine plane $X = \mathbb{C}$, via matrix multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}. \tag{3.6.1}$$

Thus by equation (3.1.1), $A = O(G)$ is a coalgebra and the $\mathrm{SL}_2(\mathbb{C})$-action on $\mathbb{C}$ induces a left A-comodule algebra structure

$$\begin{aligned}\rho : B &\longrightarrow A \otimes B\\ x &\mapsto a \otimes x + b \otimes y,\\ y &\mapsto c \otimes x + d \otimes y,\end{aligned}$$

on $B := O(X) = \mathbb{C}[x, y]$ since the coalgebra structure (3.1.1) of A ensures that $(\mathrm{id} \otimes \rho) \circ \rho = (\Delta \otimes id) \circ \rho$ and $(\varepsilon \otimes \mathrm{id}) \circ \rho = \mathrm{id}$.

However, this comodule structure does not extend to the field $\mathbb{C}(x, y)$ of rational functions.

Proposition 3.6.3. *Let*

$$e_{ijk} := \begin{cases} a^i b^j c^k & i \geq 0 \\ d^i b^j c^k & i < 0, \end{cases}$$

then the set $\{e_{ijk} : i \in \mathbb{Z},\ j, k \in \mathbb{N}\}$ *is a vector space basis for* $O(SL_2(\mathbb{C}))$.

Proof. We define an order on the set $\{e_{ijk} : i \in \mathbb{Z},\ j, k \in \mathbb{N}\} \subseteq O(SL_2(\mathbb{C}))$ by

$$e_{ijk} < e_{rst} \iff \begin{cases} i < r, & \text{or} \\ i = r,\ j < s, & \text{or} \\ i = r,\ j = s,\ k < t, \end{cases}$$

for all $i, r \in \mathbb{Z},\ j, k, s, t \in \mathbb{N}$. Following the setup of the diamond Lemma 3.3.1, take X to be the set $\{a, b, c, d\}$ of generators and I to be the ideal generated by $ad - bc - 1$. The result follows by Theorem 3.3.1. □

As the last ingredient, we define

Definition 3.6.4. Let V be a vector space. Then, for $v \in A \otimes V$, we write

$$v = \sum_{i \in \mathbb{Z},\, j,k \in \mathbb{N}} e_{ijk} \otimes v_{ijk}.$$

The *leading coefficient* of v is the vector v_{ijk} for which e_{ijk} is maximal among all $\{e_{rst} : v_{rst} \neq 0\}$.

We are now ready to prove that the SL_2-action (3.6.1) does not extend to the field of rational functions on $\mathbb{C}$.

Proposition 3.6.5. *Let A and ρ be the $\mathbb{C}$-coalgebra respectively coaction map as in Example 3.6.2. Then the comodule structure defined by ρ on $\mathbb{C}[x, y]$ does not extend to $\mathbb{C}(x, y)$.*

Proof. We will prove this by contradiction. Suppose there is such a left coaction ρ on $\mathbb{C}(x, y)$ given by

$$\rho(1/x) = \sum_{i,j,k} e_{ijk} \otimes v_{ijk},$$

where $v_{ijk} \in \mathbb{C}(x, y)$, then we have

$$1 \otimes 1 = \rho(1) = \rho(x)\rho(1/x) = \sum_{ijk} a\, e_{ijk} \otimes x\, v_{ijk} + b\, e_{ijk} \otimes y\, v_{ijk} =: w$$

Case 1: $i \geq 0$
The leading term of w is $e_{i+1\, jk} \otimes x v_{ijk}$ but the leading term of the LHS is $1 \otimes 1$ and $e_{i+1\, jk} \neq 1$.

Case 2: $i < 0$
The leading term of w is $e_{i\, j+1\, k} \otimes y\, v_{ijk}$ since $ad = 1 + bc$, however the leading term on the LHS is $1 \otimes 1$ and since $j + 1 > 0$, $e_{i\, j+1\, k} \neq 1$.

□

Remark 3.6.6. In other words, there is no A-comodule algebra structure ρ on $\mathbb{C}(x, y)$ such that when restricted to $\mathbb{C}[x, y]$ we have $\rho(x) = a \otimes x + b \otimes y$ and $\rho(y) = c \otimes x + d \otimes y$. Thus, in general Hopf algebra (either commutative or non-commutative) (co)actions on coordinate rings of an affine varieties do not extend to their fields of rational functions. In this sense, we say quantum symmetries arising from such (co)actions are not *true quantum symmetries*.

3.6.1 Constructing the Hopf dual of $A(x^3, y^2)$

Recall that A is generated by group-like elements a, d, and twisted primitives r, s satisfying the relations:

$$[a, d] = [r, s] = [a, s] = [d, r] = 0,$$
$$a^2 = d^3, \quad d^2 s + dsd + sd^2 = 0,$$
$$r^2 = s^3, \quad ar = -ra, \quad s^2 d + sds + ds^2 = 0.$$

Constructing the full Hopf dual of a Hopf algebra is rather involved. Nevertheless, for our purpose, we will compute enough representations of A which gives us a sufficiently large Hopf subalgebra of A° to arrive at a conclusion. The question: is A° pointed will be answered by investigating the dimensions of irreducible representations of A. If all irreducible finite dimensional representations of A are 1-dimensional, then the answer is in the affirmative (see Proposition 2.3.14 and Corollary 2.3.15).

Consider the quotient $\tilde{A}$ of the Hopf algebra A by the Hopf ideal generated by $a^2 - d^3 - 1$. $\tilde{A}^\circ$ embeds into the dual A° via the dual of the quotient map $A \longrightarrow \tilde{A}$. In what follows, we compute explicitly 1-dimensional (which are simple modules) and 2- dimensional representations of $\tilde{A}$.

1-dimensional representations: Observe that the 1-dimensional representations $\pi : \tilde{A} \longrightarrow \mathbb{C}$ of $\tilde{A}$ are just complex numbers satisfying the defining relations of $\tilde{A}$. In particular, the counit ε of $\tilde{A}$ is a 1-dimensional representation defined by $a, d \mapsto 1$ and $r, s \mapsto 0$. We claim that all the 1-dimensional representations of $\tilde{A}$ are of this form: clearly $a^2 = 1 = d^3$ as ε is an algebra map and this relation give rise to six 1-dimensional representations: $\pi(a) = \pm 1$, $\pi(d) \in \{1, \xi, \xi^2\}$, where $\xi^3 = 1$, and $\pi(r) = \pi(s) = 0$.

2-dimensional Representations: These are given by 2×2 matrices which satisfy the defining relations of $\tilde{A}$. Note that the image $\pi(a)^2 = 1$ of the relation $a^2 = 1$ under π means the matrix $\pi(a)$ is diagonalizable. Thus it decomposes $\mathbb{C}^2$ into eigenspaces E_{λ_1} and E_{λ_2} corresponding to the eigenvalues $\lambda_1 = 1$ respectively $\lambda_2 = -1$. That is $\mathbb{C}^2 = E_{\lambda_1} \bigoplus E_{\lambda_2}$. Hence, with respect to any basis $\{(v_1, v_2)^T : v_1, v_2 \ \in \mathbb{C}\}$ of $\mathbb{C}^2$, we have

$$\pi(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{if } \dim E_{\lambda_1} = 1, \text{ and } \dim E_{\lambda_2} = 1,$$

or

$$\pi(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{if } \dim E_{\lambda_1} = 2, \text{ and } \dim E_{\lambda_2} = 0,$$

or

$$\pi(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{if } \dim E_{\lambda_1} = 0, \text{ and } \dim E_{\lambda_2} = 2.$$

If $\pi(a) = \pm I_2$, where I_2 is the 2×2 identity matrix, then the relation $ar = -ra$ in $\tilde{A}$ which translates to $\pi(a)\pi(r) = -\pi(r)\pi(a)$ implies that $\pi(r) = 0$. Thus it follows from $r^2 = s^3$ that $\pi(s)$ is a nilpotent matrix which in Jordan normal form is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Case (i): suppose $\pi(s) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $\pi(d) = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with $x, y, z, w \in \mathbb{C}$. Then, from the relation $s^2d + sds + ds^2 = 0$ we deduce that $z = 0$, and the relation $d^3 = 1$ yields

$$x^3 = 1 = w^3, \quad y(x^2 + wx + w^2) = 0,$$

but the image of $d^2s + dsd + sd^2 = 0$ under π implies that $x^2 + xw + w^2 = 0$. However since $x, w \in \{1, \xi, \xi^2\}$, then

$$x^2 + xw + w^2 = 0 \implies x \neq w.$$

Thus, $\pi(d)$ can assume any of the matrices:

$$\begin{pmatrix} 1 & y \\ 0 & \xi \end{pmatrix}, \begin{pmatrix} 1 & y \\ 0 & \xi^2 \end{pmatrix}, \begin{pmatrix} \xi & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \xi^2 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \xi & y \\ 0 & \xi^2 \end{pmatrix}, \begin{pmatrix} \xi^2 & y \\ 0 & \xi \end{pmatrix}.$$

These matrices up to conjugation can be brought to their standard normal form so that $y = 0$, and $\pi(d)$ can admit up to conjugation *six* different diagonal matrices with distinct 3^{rd} roots of unity on the diagonal.

Case (ii): if $\pi(s) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, the only non trivial relation to work with is $d^3 = 1$ and this means the matrix $\pi(d)$ has three distinct eigenvalues $1, \xi, \xi^2$ - third roots of unity. Thus up conjugation, $\pi(d)$ can be admit any of the following matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \xi^2 \end{pmatrix}, \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}, \begin{pmatrix} \xi^2 & 0 \\ 0 & \xi^2 \end{pmatrix}, \begin{pmatrix} \xi & 0 \\ 0 & \xi^2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $\pi(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the relation $as = sa$ implies

$$\pi(a)(\pi(s)v) = \pi(s)(\pi(a)v) = \pi(s)(\lambda v) = \lambda(\pi(s)v),$$

for eigenvalue λ and eigenvector v of matrix $\pi(a)$. In other words $\pi(s)v$ lies in the eigenspaces E_{λ_1} and E_{λ_2} of $\pi(a)$, thus $\pi(s)$ as a linear map leaves the two eigenspaces invariant and we conclude that for $\alpha, \beta \in \mathbb{C}$

$$\pi(s) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Furthermore, the relation $ar = -ra$ implies that for $\gamma, \delta \in \mathbb{C}$,

$$\pi(r) = \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix}$$

which in addition to the fact that $rs = sr$ yields

$$\gamma\beta = \alpha\gamma, \quad \alpha\delta = \beta\delta.$$

Thus, for $\gamma \neq 0$ and $\delta \neq 0$, we have $\alpha = \beta$ which further implies that $\pi(s) = \alpha I_2$. However, the relations $d^2s + dsd + sd^2 = 0$ and $s^2d + sds + ds^2 = 0$ yield $\alpha\pi(d)^2 = 0$ respectively $\alpha^2\pi(d) = 0$, but $d^3 = 1$ in the Hopf algebra A so $\alpha = 0$ and $\pi(s) = 0$ as a matrix. Therefore, $\pi(r)$ is nilpotent and in Jordan normal form,

$$\pi(r) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \pi(r) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ or } \pi(r) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, the commutation $ad = da$ similarly interprets to mean $\pi(d)$ leaves the eigenspaces of $\pi(a)$ invariant, hence $\pi(d) = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$ for $m, n \in \mathbb{C}$.

(i) If $\pi(r) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, the only non-trivial relation $d^3 = 1$ implies that $m^3 = 1 = n^3$. Thus, $\pi(d)$ admits up to conjugation *six* different diagonal matrices with the 3^{rd} roots of unity as diagonal entries

$$\begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \xi^2 \end{pmatrix}, \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}, \begin{pmatrix} \xi^2 & 0 \\ 0 & \xi^2 \end{pmatrix}, \begin{pmatrix} \xi & 0 \\ 0 & \xi^2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii): If $\pi(r) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $\pi(r) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, the non-trivial relations are $rd = dr$ and $d^3 = 1$ and they imply that $m = n$ respectively m and n are 3^{rd} roots of unity. Thus, $\pi(d)$ can admit *three* matrices;

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}, \begin{pmatrix} \xi^2 & 0 \\ 0 & \xi^2 \end{pmatrix}.$$

Remark 3.6.7. Notice that none of these 2-dimensional representations is irreducible: the vector $(1\ ,0)^T$ is an eigenvector to all the matrice representations. Hence the eigenspace $(1\ ,0)^T\mathbb{C}$ is a proper subspace of $\mathbb{C}^2$ which is invariant with respect to these representations.

One could continue in this way to find representations of dimension n however there is an irreducible 6-dimensional representation of $\tilde{A}$ as shown in the following result

Proposition 3.6.8. *Let π be a 6-dimensional representation of $\tilde{A}$, defined by*

$$\pi(a)(u_\pm) = u_\pm, \quad \pi(a)(v_\pm) = v_\pm, \quad \pi(a)(w_\pm) = w_\pm, \quad \pi(d)(u_\pm) = u_\pm,$$
$$\pi(d)(v_\pm) = \xi v_\pm, \quad \pi(d)(w_\pm) = \xi^2 w_\pm, \quad \pi(s)(u_\pm) = v_\pm, \quad \pi(s)(v_\pm) = \xi w_\pm,$$
$$\pi(s)(w_\pm) = \xi^2 u_\pm, \quad \pi(r)(u_\pm) = u_\mp, \quad \pi(r)(v_\pm) = v_\mp, \quad \pi(r)(w_\pm) = w_\mp,$$

where $\{u_\pm, v_\pm, w_\pm\}$ is a basis. π is irreducible.

Proof. Suppose π is reducible and W is a non-trivial subspace of the vector space $\mathbb{C}^6$ that is invariant with respect to π. Then, as stated earlier,the relation $a^2 = 1 = d^3$ implies that $\pi(a), \pi(d)$ decomposes W into eigenspaces $W_{\pm 1}$ and W_{1,ξ,ξ^2} corresponding to the eigenvalues of $\pi(a)$ respectively $\pi(d)$. Furthermore, the commutation relation $ad - da = 0$ in $\tilde{A}$, means $\pi(a)$ and $\pi(d)$ share a common eigenvector in W which up to rescaling is one of the basis elements. Notice that $\pi(s)$ cyclicly permutes the basis elements and since W is invariant under π then $\{u_\pm, v_\pm, w_\pm\} \subseteq W \implies W = \text{codom}(\pi)$. □

In particular A has an irreducible 6-dimensional representation thus

Corollary 3.6.9. *The Hopf dual A° of the Hopf algebra A is not pointed.*

Remark 3.6.10. The 6-dimensional representation π of A described in the above proposition motivates an example of a quantum symmetry of $k[t^2, t^3]$ as follows: Recall that the coaction of A on B is given by

$$x \mapsto a \otimes x + r \otimes 1, \quad y \mapsto d \otimes y + s \otimes 1.$$

Thus the 36 matrix coefficients $s_{(ij)(mn)}$ of the 6-dimensional representation π act on B from the right as follows:

$$x \triangleleft s_{(ij)(mn)} = \delta_{im}((-1)^j \delta_{jn} x + \delta_{j(n+1)})$$
$$y \triangleleft s_{(ij)(mn)} = \xi^m \delta_{jn}(\delta_{im} y + \delta_{i(m+1)}),$$

where $i, j, m, n \in \{1, \cdots, 6\}$. We denote by X respectively Y the 6×6 matrices

$$X = \begin{pmatrix} x & 1 & 0 & 0 & 0 & 0 \\ 1 & -x & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 1 & 0 & 0 \\ 0 & 0 & 1 & -x & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 1 \\ 0 & 0 & 0 & 0 & 1 & -x \end{pmatrix} \quad Y = \begin{pmatrix} y & 0 & 1 & 0 & 0 & 0 \\ 0 & y & 0 & 1 & 0 & 0 \\ 0 & 0 & \xi y & 0 & \xi & 0 \\ 0 & 0 & 0 & \xi y & 0 & \xi \\ \xi^2 & 0 & 0 & 0 & \xi^2 y & 0 \\ 0 & \xi^2 & 0 & 0 & 0 & \xi^2 y \end{pmatrix}$$

representing the action of the $s_{ij,mn}$ on x, y. Since $x^2 = y^3$ in B, then we must have that $X^2 = Y^3$ and therefore ($\det X \neq 0$ is easily verified) setting

$$X = T^3, \ Y = T^2 \text{ with } \ T := XY^{-1},$$

we obtain

$$T := \frac{1}{t^6 + 1} \cdot \begin{pmatrix} t^7 & t^4 & -\xi^2 t^5 & -\xi^2 t^2 & \xi t^3 & \xi \\ t^4 & -t^7 & -\xi^2 t^2 & \xi^2 t^5 & \xi & -\xi t^3 \\ t^3 & 1 & \xi^2 t^7 & \xi^2 t^4 & -\xi t^5 & -\xi t^2 \\ 1 & -t^3 & \xi^2 t^4 & -\xi^2 t^7 & -\xi t^2 & \xi t^5 \\ -t^5 & -t^2 & \xi^2 t^3 & \xi^2 & \xi t^7 & \xi t^4 \\ -t^2 & t^5 & \xi^2 & -\xi^2 t^3 & \xi t^4 & -\xi t^7 \end{pmatrix}.$$

T is a quantum symmetry on $k(t)$ in particular, when restricted to $B = k[t^2, t^3]$, it is again a quantum symmetry of $k[t^2, t^3]$ since $T^2, T^3 \in M_n(k[t^2, t^3])$.

Chapter 4

Quantum symmetries of function fields

The starting point for the main results of this thesis which are presented in this chapter is to demand that like classical symmetries, the examples of quantum symmetries of the singular curves discussed in the previous chapter extend to the field of rational functions of these curves. We in particular focus on the example of the cusp. The approach we take was maybe first applied by Manin in his construction of quantum $SL(2)$ as a Hopf algebra (co)acting on the quantum plane [37]. In this approach as detailed in Sections 4.1, 4.2, we constructed a bialgebra action on a k-algebra K from an algebra morphism $K \longrightarrow \mathrm{M}_n(K)$. The resulting bialgebra becomes a Hopf algebra if this morphism when viewed as an element of $\mathrm{M}_n(\mathrm{End}_k(K))$ is *strongly invertible* in the sense of Definition 4.1.2. In particular, if $K = k(t)$, this Hopf action does indeed restricts to $k[t^2, t^3]$.

4.1 Quantum automorphisms

4.1.1 Strongly invertible matrices

Let P be a unital associative ring. Recall that if $\sigma \in \mathrm{GL}_n(P)$ is an invertible $n \times n$-matrix with entries P, then the transpose σ^T is in general not invertible:

Example 4.1.1. If a, d are elements of a ring P with $da = 1$ but $ad \neq 1$, the matrix

$$\sigma := \begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix} \quad \text{is invertible with inverse} \quad \sigma^{-1} = \begin{pmatrix} d & -1 \\ 1 - ad & a \end{pmatrix}.$$

However, $\sigma^T = \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}$ is not invertible.

More precisely, $\sigma^T \in \mathrm{GL}_n(P)$ if and only if $\sigma \in \mathrm{GL}_n(P^{\mathrm{op}})$, where P^{op} denotes the opposite ring of P (the additive abelian group P equipped with the opposite multiplication $a \cdot_{\mathrm{op}} b := ba$). Indeed for $\sigma, \tau \in \mathrm{M}_n(P)$, suppose τ is the inverse of σ in P^{op} then we have

$$\begin{aligned}\delta_{ij} = (\sigma \cdot_{\mathrm{op}} \tau)_{ij} &= \sum_{r=1}^{n} \sigma_{ir} \cdot_{\mathrm{op}} \tau_{rj} = \sum_{r=1}^{n} \tau_{rj}\sigma_{ir} \\ &= \sum_{r=1}^{n} \tau^T_{jr}\sigma^T_{ri} = (\tau^T\sigma^T)_{ji}\end{aligned}$$

that is $I_n = (\sigma \cdot_{\mathrm{op}} \tau)^T = \tau^T\sigma^T$, where on the left hand side, $\cdot_{\mathrm{op}}$ is the multiplication in $\mathrm{M}_n(P^{\mathrm{op}})$. In other words, $\sigma,\ \sigma^T \in \mathrm{GL}_n(P)$ if $\sigma \in \mathrm{GL}_n(P) \cap \mathrm{GL}_n(P^{\mathrm{op}})$. This yields the *contragredient isomorphism* (see e.g. [26, Chapter 3] for a discussion of this map)

$$\mathrm{GL}_n(P) \to \mathrm{GL}_n(P^{\mathrm{op}}), \quad \sigma \mapsto \bar{\sigma} := (\sigma^{-1})^T$$

The theory of Hopf algebras motivates to study those matrices to which one can apply this operation arbitrarily often without leaving $\mathrm{GL}_n(P)$; we are not aware of a standard name for such matrices, so we introduce a working terminology:

Definition 4.1.2. We call $\sigma \in \mathrm{M}_n(P)$ *strongly invertible* if there exists a sequence $\{\sigma_d\}_{d\in\mathbb{Z}}$ in $\mathrm{GL}_n(P)$ with $\sigma_0 = \sigma$ and $\sigma_{d+1} = \bar{\sigma}_d$.

Note this means that also all σ_d^T are invertible with $(\sigma_d^T)^{-1} = \sigma_{d-1}$. Thus, a necessary condition for σ to be strongly invertible is for $\sigma,\ \sigma^T \in \mathrm{GL}_n(P)$. We will focus on upper triangular matrices; for these, the condition is relatively easily controlled:

Proposition 4.1.3. *If $\sigma \in \mathrm{M}_n(P)$ is upper triangular, $\sigma_{ij} = 0$ for $i > j$, then σ is strongly invertible if and only if $\sigma_{ii} \in P$ is invertible for $i = 1, \ldots, n$. In this case, σ^{-1} is upper triangular and all $(\sigma^{-1})_{ij}$ are contained in the subring of P generated by the σ_{ij} and the σ_{ii}^{-1}.*

Proof. "$\Rightarrow$": Suppose σ is strongly invertible. Then σ and σ^T are invertible. As $\sigma_{ni} = 0$ for $i < n$, we then have

$$(\sigma\sigma^{-1})_{nn} = \sigma_{nn}(\sigma^{-1})_{nn} = 1 \tag{4.1.1}$$

and similarly

$$1 = ((\sigma^T)^{-1}\sigma^T)_{nn} = ((\sigma^T)^{-1})_{nn}\sigma_{nn}.$$

Hence, σ_{nn} is invertible in P with inverse

$$(\sigma_{nn})^{-1} = (\sigma^{-1})_{nn} = ((\sigma^T)^{-1})_{nn}.$$

Moreover, (4.1.1) shows

$$\sigma_{nn}(\sigma^{-1})_{nj} = 0 \ \ \forall\, j = 1, \cdots, n-1.$$

But σ_{nn} is invertible, thus

$$(\sigma^{-1})_{nj} = 0 \ \ \forall\, j = 1, \cdots, n-1.$$

Analogously, one shows that $((\sigma^T)^{-1})_{jn}$ vanishes for $j = 1, \ldots, n-1$. So σ, σ^{-1}, and $(\sigma^T)^{-1}$ can be written in block matrix form as

$$\sigma = \begin{pmatrix} \alpha & \mu \\ 0^T & \sigma_{nn} \end{pmatrix}, \qquad \sigma^{-1} = \begin{pmatrix} \beta & \gamma \\ 0^T & \sigma_{nn}^{-1} \end{pmatrix}, \qquad (\sigma^T)^{-1} = \begin{pmatrix} \delta & 0 \\ \nu^T & \sigma_{nn}^{-1} \end{pmatrix},$$

where $\alpha, \beta, \delta \in \mathrm{M}_{n-1}(P)$, μ, γ, ν are column vectors in P^{n-1}, and 0 is the zero vector in P^{n-1}.

From $\sigma\sigma^{-1} = 1 = \sigma^{-1}\sigma$ we obtain that α is invertible with inverse β. Analogously, α^T is invertible with inverse δ. We also obtain

$$\gamma = -\alpha^{-1}\mu\sigma_{nn}^{-1},$$

so the entries γ_j are elements of the subring of P generated by the entries of σ and of α^{-1}. Continuing inductively one obtains the claim.

"$\Leftarrow$": Suppose the diagonal entries σ_{ii} of $\sigma = \sigma_0$ are invertible. We show that the equation $\sigma\tau = 1$ can be solved inductively in the ring of upper triangular matrices with entries in P. Indeed, this equation means that

$$\sum_{m=l}^{n} \sigma_{lm}\tau_{mn} = \delta_{ln}.$$

These equations can be solved by induction on $n - l$. For $n - l = 0$ we obtain

$$\tau_{ll} = \sigma_{ll}^{-1}.$$

For $n - l = i$, we obtain

$$\tau_{l\ i+l} = -\sigma_{ll}^{-1}\bigg(\sum_{m=l+1}^{n} \sigma_{lm}\tau_{mn}\bigg).$$

Furthermore, solving the equation $\tau\sigma = 1$ inductively as above, we conclude that $\tau = \sigma^{-1}$. Analogously, one can show by solving the equations $\sigma^T\rho = 1$ and $\rho\sigma^T = 1$ inductively in the ring of lower triangular matrices that σ^T is invertible with inverse ρ. □

4.1.2 A quantum Galois group

Let now $k \subseteq B$ be a ring extension and let $P = \mathrm{End}_k(B)$ be the ring of k-linear maps $B \to B$. A matrix $\sigma \in \mathrm{M}_n(\mathrm{End}_k(B))$ can alternatively be viewed as a map $B \to \mathrm{M}_n(B)$, and we can demand this to be a ring morphism:

Definition 4.1.4. A *quantum automorphism* of B over $k \subseteq B$ is a strongly invertible matrix $\sigma \in \mathrm{M}_n(\mathrm{End}_k(B))$ satisfying

$$\sigma_{ij}(1) = \delta_{ij}, \qquad \sigma_{ij}(ab) = \sum_{l=1}^{n} \sigma_{il}(a)\sigma_{lj}(b) \qquad \forall\, a, b \in B. \tag{4.1.2}$$

Given a quantum automorphism, we denote by

$$U_\sigma \subseteq \mathrm{End}_k(B)$$

the ring generated by the entries $\sigma_{d,ij}$ of the $\sigma_d \in \mathrm{M}_n(\mathrm{End}_k(B))$, $\sigma_0 = \sigma$, $\sigma_{d+1} = \bar{\sigma}_d$.

For $n = 1$ this just means that σ is a ring automorphism of B that fixes k pointwise, and U_σ is the group ring of the subgroup of the Galois group $\mathrm{Gal}(B/k)$ of B over k that is generated by σ. In this sense, the set of quantum automorphisms is a generalisation of $\mathrm{Gal}(B/k)$.

If B is noncommutative, σ^{-1} is in general not a ring morphism. However, note that the set of quantum automorphisms is closed under $\sigma \mapsto \bar{\sigma}$:

Proposition 4.1.5. *If σ is a quantum automorphism, then so is $\bar{\sigma}$.*

Proof. The key point is to show that $\bar{\sigma}$ is multiplicative. To see this, first apply σ^{-1}_{pi} to (4.1.2) and sum over i. This yields $\delta_{pj}ab = \sum_{i,l=1}^{n} \sigma^{-1}_{pi}(\sigma_{il}(a)\sigma_{lj}(b))$. Inserting into this equation $a = \sigma^{-1}_{qr}(c), b = \sigma^{-1}_{jq}(d)$ for elements $c, d \in B$ and summing over j and q yields the claim:

$$\begin{aligned}
\sum_{q=1}^{n} \bar{\sigma}_{rq}(c)\bar{\sigma}_{qp}(d) &= \sum_{q=1}^{n} \sigma^{-1}_{qr}(c)\sigma^{-1}_{pq}(d) = \sum_{q,j=1}^{n} \delta_{pj}\sigma^{-1}_{qr}(c)\sigma^{-1}_{jq}(d) \\
&= \sum_{i,j,l,q=1}^{n} \sigma^{-1}_{pi}(\sigma_{il}(\sigma^{-1}_{qr}(c))\sigma_{lj}(\sigma^{-1}_{jq}(d))) \\
&= \sum_{i,l,q=1}^{n} \sigma^{-1}_{pi}(\sigma_{il}(\sigma^{-1}_{qr}(c))\delta_{lq}d) = \sum_{i,l=1}^{n} \sigma^{-1}_{pi}(\sigma_{il}(\sigma^{-1}_{lr}(c))d) \\
&= \sum_{i=1}^{n} \sigma^{-1}_{pi}(\delta_{ir}cd) = \sigma^{-1}_{pr}(cd) = \bar{\sigma}_{rp}(cd).
\end{aligned}$$

□

However, even when B is commutative, the (matrix) product of quantum automorphisms is in general not a quantum automorphism, so quantum automorphisms do not form groups. As we will explain in Section 4.2, they instead generate quantum groups (Hopf algebras).

4.1.3 A quantum subgroup

Even for basic examples of ring extensions, the set of all quantum automorphisms is too enormous to classify. There are various subsets that one can focus on, and we will in particular be interested in the following two attributes:

Definition 4.1.6. A quantum automorphism σ is

1. *upper triangular* if $\sigma_{ij} = 0$ for $i > j$, and
2. *locally finite* if for all $a \in B$, the set $\{\sigma_{ij}(a) \mid \sigma_{ij} \in U_\sigma\}$ is contained in a finitely generated k-module.

We denote by $\mathrm{QB}_n(B/k) \subseteq \mathrm{M}_n(\mathrm{End}_k(B))$ the set of all quantum automorphisms which share these two properties.

Both conditions will be motivated and explained further in Section 4.2. For now, we only point out that the upper triangularity makes it particularly easy to find such quantum automorphisms:

Corollary 4.1.7. *A k-linear ring morphism $\sigma\colon B \to \mathrm{M}_n(B)$ with $\sigma_{ij} = 0$ for $i > j$ is a quantum automorphism if and only if its diagonal entries σ_{ii} are invertible for $i = 1, \cdots, n$. In this case, U_σ is generated by the σ_{ij} together with the σ_{ii}^{-1}.*

Proof. This follows immediately from Proposition 4.1.3. □

A typical situation in which local finiteness holds is the following:

Proposition 4.1.8. *Suppose B is a k-algebra, that $\{F_d\}_{d\in\mathbb{Z}}$ is an exhaustive k-algebra filtration of B with $\dim_k F_d < \infty$, and that $\sigma \in \mathrm{End}_k(B)$ is a quantum automorphism with $\sigma_{ij}(F_d) \subseteq F_d$. Then σ is locally finite.*

Proof. By assumption, all F_d are invariant under the action of U_σ, and if $b \in B$ is any element, then there exists $d \in \mathbb{Z}$ with $b \in F_d$, and hence $U_\sigma b \subseteq F_d$ is finite-dimensional. □

4.1.4 Quantum automorphisms of $k(t)$

We will be interested in quantum automorphisms of coordinate rings of singular plane curves whose field of fractional functions is the field $k(t)$, and we first classify the upper triangular quantum automorphisms of the latter.

Proposition 4.1.9. *For any field k, the assignment $\sigma \mapsto \sigma(t) \in \mathrm{M}_n(k(t))$ defines a bijection between upper triangular quantum automorphisms of $k(t)$ over k and upper triangular matrices in $\mathrm{M}_n(k(t))$ whose diagonal entries are of the form*

$$\sigma(t)_{ii} = \frac{\alpha_i t + \beta_i}{\gamma_i t + \delta_i}$$

for some $\alpha_i, \beta_i, \gamma_i, \delta_i \in k$, $\alpha_i \delta_i - \beta_i \gamma_i \neq 0$.

Proof. For any ring extension $k \subseteq M$, $\sigma \mapsto T := \sigma(t)$ defines a bijection between the set $\{\sigma \colon k[t] \to M\}$ of k-linear ring morphisms and M.

Such a ring morphism extends in at most one way to a ring morphism $\sigma \colon k(t) \to M$ given by $\frac{p}{q} \mapsto p(T)q(T)^{-1}$, and it does extend if and only if for any $q \in k[t] \setminus \{0\}$ the element $q(T) \in M$ is invertible in M.

Specialising these general considerations to the case $M = \mathrm{M}_n(k(t))$, we have furthermore by elementary linear algebra over fields:

1. $p(T)$ is upper triangular for all $p \in k[t]$ if and only if T is so.
2. $q(T)$ is invertible if and only if $\det(q(T)) \neq 0$.
3. If $q(T)$ is invertible and upper triangular, so is $q(T)^{-1}$.
4. If T is upper triangular, $\det(q(T)) = q(T_{11}) \cdots q(T_{nn})$.

We conclude that k-linear ring morphisms $k(t) \to \mathrm{M}_n(k(t))$ that are upper triangular correspond bijectively to upper triangular matrices $T \in \mathrm{M}_n(k(t))$ with $q(T_{ii}) \neq 0$ for all $i = 1, \ldots, n$ and $q \in k[t] \setminus \{0\}$.

Corollary 4.1.7 shows that such a ring morphism is a quantum automorphism if and only if its diagonal entries σ_{ii} are in the Galois group of $k(t)$ over k, which is well-known to be the group of Möbius transformations [17, Theorem 7.5.7]. So if we are given an upper triangular matrix T that defines an upper triangular quantum automorphism of $k(t)$, the T_{ii} are necessarily of the form as stated. Conversely, if all T_{ii} are of this form, then we also have $q(T_{ii}) \neq 0$ for all $q \in k[t] \setminus \{0\}$, as the inverse of the unique k-linear ring automorphism $\sigma_{ii} \colon k(t) \to k(t)$ that maps t to T_{ii} maps $q(T_{ii})$ to q; thus T defines a ring morphism $k(t) \to \mathrm{M}_n(k(t))$ which by Corollary 4.1.7 is a quantum automorphism. □

4.1.5 Restriction to $k[t^2, t^3]$

A quantum automorphism σ of $k(t)$ restricts to an intermediate ring $k \subseteq B \subseteq k(t)$ if and only if we have

$$b(\sigma(t)) \in \mathrm{M}_n(B) \quad \forall\, b \in B \tag{4.1.3}$$

If we consider

$$B = k[t^2, t^3] = \mathrm{span}_k\{t^i \mid i \neq 1\} \subseteq k[t],$$

then it is evidently sufficient to test (4.1.3) only for $b = t^2$ and $b = t^3$. In other words, we have:

Corollary 4.1.10. *A quantum automorphism σ of $k(t)$ restricts to $k[t^2, t^3]$ if and only if $T^2, T^3 \in \mathrm{M}_n(k[t^2, t^3])$, where $T = \sigma(t)$.*

When classifying upper triangular quantum automorphisms of $k[t^2, t^3]$ that extend to $k(t)$, it is sufficient to consider matrices whose entries are Laurent polynomials:

Proposition 4.1.11. *If an upper triangular matrix $T \in \mathrm{M}_n(k(t))$ satisfies $T^2, T^3 \in \mathrm{M}_n(k[t^2, t^3])$ then $T \in \mathrm{M}_n(k[t, t^{-1}])$ whose entries contain no terms of degree less than $-3n + 4$.*

Proof. We prove that the T_{ij} are Laurent polynomials by induction on $j - i$.

$j - i = 0$: We have shown that the diagonal entries are Möbius transformations i.e are of the form $T_{ii} = \frac{\alpha_i t + \beta_i}{\gamma_i t + \delta_i}$. The condition $T^2 \in \mathrm{M}_n(k[t^2, t^3])$ and $\alpha_i\delta_i - \beta_i\gamma_i \neq 0$ forces $\beta_i = \gamma_i = 0$, so without loss of generality, we have $T_{ii} = \alpha_i t$.

$j - i = 1$: We get the equations

$$(T^2)_{i\,i+1} = (\alpha_i + \alpha_{i+1})tT_{i\,i+1},$$

$$(T^3)_{i\,i+1} = (\alpha_i^2 + \alpha_i\alpha_{i+1} + \alpha_{i+1}^2)t^2T_{i\,i+1}$$

are in $k[t^2, t^3]$ if either $tT_{i\,i+1} \in k[t^2, t^3]$ or $t^2T_{i\,i+1} \in k[t^2, t^3]$ since it is impossible that both the scalar factors are zero hence, $T_{i\,i+1}$ is a Laurent polynomial and contains no term of degree < -2.

$j - i = n$: Assume that $T_{i\,i+r}$ is for all $r < n$ a Laurent polynomial and contains no term of degree less than $-3n + 1$.

By assumption, the elements

$$(T^2)_{i\,n+i} = (\alpha_i + \alpha_{n+i})tT_{i\,n+i} + \sum_{r=1}^{n-1} T_{i\,i+r}T_{i+r\,n+i}$$
$$(T^3)_{i\,n+i} = (\alpha_i^2 + \alpha_i\alpha_{n+i} + \alpha_{n+i}^2)t^2T_{i\,n+i} + \\ + \sum_{r=1}^{n-1} T_{i\,i+r}(T^2)_{i+r\,n+i} + \alpha_i tT_{i\,i+r}T_{i+r\,n+i}$$

must be in $k[t^2, t^3]$. As both scalars $\alpha_i + \alpha_{n+i}$ and $\alpha_i^2 + \alpha_i\alpha_{n+i} + \alpha_{n+i}^2$ cannot be simultaneously zero, it follows from the induction hypothesis that $T_{i\,n+i}$ is a Laurent polynomial which contains no term of degree less than $-3n + 1$. □

4.1.6 Complete classification for $n = 2, 3$

Recall that for $l \in \mathbb{N}$ and $\beta \in k$, one defines the quantum numbers

$$[\![l]\!]_\beta := 1 + \beta + \cdots + \beta^{l-1} = \frac{1 - \beta^l}{1 - \beta},$$

where the last equality of course only applies when $\beta \neq 1$.

Lemma 4.1.12. *If $z = \sum_{i\in\mathbb{Z}} z_i t^i$, then the matrix*

$$T = \begin{pmatrix} \alpha t & z \\ 0 & \alpha\beta t \end{pmatrix}$$

corresponds to a quantum automorphism of $k[t^2, t^3]$ if and only if

1. $[\![2]\!]_\beta = 0 \Leftrightarrow \beta = -1$ *and* $z_{-1} = z_{-3} = z_{-4} = \ldots = 0$, *or*
2. $[\![3]\!]_\beta = 0 \Leftrightarrow \beta = e^{\pm 2\pi i/3}$ *and* $z_0 = z_{-2} = z_{-3} = \ldots = 0$ *or*
3. $[\![2]\!]_\beta, [\![3]\!]_\beta \neq 0$ *and* $z_0 = z_{-1} = z_{-2} = \ldots = 0$.

Proof. We have

$$T^2 = \alpha \begin{pmatrix} \alpha t^2 & [\![2]\!]_\beta tz \\ 0 & \alpha\beta^2 t^2 \end{pmatrix}, \quad T^3 = \alpha^2 \begin{pmatrix} \alpha t^3 & [\![3]\!]_\beta t^2 z \\ 0 & \alpha\beta^3 t^3 \end{pmatrix}$$

so T corresponds to a quantum automorphism of $k[t^2, t^3]$ if and only if

$$[\![2]\!]_\beta tz, \quad [\![3]\!]_\beta t^2 z \in k[t^2, t^3].$$

That is either $[\![2]\!]_\beta = 0$, that is $\beta = -1$ which implies that $[\![3]\!]_\beta t^2 z = t^2 z \in k[t^2, t^3]$ meaning $z_{-1} = z_{-3} = z_{-4} = \cdots = 0$ or $[\![3]\!]_\beta t^2 z = 0$ which thus implies $z_0 = z_{-2} = z_{-3} = \ldots = 0$, Or $[\![2]\!]_\beta tz$ and $[\![3]\!]_\beta t^2 z$ are non zero, in this case, $z \in k[t]/k$. □

Lemma 4.1.13. *If $z = \sum_{i\in\mathbb{Z}} z_i t^i$, $y = \sum_{j\in\mathbb{Z}} y_j t^j$, and $x = \sum_{l\in\mathbb{Z}} x_l t^l$, then the matrix*

$$T = \begin{pmatrix} \alpha t & x & z \\ 0 & \alpha\beta t & y \\ 0 & 0 & \alpha\beta\gamma t \end{pmatrix}$$

corresponds to a quantum automorphism of $k[t^2, t^3]$ if and only if we have

(a) $\beta = -1$ *and* $x_{-1} = x_{-3} = x_{-4} = \ldots = 0$, *or*

(b) $\beta = e^{\pm 2\pi i/3}$ *and* $x_0 = x_{-2} = x_{-3} = \ldots = 0$, *or*

(c) $\beta \neq -1, e^{\pm 2\pi i/3}$ *and* $x_0 = x_{-1} = x_{-2} = \ldots = 0$

and

(A) $\gamma = -1$ *and* $y_{-1} = y_{-3} = y_{-4} = \ldots = 0$, *or*

(B) $\gamma = e^{\pm 2\pi i/3}$ *and* $y_0 = y_{-2} = y_{-3} = \ldots = 0$, *or*

(C) $\gamma \neq -1, e^{\pm 2\pi i/3}$ *and* $y_0 = y_{-1} = y_{-2} = \ldots = 0$,

and

(r) $\beta \neq -1 - \gamma^{-1}$ *and there are* $a, b \in k[t^2, t^3]$ *with*

$$z = (1 + \beta + \beta\gamma)t^{-1}a - t^{-2}b, \quad xy = \alpha(-[\![3]\!]_{\beta\gamma} a + [\![2]\!]_{\beta\gamma} t^{-1} b),$$

or

(s) $\beta = -1 - \gamma^{-1}$ *and*

$$c := xy - \alpha\gamma tz \in k[t^2, t^3], [\![3]\!]_\gamma tc \in k[t^2, t^3].$$

Proof. Applying Lemma 4.1.12 to the two matrices obtained by deleting from T the first respectively third row and column leads to the conditions a,b,c,A,B,C. The remaining two conditions r,s arise from considering the $(1, 3)$-entry of T^2, T^3:

$$\begin{pmatrix} \alpha[\![2]\!]_{\beta\gamma} & 1 \\ \alpha^2[\![3]\!]_{\beta\gamma} t & \alpha(1 + \beta[\![2]\!]_\gamma)t \end{pmatrix} \begin{pmatrix} tz \\ xy \end{pmatrix} =: \begin{pmatrix} a \\ b \end{pmatrix} \in k[t^2, t^3]^2$$

The determinant of the coefficient matrix is

$$\alpha^2\beta t(1+\gamma+\beta\gamma).$$

So we can invert the matrix over $k[t,t^{-1}]$ if

$$\beta \neq -1-\gamma^{-1}.$$

In this regular case, we obtain

$$\frac{1}{\alpha^2\beta(1+\gamma+\beta\gamma)}\begin{pmatrix} (1+\beta[\![2]\!]_\gamma)t^{-1} & -t^{-2} \\ -\alpha[\![3]\!]_{\beta\gamma} & \alpha[\![2]\!]_{\beta\gamma}t^{-1}\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix} z \\ xy\end{pmatrix}$$

By rescaling a,b this yields elements $a,b\in k[t^2,t^3]$ with

$$z=(1+\beta+\beta\gamma)t^{-1}a-t^{-2}b$$

and

$$xy=\alpha(-[\![3]\!]_{\beta\gamma}a+[\![2]\!]_{\beta\gamma}t^{-1}b).$$

In the singular case $1+\gamma+\beta\gamma=0$, the equation reduces to

$$c:=xy-\alpha\gamma tz\in k[t^2,t^3],\quad [\![3]\!]_\gamma tc\in k[t^2,t^3].$$

□

4.1.7 An explicit example

From now on, we assume that $2,3$ are invertible in k, and that k contains a square root i of -1. We will study in detail the following example of a quantum automorphism of $k(t)$:

$$\sigma(t)=T=\begin{pmatrix} t & t-i & -\frac{1}{3}t^{-1}-\frac{1}{2}t \\ 0 & -t & t+i \\ 0 & 0 & t\end{pmatrix}.$$

Its restriction to $B=k[t^2,t^3]$ yields:

$$\sigma(x)=\begin{pmatrix} x & 0 & \frac{1}{3} \\ 0 & x & 0 \\ 0 & 0 & x\end{pmatrix},\ \sigma(y)=\begin{pmatrix} y & y-ix & -\frac{1}{2}y \\ 0 & -y & y+ix \\ 0 & 0 & y\end{pmatrix}$$

where $x:=t^2,\ y:=t^3$.

By definition, the resulting algebra U_σ has four generators that act as follows on the elements x, y:

$$\begin{gathered}\mathsf{K} := \sigma_{22} : x \mapsto x,\ y \mapsto -y,\\ \mathsf{E} := \sigma_{12} : x \mapsto 0,\ y \mapsto y - ix, \qquad \mathsf{F} := \sigma_{23} : x \mapsto 0,\ y \mapsto y + ix, \\ \mathsf{Z} := \sigma_{13} : x \mapsto \frac{1}{3},\ y \mapsto -\frac{1}{2}y.\end{gathered} \tag{4.1.4}$$

since σ is upper triangular, the operator K is an algebra automorphism of $k[t^2, t^3]$, so for all $f, g \in k[t^2, t^3]$ we have

$$\mathsf{K}(fg) = \mathsf{K}(f)\mathsf{K}(g).$$

The operators E, F are twisted derivations satisfying for $f, g \in k[t^2, t^3]$

$$\mathsf{E}(fg) = f\mathsf{E}(g) + \mathsf{E}(f)\mathsf{K}(g), \quad \mathsf{F}(fg) = \mathsf{K}(f)\mathsf{F}(g) + \mathsf{F}(f)g.$$

Similarly, Z is a twisted differential operator of order 2,

$$\mathsf{Z}(fg) = f\mathsf{Z}(g) + \mathsf{E}(f)\mathsf{F}(g) + \mathsf{Z}(f)g.$$

This and the action (4.1.4) completely determines $\mathsf{K}, \mathsf{E}, \mathsf{F}, \mathsf{Z}$ as k-linear maps on $k[t^2, t^3]$.

Lemma 4.1.14. *For all $n \in \mathbb{N}$, we have*

$$\mathsf{K}(t^n) = (-1)^n t^n, \tag{4.1.5}$$

$$\mathsf{E}(t^n) = \begin{cases} t^n - it^{n-1} & n \text{ is odd,} \\ 0 & n \text{ is even} \end{cases}, \quad \mathsf{F}(t^n) = \begin{cases} t^n + it^{n-1} & n \text{ is odd,} \\ 0 & n \text{ is even} \end{cases},$$

$$\mathsf{Z}(t^n) = \begin{cases} \frac{n-3}{6}t^{n-2} - \frac{1}{2}t^n & n \text{ is odd,} \\ \frac{n}{6}t^{n-2} & n \text{ is even.} \end{cases}.$$

Proof. We prove this by induction on n. For $n = 1$, we have

$$\mathsf{K}(t) = -t, \quad \mathsf{E}(t) = t - i, \quad \mathsf{F}(t) = t + i, \quad \mathsf{Z}(t) = -\frac{1}{3}t^{-1} - \frac{1}{2}t.$$

Assume for $n = p$ the formulas hold true. Then, as K is an automorphism,

$$\mathsf{K}(t^{p+1}) = \mathsf{K}(t^p)\mathsf{K}(t) = (-1)^{p+1}t^{p+1},$$

and since E is a twisted derivation, when p is odd we have:

$$\mathsf{E}(t^{p+1}) = \mathsf{E}(t^p t) = \mathrm{id}_B(t^p)\mathsf{E}(t) + \mathsf{E}(t^p)\mathsf{K}(t) = 0,$$

when p is even:

$$\mathsf{E}(t^{p+1}) = \mathsf{E}(t^p t) = \mathrm{id}_B(t^p)\mathsf{E}(t) + \mathsf{E}(t^p)\mathsf{K}(t) = t^{p+1} - it^p$$

which similarly holds for operator F. Finally for the second order differential operator Z, when p is odd,

$$\mathsf{Z}(t^{p+1}) = \mathrm{id}_B(t^p)\mathsf{Z}(t) + \mathsf{E}(t^p)\mathsf{F}(t) + \mathsf{Z}(t^p)\mathrm{id}_B(t) = \frac{p+1}{6}t^{p-1}$$

and when p is even, observe that the operator E vanishes thus we have,

$$\mathsf{Z}(t^{p+1}) = \mathrm{id}_B(t^p)\mathsf{Z}(t) + \mathsf{Z}(t^p)\mathrm{id}_B(t) = \frac{p-2}{6}t^{p-1} - \frac{1}{2}t^{p+1}.$$

□

In particular, one observes:

Corollary 4.1.15. *The restriction of σ to $B = k[t^2, t^3]$ is locally finite.*

Proof. The algebra $k[t^2, t^3]$ inherits a grading from $k[t]$, so

$$\deg(x) = 2, \quad \deg(y) = 3,$$

and if we denote by

$$F_d := \mathrm{span}_k\{1, t^2, t^3, \ldots, t^d\} \tag{4.1.6}$$

the resulting filtration of $k[t^2, t^3]$, then by the above formulas all assumptions of Proposition 4.1.8 are met. □

4.2 Geometric interpretation

This section contains the interpretation and motivation for the above computations: we interpret the algebra $B = k[t^2, t^3]$ as the coordinate ring of the cusp and explain how quantum automorphisms as described above give rise to (co)actions of Hopf algebras on B. Finally, we prove that the example discussed in Section 4.1.7 turns B into a quantum homogeneous space. For simplicity, Hopf algebra here means Hopf algebra with bijective antipode

4.2.1 The algebraic curve V

If k is any field and $k \subsetneq B \subseteq k(t)$ is any intermediate ring, then B is a subring of a field, hence an integral domain, and its fraction field embeds naturally into $k(t)$; so by Lüroth's theorem, the fraction field is isomorphic to $k(t)$. If B is the coordinate ring of an algebraic set V this means that V is an irreducible curve which is birationally equivalent to the affine line. In particular, when k is algebraically closed, then this is the case if and only if B is finitely generated as a k-algebra (by Hilbert's Nullstellensatz).

For $B = k[t^2, t^3]$, the curve V is the cusp

$$V = \{(\alpha, \beta) \in k^2 \mid \alpha^3 = \beta^2\} = \{(\lambda^2, \lambda^3) \mid \lambda \in k\} \subseteq k^2,$$

so in geometric terms, the theory developed in the previous section is about quantum automorphisms of the cusp that extend from its coordinate ring to its field of rational functions.

4.2.2 The Hopf algebra H_σ

In this section we will show that a quantum automorphism $\sigma \in \mathrm{M}_n(\mathrm{End}_k(B))$ of a k-algebra B gives rise to a Hopf algebra H_σ that acts inner faithfully ([4],[21]) on B. This construction follows the approach of [37]:

1. Consider the free k-algebra $k\langle s_{d,ij}\rangle$ with generators $s_{d,ij}$, $i, j = 1, \dots, n$, $d \in \mathbb{Z}$. This carries a unique bialgebra structure whose coproduct and counit are determined by

$$\Delta(s_{d,ij}) = \sum_{r=1}^{n} s_{d,ir} \otimes s_{d,rj}, \quad \varepsilon(s_{d,ij}) = \delta_{ij}.$$

2. Define an action of this free bialgebra on B in which the generators act by the entries of the quantum automorphisms σ_d:

$$\triangleright \colon k\langle s_{d,ij}\rangle \otimes B \to B, \quad s_{d,ij} \triangleright a := \sigma_{d,ij}(a).$$

This turns B into a $k\langle s_{d,ij}\rangle$-module algebra, that is, for any $X \in k\langle s_{d,ij}\rangle$ and $a, b \in B$, we have

$$X \triangleright (ab) = (X_{(1)} \triangleright a)(X_{(2)} \triangleright b).$$

3. If $I \subseteq k\langle s_{d,ij}\rangle$ is the ideal generated by all elements of the form

$$\sum_{r=1}^{n} s_{d,ir}s_{d+1,jr} - \delta_{ij}, \quad \sum_{r=1}^{n} s_{d+1,ri}s_{d,rj} - \delta_{ij}$$

for some d, i, j, then $k\langle s_{d,ij}\rangle/I$ becomes a Hopf algebra with (invertible) antipode induced by

$$S(s_{d,ij}) := s_{d+1,ji}$$

and the action $\triangleright$ on B descends by construction to this quotient. That is, if by abuse of notation we also denote by σ the action viewed as a morphism

$$\sigma \colon k\langle s_{d,ij}\rangle \to \mathrm{End}_k(B), \quad s_{d,ij} \mapsto \sigma_{d,ij}$$

then $I \subseteq \ker\sigma$ since for $i = j$ we have

$$\sum_{r=1}^{n} \sigma_{d,ir}\sigma_{d,ir}^{-1}(b) - b = 0 = \sum_{r=1}^{n} \sigma_{d,ri}^{-1}\sigma_{d,ri}(b) - b.$$

Thus σ induces a morphism

$$k\langle s_{d,ij}\rangle/I \to \mathrm{End}_k(B)$$

that we still denote by σ.

4. Up to here, the Hopf algebra $k\langle s_{d,ij}\rangle/I$ is not very interesting and depends only on the size n of the matrix $\sigma \in \mathrm{M}_n(\mathrm{End}_k(B))$ and not on the actual choice of σ or B.

Recall from Section 2.3.4 that the Hopf image H_σ of the representation σ is the universal quotient Hopf algebra that acts on B. In other words, H_σ is the quotient of $k\langle s_{d,ij}\rangle/I$ by the sum J_σ of all Hopf ideals contained in $\ker\sigma$.

Remark 4.2.1. More abstractly, steps (1)-(3) construct the free Hopf algebra with invertible antipode on the coalgebra $C := \mathrm{M}_n(k)^\star$, the dual of the algebra $\mathrm{M}_n(k)$. As discussed in Theorem 2.2.10, the free Hopf algebra on C was constructed, not forcing the antipode to be invertible. The corresponding version of the above construction would use only non-negative d in $s_{d,ij}$. The transpose is used to identify the coalgebra spanned by $s_{d,ij}$ for a fixed odd d with its coopposite. That is, the choice of a quantum automorphism σ of B turns B into a module algebra over this free Hopf algebra.

Proposition 4.2.2. *If σ is upper triangular, then we have:*

1. *H_σ is generated as an algebra by the classes $[s_{0,ij}]$ with $i \leq j$, together with $[s_{1,ii}] = [s_{0,ii}]^{-1}$.*

2. *H_σ is pointed.*

Proof. The first claim is shown in the same way as Proposition 4.1.3. The second claim uses a standard argument: define a Hopf algebra filtration $\{C_f\}$ of H_σ by assigning to $[s_{d,ij}]$ the filtration degree $j - i$,

$$C_f = \mathrm{span}_k\{[s_{d_1,i_1j_1}] \cdots [s_{d_l,i_lj_l}] \mid \sum_q j_q - i_q \leq f\}.$$

This is an algebra filtration by definition and a coalgebra filtration as $[s_{d,ij}] = 0$ if $i > j$. As the $[s_{d,ij}]$ generate H_σ as an algebra, it is exhaustive. If $S \subseteq H_\sigma$ is a simple subcoalgebra, then $\dim_k S < \infty$, so there exists a minimal $f \geq 0$ with $S \subseteq C_f$, $S \nsubseteq C_{f-1}$, and if $f > 0$, it is immediately verified that $S \cap C_{f-1}$ is a proper non-zero subcoalgebra of S, contradicting the fact that S is simple. Finally, if $S \subseteq C_0$ then S is spanned by group-likes and the span of any group-like is a subcoalgebra. So as S is simple, it is one-dimensional. □

4.2.3 Application to $k[t^2, t^3]$

For $B = k[t^2, t^3]$, the curve V is the cusp

$$V = \{(\alpha, \beta) \in k^2 \mid \alpha^3 = \beta^2\} = \{(\lambda^2, \lambda^3) \mid \lambda \in k\} \subseteq k^2,$$

so in geometric terms, the theory developed in the previous section is about quantum automorphisms of the cusp that extend from its coordinate ring to its field of rational functions. For the quantum automorphism described in Section 4.1.7, we abbreviate

$$K := [s_{0,22}], \quad E := [s_{0,12}], \quad F := [s_{0,23}], \quad Z := [s_{0,13}] \in H_\sigma.$$

By Proposition 4.2.2, H_σ is generated as an algebra by these elements whose image in U_σ are the operators $\mathsf{K}, \mathsf{E}, \mathsf{F}, \mathsf{Z}$, respectively. Furthermore, the fact that $[s_{0,ij}] = 0$ for $i > j$ implies that K is group-like, that is, its coproduct is given by

$$\Delta(K) = K \otimes K,$$

that E, F are $(1, K)$- respectively $(K, 1)$-twisted primitive,

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K \otimes F + F \otimes 1,$$

and Z is of degree 2 with respect to the coradical filtration of H_σ,

$$\Delta(Z) = 1 \otimes Z + E \otimes F + Z \otimes 1.$$

We will now obtain a presentation of H_σ as an algebra. First, we observe that the definition of H_σ implies:

Lemma 4.2.3. *We have $K^2 = 1$ and $F = -KE$.*

Proof. $K^2 - 1$ and $KE + F$ are in the kernel of the representation $\sigma\colon H_\sigma \to U_\sigma$. The coproduct of these elements are

$$\Delta(K^2 - 1) = K^2 \otimes K^2 - 1 \otimes 1 = K^2 \otimes (K^2 - 1) + (K^2 - 1) \otimes 1,$$

$$\Delta(KE + F) = K \otimes (KE + F) + (KE + F) \otimes 1,$$

It follows that each one generates a Hopf ideal in H_σ which is in the kernel of σ, so by definition of H_σ, these elements vanish. □

Thus H_σ is generated as an algebra by K, E, Z.

Second, we decompose E and Z into eigenvectors of the map given by conjugation by K; that is, we define

$$E_\pm := \frac{1}{2}(E \pm KEK), \qquad Z_\pm := \frac{1}{2}(Z \pm KZK).$$

By the definition of these elements, they (anti)commute with K:

Lemma 4.2.4. *We have $KE_\pm = \pm E_\pm K$ and $KZ_\pm = \pm Z_\pm K$.*

The coproduct of these elements and their action on B is given by

$$\begin{aligned}\Delta(E_\pm) &= 1 \otimes E_\pm + E_\pm \otimes K,\\ \Delta(Z_+) &= 1 \otimes Z_+ - E_+ \otimes E_+K - E_- \otimes E_-K + Z_+ \otimes 1,\\ \Delta(Z_-) &= 1 \otimes Z_- - E_+ \otimes E_-K - E_- \otimes E_+K + Z_- \otimes 1,\end{aligned}$$

$$\sigma(E_+)(t^n) = \begin{cases} t^n & n \text{ is odd,} \\ 0 & n \text{ is even,}\end{cases} \quad \sigma(E_-)(t^n) = \begin{cases} -it^{n-1} & n \text{ is odd,} \\ 0 & n \text{ is even,}\end{cases}$$

$$\sigma(Z_+)(t^n) = \begin{cases} \frac{n-3}{6}t^{n-2} - \frac{1}{2}t^n & n \text{ is odd,} \\ \frac{n}{6}t^{n-1} & n \text{ is even,}\end{cases} \quad \sigma(Z_-)(t^n) = \begin{cases} 0 & n \text{ is odd,} \\ 0 & n \text{ is even,}\end{cases}$$

From this we obtain in a similar manner as in Lemma 4.2.3:

Lemma 4.2.5. *We have* $Z_- = -E_+E_-$, $E_-^2 = 0$, $E_+ = -\frac{1}{2}(K-1)$.

Proof. It follows from the above and the relation $K^2 = 1$ that the elements $Z_- + E_+E_-$ and E_-^2 are primitive while $E_+ + \frac{1}{2}(K-1)$ is $(1, K)$-twisted primitive, and they are straightforwardly verified to be in $\ker\sigma$, hence as in Lemma 4.2.3 it follows that they vanish in H_σ. □

So H_σ is generated as an algebra by K, E_- and Z_+.

Finally, we abbreviate

$$Y := 6Z_+ - \frac{3}{2}(K-1), \quad D := iE_-, \quad C := YD - DY.$$

Their coproduct is given by

$$\Delta(Y) = 1 \otimes Y - 6D \otimes DK + Y \otimes 1, \quad \Delta(C) = 1 \otimes C + C \otimes K$$

and they act on B by the operators

$$\mathsf{Y}(t^n) := \begin{cases} (n-3)t^{n-2}, & n \text{ odd}, \\ nt^{n-2}, & n \text{ even}, \end{cases} \quad \mathsf{C}(t^n) := \begin{cases} 2t^{n-3}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases} \tag{4.2.1}$$

$$\mathsf{D}(t^n) := \begin{cases} t^{n-1} & n \text{ is odd}, \\ 0 & n \text{ is even}. \end{cases}$$

Their commutation relations (as elements in H_σ) are as follows:

Lemma 4.2.6. *We have* $YK = KY$, $KC = -CK$, $DC = -CD$, *and*

$$YC = CY, \quad C^2 = 0.$$

Proof. The relations $KY = YK, KC = -CK, DC = -CD$ follow from the definition of Y, C, D and the commutation relations already obtained. The remaining two relations follow as in Lemma 4.2.3; $YC - CY$ is $(1, K)$-twisted primitive while C^2 is primitive. □

Remark 4.2.7. Note that we can express the operators Y and D in terms of K, the differential operator $\frac{d}{dt}$, and the multiplication operators by t^{-m} as

$$\mathsf{D} = -\frac{1}{2}t^{-1}(\mathsf{K}-1), \qquad \mathsf{Y} = t^{-1}\frac{d}{dt} + \frac{3}{2}t^{-2}(\mathsf{K}-1).$$

The two summands in Y can be considered separately as operators on $k(t)$ that both restrict to $k[t,t^{-1}]$, but only the sum restricts to $k[t^2,t^3]$. As operators on $k(t)$, $\mathsf{Y}_0 := t^{-1}\frac{d}{dt}$ is a derivation and $\mathsf{Y}_1 := \frac{3}{2}t^{-2}(\mathsf{K}-1)$ is a twisted derivation,

$$\mathsf{Y}_0(fg) = f\mathsf{Y}_0(g) + \mathsf{Y}_0(f)g, \quad \mathsf{Y}_1(fg) = f\mathsf{Y}_1(g) + \mathsf{Y}_1(f)\mathsf{K}(g),$$

so it is a rather non-trivial fact that their sum is a twisted differential operator of order 2 on $k[t^2,t^3]$. The other generator D is a twisted derivation,

$$\mathsf{D}(fg) = f\mathsf{D}(g) + \mathsf{D}(f)\mathsf{K}(g).$$

Our aim is to prove that the relations we have found are complete. In order to do so, we define the auxiliary Hopf algebra

$$\begin{aligned}\tilde{H}_\sigma &:= k\langle \tilde{K}, \tilde{D}, \tilde{Y}\rangle / I,\\ I &:= \langle \tilde{K}^2 - 1, \tilde{K}\tilde{D} + \tilde{D}\tilde{K}, \tilde{K}\tilde{Y} - \tilde{Y}\tilde{K}, \tilde{Y}^2\tilde{D} - 2\tilde{Y}\tilde{D}\tilde{Y} + \tilde{D}\tilde{Y}^2, \tilde{D}^2\rangle\end{aligned}$$

as the algebra generated by $\tilde{K}, \tilde{D}, \tilde{Y}$ satisfying the relations established in the lemmata in this subsection, equipped with the coproduct given on generators by the same formulas as in H_σ. Bergman's diamond lemma 3.3.1 immediately yields:

Lemma 4.2.8. *If $\tilde{C} := \tilde{Y}\tilde{D} - \tilde{D}\tilde{Y}$, then the set*

$$\{\tilde{C}^a\tilde{D}^b\tilde{K}^c\tilde{Y}^d \mid a,b,c \in \{0,1\}, d \in \mathbb{N}\}$$

is a k-vector space basis of $\tilde{H}_\sigma$.

Proof. Following the setup of Bergman's diamond lemma 3.3.1, $X = \{\tilde{C}, \tilde{D}, \tilde{K}, \tilde{Y}\}$ and I is the two sided ideal spanned by the elements

$$\tilde{K}^2 - 1, \quad \tilde{K}\tilde{D} + \tilde{D}\tilde{K}, \quad \tilde{K}\tilde{Y} - \tilde{Y}\tilde{K}, \quad \tilde{D}^2, \quad \tilde{Y}\tilde{C} - \tilde{C}\tilde{Y}.$$

Clearly, $\tilde{H}_\sigma \cong k\langle X\rangle/I$ and there are no ambiguities, a close inspection reveals that the set described in the Lemma is vector space basis of $\tilde{H}_\sigma$. □

We now describe the algebra morphism $\sigma\colon \tilde{H}_\sigma \to U_\sigma$; by showing that its kernel contains no Hopf ideal, we will then prove that $\tilde{H}_\sigma = H_\sigma$.

By direct computation, one establishes that the generators $\mathsf{K}, \mathsf{C}, \mathsf{D}, \mathsf{Y}$ of U_σ satisfy the following relations in addition to those satisfied by $\tilde{K}, \tilde{C}, \tilde{D}$ and $\tilde{Y}$:

Lemma 4.2.9. *We have $\mathsf{CD} = 0$, $\mathsf{KC} = \mathsf{C}$, $\mathsf{KD} = \mathsf{D}$.*

A moment's thought tells that this is a complete presentation of U_σ:

Proposition 4.2.10. *The above relations define a presentation of U_σ.*

Proof. The claim is that if we define an abstract algebra $k\langle \mathsf{K}, \mathsf{C}, \mathsf{D}, \mathsf{Y}\rangle/R$, where R is the ideal generated by the above relations and those that follow from the ones between $\tilde{K}, \tilde{C}, \tilde{D}, \tilde{Y}$ in I, then the resulting algebra morphism $k\langle \mathsf{K}, \mathsf{C}, \mathsf{D}, \mathsf{Y}\rangle/R \to U_\sigma$ is an isomorphism. To do so, observe that using the k-vector space basis of $\tilde{H}_\sigma$ and the relations stated in the current proposition, we obtain a k-vector space basis of $k\langle \mathsf{K}, \mathsf{C}, \mathsf{D}, \mathsf{Y}\rangle/R$ of the form

$$\{\mathsf{Y}^a, \mathsf{C}\mathsf{Y}^b, \mathsf{D}\mathsf{Y}^c, \mathsf{K}\mathsf{Y}^d \mid a, b, c, d \in \mathbb{N}\}. \tag{4.2.2}$$

It is now straightforward to show that these operators are mapped to linearly independent elements of $\operatorname{End}_k(B)$, that is

$$\mathsf{Y}^a(t^n) = \begin{cases} (n-3)\cdots(n-(2a+1))t^{n-2a} & n \text{ is odd} \\ n\cdots(n-(2a-2))t^{n-2a} & n \text{ is even} \end{cases} \tag{4.2.3}$$

$$\mathsf{C}\mathsf{Y}^b(t^n) = \begin{cases} (n-3)\cdots(n-(2b+1))t^{n-3-2b} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

$$\mathsf{K}\mathsf{Y}^d(t^n) = \begin{cases} -(n-3)\cdots(n-(2d+1))t^{n-2d} & n \text{ is odd} \\ n\cdots(n-(2d-2))t^{n-2d} & n \text{ is even} \end{cases}$$

$$\mathsf{D}\mathsf{Y}^c(t^n) = \begin{cases} (n-3)\cdots(n-(2c+1))t^{n-1-2c} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}.$$

□

Remark 4.2.11. Note that U_σ carries a natural grading in which

$$\deg \mathsf{K} = 0, \quad \deg \mathsf{C} = 3, \quad \deg \mathsf{D} = 1, \quad \deg \mathsf{Y} = 2,$$

and that B becomes a graded U_σ-module, $(U_\sigma)_i B_j \subseteq B_{j-i}$.

In order to proceed, note that by Lemma 4.2.8 the subalgebra of $\tilde{H}_\sigma$ generated by $\tilde{Y}$ is as an abstract algebra the polynomial algebra $k[\tilde{Y}]$ and that $\{\tilde{C}^a\tilde{D}^b\tilde{K}^c \mid a, b, c \in \{0, 1\}\}$ is a basis of $\tilde{H}_\sigma$ as a right $k[\tilde{Y}]$-module, so as such, $\tilde{H}_\sigma$ has rank 8. Similarly, U_σ becomes a right $k[\tilde{Y}]$-module where $\tilde{Y}$ acts via right multiplication by Y, and by the above proposition, $\{1, \mathsf{C}, \mathsf{D}, \mathsf{K}\}$ is a basis of this $k[\tilde{Y}]$-module, so this has rank 4. The map $\sigma\colon \tilde{H}_\sigma \to U_\sigma$ is right $k[\tilde{Y}]$-linear, and we have:

Corollary 4.2.12. *As a right* $k[\tilde{Y}]$*-module,* $\ker\sigma$ *is free with basis given by*

$$\{\tilde{C}\tilde{K}+\tilde{C}, \tilde{D}\tilde{K}+\tilde{D}, \tilde{C}\tilde{D}, \tilde{C}\tilde{D}\tilde{K}\}.$$

Proof. As mentioned in the remark above $\tilde{H}_\sigma$ as a $k[\tilde{Y}]$-module has basis

$$\{1, \tilde{C}, \tilde{D}, \tilde{K}, \tilde{C}\tilde{D}, \tilde{C}\tilde{K}, \tilde{D}\tilde{K}, \tilde{C}\tilde{D}\tilde{K}\}.$$

Following the relations in Lemma 4.2.9, it follows that the set described in this corollary is a basis for ker σ. □

As a last ingredient, we list the one-dimensional representations of $\tilde{H}_\sigma$ and their left and right hit actions on $\tilde{H}_\sigma$:

Lemma 4.2.13. *For any* $s \in \{-1, 1\}, \lambda \in k$*, there is an algebra morphism*

$$\chi_{s,\lambda} \colon \tilde{H}_\sigma \to k, \quad \tilde{K} \mapsto s, \quad \tilde{D} \mapsto 0, \quad \tilde{Y} \mapsto \lambda$$

and any algebra morphism $\tilde{H}_\sigma \to k$ *is of this form.*

Proof. The defining relations of $\tilde{H}_\sigma$ motivate the definition of $\chi_{s,\lambda}$ and any algebra map $\tilde{H}_\sigma \longrightarrow k$ must respect these relations. □

These 1-dimensional representations define algebra automorphisms $\mathsf{L}_{s,\lambda}, \mathsf{R}_{s,\lambda} \colon \tilde{H}_\sigma \to \tilde{H}_\sigma$ given by

$$\mathsf{R}_{s,\lambda}(h) := \chi_{s,\lambda}(h_{(1)})h_{(2)}, \quad \mathsf{L}_{s,\lambda}(h) := h_{(1)}\chi_{s,\lambda}(h_{(2)}).$$

That is $\mathsf{R}_{s,\lambda} := (\chi_{s,\lambda} \otimes id) \circ \Delta$ and $\mathsf{L}_{s,\lambda} := (id \otimes \chi_{s,\lambda}) \circ \Delta$ which are composites of algebra maps. On the generators of $\tilde{H}_\sigma$, these automorphisms are given by

$$\mathsf{L}_{s,\lambda}(\tilde{K}) = \mathsf{R}_{s,\lambda}(\tilde{K}) = s\tilde{K},$$
$$\mathsf{L}_{s,\lambda}(\tilde{D}) = s\tilde{D}, \quad \mathsf{R}_{s,\lambda}(\tilde{D}) = \tilde{D},$$
$$\mathsf{L}_{s,\lambda}(\tilde{Y}) = \mathsf{R}_{s,\lambda}(\tilde{Y}) = \tilde{Y} + \lambda.$$

Note further that all one-dimensional representations of $\tilde{H}_\sigma$ descend to U_σ:

Lemma 4.2.14. *We have* $\ker\sigma \subseteq \bigcap_{s,\lambda} \ker\chi_{s,\lambda}$.

Proof. By the definition of $\chi_{s,\lambda}$, C, D, and any word $X \in \tilde{H}_\sigma$ containing alphabets C, D lie in ker $\chi_{s,\lambda}$. The claim thus follows from Lemma 4.2.12. □

We are now ready to prove:

Theorem 4.2.15. *The quotient $\tilde{H}_\sigma \to H_\sigma$ is an isomorphism.*

Proof. Assume that $J \subseteq \ker\sigma$ is a Hopf ideal. Then for all $h \in J$, we have $\Delta(h) \in J \otimes \tilde{H}_\sigma + \tilde{H}_\sigma \otimes J$; by the last lemma, applying $\chi_{s,\lambda}$ to the left or to the right tensor component yields an element in J. That is, we have

$$\mathsf{L}_{s,\lambda}(J) = \mathsf{R}_{s,\lambda}(J) = J.$$

The maps $\mathsf{R}_{s,\lambda}, \mathsf{L}_{s,\lambda}$ act on the basis elements from Lemma 4.2.8 by

$$\mathsf{L}_{s,\lambda}(\tilde{C}^a\tilde{D}^b\tilde{K}^c\tilde{Y}^d) = s^{a+b+c}\tilde{C}^a\tilde{D}^b\tilde{K}^c(\tilde{Y}+\lambda)^d,$$
$$\mathsf{R}_{s,\lambda}(\tilde{C}^a\tilde{D}^b\tilde{K}^c\tilde{Y}^d) = s^{c}\tilde{C}^a\tilde{D}^b\tilde{K}^c(\tilde{Y}+\lambda)^d.$$

Thus if

$$X = \sum_{abcd} \iota_{abcd}\tilde{C}^a\tilde{D}^b\tilde{K}^c\tilde{Y}^d \in J, \quad \iota_{abcd} \in k$$

and $d_{\max}(X)$ is the largest d such that $\iota_{abcd} \neq 0$ for some a, b, c, then unless $d_{\max} = 0$,

$$X' := X - \mathsf{R}_{1,1}(X) \in J$$

is a non-zero element with $d_{\max}(X') = d_{\max}(X) - 1$. So if $J \neq 0$, it necessarily contains a non-zero elment of the form

$$X = \sum_{abc} \iota_{abc}\tilde{C}^a\tilde{D}^b\tilde{K}^c.$$

Using now $\mathsf{R}_{-1,0}$ instead of $\mathsf{R}_{1,1}$, the analogous argument shows that J contains a non-zero element of the form

$$X = \sum_{ab} \iota_{ab}\tilde{C}^a\tilde{D}^b.$$

Considering finally

$$X \pm \mathsf{L}_{-1,0}(X)$$

we find that

$$\iota_{00} + \iota_{11}\tilde{C}\tilde{D} \in J, \quad \iota_{01}\tilde{C} + \iota_{10}\tilde{D} \in J.$$

Since $\mathsf{C}, \mathsf{D} \in U_\sigma$ are linearly independent and $J \subseteq \ker\sigma$, the second element vanishes. Using that the coproduct of $\tilde{C}\tilde{D}$ is

$$\Delta(\tilde{C}\tilde{D}) = 1 \otimes \tilde{C}\tilde{D} + \tilde{C} \otimes \tilde{K}\tilde{D} + \tilde{D} \otimes \tilde{C}\tilde{K} + \tilde{C}\tilde{D} \otimes 1$$

and $\tilde{C}, \tilde{D}$ are linearly independent modulo $\ker\sigma$, one also concludes that the first element vanishes, a contradiction. □

Remark 4.2.16. Thus H_σ does not act faithfully on B – for example, we now know that $CD \in H_\sigma$ is a non-zero element while $\mathsf{CD} = \sigma(CD) = 0$. However, H_σ acts by definition inner faithfully on B, that is, the action does descend to algebra, but not to Hopf algebra quotients of H_σ.

Remark 4.2.17. Thus H_σ is the Ore extension of the subalgebra generated by K, C, D by the derivation ∂ given by

$$K \mapsto 0, \quad D \mapsto C, \quad C \mapsto 0.$$

which is induced by the presentation of H_σ, for example: since $K^2 = 1$ in H_σ, we thus have $0 = \partial(1) = \partial(K^2) = K\partial(K) + \partial(K)K$ that is $K\partial(K) = -\partial(K)K$ which yields $\partial(K) = 0$. In particular, H_σ has Gelfan'd-Kirillov dimension 1, but note that it is not semiprime (the right ideal generated by C is a nonzero ideal that squares to zero) so is not part of the recent classification of these Hopf algebras (see e.g. [13, 36] and the references therein).

Remark 4.2.18. Note that the subalgebra generated by K and D (and similarly the subalgebra generated by K and C) is isomorphic as Hopf algebra to Sweedler's 4-dimensional Hopf algebra. Note that for any $c \in k$,

$$\begin{aligned} R_c := &\frac{1}{2}(1 \otimes 1 + 1 \otimes K + K \otimes 1 - K \otimes K) \\ &+ \frac{c}{2}(D \otimes D - D \otimes KD + KD \otimes D + KD \otimes KD) \end{aligned}$$

is a universal R-matrix for these Hopf subalgebras (see [44, Exercise 12.2.11]) that is they are cocommutative up to conjugation by R_c, more precisely,

$$R_c \Delta R_c^{-1} = \Delta^{\mathrm{cop}}.$$

The Hopf subalgebra generated by K and D(similarly by K and C) is quasitriangular. However, R_c does not define a quasitriangular structure on H_σ as

$$R_c \Delta(Y) R_c^{-1} \neq \Delta^{\mathrm{cop}}(Y),$$

but at least for $c = 0$, the corresponding braiding on $B \otimes B$ is a morphism of H_σ-modules. This braiding is simply the standard nontrivial symmetric braiding on the category of graded vector spaces,

$$t^i \otimes t^j \mapsto (-1)^{ij} t^j \otimes t^i.$$

4.2.4 The Hopf algebra A_σ

We now pass to a dual picture: assume that σ is a locally finite quantum automorphism of a k-algebra B. Then the H_σ-action on B arises by dualisation from an H_σ°-coaction, where H_σ° denotes the Hopf dual of H_σ. In other words, B is a right H_σ°-comodule algebra with a coaction that we denote by

$$\rho\colon B \to B \otimes H_\sigma^\circ, \quad b \mapsto b_{(0)} \otimes b_{(1)}.$$

Definition 4.2.19. We denote by $A_\sigma \subseteq H_\sigma^\circ$ the Hopf subalgebra generated by the matrix coefficients $\{f(b_{(0)})b_{(1)} \mid b \in B, f \in B^\star\}$ of ρ.

When H_σ is infinite-dimensional, A_σ could be a proper Hopf subalgebra of H_σ°, but note that it is always dense:

Proposition 4.2.20. *The restriction of the dual pairing of H_σ° and H_σ to $A_\sigma \otimes H_\sigma$ is non-degenerate.*

Proof. The degeneration space

$$\{X \in H_\sigma \mid a(X) = 0 \; \forall \, a \in A_\sigma\}$$

is the kernel of the Hopf algebra morphism, $\langle -, a\rangle : H_\sigma \longrightarrow k$ for each $a \in A_\sigma$. Thus it is a Hopf ideal of H_σ that acts trivially on B, hence vanishes by the definition of H_σ. □

Note that if $M \subseteq B$ is any finite-dimensional H_σ-submodule that generates B as an algebra, then we have

$$B \cong TM/R$$

as an H_σ-module algebra, where TM is the tensor algebra of M (over k) and R is the 2-sided ideal of relations that hold among the elements of M in the algebra B. Then we have:

Lemma 4.2.21. *A_σ is generated as a Hopf algebra by the matrix coefficients of M.*

Proof. The matrix coefficients of $M^{\otimes n}$ are sums of products of n matrix coefficients of M, and the space of matrix coefficients of a quotient comodule $M^{\otimes n}/(R \cap M^{\otimes n})$ is a subspace of the space of matrix coefficients of $M^{\otimes n}$. □

In coordinates, if $e_1, \dots, e_{\dim_k M}$ is a vector space basis of M, then A_σ is generated as a Hopf algebra by the functionals $a_{ij} \in H_\sigma^\circ$, $i, j = 1, \dots, \dim_k M$, for which

$$X \triangleright e_i = \sigma(X)(e_i) = \sum_{j=1}^{\dim_k M} a_{ji}(X) e_j, \quad X \in H_\sigma,$$

so the coaction ρ is given by

$$M \to M \otimes A_\sigma, \quad e_i \mapsto \sum_{j=1}^{\dim_k M} e_j \otimes a_{ji} \tag{4.2.4}$$

Remark 4.2.22. In general, the matrix coefficients of M do not generate A_σ as an algebra – the subalgebra that they generate is a subbialgebra of A_σ as the span of the matrix coefficients is a subcoalgebra (with coalgebra structure given by (2.3.1)), but this subbialgebra is not closed under the antipode in general. However, if the matrix coefficients can be chosen to be upper triangular (that is, there is a vector space basis of M such that $a_{ij} = 0$ for $i > j$), then the same arguments that were used in Proposition 4.2.2 show that A_σ is generated by the a_{ij} together with the a_{ii}^{-1}, and that A_σ is a pointed Hopf algebra.

4.2.5 The map ι

Now assume that $\chi\colon B \to k$ is an algebra map, that is, a one-dimensional representation of B . This induces a map (see [21, Section 3] for a more detailed discussion of this map)

$$\iota := (\chi \otimes \mathrm{id}_{A_\sigma}) \circ \rho\colon B \to A_\sigma, \quad b \mapsto \chi(b_{(0)}) b_{(1)}.$$

By construction, this is a morphism of algebras: $\forall\, b, \tilde{b} \in B$,

$$\iota(b\tilde{b}) = \chi(b_{(0)}\tilde{b}_{(0)}) b_{(1)}\tilde{b}_{(1)} = \chi(b_{(0)})\chi(\tilde{b}_{(0)}) b_{(1)}\tilde{b}_{(1)} = \iota(b)\iota(\tilde{b}),$$

and of right A_σ-comodules: using the coassociativity of ρ, we obtain

$$(\iota \otimes \mathrm{id}) \circ \rho = (\chi \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\rho \otimes \mathrm{id}) \circ \rho = (\chi \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ \rho,$$

which as morphisms is equal to

$$\Delta \circ (\chi \otimes \mathrm{id}) \circ \rho = \Delta \circ i,$$

that is $\Delta(\iota(B)) \subseteq \iota(B) \otimes A_\sigma$. Hence ι maps B to a right coideal subalgebra of A_σ.

Proposition 4.2.23. *ι is injective if and only if for all $b \in B$, $b \neq 0$, there exists $X \in H_\sigma$ with $\chi(X \triangleright b) \neq 0$.*

Proof. The map ι is not injective if there exists $b \in B$, $b \neq 0$ with

$$\iota(b) = \chi(b_{(0)})b_{(1)} = 0.$$

This is an element in $A_\sigma \subseteq H_\sigma^\circ$, so it is zero if and only if it pairs trivially with all elements $X \in H_\sigma$. Thus ι is not injective if and only if there exists $b \in B$, $b \neq 0$, such that

$$X(\iota(b)) = \chi(b_{(0)})b_{(1)}(X) = \chi(b_{(0)}b_{(1)}(X)) = \chi(X \triangleright b) = 0$$

for all $X \in H_\sigma$. □

If this condition is satisfied, then ι embeds B as a right coideal subalgebra into A_σ.

In particular, when $B = k[V]$ is the coordinate ring of an algebraic set V, then χ corresponds to a point $p \in V$ and the above proposition states that B can be embedded as a right coideal subalgebra into A_σ provided that there exists a point $p \in V$ such that for any non-zero regular function $b \colon V \to k$ there exists some $X \in H_\sigma$ such that the function $X \triangleright b$ does not vanish at p.

4.2.6 Application to $k[t^2, t^3]$

To compute a full presentation of A_σ in a given example is tedious, but relatively straightforward. Like elsewhere, we illustrate the theory with our main example:

Proposition 4.2.24. *If $B = k[t^2, t^3]$ and M is the H_σ-module F_3 from (4.1.6) with basis $e_1 = 1, e_2 = t^2, e_3 = t^3$, then:*

(1) There is a surjective algebra morphism

$$\pi \colon k\langle \gamma, \varphi, \psi \rangle \to A_\sigma$$

given by $\pi(\gamma) = a_{13}$, $\pi(\varphi) = a_{23}$, $\pi(\psi) = a_{33}$ whose kernel is the ideal generated by

$$\begin{aligned} &\psi^2 - 1, \quad \gamma\psi + \psi\gamma, \quad \varphi\psi + \psi\varphi, \\ &27\gamma^2 - \varphi^6, \quad 3(\gamma\varphi + \varphi\gamma) - \varphi^4. \end{aligned} \tag{4.2.5}$$

(2) In this presentation, the coalgebra structure of A_σ is given by

$$\begin{gathered}\Delta(\psi) = \psi \otimes \psi, \quad \Delta(\varphi) = 1 \otimes \varphi + \varphi \otimes \psi, \\ \Delta(\gamma) = 1 \otimes \gamma + \frac{1}{3}\varphi^2 \otimes \varphi + \gamma \otimes \psi, \\ \varepsilon(\psi) = 1, \quad \varepsilon(\gamma) = \varepsilon(\varphi) = 0,\end{gathered} \tag{4.2.6}$$

and its antipode is given by

$$S(\psi) = \psi, \quad S(\varphi) = -\varphi\psi, \quad S(\gamma) = (\frac{1}{3}\varphi^3 - \gamma)\psi. \tag{4.2.7}$$

For the proof that will be split into several lemmata, we introduce a redundant generator δ and first observe:

Lemma 4.2.25. *Let $J \lhd k\langle\gamma, \varphi, \psi, \delta\rangle$ be the ideal generated by the elements (4.2.5) together with $\delta - \frac{1}{3}\varphi^2$. Then we have:*

(1) $k\langle\gamma, \varphi, \psi, \delta\rangle$ carries a unique bialgebra structure such that

$$\begin{gathered}\Delta(\psi) = \psi \otimes \psi, \quad \Delta(\varphi) = 1 \otimes \varphi + \varphi \otimes \psi, \\ \Delta(\gamma) = 1 \otimes \gamma + \delta \otimes \varphi + \gamma \otimes \psi, \quad \Delta(\delta) = 1 \otimes \delta + \delta \otimes 1, \\ \varepsilon(\psi) = 1, \quad \varepsilon(\gamma) = \varepsilon(\varphi) = \varepsilon(\delta) = 0.\end{gathered} \tag{4.2.8}$$

(2) The ideal J is a coideal, so this bialgebra structure descends to

$$\tilde{A}_\sigma := k\langle\gamma, \varphi, \psi\rangle/J.$$

(3) The bialgebra $\tilde{A}_\sigma$ is a Hopf algebra whose antipode is given on the generators by (4.2.7) and $S(\delta) = -\delta$.

Proof. Define a coproduct and a counit on the free algebra $k\langle\gamma, \varphi, \psi, \delta\rangle$ as given in the lemma. We claim every coalgebra structure on $k\langle\gamma, \varphi, \psi, \delta\rangle$ must be of this form since for instance $\Delta(1) = \Delta(\psi^2) = \psi^2 \otimes \psi^2 = 1 \otimes 1$ which is compatible with the relator $\psi^2 - 1$ and this proves (1). By straightforward computations, we see that the coproduct of the elements in J lie in $J \otimes k\langle\gamma, \varphi, \psi, \delta\rangle + k\langle\gamma, \varphi, \psi, \delta\rangle \otimes J$ and $\varepsilon(J) = 0$, for example:

$$\Delta(\psi^2 - 1) = (\psi^2 - 1) \otimes \psi^2 + 1 \otimes (\psi^2 - 1), \qquad \varepsilon(\psi^2 - 1) = 0,$$

$$\Delta(\varphi\psi + \psi\varphi) = (\varphi\psi + \psi\varphi) \otimes i + \psi \otimes (\varphi\psi + \psi\varphi), \qquad \varepsilon(\varphi\psi + \psi\varphi) = 0.$$

This proves (2), and the third assertion follows from (1) by using the coproduct and counit formulas in the antipode equation. □

Next, we note:

Lemma 4.2.26. *There is a surjective bialgebra morphism*

$$\pi\colon k\langle \gamma, \varphi, \psi, \delta\rangle \to A_\sigma$$

satisfying $\pi(\gamma) = a_{13}$, $\pi(\varphi) = a_{23}$, $\pi(\psi) = a_{33}$ *and* $\pi(\delta) = a_{12}$.

Proof. Recall first that the values of the functionals a_{ji} on the generators K, D, Y of H_σ are by (4.2.1) and (4.1.5) given by the following matrices (for later use, we also list the values on C):

$$\begin{aligned} a(K) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad & a(D) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ a(Y) &= \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad & a(C) &= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.2.9}$$

A way to read these matrices is to know that each column is the image under (4.2.4) of each basis element. For instance, $K \triangleright t^3 = -t^3$ thus $a_{11}(K) = a_{12}(K) = 0$ and $a_{13} = -1$. As the generators of H_σ act by upper triangular matrices, all elements in H_σ act by upper triangular matrices, hence as elements of $H_\sigma^\star$, the a_{ij} with $i > j$ vanish (this is just a restatement of the fact that F_i is a filtration of B by H_σ-submodules).

π is a coalgebra map since for instance

$$\Delta_{A_\sigma}(\pi(\psi)) = \Delta_{A_\sigma}(a_{33}) = a_{33} \otimes a_{33} = \pi(\psi) \otimes \pi(\psi) = (\pi \otimes \pi) \circ \Delta(\psi),$$

and $\varepsilon_{A_\sigma} \circ \pi(\psi) = 1 = \varepsilon(\psi)$. Finally, $a_{11} = a_{22} = \varepsilon_{H_\sigma} = 1_{A_\sigma}$ is the counit of the coalgebra H_σ hence the unit of the algebra A_σ. Using this we conclude that the algebra morphism π defined on the generators $\gamma, \varphi, \psi, \delta$ as in the lemma is a bialgebra morphism, and Lemma 4.2.21 implies that π is surjective. □

Next, we discuss that π descends to a surjective Hopf algebra morphism $\tilde{A}_\sigma \to A_\sigma$:

Lemma 4.2.27. $J \subseteq \ker \pi$.

Proof. The bialgebra map π induces a pairing of bialgebras

$$\langle -, - \rangle\colon H_\sigma \otimes k\langle \gamma, \varphi, \psi, \delta\rangle \to k, \quad \langle X, \xi\rangle = \pi(\xi)(X).$$

To prove the lemma, one has to show that this pairing descends to a pairing between H_σ and $\tilde{A}_\sigma$. This is done by a long but straightforward computation which for us, it seemed to be the easiest way to organize this computation by showing that for each of the six relators ξ that generate J and for all $i, j, k \in \{0, 1\}, l \in \mathbb{N}$, we have

$$\langle C^i D^j K^k Y^l, \xi \rangle = 0.$$

For example, using that $\langle -, - \rangle$ is a pairing of bialgebras, one obtains

$$\begin{aligned}\langle C^i D^j K^k Y^l, \varphi^2 \rangle &= \langle (C^i D^j K^k Y^l)_{(1)}, \varphi \rangle \langle (C^i D^j K^k Y^l)_{(2)}, \varphi \rangle \\ &= \langle C^i_{(1)} D^j_{(1)} K^k_{(1)} Y^l_{(1)}, \varphi \rangle \langle C^i_{(2)} D^j_{(2)} K^k_{(2)} Y^l_{(2)}, \varphi \rangle \\ &= (\langle C^i_{(1)}, \varphi_{(1)} \rangle \langle D^j_{(1)}, \varphi_{(2)} \rangle \langle K^k_{(1)}, \varphi_{(3)} \rangle \langle Y^l_{(1)}, \varphi_{(4)} \rangle) \\ &\quad (\langle C^i_{(2)}, \varphi_{(1)} \rangle \langle D^j_{(2)}, \varphi_{(2)} \rangle \langle K^k_{(2)}, \varphi_{(3)} \rangle \langle Y^l_{(2)}, \varphi_{(4)} \rangle),\end{aligned}$$

where the $\varphi_{(i)}$ for $i = 1, \cdots, 4$ are the tensorands of the iterated coproduct

$$\begin{aligned}\Delta^{(4,1)}(\varphi) = &(1 \otimes 1 \otimes 1 \otimes \varphi) + (1 \otimes 1 \otimes \varphi \otimes \psi) \\ &+ (1 \otimes \varphi \otimes \psi \otimes \psi) + (\varphi \otimes \psi \otimes \psi \otimes \psi)\end{aligned}$$

of φ. Inserting the explicit coproducts of C^i, D^j, K^k, Y^l and at last the values (4.2.9) of the pairings of the generators, one obtains

$$(\langle 1, 1 \rangle \langle 1, 1 \rangle \langle K, 1 \rangle \langle -6D, \varphi \rangle)(\langle K, 1 \rangle \langle K, 1 \rangle \langle K, 1 \rangle \langle DK, \varphi \rangle) = 6$$

as the only surviving term and is equal to

$$\langle C^i D^j K^k Y^l, 3\delta \rangle = 3\delta_{i0}\delta_{j0}\delta_{l1}$$

so that $\langle -, \delta - \frac{1}{3}\varphi^2 \rangle$ vanishes as a k-linear functional on H_σ. The other five relators are treated in a similar way. □

As a preparation for showing that $\pi\colon \tilde{A}_\sigma \to A_\sigma$ is also injective, we first note:

Lemma 4.2.28. *The set*

$$\{\gamma^a \varphi^b \psi^c \mid a, c \in \{0, 1\}, b \in \mathbb{N}\}$$

is a k-vector space basis of $\tilde{A}_\sigma$.

Proof. This is shown like the analogous statements for H_σ using Bergman's diamond lemma. □

Thus to prove the injectivity of π, one has to show that the elements $\pi(\gamma^a\varphi^b\psi^c) \in A_\sigma$ are linearly independent over k. This is maybe shown most easily by explicitly computing the values of the functionals:

Lemma 4.2.29. *The dual pairing* $\langle -,-\rangle \colon H_\sigma \otimes \tilde{A}_\sigma \to k$ *satisfies*

$$\langle C^i D^j K^k Y^l, \gamma^a\varphi^b\psi^c\rangle = \delta_{j+2l,b}\delta_{ia}(-1)^{j(a+c)+ic+ab+k(a+b+c)}2^a 6^l l!$$

Proof. Using the q-binomial formula [30, Proposition 4.2.2], a direct computation and a nested induction on b and l shows

$$\Delta(\gamma^a\varphi^b\psi^c) = (1\otimes\gamma + \frac{1}{3}\varphi^2\otimes\varphi + \gamma\otimes\psi)^a \cdot \sum_{l=0}^{b}(1-[\![(b+1)l]\!]_{-1})\binom{[b/2]}{[l/2]}\varphi^l\psi^c\otimes\varphi^{b-l}\psi^{[\![bl]\!]_{-1}+c},$$

where as before $[\![n]\!]_q = 1+q+\cdots+q^{n-1}$, which for $q=-1$ is 0 if n is even and 1 if n is odd, and

$$[n/2] := \begin{cases}(n-1)/2 & n \text{ odd},\\ n/2 & n \text{ even}.\end{cases}$$

Using this and the formulas for the coproduct of and the relations between the generators of H_σ respectively $\tilde{A}_\sigma$ as well as their pairing (4.2.9), one computes

$$\begin{aligned}\langle C^iD^jY^lK^k, \gamma^a\varphi^b\psi^c\rangle &= \langle C^iD^jY^l\otimes K^k, \Delta(\gamma^a\varphi^b\psi^c)\rangle\\ &= \langle C^iD^jY^l, \gamma^a\varphi^b\psi^c\rangle\langle K^k, \psi^a\psi^{[\![b^2]\!]_{-1}+c}\rangle\\ &= \langle C^i\otimes D^jY^l, \Delta(\gamma^a\varphi^b\psi^c)\rangle(-1)^{k(a+c+[\![b^2]\!]_{-1})}.\end{aligned}$$

If $i=0$, the above is equal to

$$\begin{aligned}\ldots &= \langle D^jY^l, \gamma^a\varphi^b\psi^c\rangle(-1)^{k(a+c+[\![b^2]\!]_{-1})}\\ &= \langle\Delta(D^jY^l), \gamma^a\otimes\varphi^b\psi^c\rangle(-1)^{k(a+b+c)}\\ &= \delta_{ia}\langle D^jY^l, \varphi^b\psi^c\rangle(-1)^{k(a+b+c)}.\end{aligned}$$

The last equality follows since the pairing of the first tensor component of $\Delta(D^jY^l)$ with γ vanishes.

If $i=1$, we have instead

$$\begin{aligned}\ldots &= \langle C\otimes D^jY^l, \Delta(\gamma^a\varphi^b\psi^c)\rangle(-1)^{k(a+c+[\![b^2]\!]_{-1})}\\ &= \langle D^jY^l, \varphi^b\psi^{a+c}\rangle 2\delta_{ia}(-1)^{ic}(-1)^{ac}(-1)^{k(a+c+[\![b^2]\!]_{-1})}\\ &= \langle D^jY^l, \varphi^b\psi^{a+c}\rangle 2\delta_{ia}(-1)^{ic+ab+k(a+b+c)}.\end{aligned}$$

We deliberately wrote i instead of 0 respectively 1 in the above two cases as we now can merge them again:

$$\begin{aligned}\langle C^iD^jY^lK^k, \gamma^a\varphi^b\psi^c\rangle &= \langle D^jY^l, \varphi^b\psi^{a+c}\rangle 2^a\delta_{ia}(-1)^{ic+ab+k(a+b+c)}\\ &= \langle \Delta(D^jY^l), \varphi^c \otimes \psi^{a+c}\rangle 2^a\delta_{ia}(-1)^{ic+ab+k(a+b+c)}\\ &= \langle D^j \otimes Y^l, \Delta(\varphi^b)\rangle 2^a\delta_{ia}(-1)^{j(a+c)+ic+ab+k(a+b+c)}\\ &= 6^l l!(1-[\![j+2l+1]\!]_{-1})^j 2^a\delta_{j+2l,b}\delta_{ia}\\ &\quad \cdot (-1)^{j(a+c)+ic+ab+k(a+b+c)}\\ &= \delta_{j+2l,b}\delta_{ia}(-1)^{j(a+c)+ic+ab+k(a+b+c)}6^l l! 2^a.\end{aligned}$$

□

Therefore, if we define

$$E_{uvw} := \frac{(-1)^{2[v/2](u+v)-v-uw-uv}}{6^{[v/2]}[v/2]!2^{u+1}} C^u D^{v-2[v/2]} Y^{[v/2]}(1+(-1)^{u+v+w}K),$$

then these elements also form a basis of H_σ, and we have

$$\langle E_{uvw}, \gamma^a\varphi^b\psi^c\rangle = \delta_{ua}\delta_{vb}\delta_{wc}.$$

Recall that $\langle X, \xi\rangle = \pi(\xi)(X)$ for all $X \in H_\sigma$ and $\xi \in \tilde{A}_\sigma$. Thus $\pi(\gamma^a\varphi^b\psi^c)(E_{uvw}) = \delta_{ua}\delta_{vb}\delta_{wc}$, hence $\pi(\gamma^a\varphi^b\psi^c)$ form a dual set to E_{uvw} hence is linearly independent in A_σ and this finishes the proof of Proposition 4.2.24.

Remark 4.2.30. In our example, σ is a 3×3-matrix with entries in $\mathrm{End}_k(B)$ and (a_{ji}) is a 3×3-matrix with entries in H_σ°, but be aware it is a pure coincidence that these sizes match – in general, if σ is an $n \times n$-matrix, then H_σ has n^2 generators (some of which might be zero such as in the upper triangular case), and if B can be generated by a d-dimensional H_σ-submodule, then A_σ is generated as a Hopf algebra by the d^2 matrix coefficients a_{ij} (and in good cases they even generate A_σ as an algebra).

Finally, we prove that the map $\iota\colon B \to A_\sigma$ is injective for every point p on the cusp: the algebra morphisms $\chi\colon B = k[t^2, t^3] \to k$ are in bijection with the points $p = (\lambda^2, \lambda^3)$ on the cusp, $\lambda \in k$, and the algebra morphism ι is given on generators by

$$t^2 \mapsto \lambda^2 1 + \frac{1}{3}\varphi^2, \quad t^3 \mapsto \gamma + \lambda^2\varphi + \lambda^3\psi. \tag{4.2.10}$$

This is obtained as follows: since $i(B)$ is already a right coideal subalgebra then we can assume that $\Delta(t^2) = \sum_{j=0}^{\infty} t^j \otimes a_j$ and $\Delta(t^3) = \sum_{j=0}^{\infty} t^j \otimes v_j$ where $a_1 = 0 = v_1$. However the orbits of the action of H_σ on t^2 and t^3 are spanned by $\{1, t^2\}$ respectively $\{1, t^2, t^3\}$ thus we can assume $\Delta(t^2) = 1 \otimes a + t^2 \otimes b$ and $\Delta(t^3) = 1 \otimes u + t^2 \otimes v + t^3 \otimes w$. Using the coassociativity of Δ one obtains

$$\Delta(a) = 1 \otimes a + a \otimes b, \quad \Delta(b) = b \otimes b,$$
$$\Delta(u) = 1 \otimes u + a \otimes v + u \otimes w,$$
$$\Delta(w) = w \otimes w, \quad \Delta(v) = b \otimes v + v \otimes w.$$

Comparing these coproduct formulas with those of the generators of A_σ (4.2.5)and observing that

$$a : K, D \mapsto 0, \;\; Y \mapsto 2 \quad b : K \mapsto 1, \;\; D, Y \mapsto 0$$
$$u : K, D, Y \mapsto 0, \quad v : K, Y \mapsto 0, \;\; D \mapsto 1, \quad w : K \mapsto -1, \;\; D, Y \mapsto 0$$

we see that the group-like elements b and w agree with ε_{H_σ} respectively ψ on the generators, the twisted primitive a and v agree with $\delta(= \frac{1}{3}\varphi^2)$ respectively φ. Finally notice that the second order operator agree with γ on the generators and in particular $u(C) = 2 = \gamma(C)$. Substituting these into the coproduct formula of t^2, t^3 and applying the counitality condition $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$ yeild (4.2.10).

Now we observe:

Lemma 4.2.31. *The element $X := Y + D \in H_\sigma$ acts by*

$$\mathsf{X}(t^n) := \begin{cases} (n-3)t^{n-2} + t^{n-1}, & n \text{ is odd,} \\ nt^{n-2} & n \text{ is even.} \end{cases}$$

In particular, if $b \in F_d \setminus F_{d-1}$ is a polynomial of degree d, then

$$\mathsf{X}^{\lceil d/2 \rceil}(b) \in k \setminus \{0\}, \quad \lceil d/2 \rceil := \begin{cases} (d+1)/2, & d \text{ odd,} \\ d/2, & d \text{ even.} \end{cases}$$

Proof. The definition of X follows from the definition of the operators Y and D. Clearly when $d = 2$ and $d = 3$, $\mathsf{X}^{\left\lceil d/2 \right\rceil}(b) = 2$. Suppose for $3 < d \in \mathbb{N}$, $\mathsf{X}^{\lceil d/2 \rceil}(b) = \lambda \in k \setminus \{0\}$, then for $\deg(b) = d + 1$ we have

$$\mathsf{X}^{\lceil (d+1)/2 \rceil}(b) = \begin{cases} \mathsf{X}^{(d+1)/2)}(b) = \lambda, & d \text{ odd,} \\ \mathsf{X}^{((d/2)+1)}(b), & d \text{ even,} \end{cases}$$

where for d even, $\mathsf{X}^{((d/2)+1)}(b) \neq 0$. ∎

Corollary 4.2.32. *The map $\iota\colon B \to A_\sigma$ is injective for all algebra maps $\chi\colon B \to k$.*

Proof. The lemma above produces for any non-zero regular function b on the cusp a non-zero constant regular function $X^{\lceil d/2 \rceil} \triangleright b$ which vanishes in no point p on the $k[t^2, t^3]$. □

4.2.7 Faithful flatness

Finally, we remark that a result of Masuoka (see Theorem 3.4.9) implies that A_σ is a faithfully flat B-module. As a preliminary result, we need to compute the coradical of A_σ:

Proposition 4.2.33. *The Hopf algebra A_σ is pointed and contains two group-like elements, $\chi_{1,0} = 1$ and $\chi_{-1,0} = \psi$.*

Proof. We are in the situation mentioned in Remark 4.2.22 – A_σ is generated as an algebra by a_{ij}, $i \leq j$, and in this case we do not even have to add inverses of the a_{ii} as they all are their own inverses. Thus we are in the situation described in Remark 4.2.22 and the pointedness of A_σ follows as in Proposition 4.2.2.

Since $A_\sigma \subseteq H_\sigma^\circ$, the group-likes in A_σ are by definition one-dimensional representations of H_σ. We have listed all one-dimensional representations of H_σ in Lemma 4.2.13, and by definition of A_σ, $\chi_{s,\lambda} \in A_\sigma$ if and only if there exists $b \in B$ and $f \in B^\star$ such that

$$\chi_{s,\lambda}(h) = f(h \triangleright b) = f(b_{(0)})b_{(1)}(h)$$

holds for all $h \in H_\sigma$. In particular, this must be true for $h = Y^d$ for any $d \geq 0$,

$$f(\mathsf{Y}^d(b)) = \lambda^d.$$

However, if $2d$ exceeds the degree of $b \in B = k[t^2, t^3]$ (as a polynomial in $k[t]$)(see 4.2.3), then $\mathsf{Y}^d(b) = 0$, so $\lambda = 0$. Since we know already that $\varepsilon_{H_\sigma}, \psi \in A_\sigma$, this finishes the proof. □

Proposition 4.2.34. *For any $p \in V$, A_σ is faithfully flat over $\iota(B)$.*

Proof. By Theorem 3.4.9, we only need to prove that the intersection of $\iota(B)$ with the coradical $\mathrm{span}_k\{1, \psi\}$ is invariant under the antipode S of A_σ. However, this coradical is as a Hopf algebra isomorphic to the group algebra $k\mathbb{Z}_2$ and the restriction of the antipode S to it is the identity map (1 and ψ are their own inverses), so there is nothing to prove. □

Remark 4.2.35. Thus the coordinate ring $k[t^2, t^3]$ of the cusp is a quantum homogeneous space and in particular the quantum symmetries defined by A_σ restrict from the field of rational functions $k(t)$ to the $k[t^2, t^3]$.

4.2.8 $*$-structures and involutions

As we have seen in Lemma 4.1.13, even the classification of the 3×3-upper triangular quantum automorphisms of $B = k[t^2,t^3]$ is rather involved. The explicit example studied since Section 4.1.7 was obtained by demanding in addition that the quantum automorphism is compatible with a chosen $*$-structure on B. This might be of interest in its own right, for example as it is the starting point for the transition from the algebraic theory of Hopf algebras acting on rings to the analytic theory of locally compact quantum groups acting on C^*-algebras.

Recalling the definitions given in Section 2.2.3, the following is immediate:

Lemma 4.2.36. *A $*$-structure on an algebra B induces an involution on $P = \mathrm{End}_k(B)$ given by $\theta(f) := * \circ f \circ *$ for all $f \in P$.*

Proof. Every other property of the $*$-structure on B carry on to P and the anti-symmetry of $*$ yields the involutive property of θ by definition. □

If H is a Hopf $*$-algebra, then an H-module algebra B which is also a $*$-algebra is called a *module $*$-algebra* if $(hb)^* = \theta(h)(b^*)$ holds for all $h \in H, b \in B$, that is, if the resulting map $H \to \mathrm{End}_k(B)$ is a morphism of algebras with involution. The question we want to address in this section is whether for a given quantum automorphism σ of a $*$-algebra B the Hopf algebra H_σ becomes naturally a $*$-algebra in such a way that B is a module $*$-algebra over H_σ.

In order to do so, we first extend the $*$-structure $*$ respectively the associated involution θ to $\mathrm{M}_n(B)$ respectively $\mathrm{M}_n(\mathrm{End}_k(B))$. This depends on the choice of an involutive permutation

$$\{1,\ldots,n\} \to \{1,\ldots,n\}, \quad i \mapsto \bar{i}, \quad \bar{\bar{i}} = i$$

that will be used afterwards to be able to restrict the resulting $*$-structure on $\mathrm{M}_n(B)$ to upper triangular matrices.

Proposition 4.2.37. *Let B be a $*$-algebra and assume $s \in S_n$, $s^2 = 1$. We abbreviate $\bar{i} := s(i)$.*

(1) Setting $(\sigma^\dagger)_{ij} := \sigma^*_{\bar{j}\bar{i}}$ defines a $*$-structure $\dagger$ on $\mathrm{M}_n(B)$.

(2) Setting $\vartheta(\sigma)_{ij} := \theta(\sigma_{\bar{i}\bar{j}})$ defines an involution on $\mathrm{M}_n(\mathrm{End}_k(B))$.

Proof. Clearly, $\dagger$ is involutive:

$$(\sigma^{\dagger\dagger})_{ij} = (\sigma^\dagger)^*_{\bar{j}\bar{i}} = \sigma^{**}_{\bar{\bar{i}}\bar{\bar{j}}} = \sigma^{**}_{ij} = \sigma_{ij},$$

where the last equality follows as $*$ is a $*$-structure on B. It is also a ring morphism $\mathrm{M}_n(B) \to \mathrm{M}_n(B)^{\mathrm{op}}$: let $\sigma, \tau \in \mathrm{M}_n(B)$ then

$$\begin{aligned} ((\sigma\tau)^\dagger)_{ij} &= ((\sigma\tau)_{\bar{j}\bar{i}})^* = \big(\sum_{r=1}^n \sigma_{\bar{j}r}\tau_{r\bar{i}}\big)^* = \sum_{r=1}^n \tau^*_{r\bar{i}}\sigma^*_{\bar{j}r} \\ &= \sum_{r=1}^n (\tau^\dagger)_{i\bar{r}}(\sigma^\dagger)_{\bar{r}j} = (\tau^\dagger\sigma^\dagger)_{ij}. \end{aligned}$$

That $\dagger$ is a k-linear map $\mathrm{M}_n(B) \to \overline{\mathrm{M}_n(B)}$ follows from the fact that $*\colon B \to \bar{B}$ is linear. The second claim can be shown analogously:

$$\vartheta(\vartheta(\sigma))_{ij} = \theta(\vartheta(\sigma)_{\bar{i}\bar{j}}) = \theta(\theta(\sigma_{\bar{\bar{i}}\bar{\bar{j}}})) = \sigma_{ij},$$

where the last equality follows from the involutive property of θ. Furthermore, ϑ is also an endomorphism on $\mathrm{M}_n(\mathrm{End}_k(B))$:

$$\begin{aligned} \vartheta(\sigma\tau)_{ij} &= \theta((\sigma\tau)_{\bar{i}\bar{j}}) = \theta\left(\sum_{r=1}^n \sigma_{\bar{i}r}\tau_{r\bar{j}}\right) \\ &= \sum_{r=1}^n \theta(\sigma_{\bar{i}r})\theta(\tau_{r\bar{j}}) = \sum_{r=1}^n \vartheta(\sigma_{ir})\vartheta(\tau_{rj}) \\ &= \vartheta(\sigma)\vartheta(\tau)_{ij}, \end{aligned}$$

where we have used the ring map property of θ. Likewise that ϑ is a k-linear map $\mathrm{M}_n(\mathrm{End}_k(B)) \to \overline{\mathrm{M}_n(\mathrm{End}_k(B))}$ follows from the fact that $\theta\colon \mathrm{End}_k(B) \to \overline{\mathrm{End}_k(B)}$ is linear. □

The definition of this $*$-structure and of this involution is made in order to have the following:

Lemma 4.2.38. *A k-linear map $\sigma\colon B \to \mathrm{M}_n(B)$ satisfies*

$$\sigma(b^*) = \sigma(b)^\dagger$$

if and only if $\vartheta(\sigma) = \sigma^T$ when σ is viewed as an element in $\mathrm{M}_n(\mathrm{End}_k(B))$.

Proof. This result follows from observing that

$$(\sigma^\dagger)_{ij}(b) = \sigma^*_{\bar{j}\bar{i}}(b) = \theta(\sigma_{\bar{j}\bar{i}})(b^*) = \vartheta(\sigma)_{ji}(b^*).$$

That is σ is a morphism of $*$-algebras if and only if $\vartheta(\sigma) = \sigma^T$. □

Now we apply the above to the study of quantum automorphisms:

Proposition 4.2.39. *Let $\sigma\colon B \to \mathrm{M}_n(B)$ be a quantum automorphism.*

(1) $\vartheta(\sigma)^T = \vartheta(\sigma^T)$ is a quantum automorphism.

(2) If $\vartheta(\sigma)^T = \sigma$, then H_σ is a Hopf $*$-algebra with $*$-structure given by

$$s_{d,ij} \mapsto s_{1-d,\bar{i}\bar{j}}$$

and B is a module $*$-algebra over H_σ.

Proof. (1): First note that from the definition of ϑ we have

$$\vartheta(\sigma^T)_{ij} = \theta(\sigma_{\bar{j}\bar{i}}) = \vartheta(\sigma)_{ji} = \vartheta(\sigma)^T_{ij}.$$

If $\sigma\colon B \to \mathrm{M}_n(B)$ is an algebra morphism, then

$$\begin{aligned}(\vartheta(\sigma)^T)_{ij}(ab) &= \theta(\sigma_{\bar{j}\bar{i}})(ab) = \sigma_{\bar{j}\bar{i}}((ab)^*)^* = \sigma_{\bar{j}\bar{i}}(b^*a^*)^* \\ &= (\sum_r \sigma_{\bar{j}r}(b^*)\sigma_{r\bar{i}}(a^*))^* \\ &= \sum_r (\sigma_{r\bar{i}}(a^*))^*(\sigma_{\bar{j}r}(b^*))^* \\ &= \sum_r \theta(\sigma_{r\bar{i}})(a)\theta(\sigma_{\bar{j}r})(b) \\ &= \sum_r (\vartheta(\sigma)^T)_{ir}(a)(\vartheta(\sigma)^T)_{rj}(b).\end{aligned}$$

In order to the show that $\vartheta(\sigma)^T$ is strongly invertible note first that as ϑ is an involution on $\mathrm{M}_n(\mathrm{End}_k(B))$, it is in particular multiplicative (see Proposition 4.2.37(b)), so if $\sigma \in \mathrm{M}_n(\mathrm{End}_k(B))$ is invertible, then so is $\vartheta(\sigma)$

$$\vartheta(\sigma)\vartheta(\sigma^{-1}) = \vartheta(\sigma\sigma^{-1}) = 1 = \vartheta(\sigma^{-1}\sigma) = \vartheta(\sigma^{-1})\vartheta(\sigma)$$

with inverse given by $\vartheta(\sigma)^{-1} = \vartheta(\sigma^{-1})$. Furthermore, since ϑ commutes with taking transposes, we have

$$\vartheta(\bar{\sigma}) = \vartheta((\sigma^{-1})^T) = \vartheta(\sigma^{-1})^T = (\vartheta(\sigma)^{-1})^T = \overline{\vartheta(\sigma)}.$$

It follows that if $\{\sigma_d\}$ is a sequence of invertible matrices with $\bar{\sigma}_d = \sigma_{d+1}$ then $\{\vartheta(\sigma_d)\}$ is a sequence of invertible matrices with $\overline{\vartheta(\sigma_d)} = \vartheta(\sigma_{d+1})$.

(2): Since $k\langle s_{d,ij}\rangle$ is a free algebra, there is a unique algebra morphism

$$\theta\colon k\langle s_{d,ij}\rangle \to \overline{k\langle s_{d,ij}\rangle}$$

which maps $s_{d,ij}$ to $s_{-d,\bar{j}\bar{i}}$ which is also a coalgebra morphism $k\langle s_{d,ij}\rangle \to \overline{k\langle s_{d,ij}\rangle}^{\mathrm{cop}}$:

$$\begin{aligned}(\theta\otimes\theta)\circ\Delta^{\mathrm{cop}}(s_{d,ij}) = \theta\otimes\theta\left(\sum_{r=1}^{n} s_{d,rj}\otimes s_{d,ir}\right) = \sum_{r=1}^{n}\theta(s_{d,rj})\otimes\theta(s_{d,ir})\\ = \sum_{r=1}^{n} s_{-d,\bar{j}\bar{r}}\otimes s_{-d,\bar{r}\bar{i}} = \Delta\circ\theta(s_{d,ij}),\end{aligned}$$

and $\varepsilon(\theta(\sigma_{d,ij})) = \varepsilon(\sigma_{-d,\bar{j}\bar{i}}) = \delta_{\bar{j}\bar{i}} = \bar{\delta}_{ij} = \overline{\varepsilon(\sigma_{ij})}$.

Our aim is to show that θ descends to a Cartan involution on H_σ. For this we first prove by induction on d that

$$\theta(\sigma_{d,\bar{i}\bar{j}}) = \vartheta(\sigma_d)_{ij} = \sigma^T_{-d,ij} = \sigma_{-d,ji}. \tag{4.2.11}$$

Note that the first and last equality holds by definition, so we will only prove the middle equality. Indeed, for $d = 0$ this holds by the assumption that $\vartheta(\sigma)^T = \sigma$. In the induction step we compute

$$\vartheta(\sigma_d) = \sigma^T_{-d} \Rightarrow \vartheta(\sigma_d^{-1}) = (\sigma^T_{-d})^{-1} = \sigma_{-d-1},$$

recall the comment following Definition 4.1.2. Hence

$$\vartheta(\sigma_{d+1}) = \overline{\vartheta(\sigma_d)} = \vartheta(\bar{\sigma}_d) = \vartheta(\sigma_d^{-1})^T = \sigma^T_{-d-1}$$

which proves (4.2.11) for $d+1$.

This means that by the definition of the involution θ on $k\langle s_{d,ij}\rangle$, the map

$$f : k\langle s_{d,ij}\rangle \to \mathrm{End}_k(B) \quad s_{d,ij}\mapsto \sigma_{d,ij}$$

is a morphism of algebras with involution: that is $f\circ\theta = \theta\circ f$. In particular, θ descends to $k\langle s_{d,ij}\rangle/I$ and to H_σ and defines a Cartan involution and hence a Hopf $*$-algebra structure on H_σ by Lemma 2.2.20. It also follows that B is a module $*$-algebra over this Hopf $*$-algebra. □

Remark 4.2.40. To obtain the $*$-structure on H_σ defined in the proposition, precompose θ with the inverse of the (bijective) antipode S on H_σ which is given by $S^{-1}(\sigma_{d,ij}) = \sigma_{d-1ji}$.

Example 4.2.41. Throughout we fixed an involutive permutation $s \in S_n$. The most obvious choice is the identity, $\bar{i} = i$. In this case, $\sigma^\dagger$ is the usual adjoint of a matrix $\sigma \in \mathrm{M}_n(B)$ (transpose and apply $*$ to the entries). However, if σ is upper triangular, the condition $\vartheta(\sigma)^T = \sigma$ can not hold with respect to this involution, as we then have $\vartheta(\sigma)_{ij} = \theta(\sigma_{ij})$ so $\vartheta(\sigma)^T$ is a lower triangular matrix. Hence for upper triangular quantum automorphisms one focuses on the permutation $\bar{i} := n + 1 - i$.

4.2.9 Application to $k[t^2, t^3]$

Assume now that k is a subfield of $\mathbb{C}$ with involution given by complex conjugation. Then $B = k[t^2, t^3]$ becomes a $*$-algebra via

$$*\colon B = k[t^2, t^3] \longrightarrow B, \quad (\lambda t^n)^* := \bar{\lambda} t^n.$$

Geometrically, this $*$-structure on B describes the real points of the singular curve $V \subseteq k^2$ in the sense that the points of the curve $V_{\mathbb{R}} = V \cap \mathbb{R}^2$ correspond to the one-dimensional $*$-representations of B and if k is algebraically closed, they correspond to the maximal ideals in $k[t^2, t^3]$ which are invariant under $*$.

The quantum automorphism $\sigma\colon B \to \mathrm{M}_3(B)$ that we study since Section 4.1.7 satisfies $\vartheta(\sigma)^T = \sigma$ provided that we work with $\bar{i} := 4 - i$ as in Example 4.2.41. The Hopf $*$-structure on H_σ is computed as follows: Recall from Section 4.2.3 that $K = s_{0,22}$ thus

$$K^* = s_{1,22} = S(K) = K.$$

Similarly recalling that $D = iE_-$ and $E^* = s^*_{0,12} = S(F) = E$ we obtain that $E^*_- = E_-$ thus

$$D^* = -D.$$

Finally using $Z^* = s^*_{0,13} = S(Z) = -Z - E^2$ and $Z^*_+ = -Z_+ - E^2$, we have

$$Y^* = -Y + 6iD.$$

References

[1] Andrés, G. (2009). *Quantum Groups and Hopf Algebras.*

[2] Andruskiewitsch, N. and Schneider, H.-J. (2010). *On the classification of finite-dimensional pointed Hopf algebras.* Ann. Math. (2), 171(1):375–417.

[3] Banica, T., Bhowmick, J., and De Commer, K. (2012). *Quantum isometries and group dual subgroups.* Ann. Math. Blaise Pascal, 19(1):1–27.

[4] Banica, T. and Bichon, J. (2010). *Hopf images and inner faithful representations.* Glasg. Math. J., 52(3):677–703.

[5] Banica, T., Bichon, J., and Collins, B. (2008). *Quantum permutation groups: a survey.* In *Noncommutative harmonic analysis with applications to probability. Papers presented at the 9th workshop, Będlewo, Poland, September 29–October 10, 2006*, pages 13–34. Warsaw: Polish Academy of Sciences, Institute of Mathematics.

[6] Banica, T. and Goswami, D. (2010). *Quantum isometries and noncommutative spheres.* Commun. Math. Phys., 298(2):343–356.

[7] Banica, T. and Speicher, R. (2009). *Liberation of orthogonal Lie groups.* Adv. Math., 222(4):1461–1501.

[8] Bergman, G. M. (1977). *The diamond lemma for ring theory.* Adv. Math., 29:178–218.

[9] Bhowmick, J., Skalski, A., and Sołtan, P. M. (2015). *Quantum group of automorphisms of a finite quantum group.* J. Algebra, 423:514–537.

[10] Bichon, J. and Riche, S. (2016). *Hopf algebras having a dense big cell.* Trans. Am. Math. Soc., 368(1):515–538.

[11] Brown, K. and Gilmartin, P. (2016). *Quantum homogeneous spaces of connected Hopf algebras.* J. Algebra, 454:400–432.

[12] Brown, K. and Tabiri, A. A. (2021). *Plane curves which are quantum homogeneous spaces.* Algebras and Representation Theory, pages 1–34.

[13] Brown, K. A. and Zhang, J. J. (2010). *Prime regular Hopf algebras of GK-dimension one.* Proc. Lond. Math. Soc. (3), 101(1):260–302.

[14] Brzeziński, T. (1996). *Quantum homogeneous spaces as quantum quotient spaces.* J. Math. Phys., 37(5):2388–2399.

[15] Cartier, P. (2007). *A primer of Hopf algebras.* In *Frontiers in number theory, physics, and geometry II. On conformal field theories, discrete groups and renormalization. Papers from the meeting, Les Houches, France, March 9–21, 2003*, pages 537–615. Berlin: Springer.

[16] Cohen, M. (1994). *Quantum commutativity and central invariants.* In *Advances in Hopf algebras. Conference, August 10-14, 1992, Chicago, IL, USA*, pages 25–38. New York, NY: Marcel Dekker.

[17] Cox, D. A. (2012). *Galois theory.* Hoboken, NJ: John Wiley & Sons.

[18] Crespo, T., Rio, A., and Vela, M. (2016). *On the Galois correspondence theorem in separable Hopf Galois theory.* Publ. Mat., Barc., 60(1):221–234.

[19] Drinfel'd, V. G. (1988). *Quantum groups.* J. Sov. Math., 41(2):898–915.

[20] Etingof, P., Goswami, D., Mandal, A., and Walton, C. (2017). *Hopf coactions on commutative algebras generated by a quadratically independent comodule.* Commun. Algebra, 45(8):3410–3412.

[21] Etingof, P. and Walton, C. (2014). *Semisimple Hopf actions on commutative domains.* Adv. Math., 251:47–61.

[22] Etingof, P. and Walton, C. (2015). *Pointed Hopf actions on fields. I.* Transform. Groups, 20(4):985–1013.

[23] Etingof, P. and Walton, C. (2016). *Pointed Hopf actions on fields. II.* J. Algebra, 460:253–283.

[24] Goodearl, K. R. and Zhang, J. J. (2010). *Noetherian Hopf algebra domains of Gelfand-Kirillov dimension two.* J. Algebra, 324(11):3131–3168.

[25] Greither, C. and Pareigis, B. (1987). *Hopf Galois theory for separable field extensions.* J. Algebra, 106:239–258.

[26] Hahn, A. and O'Meara, O. T. (1989). *The classical groups and K-theory. Foreword by J. Dieudonné*, volume 291 of *Grundlehren Math. Wiss.* Berlin etc.: Springer-Verlag.

[27] Hartshorne, R. (1983). *Algebraic geometry. Corr. 3rd printing*, volume 52 of *Grad. Texts Math.* Springer, Cham.

[28] Joseph, A. (1995). *Quantum groups and their primitive ideals*, volume 29 of *Ergeb. Math. Grenzgeb., 3. Folge.* Berlin: Springer-Verlag.

[29] Kasprzak, P. and Sołtan, P. M. (2014). *Embeddable quantum homogeneous spaces.* J. Math. Anal. Appl., 411(2):574–591.

[30] Kassel, C. (1995). *Quantum groups*, volume 155. New York, NY: Springer-Verlag.

[31] Klimyk, A. and Schmüdgen, K. (1997). *Quantum groups and their representations.* Berlin: Springer.

[32] Krähmer, U. (2012). *On the Hochschild (co)homology of quantum homogeneous spaces.* Isr. J. Math., 189:237–266.

[33] Krähmer, U. and Martins, M. (2020). *The nodal cubic and quantum groups at roots of unity.* J. Noncommut. Geom., 14(2):667–680.

[34] Krähmer, U. and Tabiri, A. A. (2017). *The nodal cubic is a quantum homogeneous space.* Algebr. Represent. Theory, 20(3):655–658.

[35] Lam, T. Y. (1999). *Lectures on modules and rings*, volume 189 of *Grad. Texts Math.* New York, NY: Springer.

[36] Liu, G. (2009). *On Noetherian affine prime regular Hopf algebras of Gelfand-Kirillov dimension 1.* Proc. Am. Math. Soc., 137(3):777–785.

[37] Manin, Y. I. (1987). *Some remarks on Koszul algebras and quantum groups.* Ann. Inst. Fourier, 37(4):191–205.

[38] Masuoka, A. (1991). *On Hopf algebras with cocommutative coradicals.* J. Algebra, 144(2):451–466.

[39] Masuoka, A. and Wigner, D. (1994). *Faithful flatness of Hopf algebras.* J. Algebra, 170(1):156–164.

[40] Montgomery, S. (1993). *Hopf algebras and their actions on rings, CBMS*, volume 82 of *Regional Conference Series in Mathematics*. Providence, RI: American Mathematical Society.

[41] Müller, E. F. and Schneider, H.-J. (1999). *Quantum homogeneous spaces with faithfully flat module structures*. Isr. J. Math., 111:157–190.

[42] Podleś, P. (1987). *Quantum spheres*. Letters in Mathematical Physics, 14(3):193–202.

[43] Podleś, P. (1995). *Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups*. Commun. Math. Phys., 170(1):1–20.

[44] Radford, D. E. (2012). *Hopf algebras.*, volume 49. Hackensack, NJ: World Scientific.

[45] Rotman, J. J. (2009). *An introduction to homological algebra*. Berlin: Springer.

[46] Schneider, H.-J. (1990). *Principal homogeneous spaces for arbitrary Hopf algebras*. Isr. J. Math., 72(1-2):167–195.

[47] Sweedler, M. E. (1969). *Hopf algebras*. New York: W.A. Benjamin, Inc. 1969, 336 p. (1969).

[48] Takeuchi, M. (1979). *Relative Hopf modules - equivalences and freeness criteria*. J. Algebra, 60:452–471.

[49] Taylor, S. and Truman, P. J. (2019). *The structure of Hopf algebras giving Hopf-Galois structures on quaternionic extensions*. New York J. Math., 25:219–237.

[50] Tsang, C. (2019). *Non-existence of Hopf-Galois structures and bijective crossed homomorphisms*. J. Pure Appl. Algebra, 223(7):2804–2821.

[51] Vaes, S. (2005). *A new approach to induction and imprimitivity results*. J. Funct. Anal., 229(2):317–374.

[52] Weber, M. (2017). *Introduction to compact (matrix) quantum groups and Banica-Speicher (easy) quantum groups*. Proc. Indian Acad. Sci., Math. Sci., 127(5):881–933.