SISSA
Scuola Internazionale Superiore di Studi Avanzati

Mathematics Area - PhD course in
Geometry and Mathematical Physics

Ricci-Flat Metrics on Canonical Bundles

Candidate:
Umar Shahzad

Advisor:
Prof. Ugo Bruzzo

Co-advisor:
Prof. Pietro Fré

Academic Year 2022-23

To my baby who will be among us soon!

Acknowledgments

I am thankful to my family for their constant support throughout my life. To my *Father*, who despite having no formal education gave his best to help me to get where I am now and continues supporting me. To my Mother, who showed me to be kind and brave. To my brothers and sisters *Qumar, Misbah, Tanzila, and Khurram* for their love and support. To my wife *Sadaf* for her love and understanding.

I am deeply thankful to all of my teachers for their valuable guidance and support. Especially, to my advisor *Prof. Ugo Bruzzo* for his support, wisdom, and guidance which helped me complete this thesis. To co-advisor *Prof. Pietro Fré* for his wonderful discussions and suggestions which helped me a lot in writing this thesis. To my master's advisor *Prof. Amer Iqbal* for his valuable guidance in shaping my career. To my college maths teacher *Mr. Shoaib Ghumman*, who expanded my interest in mathematics.

I am also thankful to my friends *Michele, Massimo, Felipe, Mario, Leo, Stefano, Fran, and Asem.* We spent a fun time together and learned a lot from each other. I am also thankful to the administration of SISSA for their help.

Abstract

In this thesis, we studied a new two-parametric class of Kähler-Einstein surfaces $\mathcal{M}^{[\lambda_1,\lambda_2]}$ with explicit KE metrics with $SU(2) \times U(1)$ isometries, and have conical singularities. Topologically, every $\mathcal{M}^{[\lambda_1,\lambda_2]}$ is homeomorphic to $\mathbb{F}_2$, the second Hirzebruch surface, but are different as complex manifolds. We studied their differential geometry in detail regarding the behavior of the associated Riemannian curvature, geodesics, contact structure, and the nature of singularities. We used Calabi's ansatz to put explicit Ricci-flat metrics on $tot(K_{\mathcal{M}}^{[\lambda_1,\lambda_2]})$. These Ricci-flat metrics are $D3$-brane solutions of type IIB supergravity theories.

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Chapter 1

Introduction

In a series of papers [6, 7, 5], the gauge/gravity correspondence was studied by considering the quiver gauge theories associated with McKay quivers and the generalized Kronheimer construction of the resolution of $\mathbb{C}^3/\Gamma$ singularities. McKay quivers are certain kinds of quivers that are associated with the resolutions of $\mathbb{C}^n/\Gamma$ quotient singularities where $\Gamma \subset SU(n)$ is a finite subgroup. J. McKay [39] proved a correspondence between ADE-type quivers and the resolutions of du-Val singularities in dimension two.

In [37], Kronheimer constructed the resolutions of du-Val singularities by using the Kähler quotient data associated with ADE-type quivers and found hyperKähler metrics on all resolutions.

In general, if we consider a finite subgroup $\Gamma \subset SU(n)$ with an action on $\mathbb{C}^n$, the quotient space $\mathbb{C}^n/\Gamma$ has singularities and resolving those singularities can lead to a special kind of resolutions called "**crepant**", which means the resolution space is a noncompact Calabi-Yau manifold. These crepant resolutions do not always exist in dimensions greater than three. In dimension two, they exist and are unique. In dimension three, they exist but they may not be unique.

In [33, 32, 36, 18, 19, 20, 45, 46, 47], the three dimensional McKay correspondence was studied. It provides a group theoretical prediction of the structure of the cohomological ring $H^*(Y_\Gamma, \mathbb{Z})$ in terms of irreducible representations of Γ, where Y_Γ

is a crepant resolution of $\mathbb{C}^3/\Gamma$.

Since these crepant resolutions are noncompact Calabi-Yau's, one might expect Ricci-flat metrics on them. Indeed, in [34, 35], the existence of Ricci flat metrics on Y_Γ was proven by D. Joyce in every class of ALE or $Quasi - ALE$ Kähler two forms depending upon the nature of the quotient singularity $\mathbb{C}^3/\Gamma$ being isolated or non-isolated respectively.

From the differential geometric viewpoint, the crepant resolutions are constructed via generalized Kronheimer construction. In [6], the classical Kronheimer construction was generalized to the case of $\mathbb{C}^3/\Gamma$ quotient singularities via the McKay quiver associated with Γ. A Kähler metric was also associated via this construction to the crepant resolution Y_Γ. In [7], the case of $\mathbb{C}^3/\mathbb{Z}_4$ quotient singularity was considered, where the crepant resolution is unique, and it is the total space of the canonical bundle of the second Hirzebruch surface. The Kronheimer Kähler metric on the resolution $Y_{\mathbb{Z}_4}$ coming from the generalized Kronheimer construction turns out to be not Ricci flat. On the other hand, in order to establish the gauge/gravity correspondence, we need a Ricci-flat metric on the $Y_{\mathbb{Z}_4}$. A quest to find Ricci-flat metric on the crepant resolution of $\mathbb{C}^3/\mathbb{Z}_4$ began in this regard.

Although the Kronheimer metric on the resolved space of $\mathbb{C}^3/\mathbb{Z}_4$ is not Ricci flat its restriction to the second Hirzebruch surface provided a one-parametric family of Kähler metrics parameterized by $\alpha \in \mathbb{R}_+$, and with $SU(2) \times U(1)$ isometry.

In general, whenever Γ is abelian, the quotient variety $\mathbb{C}^3/\Gamma$ and their resolutions are toric varieties. In some cases, like $\mathbb{C}^3/\mathbb{Z}_4$, there may exist a crepant resolution which is the total space of the canonical bundle of a compact toric divisor of the resolution $\phi : Y_\Gamma \longrightarrow \mathbb{C}^3/\Gamma$. In [10], Calabi provided an ansatz to construct Kähler-Einstein metrics on the total space of a holomorphic vector bundle over a compact Kähler-Einstein manifold. In particular, Calabi's ansatz provides explicit Ricci flat metrics on the total space of canonical bundle over compact Kähler-Einstein manifolds.

In [10], Calabi provides a recipe to construct a Kähler metric g_E on the total space of a holomorphic vector bundle $E \to \mathcal{M}$ in the standard complex formalism, where $\mathcal{M}$ is a compact Kähler manifold, satisfying the following conditions:

C1: the restriction of g_E to the space tangent to the zero section of E coincides with a given Kähler-Einstein (KE) metric $g_{\mathcal{M}}$ on $\mathcal{M}$;

C2: the horizontal spaces given by the Chern connection of the metric g_E are the orthogonal complement of the tangent spaces to the fibers of E with respect to g_E;

C3: g_E restricts on every fiber of E to a hermitian metric on the fiber (as a vector space).

In the case of $\mathbb{C}^3/\mathbb{Z}_4$, the resolved variety is the total space of the canonical bundle of the second Hirzebruch surface $\mathbb{F}_2$, and since $\mathbb{F}_2$ admits no KE metric, the Calabi ansatz does not work.

In [21, 1, 30, 29], the study of symplectic geometry on the compact toric varieties was established. In [21], Delzant proved a one-to-one correspondence between compact connected symplectic toric manifolds and so-called Delzant polytopes. In [30], Guillemin provided a canonical symplectic potential on toric manifold $\mathcal{M}_P$ associated with the Delzant polytope P.

In this thesis, we explore the possibility of finding a new class of Kähler-Einstein metrics, with isometries $SU(2)\times U(1)$, such that the corresponding KE manifolds are topologically homeomorphic to $\mathbb{F}_2$ but have singularities in their differential structures.

To get these manifolds, we start from the fact that Kronheimer Kähler potential $\mathcal{K}_{\mathbb{F}_2}$ on $\mathbb{F}_2$ found in [7] depends upon a single real variable

$$\varpi = (1 + |u|^2)^2|v|^2,$$

which insures $SU(2) \times U(1)$ isometry. We consider a generic Kähler potential $\mathcal{K}_{\mathcal{M}_B}$ which only depends upon ϖ. Furthermore, ϖ depends on the modules of the two complex variables u and v on $\mathcal{M}_B$. Using so-called AMSY(Abreu-Martilli-Sparks-Yau) formalism [1, 38], we were able to find an entire 2-parametric family of Kähler-Einstein metrics.

Furthermore, we transform Calabi's ansatz, originally formulated in standard complex formalism, into the AMSY formalism which yields a simple and elegant transcription. Using this, we are able to find a general expression for the Ricci flat metrics on the total spaces of canonical bundles of the 2-parameter class of KE manifolds $\mathcal{M}_B^{[\lambda_1,\lambda_2]}$, each of which we label with two real numbers:

$$0 \leq \lambda_1 < \lambda_2 < \infty$$

.

The geometry of the 4-folds. From a geometric point of view, it should be stressed that all 4-manifolds we find via AMSY formalism in our analysis are homeomorphic to the product $S^2 \times S^2$. Concerning the complex structure, for some choices of the potential $\mathcal{K}_{\mathcal{M}_B}$ we obtain the second Hirzebruch surface $\mathbb{F}_2$; for other choices, we obtain a large family of KE manifolds, which of course must be singular. Our analysis in Section 4.6 shows indeed that they have conical singularities which we studied in detail.

Gauge/gravity correspondence. The Ricci-flat metric on the total space of canonical bundle of $\mathcal{M}_B^{[\lambda_1,\lambda_2]}$ allows us to write explicit exact $D3$-brane solution of Type IIB supergravity for each chosen manifold $\mathcal{M}_B^{[\lambda_1,\lambda_2]}$ (see **Appendix A**5.2.1).

The problem of the dual pair of theories is now reversed. We have, by construction, the exact classical solution of supergravity based on the Ricci flat metric. In order to find the other member of the pair, namely the corresponding 4-dimensional gauge theory, we should be able to find the corresponding quotient singularity $\mathbb{C}^3/\Gamma$ or its mass-deformation so as to derive the spectrum of the theory from a suitable quiver. This part of the problem is still unexplored.

Structure of the thesis. In the first part of the chapter 2, we begin by revising quotient singularities and the McKay correspondence in dimension three. We restrict ourselves to the case of abelian quotient singularities, which are toric singular varieties, and their crepant resolutions are toric Calabi-Yau manifolds. In subsection 2.3.2, we recalled the well-known toric description of Hirzebruch surfaces and specialized it to the case of the second Hirzebruch surface.

In the second part of chapter 2, we recall the Generalized Kronheimer construction. The McKay quiver associated with finite group $\Gamma \subset SU(3)$ singles out:

1. the gauge group $\mathcal{F}_\Gamma$,

2. The moment maps μ_I corresponding to each non-trivial irreducible representation,

3. the so-called level sets denoted by $\mu^{-1}(\zeta)$ for a vector ζ,

4. The crepant resolution $\mathcal{M}_\zeta$ for a genric ζ defined via the Kähler quotient.

5. A Kähler potential $\mathcal{K}_{\mathcal{M}_\zeta}$ on $\mathcal{M}_\zeta$.

In chapter 3, we recall the toric symplectic geometry, formulated by Abreu, Delzant, and Guillemin. We calculated a one-parametric family of Guillemin's symplectic potentials for Delzant polytopes $P_{a,a+2}$ associated with $\mathbb{F}_2$, where every given $P_{a,a+2}$ provides an embedding of $\mathbb{F}_2$ in $\mathbb{CP}^{2a+2}$.

chapter 4 and chapter 5 are based on [8].

Structure of chapter 4. We start by considering a generic Kähler potential $\mathcal{K}_{\mathcal{M}_B}(\varpi)$ on 4D compact Kähler manifolds $\mathcal{M}_B$ with isometry $SU(2) \times U(1)$. Using AMSY formalism, we get a generic symplectic potential of the form:

$$G_{\mathcal{M}_B} = G_0(\mathfrak{u}, \mathfrak{v}) \, + \, \mathcal{D}(\mathfrak{v})$$

where

$$G_0\left(\mathfrak{u}, \mathfrak{v}\right) = \left(\mathfrak{v} - \frac{\mathfrak{u}}{2}\right) \log(2\mathfrak{v} - \mathfrak{u}) + \frac{1}{2}\mathfrak{u} \log(\mathfrak{u}) - \frac{1}{2}\mathfrak{v} \log(\mathfrak{v})$$

and $\mathcal{D}(\mathfrak{v})$ is an arbitrary smooth function. We established the fact that the corresponding general form of the Riemannian metric is given by:

$$ds^2_{\mathcal{M}_B} = F(\mathfrak{v}) \left[d\phi(1 - \cos\theta) + d\tau\right]^2 + \frac{d\mathfrak{v}^2}{F(\mathfrak{v})} + \mathfrak{v} \underbrace{\left(d\phi^2 \sin^2\theta + d\theta^2\right)}_{S^2 \text{ metric}}$$

These metrics are parameterized by choosing a function $F(\mathfrak{v})$, and for a finite interval of $\mathfrak{v}$, this class of metrics represents metrics on $S^2 \times S^2$. These metrics also contain the Kronheimer metrics on $\mathbb{F}_2$ and the metric on the weighted projective plane $\mathbb{WP}[1,1,2]$ found in [7]. Using the KE condition, we find a 2-parametric class of functions $F(\mathfrak{v})$ such that the corresponding manifolds $\mathcal{M}_B$ are Kähler-Einstein. We summarize our findings in 1.1.

From subsection 4.5.1, we study the nature of the metrics given by the different choices of $F(\mathfrak{v})$. To do so, we first find the expressions for the Riemann tensor constructed in terms of three functions:

$$\begin{aligned}
\mathcal{CF}_1(\mathfrak{v}) &= F''(\mathfrak{v}) \quad ; \\
\mathcal{CF}_2(\mathfrak{v}) &= \frac{(\mathfrak{v}F'(\mathfrak{v}) - F(\mathfrak{v}))}{\mathfrak{v}^2} \quad ; \\
\mathcal{CF}_3(\mathfrak{v}) &= \frac{(\mathfrak{v} - F(\mathfrak{v}))}{\mathfrak{v}^2}
\end{aligned}$$

If these functions are regular in the interval $[\mathfrak{v}_{min}, \mathfrak{v}_{max}]$, the Riemann tensor is well-defined and finite, and the manifold $\mathcal{M}_B$ is a smooth compact manifold. Our analysis of the above functions for different choices of $F(\mathfrak{v})$ shows that the 2-parametric family of metrics in the KE case have conical singularities.

In subsection 4.5.2, we establish the complex structures on KE manifolds and the homeomorphism of these manifolds with $S^2 \times S^2$. In section 4.8, we describe the contact structure on 3-dimensional submanifold $\mathcal{M}_3$ by fixing $\mathfrak{v} = const.$ and choosing a Liouville vector field $\mathbf{L}$ to which all $\mathcal{M}_3$ are transversal for every fixed value of $\mathfrak{v}$.

We conclude chapter 4 by calculating different Geodesics for the family of manifolds $\mathcal{M}_B$.

Structure of chapter 5. Once we have found a new class of KE manifolds in chapter 4, our next step is to find the Ricci-flat metrics on $tot(K_{\mathcal{M}_B^{[\lambda_1,\lambda_2]}})$ by using Calabi's ansatz. This is done in chapter 5. We started by describing Calabi's ansatz for the 4D KE manifolds with $SU(2)\times U(1)$ isometries, and we found the consistency

$F^{\mathbb{F}_2}(\mathfrak{v}) = \frac{\left(1024\mathfrak{v}^2-81\alpha^2\right)(32\mathfrak{v}-9(3\alpha+4))}{16(81\alpha^2+1024\mathfrak{v}^2-576(3\alpha+4)\mathfrak{v})}$	$\mathfrak{v}_{min} = \frac{9\alpha}{32}$	$\mathfrak{v}_{max} = \frac{9}{32}(3\alpha+4)$	$\alpha > 0$
$F^{\mathbb{WP}[1,1,2]}(\mathfrak{v}) = \frac{\mathfrak{v}(8\mathfrak{v}-9)}{4\mathfrak{v}-9}$	$\mathfrak{v}_{min} = 0$	$\mathfrak{v}_{max} = \frac{9}{8}$	$\alpha = 0$
$F^{KE}(\mathfrak{v}) = -\frac{(\mathfrak{v}-\lambda_1)(\mathfrak{v}-\lambda_2)(\lambda_2\mathfrak{v}+\lambda_1(\lambda_2+\mathfrak{v}))}{\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)\mathfrak{v}}$	$\mathfrak{v}_{min} = \lambda_1$	$\mathfrak{v}_{max} = \lambda_2$	$0 < \lambda_1 < \lambda_2$
$F^{0}(\mathfrak{v}) = \frac{\mathfrak{v}(\lambda_2-\mathfrak{v})}{\lambda_2}$	$\mathfrak{v}_{min} = 0$	$\mathfrak{v}_{max} = \lambda_2$	$\lambda_2 > 0$
$F^{cone}(\mathfrak{v}) = \mathfrak{v}$	$\mathfrak{v}_{min} = 0$	$\mathfrak{v}_{max} = \infty$	

Table 1.1: Different possibilities for the function F.

conditions for the ansatz. Then we calculated the generalized form of the symplectic potential on $tot(K_{\mathcal{M}_B^{[\lambda_1,\lambda_2]}})$ with $SU(2)\times U(1)\times U(1)$ isometry via AMSY formulation. The generic form of the symplectic potential is:

$$G\left(\mathfrak{u},\mathfrak{v},\mathfrak{w}\right) \;=\; \underbrace{\left(\mathfrak{v}-\frac{\mathfrak{u}}{2}\right)\log(2\mathfrak{v}-\mathfrak{u})+\frac{1}{2}\mathfrak{u}\log(\mathfrak{u})\;-\;\frac{1}{2}\,\mathfrak{v}\log(\mathfrak{v})}_{\text{universal part } G_0(\mathfrak{u},\mathfrak{v})}+\;\underbrace{\mathcal{G}(\mathfrak{v},\mathfrak{w})}_{\text{variable part}}$$

where $\mathcal{G}(\mathfrak{v},\mathfrak{w})$ is a function of two variables that encode the specific structure of the metric. We write down an explicit expression in the case of our 4D KE manifolds, which takes the following form:

$$\begin{aligned}
G_{\mathcal{M}_T^{KE}}\left(\mathfrak{u},\mathfrak{v},\mathfrak{w}\right) \;&=\; G_0\left(\mathfrak{u},\mathfrak{v}\right)\,+\,\mathcal{G}^{KE}\left(\mathfrak{v},\mathfrak{w}\right)\\
G_0\left(\mathfrak{u},\mathfrak{v}\right) \;&=\; \left(\mathfrak{v}-\tfrac{\mathfrak{u}}{2}\right)\log[2\mathfrak{v}-\mathfrak{u}]+\tfrac{1}{2}\mathfrak{u}\log[\mathfrak{u}]-\tfrac{1}{2}\mathfrak{v}\log[\mathfrak{v}]\\
\mathcal{G}^{KE}\left(\mathfrak{v},\mathfrak{w}\right) \;&=\; \left(\tfrac{\kappa\mathfrak{w}}{2}+1\right)\mathcal{D}^{KE}\left(\tfrac{2\mathfrak{v}}{\kappa\mathfrak{w}+2}\right)-\tfrac{1}{2}\mathfrak{v}\log\left(\tfrac{\kappa\mathfrak{w}}{2}+1\right)+\tfrac{1}{2}\mathfrak{w}\log(\mathfrak{w})\\
&\quad+\frac{(\kappa\mathfrak{w}+3)\log(\kappa\mathfrak{w}(\kappa\mathfrak{w}+6)+12)}{2\kappa}+\frac{\sqrt{3}\arctan\left(\frac{\kappa\mathfrak{w}+3}{\sqrt{3}}\right)}{\kappa}
\end{aligned}$$

Some future directions. As in general, Calabi's ansatz only works when the base manifold is KE, a generalization of it is needed. There are different approaches adopted in [26, 31, 41] to study different specialized metrics on the vector bundles. But little is known to find the Ricci flat metrics explicitly beyond Calabi's ansatz. On the other hand, $tot(K_{\mathcal{M}})$ is always a noncompact Calabi-Yau manifold, hence one must be able to generalize Calabi's ansatz by considering nonKähler-Einstein metrics on $\mathcal{M}$. Calabi's potential

$$\mathcal{K}_E = \pi^*(\mathcal{K}_{\mathcal{M}}) + U(\lambda)$$

will always provide a Kähler metric on the total space $tot(E \longrightarrow \mathcal{M})$, and one needs to understand what other kinds of Kähler metrics on $\mathcal{M}$ would bring us to a nice Monge-Amperè equation. This has proven to be a hard problem as it is evident from the case of $tot(K_{\mathbb{F}_2})$. But it is essential for example to establish gauge/gravity correspondence in quiver gauge theories.

A possible direction is replacing KE metrics with extremal metrics introduced by Calabi[11, 12].

Sasakian structure. Sasakian geometry [48] (see **Appendix B**5.2.1) is a powerful tool to get explicit examples of Ricci-flat metrics. A Sasakian manifold S is an odd-dimensional Riemannian manifold with a given metric g such that the manifold $C(S) = \mathbb{R}_+ \times S$ together with the metric $\bar{g} = dr^2 + r^2 g$ is Kähler. A well-known result in Sasakian geometry says that $(C(S), \bar{g})$ is Calabi-Yau if and only if (S, g) is Einstein.

Given a holomorphic line bundle $L \longrightarrow \mathcal{M}$ where $\mathcal{M}$ is Kähler, we can always consider a $U(1)$-principal bundle over $\mathcal{M}$ inside L. The total space of such a principal bundle is an odd-dimensional Riemannian manifold. We can try to find an Einstein metric on it so that $tot(L)$ minus the zero section corresponds to the Sasakian cone over the odd-dimensional manifold with Ricci-flat metrics.

Appropriate Quiver Gauge theory for the class of 4D KE manifolds To complete gauge/gravity correspondence, we need the field contents on the gauge side of the theory. The generalized Kronheimer construction is a powerful tool to get those fields. It is not clear to us yet what kind of quivers we can associate with our manifolds to apply the generalized Kronheimer construction. Remember, away from the singularities both $tot(K_{\mathbb{F}_2})$ and $tot(K_{\mathcal{M}^{[\lambda_1,\lambda_2]}})$ are the same.

Chapter 2

Preliminaries

This chapter deals with the basic mathematical tools available in the literature upon which I am relying throughout this thesis.

2.1 Quotient Singularities

Quotient singularities are being studied since the 19th century in different aspects. Here we are interested in a very special kind of quotient singularity, namely, given a finite subgroup $\Gamma \subset SU(n)$, we consider the corresponding quotient variety $\mathbb{C}^n/\Gamma$ with an appropriate Γ-action on $\mathbb{C}^n$.

2.1.1 du Val Singularities

For $n = 2$, these singularities are classified and known as the du Val singularities or ADE-type of singularities. They were first studied by Felix Klein in the late 19th century and then by Patrick du Val in mid 20th century [22]. A modern treatment of du Val singularities can be seen in [42]. The purpose of this thesis is to study the nature of Kähler metrics on the resolved spaces of such singularities, hence I will just borrow the classification table in [42] and for more details, the reader is encouraged to have a look at Reid's article.
All the du Val singularities listed in the table 2.1 have only one isolated singularity

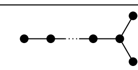
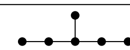
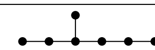
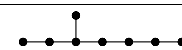
Name	Equation	Group	Dynkin Diagrams
A_r	$x^2 + y^2 + z^{r+1} = 0$	Cyclic $\mathbb{Z}/(r+1)$	A_r
D_r	$x^2 + y^2 z + z^{r-1} = 0$	Binary dihedral $BD_{4(r-2)}$	D_r 
E_6	$x^2 + y^3 + z^4 = 0$	Binary tetrahedral	E_6 
E_7	$x^2 + y^3 + yz^3 = 0$	Binary octahedral	E_7 
E_8	$x^2 + y^3 + z^5 = 0$	Binary icosahedral	E_8 

Table 2.1: du Val singularities

at the origin and a blow-up at the origin provides a resolution of singularities.
The key fact about du Val singularities is the existence of crepant resolutions (the word was introduced by M. Reid, as resolution has no discrepancy). The crepant resolution $\varphi : Y \to X$ is such that $K_Y = \varphi^* K_X$. For further details, we refer to [42, 43].

2.1.2 Canonical singularities

Before going into the higher dimensional version of du Val singularities, I think it is important to introduce the nature of singularities we are dealing with, namely the canonical singularities of a normal variety X. We say a normal variety X has **canonical singularities** if the canonical divisor K_X has an integral multiple mK_X for some positive integer m such that mK_X is Cartier and it has a resolution $\varphi : Y \to X$ with the family of exceptional divisors $\{E_i\}$ such that:

$$m\ K_Y = \varphi^*(m\ K_X) + \sum a_i\ E_i \qquad \text{with} \qquad a_i \geq 0 \tag{2.1.1}$$

The singularities are further called **terminal** if all a_i's are positive.
A very nice introduction to such singularities is given in [43]. The $\mathbb{Q}$-divisor $D = 1/m\ \sum a_i\ E_i$ is called the discrepancy of the resolution, and a resolution $\varphi : Y \to X$ is called crepant if $D = 0$, in which case $K_Y = \varphi^* K_X$. It means the resolved space Y is a Calabi-Yau variety. For the rest of this thesis, we will only be interested in

the crepant resolutions of normal varieties with canonical singularities.
In dimension 2, all the canonical singularities are classified and they are exactly du Val singularities listed in 2.1.

2.1.3 Canonical singularities in Higher dimensions:

The case of surface canonical singularities has been under the microscope for almost two centuries and they are well understood. The higher dimensional case is nowadays a very active field of research from both algebraic and differential geometric points of view. From the algebraic point of view, we now know that even the existence of crepant resolutions of quotient singularities $\mathbb{C}^n/\Gamma$ is not guaranteed for $n \geq 4$. In dimension 3, the crepant resolutions do exist thanks to the case-by-case analysis, but they need not be unique.

In this thesis, we will be interested in rather a very special kind of quotient singularities known as abelian quotient singularities in dimension 3. Here, the finite group $\Gamma \subset SU(3)$ is abelian. It has been known that all abelian quotient singular spaces for $\Gamma \subset SU(n)$ are toric varieties. Thanks to this, the study of the nature of such varieties becomes much simpler than the non-abelian cases because of the associated toric data which encodes the information of all geometric properties of the varieties. Before going to a brief introduction to toric varieties, we would like to describe a useful correspondence between the representation of the group Γ and the homology of the crepant resolutions.

2.2 McKay-Correspondence:

In 1980, John McKay [39] identified a correspondence between the McKay quiver associated with a finite subgroup $\Gamma \subset SU(2)$ and the extended Dynkin diagram associated with the root system of ADE-type. This led to the discovery of McKay-correspondence between the homology of the crepant resolution Y and the irreducible representations of the group Γ in higher dimensions, studied by A. Craw, Y. Ito, M. Ried, and many others (see [33, 44, 18]). It is appropriate to introduce McKay quivers here before describing the McKay-correspondence.

Let Γ be a finite group and V be an n-dimensional complex vector space. Let $\rho : \Gamma \to GL(n, V)$ be a given representation of Γ. Further, let $\{\rho_i\}$ be the irreducible representations of Γ. The tensor product of the representation ρ with irreducible representations satisfies the following decomposition

$$V \otimes V_i \cong \bigoplus_j a_{ij} \cdot V_j \tag{2.2.1}$$

for each i and $a_{ij} \in \mathbb{N}$. Here V_i is the respected vector space for the irreducible representation ρ_i for every i. The **McKay quiver** associated with Γ is a quiver that consists of the following data:

Vertices: Irreducible representations ρ_i
Edges: a_{ij} edges between ith and jth vertex

McKay in [39] gave a correspondence between the McKay quiver associated with finite subgroup $\Gamma \subset SU(2)$ with its natural representation $V \cong \mathbb{C}^2$ and the extended ADE-Dynkin diagrams including the trivial representation, which provides the following one-to-one correspondence

$$\{\text{irreducible representations of } \Gamma\} \leftrightarrow \text{basis of } H_*(Y, \mathbb{Z}) \tag{2.2.2}$$

where Y is the crepant resolution of $\mathbb{C}^2/\Gamma$. This led to the following conjecture in every dimension back in the mid-'90s by Ito and Reid [33].

Conjecture 2.2.1. *Let $\Gamma \subset SL(n, \mathbb{C})$ is a finite subgroup and $X = \mathbb{C}^n/\Gamma$ the quotient space. Let $\varphi : Y \to X$ be a crepant resolution of X then there is a bijective correspondence*

$$\{\textit{algebraic basis of } H^*(Y, \mathbb{Q})\} \leftrightarrow \{\textit{Conjugacy classes of } \Gamma\} \tag{2.2.3}$$

They proved the conjecture for $n = 3$. A stronger version of the conjecture was

given by Batyrev and Dais in [2] soon after the Ito-Reid conjecture. This new version is now known as strong McKay correspondence. We first define here the notion of **age** associated with every element $g \in \Gamma$ introduced by Reid.
Since every $g \in \Gamma$ is conjugate to a diagonal matrix

$$g = \begin{pmatrix} e^{2\pi i a_1(g)/o(g)} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{2\pi i a_n(g)/o(g)} \end{pmatrix}$$

where $o(g)$ is the order of g and $0 \leq a_j(g) < o(g)$. Then the **age** of $g \in \Gamma$ is

$$age(g) = \frac{1}{o(g)} \sum_{j=1}^{n} a_j(g) \tag{2.2.4}$$

Age 1 elements of Γ are called the **junior elements** and the conjugacy classes associated with them are called the junior conjugacy classes.

Conjecture 2.2.2. *Let $\varphi : Y \to X$ a crepant resolution, then*

$$dim_{\mathbb{C}} H^{2k}(Y, \mathbb{C}) = \#\{age\ k\ conjugacy\ classes\ of\ \Gamma\} \tag{2.2.5}$$

Batyrev and Dais proved this in the abelian case. In particular, in abelian case, we have the following very useful corollary:

Corollary 2.2.3.

$$dim_{\mathbb{C}} H^{2}(Y, \mathbb{C}) = \#\{junior\ conjugacy\ classes\ of\ \Gamma\} \tag{2.2.6}$$

$$\{exceptional\ divisors\ of\ \varphi : Y \to X\} \leftrightarrow \{junior\ conjugacy\ classes\ of\ \Gamma\} \tag{2.2.7}$$

Since this thesis deals with differential geometric questions associated with crepant resolutions, we do not dig deeply into the beautiful algebraic geometric side, which on its own, is a very much active field of research.

2.3 Toric Calabi-Yau manifolds

We are interested in studying the abelian quotient spaces and their crepant resolutions, which are all toric spaces. So we give below a brief introduction to the toric varieties. A very beautiful treatment of the subject is given in Cox-Little-Schenck's book [17].

2.3.1 Toric Recipe

Definition 2.3.1. *An affine variety X is called toric if there exists a dense open subset $T \subseteq X$ which is isomorphic to algebraic torus $(\mathbb{C}^*)^n$ and the action of the torus on itself can be extended to an action on the entire affine variety X.*

Normal Toric varieties can be studied by associating combinatorial data encoded in a "fan" which consists of "strongly convex rational polyhedral cones" in $N_{\mathbb{R}} \cong N \otimes_{\mathbb{Z}} \mathbb{R}$. Here the lattice $N \cong \mathbb{Z}^n$ is the family of one-parameter subgroups of the algebraic torus $T \subseteq X$. Its dual is denoted by M and it consists of characters associated with T *i.e.* group morphisms $\chi : T \to \mathbb{C}^*$. In practice, to create toric varieties we start by fixing a lattice and then associate with it some combinatorial data which leads us to a variety or vice versa.

Definition 2.3.2. *Let $N \cong \mathbb{Z}^n$ be a lattice than a* ***rational polyhedral cone*** *is a convex hull of a finite set $S \subset N$, i.e*

$$\sigma = cone(S) = \{\sum \lambda_i u_i | u_i \in S,\ \lambda_i \geq 0\} \subseteq N_{\mathbb{R}} = N \otimes \mathbb{R} \cong \mathbb{R}^n$$

We call $M = Hom(N, \mathbb{Z}) \cong \mathbb{Z}^n$ the dual lattice of N and we define the **dual cone** of σ in the following way

$$\sigma^\vee = \{m \in M_{\mathbb{R}} | \langle m, u\rangle \geq 0\ \forall\ u \in \sigma\} \subseteq M_{\mathbb{R}} = M \otimes \mathbb{R} \cong \mathbb{R}^n$$

where $\langle \cdot, \cdot \rangle$ is the natural bilinear pairing.
The **dimension** of the cone σ is equal to the dimension of the real vector space

spanned by σ. We say a rational polyhedral cone σ is **strongly convex** if the dimension of $\sigma^\vee$ is n.
Given a strongly convex rational polyhedral cone σ, $\mathbb{C}[\sigma^\vee \cap M]$ happens to be a finitely generated $\mathbb{C}-$algebra.

Definition 2.3.3. *Let σ be a strongly convex rational polyhedral cone, then an* ***affine toric variety*** *X_σ is defined as*

$$X_\sigma = Spec(\mathbb{C}[^\vee \cap M]) \tag{2.3.1}$$

The fact that σ is strongly convex implies that the corresponding affine toric variety X_σ is **normal**. Since we are only concerned with normal toric varieties, for us σ will always be strongly convex, and from now on we refer to it just as a cone.

Definition 2.3.4. *A fan $\Sigma \subset N_\mathbb{R}$ is a finite collection of cones such that:*

a) the intersection $\sigma_1 \cap \sigma_2$ of two cones in Σ is a face of both cones, and Σ is closed under the intersection operation;

b) every face of $\sigma \in \Sigma$ is also in Σ.

Given a fan Σ we construct an abstract normal toric variety X_Σ in the following way: To every cone $\sigma \in \Sigma$ we consider the affine normal piece X_σ and if $\tau = \sigma_1 \cap \sigma_2$ is a common face then we glue the two affine pieces X_{σ_1} and X_{σ_2} by identifying along subvariety X_τ.

The smoothness of the variety X_σ (or X_Σ) comes from the fact whether the generating vectors of σ form part of an integral basis for the lattice N or not. If they do then X_σ is smooth. Otherwise in order to resolve the singularity we perform a well-known surgical technique in toric geometry called the **barycentric subdivision**, where we add a new ray generated by an integral point in such a way that the new fan Σ_{res} consists of new cones which are smooth and so is the variety $X_{\Sigma_{res}}$. This technique of resolving the toric singularities makes the study of toric varieties very useful.

The dense open subset T in the toric variety X is a group and the group action extends to the whole X which allows us to look for the orbits under this action. A well-known result in toric geometry gives a one-to-one correspondence between the cones in the toric fan Σ and the orbits.

$$\{r-\dim \text{ cones } \sigma \in \Sigma\} \leftrightarrow \{(n-r)-\dim \text{ orbits in } X\} \tag{2.3.2}$$

This correspondence is called "**Orbit-Cone correspondence**". In particular, one-dimensional cones called edges give us codimension 1 orbit. Their closures are called the **toric divisors**. $n-$dimensional cones correspond to zero-dimensional orbits and we call them **toric points** of X.

Theorem 2.3.1. *Let X_Σ be a toric variety with fan Σ. The **canonical divisor** of X_Σ is given by*

$$K_{X_\Sigma} = -\sum_{\rho\in\Sigma(1)} D_\rho \tag{2.3.3}$$

where $\Sigma(1)$ consists of all one-dimensional cones in Σ.

The total space of the canonical bundle over a toric variety is itself a toric variety and the fan Σ_{can} associated with $L_{can} = tot(K_{X_\Sigma} \to X_\Sigma)$ is defined in the following way:
Let $\Sigma(n)$ be the set of $n-$dim cones in Σ. We define new cones $\sigma_{can} = span(\sigma(1) \times \{1\}, (0, 0, \cdots, 1))$ corresponding to each $\sigma \in \Sigma(n)$. Then Σ_{can} is the fan associated with L_{can} whose $(n+1)-$dim cones are σ_{can}.
Let $\mathcal{H}_\Sigma$ be the set of ray generators of Σ, *i.e.* for every $\rho \in \Sigma(1)$ there exists a $u \in \mathcal{H}_\Sigma$ such that $\rho = span(u)$. The Calabi-Yau condition on toric varieties is given as:

Theorem 2.3.2. *A toric variety X_Σ is **Calabi-Yau** if $\mathcal{H}_\Sigma$ is contained in a hyperplane in $\mathbb{R}^n$.*

Remark 2.3.3. *Note that L_{can} is Calabi-Yau.*

2.3.2 Toric description of Hirzebruch Surfaces

One of the important classes of toric surfaces is Hirzebruch surfaces denoted by $\mathbb{F}_r$ for $r \in \mathbb{N}$. The following result about toric surfaces in [17] classifies all possible smooth compact toric surfaces.

Theorem 2.3.4. *Every smooth compact toric surface X_Σ is obtained from one of the following*

$$\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } \mathbb{F}_r \text{ for } r \geq 2$$

by a finite sequence of blowups at fixed points of the torus action.

In order to study the geometry of smooth compact toric surfaces it is enough to understand the toric data associated with the toric surfaces in the list above theorem. For the purpose of this thesis, we are mainly interested in Hirzebruch surfaces. So it is only appropriate to describe the toric geometry of such surfaces.
The fan Σ_r associated with $\mathbb{F}_r$ is shown below. Note that the fan consists of four

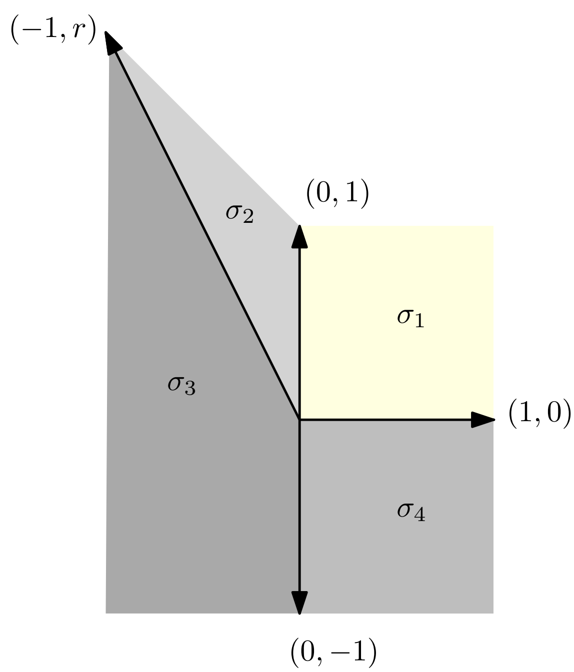


Figure 2.1: Σ_r associated with $\mathbb{F}_r$

2−dim cones $\sigma_1, \sigma_2, \sigma_3$ and σ_4 and four 1−dim edges generated by the integral vectors $u_1 = (1,0), u_2 = (0,1), u_3 = (-1,r)$ and $u_4 = (0,-1)$. By orbit-cone correspondence, we can associate toric divisors D_1, D_2, D_3 and D_4 corresponding to each u_i. An elementary calculation shows us that the class group of $\mathbb{F}_r$ is isomorphic to $\mathbb{Z}^2$ and it is generated by the classes $[D_2]$ and $[D_3]$. Since $\mathbb{F}_r$ are smooth toric varieties hence the class group is also the Picard group of $\mathbb{F}_r$. The canonical divisor associated with $\mathbb{F}_r$ is then $-(D_1 + D_2 + D_3 + D_4)$.

2.3.3 Second Hirzebruch surface

If we take $r = 2$, the corresponding Hirzebruch surface $\mathbb{F}_2$ is called the second Hirzebruch surface. The toric fan consists of four two-dimensional cones as shown in figure 2.1. The generators of the corresponding cones are

$$u_1 = (1,0),\ u_2 = (0,1),\ u_3 = (-1,2) \text{ and } u_4 = (0,-1)$$

Among these generators note that $u_1,\ u_2,$ and u_3 lie on the same line. This fact implies that the corresponding 3-dimensional fan for the canonical bundle can be described as a planar triangular graph 2.2 An interesting fact about the crepant resolutions discovered by Craw [18] is that only those smooth toric varieties can be crepant resolutions of some abelian quotient singularities in dimension three, whose three-dimensional fan can be described as a planar triangulated graph. $\mathbb{F}_2$ is the only Hirzebruch surface whose corresponding fan for the canonical bundle can be described as a triangulated graph. All the other fans of canonical bundles of Hirzebruch surfaces can only be described by quadrilateral planar graphs. This makes $\mathbb{F}_2$ more interesting at least in this setup.
In fact, it has been shown in [7] that the total space of the canonical bundle of $\mathbb{F}_2$ is the unique crepant resolution of $\mathbb{C}^3/\mathbb{Z}_4$ quotient singular variety, where the action of $\mathbb{Z}_4$ on $\mathbb{C}^3$ is given by:

$$\omega \cdot (z_1, z_2, z_3) = (\omega\, z_1, \omega\, z_2, \omega^2 z_3) \quad \text{for } \omega \in \mathbb{Z}_4 \tag{2.3.4}$$

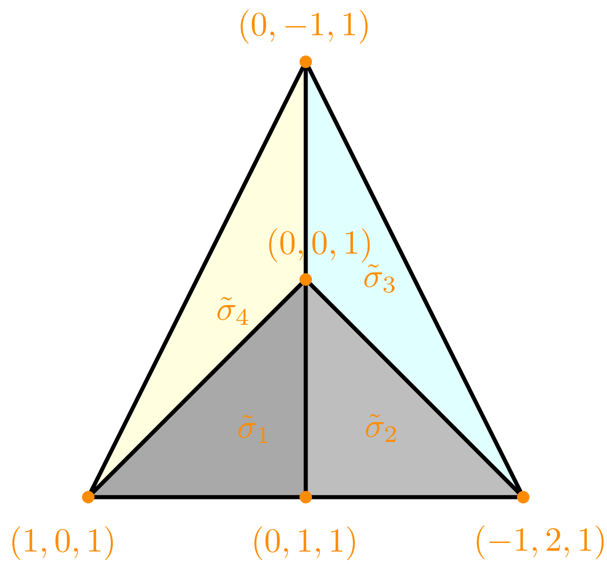


Figure 2.2: Planar triangulated graph of the fan of Canonical bundle of $\mathbb{F}_2$

In this thesis, we were initially interested in a Ricci-flat metric on the total space of the canonical bundle of $\mathbb{F}_2$. In chapter 4 we will describe a new family of Kähler-Einstein spaces which topologically are homeomorphic to $\mathbb{F}_2$ and in chapter 5 we will describe a way to put Ricci-flat metrics on the canonical bundle of those spaces. This is mainly because the only known ansatz to do so as far I know is due to Calabi [10]. In Calabi's ansatz, whenever the base manifold is Kähler-Einstein, the Monge-Ampère equation for the Ricci-flat metric becomes an ODE with constant coefficients. On the other hand, It is a well-known fact due to Tian that $\mathbb{F}_2$ is not Kähler-Einstein, and the Ricci-flat condition on the total space of canonical bundle of $\mathbb{F}_2$ happens to be a non-linear second-order PDE with coefficients depending upon a real variable coming from $\mathbb{F}_2$.

2.4 Generalized Kronheimer Construction:

In 1989, Kronheimer in [37] developed an ansatz to find explicit ALE (asymptotically locally euclidean)-hyperKähler metrics on the resolution of du Val singular varieties.

This ansatz is known as the Kronheimer construction. In [7], a generalized version of Kronheimer construction was described in an attempt to find the Ricci-flat metric on the crepant resolution in dimension three. It turns out that the generalized Kronheimer construction fails to provide a canonical Ricci-flat metric impose by the ansatz. Nevertheless, the construction is still vital to understand the kind of isometries we want to impose on our required metric. As will be the case in our situation, where we are looking at Ricci-flat metrics on the canonical bundles of two complex dimensional Kähler manifolds $\mathcal{M}_B$ with isometry $SU(2) \times U(1) \times U(1)$.

It turns out that the Kronheimer construction provides one parametric family of Kähler metrics on $\mathbb{F}_2$ with the isometry $SU(2) \times U(1)$ in the case of $\mathbb{C}^3/Z_4$ singular variety. This one-parametric family comes by restricting the Kronheimer metric on the total space of the canonical bundle of $\mathbb{F}_2$ to $\mathbb{F}_2$ (see [7]). We will come to this point later. First, let me start by defining ALE and QALE-metrics first.

Definition 2.4.1. *Let $(\mathcal{M}, g)$ be a non-compact Riemannian manifold of dimension n. Let $\Gamma \subset SO(n)$ be a finite subgroup that acts freely on $\mathbb{R}^n \setminus \{0\}$. We say $(\mathcal{M}, g)$ is an ALE-manifold asymptotic to $\mathbb{R}^n/\Gamma$ if*

i) there $\exists$ a compact subset $\mathcal{D}_c \subset \mathcal{M}$, a map $\pi : \mathcal{M} \setminus \mathcal{D}_c \longrightarrow \mathbb{R}^n/\Gamma$ and a map $\mathfrak{r} : \mathbb{R}^n/\Gamma \longrightarrow \mathbb{R}_+$ such that

$$\mathcal{M} \setminus \mathcal{D}_c \simeq_{diff} \{x \in \mathbb{R}^n/\Gamma \mid \mathfrak{r}(x) > R\} \tag{2.4.1}$$

for some fixed $R > 0$;

ii) The push-forward metric $\pi_*(g)$ satisfies

$$\nabla^k(\pi_*(g) - h) = O(\mathfrak{r}^{-m-k}) \tag{2.4.2}$$

where $k \geq 0$.
Here ∇ is the Levi-Civita connection of Euclidean metric h on $\mathbb{R}^n$.

The Eguchi-Hanson metric [23] on the resolution of the A_1 du Val singular variety is among the first examples of Ricci-flat ALE-metric. Kronheimer metrics on the resolution spaces of all du Val singular spaces are also ALE. In [34], Joyce proved the existence of Ricci-flat ALE metrics on the crepant resolutions of isolated quotient singularities. In [35], he confirmed the existence of Ricci-flat metrics on the crepant resolutions of non-isolated quotient singularities. Although Joyce has proved its existence, there is no proper technique to get explicit metrics other than Calabi's ansatz. The precise definition of $QALE$-metrics is a bit longer and we refer the reader to [35].

2.4.1 Generalized Kronheimer Construction of Crepant resolutions:

To describe the generalized Kronheimer construction (see for more details [6, 7]) we follow the following steps:

McKay quiver associated with $\Gamma \subset SU(n)$: We start by associating McKay quiver $\mathcal{Q}_\Gamma$:

- Let $\mathcal{N}$ be the $n-$dimensional complex representation of $\Gamma \subset SU(n)$;
- Let $\{\mathcal{N}_i\}_{i=1}^{s+1}$ be the $s+1$ irreducible representations of Γ associated with each conjugacy class;
- Define
$$\mathcal{N} \otimes \mathcal{N}_i = \bigoplus_{j=1}^{s+1} Q_{ij}\mathcal{N}_j \quad \text{for each } \; i = 1, \cdots, s+1 \tag{2.4.3}$$
via the Schur lemma;
- each vertex of $\mathcal{Q}$ is labeled by $\mathcal{N}_i$ and a number n_i, where n_i is the dimension of the irreducible representation $\mathcal{N}_i$. Q_{ij} is the number of arrows from vertex i to j.

By this, we can associate a McKay quiver to any $\Gamma \subset SU(n)$. When Γ is abelian, we have exactly $\mid \Gamma \mid$ $1-$dimensional irreducible representations and we drop the integers n_i to each vertex with the understanding that it is 1 at every vertex. Figure 2.3 is the McKay quiver associated with $\mathbb{Z}_4$. Note that

$$Q = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \end{pmatrix}$$

is the matrix of (2.4.3)

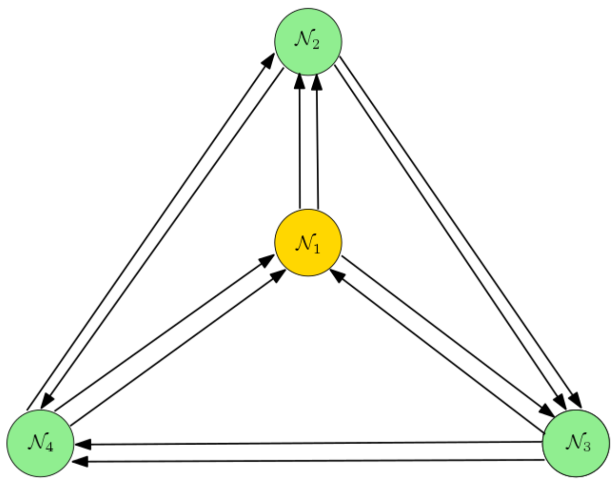


Figure 2.3: McKay quiver $\mathcal{Q}_{\mathbb{Z}_4}$ associated to $\mathbb{Z}_4$. Here the central vertex $\mathcal{N}_1$ denotes the trivial representation.

Generalized Kronheimer construction in dimension three: The detailed description of generalized Kronheimer construction is a bit longer (see [6]). Here I will try to summarize the key ingredients without much explanation. We need the following data to get the crepant resolution Y corresponding to quotient singular space $\mathbb{C}^3/\Gamma$ with a bonafide Kähler metric g_Y:

- Let $\mathcal{R}$ be the regular representation of Γ and define

$$\mathcal{P}_\Gamma = \mathrm{Hom}(\mathcal{R}, \mathcal{N} \otimes \mathcal{R}) = \mathcal{N} \otimes \mathrm{Hom}(\mathcal{R}, \mathcal{R}) \tag{2.4.4}$$

an element $p \in \mathcal{P}_\Gamma$ is given by

$$p = \begin{pmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \mathcal{A}_3 \end{pmatrix} \tag{2.4.5}$$

where $\mathcal{A}_i$ is $\mid \Gamma \mid \times \mid \Gamma \mid$ matrix for $i = 1, 2, 3$;

- The group Γ acts on $\mathcal{P}_\Gamma$ in the following way

$$\forall\ \gamma \in \Gamma : \quad \gamma\ \cdot\ p = \mathcal{N}(\gamma) \begin{pmatrix} \mathcal{R}(\gamma)\mathcal{A}_1\mathcal{R}(\gamma^{-1}) \\ \mathcal{R}(\gamma)\mathcal{A}_2\mathcal{R}(\gamma^{-1}) \\ \mathcal{R}(\gamma)\mathcal{A}_3\mathcal{R}(\gamma^{-1}) \end{pmatrix} \tag{2.4.6}$$

and consider

$$\mathcal{P}_\Gamma^\Gamma = \left\{ p \in \mathcal{P}_\Gamma \ \mid\ \forall\, \gamma \in \Gamma,\ \gamma \cdot p = p \right\} \tag{2.4.7}$$

be the invariant subspace of $\mathcal{P}_\Gamma$. It has a complex dimension $3 \mid \Gamma \mid$. In the generalized Kronheimer construction, we are mainly interested in different subspaces within $\mathcal{P}_\Gamma^\Gamma$;

- A $\mid \Gamma \mid \times \mid \Gamma \mid$ matrix A which commutes with $\mathcal{R}(\gamma)$ for every $\gamma \in \Gamma$ gives us an automorphism of $\mathcal{P}_\Gamma^\Gamma$ and we denote by $\mathcal{G}_\Gamma$ the group of such automorphisms of $\mathcal{P}_\Gamma^\Gamma$. It can also be realized as

$$\mathcal{G}_\Gamma = \bigotimes_{i=1}^{s+1} GL(n_i, \mathbb{C}) \bigcap SL(\mid \Gamma \mid, \mathbb{C}) \tag{2.4.8}$$

;

- We are interested in Kähler structures so we can further restrict ourselves to the possible group of isometries of the Kähler structure on $\mathcal{P}_\Gamma^\Gamma$, *i.e.*

$$\mathcal{F}_\Gamma = \bigotimes_{i=1}^{s+1} U(n_i) \bigcap SU(|\Gamma|) \subset \mathcal{G}_\Gamma \tag{2.4.9}$$

- We associate two subspaces denoted by $\mathcal{D}_\Gamma$ and $\mathcal{L}_\Gamma$ in $\mathcal{P}_\Gamma^\Gamma$ as following

$$\mathcal{D}_\Gamma = \{p \in \mathcal{P}_\Gamma^\Gamma \mid [\mathcal{A}_1, \mathcal{A}_2] = [\mathcal{A}_2, \mathcal{A}_3] = [\mathcal{A}_3, \mathcal{A}_1] = 0\} \tag{2.4.10}$$

$$\begin{aligned} \mathcal{L}_\Gamma = \{p \in \mathcal{P}_\Gamma^\Gamma \mid \mathcal{A}_i \text{ for every } i = 1, 2, 3 \text{ is diagonal} \\ \text{in the natural basis of } \mathcal{R}\} \end{aligned} \tag{2.4.11}$$

  In fact, $\mathcal{L}_\Gamma$ is isomorphic to the singular variety $\mathbb{C}^3/\Gamma$ and $\mathcal{D}_\Gamma$ is invariant under the action of $\mathcal{F}_\Gamma$;

- We define a Kähler potential on $\mathcal{P}_\Gamma^\Gamma$ by

$$\mathcal{K}_{\mathcal{P}_\Gamma^\Gamma} = \mathrm{Tr}(\mathcal{A}_1^* \mathcal{A}_1) + \mathrm{Tr}(\mathcal{A}_2^* \mathcal{A}_2) + \mathrm{Tr}(\mathcal{A}_3^* \mathcal{A}_3) \tag{2.4.12}$$

  where $\mathcal{A}^*$ is the conjugate transpose of the matrix $\mathcal{A}$;

- Corresponding to the action of $\mathcal{F}_\Gamma$ on $\mathcal{D}_\Gamma$, we can associate s number of real moment maps μ_I corresponding to each non-trivial irreducible representation of Γ and we can define level sets denoted by $\mu^{-1}(\zeta) \subset \mathcal{D}_\Gamma$ for any vector $\zeta = (\zeta^2, \cdots, \zeta^{s+1})$;

- We define the spaces

$$\mathcal{M}_\zeta = \mu^{-1}(\zeta)//\mathcal{F}_\Gamma \tag{2.4.13}$$

  via the Kähler quotient construction. For generic values of ζ, $\mathcal{M}_\zeta$ are smooth 3-complex dimensional manifolds and for $\zeta = 0$ we get our singular variety back, *i.e.* for a generic value we can regard $\mathcal{M}_\zeta$ as a crepant resolution of our quotient singular variety $\mathbb{C}^3/\Gamma$. Furthermore, under this quotient construction, note that $\mu^{-1}(\zeta) \longrightarrow \mathcal{M}_\zeta$ can be regarded as $\mathcal{F}_\Gamma$ principal line bundle;

- The $\mathcal{F}_\Gamma$ principal line bundle $\mu^{-1}(\zeta) \longrightarrow \mathcal{M}_\zeta$ induces holomorphic vector bundles of rank n_i for $i = 2, \cdots, s+1$, and one can associate hermitian fiber metrics

$\mathfrak{h}_i$ for every bundle and we define a Kähler potential on $\mathcal{M}_\zeta$ as

$$\mathcal{K}_{\mathcal{M}_\zeta} = \mathcal{K}_{\mathcal{P}^\Gamma_\Gamma} \mid_{\mu^{-1}(\zeta)} + \zeta^i M_{ij} log\big(Det[h_j]\big) \tag{2.4.14}$$

where M is a constant $s \times s$ matrix.

For a much more detailed description see [6, 7]. In [7], following the generalized Kronheimer construction, the authors found a Kähler metric on the crepant resolution of $X = \mathbb{C}^3/\mathbb{Z}_4$ quotient space. It turns out that the metric was not the desired Ricci-flat as was the case in classical Kronheimer construction but still it provided two fruitful results namely:

a) The process provided a Ricci-flat metric on a partial resolution of X;

b) Since the full resolution of X is the total space of the canonical bundle of the second Hirzebruch surface, the Kronheimer metric when restricted to $\mathbb{F}_2$ provided a one-parametric family of potentials on $\mathbb{F}_2$.

I will come to these last two findings in chapter 4.

Chapter 3

Toric Symplectic Geometry

In this chapter, we will review the symplectic geometry of the toric manifolds developed by Delzant, Guillemin, and Abreu in [1, 21, 30]. In the last section, we describe Guillemins's symplectic potential on the second Hirzebruch surface.

3.1 Delzant Correspondence

Delzant in [21] provided a wonderful one-to-one correspondence between symplectic toric manifolds and so-called Delzant polytopes up to symplectomorphisms, *i.e.*

$$\begin{gathered}\big\{\text{Compact toric symplectic manifolds up to symplectomorphisms}\big\}\\ \updownarrow \\ \big\{\text{Delzant polytopes}\big\}\end{gathered}$$

Definition 3.1.1. *A symplectic toric manifold of real $2n$ dimension is a symplectic manifold $(\mathcal{M}^{2n}, \omega)$ together with an effective Hamiltonian action*

$$\bullet : T^n \times \mathcal{M} \longrightarrow \mathcal{M} \tag{3.1.1}$$

of the real torus $T^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$.

Definition 3.1.2. *A convex polytope $P \subset \mathbb{R}^n$ is called **Delzant** if:*

I) at each vertex v of P exactly n edges meet;

II) every edge meeting at vertex v is rational in the sense that the line containing the edge and passing through v has the form $v + t\,u_i$ for some $u_i \in \mathbb{Z}^n$ and $t \in [0, \infty)$;

III) the vectors $u_1, \cdots, u_n$ in (II) form a basis of $\mathbb{Z}^n$.

Delzant associated with every such polytope P a closed connected symplectic manifold $(\mathcal{M}_P, \omega_P, \bullet_P)$ of dimension $2n$ together with a Hamiltonian T^n-action denoted by $\bullet_P$, and proved the following:

Theorem 3.1.1. *Let $(\mathcal{M}^{2n}, \omega)$ be a compact, connected symplectic manifold along with an effective torus action of real torus T^n. Let $\mu : \mathcal{M} \longrightarrow \mathbb{R}^n$ be the corresponding moment map then:*

- *the image of μ is a Delzant polytope denoted by P, i.e.*

$$\mu(\mathcal{M}) = P$$

- *$(\mathcal{M}, \omega, \bullet)$ is equivariantly symplectomorphic to $(\mathcal{M}_P, \omega_P, \bullet_P)$.*

3.2 Kähler and symplectic geometry

Let

$$\mathcal{M}^\circ = \big\{x \in \mathcal{M} \ \mid \text{the } T^n - \text{action is free at } \ x\big\}$$

It is a dense open subset of $\mathcal{M}$. Let us consider the complex holomorphic coordinates on $\mathcal{M}^\circ$ in action-angle form:

$$z_j = e^{u_j + i\Theta_j}, \quad \text{for } \ j = 1, \cdots, n \tag{3.2.1}$$

we can realize $\mathcal{M}^\circ$ as $\mathbb{R}^n \times iT^n$ space. The action of T^n on $\mathcal{M}^\circ$ is given by

$$g \bullet z = \big(e^{u_1 + i(\Theta_1 + \phi_1)}, \cdots, e^{u_n + i(\Theta_n + \phi_n)}\big)$$

where $g = (e^{i\phi_1}, \cdots, e^{i\phi_n}) \in T^n$.

T^n-invariant Kähler potential. Let $\mathcal{K}_{\mathcal{M}}$ be a Kähler potential on $\mathcal{M}^\circ$ such that $\omega = i\partial\bar{\partial}\mathcal{K}_{\mathcal{M}}$. We need ω to be invariant under the T^n action so the Kähler potential $\mathcal{K}_{\mathcal{M}}$ must only depend on the real coordinates $u = \{u_1, \cdots, u_n\}$. We denote by

$$\mathcal{F} = Hess(K_{\mathcal{M}}) \tag{3.2.2}$$

the Hessian of $\mathcal{K}_{\mathcal{M}}$ with respect to the coordinates u. The corresponding Riemannian metric is given by:

$$g = \left(\begin{array}{c|c} \mathcal{F} & \mathbf{0} \\ \hline \mathbf{0} & \mathcal{F} \end{array}\right) \tag{3.2.3}$$

Legendre transform and symplectic potential. The moment map $\mu : \mathcal{M} \longrightarrow \mathbb{R}^n$ is of the form $(\mu_1, \cdots, \mu_n)$, where,

$$\mu_j = \frac{\partial \mathcal{K}_{\mathcal{M}}}{\partial u_j} \tag{3.2.4}$$

Via Delzant correspondence theorem, we have $\mu(\mathcal{M}) = P$ and we have the following identification:

$$\mathcal{M}^\circ \cong P^\circ \times T^n \tag{3.2.5}$$

Here, P° is the interior of the Delzant polytope P. Geometrically, every point in the interior of the Delzant polytope can be thought of as a copy of the torus T^n. In general, a point in the interior of an r-dimensional face of the polytope can be thought of as a copy of r dimensional orbit under T^n action. Vertices of P are regarded as the fixed points under T^n action (see figure 3.1).

Via the Legendre transform we consider the symplectic coordinates $\{\mathfrak{u}_1, \cdots, \mathfrak{u}_n; \Theta_1, \cdots, \Theta_n\}$, where we set:

$$\mathfrak{u}_j = \frac{\partial K_{\mathcal{M}}}{\partial u_j} \tag{3.2.6}$$

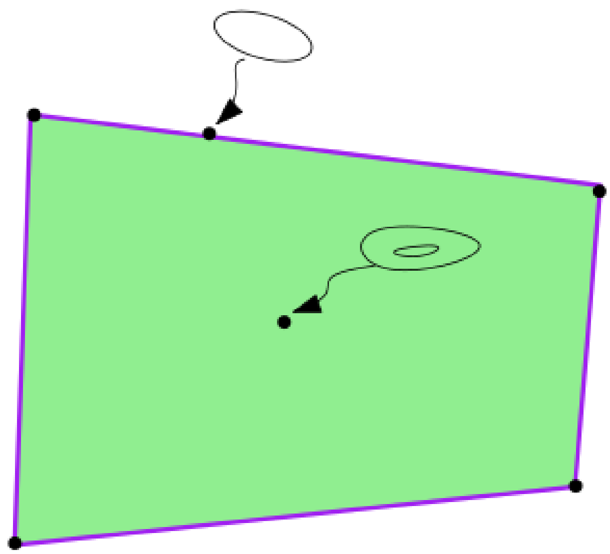


Figure 3.1: A generic Delzant polytope in dimension 2.

Under this change of coordinates, the Kähler form $\omega = i\partial\bar{\partial}\mathcal{K}_{\mathcal{M}}$ becomes

$$\omega = d\mathfrak{u}_j \wedge d\Theta_j \tag{3.2.7}$$

The Riemannian metric g takes the form

$$g = \left(\begin{array}{c|c} \mathcal{G} & \mathbf{0} \\ \hline \mathbf{0} & \mathcal{G}^{-1} \end{array}\right) \tag{3.2.8}$$

where $\mathcal{G} = Hess(\mathcal{G}_{\mathcal{M}})$ with respect to the real coordinates $\mathfrak{u}$ for a smooth function $\mathcal{G}_{\mathcal{M}}$ defined on P°, and we call $\mathcal{G}_{\mathcal{M}}$ the symplectic potential.

The Kähler and symplectic potentials are in fact Legendre dual to each other, *i.e.*

$$\mathcal{K}_{\mathcal{M}} + \mathcal{G}_{\mathcal{M}} = \sum_{j=1}^{n} \frac{\partial \mathcal{K}_{\mathcal{M}}}{\partial u_j} \frac{\partial \mathcal{G}_{\mathcal{M}}}{\partial \mathfrak{u}_j} \tag{3.2.9}$$

and the corresponding Hessians $\mathcal{F}$ at u and $\mathcal{G}$ at $\mathfrak{u}$ are inverse to each other.

3.3 Guillemin's symplectic potential

V. Guillemin in [30] found a canonical expression for the symplectic potential associated with a given Delzant polytope P by associating affine functions $l_\alpha : \mathbb{P}^\circ \longrightarrow \mathbb{R}$ to every $(n-1)-$dimensional face α, defined as:

$$l_\alpha(\mathfrak{u}) = \langle \mathfrak{u}, \nu_\alpha \rangle - \lambda_\alpha \tag{3.3.1}$$

where $\nu_\alpha \in \mathbb{Z}^n$ is a primitive inward normal vector to the face α for every α and λ_α is a constant. Furthermore, the set of inequalities $\{l_\alpha \geq 0\}$ defines the polytope P. The canonical symplectic potential $\mathcal{G}_P : P^\circ \longrightarrow \mathbb{R}$ defined by Guillemin is:

$$\mathcal{G}_P = \frac{1}{2}\sum_{\alpha=1}^{r} l_\alpha(\mathfrak{u})\mathrm{Log}[l_\alpha(\mathfrak{u})] \tag{3.3.2}$$

Transforming from the complex holomorphic coordinates (u, Θ) to symplectic coordinates $(\mathfrak{u}, \Theta)$ via the Legendre transform often requires solving the system (3.2.6) for u_j. This might prove to be a difficult step in general but once we get past this and transform our Kähler potential into symplectic potential, the further calculations turn out to be much simpler. On the other hand, one can always start from a Delzant polytope associated with any compact toric variety and we consider Guillemin's symplectic potential.

Remark 3.3.1. *Remember that to every toric fan Σ of a compact toric variety we can associate a polytope such that the maximal cones in Σ are generated by the inward normals of the edges meeting at each vertex. This dual description of a compact toric variety is very useful in toric geometry. In fact, given a toric divisor D on a toric variety X, we can always associate a polytope, so in a way, Delzant polytope also comes equipped with a bonafide toric divisor.*

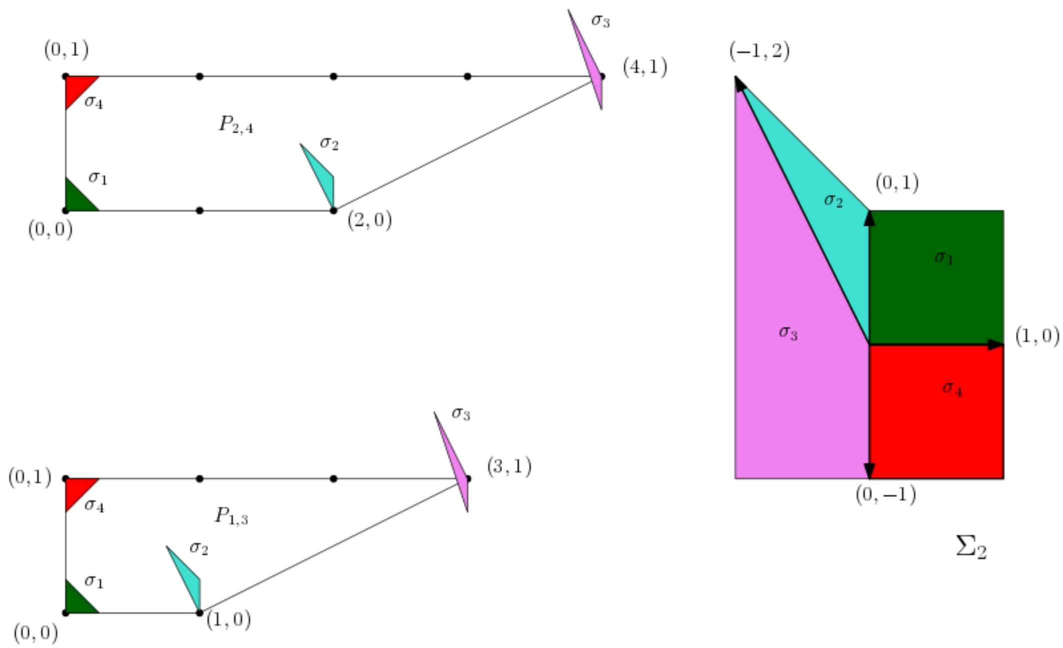


Figure 3.2: Two polytopes $P_{2,4}$ and $P_{1,3}$ with $r = 2$ and their corresponding toric fan Σ_2

3.4 Generic Delzant polytope associated with $\mathbb{F}_2$

Let us consider a generic polytope of the form

$$P_{a,b} = \mathrm{Conv}(0, a\,e_1, e_2, b\,e_1 + e_2) \subset \mathbb{R}^2 \tag{3.4.1}$$

where $a, b \in \mathbb{N}$ satisfy $1 \leq a \leq b$. Let $\Sigma_{a,b}$ be the normal fan associated with $P_{a,b}$ and $X_{a,b}$ be the normal toric variety associated with $\Sigma_{a,b}$. Note that the polytope $P_{a,b}$ has only one slanted edge joining the vertices $(a, 0)$ and $(b, 1)$. The difference $r = b - a$ determines the slope of this slanted edge. Furthermore, the polytopes $P_{a,a+r}$ correspond to the same normal fan denoted by Σ_r, and consequently the same toric variety X_r. A simple calculation reveals that the variety X_r is the $rth-$ Hirzebruch surface $\mathbb{F}_r$.

In figure 3.2, we showed two polytopes $P_{2,4}$ and $P_{1,3}$. Their normal fan is Σ_2, which is the fan of the second Hirzebruch surface $\mathbb{F}_2$. $P_{2,4}$ describes $\mathbb{F}_2$ as an embedded surface in $\mathbb{CP}^7$, while $P_{1,3}$ describe it as an embedded surface in $\mathbb{CP}^5$. Furthermore, every $P_{a,2+a}$ is a Delzant polytope.

A generic polytope $P_{a,2+a}$ associated with second Hirzebruch surface $\mathbb{F}_2$, in this terminology, is defined by the following inequalities:

$$a - \mathfrak{u} + 2\mathfrak{v} \geq 0; \quad \mathfrak{v} \geq 0; \quad 1 - \mathfrak{v} \geq 0 \text{ and } \mathfrak{u} \geq 0 \tag{3.4.2}$$

Guillemin's symplectic potential associated with $P_{a,2+a}$ is of the form:

$$\mathcal{G}_{\mathbb{F}_2} = \frac{1}{2}\Big((a-\mathfrak{u}+2\mathfrak{v})log(a-\mathfrak{u}+2\mathfrak{v})+\mathfrak{u}log(\mathfrak{u})+\mathfrak{v}log(\mathfrak{v})+(1-\mathfrak{v})log(1-\mathfrak{v})\Big) \tag{3.4.3}$$

Note that, by fixing the slope of the slanted edge we always get a unique Hirzebruch surface. In chapter 4, we will fix the slope of the slanted edge equal to the slope of the Delzant polytopes associated with $\mathbb{F}_2$ and we will vary the edge opposite to the slanted one. This variation turns out to be a very productive technique to create different classes of Kähler manifolds which are homeomorphic to $\mathbb{F}_2$ but are different as complex manifolds.

Chapter 4

New Kähler-Einstein Manifolds

In this chapter, we will restrict ourselves to the case of two-dimensional toric compact Kähler manifolds which we will denote by $\mathcal{M}_B$. Furthermore, we want to study Kähler metrics $g_{\mathcal{M}_B}$ on $\mathcal{M}_B$ with $SU(2) \times U(1)$ isometries. We will use the AMSY formulation developed by Abreu, Martelli, Sparks and Yau (AMSY) in [1, 38, 9] to transform the Kähler potential $\mathcal{K}$ corresponding to $g_{\mathcal{M}_B}$ to get the symplectic potential $\mathcal{G}$. First, we recall the AMSY formulation in general.

4.1 The AMSY symplectic formulation

Given a Kähler potential $\mathcal{K}(|z_1|, .., |z_n|)$ on a toric complex n-dimensional Kähler manifold, where $z_i = x_i + i\Theta_i$ are the complex coordinates. The dependence of $\mathcal{K}$ on the modules of the complex coordinates z is due to the fact that $\mathcal{K}$ must be invariant under the $U(1)^n$ action. Introducing the moment variables

$$\mu^i = \partial_{x_i}\mathcal{K} \tag{4.1.1}$$

we can obtain the symplectic potential by means of the Legendre transform:

$$G\,(\mu_i) = \sum_i^n x_i\,\mu^i \;-\; \mathcal{K}(|z_1|, .., |z_n|) \tag{4.1.2}$$

The main issue involved in the use of eqn. (4.1.2) is the inverse transformation that expresses the coordinates x_i in terms of the moments μ^i. Once this is done one can calculate the metric in moment variables utilizing the Hessian:

$$G_{ij} = \frac{\partial^2}{\partial\mu^i\partial\mu^j} G\left(\mu\right) \tag{4.1.3}$$

and its matrix inverse. Complex coordinates better adapted to the complex structure tensor can be defined as

$$u_i = e^{z_i} = \exp[x_i \, + \, \mathrm{i}\Theta_i] \tag{4.1.4}$$

The Kähler 2-form has the following universal structure:

$$\mathbb{K} \, = \, \sum_{i=1}^{n} \mathrm{d}\mu^i \wedge \mathrm{d}\Theta_i \tag{4.1.5}$$

and the metric is expressed as

$$ds^2_{symp} = \mathbf{G}_{ij} d\mu^i \, d\mu^j \, + \, \mathbf{G}^{-1}_{ij} d\Theta^i \, d\Theta^j \tag{4.1.6}$$

4.2 Kähler metrics with SU(2) × U(1) isometry

In [7],[5], it was remarked that in order to study two complex dimensional compact toric Kähler manifold $\mathcal{M}_B$ equipped with a Kähler metric $g_{\mathcal{M}_B}$ with isometry $SU(2)\times U(1)$, the Kähler potential $\mathcal{K}_0(\varpi)$ depends only on the real combination,

$$\varpi = \left(1+ \mid u \mid^2\right)^2 \, \mid v \mid^2, \tag{4.2.1}$$

where u and v are the complex coordinates, which guarantees invariance under $\mathrm{SU}(2) \times \mathrm{U}(1)$ transformations realized as

$$\begin{aligned} \text{if } \mathbf{g} \; &= \; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2) \quad \text{then} \quad \mathbf{g}\,(u,v) = \left(\tfrac{a\,u+b}{c\,u+d}, \quad v\;(c\,u+d)^2\right); \\ \text{if } \mathbf{g} \; &= \; \exp(i\,\theta_1) \in \mathrm{U}(1) \quad \text{then} \quad \mathbf{g}\,(u,v) = (u, \quad \exp(i\,\theta_1)\,v)\,. \end{aligned} \tag{4.2.2}$$

The above realization of the isometry captures the idea that, at least topologically, the manifold is an S^2 fibration over S^2 (u being a coordinate on the base and v a fiber coordinate). Even a posteriori, $\mathcal{M}_B$ will be an S^2 fibration over S^2, but only topologically, with possible singularities in the differential structures(in fact, this will be the case for the new class of Kähler-Einstein manifolds discussed later in this chapter).

Within such a framework, in [7],[5] two cases are discussed in detail, namely

a) the singular weighted projective plane $\mathbb{WP}[1,1,2]$;

b) the second Hirzebruch surface $\mathbb{F}_2$.

In the case of the singular variety $\mathbb{WP}[1,1,2]$ there is a nonKähler–Einstein metric that emerges from a partial resolution of the $\mathbb{C}^3/\mathbb{Z}_4$ singularity within the generalized Kronheimer construction, whose explicit Kähler potential is the following one:

$$\mathcal{K}_0^{Kr}(\varpi) = \frac{9}{4}\left(\frac{3\varpi + \sqrt{\varpi(\varpi+8)}}{\varpi + \sqrt{\varpi(\varpi+8)}} + \log\left(\varpi + \sqrt{\varpi(\varpi+8)} + 4\right)\right) \tag{4.2.3}$$

On the other hand the Kähler potential (4.2.3) is the particular case $\alpha = 0$ of a one-parameter family of Kähler potentials obtained from the Kronheimer construction:

$$\begin{aligned} K^{Kr}[\varpi,\alpha] = -\frac{9}{16}\Bigg(& -4(\alpha+1)\log\left[\sqrt{\alpha^2+6\alpha\varpi+\varpi^2+8\varpi}+3\alpha+\varpi+4\right] \\ & -\mathcal{F}[\varpi,\alpha] + 4\alpha\log\sqrt{\frac{\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha+\varpi}{\sqrt{\varpi}}}\Bigg) \end{aligned} \tag{4.2.4}$$

where

$$\mathcal{F}[\varpi,\alpha] = \frac{\mathcal{F}_1(\varpi,\alpha)}{\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha+\varpi}$$

and

$$\begin{aligned} \mathcal{F}_1(\varpi,\alpha) = 4\Big(&\alpha\left(\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+2\varpi+1\right) + \\ & \sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha^2+3\varpi\Big) \end{aligned}$$

that, for $\alpha > 0$ generate *bona fide* Kähler metrics on the second Hirzebruch surface $\mathbb{F}_2$.

4.2.1 A family of 4D Kähler metrics

In this thesis, we extend the work presented in [7],[5] further by restricting ourselves within the same framework presented there and by using the AMSY formalism we will find a new class of Kähler-Einstein manifolds $\mathcal{M}_B$. These are endowed with a metric invariant under the $\mathrm{SU}(2) \times \mathrm{U}(1)$ isometry group acting as in eqn. (4.2.2). The class of these manifolds is singled out by the above assumption that, in the complex formalism, their Kähler potential $\mathcal{K}_0(\varpi)$ is a function only of the invariant ϖ defined in eqn. (4.2.1).

Via the Legendre transform we can associate a symplectic potential $\mathcal{G}_0$ using the dependence of the Kähler potential $\mathcal{K}_0(\varpi)$ but the explicit form of the Kähler potential $\mathcal{K}_0(\varpi)$ cannot be worked out analytically in all cases since the inverse Legendre transform involves the roots of higher-order algebraic equations; yet, using the ϖ-dependence assumption, the Kähler metric can be explicitly worked out in the symplectic coordinates and has a simple and very elegant form – actually the metric depends on a single function of one variable $F(\mathfrak{v})$ which encodes all the geometric properties and substitutes $\mathcal{K}_0(\varpi)$. Posing all the questions in this symplectic language allow one to calculate all the geometric properties of the spaces in the class under consideration and leads also to some new intriguing results. We choose to treat the matter in general, by utilizing a *local approach* where we discuss the differential equations in a given open dense coordinate patch u, v, and we address the question of its global topological and algebraic structure only a posteriori, once the metric has been found in the considered chart.

In order to get the symplectic structure of the metric on $\mathcal{M}_B$ we would like to start with the toric data associated with these manifolds. Following the AMSY formulation of toric manifolds and keeping in mind that we want manifolds $\mathcal{M}_B$ topologically homeomorphic to $\mathbf{S}^2 \times \mathbf{S}^2$ and the metric should have $SU(2) \times U(1)$ isometry, we get the following general ansatz:

The symplectic structure of the metric on $\mathcal{M}_B$ is exhibited in the following way:

$$ds^2_{\mathcal{M}_B} = \mathbf{g}_{\mathcal{M}_B|\mu\nu}\, dq^\mu\, dq^\nu \quad ; \quad q^\mu = \{\mathfrak{u}, \mathfrak{v}, \phi, \tau\} \quad ;$$
$$\mathbf{g}_{\mathcal{M}_B} = \left(\begin{array}{c|c} \mathbf{G}_{\mathcal{M}_B} & \mathbf{0}_{2\times 2} \\ \hline \mathbf{0}_{2\times 2} & \mathbf{G}^{-1}_{\mathcal{M}_B} \end{array}\right) \tag{4.2.5}$$

where the Hessian $\mathbf{G}_{\mathcal{M}_B}$ is defined by:

$$\mathbf{G}_{\mathcal{M}_B} = \partial_{\mu^i}\, \partial_{\mu^j}\, G_{\mathcal{M}_B} \quad ; \quad \mu^i = \{\mathfrak{u}, \mathfrak{v}\} \tag{4.2.6}$$

and

$$G_{\mathcal{M}_B} = G_0(\mathfrak{u}, \mathfrak{v}) \;+\; \mathcal{D}(\mathfrak{v}) \tag{4.2.7}$$

$$G_0\,(\mathfrak{u}, \mathfrak{v}) = \left(\mathfrak{v} - \frac{\mathfrak{u}}{2}\right) \log(2\mathfrak{v} - \mathfrak{u}) + \frac{1}{2}\mathfrak{u}\log(\mathfrak{u}) - \frac{1}{2}\mathfrak{v}\log(\mathfrak{v}) \tag{4.2.8}$$

The specific structure (4.2.7),(4.2.8) is the counterpart within the symplectic formalism, via Legendre transform, of the assumption that the Kähler potential $\mathcal{K}_0(\varpi)$ depends only on the ϖ variable.

After noting this important point, we go back to the discussion of $\mathcal{M}_B$ geometry and stress that with the given isometries its Riemannian structure is completely encoded in the boundary function $\mathcal{D}(\mathfrak{v})$. All the other items in the construction are as follows. For the Kähler form we have

$$\mathbb{K}^{\mathcal{M}_B} = 2\;(\,d\mathfrak{u} \wedge d\phi + d\mathfrak{v} \wedge d\tau) \;=\; \mathbf{K}^{\mathcal{M}_B}_{\mu\nu}\, dq^\mu \wedge dq^\nu \quad ;$$
$$\mathbf{K}^{\mathcal{M}_B} = \left(\begin{array}{c|c} \mathbf{0}_{2\times 2} & \mathbf{1}_{2\times 2} \\ \hline -\mathbf{1}_{2\times 2} & \mathbf{0}_{2\times 2} \end{array}\right) \tag{4.2.9}$$

and for the complex structure, we obtain

$$\mathfrak{J}^{\mathcal{M}_B} = \mathbf{K}^{\mathcal{M}_B}\, \mathbf{g}^{-1}_{\mathcal{M}_B} = \left(\begin{array}{c|c} \mathbf{0}_{2\times 2} & \mathbf{G}_{\mathcal{M}_B} \\ \hline -\mathbf{G}^{-1}_{\mathcal{M}_B} & \mathbf{0}_{2\times 2} \end{array}\right) \tag{4.2.10}$$

Explicitly the 2×2 Hessian is the following:

$$
\begin{aligned}
\mathbf{G}_{\mathcal{M}_B} &= \begin{pmatrix} -\frac{\mathfrak{v}}{\mathfrak{u}^2-2\mathfrak{u}\mathfrak{v}} & \frac{1}{\mathfrak{u}-2\mathfrak{v}} \\ \frac{1}{\mathfrak{u}-2\mathfrak{v}} & \frac{-2\mathfrak{v}(\mathfrak{u}-2\mathfrak{v})\mathcal{D}''(\mathfrak{v})+\mathfrak{u}+2\mathfrak{v}}{2\mathfrak{v}(2\mathfrak{v}-\mathfrak{u})} \end{pmatrix} \\
\mathbf{G}^{-1}_{\mathcal{M}_B} &= \begin{pmatrix} \frac{\mathfrak{u}(-2\mathfrak{v}(\mathfrak{u}-2\mathfrak{v})\mathcal{D}''(\mathfrak{v})+\mathfrak{u}+2\mathfrak{v})}{\mathfrak{v}(2\mathfrak{v}\mathcal{D}''(\mathfrak{v})+1)} & \frac{2\mathfrak{u}}{2\mathfrak{v}\mathcal{D}''(\mathfrak{v})+1} \\ \frac{2\mathfrak{u}}{2\mathfrak{v}\mathcal{D}''(\mathfrak{v})+1} & \frac{2\mathfrak{v}}{2\mathfrak{v}\mathcal{D}''(\mathfrak{v})+1} \end{pmatrix}
\end{aligned}
\tag{4.2.11}
$$

The family of metrics (4.2.5) is parameterized by the choice of a unique one-variable function:

$$
f(\mathfrak{v}) = \mathcal{D}''(\mathfrak{v}) \tag{4.2.12}
$$

and is worth being considered in its own right.

4.2.2 The inverse Legendre transform

Before proceeding further with the analysis of this class of metrics, it is convenient to consider the inverse Legendre transform and see how one reconstructs the Kähler potential on $\mathcal{M}_B$. The inverse Legendre transform provides the Kähler potential through the formula:

$$
\mathcal{K}_0 = x_u\, \mathfrak{u} \,+\, x_v\, \mathfrak{v} \,-\, G_{\mathcal{M}_B}(\mathfrak{u}, \mathfrak{v}) \tag{4.2.13}
$$

where $G_{\mathcal{M}_B}(\mathfrak{u}, \mathfrak{v})$ is the symplectic potential on the $\mathcal{M}_B$ defined in eqn. (4.2.7), and

$$
x_u = \partial_{\mathfrak{u}}\, G_{\mathcal{M}_B}(\mathfrak{u}, \mathfrak{v}) \quad ; \quad x_v = \partial_{\mathfrak{v}}\, G_{\mathcal{M}_B}(\mathfrak{u}, \mathfrak{v}) \tag{4.2.14}
$$

which explicitly yields:

$$
\begin{aligned}
x_u &= \frac{1}{2}\left(\log(\mathfrak{u}) - \log(2\mathfrak{v} - \mathfrak{u})\right) \quad ; \\
x_v &= \mathcal{D}'(\mathfrak{v}) + \log(2\mathfrak{v} - \mathfrak{u}) - \frac{1}{2}\log(\mathfrak{v}) + \frac{1}{2}
\end{aligned}
\tag{4.2.15}
$$

Using eqn. (4.2.15)) in eqn. (4.2.13) we immediately obtain the explicit form of the Kähler potential on M_B as a function of the moment $\mathfrak{v}$:

$$\mathcal{K}_0 = \mathfrak{K}_0\left(\mathfrak{v}\right) = \mathfrak{v}\left(\mathcal{D}'\left(\mathfrak{v}\right) + \frac{1}{2}\right) - \mathcal{D}\left(\mathfrak{v}\right) \tag{4.2.16}$$

The problem is that we need the Kähler potential $\mathcal{K}_0$ as a function of the invariant ϖ. Utilizing eqn. (4.2.15) it is fairly easy to obtain the expression of ϖ in terms of the moment $\mathfrak{v}$ for a generic function $\mathcal{D}(\mathfrak{v})$ that codifies the geometry of the base manifold, obtaining

$$\varpi = \left(1 + \exp\left[2\,x_u\right]\right)^2 \exp\left[2\,x_u\right] = \Omega\left(\mathfrak{v}\right) = 4\mathfrak{v}\exp\left[\,2\,\partial_{\mathfrak{v}}\mathcal{D}\left(\mathfrak{v}\right) + 1\right] \tag{4.2.17}$$

If one is able to invert the function $\Omega\left(\mathfrak{v}\right)$, the original Kähler potential of the base manifold can be written as:

$$\mathcal{K}_0\left(\varpi\right) = \mathfrak{K}_0 \circ \Omega^{-1}\left(\varpi\right) \tag{4.2.18}$$

The inverse function $\Omega^{-1}\left(\varpi\right)$ can be written explicitly in some simple cases, but not always, and this inverse is the main reason why certain Kähler metrics can be much more easily found in the AMSY symplectic formalism which deals only with real variables than in complex formalism. Since nothing good comes without paying a price, the metrics found in the symplectic approach requires that the ranges of the variables $\mathfrak{u}$ and $\mathfrak{v}$ should be determined, since it is just in those ranges that the topology and algebraic structure of the underlying manifold is hidden; indeed the ranges of $\mathfrak{u}$ and $\mathfrak{v}$ define a convex closed *polytope* in the $\mathbb{R}^2$ plane that encodes very precious information about the structure of the underlying manifold.

4.3 Conversion rules from Kähler to Symplectic geometry

Before we start calculating the different tensors related to the family of metrics (4.2.5) it is appropriate to translate some basic operators in complex geometry to symplectic geometry. In general, let $\mathcal{M}$ be a toric compact Kähler manifold of complex dimension n and $\mathcal{K}(x_1, \cdots, x_n)$ be a Kähler potential on $\mathcal{M}$ which depends on real variables x_i. Let us consider the complex coordinates given as in (4.1.4).

In general, in complex geometry we have

$$\begin{aligned} \frac{\partial}{\partial z_i} &= \frac{1}{2}\left(\frac{\partial}{\partial \tilde{x}_i} - i\frac{\partial}{\partial \tilde{y}_i}\right) \\ \frac{\partial}{\partial \bar{z}_i} &= \frac{1}{2}\left(\frac{\partial}{\partial \tilde{x}_i} + i\frac{\partial}{\partial \tilde{y}_i}\right) \end{aligned} \tag{4.3.1}$$

where $z_i = \tilde{x}_\alpha + i\tilde{y}_\alpha$. In our action-angle coordinates (4.1.4) we have:

$$\begin{aligned} \tilde{x}_i &= e^{x_i} cos(\Theta_i) \\ \tilde{y}_i &= e^{x_i} sin(\Theta_i) \end{aligned} \tag{4.3.2}$$

and

$$\begin{aligned} x_i &= log(\tilde{x}_i^2 + \tilde{y}_i^2) \\ \Theta_1 &= arctan(\frac{\tilde{y}_i}{\tilde{x}_i}) \end{aligned} \tag{4.3.3}$$

First, we convert (4.3.1) into the action-angle coordinates and we get:

$$\begin{aligned} \frac{\partial}{\partial z_i} &= \frac{1}{2}e^{-x_i - i\Theta_i}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial \Theta_i}\right) \\ \frac{\partial}{\partial \bar{z}_i} &= \frac{1}{2}e^{-x_i + i\Theta_i}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial \Theta_i}\right) \end{aligned} \tag{4.3.4}$$

Now to translate from Kähler to symplectic geometry via AMSY formulation, we introduced new coordinates called $\{\mathfrak{u}_1, \cdots, \mathfrak{u}_n; \Theta_1, \cdots, \Theta_n\}$ where the relationship between x_i's and $\mathfrak{u}_i$'s is defined as

$$\mathfrak{u}_i = \frac{\partial \mathcal{K}}{\partial x_i} \quad \text{and} \quad x_i = \frac{\partial \mathcal{G}}{\partial \mathfrak{u}_i}$$

where $\mathcal{G}(\mathfrak{u}_1, \cdots, \mathfrak{u}_n)$ is the corresponding symplectic potential. In these symplectic coordinates (4.3.4) converts as

$$\begin{aligned}\frac{\partial}{\partial z_i} &= \frac{1}{2} e^{-x_i - i\Theta_i} \left(\sum_j \left(\frac{\partial \mathfrak{u}_j}{\partial x_i} \frac{\partial}{\partial \mathfrak{u}_j} \right) - i \frac{\partial}{\partial \Theta_i} \right) \\ \frac{\partial}{\partial \bar{z}_i} &= \frac{1}{2} e^{-x_i + i\Theta_i} \left(\sum_j \left(\frac{\partial \mathfrak{u}_j}{\partial x_i} \frac{\partial}{\partial \mathfrak{u}_j} \right) + i \frac{\partial}{\partial \Theta_i} \right)\end{aligned} \tag{4.3.5}$$

To get the final expression one needs to find the inverse relationships between x_i's and $\mathfrak{u}_i$'s using the above expressions. As we can guess easily that providing either a Kähler potential or symplectic potential we can use ASMY formulation to go from one side to another. But as we have mentioned earlier, every time we use the AMSY formulation one needs to write down the coordinates of one side in the form of the other side, which often poses a significant challenge.

4.4 The Ricci tensor and the Ricci form

Now we come back to our problem. Calculating the Ricci tensor for the family of metrics (4.2.5) we obtain the following structure:

$$\mathrm{Ric}^{\mathcal{M}_B}_{\mu\nu} = \left(\begin{array}{c|c} \mathbf{P}_\mathrm{U} & \mathbf{0}_{2\times 2} \\ \hline \mathbf{0}_{2\times 2} & \mathbf{P}_\mathrm{D} \end{array} \right) \tag{4.4.1}$$

The expressions for $\mathbf{P}_\mathrm{U}$ and $\mathbf{P}_\mathrm{D}$ are quite lengthy and we omit them. We rather consider the Ricci 2-form defined by:

$$\mathbb{R}\mathrm{ic}_{\mathcal{M}_B} = \mathbf{Ric}^{\mathcal{M}_B}_{\mu\nu}\, dq^\mu \wedge dq^\nu \tag{4.4.2}$$

where:

$$\mathbf{Ric}^{\mathcal{M}_B} = \mathrm{Ric}^{\mathcal{M}_B}\,\mathfrak{J}^{\mathcal{M}_B} = \left(\begin{array}{c|c} \mathbf{0}_{2\times 2} & \mathbf{R} \\ \hline -\mathbf{R}^T & \mathbf{0}_{2\times 2} \end{array}\right) \tag{4.4.3}$$

and

$$\begin{aligned} \mathbf{R} &= \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \\ r_{11} &= \tfrac{2\mathfrak{v}\left(\mathfrak{v}f'(\mathfrak{v})+2\mathfrak{v}f(\mathfrak{v})^2+f(\mathfrak{v})\right)-1}{2\mathfrak{v}(2\mathfrak{v}f(\mathfrak{v})+1)^2} \\ r_{12} &= 0 \\ r_{21} &= \frac{A}{2\mathfrak{v}^2(2\mathfrak{v}f(\mathfrak{v})+1)^3} \\ r_{22} &= \frac{B}{(2\mathfrak{v}f(\mathfrak{v})+1)^3} \\ \mathbf{R}^T &= \mathbf{P}_{\mathrm{D}}\,\mathbf{G}^{-1}_{\mathcal{M}_B} \end{aligned} \tag{4.4.4}$$

where

$$\begin{aligned} A = 2\mathfrak{u}\mathfrak{v}\,\big(&\mathfrak{v}^2(2\mathfrak{v}f(\mathfrak{v})+1)f''(\mathfrak{v}) - 4\mathfrak{v}^3f'(\mathfrak{v})^2 + 4\mathfrak{v}f'(\mathfrak{v}) - \\ &4\mathfrak{v}^2f(\mathfrak{v})^3 - 2\mathfrak{v}f(\mathfrak{v})^2 + 3f(\mathfrak{v}) + \mathfrak{u} \end{aligned}$$

and

$$\begin{aligned} B = f(\mathfrak{v})\left(2\mathfrak{v}^2\left(\mathfrak{v}f''(\mathfrak{v})+f'(\mathfrak{v})\right)+3\right) + \mathfrak{v}\,(&\mathfrak{v}f''(\mathfrak{v}) - \\ &4\mathfrak{v}^2f'(\mathfrak{v})^2 + 5f'(\mathfrak{v}) + 2\mathfrak{v}f(\mathfrak{v})^2 \end{aligned}$$

The last equation is not a definition but rather a consistency constraint (the Ricci tensor must be skew-symmetric).

4.4.1 A two-parameter family of KE metrics for $\mathcal{M}_B$

An interesting and legitimate question is whether this family of metrics contains KE ones. Quite surprisingly, the answer is positive, and they make up a two-parameter subfamily. A metric is KE if the Ricci 2-form is proportional to the Kähler 2-form:

$$\mathbb{R}\mathrm{ic}^{\mathcal{M}_B} = \frac{k}{4}\,\mathbb{K}^{\mathcal{M}_B} \tag{4.4.5}$$

where k is a constant. This amounts to requiring that the 2×2 matrix $\mathbf{R}$ displayed in eqn. (4.4.4) be proportional via $\frac{k}{4}$ to the identity matrix $\mathbf{1}_{2\times 2}$. This condition implies differential constraints on the function $F(\mathfrak{v})$ that are uniquely solved by the following function:

$$f(\mathfrak{v}) = \frac{-3\beta + k\mathfrak{v}^3 + 3\mathfrak{v}^2}{-2k\mathfrak{v}^4 + 6\mathfrak{v}^3 + 6\beta\mathfrak{v}} \tag{4.4.6}$$

the parameter β being the additional integration constant, while k is defined by equation (4.4.5). To retrieve the original symplectic potential $\mathcal{D}(\mathfrak{v})$ one has to perform a double integration in the variable $\mathfrak{v}$. The explicit calculation of the integral requires a summation over the three roots $\lambda_{1,2,3}$ of the following cubic polynomial:

$$P(x) = x^3 - \frac{3x^2}{k} - \frac{3\beta}{k} \tag{4.4.7}$$

whose main feature is the absence of the linear term. Hence a beautiful way of parameterizing the family of KE metrics is achieved by using as parameters two of the three roots of the polynomial (4.4.7). Let us call the independent roots λ_1 and λ_2. The polynomial (4.4.7) is reproduced by setting:

$$k = \frac{3(\lambda_1 + \lambda_2)}{\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2^2}, \qquad \beta = -\frac{\lambda_1^2\lambda_2^2}{\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2^2}, \quad \lambda_3 = -\frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2} \tag{4.4.8}$$

Substituting (4.4.8) in eqn. (4.4.6) we obtain

$$f(\mathfrak{v}) = -\frac{\lambda_1\mathfrak{v}^2(\lambda_2+\mathfrak{v})+\lambda_2\mathfrak{v}^2(\lambda_2+\mathfrak{v})+\lambda_1^2\left(\lambda_2^2+\mathfrak{v}^2\right)}{2\mathfrak{v}(\mathfrak{v}-\lambda_1)(\mathfrak{v}-\lambda_2)(\lambda_2\mathfrak{v}+\lambda_1(\lambda_2+\mathfrak{v}))} \tag{4.4.9}$$

which is completely symmetrical in the exchange of the two independent roots λ_1, λ_2. Utilizing the expression (4.4.9) the double integration is easily performed, and we obtain the explicit result, where we omitted irrelevant linear terms:

$$\begin{aligned} \mathcal{D}^{KE}(\mathfrak{v}) = & -\frac{\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)(\mathfrak{v}-\lambda_1)\log(\mathfrak{v}-\lambda_1)}{\lambda_1^2+\lambda_2\lambda_1-2\lambda_2^2} \\ & -\frac{\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)(\mathfrak{v}-\lambda_2)\log(\mathfrak{v}-\lambda_2)}{-2\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2} \\ & +\frac{\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)(\lambda_2\mathfrak{v}+\lambda_1(\lambda_2+\mathfrak{v}))\log(\lambda_2\mathfrak{v}+\lambda_1(\lambda_2+\mathfrak{v}))}{(\lambda_1+\lambda_2)\left(2\lambda_1^2+5\lambda_2\lambda_1+2\lambda_2^2\right)} - \frac{1}{2}\mathfrak{v}\log(\mathfrak{v}) \end{aligned} \tag{4.4.10}$$

Comparing with the original papers on the AMSY formalism [1, 38], we note that the full symplectic potential for the 4-manifold $\mathcal{M}_B$ has precisely the structure of what is there called *natural symplectic potential*

$$\begin{aligned} G_{natural} &= \sum_{\ell=1}^{r} c_\ell \, p_\ell \left(\mathfrak{u}, \mathfrak{v}\right) \times \log\left[p_\ell \left(\mathfrak{u}, \mathfrak{v}\right)\right] \quad ; \\ p_\ell \left(\mathfrak{u}, \mathfrak{v}\right) &= \text{linear functions of the moments} \end{aligned} \tag{4.4.11}$$

The only difference is that in [1] the coefficients c_ℓ are all equal while here they differ one from the other in a precise way that depends on the parameters λ_1, λ_2 defining the metric and the argument of the logarithms. As we are going to discuss later on, the same thing happens also for the nonKE metric on the second Hirzebruch surface $\mathbb{F}_2$ derived from the Kronheimer construction.

4.5 Properties of the complete family of metrics

In this section, we perform a complete and in-depth study of the considered class of 4-dimensional metrics. We do not start from a given manifold but rather from a family of metrics $\mathbf{g}_B$ parameterized by the choice of a function $F(\mathfrak{v})$ of one variable $\mathfrak{v}$, given explicitly in coordinate form.

I) The first tasks we are confronted with are the definition of the maximal extension of our coordinates, and the search for possible singularities in the metric and/or in the Riemannian curvature, which happens to be the cleanest probing tool;

II) Secondly, we can calculate integrals of the curvature 2-form, of the Ricci and Kähler 2-forms, Chern characters and Chern classes. All this information is easily computed since everything reduces to the evaluation of a few integral-differential functionals of the function $F(\mathfrak{v})$;

III) Thirdly, we can construct geodesics relative to the given metric and try to explore their behavior. This is probably the finest and most accurate tool to visualize the geometry of a manifold and we will pursue it hoping that by this tool we can finally enucleate the difference between the algebraic manifold $\mathbb{F}_2$ and its KE cognates.

We can also integrate the complex structure and find explicitly the complex coordinates.

We begin by observing that all the metrics deriving from the symplectic potential[1]

$$\begin{aligned} G_B(\mathfrak{u},\mathfrak{v}) &= G_0(\mathfrak{u},\mathfrak{v}) + \mathcal{D}(\mathfrak{v}) \\ G_0(\mathfrak{u},\mathfrak{v}) &= \left(\mathfrak{v} - \tfrac{\mathfrak{u}}{2}\right)\log(2\mathfrak{v}-\mathfrak{u}) + \tfrac{1}{2}\mathfrak{u}\log(\mathfrak{u}) - \tfrac{1}{2}\mathfrak{v}\log(\mathfrak{v}) \end{aligned} \tag{4.5.1}$$

admit a general form which we display below:

$$ds^2_{\mathcal{M}_B} = F(\mathfrak{v})\left[d\phi(1-\cos\theta) + d\tau\right]^2 + \frac{d\mathfrak{v}^2}{F(\mathfrak{v})} + \mathfrak{v}\,\underbrace{\left(d\phi^2\sin^2\theta + d\theta^2\right)}_{S^2\text{ metric}} \tag{4.5.2}$$

where we have defined

$$F(\mathfrak{v}) = \frac{2\mathfrak{v}}{2\mathfrak{v}\mathcal{D}''(\mathfrak{v}) + 1} \tag{4.5.3}$$

This expression for the metric is obtained performing a convenient change of variable:

$$\mathfrak{u} \to (1-\cos\theta)\,\mathfrak{v} \quad ; \quad \theta \in [0,\pi] \tag{4.5.4}$$

which automatically takes into account that $\mathfrak{u} \leq 2\,\mathfrak{v}$.

Three-dimensional sections. Furthermore, this change of variables clearly reveals that all the three-dimensional sections of $\mathcal{M}_B$ obtained by fixing $\mathfrak{v} = \text{const.}$ are S^1 fibrations on S^2 which is consistent with the isometry $\mathrm{SU}(2)\times\mathrm{U}(1)$. Indeed all the spaces $\mathcal{M}_B$ have cohomogeneity equal to one and the moment variable $\mathfrak{v}$ is the only one whose dependence is not fixed by isometries.

The next important point is that the metric (4.5.2) is positive definite only in the interval of the positive $\mathfrak{v}$-axis where $F(\mathfrak{v}) \geq 0$. Let us name the lower and upper

[1] In this section which deals only with the base manifold $\mathcal{M}_B$ and where there is no risk of the confusion we drop the suffix 0 in the moment variables, in order to make formulas simpler.

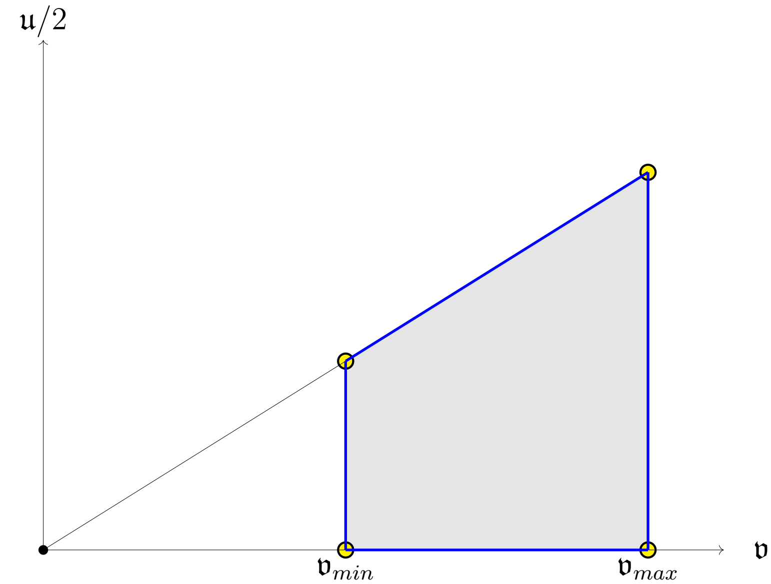


Figure 4.1: The universal polytope in the $\mathfrak{v}, \frac{\mathfrak{u}}{2}$ plane for all the metrics of the $\mathcal{M}_B$ manifolds considered in this paper and defined in equation (4.5.2)

endpoints of such an interval $\mathfrak{v}_{min}$ and $\mathfrak{v}_{max}$, respectively. If the interval $[\mathfrak{v}_{min}, \mathfrak{v}_{max}]$ is finite, then the space $\mathcal{M}_B$ is compact and the domain of the coordinates $\mathfrak{u}, \mathfrak{v}$ is provided by the trapezoidal polytope displayed in Figure 4.1.

Our two main examples both correspond to the same universal polytope of Figure 4.1, are provided by the case of a one-parameter family of Kronheimer metrics on the $\mathbb{F}_2$-surface, studied in [5, 7], whose Kähler potential was recalled in eqn. (4.2.4)), and by the KE metrics discovered in this thesis. In addition, within the first class, we have the degenerate case where the parameter α goes to zero and the trapezoid degenerates into a triangle. That case corresponds to the singular space $\mathcal{M}_B = \mathbb{WP}[1, 1, 2]$ (a weighted projective plane).

Inspecting Table 4.1 we see that the functions $F^{\mathbb{F}_2}(\mathfrak{v})$ and $F^{KE}(\mathfrak{v})$ show strict similarities but also a difference that is expected to account for different topologies. In both cases the function $F(\mathfrak{v})$ is the ratio of a cubic polynomial having three real roots, two positive and one negative, and of a denominator that has no zeros in the

$F^{\mathbb{F}_2}(\mathfrak{v}) = \frac{\left(1024\mathfrak{v}^2-81\alpha^2\right)(32\mathfrak{v}-9(3\alpha+4))}{16(81\alpha^2+1024\mathfrak{v}^2-576(3\alpha+4)\mathfrak{v})}$	$\mathfrak{v}_{min} = \frac{9\alpha}{32}$	$\mathfrak{v}_{max} = \frac{9}{32}(3\alpha+4)$	$\alpha > 0$
$F^{\mathbb{WP}[1,1,2]}(\mathfrak{v}) = \frac{\mathfrak{v}(8\mathfrak{v}-9)}{4\mathfrak{v}-9}$	$\mathfrak{v}_{min} = 0$	$\mathfrak{v}_{max} = \frac{9}{8}$	$\alpha = 0$
$F^{KE}(\mathfrak{v}) = -\frac{(\mathfrak{v}-\lambda_1)(\mathfrak{v}-\lambda_2)(\lambda_2\mathfrak{v}+\lambda_1(\lambda_2+\mathfrak{v}))}{\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)\mathfrak{v}}$	$\mathfrak{v}_{min} = \lambda_1$	$\mathfrak{v}_{max} = \lambda_2$	$0 < \lambda_1 < \lambda_2$
$F^{0}(\mathfrak{v}) = \frac{\mathfrak{v}(\lambda_2-\mathfrak{v})}{\lambda_2}$	$\mathfrak{v}_{min} = 0$	$\mathfrak{v}_{max} = \lambda_2$	$\lambda_2 > 0$
$F^{cone}(\mathfrak{v}) = \mathfrak{v}$	$\mathfrak{v}_{min} = 0$	$\mathfrak{v}_{max} = \infty$	

Table 4.1: Different possibilities for the function F.

$[\mathfrak{v}_{min}, \mathfrak{v}_{max}]$ interval. In the KE case there is a simple pole at $\mathfrak{v} = 0$ while for $\mathbb{F}_2$ (which is not KE) the denominator has two zeros and therefore $F(\mathfrak{v})$ has two simple poles at

$$\mathfrak{v}_{poles} = \frac{9}{32}\left[(3\alpha + 4) \pm 2\sqrt{2}\sqrt{\alpha^2 + 3\alpha + 2}\right] \tag{4.5.5}$$

These poles are out of the interval $[\mathfrak{v}_{min}, \mathfrak{v}_{max}]$ for any positive $\alpha > 0$, namely these poles do not correspond to points of the manifold $\mathcal{M}_B$, just as is the case for the single pole $\mathfrak{v} = 0$ in the KE case.

The last case ($F(\mathfrak{v}) = \mathfrak{v}$) corresponds to a metric cone on the 3-sphere, i.e., $\mathbb{C}^2/\mathbb{Z}_2$ with a flat metric. The case F^0 will be discussed in Section 4.6 .

4.5.1 Vielbein formalism and the curvature 2-form of $\mathcal{M}_B$

The metric (4.5.2) is in diagonal form so it is easy to write a set of vierbein 1-forms. Indeed if we set

$$\mathbf{e}^i = \left\{\frac{d\mathfrak{v}}{\sqrt{F(\mathfrak{v})}},\ \sqrt{F(\mathfrak{v})}\ [d\phi(1 - \cos\theta) + d\tau]\,,\ \sqrt{\mathfrak{v}}\, d\theta,\ \sqrt{\mathfrak{v}}\, d\phi\ \sin\theta\right\} \tag{4.5.6}$$

the line element (4.5.2) reads

$$ds_B^2 = \sum_{i=1}^{4} \mathbf{e}^i \otimes \mathbf{e}^i \tag{4.5.7}$$

Furthermore, we can calculate the matrix vielbein and its inverse quite easily, obtaining:

$$
\begin{aligned}
\mathbf{e}^i &= E^i_\mu \, dy^\mu \quad ; \quad y^\mu = \{\mathfrak{v}, \theta, \phi, \tau\} \\
E^i_\mu &= \begin{pmatrix} \frac{1}{\sqrt{F(\mathfrak{v})}} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{F(\mathfrak{v})}(1-\cos\theta) & \sqrt{F(\mathfrak{v})} \\ 0 & \sqrt{\mathfrak{v}} & 0 & 0 \\ 0 & 0 & \sqrt{\mathfrak{v}}\sin\theta & 0 \end{pmatrix} \qquad (4.5.8) \\
E^\nu_j &= \begin{pmatrix} \sqrt{F(\mathfrak{v})} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\mathfrak{v}}} & 0 \\ 0 & 0 & 0 & \frac{\csc(\theta)}{\sqrt{\mathfrak{v}}} \\ 0 & \frac{1}{\sqrt{F(\mathfrak{v})}} & 0 & \frac{(\cos\theta-1)\csc\theta}{\sqrt{\mathfrak{v}}} \end{pmatrix}
\end{aligned}
$$

By means of the MATHEMATICA package NOVAMANIFOLDA[2] we can easily calculate the Levi-Civita spin connection and the curvature 2-form from the definitions

$$
0 = \mathfrak{T}^i = d\mathbf{e}^i + \omega^{ij} \wedge \mathbf{e}^j \quad ; \quad \mathfrak{R}^{ij} = d\omega^{ij} + \omega^{ik} \wedge \omega^{kj} = \mathcal{R}^{ij}{}_{k\ell}\, \mathbf{e}^k \wedge \mathbf{e}^\ell \qquad (4.5.9)
$$

obtaining

$$
\begin{aligned}
\mathfrak{R}^{12} &= -\frac{F''(\mathfrak{v})}{2}\mathbf{e}^1 \wedge \mathbf{e}^2 - \frac{(\mathfrak{v}F'(\mathfrak{v})-F(\mathfrak{v}))}{2\mathfrak{v}^2}\,\mathbf{e}^3 \wedge \mathbf{e}^4 \\
\mathfrak{R}^{13} &= -\frac{(\mathfrak{v}F'(\mathfrak{v})-F(\mathfrak{v}))}{4\mathfrak{v}^2}\,\mathbf{e}^1 \wedge \mathbf{e}^3 - \frac{(\mathfrak{v}F'(\mathfrak{v})-F(\mathfrak{v}))}{4\mathfrak{v}^2}\,\mathbf{e}^2 \wedge \mathbf{e}^4 \\
\mathfrak{R}^{14} &= \frac{(\mathfrak{v}F'(\mathfrak{v})-F(\mathfrak{v}))}{4\mathfrak{v}^2}\,\mathbf{e}^2 \wedge \mathbf{e}^3 - \frac{(\mathfrak{v}F'(\mathfrak{v})-F(\mathfrak{v}))}{4\mathfrak{v}^2}\,\mathbf{e}^1 \wedge \mathbf{e}^4 \\
\mathfrak{R}^{23} &= \frac{(\mathfrak{v}F'(\mathfrak{v})-F(\mathfrak{v}))}{4\mathfrak{v}^2}\,\mathbf{e}^1 \wedge \mathbf{e}^4 - \frac{(\mathfrak{v}F'(\mathfrak{v})-F(\mathfrak{v}))}{4\mathfrak{v}^2}\,\mathbf{e}^2 \wedge \mathbf{e}^3 \\
\mathfrak{R}^{24} &= -\frac{(\mathfrak{v}F'(\mathfrak{v})-F(\mathfrak{v}))}{4\mathfrak{v}^2}\,\mathbf{e}^1 \wedge \mathbf{e}^3 - \frac{(\mathfrak{v}F'(\mathfrak{v})-F(\mathfrak{v}))}{4\mathfrak{v}^2}\,\mathbf{e}^2 \wedge \mathbf{e}^4 \\
\mathfrak{R}^{34} &= \frac{(\mathfrak{v}-F(\mathfrak{v}))}{\mathfrak{v}^2}\,\mathbf{e}^3 \wedge \mathbf{e}^4 - \frac{(\mathfrak{v}F'(\mathfrak{v})-F(\mathfrak{v}))}{2\mathfrak{v}^2}\,\mathbf{e}^1 \wedge \mathbf{e}^2
\end{aligned} \qquad (4.5.10)
$$

[2]NOVAMANIFOLDA is a personal mathematica package of Pietro Fré, which calculates different tensors of a given Riemannian metric

Equation (4.5.10) shows that the Riemann tensor $\mathcal{R}^{ij}{}_{k\ell}$ is constructed in terms of only three functions:

$$\begin{aligned}
\mathcal{CF}_1(\mathfrak{v}) &= F''(\mathfrak{v}) \quad ; \\
\mathcal{CF}_2(\mathfrak{v}) &= \frac{(\mathfrak{v}F'(\mathfrak{v}) - F(\mathfrak{v}))}{\mathfrak{v}^2} \quad ; \\
\mathcal{CF}_3(\mathfrak{v}) &= \frac{(\mathfrak{v} - F(\mathfrak{v}))}{\mathfrak{v}^2}
\end{aligned} \tag{4.5.11}$$

If these functions are regular in the interval $[\mathfrak{v}_{min}, \mathfrak{v}_{max}]$ the Riemann tensor is well defined and finite in the entire polytope of Figure 4.1 and $\mathcal{M}_B$ should be a smooth compact manifold.

The $\mathbb{F}_2$ case. In the $\mathbb{F}_2$ case with the "Kronheimer" metric we have:

$$\begin{aligned}
\mathcal{CF}_1^{\mathbb{F}_2}(\mathfrak{v}) &= \frac{331776(\alpha+1)(\alpha+2)\left(729\alpha^2(3\alpha+4)+16384\mathfrak{v}^3-3888\alpha^2\mathfrak{v}\right)}{\left(81\alpha^2+1024\mathfrak{v}^2-576(3\alpha+4)\mathfrak{v}\right)^3} \\
\mathcal{CF}_2^{\mathbb{F}_2}(\mathfrak{v}) &= -\frac{9}{32\mathfrak{v}^2\left(81\alpha^2+1024\mathfrak{v}^2-576(3\alpha+4)\mathfrak{v}\right)^2} \times A \\
\mathcal{CF}_3^{\mathbb{F}_2}(\mathfrak{v}) &= \frac{\mathfrak{v} - B}{\mathfrak{v}^2}
\end{aligned} \tag{4.5.12}$$

where

$$\begin{aligned}
A = {} & 6561\alpha^4(3\alpha+4) + 1048576(3\alpha+4)\mathfrak{v}^4 - 1179648\alpha^2\mathfrak{v}^3 + \\
& 497664\alpha^2(3\alpha+4)\mathfrak{v}^2 - 93312\alpha^2(3\alpha+4)^2\mathfrak{v}
\end{aligned}$$

and

$$B = \frac{(1024\mathfrak{v}^2 - 81\alpha^2)\,(32\mathfrak{v} - 9(3\alpha+4))}{16\,(81\alpha^2 + 1024\mathfrak{v}^2 - 576(3\alpha+4)\mathfrak{v})}$$

The three functions $\mathcal{CF}^{\mathbb{F}_2}_{1,2,3}(\mathfrak{v})$ are smooth in the interval $\left(\frac{9\alpha}{32}, \frac{9}{32}(3\alpha+4)\right)$ and they are defined at the endpoints: see for instance Figure 4.2.

Indeed the values of the three functions at the endpoints are

$$\begin{aligned}
\mathcal{CF}^{\mathbb{F}_2}_{1,2,3}(\mathfrak{v}_{min}) &= \left\{-\frac{128(\alpha+1)}{9\alpha(\alpha+2)}, \frac{32}{9\alpha}, \frac{32}{9\alpha}\right\} \\
\mathcal{CF}^{\mathbb{F}_2}_{1,2,3}(\mathfrak{v}_{max}) &= \left\{-\frac{32(3\alpha+4)}{9(\alpha^2+3\alpha+2)}, -\frac{32}{27\alpha+36}, \frac{32}{9(3\alpha+4)}\right\}
\end{aligned} \tag{4.5.13}$$

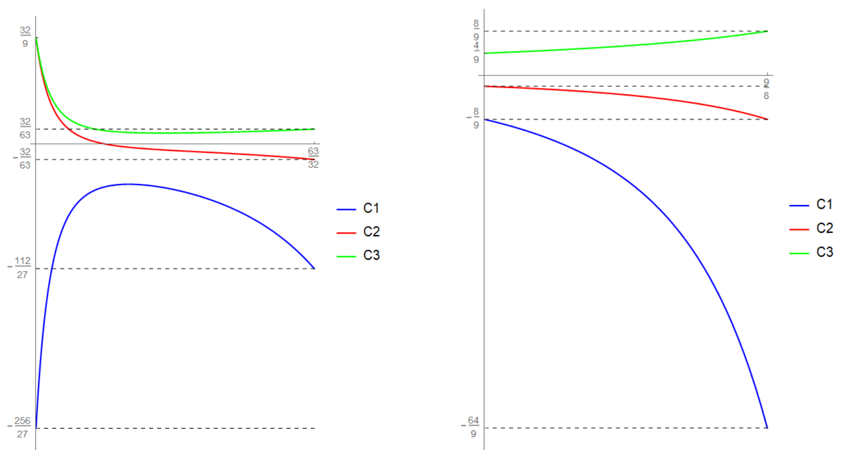


Figure 4.2: A (left): Plot of the three functions $\mathcal{CF}^{\mathbb{F}_2}_{1,2,3}(\mathfrak{v})$ entering the intrinsic Riemann curvature tensor for the "Kronheimer" metric on $\mathbb{F}_2$ with the choice of the parameter $\alpha = 1$. B (right): Plot of the three functions $\mathcal{CF}^{\mathbb{WW}_{112}}_{1,2,3}(\mathfrak{v})$ entering the intrinsic Riemann curvature tensor for the Kronheimer metric on $\mathbb{WP}[1,1,2]$ with the choice of the parameter $\alpha = 0$. Comparing this picture with the one on the upper we see the discontinuity. In all smooth cases the functions $\mathcal{CF}^{\mathbb{F}_2}_{2,3}$ attain the same value in the lower endpoint of the interval while for the singular case of the weighted projective space, the initial values of $\mathcal{CF}^{\mathbb{WP}[1,1,2]}_{2,3}(\mathfrak{v})$ are different.

The singularity which might be developed by the space corresponding to the value $\alpha = 0$ is evident from eqns. (4.5.13). The intrinsic components of the Riemann curvature seem to have a singularity in the lower endpoint of the interval, for $\alpha = 0$.

The case of the singular manifold $\mathbb{WP}[1,1,2]$. In the previous section, we utilized the wording *seem to have a singularity* for the components of the Riemann curvature in the case of the of the space $\mathbb{WP}[1,1,2]$ since such a singularity in the curvature actually does not exist. The space $\mathbb{WP}[1,1,2]$ has indeed a singularity at $\mathfrak{v} = 0$ but it is very mild since the intrinsic components of the Riemann curvature are well-behaved in $\mathfrak{v} = 0$ and have a finite limit. It depends on the way one does the limit $\alpha \to 0$. If we first compute the value of the curvature 2-form at the endpoints for generic α and then we do the limit $\alpha \to 0$ we see the singularity that is evident from equations (4.5.13). On the other hands, if we first reduce the function $F(\mathfrak{v})$ to its $\alpha = 0$ form we obtain:

$$F^{\mathbb{WP}[1,1,2]}(\mathfrak{v}) = \frac{\mathfrak{v}(8\mathfrak{v}-9)}{4\mathfrak{v}-9} \tag{4.5.14}$$

and the corresponding functions appearing in the curvature are:

$$\mathcal{CF}^{\mathbb{WP}[1,1,2]}_{1,2,3}(\mathfrak{v}) = \left\{\frac{648}{(4\mathfrak{v}-9)^3}, -\frac{18}{(9-4\mathfrak{v})^2}, \frac{4}{9-4\mathfrak{v}}\right\} \tag{4.5.15}$$

which are perfectly regular in the interval $[0, 9/8]$ and have finite value at the endpoints (see Figure 4.2B).

The case of the KE manifolds. In the case of the KE metrics the function $F(\mathfrak{v})$ is

$$F^{KE}(\mathfrak{v}) = -\frac{(\mathfrak{v}-\lambda_1)(\mathfrak{v}-\lambda_2)(\lambda_1\lambda_2 + (\lambda_1+\lambda_2)\mathfrak{v})}{(\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2^2)\mathfrak{v}} \tag{4.5.16}$$

and the corresponding functions entering the intrinsic components of the Riemann curvature are

$$\mathcal{CF}^{KE}_{1,2,3}(\mathfrak{v}) = \left\{ -\frac{2\left(\lambda_1^2\lambda_2^2 + (\lambda_1+\lambda_2)\,\mathfrak{v}^3\right)}{\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)\mathfrak{v}^3}, \frac{2\lambda_1^2\lambda_2^2 - (\lambda_1+\lambda_2)\,\mathfrak{v}^3}{2\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)\mathfrak{v}^3}, \right. \\ \left. \frac{\lambda_1^2\lambda_2^2 + (\lambda_1+\lambda_2)\,\mathfrak{v}^3}{\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)\mathfrak{v}^3} \right\} \tag{4.5.17}$$

and the interval of variability of the moment coordinate $\mathfrak{v}$ is the following $\mathfrak{v} \in [\lambda_1, \lambda_2]$. Correspondingly the boundary values are

$$\begin{aligned} \mathcal{CF}^{KE}_{1,2,3}(\mathfrak{v}_{min}) &= \left\{ -\tfrac{2}{\lambda_1}, \tfrac{1}{\lambda_1} - \tfrac{3(\lambda_1+\lambda_2)}{2\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)}, \tfrac{1}{\lambda_1} \right\} \\ \mathcal{CF}^{KE}_{1,2,3}(\mathfrak{v}_{max}) &= \left\{ -\tfrac{2}{\lambda_2}, \tfrac{1}{\lambda_2} - \tfrac{3(\lambda_1+\lambda_2)}{2\left(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2\right)}, \tfrac{1}{\lambda_2} \right\} \end{aligned} \tag{4.5.18}$$

We can use the case $\lambda_1 = 1, \lambda = 2$ as a standard example. In this case, the behavior of the three functions is displayed in Figure 4.3

4.5.2 The complex structure and its integration

All the metrics in the considered family are of cohomogeneity one and have the same isometry, furthermore, they are all Kähler and share the same Kähler 2-form that can be written as it follows:

$$\mathbb{K} = d\mathfrak{u} \wedge d\phi + d\mathfrak{v} \wedge d\tau = \mathbf{e}^1 \wedge \mathbf{e}^2 + \mathbf{e}^3 \wedge \mathbf{e}^4 = \frac{1}{2}\,\mathfrak{J}_{ij}\,\mathbf{e}^i \wedge \mathbf{e}^j \tag{4.5.19}$$

where:

$$\mathfrak{J}^i{}_j = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \delta^{ik}\,\mathfrak{J}_{kj} \tag{4.5.20}$$

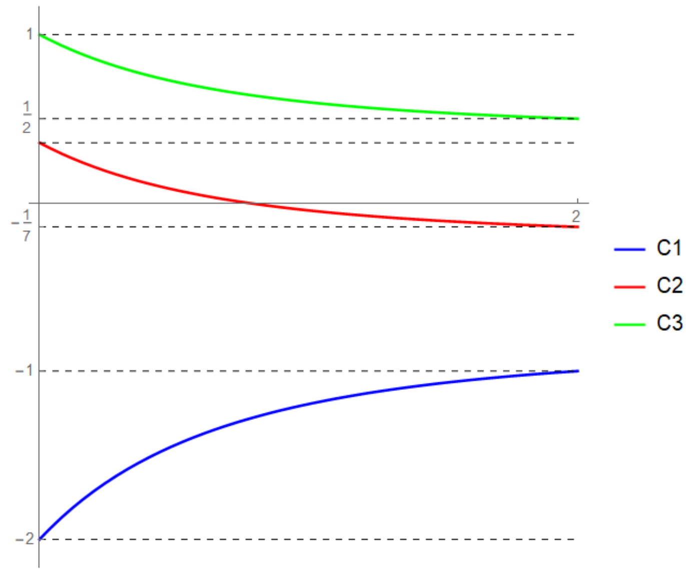


Figure 4.3: Plot of the three functions $\mathcal{CF}^{KE}_{1,2,3}(\mathfrak{v})$ entering the intrinsic Riemann curvature tensor for the KE metric with the choice of the parameter $\lambda_1 = 1$, $\lambda_2 = 2$.

is the complex structure in flat indices. We can easily convert the complex structure into curved indices using the vierbein and its inverse:

$$\mathfrak{J}^{\mu}{}_{\nu} = E^{\mu}_{i}\,\mathfrak{J}^{i}{}_{j}\,E^{j}_{\nu} = \begin{pmatrix} 0 & 0 & -F(\mathfrak{v})\,(\cos\theta + 1) & F(\mathfrak{v}) \\ 0 & 0 & \sin\theta & 0 \\ 0 & -\csc\theta & 0 & 0 \\ -\frac{1}{F(\mathfrak{v})} & \tan\frac{\theta}{2} & 0 & 0 \end{pmatrix} \tag{4.5.21}$$

Since $\mathfrak{J}^2 = -\mathbf{1}_{4\times 4}$ the eigenvalues of $\mathfrak{J}$ are $\pm i$ and the eigenvectors are the rows of the following matrix:

$$\mathfrak{a}^{i}_{\mu} = \begin{pmatrix} \frac{i}{F(\mathfrak{v})} & -i\tan\frac{\theta}{2} & 0 & 1 \\ 0 & i\csc\theta & 1 & 0 \\ -\frac{i}{F(\mathfrak{v})} & i\tan\frac{\theta}{2} & 0 & 1 \\ 0 & -i\csc\theta & 1 & 0 \end{pmatrix} \tag{4.5.22}$$

We obtain the eigen-differentials by defining:

$$\mathrm{da}^i \; = i \, \mathfrak{a}^i_\mu \, dx^\mu \quad ; \quad dx^\mu = \{d\mathfrak{v}, \, d\theta, \, d\phi, \, d\tau\} \tag{4.5.23}$$

The essential thing is that the eigen-differentials are all closed and that the first two are the complex conjugate of the second two:

$$\mathrm{dda}^i \; = \; 0 \quad (i = 1, \ldots, 4) \quad ; \quad \mathrm{da}^1 = \overline{\mathrm{da}^3} \quad ; \quad \mathrm{da}^2 = \overline{\mathrm{da}^4} \tag{4.5.24}$$

This allows us to define the two complex variables u and v, by setting:

$$\begin{aligned} \mathrm{da}^3 &= \mathrm{d}\log[v] \\ \mathrm{da}^4 &= \mathrm{d}\log[u] \end{aligned} \tag{4.5.25}$$

In this way one obtains the universal result:

$$u = e^{i\phi} \tan\frac{\theta}{2} \quad ; \quad v = \frac{1}{2} e^{i\tau} \left(\cos\theta + 1\right) \, H(\mathfrak{v}) \tag{4.5.26}$$

where:

$$H(\mathfrak{v}) = \exp\left[\int \frac{1}{F(\mathfrak{v})} \, d\mathfrak{v}\right] \tag{4.5.27}$$

Hence the whole difference between the various spaces is encoded in the properties of the function $H(\mathfrak{v})$ which is obviously defined up to a multiplicative constant due to the additive integration constant in the exponential.

The function $H(\mathfrak{v})$ for the Kronheimer metric on the smooth $\mathbb{F}_2$ surface. In the case of the metric on $\mathbb{F}_2$ we obtain

$$H^{\mathbb{F}_2}(\mathfrak{v}) = i \sqrt{\frac{1024\mathfrak{v}^2 - 81\alpha^2}{-32\mathfrak{v} + 9(3\alpha + 4)}} \tag{4.5.28}$$

The factor i can always be reabsorbed into a shift of $\pi/2$ of the phase τ and the function $\mathcal{H}^{\mathbb{F}_2}(\mathfrak{v})$ is positive definite in the finite interval $\left[\frac{9\alpha}{32}\,,\, \frac{9}{32}(3\alpha + 4)\right]$ and goes from

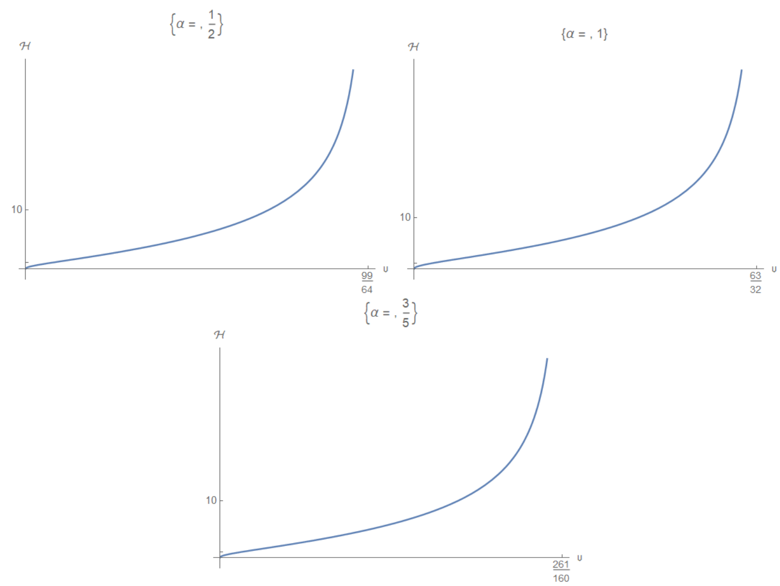


Figure 4.4: Plot of three examples of the $H^{\mathbb{F}_2}(\mathfrak{v})$ function for three different choices of the parameter α.

0 to $+\infty$ for all positive values of $\alpha > 0$. See Figure 4.4 for some examples. What is important is the monotonic behavior of the function $H^{\mathbb{F}_2}(\mathfrak{v})$, which guarantees that the two coordinates u, v describe a copy of $\mathbb{C}^2$ and hence define a dense open chart in the compact manifold $\mathbb{F}_2$.

The function $H(\mathfrak{v})$ for the KE metrics. In the case of the KE metrics in an equally easy way we obtain the following result:

$$\begin{aligned} H^{KE}(\mathfrak{v}) = \exp\Bigg[- \left(\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2^2\right) \Bigg(& \frac{\log(\mathfrak{v} - \lambda_1)}{\lambda_1^2 + \lambda_2\lambda_1 - 2\lambda_2^2} \\ & + \frac{\log(\mathfrak{v} - \lambda_2)}{-2\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2^2} - \frac{\log(\lambda_2\mathfrak{v} + \lambda_1(\lambda_2 + \mathfrak{v}))}{2\lambda_1^2 + 5\lambda_2\lambda_1 + 2\lambda_2^2} \Bigg) \Bigg] \end{aligned} \tag{4.5.29}$$

The structure of the function is similar to that of the $\mathbb{F}_2$ case since there is a zero of the function in the lower limit $\mathfrak{v} \to \lambda_1$ and a pole in the upper limit $\mathfrak{v} \to \lambda_2$, yet this time the exponents of the pole and of the zero are rational numbers depending

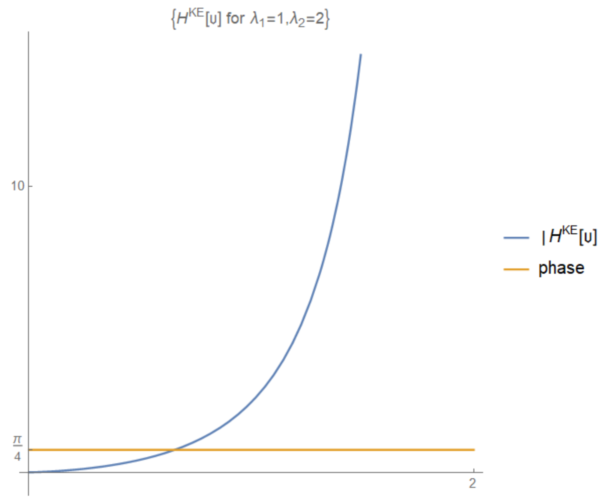


Figure 4.5: Plot of the $H(\mathfrak{v})$ function in the KE case with the choice $\lambda_1 = 1, \lambda_2 = 2$.

on the choice of the roots $\lambda_{1,2}$; similarly, it happens for the third factor associated with the third root which is located out of the basic polytope (see Figure 4.1). Our canonical example $\lambda_1 = 1, \lambda_2 = 2$ helps to illustrate the general case; with this choice we obtain

$$H^{KE}(\mathfrak{v}) \mid_{\lambda_1=1,\lambda_2=2} = \frac{(\mathfrak{v}-1)^{7/5}(3\mathfrak{v}+2)^{7/20}}{(\mathfrak{v}-2)^{7/4}} = e^{i\frac{\pi}{4}} \times \underbrace{\frac{(\mathfrak{v}-1)^{7/5}(3\mathfrak{v}+2)^{7/20}}{(2-\mathfrak{v})^{7/4}}}_{\mathcal{H}^{KE}(\mathfrak{v})} \quad (4.5.30)$$

where, once again, the constant phase factor can be reabsorbed by a constant shift of the angular variable τ and what remains of $\mathcal{H}^{KE}(\mathfrak{v})$ is a positive definite function of $\mathfrak{v}$ in the interval $[1, 2]$ that has the same feature of its analogue in the $\mathbb{F}_2$ case, namely, it maps, smoothly and monotonically, the finite interval $(1, 2)$ into the infinite interval $(0, +\infty)$. The behavior of this function is displayed in Figure 4.5.

4.6 The structure of $\mathcal{M}_3$ and the conical singularity

Let us anticipate the main argument which we will develop further on. The two real manifolds defined by the restriction to the dense chart $\mathfrak{u}, \mathfrak{v}, \phi, \tau$, of the surface $\mathbb{F}_2$ and of the manifold $\mathcal{M}_B^{KE}$ are fully analogous. Cutting the compact four manifold into $\mathfrak{v} =$ const slices we always obtain the same result, namely a three-manifold $\mathcal{M}_3$ with the structure of a circle fibration on S^2:

$$\mathcal{M}_B \supset \mathcal{M}_3 \xrightarrow{\pi} S^2 \quad ; \quad \forall p \in S^2 \quad \pi^{-1}(p) \sim S^1 \tag{4.6.1}$$

The metric on $\mathcal{M}_3$ is the standard one for fibrations:

$$ds^2_{\mathcal{M}_3} = \mathfrak{v} \left(d\phi^2 \sin^2\theta + d\theta^2\right) + F(\mathfrak{v}) \left[d\phi(1-\cos\theta) + d\tau\right]^2 \tag{4.6.2}$$

The easiest way to understand $\mathcal{M}_3$ is to study its intrinsic curvature by using the dreibein formalism. Referring to equation (4.6.2) we introduce the following dreiben 1-forms:

$$\boldsymbol{\epsilon}^1 = \sqrt{\mathfrak{v}}\, d\theta \quad ; \quad \boldsymbol{\epsilon}^2 = \sqrt{\mathfrak{v}}\, \sin\theta d\phi \quad ; \quad \boldsymbol{\epsilon}^3 = \sqrt{F(\mathfrak{v})}\, \left[d\phi(1-\cos\theta) + d\tau\right] \tag{4.6.3}$$

The fixed parameter $\mathfrak{v}$ plays the role of the squared radius of the sphere S^2 while $\sqrt{F(\mathfrak{v})}$ weights the contribution of the circle fiber defined over each point $p \in S^2$. At the endpoints of the intervals $F(\mathfrak{v}_{min}) = F(\mathfrak{v}_{max}) = 0$ the fiber shrinks to zero.

Using the standard formulas of differential geometry and once again the MATHEMATICA package NOVAMANIFOLDA we calculate the spin connection and the curvature 2-form. We obtain:

$$\mathfrak{R} = \left(\begin{array}{cc||c} 0 & \frac{(4\mathfrak{v}-3F(\mathfrak{v}))}{4\mathfrak{v}^2}\, \boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2 & \frac{F(\mathfrak{v})}{4\mathfrak{v}^2}\, \boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^3 \\ -\frac{(4\mathfrak{v}-3F(\mathfrak{v}))}{4\mathfrak{v}^2}\, \boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2 & 0 & \frac{F(\mathfrak{v})}{4\mathfrak{v}^2}\, \boldsymbol{\epsilon}^2 \wedge \boldsymbol{\epsilon}^3 \\ \hline -\frac{F(\mathfrak{v})}{4\mathfrak{v}^2}\, \boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^3 & -\frac{F(\mathfrak{v})}{4\mathfrak{v}^2}\, \boldsymbol{\epsilon}^2 \wedge \boldsymbol{\epsilon}^3 & 0 \end{array}\right) \tag{4.6.4}$$

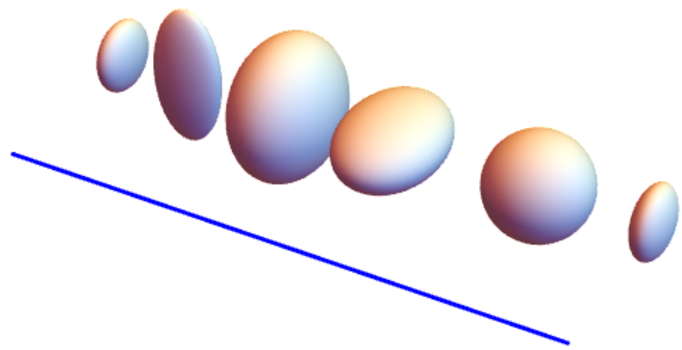


Figure 4.6: A conceptual picture of the $\mathcal{M}_B$ spaces that include also the second Hirzebruch surface. The finite blue segment represents the $\mathfrak{v}$-variable varying from its minimum to its maximum value. Over each point of the line, we have a three-dimensional space $\mathcal{M}_3$ which is homeomorphic to a 3-sphere but is variously deformed at each different value $\mathfrak{v}$. At the initial and final points of the blue segment the three dimensional space degenerates into an S^2 sphere. Graphically we represent the deformed 3-sphere as an ellipsoid and the 2-sphere as a flat-filled circle.

The Riemann curvature 2-form in flat indices has constant components and if the coefficient $\frac{(4\mathfrak{v}-3F(\mathfrak{v}))}{4\mathfrak{v}^2}$ were equal to the coefficient $\frac{F(\mathfrak{v})}{4\mathfrak{v}^2}$ the 2-form in eqn. (4.6.4) would be the standard Riemann 2-curvature of the homogeneous space SO(4)/SO(3), namely the the 3-sphere S^3. What we learn from this easy calculation is that every section $\mathfrak{v} =$ constant of $\mathcal{M}_B$ is homeomorphic to a 3-sphere endowed with a metric that is not the maximal symmetric one with isometry $\mathrm{SU}(2) \times \mathrm{SU}(2)$ but a slightly deformed one with isometry $\mathrm{SU}(2) \times \mathrm{U}(1)$: in other words, we deal with a 3-sphere deformed into the 3-dimensional analog of an ellipsoid. At the endpoints of the $\mathfrak{v}$-interval the ellipsoid degenerates into a sphere since the third dreibein $\boldsymbol{\epsilon}$ vanishes. A conceptual picture of the full space $\mathcal{M}_B$ is provided in picture Figure 4.6.

4.6.1 Global properties of $\mathcal{M}_3$

Expanding on the global properties of $\mathcal{M}_3$, we describe it as a magnetic monopole bundle over S^2 and prove that the corresponding monopole strength is $n = 2$. We start from the definition of the action of the $SU(2)$ isometry (4.2.2) and describe the 2-sphere S^2 spanned by θ and ϕ as $\mathbb{CP}^1$ with projective coordinates U^0, U^1:

$$U^0 = r \sin\left(\frac{\theta}{2}\right) e^{i\,\frac{\gamma+\phi}{2}}\,, \quad U^1 = r\cos\left(\frac{\theta}{2}\right) e^{i\,\frac{\gamma-\phi}{2}} \tag{4.6.5}$$

where $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$, $0 \le \gamma < 4\pi$. In the North patch $\mathcal{U}_N$, $U^1 \neq 0$ and the sphere is spanned by the stereographic coordinate $u_N = U^0/U^1$, while in the south patch $\mathcal{U}_S$, $U^0 \neq 0$ and the stereographic coordinate $u_S = U^1/U^0$. The transformation properties (4.2.2) define a line bundle whose local trivializations about the two poles are:

$$\phi_N^{-1}(\mathcal{U}_N) = (u_N,\, v_N) = \left(\frac{U^0}{U^1},\, \xi\,(U^1)^2\right)\,, \tag{4.6.6}$$

$$\phi_S^{-1}(\mathcal{U}_S) = (u_S,\, v_S) = \left(\frac{U^1}{U^0},\, \xi\,(U^0)^2\right)\,, \tag{4.6.7}$$

where ξ is a complex number in the fiber not depending on the patch. As $(U^0,\, U^1)$ transform linearly under the an $SU(2)$-transformation:

$$\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} U^0 \\ U^1 \end{pmatrix}\,, \tag{4.6.8}$$

the fiber coordinate v transforms so that $(1+|u_N|^2)^2|v_N|^2$ and $(1+|u_S|^2)^2|v_S|^2$, in $\mathcal{U}_N$ and $\mathcal{U}_S$, respectively, are invariant. The transition function on the fiber reads, at the equator $\theta = \pi/2$:

$$t_{NS} = \left(\frac{U^1}{U^0}\right)^2 = e^{-2i\phi} = e^{-in\,\phi}\,, \tag{4.6.9}$$

implying that the $U(1)$-bundle associated with the phase of v (i.e. the submanifold of the Kähler-Einstein space at constant $|v|$), is a monopole bundle with monopole strength $n = 2$. This has to be contrasted with the Hopf-fiber description of S^3, for which the local trivializations have fiber components $U^1/|U^1|$ and $U^0/|U^0|$ in the two patches, respectively, and t_{NS} at the equator is $U^1/U^0 = e^{-i\phi}$. In this case, the monopole strength is $n = 1$.We can verify that the manifold at constant $|v|$ is a Lens space $S^3/\mathbb{Z}_2$ also by direct inspection of the metric.

4.6.2 Conical singularities and regularity of $\mathbb{F}_2$

Let us prove, by focusing on the fiber spanned by $\mathfrak{v}$ and $\tau \in (0, 2\pi)$, that the KE manifolds feature

$$ds^2 = \frac{d\mathfrak{v}^2}{F(\mathfrak{v})} + F(\mathfrak{v})\, d\tau^2 \,. \tag{4.6.10}$$

Let λ denote one of the two roots λ_1, λ_2 of $F(\mathfrak{v})$. Close to λ, to first order in $\mathfrak{v}$, in the KE case, the metric (4.6.10) is flat and features a deficit angle signaling a conifold singularity. This singularity is absent in the $\mathbb{F}_2$, as expected. To show this let us Taylor expand $F(\mathfrak{v})$ about λ:

$$F(\mathfrak{v}) = F'(\lambda)(\mathfrak{v} - \lambda) + O((\mathfrak{v} - \lambda)^2). \tag{4.6.11}$$

We can verify that:

$$\text{KE} : F'(\lambda_1) = \frac{(\lambda_2 - \lambda_1)(\lambda_1 + 2\lambda_2)}{\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2} \;,\;\; F'(\lambda_2) = \frac{(\lambda_1 - \lambda_2)(2\lambda_1 + \lambda_2)}{\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2} \,, \tag{4.6.12}$$

$$\mathbb{F}_2 : F'(\lambda_1) = 2 \;,\;\; F'(\lambda_2) = -2 \,. \tag{4.6.13}$$

Next, we replace the first-order expansion of this function in the fiber metric:

$$ds^2 = \frac{d\mathfrak{v}^2}{F'(\lambda)(\mathfrak{v} - \lambda)} + F'(\lambda)(\mathfrak{v} - \lambda)\, d\tau^2 \,, \tag{4.6.14}$$

and write it as a flat metric in polar coordinates:

$$ds^2 = dr^2 + \beta^2\, r^2\, d\tau^2 \,. \tag{4.6.15}$$

One can easily verify that:

$$r = 2\sqrt{\frac{\mathfrak{v}-\lambda}{F'(\lambda)}}\ , \quad \beta = \frac{|F'(\lambda)|}{2}\,. \tag{4.6.16}$$

Defining $\tilde{\varphi} = \beta\,\tau$, we can write the fiber metric as follows:

$$ds^2 = dr^2 + r^2\,d\tilde{\varphi}^2\,.$$

Now the polar angle varies in the range: $\tilde{\varphi} \in [0,\, 2\pi\,\beta]$. If $\beta < 1$ we have a deficit angle:

$$\Delta\phi = 2\pi(1-\beta)\,.$$

Let us see what this implies in the five possible cases of Table 4.1.

1. In the case of the $\mathbb{F}_2$ manifold one has $|F'(\lambda)| = 2$ and $\beta = 1$, so there is no conical singularity, as expected.

2. In the case of $\mathbb{WP}[1,1,2]$ we have

$$F'(\lambda) = \frac{32\lambda^2 - 144\lambda + 81}{(4\lambda-9)^2}.$$

   For the limiting value $\lambda = 0$ we obtain $\beta = \frac{1}{2}$, i.e., a $\mathbb{C}^2/\mathbb{Z}_2$ singularity, while for $\lambda = \frac{9}{8}$ we have $\beta = 1$, i.e., no singularity, as we expected as $\mathbb{WP}[1,1,2]$ is an orbifold $\mathbb{P}^2/\mathbb{Z}_2$ with one singular point.

3. In the KE manifold case considering $\lambda = \lambda_1$, we have:

$$F'(\lambda_1) = \frac{(\lambda_2-\lambda_1)(\lambda_1+2\lambda_2)}{\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2} = -1 + \frac{3\lambda_2^2}{\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2} < -1 + \frac{3\lambda_2^2}{\lambda_2^2} = 2 \quad \Rightarrow\ \beta < 1\,, \tag{4.6.17}$$

   and $|F'(\lambda_2)| < |F'(\lambda_1)|$, so that $\beta < 1$ also at λ_2. The manifold has two conical singularities, both in the same fiber of the projection to one of the S^2's.

One of the singularities will be an orbifold singularity of type $\mathbb{C}^2/\mathbb{Z}_n$ if the corresponding value of β is

$$\beta = 1 - \frac{1}{n}. \tag{4.6.18}$$

It is interesting to note that when this happens, the form of the function F does not allow the other singularity to be of this type as well, as the corresponding integer m should satisfy

$$m = \frac{4n}{2 + 5n \pm \sqrt{9n^2 + 12n - 12}}$$

which is not satisfied by any pair (m, n) where both m, and n are integers greater than 1; so the singular fiber can never be a football or a spindle.

4. We discuss the case $\lambda_1 = 0$. In this case

$$F^0(\mathfrak{v}) = \frac{\mathfrak{v}(\lambda_2 - \mathfrak{v})}{\lambda_2}\,. \tag{4.6.19}$$

If we focus on the fiber metric:

$$ds^2 = \frac{d\mathfrak{v}^2}{F_0(\mathfrak{v})} + F_0(\mathfrak{v})\, d\tau^2\,. \tag{4.6.20}$$

We can easily verify that in the coordinates $\tilde{\theta} \in [0, \pi]$ and $\tilde{\varphi} \in [0, \pi)$ defined by

$$\mathfrak{v}(\tilde{\theta}) = R^2 \sin^2\left(\frac{\tilde{\theta}}{2}\right) \leq R^2 = \lambda_2\ ,\ \ \tilde{\varphi} = \frac{\tau}{2}\,,$$

where $R = \sqrt{\lambda_2}$, the fiber metric (4.6.20) becomes

$$ds^2 = R^2\left(d\tilde{\theta}^2 + \sin^2(\tilde{\theta})\, d\tilde{\varphi}^2\right)\,. \tag{4.6.21}$$

Since $\tilde{\varphi} = \tau/2 \in [0, \pi)$, the fiber is the splindle $S^2/\mathbb{Z}_2$. Topologically, the entire 4-manifold is still $S^2 \times S^2$.

5. For $F(\mathfrak{v}) = \mathfrak{v}$ we get $\beta = \frac{1}{2}$, in accordance with the fact that the variety, in this case, is $\mathbb{C}^2/\mathbb{Z}_2$.

4.7 Complex structures

In this section we study the complex structures corresponding to the KE case, i.e., the cases corresponding to the function F^{KE}. These are singular KE manifolds of complex dimension 2, homeomorphic to $S^2 \times S^2$. To make the analysis completely quantitative, let us choose a value of the parameter α and two values of λ_1, λ_2 so that the basic polytope becomes exactly identical in the two cases.

We choose the value $\alpha = \frac{4}{9}$ so that the endpoints of the interval in the pure $\mathbb{F}_2$ case are:

$$\mathfrak{v}_{min} = \frac{1}{8} \quad ; \quad \mathfrak{v}_{max} = \frac{3}{2} \tag{4.7.1}$$

and using the previously discussed procedure we obtain the complex v coordinate for the $\mathbb{F}_2$ pure case:

$$v_{\mathbb{F}_2} = \exp\left[i\left(\tau + \frac{\pi}{4}\right)\right] \times \frac{1+\cos\theta}{2} \times \sqrt{\frac{64\mathfrak{v}^2 - 1}{3\,-\,2\,\mathfrak{v}}} \tag{4.7.2}$$

In the same way, we obtain the complex v variable for the Kähler Einstein case:

$$v_{KE} = \exp\left[i\left(\tau + \frac{157}{154}\pi\right)\right] \times \frac{1+\cos\theta}{2} \times \frac{\left(\mathfrak{v}-\frac{1}{8}\right)^{157/275}\left(\frac{3\mathfrak{v}}{2}+\frac{1}{8}\left(\mathfrak{v}+\frac{3}{2}\right)\right)^{157/350}}{\left(\frac{3}{2}-\mathfrak{v}\right)^{157/154}} \tag{4.7.3}$$

Obviously there is no holomorphic way of writing v_{KE} in terms of $v_{\mathbb{F}_2}$ or viceversa. This can be immediately seen in the following way. Taking the ratio of the two coordinates v we obtain:

$$(i)^{\frac{237}{154}} \times \frac{v_{\mathbb{F}_2}}{v_{KE}} = 4 \times \frac{2^{1877/3850}(3-2\mathfrak{v})^{40/77}\sqrt{64\mathfrak{v}^2-1}}{(8\mathfrak{v}-1)^{157/275}(26\mathfrak{v}+3)^{157/350}} \tag{4.7.4}$$

Moreover, with some manipulations, we can write

$$\mathfrak{v} = \frac{1}{64}\left(-\varpi_{\mathbb{F}_2} \pm \sqrt{\varpi^2_{\mathbb{F}_2} + 192\,\varpi_{\mathbb{F}_2} + 64}\right) \tag{4.7.5}$$

Inserting eqn. (4.7.5) into eqn. (4.7.4) we immediately arrive at the conclusion that the complex coordinate v_{KE} is not a holomorphic function of the complex coordinates $u_{\mathbb{F}_2}, v_{\mathbb{F}_2}$.

The argument that we have utilized here can be used for all values of α. We can always choose the independent roots $\lambda_{1,2}$ so that the interval of the moment variable $\mathfrak{v}_{min}$,$\mathfrak{v}_{max}$ coincides in the Kähler Einstein case and in the $\mathbb{F}_2$ case. In the dense open chart that we are using $\mathbb{F}_2$ and the manifold that admits a KE metric have different complex structures. The plot of the two functions $\mathcal{H}(\mathfrak{v})$ is displayed in Figure 4.7.

The remaining problem is therefore the following. When we utilize the Kronheimer metric for $\mathbb{F}_2$ written in real variables in the open chart provided by the coordinates $\mathfrak{u}, \mathfrak{v}, \phi, \tau$ we know that the closure of such a dense chart is the second Hirzebruch surface. In the same open real chart, we have also a KE metric: the question is, *What is the closure of such an open real chart compatible with the KE metric?*. The important point to keep in mind while trying to answer such a question is that, topologically, the Hirzebruch surface is just $S^2 \times S^2$. What makes this real manifold Hirzebruch is the complex structure induced by the holomorphic embedding in $\mathbb{P}^1 \times \mathbb{P}^2$ as an algebraic variety. Yet the complex structure of $\mathbb{F}_2$ is different and incompatible with the complex structure compatible with the KE metric defined in the same open chart. This must be the guiding principle.

4.7.1 The homeomorphism with $S^2 \times S^2$

We want to analyze in detail the homeomorphism with the space $S^2 \times S^2$ in the KE case. Prior to that let us stress, once again, that all the manifolds we are considering are KE by construction, as:

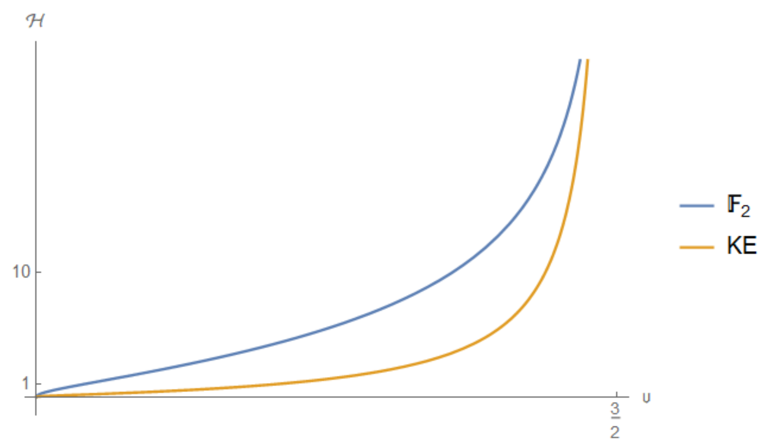


Figure 4.7: Comparison of the plots of $\mathcal{H}^{\mathbb{F}_2}(\mathfrak{v})$ for the pure $\mathbb{F}_2$ case with $\mathcal{H}^{KE}(\mathfrak{v})$ for the KE case, when they are calibrated to insist on the same interval $\left[\frac{1}{8}, \frac{3}{2}\right]$

a) There is a candidate Kähler form written as

$$\mathbb{K} = K_{ij}\mathbf{e}^i \wedge \mathbf{e}^j \tag{4.7.6}$$

where the one forms $\mathbf{e}^i$ are a tetrad representation of the considered metric:

$$ds^2_{\mathcal{M}_B} = g_{\mu\nu}dx^\mu \, dx^\nu = \delta_{ij} \, \mathbf{e}^i \mathbf{e}^j \tag{4.7.7}$$

b) The candidate form is closed:

$$\mathrm{d}\mathbb{K} = 0 \tag{4.7.8}$$

c) The component tensor of the Kähler form in flat indices K_{ij} satisfies the condition:

$$K_{ij} \, K_{jk} = - \, \delta_{ik} \tag{4.7.9}$$

This guarantees that for each metric we can construct the corresponding complex structure tensor:

$$\mathfrak{J}_\nu{}^\mu = E^i_\mu \, K_{ij} \, \delta^{jk} \, E^\nu_k \quad ; \quad \mathfrak{J}^2 = - \, \mathrm{Id} \tag{4.7.10}$$

and by construction, the metric is hermitian with respect to that contact structure.

The Kähler form is the same for all the families of metrics (4.5.2) and each metric chooses the complex structure with respect to which it is hermitian. In principle these complex structures are all different, yet some of them might be compatible, as is the case for the one-parameter family of metrics on the Hirzebruch surface, that share the same complex structure and can be described in terms of the same complex coordinates. However, in the previous section, we have already shown that the complex structure selected by one of the KE metrics is certainly incompatible with that of the Hirzebruch surface and this removes any possible conceptual clash. On the other hand there is no obstacle to the fact that the underlying real manifold of the Hirzebruch case and of the (singular) KE case might be homeomorphic and this is what we want to show.

In the next lines, we argue how to construct explicitly such a homeomorphism. First of all, by looking at the metric in eqn. (4.5.2) we see that the first 2-sphere is already singled out in the standard coordinates θ and ϕ. As for the second sphere, the azimuthal angle is already identified in the coordinate τ. It remains to be seen that the coordinate $\mathfrak{v}$ in the finite closed range $[\mathfrak{v}_{min}\,,\,\mathfrak{v}_{max}]$ is in one-to-one continuous correspondence with a new right ascension angle χ.

Behavior of the function $\sqrt{F(\mathfrak{v})}$. To this effect the main point is that the function $\sqrt{F(\mathfrak{v})}$ should be upper limited by the value 1 in the interval $[\mathfrak{v}_{min}\,,\,\mathfrak{v}_{max}]$, it should grow monotonically from 0 to a maximum value $a_0 \le 1$, attained at $\mathfrak{v} = \mathfrak{v}_0$ and then it should decrease monotonically from a_0 to 0 in the second part of the interval $[\mathfrak{v}_0\,,\,\mathfrak{v}_{max}]$. Under such conditions the inverse function arcsin can be applied unambiguously to $\sqrt{F(\mathfrak{v})}$ and we can obtain a one-to-one continuous map between the coordinate $\mathfrak{v}$ and a new right ascension angle χ. The homeomorphism is encoded in the following relation where the function $\mathfrak{h}(\mathfrak{v})$ is continuous and monotonous only under the above carefully specified conditions.

$$\chi = \mathfrak{h}(\mathfrak{v}) = \frac{\pi \arcsin\left(\sqrt{F(\mathfrak{v})}\right)}{2\arcsin\left(a_0\right)} + \Theta\left(\mathfrak{v}-\mathfrak{v}_0\right)\pi\left(1-\frac{\arcsin\left(\sqrt{F(\mathfrak{v})}\right)}{\arcsin\left(a_0\right)}\right) \quad (4.7.11)$$

In the above formula the symbol $\Theta(x)$ denotes the well known Heaviside step function that vanishes for $x < 0$ and evaluates to 1 for $x > 0$.

The relevant fact is that for both the case of the Hirzebruch surface metric and the KE ones the above specified conditions are verified and the homeomorphism (4.7.11) can be written. We examine in detail one instance of the first and one instance of the second case, having verified that in each class the chosen examples represent the behavior of all members of the same class. For the Hirzebruch case, we set the parameter $\alpha = 1$ and we obtain:

$$F^{\mathbb{F}_2}\left(\mathfrak{v}\right) = \frac{\left(32\mathfrak{v}-63\right)\left(1024\mathfrak{v}^2-81\right)}{16\left(1024\mathfrak{v}^2-4032\mathfrak{v}+81\right)} \quad (4.7.12)$$

For the KE case, we choose as we already did in previous section $\lambda_1 = 1$, $\lambda_2 = 2$ and we obtain:

$$F^{KE}\left(\mathfrak{v}\right) = -\frac{(\mathfrak{v}-2)(\mathfrak{v}-1)(3\mathfrak{v}+2)}{7\mathfrak{v}} \quad (4.7.13)$$

The behavior of the function $\sqrt{F(\mathfrak{v})}$ in the two cases and the associated homeomorphism on the right ascension angle is shown in Figure s 4.8,4.9 On the basis of the above lore, in order to explore the behavior of a function, a vector field, or whatever different geometric object in the neighborhood of the North pole of the second sphere, one has at their disposal a well-defined transition function from the coordinates $\mathfrak{v}_N, \tau_N$ to the coordinates $\mathfrak{v}_S, \tau_S$:

$$\begin{aligned} \mathfrak{v}_S &= \mathfrak{h}^{-1}\left(\mathfrak{h}(\mathfrak{v}_N)-\pi\right) \\ \tau_S &= -\tau_N + \pi \end{aligned} \quad (4.7.14)$$

The inverse of the function $\mathfrak{h}$ is highly nontrivial since it is a combination of transcendental and algebraic functions, yet it can always be done numerically if needed.

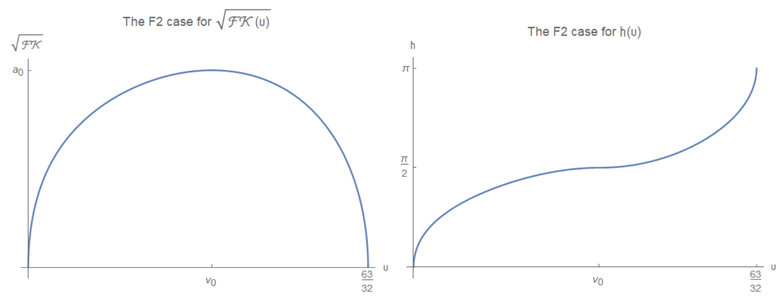


Figure 4.8: On the left the plot of the function $\sqrt{F^{\mathbb{F}_2}(\mathfrak{v})}$ corresponding to $\alpha = 1$ and explicitly displayed in eqn. (4.7.12). On the right the corresponding function $\mathfrak{h}(\mathfrak{v})$ providing the homeomorphism to the right ascension angle.

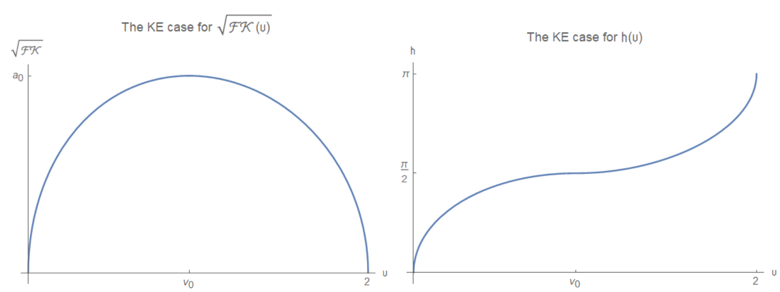


Figure 4.9: On the left the plot of the function $\sqrt{F^{KE}(\mathfrak{v})}$ corresponding to $\lambda_1 = 1, \lambda_2 = 2$ and explicitly displayed in eqn. (4.7.13). On the right the corresponding function $\mathfrak{h}(\mathfrak{v})$ providing the homeomorphism to the right ascension angle.

As for the metric itself, there is no need since the curvature is nonsingular in the point $\mathfrak{v}_{max}$ and the geodesics are also well-behaved in all limits.

The conclusion is that the underlying manifold of the Kähler Einstein metrics is $S^2 \times S^2$, as in the case of $\mathbb{F}_2$.

4.8 Liouville vector field on $\mathcal{M}_B$ and the contact structure on $\mathcal{M}_3$

Next we go back to consider general properties of the metric (4.5.2), and using eqn. (4.5.4) we also rewrite in terms of the coordinates $\mathfrak{u}, \mathfrak{v}, \phi, \tau$:

$$ds^2_{\mathcal{M}_B} = \frac{\mathrm{d}\mathfrak{v}^2}{F(\mathfrak{v})} + \frac{\mathfrak{u}(2\mathfrak{v}-\mathfrak{u})}{\mathfrak{v}}\mathrm{d}\phi^2 + \frac{(\mathfrak{v}\mathrm{d}\mathfrak{u} - \mathfrak{u}\mathrm{d}\mathfrak{v})^2}{\mathfrak{u}\mathfrak{v}(2\mathfrak{v}-\mathfrak{u})} + \frac{F(\mathfrak{v})}{\mathfrak{v}^2}\left(\mathfrak{u}\,\mathrm{d}\phi + \mathfrak{v}\,\mathrm{d}\tau\right)^2 \tag{4.8.1}$$

The corresponding Kähler 2-form is provided by the equation (4.5.19) that for reader's convenience, we copy here:

$$\mathbb{K} = d\mathfrak{u} \wedge d\phi \,+\, d\mathfrak{v} \wedge d\tau \tag{4.8.2}$$

The pair $(\mathcal{M}_B, \mathbb{K})$ constitutes a symplectic manifold, independently from the Riemannian structure provided by the metric (4.8.1). For symplectic manifolds there exists the notion of Liouville vector fields (see for instance [28, 25]) defined as follows. The vector field $\mathbf{L} \in \Gamma\left[T\mathcal{M}_B, \mathcal{M}_B\right]$ is a Liouville vector field if

$$\mathcal{L}_{\mathbf{L}}\,\mathbb{K} = \mathbb{K} \tag{4.8.3}$$

where $\mathcal{L}_{\mathbf{V}}$ denotes the Lie derivative along the specified vector field $\mathbf{V}$. Utilizing Cartan's formula for the Lie derivative, we get:

$$\mathcal{L}_{\mathbf{L}}\mathbb{K} = i_{\mathbf{L}}\,\mathrm{d}\mathbb{K} \,+\, \mathrm{d}\left(i_{\mathbf{L}}\,\mathbb{K}\right) \,=\, \mathrm{d}\left(i_{\mathbf{L}}\,\mathbb{K}\right) \,=\, \mathbb{K} \tag{4.8.4}$$

A very simple Liouville field for the symplectic manifold $(\mathcal{M}_B, \mathbb{K})$ is the following one:

$$\mathbf{L} = \mathfrak{u}\,\frac{\partial}{\partial \mathfrak{u}} \,+\, \mathfrak{v}\,\frac{\partial}{\partial \mathfrak{v}} \tag{4.8.5}$$

as one can immediately verify.

Another result in symplectic geometry, (see [28, 24, 40, 15, 13, 14, 25]) states that a $(2n+1)$-submanifold $\mathcal{Z} \subset \mathcal{M}_B$ of a $(2n+2)$-dimensional symplectic manifold $(\mathcal{M}_B, \mathbb{K})$ that is transverse to a Liouville field $\mathbf{L}$ is a contact manifold with the contact structure

$$\Omega = i_{\mathbf{L}}\,\mathbb{K} \tag{4.8.6}$$

In view of this theorem, the interpretation of the $\mathcal{M}_3^{\mathfrak{v}}$ manifolds, extensively discussed in previous sections, that have the topology of S^3 and are all transverse to the Liouville field since they correspond to fixed values of the coordinate $\mathfrak{v}$, becomes clear. They constitute the leaves of a foliation of the symplectic manifold $(\mathcal{M}_B, \mathbb{K})$ in diffeomorphic contact manifolds whose contact form is:

$$\Omega = \mathfrak{u}\,\mathrm{d}\phi \,+\, \mathfrak{v}\,\mathrm{d}\tau \tag{4.8.7}$$

On each leave $\mathfrak{v} = \mathrm{const}$ we have:

$$\mathrm{d}\Omega = \mathrm{d}\mathfrak{u} \wedge \mathrm{d}\phi \quad ; \quad \mathrm{d}\Omega \,\wedge\, \Omega = \mathfrak{v}\,\mathrm{d}\mathfrak{u} \wedge \mathrm{d}\phi \wedge \mathrm{d}\tau = \mathrm{const} \times \mathrm{Vol}_3^{\mathfrak{v}} \tag{4.8.8}$$

4.8.1 The Reeb field and Beltrami equation

It is now interesting to calculate the normalized Reeb field associated with the contact form Ω. This is possible since the symplectic manifold $(\mathcal{M}_B, \mathbb{K})$ is endowed with the Riemannian structure provided by the metric (4.6.2). Expanding the one-form Ω along the coordinate differentials:

$$\Omega = \Omega_\mu\, dy^\mu \quad ; \quad y^\mu = \{\theta, \phi, \tau\} \tag{4.8.9}$$

we find:

$$\Omega_\mu = \{0, \mathfrak{v}(1-\cos(\theta)), \mathfrak{v}\} \tag{4.8.10}$$

Utilizing the inverse of the metric tensor defined by the line element (4.6.2) namely:

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{\mathfrak{v}} & 0 & 0 \\ 0 & \frac{\csc^2(\theta)}{\mathfrak{v}} & \frac{(\cos(\theta)-1)\csc^2(\theta)}{\mathfrak{v}} \\ 0 & \frac{(\cos(\theta)-1)\csc^2(\theta)}{\mathfrak{v}} & \frac{1}{F(\mathfrak{v})} + \frac{\tan^2\left(\frac{\theta}{2}\right)}{\mathfrak{v}} \end{pmatrix} \tag{4.8.11}$$

we can raise the index of Ω_μ and we obtain the components of a normalized Reeb vector field:

$$U^\mu \;=\; g^{\mu\nu}\Omega_\nu = \left\{0, 0, \frac{\mathfrak{v}}{F(\mathfrak{v})}\right\} \quad \Rightarrow \quad \mathbf{U} \;=\; \frac{\mathfrak{v}}{F(\mathfrak{v})}\,\partial_\tau \tag{4.8.12}$$

such that:

$$\Omega\left(\mathbf{U}\right) = 1 \quad ; \quad i_{\mathbf{U}}\,\mathrm{d}\Omega = 0 \tag{4.8.13}$$

It is a notable fact that the above The Reeb vector field automatically satisfies the Beltrami equation. Indeed it is known that every contact structure in 3 dimensions admits a contact form and an associated Reeb field that satisfies Beltrami equation (see for instance [24, 25]) yet it is remarkable that the choice of the Liouville vector field (4.8.3) immediately selects a Beltrami Reeb field. The verification of our statement is almost immediate if we utilize the formulation of Beltrami equation introduced in [16] (see also [25]), namely:

$$\mathrm{d}\Omega^{\mathbf{U}} = \lambda\, i_{\mathbf{U}}\,\mathrm{Vol}_3 \tag{4.8.14}$$

where Vol_3 denotes the volume 3-form of the considered 3-manifold, $\Omega^{\mathbf{U}}$ is the contact form that admits $\mathbf{U}$ as normalized Reeb field and $\lambda \in \mathbb{R}$ is the Beltrami eigenvalue. In our case the volume form is:

$$\mathrm{Vol}_3 = \boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2 \wedge \boldsymbol{\epsilon}^3 \;=\; \mathfrak{v}\,\sqrt{F(\mathfrak{v})}\,\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi \wedge \mathrm{d}\tau \tag{4.8.15}$$

and equation (4.8.14) is satisfied with eigenvalue:

$$\lambda = -\frac{1}{\mathfrak{v}}. \tag{4.8.16}$$

4.9 Geodesics for the family of manifolds $\mathcal{M}_B$

In this section, we study the general problem of calculating the geodesics for the class of metrics (4.5.2). Sometimes the differential system determining the geodesics is completely integrable and this allows one to reduce it to first order and to quadratures, obtaining in this way the complete set of all geodesics — leaving apart the practical problem of inverting transcendental functions, which possibly can be accomplished with numerical methods. An example is the Kerr metric where there is a hidden first integral (the Carter constant) which can be revealed by the use of the Hamilton-Jacobi formulation, and allows for complete integration.

We show in this section that for any choice of the function $F(\mathfrak{v})$ the geodesic dynamical system associated with the metrics (4.5.2) is fully integrable and admits a hidden Carter constant, another first integral in addition to the Hamiltonian, which allows to write the full system of geodesic lines for all the metrics in the class, in particular for the $\mathbb{F}_2$ surface and for the KE manifolds brought to attention in this thesis.

4.9.1 The geodesic equation

We take for Lagrangian functional the square of the arc length

$$\mathcal{L} = \frac{1}{2}\,F(\mathfrak{v})\left(\dot{\phi}\,(1-\cos\theta)+\dot{\tau}\right)^2 + \frac{\dot{\mathfrak{v}}^2}{F(\mathfrak{v})} + \mathfrak{v}\left(\dot{\phi}^2\sin^2\theta+\dot{\theta}^2\right). \tag{4.9.1}$$

As usual, the Euler-Lagrange equations take the standard form if the Lagrangian satisfies the constraint

$$\mathcal{L}\mid_{\text{on geodesics}} = \frac{k}{4} \tag{4.9.2}$$

for some $k > 0$; in this way parameter along the curves is the arc length s. In a mechanical analogy, k is the energy.

Cyclic variables and conserved momenta. The angles ϕ and τ are cyclic variables (due to toric symmetry) which leads to two first integrals of the motion, which we call $\ell,\ m$, and can be represented in a synthetic way:

$$\begin{pmatrix} \ell \\ m \end{pmatrix} = \begin{pmatrix} p_\phi \\ p_\tau \end{pmatrix} = \mathfrak{M} \begin{pmatrix} \dot\phi \\ \dot\tau \end{pmatrix}, \qquad (4.9.3)$$

$$\mathfrak{M} = \tfrac{1}{2} \begin{pmatrix} 8F(\mathfrak{v}) \sin^4 \frac{\theta}{2} + 2\mathfrak{v} \sin^2\theta & -2F(\mathfrak{v})(\cos\theta - 1) \\ -2F(\mathfrak{v})(\cos\theta - 1) & 2F(\mathfrak{v}) \end{pmatrix}$$

The Hamiltonian. We perform the Legendre transform in order to obtain the Hamiltonian H:

$$H = \dot\phi\, p_\phi + \dot\tau p_\tau + \dot\theta\, p_\theta + \dot{\mathfrak{v}} p_{\mathfrak{v}} - \mathcal{L} \qquad (4.9.4)$$

getting

$$H(q,p) = \frac{1}{2} \left(\frac{p_\tau^2}{F(\mathfrak{v})} + F(\mathfrak{v}) p_{\mathfrak{v}}^2 + \frac{\csc^2\theta\, [(\cos\theta - 1)p_\tau + p_\phi]^{\,2} + p_\theta^2}{\mathfrak{v}} \right) \qquad (4.9.5)$$

where

$$p = \{p_\phi, p_\tau, p_\theta, p_{\mathfrak{v}}\} \quad ; \quad q = \{\phi,\, \tau,\, \theta,\, \mathfrak{v}\} \qquad (4.9.6)$$

are the momenta and coordinates.

As it is always the case in the geodesic problem, the Hamiltonian has the structure

$$H = g^{ij}(q)\, p_i\, p_j \qquad (4.9.7)$$

having denoted by $g^{ij}(q)$ the inverse metric tensor.

The reduced Lagrangian and the reduced Hamiltonian. Having singled out two first integrals of the motion $\ell, \mathfrak{m}$, it is convenient to introduce a reduced Lagrangian for the two residual degrees of freedom $\mathfrak{v}, \theta$ that, geometrically, correspond to the two angles of *right ascension* of the 2-spheres composing the underlying differentiable manifold (see section 4.7.1). The reduction of the Lagrangian is obtained by replacing the velocities of the cyclic coordinates $q^{\mathfrak{c}}$ with the corresponding momenta $p_{\mathfrak{c}}$ that are constant of the motion, namely ℓ, m:

$$\mathcal{L}_{red} = \frac{F(\mathfrak{v})\left(\dot{\theta}^2\,\mathfrak{v}^2 \;+\; \csc^2\theta\,(m\cos\theta - m + \ell)^2\right) \;+\; m^2\mathfrak{v} \;+\; \mathfrak{v}\,\dot{\mathfrak{v}}^2}{2\mathfrak{v}F(\mathfrak{v})} \tag{4.9.8}$$

Performing the Legendre transform we obtain the reduced Hamiltonian

$$\begin{aligned} H_{red} &= p_{\mathfrak{v}}\,\dot{\mathfrak{v}} + p_\theta\,\dot{\theta} - \mathcal{L}_{red} \\ &= \frac{1}{2}\left(-\frac{m^2}{F(\mathfrak{v})} + F(\mathfrak{v})p_{\mathfrak{v}}^2 - \frac{\csc^2\theta(m\cos\theta - m + \ell)^2 - p_\theta^2}{\mathfrak{v}}\right) \end{aligned} \tag{4.9.9}$$

where

$$p_{\mathfrak{v}} = \frac{\dot{\mathfrak{v}}}{F(\mathfrak{v})}, \qquad p_\theta = \mathfrak{v}\,\dot{\theta} \tag{4.9.10}$$

The Carter constant and the reduction to quadratures. Considering now the reduced system with four Hamiltonian variables we have a nice surprise: there is an additional function of the q and p that is in involution with the Hamiltonian and therefore constitutes an additional conserved quantity, yielding in this way the complete integrability of the system. Since it is the analog of the Carter constant for the Kerr metric we call it the *Carter Hamiltonian* and we denote it with the letter $\mathcal{C}$:

$$\mathcal{C} = \csc^2\theta\;(m\cos\theta - m + \ell)^2 \;-\; p_\theta^2 \tag{4.9.11}$$

An immediate calculation shows that the Carter function has vanishing Poisson bracket with the reduced Hamiltonian:

$$\{\mathcal{C}\,,\,H_{red}\} = \sum_{i=1}^{2}\left(\frac{\partial\mathcal{C}}{\partial q^i}\,\frac{\partial H_{red}}{\partial p_i} \;-\; \frac{\partial\mathcal{C}}{\partial p_i}\,\frac{\partial H_{red}}{\partial q^i}\right) = 0 \tag{4.9.12}$$

Hence on any solution of the equations motion (that is along geodesics) both the principal Hamiltonian H_{red} and $\mathcal{C}$ must assume constant values that we call respectively $\mathcal{E}$ (the energy) and K (the Carter constant):

$$H_{red} = \mathcal{E} \quad ; \quad \mathcal{C} \, = \, K \tag{4.9.13}$$

Using equations (4.9.11),(4.9.9) we can solve algebraically eqn. (4.9.13) for the two momenta $p_{\mathfrak{v}}$ and p_θ and we get the following two first-order differential equations:

$$\begin{aligned}
\tfrac{\mathrm{d}\theta}{\mathrm{d}s} &= \frac{\sqrt{\csc^2\theta\,\left(\cos 2\theta\,\left(K+m^2\right)-K+3m^2+2\ell^2+4m\cos\theta(\ell-m)-4m\ell\right)}}{\sqrt{2}\mathfrak{v}} \\
\tfrac{\mathrm{d}\mathfrak{v}}{\mathrm{d}s} &= \frac{\sqrt{F(\mathfrak{v})(K+2\mathfrak{v}\mathcal{E})+m^2\mathfrak{v}}}{\sqrt{\mathfrak{v}}}
\end{aligned} \tag{4.9.14}$$

Eliminating the derivatives with respect to s we finally obtain the differential equation of the "orbit"

$$\frac{d\theta}{d\mathfrak{v}} = \frac{\sqrt{\csc^2(\theta)\left(\cos(2\theta)\left(K+m^2\right)-K+3m^2+2\ell^2+4m\cos(\theta)(\ell-m)-4m\ell\right)}}{\sqrt{2}\sqrt{\mathfrak{v}}\sqrt{F(\mathfrak{v})(K+2\mathfrak{v}\mathcal{E})+m^2\mathfrak{v}}} \tag{4.9.15}$$

which can be reduced to quadratures:

$$\begin{aligned}
\Lambda\left(\theta\right) &= \int \frac{d\theta}{\mathcal{A}} \\
\Sigma\left(\mathfrak{v}\right) &= \int \frac{d\mathfrak{v}}{\sqrt{2}\sqrt{\mathfrak{v}}\sqrt{F(\mathfrak{v})(K+2\mathfrak{v}\mathcal{E})+m^2\mathfrak{v}}}
\end{aligned} \tag{4.9.16}$$

where

$$\mathcal{A} = \sqrt{\csc^2(\theta)\left(\cos(2\theta)\left(K+m^2\right)-K+3m^2+2\ell^2+4m\cos(\theta)(\ell-m)-4m\ell\right)}$$

The solution to the geodesic problem is provided by giving the dependence of the variable $\mathfrak{v}$ on the *right ascension angle* θ of the first sphere:

$$\mathfrak{v} = \Sigma^{-1}\circ\Lambda\left(\theta\right), \quad \theta = \Lambda^{-1}\circ\Sigma\left(\mathfrak{v}\right) \tag{4.9.17}$$

Both functions Λ and Σ are transcendental and the inverse problem can be solved only numerically, except for some special cases as we are going to illustrate in the next section.

We conclude this section by noting that the existence of the Carter conserved Hamiltonian is probably an implicit consequence of the larger nonabelian isometry of the original metric. The two first integrals ℓ, $\mathfrak{m}$ follow from the toric symmetry $\mathrm{U}(1) \times \mathrm{U}(1)$. The Carter constant is indirectly linked to the extension to $\mathrm{SU}(2)$ of one of the two $\mathrm{U}(1)$'s. If we had $\mathrm{SU}(2) \times \mathrm{SU}(2)$ isometry, then the metric would be the direct product of two Fubini-Study metrics. With $\mathrm{SU}(2) \times \mathrm{U}(1)$ isometry we have the hybrid case where one sphere is the fiber and the other is the base manifold.

4.9.2 Irrotational geodesics

The function $\Lambda(\theta)$ can be calculated explicitly in the general case $(\ell \leq 0, (m) \leq 0)$ and we obtain:

$$
\begin{aligned}
\Lambda(\theta) &= \tfrac{N(\theta)}{D(\theta)} \\
N(\theta) &= \mathcal{B} \times \mathcal{C} \\
D(\theta) &= (\cos(\theta)+1)\sqrt{K+m^2}\;\mathcal{D}
\end{aligned}
\tag{4.9.18}
$$

where

$$
\mathcal{B} = \sqrt{\cos(2\theta)\left(K+m^2\right) - K + 3m^2 + 2\ell^2 + 4m\cos(\theta)(\ell - m) - 4m\ell}
$$

$$
\mathcal{C} = -\arctan\left[\frac{\sec^2\left(\frac{\theta}{2}\right)\left(\cos(\theta)\left(K+m^2\right) + m(\ell - m)\right)}{\sqrt{K+m^2}\sqrt{\sec^4\left(\frac{\theta}{2}\right)(m\cos(\theta) - m + \ell)^2 - 4K\tan^2\left(\frac{\theta}{2}\right)}}\right]
$$

and

$$
\mathcal{D} = \sqrt{\sec^4\left(\frac{\theta}{2}\right)(m\cos(\theta) - m + \ell)^2 - 4K\tan^2\left(\frac{\theta}{2}\right)}
$$

while the integral defining the function $\Sigma(\mathfrak{v})$ in the general case does not evaluate to a combination of known special functions — neither for f the $\mathbb{F}_2$ metric nor for KE metrics.

Although it can be done, it is rather cumbersome to write explicit computer codes for the numerical computation of the function Σ in the general case and for the needed inverse of the function Λ. Hence, at this stage, it is difficult to present explicit geodesics with trivial angular momenta. The alternative is the numerical integration of the pair of first-order equations (4.9.14) but also here we meet some difficulties since the differential system is *stiff*[3] and without special care and an in-depth study of the phase space, the standard integration routines run into divergences and fail to provide solutions both for the KE and the Hirzebruch case. This is not surprising given the analogy with the Kerr metric. Indeed the study of Kerr geodesics is a wide field, there is a large variety of types of geodesics and each requires nontrivial computational efforts to be worked out.

Yet in our case, things enormously simplify if we consider *irrotational geodesics* defined as those where $\ell = m = 0$ and only the Carter constant $\mathcal{C}$ and the energy $\mathcal{E}$ label the curve. Geometrically this corresponds to the fact that the azimuthal angles ϕ, τ span a two-dimensional torus T^2. Pursuing the analogy with General Relativity, irrotational geodesics are the analogues of the *radial geodesics* utilized in cosmology and in the study of the causal structure of spacetimes where one preserves only time t and radial distance r. The analogues of t and r are in our case the variables $\mathfrak{v}$ and θ, namely, the two ascension angles of $S^2 \times S^2$.

By suppressing angular momenta things simplify drastically. The orbit equation (4.9.15) reduces to

$$\frac{d\theta}{d\mathfrak{v}} = \frac{\sqrt{-K}}{\sqrt{\mathfrak{v}}\sqrt{F(\mathfrak{v})(K + 2\mathfrak{v}\mathcal{E})}}, \tag{4.9.19}$$

which implies

$$\theta = \mathcal{FM}(\mathfrak{v}, K, \mathcal{E}) \; = \; \int \frac{\sqrt{-K}}{\sqrt{\mathfrak{v}}\sqrt{F(\mathfrak{v})(K + 2\mathfrak{v}\mathcal{E})}}\, \mathrm{d}\mathfrak{v}. \tag{4.9.20}$$

The good news is that in the KE case (with the choice $\lambda_1 = 1$, $\lambda_2 = 2$), the integral

[3] The term "stiff" comes from numerical analysis and denotes a differential equation or differential system whose numerical solution is unstable unless the step size is taken to be very small.

of eqn. (4.9.20) can be explicitly evaluated by obtaining

$$
\begin{aligned}
\mathcal{FM}^{KE}(\mathfrak{v},K,\mathcal{E}) &= \frac{N(\mathfrak{v},K,\mathcal{E})}{D(\mathfrak{v},K,\mathcal{E})} \\
N(\mathfrak{v},K,\mathcal{E}) &= \sqrt{7}\sqrt{-K}(\mathfrak{v}-2)(\mathfrak{v}-1)\sqrt{\frac{(3\mathfrak{v}+2)(K+4\mathcal{E})}{K+2\mathfrak{v}\mathcal{E}}} \\
&\quad \times \mathcal{F}\left(\arcsin\left(\frac{\sqrt{-\frac{(3K-4\mathcal{E})(\mathfrak{v}-2)}{K+2\mathcal{E}\mathfrak{v}}}}{2\sqrt{2}}\right)|\frac{8(K+2\mathcal{E})}{3K-4\mathcal{E}}\right) \\
D(\mathfrak{v},K,\mathcal{E}) &= \sqrt{\mathfrak{v}}\sqrt{-\frac{(\mathfrak{v}-2)(3K-4\mathcal{E})}{K+2\mathfrak{v}\mathcal{E}}}\sqrt{\frac{(\mathfrak{v}-1)(K+4\mathcal{E})}{K+2\mathfrak{v}\mathcal{E}}} \\
&\quad \times\sqrt{-\frac{(3\mathfrak{v}^3-7\mathfrak{v}^2+4)(K+2\mathfrak{v}\mathcal{E})}{\mathfrak{v}}}
\end{aligned}
\tag{4.9.21}
$$

where by $\mathcal{F}(z|h)$ we have denoted the $\mathcal{F}$ elliptic function.

In the case of the Hirzebruch surface metric, the integral in eqn. (4.9.20) does not evaluate to known special functions, yet it can be easily computed numerically, allowing one to to draw the geodesic curves in the $\mathfrak{u},\mathfrak{v}$-plane: the parametric form is

$$\{(1-\cos[\mathcal{FM}(\mathfrak{v},K,\mathcal{E})])\ \mathfrak{v}\ ,\ \mathfrak{v}\} \tag{4.9.22}$$

as follows from eqn. (4.5.4). Choosing various different values of the energy and of the Carter constant we obtain the curves shown in Figure 4.10. In both cases the irrotational geodesics are smooth and approach value $\mathfrak{v}_{max}$, that is, the North pole of the second sphere. The only difference is that in the Hirzebruch case they reach $\mathfrak{v}_{max}$ at various values of the right ascension of the first 2-sphere while in the KE case they tend to reach the North Pole of the second sphere arriving simultaneously at the North Pole of the first sphere.

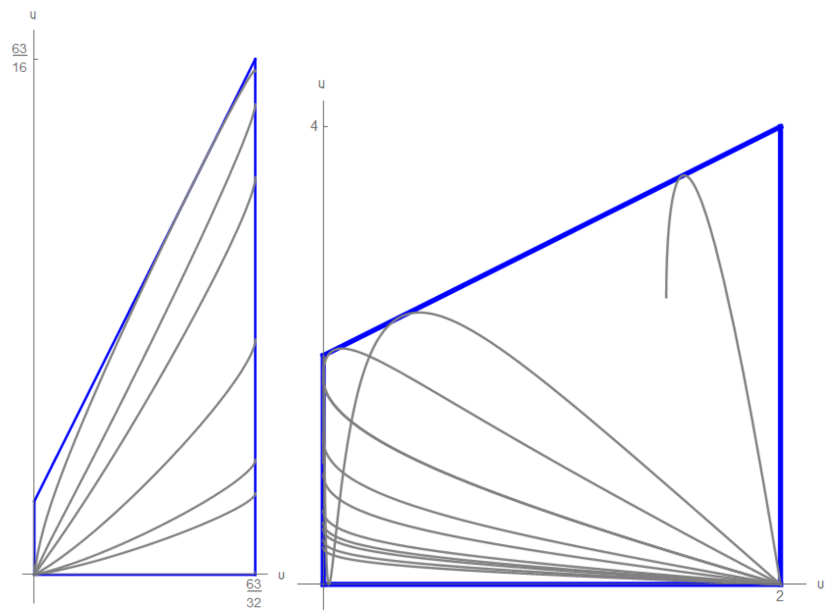


Figure 4.10: On the left is a plot of some irrotational geodesics for the case of the Hirzebruch surface. On the right plot of the same type of geodesics for the KE metric.

Chapter 5

New Class of Calabi-Yau Metrics

5.1 The Calabi Ansatz and the AMSY symplectic formalism

Having studied in some detail the KE base manifolds $\mathcal{M}_B^{KE}$, we turn now to the main issue of this thesis, namely, the construction of a Ricci flat metric on the total space canonical bundle. We use Calabi's ansatz, which however only works for KE base manifolds.

Calabi's Ansatz Calabi's paper [10] introduces the following Ansatz for the local Kähler potential $\mathcal{K}(z, \bar{z}, w, \bar{w})$ of a Kähler metric g_E on the total space of a holomorphic vector bundle $E \to \mathcal{M}$, where $\mathcal{M}$ is a compact Kähler manifold satisfying the conditions already stated in the introduction. In particular, one has a Kähler potential of the form:

$$\mathcal{K}(z, \bar{z}, w, \bar{w}) = \mathcal{K}_0(z, \bar{z}) + U(\lambda) \tag{5.1.1}$$

where $\mathcal{K}_0(z, \bar{z})$ is a Kähler potential for $g_{\mathcal{M}}$, (z^i, $i = 1, \dots \dim_{\mathbb{C}}\mathcal{M}$) being the complex coordinates of the base manifold) and U is a function of a real variable λ, which we shall identify with the function

$$\lambda = \mathcal{H}_{\mu\bar{\nu}}(z,\bar{z})\, w^{\mu}\, w^{\bar{\nu}} \; = \parallel w \parallel^2 \tag{5.1.2}$$

(the square norm of a section of the bundle with respect to a fiber metric $\mathcal{H}_{\mu\bar{\nu}}(z,\bar{z})$). If θ is the Chern connection on E, canonically determined by the Hermitian structure $\mathcal{H}$ and the holomorphic structure of E, its local connections forms can be written as

$$\theta_{\nu}{}^{\lambda} = \sum_{i} dz^{i}\, L_{i|\nu}{}^{\lambda} \tag{5.1.3}$$

where

$$L_{i|\nu}{}^{\lambda} = \sum_{\bar{\mu}} \mathcal{H}^{\lambda\bar{\mu}} \frac{\partial}{\partial z^{i}} \mathcal{H}_{\nu\bar{\mu}} \qquad ; \qquad [\mathcal{H}^{\lambda\bar{\mu}}] = ([\mathcal{H}_{\lambda\bar{\mu}}]^{-1})^{T} \tag{5.1.4}$$

The curvature 2-form Θ of the connection θ is given by:

$$\Theta_{\nu}{}^{\lambda} = \sum_{i,\bar{\jmath}} dz^{i} \wedge d\bar{z}^{\bar{\jmath}}\, S_{i\bar{\jmath}|\nu}{}^{\lambda} \qquad ; \qquad S_{i\bar{\jmath}|\nu}{}^{\lambda} \;=\; \frac{\partial}{\partial \bar{z}^{\bar{\jmath}}}\, L_{i|\nu}{}^{\lambda} \tag{5.1.5}$$

The Kähler metric g_E corresponding to the Kähler potential $\mathcal{K}$ can be written as follows:

$$\begin{aligned} \partial\bar{\partial}\mathcal{K} \;=\; & \sum_{i,\bar{\jmath}} \Big[g_{i\bar{\jmath}} + \lambda\, U'(\lambda) \sum_{\lambda,\nu,\bar{\mu}} \mathcal{H}_{\sigma\bar{\mu}} S_{i\bar{\jmath}|\rho}{}^{\sigma} w^{\rho} \bar{w}^{\bar{\mu}} \Big] dz^{i} d\bar{z}^{\jmath} + \\ & \sum_{\sigma,\bar{\mu}} \Big[U'(\lambda) \;+\; \lambda\, U''(\lambda) \Big]\, \mathcal{H}_{\sigma\bar{\mu}}\, \nabla w^{\sigma}\, \nabla \bar{w}^{\bar{\mu}}. \end{aligned} \tag{5.1.6}$$

If E is a line bundle then the above equation reduces to

$$\partial\bar{\partial}\mathcal{K} \;=\; \sum_{i,\bar{\jmath}} [g_{i\bar{\jmath}} + \lambda U'(\lambda)\, S_{i\bar{\jmath}}] dz^{i} d\bar{z}^{\bar{\jmath}} + [U'(\lambda) \;+\; \lambda U''(\lambda)] \mathcal{H}(z,\bar{z}) \nabla w \nabla \bar{w} \tag{5.1.7}$$

where $\lambda = \mathcal{H}(z,\bar{z})\, w\bar{w}$ is the nonnegative real quantity defined in equation (5.1.2) and ∇w denotes the covariant derivative of the fiber-coordinate with respect to Chern connection θ:

$$\nabla w = dw + \theta\, w \tag{5.1.8}$$

5.1.1 Ricci-flat metrics on canonical bundles

Now we assume that E is the canonical bundle $K_{\mathcal{M}}$ of a Kähler surface $\mathcal{M}$ ($\dim_{\mathbb{C}} M = 2$). The total space of $K_{\mathcal{M}}$ has vanishing first Chern class, i.e., it is a noncompact Calabi-Yau manifold, and we may try to construct explicitly a Ricci-flat metric on it. Actually, following Calabi, we can reduce the condition that g_E is Ricci-flat to a differential equation for the function $U(\lambda)$ introduced in equation (5.1.1). Note that under the present assumptions, S is a scalar 2-form on $\mathcal{M}$.

Since our main target is the construction of a Ricci flat metric on the space $\mathrm{tot}(\mathcal{K}_{\mathcal{M}_B^{KE}})$, where $\mathcal{M}_B^{KE}$ denotes any of the KE manifolds discussed at length in previous sections, we begin precisely with an analysis of that case which will allow us to drive a general form of $U(\lambda)$ as a function of the moment $\mathfrak{w}$ associated with the U(1) group acting by phase transformations of the fiber coordinate w, and of certain coefficients A, B, F that are determined in terms of the Kähler potential $\mathcal{K}_0$ of the base manifold $\mathcal{M}$. Consistency of the Calabi Ansatz requires that these coefficients should be constant, which happens in the case of base manifolds equipped with Kähler Einstein metrics. KE metrics do not exist on Hirzebruch surfaces and the Calabi Ansatz is not applicable in this case. As we discuss in the sequel, there exists a Ricci flat metric on the canonical bundle of a singular blow-down of $\mathbb{F}_2$, namely the weighted projective plane $\mathbb{WP}[1,1,2]$, which is known in the AMSY symplectic toric formalism of [1] and [38]. If we were able to invert the Legendre transform we might reconstruct the so far missing Kähler potential and get inspiration on possible generalizations of the Calabi Ansatz. Hence we are going to pay attention to both formulations, the Kähler one and the symplectic one.

5.1.2 Calabi Ansatz for 4D Kähler metrics with $SU(2)\times U(1)$ isometry

The Calabi Ansatz can be applied with success or not according to the structure of the Kähler potential $\mathcal{K}_0$ for the base manifold $\mathcal{M}$ and the algebraic form of the invariant combination Ω of the complex coordinates u, v which is the only real variable from which the Kähler potential $\mathcal{K}_0 = \mathcal{K}_0(\Omega)$ is assumed to depend. On the other hand, Ω encodes the group of isometries which is imposed on the Kähler metric of $\mathcal{M}$.

In the case of the metrics discussed in section 4.2, having $SU(2)\times U(1)$ isometry, the invariant is chosen to be

$$\Omega = \varpi \tag{5.1.9}$$

where ϖ was defined in eqn. (4.2.1). This choice guarantees the isometry of the Kähler metric $g_{\mathcal{M}}$ to be the group $\mathrm{SU}(2)\times\mathrm{U}(1)$ with the action described in eqn. (4.2.2). Hence we focus on such manifolds and we consider a Kähler potential for $\mathcal{M}$ that for the time being is a generic function $\mathcal{K}_0(\varpi)$ of the invariant variable. In this case, the determinant of the Kähler metric $g_{\mathcal{M}}$ has an explicit expression in terms of $\mathcal{K}_0(\varpi)$

$$\det\left(g_{\mathcal{M}}\right) \;=\; 2\varpi\mathcal{K}_0{}'(\varpi)\left(\varpi\mathcal{K}_0{}''(\varpi)+\mathcal{K}_0{}'(\varpi)\right) \tag{5.1.10}$$

while the determinant of the Ricci tensor takes the form

$$\begin{array}{c}
det\left(\mathrm{Ric}_{\mathcal{M}}\right) = \frac{N_{Ric}}{D_{Ric}} \\
N_{Ric} = 2\left(\varpi^2\mathcal{K}_0{}''(\varpi)^2+\mathcal{K}_0{}'(\varpi)^2+\varpi\mathcal{K}_0{}'(\varpi)\left(\varpi\mathcal{K}_0{}^{(3)}(\varpi)+4\mathcal{K}_0{}''(\varpi)\right)\right)\times \\
\times\left(-\varpi^3\mathcal{K}_0{}''(\varpi)^4+\varpi^2\mathcal{K}_0{}'(\varpi)\left(\varpi\mathcal{K}_0{}^{(3)}(\varpi)-\mathcal{K}_0{}''(\varpi)\right)\mathcal{K}_0{}''(\varpi)^2\right. \\
+\varpi\mathcal{K}_0{}'(\varpi)^2\left(-\varpi^2\mathcal{K}_0{}^{(3)}(\varpi)^2-\mathcal{K}_0{}''(\varpi)^2+\right. \\
\left.\varpi\left(\varpi\mathcal{K}_0{}^{(4)}(\varpi)+2\mathcal{K}_0{}^{(3)}(\varpi)\right)\mathcal{K}_0{}''(\varpi)\right. \\
\left.+\mathcal{K}_0{}'(\varpi)^3\left(3\mathcal{K}_0{}''(\varpi)+\varpi\left(\varpi\mathcal{K}_0{}^{(4)}(\varpi)+5\mathcal{K}_0{}^{(3)}(\varpi)\right)\right)\right) \\
\\
D_{Ric} = \mathcal{K}_0{}'(\varpi)^3\left(\varpi\mathcal{K}_0{}''(\varpi)+\mathcal{K}_0{}'(\varpi)\right)^3
\end{array} \tag{5.1.11}$$

and the scalar curvature

$$R_s = \mathrm{Tr}\left(\mathrm{Ric}_{\mathcal{M}}\, g_{\mathcal{M}}^{-1}\right) \tag{5.1.12}$$

is

$$\begin{aligned} R_s &= \tfrac{N_s}{D_s} \\ N_s &= {\mathcal{K}_0}'(\varpi)^3 + \varpi^3 {\mathcal{K}_0}''(\varpi)^2 \left(2\varpi {\mathcal{K}_0}^{(3)}(\varpi) + 5{\mathcal{K}_0}''(\varpi)\right) + \varpi^2 {\mathcal{K}_0}'(\varpi)\mathcal{A} + \\ &\quad \varpi {\mathcal{K}_0}'(\varpi)^2 \mathcal{B} \\ D_s &= \varpi {\mathcal{K}_0}'(\varpi)\left(\varpi {\mathcal{K}_0}''(\varpi) + {\mathcal{K}_0}'(\varpi)\right)^3 \end{aligned} \tag{5.1.13}$$

where

$$\mathcal{A} = -\varpi^2 {\mathcal{K}_0}^{(3)}(\varpi)^2 + 9{\mathcal{K}_0}''(\varpi)^2 + \varpi\left(\varpi {\mathcal{K}_0}^{(4)}(\varpi) + 4{\mathcal{K}_0}^{(3)}(\varpi)\right){\mathcal{K}_0}''(\varpi)$$

and

$$\mathcal{B} = 9{\mathcal{K}_0}''(\varpi) + \varpi\left(\varpi {\mathcal{K}_0}^{(4)}(\varpi) + 6{\mathcal{K}_0}^{(3)}(\varpi)\right)$$

Given these base manifold data, we introduce a Kähler potential for a metric on the canonical bundle $\mathrm{tot}(\mathcal{K}_{\mathcal{M}})$ in accordance with the Calabi Ansatz, namely

$$\mathcal{K}(\varpi,\lambda) = \mathcal{K}_0(\varpi) + U(\lambda); \quad \lambda = \underbrace{\exp\left[\mathcal{P}(\varpi)\right]}_{\text{fiber metric } \mathcal{H}(\varpi)} \mid w \mid^2 = \parallel w \parallel^2 \tag{5.1.14}$$

where λ is the square norm of a section of the canonical bundle and $\exp\left[\mathcal{P}(\varpi)\right]$ is some fiber metric. The determinant of the corresponding Kähler metric g_E on the total space of the canonical bundle is

$$\begin{aligned} \det g_E &= 2\varpi\Sigma(\lambda)e^{\mathcal{P}(\varpi)}\Sigma'(\lambda)\left(\varpi\mathcal{K}_0''(\varpi)\mathcal{P}'(\varpi) + \mathcal{K}_0'(\varpi)\left(\varpi\mathcal{P}''(\varpi) + 2\mathcal{P}'(\varpi)\right)\right) \\ &\quad + 2\varpi e^{\mathcal{P}(\varpi)}\Sigma'(\lambda)\mathcal{K}_0'(\varpi)\left(\mathcal{K}_0'(\varpi) + \varpi\mathcal{K}_0''(\varpi)\right) + \\ &\quad 2\varpi\Sigma(\lambda)^2 e^{\mathcal{P}(\varpi)}\Sigma'(\lambda)\mathcal{P}'(\varpi)\left(\varpi\mathcal{P}''(\varpi) + \mathcal{P}'(\varpi)\right) \end{aligned} \tag{5.1.15}$$

where we set

$$\Sigma(\lambda) = \lambda\, U'(\lambda) \tag{5.1.16}$$

If we impose the Ricci flatness condition, namely, the determinant of the metric g_E is a constant which we can always assume to be one since any other number can be reabsorbed into the normalization of the fiber coordinate w, by integration we get

$$\lambda = \frac{1}{48}\left(A\,\mathfrak{w}^3 + 2B\,\mathfrak{w}^2 + 4\,F\,\mathfrak{w}\right) \tag{5.1.17}$$

where we have set

$$\begin{array}{rcl} \Sigma(\lambda) & = & 2\,\mathfrak{w} \\ A & = & 4\varpi e^{\mathcal{P}(\varpi)}\mathcal{P}'(\varpi)\left[\varpi\mathcal{P}''(\varpi) + \mathcal{P}'(\varpi)\right] \\ B & = & 6\varpi e^{\mathcal{P}(\varpi)}\left[\varpi\mathcal{K}_0''(\varpi)\mathcal{P}'(\varpi) + \mathcal{K}_0'(\varpi)\left(\varpi\mathcal{P}''(\varpi) + 2\mathcal{P}'(\varpi)\right)\right] \\ F & = & 12\varpi e^{\mathcal{P}(\varpi)}\mathcal{K}_0'(\varpi)\left[\mathcal{K}_0'(\varpi) + \varpi\mathcal{K}_0''(\varpi)\right] \end{array} \tag{5.1.18}$$

In our complex three-dimensional case, setting

$$x_u = \log \mid u \mid, \quad x_v = \log \mid v \mid, \qquad x_w = \log \mid w \mid, \tag{5.1.19}$$

the corresponding three moments can be named with the corresponding gothic letters, and we have

$$\mathfrak{u} = \partial_{x_u}\mathcal{K}\left(\varpi,\lambda\right), \quad \mathfrak{v} = \partial_{x_v}\mathcal{K}\left(\varpi,\lambda\right), \quad \mathfrak{w} = \partial_{x_w}\mathcal{K}\left(\varpi,\lambda\right). \tag{5.1.20}$$

As the fiber coordinate w appears only in the function $U(\lambda)$ via the squared norm λ, we have

$$\mathfrak{w} = 2\,\lambda\,U'(\lambda) = \Sigma(\lambda) \tag{5.1.21}$$

and this justifies the position (5.1.18). At this point the function $U(\lambda)$ can be easily determined by first observing that, in view of eqn. (5.1.17) we can also set

$$U(\lambda) = \mathrm{U}\left(\mathfrak{w}\right) \tag{5.1.22}$$

and we can use the chain rule

$$\partial_{\mathfrak{w}}\,\mathrm{U}\,(\mathfrak{w}) = \frac{\mathfrak{w}\lambda'(\mathfrak{w})}{2\lambda(\mathfrak{w})} = \frac{3A\mathfrak{w}^2 + 4B\mathfrak{w} + 4F}{2A\mathfrak{w}^2 + 4B\mathfrak{w} + 8F} \tag{5.1.23}$$

which by integration yields the universal function

$$\mathrm{U}(\mathfrak{w}) = -\frac{\mathcal{C} + \mathcal{D}}{2\,A} \tag{5.1.24}$$

where

$$\mathcal{C} = 2\sqrt{4\,A\,F - B^2}\ \arctan\left(\frac{A\,\mathfrak{w} + B}{\sqrt{4\,A\,F - B^2}}\right)$$

and

$$\mathcal{D} = B\ \log\left(A\,\mathfrak{w}^2\ +\ 2\,B\mathfrak{w}\ +\ 4\,F\right) - 3\,A\,\mathfrak{w}$$

The function $U(\lambda)$ appearing in the Kähler potential can be obtained by substituting for the argument $\mathfrak{w}$ in (5.1.24) the unique real root of the cubic equation (5.1.17), namely:

$$\mathfrak{w} \;\;=\;\; \frac{\mathcal{B}_1}{3\sqrt[3]{2}A} + \frac{4\sqrt[3]{2}\,(B^2 - 3AF)}{3A\mathcal{B}_1} - \frac{2B}{3A} \tag{5.1.25}$$

where

$$\mathcal{B}_1 = \sqrt[3]{8\mathcal{B}_0 + 1296A^2\lambda + 72ABF - 16B^3}$$

and

$$\mathcal{B}_0 = \sqrt{\left(162A^2\lambda + 9ABF - 2B^3\right)^2 - 4\left(B^2 - 3AF\right)^3}$$

5.1.3 Consistency conditions for the Calabi Ansatz

In order for the Calabi Ansatz to yield a *a solution* of the Ricci flatness condition, it is necessary that the universal function $\mathrm{U}(\mathfrak{w})$ in eqn. (5.1.24) should depend only on $\mathfrak{w}$, which happens if and only if the coefficients A, B, F are constants. In the case under consideration, where the invariant combination of the complex coordinates u, v is the one provided by ϖ as defined in eqn. (4.2.1), imposing such a consistency condition would require the solution of three ordinary differential equations for two functions $\mathcal{P}(\varpi)$ and $\mathcal{K}_0(\varpi)$, namely:

$$\begin{aligned}
k_1 &= 4\varpi e^{\mathcal{P}(\varpi)}\mathcal{P}'(\varpi)\left[\varpi\mathcal{P}''(\varpi)+\mathcal{P}'(\varpi)\right] \\
k_2 &= 6\varpi e^{\mathcal{P}(\varpi)}\left[\varpi\mathcal{K}_0''(\varpi)\mathcal{P}'(\varpi)+\mathcal{K}_0'(\varpi)\left(\varpi\mathcal{P}''(\varpi)+2\mathcal{P}'(\varpi)\right)\right] \\
k_3 &= 12\varpi e^{\mathcal{P}(\varpi)}\mathcal{K}_0'(\varpi)\left[\mathcal{K}_0'(\varpi)+\varpi\mathcal{K}_0''(\varpi)\right]
\end{aligned} \tag{5.1.26}$$

where $k_{1,2,3}$ are three constants. It is clear from their structure that the crucial differential equation is the first one. If we could find a solution for it then it would suffice to identify the original Kähler potential $\mathcal{K}_0(\varpi)$ with a linear function of $\mathcal{P}(\varpi)$ and we could solve the three equations. So far we were not able to find any analytical solution of these equations but if we could find one, we still should verify that the Kähler metric following from such $\mathcal{K}_0$ is a good metric on the Hirzebruch surface $\mathbb{F}_2$.

On the contrary, for the Kähler potentials obtained from the Kronheimer construction that defines a one-parameter family of Kähler metrics on $\mathbb{F}_2$ and were discussed in [6, 7], namely those presented in eqn. (4.2.4), equations (5.1.26) cannot be solved and no Ricci-flat metric on the canonical bundle can be obtained by means of the Calabi Ansatz.

The general case with the natural fiber metric $\mathcal{H} = \frac{1}{\det(g_{\mathcal{M}})}$. If we consider the general case of a toric two-dimensional compact manifold $\mathcal{M}$ with a Kähler metric $g_{\mathcal{M}}$ derived from a Kähler potential of the form:

$$\mathcal{K}_0 = \mathcal{K}_0\left(|u|^2, |v|^2\right) \tag{5.1.27}$$

choosing as fiber metric the natural one for the canonical bundle, namely setting:

$$\lambda = \mathcal{H}\,|w|^2 \;=\; \frac{1}{\det\left(g_{\mathcal{M}}\right)}\,|w|^2 \tag{5.1.28}$$

and going through the same steps as in section 5.1.2, we arrive at an identical result for the function $\mathrm{U}(\mathfrak{w})$ as in equation (5.1.24) but with the following coefficients:

$$A = 2\,\frac{\det\left(\mathrm{Ric}_{\mathcal{M}}\right)}{\det\left(g_{\mathcal{M}}\right)}, \quad B = 3\,\mathrm{Tr}\left(\mathrm{Ric}_{\mathcal{M}}\, g_{\mathcal{M}}^{-1}\right), \quad F = 6 \tag{5.1.29}$$

It clearly appears why the Calabi Ansatz works perfectly if the starting metric on the base manifold is KE. In that case, the Ricci tensor is proportional to the metric tensor:

$$\mathrm{Ric}_{i\bar{\jmath}} = \kappa\, g_{i\bar{\jmath}} \tag{5.1.30}$$

and we get:

$$\det\left(\mathrm{Ric}_{\mathcal{M}}\right) = \kappa^2 \det\left(g_{\mathcal{M}}\right), \quad \mathrm{Tr}\left(\mathrm{Ric}_{\mathcal{M}}\, g_{\mathcal{M}}^{-1}\right) = 2\,\kappa \tag{5.1.31}$$

which implies:

$$A = 2\,\kappa^2, \quad B = 6\,\kappa, \quad F = 6. \tag{5.1.32}$$

5.1.4 The AMSY symplectic formulation for the Ricci flat metric on $\mathrm{tot}(\mathcal{K}_{\mathcal{M}_B})$

According to the discussion of the AMSY symplectic formalism presented in section chapter 4, given the Kähler potential of a toric complex three manifold $\mathcal{K}(|u|,|v|,|w|)$, we can define the moments

$$\mathfrak{u} = \partial_{x_u}\mathcal{K}, \quad \mathfrak{v} = \partial_{x_v}\mathcal{K}, \quad \mathfrak{w} = \partial_{x_w}\mathcal{K} \tag{5.1.33}$$

and we can obtain the symplectic potential by means of the Legendre transform:

$$G\left(\mathfrak{u},\mathfrak{v},\mathfrak{w}\right) = x_u\,\mathfrak{u} \,+\, x_v\,\mathfrak{v} + \,x_w\,\mathfrak{w} \,-\, \mathcal{K}(|u|,|v|,|w|) \tag{5.1.34}$$

The main issue in the use of eqn. (5.1.34) is the inverse transformation that expresses the coordinates $x_i = \{x_u, x_v, x_w\}$ in terms of the three moments $\mu^i = \{\mathfrak{u},\mathfrak{v},\mathfrak{w}\}$. Once this is done one can calculate the metric in moment variables utilizing the Hessian as explained in section chapter 4. Relying once again on the results of that section we know that the Kähler 2-form has the following universal structure:

$$\mathbb{K} \;=\; \mathrm{d}\mathfrak{u} \wedge \mathrm{d}\phi \,+\, \mathrm{d}\mathfrak{v} \wedge \mathrm{d}\tau \,+\, \mathrm{d}\mathfrak{w} \wedge \mathrm{d}\chi \tag{5.1.35}$$

and the metric is expressed as displayed in eqn. (4.1.6))

The symplectic potential in the case with $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$ **isometries** In the case where the Kähler potential has the special structure which guarantees an $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$ isometry, namely it depends only on the two variables ϖ (see eqn. (4.2.1))) and $|w|^2$, also the symplectic potential takes a more restricted form. Indeed we find

$$G\left(\mathfrak{u},\mathfrak{v},\mathfrak{w}\right) \;=\; \underbrace{\left(\mathfrak{v}-\frac{\mathfrak{u}}{2}\right)\log(2\mathfrak{v}-\mathfrak{u})+\frac{1}{2}\mathfrak{u}\log(\mathfrak{u}) \,-\, \frac{1}{2}\,\mathfrak{v}\log(\mathfrak{v})}_{\text{universal part } G_0(\mathfrak{u},\mathfrak{v})} + \underbrace{\mathcal{G}(\mathfrak{v},\mathfrak{w})}_{\text{variable part}} \tag{5.1.36}$$

where $\mathcal{G}(\mathfrak{v},\mathfrak{w})$ is a function of two variables that encode the specific structure of the metric. Note that when we freeze the fiber moment coordinate $\mathfrak{w}$ to some fixed constant value, for instance, 0, the function $\mathcal{G}(\mathfrak{v},0) = \mathcal{D}(\mathfrak{v})$ can be identified with the boundary function that appears in eqns. (4.2.7),(4.2.8), namely in the symplectic potential for the Kähler metric of the base manifold.

With the specific structure (5.1.36) of the symplectic potential we obtain the following form for the Hessian (4.1.3):

$$\mathbf{G} = \begin{pmatrix} -\frac{\mathfrak{v}}{\mathfrak{u}^2-2\mathfrak{u}\mathfrak{v}} & \frac{1}{\mathfrak{u}-2\mathfrak{v}} & 0 \\ \frac{1}{\mathfrak{u}-2\mathfrak{v}} & \frac{-2\mathfrak{v}(\mathfrak{u}-2\mathfrak{v})\mathcal{G}^{(2,0)}(\mathfrak{v},\mathfrak{w})+\mathfrak{u}+2\mathfrak{v}}{2\mathfrak{v}(2\mathfrak{v}-\mathfrak{u})} & \mathcal{G}^{(1,1)}(\mathfrak{v},\mathfrak{w}) \\ 0 & \mathcal{G}^{(1,1)}(\mathfrak{v},\mathfrak{w}) & \mathcal{G}^{(0,2)}(\mathfrak{v},\mathfrak{w}) \end{pmatrix} \tag{5.1.37}$$

5.2 The general form of the symplectic potential for the Ricci flat metric on $\mathrm{tot}(\mathcal{M}_B^{KE})$

Having seen that KE metrics do indeed exist, in the form described in eqs.(4.2.5), (4.2.6), it is natural to inquire how we can utilize the Calabi Ansatz to write immediately the symplectic potential for a Ricci-flat metric on the canonical bundle of $\mathcal{M}_B^{KE}$ without going through the process of inverting the Legendre transform. Namely, we would like to make the back-and-forth trip via inverse and direct Legendre transform only once and in full generality rather than case by case. Our goal is not only a simplification of the computational steps but also involves a conceptual issue. Indeed, when we introduce intermediate steps that rely on the variable ϖ whose range is $[0, +\infty)$ we necessarily have to choose a branch of a cubic equation whose coefficients are determined by the root parameters $\lambda_{1,2}$. On the contrary, if we are able to determine directly the symplectic potential in terms of the symplectic coordinates, then we can explore the behavior of the metric and of its curvature on the full available range of variability of these latter and we learn more about the algebraic and topological structure of the underlying manifold.

So let us anticipate the final result of our general procedure. As we did in the previous section we assume that the Ricci form of $\mathcal{M}_B$ is proportional to the Kähler form via a coefficient

$$\kappa = \frac{k}{4} \tag{5.2.1}$$

as in eqs.(4.4.5),(5.1.30). The complete symplectic potential for the Ricci flat metric on $\mathcal{M}_T = \mathrm{tot}(\mathcal{K}_{\mathcal{M}_B})$ has the following structure:

$$\begin{aligned}
G_{\mathcal{M}_T^{KE}}(\mathfrak{u},\mathfrak{v},\mathfrak{w}) &= G_0(\mathfrak{u},\mathfrak{v}) + \mathcal{G}^{KE}(\mathfrak{v},\mathfrak{w}) \\
G_0(\mathfrak{u},\mathfrak{v}) &= \left(\mathfrak{v} - \tfrac{\mathfrak{u}}{2}\right)\log[2\mathfrak{v} - \mathfrak{u}] + \tfrac{1}{2}\mathfrak{u}\log[\mathfrak{u}] - \tfrac{1}{2}\mathfrak{v}\log[\mathfrak{v}] \\
\mathcal{G}^{KE}(\mathfrak{v},\mathfrak{w}) &= \left(\tfrac{\kappa\mathfrak{w}}{2} + 1\right)\mathcal{D}^{KE}\left(\tfrac{2\mathfrak{v}}{\kappa\mathfrak{w}+2}\right) - \tfrac{1}{2}\mathfrak{v}\log\left(\tfrac{\kappa\mathfrak{w}}{2} + 1\right) + \tfrac{1}{2}\mathfrak{w}\log(\mathfrak{w}) \\
&\quad + \frac{(\kappa\mathfrak{w}+3)\log(\kappa\mathfrak{w}(\kappa\mathfrak{w}+6)+12)}{2\kappa} + \frac{\sqrt{3}\arctan\left(\frac{\kappa\mathfrak{w}+3}{\sqrt{3}}\right)}{\kappa}
\end{aligned} \tag{5.2.2}$$

where the second equation is a repetition for the reader's convenience of eqn. (4.2.8) and $\mathcal{D}^{KE}(\mathfrak{v}_0)$ is the boundary function defined in equation (4.4.10); the relation between the two independent roots $\lambda_{1,2}$ and the parameter κ is provided by equations (4.4.8),(5.2.1). The reason why we have used the argument

$$\mathfrak{v}_0 = \frac{2\mathfrak{v}}{\kappa\mathfrak{w} + 2} \tag{5.2.3}$$

is that the symplectic variable $\mathfrak{v}_0$ associated with the base-manifold metric and the symplectic variable $\mathfrak{v}$ associated with the metric on the canonical bundle $\mathcal{M}_T^{KE}$ are not the same; their relation is precisely that in eqn. (5.2.3) which is a direct consequence of the Calabi Ansatz as we explain below.

5.2.1 Derivation of the formula for $\mathcal{G}^{KE}(\mathfrak{v}, \mathfrak{w})$

The general formula (5.2.2) is a straightforward yield of the direct Legendre transform after the Calabi Ansatz:

$$G_{\mathcal{M}_T^{KE}}(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = x_u\,\mathfrak{u} + x_v\,\mathfrak{v} + x_w\,\mathfrak{w} - \mathcal{K}_0(\mathfrak{v}_0) - U(\lambda) \tag{5.2.4}$$

where

$$\begin{aligned}
\lambda &= \tfrac{w\,\bar{w}}{\det g_{\mathcal{M}_B}} = \text{const} \times w\,\bar{w}\;\exp\left[\kappa\,\mathcal{K}_0(\mathfrak{v}_0)\right] = \\
&\quad \Lambda(\mathfrak{w}) = \tfrac{1}{24}\mathfrak{w}\left(\kappa^2\mathfrak{w}^2 + 6\kappa\mathfrak{w} + 12\right) \\
\frac{\mathfrak{w}}{2} &= \lambda\,U'(\lambda) \\
U(\lambda) &= \mathbb{U}(\mathfrak{w}) = \frac{-3\log\left(2\left(\kappa^2\mathfrak{w}^2 + 6\kappa\mathfrak{w} + 12\right)\right) + 3\kappa\mathfrak{w} - 2\sqrt{3}\arctan\left(\frac{\kappa\mathfrak{w}+3}{\sqrt{3}}\right)}{2\kappa} \\
\mathcal{K}_0(\mathfrak{v}_0) &= \mathfrak{v}_0\mathcal{D}'(\mathfrak{v}_0) - \mathcal{D}(\mathfrak{v}_0) + \tfrac{\mathfrak{v}_0}{2}
\end{aligned} \tag{5.2.5}$$

The last two lines in eqns. (5.2.5) were derived earlier, respectively in eqns. (5.1.24),(4.2.16). The explicit form of $\mathbb{U}(\mathfrak{w})$ follows from eqn. (5.1.24) using the KE condition, namely eqn. (5.1.32). From the above relations, one easily obtains the relations

$$\begin{aligned} \mathfrak{u}_0 &= \frac{2\mathfrak{u}}{k\mathfrak{w}+2}, \quad \mathfrak{v}_0 = \frac{2\mathfrak{v}}{k\mathfrak{w}+2} \\ x_w &= \tfrac{1}{2}\left\{\log\left[\Lambda(\mathfrak{w})\right] - \kappa\,\mathcal{K}_0(\mathfrak{v}_0)\right\} \end{aligned} \tag{5.2.6}$$

The first two relations can be understood as follows. The momenta $\mathfrak{u}_0, \mathfrak{v}_0$ are, by definition

$$\mathfrak{u}_0 = \partial_{x_u}\mathcal{K}_0 \quad ; \quad \mathfrak{v}_0 = \partial_{x_v u}\mathcal{K}_0 \tag{5.2.7}$$

while we have

$$\mathfrak{u} = \partial_{x_u}\mathcal{K} \quad ; \quad \mathfrak{v} = \partial_{x_v}\mathcal{K} \tag{5.2.8}$$

By the Calabi Ansatz we get:

$$\mathfrak{u} = \mathfrak{u}_0 + \partial_{x_u}U(\lambda) \; = \; \mathfrak{u}_0 + \partial_{xu}\lambda\,\partial_\lambda U(\lambda) = \mathfrak{u}_0 + \kappa\partial_{xu}\mathcal{K}_0\,\lambda\partial_\lambda U(\lambda) \; = \; \mathfrak{u}_0\left(1 + \frac{\kappa}{2}\,\mathfrak{w}\right) \tag{5.2.9}$$

A completely analogous calculation can be done for the case of $\mathfrak{v}$. Finally, let us note that the coordinates x_u, x_v were already resolved in terms of u_0, v_0 in eqns. (4.2.15):

$$\begin{aligned} x_u &= \tfrac{1}{2}\left(\log\left(\mathfrak{u}_0\right) - \log\left(2\mathfrak{v}_0 - \mathfrak{u}_0\right)\right) \quad ; \\ x_v &= \mathcal{D}'\left(\mathfrak{v}_0\right) + \log\left(2\mathfrak{v}_0 - \mathfrak{u}_0\right) - \tfrac{1}{2}\log\left(\mathfrak{v}_0\right) + \tfrac{1}{2} \end{aligned} \tag{5.2.10}$$

The information provided in the above equations (5.2.5) - 5.2.10 is sufficient to complete the Legendre transform (5.2.4) and retrieve the very simple and elegant result encoded in eqn. (5.2.2).

To check the correctness of the general formula (5.2.2) we have explicitly calculated, by means of the MATHEMATICA package METRICGRAV[1], the Ricci tensor for a few cases of $\mathcal{M}_T^{[\lambda_1,\lambda_2]}$, always finding zero.

The example of the metric $[2,1]$**.** Here we present the explicit form in symplectic coordinates of the Ricci flat metric on the canonical bundle of the KE manifold $\mathcal{M}_B^{[1,2]}$,

[1]METRICGRAV is a personal package of Pietro Fré which calculates Ricci tensors etc

namely that determined by the choice: $\lambda_1 = 1$, $\lambda_2 = 2$. We get:

$$\begin{aligned}
ds^2_{\mathcal{M}_T^{[1,2]}} &= d\phi^2 \left(-\tfrac{\mathfrak{u}^2(9\mathfrak{w}+14)^2}{343\mathfrak{v}^3} - \tfrac{16464\mathfrak{u}^2}{(9\mathfrak{w}+14)^4} + 2\mathfrak{u}\right) \\
&+ \frac{d\mathfrak{v}^2\left(\mathfrak{u}\left(2058\mathfrak{v}^3 + (9\mathfrak{w}+14)^3\right) - 686\mathfrak{v}^3(9\mathfrak{w}+14)\right)}{\mathfrak{v}(2\mathfrak{v}-\mathfrak{u})(7\mathfrak{v}-9\mathfrak{w}-14)(14\mathfrak{v}-9\mathfrak{w}-14)(21\mathfrak{v}+9\mathfrak{w}+14)} \\
&+ 2d\tau d\phi \left(-\frac{\mathfrak{u}(9\mathfrak{w}+14)^2}{343\mathfrak{v}^2} - \frac{16464\mathfrak{u}\mathfrak{v}}{(9\mathfrak{w}+14)^4} + \mathfrak{u}\right) + \frac{d\mathfrak{u}d\mathfrak{v}}{\mathfrak{u}-2\mathfrak{v}} \\
&+ \frac{d\mathfrak{u}(\mathfrak{u}d\mathfrak{v} - \mathfrak{v}d\mathfrak{u})}{\mathfrak{u}(\mathfrak{u}-2\mathfrak{v})} + \frac{36\mathfrak{u}\mathfrak{w}\left(27\mathfrak{w}^2 + 126\mathfrak{w} + 196\right)d\chi d\phi}{(9\mathfrak{w}+14)^3} + \\
& d\tau^2 \left(-\frac{16464\mathfrak{v}^2}{(9\mathfrak{w}+14)^4} - \frac{(9\mathfrak{w}+14)^2}{343\mathfrak{v}} + \mathfrak{v}\right) + \frac{6174\mathfrak{v}^2 d\mathfrak{v}d\mathfrak{w}}{\mathcal{C}_0} \\
&+ \frac{d\mathfrak{w}^2\mathcal{C}_1}{\mathcal{C}_2} + \frac{36\mathfrak{v}\mathfrak{w}\left(27\mathfrak{w}^2 + 126\mathfrak{w} + 196\right)d\tau d\chi}{(9\mathfrak{w}+14)^3} + \\
& \frac{2\mathfrak{w}\left(27\mathfrak{w}^2 + 126\mathfrak{w} + 196\right)d\chi^2}{(9\mathfrak{w}+14)^2}
\end{aligned} \tag{5.2.11}$$

Where

$$\mathcal{C}_0 = (7\mathfrak{v} - 9\mathfrak{w} - 14)(14\mathfrak{v} - 9\mathfrak{w} - 14)(21\mathfrak{v} + 9\mathfrak{w} + 14)$$

$$\mathcal{C}_1 = \left(5647152\mathfrak{v}^3 - 343\mathfrak{v}^2(9\mathfrak{w}+14)^4 + (9\mathfrak{w}+14)^6\right)$$

and

$$\mathcal{C}_2 = 2\mathfrak{w}(9\mathfrak{w}+14)\left(27\mathfrak{w}^2 + 126\mathfrak{w} + 196\right)(7\mathfrak{v}-9\mathfrak{w}-14)(14\mathfrak{v}-9\mathfrak{w}-14)(21\mathfrak{v}+9\mathfrak{w}+14)$$

These Ricci-flat metrics provide $D3$-brane type IIB supergravity solutions, which we show in Appendix A5.2.1 . As we know, the generalized Kronheimer construction captures the field contents on the gauge side of the gauge/gravity correspondence. It is not known yet what kind of quiver gauge models we can associate with our new class of KE surfaces in order to complete the correspondence. We suspect them to be closely related to the quiver gauge theory associated with $\mathcal{Q}_{\mathbb{Z}_4}$.

Also, we started this project to put explicit Ricci-flat metrics with $SU(2)\times U(1)\times U(1)$ isometry on $tot(K_{\mathbb{F}_2})$ which completes the gauge/gravity correspondence of the quiver gauge theory associated with $\mathcal{Q}_{\mathbb{Z}_4}$. In [7], the field contents were found, and if we are able to find a Ricci-flat metric then we are done. This has proven to be a hard nut to crack. A thorough study is needed to understand how to get explicit Ricci-flat metrics beyond Calabi's ansatz. That will involve solving some non-linear Monge-Ampère equations with non-constant coefficients. Perhaps, due to the complexity of the MA equations, in general, and the lack of explicit techniques to solve PDE's, we will not be able to do so. but, at least in cases like canonical bundles over toric compact surfaces, there might be some improvements that can be made to get results, due to the presence of combinatorial data associated with toric manifolds.

Appendices

Appendix A

For the reader's convenience in this appendix we concisely collect the main formulas related with the D3-brane solution of Type IIB supergravity. For more explanations, the reader is referred to section 2 of [5].

We separate the ten coordinates of space-time into the following subsets:[2]

$$x^M = \begin{cases} x^\mu, & \mu = 0, \ldots, 3 \quad \text{real coordinates of the 3-brane world volume} \\ y^\tau, & \alpha = 1,2,3 \quad\ \ \text{complex coordinates of the } Y \text{ variety} \end{cases} \tag{.0.12}$$

and we make the following Ansatz for the metric:

$$\mathrm{ds}^2_{[10]} = H(\boldsymbol{y}, \bar{\boldsymbol{y}})^{-\frac{1}{2}} \left(-\eta_{\mu\nu}\, dx^\mu \otimes dx^\nu\right) + H(\boldsymbol{y}, \bar{\boldsymbol{y}})^{\frac{1}{2}} \left(\mathbf{g}^{\mathrm{RFK}}_{\alpha\bar{\beta}}\, dy^\alpha \otimes d\bar{y}^{\bar{\beta}}\right) \tag{.0.13}$$

$$\mathrm{ds}^2_Y = \mathbf{g}^{\mathrm{RFK}}_{\alpha\bar{\beta}}\, dy^\alpha \otimes d\bar{y}^{\bar{\beta}} \tag{.0.14}$$

$$\eta_{\mu\nu} = diag(+,-,-,-) \tag{.0.15}$$

where $\mathbf{g}^{\mathrm{RFK}}$ is the Kähler metric of the manifold Y:

$$\mathbf{g}^{\mathrm{RFK}}_{\alpha\bar{\beta}} = \partial_\alpha\, \partial_{\bar{\beta}}\, \mathcal{K}^{\mathrm{RFK}}\left(\boldsymbol{y}, \bar{\boldsymbol{y}}\right), \tag{.0.16}$$

the real function $\mathcal{K}^{\mathrm{RFK}}\left(\boldsymbol{y}, \bar{\boldsymbol{y}}\right)$ being a suitable Kähler potential. It follows that

$$det(g_{[10]}) = H(\boldsymbol{y}, \bar{\boldsymbol{y}})\, det(\mathbf{g}^{\mathrm{RFK}}).$$

Actually the formalism which is best suited for our aims is the AMSY symplectic one, rather than using holomorphic coordinates. In terms of the vielbein the Ansatz

[2] Latin indices are always frame indices referring to the vielbein formalism. Furthermore, we distinguish the 4 directions of the brane volume by using Latin letters from the beginning of the alphabet while the 3 complex transversal directions are denoted by Latin letters from the middle and the end of the alphabet. For the coordinate indices we utilize Greek letters and we do exactly the reverse: early Greek letters $\alpha, \beta, \gamma, \delta, \ldots$ refer to the 3 complex transverse directions while Greek letters from the second half of the alphabet $\mu, \nu, \rho, \sigma, \ldots$ refer to the D3 brane world volume directions as it is customary in $D = 4$ field theories.

(.0.13) corresponds to

$$V^A = \left\{ \begin{array}{lclll} V^a & = & H(\boldsymbol{y},\bar{\boldsymbol{y}})^{-1/4}\, dx^a & a = 0,1,2,3 \\ V^\ell & = & H(\boldsymbol{y},\bar{\boldsymbol{y}})^{1/4}\, \mathbf{e}^\ell & \ell = 4,5,6,7,8,9 \end{array} \right. \tag{.0.17}$$

where $\mathbf{e}^\ell$ are the vielbein 1-forms of the manifold Y. The structure equations are (the hats denote quantities computed without the warp factor, i.e., with $H = 1$)

$$\begin{array}{rcl} 0 & = & d\,\mathbf{e}^i - \widehat{\omega}^{ij} \wedge \mathbf{e}^k\, \eta_{jk} \\ \widehat{R}^{ij} & = & d\widehat{\omega}^{ij} - \widehat{\omega}^{ik} \wedge \widehat{\omega}^{\ell j}\, \eta_{k\ell} = \widehat{R}^{ij}{}_{\ell m}\, \mathbf{e}^\ell \wedge \mathbf{e}^m . \end{array} \tag{.0.18}$$

The relevant property of the Y metric that we use in solving Einstein's equation is that is Ricci-flat:

$$\widehat{R}^{im}{}_{\ell m} = 0. \tag{.0.19}$$

To derive our solution and discuss its supersymmetry properties we need the explicit form of the spin connection for the full 10-dimensional metric (.0.13) and the corresponding Ricci tensor. From the torsion equation one can uniquely determine the solution:

$$\begin{array}{rcl} \omega^{ab} & = & 0 \\ \omega^{a\ell} & = & \frac{1}{4}\, H^{-3/2}\, dx^a \eta^{\ell k}\, \partial_k\, H \\ \omega^{\ell m} & = & \widehat{\omega}^{\ell m} + \Delta\omega^{\ell m} \quad ; \quad \Delta\omega^{\ell m} = -\frac{1}{2}\, H^{-1}\, \mathbf{e}^\ell\, \eta^{mk}\, \partial_k H \end{array} \tag{.0.20}$$

Inserting this result into the definition of the curvature 2-form we obtain

$$\begin{array}{rcl} R^a_b & = & -\frac{1}{8}\, \left[H^{-3/2} \Box_{\mathbf{g}}\, H - H^{-5/2}\, \partial_k H \partial^k H \right]\, \delta^a_b \\ R^a_\ell & = & 0 \\ R^m_\ell & = & \frac{1}{8} H^{-3/2} \Box_{\mathbf{g}} H \delta^m_\ell - \frac{1}{8} H^{-5/2} \partial_s H \partial^s H \delta^m_\ell + \frac{1}{4} H^{-5/2} \partial_\ell H \partial^m H \end{array} \tag{.0.21}$$

where for any function $f\,(\boldsymbol{y},\bar{\boldsymbol{y}})$ on Y the equation

$$\Box_{\mathbf{g}}\, f\,(\boldsymbol{y},\bar{\boldsymbol{y}}) \;=\; \frac{1}{\sqrt{\det \mathbf{g}}}\, \left(\partial_\alpha \left(\sqrt{\det \mathbf{g}}\; \mathbf{g}^{\alpha\bar{\beta}}\, \partial_{\bar{\beta}}\, f \right) \right) \tag{.0.22}$$

defines the Laplace–Beltrami operator with respect to the Ricci-flat metric (.0.16);

we have omitted the superscript RFK just for simplicity — on the supergravity side of the correspondence we shall only use the Ricci-flat metric and there will be no ambiguity.

The equations of motion for the scalar fields φ and $C_{[0]}$ and for the 3-form field strengths $F^{NS}_{[3]}$ and $F^{RR}_{[3]}$ can be better analyzed using the complex notation. Defining, as it is explained in [5] above:

$$\begin{array}{rcl} \mathcal{H}_{\pm} & = & \pm 2\, e^{-\varphi/2} F^{NS}_{[3]} + i2\, e^{\varphi/2}\, F^{RR}_{[3]} \quad \Rightarrow \quad \overline{\mathcal{H}_{+}} = -\,\mathcal{H}_{-} \\ P & = & \frac{1}{2}\, d\varphi - i\frac{1}{2}\, e^{\varphi}\, F^{RR}_{[1]} \end{array} \tag{.0.23}$$

but also setting in our Ansatz

$$\varphi = 0 \quad ; \quad C_{[0]} = 0 \tag{.0.24}$$

we reduce the equations for the complex 3-forms to

$$\begin{array}{r} \mathcal{H}_{+} \wedge \star\mathcal{H}_{+} = 0 \\ d \star \mathcal{H}_{+} = i\, F^{RR}_{[5]} \wedge \mathcal{H}_{+} \end{array} \tag{.0.25}$$

while the equation for the 5-form becomes

$$d \star F^{RR}_{[5]} = i\, \frac{1}{8}\, \mathcal{H}_{+} \wedge \mathcal{H}_{-} \tag{.0.26}$$

The Ansatz for the complex 3-forms of type IIB supergravity is given below and is inspired by what was done in [4, 3] in the case where $Y = \mathbb{C} \times \mathrm{ALE}_{\Gamma}$:

$$\mathcal{H}_{+} = \Omega^{(2,1)} \tag{.0.27}$$

where $\Omega^{(2,1)}$ lives on Y and satisfies

$$\star_{\mathbf{g}}\, \mathrm{Q}^{(2,1)} = -i\, \mathrm{Q}^{(2,1)} \tag{.0.28}$$

As shown in [5] this guarantees that

$$\mathcal{H}_+ \wedge \star_{10} \mathcal{H}_+ = 0. \tag{.0.29}$$

The Ansatz for $F^{RR}_{[5]}$ is

$$\begin{aligned} F^{RR}_{[5]} &= \alpha \left(U + \star_{10} U \right) \\ U &= d \left(H^{-1} \, \mathrm{Vol}_{\mathbb{R}^{(1,3)}} \right) \end{aligned} \tag{.0.30}$$

where α is a constant to be determined later. By construction $F^{RR}_{[5]}$ is self-dual and its equation of motion is trivially satisfied. What is not guaranteed is that also the Bianchi identity is fulfilled. Imposing it results into a differential equation for the function $H\left(\boldsymbol{y}, \bar{\boldsymbol{y}}\right)$. Indeed we obtain

$$d \, F^{RR}_{[5]} \;=\; \alpha \, \Box_{\mathbf{g}} \, H(\boldsymbol{y}, \bar{\boldsymbol{y}}) \; \times \; \mathrm{Vol}_Y \tag{.0.31}$$

where

$$\mathrm{Vol}_Y \;=\; \sqrt{\det \mathbf{g}} \, \tfrac{1}{(3!)^2} \, \epsilon_{\alpha\beta\gamma} \, dy^{\alpha} \wedge dy^{\beta} \wedge dy^{\gamma} \, \wedge \epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \, d\bar{y}^{\bar{\alpha}} \wedge d\bar{y}^{\bar{\beta}} \wedge d\bar{y}^{\bar{\gamma}} \tag{.0.32}$$

is the volume form of the transverse six-dimensional manifold *i.e.* the total space of the canonical bundle $K\left[\mathcal{M}_B\right]$. With our Ansatz we obtain

$$\begin{aligned} \tfrac{1}{8} \, \mathcal{H}_+ \, \wedge \, \mathcal{H}_- &= \mathbb{J}\left(\boldsymbol{y}, \bar{\boldsymbol{y}}\right) \, \times \, \mathrm{Vol}_Y \\ \mathbb{J}\left(\boldsymbol{y}, \bar{\boldsymbol{y}}\right) &= - \, \frac{1}{72 \, \sqrt{\det \mathbf{g}}} \; \Omega_{\alpha\beta\bar{\eta}} \; \bar{\Omega}_{\bar{\delta}\bar{\theta}\gamma} \; \epsilon^{\alpha\beta\gamma} \; \epsilon^{\bar{\eta}\bar{\delta}\bar{\theta}} \end{aligned} \tag{.0.33}$$

and we conclude that

$$\Box_{\mathbf{g}} \, H = - \frac{1}{\alpha} \, \mathbb{J}\left(\boldsymbol{y}, \bar{\boldsymbol{y}}\right) \tag{.0.34}$$

This is the main differential equation to which the entire construction of the D3-brane solution can be reduced. In [5] it was shown that the parameter α is determined by Einstein's equations and is fixed to $\alpha = 1$. With this value the field equations for the complex three forms simplify and reduce to the condition that $\Omega^{2,1}$ should be closed, and then, being anti-selfdual also co-closed, namely harmonic:

$$\tilde{\Omega}^{(2,1)} = \star_{\mathbf{g}}\Omega^{(2,1)} = -\,i\,\Omega^{(2,1)} \quad ; \quad d\star_{\mathbf{g}}\Omega^{(2,1)} = 0 \quad ; \quad d\Omega^{(2,1)} = 0 \tag{.0.35}$$

In other words the solution of type IIB supergravity with 3-form fluxes exists *if and only if* the transverse space admits *closed and imaginary anti-self-dual forms* $\Omega^{(2,1)}$, as we already stated.

Summarizing, in order to construct a D3-brane solution of type IIB supergravity we need:

a) to find a Ricci flat Kähler metric $\mathbf{g}_{RFK}$ on the transverse 6D space Y;

b) to verify if in the background of the metric $\mathbf{g}_{RFK}$ there exists a nonvanishing linear space of anti-self-dual (2,1)-forms $\Omega^{(2,1)}$. In the case of a positive answer, the 3-form $\mathcal{H}_+$ will be a linear combination of such forms; otherwise it will be zero.

c) to solve the Laplacian equation for the harmonic function H which is homogeneous if there are no 3-form fluxes, otherwise it is inhomogeneous as in eqn. (.0.34).

The issue of (2,1)-forms We come to the issue of (2,1)-forms, giving a proof that no (anti)-self-dual $(2,1)$ forms exist in the KE manifolds previously studied. A (2,1)-form can be written as

$$\Omega_{ij\bar{k}}\,(z,\bar{z})\,dz^i \wedge dz^j \wedge d\bar{z}^{\bar{k}} \tag{.0.36}$$

The dual (2,1)-form is

$$\star_g\Omega \;=\; \tilde{\Omega}_{\ell p\bar{q}}\,(z,\bar{z})\,dz^\ell \wedge dz^p \wedge d\bar{z}^{\bar{q}} \tag{.0.37}$$

where:

$$\tilde{\Omega}_{\ell p\bar{q}}\,(z,\bar{z}) \;=\; \frac{1}{\sqrt{\det g}}\,g_{\ell\bar{m}}\,g_{p\bar{n}}\,g_{\bar{q}s}\,\epsilon^{\bar{m}\bar{n}\bar{k}}\epsilon^{ijs}\,\Omega_{ij\bar{k}}\,(z,\bar{z}) \tag{.0.38}$$

Hence the (anti)-selfduality condition is expressed by the equation:

$$\pm\, i\, \tilde{\Omega}_{\ell p \bar{q}}\,(z, \bar{z}) \;=\; \Omega_{\ell p \bar{q}}\,(z, \bar{z}) \tag{.0.39}$$

Given the complex structure tensor $\mathfrak{J}$ and its eigenvectors one writes the complex differentials:

$$dz^i \;=\; \omega^i_\ell(\mu) d\mu^\ell + i\, d\Theta^i \quad ; \quad d\bar{z}^i \;=\; \omega^i_\ell(\mu) d\mu^\ell - i\, d\Theta^i \tag{.0.40}$$

where the real 1-forms $\omega^i \;\equiv\; \omega^i_\ell(\mu) d\mu^\ell$ depending only on moment variables are defined by the complex structure tensor and hence by the explicit form of the metric in terms of the Hessian. Using this formalism a (2,1)-form is written as

$$\begin{aligned} \Omega^{(2,1)} = &\;\; \Omega_{ij|k}\,(\mu)\; (\omega^i + i\, d\Theta^i) \wedge (\omega^j + i\, d\Theta^j) \wedge (\omega^k - i\, d\Theta^k) \\ &= Q_{IJK}\,(\mu)\; dy^I \wedge dy^J \wedge dy^K \end{aligned} \tag{.0.41}$$

where $y^I \;=\; \{\mu^i, \Theta^j\}$ is the complete set of the $2n$ real coordinates (moments and angles).The complex functions $Q_{IJK}\,(\mu)$ depend on the real variables μ. The (anti)selfduality condition is most easily written in the symplectic formalism as the determinant of the metric tensor in symplectic coordinates is just 1. One gets

$$Q_{IJK} \;=\; \pm i\, \epsilon_{IJKPQR}\, Q_{PQR} \tag{.0.42}$$

The original components $\Omega_{ij|k}\,(\mu)$ are supposed to be complex valued functions of their real arguments which means that we have a total of 9-complex valued functions, namely a total of 18 real functions:

$$r(\mu) \;=\; \{f_1(\mu), \ldots, f_9(\mu), g_1(\mu), \ldots, g_9(\mu)\} \tag{.0.43}$$

Explicitly we obtain

$$
\begin{aligned}
\Omega \;=\;& 2\left(\left(f_3-f_5-f_7+ig_3-ig_5-ig_7\right)d\tau\wedge d\chi\wedge\omega^1-\right.\\
&2\left(f_8+ig_8\right)d\tau\wedge d\chi\wedge\omega^2-2\left(f_9+ig_9\right)d\tau\wedge d\chi\wedge\omega^3+\\
&2\left(f_1+ig_1\right)d\tau\wedge d\phi\wedge\omega^1+2\left(f_2+ig_2\right)d\tau\wedge d\phi\wedge\omega^2\\
&+\left(f_3+f_5-f_7+ig_3+ig_5-ig_7\right)d\tau\wedge d\phi\wedge\omega^3+\\
&2\left(g_2-if_2\right)d\tau\wedge\omega^1\wedge\omega^2+\left(-if_3-if_5-if_7+g_3+g_5+g_7\right)d\tau\wedge\omega^1\wedge\omega^3\\
&+2\left(g_8-if_8\right)d\tau\wedge\omega^2\wedge\omega^3+\left(-if_3-if_5+if_7+g_3+g_5-g_7\right)d\chi\wedge\omega^1\wedge\omega^2\\
&+2\left(g_6-if_6\right)d\chi\wedge\omega^1\wedge\omega^3+2\left(g_9-if_9\right)d\chi\wedge\omega^2\wedge\omega^3\\
&-2\left(f_4+ig_4\right)d\phi\wedge d\chi\wedge\omega^1-\left(f_3+f_5+f_7+ig_3+ig_5+ig_7\right)d\phi\wedge d\chi\wedge\omega^2\\
&-2\left(f_6+ig_6\right)d\phi\wedge d\chi\wedge\omega^3+2\left(g_1-if_1\right)d\phi\wedge\omega^1\wedge\omega^2+\\
&2\left(g_4-if_4\right)d\phi\wedge\omega^1\wedge\omega^3+\left(if_3-if_5-if_7-g_3+g_5+g_7\right)d\phi\wedge\omega^2\wedge\omega^3\\
&+\left(-if_3+if_5-if_7+g_3-g_5+g_7\right)d\tau\wedge d\phi\wedge d\chi\\
&\left.+\omega^1\wedge\omega^2\wedge\omega^3\left(f_3-f_5+f_7+ig_3-ig_5+ig_7\right)\right)
\end{aligned}
\tag{.0.44}
$$

which is the most general expression for a (2,1)-form expressed in the real symplectic coordinate basis. Expanding each of the closed one-forms in the differentials $d\mu^i$ of the moments one obtains the explicit form of the 20 components $Q_{IJK}(\mu)$ mentioned in (.0.41). For instance in our standard example $\lambda_1=1,\lambda_2=2$ we have:

$$
\begin{aligned}
\omega^1 &= \tfrac{d\mathfrak{v}-\mathfrak{v}d\mathfrak{u}}{\mathfrak{u}(\mathfrak{u}-2\mathfrak{v})}\\
\omega^2 &= \tfrac{N_2}{D_2}\\
N_2 &= 3087\mathfrak{v}^3 d\mathfrak{w}(\mathfrak{u}-2\mathfrak{v})-\mathfrak{u}d\mathfrak{v}\left(2058\mathfrak{v}^3+(9\mathfrak{w}+14)^3\right)+\mathfrak{v}d\mathfrak{u}\left(2058\mathfrak{v}^3\right.\\
&\quad\left.-343\mathfrak{v}^2(9\mathfrak{w}+14)+(9\mathfrak{w}+14)^3\right)+686\mathfrak{v}^3(9\mathfrak{w}+14)d\mathfrak{v}\\
D_2 &= \mathfrak{v}(\mathfrak{u}-2\mathfrak{v})\left(2058\mathfrak{v}^3-343\mathfrak{v}^2(9\mathfrak{w}+14)+(9\mathfrak{w}+14)^3\right)\\
\omega^3 &= \tfrac{N_3}{D_3}\\
N_3 &= 6174\mathfrak{v}^2 d\mathfrak{v}+\tfrac{d\mathfrak{w}\left(5647152\mathfrak{v}^3-343\mathfrak{v}^2(9\mathfrak{w}+14)^4+(9\mathfrak{w}+14)^6\right)}{\mathfrak{w}(243\mathfrak{w}^3+1512\mathfrak{w}^2+3528\mathfrak{w}+2744)}\\
D_3 &= 2\left(2058\mathfrak{v}^3-343\mathfrak{v}^2(9\mathfrak{w}+14)+(9\mathfrak{w}+14)^3\right)
\end{aligned}
\tag{.0.45}
$$

and by substitution one straightforwardly obtains the Q_{IJK} components whose expression is too lengthy to be displayed. In general a complex valued 3-form has the following structure:

$$
\begin{aligned}
\Omega^{[3]} \;=\; & (X_{20}+iY_{20})\, d\tau\wedge d\phi\wedge d\chi + (X_{10}+iY_{10})\, d\mathfrak{u}\wedge d\tau\wedge d\chi \\
& + (X_8+iY_8)\, d\mathfrak{u}\wedge d\tau\wedge d\phi + (X_3+iY_3)\, d\mathfrak{u}\wedge d\mathfrak{v}\wedge d\tau \\
& + (X_1+iY_1)\, d\mathfrak{u}\wedge d\mathfrak{v}\wedge d\mathfrak{w} + (X_4+iY_4)\, d\mathfrak{u}\wedge d\mathfrak{v}\wedge d\chi \\
& + (X_2+iY_2)\, d\mathfrak{u}\wedge d\mathfrak{v}\wedge d\phi + (X_6+iY_6)\, d\mathfrak{u}\wedge d\mathfrak{w}\wedge d\tau \\
& + (X_7+iY_7)\, d\mathfrak{u}\wedge d\mathfrak{w}\wedge d\chi + (X_5+iY_5)\, d\mathfrak{u}\wedge d\mathfrak{w}\wedge d\phi \\
& + (X_9+iY_9)\, d\mathfrak{u}\wedge d\phi\wedge d\chi + (X_{16}+iY_{16})\, d\mathfrak{v}\wedge d\tau\wedge d\chi \\
& + (X_{14}+iY_{14})\, d\mathfrak{v}\wedge d\tau\wedge d\phi + (X_{12}+iY_{12})\, d\mathfrak{v}\wedge d\mathfrak{w}\wedge d\tau \\
& + (X_{13}+iY_{13})\, d\mathfrak{v}\wedge d\mathfrak{w}\wedge d\chi + (X_{11}+iY_{11})\, d\mathfrak{v}\wedge d\mathfrak{w}\wedge d\phi \\
& + (X_{15}+iY_{15})\, d\mathfrak{v}\wedge d\phi\wedge d\chi + (X_{19}+iY_{19})\, d\mathfrak{w}\wedge d\tau\wedge d\chi \\
& + (X_{17}+iY_{17})\, d\mathfrak{w}\wedge d\tau\wedge d\phi + (X_{18}+iY_{18})\, d\mathfrak{w}\wedge d\phi\wedge d\chi
\end{aligned}
\tag{.0.46}
$$

where the X_i and Y_i are real functions of the momenta μ. The self-duality condition (.0.42) reduces to an algebraic relation that expresses all the Y_i in terms of the X_i, precisely:

$$
\begin{array}{rcrr@{\qquad}rcrr}
Y_1 &=& \pm & X_{20} & Y_{11} &=& \pm & X_{10} \\
Y_2 &=& \pm & X_{19} & Y_{12} &=& \pm & -X_9 \\
Y_3 &=& \pm & -X_{18} & Y_{13} &=& \pm & -X_8 \\
Y_4 &=& \pm & -X_{17} & Y_{14} &=& \pm & -X_7 \\
Y_5 &=& \pm & -X_{16} & Y_{15} &=& \pm & -X_6 \\
Y_6 &=& \pm & X_{15} & Y_{16} &=& \pm & X_5 \\
Y_7 &=& \pm & X_{14} & Y_{17} &=& \pm & X_4 \\
Y_8 &=& \pm & X_{13} & Y_{18} &=& \pm & X_3 \\
Y_9 &=& \pm & X_{12} & Y_{19} &=& \pm & -X_2 \\
Y_{10} &=& \pm & -X_{11} & Y_{20} &=& \pm & -X_1
\end{array}
\tag{.0.47}
$$

The choice of the $\pm$ sign corresponding to self/anti-self duality, respectively. Comparing eqn. (.0.44) with eqn. (.0.46) and using eqn. (.0.45) one obtains the 20 X_i and the 20 Y_i of a generic (2,1)-form as linear combination of the 18 free parameter functions (.0.43) with coefficients that are rational functions of the moment μ. The self-duality constraint is a set of 20 linear equations on the 18 parameters. Obviously unless the rank of the 20×18 matrix is less than 18, there are no nontrivial solutions. We have indeed verified that the 20 equations do not have nontrivial solutions for the standard case $\lambda_1 = 1, \lambda_2 = 2$ and for some other choices of the parameters. Hence no harmonic self-dual (2,1) forms exist on this KE background and we have exact D3-brane solutions without 3-form fluxes.

Appendix B

Sasakian manifold A $(2n-1)$-dimensional Riemannian manifold (S, g) is called Sasakian if and only if $(2n)$-dimensional manifold $(C(S) = S \times \mathbb{R}_+\,,\, \bar{g} = dr^2 + r^2 g)$ is Kähler. Here r is the radial coordinate.

Given a Sasakian manifold (S, g), we can associate a unique vector field known as **Reeb vector field** $\xi = r\partial_r$ such that $\mathcal{L}_\xi \bar{g} = 0$. Its dual is called a contact 1-form η and satisfies:

$$\eta(\xi) = 1, \qquad i_\xi d\eta = 0$$

The existence of contact 1-form allows us to split the tangent bundle TS as:

$$TS = E \oplus L_\xi$$

where $E = \ker \eta$ and L_ξ is the line tangent to ξ.

ξ and η define a foliation $\mathcal{F}_\xi$ called the **Reeb foliation** such that E is the normal bundle of $\mathcal{F}_\xi$. The global properties of $\mathcal{F}_\xi$ provides classification of Sasakian manifolds. We have three classifictaions of Sasakian manifolds, namely:

1. if the orbits of ξ are all closed then ξ provides an isometric $U(1)$-action on (S, g). If the action is free we say that (S, g) is regular.

2. if the orbits are closed but the action is only locally free then (S, g) is called quasi-regular.

3. if the orbits are no closed then (S, g) is called irregular.

A well known result in Sasakian geometry says: $(C(S), \bar{g})$ is Calabi-Yau manifold if and only if (S, g) is Einstein. In the case of regular and quasi-regular Sasakian manifolds the situation is much nicer as (S, g) happens to be $U(1)$-bundle over a Kähler-Einstein manifold in the regular case and over a Kähler-Einstein orbifold in the case of quasi-regular.

In [27], a class of Sasakian-Einstein manifolds $Y^{(p,q)}$ was found which is diffeomorphic to $S^2 \times S^3$. In this class there exists infinitly manly irregular Sasakian manifolds as well. we suspect in our case the Sasakian manifold which we are looking for is closely related to these manifolds. This needs to be explored in detail.

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