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EXPANSIVE FLOWS ON THREE - MANIFOLDS

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TRIESTE

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CONTENTS

PREFACE

Der Beweis muss
übersehbar sein.
(L. Wittgenstein)

In this work we shall study expansive flows defined on a closed manifold of dimension 3, from the point of wiew of the topological equivalence. Expansive flows were introduced in the 70's, their definition being motivated by the theory of Anosov flows: expansive flows may be seen as the topological counterpart of Anosov flows, which belong to the domain of differentiable dynamics. Recent results (Lewowicz, Hiraide, Paternain, Inaba - Matsumoto,...) show that expansive flows, at least in the smallest nontrivial dimension (=3), are in fact a "slight" generalization of Anosov flows: as the latter, the former posses stable and unstable foliations, but now these foliations may have singularities. Expansive flows on 3-manifolds could be appropriately renamed *Pseudo-Anosov flows.*

It follows that an analysis of expansive flows may use (and must use) the theory of foliations. In dimension 3 this theory is a well developed subject (Novikov, Roussarie, Thurston, Ghys,...), and it allows to prove nontrivial results when applied to dynamical systems.

Hence our work belongs to the intersection of two rather classical domains: hyperbolic dynamical systems and foliations on 3-manifolds. We shall assume that the reader is acquainted with both these domains.

We now give a short outline of what is written in the following pages. After recalling in chapter 1 the above recent results about expansive flows in dimension 3, we extend in chapter 2 the theorem of Fried about surfaces of section for Anosov flows to cover the case of expansive flows. This fact has some consequences (ergodic theory, smooth models, pseudo-Anosov flows...) and it is also a justification for the next results.

Chapter 3 is the central one. There is proven a theorem which gives a sufficient condition for the topological equivalence of two expansive flows. This condition involves surfaces of section (of one of the two flows) and homotopic properties of the singular sets of the stable and unstable foliations (of both the two flows). We remark that such a theorem is new and nontrivial even in the case of Anosov flows. A first corollary is drawn at the end of chapter 3, a second one at the end of chapter 4, where we study what happens if the 3-manifold supporting the expansive flow is a Seifert fibration.

The methods of chapter 3 and 4 have a foliated flavour (in fact, we shall forget frequently the flows and we shall consider only their stable foliations), whereas chapter 2 belongs mainly to the hyperbolic world. The last chapter, in which we will study only Anosov flows and we will be concerned with the problem of transitivity, mixes hyperbolic arguments with topological and foliated aspects.

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PRELIMINARIES

We recall some basic definitions and results about expansive homeomorphisms and flows, with emphasis on the low-dimensional cases and on the relations with (pseudo-) Anosov diffeomorphisms and flows. We assume some familiarity with the theory of Anosov systems ([Ano], [A-A], [Sma], [Fra1], [Pla1]) and with the qualitative theory of foliations ([C-L], [God], [Nov]).

Let M be a compact manifold, possibly with boundary, and let $d(\cdot,\cdot)$ be the distance on M induced by any riemannian metric.

Definition 1.1 ([B-W]). A homeomorphism $f : M \to M$ is *expansive* if $\exists \delta > 0$ such that $d(f^n(x), f^n(y)) < \delta \ \forall n \in \mathbf{Z}$ implies $x = y$. A continuous flow $\phi_t : M \to M$ is *expansive* if it is without singularities and $\forall \epsilon > 0 \ \exists \delta > 0$ such that $d(\phi_t(x), \phi_{h(t)}(y)) < \delta$ $\forall t \in \mathbf{R}$ and for some homeomorphism $h : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ implies $y = \phi_s(x)$ for some $s \in (-\epsilon, \epsilon)$.

The most classical and interesting examples of expansive homeomorphisms and flows are Anosov diffeomorphisms and flows ([Ano]); the above definition of expansiveness was motivated by that examples.

If $f : M \to M$ is an Anosov diffeomorphism, we denote by $\mathcal{F}^s$, $\mathcal{F}^u$ its stable and unstable foliations. The leaves of these foliations are smooth, and diffeomorphic to euclidean spaces. The foliations themselves are only of class C^α, $\forall \alpha \in (0,1)$. If f is transitive (an old conjecture, still open, says that any Anosov diffeomorphism is transitive) then every leaf of $\mathcal{F}^s$, $\mathcal{F}^u$ is dense in M ([Fra1]).

If $\phi_t : M \to M$ is an Anosov flow, we denote by $\mathcal{F}^s$, $\mathcal{F}^u$ its (weak) stable and unstable foliations, and by $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$ the strong ones. The leaves of $\mathcal{F}^s$, $\mathcal{F}^u$ are smooth and are diffeomorphic either to $\mathbf{R}^{k+1}$ or to $\mathbf{S}^1 \times \mathbf{R}^k$ or to $\mathbf{S}^1 \bowtie \mathbf{R} \times \mathbf{R}^{k-1}$ (for the appropriate k), where $\mathbf{S}^1 \bowtie \mathbf{R}$ denotes the (open) Moebius strip. $\mathcal{F}^s$ and $\mathcal{F}^u$ are, in general, only of class C^α, $\forall \alpha \in (0,1)$. If the flow is transitive the leaves of $\mathcal{F}^s$, $\mathcal{F}^u$ are dense in M ([Pla1]), but now there are examples ([F-W]) of nontransitive Anosov flows.

If $\mathcal{F}^s$ or $\mathcal{F}^u$ is a codimension-one foliation, then it is of class C^1 ([HPS]); and in the case of Anosov diffeomorphisms on 2-manifolds and Anosov flows on 3-manifolds $\mathcal{F}^s$ and $\mathcal{F}^u$ are of class $C^{1+\alpha}$, $\forall\alpha \in (0,1)$ ([H-K]).

The existence of $\mathcal{F}^u$, $\mathcal{F}^s$ implies that the only surface which admits Anosov diffeomorphisms is the torus. On compact surfaces different from $\mathbf{T}^2$, and possibly with boundary, it is possible to consider a class of diffeomorphisms whose properties are very near to those enjoied by Anosov ones on $\mathbf{T}^2$. These maps, introduced by Thurston in his work on mapping class groups under the name *pseudo-Anosov*, still have stable and unstable foliations, but now these foliations may have singularities.

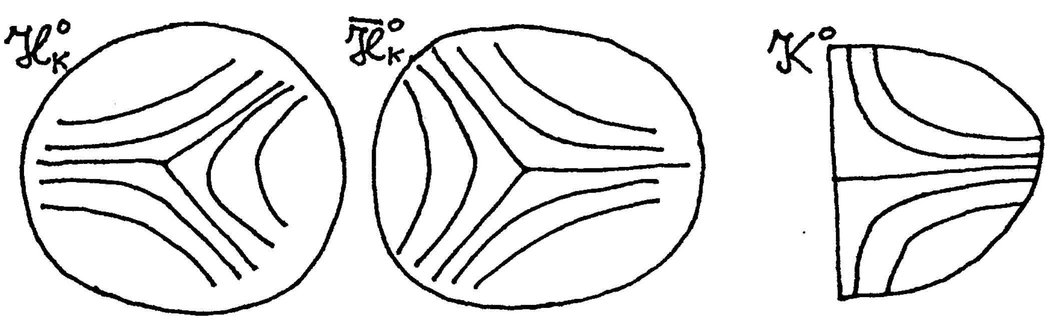
Let Σ be a compact surface, possibly with boundary.

We will consider on $\mathbf{D}^2 = \{z \in \mathbf{C} |\ |z| < 1\}$ the (singular) foliations $\mathcal{H}_k^0$, $\bar{\mathcal{H}}_k^0$, $k \in \mathbf{N} \setminus \{2\}$, given by

$$\mathcal{H}_k^0 = \{d(\Re e z^{\frac{k}{2}}) = 0\}, \qquad \bar{\mathcal{H}}_k^0 = \{d(\Im m z^{\frac{k}{2}}) = 0\},$$

and on $S\mathbf{D}^2 = \{z \in \mathbf{C} |\ |z| < 1, \Re e z \geq 0\}$ the (singular) foliation $\mathcal{K}^0$ given by

$$\mathcal{K}^0 = \{d(\Re e z \cdot \Im m z) = 0\}.$$



Definition 1.2 ([FLP], [Thu]). A *foliation with prongs* on Σ is a C^0-foliation with singularities $\mathcal{F}$ such that if $x \in Sing(\mathcal{F})$ then $\mathcal{F}$ is topologically equivalent in a neighborhood of x to $\mathcal{K}^0$ (if $x \in \partial\Sigma$) or to $\mathcal{H}_k^0$ (if $x \in int\Sigma$), for some $k \in \mathbf{N} \setminus \{2\}$. Two foliations with prongs $\mathcal{F}$ and $\mathcal{G}$ are *transverse* if $Sing(\mathcal{F}) \cap Sing(\mathcal{G}) \cap \partial\Sigma = \emptyset$, $Sing(\mathcal{F}) \cap int\Sigma = Sing(\mathcal{G}) \cap int\Sigma$, they are transverse in $int\Sigma \setminus Sing(\mathcal{F})$, and if $x \in int\Sigma \cap Sing(\mathcal{F})$ then the pair $(\mathcal{F}, \mathcal{G})$ is topologically equivalent in a neighborhood of x to $(\mathcal{H}_k^0, \bar{\mathcal{H}}_k^0)$, for some $k \in \mathbf{N} \setminus \{2\}$.

The integer k is called *number of prongs* of the singularity.

The usual definition of measured foliation ([God]), or foliation with a transverse measure, extends naturally to foliations with prongs.

Definition 1.3 ([FLP], [Thu]). A homeomorphism $f : \Sigma \to \Sigma$ is *pseudo-Anosov* if there exist two measured foliations with prongs $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$ and a constant $\lambda > 1$ such that:

i) $\mathcal{F}^s$, $\mathcal{F}^u$ are transverse and every singularity in $int\Sigma$ has at least 3 prongs

ii) μ^s, μ^u are without atoms and positive on nonempty open sets

iii) f preserves $\mathcal{F}^s$, $\mathcal{F}^u$ and

$$f_*(\mu^s) = \frac{1}{\lambda}\mu^s \qquad\qquad f_*(\mu^u) = \lambda\mu^u.$$

The meaning of the condition $f_*(\mu^s) = \frac{1}{\lambda}\mu^s$ is that f expands uniformly (w.r. to μ^s) the leaves of $\mathcal{F}^u$: if l is a segment contained in a leaf of $\mathcal{F}^u$, then l is transverse to $\mathcal{F}^s$ and its length can be measured with μ^s; $f(l)$ is also a segment contained in a leaf of $\mathcal{F}^u$, and the above condition shows that $\mu^s(f(l)) = \lambda\mu^s(l)$. Similar interpretation for $f_*(\mu^u) = \lambda\mu^u$.

If in 1.3 we don't require the restriction on the number of prongs, then we obtain the so called *generalized* pseudo-Anosov homeomorphisms.

Remark: the definition given in [FLP] is more restrictive: foliations and measures are assumed to be smooth, and f is smooth outside $Sing(\mathcal{F}^s)$ and $Sing(\mathcal{F}^u)$. But, using the "coordinates charts" given by the measures μ^s, μ^u, it is easy to verify that a pseudo-Anosov homeomorphism in the sense of 1.3 is C^0-conjugate to a pseudo-Anosov diffeomorphism in the sense of [FLP], unique up to smooth conjugation.

An Anosov diffeomorphism on $\mathbf{T}^2$ is an example of pseudo-Anosov homeomorphism; the existence of the transverse measures μ^s and μ^u is a particular case of a result of Sinai ([Sin], it is also a consequence of [Man]). Conversely, any pseudo-Anosov homeomorphism of $\mathbf{T}^2$ is C^0-conjugate to a (linear) Anosov diffeomorphism.

The foliations $\mathcal{F}^s$, $\mathcal{F}^u$ (called, naturally, stable and unstable foliations) enjoy many properties of the stable and unstable foliations of a toral Anosov diffeomorphism. Every leaf of $\mathcal{F}^s$, $\mathcal{F}^u$, different from a leaf contained in $\partial\Sigma$, is homeomorphic to $\mathbf{R}$ and is dense in Σ. Every pseudo-Anosov homeomorphism is transitive, and its periodic points form a dense subset of Σ.

The following "rigidity theorem" is a central result in the theory of pseudo-Anosov homeomorphisms.

Theorem 1.4 ([FLP], [Thu]). *Let $f, g : \Sigma \to \Sigma$ be pseudo-Anosov homeomorphisms. If f and g are homotopic, then they are C^0-conjugate, through a homeomorphism isotopic to the identity.*

Pseudo-Anosov homeomorphisms on closed surfaces are expansive (but not the generalized ones). A deep theorem of Lewowicz and Hiraide shows that the converse is also true.

Theorem 1.5 ([Lew1], [Hir]). *Every expansive homeomorphism f of a closed surface Σ is pseudo-Anosov.*

Remark that every orientable closed surface of positive genus admits expansive homeomorphisms ([O-R]); the requirement of orientability seems inessential ([Pen]).

The main step in the proof of theorem 1.5 is to show that the stable and unstable sets

$$W^s_\epsilon(x) = \{y \in \Sigma | d(f^n(x), f^n(y)) < \epsilon \ \ \forall n \geq 0\}$$

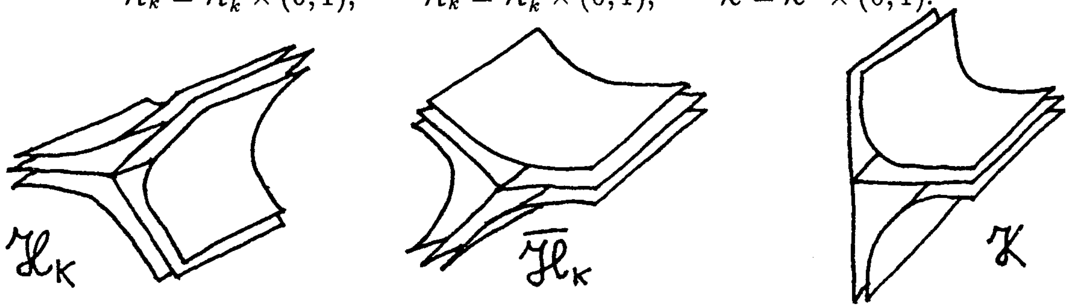
$$W^u_\epsilon(x) = \{y \in \Sigma | d(f^n(x), f^n(y)) < \epsilon \ \ \forall n \leq 0\}$$

"glue" together giving origin to two transverse foliations with prongs $\mathcal{F}^s$, $\mathcal{F}^u$. This step has been carried out also in the case of expansive flows on closed 3-manifolds.

Let M be a compact three-manifold, possibly with boundary.

We will consider on $\mathbf{D}^2 \times (0,1)$ and $S\mathbf{D}^2 \times (0,1)$ the singular codimension-one foliations given by

$$\mathcal{H}_k = \mathcal{H}^0_k \times (0,1), \qquad \bar{\mathcal{H}}_k = \bar{\mathcal{H}}^0_k \times (0,1), \qquad \mathcal{K} = \mathcal{K}^0 \times (0,1).$$



Definition 1.6 ([I-M], [Pat1]). A *foliation with circle-prongs* on M is a C^0-foliation with singularities $\mathcal{F}$ such that if $x \in Sing(\mathcal{F})$ then $\mathcal{F}$ is topologically equivalent in a neighborhood of x to $\mathcal{K}$ (if $x \in \partial M$) or to $\mathcal{H}_k$ (if $x \in intM$), for some $k \in \mathbf{N} \setminus \{2\}$. Two foliations with circle-prongs $\mathcal{F}$ and $\mathcal{G}$ are *transverse* if $Sing(\mathcal{F}) \cap Sing(\mathcal{G}) \cap \partial M = \emptyset$, $Sing(\mathcal{F}) \cap intM = Sing(\mathcal{G}) \cap intM$, they are transverse in $intM \setminus Sing(\mathcal{F})$, and if $x \in$

$intM \cap Sing(\mathcal{F})$ then the pair $(\mathcal{F}, \mathcal{G})$ is topologically equivalent in a neighborhood of x to $(\mathcal{H}_k, \bar{\mathcal{H}}_k)$, for some $k \in \mathbf{N} \setminus \{2\}$.

In other words, a foliation with circle-prongs is locally the product of a foliation with prongs with the interval $(0,1)$. From the compactness of M we deduce that $Sing(\mathcal{F})$ is union of a finite number of circles. If $S \subset Sing(\mathcal{F})$ is a singular circle, the integer k which appears in 1.6 is called *number of prongs at S*. Remark that the number of prongs at S may be strictly greater then the number of separatrices: the foliation $\mathcal{F}$ may be twisted around S. If we glue together a singular circle S and all the separatrices at S, we obtain a new decomposition of M, whose pieces are called *extended leaves.*

Assume now that M is closed, and let $\phi_t : M \to M$ be an expansive flow. For any $x \in M$ and any $\epsilon > 0$ define

$$W^s_\epsilon(x) = \{y \in M | \exists h \in Homeo([0,+\infty)) s.t.\ d(\phi_t(x), \phi_{h(t)}(y)) < \epsilon \ \forall t \in [0,+\infty)\}$$

$$W^u_\epsilon(x) = \{y \in M | \exists h \in Homeo((-\infty,0]) s.t.\ d(\phi_t(x), \phi_{h(t)}(y)) < \epsilon \ \forall t \in (-\infty,0]\}.$$

The following result of Inaba - Matsumoto and Paternain must be compared with theorem 1.5.

Theorem 1.7 ([I-M], [Pat1]). *Let $\phi_t : M \to M$ be an expansive flow on a closed 3-manifold. Then there exist two foliations with circle-prongs $\mathcal{F}^s$, $\mathcal{F}^u$ on M such that:*

i) $\mathcal{F}^s$, $\mathcal{F}^u$ are transverse and every singular circle has at least three prongs

ii) $\mathcal{F}^s$, $\mathcal{F}^u$ are invariant by ϕ_t

iii) for $\epsilon > 0$ sufficiently small and for any $x \in M$, the stable set $W^s_\epsilon(x)$ (the unstable set $W^u_\epsilon(x)$) is a neighborhood of x in the extended leaf of $\mathcal{F}^s$ ($\mathcal{F}^u$) through x.

As an example, suppose that $\phi_t : M \to M$ has a global cross-section $\Sigma \subset M$. Then the first return map $f : \Sigma \to \Sigma$ is expansive, hence by Lewowicz - Hiraide theorem there are f-invariant foliations $\mathcal{F}^s_0$, $\mathcal{F}^u_0$ on Σ. The ϕ_t-invariant foliations in the theorem of Inaba - Matsumoto - Paternain are obtained by suspending $\mathcal{F}^s_0$ and $\mathcal{F}^u_0$ via f.

Let us observe that the circles that compose the singular set $Sing(\mathcal{F}^s) = Sing(\mathcal{F}^u)$ are closed orbits of ϕ_t. They are called *singular closed orbits.* If γ is such an orbit, then the number of prongs (≥ 3) of $\mathcal{F}^s$ (or $\mathcal{F}^u$) at γ will be denoted by $np(\gamma)$. If γ is a regular (= non singular) closed orbit of ϕ_t, we define $np(\gamma) = 2$. The *irregularity* of a singular closed orbit is the positive number $irr(\gamma) = np(\gamma) - 2$.

We now state some properties of $\mathcal{F}^s$, $\mathcal{F}^u$. These properties are proved in [I-M], [Pat1], [K-S], or may be proved along the same lines as in the hyperbolic (Anosov) context ([Ano], [Pla1]).

First of all, these foliations are dynamically characterized: two points $x, y \in M$ belong to the same extended leaf of $\mathcal{F}^s$ ($\mathcal{F}^u$) if and only if $d(\phi_t(x), \phi_{h(t)}(y)) \to 0$ as $t \to +\infty$ ($t \to -\infty$), for some $h \in Homeo([0, +\infty))$ ($h \in Homeo((-\infty, 0]))$. Every leaf is a plane, or an open cylinder, or an open Moebius strip. If $L \in \mathcal{F}^s$ is a plane then $\phi_t|_L$ is C^0-equivalent to a translation. If L is a Moebius strip then $\phi_t|_L$ has one and only one closed orbit, which attracts every other trajectory. If L is a cylinder then $\phi_t|_L$ has again one and only one attracting closed orbit, except in the case where L is a separatrix of some singular closed orbit γ: in this case every trajectory of $\phi_t|_L$ tends to γ (which is the limit set of one end of L). An important related property is that there are not connections between singular closed orbits: if one end of $L \in \mathcal{F}^s$ abuts on a singular circle γ (i.e., L is a separatrix at γ) then the other end of L cannot abuts on another (or on the same) singular circle.

Inaba and Matsumoto proved a Novikov-type theorem for foliations with circle-prongs, from which one deduces that every leaf of $\mathcal{F}^s$ and $\mathcal{F}^u$ injects its fundamental group in $\pi_1(M)$ and there are not closed transversals to $\mathcal{F}^s$ or $\mathcal{F}^u$ homotopic to zero. Moreover, M must be irreducible (every embedded sphere in M bounds an embedded 3-disk , see [Hem]).

More precisely, the universal covering $\tilde{M}$ of M is the euclidean space $\mathbf{R}^3$. This is a consequence of Palmeira's theorem ([Pal]) and of the fact that the 1-dimensional foliation given by the flow ϕ_t lifts on $\tilde{M}$ to a foliation by lines (because every leaf injects its fundamental group), whose space of leaves is Hausdorff and hence homeomorphic to $\mathbf{R}^2$ (because of the expansiveness). Another way to prove $\tilde{M} = \mathbf{R}^3$ is the following: we may construct on M, starting from $\mathcal{F}^s$, an "essential lamination" ([G-O]), and theorem 6.1 of [G-O] tell us that the universal covering of M is the euclidean space.

An important property of a 3-manifold admitting an expansive flow has been proved by Paternain: the fundamental group of the manifold must have exponential growth ([Pat]).

From the existence of stable and unstable foliations, working as in the Anosov or Axiom A case ([Sma], [Shu]), it is immediate to prove a *spectral decomposition theorem*:

the nonwandering set $\Omega(\phi_t)$ is union of a finite number of pairwise disjoint *basic sets* Ω_j, $j = 1, ..., N$. Every basic set is closed, ϕ_t-invariant, transitive, and the closed orbits are dense in $\Omega(\phi_t)$. If the flow is transitive, then every leaf of $\mathcal{F}^s$ and $\mathcal{F}^u$ is dense in M (proof: as in [Pla1]).

All these properties suggest that the theory of expansive flows on 3-manifolds is completely parallel to the theory of Anosov flows, in the same sense as the theory of pseudo-Anosov homeomorphisms is parallel to the theory of toral Anosov diffeomorphisms. The only "essential" difference is the presence of singularities in the stable and unstable foliations. In the next chapter we develop this parallelism, under the hypothesis of transitivity.

Let us also remark that the expansive property may be seen (at least in low dimension) as the topological version of the Anosov property. Expansive systems may be only continuous, and the definition of expansiveness is invariant by C^0- equivalence, whereas Anosovness is not. The structural stability of Anosov systems is replaced in the expansive context by the persistence property ([Lew2]).

SURFACES OF SECTION

Let $\phi_t : M \to M$ be a continuous nonsingular flow on a closed connected 3-manifold M. A ***surface of section*** for ϕ_t ([Bir], [Fri1]) is an embedded compact connected surface $\Sigma \hookrightarrow M$ such that:

i) $\partial\Sigma$ is union of closed orbits $\gamma_1, ..., \gamma_N$ of ϕ_t

ii) $int\Sigma$ is transverse to ϕ_t (in the topological sense, [Whi])

iii) every trajectory of ϕ_t intersects Σ in a uniformly bounded time: $\exists T > 0$ s.t. $\phi_{[0,T]}(x) \cap \Sigma \neq \emptyset \ \forall x \in M$.

If $\partial\Sigma = \emptyset$, then a surface of section is nothing else that a global cross section for ϕ_t. In this case ϕ_t is topologically equivalent to the suspension of the first return map $f : \Sigma \to \Sigma$. In any case, a surface of section $\Sigma \hookrightarrow M$ induces a first return map $f^o : int\Sigma \to int\Sigma$. We will require the following property (which is not restrictive, at least if ϕ_t is C^1 near $\partial\Sigma$, see [Fri1]):

iv) f^o extends to a homeomorphism $f : \Sigma \to \Sigma$.

Remark that f preserves the components of the boundary of Σ, and on every such a component f is an orientation preserving homeomorphism ([Fri1]). Remark also that a surface of section Σ induces on M an *open book decomposition*, with pages homeomorphic to Σ and with monodromy given by the isotopy class of f. This open book decomposition is "adapted" to ϕ_t, in the sense that the interiors of the pages are transverse to ϕ_t and the binding is a union of closed orbits. The existence of an open book decomposition is not a restriction on the manifold: by a classical theorem of Alexander, every closed 3-manifold has such a decomposition.

Consider now the suspension flow ψ_t of f,

$$\psi_t : N \to N, \qquad N = \frac{\Sigma \times [0,1]}{(x,0) \sim (f(x),1)}.$$

The manifold N is the blow-up of M along the collection of closed curves $\{\gamma_1, ..., \gamma_N\}$ (every point of γ_j is replaced by a circle), and the blow-down projection $p : N \to M$ maps the orbits of ψ_t to the orbits of ϕ_t. Remark that p restricts to a diffeomorphism between $N \setminus \partial N$ and $M \setminus \partial\Sigma$, and maps every component of ∂N to one of the closed orbits $\gamma_1, ..., \gamma_N$.

We may collapse every component of $\partial\Sigma$ to a point, obtaining a closed surface $\hat{\Sigma}$. The map f induces a homeomorphism $\hat{f} : \hat{\Sigma} \to \hat{\Sigma}$; denote by $\hat{\phi}_t : \hat{M} \to \hat{M}$ the suspension flow, and by $\hat{\gamma}_1, ..., \hat{\gamma}_N$ the closed orbits of $\hat{\phi}_t$ which derive from the fixed points of $\hat{f}$ corresponding to the collapsed components of $\partial\Sigma$. The above manifold N may be again identified with the blow-up of $\hat{M}$ along $\{\hat{\gamma}_1, ..., \hat{\gamma}_N\}$, and the blow-down projection $\hat{p} : N \to \hat{M}$ maps orbits of ψ_t to orbits of $\hat{\phi}_t$. The map $\hat{p}$ induces a diffeomorphism between $N \setminus \partial N$ and $\hat{M} \setminus \cup_{j=1}^{N} \hat{\gamma}_j$, and maps every component of ∂N to some $\hat{\gamma}_j$.

In other words, both the flows $\phi_t : M \to M$ and $\hat{\phi}_t : \hat{M} \to \hat{M}$ are obtained from the flow $\psi_t : N \to N$ by collapsing the components of ∂N to circles: there are on ∂N two transverse circle fibrations $\mathcal{L}$ and $\hat{\mathcal{L}}$, both transverse to $\psi_t|_{\partial N}$, such that if we collapse every fibre of $\mathcal{L}$ ($\hat{\mathcal{L}}$) to a point we obtain the manifold M ($\hat{M}$) and ψ_t projects to ϕ_t ($\hat{\phi}_t$). In the language of three-manifold topology, this may be rephrased by saying that the flow ϕ_t is constructed from $\hat{\phi}_t$ with the help of a Dehn surgery along the closed orbits $\{\hat{\gamma}_1, ..., \hat{\gamma}_N\}$. Hence a flow with a surface of section is a suspension flow "modulo Dehn surgery" ([Fri1], [Fri2]).

Birkhoff proved in [Bir] that the geodesic flow on a closed surface of constant negative curvature S admits surfaces of section. Moreover, he showed explicitly how to construct a particular surface of section; we shall use later his construction. Fried ([Fri2]) generalized the theorem of Birkhoff to any transitive Anosov flow on a closed 3-manifold. The first return map $f : \Sigma \to \Sigma$ associated to such a surface of section is a pseudo-Anosov diffeomorphism, the only singularities of which are on the boundary $\partial\Sigma$.

Here we adapt the proof of Fried to extend his result to the expansive case, thanks to the result of Inaba - Matsumoto - Paternain.

Theorem 2.1. *Let $\phi_t : M \to M$ be a transitive expansive flow on a closed connected three-manifold M, then there exists a surface of section $\Sigma \hookrightarrow M$, with pseudo-Anosov first return map $f : \Sigma \to \Sigma$.*

The requirement of transitivity is necessary: every pseudo-Anosov homeomorphism is transitive.

We like to think to the above theorem as to an "expansive" version of Alexander theorem: every closed 3-manifold equipped with a transitive expansive flow has an open book decomposition adapted to the flow.

A consequence of theorem 2.1 is that if the stable and unstable foliations of a transitive expansive flow are without singularities, then the flow is topologically equivalent to an Anosov flow ([Fri2]). It seems that such a result holds also in the nontransitive case.

After the proof of 2.1 we shall outline some consequences, in two directions: ergodic theory and construction of smooth models.

Proof of theorem 2.1

Let $\mathcal{F}^s$, $\mathcal{F}^u$ be the stable and unstable foliations with circle-prongs of ϕ_t, given by theorem 1.7. Because of the transitivity of ϕ_t, the set of closed orbits is dense in M, and every leaf of $\mathcal{F}^s$ and $\mathcal{F}^u$ is dense in M. We denote by $C_1, ..., C_N$ the closed orbits of ϕ_t which form the singular set $\mathcal{S} = Sing(\mathcal{F}^s) = Sing(\mathcal{F}^u)$, and by $np(C_j)$ the number of prongs at C_j.

As in Fried's proof, the following "local" result is the key for the proof.

Lemma 2.2. *For any $x \in M$ there exists an immersion $j : D \to M$ of a compact surface with boundary D such that:*

i) $j(\partial D)$ is union of closed orbits of ϕ_t

ii) $j(intD)$ is transverse to ϕ_t

iii) $j|_{\partial D}$ is an embedding

iv) $x \in j(intD) \setminus [j(intD) \cap j(\partial D)]$.

Proof.

Suppose firstly that $x \in C_j \subset \mathcal{S}$, and let $i : \mathbf{D}^2 \hookrightarrow M$ be an embedding of the disk transverse to ϕ_t, with $i(0) = x$. Such an embedding exists, by a result of Whitney ([Whi]). Choosing i with image sufficiently small, we may assume that the pair of singular foliations $i^*(\mathcal{F}^s)$, $i^*(\mathcal{F}^u)$ is C^0- conjugate to the pair $\mathcal{H}_k$, $\bar{\mathcal{H}}_k$ (see chap. 1) for $k = np(C_j)$.

The separatrices of $\mathcal{H}_k$ and $\bar{\mathcal{H}}_k$ divide $\mathbf{D}^2$ in $2k$ open regions $A_1, B_1, ..., A_k, B_k$ (fig. 1), with $(\bar{A}_j \cap \bar{B}_j)$ = a separatrix of $\mathcal{H}_k$, $(\bar{B}_j \cap \bar{A}_{j+1})$ = a separatrix of $\bar{\mathcal{H}}_k$, $\forall j = 1, ..., k$ $(k+1 = 1)$. Take points $p_j \in A_j$, $j = 1...k$, such that $i(p_j)$ is a point on a closed orbit γ_j of ϕ_t $\forall j = 1, ..., k$, and assume that $\gamma_j \neq \gamma_i$ for $j \neq i$. Assume also that the leaves $\bar{\mathcal{H}}_k(p_j)$ and $\mathcal{H}_k(p_{j+1})$ intersect in a point $r_j \in B_j$, $\forall j = 1, ..., k$. These choices are possible because of the transitivity of ϕ_t.

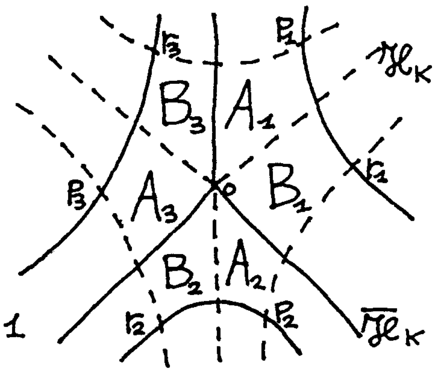


FIG. 1

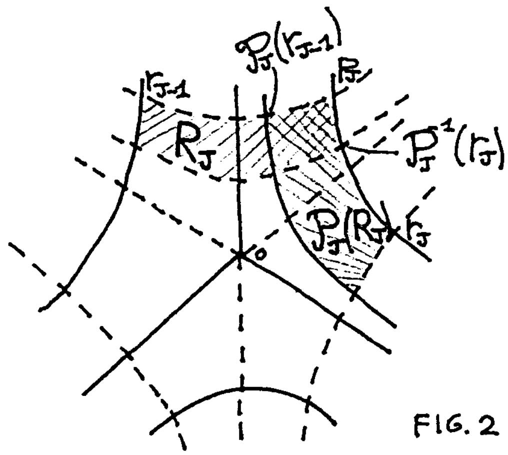


FIG. 2

For any $j = 1, ..., k$ let $\mathcal{P}_j : \mathcal{D}_j \to \tilde{\mathcal{D}}_j$ be the first return map corresponding to the closed orbit γ_j and to the transversal $i : \mathbf{D}^2 \to M$; $\mathcal{D}_j$ and $\tilde{\mathcal{D}}_j$ are subset of $\mathbf{D}^2$, $\mathcal{D}_j =$ maximal connected domain of definition of $\mathcal{P}_j$ on which the first return time is continuous, $\tilde{\mathcal{D}}_j = \mathcal{P}_j(\mathcal{D}_j)$. We may assume that $\mathcal{P}_j$ preserves the orientations of the stable and unstable leaves through p_j, because the closed orbits with this property are dense in M; we postpone the verification of this fact to the end of the proof.

Clearly, the segment of $\mathcal{H}_k(p_j)$ between p_j and r_{j-1} is contained in $\mathcal{D}_j$, and the segment of $\bar{\mathcal{H}}_k(p_j)$ between p_j and r_j is contained $\tilde{\mathcal{D}}_j$. Moreover, $\mathcal{P}_j^{-1}(r_j)$ belongs to A_j (and not to B_j), since otherwise the point $\bar{\mathcal{H}}_k(p_j) \cap \mathcal{H}_k(0)$ would belong to $\mathcal{D}_j$ and would be mapped by $\mathcal{P}_j$ to a point in B_j, which is impossible because points of $\mathcal{H}_k(0)$ correspond to orbits of ϕ_t positively asimptotic to the closed orbit C_j and so the positive semitrajectory $\phi_{[0,+\infty)}(i(\bar{\mathcal{H}}_k(p_j) \cap \mathcal{H}_k(0)))$ intersects $i(\mathbf{D}^2)$ only in points belonging to $i(\mathcal{H}_k(0))$. A similar argument (with time reversed) show that $\mathcal{P}_j(r_{j-1}) \in A_j$.

We deduce the existence of a "rectangle" $R_j \subset \mathcal{D}_j$, bounded by leaves of $\mathcal{H}_k$ and $\bar{\mathcal{H}}_k$, and with $r_{j-1}, p_j, \mathcal{P}_j^{-1}(r_j)$ as vertices (see fig. 2). Remark that $\forall j = 1, ..., k$ $\mathcal{P}_j(R_j) \cap R_{j+1}$ is a non-empty rectangle.

Define:

$$Q_1 = \mathcal{P}_1^{-1}(\mathcal{P}_1(R_1) \cap R_2 = \{z \in R_1 | \mathcal{P}_1(z) \in R_2\}$$

and for $l = 2, ..., k-1$:

$$Q_l = (\mathcal{P}_l \circ \mathcal{P}_{l-1} \circ ... \mathcal{P}_1)^{-1}((\mathcal{P}_l \circ \mathcal{P}_{l-1} \circ ... \mathcal{P}_1)(Q_{l-1}) \cap R_{l+1}) =$$

$$= \{z \in R_1 | \mathcal{P}_1(z) \in R_2, \mathcal{P}_2 \circ \mathcal{P}_1(z) \in R_3, ..., \mathcal{P}_l \circ ... \mathcal{P}_1(z) \in R_{l+1}\}$$

then Q_{k-1} is a non-empty subrectangle of R_1 and $\mathcal{P}_k \circ ...\mathcal{P}_1$ is defined on it. Moreover, $(\mathcal{P}_k \circ ...\mathcal{P}_1)(Q_{k-1})$ is a subrectangle of $\mathcal{P}_k(R_k)$ which intersects Q_{k-1} along a subrectangle Q of $\mathcal{P}_k(R_k) \cap R_1$, as in fig. 3.

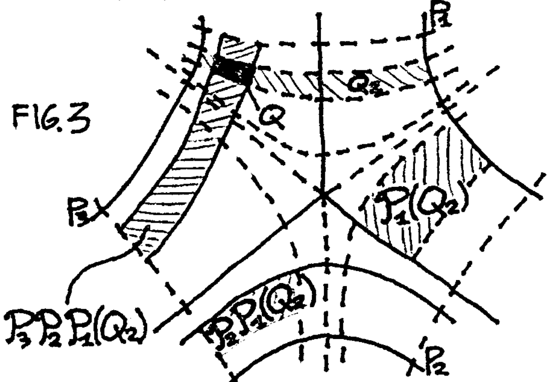


FIG.3

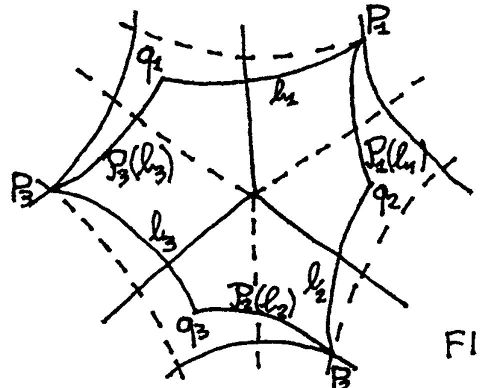


FIG.4

We deduce the existence in Q of a fixed point q_1 of $\mathcal{P}_k \circ ...\mathcal{P}_1$, from the usual "hyperbolic fixed point argument". Let us remark that $q_j = (\mathcal{P}_{j-1} \circ ...\mathcal{P}_1)(q_1)$ belongs to $R_j \cap \mathcal{P}_{j-1}(R_{j-1})\ \forall j = 1, ..., k$ and so, intuitively, the closed orbit γ of ϕ_t that corresponds to q_1 "follows" cyclically $\gamma_1, ..., \gamma_k$.

Now, as in [Fri2], we consider segments $l_j \subset \mathbf{D}^2$ joining q_j with p_j, transverse to $\mathcal{H}_k$, $\bar{\mathcal{H}}_k$, such that $l_1 \cup \mathcal{P}_1(l_1) \cup ...l_k \cup \mathcal{P}_k(l_k)$ bounds a $2k$-gon $D_0 \subset \mathbf{D}^2$ (fig. 4). The union of $i(D_0)$ and the segments of trajectories of ϕ_t from $y \in i(l_j)$ to $i(\mathcal{P}_j(y)) \in i(\mathcal{P}_j(l_j))$ may be deformed to an immersed surface $D \to M$ with the required properties ([Fri2]). Observe that $\partial D = \{\gamma_1, ..., \gamma_k, \gamma\}$ and D is a disk with k holes.

This complete the proof in the case $x \in \mathcal{S}$; the case $x \in M \setminus \mathcal{S}$ is completely similar (in fact, simpler), and formally corresponds to the case $k = 2$.

It remains only to prove the above statement about the density of closed orbits with first return map which preserves the orientations of the stable and unstable leaves. We repeat the above construction with only the following changement: if $\mathcal{P}_j : \mathcal{D}_j \to \tilde{\mathcal{D}}_j$ does not preserve the orientations of $\mathcal{H}_k(p_j)$ and $\bar{\mathcal{H}}_k(p_j)$, then we substitute it with $\mathcal{P}_j^2$ (which preserves the orientations). Then we obtain again a closed orbit γ for ϕ_t, which "follows" $\gamma_1, ..., \gamma_k$ but some γ_j are now "followed" two times; this orbit γ has first return map with the desired property, and the arbitrarity in the choice of the initial embedding $i : \mathbf{D}^2 \to M$ shows the density of the orbits of this type. $\triangle$

The proof of the existence of a surface of section is achieved as in [Fri2]: we take, by compactness, a finite union of immersed surfaces $D_1, ..., D_m$ as in the lemma, such that $\{\partial D_j\}$ are pairwise disjoint and every flowline of ϕ_t intersects $\cup_{j=1}^m D_j$ in a uniformly bounded time; $\cup_{j=1}^m D_j$ is then an "immersed surface of section", and a surgery along its self-intersections produces an embedded surface of section Σ for ϕ_t. The local structure of an expansive flow near a closed orbit garantees that the first return map $int\Sigma \to int\Sigma$ extends to Σ.

Let us remark that the above construction gives a lot of surfaces of section: the proof of lemma 2.2 shows that any closed orbit, which is nonsingular and with orientable stable and unstable leaves, may be a component of the boundary of a surface of section. On the other hand, if $\Sigma \hookrightarrow M$ is a surface of section then $\partial\Sigma$ has orientable tubular neighborhood; this implies that (at least on nonorientable manifolds) not every closed orbit can belong to the boundary of a surface of section. Remark also that if $\{\gamma_1, ..., \gamma_N\}$ is any collection of closed orbits, then there exists a surface of section whose boundary is disjoint from $\cup_{j=1}^N \gamma_j$.

We have constructed a surface of section Σ such that $\partial\Sigma \cap S = \emptyset$; however, it is clearly possible that an expansive flow admits surfaces of section whose boundary contains some singular closed orbit (take a flow obtained by Dehn surgery along a singular closed orbit of the suspension of a pseudo-Anosov homeomorphism).

We now have to prove the statement about the pseudo-Anosov character of the first return map. So, let $\Sigma \hookrightarrow M$ be any surface of section for ϕ_t (not necessarily the one above constructed), and let $f : \Sigma \to \Sigma$ be the first return map. We may assume, up to topological equivalence, that the stable and unstable manifolds of the closed orbits in $\partial\Sigma$ are smooth near $\partial\Sigma$ and their different branches intersect transversally, and that the angle of incidence between Σ and these branches varies with non-zero velocity along these closed orbits. Then the foliations with circle-prongs $\mathcal{F}^s$, $\mathcal{F}^u$ restricted to Σ give foliations with prongs $\mathcal{G}^s$, $\mathcal{G}^u$, transverse and invariant by f. Any component of $\partial\Sigma$ contains at least one prong of $\mathcal{G}^s$ and of $\mathcal{G}^u$; it may happen the case where there is only one prong, if the stable and unstable manifolds of the corresponding closed orbit are Moebius strips. Observe that $f|_{int\Sigma}$ is expansive with respect to a distance which degenerates on $\partial\Sigma$, but recall that Lewowicz - Hiraide theorem has been proved only for closed surfaces. We have to construct transverse measures μ^s, μ^u as in the definition of pseudo-Anosov homeomorphism.

We first prove by a classical argument ([Fra1]) that $\mathcal{G}^s$ and $\mathcal{G}^u$ are minimal, in the sense that every leaf in $int\Sigma$ is dense in Σ. Let $L_0 \in \mathcal{G}^u$ be a the unstable leaf through a periodic point of period k and let $L_j = f^j(L_0)$, $L = \cup_{j=0}^{k-1} L_j$. Take $x \in \bar{L}$ regular point and let $U \subset int\Sigma$ be a product neighborhood for $\mathcal{G}^s$, $\mathcal{G}^u$; if $y \in U$ is l-periodic then its stable leaf $\mathcal{G}^s(y)$ intersects $\mathcal{G}^u(x) \subset \bar{L}$ in a point $z \in U$ and the f-invariance of $\bar{L}$, together with the property $f^{ln}(z) \to y$ as $n \to +\infty$, implies that $y \in \bar{L}$; the density of periodic orbits and the closeness of $\bar{L}$ imply that $U \subset \bar{L}$, hence $\bar{L} \cap int\Sigma$ is open, i.e. $\bar{L} = \Sigma$. This means that every L_j is also dense in Σ, i.e. the leaves of $\mathcal{G}^u$ through periodic points are dense in Σ. Similarly, the leaves of $\mathcal{G}^s$ through periodic points are dense in Σ.

Let now $V \subset int\Sigma$ be any open set and take $y \in V$ k-periodic; let $l \subset \mathcal{G}^s(y) \cap V$ be a segment containing y. Then for N sufficiently large $f^{-kN}(l)$ is a segment in $\mathcal{G}^s(y)$ with the property that every leaf of $\mathcal{G}^u|_{int\Sigma}$ intersects $f^{-kN}(l)$ (because $\mathcal{G}^s(y)$ is dense in Σ and $\mathcal{G}^s(y) = \cup_{n=1}^{+\infty} f^{-kn}(l)$). If $x \in int\Sigma$ is any point, then $\mathcal{G}^u(f^{-kN}(x))$ intersects $f^{-kN}(l)$ and hence $\mathcal{G}^u(x)$ intersects l, i.e. $\mathcal{G}^u(x) \cap V \neq \emptyset$. This shows that every leaf of $\mathcal{G}^u|_{int\Sigma}$ is dense in Σ, and $\mathcal{G}^u$ is minimal. Similarly, $\mathcal{G}^s$ is minimal.

Let now Σ' denote the closed surface obtained from Σ by collapsing to a point every component of $\partial\Sigma$, let $f' : \Sigma' \to \Sigma'$ be the homeomorphism naturally induced by f, and let $\mathcal{G}^{s\prime}$, $\mathcal{G}^{u\prime}$ be the f'-invariant foliations induced by $\mathcal{G}^s$, $\mathcal{G}^u$. Clearly every leaf of $\mathcal{G}^{s\prime}$, $\mathcal{G}^{u\prime}$ is dense in Σ'. Let $\pi : \Sigma \to \Sigma'$ be the natural projection.

Remark that f' is not necessarily expansive, because $\mathcal{G}^{s\prime}$ and $\mathcal{G}^{u\prime}$ may have mono-prong singularities in points of Σ' arising from components of $\partial\Sigma$ which contain a single prong; however, proposition B of [Hir] applies also to this situation and gives two transverse Borel measures $\mu^{s\prime}$, $\mu^{u\prime}$ on $\mathcal{G}^{s\prime}$, $\mathcal{G}^{u\prime}$, which are non-atomic, positive on open non-empty sets, and such that for some $\lambda > 1$:

$$f'_*(\mathcal{G}^{u\prime}, \mu^{u\prime}) = (\mathcal{G}^{u\prime}, \lambda\mu^{u\prime}) \qquad f'_*(\mathcal{G}^{s\prime}, \mu^{s\prime}) = (\mathcal{G}^{s\prime}, \lambda^{-1}\mu^{s\prime})$$

$\mu^s \overset{def}{=} \pi^*(\mu^{s\prime})$ and $\mu^u \overset{def}{=} \pi^*(\mu^{u\prime})$ are then transverse Borel measures on $\mathcal{G}^s$, $\mathcal{G}^u$, non-atomic, positive on open non-empty sets, and such that:

$$f_*(\mathcal{G}^u, \mu^u) = (\mathcal{G}^u, \lambda\mu^u) \qquad f_*(\mathcal{G}^s, \mu^s) = (\mathcal{G}^s, \lambda^{-1}\mu^s)$$

this means that f is a pseudo-Anosov homeomorphism, and the proof of theorem 2.1 is now complete.$\triangle$

<u>Ergodic theory of expansive flows</u>

Bowen and Walters ([B-W]) proved that any transitive expansive flow (on any compact metric space) is semi-conjugate with the suspension of some subshift. We show here that, under the hypotheses of theorem 2.1, this result may be refined, and an ergodic theory similar to that of Anosov flows may be developed.

The key fact is that any transitive expansive flow in dimension 3 has Markov partitions. Such a property may be directly verified, following the same arguments as in the hyperbolic case ([Bow1], [Rat]), but it seems convenient to give a proof using the existence of surfaces of sections and the corresponding result for pseudo-Anosov homeomorphisms ([F-S]).

Let $\phi_t : M \to M$ be an expansive flow on a closed 3-manifold, with stable and unstable foliations $\mathcal{F}^s$, $\mathcal{F}^u$. The following definitions are given in analogy with [F-S] and [Bow1].

Definition 2.3. A *rectangle* is a closed subset $R \subset M$, contained in the image of an embedding $\mathbf{D}^2 \hookrightarrow M$ transverse to ϕ_t, such that there exists a homeomorphism $h : [0,1] \times [0,1] \to R$ mapping $\{0\} \times (0,1)$, $\{1\} \times (0,1)$, $\{s\} \times [0,1]$ $\forall s \in (0,1)$ to leaves of $\mathcal{F}^s$, and $(0,1) \times \{0\}$, $(0,1) \times \{1\}$, $[0,1] \times \{t\}$ $\forall t \in (0,1)$ to leaves of $\mathcal{F}^u$.

If $R = h([0,1] \times [0,1])$ is a rectangle, define $\overset{\circ}{R} = h((0,1) \times (0,1))$, and if $x = h(s,t) \in R$ define $W^s(x,R) = h(\{s\} \times [0,1])$, $W^u(x,R) = h([0,1] \times \{t\})$.

Definition 2.4: a *Markov partition* is a finite union of disjoint rectangles $\mathcal{R} = \{R_1, ..., R_m\}$ such that for some $\alpha > 0$:

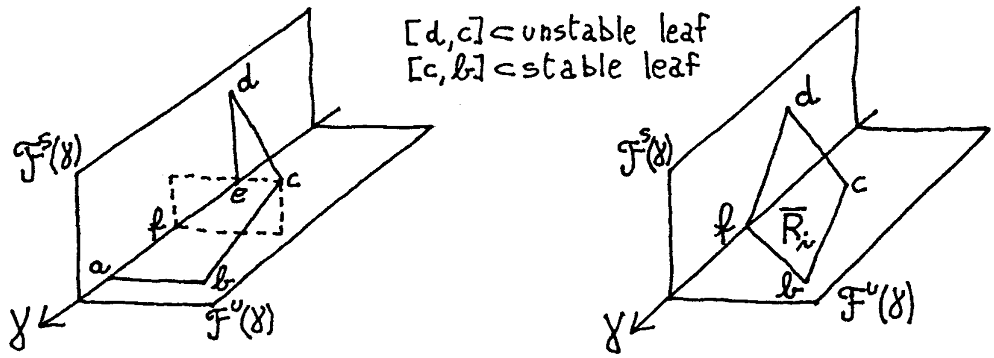
1) $\phi_{[0,\alpha]}(\cup_j R_j) = \phi_{[-\alpha,0]}(\cup_j R_j) = M$

2) if $x \in \overset{\circ}{R}_j$ and $y = \phi_t(x) \in \overset{\circ}{R}_i$ for some $t > 0$, then there exists a continuous function $\beta : W^s(x,R_j) \to \mathbf{R}$, $\beta(x) = t$, such that $\phi_{\beta(z)}(z) \in W^s(y,R_i)$ $\forall z \in W^s(x,R_j)$; and there exists a continuous function $\gamma : W^u(y,R_i) \to \mathbf{R}$, $\gamma(y) = -t$, such that $\phi_{\gamma(z)}(z) \in W^u(x,R_j)$ $\forall z \in W^u(y,R_i)$.

Proposition 2.5. *Any transitive expansive flow ϕ_t on a closed 3-manifold M has a Markov partition.*

Proof.

Let $\Sigma \overset{j}{\hookrightarrow} M$ be a surface of section and let $f : \Sigma \to \Sigma$ be the first return map. Because f is a pseudo-Anosov homeomorphism, it admits thanks to [F-S] a Markov partition $\mathcal{R} = \{R_1, ..., R_m\}$ (see [F-L-P], exposé 11, the modifications needed for the case of surfaces with boundary). This Markov partition is formed by rectangles in $int\Sigma$ and pentagons intersecting $\partial\Sigma$ along one of their sides. Let R_i be one of such pentagons, then the embedding $j : \Sigma \to M$ induces an embedding $j_i : R_i \to M$, which maps one side of R_i to a segment of a closed orbit γ and the two adiacent sides to two segments contained in the stable and the unstable leaf through γ; moreover, $j_i(R_i)$ is transverse to ϕ_t except along γ. Clearly, we may move $j_i(R_i)$ along the flowlines of ϕ_t in order to produce a rectangle $\bar{R}_i$:



If, on the contrary, R_j is a rectangle, then its image in M is also a rectangle $\bar{R}_j$. Deforming along the flowlines all the rectangles so obtained from $\mathcal{R}$ we obtain a disjoint collection of rectangles, which is the desired Markov partition.△

Let now $\mathcal{R} = \{R_1, ..., R_m\}$ be a Markov partition for ϕ_t and let us consider the matrix $A = (a_{ij})_{1 \le i,j \le m}$ defined by

$$a_{ij} = \begin{cases} 1 \text{ if } \exists x \in \overset{\circ}{R}_i \ \ s.t. \ \phi_t(x) \in \overset{\circ}{R}_j \ \text{ for some } t > 0 \text{ and } \phi_s(x) \notin \cup_j R_j \ \forall s \in (0,t) \\ 0 \text{ otherwise} \end{cases}$$

Let $\sigma_A : \Sigma_A \to \Sigma_A$ be the subshift of finite type associated to A: Σ_A is the set of sequences $\{c_n\}_{n \in \mathbf{Z}} \subset \{1, 2, ..., m\}^{\mathbf{Z}}$ such that $a_{c_n c_{n+1}} = 1 \ \forall n$, equipped with the product topology, and σ_A is defined by $\sigma_A(\{c_n\}) = \{c'_n\}$, $c'_n = c_{n+1}$. If $\omega : \Sigma_A \to \mathbf{R}$ is a continuous positive function let $\psi_t : \Sigma(\sigma_A, \omega) \to \Sigma(\sigma_A, \omega)$ be the suspension of σ_A with first return time ω, where $\Sigma(\sigma_A, \omega) = \frac{\Sigma_A \times \mathbf{R}}{(x,t) \simeq (\sigma_A(x), t + \omega(x))}$ with the usual metric, see [Bow1] or [B-W]. Then the usual arguments ([Bow1], [B-W], [F-S]) give:

<u>Corollary 2.6</u>. *There exists for an appropriate ω a continuous, surjective, finite-to-one map $h : \Sigma(\sigma_A, \omega) \to M$ such that $h \circ \psi_t = \phi_t \circ h \ \forall t \in \mathbf{R}$.* △

Starting from this corollary we may repeat in our context the same results obtained by Bowen, Sinai and others in the hyperbolic context.

For example, we may construct on M a ϕ_t-invariant probability measure which is, roughly speaking, the limit of measures concentrated on T-periodic orbits, as $T \to +\infty$

([Bow1]). Then the above semiconjugacy h becomes a measure theoretic isomorphism between Σ_A and M, if we put on Σ_A the (natural) measure described in [Bow1].

On the other hand, a Dehn surgery on a flow is an operation with no essential influence on the measure-theoretic properties of the flow. Hence it is not surprising that ergodic properties of pseudo-Anosov homeomorphisms ([F-S]) can be translated in the context of transitive expansive flows.

Smooth models

In [G-K], [Ger], [L-L] it is proved that any expansive homeomorphism of a surface is topologically conjugate a smooth (or even analytic) one, which moreover preserves a s-mooth measure (= measure given by a smooth everywhere positive density). Using surfaces of section, such a result may be extended to transitive expansive flows on 3-manifolds.

Proposition 2.7. *Any transitive expansive flow ϕ_t on a closed three-manifold M is topologically equivalent to a smooth flow, which preserves a smooth measure.*

Proof.

The flow is obtained by Dehn surgery along closed orbits $\gamma_1,...,\gamma_N$ of the suspension $\psi_t : N \to N$ of a generalized pseudo-Anosov homeomorphism $f : \Sigma \to \Sigma$, Σ a closed surface. By [G-K], [Ger], [L-L], we may assume that f is smooth and preserves a smooth measure ω; hence ψ_t also is smooth and preserves a smooth measure Ω. After the surgery, the orbits $\gamma_1,...,\gamma_N$ of ψ_t become closed orbits $\hat{\gamma}_1,...,\hat{\gamma}_N$ of ϕ_t, and the measure Ω transforms to a ϕ_t-invariant measure $\hat{\Omega}$. But ϕ_t and $\hat{\Omega}$ are not smooth along $\cup_j\hat{\gamma}_j$.

Consider now a smooth surface of section $\Sigma_0 \hookrightarrow M$ for ϕ_t, such that

$$\partial\Sigma_0 \cap \{\hat{\gamma}_1,...,\hat{\gamma}_N\} = \emptyset$$

(Σ_0 exists, see proof of 2.1). The first return map $f_0 : \Sigma_0 \to \Sigma_0$ is smooth outside the periodic points corresponding to the closed orbits $\{\hat{\gamma}_1,...,\hat{\gamma}_N\}$; moreover f preserves a measure $\hat{\omega}$ (induced by $\hat{\Omega}$) which is smooth outside the same set of periodic points.

Because the smoothing technique of [G-K] and [L-L] is local, we may perturb f_0 in a neighborhood of these points in order to produce a map $\tilde{f}_0 : \Sigma_0 \to \Sigma_0$ which is smooth, preserves a smooth measure, and is C^0-conjugate to f_0. To such a perturbation there corresponds on M a smoothing of ϕ_t and $\hat{\Omega}$ along $\{\hat{\gamma}_1,...,\hat{\gamma}_N\}$, and the flow so obtained is topologically equivalent to ϕ_t. $\triangle$

Remark: these smooth models are *conditionally stable* ([Ger]), i.e. they are structurally stable with respect to perturbations whose k-jets vanish along the singular closed orbits (k depending on the closed orbit). It follows the existence of analytic models ([Ger], [L-L]).

A CRITERION FOR THE TOPOLOGICAL EQUIVALENCE OF EXPANSIVE FLOWS

Let $\phi_t : M \to M$ be a transitive expansive flow on a closed connected 3-manifold M, and let $\Sigma \hookrightarrow M$ be a surface of section, with (pseudo-Anosov) first return map $f : M \to M$.

Recall that there exists a natural fibration

$$\Sigma \hookrightarrow \tilde{M} \xrightarrow{p} \mathbf{S}^1,$$

where $\tilde{M}$ is the blow-up of M along $\partial\Sigma = \{\gamma_1, ..., \gamma_N\}$, and that ϕ_t lifts to a flow $\tilde{\phi}_t : \tilde{M} \to \tilde{M}$ which is transverse to the fibration and may by identified, modulo reparametrization, to the suspension of f.

If $\gamma \subset M \setminus \partial\Sigma$ is a closed orbit of ϕ_t and if $\tilde{\gamma} \subset \tilde{M} \setminus \partial\tilde{M}$ denotes the corresponding closed orbit of $\tilde{\phi}_t$, then the map $p|_{\tilde{\gamma}} : \tilde{\gamma} \to \mathbf{S}^1$ (which is a covering) has a well defined degree, whose absolute value coincides with the period of the orbit of f corresponding to $\tilde{\gamma}$. We define

$$deg(\gamma; \Sigma) = deg(p|_{\tilde{\gamma}}).$$

Let now $\{\bar{\gamma}_1, ..., \bar{\gamma}_k\}$ be the collection of all singular closed orbits of ϕ_t which are disjoint from $\partial\Sigma$. The *total irregularity of ϕ_t with respect to* Σ is the natural number

$$I(\phi_t; \Sigma) = \sum_{j=1}^{k} |deg(\bar{\gamma}_j; \Sigma)| \cdot irr(\bar{\gamma}_j).$$

Suppose that $\psi_t : M \to M$ is another transitive expansive flow on the same manifold M. Suppose that there exists a neighborhood U of $\partial\Sigma$ such that ψ_t and ϕ_t define on U the same 1-dimensional oriented foliation (in particular, $\gamma_1, ..., \gamma_N$ are closed orbits also for ψ_t). Then ψ_t lifts to a flow $\tilde{\psi}_t : \tilde{M} \to \tilde{M}$, equal to $\tilde{\phi}_t$ on a neighborhood of $\partial\tilde{M}$.

If $\gamma' \subset M \setminus \partial\Sigma$ is a closed orbit of ψ_t and $\tilde{\gamma}'$ denotes its lifting to $\tilde{M} \setminus \partial\tilde{M}$, then it makes sense to set $deg(\gamma'; \Sigma) = deg(p|_{\tilde{\gamma}'})$ and to define the total irregularity of ψ_t with respect to Σ as

$$I(\psi_t;\Sigma) = \sum_{j=1}^{h} |deg(\bar{\gamma}'_j;\Sigma)| \cdot irr(\bar{\gamma}'_j)$$

where $\{\bar{\gamma}'_1, ..., \bar{\gamma}'_h\}$ are the singular closed orbits of ψ_t disjoint from $\partial\Sigma$.

Our main result is the following.

Theorem 3.1. *Let $\phi_t : M \to M$ be a transitive expansive flow on a closed connected 3-manifold M and let $\Sigma \hookrightarrow M$ be a surface of section, $\partial\Sigma \neq \emptyset$. Let $\psi_t : M \to M$ be another transitive expansive flow on the same manifold M, such that:*

i) there exists a neighborhood U of $\partial\Sigma$ such that ψ_t and ϕ_t define on U the same 1-dimensional oriented foliation;

ii)

$$I(\psi_t;\Sigma) \geq I(\phi_t;\Sigma).$$

Then ϕ_t and ψ_t are topologically equivalent.

This theorem (but not its proof) may be seen as a generalization of a theorem of Plante ([Pla2], see also [Pla3], [G-S], [Arm] for related results), asserting that any Anosov flow on a 3-manifold which is a torus bundle over $\mathbf{S}^1$ is topologically equivalent to the suspension of a toral Anosov diffeomorphism A. In this case the "model flow" ϕ_t is the suspension of A, that is a flow with a surface of section with empty boundary ($\Sigma \simeq \mathbf{T}^2$). We shall return on that at the end of the chapter, where we will comment also the hypothesis $\partial\Sigma \neq \emptyset$: if Σ is a surface of section without boundary (i.e., a global cross section) then the theorem is still true modulo reversing of the time (ψ_t is topologically equivalent either to ϕ_t or to ϕ_{-t}).

Let us comment a little the hypotheses i) and ii).

About i): up to topological equivalence, this is equivalent to require that $\{\gamma_1, ..., \gamma_N\} = \partial\Sigma$ are closed orbits also for ψ_t, with the same number of prongs and with the same twisting of the invariant foliations as for ϕ_t.

About ii): because the total irregularity is always ≥ 0, this hypothesis is trivially satisfied if $I(\phi_t;\Sigma) = 0$. This happens if ϕ_t is an Anosov flow, or if Σ is choosen so that $\partial\Sigma$ contains all the singular closed orbits of ϕ_t. On the other hand, this hypothesis seems essential: if it were unnecessary then we would obtain as a corollary that any transitive expansive flow on $\Sigma_A \stackrel{def}{=} \frac{\Sigma\times[0,1]}{(x,0)\sim(f(x),1)}$, $f : \Sigma \to \Sigma$ pseudo-Anosov, would be topologically equivalent to the suspension of f or f^{-1}. But such a corollary seems false, if $dim H^1(\Sigma_A, \mathbf{R}) > 1$ (these problems are related to the Thurston norm, [Fri3]).

Here is a sketch of the proof of theorem 3.1. We take a fibre $\Sigma \hookrightarrow \tilde{M}$ and, using the classical method of Roussarie ([Rou1], [Rou2]) and following ideas of [G-S], [Pla3], we isotope Σ $(rel \partial \Sigma)$ to a surface transverse to the stable foliation of $\tilde{\psi}_t$. It is here that ii) plays a rôle. Next, we cut $\tilde{M}$ along this surface, obtaining a manifold diffeomorphic to $\Sigma \times [0,1]$ equipped with a foliation induced by the stable foliation of $\tilde{\psi}_t$. The analysis of this foliation through Novikov's theorem ([Nov], [I-M], but we will need a "singular" version of these results) shows that every closed orbit of $\tilde{\psi}_t$ projects by $\tilde{M} \xrightarrow{p} \mathbf{S}^1$ to a nontrivial element of $\pi_1(\mathbf{S}^1)$. It is then possible, using a Baire argument ([Ver], [Ful]), to prove that the fibre Σ, which is a global cross section for $\tilde{\phi}_t$, is isotopic to a global cross section for $\tilde{\psi}_t$. The rigidity theorem for pseudo-Anosov homeomorphisms shows that the first return maps of $\tilde{\phi}_t$ and $\tilde{\psi}_t$ are C^0- conjugate, hence $\tilde{\phi}_t$ and $\tilde{\psi}_t$ are topologically equivalent. The proof is completed by showing that this topological equivalence on $\tilde{M}$ "blow-down" to a topological equivalence between ϕ_t and ψ_t.

Proof of theorem 3.1. Isotopies of Σ

We denote by $\mathcal{F}^s$, $\mathcal{F}^u$ the stable and unstable foliations with circle-prongs associated to ϕ_t, and by $\mathcal{G}^s$, $\mathcal{G}^u$ those associated to ψ_t. The lifted flows $\tilde{\phi}_t, \tilde{\psi}_t : \tilde{M} \to \tilde{M}$ are equal on a neighborhood of $\partial \tilde{M}$ and have invariant foliations with circle-prongs $\tilde{\mathcal{F}}^s$, $\tilde{\mathcal{F}}^u$ (for $\tilde{\phi}_t$) and $\tilde{\mathcal{G}}^s$, $\tilde{\mathcal{G}}^u$ (for $\tilde{\psi}_t$).

Consider a fibre $\Sigma \subset \tilde{M}$, e.g. Σ is the lift of the surface of section for ϕ_t. Remark that $\tilde{\psi}_t$ is transverse to Σ near $\partial\Sigma$. The stable foliation $\tilde{\mathcal{G}}^s$ induces on Σ a 1-dimensional foliation with singularities $\mathcal{H}$. A small perturbation ([Sol1], [H-H]) ensures that all the singularities of $\mathcal{H}$ belong to one of the following classes:

a) a *saddle* or a *centre* in $int\Sigma$, corresponding to a tangency of $\tilde{\mathcal{G}}^s$ with Σ of "Morse-type";

b) a singularity due to the transverse intersection of Σ with a circle-prong of $\tilde{\mathcal{G}}^s$; such a singularity will be called *prong* if it belongs to $int\Sigma$, and *semi-saddle* if it belongs to $\partial\Sigma$.

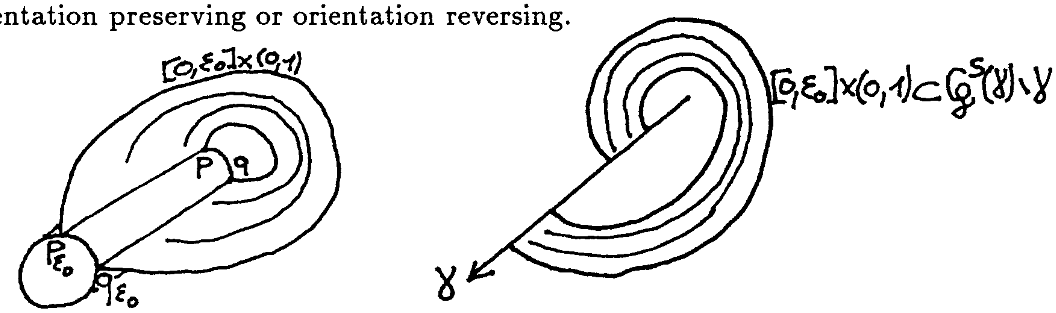
Lemma 3.2. *Σ is isotopic $(rel\partial\Sigma)$ to a surface Σ' such that the induced foliation $\mathcal{H}' = \tilde{\mathcal{G}}^s|_{\Sigma'}$ has only generic singularities (of the classes a) and b)) and there are not* *connections between two different singularities nor prong-selfconnections.*

Proof.

A connection between two different saddles or a saddle and a prong or a saddle and a semi-saddle may be removed through a small perturbation of Σ ([Sol1], [H-H]).

Because $\mathcal{G}^s$ has no connection between two circle-prongs nor circle-prong selfconnection, a connection between two singularities of the class b) (possibly coincident) is either a connection between two semi-saddles on the same connected component of $\partial\Sigma$, or a connection between two different prongs arising from the same circle-prong.

The first case must be excluded for the following argument. If $l \subset \Sigma$ is a connection between the semi-saddles $p, q \in \partial\Sigma$, then any other fibre Σ_ϵ near Σ admits an induced foliation $\mathcal{H}_\epsilon$ with a connection l_ϵ between two semi-saddles $p_\epsilon, q_\epsilon \in \partial\Sigma_\epsilon$, p_ϵ near p and q_ϵ near q. Glueing together these connections l_ϵ $\forall\epsilon \in [0, \epsilon_0]$, ϵ_0 small, we obtain a strip $[0,\epsilon_0] \times (0,1)$ embedded in a leaf of $\tilde{\mathcal{G}}^s$. Projecting on M we find an embedded strip $[0,\epsilon_0] \times (0,1)$ contained in $\mathcal{G}^s(\gamma) \setminus \gamma$ for some closed orbit γ, such that the ends $[0,\epsilon_0] \times \{0\}$ and $[0,\epsilon_0] \times \{1\}$ are on γ. Moreover, giving to γ any orientation and to $[0,\epsilon_0] \times \{0\}$, $[0,\epsilon_0] \times \{1\}$ the orientations induced by $[0,\epsilon_0]$, the inclusions $[0,\epsilon_0] \times \{0\} \subset \gamma$ and $[0,\epsilon_0] \times \{1\} \subset \gamma$ are both orientation preserving or orientation reversing.



This situation is in contradiction with the structure of $\mathcal{G}^s$: every connected component of $\mathcal{G}^s(\gamma) \setminus \gamma$ is a cylinder.

The second case (a connection $l \subset \Sigma$ between two prongs p, q arising from the same circle-prong C) may be avoided with an isotopy of Σ ([I-M]). There exists a disk D embedded in the leaf of $\tilde{\mathcal{G}}^s$ containing l, bounded by l and a segment $m \subset C$; the "Whitney trick" allows to remove the connection l, through an isotopy localized around such a disk. A last perturbation will give Σ' with the desired properties.$\triangle$

Consider now the foliation $\mathcal{K}$ induced by $\tilde{\mathcal{F}}^s$ on Σ. It is a foliation with prongs and semi-saddles, because Σ is transverse to $\tilde{\mathcal{F}}^s$. If $\mathcal{P}(\mathcal{K})$ denotes the set of prongs of $\mathcal{K}$, we have the relation

$$I(\phi_t;\Sigma) = -2 \sum_{p_j \in \mathcal{P}(\mathcal{K})} index(p_j)$$

where $index(\cdot)$ is the Poincare'-Hopf index. This equality is obtained by recalling that the Poincare'-Hopf index of a prong with k separatrices is $-\frac{1}{2}(k-2)$, and by observing that a closed orbit of $\tilde{\phi}_t$ with degree h intersects Σ in $|h|$ points.

Let now $\tilde{\Sigma}$ be any surface isotopic to $\Sigma(rel\partial\Sigma)$, such that $\tilde{\mathcal{H}} = \tilde{\mathcal{G}}^s|_{\tilde{\Sigma}}$ has only generic singularities, and let $\mathcal{P}(\tilde{\mathcal{H}})$ be the set of prongs of $\tilde{\mathcal{H}}$. Because a closed orbit of $\tilde{\psi}_t$ with degree h intersects $\tilde{\Sigma}$ in at least $|h|$ points, we have the inequality

$$I(\psi_t;\Sigma) \leq -2 \sum_{q_j \in \mathcal{P}(\tilde{\mathcal{H}})} index(q_j).$$

Hence the hypothesis $I(\psi_t;\Sigma) \geq I(\phi_t;\Sigma)$ translates into

$$-\sum_{q_j \in \mathcal{P}(\tilde{\mathcal{H}})} index(q_j) \geq -\sum_{p_j \in \mathcal{P}(\mathcal{K})} index(p_j)$$

(roughly speaking, $\tilde{\mathcal{H}}$ has more prongs then $\mathcal{K}$).

On the other hand, the fact that $\tilde{\phi}_t = \tilde{\psi}_t$ near $\partial\tilde{M}$ implies that $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{G}}^s$ are topologically equivalent (not equal!) near $\partial\tilde{M}$, hence $\mathcal{K}$ and $\tilde{\mathcal{H}}$ are topologically equivalent near $\partial\Sigma = \partial\tilde{\Sigma}$ and, in particular, $\mathcal{K}$ and $\tilde{\mathcal{H}}$ have the same number of semi-saddles.

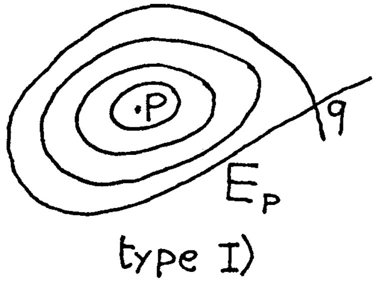
Lemma 3.3. *Σ' is isotopic $(rel\partial\Sigma)$ to a surface Σ'' which is transverse to $\tilde{\mathcal{G}}^s$.*

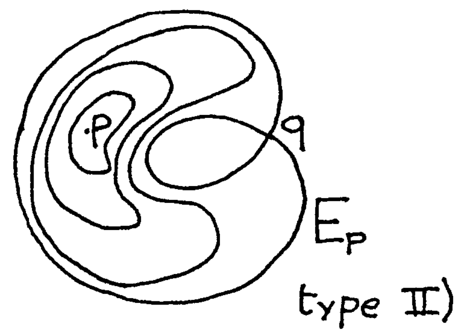
Proof.

The above remarks and Poincare'-Hopf formula show that if we are able to isotope Σ' $(rel\partial\Sigma)$ to a surface Σ'' such that the induced foliation $\mathcal{H}'' = \tilde{\mathcal{G}}^s|_{\Sigma''}$ has generic singularities and no centre, then Σ'' is transverse to $\tilde{\mathcal{G}}^s$ ($\mathcal{H}''$ has more prongs and hence less saddles then $\mathcal{K}$, but $\mathcal{K}$ has no saddle...).

To eliminate the centres we use the method of Roussarie ([Rou1], [Rou2]).

Let $p \in \Sigma'$ be a centre of $\mathcal{H}'$, and let $E_p \subset \Sigma'$ be the closure of the union of leaves of $\mathcal{H}'$ which are circles bounding on Σ' a (unique) disk containing p. The properties of Σ' stated in lemma 3.2, and the arguments of [Rou1], imply that E_p is contained in $int\Sigma'$ and it is bounded by one or two homoclinic trajectories ending to the same saddle $q \in int\Sigma'$:





The embedding $\Sigma' \hookrightarrow \tilde{M}$ is incompressible, i.e. it induces an injection $\pi_1(\Sigma') \to \pi_1(\tilde{M})$. The manifold $\tilde{M}$ is irreducible, because M is. The foliation $\tilde{\mathcal{G}}^s$ is without vanishing cycles ([God], [C-L], [H-H]). These three properties are sufficient to repeat in our (singular) context the arguments of [Rou1,49-52], and to prove that Σ' is isotopic to a surface (again denoted Σ') such that the induced foliation $\mathcal{H}'$ satisfies again the conclusions of lemma 3.2 and, moreover, has no centre of type I).

Let now $p \in \Sigma'$ be a centre of type II). Because there are not centres of type I), the two homoclinic trajectories $l_1, l_2 \subset \Sigma'$ which bound E_p are both noncontractible on the leaf $L \in \tilde{\mathcal{G}}^s$ which contains them ([Rou2,109]). In particular, L is a leaf with nontrivial fundamental group, i.e. $L \simeq$ cylinder or $L \simeq$ Moebius strip. A small perturbation of Σ' pushes the saddle $q \in \partial E_p$ to a leaf homeomorphic to $\mathbf{R}^2$, because the leaves with this property are dense in $\tilde{M}$. Now the centre of type II) has been replaced by a centre of type I), which is removed through an isotopy as before. After a finite number of steps the proof of the lemma is completed. $\triangle$

Again the Poincare'-Hopf formula shows that, now, we must have

$$I(\phi_t; \Sigma) = I(\psi_t; \Sigma)$$

and every closed orbit of $\tilde{\psi}_t$ with degree h intersects Σ'' in exactly $|h|$ points. However, there may be, a priori, closed orbits with degree 0 and hence disjoint from Σ''.

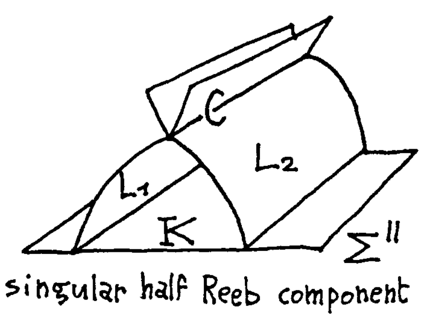
Cut $\tilde{M}$ along this surface Σ''. Because Σ'' is isotopic to a fibre of $\tilde{M} \xrightarrow{p} \mathbf{S}^1$, the result of the cutting is a manifold $\bar{M}$ with boundary and corners diffeomorphic to $\Sigma'' \times [0,1]$. The foliation $\tilde{\mathcal{G}}^s$ induces a foliation $\bar{\mathcal{G}}^s$ with circle-prongs and line-prongs (obvious the definition), transverse to $\partial_0 \bar{M} \overset{def}{=} \Sigma'' \times \{0,1\}$.

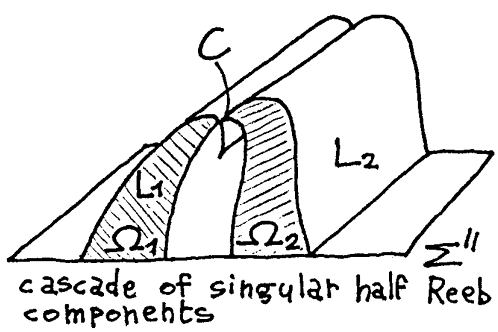
It may happen that there exists a continuous map $\gamma : [0,1] \to \bar{M}$ with image in a leaf $L \in \bar{\mathcal{G}}^s$ such that $\gamma(0), \gamma(1)$ belong to the same component of $\partial_0 \bar{M}$ and γ represents a nontrivial element of $\pi_1(L, \partial L)$. We shall say that γ is an *arch map*. It will be useful to choose carefully the surface Σ'' transverse to $\bar{\mathcal{G}}^s$, in such a way that $\bar{\mathcal{G}}^s$ does not admit arch maps.

To this end, recall that an *orientable (nonorientable) half Reeb component* ([M-R], [G-S]) of $\bar{\mathcal{G}}^s$ is a closed saturated subset $\Omega \subset (int\Sigma'') \times [0,1]$, bounded by a leaf $L \in \bar{\mathcal{G}}^s$ homeomorphic to $\mathbf{S}^1 \times [0,1]$ ($\mathbf{S}^1 \bowtie [0,1]$) and a subset $K \subset (int\Sigma'') \times \{0,1\}$ homeomorphic to $\mathbf{S}^1 \times [0,1]$ ($\mathbf{S}^1 \bowtie [0,1]$), such that its double $2 \cdot \Omega$ is an orientable (nonorientable) Reeb component of $2 \cdot \bar{\mathcal{G}}^s$.

By a *singular half Reeb component* of $\bar{\mathcal{G}}^s$ we mean a closed saturated subset $\Omega \subset (int\Sigma'') \times [0,1]$, bounded by a subset $K \subset (int\Sigma'') \times \{0,1\}$ homeomorphic to $\mathbf{S}^1 \times [0,1]$, two leaves $L_1, L_2 \in \bar{\mathcal{G}}^s$ homeomorphic to $\mathbf{S}^1 \times [0,1]$, and a singular circle C for which L_1 and L_2 are separatrices, such that the restriction of $\bar{\mathcal{G}}^s$ to $int\Omega$ is C^0- conjugate to the restriction to the interior of an orientable half Reeb component.

A *cascade of singular half Reeb components* is a closed saturated subset $\Omega \subset (int\Sigma'') \times [0,1]$ bounded by an annulus $K \subset (int\Sigma'') \times \{0,1\}$, a singular circle C, and two separatrices $L_1, L_2 \simeq \mathbf{S}^1 \times [0,1]$ at C, such that L_1, L_2 are in the boundary of singular half Reeb components Ω_1, Ω_2, and every separatrix at C is contained in Ω.



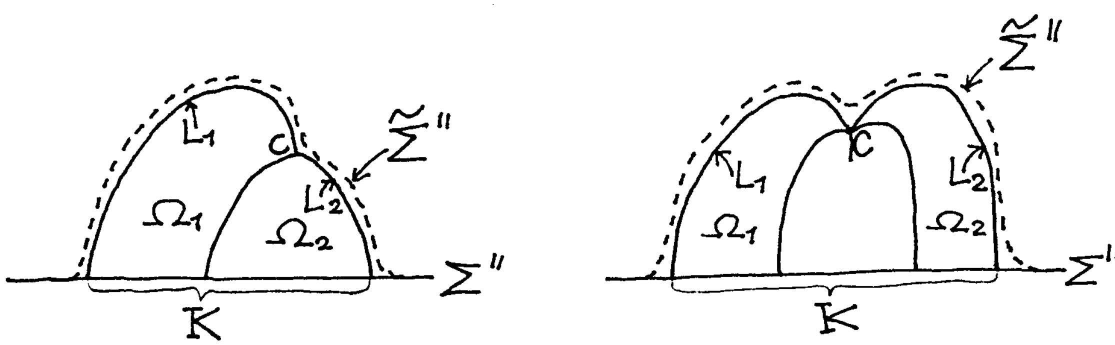


Lemma 3.4. *Σ'' is isotopic ($rel\partial\Sigma$) to a surface Σ''' which is again transverse to $\tilde{\mathcal{G}}^s$ and moreover, if $\hat{\mathcal{G}}^s$ denotes the foliation induced by $\tilde{\mathcal{G}}^s$ on the manifold $\hat{M} \simeq \Sigma''' \times [0,1]$ obtained by cutting $\tilde{M}$ along Σ''', then $\hat{\mathcal{G}}^s$ has not half Reeb components nor cascades of singular half Reeb components.*

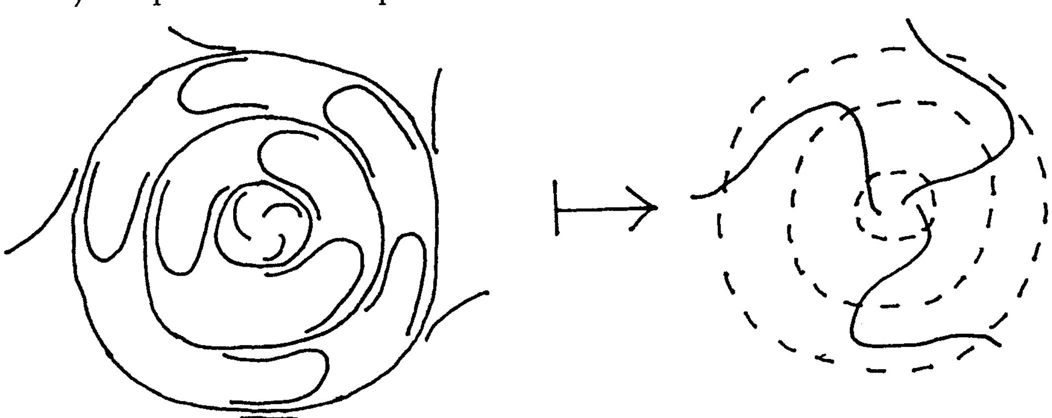
Proof.

Let $\Omega \subset \bar{M}$ be a cascade of singular half Reeb components for $\bar{\mathcal{G}}^s$, and let L_1, L_2 be the corresponding boundary separatrices, Ω_1, Ω_2 the singular half Reeb components which contain L_1, L_2, C the singular circle, $K \subset \partial_0 \bar{M}$ the boundary annulus. Remark that $\mathcal{H}'' = \bar{\mathcal{G}}^s|_{\Sigma''}$ has (at least) two parallel planar Reeb components in correspondence of $K \cap (\Omega_1 \cup \Omega_2)$.

The "leaf" $L_1 \cup C \cup L_2$ has contracting (or repelling) holonomy on the side exterior to Ω. This is sufficient to find an isotopy of Σ'' which produces a surface $\tilde{\Sigma}''$ equal to Σ'' outside a small neighborhood of K, disjoint from Ω, and still transverse to $\bar{\mathcal{G}}^s$:

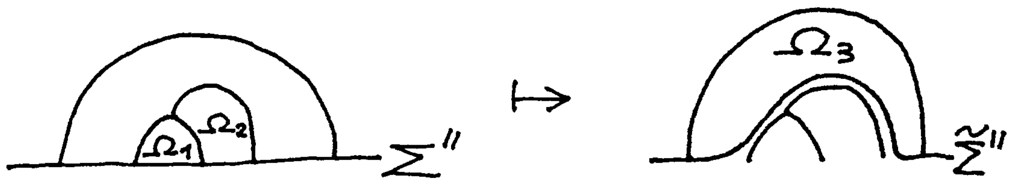


This isotopy "cancels" a cascade of singular half Reeb components. From the point of view of the foliation $\mathcal{H}''$, this corresponds to a transformation $\mathcal{H}'' \mapsto \tilde{\mathcal{H}}''$ which eliminates (at least) two planar Reeb components:



A similar isotopy cancels a half Reeb component.

It may happen that an elimination of an half Reeb component or of a cascade of singular half Reeb components produces a new half Reeb component or a new cascade of singular half Reeb components, because (singular) half Reeb components may be, roughly speaking, "enclosed" one inside the other:



However, limit cycles that bound planar Reeb components cannot accumulate and hence are finite, so after a finite number of isotopies of the previous type we arrive to a surface Σ''' satisfying the conclusion of the lemma. To this regard, observe also that the elimination of planar Reeb components in $\mathcal{H}''$ may produce new limit cycles, but these cycles are surely not in the boundary of half Reeb components, because they admit closed transversals. △

<u>**Proof of theorem 3.1. Closed orbits of $\tilde{\psi}_t$**</u>

Let $\Sigma''' \subset \tilde{M}$ be the surface isotopic to Σ ($rel\partial\Sigma$) obtained in lemma 3.4: $\hat{\mathcal{G}}^s$ has no half Reeb component and no cascade of half Reeb components. The following lemma may be seen as a singular (and relative) version of Novikov's theorem ([Nov], [I-M]).

<u>**Lemma 3.5**</u>. *$\hat{\mathcal{G}}^s$ does not admit arch maps.*

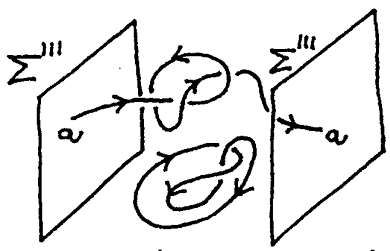
Proof.

Assume, by contradiction, that there is an arch map

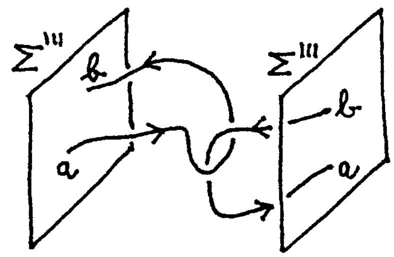
$$\gamma : [0,1] \to \hat{M} \simeq \Sigma''' \times [0,1];$$

to fix ideas, suppose $\gamma(0), \gamma(1) \in \Sigma''' \times \{0\}$, and let $L \in \hat{\mathcal{G}}^s$ be the leaf containing $\gamma([0,1])$.

We collapse every circle $c \times \{t\}$, c a connected component of $\partial\Sigma'''$, $t \in [0,1]$, to a point; then we double the result. We obtain a closed 3-manifold N, diffeomorphic to $S \times \mathbf{S}^1$ where S is a closed surface. The foliation $\hat{\mathcal{G}}^s$ yields a foliation $\mathcal{G}$ on N with circle-prongs. There may be circle-prongs with only 1 prong (they come from connected components of $\partial\Sigma''' \times [0,1]$), and there may be connections between two circle-prongs C_1, C_2 (C_2 is then the doubling of C_1, and the connection is the doubling of a leaf of $\hat{\mathcal{G}}^s$ which joins $C_1 \subset int\hat{M}$ with $\partial_0 \hat{M} = \Sigma''' \times \{0,1\}$). In the following discussion it will be useful to recall that every closed orbit of $\tilde{\psi}_t$ intersects Σ''' in $|h|$ points, where h is the degree of the closed orbit.



allowed closed orbits



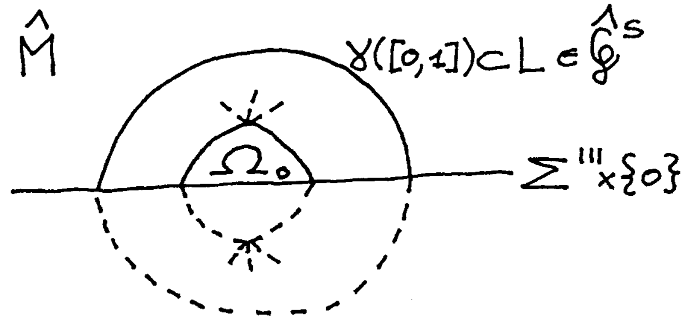
prohibited closed orbit

The arch map γ generates a continuous map $\omega : \mathbf{S}^1 \to N$, with image in the leaf $2 \cdot L \in \mathcal{G}$, which is homotopic to zero in N but not in $2 \cdot L$.

Let $\Gamma : \mathbf{D}^2 \to N$ be a continuous symmetric extension of ω. We may suppose that $\Gamma(\mathbf{D}^2)$ does not intersect the (eventual) circle-mono-prongs of $\mathcal{G}$, and (after perturbation) that Γ satisfies the usual general position properties: $\Gamma^*(\mathcal{G})$ has only singularities of the type saddle, prong, centre; there are not connections between two saddles or a saddle and a prong. We remove through a Whitney disk the eventual connections between two prongs arising from the same circle-prong. However, there may be "irremovable" connections between two prongs p_1, p_2 lying on two different circle-prongs C_1, C_2.

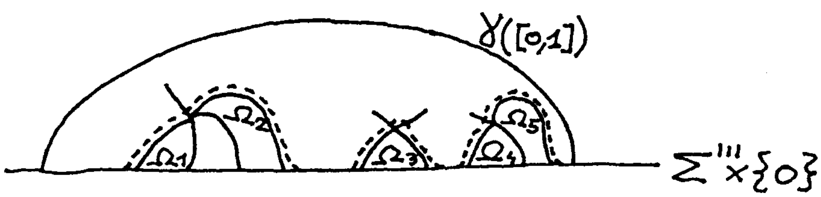
The arguments of Inaba - Matsumoto ([I-M, 336-337]) work also in our situation: the only difference is that the vanishing cycle $v : \mathbf{S}^1 \to \mathbf{D}^2$ that we find may be "singular", i.e. composed by two connections l_1, l_2 between two prongs p_1, p_2 lying on two different circle-prongs C_1, C_2. See also [God] and [C-L] about the techniques to find vanishing cycles.

The foliation $\hat{\mathcal{G}}^s$ has no half Reeb component (and, obviously, no Reeb component), hence $\mathcal{G}$ has no Reeb component. The vanishing cycle v generates, as in [I-M, 337-339], a compact extended leaf which is the boundary of a singular Reeb component (= the double of a singular half Reeb component). This singular Reeb component is symmetric, hence the initial foliation $\hat{\mathcal{G}}^s$ on $\hat{M}$ has a singular half Reeb component Ω_0:

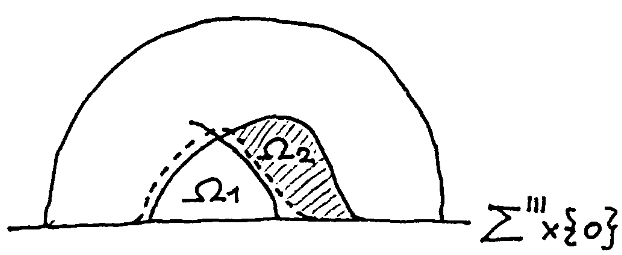


Briefly, we have relativized Inaba - Matsumoto theorem.

The foliation $\hat{\mathcal{G}}^s$ may have several singular half Reeb components $\Omega_1, ..., \Omega_k$ generated by vanishimg cycles in $\Gamma(\mathbf{D}^2)$. We remove all these components, "pushing" $\Sigma''' \times \{0\}$ inside $\hat{M}$ preserving the transversality (as was done in lemma 3.4):



Remark that, by hypothesis, there are not cascades of singular half Reeb components, so any annulus of $\Sigma''' \times \{0\}$ which has been pushed inside $\hat{M}$ must intersect at least one separatrix of a circle-prong. This fact ensures that the map γ is still an arch map, even after the pushing of $\Sigma''' \times \{0\}$. We repeat the above arguments and we find another (perhaps singular) half Reeb component. The only possibility is that this component contains as a leaf one of the above separatrices:



(in particular, it is a nonsingular half Reeb component).

This new half Reeb component corresponds to a singular half Reeb component of $\hat{\mathcal{G}}^s$, and we arrive to a contradiction with the fact that all the singular half Reeb component were pushed-off $\hat{M}$. In conclusion, we arrive to a contradiction with the supposed existence of an arch map. $\triangle$

We are finally able to prove the central step of the proof of theorem 3.1.

Proposition 3.6. *Every closed orbit γ of $\tilde{\psi}_t$ projects under $\tilde{M} \xrightarrow{p} \mathbf{S}^1$ to a noncontractible curve.*

Proof.

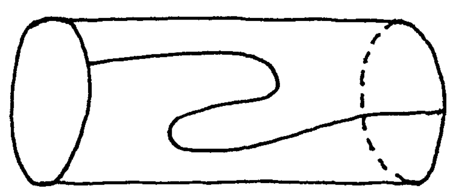
Suppose firstly that γ is a regular closed orbit, and let $L \in \tilde{\mathcal{G}}^s$ be the leaf containing γ; L is either a cylinder or a Moebius strip, but we will consider only the former case because the latter is completely similar (alternatively: pass to a double covering...).

Consider the intersection $L \cap \Sigma''' \subset L$. By transversality, it is a closed 1-dimensional submanifold, consisting of:

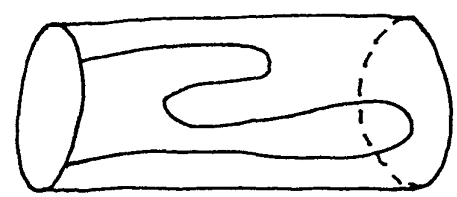
i) circles, non homotopic to zero on L (because they are non homotopic to zero in Σ''' and hence in $\tilde{M}$, being $\pi_1(\Sigma''') \to \pi_1(\tilde{M})$ injective);

ii) lines from $+\infty$ to $-\infty$, where $\pm\infty$ denote the two ends of L;

iii) lines from $+\infty$ to $+\infty$ or from $-\infty$ to $-\infty$.



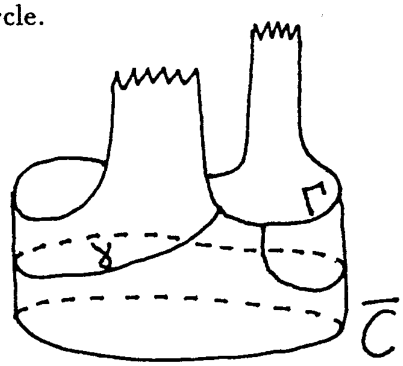
line of class ii)

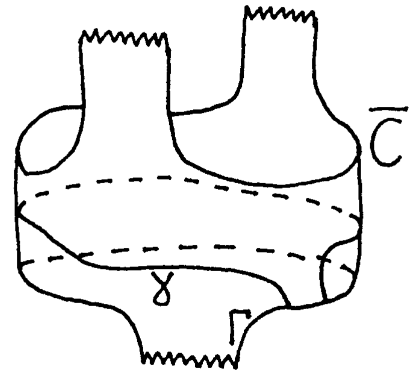


line of class iii)

The transitivity of ψ_t implies that L is dense in $\tilde{M}$, hence $L \cap \Sigma'''$ is dense in Σ'''. In particular, $L \cap \Sigma'''$ cannot reduce only to circles: every circle in $L \cap \Sigma'''$ is a limit cycle for $\mathcal{H}''' = \tilde{\mathcal{G}}^s|_{\Sigma'''}$.

Suppose that there are not lines of class ii). Then there is a connected component C of $L \setminus (L \cap \Sigma''')$ which is not simply connected and is bounded by lines of class iii) and, possibly, one circle.





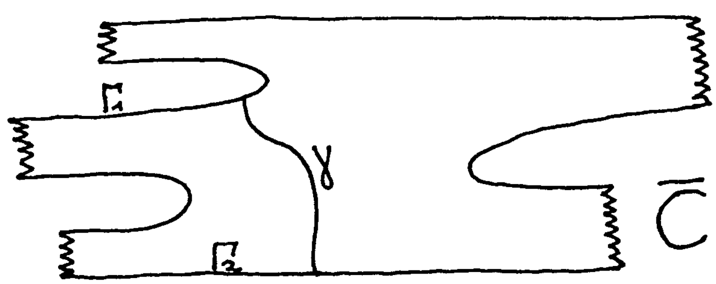
Cutting $\tilde{M}$ along Σ''', this component C gives origin to a leaf $\bar{C} \in \hat{\mathcal{G}}^s$, with $\partial\bar{C} \subset \partial_0\hat{M} = \Sigma''' \times \{0,1\}$. We easily find a path $\gamma : [0,1] \to \bar{C}$ such that:

a) $\gamma(0), \gamma(1)$ are on the same line $\Gamma \subset \partial\bar{C}$

b) γ defines a nontrivial element of $\pi_1(\bar{C}, \partial\bar{C})$.

This γ is an arch map, and we arrive to a contradiction with lemma 3.5. This means that $L \cap \Sigma'''$ contains at least one line of class ii) (and, consequently, no circle).

Suppose now that there are lines of class iii). We can find a connected component C of $L \setminus (L \cap \Sigma''')$ bounded by two lines of class ii) (possibly coincident) and one or more lines of class iii).



Let $\bar{C}$ be the corresponding leaf of $\hat{\mathcal{G}}^s$. Because $\partial\bar{C}$ contains at least 3 connected components, we may find a path $\gamma : [0,1] \to \bar{C}$ such that:

a) $\gamma(0), \gamma(1)$ are both on $\Sigma''' \times \{0\}$ or both on $\Sigma''' \times \{1\}$

b) $\gamma(0), \gamma(1)$ are on two different lines $\Gamma_1, \Gamma_2 \subset \partial\bar{C}$.

Again, γ is an arch map and we arrive to a contradiction.

We have proved that $L \cap \Sigma'''$ is composed only by lines of class ii), and hence every connected component of $L \setminus (L \cap \Sigma''')$ gives origin to a leaf of $\hat{\mathcal{G}}^s$ homeomorphic to $\mathbf{R} \times [0,1]$. The absence of arch maps shows again that every such leaf has $\mathbf{R} \times \{0\}$ on a connected component of $\partial_0\hat{M}$ and $\mathbf{R} \times \{1\}$ on the other connected component. This implies that γ

intersects Σ''' a number of times (> 0) equal to the modulus of the degree of $p|_\gamma : \gamma \to \mathbf{S}^1$, from which the conclusion of the proposition.

It remains only the case of γ singular. The remark after the proof of lemma 3.3 reduces the problem only to the case $\gamma \cap \Sigma''' = \emptyset$. We take a separatrix L of γ (L is dense in $\tilde{M}$), then $L \cap \Sigma'''$ is formed only by circles and lines from $+\infty$ to $+\infty$ ($+\infty$ is the unique nonsingular end of L), and there is at least one line. We arrive easily to a contradiction working as in the nonsingular case. $\triangle$

Proof of theorem 3.1. Conclusion

The only result of the previous work that we need is proposition 3.6.

The fibre $\Sigma \subset \tilde{M}$ represents a cohomology class $\omega \in H^1(\tilde{M}, \mathbf{Z})$: if $\gamma \subset \tilde{M}$ is an oriented closed curve then $\omega([\gamma])$ is the degree of $p|_\gamma : \gamma \to \mathbf{S}^1$. Proposition 3.6 tell us that for every closed orbit γ of $\tilde{\psi}_t$ the value $\omega([\gamma])$ is different from 0. Because closed orbits are dense, the arguments of [Ver, 74] shows that either $\omega([\gamma]) > 0$ for all closed orbits γ, or $\omega([\gamma]) < 0$ for all closed orbits γ (a closed orbit is oriented by the flow).

Then ([Ful], [Ver], [Fri1]) there exists a fibration

$$q : \tilde{M} \to \mathbf{S}^1$$

homotopic to p and with fibres transverse to $\tilde{\psi}_t$.

Lemma 3.7. *The fibrations p and q are isotopic.*

Proof.

It is a typical (and known) result "homotopy $\Rightarrow$ isotopy", which may be proved in several ways ([Wal], [Lau], [Fri3], [K-Q]). First of all, $q|_{\partial\tilde{M}}$ and $p|_{\partial\tilde{M}}$ are homotopic and hence isotopic, so the fibration defined by q is isotopic to the fibration defined by some $q' : \tilde{M} \to \mathbf{S}^1$, with $q' = p$ near $\partial\tilde{M}$. We collapse to a point every fibre of $p|_{\partial\tilde{M}} = q'|_{\partial\tilde{M}}$, obtaining a closed 3-manifold M_0 with two homotopic fibrations $p_0, q_0' : M_0 \to \mathbf{S}^1$, equal on a neighborhood of a closed (perhaps non connected) curve Γ, transverse to the fibrations.

By [K-Q], there exists an isotopy which transform q_0' to p_0. This isotopy, perhaps, does not preserve Γ, but maps Γ to some closed curve Γ', isotopic to Γ and transverse to the fibres of p_0. It is clear how to produce a second isotopy which preserves p_0 and maps Γ' to Γ. The composition is then an isotopy which transforms q_0' to p_0 preserving Γ.

This isotopy on M_0 lifts to $\tilde{M}$ and completes the proof. $\triangle$

In other words, there exists an isotopy of $\tilde{M}$ which transforms $\tilde{\psi}_t$ to a flow (again denoted by $\tilde{\psi}_t$) transverse to the fibres of $p : \tilde{M} \to \mathbf{S}^1$, as the flow $\tilde{\phi}_t$. We may assume that such an isotopy is the identity near $\partial\tilde{M}$, and hence blow-down to an isotopy of M. So, there exists an isotopy of M which transforms ψ_t to a flow (again denoted by ψ_t) such that the open book decomposition adapted to ϕ_t is adapted also to ψ_t.

Let $f, g : \Sigma \to \Sigma$ be the first return maps of $\tilde{\phi}_t$, $\tilde{\psi}_t$, relative to a fibre Σ. Because $\tilde{\phi}_t$ and $\tilde{\psi}_t$ are transverse to the same fibration, f and g are isotopic homeomorphisms. They are also pseudo-Anosov, hence by theorem 1.4 they are topologically conjugate through a homeomorphism $h : \Sigma \to \Sigma$ isotopic to the identity. The homeomorphism h generates a homeomorphism

$$H : \tilde{M} \to \tilde{M}$$

which realizes a topological equivalence between $\tilde{\phi}_t$ and $\tilde{\psi}_t$ (maps orbits of $\tilde{\phi}_t$ to orbits of $\tilde{\psi}_t$).

Recall now the two circle fibrations defined on $\partial\tilde{M}$:

1) the fibration $\mathcal{L}_0$, given by restriction of p; it is preserved by $H|_{\partial\tilde{M}}$, by construction of H;

2) the fibration $\mathcal{L}_1$ arising from the blowing up $\tilde{M} \to M$; it is preserved by $\tilde{\phi}_t|_{\partial\tilde{M}} = \tilde{\psi}_t|_{\partial\tilde{M}}$.

Both $\mathcal{L}_0$ and $\mathcal{L}_1$ are transverse to $\tilde{\phi}_t|_{\partial\tilde{M}} = \tilde{\psi}_t|_{\partial\tilde{M}}$, and they are transverse each other. Hence there exists a continuous function $T : \partial\tilde{M} \to \mathbf{R}$ such that, if we define

$$K(x) = \tilde{\psi}_{T(x)}(x) \qquad \forall x \in \partial\tilde{M},$$

then K is a homeomorphism and $K \circ H|_{\partial\tilde{M}}$ preserves the fibration $\mathcal{L}_1$ instead $\mathcal{L}_0$. A careful extension of $\tilde{T}$ to all of $\tilde{M}$ will give a homeomorphism

$$\tilde{K}(x) = \tilde{\psi}_{\tilde{T}(x)}(x) \qquad \forall x \in \tilde{M}$$

which extends the previous K.

The composition $\tilde{K} \circ H$ is then a homeomorphism, which still realizes a C^0- equivalence between $\tilde{\phi}_t$ and $\tilde{\psi}_t$, and moreover preserves $\mathcal{L}_1$. Hence we may blow-down $\tilde{K} \circ H$ and obtain the required topological equivalence between ϕ_t and ψ_t. $\triangle$

Expansive flows on torus bundles

A consequence of theorem 3.1 is that any transitive expansive flow ψ_t on a torus bundle

$$\mathbf{T}^3_A = \frac{\mathbf{T}^2 \times [0,1]}{(x,0) \sim (Ax,1)}, \qquad A \in GL(2,\mathbf{Z})$$

is C^0- equivalent to the suspension of A or A^{-1} (cfr. [Pla2]). Remark that if $\mathbf{T}^3_A$ admits an expansive flow then A is necessarily hyperbolic, because $\pi_1(\mathbf{T}^3_A)$ must have exponential growth ([Pat1]). We may take as a model flow ϕ_t the Anosov flow given by suspending A, and a fibre $\mathbf{T}^2 \hookrightarrow \mathbf{T}^3_A$ as a surface of section Σ. Now $\partial\Sigma = \emptyset$, but working exactly as in 3.1 we arrive to construct an homeomorphism h of $\mathbf{T}^3_A$ which maps orbits of ϕ_t to orbits of ψ_t. If h preserves the orientations of the orbits given by the flows then ψ_t is C^0- equivalent to ϕ_t, otherwise it is C^0- equivalent to ϕ_{-t}, i.e. to the suspension of A^{-1}.

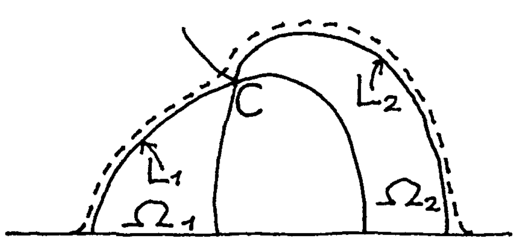
We show here how to remove the hypothesis of transitivity.

Theorem 3.8. *Any expansive flow on $\mathbf{T}^3_A$ is topologically equivalent to the suspension of A or A^{-1}.*

Proof.

We observe that in the proof of 3.1 the transitivity of ψ_t was not used until proposition 3.6. Hence, if $\psi_t : \mathbf{T}^3_A \to \mathbf{T}^3_A$ is any expansive flow, then we may isotope a fibre $\mathbf{T}^2 \subset \mathbf{T}^3_A$ to a torus $\Sigma \subset \mathbf{T}^3_A$ which is transverse to the stable foliation $\mathcal{G}^s$ of ψ_t and, moreover, the foliation $\hat{\mathcal{G}}^s$ on $\hat{M} \simeq \Sigma \times [0,1]$ obtained by cutting along Σ is without half Reeb components and without cascades of half Reeb components.

In fact, the same proof of lemma 3.4 shows that we may obtain a little more: we may obtain that $\hat{\mathcal{G}}^s$ is without *quasi-cascades* of singular half Reeb components, where with this term we denote a closed saturated subset $\Omega \subset \hat{M}$ which is like a cascade except that it is allowed that the singular circle $C \subset \Omega$ has one (but only one!) separatrix outside Ω:



We want to show that $\mathcal{G}^s$ is without circle-prongs, i.e. ψ_t has no singular closed orbit.

Remark that if there is a singular closed orbit γ, then $\gamma \subset \mathbf{T}^3_A \setminus \Sigma$, because Σ is transverse to $\mathcal{G}^s$ and it is a torus. Moreover, if L is a separatrix at γ then $L \cap \Sigma$ is reduced only to a collection of circles, because $\hat{\mathcal{G}}^s$ does not have arch maps (cfr. proof of 3.6). Because A is hyperbolic and hence cannot fix an integral cohomology class, we deduce that $L \cap \Sigma$ is either empty or consists of a single circle (if there were two circles then there would be a cylinder in $\hat{\mathcal{G}}^s$ which joins $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$...). Hence a separatrix $\hat{L}$ at $\hat{\gamma} \subset \hat{M}$ either is completely contained in $int\hat{M}$, or it realizes a cobordism between $\hat{\gamma}$ and a closed curve on $\partial\hat{M}$.

So, let γ be a singular closed orbit of ψ_t and let $\hat{\gamma} \subset \hat{M}$ be the corresponding circle-prong of $\hat{\mathcal{G}}^s$. Because $[\gamma] \neq 0$ in $H_1(\mathbf{T}^3_A, \mathbf{Z})$, we may find an embedding j of the annulus $A = \mathbf{S}^1 \times [0,1]$ into $\hat{M}$, with $j(\mathbf{S}^1 \times \{0\}) \subset \Sigma \times \{0\}$ and $j(\mathbf{S}^1 \times \{1\}) \subset \Sigma \times \{1\}$, such that the intersection number $j(A) \cdot \hat{\gamma}$ is different from zero.

We put A in general position w.r to $\hat{\mathcal{G}}^s$ ([Sol1]) and we use the Whitney trick ([I-M]) to remove (from the induced foliation on A) connections between two prongs. We obtain on A a foliation $\mathcal{H}$ with the following properties:

i) $\mathcal{H}$ has only generic singularities: saddles, centres, prongs in $intA$, half-saddles and half-centres on ∂A;

ii) $\mathcal{H}$ has no connection between two prongs, two (half-)saddles, a prong and a (half-) saddle, nor prong self-connection;

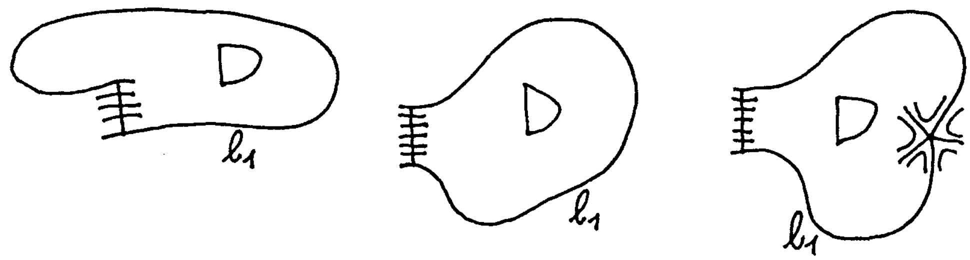
iii) $\mathcal{H}$ contains some prongs, because $j(A) \cdot \hat{\gamma} \neq 0$.

Let us observe also the following property.

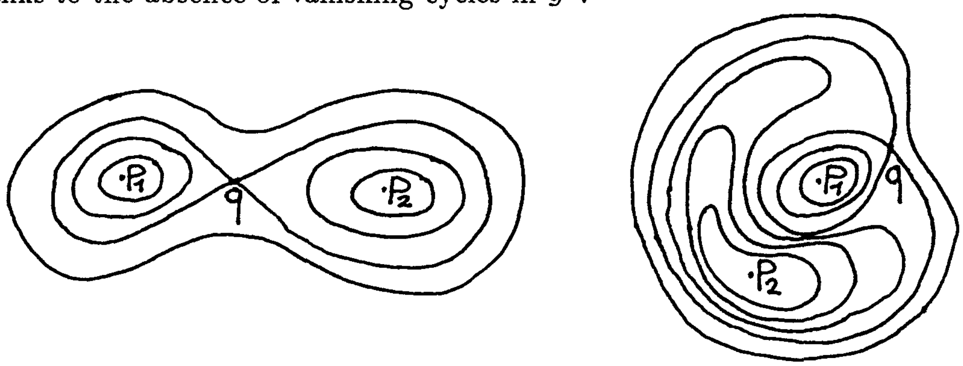
Lemma 3.9. *Let $l \simeq [0,1] \subset A$ be a segment transverse to $\mathcal{H}$ with the extrema belonging to some extended leaf $L \in \mathcal{H}$, and let $l_1 \subset L$ be a segment with extrema coinciding with those of l. Then $l \cup l_1$ is not contractible in A.*

Proof.

Assume by contradiction that the lemma is not true: $l \cup l_1$ bounds a disk $D \subset A$



The foliation $\mathcal{K} = \mathcal{H}|_D$ has Poincare'-Hopf index $\geq \frac{1}{2}$. If $p \in D$ is a centre then (cfr. lemma 3.3) the boundary of the closure E_p of the union of the circles of $\mathcal{K}$ surrounding p must contain a saddle $q \in D$. If $p_1, p_2 \in D$ are two centres associated in this way to the same saddle q, then we may "continue" the family of circles around $E_{p_1} \cup E_{p_2}$, again thanks to the absence of vanishing cycles in $\mathcal{G}^s$:



This exterior family of circles must "die" on some other saddle. This means that, for $\mathcal{K}$, $\sharp$ saddles $\geq \sharp$ centres, and hence the contribute of saddles and centres to the Poincare'-Hopf index of $\mathcal{K}$ is nonpositive.

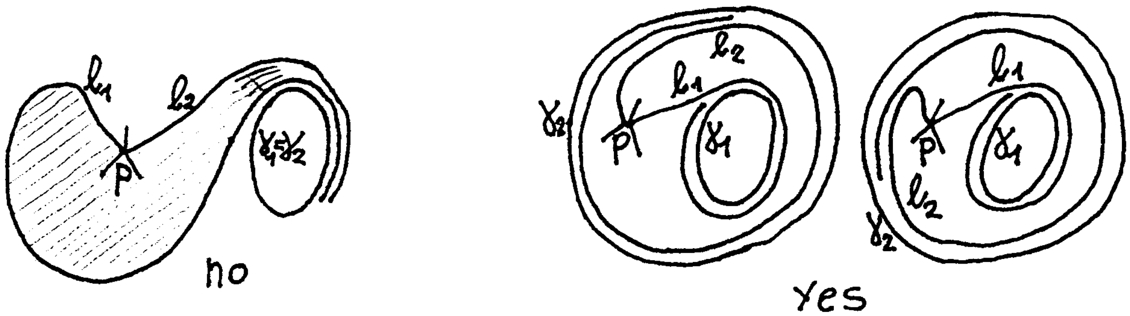
Because a prong gives a negative contribution to the Poincare'-Hopf index, we arrive to a contradiction with the fact that the total index of $\mathcal{K}$ is positive. $\triangle$

Lemma 3.9 has the consequence that Poincare'-Bendixson theorem holds also for $\mathcal{H}$, even when $\mathcal{H}$ is not orientable.

Let now $p \in A$ be a prong of $\mathcal{H}$ (originated from $\hat{\gamma}$), with separatrices $l_1, ..., l_k$ ($k \geq 3$). Every l_j either goes to ∂A, or tends to a closed orbit, or to a policycle. This latter case may be avoided by a perturbation of A.

If all except at most one separatrices go to ∂A, then all except at most one separatrices of $\hat{\gamma}$ go to $\partial \hat{M}$. These separatrices bound a saturated subset $\Omega \subset \hat{M}$ which is either a cascade, or a quasi-cascade, or it contains a cascade or a quasi-cascade or a half Reeb component. Contradiction with the choice of Σ.

Hence at least 2 separatrices at p (say, l_1 and l_2) tend to closed orbits $\gamma_1, \gamma_2 \subset A$. These two closed orbits are not homotopic to zero in A, and they must be different by lemma 3.9:



One sees easily that now there is no room for a third separatrix l_3: l_3 cannot go to ∂A, because p is between γ_1 and γ_2, and cannot tend to a third closed orbit γ_3.

Briefly, $\mathcal{H}$ has no prong, and $\mathcal{G}^s$ has no circle-prong.

To complete the proof we have several choices:

first) once we know that $\mathcal{G}^s$ is regular, we may use the theory of holonomy invariant measures as in [Pla2] to obtain that ψ_t is transitive; then we apply 3.1, or we reduce to [Pla2] after recalling that (via surfaces of section) $\mathcal{G}^s$ regular + ψ_t transitive $\Rightarrow$ ψ_t is topologically conjugate to an Anosov flow.

second) the lifting of $\mathcal{G}^s$ to the universal covering $\mathbf{R}^3$ is a trivial foliation by planes ([Sol2]); in the next chapter we will show that this implies the transitivity; then we conclude as in *first)*.

third) in the last chapter we shall study nontransitive Anosov flows, but the results therein are still true in the expansive nonsingular case; those results show that ψ_t is transitive, and we conclude as before. $\triangle$

In the next chapter we will see more interesting examples where theorem 3.1 applies.

EXPANSIVE FLOWS ON SEIFERT MANIFOLDS

In this chapter we concentrate our attention to expansive flows defined on a 3-manifold M which is a Seifert fibration. We recall that a closed connected 3-manifold M is said to be a ***Seifert manifold*** or ***Seifert fibration*** if there exists on M a foliation $\mathcal{L}$ by circles, such that every circle has an orientable tubular neighborhood ([Hem], [Eps]).

We may endow the leaf space $B = \frac{M}{\mathcal{L}}$ with a structure of closed two dimensional orbifold. The singularities of B are of conical type and they correspond to leaves of $\mathcal{L}$ with nontrivial holonomy. For many purposes, we may consider M as a circle bundle over the orbifold B; the fibres over the singularities of B are called *singular fibres*. In particular, there exists an exact sequence

$$1 \to \mathcal{Z} \to \pi_1(M) \to \pi_1(B) \to 1$$

where $\mathcal{Z}$ is the cyclic normal subgroup generated by a regular fibre. A singular fibre with holonomy of period p represents in $\pi_1(M)$ a p-root of the element represented by a regular fibre.

An example of Seifert manifold is a 3-manifold which is a circle-bundle, in this case B is smooth (without singularities).

E. Ghys classified in [Ghy1] Anosov flows on circle-bundles up to topological equivalence and up to finite covering. It seems that his classification extends to the case of Seifert manifolds: any Anosov flow on a Seifert manifold M with base B is, up to topological equivalence, a covering of the geodesic flow on B with respect to a metric of constant negative curvature (remark: B has singularities, but its unit tangent bundle T_1B is smooth, and the geodesic flow on B is a smooth flow on T_1B, of Anosov type if the curvature is negative).

Our results allows to extend Ghys' classification to cover the case of expansive flows.

<u>**Theorem 4.1**</u>. *Let $\phi_t : M \to M$ be an expansive flow on a Seifert manifold M. Then ϕ_t is topologically equivalent to an Anosov flow.*

As a corollary we obtain the theorem of [Pat2]: a geodesic expansive flow on a surface is topologically equivalent to the geodesic (Anosov) flow with respect to a metric of constant negative curvature.

The proof of theorem 4.1 is in two steps. Firstly we show that the stable and unstable foliations of ϕ_t, $\mathcal{F}^s$ and $\mathcal{F}^u$, are without singularities and their liftings $\tilde{\mathcal{F}}^s$, $\tilde{\mathcal{F}}^u$ in the universal covering $\tilde{M} \simeq \mathbf{R}^3$ are product foliations (we follow here some ideas of [Ghy1]). Secondly we prove that the above fact implies that ϕ_t is transitive (see [Sol3] for the Anosov case). The proof of 4.1 is completed by the existence of surfaces of section (chap. 2): a transitive expansive flow with nonsingular invariant foliations is topologically equivalent to an Anosov flow.

According to recent developments in the topology of 3-manifolds ([J-S]), the "simplest" 3-manifolds are the so called *graph manifolds* (introduced by Waldhausen) which are obtained by glueing together Seifert manifolds along boundary tori. Examples of Anosov flows on graph manifolds (different from Seifert manifolds) were found by Haendel and Thurston ([H-T]). It would be interesting to extend theorem 4.1 to this class of manifolds, or at least to prove that an expansive flow on a graph manifold is necessarily transitive.

After the proof of theorem 4.1 we will specialize to the case where M is the unit tangent bundle of a closed surface.

Structure of the stable and unstable foliations

Let M be a Seifert manifold, with base orbifold B and projection $p : M \to B$. Let $\phi_t : M \to M$ be an expansive flow, with stable and unstable foliations $\mathcal{F}^s$, $\mathcal{F}^u$, and let $\mathcal{S}$ be the singular set of $\mathcal{F}^s$, $\mathcal{F}^u$ ($\mathcal{S}$= union of circles).

Let $\pi : \tilde{M} \to M$ be the universal covering of M, and let

$$\tilde{\mathcal{S}} = \pi^{-1}(\mathcal{S}), \qquad \tilde{\mathcal{F}}^s = \pi^*(\mathcal{F}^s), \qquad \tilde{\mathcal{F}}^u = \pi^*(\mathcal{F}^u).$$

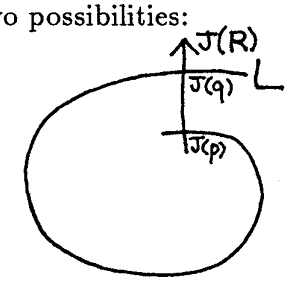
The set $\tilde{\mathcal{S}}$ is a countable and discrete set of lines, closed in $\tilde{M}$, and $\tilde{\mathcal{F}}^s$, $\tilde{\mathcal{F}}^u$ are foliations with line-prongs, singular along $\tilde{\mathcal{S}}$. Denote by $\tilde{\mathcal{F}}_0^s$, $\tilde{\mathcal{F}}_0^u$ the (regular) foliations on $\tilde{M} \setminus \tilde{\mathcal{S}}$ obtained by restriction of $\tilde{\mathcal{F}}^s$, $\tilde{\mathcal{F}}^u$. Every leaf of $\tilde{\mathcal{F}}_0^s$ or $\tilde{\mathcal{F}}_0^u$ is a plane, closed in $\tilde{M} \setminus \mathcal{S}$ (for Inaba - Matsumoto - Paternain results).

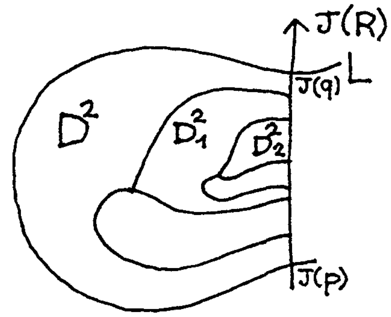
The foliations $\tilde{\mathcal{F}}_0^s$ and $\tilde{\mathcal{F}}_0^u$ may be not transversely orientable, because there may be circle-prongs in $\mathcal{S}$ with an odd number of prongs. Hence the following lemma (which does not use the Seifert structure) is not completely trivial.

<u>**Lemma 4.2**</u>. *Let $j : \mathbf{R} \to \tilde{M} \setminus \tilde{S}$ be an embedding transverse to $\tilde{\mathcal{F}}_0^s$. Then every leaf of $\tilde{\mathcal{F}}_0^s$ intersects $j(\mathbf{R})$ in no more then one point.*

Proof.

Assume that a leaf L intersects $j(\mathbf{R})$ in two points $j(q), j(p)$, $q > p$, and let $\gamma : [0,1] \to L$ be a curve joining $j(q)$ and $j(p)$. Fix a continuous coorientation of L in $\tilde{M}$ along γ, then there are two possibilities:





i) the coorientations in $\gamma(0)$ and $\gamma(1)$ are both compatible or both incompatible with the coorientations induced by $j(\mathbf{R})$: then it is easy to perturb the cycle $\gamma([0,1]) \cup j([p,q])$ to obtain a closed curve transverse to $\tilde{\mathcal{F}}_0^s$, but this is absurd because $\mathcal{F}^s$ does not admit closed transversals homotopic to zero.

ii) the coorientation in $\gamma(0)$ is compatible with the one given by $j(\mathbf{R})$ and the coorientation in $\gamma(1)$ is incompatible (or viceversa): take a disk $\mathbf{D}^2 \hookrightarrow \tilde{M}$ with boundary $\gamma([0,1]) \cup j([p,q])$ such that ([I-M]) $\tilde{\mathcal{F}}_0^s$ induces on it a foliation $\mathcal{G}$ with centres, saddles, prongs, without connections between two prongs or a saddle and a prong, and without prong self-connections. Then a Poincaré-Hopf argument shows that $\mathcal{G}$ has at least one prong (with an odd,$\geq$ 3, number of separatrices), and all the separatrices of this prong must intersects $j([p,q])$, because the leaves of $\mathcal{G}$ are closed in $\mathbf{D}^2 \setminus Sing(\mathcal{G})$. Two of these separatrices separate a sub-disk $\mathbf{D}_1^2 \subset \mathbf{D}^2$ with a piece of boundary on $j((p,q))$, and we may repeat the above argument for $\mathbf{D}_1^2$. An iteration of this construction would produce an infinite number of prongs on $\mathbf{D}^2$, which is absurd. $\triangle$

In particular, $\tilde{\mathcal{F}}_0^s$ (and similarly $\tilde{\mathcal{F}}_0^u$) is a *simple* foliation ([God]). As a consequence, the leaf spaces

$$V^s = \frac{\tilde{M} \setminus \tilde{S}}{\tilde{\mathcal{F}}_0^s} \qquad V^u = \frac{\tilde{M} \setminus \tilde{S}}{\tilde{\mathcal{F}}_0^u}$$

are one-dimensional connected manifold, with countable base, but perhaps non-Hausdorff.

The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ preserving $\tilde{S}$ and $\tilde{\mathcal{F}}^s$, and so there is an induced action on V^s. Fixed points of $\alpha \in \pi_1(M)$ on V^s correspond to leaves which project on M to leaves containing a closed orbit representing α, or to separatrices of some singular closed orbit representing α.

The following lemma also does not use the Seifert structure.

Lemma 4.3. *Let $\delta \in \pi_1(M)$ be a central element, different from the identity, then δ acts on V^s without fixed points.*

Proof.

It is exactly the same of lemme 2.4 of [Ghy1]: the set

$$Fix^{\approx}(\delta) = \{x \in V^s | x \text{ and } \delta(x) \text{ are not separated}\}$$

is closed, countable (because $Fix(\delta)$ is countable and $Fix^{\approx}(\delta) \setminus Fix(\delta)$ is contained in the countable set of branching points) and $\pi_1(M)$- invariant (because δ is central), and if it is not empty then the foliation $\mathcal{F}_0^s = \mathcal{F}^s|_{M \setminus \mathcal{S}}$ has a closed saturated set K which is transversely countable. It is easy to see that if $\{K_\alpha\}_{\alpha \in I}$ is a collection of closed $\mathcal{F}_0^s$-saturated non-empty sets, totally ordered with respect to the inclusion, then $\cap_{\alpha \in I} K_\alpha$ is non-empty (in spite of the non-compactness of $M \setminus \mathcal{S}$: we use here the structure of $\mathcal{F}_0^s$ around the singular set $\mathcal{S}$, in particular the fact that there is a finite number of leaves of $\mathcal{F}^s$ with one end on a circle-prong). Hence we may apply Zorn lemma to deduce that K contains a minimal set, which is a closed leaf because of the transverse countability. But this is in contradiction with Inaba-Matsumoto-Paternain theorem, hence $Fix^{\approx}(\delta) = \emptyset$. $\triangle$

Clearly lemma 4.3 holds also for roots of central elements: if $\beta \in \pi_1(M)$ is such that β^k is a nontrivial central element for some $k \neq 0$, then β has no fixed point on V^s.

Remark: if we could know a priori that ϕ_t is transitive, then lemma 4.3 could be proven in the following way. If $\gamma \subset M$ is a closed orbit of ϕ_t, representing $\alpha \in \pi_1(M)$, then using the fact that the stable leaf of γ accumulates on itself it is easy to construct a closed transversal to $\mathcal{F}^s$ representing $\alpha * \beta * \alpha^{-1} * \beta^{-1}$ for some $\beta \in \pi_1(M)$, and the nontriviality of this element (by [I-M]) implies that α is not a central element.

Now we start to use the seiferticity of M.

The results of Paternain ($\pi_1(M)$ has exponential growth) and Inaba - Matsumoto (M is aspherical) imply that $\tilde{M}$ is the euclidean space (this is, in fact, always true) and that the base orbifold B has genus g greater or equal than 2. We want to prove that $\mathcal{F}^s$, $\mathcal{F}^u$ are without singularities, hence we may work up to finite covering. So we may assume that M and B are orientable, and in particular that the cyclic subgroup $\mathcal{Z} \subset \pi_1(M)$ generated by a regular fibre of $M \xrightarrow{p} B$ is a central subgroup ([Hem]).

Lemma 4.3 tell us that no closed orbit γ of ϕ_t is freely homotopic to a fibre (regular or singular). In particular, every closed orbit γ of ϕ_t projects on B to a curve $p(\gamma)$ which is not homotopic to zero, thanks to the exact sequence $1 \to \mathcal{Z} \to \pi_1(M) \to \pi_1(B) \to 1$.

Let $\omega \subset B$ be a closed, simple, smooth curve, non homotopic to zero, and let $\mathbf{T}^2_\omega \subset M$ be the incompressible torus $\pi^{-1}(\omega)$.

<u>**Lemma 4.4**</u>. *The torus $\mathbf{T}^2_\omega$ is isotopic to a torus disjoint from $\mathcal{S}$.*

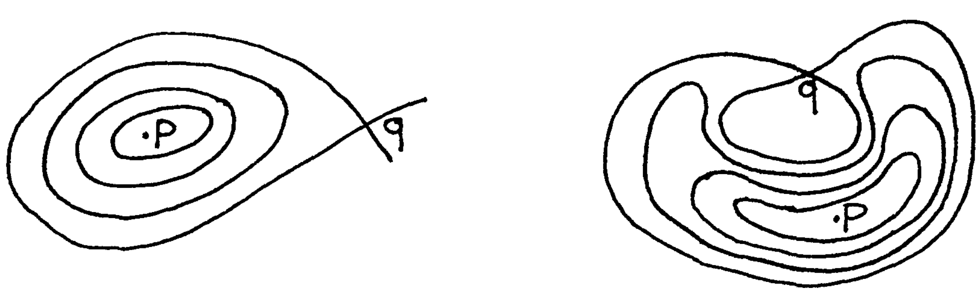
Proof.

We use arguments similar to those of theorem 3.8 (the ubiquitous Poincaré - Hopf formula...). First of all we put $\mathbf{T}^2_\omega$ in general position with respect to the foliation $\mathcal{F}^s$ ([Sol1]): the induced foliation $\mathcal{G}$ has only a finite number of centres and saddles due to tangency points, and a finite number of prongs due to the transverse intersection $\mathbf{T}^2_\omega \cap \mathcal{S}$; there are no connections between two different saddles, and no saddle-prong connections. An isotopy of $\mathbf{T}^2_\omega$ will ensure that, moreover, there are no connections between two different prongs and no prong self-connections ([I-M]). We claim that, at this point, we have $\mathbf{T}^2_\omega \cap \mathcal{S} = \emptyset$.

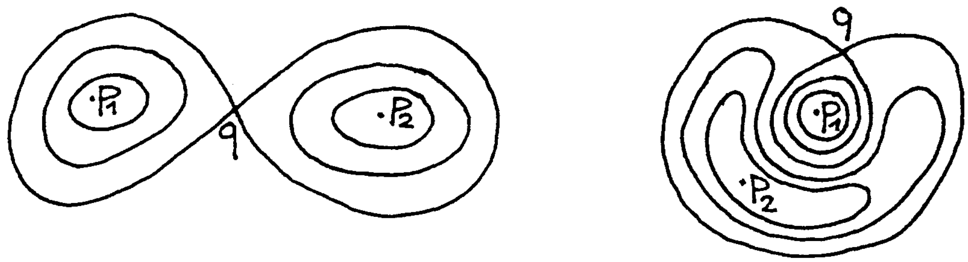
Consider a centre $p \in \mathbf{T}^2_\omega$ and define (cfr. [Rou1], [Rou2]):

$$E_p = \text{closure of } \bigcup \{\mathbf{D}^2 \subset \mathbf{T}^2_\omega | p \in \mathbf{D}^2 \text{ and } \mathcal{G}|_{\mathbf{D}^2 \setminus p} \text{ is a foliation by circles}\}$$

as in Roussarie's works, the absence of vanishing cycles in $\mathcal{F}^s$ implies that ∂E_p is formed by one or two separatrices and a saddle point q:



In this way (cfr. lemma 3.9) we associate to every centre p a saddle $q = q(p)$. It may happen that two centres p_1, p_2 are associated to the same saddle q:



but the absence of vanishing cycles in $\mathcal{F}^s$ implies that there exist embeddings $\mathbf{D}^2 \overset{i}{\hookrightarrow} \mathbf{T}^2_\omega$ such that $E_{p_1} \cup E_{p_2} \subset i(\mathbf{D}^2)$ and $\mathcal{G}|_{i(\mathbf{D}^2)\setminus(E_{p_1}\cup E_{p_2})}$ is a foliation by circles. The union of all these embeddings is again a region bounded by one or two separatrices and a saddle point.

In conclusion we see that the number of saddles must be greater or equal then the number of centres, and the Poincaré-Hopf formula shows that these two numbers are in fact equal, and the number of prongs is zero, i.e. $\mathbf{T}^2_\omega \cap \mathcal{S} = \emptyset$. $\triangle$

Suppose now by contradiction that $\mathcal{S} \neq \emptyset$, and let $\gamma \subset \mathcal{S}$ be a singular closed orbit. The previous lemma, and the arbitrarity of ω, shows that $p(\gamma) \subset B$ is homotopic to a curve disjoint from ω, for any $\omega \subset B$ as above. This implies that $[p(\gamma)] = 0$ in $\pi_1(B)$, and we arrive to a contradiction with lemma 4.3. Briefly, we must have

$$\mathcal{S} = \emptyset.$$

Finally, we may at this point repeat the proof of Ghys about the triviality of $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ ([Ghy1], lemmes 2.4-2.7). We resume these facts in the next proposition.

Proposition 4.5. *Let $\phi_t : M \to M$ be as in theorem 4.1. Then $\mathcal{F}^s$, $\mathcal{F}^u$ are regular foliations and their universal coverings $\tilde{\mathcal{F}}^s$, $\tilde{\mathcal{F}}^u$ are product foliations (i.e., $\frac{\tilde{M}}{\tilde{\mathcal{F}}^s} \simeq \frac{\tilde{M}}{\tilde{\mathcal{F}}^u} \simeq \mathbf{R}$).*
$\triangle$

Transitivity

The proof of theorem 4.1 is completed by the following result.

Proposition 4.6. *Let $\phi_t : M \to M$ be an expansive flow on a closed 3-manifold M, such that $\mathcal{F}^s$, $\mathcal{F}^u$ are regular and $\tilde{\mathcal{F}}^s$, $\tilde{\mathcal{F}}^u$ are product foliations. Then ϕ_t is transitive.*

Proof.

It is sufficient to prove the transitivity of some finite covering of (M, ϕ_t), so we may assume that $\mathcal{F}^s$, $\mathcal{F}^u$ are orientable and transversely orientable. A transverse orientation of $\mathcal{F}^u$ allows to distinguish between a "positive side" $\mathcal{F}_x^{s+}$ and a "negative side" $\mathcal{F}_x^{s-}$ of every leaf $\mathcal{F}_x^s$: $\mathcal{F}_x^s = \mathcal{F}_x^{s+} \cup [\phi_{\mathbf{R}}(x)] \cup \mathcal{F}_x^{s-}$ (disjoint union). Similarly, we decompose $\mathcal{F}_x^u = \mathcal{F}_x^{u+} \cup [\phi_{\mathbf{R}}(x)] \cup \mathcal{F}_x^{u-}$.

Define (cfr. [Ver]):

$$C_+ = \{x \in M | \mathcal{F}_x^{s+} \cap \mathcal{F}_x^u = \emptyset\}$$

$$C_- = \{x \in M | \mathcal{F}_x^{s-} \cap \mathcal{F}_x^u = \emptyset\}.$$

As in [Ver, 54], C_+ and C_- are finite unions of closed orbits of ϕ_t. In particular, if x is not periodic then

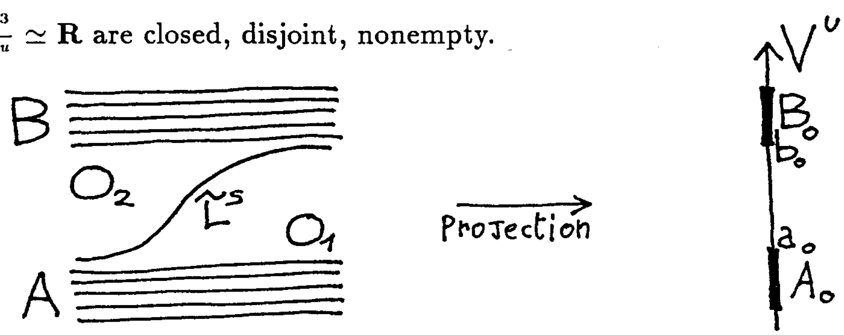
$$\mathcal{F}_x^{s+} \cap \mathcal{F}_x^u \neq \emptyset \qquad \mathcal{F}_x^{s-} \cap \mathcal{F}_x^u \neq \emptyset.$$

Remark that every leaf contains nonperiodic points.

<u>**Lemma 4.7**</u>. *Every leaf of $\mathcal{F}^s$ intersects every leaf of $\mathcal{F}^u$.*

Proof.

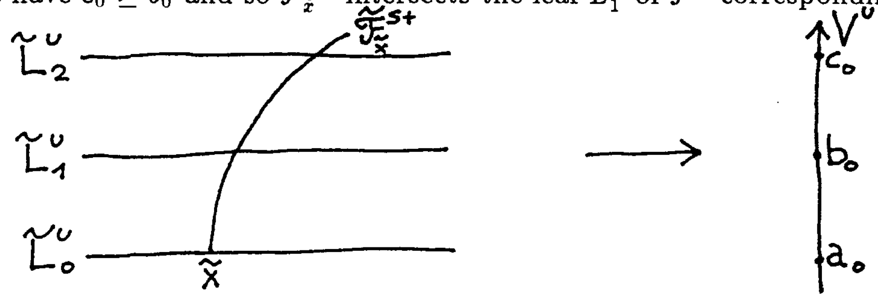
Assume by contradiction that $L^s \in \mathcal{F}^s$, $L^u \in \mathcal{F}^u$ do not intersect; then L^s and $\overline{L^u}$ also do not intersect. Let $\tilde{L}^s \subset \tilde{M} = \mathbf{R}^3$ be any lifting of L^s, and let $K = \pi^{-1}(\overline{L^u})$ ($\pi : \mathbf{R}^3 \to M$ is the universal covering). The leaf $\tilde{L}^s \in \tilde{\mathcal{F}}^s$ is a closed plane which divide $\mathbf{R}^3$ into two open subset O_1, O_2: $\mathbf{R}^3 = O_1 \cup \tilde{L}^s \cup O_2$ (disjoint union). Because $K \cap \tilde{L}^s = \emptyset$ and K is closed, the sets $A = K \cap O_1$, $B = K \cap O_2$ are closed, disjoint, saturated by $\tilde{\mathcal{F}}^u$, and both nonempty. The projections A_0, B_0, $K_0 = A_0 \cup B_0$ of A, B, K on the leaf space $V^u = \frac{\mathbf{R}^3}{\mathcal{F}^u} \simeq \mathbf{R}$ are closed, disjoint, nonempty.



Take $a \in A_0$, $b \in B_0$, and suppose $a < b$ (the opposite case being completely analogous). Define

$$a_0 = Max\{r \in A_0 | r < b\} \qquad b_0 = Min\{s \in B_0 | s > a_0\}.$$

Because A_0, B_0 are closed and disjoint, we have $a_0 < b_0$ and $(a_0, b_0) \cap K = \emptyset$. Let $\tilde{L}_0^u$ be the leaf of $\tilde{\mathcal{F}}^u$ corresponding to a_0 and let $\tilde{x} \in \tilde{L}_0^u$ be a point such that $x = \pi(\tilde{x})$ is not periodic. The above remarks about C_+ means that $\tilde{\mathcal{F}}^{s+}_{\tilde{x}}$ (obviously defined) intersects $\pi_1(M)(\tilde{L}^u_0)$, because $\mathcal{F}^{s+}_x$ intersects $\mathcal{F}^u_x = \pi(\tilde{L}^u_0)$. Hence there exists $c_0 \in K$, $c_0 > a_0$, such that $\tilde{\mathcal{F}}^{s+}_{\tilde{x}}$ intersects the leaf $\tilde{L}^u_2$ of $\tilde{\mathcal{F}}^u$ corresponding to c_0. Because $(a_0, b_0) \cap K = \emptyset$, we must have $c_0 \geq b_0$ and so $\tilde{\mathcal{F}}^{s+}_{\tilde{x}}$ intersects the leaf $\tilde{L}^u_1$ of $\tilde{\mathcal{F}}^u$ corresponding to b_0.



But $\tilde{L}^s$ separates A and B, and hence every other leaf of $\tilde{\mathcal{F}}^s$ cannot intersect both A and B. This contradiction shows that $L^s \cap L^u \neq \emptyset$. $\triangle$

The proof of 4.6 can be completed with the spectral decomposition theorem (see introduction): if ϕ_t were not transitive, then among the basic sets $\{\Omega_j\}_{j=1}^N$ there would be an attractor Ω_{j_1} $(\cup_{x\in\Omega_{j_1}} \mathcal{F}^u_x = \Omega_{j_1})$ and a repellor Ω_{j_2} $(\cup_{x\in\Omega_{j_2}} \mathcal{F}^s_x = \Omega_{j_2})$, in particular $\mathcal{F}^s_x \cap \mathcal{F}^u_y = \emptyset$ if $x \in \Omega_{j_2}$, $y \in \Omega_{j_1}$. But this is in contradiction with lemma 4.7. $\triangle$

Anosov flows on unit tangent bundles

Let $M = T_1 S$ be the unit tangent bundle of a closed orientable surface S and let $\phi_t : M \to M$ be an Anosov flow. The exponential growth of $\pi_1(M)$ implies that the genus g of S is greater or equal than two; we may put on S a metric of constant negative curvature and we may consider on M the corresponding geodesic flow $\psi_t : M \to M$, which is of Anosov type.

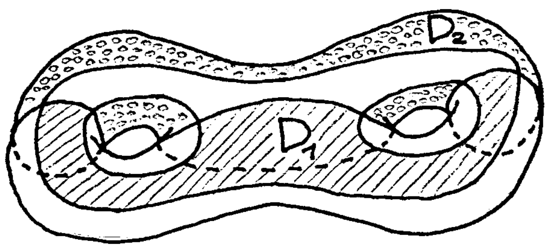
Theorem 4.8. *The flows ϕ_t and ψ_t are topologically equivalent.*

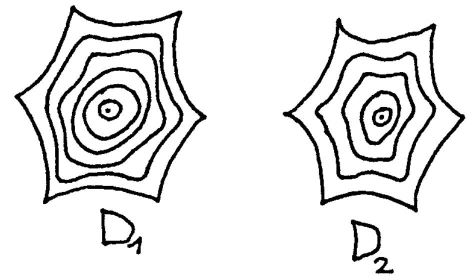
Using theorem 4.1 we obtain that the same conclusion holds even if ϕ_t is supposed only expansive.

Theorem 4.8 is not new. It may be deduced from Ghys' classification ([Ghy1]) by observing that any covering $T_1S \to T_1S$ is in fact a diffeomorphism (why?). It may be also deduced from the results of Matsumoto in [Mat]. However we like to give a direct proof of 4.8, using theorem 3.1. Our method is very near to that of Matsumoto.

First of all we recall Birkhoff's construction of surfaces of section for the geodesic flow

ψ_t ([Bir], [Fri2]). We take $2g+2$ closed, simple geodesics on S, such that the complement is simply connected (four disks):





Two "opposite" disks D_1, D_2 in the complement are foliated by circles, with a singularity of type centre, such that every circle is strictly convex. Let Σ be the surface in T_1S constructed taking the closure of all the unit tangent vectors to the leaves of these foliations. This surface Σ has boundary composed by $4g+4$ closed orbits of ψ_t (corresponding to the initial $2g+2$ closed geodesics: every geodesic may be percorred in two directions), and $int\Sigma$ is transverse to ψ_t because the circles on D_1, D_2 are strictly convex. It is not difficult to see that Σ is, in fact, a surface of section for ψ_t.

The stable foliation $\mathcal{G}^s$ of ψ_t is transverse to the fibres of $T_1S \xrightarrow{p} S$ and hence ([God], [C-L]) is given by a suspension of a representation of $\pi_1(S)$ into $Homeo_+(\mathbf{S}^1)$:

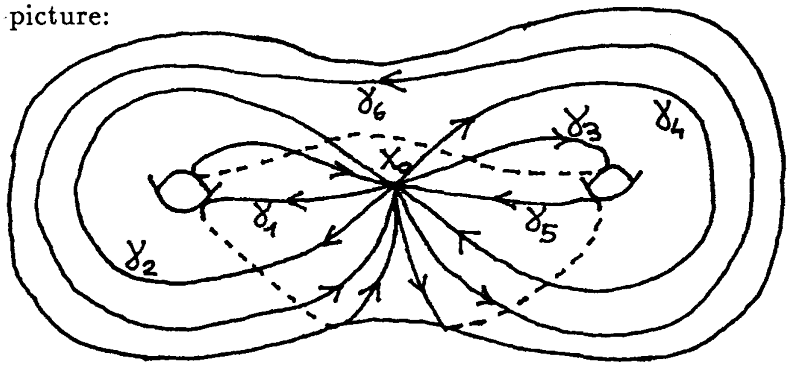
$$\Psi : \pi_1(S) \to Homeo_+(\mathbf{S}^1).$$

We may assume ([Ghy1]) that also the stable foliation $\mathcal{F}^s$ of ϕ_t is transverse to the fibres and hence given by a suspension of a representation Φ:

$$\Phi : \pi_1(S) \to Homeo_+(\mathbf{S}^1).$$

The representations Φ and Ψ contain all the information that we need about the flows ϕ_t and ψ_t.

Fix a base point $x_0 \in S$ and consider oriented closed curves $\gamma_1, ..., \gamma_{2g+2}$ as in the following picture:



Every $\Psi([\gamma_j]) \in Homeo_+(\mathbf{S}^1)$ has exactly two fixed points $x_j^+, x_j^- \in \mathbf{S}^1$ (x_j^+ is attractive, x_j^- is repulsive), which correspond to two closed orbits $\gamma_j^+, \gamma_j^- \in \partial\Sigma$ whose projections on S are homotopic to γ_j.

The conclusion of the following proposition is a sort of translation of hypotheses of theorem 3.1 from the language of flows ϕ_t, ψ_t to the language of representations Φ, Ψ.

<u>**Proposition 4.9**</u>. *The representation Φ is topologically conjugate to a representation $\bar{\Phi}$ such that $\forall j = 1, ..., 2g+2$ the point x_j^+ (x_j^-) is an attractive (repulsive) fixed point of $\bar{\Phi}([\gamma_j])$.*

From this proposition it follows that, up to C^0-equivalence, the closed orbits $\gamma_j^\pm$, $j = 1, ..., 2g+2$, of ψ_t are closed orbits also for ϕ_t. Moreover, because $\mathcal{F}^s$ and $\mathcal{G}^s$ are both transverse to the fibres of $T_1 S \to S$, the twisting of $\mathcal{F}^s$ and $\mathcal{G}^s$ around each $\gamma_j^\pm$ is the same. So, we may assume that ϕ_t is equal to ψ_t in a neighborhood of $\cup_{j=1}^{2g+2}(\gamma_j^+ \cup \gamma_j^-) = \partial\Sigma$, and theorem 3.1 says that ϕ_t and ψ_t are topologically equivalent.

<u>**Proof of proposition 4.9**</u>

To prove proposition 4.9 we need some facts about the Euler class ([Ghy2], [Mat]). Let S_0 be a compact, connected, orientable surface, possibly with boundary, and let

$$\Phi_0 : \pi_1(S_0) \to Homeo_+(\mathbf{S}^1)$$

be a representation such that $\forall \alpha \in \pi_1(S_0)$ representing a boundary component the homeomorphism $\Phi_0(\alpha)$ has rotation number equal to zero (i.e., $\Phi_0(\alpha)$ has fixed points). Then it is possible to define the *Euler class*

$$eu(\Phi_0) \in H^2(S_0, \partial S_0; \mathbf{Z}) \simeq \mathbf{Z}$$

([Ghy2]): the suspension of Φ_0 gives rise to a circle-bundle over S_0 together with a section over ∂S_0, and $eu(\Phi_0)$ is the obstruction to extend this section to all of S_0.

There is the Milnor - Wood inequality:

$$|eu(\Phi_0)| \leq |\chi(S_0)|.$$

Let $C_1, ..., C_r \subset S_0$ be simple closed curves, pairwise disjoint, and let $S_{0,1}, ..., S_{0,l}$ be the closures of the connected components of $S_0 \setminus \cup_{j=1}^r C_j$. Suppose that every $\Phi_0([C_j])$ has rotation number zero, then it is possible to consider the Euler classes $eu(\Phi_{0,j})$, $j = 1, ..., l$, of the restricted representations

$$\Phi_{0,j} : \pi_1(S_{0,j}) \to Homeo_+(\mathbf{S}^1).$$

Then the following localization formula holds ([Ghy2]):

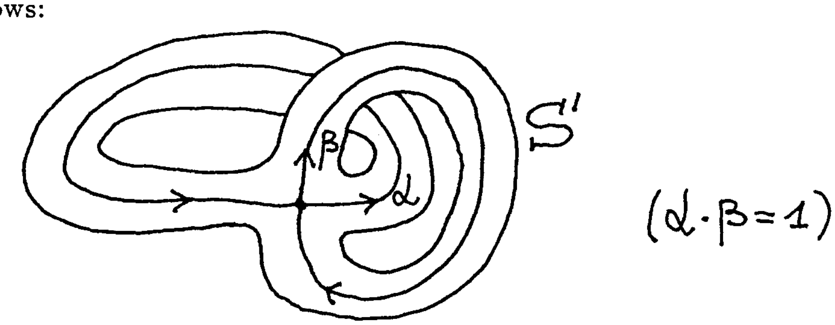
$$eu(\Phi_0) = \sum_{j=1}^{l} eu(\Phi_{0,j}).$$

Our representations Φ and Ψ originates from foliations on $T_1 S$, hence they satisfy $|eu(\Phi)| = |eu(\Psi)| = |\chi(S)|$. We may assume, without loss of generality, that

$$eu(\Phi) = eu(\Psi) = \chi(S).$$

It was remarked by Matsumoto ([Mat, 346-347]) that if Φ_0 is a representation of $\pi_1(S_0)$ such that $eu(\Phi_0) = \chi(S_0)$, then every $\Phi_0(\alpha)$ has rotation number zero, and hence we may define without problems $eu(\Phi_0')$ for every connected subsurface $S_0' \subset S_0$. The Milnor - Wood inequality and the localization formula imply that $eu(\Phi_0') = \chi(S_0')$.

In particular, let $S' \subset S$ be a subsurface diffeomorphic to the torus with one hole, and let $\Phi' : \pi_1(S') \to Homeo_+(\mathbf{S}^1)$ be the restriction of Φ, so that $eu(\Phi') = -1$. Let $\alpha, \beta \in \pi_1(S')$ as follows:



Define $f = \Phi'(\alpha)$, $g = \Phi'(\beta)$, and let $\bar{f}, \bar{g} \in Homeo_+(\mathbf{R})$ be liftings of f, g with the property that $Fix(\bar{f}) \neq \emptyset$, $Fix(\bar{g}) \neq \emptyset$ (it is possible, because $Fix(f) \neq \emptyset$, $Fix(g) \neq \emptyset$).

Define $h = \Phi'([\alpha,\beta]^{-1}) = [f,g]^{-1}$ ($[f,g]h = id$) and let $\bar{h} \in Homeo_+(\mathbf{R})$ be a lifting with fixed points. The Milnor algorithm ([Ghy2], [Mat]) implies: $[\bar{f},\bar{g}]\bar{h} = T_{-1}$, where $T_m(x) = x + m$, $m \in \mathbf{Z}$. Starting from this relation (i.e., $[\bar{f},\bar{g}](x) = \bar{h}^{-1}(x) - 1$, where $\bar{f},\bar{g},\bar{h}$ have fixed points) a straightforward analysis shows:

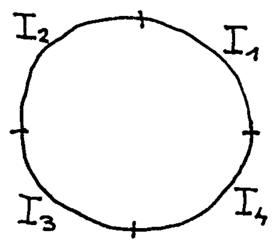
__Lemma 4.10__. *We may decompose* $\mathbf{S}^1$ *as union of four consecutive intervals*

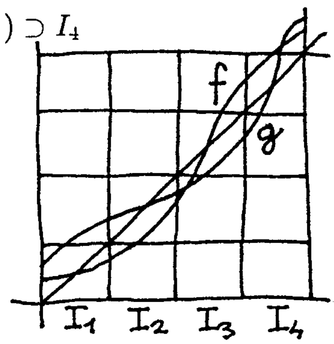
$$I_1 \cup I_2 \cup I_3 \cup I_4$$

such that:

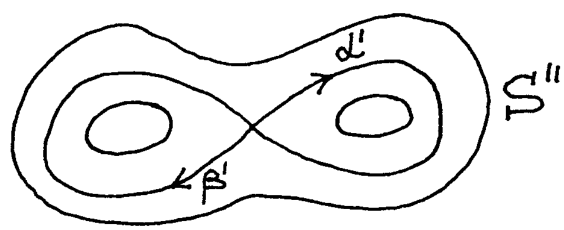
i) $Fix(f) \subset I_1 \cup I_3$, $f(I_1) \subset intI_1$, $f(intI_3) \supset I_3$

ii) $Fix(g) \subset I_2 \cup I_4$, $g(I_2) \subset intI_2$, $g(intI_4) \supset I_4$





Similarly, let $S'' \subset S$ be a subsurface diffeomorphic to the disk with two holes, and let $\Phi'' : \pi_1(S'') \to Homeo_+(\mathbf{S}^1)$ be the restriction of Φ, so that $eu(\Phi'') = -1$. Let $\alpha', \beta' \in \pi_1(S'')$ as follows:



Define $f' = \Phi''(\alpha')$, $g' = \Phi''(\beta')$, then we have (again starting from Milnor's algorithm):

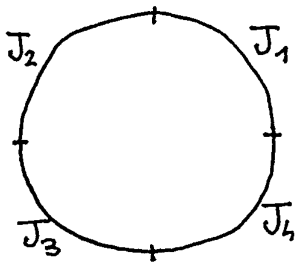
__Lemma 4.11__. *We may decompose* $\mathbf{S}^1$ *as union of four consecutive intervals*

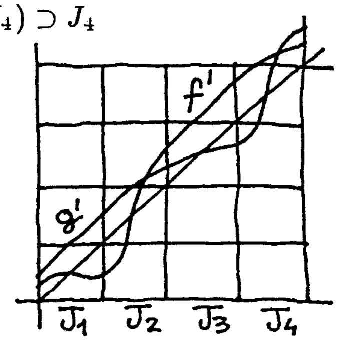
$$J_1 \cup J_2 \cup J_3 \cup J_4$$

such that:

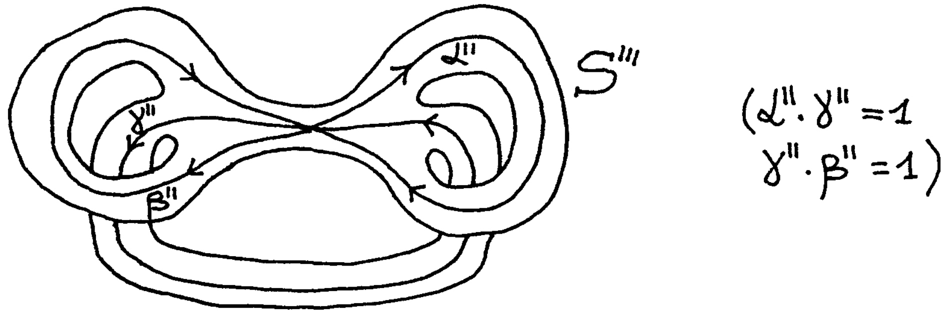
i) $Fix(f') \subset J_1 \cup J_2$, $f'(J_1) \subset intJ_1$, $f'(intJ_2) \supset J_2$

ii) $Fix(g') \subset J_3 \cup J_4$, $g'(J_3) \subset intJ_3$, $g'(intJ_4) \supset J_4$





Finally, let $S''' \subset S$ be a subsurface diffeomorphic to the torus with two holes, let $\Phi''' : \pi_1(S''') \to Homeo_+(\mathbf{S}^1)$ be the restriction of Φ ($eu(\Phi''') = -2$), let $\alpha'', \beta'', \gamma'' \in \pi_1(S''')$ as follows:



and let $F = \Phi'''(\alpha'')$, $G = \Phi'''(\beta'')$, $H = \Phi'''(\gamma'')$. From lemmata 4.10, 4.11 one obtains:

Lemma 4.12. *We may decompose* $\mathbf{S}^1$ *as union of eight consecutive intervals*

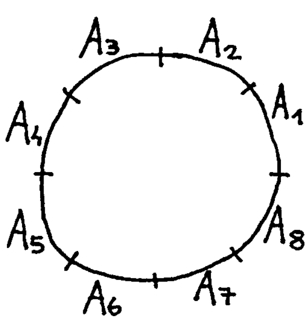
$$A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 \cup A_7 \cup A_8$$

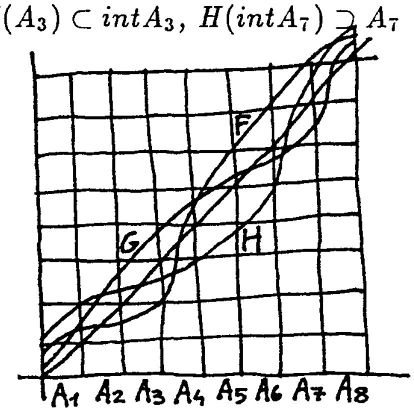
such that:

i) $Fix(F) \subset A_2 \cup A_4$, $F(A_2) \subset intA_2$, $F(intA_4) \supset A_4$

ii) $Fix(G) \subset A_6 \cup A_8$, $G(A_6) \subset intA_6$, $G(intA_8) \supset A_8$

iii) $Fix(H) \subset A_1 \cup A_3 \cup A_5 \cup A_7$, $H(A_3) \subset intA_3$, $H(intA_7) \supset A_7$



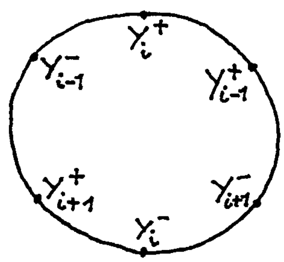


We return now to the curves $\gamma_1, ..., \gamma_{2g+2}$ initially choosen on S. Take 3 consecutive curves $\gamma_{i-1}, \gamma_i, \gamma_{i+1}$ $(2g+3 \equiv 1)$ and take a tubular neighborhood V of $\gamma_{i-1} \cup \gamma_i \cup \gamma_{i+1}$ diffeomorphic to the torus with two holes. We may apply 4.12 to $S''' = V$, $\alpha'' = [\gamma_{i-1}]$, $\beta'' = [\gamma_{i+1}]$, $\gamma'' = [\gamma_i]$. It follows that we may find two fixed points $y_i^+, y_i^- \in \mathbf{S}^1$ of $\Phi([\gamma_i])$ ($= H$ in the lemma) such that y_i^+ is attractive and belongs to A_3, y_i^- is repulsive and belongs to A_7. Remark that every connected component of $\mathbf{S}^1 \setminus \{y_i^+, y_i^-\}$ contains fixed points of $\Phi([\gamma_{i-1}])$ and $\Phi([\gamma_{i+1}])$.

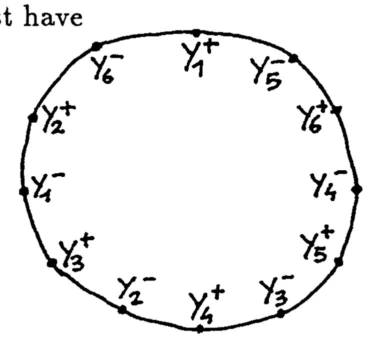
We repeat this construction for all $i = 1, ..., 2g+2$, so that we find points $\{y_i^+, y_i^-\}_{i=1}^{2g+2}$ on $\mathbf{S}^1$ such that:

a) $y_i^\pm \in Fix\Phi([\gamma_i])$, y_i^+ is attractive, y_i^- is repulsive;

b) the relative positions of $y_{i-1}^\pm, y_i^\pm, y_{i+1}^\pm$ are



The property *b)* implies that the relative positions of $\{y_i^\pm\}_{i=1}^{2g+2}$ are uniquely defined. For example, for $g = 2$ we must have



But this is also the relative disposition of the fixed points $\{x_i^\pm\}_{i=1}^{2g+2}$ of the homeomorphisms $\{\Psi([\gamma_i^\pm])\}_{i=1}^{2g+2}$ (remark that every $\Psi([\gamma_i])$ has exactly two fixed points, and that we may repeat the previous considerations for Ψ instead of Φ).

Hence we may take a homeomorphism $\omega : \mathbf{S}^1 \to \mathbf{S}^1$ such that

$$\omega(x_i^+) = y_i^+, \qquad \omega(x_i^-) = y_i^- \qquad \forall i = 1, ..., 2g+2.$$

This homeomorphism conjugate Φ to a representation $\tilde{\Phi}$ with the properties required in proposition 4.9. $\triangle$

Remark: the result of Matsumoto ([Mat]) shows directly that Φ is topologically conjugate to Ψ. But his proof is far from being elementary, because it relies on deep theorems of Ghys, Matsumoto and others about "Euler class and bounded cohomology".

NONTRANSITIVE ANOSOV FLOWS

Examples of Anosov flows on a 3-manifold with nonwandering set different from all of the manifold were found by J. Franks and R. Williams in 1979 ([F-W]), answering negatively to an old conjecture.

We recall that a basic set Ω of a nonsingular smooth flow $\phi_t : M \to M$ on a 3-manifold M belongs necessarily to one of the following four classes:

i) expanding attractor: $dim\Omega = 2$, $\cup_{x\in\Omega}\mathcal{F}^u_x = \Omega$;

ii) contracting repellor: $dim\Omega = 2$, $\cup_{x\in\Omega}\mathcal{F}^s_x = \Omega$;

iii) cantorian saddle: $dim\Omega = 1$, Ω different from a single closed orbit;

iv) hyperbolic closed orbit.

Using a sort of "connected sum" of (transitive) Anosov flows, Franks and Williams constructed an Anosov flow with two basic sets Ω_1 and Ω_2, Ω_1 an expanding attractor, Ω_2 a contracting repellor. Then they showed that it may happen that an Anosov flow has a basic set reduced to a single hyperbolic closed orbit (necessarily of saddle type). We will show later how to construct an Anosov flow with a basic set of the type cantorian saddle, more precisely a basic set isomorphic to the suspension of Smale's horseshoe.

In all these examples the basic sets of the flow are separated by incompressible tori. This is a general fact.

Theorem 5.1. *Let M be a closed orientable 3-manifold and let $\phi_t : M \to M$ be a nontransitive Anosov flow, with basic sets $\Omega_1, ..., \Omega_N$. Then there exist embedded, pairwise disjoint tori $T_1, ..., T_k \subset M$ such that:*

i) the nonwandering set $\Omega(\phi_t) = \cup_{j=1}^N \Omega_j$ is disjoint from $\cup_{i=1}^k T_i$;

ii) each connected component of $M \setminus \left[\cup_{i=1}^k T_i\right]$ contains one and only one basic set of ϕ_t;

iii) each T_i is incompressible in M ($\pi_1(T_i) \to \pi_1(M)$ is injective) and T_i, T_j are not isotopic for $j \neq i$.

The requirement of orientability of M is not essential (otherwise, there would be Klein bottles among the $\{T_i\}$...).

Recall that a 3-manifold supporting an Anosov flow is irreducible ([Pal]). Using Dehn lemma and other standard tools of three-dimensional topology ([Lau]), it is not difficult to see that two embedded, disjoint, incompressible, isotopic tori in an irreducible 3-manifold M are also *parallel*, i.e. they bound in M an embedded copy of $\mathbf{T}^2 \times [0,1]$. A theorem of W. Haken ([Hem], page 140) asserts that, given a closed irreducible 3-manifold M, there exists a natural number $h(M)$ such that if $T_1, ..., T_k$ are embedded, incompressible, pairwise disjoint and nonisotopic tori in M then $k \leq h(M)$. It follows that there exists $h_0(M) \in \mathbf{N}$, $h_0(M) \leq h(M) + 1$, such that if $T_1, ..., T_k$ are as before then $M \setminus \left[\, \cup_{j=1}^{k} T_j \right]$ has at most $h_0(M)$ connected components.

Hence theorem 5.1 has the following consequence.

<u>**Corollary 5.2**</u>. *Let ϕ_t be a nontransitive Anosov flow on the closed orientable 3-manifold M. Then the number of basic sets of ϕ_t is at most $h_0(M)$.*

Example 1: if M is atoroidal then $h(M) = 0$ and hence M cannot support a nontransitive Anosov flow. On the other hand, S. Goodman constructed in [Goo1] examples of transitive Anosov flows on atoroidal 3-manifolds.

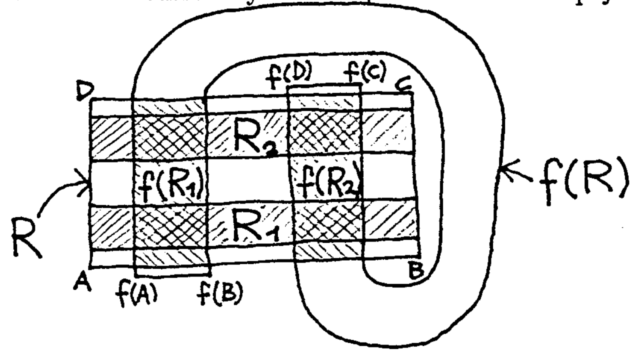
Example 2: if $M = \frac{\mathbf{T}^2 \times [0,1]}{(x,0) \sim (Ax,1)}$, $A \in SL(2, \mathbf{Z})$ hyperbolic, then $h(M) = 1$ (every incompressible torus is isotopic to a fibre) and $h_0(M) = 1$ (a fibre does not separate M); hence every Anosov flow on M is transitive. This fact was observed also by Plante ([Pla2]), using the theory of holonomy invariant measures. It can be also deduced from proposition 4.6, via [Sol2] asserting that the liftings of the stable and unstable foliations to the universal covering $\mathbf{R}^3$ are product foliations.

On the other hand, let us observe that the previous "Haken estimate" is not the best one. For example, we have seen in the last chapter that an Anosov flow on a Seifert manifold is always transitive, whereas $h_0(M)$ can be arbitrarily large for such a manifold (for example, $h_0(T_1 S) = g$, where g is the genus of S).

<u>Embedding the Smale's horseshoe</u>

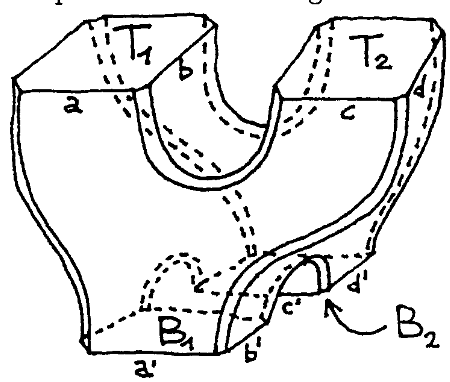
The basic set Ω_{sh} defined by suspending the Smale's horseshoe ([Sma], [Shu]) is topologically equivalent to the suspension of the subshift defined by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Consider the basic set Ω defined by the suspension of a map f of the following type:



(as usual, f is "hyperbolic and linear" on $f^{-1}(f(R) \cap R)$, preserves horizontal and vertical lines, etc., and $\Omega = \cap_{n \in \mathbf{Z}} f^n(R)$; see [Sma] or [Shu] for details about these constructions). This basic set also is topologically equivalent to the suspension of the subshift defined by $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and hence to Ω_{sh}. We can extend f to a diffeomorphism of $\mathbf{S}^2$ and hence Ω becomes a basic set of a flow $\hat{\psi}_t : \mathbf{S}^2 \times \mathbf{S}^1 \to \mathbf{S}^2 \times \mathbf{S}^1$.

According to [Bi-W] (see also [Goo2], [Fra2]) there exists a compact neighborhood K of Ω in $\mathbf{S}^2 \times \mathbf{S}^1$ diffeomorphic to the following set:



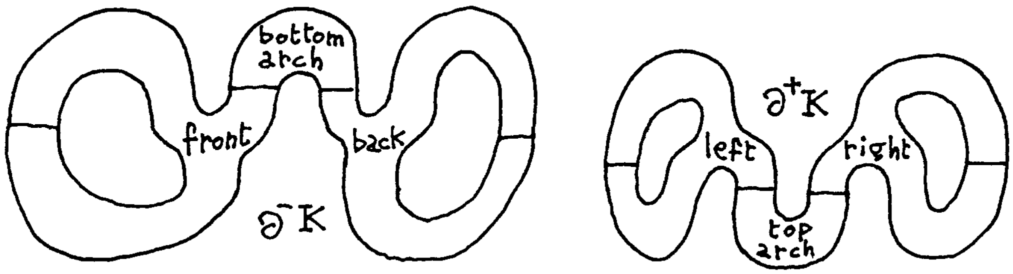
where T_1 is identified with B_1 through a diffeomorphism mapping a to a', b to b', and T_2 is identified with B_2 through a diffeomorphism mapping c to c', d to d'. The boundary ∂K is decomposed in three pieces, $\partial K = \partial^- K \cup \partial^+ K \cup H$, where:

- $\partial^- K$ is the part of ∂K where $\hat{\psi}_t$ exits from K, and it is composed by the two "Y" pieces (front and back) and the bottom arch;

- $\partial^+ K$ is the part of ∂K where $\hat{\psi}_t$ enters into K, and it is composed by the two "inverted Y" pieces (left and right) and the top arch;

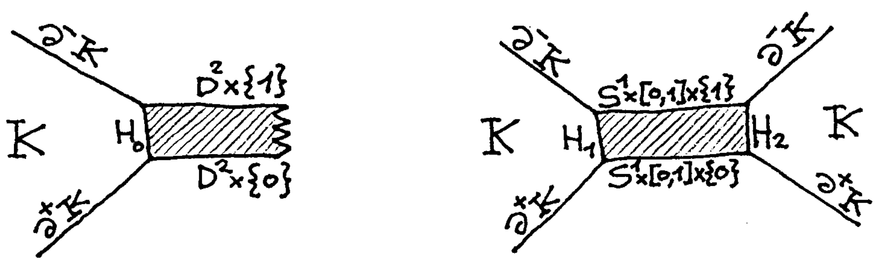
- H is the part of ∂K where $\hat{\psi}_t$ is tangent to ∂K, and it is composed by narrow strips between $\partial^- K$ and $\partial^+ K$.

The sets $\partial^- K$ and $\partial^+ K$ are both diffeomorphic to disks with two holes



whereas H is composed by 3 annuli, a "long" one (which is adiacent to the component of $\partial(\partial^- K)$ intersecting the bottom arch and to the component of $\partial(\partial^+ K)$ intersecting the top arch) and 2 smaller ones. Denote by H_0 the long annulus, by H_1, H_2 the other two annuli.

We glue $\mathbf{D}^2 \times [0,1]$ to K by identifying $\partial \mathbf{D}^2 \times [0,1]$ with H_0 in such a way that $\partial \mathbf{D}^2 \times \{0\}$ goes to $\partial H_0 \cap \partial(\partial^+ K)$ and $\partial \mathbf{D}^2 \times \{1\}$ goes to $\partial H_0 \cap \partial(\partial^- K)$. We also glue $\mathbf{S}^1 \times [0,1] \times [0,1]$ to K by identifying $\mathbf{S}^1 \times \{0,1\} \times [0,1]$ with $H_1 \cup H_2$ in such a way that $\mathbf{S}^1 \times \{0,1\} \times \{0\}$ goes to $\partial(H_1 \cup H_2) \cap \partial(\partial^+ K)$ and $\mathbf{S}^1 \times \{0,1\} \times \{1\}$ goes to $\partial(H_1 \cup H_2) \cap \partial(\partial^- K)$:

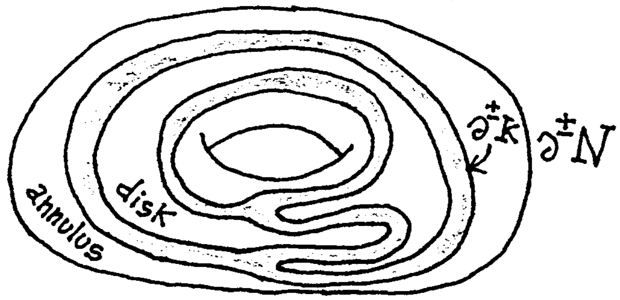


The result is a manifold $N = K \cup (\mathbf{D}^2 \times [0,1]) \cup (\mathbf{S}^1 \times [0,1] \times [0,1])$. The flow $\hat{\psi}_t|_K$ extends naturally to a smooth flow

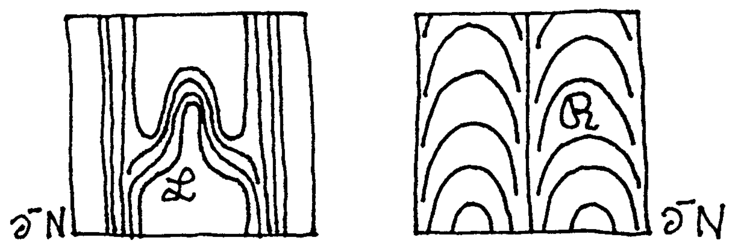
$$\psi_t : N \to N,$$

which is transverse to ∂N and for which $\Omega(\psi_t) = \Omega$: $\psi_t|_{(\mathbf{D}^2 \cup \mathbf{S}^1 \times [0,1]) \times [0,1]}$ is a translation along the second component, in the positive direction.

Define $\partial^+ N = \{x \in \partial N | \psi_t \text{ enters in } N \text{ through } x\}$, $\partial^- N = \{x \in \partial N | \psi_t \text{ exits from } N \text{ through } x\}$. Then $\partial^+ N$, $\partial^- N$ are obtained from $\partial^+ K$, $\partial^- K$ by glueing a disk and an annulus. They are tori, and the inclusions $\partial^+ K \subset \partial^+ N$, $\partial^- K \subset \partial^- N$ are as follows:



Consider now the unstable lamination $\mathcal{F}^u(\Omega)$ associated to the basic set Ω of ψ_t. Then $\mathcal{F}^u(\Omega) \cap \partial^+ N = \emptyset$ and $\mathcal{F}^u(\Omega) \cap \partial^- N$ is a one-dimensional lamination $\mathcal{L}$ whose leaves are contained in $\partial^- K$ and disposed in "longitudinal" way ([Bi-W], [Goo2], [Fra2]); it follows that there exists a foliation $\mathcal{R}$ of $\partial^- N$, formed by two Reeb components, transverse to $\mathcal{L}$:



Let N' be the manifold obtained from $\frac{\mathbf{T}^2 \times [0,1]}{(x,0)\sim(Ax,1)}$, $A \in SL(2, \mathbf{Z})$ hyperbolic, by deleting a tubular neighborhood of $\{0\} \times [0,1]$, and let

$$\psi'_t : N' \to N'$$

be the DA flow considered in [F-W]: ψ'_t enters in N' through $\partial N' \simeq \mathbf{T}^2$, and $\Omega(\psi'_t)$ is a single basic set Ω' of the type expanding attractor. The stable foliation $\mathcal{F}^s(\Omega')$ intersects $\partial N'$ along a foliation $\mathcal{S}$ by two Reeb components.

We glue (N, ψ_t) and (N', ψ'_t) by identifying $(\partial^- N, \mathcal{R})$ with $(\partial N', \mathcal{S})$. The result is an hyperbolic flow

$$\bar{\psi}_t : \bar{N} \to \bar{N}, \qquad \bar{N} \stackrel{def}{=} N \cup N',$$

with the following properties:

a) $\bar{\psi}_t$ enters into $\bar{N}$ through $\partial \bar{N} = \partial^+ N$;

b) $\bar{\psi}_t$ has two basic sets: a cantorian saddle Ω and an expanding attractor Ω';

c) $\mathcal{F}^s(\Omega')$ is transverse to $\mathcal{F}^u(\Omega)$.

The foliation $\mathcal{F}^s(\Omega \cup \Omega')$ of $\bar{N}$ intersects the boundary $\partial^+ N$ along a foliation $\mathcal{G}$, which extends the lamination given by $\mathcal{F}^s(\Omega) \cap \partial^+ N$. This lamination is similar to the previous lamination $\mathcal{L}$, in particular the extension $\mathcal{G}$ is without global cross sections. This means that $\mathcal{G}$ has compact leaves and, consequently, it is structurally stable (by hyperbolicity). As a consequence, there exists a diffeomorphism $\varphi : \partial^+ N \to \partial^+ N$ such that $\varphi^*(\mathcal{G})$ is transverse to $\mathcal{G}$.

We now repeat the construction of [F-W]: we glue $(\bar{N}, \bar{\psi}_t)$ with $(\bar{N}, \bar{\psi}_{-t})$ via the diffeomorphism φ. The result is a nontransitive Anosov flow $\phi_t : M \to M$ with four basic sets: an expanding attractor, a contracting repellor, and two cantorian saddles, each of which is isomorphic to the suspension of the subshift defined by $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Remark 1: we don't know the shape of the previous foliation $\mathcal{G}$ on $\partial^+ N$; if we would be able to prove that $\mathcal{G}$ admits a transverse foliation formed by two Reeb components, then we could be able to construct a nontransitive Anosov flow with 3 basic sets: an attractor, a repellor, and a cantorian saddle.

Remark 2: Pugh and Shub showed that any suspension of subshift of finite type may be realized as a basic set of some flow on a 3-manifold. Is this result still true if we require that such a flow is Anosov?

Proof of theorem 5.1

Let $f : M \to \mathbf{R}$ be a Lyapunov function for ϕ_t ([Shu], page 18). This means that f is smooth,

$$df(x) = 0 \qquad \forall x \in \Omega(\phi_t),$$

$$\frac{d}{dt}|_{t=0} f(\phi_t(x)) < 0 \qquad \forall x \in M \setminus \Omega(\phi_t).$$

Because of the transitivity of $\phi_t|_{\Omega_j}$, f is constant on every basic set. We denote by c_j the value of f on Ω_j, $j = 1, ..., N$; we may assume that $c_1, ..., c_N$ are all distinct and (modulo a reindexing of the basic sets) $c_1 < c_2 < ... < c_N$.

Take real numbers $r_1, ..., r_{N-1}$, with $r_j \in (c_j, c_{j+1})$. Denote by Σ_j the surface $f^{-1}(r_j) \subset M$, $j = 1, ..., N-1$. The surfaces Σ_j are transverse to the flow, because r_j are regular values, hence they are transverse also to the stable and unstable foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. This fact, together with the orientability of M and the transverse orientability of Σ_j, implies that every Σ_j is a union of tori.

Lemma 5.3. *Σ_j is incompressible in M, $\forall j = 1, ..., N-1$.*

Proof.

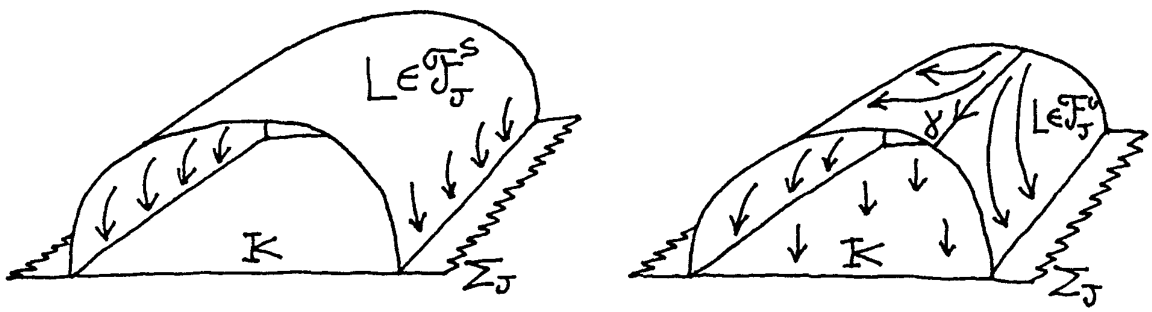
Define

$$M_j^+ = f^{-1}([r_j, c_N]) \qquad M_j^- = f^{-1}([c_1, r_j]).$$

The foliation $\mathcal{F}_j^s \stackrel{def}{=} \mathcal{F}^s|_{M_j^+}$ is without Reeb components ([C-L], [Nov]) and without half-Reeb components.

The non-existence of half-Reeb components may be seen as follows. Suppose that H is a half-Reeb component of $\mathcal{F}_j^s$, $\partial H = L \cup K$, $L \in \mathcal{F}_j^s$ (a cylindrical leaf), $K \subset \partial M_j^+ = \Sigma_j$ (a planar Reeb component for $\mathcal{F}^s|_{\Sigma_j}$). The flow ϕ_t is outward M_j^+ through Σ_j, hence $\phi_t|_L$ is outward $L \simeq \mathbf{S}^1 \times [0,1]$ through $\partial L \simeq \mathbf{S}^1 \times \{0,1\}$. This is impossible, because L is a piece of a stable manifold of ϕ_t.

Remark that also $\mathcal{F}_j^u \stackrel{def}{=} \mathcal{F}^u|_{M_j^+}$ is without half-Reeb components (and, of course, without Reeb components): if $H \subset M_j^+$ is such a component, $\partial H = L \cup K$, then the same argument as before shows that L must contain a closed orbit γ of ϕ_t. Moreover, ϕ_t is outward H through K, and we arrive to an absurd by considering the α-limits of points $x \in intH$: $\alpha(x) \subset intH$, but $intH \cap \Omega(\phi_t) = \emptyset$.



Now we consider the double foliation $2\mathcal{F}_j^s$ on $M_j^+ \cup_{\Sigma_j} M_j^+$. It is a foliation without Reeb components, hence by Novikov's theorem ([Nov], [C-L])

$$\pi_2(M_j^+ \cup_{\Sigma_j} M_j^+) = 0.$$

It follows that $\Sigma_j \hookrightarrow M_j^+$ induces an injection $\pi_1(\Sigma_j) \to \pi_1(M_j^+)$. Similarly, $\pi_1(\Sigma_j) \to \pi_1(M_j^-)$ is also injective. The proof of the lemma is completed by the following topological fact, which seems well known but we have not been able to find a reference and hence we give a proof. $\triangle$

Sublemma 5.4. *If Σ is a separating surface in a 3-manifold M, $M = M' \cup_\Sigma M''$,* *then the injectivity of* $\pi_1(\Sigma) \to \pi_1(M')$ *and of* $\pi_1(\Sigma) \to \pi_1(M'')$ *implies the injectivity of* $\pi_1(\Sigma) \to \pi_1(M)$.

Proof.

Suppose, by contradiction, that $\pi_1(\Sigma) \to \pi_1(M)$ is not injective; then there exists a smooth map

$$j : (\mathbf{D}^2, \partial\mathbf{D}^2) \to (M, \Sigma)$$

such that $j(\partial\mathbf{D}^2)$ defines a nontrivial element of $\pi_1(\Sigma)$. We may assume, by a general position argument, that j is transverse to Σ, and hence $j^{-1}(\Sigma)$ is a finite union of disjoint circles on $\mathbf{D}^2$, one of which is $\partial\mathbf{D}^2$.

Let $\Gamma \subset \mathbf{D}^2$ be one of these circles, such that $j(\Gamma)$ is homotopic to zero in Σ. Let $\Delta \subset \mathbf{D}^2$ be the disk bounded by Γ, and let $h : \Delta \to \Sigma$ be a smooth map such that $h|_\Gamma = j|_\Gamma$. Let $j' : \mathbf{D}^2 \to M$ be defined by $j'|_{\mathbf{D}^2\setminus\Delta} = j$, $j'|_\Delta = h$. Then j' maps a neighborhood V of Δ to one of the two pieces M' or M'', and we may modify j' in V in such a way that, denoting with j'' the result of the modification, we have $j''(V) \cap \Sigma = \emptyset$ and $j''^{-1}(\Sigma) = j^{-1}(\Sigma) \cap V^c$. After a finite number of these modifications we obtain a map $i : \mathbf{D}^2 \to M$ such that $i|_{\partial\mathbf{D}^2} = j|_{\partial\mathbf{D}^2}$, i is transverse to Σ, and every circle in $i^{-1}(\Sigma)$ is mapped by i to a noncontractible circle on Σ.

Let now $\hat{\Gamma}$ be one of these circles, such that the disk $\hat{\Delta} \subset \mathbf{D}^2$ bounded by $\hat{\Gamma}$ does not contain other circles in $i^{-1}(\Sigma)$ (clearly, such a $\hat{\Gamma}$ exists). The map

$$i|_{\hat{\Delta}} : (\hat{\Delta}, \hat{\Gamma}) \to (M, \Sigma)$$

has image entirely contained in M' or M'', and this contradicts the hypotheses on $\pi_1(\Sigma) \to \pi_1(M')$ and $\pi_1(\Sigma) \to \pi_1(M'')$. $\triangle$

By construction, the surfaces $\{\Sigma_j\}$ have the property that every connected component of $M \setminus \left[\cup_{j=1}^{N-1} \Sigma_j\right]$ contains at most one basic set. Let N be the closure of a component which contains no basic set. Then

$$\partial^+ N = \{x \in \partial N |\ \text{the trajectory through } x \text{ enters in } N\}$$

is isotopic to

$$\partial^- N = \{x \in \partial N |\ \text{the trajectory through } x \text{ exit from } N\}$$

the isotopy being realized by the flow itself; it follows that $N \simeq \mathbf{T}^2 \times [0,1]$. If we suppress the torus $\partial^+ N$ (or $\partial^- N$) we obtain a new collection of tori $(\cup_{j=1}^{N-1} \Sigma_j) \setminus \partial^+ N$ which still separate the basic sets of ϕ_t. Repeating this semplification a finite number of times, we finally obtain a collection of embedded, disjoint, incompressible tori $T_1, ..., T_k \subset M$, which satisfy *i)* and *ii)* of theorem 5.1.

It remains only to prove that, for $j \neq i$, T_j and T_i are not isotopic. Because incompressible isotopic tori are parallel, it suffices to prove the following lemma.

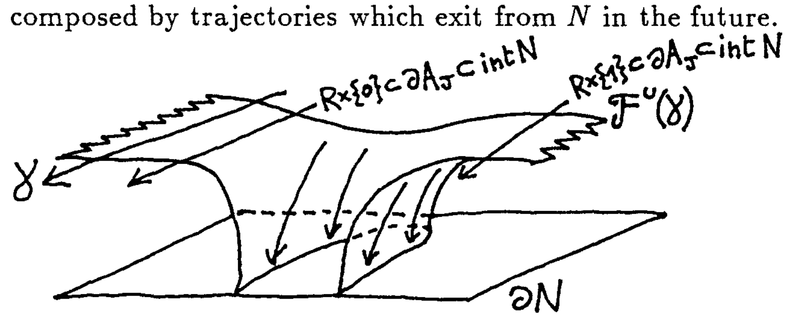
Lemma 5.5. *Let $N \subset M$ be a submanifold diffeomorphic to $\mathbf{T}^2 \times [0,1]$, such that ∂N is disjoint from $\Omega(\phi_t)$ and transverse to ϕ_t. Then $N \cap \Omega(\phi_t) = \emptyset$.*

Proof.

A basic set Ω of a nontransitive Anosov flow on a 3-manifold belongs necessarily to one of the following classes: a) expanding attractor; b) contracting repellor; c) cantorian saddle; d) hyperbolic closed orbit, of saddle type.

Suppose that N contains an expanding attractor Ω. As in lemma 5.3, the foliation $\mathcal{F}^u|_N$ is without Reeb components and half-Reeb components. The unstable leaves of Ω are contained in Ω and hence in $intN$; they have exponential growth ([Pla4]). Passing to the double $N \cup_{\partial N} N \simeq \mathbf{T}^3$ we obtain a foliation without Reeb components, with some leaves of exponential growth, on a manifold with fundamental group of polynomial growth. This contradicts a theorem of Plante ([Pla4]), hence N cannot contain expanding attractors (and, similarly, contracting repellors).

Suppose now that N contains a cantorian saddle Ω. Let $\gamma \subset \Omega$ be a closed orbit, then there are infinitely many disjoint strips $A_j \simeq \mathbf{R} \times [0,1] \subset \mathcal{F}^u(\gamma)$, $j \in \mathbf{N}$, such that $\partial A_j \simeq \mathbf{R} \times \{0,1\}$ is composed by two trajectories of ϕ_t which remain in N $\forall t \in \mathbf{R}$, and $intA_j \simeq \mathbf{R} \times (0,1)$ is composed by trajectories which exit from N in the future.



We use here the fact that N does not contain attractors. Passing to the double, $\mathcal{F}^u(\gamma)$ becomes a leaf L with non-abelian (in fact, non-finitely generated) fundamental group; hence $\pi_1(L) \to \pi_1(N \cup_{\partial N} N) = \mathbf{Z}^3$ is not injective and we arrive to a contradiction with the theorem of Novikov. This means that N does not contain cantorian saddles.

Finally, suppose that $N \cap \Omega(\phi_t) = \{\gamma_1, ..., \gamma_n\}$, γ_j = hyperbolic closed orbit. The "no cycle condition" ([Sma], [Shu]) ensures that at least one of these closed orbits (say, γ_1) has the property that $\forall x \in \mathcal{F}^u(\gamma_1) \cap N$ there exists $t > 0$ such that $\phi_t(x) \in \partial N$. Hence $\mathcal{F}^u(\gamma_1) \cap N \simeq \mathbf{S}^1 \times [0,1]$. Remark that, because N does not contain attractors and repellors, $\partial^+ N$ and $\partial^- N$ are both non empty and consequently diffeomorphic to $\mathbf{T}^2$. This means that $\mathcal{F}^u(\gamma_1) \cap \partial N$ is formed by two parallel circles Γ_1, Γ_2 on $\partial^- N$. Moreover, one of the two annuli composing $\partial^- N \setminus (\Gamma_1 \cup \Gamma_2)$ together with $\mathcal{F}^u(\gamma_1) \cap N$ bound a subset $H \subset N$ homeomorphic to $\mathbf{D}^2 \times \mathbf{S}^1$. We arrive to a contradiction by considering, as in lemma 5.3, the α-limits of points in $intH$. This completes the proof of lemma 5.5 and of theorem 5.1. $\triangle$

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