

Free energy on the sphere for non-abelian gauge theories

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ABSTRACT: We compute the S^d partition function of the fixed point of non-abelian gauge theories in continuous d , using the ϵ -expansion around $d = 4$. We illustrate in detail the technical aspects of the calculation, including all the factors arising from the gauge-fixing procedure, and the method to deal with the zero-modes of the ghosts. We obtain the result up to NLO, i.e. including two-loop vacuum diagrams. Depending on the sign of the one-loop beta function, there is a fixed point with real gauge coupling in $d > 4$ or $d < 4$. In the first case we extrapolate to $d = 5$ to test a recently proposed construction of the UV fixed point of $5d$ SU(2) Yang-Mills via a susy-breaking deformation of the E_1 SCFT. We find that the F theorem allows the proposed RG flow. In the second case we extrapolate to $d = 3$ to test whether QCD₃ with gauge group SU(n_c) and n_f fundamental matter fields flows to a CFT or to a symmetry-breaking case. We find that within the regime with a real gauge coupling near $d = 4$ the CFT phase is always favored. For lower values of n_f we compare the average of F between the two complex fixed points with its value at the symmetry-breaking phase to give an upper bound of the critical value n_f^* below which the symmetry-breaking phase takes over.

KEYWORDS: Field Theories in Higher Dimensions, Field Theories in Lower Dimensions, Renormalization Group, Spontaneous Symmetry Breaking

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1 Introduction

The intuitive idea that the number of degrees of freedom should decrease along a Renormalization Group (RG) flow can be made precise in Quantum Field Theory (QFT). This is done by assigning a quantity to Conformal Field Theories (CFT) –which typically are the endpoints of RG flows– with the property of monotonicity along the flow, i.e. its value is always larger for the UV CFT than it is for the IR CFT. Such quantities were first found in even spacetime dimensions d [1–4] and identified with the a coefficient in the Weyl anomaly

$$\langle T^\mu_\mu \rangle \sim (-1)^{d/2} a E_d + \sum_i c_i I_i, \tag{1.1}$$

where E_d is the Euler density term, and c_i are the coefficients of other Weyl invariant curvature terms I_i . The inequality $a_{\text{UV}} > a_{\text{IR}}$ was proved in $d = 2$ [1], in which case it reduces to the celebrated c -theorem because $a = c/3$, and in $d = 4$ [3, 4]. Attempts to generalize the proof to $d = 6$ [5] so far have only succeeded in the supersymmetric case [6].

In odd dimensions, even though there are no Weyl anomalies, a monotonic quantity can still be defined using the (renormalized) free energy on the sphere

$$F = -\log Z_{S^d}. \tag{1.2}$$

The inequality $F_{\text{UV}} > F_{\text{IR}}$, known as the F -theorem, was proposed and checked in [7, 8] and then proved in $d = 3$ using the relation to entanglement entropy across a spherical entangling surface [9, 10], which can also be used to prove monotonicity of the a coefficient in $d = 2, 4$ [11, 12]. For other odd dimensions it is conjectured that the decreasing quantity becomes $(-1)^{\frac{d+1}{2}} F$ [8]. This has not been proved yet but is motivated by several examples.

It is sometimes possible to continue RG flows to non-integer dimensions, at least formally. When a flow to an interacting fixed point can be continued to the vicinity of its upper or lower critical dimension, it becomes short and controllable in perturbation theory. This strategy, known as ϵ -expansion [13, 14], can lead to a useful approximation of strongly coupled fixed points. Motivated by this method, ref. [15] proposed to unify all the previously mentioned inequalities in a single relation valid in continuous dimensions. The authors defined

$$\tilde{F} = -\sin\left(\frac{\pi d}{2}\right) F, \tag{1.3}$$

which in odd d exactly reproduces the $(-1)^{(d+1)/2} F$ term, while in even d provides a smooth limit proportional to the a anomaly: the factor $\sin(\frac{\pi d}{2})$ cancels the pole in the free energy leading to the finite limit $\tilde{F} = \pi a/2$. Therefore, the inequality

$$\tilde{F}_{\text{UV}} > \tilde{F}_{\text{IR}} \tag{1.4}$$

automatically encodes all the previous relations, and extends them to non-integer values of d .

In this paper we compute the quantity \tilde{F} for the fixed point in non-abelian gauge theories, in an expansion around $d = 4$. The existence of such fixed points can be inferred from the leading terms in the β function for the gauge coupling, which in $d = 4 + 2\epsilon$ read

$$\beta_g = \epsilon g + \beta_0 g^3 + \mathcal{O}(g^5). \tag{1.5}$$

For a small number of matter fields the one-loop coefficient β_0 is negative and leads to a one-loop fixed point $g_{1\text{-loop}}^{*2} > 0$ for $d > 4$, while for $d < 4$ a minimal number of matter fields is required to have $\beta_0 > 0$, so that $g_{1\text{-loop}}^{*2} > 0$. The quantity \tilde{F} was computed in [16] for d -dimensional QED with n_f four-component fermionic matter fields, for which β_0 is always positive. It was then extrapolated to $d = 3$ to study the existence of an interacting IR CFT for QED in 3 spacetime dimensions, by comparing with the quantity F for the spontaneously broken phase of $2n_f^2 + 1$ massless Goldstone bosons.

The calculation in non-abelian gauge theories presents several new challenges compared to the abelian case. Firstly, the gauge fixing requires a more careful analysis, because it becomes unavoidable to include the interaction with the ghost fields. On the sphere massless scalar fields like the ghosts have zero modes. Due to the fermionic nature of ghosts, this naively leads to a zero in the partition function, which manifests as an IR divergence in \tilde{F} . This divergence needs to be cured by an appropriate regulator (or alternatively by appropriately modifying the gauge-fixing procedure, as we describe in an appendix). Note that, in order to obtain \tilde{F} , it is crucial to carefully keep track of the normalization of the path integral on S^d when implementing the gauge-fixing through the Faddeev-Popov procedure [17]. Secondly, the derivative self-interaction of the gluon leads to diagrams with two derivatives acting on the propagator, and it is important to include also the contact-term contributions in order to evaluate correctly the integrals over the positions of the vertices. Thirdly, unlike QED the renormalization in the gauge sector is not simply encoded in the definition of a renormalized gauge coupling, instead one needs to consider also wave-function counterterms for the gluons and the ghosts. We perform the calculation, taking care of all these issues, up to the next-to-leading (NLO) order, i.e. including up to two-loop vacuum diagrams. The result is in eq. (4.49). Note that, while we compute the two-loop diagrams in generic ξ -gauge, which allows us to compare with heat-kernel results for generic background [18], we keep track of the normalization of the path integral only in the special case of the Landau gauge, i.e. $\xi = 0$.

We then apply this result to the fixed points of $SU(n_c)$ non-abelian gauge theories in $d = 3$ and in $d = 5$. In $d = 3$, just like in the QED case mentioned above, the theory is known to flow to a CFT in the IR for a sufficiently large number of matter flavors n_f [19], and it is conjectured to change its behavior for n_f smaller than an unknown critical value n_f^* , flowing instead to a phase with spontaneous breaking of the global symmetry [19–22]. We adopt the same logic as in [16], and compare F of the fixed point to that of the putative Goldstone bosons phase. We find that when $\beta_0 > 0$, so that $g_{1\text{-loop}}^{*2} > 0$, the conformal phase is always favored compared to the symmetry-breaking phase. For $\beta_0 < 0$ the fixed point is complex in the ϵ -expansion, but a unitary fixed point in $d = 3$ can still exist.¹ We propose a more speculative approach to estimate F of the $3d$ CFT in this case, by taking an average value of \tilde{F} among the two complex fixed points. With this method we find that the Goldstone boson phase becomes favored for small n_f , allowing us to put an upper bound on n_f^* . The values found for $2 \leq n_c \leq 5$ are reported in eq. (5.14). The result for $n_c = 2$

¹The opposite situation can also occur, a fixed point for $\epsilon \ll 1$ which disappears in physical integer dimensions.

favorably compares with previous bounds found using again the F -theorem combined with supersymmetry [23], or lattice methods [24]. We also give an estimate for the upper bound on x^* in the Veneziano limit in eq. (5.15), where $x = n_f/n_c$.

In $d = 5$ we use our calculation to investigate the existence of interacting CFTs that UV complete $5d$ non-abelian gauge theories. If such CFTs exist they would be an example of a non-supersymmetric interacting CFT in $d > 4$. An interesting construction in the case of $SU(2)$ Yang-Mills theory was recently proposed in [25], and further refined in [26], using the E_1 superconformal field theory that UV completes $SU(2)$ Super Yang-Mills. The putative non-supersymmetric CFT is obtained as the IR endpoint of the RG flow triggered by a certain non-supersymmetric deformation of E_1 , and by construction it is endowed with a relevant deformation that flows to ordinary $SU(2)$ Yang-Mills theory. Using our extrapolation to $5d$ we can compare the quantity F of the non-supersymmetric CFT with that of the E_1 SCFT, known from supersymmetric localization [27], and test if the RG flow is allowed. We can also easily repeat this check in the case with fundamental flavors n_f and compare with the F quantity of the E_{n_f+1} SCFT that UV completes the supersymmetric gauge theory with flavors. In all cases in which we have evidence for a fixed point in $d = 5$, namely $n_f \leq 4$ [28], we obtain that the F -theorem allows the proposed RG flow.

The rest of the paper is organized as follows: in section 2 we explain some generalities about the calculation of the sphere partition function, we perform the gauge-fixing and compute the one-loop determinants for non-abelian gauge theories. In section 3 we derive the Feynman rules on the sphere, including the gauge field propagator in an arbitrary ξ -gauge. In section 4 we compute the two-loop vacuum-vacuum diagrams and obtain our main result. In section 5 we apply the result to the $d = 3$ and $d = 5$ models described above. In section 6 we draw our conclusions and outline some possible future directions. Most of the technical points of the calculation are relegated to the first three appendices. In appendix D we show a sanity check of our results, by comparing in detail the UV divergences obtained for pure Yang-Mills theory in ref. [18] in the Feynman gauge $\xi = 1$ with our results. In appendix E we explain a possible alternative gauge-fixing procedure (used already in [29]) where ghost zero modes are treated more carefully by introducing ghosts for ghosts, which we also use to partially check the results in the main body.

Finally, a comment on notation: in this paper n_f always refers to the number of $4d$ Dirac fermions. Given the way we analytically continue fermions, n_f $4d$ Dirac fermions give rise to $2n_f$ Dirac fermions in $3d$ and n_f Dirac fermions in $5d$.

2 Free energy of gauge theories on the sphere: leading order

Let us consider a non-abelian gauge theory with n_f massless Dirac fermions in the fundamental representation. We want to compute the sphere free energy in $d = 4 + 2\epsilon$, defined as

$$F = -\log Z_{S^d}, \quad Z_{S^d} = \frac{1}{\text{vol}(\mathcal{G})} \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-S[A, \psi, \bar{\psi}, h]\right). \quad (2.1)$$

Here h denotes the round metric $h_{\mu\nu}$ on S^d with radius R and coordinate x , while $\text{vol}(\mathcal{G})$ is the volume of the space of all gauge transformations, which in our choice of normalization

does not depend on the gauge coupling g . We can split the action on the sphere in

$$S = S_{\text{YM}} + S_{\text{Ferm}} + S_{\text{curv}}, \tag{2.2}$$

with

$$S_{\text{YM}} = \int d^d x \sqrt{h} \left(\frac{1}{4g_0^2} \text{Tr}[F_{\mu\nu}(x)F^{\mu\nu}(x)] \right), \tag{2.3}$$

$$S_{\text{Ferm}} = \int d^d x \sqrt{h} \left(- \sum_{i=1}^{n_f} \bar{\psi}_i \gamma^\mu (\nabla_\mu + iA_\mu) \psi^i \right), \tag{2.4}$$

$$S_{\text{curv}} = \int d^d x \sqrt{h} \left(b_0 E + c_0 \mathcal{R}^2 / (d-1)^2 \right), \tag{2.5}$$

where $h = \det h_{\mu\nu}$,² g_0 is the bare gauge coupling constant, ψ^i are n_f four-component Dirac fermions and ∇_μ is the curved space covariant derivative which includes the spin connection term when acting on fermions. As the action should contain all operators that are marginal in $d = 4$, we have added the curvature terms together with their bare coupling parameters b_0 and c_0 .³ For future purposes, we recall the expression for the Ricci scalar \mathcal{R} and the Euler density E on S^d :

$$\mathcal{R} = \frac{d(d-1)}{R^2}, \quad E = \mathcal{R}_{\mu\nu\lambda\rho} \mathcal{R}^{\mu\nu\lambda\rho} - 4\mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \mathcal{R}^2 = \frac{d(d-1)(d-2)(d-3)}{R^4}. \tag{2.6}$$

2.1 One-loop determinants

At leading order in a loopwise expansion the free energy is determined by one loop determinants. As a consequence of the splitting in eq. (2.2), we can divide the leading term of the sphere free energy F_{Free} in three parts:

$$F_{\text{Free}} = F_{\text{free-YM}} + F_{\text{free-ferm}} + F_{\text{curv}}, \tag{2.7}$$

with

$$F_{\text{free-YM}} = -\log \left(\frac{1}{\text{vol}(\mathcal{G})} \int \mathcal{D}A e^{-S_{\text{free-YM}}[A,h]} \right), \tag{2.8}$$

$$F_{\text{free-ferm}} = -\log \left(\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_{\text{free-ferm}}[\psi,h]} \right), \tag{2.9}$$

$$F_{\text{curv}} = \Omega_d R^{d-4} (d(d-1)(d-2)(d-3)) b_0 + d^2 c_0, \tag{2.10}$$

where $S_{\text{free-YM}}$ is the quadratic part of the Yang-Mills action, $S_{\text{free-ferm}}$ the free fermion action and $\Omega_d = 2\pi^{\frac{d+1}{2}} / \Gamma(\frac{d+1}{2})$ is the volume of the d -dimensional sphere with unit radius.

The expression for $F_{\text{free-ferm}}$ was found in ref. [15]. The result for a single four-component Dirac spinor is

$$F_{\text{free-ferm}}(d) = -\frac{4}{\sin(\frac{\pi d}{2}) \Gamma(1+d)} \int_0^1 du \cos\left(\frac{\pi u}{2}\right) \Gamma\left(\frac{1+d+u}{2}\right) \Gamma\left(\frac{1+d-u}{2}\right). \tag{2.11}$$

²We use the same symbol h for the metric determinant $\det h_{\mu\nu}$ and for the metric tensor $h_{\mu\nu}$ whenever indices are omitted. The difference should be clear from the context.

³In a generic Euclidean manifold we should also include a term with the square of the Weyl tensor, omitted here as it vanishes on the sphere.

Let us now focus on the computation of $F_{\text{free-YM}}$. The gauge field A^μ on the sphere can be written as the sum of a longitudinal part $A_{(0)}^\mu$ and a transverse part $A_{(1)}^\mu$, which can be separately decomposed in orthonormal eigenvectors of the sphere Laplacian $-\nabla^2$:

$$\begin{aligned} A^\mu &= A_{(0)}^\mu + A_{(1)}^\mu, \quad \text{such that} \quad \nabla_\mu A_{(1)}^\mu = 0, \\ A_{(0)}^\mu &= \sum_{\ell > 0} a_{(0)}^\ell A_{(0)}^{\mu \ell}, \quad A_{(1)}^\mu = \sum_{\ell > 0} a_{(1)}^\ell A_{(1)}^{\mu \ell}, \end{aligned} \tag{2.12}$$

with corresponding eigenvalues $\lambda_\ell^{(1)}$, $\lambda_\ell^{(0)}$ and degeneracies $g_\ell^{(1)}$, $g_\ell^{(0)}$ given by [30]

$$\begin{aligned} \lambda_\ell^{(1)} &= \frac{(\ell(\ell + d - 1) - 1)}{R^2}, & g_\ell^{(1)} &= \frac{\ell(\ell + d - 1)(2\ell + d - 1)\Gamma(\ell + d - 2)}{\Gamma(\ell + 2)\Gamma(d - 1)}, \quad \ell > 0, \\ \lambda_\ell^{(0)} &= \frac{\ell(\ell + d - 1) - (d - 1)}{R^2}, & g_\ell^{(0)} &= \frac{(2\ell + d - 1)\Gamma(\ell + d - 1)}{\Gamma(\ell + 1)\Gamma(d)}, \quad \ell > 0. \end{aligned} \tag{2.13}$$

Note that the eigenfunctions of the longitudinal part can be rewritten in terms of the covariant derivative of the spherical harmonics $Y_\ell(x)$

$$A_{(0)}^{\mu \ell} = \frac{1}{\sqrt{\lambda_\ell^{(S)}}} \nabla^\mu Y_\ell(x), \quad \text{for } \ell \geq 1. \tag{2.14}$$

We take the spherical harmonics to be normalized as

$$\int d^d x \sqrt{h} Y_\ell(x) Y_{\ell'}(x) = \delta_{\ell\ell'}. \tag{2.15}$$

In order to make the basis $A_{(0)}^{\mu \ell}$ orthonormal, we have fixed the normalization factor in terms of the eigenvalue of the laplacian operator associated to $Y_\ell(x)$

$$\lambda_\ell^{(S)} = \frac{\ell(\ell + d - 1)}{R^2}, \tag{2.16}$$

which has degeneracy $g_\ell^{(0)}$. Note a crucial difference between the spectrum for a scalar and for the longitudinal modes of a vector: the former includes a constant mode with eigenvalue $\lambda_0^{(S)} = 0$ and degeneracy $g_0^{(0)} = 1$, while for the latter the modes are restricted to $\ell > 0$ and as a result the constant is excluded.

In dimensional regularization the following identities are valid, which will be useful later in the computation:

$$\sum_{\ell=1}^{\infty} g_\ell^{(1)} = 1 \quad \text{and} \quad \sum_{\ell=1}^{\infty} g_\ell^{(0)} = -1. \tag{2.17}$$

With this decomposition in longitudinal and transverse mode the path integral measure can be rewritten as

$$\int \mathcal{D}A = \int \prod_{\ell=1}^{\infty} da_{(0)}^\ell \int \prod_{\ell=1}^{\infty} da_{(1)}^\ell. \tag{2.18}$$

2.2 Computation in Landau gauge

We want to compute

$$F_{\text{free-YM}} = -\log \left(\frac{1}{\text{vol}(\mathcal{G})} \int \mathcal{D}A e^{-S_{\text{free-YM}}[A,h]} \right), \quad (2.19)$$

with

$$S_{\text{free-YM}} = \int d^d x \sqrt{h} \frac{1}{2g_0^2} \text{Tr} \left[A_\nu (-\delta_\mu^\nu \nabla^2 + R_\mu^\nu + \nabla^\nu \nabla_\mu) A^\mu \right] \quad (2.20)$$

and $R_\mu^\nu = \frac{d-1}{R^2} \delta_\mu^\nu$ on S^d . In order to perform the explicit computation it is convenient to add a gauge-fixing term to the action. We work in Landau gauge and set to zero the longitudinal component of the gauge field. In order to do that we insert in the path integral of eq. (2.1) the following functional identity, valid for any fixed $A_\mu(x)$:

$$1 = \int_{\mathcal{G}'} \mathcal{D}\mu_g(U) \delta(\nabla^\mu A_\mu^U) \left| \det \frac{\delta \nabla^\mu A_\mu^U}{\delta \epsilon} \right|, \quad (2.21)$$

where $A_\mu^U(x)$ is the gauge-transformed field under $U(x)$

$$A_\mu(x) \rightarrow A_\mu^U(x) = U(x)(\nabla_\mu + iA_\mu(x))U^{-1}(x) \equiv U(x)D_\mu^A U^{-1}(x). \quad (2.22)$$

Taking the components in the Lie Algebra, denoted with indices a, b, c, \dots , and also writing $U = \exp(i\epsilon^a T^a)$ in terms of the parameter ϵ^a and the generators T^a , we get the infinitesimal transformation

$$\delta A_\mu^a(x) = (D_\mu^A \epsilon)^a(x) = \nabla_\mu \epsilon^a(x) + i f^{abc} A_\mu^b(x) \epsilon^c(x). \quad (2.23)$$

The integration in eq. (2.21) is performed over the functional Haar measure μ_g and is restricted to the set of gauge transformations \mathcal{G}' that act non-trivially on $A_\mu(x)$, i.e. those that give a non-zero functional determinant. In the functional derivative the variation $\delta\epsilon$ is an infinitesimal variation away from U (the integration variable) and tangential to \mathcal{G}' , hence $\delta\epsilon$ is any fluctuation not annihilated by the covariant derivative with connection A_μ^U . So we have

$$\left| \det \frac{\delta \nabla^\mu A_\mu^U}{\delta \epsilon} \right| = \det' \left(-\nabla^\mu D_\mu^{A^U} \right), \quad (2.24)$$

where the prime denotes that we need to exclude the zero eigenvalue and the minus sign is taken to ensure positivity of the determinant, at least perturbatively. At this point in order to proceed we restrict ourselves to the case of Landau gauge, and use that in Landau gauge the operator is self-adjoint as ∇^μ and $D_\mu^{A^U}$ commute. Therefore, we can implement the prime by excluding constant modes instead of covariantly constant ones. We will always assume this meaning of the prime from now on, as this will lead to a great simplification in the following manipulations.

Inserting the identity in the path integral and exchanging the order of the integrals we obtain

$$F = -\log \left(\frac{1}{\text{vol}(\mathcal{G})} \int_{\mathcal{G}'} \mathcal{D}\mu_g(U) \int \mathcal{D}A \exp \left(-S[A, \psi, \bar{\psi}, h] \right) \delta(\nabla^\mu A_\mu^U) \det' \left(-\nabla^\mu D_\mu^{A^U} \right) \right). \quad (2.25)$$

Using gauge invariance of the integration measure and of the action the integral in A can be rewritten in terms of the variable A^U , renamed A . As a result the integral over μ_g yields just the volume of \mathcal{G}' and we get

$$F = -\log \left(\frac{\text{vol}(\mathcal{G}')}{\text{vol}(\mathcal{G})} \int \mathcal{D}A \exp \left(-S[A, \psi, \bar{\psi}, h] \right) \delta(-\nabla^\mu A_\mu) \det' \left(-\nabla^\mu D_\mu^A \right) \right). \quad (2.26)$$

The ratio of the two infinite-dimensional volumes gives the volume of the constant gauge transformations, i.e. the volume of the group G , multiplied by an additional factor that arises by requiring an orthonormal mode decomposition in the path integral.⁴ In order to explain this factor, consider separating a generic gauge transformation $f : S^d \rightarrow G$ in a constant and a non-constant part $f(x) = f_0 + f'(x)$. This can be done via the decomposition in spherical harmonics: $f(x) = \sum_{\ell=0}^{\infty} F_\ell Y_\ell(x)$. In terms of this decomposition the measure of the path integral is

$$\int \mathcal{D}f = \int \prod_{\ell=0}^{\infty} dF_\ell. \quad (2.27)$$

Because of the normalization in (2.15) we have $Y_0 = 1/\sqrt{\text{vol}(S^d)}$, which implies $f_0 = F_0/\sqrt{\text{vol}(S^d)}$ and

$$\text{vol}(\mathcal{G}) = \int dF_0 \text{vol}(\mathcal{G}') = \text{vol}(S^d)^{\frac{\dim(G)}{2}} \text{vol}(G) \text{vol}(\mathcal{G}'). \quad (2.28)$$

This leads to

$$F = -\log \left(\frac{\text{vol}(S^d)^{\frac{-\dim(G)}{2}}}{\text{vol}(G)} \int \mathcal{D}A \exp \left(-S[A, \psi, \bar{\psi}, h] \right) \delta(\nabla^\mu A_\mu) \det' \left(-\nabla^\mu D_\mu^A \right) \right). \quad (2.29)$$

We then introduce non-constant c' and \bar{c}' ghost modes to rewrite the \det' as

$$\det' \left(-\nabla^\mu D_\mu^A \right) = \int \mathcal{D}c' \mathcal{D}\bar{c}' \exp \left(- \int d^d x \sqrt{h(x)} \text{Tr}[\bar{c}'(x) \nabla^\mu D_\mu^A c'(x)] \right). \quad (2.30)$$

The final step is to use the decomposition (2.12) to rewrite the δ -functional in eq. (2.29) in terms of the coefficients of the decomposition

$$\delta(\nabla^\mu A_\mu) = \delta \left(\sum_{\ell=1}^{\infty} \frac{a_{(0)}^\ell}{\sqrt{\lambda_\ell^{(S)}}} \nabla^2 Y_\ell(x) \right) = \prod_{\ell=1}^{\infty} \left(\frac{\ell(\ell+d-1)}{R^2} \right)^{-\frac{g_\ell^{(0)}}{2} \dim(G)} \delta \left(a_{(0)}^\ell \right). \quad (2.31)$$

⁴The normalization of the path integral is chosen following ref. [29]. There is however a difference in the computation of the volume of the gauge group as in our notation the coupling does not appear in the volume expression.

This sets to zero the longitudinal modes and provides a crucial factor in the path integral. Plugging eq. (2.31) in eq. (2.29) and focusing on the Yang-Mills leading contribution gives

$$F_{\text{free-YM}} = -\log \left(\frac{1}{\text{vol}(G)\sqrt{\text{vol}(S^d)^{\dim(G)}}} \prod_{\ell=1}^{\infty} \left(\frac{\ell(\ell+d-1)}{R^2} \right)^{-\frac{g_\ell^{(0)}}{2}\dim(G)} \int \mathcal{D}A_{(1)} \mathcal{D}c' \mathcal{D}\bar{c}' \exp \left(-S_{\text{YM-Free}}[A_{(1)}, h] - \int d^d x \sqrt{h(x)} \text{Tr}[\bar{c}'(x) \nabla^2 c'(x)] \right) \right). \quad (2.32)$$

We are finally ready to compute the integral. We start from the integration over $A_{(1)}$. Using the decomposition in eqs. (2.12)–(2.13) and the normalization in eq. (2.27) we get

$$\begin{aligned} & \int \mathcal{D}A_{(1)} \exp \left(-\int d^d x \sqrt{h} \frac{1}{2g_0^2} \left(A_{(1)\mu}^a (-\nabla^2 + (d-1)) A_{(1)a}^\mu \right) \right) \\ &= \prod_{\ell=1}^{\infty} \left(\frac{2\pi g_0^2 R^2}{(\ell+1)(\ell+d-2)} \right)^{\frac{g_\ell^{(1)}}{2}\dim(G)}. \end{aligned} \quad (2.33)$$

For the computation of the ghost path integral we again decompose in spherical harmonics:

$$c'(x) = \sum_{\ell=1}^{\infty} C_\ell Y_\ell(x), \quad \int \mathcal{D}c' = \int \prod_{\ell=1}^{\infty} dC_\ell. \quad (2.34)$$

As we are dealing with Grassmann variables, we have

$$\int \mathcal{D}C_\ell \mathcal{D}\bar{C}_\ell \exp(\bar{C}_\ell C_\ell) = 1, \quad (2.35)$$

implying

$$\int \mathcal{D}c' \mathcal{D}\bar{c}' \exp \left(-\int d^d x \sqrt{h(x)} \text{Tr}[\bar{c}'(x) \nabla^2 c'(x)] \right) = \prod_{\ell=1}^{\infty} \left(\frac{\ell(\ell+d-1)}{R^2} \right)^{g_\ell^{(0)}\dim(G)}. \quad (2.36)$$

Replacing in eq. (2.32), we get

$$\begin{aligned} F_{\text{free-YM}} &= \log \text{vol}(G) + \frac{\dim(G)}{2} \left(\log \text{vol}(S^d) + \sum_{\ell=1}^{\infty} g_\ell^{(1)} \log \left(\frac{(\ell+1)(\ell+d-2)}{2\pi g_0^2 R^2} \right) \right. \\ &\quad \left. - \sum_{\ell=1}^{\infty} g_\ell^{(0)} \log \left(\frac{\ell(\ell+d-1)}{R^2} \right) \right). \end{aligned} \quad (2.37)$$

In order to find an explicit expression for these series one can follow [16], who performed the same computation in the abelian case. Their procedure is based on the rewriting of the logarithms appearing in eq. (2.37) with the identities

$$\log(y) = \int_0^\infty \frac{dt}{t} (e^{-t} - e^{-yt}), \quad \frac{1}{t} = \frac{1}{1-e^{-t}} \int_0^1 du e^{-ut}. \quad (2.38)$$

Then, using gamma function identities, eq. (2.17), and performing the t -integrals, one can find an analytical expression for $F_{\text{free-YM}}$. The only subtle point regards the ghost determinant. It is necessary to add and remove the zero mode, regulating with a mass parameter δ which is set to zero in the end. This provides

$$-\sum_{\ell=1}^{\infty} g_{\ell}^{(0)} \log(\ell(\ell+d-1)) = \lim_{\delta \rightarrow 0} \left[-\sum_{\ell=0}^{\infty} g_{\ell}^{(0)} \log((\ell+\delta)(\ell+d-1)) + \log(\delta(d-1)) \right], \quad (2.39)$$

For the sum over ℓ we use again eq. (2.38), while for the $\log(\delta(d-1))$ we use [16]

$$\log(\delta) = -\int_0^1 \frac{1}{u+\delta} + \log(1+\delta). \quad (2.40)$$

Putting everything together we find a smooth limit $\delta \rightarrow 0$, which reads

$$F_{\text{free-YM}}(d) = \dim(G) F_{\text{Max}}(d) - \frac{\dim(G)}{2} \log(g_0^2 R^{4-d}) + \log \frac{\text{vol}(G)}{(2\pi)^{\dim(G)}}. \quad (2.41)$$

where $F_{\text{Max}}(d)$ reads

$$\begin{aligned} F_{\text{Max}}(d) = & \frac{1}{2} \log(2\pi(d-1)^2 \Omega_d) - \frac{1}{\sin\left(\frac{\pi d}{2}\right)} \int_0^1 du \left((2u-d) \sin\left(\frac{\pi}{2}(d-2u)\right) \frac{\Gamma(d-u)\Gamma(u)}{\Gamma(d+1)} \right. \\ & + \left(d^2 + 1 - 3d(1+u) + 2u(u+2) \right) \frac{\sin\left(\frac{\pi}{2}(2u-d)\right) \Gamma(d-2-u)\Gamma(1+u)}{2\Gamma(d)} \\ & \left. + \frac{\sin\left(\frac{\pi d}{2}\right)(d-2)}{(d-2)^2 - u^2} + \frac{\sin\left(\frac{\pi d}{2}\right)}{u} \right). \end{aligned} \quad (2.42)$$

3 Feynman rules on the sphere

In this section we discuss the Feynman rules on S^d for non-abelian gauge theories. We start by reviewing some preliminary notion on maximally symmetric spaces. We then generalize the computation of the vector propagator presented in [31] in the Feynman gauge to an arbitrary ξ -gauge. The ghost propagator requires some care in order to remove the zero mode, while the propagator of the Dirac fermion is computed by a Weyl transformation from flat space. We then derive the Feynman rules for the vertices.

3.1 Bitensors in maximally symmetric spaces

The two-point function of a spinning operator in a curved space M defines a bitensor, namely a bilocal function that is a tensor with respect to both of its arguments. In maximally symmetric spaces bitensors can be expressed as sums and products of a few building blocks. Let us briefly review these building blocks following ref. [31]. Starting with the geodesic distance $\mu(x, x')$, which is a biscalar, other basic geometric objects are the parallel propagator $h^{\nu'}_{\nu}(x, x')$ transporting vectors along geodesics from x to x' , and the unit vectors $n_{\nu}(x, x')$ and $n_{\nu'}(x, x')$ tangent to the geodesic at x and x' respectively:

$$n_{\nu}(x, x') = \nabla_{\nu} \mu(x, x') \quad \text{and} \quad n_{\nu'}(x, x') = \nabla_{\nu'} \mu(x, x'). \quad (3.1)$$

$h_{\nu'}^{\nu}(x, x')$, $n_{\nu}(x, x')$ and $n_{\nu'}(x, x')$ are examples of bitensors. We use the following notation: a bitensor (n, m) is a rank n tensor at x and a rank m tensor at x' . So for instance $h_{\nu'}^{\nu}(x, x')$, $n_{\nu}(x, x')$ and $n_{\nu'}(x, x')$ are respectively $(1, 1)$, $(1, 0)$ and $(0, 1)$ bitensors. In general objects written as the contraction of two bitensors depend on both x and x' , even if they contain only primed or unprimed indices. An exception is the following identity relating the metric $h_{\nu\lambda}$ to the parallel propagator

$$h_{\nu\lambda}(x) = h_{\nu'}^{\rho'}(x, x')h_{\rho'\lambda}(x', x) , \tag{3.2}$$

and similarly for $h_{\nu'\lambda'}(x')$. Covariant derivatives of bitensors can be taken with respect to either x or x' and are denoted by ∇_{ν} and $\nabla_{\nu'}$ respectively.

It is possible to prove that any bitensor in a maximally symmetric space can be expressed as sums and products of the building blocks $h_{\nu\lambda}$, $h_{\nu'\lambda'}$, n_{ν} , $n_{\nu'}$ and $h_{\nu\lambda'}$, with coefficients that are only functions of μ . This provides a remarkable simplification in finding the structure of propagators and their explicit expressions.

Let us list some properties, useful for the derivation of propagators:

$$\begin{aligned} \nabla_{\nu}n_{\lambda} &= A(h_{\nu\lambda} - n_{\nu}n_{\lambda}) , \\ \nabla_{\nu}n_{\lambda'} &= C(h_{\nu\lambda'} + n_{\nu}n_{\lambda'}) , \\ \nabla_{\nu}h_{\lambda\rho'} &= -(A + C)(h_{\nu\lambda}n_{\rho'} + h_{\nu\rho}n_{\lambda}) , \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} A(\mu) &= \frac{1}{R} \cot(\mu/R) , \\ C(\mu) &= -\frac{1}{R} \frac{1}{\sin(\mu/R)} , \end{aligned} \tag{3.4}$$

where R is the radius, defined in terms the constant value of the Ricci curvature scalar in eq. (2.6). For future convenience it is useful to introduce the variable

$$z(x, x') \equiv \cos^2\left(\frac{\mu(x, x')}{2R}\right) . \tag{3.5}$$

which is the chordal distance between the points.

Let us now specify to a sphere S_R^d of radius R . Using stereographic coordinates x^{μ} we write the metric as

$$ds^2 = h_{\mu\nu}dx^{\mu}dx^{\nu} , \quad h_{\mu\nu} = \frac{4R^4}{(R^2 + |x|^2)^2}\delta_{\mu\nu} . \tag{3.6}$$

The geodesic distance is given by the following identity

$$\cos\left(\frac{\mu(x, x')}{R}\right) = 1 - \frac{2R^2|x - x'|^2}{(R^2 + |x|^2)(R^2 + |x'|^2)} = 2z(x, x') - 1 . \tag{3.7}$$

When $x' = 0$, we denote for simplicity

$$z \equiv z(x, 0) = \frac{R^2}{R^2 + x^2} . \tag{3.8}$$

The variable z will be useful to write propagator expressions and, in particular, their expansion around coincident points.

3.2 Vector propagator on S^d

Vector propagators for maximally d -dimensional symmetric spaces have been computed in [31]. For our purpose we need the expression of the massless vector field on the sphere. It follows from the quadratic part of the gauge action

$$S_{\text{free-YM}} = \int d^d x \sqrt{h} \frac{1}{2g_0^2} \text{Tr} \left[A_\nu (-\delta_\mu^\nu \nabla^2 + R_\mu^\nu + \left(1 - \frac{1}{\xi}\right) \nabla^\nu \nabla_\mu) A^\mu \right], \quad (3.9)$$

that the vector propagator $Q_{\nu\lambda'}^{ab}(x, x') = \langle A_\nu^a(x) A_{\lambda'}^b(x') \rangle = g_0^2 \delta^{ab} Q_{\nu\lambda'}(x, x')$ satisfies the equation

$$\left(-h^{\mu\nu} \nabla^2 - R^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) \nabla^\mu \nabla^\nu \right) Q_{\nu\lambda'}(x, x') = \delta(x - x') h_{\lambda'}^\mu. \quad (3.10)$$

The propagator $Q_{\nu\lambda'}(x, x')$ is a maximally symmetric (1,1) bitensor and can be decomposed as

$$Q_{\nu\lambda'}(x, x') = \alpha(\mu) h_{\nu\lambda'} + \beta(\mu) n_\nu n_{\lambda'}, \quad (3.11)$$

where α and β are generic functions of the geodesic distance. Their expression is found in eqs. (A.14), (A.19) and (A.21) in appendix A, where the interested reader can also find their detailed derivation.

3.3 Ghost propagator on S^d

The ghost propagator $G^{ab}(x, x') = \langle c'^a(x) \bar{c}'^b(x') \rangle$ satisfies

$$\nabla^2 G^{ab}(x, x') = \delta(x - x') \delta^{ab}. \quad (3.12)$$

As explained in section 2.2, c' has the zero mode removed, so we need to subtract the constant part from this propagator. This is also clear from the expansion of the propagator in terms of the spherical harmonics (2.16):

$$G^{ab}(x, x') = \sum_{\ell > 0} \frac{R^2}{-\ell(\ell + d - 1)} Y_\ell(x) Y_\ell(x') \delta^{ab}, \quad (3.13)$$

where the constant mode $\ell = 0$ is excluded from the sum, otherwise giving a divergence. In order to resum this expression we need to introduce a small regulator, as we did for the one-loop computation of the free energy:

$$G^{ab}(x, x') = \lim_{\delta \rightarrow 0} \left[\sum_{\ell > 0} \frac{R^2 Y_\ell(x) Y_\ell(x')}{-\ell(\ell + d - 1) + \delta(d - 1 + \delta)} - \frac{R^2 Y_0^2}{\delta(d - 1 + \delta)} \right] \delta^{ab}. \quad (3.14)$$

The first term corresponds to the propagator $G_{\text{reg}}(x, x')$ associated to a scalar field with mass $m^2 = \delta(d - 1 + \delta)/R^2$, whose expression as a function of z is

$$G_{\text{reg}}(z; \delta) = -\frac{\Gamma(d - 1 + \delta)}{4(4\pi)^{\frac{d}{2}-1} R^{d-2} \Gamma(1 + \delta) \sin(\pi\delta) \Gamma(\frac{d}{2})} {}_2F_1\left(-\delta, -1 + d - \delta, \frac{d}{2}, z\right). \quad (3.15)$$

Plugging in eq. (3.14) and taking the limit $\delta \rightarrow 0$, we find a well-defined expression for the ghost propagator:

$$G^{ab}(z) = \delta^{ab}G(z) = \frac{\delta^{ab}}{4(4\pi)^{\frac{d}{2}-1}R^d \sin(\pi d)\Gamma(2-d)\Gamma(\frac{d}{2})} \left(H(d-2) - \frac{2(d-1)z}{d} {}_3F_2\left(1, 1, d; 2, 1 + \frac{d}{2}; z\right) \right), \quad (3.16)$$

where H denotes the harmonic number, which can be written in terms of the digamma function ψ and the Euler constant γ as

$$H(x) = \gamma + \psi(x+1). \quad (3.17)$$

3.4 Fermion propagator on S^d

The fermion propagator on S^d is easily computed from its known expression in flat space by performing a Weyl rescaling, see eq. (3.6). We have

$$S_j^i(x, 0) = \langle \psi^i(x)\bar{\psi}_j(0) \rangle_{\text{sphere}} = \frac{\langle \psi^i(x)\bar{\psi}_j(0) \rangle_{\text{flat}}}{\Omega(x)^{\frac{d}{2}}\Omega(0)^{\frac{d}{2}}} = \delta_j^i \frac{\Gamma\left(\frac{d}{2}\right)(R^2+x^2)^{\frac{d}{2}}\gamma^\mu x_\mu}{2^{(d+1)}\pi^{\frac{d}{2}}(x^2)^{\frac{d}{2}}R^d}, \quad (3.18)$$

where in the last equality we used

$$\langle \psi^i(x)\bar{\psi}_j(0) \rangle_{\text{flat}} = \delta_j^i \frac{\Gamma\left(\frac{d}{2}\right)\gamma^\mu x_\mu}{2\pi^{\frac{d}{2}}(x^2)^{\frac{d}{2}}}, \quad \Omega(x) = \frac{2R^2}{R^2+x^2}. \quad (3.19)$$

3.5 Vertices on the sphere

The Feynman rules for the vertices can be read from the interacting part of the gauge-fixed action. Namely, we have four possible interactions defined as

$$g_0\Gamma^{\text{TR}}(x) = -\frac{1}{g_0^2}f^{abc}\nabla_\nu A_\lambda^a A_\nu^b A_\lambda^c(x), \quad (3.20)$$

$$g_0^2\Gamma^{\text{QU}}(x) = -\frac{1}{4g_0^2}f^{abc}f^{ade}g_0A_\nu^b A_\lambda^c A_\nu^d A_\lambda^e(x), \quad (3.21)$$

$$g_0\Gamma^{\text{GH}}(x) = f^{abc}\nabla_\nu \bar{c}^a A_\nu^b c^c(x), \quad (3.22)$$

$$g_0\Gamma^{\text{FE}}(x) = T_{\alpha\beta}^a \bar{\psi}_i^\alpha \gamma^\mu \psi_i^\beta A_\mu^a(x), \quad (3.23)$$

respectively the triple gluon, the quartic gluon, the ghost-gluon and the fermion-gluon interactions.

4 Next to leading contribution

In the previous section we have obtained the Feynman rules for gauge theories on the sphere. We now have all the ingredients to compute the free energy at the next-to-leading order.

For n_f Dirac fermions in the fundamental representation of the gauge group $G = \text{SU}(n_c)$ we have

$$\begin{aligned}
 F = & (n_c^2 - 1)F_{\text{Max}}(d) - \frac{1}{2}(n_c^2 - 1) \log \left(g_0^2 R^{4-d} \right) + \log \left(\frac{\text{vol}(\text{SU}(n_c))}{(2\pi)^{n_c^2-1}} \right) \\
 & + n_f n_c F_{\text{free-ferm}} + F_{\text{curv}} - \frac{1}{2} g_0^2 G_2 + \dots,
 \end{aligned} \tag{4.1}$$

where G_2 includes all the two-loop vacuum diagrams. Note that we have kept all the couplings bare. In section 4.1 we compute the various diagrams contributing to G_2 in eq. (4.1): the divergent terms in a generic ξ -gauge and the finite pieces in the Landau gauge $\xi = 0$. Renormalization is discussed in section 4.2. As a check of the validity of our results we verify in appendix D that the divergences that we obtain match with those computed with heath-kernel methods in ref. [18] in the Feynman gauge $\xi = 1$.

4.1 Computation of the diagrams

The leading interacting part of the free energy is given by connected vacuum diagrams up to order g_0^2 . The corresponding contribution, which we will call G_2 , is composed by the following two-loop diagrams:

$$G_2 = G_2^{\text{triple}} + G_2^{\text{ghost}} + G_2^{\text{ferm}} + G_2^{\text{quart}} + G_2^{\text{CT-vec}} + G_2^{\text{CT-gh}}. \tag{4.2}$$

The first four terms in (4.2) are genuine two-loop graphs:

$$\begin{aligned}
 G_2^{\text{triple}} &= \langle \text{triple diagram} \rangle = \int d^d x d^d x' \sqrt{h} \sqrt{h'} \langle \Gamma^{\text{triple}}(x) \Gamma^{\text{triple}}(x') \rangle, \\
 G_2^{\text{ghost}} &= \langle \text{ghost diagram} \rangle = \int d^d x d^d x' \sqrt{h} \sqrt{h'} \langle \Gamma^{\text{ghost}}(x) \Gamma^{\text{ghost}}(x') \rangle, \\
 G_2^{\text{ferm}} &= \langle \text{fermion diagram} \rangle = \int d^d x d^d x' \sqrt{h} \sqrt{h'} \langle \Gamma^{\text{fermion}}(x) \Gamma^{\text{fermion}}(x') \rangle, \\
 G_2^{\text{quart}} &= \langle \text{quartic diagram} \rangle = 2 \int d^d x \sqrt{h} \langle \Gamma^{\text{quart}}(x) \rangle.
 \end{aligned} \tag{4.3}$$

The last two ones are instead one-loop graphs with (one-loop) counterterm insertions:

$$\begin{aligned}
 G_2^{\text{CT-vec}} &= \langle \text{vec diagram} \rangle = -2\delta_L \int d^d x \langle \frac{1}{2\xi} (\nabla_\mu A^{\mu a}(x))^2 \rangle - 2\delta_T \int d^d x \langle \frac{1}{4} (\nabla_\mu A_\nu^a(x) - \nabla_\nu A_\mu^a(x))^2 \rangle, \\
 G_2^{\text{CT-gh}} &= \langle \text{ghost diagram} \rangle = -2\delta_c \int d^d x \langle (\bar{c}_a(x) \nabla^2 c^a(x)) \rangle.
 \end{aligned} \tag{4.4}$$

These counterterms are defined from the renormalized Lagrangian

$$\frac{Z_T}{Z_{g^2} g^2} \frac{1}{4} (\nabla_\mu A_\nu^a(x) - \nabla_\nu A_\mu^a(x))^2 + \frac{Z_L}{2g^2 \xi} (\nabla_\mu A^{\mu a}(x))^2 - \frac{Z_T^{3/2}}{Z_{g^2} g^2} f^{abc} \nabla_\mu A_\nu^a A^{\mu b} A^{\nu c} + \dots, \tag{4.5}$$

where $g_0^2 = Z_g^2 g^2$ is the relation between the bare and the renormalized coupling and $Z_\bullet = 1 + \delta_\bullet g^2$. Thanks to the vector equations of motion, we have

$$\int d^d x \langle \frac{1}{4} (\nabla_\mu A_\nu^a(x) - \nabla_\nu A_\mu^a(x))^2 \rangle = - \int d^d x \langle \frac{1}{2\xi} (\nabla_\mu A_a^\mu(x))^2 \rangle + \mathcal{O}(g), \quad (4.6)$$

modulo a $\delta^{(d)}(0)$ factor, which vanishes in dimensional regularization. The counterterms δ_T and δ_L entering the vector propagator can be computed in flat space and they read (see e.g. [32]⁵)

$$\delta_L = 0, \quad \delta_T = C_A \frac{3 + \xi}{32\pi^2 \epsilon} (1 + \mathcal{O}(g_0^2)), \quad (4.7)$$

with $C_A = n_c$ for the $SU(n_c)$ group. The presence of the ghost counterterm is instead a peculiarity of S^d , consequence of the removal of zero modes from the propagator. We refer to appendix B for its computation. The final result is

$$\delta_c = -C_A \frac{3 - \xi}{64\pi^2 \epsilon} (1 + \mathcal{O}(g_0^2)). \quad (4.8)$$

Applying Wick's contraction and the previously listed Feynman rules to eq. (4.3), we get

$$G_2^{\text{triple}} = \kappa \int d^d x d^d x' \sqrt{h} \sqrt{h'} \left(\nabla^\mu \nabla^{\mu'} Q^{\nu\nu'} (Q_{\mu\mu'} Q_{\nu\nu'} - Q_{\mu\nu'} Q_{\nu\mu'}) \right. \\ \left. + \nabla^\nu Q^{\mu\mu'} (\nabla^{\nu'} Q_{\nu\mu'} Q_{\mu\nu'} - Q_{\nu\nu'} \nabla^{\nu'} Q_{\mu\mu'}) + \nabla_\nu Q^{\mu\mu'} (\nabla_{\nu'} Q^{\nu\mu'} Q_{\mu\mu'} - Q^{\nu\mu'} \nabla_{\nu'} Q_{\mu\nu'}) \right), \quad (4.9)$$

$$G_2^{\text{ghost}} = \kappa \int d^d x d^d x' \sqrt{h} \sqrt{h'} (\nabla_\mu G \nabla_{\mu'} G Q^{\mu\mu'}), \quad (4.10)$$

$$G_2^{\text{ferm}} = -n_f T_f (n_c^2 - 1) \int d^d x d^d x' \sqrt{h} \sqrt{h'} (\text{Tr} [\gamma_\mu S \gamma_{\mu'} S] Q^{\mu\mu'}), \quad (4.11)$$

$$G_2^{\text{quart}} = -\frac{\kappa}{2} \int d^d x \sqrt{h} (Q^\mu_\mu Q^\nu_\nu - Q_{\mu\nu} Q^{\mu\nu}), \quad (4.12)$$

$$G_2^{\text{CT-vec}} = \kappa \frac{3 + \xi}{16\pi^2 \epsilon} \int d^d x \sqrt{h} \left(\frac{1}{2\xi} \nabla^\mu \nabla^\nu Q^\mu_\nu \right), \quad (4.13)$$

$$G_2^{\text{CT-gh}} = \kappa \frac{3 - \xi}{32\pi^2 \epsilon} \int d^d x \sqrt{h} (\nabla^2 G). \quad (4.14)$$

where $T_f = 1/2$ for the fundamental representation and we have defined

$$\kappa = C_A (n_c^2 - 1). \quad (4.15)$$

Note that the first term in the triple diagram (4.9) includes a double derivative of the vector propagator, which should be treated with care, because it contains a term proportional to a δ -function at coincident points, which contributes to the integral. A simple way to circumvent this problem consists in integrating by parts the first term of eq. (4.9) getting

$$G_2^{\text{triple}} = \kappa \int d^d x d^d x' \sqrt{h} \sqrt{h'} \left(\nabla^{\mu'} Q^{\nu\nu'} \nabla^\mu (-Q_{\mu\mu'} Q_{\nu\nu'} + Q_{\mu\nu'} Q_{\nu\mu'}) \right. \\ \left. + \nabla^\nu Q^{\mu\mu'} (\nabla^{\nu'} Q_{\nu\mu'} Q_{\mu\nu'} - Q_{\nu\nu'} \nabla^{\nu'} Q_{\mu\mu'}) + \nabla_\nu Q^{\mu\mu'} (\nabla_{\nu'} Q^{\nu\mu'} Q_{\mu\mu'} - Q^{\nu\mu'} \nabla_{\nu'} Q_{\mu\nu'}) \right). \quad (4.16)$$

⁵Comparing our Lagrangian (4.5) with the definitions in section 26.5 of [32], we see that the relation between our counterterms and the counterterms δ_3 and δ_{A3} defined there are: $g^2 \delta_T = \delta_{A3} - \delta_3$, and $g^2 \delta_{g^2} = \delta_{A3} - \frac{3}{2} \delta_3$. Moreover since there is no correction proportional to the longitudinal part of the propagator, $\delta_L = 0$. Note also that $\epsilon_{\text{there}} = -2\epsilon_{\text{here}}$.

We refer to appendix C for more details on how to treat contact terms and integration by parts on S^d in presence of delta function singularities.

For the first three integrals ($t = \text{triple}$, $g = \text{ghost}$, $f = \text{fermion}$) we proceed as follows. As the integrals only depend on the geodesic distance, or equivalently on z , we can use spherical invariance to put x' to zero and reduce the integration over x' to a volume factor:

$$G_2^i = \int d^d x d^d x' \sqrt{h} \sqrt{h'} g^i(z) = \Omega_d R^d \int d^d x \sqrt{h} g^i(z), \quad i = t, g, f. \quad (4.17)$$

Then we use stereographic coordinates to convert the remaining integral in x to a one-dimensional integral in the variable z defined in eq. (3.8):

$$\int d^d x \sqrt{h} = \Omega_{d-1} R^{2d} \int_0^\infty dx \frac{2^d x^{d-1}}{(R^2 + x^2)^d}. \quad (4.18)$$

In this way we write

$$G_2^i = \int_0^1 dz f^i(z), \quad i = t, g, f, \quad (4.19)$$

for some functions $f^i(z)$. The integral (4.19) cannot be computed directly as it contains UV divergences in $d = 4$. We isolate them by expanding $f^i(z)$ around coincident points, i.e. $z = 1$:

$$f^i(z) = \sum_{k=n_i}^{N_i} \left(f_{1k}^i(d)(1-z)^{k-1} + f_{2k}^i(d)(1-z)^{k-d/2+1} + f_{3k}^i(d)(1-z)^{k-d+3} \right) + \tilde{f}^i(z), \quad (4.20)$$

where $f_{jk}^i(d)$ are analytic functions of d and \tilde{f}^i remainder terms. The lower bound n_i in the sum appearing in eq. (4.20) is the leading UV divergence of the integrand, and the upper bound N_i is chosen in such a way that the integral of $\tilde{f}^i(z)$ over z between 0 and 1 is finite. We write

$$G_2^i = (G_2^i)_{N_i} + \tilde{G}_2^i, \quad (4.21)$$

with

$$(G_2^i)_{N_i} = \int_0^1 dz \sum_{k=n_i}^{N_i} \left(f_{1k}^i(d)(1-z)^{k-1} + f_{2k}^i(d)(1-z)^{k-d/2+1} + f_{3k}^i(d)(1-z)^{k-d+3} \right) \quad (4.22)$$

and

$$\tilde{G}_2^i = \int_0^1 dz \tilde{f}^i(z), \quad (4.23)$$

with \tilde{G}_2^i finite. The integral $(G_2^i)_{N_i}$ can be computed analytically using

$$\int_0^1 (1-z)^{a-1} = \frac{1}{a}, \quad (4.24)$$

which is valid for $a > 0$, but is extendable to any d -dependent a by analytic continuation in d .⁶ We then set $d = 4 + 2\epsilon$ and extract the divergent part of eq. (4.22) by expanding

⁶Luckily, $f_{1k}^i(d)$ is zero for $k \leq 0$ in all the integrals that we have computed. Otherwise, analytic continuation of the dimension would not be sufficient to regulate the integral of eq. (4.24).

the result in powers of ϵ and isolating the negative powers of ϵ . Note that the divergence of the integral has a double source: it comes from both integration over z when $k = 0$ and from the expansion of the functions $f_{jk}^i(d)$ around $d = 4$.⁷ This explains the presence of double poles in the final result.

For the quartic and the counterterm diagrams the situation is simpler, as we have an integration over a single variable. Spherical invariance then means that we need to compute the integrand at coincident points and multiply it by a volume factor. We work out the procedure for the quartic case (4.12) as example. We have

$$G_2^{\text{quart}} = 2\kappa R^d \Omega_d \frac{\alpha(z)(d-1)(2\beta(z) - d\alpha(z))}{4z^2} \Big|_{z \rightarrow 1}, \quad (4.25)$$

where α and β are the coefficients of the two components of the vector propagator defined in eq. (3.11). For physical values of d , eq. (4.25) is UV divergent. We expand it around coincident points for generic d , obtaining

$$G_2^{\text{quart}} = \sum_{k=0}^N \left(g_{1k}^{\text{q}}(d)(1-z)^k + g_{2k}^{\text{q}}(d)(1-z)^{-d/2+1+k} + g_{3k}^{\text{q}}(d)(1-z)^{-d+2+k} \right) \Big|_{z \rightarrow 1}, \quad (4.26)$$

where $N \geq 1$ and g_{jk}^{q} are analytic functions of d . For sufficiently small d all terms in the expansion vanish, except g_{1k}^{q} , with $k = 0$. We then get

$$G_2^{\text{quart}} = g_{10}^{\text{q}}(d) = -\frac{\kappa R^{d-4} \Gamma(d-1)}{2^{d+2} \pi^{\frac{d}{2}} (d-3)^2 \Gamma\left(\frac{d}{2} + 1\right)} \left(\gamma(d-3)\xi + \pi((d-3)\xi - d + 1) \cot\left(\frac{\pi d}{2}\right) + (d(\xi-1) - 3\xi + 1)\psi(d) - \gamma d + d + \gamma \right)^2. \quad (4.27)$$

The analytic continuation of eq. (4.27) for any d gives us the final result. A similar computation of the integrals in eqs. (4.13) and (4.14) gives just -1 and $-1/2$, respectively, for any d .

We finally expand eqs. (4.9)–(4.14) around $\epsilon = 0$ with $d = 4 + 2\epsilon$, keeping terms up to constant order, and we get:

$$G_2^{\text{triple}} \Big|_{\text{div.}} = \kappa \left(\frac{(\xi-3)(3\xi-7)}{192\pi^2\epsilon^2} + \frac{\xi(31\xi-64) - 71 - 2(\xi-3)(3\xi-7)(\gamma + \log(4\pi R^2))}{384\pi^2\epsilon} \right), \quad (4.28)$$

$$G_2^{\text{ghost}} \Big|_{\text{div.}} = \kappa \left(\frac{3-\xi}{96\pi^2\epsilon^2} + \frac{-\xi-13 + 2(\xi-3)(\gamma + \log(4\pi R^2))}{192\pi^2\epsilon} \right), \quad (4.29)$$

$$G_2^{\text{ferm}} \Big|_{\text{div.}} = (n_c^2 - 1) \frac{n_f T_f}{6\pi^2\epsilon}, \quad (4.30)$$

$$G_2^{\text{quart}} \Big|_{\text{div.}} = \kappa \left(-\frac{(\xi-3)^2}{64\pi^2\epsilon^2} + \frac{(3-\xi)(3+31\xi) + 6(\xi-3)^2(\gamma + \log(4\pi R^2))}{384\pi^2\epsilon} \right), \quad (4.31)$$

⁷The functions $f_{jk}^i(d)$ remain separately divergent $k > 0$, but for $k > N_i$ these divergences cancel when the $j = 1, 2, 3$ contributions are summed up.

$$G_2^{\text{CT-vec}}|_{\text{div.}} = -\kappa \frac{3 + \xi}{32\pi^2\epsilon}, \quad (4.32)$$

$$G_2^{\text{CT-gh}}|_{\text{div.}} = \kappa \frac{3 - \xi}{32\pi^2\epsilon}. \quad (4.33)$$

Summing all the contributions gives

$$G_2|_{\text{div.}} = -(n_c^2 - 1) \frac{11C_A - 4n_f T_f}{48\pi^2\epsilon}. \quad (4.34)$$

Note that the results in eq. (4.28)–(4.31) have double poles, which cancel in the sum. Moreover, after summation the ξ -dependence of G_2 cancels, as required from gauge invariance of the total free energy.

As explained before, we compute finite terms only in the Landau gauge $\xi \rightarrow 0$. These are computed numerically. However, thanks to the integer-relation finding algorithm PSLQ [33], we can obtain the exact result from the approximated one:

$$G_2^{\text{triple}}|_{\text{fin.}} = \kappa \frac{-562 + 63\pi^2 + 6(\gamma + \log(4\pi R^2))(71 + 21(\gamma + \log(4\pi R^2)))}{2304\pi^2}, \quad (4.35)$$

$$G_2^{\text{ghost}}|_{\text{fin.}} = \kappa \frac{97 + 9\pi^2 + 6(\gamma + \log(4\pi R^2))(13 + 3(\gamma + \log(4\pi R^2)))}{1152\pi^2}, \quad (4.36)$$

$$G_2^{\text{ferm}}|_{\text{fin.}} = (n_c^2 - 1)n_f T_f \frac{5 + 3(\gamma + \log(4\pi R^2))}{36\pi^2}, \quad (4.37)$$

$$G_2^{\text{quart}}|_{\text{fin.}} = \kappa \frac{128 - 9\pi^2 - 6(\gamma + \log(4\pi R^2))(1 + 3(\gamma + \log(4\pi R^2)))}{256\pi^2}, \quad (4.38)$$

and zero for the counterterms, leading to

$$G_2|_{\text{fin.}} = (n_c^2 - 1) \left(C_A \frac{49 + 33(\gamma + \log(4\pi R^2))}{144\pi^2} - n_f T_f \frac{5 + 3(\gamma + \log(4\pi R^2))}{36\pi^2} \right). \quad (4.39)$$

4.2 Renormalization

Let us now check that the free-energy (4.1) is UV finite up to order g^2 , when expressed in terms of renormalized couplings. The bare curvature couplings in eq. (2.5) renormalize as follows [34]:

$$b_0 = \mu^{2\epsilon} \left(b + \frac{62(n_c^2 - 1) + 11n_f n_c}{720(4\pi)^2\epsilon} + \mathcal{O}(g^4) \right), \quad (4.40)$$

$$c_0 = \mu^{2\epsilon} \left(c + \mathcal{O}(g^6) \right), \quad (4.41)$$

while for the gauge coupling we have the well-known relation

$$g_0^2 = \mu^{-2\epsilon} \left(g^2 + \frac{11C_A - 4n_f T_f}{3\epsilon} \frac{g^4}{(4\pi)^2} + \mathcal{O}(g^6) \right), \quad (4.42)$$

where μ is the RG sliding scale. Expanding in ϵ for $d = 4 + 2\epsilon$, we get the following divergent contribution from eq. (4.1) at $\mathcal{O}(g^0)$:

$$\begin{aligned} F_{\text{free-YM}}|_{\text{div.}} &= -\frac{31(n_c^2 - 1)}{90\epsilon}, \\ n_f n_c F_{\text{free-ferm}}|_{\text{div.}} &= -\frac{11n_f n_c}{180\epsilon}, \\ F_{\text{curv}}|_{\text{div.}} &= \frac{31(n_c^2 - 1)}{90\epsilon} + \frac{11n_f n_c}{180\epsilon}, \end{aligned} \quad (4.43)$$

which cancel in the sum. At $\mathcal{O}(g^2)$ we have

$$\begin{aligned}
 -\frac{1}{2}(n_c^2 - 1) \log(g_0^2)|_{\text{div.}} &= -g^2(n_c^2 - 1) \frac{11C_A - 4n_f T_f}{96\pi^2 \epsilon} + \mathcal{O}(g^4), \\
 -\frac{1}{2}g_0^2 G_2|_{\text{div.}} &= g^2(n_c^2 - 1) \frac{11C_A - 4n_f T_f}{96\pi^2 \epsilon} + \mathcal{O}(g^4),
 \end{aligned} \tag{4.44}$$

which also cancel in the sum. Therefore we obtained, as expected, a finite result for the total free-energy at order $\mathcal{O}(g^2)$, and in any ξ -gauge.

4.3 Free energy at the fixed point

We determine here the final form of the free-energy at the fixed point obtained in the ϵ expansion up to $\mathcal{O}(\epsilon)$. The fixed point is obtained by setting to zero the gauge and the curvature beta-functions β_g , β_b and β_c . β_b and β_c , computed in [34]. At the fixed point g^*, b^*, c^* we have

$$F_{\text{conf}}(\epsilon) = F(g^*, b^*, c^*, \mu R), \tag{4.45}$$

of order ϵ up to two loops. Note that F_{conf} has to be conformal invariant and therefore the dependence on R has to cancel in the final result. The expressions for β_g , β_b and β_c – up to the order required to get $F_{\text{conf}}(\epsilon)$ to order ϵ – are

$$\begin{aligned}
 \beta_g &= \epsilon g - \left(\frac{11}{3}C_A - \frac{4}{3}T_f n_f \right) \frac{g^3}{(4\pi)^2} - \left(\frac{34}{3}C_A^2 - \frac{20}{3}C_A T_f n_f - 4C_f T_f n_f \right) \frac{g^5}{(4\pi)^4} + \mathcal{O}(g^7), \\
 \beta_b &= -2\epsilon b - \frac{62(n_c^2 - 1) + 11n_f n_c}{360(4\pi)^2} - \frac{(n_c^2 - 1)}{8} \left(\frac{34}{3}C_A^2 - \frac{20}{3}C_A T_f n_f - 4C_f T_f n_f \right) \frac{g^4}{(4\pi)^6} \\
 &\quad + \mathcal{O}(g^6), \\
 \beta_c &= -2\epsilon c + \mathcal{O}(g^6),
 \end{aligned} \tag{4.46}$$

from which we get

$$\begin{aligned}
 g_* &= 4\pi \sqrt{\frac{3\epsilon}{11C_A - 4n_f T_f}} \left(1 - \frac{3(17C_A^2 - 10C_A n_f T_f - 6C_f n_f T_f)}{(11C_A - 4n_f T_f)^2} \epsilon + \mathcal{O}(\epsilon^2) \right), \\
 b_* &= - \left(\frac{62(n_c^2 - 1) + 11n_f n_c}{720(4\pi)^2 \epsilon} + \frac{(n_c^2 - 1)(17C_A^2 - 10C_A n_f T_f - 6C_f n_f T_f)}{24\epsilon} \frac{g_*^4}{(4\pi)^6} \right) + \mathcal{O}(\epsilon^2), \\
 c_* &= \mathcal{O}(\epsilon^2),
 \end{aligned} \tag{4.47}$$

where

$$C_f = \frac{n_c^2 - 1}{2n_c}. \tag{4.48}$$

Note that, since β_b contains a constant term, b^* starts at order $1/\epsilon$.

Plugging eq. (4.47) in the free energy (4.1) and using the results for G_2 obtained in section 4.1, including the finite pieces computed in the $\xi = 0$ gauge, we obtain

$$\begin{aligned}
 F_{\text{conf}} &= (n_c^2 - 1) \left(F_{\text{Max}}(d) - \frac{1}{2} \log \left(\frac{48\pi^2 \epsilon}{11C_A - 4n_f T_f} \right) \right) + n_f n_c F_{\text{free-ferm}}(d) + \log \left(\frac{\text{vol}(\text{SU}(n_c))}{(2\pi)^{n_c^2 - 1}} \right) \\
 &\quad + (n_c^2 - 1) \left(\frac{-n_f T_f (1089C_f - 913C_A + 584n_f T_f)}{121(11C_A - 4n_f T_f)^2} - \frac{386 + 363(\gamma + \log(4\pi))}{726} \right) \epsilon + \mathcal{O}(\epsilon^2),
 \end{aligned} \tag{4.49}$$

where F_{Max} and $F_{\text{free-ferm}}$ are given in eqs. (2.42) and (2.11), respectively. The volume of the $SU(n)$ group reads (see e.g. [35])

$$\text{vol}(SU(n)) = \frac{(2\pi)^{\frac{n(n+1)-2}{2}}}{\prod_{k=1}^{n-1} k!}. \tag{4.50}$$

The cancellation of the $\log(\mu R)$ term⁸ present in the two loop correction (4.39) with those arising from the replacement of the bare coupling b_0 in eq. (2.10) and g_0 in the log term in eq. (4.1) is a check of the result. Equation (4.49) is the main result of this work.

As discussed in the introduction, the conjectured generalized F -theorem (1.4) involves the modified free energy (1.3). Using the expression for F_{conf} we get

$$\begin{aligned} \tilde{F}_{\text{conf}} &= (n_c^2 - 1) \left(\tilde{F}_{\text{Max}}(d) + \frac{1}{2} \sin\left(\frac{\pi d}{2}\right) \log\left(\frac{48\pi^2 \epsilon}{11C_A - 4n_f T_f}\right) \right) + n_f n_c \tilde{F}_{\text{free-ferm}}(d) \\ &\quad - \frac{1}{2} \sin\left(\frac{\pi d}{2}\right) \log\left(\frac{\text{vol}(SU(n_c))}{(2\pi)^{n_c^2-1}}\right) \\ &\quad + (n_c^2 - 1) \left(\frac{n_f T_f (1089C_f - 913C_A + 584n_f T_f)}{121(11C_A - 4n_f T_f)^2} + \frac{386 + 363(\gamma + \log(4\pi))}{726} \right) \pi \epsilon^2 \\ &\quad + \mathcal{O}(\epsilon^3), \end{aligned} \tag{4.51}$$

where

$$\tilde{F}_{\text{Max}} = -\sin\left(\frac{\pi d}{2}\right) F_{\text{Max}}, \quad \tilde{F}_{\text{free-ferm}} = -\sin\left(\frac{\pi d}{2}\right) F_{\text{free-ferm}}. \tag{4.52}$$

For completeness we report its expression in the Veneziano limit, where $n_c, n_f \rightarrow \infty$ with $x = n_f/n_c$ fixed. We get

$$\begin{aligned} F_{\text{conf}} &= n_c^2 \left(F_{\text{Max}}(d) - \frac{1}{2} \log\left(\frac{48\pi^2 \epsilon}{11 - 2x}\right) + x F_{\text{free-ferm}}(d) + \frac{3}{4} - \frac{1}{2} \log(2\pi) \right. \\ &\quad \left. - \left(\frac{193}{363} - \frac{737x - 584x^2}{484(11 - 2x)^2} + \frac{1}{2}(\gamma + \log(4\pi)) \right) \epsilon \right) + \mathcal{O}(n_c). \end{aligned} \tag{4.53}$$

Note that $n_c^2 \log(n_c)$ terms are induced from both log terms appearing in eq. (4.49) and they precisely cancel. The same cancellation occurs in the t' Hooft limit. This cancellation is expected from large n_c considerations and the fact that a log term is not expected in the genus expansion.

5 Applications

In this section we are going to use the conjectured monotonicity of \tilde{F} along RG flows [15] to test some proposed RG flows in $d = 3$ and $d = 5$, using our result (4.49). The perturbative expression in eq. (4.49) is not adequate to extrapolate to physical dimensions with $|\epsilon| = 1/2$.

⁸All the $\log R$ terms appearing in the loop computations of section 4.1 arise from the expansion of an overall R^{d-4} factor present in all the contributions. When moving from g_0 to g via eq. (4.42) we effectively have $R \rightarrow \mu R$.

The number of available terms (three) is too limited to attempt a Borel resummation. In the same spirit of ref. [16], we will instead look for Padé approximants for \tilde{F} . We also use the knowledge of \tilde{F} for special values of d to effectively increase by one order the expansion in ϵ .

Note that \tilde{F} contains a $\log(\epsilon)$ term, which, being non-analytic, prevents the application of standard Padé approximants. Moreover, the free-fermion one-loop determinant is known exactly as a function of d and it is convenient to keep it not expanded in ϵ . For these reasons, we split the total \tilde{F} in two parts, one that we keep in d dimensions and contains the non-analytic term, and one that is a series in ϵ . Following ref. [16], we split \tilde{F}_{conf} as

$$\tilde{F}_{\text{conf}} = n_f n_c \tilde{F}_{\text{free-ferm}} + \frac{1}{2} \sin\left(\frac{\pi d}{2}\right) (n_c^2 - 1) \log\left(\frac{2\epsilon}{11C_A - 4n_f T_f}\right) + \delta\tilde{F}(\epsilon), \quad (5.1)$$

and we use Padé approximants only on the $\delta\tilde{F}(\epsilon)$ term. The latter includes the free photon contribution, which is evaluated numerically, and reads

$$\begin{aligned} \delta\tilde{F}(\epsilon) &= (n_c^2 - 1) \frac{31\pi}{90} + \left((n_c^2 - 1) 4.696 - \pi \log\left(\frac{\text{vol}(\text{SU}(n_c))}{(2\pi)^{n_c^2 - 1}}\right) \right) \epsilon \\ &+ (n_c^2 - 1) \left(\frac{n_f \pi (584n_f n_c - 1089 - 737n_c^2)}{484n_c(11n_c - 2n_f)^2} + \frac{386\pi + 363\pi(\gamma + \log(4\pi))}{726} - 10.098 \right) \epsilon^2 \\ &+ \mathcal{O}(\epsilon^3). \end{aligned} \quad (5.2)$$

For presentation purposes we rounded to the first 4 digits the $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ contribution coming from the photon free energy, but the result is available to higher precision. Let us stress the fact that the above splitting is arbitrary and that the corresponding choice significantly affects the final results. This is a signal of the poor knowledge that we have on the series. For the same reason we have not attempted to estimate an error bar in our results.

The fixed points we get in $d = 4 + 2\epsilon$ of QCD_d with gauge group $\text{SU}(n_c)$ and n_f massless Dirac fermions in the fundamental representation are expected to match two known CFTs:

- For $\epsilon = -1$ ($d = 2$) the IR fixed point of QCD_d with gauge group $\text{SU}(n_c)$ and $2n_f$ massless Dirac fermions in the fundamental representation is an $\text{SU}(2n_f)_{n_c}$ Wess-Zumino-Witten model with an additional decoupled free boson [36, 37]. This CFT has central charge

$$c = \frac{n_c(4n_f^2 - 1)}{2n_f + n_c} + 1, \quad (5.3)$$

and

$$\tilde{F}_{\text{WZW}}(d = 2) = \frac{\pi}{6} c. \quad (5.4)$$

Plugging $d = 2$ in eq. (5.1) and identifying \tilde{F}_{conf} with \tilde{F}_{WZW} gives

$$\delta\tilde{F}(\epsilon = -1) = \tilde{F}_{\text{WZW}} - n_c n_f \tilde{F}_{\text{free-ferm}} = -\frac{\pi}{3} \frac{n_f(n_c^2 - 1)}{2n_f + n_c}. \quad (5.5)$$

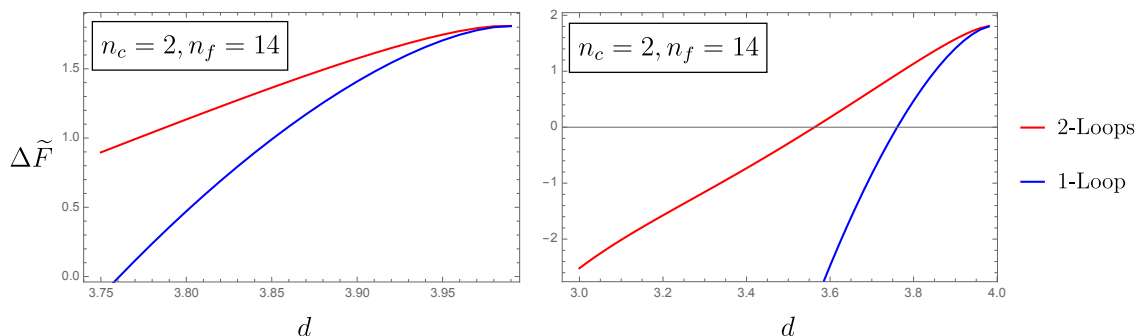


Figure 1. Left panel: comparison between $\Delta\tilde{F}$ as a function of the dimension d for small ϵ computed by using the result for \tilde{F} in eq. (4.51) (red) or only its free part given by the first two rows of eq. (4.51) (blue). Right panel: same comparison extended up to $d = 3$.

- For $\epsilon = 1$ ($d = 6$) the theory is conjectured to have a non-unitary UV fixed point described by a Lagrangian with a higher-derivative kinetic term $F_{\mu\nu}^a \nabla^2 F_a^{\mu\nu}$ [38, 39], whose anomaly coefficient is $a = -(n_c^2 - 1) \frac{55}{84}$ [16]. This leads to

$$\delta\tilde{F}_{d=6} = \frac{\pi}{2} a = -\frac{55\pi}{168} (n_c^2 - 1). \tag{5.6}$$

To improve the numerical estimate of our result we constrain the Padé approximants of $\delta\tilde{F}$ to these known points. In order to avoid misleading results, we exclude approximants with poles in the range between the constraint and $d = 4$.

5.1 F-Theorem in $d = 3$

Non-abelian $3d$ gauge theories have received particular attention in the last years due to their possible emergence in quantum phase transitions with deconfined criticality [40] and as theories governing domain walls among different vacua in non-abelian $4d$ gauge theories [41]. Theoretically, they are of course also interesting theories by themselves.

It is known since the early work [19] that at large n_f QCD₃ flows in the IR to a CFT. For $n_f \leq n_f^*$, with n_f^* an unknown parameter, a phase with spontaneous symmetry breaking (SB) of the $U(2n_f)$ global symmetry is expected. The only pattern of spontaneous breaking of the global symmetry $U(2n_f)$ compatible with the results of [20, 21] is

$$U(2n_f) \rightarrow U(n_f) \times U(n_f). \tag{5.7}$$

More recently, a qualitative phase diagram of the theory as a function of the number of flavors n_f , a fermion mass term, and the level k of a possible Chern-Simons term has been suggested [22]. We will focus on $k = 0$ in the following and use the F -theorem to put an upper bound on n_f^* . A naive way to check if the spontaneous symmetry breaking phase (5.7) can be realized would be to compare $F_{\text{IR}} = F_{\text{SB}}$ as given by $2n_f^2$ Goldstone bosons (free in the deep IR), with F_{UV} given in the deep UV by $n_c^2 - 1$ free photons and $n_f n_c$ free fermions. Unfortunately, due to the log term in (2.41), F_{UV} diverges and no useful information can be extracted. We overcome this problem by assuming that conformality

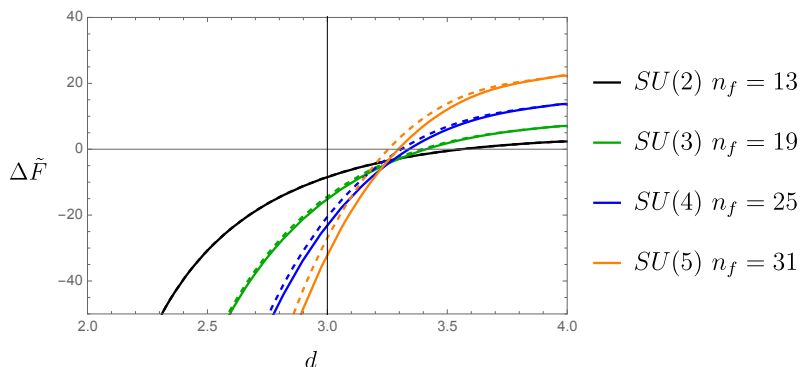


Figure 2. Values of $\Delta\tilde{F}$ for $SU(n_c)$ as a function of the dimension d computed with Padé-approximants $[2/1]$ (continuous line) and $[1/2]$ (dashed line) at $n_c = 2, 3, 4, 5$. The value of n_f is set to the smallest integer without poles in both approximants in $2 < d < 4$ satisfying $g^{*2} > 0$.

is lost at $n_f = n_f^*$ by annihilation between the critical QCD₃ fixed point with another one, known as QCD₃^{*} [42]. A similar analysis for QED₃ has been performed in [16]. Treating n_f as a continuous parameter, for $n_f = n_f^* + \eta$ and $0 < \eta \ll 1$, the theory flows to the IR fixed point QCD₃. On the other hand, for $n_f = n_f^* - \eta$ the theory is expected to undergo a weak first-order phase transition [43] (i.e. a walking regime, see [44] for an explicit realization in 4d gauge theories) with a slow RG passing close to the (now complex) fixed points, reaching eventually the spontaneously broken phase (5.7). By continuity and the generalized F -theorem, we then expect that

$$\Delta\tilde{F}(n_f^*) = \tilde{F}_{\text{conf}}(n_f^*) - \tilde{F}_{\text{SB}}(n_f^*) > 0. \tag{5.8}$$

Note that values of n_f such that $\Delta\tilde{F}(n_f) < 0$ are incompatible with a symmetry breaking phase. On the other hand, values of n_f with $\Delta\tilde{F}(n_f) > 0$ are compatible with either a CFT or a symmetry breaking phase. For this reason we can only determine an upper bound $n_f^* \leq n_f^0$, where $\Delta\tilde{F}(n_f^0) = 0$.

An early previous estimate of n_f^* was based on Schwinger-Dyson gap equations [19] and resulted in $n_f^* \approx 128(n_c^2 - 1)/(3\pi^2 n_c)$. More recently, a lattice analysis [24] found $n_f^* \leq 4$ for $n_c = 2$. An estimate based on the F -theorem already appeared in [23], where as UV theory it was used a SUSY version of QCD₃, a genuine CFT with finite F which can flow to QCD₃ by appropriate deformations. By comparing F_{SUSY} computed by means of supersymmetric localization with F_{SB} (and assuming that we can flow from the IR SCQD₃ fixed point to the IR QCD₃ fixed point), it was found $n_f^* < 13/2$ for $n_c = 2$.

The value of $\tilde{F}_{\text{SB}}(n_f)$ is easily computed by noting that the $2n_f^2$ Goldstone bosons associated to the breaking pattern (5.7) become free in the deep IR. The contribution to the free energy for a single real scalar reads [15]

$$F_{\text{free-sc}} = -\frac{1}{\sin(\frac{\pi d}{2})\Gamma(1+d)} \int_0^1 du u \sin(\pi u) \Gamma\left(\frac{d}{2} + u\right) \Gamma\left(\frac{d}{2} - u\right), \tag{5.9}$$

$$\tilde{F}_{\text{free-sc}} = -\sin\left(\frac{\pi d}{2}\right) F_{\text{free-sc}}.$$

| n_f | 12 | 13 | 14 | 15 | 16 |
|-------|-------|-------|-------|-------|-------|
| SB | 18.38 | 21.57 | 25.01 | 28.71 | 32.67 |
| [2/1] | 12.1 | 13.1 | 13.6 | 13.9 | 14.16 |
| [1/2] | – | 13.2 | 13.9 | 15.01 | 16.10 |

Table 1. Comparison between the $3d$ values of \tilde{F} in the broken phase \tilde{F}_{SB} (red) with those obtained from Padé-approximants [2/1] and [1/2] of \tilde{F}_{conf} for QCD₃ with $n_c = 2$. In all cases $\Delta\tilde{F} < 0$.

We then have

$$\tilde{F}_{\text{SB}}(n_f) = 2n_f^2 \tilde{F}_{\text{free-sc}}. \tag{5.10}$$

For $d = 3$ it reads

$$\tilde{F}_{\text{SB}} = 2n_f^2 \left(\frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2} \right). \tag{5.11}$$

Before presenting the results of our extrapolations to $d = 3$, it is useful to see the effect of the 2-loop correction to the free energy with respect to the one-loop free theory contribution in the controlled regime with $|\epsilon| \ll 1$. This is shown in figure 1 where we plot $\Delta\tilde{F}$ (for $n_c = 2$ and $n_f = 14$) defined as in eq. (5.8) as a function of the dimension d . We compare the result for \tilde{F}_{conf} obtained using eq. (4.51) (red line) with the one obtained using only the first two rows of the same equation (blue line), i.e. only its free part. We note that the effect of the interactions is to favor the SB phase with respect to the conformal one and that the latter is more favored as we lower the space-time dimensions. More importantly, we see from the left panel in the figure that when $|\epsilon| \approx 0.1$ the one and two-loop results differ significantly and that there is no hope to get reliable results from perturbation theory in $d = 3$ (for illustration purposes we report in the right panel of figure 1 the same plot extended up to $d = 3$). As anticipated at the beginning of the section, we then consider Padé approximants of (5.2). For $d < 4$ we augment the approximant by one more term by imposing the constraint (5.5).

In figure 2 we show the value of $\Delta\tilde{F}$ as a function of the dimension d for $n_c = 2, 3, 4, 5$ and n_f equal to the smallest integer without poles in approximants [1/2] and [2/1] satisfying $g^{*2} > 0$, i.e. $n_f = 13, 19, 25, 31$ respectively.⁹ We see that at $d = 3$ $\Delta\tilde{F} < 0$ in all these cases, indicating the presence of the conformal phase. As expected, this behavior persists for higher values of n_f : we report in table 1 the comparison between the free energy \tilde{F}_{conf} and that of the broken phase for $n_c = 2$, $12 \leq n_f \leq 16$. Not only the value of \tilde{F}_{SB} remains above \tilde{F}_{conf} , but also the gap between the two values gets larger and larger.

5.1.1 Small n_f

The one-loop beta-function of the gauge coupling vanishes at $n_f = 11n_c/2$ and changes sign below that, making $g_{\text{one-loop}}^{*2} < 0$. Of course, a unitary fixed point in $d = 3$ does

⁹Note that regions in n_f close to $11n_c/2$ are more subject to instabilities as g^{*2} blows up there, producing a pole of order two in the free energy. This is another reason to avoid smaller values of n_f which still satisfy $g^{*2} > 0$ (i.e. $n_f = 12$ for $n_c = 2$).

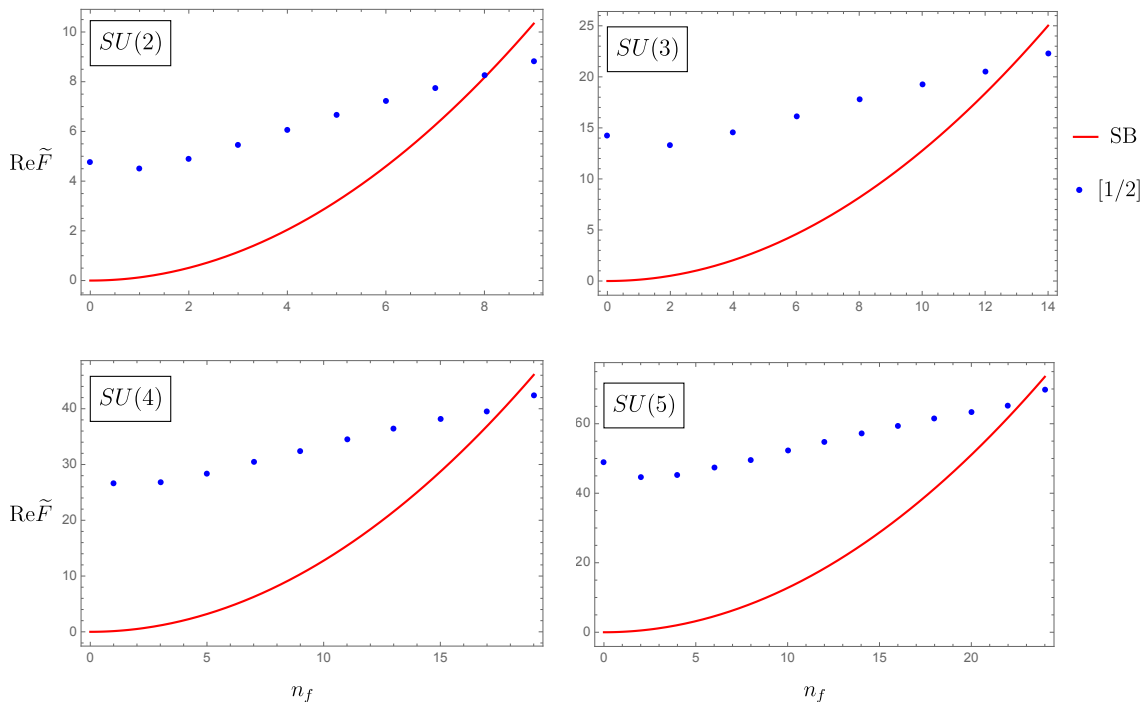


Figure 3. Comparison between the 3d value of \tilde{F}_{SB} (red line) and of the real part of \tilde{F}_{conf} (blue points) as a function of n_f for $n_c = 2, 3, 4, 5$. The $[1/2]$ approximants provide $\text{Re}[\tilde{F}_{\text{conf}}] > \tilde{F}_{\text{SB}}$ for $n_f \leq 8, 12, 17, 22$ suggesting that a chiral symmetry breaking may occur in these ranges of values.

not necessarily appear as a real *one-loop* fixed point when $\epsilon \ll 1$.¹⁰ As mentioned, lattice results for SU(2) find that $n_f^* \leq 4$, suggesting that even if $g_{\text{one-loop}}^{*2} < 0$, there exists a range in n_f where the 3d theory is conformal in the IR. For $n_f < 11n_c/2$ we could still use the free energy to extract information on the RG flow. For $g^{*2} < 0$, the free energy becomes complex, due to the log term in eq. (5.1), with an opposite phase depending on which of the two imaginary fixed points is chosen:

$$\log(g^{*2}) = \log(|g^{*2}|) \pm i \log(\pi). \quad (5.12)$$

We propose to estimate the value of F at the strongly coupled real fixed point by an extrapolation of the half-sum of the two complex values obtained with the ϵ -expansion, i.e. of their real part. The stability of the conformal phase then requires this value to be smaller than \tilde{F}_{SB} . As a result, our more speculative criterion in the range $n_f < 11n_c/2$ is

$$\text{Re} \Delta \tilde{F}(n_f^*) = \tilde{F}_{\text{conf}}(n_f^*) - \text{Re} \tilde{F}_{\text{SB}}(n_f^*) > 0. \quad (5.13)$$

We report in figure 3 the real part of \tilde{F}_{conf} compared to \tilde{F}_{SB} for $n_c = 2, 3, 4, 5$ computed with the Padé approximant $[1/2]$. We see that in all cases there is a wide range of n_f for

¹⁰A notable example of this sort is provided by the abelian Higgs model of n complex scalar fields. It is known that in this theory a real one-loop Wilson-Fisher fixed-point appears for $n > 183$ [45] and this number greatly varies with the order, see e.g. [46]. It is in fact likely that the 3d abelian Higgs theory has an IR conformal phase for values of n well below 183.

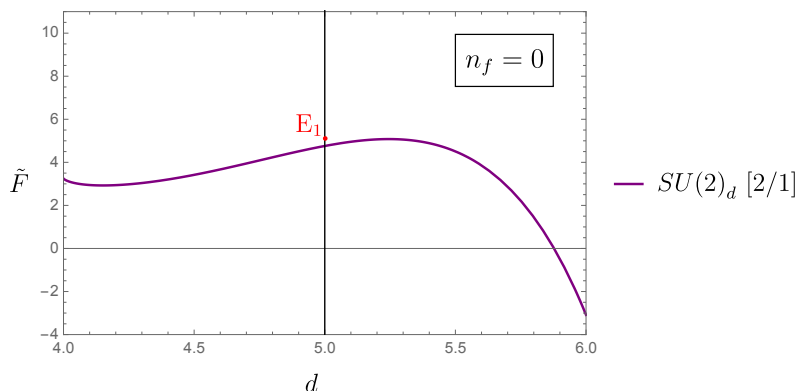


Figure 4. Values of \tilde{F}_{conf} for pure SU(2) YM as a function of the dimension d computed with the Padé-approximant [2/1] (purple line), compared to the value of the $5d$ supersymmetric fixed point E_1 , the UV completion of SU(2) SYM gauge theory (red point).

which the conformal phase appears to be unstable. We have

$$\begin{aligned}
 n_f^* &\lesssim 8, & \text{SU}(2), \\
 n_f^* &\lesssim 12, & \text{SU}(3), \\
 n_f^* &\lesssim 17, & \text{SU}(4), \\
 n_f^* &\lesssim 22, & \text{SU}(5).
 \end{aligned}
 \tag{5.14}$$

The upper bound for SU(2) is consistent with the bound $n_f^* < 13/2$ of [23], and $n_f^* \leq 4$ of [24]. A similar analysis can be done in the Veneziano limit, by taking the large n_c, n_f limit of eq. (5.13). The resulting bound is

$$x^* \lesssim 4.5.
 \tag{5.15}$$

5.2 F-Theorem in $d = 5$

In this section we extrapolate \tilde{F} to $5d$ to test a proposed construction of an interacting CFT that provides a UV completion of $5d$ SU(2) YM theory. Ref. [25] proposed to construct this CFT as the IR fixed point of a supersymmetry-breaking deformation of the interacting superconformal field theory known as E_1 theory [47]. The latter is known to provide the UV completion of SU(2) supersymmetric YM theory (SYM). Ref. [25] studied the various phases in the two-dimensional space of relevant deformations of the E_1 theory, which includes both the supersymmetric deformation to SYM and the non-supersymmetric one, and suggested the existence of a second-order transition between two phases that are described by SU(2) YM theory and a different symmetry-protected topological order. The CFT capturing this phase transition would therefore be a UV completion of YM, and provide an example of a non-supersymmetric interacting CFT in $d > 4$. This scenario was further explored in [26], that showed that actually the phase transition should be viewed as separating the YM phase from a phase with spontaneous breaking of the instantonic

| n_f | 0 | 1 | 2 | 3 | 4 |
|-------------|-------|-------|-------|-------|--------|
| E_{n_f+1} | 5.097 | 6.140 | 7.395 | 8.959 | 11.007 |
| [2,1] | 4.8 | 5.1 | 5.4 | 5.7 | 6.2 |

Table 2. Comparison between the value of $\tilde{F}_{E_{n_f+1}}$ (red) and the [2, 1] Padé approximant of \tilde{F}_{conf} in $d = 5$ (black) as a function of n_f for $0 \leq n_f \leq 4$.

U(1), and in [48] where a certain generalization of the theory admitting a large N limit was argued to have a second order transition in that limit.

A possible test for the proposal of refs. [25, 26] relies on the F -theorem: the sphere free energy \tilde{F}_{E_1} of the SCFT and that of the non-supersymmetric CFT \tilde{F}_{CFT} should satisfy $\tilde{F}_{E_1} > \tilde{F}_{\text{CFT}}$. The quantity \tilde{F}_{E_1} has been computed using localization in [27]. It is natural to conjecture that the non-supersymmetric fixed point is the continuation to $d = 5$ of the UV fixed point visible in the ϵ expansion in $d = 4 + 2\epsilon$, and therefore to estimate \tilde{F}_{CFT} by an extrapolation of our result (4.49). An evidence for the persistence of the $d = 4 + 2\epsilon$ fixed point up to $d = 5$ was obtained in ref. [28] using the five-loop $\overline{\text{MS}}$ β -function and Padé-Borel resummation techniques, both for the pure SU(2) YM theory and for the theory with n_f fundamental Dirac fermions, with $n_f \leq 4$. Note that the continuation from $d = 4 + 2\epsilon$ suggests that the critical point should separate a free YM phase from a confined phase (the only phase realized in $d = 4$) rather than a second YM phase, similarly to the refined proposal of [26] and in agreement with a recent lattice study that sees hints of a second order confinement/deconfinement transition [49]. We therefore proceed to extrapolate \tilde{F}_{conf} using the only available Padé approximant that is constrained also by the $d = 6$ boundary condition (5.6) and without poles in the interval $4 \leq d \leq 6$. In figure 4 we plot the resulting extrapolation of \tilde{F}_{conf} as a function of the dimension. The value ranges between a local minimum of ~ 2.9 and a maximum of ~ 5.0 , before turning negative in the vicinity of $d = 6$. The value in $d = 5$ is ~ 4.8 , remarkably close to the known value ~ 5.1 of \tilde{F} in the E_1 theory, and below it consistently with the proposals of [25, 26].

The UV completion of the supersymmetric theory is also known in the case with $0 < n_f \leq 7$ flavors and is given by the E_{n_f+1} SCFT [47]. The value of \tilde{F} can be obtained from localization similarly to the E_1 case [27]. It is possible that also these theories flow to a non-supersymmetric fixed point when perturbed by a susy-breaking deformation. This fixed point would then provide a UV completion of the non-supersymmetric SU(2) gauge theory with n_f flavors. We test this possibility by comparing our extrapolation of \tilde{F}_{conf} to $\tilde{F}_{E_{n_f+1}}$. We limit ourselves to the range $n_f \leq 4$ in which the fixed point in $d = 4 + 2\epsilon$ was seen to persist up to $d = 5$ in [28]. We collect the values of the two \tilde{F} 's in table 2. We always find $\tilde{F}_{E_{n_f+1}} > \tilde{F}_{\text{conf}}$, consistently with the existence of the RG flow.

6 Conclusion

In this paper we obtained the NLO result for the free energy on S^d in non-abelian gauge theories in Euclidean d dimensions evaluated at their perturbative fixed point. We extrapolated the result to compute the quantity F for the corresponding CFTs in $d = 3$ or

$d = 5$ and used our best estimates together with the monotonicity property of F to test the existence and/or proposed constructions of these CFTs.

While successful in many contexts, the ϵ expansion is not a rigorous method. Going forward, it would be interesting to assess its reliability in the context of gauge theories. A possible verification could come from comparison with lattice and/or conformal bootstrap results. To that end it would be useful to compute scaling dimensions of operators in addition to those obtained in [28], or to improve the precision of the predictions by computing at higher loop order.

Another possibility is to apply the ϵ expansion to cases in which the existence of a fixed point, and the associated data, are known from other methods such as supersymmetry or holography. For instance, one could apply it to the $4d$ theory with the same matter content as $5d$ $\mathcal{N} = 1$ $SU(2)$ SYM with n_f fundamental flavors, and check if ϵ expansion finds a UV fixed point that extrapolates to the E_{n_f} SCFT in $5d$. Note that when continuing the fields to $4d$ one does not land on a supersymmetric theory: the $5d$ vector multiplet contains a real scalar, a $5d$ vector, and a symplectic Majorana fermion, all in the adjoint representation, and their continuation to $4d$ gives rise to a real scalar, a $4d$ vector, and a Dirac fermion, which is not the content of a supersymmetric theory in $4d$.¹¹ As a result, supersymmetry is expected to emerge only in the limit $d \rightarrow 5$. To check the existence of fixed points in $d = 4 + 2\epsilon$ one then needs the coupled system of β functions for the gauge coupling in the presence of both fermionic and bosonic adjoint matter, and of the Yukawa coupling, see e.g. the Lagrangian (15) in [51]. Note that these β functions are known at lower loop order compared to the case with only fermionic matter that was used in [28], see [52, 53]. We leave this as direction for future studies.

The perturbative expansion of the free energy is insensitive to the global structure of the gauge group, except the log term in eq. (4.49) where the volume of the gauge group appears. It would be interesting to compare our results for F with those computed using localization (or some other method) in SCFTs based on gauge theories such as $PSU(n_c) = SU(n_c)/\mathbf{Z}_{n_c}$. This analysis might be useful to shed some light on the nature of the transition delimited by our fixed points for $d > 4$, since a confinement/deconfinement transition has the one-form symmetry $\mathbf{Z}_{n_c}^{(1)}$ as order parameter, and the latter is gauged in $PSU(n_c)$ theories.

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¹¹At least, this is the case with the method we are currently using to continue vector fields. One could imagine a different continuation, more in the spirit of DRED [50], in which the number of components of the vector is kept fixed. In this putative approach, $3d$ gauge theories with matter would not be obtainable with ϵ expansion, because the vector in $4d$ would give rise to additional scalars coupled to matter fields (and gauge fields as well, in the non-abelian case).

A Computation of the vector propagator

In this appendix we follow ref. [31] for the computation of the vector propagator. We report the main steps, generalizing the computation to an arbitrary choice of the gauge.

We have seen in the main text that the vector propagator $Q_{\nu\lambda}^{ab}(x, x') = \delta^{ab} g_0^2 Q_{\nu\lambda}(x, x')$ satisfies eq. (3.10) and can be written as in eq. (3.11) where α and β are generic functions of the geodesic distance. Using the relations in eq. (3.3) we can decompose eq. (3.11) in two parts, respectively proportional to $g_{\nu\lambda'}$ and $n_\nu n_{\lambda'}$:

$$\begin{aligned} & \alpha'' + (d-1)A\alpha' + \left((A+C)^2 + (d-1) \right) \alpha + 2AC\beta \\ & - \left(1 - \frac{1}{\xi} \right) C (\beta' - \alpha' - (d-1)(A+C)\alpha + (d-1)A\beta) = -\delta(x, x'), \\ \beta'' + (d-1)A\beta' + \left((A+C)^2 - d(A^2 + C^2) - (d-1) \right) \beta + (d-2)(A+C)^2\alpha \\ & + \left(1 - \frac{1}{\xi} \right) \left(\beta'' - \alpha'' + ((d-1)A+C)\beta' - (d(A+C) - A)\alpha' + ((d-1)AC \right. \\ & \left. + (d-1)A')\beta + A'(- (d-1)C(A+C) + (1-d)A' + (1-d)C')\alpha \right) = 0. \end{aligned} \quad (\text{A.1})$$

This is a system of two coupled second order differential equations, which is in general hard to solve. To make the computation easier it is convenient to introduce a new maximally symmetric gauge invariant bitensor defined as

$$\langle F_{\mu\nu}^a F_b^{\mu'\nu'} \rangle = 4\delta_b^a \nabla_{[\mu} \nabla^{[\mu'} Q_{\nu]}^{\nu']} = \delta_b^a \left(\sigma(\mu) h_{[\mu}^{[\mu'} h_{\nu]}^{\nu']} + \tau(\mu) n_{[\mu} h_{\nu]}^{[\nu'} n^{\mu']}] \right), \quad (\text{A.2})$$

with square brackets meaning antisymmetrized indices and τ and σ being generic functions of the geodesic distance. From the definition of $Q_{\nu\lambda'}$ in terms of α and β and eq. (3.3), we get

$$\sigma = 4C [\alpha' + (A+C)\alpha - C\beta], \quad (\text{A.3})$$

$$\tau = C^{-1} [\sigma' + 2(A+C)\sigma]. \quad (\text{A.4})$$

Now, taking the covariant derivative of eq. (A.2) and using eq. (3.10) properly antisymmetrized, one can find the equation of motion for σ and τ :

$$\nabla^\mu \nabla_{[\mu} \nabla^{[\mu'} Q_{\nu]}^{\nu']} = \frac{1}{\xi} \nabla_\nu \left(\nabla^\mu \nabla^{[\mu'} Q_{\mu}^{\nu']}] \right) = 0. \quad (\text{A.5})$$

The last equality in (A.5) derives from the fact that the bitensor in parenthesis has two primed antisymmetrized indices, while the only (0,2) bitensor structures are symmetric. In terms of σ and τ defined in eq. (A.2), eq. (A.5) reads

$$\sigma' - \frac{1}{2}\tau' + (d-2)(A+C)\sigma - \frac{1}{2}(d-2)A\tau = 0. \quad (\text{A.6})$$

Plugging the expression for τ in eq. (A.4), we get a second order differential equation for σ , which will be useful in the following to solve the system for α and β :

$$\sigma'' + (d+1)A\sigma' - 2(d-1)\sigma = 0. \quad (\text{A.7})$$

This equation can be rewritten as a function of the variable z defined in eq. (3.5):

$$z(1-z)\frac{d^2\sigma}{dz^2} + \frac{1}{2}(d+2)(1-2z)\frac{d\sigma}{dz} - 2(d-1)\sigma = 0, \quad (\text{A.8})$$

which is solved by two linearly independent hypergeometric functions. The correct solution is chosen by imposing regularity at antipodals point ($z = 0$) and the correct limit of coincident points ($z = 1$). The last condition can be computed by starting from the expression in coordinate space of the gauge propagator in flat space

$$\langle A_a^\mu(x)A_{\nu'}^b(x') \rangle_{\text{flat}} = \delta_a^b \left(\frac{\Gamma\left(\frac{d}{2}-1\right)(1+\xi)}{2(4\pi)^{\frac{d}{2}}|x-x'|^{d-2}}\delta_{\nu'}^\mu + \frac{\Gamma\left(\frac{d}{2}\right)(1-\xi)}{(4\pi)^{\frac{d}{2}}|x-x'|^d}x^\mu x_{\nu'} \right) \quad (\text{A.9})$$

and the flat space expression for σ :

$$\sigma(z)_{\text{flat}} = \frac{2\Gamma\left(\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}(1-z)^{\frac{d}{2}}}. \quad (\text{A.10})$$

We find then

$$\sigma(z) = p {}_2F_1\left(d-1, 2, \frac{d}{2}+1, z\right), \quad (\text{A.11})$$

with

$$p = \frac{\Gamma(d-1)}{\Gamma\left(\frac{d}{2}+1\right)2^{d-1}\pi^{\frac{d}{2}}}. \quad (\text{A.12})$$

We can use this result to compute α , proceeding as follows: we compute β as a function of α and σ from eq. (A.3) and we replace the result in eq. (A.1). This leads to the following inhomogenous equation for α :

$$\begin{aligned} \alpha'' + (d+1)A\alpha' - d\alpha - \frac{A}{2C}\sigma - \left(1 - \frac{1}{\xi}\right)\left(\alpha'' + \left(\frac{C'}{C} - Ad\right)\alpha'\right) \\ - \left(d-1 - A' + \frac{AC}{C'}\right)\alpha - \frac{1}{4C}\sigma + \left(\frac{C'}{2C} - \frac{A}{4C}(d-1)\right)\sigma = -\delta(x, x'). \end{aligned} \quad (\text{A.13})$$

The solution is given by the sum of the solution of the corresponding homogenous equation (again we should impose the flat space limit and regularity at $z = 0$) and a particular solution to reproduce the correct source term. We have

$$\alpha(z) = q {}_2F_1\left(d, 1, \frac{d}{2}+1, 1-z\right) + \tilde{\alpha}(z), \quad (\text{A.14})$$

where the first term is the solution of the homogeneous equation, with a normalization q to be fixed, and $\tilde{\alpha}$ is a particular solution of the full equation, that plugging the expression for σ , C and A takes the form

$$\begin{aligned} z(1-z)\frac{d^2\tilde{\alpha}}{dz^2} + \left(\frac{d}{2}+1 - (d+2)z\right)\frac{d\tilde{\alpha}}{dz} - d\tilde{\alpha} = \frac{\pi^{-\frac{d}{2}}\Gamma(d-1)}{2^{d+1}} \\ \left((d+2-\xi)(1-2z) {}_2F_1\left(2, d-1; \frac{d}{2}+1; z\right) - 4(d-\xi)(z-1)z {}_2F_1\left(3, d; \frac{d}{2}+2; z\right) \right). \end{aligned} \quad (\text{A.15})$$

A solution for $\tilde{\alpha}$ can be found as follows [31]. We introduce the hypergeometric operator

$$H(a, b, c) = z(1-z) \frac{d^2}{dz^2} + \left(\left(c - (a+b+1)z \frac{d}{dz} \right) - ab \right), \quad (\text{A.16})$$

in order to rewrite the left-hand side of eq. (A.15) as

$$H(a_1 + 1, b_1 - 1, c_1) \tilde{\alpha}, \quad (\text{A.17})$$

with $a_1 = d - 1$, $b_1 = 2$, $c_1 = \frac{d}{2} + 1$. Then, we rewrite the right-hand-side of eq. (A.15) as $H(a_1 + 1, b_1 - 1, c_1)f$, with f a function to be determined, using identities among hypergeometric functions (see e.g. chapter 15 of ref. [54]). A particular solution would then be $\tilde{\alpha} = f$.

The right-hand-side of eq. (A.15) is first rewritten as a function of ${}_2F_1(a_1, b_1, c_1, z)$, ${}_2F_1(a_1 - 1, b_1, c_1, z)$ and ${}_2F_1(a_1 + 1, b_1 - 1, c_1, z)$ only. Then, the following identities are used:

$$\begin{aligned} {}_2F_1(a_1, b_1, c_1, z) &= \frac{1}{d-2} H(a_1+1, b_1-1, c_1) {}_2F_1(a_1, b_1, c_1, z), & (\text{A.18}) \\ {}_2F_1(a_1-1, b_1, c_1, z) &= \frac{1}{2(d-3)} H(a_1+1, b_1-1, c_1) ({}_2F_1(a_1-1, b_1, c_1, z) + {}_2F_1(a_1, b_1, c_1, z)), \\ {}_2F_1(a_1+1, b_1-1, c_1, z) &= \frac{1}{b_1-a_1-2} H(a_1+1, b_1-1, c_1) \left(\frac{\partial}{\partial a} - \frac{\partial}{\partial b} \right) {}_2F_1(a, b, c_1, z) \Big|_{\substack{a=a_1+1 \\ b=b_1-1}}. \end{aligned}$$

Matching with the left hand side of eq. (A.15) gives the particular solution $\tilde{\alpha} = f$:

$$\begin{aligned} \tilde{\alpha} &= \frac{p}{4(d-3)^2} \left(-2 {}_2F_1(a_1-1, b_1+1, c_1, z) + (d-4) {}_2F_1(a_1-1, b_1, c_1, z) \right. \\ &\quad \left. + (2 + (d-3)(1-\xi))(3-d) \left(\frac{\partial}{\partial a} - \frac{\partial}{\partial b} \right) {}_2F_1(a, b, c_1, z) \Big|_{\substack{a=a_1+1 \\ b=b_1-1}} \right). \end{aligned} \quad (\text{A.19})$$

The value of the coefficient q appearing in eq. (A.14) is determined by imposing the correct flat space limit of α . which is the term proportional to $\delta_{\nu'}^{\mu}$ in eq. (A.9). We get

$$q = p \frac{(d-1)(d-2) - (2 + (d-3)(1-\xi))(d-3)(\psi(d) - \psi(1))}{4(d-3)^2}. \quad (\text{A.20})$$

Finally, we obtain the expression for β by replacing α and σ in eq. (A.3):

$$\begin{aligned} \beta &= - \frac{(z-1)\Gamma(d-1)}{2^d \pi^{\frac{d}{2}} (d-3)\Gamma\left(\frac{d}{2}+1\right)} \\ &\quad \left(\frac{d}{dz} \left(-z(2+(d-3)(1-\xi))(3-d) \left(\frac{\partial}{\partial a} - \frac{\partial}{\partial b} \right) {}_2F_1(a, b, c_1, z) \Big|_{\substack{a=a_1+1 \\ b=b_1-1}} \right) \right. \\ &\quad \left. + \Gamma\left(\frac{d}{2}+1\right) \left({}_2F_1\left(2, d; \frac{d}{2}+1; z\right) \left(-((d-3)(1-\xi)+2)(\psi^{(0)}(d)+\gamma) - 2dz + d + 2z - 4 \right) \right. \right. \\ &\quad \left. \left. - 4(z-1) {}_2F_1\left(3, d; \frac{d}{2}+1; z\right) \right) \right). \end{aligned} \quad (\text{A.21})$$

Summarizing, the gauge propagator on S^d is obtained by replacing in eq. (3.11) the expression for α in eqs. (A.14), (A.19), the one for β just reported, together with the expressions for the coefficients p and q in eqs. (A.12), (A.20).

Let us now explain how to expand α and β around coincident points ($z = 1$). First, note that the hypergeometrics have branch points in $z = 1$. In order to expand in powers of $(z - 1)$ it is then convenient to use an identity to obtain only hypergeometric functions with argument $1 - z$. In this way the non-analytic dependence on $z - 1$ will be captured completely by the power-law prefactors. The identity that we will use for this purpose is

$$\begin{aligned}
 {}_2F_1(a, b, c, z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, a+b+1-c, 1-z) \\
 &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b, 1+c-a-b, 1-z).
 \end{aligned}
 \tag{A.22}$$

Derivatives of the hypergeometrics with respect to the parameter a and b appear in both α and β . In order to obtain the expansion in $(1 - z)$ for these derivatives, we first apply the identity in eq. (A.22) and then we expand the hypergeometric as

$${}_2F_1(a, b, c, 1-z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{(1-z)^n}{n!}, \tag{A.23}$$

where $(x)_n$ are the Pochhammer symbols. We truncate the series at a sufficiently high order and then we apply the derivatives with respect to a and b to this truncated series. In order to improve the efficiency of the numerical integration of hypergeometrics needed to get the finite terms (4.35)–(4.37), it is useful to split the interval of integration $0 \leq z \leq 1$ in two parts (i.e. $[0, 1/2]$ and $[1/2, 1]$) and expand respectively around 0 and around 1 the hypergeometrics.

B Computation of the ghost counterterm

The ghost wave function renormalization can be computed by imposing finiteness of the ghost propagator at one loop. We compute the divergence in configuration space:

$$\left. \text{---} \blacktriangleright \text{---} \begin{array}{c} \text{wavy line} \\ \text{---} \blacktriangleright \text{---} \end{array} \text{---} \blacktriangleright \text{---} \right|_{\text{div.}} = \delta_{ab} g_0^2 C_A \int d^d x_1 d^d x_2 \sqrt{h} \sqrt{h'} G(x, x_1) \nabla_{\mu_1} G(x_1, x_2) \tag{B.1}$$

$$Q^{\mu_1 \mu_2}(x_1, x_2) \nabla_{\mu_2} G(x_2, 0).$$

Since we are dealing with UV divergences, we can take the limit of coincident points $x_1 \sim x_2$. Taylor expanding the propagator $G(x_2, 0)$ around x_1 , we get

$$G(x_2, 0) = G(x_1, 0) + (x_2^\mu - x_1^\mu) \nabla_\mu G(x_1, 0) + \frac{1}{2} (x_2^\mu - x_1^\mu) (x_2^\nu - x_1^\nu) \nabla_\nu \nabla_\mu G(x_1, 0) + \dots \tag{B.2}$$

Replacing this expression in eq. (B.1) we find that the only non-vanishing contribution comes from the third term: the first vanishes when derived with respect to x_2 , while the second is zero because of Lorentz invariance. All other terms in the expansion provide

convergent result and are therefore not relevant for the computation of counterterms. We have then

$$\delta_{ab} \frac{g_0^2}{2} C_A \int d^d x_1 \sqrt{h} G(x, x_1) I^{\mu\nu} \nabla_\nu \nabla_\mu G(x_1, 0), \tag{B.3}$$

with

$$I^{\mu\nu} = \int d^d x_2 \sqrt{h'} (\nabla_{\mu_1} G(x_1, x_2) Q^{\mu_1 \mu_2}(x_1, x_2) \nabla_{\mu_2} (x_2^\mu - x_1^\mu)(x_2^\nu - x_1^\nu)). \tag{B.4}$$

By spherical invariance this integral does not depend on the position of x_1 , which can be set to zero. We can use Lorentz invariance to rewrite the integral as

$$I^{\mu\nu} = \frac{g^{\mu\nu}}{d} (I^{\lambda\sigma} g_{\lambda\sigma}). \tag{B.5}$$

The divergence can be computed by using stereographic coordinates and expanding around coincident points, as done in section 4.1: we get

$$\left. \text{---} \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \text{---} \right|_{\text{div.}} = \delta_{ab} \frac{3-\xi}{64\pi^2\epsilon} g_0^2 C_A \int d^d x_1 \sqrt{h} G(x, x_1) \nabla^2 G(x_1, 0). \tag{B.6}$$

This divergence can be removed by taking the following wave function renormalization:

$$c = Z_c^{\frac{1}{2}} c_R, \quad \bar{c} = Z_c^{\frac{1}{2}} \bar{c}_R, \tag{B.7}$$

with

$$Z_c = 1 + \delta_c = 1 - g_0^2 \frac{3-\xi}{64\pi^2\epsilon} C_A + \mathcal{O}(g^4), \tag{B.8}$$

which reproduces eq. (4.8). Note that since there is no divergence proportional to

$$\int d^d x_1 \sqrt{h} G(x, x_1) G(x_1, 0), \tag{B.9}$$

there is no mass renormalization, as expected.

C Subtleties on contact terms and integration by parts

In this appendix we show the subtleties that can arise when integrating propagators derived multiple times on S^d . This analysis is relevant for our purposes in presence of two derivatives acting on the same propagator. For simplicity we will consider a scalar propagator satisfying the equation

$$(-\nabla^2 + m^2)G(x, x') = \delta(x, x'), \tag{C.1}$$

but the same remarks hold for the vector propagator and can be applied to eq. (4.9). Let us consider the integral

$$\int_{S^d} d^d x \sqrt{h} f(\mu) \nabla^2 G(x, 0), \tag{C.2}$$

where f is a function of the geodesic distance $\mu = \mu(x, 0)$, which is taken to be smooth and bounded on S^d . If one tries to compute this integral by specifying some coordinate system and writing explicitly the action of the laplacian on the resulting function in the chosen

coordinates, one gets a wrong answer. This is because the resulting expression for $\nabla^2 G(x, 0)$ misses the contact term, and the answer one gets would correspond to substituting simply $\nabla^2 G(x, 0) = m^2 G(x, 0)$ inside the integral.

A strategy to obtain the correct answer is to integrate by parts

$$\int_{S^d} d^d x \sqrt{h} f(\mu(x, 0)) \nabla^2 G(x, 0) = - \int_{S^d} d^d x \sqrt{h} \nabla^\nu f(\mu(x, 0)) \nabla_\nu G(x, 0). \quad (\text{C.3})$$

To check that this works, let us start by separating two regions in the integral

$$\int_{S^d \setminus B_\delta} d^d x \sqrt{h} f(\mu) \nabla^2 G(x, 0) + \int_{B_\delta} d^d x \sqrt{h} f(\mu) \nabla^2 G(x, 0), \quad (\text{C.4})$$

where B_δ is defined as a small d -dimensional ball of radius δ centered at the origin. In the second integral, for $\delta \rightarrow 0$, we get the contact term $-f(0)$. In the first term, we integrate by parts

$$\begin{aligned} \int_{S^d \setminus B_\delta} d^d x \sqrt{h} f(\mu) \nabla^2 G(x, 0) &= - \int_{S^d \setminus B_\delta} d^d x \sqrt{h} \nabla^\nu f(\mu) \nabla_\nu G(x, 0) \\ &+ \int_{S^d \setminus B_\delta} d^d x \sqrt{h} \nabla^\nu (f(\mu) \nabla_\nu G(x, 0)). \end{aligned} \quad (\text{C.5})$$

In the first integral the limit $\delta \rightarrow 0$ is straightforward, while the second integral requires more care. It is a boundary term that we can rewrite using the first relations in eqs. (3.3), (3.4) and the chain rule as

$$\begin{aligned} \int_{S^d \setminus B_\delta} d^d x \sqrt{h} \nabla^\nu (f(\mu) \nabla_\nu G(x, 0)) &= \int_{S^d \setminus B_\delta} d^d x \sqrt{h} \nabla^\nu \left(f(\mu(z)) G'(z) \frac{\partial z}{\partial \mu} n_\nu \right) \\ &= \int_{S^d \setminus B_\delta} d^d x \sqrt{h} \left(A(d-1) f(\mu(z)) G'(z) \frac{\partial z}{\partial \mu} + \frac{\partial}{\partial z} \left(f(\mu(z)) G'(z) \frac{\partial z}{\partial \mu} \right) \frac{\partial z}{\partial \mu} \right). \end{aligned} \quad (\text{C.6})$$

Here we used the variable z defined in (3.7). By changing the integration variable to z we get

$$\lim_{\delta' \rightarrow 0} \frac{2^d \pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^{1-\delta'} dz \frac{\partial}{\partial z} \left((z(1-z))^{\frac{d-1}{2}} f(\mu(z)) G'(z) \frac{\partial z}{\partial \mu} \right). \quad (\text{C.7})$$

The above integral would vanish for well-defined functions on S^d , as expected from Stokes theorem, but the propagator is actually a distribution which is singular at coincident points $z \rightarrow 1$, so care is required. In the limit $z \rightarrow 1$ the scalar propagator can be approximated to

$$G(z) \simeq \frac{\pi^{1-\frac{d}{2}}}{2^d \Gamma\left(2 - \frac{d}{2}\right) \sin\left(\frac{d\pi}{2}\right)} (1-z)^{1-\frac{d}{2}} + \dots \quad (\text{C.8})$$

Replacing eq. (C.8) in eq. (C.7) gives a non-vanishing result:

$$\lim_{\delta' \rightarrow 0} f(\mu(z)) z^{\frac{d}{2}} \Big|_{z=1-\delta'} = f(\mu=0). \quad (\text{C.9})$$

This boundary term exactly cancels the contribution coming from the second term in eq. (C.4), proving eq. (C.3). Summarizing, the evaluation of eq. (C.2) without integrating by parts would require to pay attention to contact terms by introducing a regulator, while upon integrating by parts the contact term contribution is compensated by another contact term arising from a total derivative contribution.

D Check with Jack [18]

The poles of the diagrams (4.28), (4.29), (4.31), and (4.32) can also be computed with a different procedure. This procedure, which is based on the heat-kernel expansion of the propagators, is more general because it works on any curved background. We show in this section how the divergences which were obtained in this way for pure Yang-Mills theory in ref. [18] in the Feynman gauge $\xi = 1$ agree with our previous results. Matching with the results of [18] requires a bit of manipulations. It is then useful to briefly recall the key results found in [18, 55] using heat kernel methods.

Let us consider an elliptic differential operator of the form

$$M(x) = -\nabla^2 + Y(x) \tag{D.1}$$

and the corresponding propagator satisfying

$$M(x)G_M(x, x') = \delta(x - x'). \tag{D.2}$$

Around coincident points $x \sim x'$ the following expansion holds [55]:

$$G_M \sim -\frac{1}{16\pi^2\epsilon} a_{1\text{diag}}^M + H_{\text{diag}}^M, \tag{D.3}$$

where H_{diag}^M is in general a complicated non-local expression satisfying

$$M(x)H_{\text{diag}}^M = \frac{1}{16\pi^2} a_{2\text{diag}}^M. \tag{D.4}$$

The coefficients $a_{1\text{diag}}^M$ and $a_{2\text{diag}}^M$ admit instead a local expression in terms of the curvature tensors and they can be computed for any elliptic operator M . From the propagator equations in section 3, we see that the ghost differential operator is indeed of the form (D.1), while the vector one is not for a generic choice of the gauge. This is why ref. [55] provides results only in the Feynman gauge $\xi = 1$, for which also the vector operator is of the form (D.1). The coefficients then read

$$\begin{aligned} a_{1\text{diag}}^{\text{gh}} &= \frac{1}{6} \mathcal{R}, \\ a_{2\text{diag}}^{\text{gh}} &= \frac{1}{180} (\mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma} - \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu}) + \frac{1}{72} \mathcal{R}^2, \end{aligned} \tag{D.5}$$

for the ghost and

$$\begin{aligned} a_{1\text{diag}}^{\text{vec}}{}_{\mu\nu} &= \frac{1}{6} \mathcal{R} g_{\mu\nu} - \mathcal{R}_{\mu\nu}, \\ a_{2\text{diag}}^{\text{vec}} &= \frac{1}{360} (2(d-15) \mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma} - 2(d-90) \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + 5(d-12) \mathcal{R}^2), \end{aligned} \tag{D.6}$$

for the vector. Ref. [18] provides the expressions for the poles of diagrams as a function of the curvature tensors and of the derivatives of H_{diag} , more specifically, $\nabla^2 H_{\text{diag}}^{\text{gh}}$, $H_{\text{diag}}^{\text{vec}}{}_{\mu\nu}$,

$g_{\mu\nu}\nabla^2 H_{\text{diag}}^{\text{vec } \mu\nu}$ and $\nabla_\mu\nabla_\nu H_{\text{diag}}^{\text{vec } \mu\nu}$.¹² Now, from eq. (D.4) we have

$$-\nabla^2 H_{\text{diag}}^{\text{gh}} = \frac{1}{16\pi^2} a_{2\text{diag}}^{\text{gh}}, \quad (\text{D.7})$$

$$-\nabla^2 H_{\text{diag}}^{\text{vec } \mu\nu} g_{\mu\nu} = \frac{1}{16\pi^2} a_{2\text{diag}}^{\text{vec}} - H_{\text{diag}}^{\text{vec } \mu\nu} \mathcal{R}_{\mu\nu}. \quad (\text{D.8})$$

The first equation allows us to find a simple expression for $\nabla^2 H_{\text{diag}}^{\text{gh}}$ in terms of the curvature tensors. However we cannot solve the second equation to obtain a similar simple expression for $H_{\text{diag}}^{\text{vec } \mu\nu}$. A way to compute $\nabla_\mu\nabla_\nu H_{\text{diag}}^{\text{vec } \mu\nu}$ is by imposing the cancellation of poles in the total free-energy inserting the expression for the diagrams obtained in ref. [18] (detailed in footnote 13 below) in eq. (4.1). Note that only $\nabla_\mu\nabla_\nu H_{\text{diag}}^{\text{vec } \mu\nu}$ and $\nabla^2 H_{\text{diag}}^{\text{gh}}$ enter the expression for these diagrams, not $H_{\text{diag}}^{\text{vec } \mu\nu}$. In order to obtain a result valid on a generic manifold, we use the one-loop free energies computed in refs. [18, 55] and the renormalization of the curvature coefficient a . We get

$$\nabla_\mu\nabla_\nu H_{\text{diag}}^{\text{vec } \mu\nu} = \frac{1}{8\pi^2} \left(\frac{109}{3960} \mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma} - \frac{229}{3960} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \frac{5}{1584} \mathcal{R}^2 + 3 \right). \quad (\text{D.9})$$

The above relations apply on any manifold.

We can now focus on S^d to get explicit results. The term $H_{\text{diag}}^{\text{vec } \mu\nu}$ can be computed by expanding the propagator around coincident points and using eq. (D.3). Taking the expression for the gauge propagator of eq. (3.11) for $\xi = 1$ we get

$$Q_{\nu}^{\mu}(z) = R^{2-d} \left(\frac{\Gamma\left(\frac{d}{2}-1\right)}{2^d \pi^{\frac{d}{2}}} (1-z)^{1-\frac{d}{2}} + \frac{\Gamma\left(\frac{d}{2}-2\right)(d^2-6d+4)}{2^{d+2} \pi^{\frac{d}{2}}} (1-z)^{2-\frac{d}{2}} \right. \\ \left. + \frac{\Gamma\left(\frac{d-3}{2}\right) \left(-d+2\pi \cot\left(\frac{\pi d}{2}\right) + 2(\psi(d)+\gamma)\right)}{8\pi^{\frac{d+1}{2}} d} \right) \delta_{\nu}^{\mu} + \dots \quad (\text{D.10})$$

Using analytic continuation in d , we can set to zero the powers $(1-z)^{1-\frac{d}{2}}$ and $(1-z)^{2-\frac{d}{2}}$ of the expansion. The remaining part can be computed at $d = 4 + 2\epsilon$ and expanded around $\epsilon = 0$. Plugging the result in eq. (D.3) gives

$$H_{\text{diag}}^{\text{vec } \mu\nu}(S^4) = -\frac{1+3\gamma+3\log(4\pi\mu^2 R^2)}{48\pi^2 R^2} \delta^{\mu\nu}. \quad (\text{D.11})$$

From eq. (D.9) we have

$$\nabla_\mu\nabla_\nu H_{\text{diag}}^{\text{vec } \mu\nu}(S^4) = \frac{61}{240\pi^2 R^4}. \quad (\text{D.12})$$

Using eq. (D.8) we can similarly get the expressions for $\nabla^2 H_{\text{diag}}^{\text{gh}}$ and $\nabla_\mu\nabla_\nu H_{\text{diag}}^{\text{vec } \mu\nu}$:

$$g_{\mu\nu}\nabla^2 H_{\text{diag}}^{\text{vec } \mu\nu}(S^4) = -\frac{232+120(-1+3\gamma+\log(4\pi\mu^2 R^2))}{480\pi^2 R^4}, \quad (\text{D.13})$$

$$\nabla^2 H_{\text{diag}}^{\text{gh}}(S^4) = -\frac{29}{240\pi^2 R^4}. \quad (\text{D.14})$$

¹²In ref. [18] $H_{\text{diag}}^{\text{gh}}$ is denoted H_{diag}^0 , while $H_{\text{diag}}^{\text{vec } \mu\nu}$ is denoted $H_{\text{diag}}^{1\ \mu\nu}$.

Substituting in the results of ref. [18], we find¹³

$$G_{2J}^{\text{triple}} \Big|_{\text{div.}} = \kappa \left(\frac{1}{24\pi^2\epsilon^2} - \frac{13 + 2(\gamma + \log(4\pi\mu^2 R^2))}{48\pi^2\epsilon} \right), \quad (\text{D.15})$$

$$G_{2J}^{\text{ghost}} \Big|_{\text{div.}} = \kappa \left(\frac{1}{48\pi^2\epsilon^2} - \frac{1 + 2(\gamma + \log(4\pi\mu^2 R^2))}{96\pi^2\epsilon} \right), \quad (\text{D.16})$$

$$G_{2J}^{\text{quart}} \Big|_{\text{div.}} = \kappa \left(-\frac{1}{16\pi^2\epsilon^2} + \frac{17 + 6(\gamma + \log(4\pi\mu^2 R^2))}{96\pi^2\epsilon} \right), \quad (\text{D.17})$$

$$G_{2J}^{\text{CT-vect}} \Big|_{\text{div.}} = \kappa \left(-\frac{1}{8\pi^2\epsilon} \right). \quad (\text{D.18})$$

Eqs. (D.15), (D.17) and (D.18) match respectively eqs. (4.28), (4.31) and (4.32) evaluated at $\xi = 1$. As explained, the ghost counterterm (4.29) arises because of ghost zero modes, specific for S^d . Heat kernel methods apply to generic manifolds and therefore there is no ghost counterterm in ref. [18]. The ghost contribution (D.16) should then match the sum of eq. (4.29) and the counterterm (4.33) for $\xi = 1$, and this is indeed the case. We then have a check diagram by diagram of our computation.

E Alternative gauge-fixing procedure

In section 2.2 we have seen that the quantization of non-abelian gauge theories on S^d using an ordinary Faddeev-Popov formalism leads to ghost zero modes. In this appendix we would like to show that our heuristic treatment of the zero modes is confirmed by a more rigorous treatment using a Batalin-Vilkovisky formalism and ghosts for ghosts, see e.g. ref. [56] for a nice introduction or ref. [57] for a more detailed treatment. We start by briefly recalling the method in Yang-Mills theories on flat space and then apply it on S^d , where we reproduce the action presented in ref. [29]. We then compute the ghost contribution in eq. (4.2) using the new action and show that it matches with eq. (4.10) obtained with the more heuristic treatment discussed in the main text.

E.1 Gauge theories on S^d

Yang-Mills theories on flat space do not require ghosts for ghosts and can be treated with the Faddeev-Popov method. Let us briefly review how the same gauge-fixing can be obtained with the Batalin-Vilkovisky formalism. Recall that in this formalism for each field ϕ_A we introduce an antifield ϕ_A^* and we require the master equation

$$(S, S) = 0, \quad (\text{E.1})$$

where

$$(F, G) \equiv \frac{\delta_R F}{\delta \phi^A} \frac{\delta_L G}{\delta \phi_A^*} - \frac{\delta_R F}{\delta \phi_A^*} \frac{\delta_L G}{\delta \phi^A}. \quad (\text{E.2})$$

¹³See eq. (2.55) of ref. [18] for G_{2J}^{triple} , eq. (2.52) for G_{2J}^{ghost} , eq. (2.33) for G_{2J}^{quart} and eqs. (2.31), (2.59) for $G_{2J}^{\text{CT-vect}}$. Note that in our convention $d = 4 + 2\epsilon$, while in ref. [18] the authors used $d = 4 - \epsilon$. Moreover, all diagrams in ref. [18] are multiplied by a factor 1/2, which we factorized instead outside G_2 .

In the Yang-Mills theory case, $\phi^A = \{A, c\}$, where c are the ghost fields needed to take into account of the gauge redundancy of the classical action. The action satisfying the master equation (E.1) reads

$$S_{\text{flat}} = S_{\text{YM}}[A] + \int d^d x \left(A^* Dc - ic^* c^2 + \bar{c}^* B \right), \quad (\text{E.3})$$

where $D = \partial - i[A, \cdot]$, $[\phi_1, \phi_2] = if_{abc} T^a \phi_1^b \phi_2^c$, with T^a and f_{abc} the generators in the fundamental representation and the structure constants of the Lie algebra, respectively. In eq. (E.3) trace over group indices and Lorentz indices are implicit and we have added an auxiliary pair of fields \bar{c}/B and their corresponding antifields, which do not affect the master equation. Note that only \bar{c}^* and B enter the action but also \bar{c} and B^* are integrated over in the path integral. A gauge-fixing is introduced through a fermionic functional $\Psi[\phi]$ which fixes the value of the antifields:

$$\phi_A^* = \frac{\delta \Psi[\phi]}{\delta \phi^A}, \quad (\text{E.4})$$

where now $\phi^A = \{A, c, \bar{c}, B\}$. An appropriate choice for the gauge-fixing functional is

$$\Psi = \int d^d x \bar{c} \left(-\frac{\xi}{2} B - \partial A \right), \quad (\text{E.5})$$

which leads to

$$S_{\text{flat}}^{\text{g.f.}} = S_{\text{YM}}[A] + \int d^d x \left(\bar{c} \partial Dc - B \left(\frac{\xi}{2} B + \partial A \right) \right), \quad (\text{E.6})$$

which is the usual R_ξ gauge fixing of the Yang-Mills action.

On S^d an important difference arises. Covariantly constant modes leave the gauge field invariant, so the transformation $c \rightarrow c + \theta \tilde{a}_0$, with θ a Grassmann constant parameter, leaves the gauge field invariant, provided that

$$D\tilde{a}_0 = 0. \quad (\text{E.7})$$

The mode \tilde{a}_0 is a (bosonic) ghost for ghost. We should then add \tilde{a}_0 to the set of fields in the action, together with its antifield. \tilde{a}_0 is actually not a field, but a single mode of a field, the covariantly constant one. For simplicity we keep this implicit. The solution to the master equation reads now

$$S = S_{\text{YM}}[A] + \int d^d x \sqrt{\hbar} \left(A^* Dc + c^* a_0 - ic^* c^2 + ia_0^* [c, a_0] + \bar{c}^* B + \bar{a}_0^* \bar{c}_0 + b_0^* c_0 \right), \quad (\text{E.8})$$

where we have added two pairs \bar{a}_0/\bar{c}_0 and b_0/c_0 of fields (and their antifields), composed only of a covariantly constant mode, like \tilde{a}_0 . In a perturbative treatment, where we expand in modes the quadratic action, the covariantly constant mode \tilde{a}_0 should be replaced by a constant mode a_0 satisfying

$$\nabla a_0 = 0, \quad (\text{E.9})$$

which corresponds to the ghost zero modes found in the ordinary Faddeev-Popov procedure followed in section 2.2. The action (E.8) no longer solves the classical master equation if $\tilde{a}_0 \rightarrow a_0$. We now have

$$(S, S) = 2A^* D a_0 = 2iA^* [a_0, A] \neq 0, \quad (\text{E.10})$$

However, adding appropriate terms to the action we can introduce a new action $\tilde{S} = S + \delta S$ such that

$$(\tilde{S}, \tilde{S}) = 2i\phi_A^*[a_0, \phi^A]. \quad (\text{E.11})$$

In this way, after gauge-fixing we have

$$(\tilde{S}, \tilde{S}) = 2i \frac{\delta \Psi}{\delta \phi^A} [a_0, \phi^A] = 2i[a_0, \Psi[\phi]]. \quad (\text{E.12})$$

For appropriate choices of the fermionic functional Ψ (gauge-fixing), the last term in eq. (E.12) vanishes and the master equations are satisfied, together with gauge-fixing independence of correlations function of gauge-invariant operators. In order to satisfy eq. (E.11), we add to S in eq. (E.8) a term

$$\delta S = i \int d^d x \sqrt{h} \left(B^*[a_0, \bar{c}] + \bar{c}_0^*[a_0, \bar{a}_0] + c_0^*[a_0, b_0] + a_0^*[a_0, c] \right). \quad (\text{E.13})$$

The BRST transformation of fields is given by $\delta_\theta \phi^A = \theta(-1)^{\epsilon_A}(S, \phi^A)$, $\delta_\theta \phi_A^* = -\theta(-1)^{\epsilon_A}(S, \phi_A^*)$, with $\epsilon_A = 0, 1$ depending on the statistics of ϕ_A . Explicitly we get

$$\begin{aligned} \delta_\theta A &= \theta Dc, & \delta_\theta c &= \theta(-a_0 + ic^2), & \delta_\theta \bar{c} &= -\theta B, & \delta_\theta B &= i\theta[a_0, \bar{c}], \\ \delta_\theta a_0 &= 0, & \delta_\theta \bar{c}_0 &= -i\theta[a_0, \bar{a}_0], & \delta_\theta c_0 &= -i\theta[a_0, b_0], & \delta_\theta \bar{a}_0 &= \theta \bar{c}_0, & \delta_\theta b_0 &= \theta c_0, \end{aligned}$$

so that $\delta_{\theta_1} \delta_{\theta_2} \phi = -i\theta_1 \theta_2 [a_0, \phi]$ for any field ϕ . The gauge-fixing fermionic functional is taken as

$$\Psi = \int d^d x \sqrt{h} \left(\bar{c} \left(-\frac{\xi}{2} B - \nabla A - b_0 \right) + \bar{a}_0 c \right), \quad (\text{E.14})$$

providing

$$\begin{aligned} A^* &= -\nabla \bar{c}, & c^* &= \bar{a}_0, & \bar{c}^* &= -\frac{\xi}{2} B - \nabla A - b_0, \\ B^* &= -\frac{\xi}{2} \bar{c}, & b_0^* &= \bar{c}, & \bar{a}_0^* &= c. \end{aligned} \quad (\text{E.15})$$

We then get

$$S^{\text{g.f.}} = S_{\text{YM}}[A] + \int d^d x \sqrt{h} \left(\bar{c} \nabla Dc + B \left(-\frac{\xi}{2} B + \nabla A + b_0 \right) + \bar{a}_0 a_0 + c \bar{c}_0 + \bar{c} c_0 + i\xi \bar{c}^2 a_0 - ic^2 \bar{a}_0 \right), \quad (\text{E.16})$$

which is the same action of eq. (4.2) in ref. [29].¹⁴ We have then the following path integral:

$$\begin{aligned} Z_{\text{YM}} &= \frac{1}{\text{vol}(\mathcal{G}, g)} \int \mathcal{D}A \exp(-S_{\text{YM}}) \\ &= \frac{1}{\text{vol}(G) \sqrt{\text{vol}(S^d)^{\dim(G)}}} \int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}B \mathcal{D}a_0 \mathcal{D}\bar{a}_0 \mathcal{D}b_0 \mathcal{D}\bar{c}_0 \mathcal{D}c_0 \exp(-S^{\text{g.f.}}). \end{aligned} \quad (\text{E.17})$$

¹⁴The precise matching, in the notation of ref. [29], is $\bar{c} \rightarrow i\bar{c}$, $B \rightarrow -ib$, $\bar{a}_0 \rightarrow \bar{a}_0 - \xi_2 a_0/2$, where ξ_2 is another gauge fixing parameter which does not affect the total path integral. Note that ref. [29] takes the fields to be antihermitian and not hermitian as in our case.

The volume factor obtained after gauge fixing is the same found with the procedure used in the rest of this work: we can indeed verify that integrating out all fields and proceeding in reverse order to what we did in section 2.2, we reproduce the path integral in the first line of eq. (E.17). The integration of \bar{c}_0, c_0 and b_0 removes the zero modes respectively of c, \bar{c} and B . Integrating out a_0 (along an imaginary contour to have a convergent path integral) sets also \bar{a}_0 to zero, while the gaussian integration in B reproduces the usual gauge fixing term $(\nabla A)^2/(2\xi)$.

E.2 Computation of the ghost propagator

In the previous section we explained how to perform the gauge-fixing of Yang-Mills theories on S^d with the field-antifield formalism. The action that we obtained contains many fields that were not present in our main computation. As mentioned, one possibility is to integrate them out: in such a way we recover our original action and we can proceed as we already did. The other possibility is to keep the action (E.16) as it is and compute Feynman rules directly from it. We will focus in particular on the ghost action, which is

$$S_{\text{ghost}} = \int d^d x \sqrt{h} \bar{c} \nabla D c + c \bar{c}_0 + \bar{c} c_0 + i \xi \bar{c}^2 a_0 - i c^2 \bar{a}_0. \quad (\text{E.18})$$

In order to compute the propagator, we should rewrite the quadratic part of this action as

$$S_{\text{ghost}} = \int d^d x \sqrt{h} \frac{1}{2} \begin{pmatrix} \bar{c} & c \end{pmatrix} M \begin{pmatrix} c \\ \bar{c} \end{pmatrix} + c \bar{c}_0 + \bar{c} c_0, \quad (\text{E.19})$$

with

$$M^{ab} = \begin{pmatrix} \delta^{ab} \nabla^2 & -\xi f^{abc} a_{0c} \\ f^{abc} \bar{a}_{0c} & -\delta^{ab} \nabla^2 \end{pmatrix}. \quad (\text{E.20})$$

As explained before, the terms linear in \bar{c}_0 and c_0 set to zero the constant modes. As opposed to the standard ghost action, we do not have only the ghost-antighost term, but also terms quadratic in ghosts and antighosts. The propagator will then be a matrix

$$G_{ab} = \left(\begin{array}{cc} \text{---} \dashrightarrow \text{---} \dashrightarrow \text{---} \dashrightarrow \text{---} & \text{---} \dashleftarrow \text{---} \dashleftarrow \text{---} \dashleftarrow \text{---} \\ \text{---} \dashrightarrow \text{---} \dashrightarrow \text{---} \dashrightarrow \text{---} & \text{---} \dashleftarrow \text{---} \dashleftarrow \text{---} \dashleftarrow \text{---} \end{array} \right) = \begin{pmatrix} \langle \bar{c}'_a c'_b \rangle & \langle c'_a \bar{c}'_b \rangle \\ \langle \bar{c}'_a \bar{c}'_b \rangle & \langle c'_a c'_b \rangle \end{pmatrix} \quad (\text{E.21})$$

with all entries different from zero, satisfying

$$M_{ab}^{ij} G_{jk}^{bc} = -\delta_k^i \delta_a^c \delta(x - x'). \quad (\text{E.22})$$

Let us consider the following ansatz and verify if there exists such a solution:

$$G_{ab} = \begin{pmatrix} \delta_{ab} f_1(z) + a_{0b} \bar{a}_{0a} f_2(z) & f_{abc} a_0^c g_1(z) \\ f_{abc} \bar{a}_0^c h_1(z) & -\delta_{ab} f_1(z) - a_{0a} \bar{a}_{0b} f_2(z) \end{pmatrix}, \quad (\text{E.23})$$

with $f_1(z), f_2(z), g(z), h(z)$ generic functions of the stereographic coordinates. By replacing in eq. (E.22) and using the identities

$$f_{abc} f_{a'b'c} = \frac{n_c}{n_c^2 - 2} (\delta_{aa'} \delta_{bb'} - \delta_{ab'} \delta_{ba'}), \quad (\text{E.24})$$

we get

$$\begin{aligned} \nabla^2 f_1 &= -\xi \frac{n_c}{n_c^2 - 2} (a_0 \bar{a}_0) h_1(z) + \delta(x - x'), & \nabla^2 h_1 &= -f_1, \\ \nabla^2 f_2 &= \xi \frac{n_c}{n_c^2 - 2} h_1(z), & g_1 &= \xi h_1. \end{aligned} \tag{E.25}$$

This is a system of coupled ordinary differential equations. The solution is easily found by decomposing in spherical harmonics each function of z :

$$f(z) = \sum_{\ell > 0} f_\ell Y_\ell(x) Y_\ell(x'), \tag{E.26}$$

where we exclude the constant mode $\ell = 0$ because of the linear terms in eq. (E.19). We get

$$\begin{aligned} f_{1\ell} &= \frac{1}{2} \left(\frac{1}{-\ell(\ell + d - 1) + m^2} + \frac{1}{-\ell(\ell + d - 1) - m^2} \right), \\ h_{1\ell} &= \frac{1}{2m^2} \left(\frac{1}{-\ell(\ell + d - 1) + m^2} - \frac{1}{-\ell(\ell + d - 1) - m^2} \right), \\ f_{2\ell} &= \frac{\xi}{2m^4} \left(\frac{1}{-\ell(\ell + d - 1) + m^2} + \frac{1}{-\ell(\ell + d - 1) - m^2} - \frac{2}{-\ell(\ell + d - 1)} \right), \\ g_{1\ell} &= \xi f_{1\ell}, \end{aligned} \tag{E.27}$$

where

$$m^2 \equiv \sqrt{\frac{\xi n_c}{n_c^2 - 2}} (a_0 \bar{a}_0)^{\frac{1}{2}}. \tag{E.28}$$

Following the notation of section 3.3, we denote by $G_{\text{reg}}(x, m^2)$ the solution of the scalar propagator equation on S^d with zero modes removed:

$$G_{\text{reg}}(z, m^2) = \sum_{\ell > 0} \frac{1}{-\ell(\ell + d - 1) + m^2} Y_\ell(x) Y_\ell(x'). \tag{E.29}$$

Summing over the non-constant modes we then find

$$\begin{aligned} f_1 &= \frac{1}{2} \left(G_{\text{reg}}(z, m^2) + G_{\text{reg}}(z, -m^2) \right), \\ f_2 &= \frac{\xi}{2m^4} \left(G_{\text{reg}}(z, m^2) + G_{\text{reg}}(z, -m^2) - 2G_{\text{reg}}(z, 0) \right), \\ h_1 &= \frac{1}{2m^2} \left(G_{\text{reg}}(z, m^2) - G_{\text{reg}}(z, -m^2) \right), \\ g_1 &= \xi h_1. \end{aligned} \tag{E.30}$$

E.3 Match with eq. (4.10)

Let us now consider how the ghost contribution G_2^{ghost} is modified when the propagator in eq. (E.21) is used. As in this case also Wick contractions of two ghosts or two antighosts are allowed, the number of ghost diagrams increases. We have

$$\tilde{G}_2^{\text{ghost}} = \text{diagram 1} + \text{diagram 2} = \tilde{G}_2^{\text{ghost1}} + \tilde{G}_2^{\text{ghost2}}, \tag{E.31}$$

with

$$\begin{aligned}
 \tilde{G}_2^{\text{ghost1}} &= g_0^2 \int d^d x d^d x' \sqrt{h} \sqrt{h'} \left(n_c (n_c^2 - 1) \nabla_\mu f_1 \nabla_{\mu'} f_1 \right. \\
 &\quad \left. + \frac{n_c}{n_c^2 - 2} (a_0^2 \bar{a}_0^2 - (a_0 \bar{a}_0)^2) \nabla_\mu f_2 \nabla_{\mu'} f_2 \right) Q^{\mu\mu'}, \\
 \tilde{G}_2^{\text{ghost2}} &= g_0^2 \frac{n_c^2}{n_c^2 - 2} \xi (a_0 \bar{a}_0) \int d^d x d^d x' \sqrt{h} \sqrt{h'} (\nabla_{\mu'} \nabla_\mu h_1) h_1 Q^{\mu\mu'}. \tag{E.32}
 \end{aligned}$$

The evaluation of eq. (E.32) for generic ξ , which includes integrating over a_0 and \bar{a}_0 , is a non-trivial task. The computation remarkably simplifies in the Landau gauge $\xi \rightarrow 0$. In this limit $m^2 \rightarrow 0$, the functions f_1 and h_1 are of order 1, while f_2 and g_1 are subleading in ξ . The only contribution left in the limit is given by the first term in G_2^{ghost1} , the one involving the product of two f_1 . For $\xi \rightarrow 0$, $f_1 \rightarrow G_{\text{reg}}(z, 0)$, which coincides with the ghost propagator in eq. (3.16), and we reproduce exactly the result in eq. (4.10):

$$\lim_{\xi \rightarrow 0} \tilde{G}_2^{\text{ghost}} = G_2^{\text{ghost}}. \tag{E.33}$$

This is a sanity check of the validity of the heuristic Faddeev-Popov approach followed in the main text.

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