SISSA **ISAS**

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Topics On String/Gauge Theories Dualities

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Candidate:

Luis Fernando Alday

Supervisors:

Prof. Edi Gava

Prof. Kumar S. Narain

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Chapter 1

Introduction

1.1 Gauge theories as string theories

The microscopic behavior of nature, the fundamental particles and the interactions among them are, on the base of present knowledge, well described by gauge theories, where the fundamental objects are point-like and interactions are local. It is believed, however, that at small distances (of the order of the Planck scale) quantum gravity effects become important. Hence we need a quantum theory including gravity.

It turns out that a consistent quantum theory of gravity can be developed if one abandons the idea of point-like fundamental objects and use, instead, one-dimensional objects or strings. Today string theory is seen as the strongest candidate for a quantum theory of gravity.

However, string theory was originally discovered in trying to describe the large number of hadrons and mesons discovered in the 1960's. It was conjectured that all these particles where different oscillation modes of a string. The hadronic spectrum has many stringy features, for instance, the mass and the angular momentum of a relativistic rotating string of tension T satisfy $M^2 \approx TJ^2 + const.$, relation also satisfied for the lightest hadron with a given spin. Today we know that hadrons and mesons are made of quarks and these are well described by QCD.

The fact that the hadronic spectrum can be described in terms of a gauge theory (QCD) and (to some extent) in terms of a string theory, make us suspect that both descriptions may be equivalent, in other words, the existence of a string dual of QCD is more than plausible.

This idea was made precise by 't Hooft, who showed that gauge theories with $U(N)$ (or $SU(N)$) gauge group greatly simplify in the limit of a large number of colors where the theory admits a stringy description [1]. More precisely, in the so called 't Hooft limit

$$N \to \infty, \qquad \lambda = g_{YM}^2 N \quad \text{fixed} \tag{1.1}$$

non planar contributions (to a given amplitude) are suppressed with respect to planar ones, and the diagrams (in the double line notation) can be seen as string diagrams, where the interaction (splitting and joining of strings) coupling is proportional to $1/N$.

These arguments are very general and are expected to work for almost any gauge theory admitting a large N expansion. Finally, we should mention the important result that in trying to construct a string theory dual of a four dimensional gauge theory we are forced to introduce at least one additional dimension.

1.2 AdS/CFT duality

In the last 20 years string theories, as theories of quantum gravity, were extensively studied. It turns out that a quantum theory of strings is not consistent in any dimension, in particular its supersymmetric version requires a ten dimensional space time.

An explicit realization of 't Hooft ideas was found by Maldacena, who has actually shown that the string theories dual to large N gauge theories were nothing but the same as the ones proposed as quantum theories of gravity!

More precisely the Maldacena conjecture, or AdS/CFT duality, states the equivalence between $\mathcal{N}=4$ super Yang-Mills (SYM) in four dimensions and string theory on the ten-dimensional background $AdS_5 \times S^5$ [2].

Such duality has been widely studied since its original formulation. The reasons for such a big interest are several. On the one hand the conjecture allows to compute quantities in the gauge theory by doing computations on the string side (often in the supergravity limit) and vice-versa. On the other hand it mixes many interesting ideas of physics.

A holographic description of a theory is an equivalent description in terms of a lower dimensional theory. It has been argued that a quantum theory of gravity on a volume V should be described in terms of a theory on its boundary ∂V. In the AdS/CFT duality the gauge theory lives in four dimensions, that can actually be interpreted as the boundary of AdS_5, so the AdS/CFT duality is an explicit and precise realization of the holographic principle [3].

Another idea behind the Maldacena conjecture is the duality between open and closed string descriptions, called open-closed duality. As we will see, the AdS/CFT duality arises as two equivalent descriptions of the same system. The open-closed duality is thought to play an important role in the understanding of string theory.

A related duality is that between string theory on PP-waves and a specific subsector of $\mathcal{N}=4$ SYM [4]. Such duality can be seen as a particular limit of the AdS/CFT correspondence (to be described in the next chapter) and it has the advantage that string theory can be quantized in this background [5] and its spectrum can be fully computed. [1]

1.3 Summary of the thesis

In this thesis we study different topics on the AdS/CFT duality and different limits of it. We consider such study very important since the duality helps to understand many aspects of gauge and string theory and relates many areas of theoretical high energy physics.

In the next chapter we give a brief introduction to the AdS/CFT duality and related topics. In the first part of the chapter we describe the prototypical example of strings on $AdS_5 \times S^5$ dual to $\mathcal{N}=4$ SYM. In the second part we describe strings on PP-waves, as a particular limit of strings on $AdS_5 \times S^5$ and the corresponding limit on the gauge theory side. Finally, we describe some previous attempts, in the literature, aimed at understanding the duality in the limit in which the gauge theory is weakly coupled. This limit is particularly interesting and difficult to study since the string theory dual presumably lives on a highly curved background.

In chapter 3 we study D-brane solutions on a PP-wave background. D-branes are very important in string theory (being non perturbative objects) as well as very powerful tools, for instance, the allow one to engineer gauge theories. Moreover, as already mentioned, strings can be quantized on PP-waves, so the study of such background in the presence of D-branes is interesting. In particular we describe fully localized (*i.e.* the solution depends on

[1] Actually, string on pp-waves can be quantized in the ligth-cone gauge and for values of the light cone momentum different from zero.

all the transverse coordinates) supersymmetric Dp-brane solutions, as well as the $D1 - D5$ system.

In the last year, clues for the existence of integrable structures on both sides of the AdS/CFT duality have been pointed out. For instance, hints to the integrability of $\mathcal{N} = 4$ SYM in the large N limit were given by studying the dilatation operator in perturbation theory and it was shown that the one loop mixing matrix giving the anomalous dimensions, upon diagonalization, can be identified with the Hamiltonian of an integrable spin chain [6, 7, 8].

On the other hand, from the string theory side, after some clue from the bosonic theory [9] , Bena *et. al.* found an infinite set of non-local classically conserved charges for the Green-Schwarz superstring on $AdS_5 \times S^5$ [10]. This would imply that the world-sheet theory be an integrable system. Such charges satisfy an algebra called the Yangian and such algebra appears also in operators acting on spin chains, the relation between the two approaches was given in [11].

In chapter 4 we study the structure of such charges on the PP-wave limit of $AdS_5 \times S^5$. First we construct a set of non-local charges from the world-sheet sigma model of closed strings on PP-waves backgrounds and then we construct the charges on $AdS_5 \times S^5$ and take their Penrose limit. Both methods are shown to agree.

As already mentioned, the study of the AdS/CFT duality in the limit in which the gauge theory is weakly coupled is non trivial, since string theory lives on a highly curved background. In the last part of this thesis we describe some attempts to find a string theory formulation underlying free $\mathcal{N} = 4$ SYM.

In chapter 5 we describe a partial match between states on a PP-wave-like string theory and free SYM states (extending the BMN dictionary down to small J). We further develop a discretized string field theory (SFT) that we use in order to compute three-point correlation functions among various string states. Such correlation functions do not agree with those of free SYM, however the mismatch is small and seems to be systematic. We end the chapter by discussing some possible reasons for such mismatch.

In chapter 6 a more effective approach is taken in order to give a string dual of free SYM. By using a oscillator construction (which would act as string oscillators in position space) we are able to obtain the precise spectrum of SYM. Further we re-develop the discretized light cone SFT method to compute correlation functions in these new variables and match them, successfully, against some simple examples of free SYM correlations functions.

Then we focus on the question of quantum corrections to two and three point functions. In string theory we have no way to compute corrections in the 't Hooft coupling to the Hamiltonian, but once we use such corrections as a input then corrections (at that given order) to n-point functions are fixed. With such input string theory "predicts" the correct quantum corrections to two and three points functions of the closed $SO(6)$ sector. We end the chapter with some conclusions and outlook.

Chapter 2

AdS/CFT correspondence

2.1 AdS_5/CFT_4 duality

The best established duality between a gauge theory and a string theory is that between $\mathcal{N} = 4$ super Yang-Mills in four dimensions and type IIB string theory compactified on $AdS_5 \times S^5$. The relation between these two theories can be understood as follows [1].

D-branes can be defined in string perturbation theory as an hypersurface on which open strings can end. By a Dp-brane we mean an hypersurface with p spatial dimensions plus time. Even though D-branes are not necessary neither sufficient to proof the AdS/CFT correspondence, we will use them in order to give a heuristic argument supporting it.

Consider an stack of N parallel D3-branes sitting together on flat ten dimensional Minkowski space-time, as mentioned before, these D3-branes are extended along a (3+1) dimensional plane. String theory on this background contains two kinds of perturbative excitations, closed strings and open strings and admits two dual descriptions.

In the open string description the dynamics of the D-branes is described, at low energies, by the massless modes of the open strings ending on them. For such N branes we have N^2 different strings (depending on the pair of D-branes on which they end) that actually represent the N^2 massless gauge bosons, together with their superpartners, of a $U(N)$ gauge theory. As the endpoints of the strings are confined to live on the D3-branes then the gauge theory lives in four dimensions, further it can be seen that the theory is

[1] There are many nice reviews on the subject, for instance [12, 13, 14, 15, 16].

conformal and possesses $\mathcal{N}=4$ supersymmetries.

An alternative description can be given in terms of closed strings. These describe the geometry of the space-time in the presence of the stack of D3-branes. More precisely we can find a D3-brane solution of supergravity of the form

$$ds^2 = f^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2) \tag{2.1}$$

$$F_5 = -f^{-2}(1+*)dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge df \tag{2.2}$$

$$f = 1 + \frac{R^4}{r^4}, \qquad R^4 \equiv 4\pi g_s \alpha'^2 N \tag{2.3}$$

With F_5 the field strength of the four form Ramond-Ramond (RR) potential A_4, whose source are the D3-branes. The energy E_p of an object as measured by an observer at a constant position r and the energy E measured by an observer at infinity are related by the redshift factor

$$E = f^{-1/4} E_p \tag{2.4}$$

i.e. the same object brought closer and closer to $r=0$ would appear to have lower and lower energy for an outside observer. In another words, in the low energy limit we should focus on the near horizon region ($r \ll R \Rightarrow f \sim R^4/r^4$), now the geometry becomes

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2}(dr^2 + r^2 d\Omega_5^2) \tag{2.5}$$

$$F_5 = (1+*)dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge d\frac{r^4}{R^4} \tag{2.6}$$

which is the geometry of $AdS_5 \times S^5$ supported by N units of RR-five form flux

$$\int_{S^5} F_5 = N \tag{2.7}$$

As both descriptions should be equivalent, we are led to the conjecture that $\mathcal{N}=4$ super conformal Yang-Mills theory, with gauge group $U(N)$ living in $(3+1)$ dimensions be dual to type IIB string theory compactified on $AdS_5 \times S^5$ with N units of RR five form flux.

The duality also provides us with a dictionary between the parameters in the two descriptions. The parameters of the gauge theory are the coupling constant g_{YM} and the rank of the group N (or the 't Hooft coupling $\lambda = g_{YM}^2 N$) whereas on the string theory we have the radius of the AdS space R (equal to the radius of the S^5) measured in units of $\sqrt{\alpha'}$ and the string coupling constant g_s, the dictionary reads [2]:

$$4\pi g_s = g_{YM}^2, \qquad \frac{R^2}{\alpha'} = \sqrt{g_{YM}^2 N} = \sqrt{\lambda} \tag{2.8}$$

The strongest version of the duality states that the equivalence holds for all the range of these parameters.[3]

It is important to notice the range of validity of both descriptions. An analysis of loop diagrams in the field theory shows that we can trust perturbative Yang-Mills when

$$g_{YM}^2 N \sim \frac{R^4}{l_s^4} \ll 1 \tag{2.9}$$

besides $g_{YM} \ll 1$, where we have defined the string scale $l_s \approx \sqrt{\alpha'}$. On the other hand, we can rely on the classical gravity description when the radius of curvature becomes large compared to the string length

$$R \gg l_s. \tag{2.10}$$

We see that the two regimes are incompatible, avoiding any contradiction from the fact that the two theories look rather different, this makes the duality very useful, but at the same time hard to prove or hard to disprove!

From the dictionary (2.8) we can consider many limits, as particular cases of the strong version of the duality. For instance, classical string theory ($g_s \to 0$, with R^2/α' fixed) is dual to the large 't Hooft N limit of gauge theory ($N \to \infty, g_{YM} \to 0$, with λ fixed) and type IIB supergravity on $AdS_5 \times S^5$ ($g_s \to 0$, $R^2/\alpha' \to \infty$) is dual to $\mathcal{N} = 4$ SYM in four dimensions at strong 't Hooft coupling ($g_{YM} \to 0$, $\lambda \to \infty$).

Let us have a closer look at these two apparently different theories. In order to see the symmetries of the string theory background we can write AdS_5 as a hypersurface embedded in $\mathbb{R}^{2,4}$

[2] Actually the dictionary extends for the full complexified coupling constants, relating the vacuum expectation value of the axion in string theory with the theta angle of the gauge theory: $\chi = \theta$

[3] Possibly with some corrections to the dictionary for small values of λ.

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -R^2 \tag{2.11}$$

whereby it can be seen that the AdS_5 isometry group is $SO(2,4)$. By choosing an appropriate parametrization of (2.11) we can write the AdS_5 metric as

$$ds^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2) \tag{2.12}$$

these are called "global" coordinates and they cover the whole AdS space.

Besides AdS_5 we have the five dimensional sphere S^5, whose isometry group is $SO(6)$. The full $AdS_5 \times S^5$ metric written in global coordinates reads

$$ds^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + \cos^2\theta d\phi^2 + d\theta^2 + \sin^2\theta d\Omega'^2_3) \tag{2.13}$$

The gauge theory has four supersymmetries in four dimensions, *i.e.* sixteen real supercharges, and its field content in uniquely fixed, there is a unique super multiplet, transforming in the adjoint representation of the gauge group, so the only freedom is the choice of the gauge group and the coupling constant (together with the θ vacuum angle). The field content consist of a gauge boson A_μ, $\mu = 1, ..., 4$, six real scalars ϕ^I, $I = 1, ..., 6$ and four fermions $\lambda_{\alpha i}$ and $\bar{\lambda}_{\dot{\alpha}\bar{i}}$, where $\alpha = 1, 2, \dot{\alpha} = 1, 2$ are four dimensional chiral anti-chiral indexes and $i = 1, ..., 4$. The theory has a $SO(6) \simeq SU(4)$ global symmetry, called R-symmetry, that rotates the scalars and fermions, the index I belong to the fundamental representation of $SO(6)$ and $i, \bar{i}$ to the $\mathbf{4}, \bar{\mathbf{4}}$ (spinorial) representations. the supercharges are in the $\mathbf{4}, \bar{\mathbf{4}}$ as well. Hence the isometry of the S^5 manifests as the global R-symmetry on the gauge theory.

The theory is scale invariant quantum mechanically, *i.e.* the beta function is zero to all orders, so it is conformally invariant. The conformal symmetry extends the Poincaré algebra by adding the generator of scale transformations (the dilatation operator) and the conformal boosts. The result is the conformal algebra in four dimensions, that is isomorphic to SO(2,4). Actually the space-time in which the gauge theory lives can be interpreted as the boundary of AdS_5, and the isometry group $SO(2,4)$ acts on the boundary as the conformal group acting on Minkowski space. It is a general feature that local symmetries in the bulk reflect as global symmetries in the boundary.

Commutating the sixteen ordinary supercharges with the extra conformal generators gives rise to sixteen new conformal supercharges. Actually it can be also shown that the supersymmetries coincide on the two sides of the duality.

2.2 Plane wave limit of AdS/CFT

A particular limit of the AdS/CFT duality states the equivalence between type IIB string theory on the maximally supersymmetric PP-wave [17] and a specific subsector of $\mathcal{N}=4$ SYM. Since string theory can be quantized on pp-waves (at least on the light cone gauge), this duality is particularly interesting, as it is the only known duality between a gauge theory and a quantizable string theory [4].

On the string theory side the appropriate limit (also called Penrose limit [21]) is obtained by focusing on the geometry seen by a particle moving very fast on an equator of S^5. This can systematically be done by first introducing light-cone coordinates $\tilde{x}^- = t - \phi$, $x^+ = t + \phi$ and then performing the following rescaling:

$$x^- = R^2 \tilde{x}^-, \ \rho = \frac{r}{R}, \ \theta = \frac{y}{R}, \ R \to \infty \tag{2.14}$$

where x^+ is not rescaled and x^-, r and y are kept fixed. In this limit the metric (2.13) becomes

$$ds^2 = -2dx^+ dx^- - (\vec{r}^2 + \vec{y}^2)(dx^+)^2 + d\vec{r}^2 + d\vec{y}^2 \tag{2.15}$$

where $\vec{r}$ and $\vec{y}$ parameterize points on $\mathbb{R}^4 \times \mathbb{R}^4$. It can also be seen that the only components of the RR five-form which survive are those with a plus index. A mass parameter μ can be introduced by rescaling $x^- \to x^-/\mu$ and $x^+ \to \mu x^+$. The energy and angular momentum along the ϕ coordinate also scale in the limit (2.14). In global coordinates these are given by $E = i\partial_t$ and $J = -i\partial_\phi$.

What does this limit correspond to on the gauge theory side? In terms of the dual CFT the energy and angular momentum correspond to the conformal dimension Δ and R-charge J (on a particular direction). It can be seen that:

[4] Some nice reviews on the subject are [18, 19, 20]

$$P^- = i\partial_{x^+} = i(\partial_t + \partial_\phi) = \Delta - J \tag{2.16}$$

$$P^+ = -\frac{\tilde{p}_-}{R^2} = \frac{\Delta + J}{R^2} \tag{2.17}$$

So we need to consider gauge invariant operators with large conformal dimension and large R-charge ($\Delta, J \sim R^2$) while their difference has to be kept finite. Summarizing, the Penrose limit corresponds, on the gauge theory side, to the double scaling limit:

$$\Delta \to \infty,\ J \to \infty,\ N \to \infty \text{ keeping } \frac{J}{\sqrt{N}},\ \Delta - J \text{ fixed.} \tag{2.18}$$

The fact that the spectrum of string on pp-waves can be exactly computed provides us with a (almost) precise dictionary between string states and gauge invariant operators. For instance, for the first levels in $\Delta - J$ we have [5]

$$|0> \leftrightarrow Tr[Z^J] \tag{2.19}$$

$$a_0^{i\dagger}|0> \leftrightarrow Tr[\phi^i Z^J] \tag{2.20}$$

$$a_0^{i\dagger}a_0^{j\dagger}|0> \leftrightarrow \sum_{l=0}^{J} Tr[\phi^i Z^l \phi^j Z^{J-l}] \tag{2.21}$$

$$a_n^{i\dagger}\tilde{a}_n^{j\dagger}|0> \leftrightarrow \sum_{l=0}^{J} Tr[\phi^i Z^l \phi^j Z^{J-l}]e^{\frac{2\pi i l n}{J}} \tag{2.22}$$

Here we have chosen the R-charge in the direction of $Z = \phi^5 + i\phi^6$, and ϕ^i, $i = 1, ..., 4$ is one of the four scalars with 0 R-charge. The choice for the dual of the vacuum is the most natural, since on the gauge theory side we have written the only single trace gauge invariant operator with R-charge J and $\Delta - J = 0$.

For the next two supergravity states, one can easily see that they have the correct quantum numbers and index structure. On the other hand, we know from AdS/CFT that supergravity states map into BPS operators, so we insert the "impurities" ϕ^i in a completely symmetric way.

[5] All our conventions for the string oscillators are collected in appendix A.

As for the string state, Berenstein, Maldacena and Nastase (BMN) proposed that an insertion $a_n^{i\dagger}$ on the string side correspond to an insertion ϕ^i on the gauge theory, with a phase depending on the position at which such operator is inserted[4].

By quantizing the theory (in the light cone gauge) we can compute the spectrum of energies:

$$H_{lc} = \Delta - J = \sum_n N_n \sqrt{1 + \frac{n^2}{(\alpha' \mu p^+)^2}} = \sum_n N_n \sqrt{1 + \frac{4\pi g_{YM} N n^2}{J^2}} \qquad (2.23)$$

N_n being the occupation number of the oscillator a_n or $\tilde{a}_n$.

A method to describe string interactions (splitting and joining of strings), in light-cone gauge, on pp-waves has been developed in [22] and subsequent works. The method consists in introducing an independent Hilbert space for each external state involved in the process and then to describe the interaction by an state $|V_n\rangle$ living in the tensor product of these Hilbert spaces. In this thesis we will focus on three-point functions. In this case $|V_3\rangle$, commonly called the three-string vertex, will encode the three-string interaction in such a way that the coupling among three string states, $|S_i\rangle$, $i = 1, 2, 3$ be given by

$$C_{3,s_i} = (\langle S_1| \otimes \langle S_2| \otimes \langle S_3|)|V_3\rangle \qquad (2.24)$$

2.3 String dual of weak SYM

In spite of its enormous success, the AdS/CFT duality seems to be far from being completely understood. One of the reasons being that it is not known how to quantize string theory on $AdS_5 \times S^5$ in the presence of RR-fluxes (apart from its Penrose limit), so often one is obliged to consider the limit where the supergravity approximation can be trusted. This happens when the curvature radius is large in string units ($R \gg \sqrt{\alpha'}$) which corresponds to the limit of large 't Hooft coupling ($\lambda \gg 1$) on the gauge theory side, actually it is in this limit where the correspondence has been precisely formulated and most of the tests have been done.

The opposite limit, *i.e.* small 't Hooft coupling in the gauge theory and small curvature radius on the AdS_5 [6] is less understood. We will end this chapter by briefly describing some of the approaches in the literature to tackle this problem. [7]

2.3.1 String bits approach

In [25] a way to quantize string theory in the light-cone gauge on $AdS_5 \times S^5$ in the limit of small ($\frac{R}{\sqrt{\alpha'}} \ll 1$) radius was proposed. Actually, it turns out that on this background the usual light-cone gauge choice $x^+ = P^+\tau$ cannot be combined with a conformal gauge on the world-sheet metric (since that is not consistent with the equations of motion) so a slightly different gauge is chosen

$$\sqrt{-g}g^{\tau\tau} = \frac{-1}{\sqrt{-g}g^{\sigma\sigma}} = -\frac{2\pi z^2}{\sqrt{T}} \tag{2.25}$$

In the limit $T = R/\sqrt{\alpha'} = 0$ all the terms containing σ derivatives disappear from the light-cone Hamiltonian, so the string seems to separate into non-interacting bits. Every bit behaves like a superparticle on $AdS_5 \times S^5$, and it contains the full supergravity spectrum. Multi-bit states are then obtained by simply cyclically symmetrizing the tensor product of a given number of single bit states.

One can further consider the first order corrections in $R/\sqrt{\alpha'}$, which introduce interactions among the bits, and compute the corrections to the spectrum. Interestingly, potential divergences cancel for the few examples considered in [25].

The whole string spectrum, however, is not easy to find in this picture, basically due to the fact that it is not known how to take the continuum limit (infinite number of bits) keeping the energy of the string states finite. More precisely, one should find a way to regularize the energy (which otherwise grows with the number of bits) while preserving the superconformal algebra.

On the other hand, if one insists in keeping a finite number of bits then the multi bit spectrum does not coincide with that of free SYM.

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[6] As already mentioned, it is not clear whether free SYM corresponds to the zero radius limit, or rather to some small radius $R \approx \sqrt{\alpha'}$. Since the dictionary (2.8) is derived for large 't Hooft coupling, it is not clear whether it will get corrections in the weak limit.

[7] We will not describe many other interesting approaches, for instance those in relation with tensionless strings, see [23] and references therein and many regarding light-cone string theory, see [24] and references therein.

2.3.2 Free field theories and open-close duality

It is a general belief that the underlying mechanism leading to dualities between gauge and string theories is the open-closed duality. In the prototypical example of AdS/CFT, the duality is the result of the equivalence between the open string (gauge theory) and closed string (geometry) descriptions of the same system.

Nevertheless, the open-closed duality, and its connection to AdS/CFT remains to be fully understood. A step in this direction was taken in [26] and [27] were the open-closed duality was used as a guide in trying to construct systematically a closed string theory starting from a (free) field theory. Since the simplest gauge theory is a free field theory then it seems the natural starting point for such program.

The general idea consists in rewriting a gauge theory correlation function, such as $\rangle \prod_{i=1}^{n} Tr[\phi^{J_i}(k_i)]\langle$ as an string amplitude on AdS. In figure (2.1) we can see a planar contribution to a 5-points correlation function (A). The Schwinger parameters associated to the propagators linking two vertices can be replaced by an "effective" Schwinger parameter [8], hence the diagram glues up into an equivalent "skeleton" diagram (B). It was argued (by counting of moduli) in [27] that the moduli space of the planar skeleton graphs is basically the same as the moduli space of genus zero Riemann surfaces with n holes (C). In other words, the n point (planar) field theory correlation can be written as an integral over the moduli space of a sphere with n holes. As the radius of the holes go to zero, the sphere becomes an sphere with n punctures (D).

This program was carried out for two and three point functions of scalars, whereas the counting was done for any correlation function of scalars, but was conjectured to work also for other fields, and shown to work also for higher genus diagrams. At the moment, however, the picture is too general, and is not clear how the properties of free $\mathcal{N} = 4$ SYM in four dimensions reflect on the string theory.

[8] It is very useful to think of the diagram as an electrical network and of the propagators as resistors, it is well known that a set of parallel resistors can be replaced by a equivalent one.

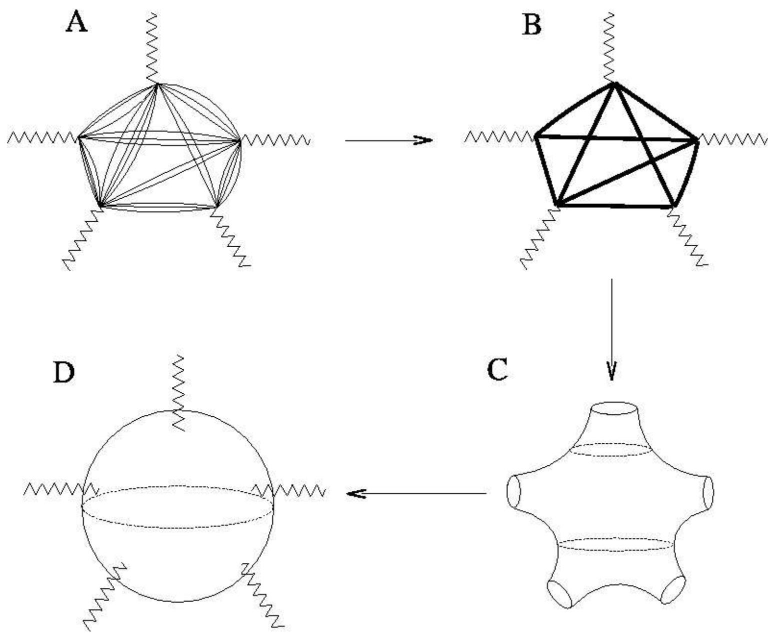

Figure 2.1: Path from a gauge theory correlator to a string theory correlator

2.3.3 Stringy AdS and higher spin symmetry

An important property not exploited by previous approaches is the fact that at the $\lambda = 0$ point $\mathcal{N} = 4$ SYM possesses an infinite set of higher spin symmetries, *i.e.* an infinite set of conserved currents with arbitrary high spin.

In [28] the spectrum of Kaluza-Klein (KK) descendants of string excitations on $AdS_5 \times S^5$ was derived. The basic assumption in such derivation was that the ground floors in the towers of KK descendants were basically given by the flat string excitations, rearranged in AdS representations. In other words to each string excitation in flat space one associateds a tower of KK descendants on S^5. In such procedure the $SO(6) \times SO(4)$ quantum numbers of the spectrum are fixed, but the conformal dimensions remain unknown.

By exploiting higher spin symmetry a simple formula for the dimension of the highest weight state (HWS) of every multiplet was conjectured (in order to satisfy certain unitary bounds)

$$\Delta_0 = 2l + n \tag{2.26}$$

With Δ_0 the conformal dimension at the $\lambda = 0$ point, l the string level and n the KK level, then by supersymmetry the conformal dimension of the other components of the supermultiplet could be fixed. With this assumptions the string spectrum was shown to match that of free $\mathcal{N} = 4$ SYM up to $\Delta_0 = 4$.

In [29] the conformal dimension formula was refined, basically by setting $g_{YM} = 0$ on the plane wave limit formula (2.23), *i.e.* conformal dimensions were assigned in such a way that

$$\Delta - J = \nu \tag{2.27}$$

with $\nu = \sum_n N_n$ the occupation number. Such relation was proposed to hold for the entire massive string spectrum (at the $\lambda = 0$ point), in particular the quantum number J could be small. In order to determine such quantum number we take the massive spectrum of strings in flat space, which assembles into $SO(9)$ representations, and lift it to $SO(10)$, then by breaking $SO(10) \to SO(8) \times SO(2)_J$ we get J.

With such conformal dimensions the spectrum of strings on $AdS_5 \times S^5$ was shown to match that of free $\mathcal{N} = 4$ SYM up to $\Delta = 10!$. As a result, the SYM spectrum can be written as

$$\mathcal{Z}_{SYM} = \mathcal{Z}_{BPS} + \sum_{\Delta} \mathcal{Z}^{\Delta}_{SO(10)} \tag{2.28}$$

In [29] it was also predicted (and verified up to $\Delta = 10$) that, once 1/2-BPS states, dual to supergravity, are subtracted, the SYM spectrum arranges, for a given conformal dimension Δ, into $SO(10)$ representations, for instance, for the lower conformal dimensions, one can see

$$\mathcal{Z}^{2}_{SO(10)} = \mathbf{1} \tag{2.29}$$
$$\mathcal{Z}^{5/2}_{SO(10)} = \mathbf{16} \tag{2.30}$$
$$\mathcal{Z}^{3}_{SO(10)} = \mathbf{10} + \mathbf{120} \tag{2.31}$$
$$\mathcal{Z}^{7/2}_{SO(10)} = \mathbf{16} + \mathbf{144} + \mathbf{560} \tag{2.32}$$

And so on, where bold face denote $SO(10)$ representations.

Chapter 3

D-branes on pp-waves

As already mentioned, D-branes play an important role in string theory, for instance, they are a very powerful tools to understand dualities between string and gauge theories. On the other hand, starting [5, 17, 4] and related work a lot of attention has been paid to string theory on PP-waves and in particular to the presence of D-branes on such backgrounds [30, 31, 32, 33].

In this chapter we describe a supersymmetric solution of type IIB supergravity equations corresponding to fully localized Dp-branes in presence of a pp-wave background by explicitly solving the equations of motion and the supersymmetry conditions. further, such results are generalized to the D1-D5-brane system [1].

In the next section we state our ansatz for a D3-brane on a pp-wave, using as guide D-brane solutions on flat space. In section 3.2 we show that our solution is supersymmetric and in section 3.3 we write down and solve the equations of motion. Finally we generalize these results to other Dp-branes (with p odd) and to the D1-D5 system. We end by summarizing the results obtained in this chapter.

3.1 Putting D-branes on the pp-wave

Let us briefly review the case of a Dp-brane in flat space-time. Due to the presence of the Dp-brane, the ten dimensional flat metric:

$$ds^2 = \eta_{MN} dx^M dx^N \tag{3.1}$$

[1]This chapter is based on [34]

is modified (in the string frame) according to:

$$ds_s^2 = H_p^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu - H_p^{1/2}dx^i dx^i \tag{3.2}$$

where $\mu = 0,\dots,p$ runs over the coordinates on the Dp-brane and $i = p+1,\dots,9$ on the transverse coordinates and H_p is a harmonic function of the transverse coordinates. In order to be consistent with the equations of motion we are obliged to turn on a $(p+2)$ field strength (whose potential couples to the world volume of the Dp-brane):

$$F_{[p+2]} = g_s^{-1}dx^0 \wedge \cdots \wedge dx^p \wedge d(H_p^{-1}) \tag{3.3}$$

(if F is a 5 form we should add also its dual). Further, the standard relation between the dilaton and the string coupling constant ($e^{2\phi} = {g_s}^2$) is modified according to:

$$e^{2\phi} = g_s^2 H_p^{\frac{3-p}{2}} \tag{3.4}$$

Note that for the special case of a D3-brane the dilaton is constant, and we can set it equal to zero. From now on we will set $g_s = 1$.

To pass to the so called Einstein frame, we perform the following rescaling:

$$g_{\mu\nu,s} = e^{\phi/2}g_{\mu\nu,e} \tag{3.5}$$

For a generic p, the metric in this frame takes the form:

$$ds_e^2 = H_p^{\frac{p-7}{8}}\eta_{\mu\nu}dx^\mu dx^\nu - H_p^{\frac{p+1}{8}}dx^i dx^i \tag{3.6}$$

Note that for $p = 3$ the metric is the same in both frames.

The standard pp-wave background reads (written in Brinkman coordinates):

$$\begin{aligned} ds^2 &= 2dudv + 2S(u,x^i)dudu - dx^i dx^i \\ F_{[5]} &= du \wedge \varphi_{[4]}(x^i) \end{aligned} \tag{3.7}$$

Where the light–cone coordinates have been introduced according to $u = (x^0 + x^9)/2$ and $v = (x^0 - x^9)/2$, and $i = 1,\dots,8$. Here $\varphi_{[4]}(x^i)$ is a four form such that the five form is self dual. This kind of backgrounds was studied previously for instance in [36] [37], where the problem of supersymmetry has been addressed.

Many people have studied the problem of putting D-branes in backgrounds of this kind [30, 31, 32, 33]. We will focus on D3-branes and consider the following pp-wave background instead of the usual one:

$$
\begin{aligned}
ds^2 &= \left(2du\,(dv + S(\vec{y})du) - d\vec{x}^{\,2}\right) - d\vec{y}^{\,2} \\
F_{[5]B} &= \frac{1}{\sqrt{2}}(\varphi_{[3]}) \wedge dz^4 \wedge du + c.c. \\
\varphi_{[3]} &= W_1 d\bar{z}^1 \wedge dz^2 \wedge dz^3 + W_2 dz^1 \wedge d\bar{z}^2 \wedge dz^3 + W_3 dz^1 \wedge dz^2 \wedge d\bar{z}^3
\end{aligned}
\tag{3.8}
$$

where now $\vec{x}$ (z^4 in complex notation) denotes points on $\mathbb{R}^2$ and $\vec{y}$ ($z^{1,2,3}$ in complex notation) points on $\mathbb{R}^6$ and try to mimic the procedure previously explained for the flat case. Note that S is only function of the $\vec{y}$ coordinates.

Inspired by the previous discussion it is natural to propose the following ansatz for a D3-brane in the background (3.8):

$$
\begin{aligned}
ds^2 &= H(\vec{y})^{-\frac{1}{2}} \left(2du\,(dv + S(\vec{y})\,du) - d\vec{x}^{\,2}\right) - H(\vec{y})^{\frac{1}{2}}\, d\vec{y}^{\,2} \\
F_{[5]D} &= \frac{1}{\sqrt{2}} du \wedge dv \wedge dx^1 \wedge dx^2 \wedge dH^{-1} \ + dual \\
F_{[5]B} &= \frac{1}{\sqrt{2}}(\varphi_{[3]}) \wedge dz^4 \wedge du + c.c. \\
\varphi_{[3]} &= W_1 d\bar{z}^1 \wedge dz^2 \wedge dz^3 + W_2 dz^1 \wedge d\bar{z}^2 \wedge dz^3 + W_3 dz^1 \wedge dz^2 \wedge d\bar{z}^3
\end{aligned}
\tag{3.9}
$$

Consistently with (3.4) we are assuming constant dilaton. Note that we added the dual of the D-brane 5 form, as before. The W's have to be chosen in such a way that the five-form is closed. Here we are placing the D3-branes along the coordinates u, v and $\vec{x}$ (or, in complex notation, in the 4^{th} complex plane) corresponding to a *longitudinal* D-brane [30]. Of course there are other possibilities, namely *transverse* (that is with the u coordinate belonging to the brane world volume but not v) or *instantonic* (neither u nor v belonging to the brane world volume) D-branes [32] [39]. Since in our case with this orientation the worldsheet scalars coming from the pp–wave are transverse to the D3-brane this choice seems to be the natural one.

The next sections are devoted to the study of this ansatz, first looking at the conditions for the preservation of some supersymmetry generators and then solving the equations of motion.

3.2 Supersymmetry conditions

In this section we will analyze the supersymmetry conditions for the background (3.9). The supersymmetries are obtained by equating the variations of the gravitino and the dilatino to zero and looking for non-trivial spinors (ϵ) satisfying these restrictions. The generic supersymmetry variations for the dilatino and gravitino in the "doubled" formulation of supergravity, wherein both electric and magnetic RR fields, except for $C_{[0]}$, are used, are [30]:

$$\begin{aligned} \delta\Psi_\mu &= D_\mu\epsilon + \frac{1}{16}e^\phi\left(2\partial\!\!\!/ C_{[0]}\left(i\sigma^2\right) + \frac{1}{3!}F\!\!\!/_{[3]}\left(\sigma^1\right) + \frac{1}{5!}F\!\!\!/_{[5]}\left(i\sigma^2\right) + \frac{1}{7!}F\!\!\!/_{[7]}\left(\sigma^1\right)\right)\gamma_\mu\epsilon \\ \delta\chi &= \partial\!\!\!/\phi\epsilon + \frac{1}{4}e^\phi\left(-4\partial\!\!\!/ C_{[0]}\left(i\sigma^2\right) - \frac{1}{3!}F\!\!\!/_{[3]}\left(\sigma^1\right) + \frac{1}{7!}F\!\!\!/_{[7]}\left(\sigma^1\right)\right)\epsilon \end{aligned} \tag{3.10}$$

With F we denote the RR forms of type IIB string theory, $C_{[0]}$ is the axion and ϕ the dilaton. We define $F\!\!\!/_{[n]} = F_{\alpha_1\ldots\alpha_n}\Gamma^{\alpha_1\cdots\alpha_n}$. The covariant derivative is defined as:

$$D_\mu = \partial_\mu - \frac{1}{4}\omega_{\mu\underline{ab}}\Gamma^{\underline{ab}} \tag{3.11}$$

and the spin connection ω can be expressed in terms of the vielbein through:

$$e_\nu{}^{\underline{m}}e_\rho{}^{\underline{n}}\omega_{\mu\underline{mn}} = \frac{1}{2}[e_{\rho\underline{p}}(\partial_\mu e_\nu{}^{\underline{p}} - \partial_\nu e_\mu{}^{\underline{p}}) - e_{\mu\underline{p}}(\partial_\nu e_\rho{}^{\underline{p}} - \partial_\rho e_\nu{}^{\underline{p}}) + e_{\nu\underline{p}}(\partial_\rho e_\mu{}^{\underline{p}} - \partial_\mu e_\rho{}^{\underline{p}})] \tag{3.12}$$

With μ, ν, etc we label space-time coordinates and with $\underline{a}, \underline{b}, etc$ coordinates on the tangent space. In the following, a will refer to coordinates transverse to the D-brane whereas i to coordinates longitudinal to the D-brane. The components of the vielbein are chosen as: $(e^{\underline{m}} = e_\mu{}^{\underline{m}}dx^\mu)$

$$\begin{aligned} &e_{\underline{v}} = e^{\underline{u}} = H^{-\frac{1}{4}}du\,, \quad e_{\underline{u}} = e^{\underline{v}} = H^{-\frac{1}{4}}(dv + Sdu)\,, \\ &-e_{\underline{i}} = e^{\underline{i}} = H^{-\frac{1}{4}}dx^i, \quad -e_{\underline{a}} = e^{\underline{a}} = H^{\frac{1}{4}}dy^a \end{aligned} \tag{3.13}$$

and those of the inverse vielbeins as: $(\theta_{\underline{m}} = e_{\underline{m}}{}^\mu\partial_\mu)$

$$\begin{aligned} &\theta_{\underline{u}} = H^{\frac{1}{4}}(\partial_u - S\partial_v)\,, \quad \theta_{\underline{v}} = H^{\frac{1}{4}}\partial_v\,, \\ &\theta_{\underline{i}} = H^{\frac{1}{4}}\partial_i, \qquad\qquad\quad \theta_{\underline{a}} = H^{-\frac{1}{4}}\partial_a \end{aligned} \tag{3.14}$$

and we use flat light cone metric $\eta^{\underline{uv}} = 1$, $\eta^{\underline{ij}} = -\delta_{ij}$ and $\eta^{\underline{ab}} = -\delta_{ab}$. The non-zero components of the spin connection are:

$$
\begin{aligned}
&\omega_{\underline{uua}} = +H^{-\frac{1}{4}}\partial_a S\,, \ \ \omega_{\underline{uva}} = -\tfrac{1}{4}H^{-\frac{5}{4}}\partial_a H\ , \\
&\omega_{\underline{vua}} = -\tfrac{1}{4}H^{-\frac{5}{4}}\partial_a H\,, \ \ \omega_{\underline{iaj}} = -\tfrac{1}{4}H^{-\frac{5}{4}}\partial_a H \delta_{ij}\,, \\
&\omega_{\underline{abc}} = -\tfrac{1}{2}H^{-\frac{5}{4}}\eta_{a[b}\partial_{c]}H
\end{aligned}
\tag{3.15}
$$

For the background (3.9) the dilatino variation is automatically zero, whereas the gravitino variation becomes:

$$
\delta\Psi_\mu = D_\mu\epsilon + \frac{1}{16}\left(\frac{1}{5!}\not{F}_{[5]}\left(i\sigma^2\right)\right)\gamma_\mu\epsilon \tag{3.16}
$$

Equating this variation to zero we obtain one equation for every direction and for each spinor component, the solutions of whose are the killing spinors. The resulting equations read:

$$
\begin{aligned}
\partial_a\epsilon + \frac{1}{8}H^{-1}(\partial_a H)\epsilon + (H^{-1/4}\mathcal{W}_{[5]}(i\sigma^2))H^{1/4}\gamma_{\underline{a}}\epsilon &= 0 && (3.17a)\\
\partial_v\epsilon &= 0 && (3.17b)\\
\partial_i\epsilon + (H^{-1/4}\mathcal{W}_{[5]}(i\sigma^2))H^{-1/4}\gamma_{\underline{i}}\epsilon &= 0 && (3.17c)\\
\partial_u\epsilon + (H^{-1/4}\mathcal{W}_{[5]}(i\sigma^2))H^{-1/4}\gamma^{\underline{v}}\epsilon - \frac{1}{2}H^{-1/2}\partial_a S\Gamma^{\underline{ua}}\epsilon &= 0 && (3.17d)
\end{aligned}
$$

With $\mathcal{W}_{[5]}$ we denote the contribution coming from the background 5-form where the H dependence has been explicitly shown. In order to obtain these equations we also assumed the standard chirality condition in the D3-brane world volume:

$$
\Gamma_{wv}(i\sigma^2)\epsilon = \epsilon \tag{3.18}
$$

where σ^2 acts on ϵ as a doublet of 16 components complex spinor. By rescaling the spinor with a factor of $H^{-1/8}$ equation (3.17a) becomes [2]:

$$
\partial_a\epsilon + (H^{-1/4}\mathcal{W}_{[5]}(i\sigma^2))H^{1/4}\gamma_{\underline{a}}\epsilon = 0 \tag{3.19}
$$

whereas, since H only depends on $\vec{y}$, the other equations remain untouched.

[2] This kind of rescaling is standard in the flat space D-brane solution.

A solution can be easily found by considering constant spinors. Let us denote them by $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$, where every sign corresponds to the chirality with respect to the corresponding complex plane (or to the u–v plane). We choose conventions such that $\gamma^{\underline{u}}(+) = 0$, $\gamma^{\underline{v}}(-) = 0$, $\Gamma^i(+) = 0$ and $\Gamma^{\bar{i}}(-) = 0$, where $\Gamma^i = \Gamma^{\bar{i}^*} = \gamma_i^x + i\gamma_i^y$ are the complex γ matrices for the complex plane i.

The condition of negative space-time chirality implies the choice of the spinor with an odd number of minus sign components [3].

Remember that equations (3.17) were obtained imposing a definite chirality in the world volume of the D-brane. This means an even number of minuses in the two planes on which the D3-brane lies (in our case, the first and the last planes) in the case of positive chirality and an odd number in the case of negative.

A trivial solution can be found by noticing that $\gamma^{\underline{u}}$ annihilates the spinors with a plus in the first position, and $\mathcal{W}_{[5]}$ annihilates the spinors of the form $(\pm\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, \pm\frac{1}{2})$ and $(\pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2})$, since a definite sign (plus or minus) is killed by either Γ^i or $\Gamma^{\bar{i}}$. So, we see that the spinor $(+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2})$ is a solution for positive world-volume chirality, and $(+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2})$ is so for negative world volume chirality.

However this is not the only solution. We can find another solution assuming that $\partial_i\epsilon = 0$ and $\partial_u\epsilon = 0$ so that equations (3.17) become:

$$\partial_a\epsilon + \mathcal{W}_{[5]}(i\sigma^2)\gamma_{\underline{a}}\epsilon = 0 \tag{3.20a}$$

$$\partial_v\epsilon = 0 \tag{3.20b}$$

$$\mathcal{W}_{[5]}(i\sigma^2)\gamma_{\underline{i}}\epsilon = 0 \tag{3.20c}$$

$$\mathcal{W}_{[5]}(i\sigma^2)\gamma^{\underline{v}}\epsilon - \frac{1}{2}\partial_a S\Gamma^{\underline{ua}}\epsilon = 0 \tag{3.20d}$$

It is remarkable that the function H has disappeared from (3.17); thank to this we obtain exactly the equations of [36] [4], with the further restriction of definite chirality on the D-brane worldvolume [5] and the condition that the spinor depends only on the transverse coordinates to the D-brane.

[3] In our conventions $\Gamma^{11} = \Gamma^{uv12345678} = \Gamma^{uv1\bar{1}2\bar{2}3\bar{3}4\bar{4}}$

[4] Equations (3.20) should be compared with (2.5) of [36] taking into account the different conventions, for instance, ϵ there is a 16 components complex spinor.

[5] As it can be checked this condition is consistent, since the chirality matrix leaves equations (3.20) invariant.

In particular our background will be a special case of the background (2.9) of [36]

$$\begin{aligned} ds^2 &= -2dudv - 32(|\partial_k \mathtt{V}|^2 + |\phi_{j\bar{k}} z^j|^2)(du)^2 + dz^i d\bar{z}^i \\ \phi_{mn} &= \partial_m \partial_n \mathtt{V}, \quad \phi_{\overline{mn}} = \partial_{\bar{m}} \partial_{\bar{n}} \overline{\mathtt{V}}, \quad \phi_{l\bar{m}} = constants \end{aligned} \tag{3.21}$$

where ϕ_{mn} and $\phi_{\overline{mn}}$ are defined in terms of $\varphi_{[4]}$ as:

$$\begin{aligned} \phi_{mn} &= \tfrac{1}{3!}(\varphi_{[4]})_{m\overline{ijk}} \epsilon^{\overline{ijkn}} g_{n\bar{n}} \\ \phi_{\overline{mn}} &= \tfrac{1}{3!}(\varphi_{[4]})_{\overline{m}ijk} \epsilon^{ijkn} g_{n\bar{n}} \end{aligned} \tag{3.22}$$

Since in our background we don't have forms of the kind (2,2) (two holomorphic and two antiholomorphic indices) $\phi_{m\bar{n}}$ are zero. Here $\mathtt{V}$ is a generic holomorphic function.

In [36] were considered spinors with ϵ_- [6]:

$$\epsilon_- = \alpha(-\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}) + \zeta(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \tag{3.23}$$

where α and ζ are complex numbers. Then, for a given $\mathtt{V}$, ϵ_+ is solved as a function of ϵ_-.

With the additional restriction of definite chirality with respect to the world-volume of the D3-brane, we have to choose one and only one of these spinors.

Finally we have (at least) 2 complex spinors (for a definite world-volume chirality), this means 1/8 of supersymmetries.

3.3 Equations of motion

For the ansatz (3.9) Einstein equations read:

$$R_{\mu\nu} = \frac{1}{2} T^{[5]}_{\mu\nu} \tag{3.24}$$

where $T^{[5]}$ is the energy momentum tensor for the RR 5-form (the only different from zero for our background) defined by:

$$T^{[5]}_{\mu\nu} = \frac{1}{4!}(F_{[5]\mu\alpha 1 \ldots \alpha 4} F_{[5]\nu}^{\alpha 1 \ldots \alpha 4} - \frac{1}{10} g_{\mu\nu}(F_{[5]\alpha 1 \ldots \alpha 5} F_{[5]}^{\alpha 1 \ldots \alpha 5})) \tag{3.25}$$

[6] Note that we use opposite convention for ϵ_+ and ϵ_-.

For our background we obtain:

$$
\begin{aligned}
T_{uu} &= \frac{1}{H}(|W_1|^2+|W_2|^2+|W_3|^2)+\frac{S}{H^3}(\partial_a H\partial_a H) && (3.26a)\\
T_{uv} &= \frac{1}{2}\left(\frac{\partial_a H\partial_a H}{H^3}\right) && (3.26b)\\
T_{vv} &= 0 && (3.26c)\\
T_{ij} &= -\delta_{ij}\frac{1}{2}\left(\frac{\partial_a H\partial_a H}{H^3}\right) && (3.26d)\\
T_{ab} &= -\left(\frac{\partial_a H\partial_b H}{H^2}\right)+\delta_{ab}\frac{1}{2}\left(\frac{\partial_c H\partial_c H}{H^2}\right) && (3.26e)
\end{aligned}
$$

with the rest of the components being zero. Whereas the components of the Ricci tensor for the metric (3.9) are:

$$
\begin{aligned}
R_{vv} &= 0 && (3.27a)\\
R_{uu} &= \frac{1}{2H^3}(S\,(\partial_a H\,\partial_a H-H\,(\partial_a\partial_a H))+2H^2(\partial_a\partial_a S)) && (3.27b)\\
R_{uv} &= \frac{1}{4H^3}(\partial_a H\,\partial_a H-H(\partial_a\partial_a H)) && (3.27c)\\
R_{ab} &= \delta_{ab}\frac{1}{4H^2}(\partial_c H\,\partial_c H-H(\partial_a\partial_a H))-\frac{1}{2H^2}(\partial_a H\,\partial_b H) && (3.27d)\\
R_{ij} &= -\delta_{ij}\frac{1}{4H^3}(\partial_a H\,\partial_a H-H({\partial_a}^2 H)) && (3.27e)
\end{aligned}
$$

In (3.26) and (3.27) the contractions are understood in the euclidean metric.

Finally, from the equations of motion we obtain the following conditions[7]:

$$
\begin{aligned}
\partial_a\partial_a H &= 0 && (3.28a)\\
\partial_a\partial_a S &= \frac{1}{2}(|W_1|^2+|W_2|^2+|W_3|^2) && (3.28b)
\end{aligned}
$$

Condition (3.28a) just says that H is harmonic in the 6 transverse directions; this fact together with the correct asymptotic behavior of H (far from the D3-brane the space time should look like the standard pp-wave) gives:

[7]Similar equations were found, for instance, in [38]

$$H = 1 + \frac{Q}{y^4} \tag{3.29}$$

with $y^2 = (y^1)^2 + \cdots + (y^6)^2$ and Q a non negative real number.

As an important difference with [30] we stress that this solution depends on all the transverse coordinates (and is also supersymmetric). This difference is basically due to our choice for the five form.

The remaining equation, (3.28b), is easily solved, for instance for a quadratic S (corresponding to mass terms in the pp-wave) and constant W's, giving just a relation among them. This equation admits, however, more general solutions (of course with the restriction on the W's coming from the closure of the five form field strength).

In the next section we will see how part of these results can be generalized to other kinds of Dp-branes.

3.4 Generalization to other Dp-branes

In this section we will extend the previous results to other Dp-branes.

First we have to take into account that for other Dp-branes the dilaton is not any more constant. It seems natural to suppose that relation (3.4) holds also for this kind of background. We will also suppose that the pp-wave background is supported by the following $(p+2)$ form:

$$F_{[p+2]} = du \wedge dv \wedge dx^1 \wedge \cdots \wedge dx^{p-1} \wedge d(H_p^{-1}) \tag{3.30}$$

Now the dilatino supersymmetry variation (3.10) becomes [30]:

$$\delta\chi = (\partial_a H \gamma^{\underline{a}})\,(1 - \Gamma_{wv}\mathcal{P})\,\epsilon = 0 \tag{3.31}$$

where Γ_{wv} is the chirality matrix on the world-volume of the Dp-brane we are considering and $\mathcal{P}$, as can be read off from (3.10), is a projector given by σ^1 or $i\sigma^2$ depending on p (recall that this projector acts on ϵ as a doublet of 16 components complex spinor).

So we see that requiring the standard chirality condition with respect to the world-volume of the Dp-brane :

$$\Gamma_{wv}\mathcal{P}\epsilon = \epsilon \tag{3.32}$$

the above equation is satisfied.

Now we will discuss the issue of the supersymmetric variation of the gravitino and the equations of motion. The background to consider for a generic Dp-brane is (in the string frame):

$$\begin{aligned}
ds^2 &= H\left(\vec{y}\right)^{-\frac{1}{2}}\left(2du\left(dv+S\left(\vec{y}\right)du\right)-d\vec{x}^{\,2}\right)-H\left(\vec{y}\right)^{\frac{1}{2}}d\vec{y}^{\,2} && \text{(3.33a)}\\
F_{[p+2]D} &= du\wedge dv\wedge dx^1\cdots\wedge dx^{p-1}\wedge dH^{-1} && \text{(3.33b)}\\
F_{[5]B} &= \frac{1}{\sqrt{2}}(\varphi_{[4]})\wedge du+c.c. && \text{(3.33c)}\\
\varphi_{[4]} &= W_1 d\bar{z}^1\wedge dz^2\wedge dz^3\wedge dz^4+W_2 dz^1\wedge d\bar{z}^2\wedge dz^3\wedge dz^4 && \text{(3.33d)}\\
&\quad +W_3 dz^1\wedge dz^2\wedge d\bar{z}^3\wedge dz^4+W_4 dz^1\wedge dz^2\wedge dz^3\wedge d\bar{z}^4
\end{aligned}$$

(plus the usual relation between the dilaton and H) where now $\vec{y}$ denotes points in $\mathbb{R}^{9-p}$ and $\vec{x}$ denotes points in $\mathbb{R}^{p-1}$. For a Dp-brane, with $k=\frac{9-p}{2}$ only $W_1\ldots W_k$ will be different from zero.

For the supersymmetry variation of the gravitino we obtain:

$$\begin{aligned}
\partial_a\epsilon+\frac{1}{8}H^{-1}(\partial_a H)\epsilon+(H^{-\frac{1}{4}}\mathcal{W}_{[5]}(i\sigma^2))H^{1/4}\gamma_{\underline{a}}\epsilon &= 0 && \text{(3.34a)}\\
\partial_v\epsilon &= 0 && \text{(3.34b)}\\
\partial_i\epsilon+(H^{-\frac{1}{4}}\mathcal{W}_{[5]}(i\sigma^2))H^{-1/4}\gamma_{\underline{i}}\epsilon &= 0 && \text{(3.34c)}\\
\partial_u\epsilon+(H^{-\frac{1}{4}}\mathcal{W}_{[5]}(i\sigma^2))H^{-1/4}\gamma^{\underline{v}}\epsilon-\frac{1}{2}H^{-1/2}\partial_a S\Gamma^{\underline{ua}}\epsilon &= 0 && \text{(3.34d)}
\end{aligned}$$

These equations are exactly the ones obtained in section 3. There, we solved them for non constant spinors referring to the techniques developed in [36] and imposing definite chirality in the world volume of the D-brane. In order to redo this analysis for a Dp-brane, we must first check that the condition of definite chirality on the D-brane can be imposed consistently. Unfortunately, this is not the case for D1 and D5 branes, since in these cases equations (3.34) are not invariant under the action of the chirality matrix Γ_{wv}: because of this we will only be able to find constant solutions. Let us study these cases separately:

D7: one can easily check that if positive chirality is imposed on the world volume of the brane, the constant spinors $(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$, $(+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$ and $(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$ are solutions. On the other hand, for negative chirality, the solutions are $(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$, $(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$ and

$(+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$. In both cases, we should add (only) one of the spinors (3.23). This means that for definite chirality, we have four complex spinors satisfying equations (3.34), or, in other words, 1/4 of the supersymmetries are preserved.

D5: as stated before, for a D5 brane, we could only find constant solutions of (3.34). If we choose negative world volume chirality, they are $(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$, $(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$, $(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2})$ and $(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2})$ meaning that only 1/4 supersymmetries are preserved. Apparently for positive chirality the D5 brane breaks all the supersymmetries.

D1: here, one can check that no u-independent Killing spinors are allowed, irrespectively of the world volume chirality (unless some W_i is set equal to zero). The D1 brane seems not supersymmetric in our background.

In order to study the equations of motion we remark that S appears only in the uu direction, so that for our ansatz the other equations are automatically satisfied, therefore we will focus only on the equation of motion for the uu direction.

In order to analyze the equations of motion we find it more convenient to work in the Einstein frame, where the metric is:

$$ds^2 = H(\vec{y})^{\frac{p-7}{8}}\left(2du\,(dv + S(\vec{y})\,du) - d\vec{x}^2\right) - H(\vec{y})^{\frac{p+1}{8}}\,d\vec{y}^2 \tag{3.35}$$

and the rest of the fields are identical to (3.33). In this frame the equations of motion are [40]:

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + S_{\mu\nu} \tag{3.36}$$

with

$$S_{\mu\nu} = \sum_p \frac{1}{2\,(p+1)!}e^{\frac{(3-p)\phi}{2}}\left(F_{[p+2]\mu\alpha_1\ldots\alpha_{p+1}}F_{[p+2]\nu}^{\alpha_1\ldots\alpha_{p+1}} - \frac{p+1}{8\,(p+2)}g_{\mu\nu}F^2_{[p+2]}\right) \tag{3.37}$$

with

$$F^2_{[p+2]} = F_{[p+2]\alpha_1\ldots\alpha_{p+2}}F_{[p+2]}^{\alpha_1\ldots\alpha_{p+2}} \tag{3.38}$$

The uu component of the Ricci tensor for generic p is:

$$R_{uu} = \frac{\partial_a \partial_a S}{H} - \frac{p-7}{8} S \frac{\partial_a H\, \partial_a H}{H^3} + \frac{p-7}{8} S \frac{\partial_a \partial_a H}{H^2} \tag{3.39}$$

From this we obtain the conditions:

$$\partial_a \partial_a H = 0 \tag{3.40a}$$

$$\partial_a \partial_a S = \frac{1}{2}(|W_1|^2 + |W_2|^2 + |W_3|^2 + |W_4|^2) \tag{3.40b}$$

As before, the first condition says that H is harmonic (in the transverse $(9-p)$–dimensional space) while the second simply states a differential relation between S and the W's. Taking into account the correct asymptotic behavior for H, one must have:

$$H = 1 + \frac{Q_p}{r^{(7-p)}} \tag{3.41}$$

for p different from 7, and:

$$H = 1 + Q_7 \ln r \tag{3.42}$$

for $p = 7$, r being the radius of the transverse space.

For the case of the D5-brane and for constant W's (when S is quadratic) our background is similar to those studied in [41] in the context of $AdS_3 \times S^3 \times \mathbb{R}^4$ and its Penrose limits. Note, however, that we are considering more general backgrounds.

We stress the fact that S doesn't have to be function of all the transverse coordinates, that is, we don't have to give mass to all the transverse scalars. So for a Dp-brane solution with a given S, there will be also a Dp'-brane solution with $p' < p$.

3.5 The D1/D5 system

In this section we extend the previous analysis to the D1/D5 system. This system plays an important role in the AdS/CFT duality since its near horizon limit is of the form $AdS_3 \times S^3 \times M$ (see, for instance [12]); its Penrose limit was considered for instance in [43]. First we solve the equations of motion (in the Einstein frame) and then we briefly discuss the issue of supersymmetry.

We propose the following ansatz:

$$\begin{aligned}
ds^2 &= H_1^{-\frac{3}{4}} H_5^{-\frac{1}{4}} (2du(dv + S(\vec{y})du)) - H_1^{\frac{1}{4}} H_5^{-\frac{1}{4}} d\vec{x}^{\,2} - H_1^{\frac{1}{4}} H_5^{\frac{3}{4}} d\vec{y}^{\,2} \\
F_{[3]D} &= du \wedge dv \wedge dH_1^{-1} \\
F_{[7]D} &= du \wedge dv \wedge dx^1 \cdots \wedge dx^4 \wedge dH_5^{-1} \\
F_{[5]B} &= \frac{1}{\sqrt{2}} (\varphi_{[4]}) \wedge du + c.c. \qquad (3.43) \\
\varphi_{[4]} &= W_1 d\bar{z}^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 + W_2 dz^1 \wedge d\bar{z}^2 \wedge dz^3 \wedge dz^4 \\
e^{2\phi} &= H_1 H_5^{-1}
\end{aligned}$$

Here $\vec{x}$ and $\vec{y}$ denote points of $\mathbb{R}^4$. H_1 and H_5 are only function of the coordinates $\vec{y}$ transverse to the D5-brane. As can be seen from the metric, we place the D1-brane on the u, v coordinates and the D5-brane on u, v and $\vec{x}$ so that the D1-brane lies inside the D5-brane.

As before, the only non trivial equation of motion is the one in the uu direction. From the metric (3.43) we have the following Ricci tensor:

$$R_{uu} = \frac{3}{4} \frac{S \partial_a H_1 \partial_a H_1}{H_5 H_1^3} - \frac{3}{4} \frac{S \partial_a \partial_a H_1}{H_5 H_1^2} + \frac{1}{4} \frac{S \partial_a H_5 \partial_a H_5}{H_1 H_5^3} - \frac{1}{4} \frac{S \partial_a \partial_a H_5}{H_5^2 H_1} + \frac{\partial_a \partial_a S}{H_1 H_5} \qquad (3.44)$$

From the equations of motion (4.17) we obtain the following conditions:

$$\begin{aligned}
\partial_a \partial_a H_1 &= 0 \qquad (3.45a) \\
\partial_a \partial_a H_5 &= 0 \qquad (3.45b) \\
\partial_a \partial_a S &= \frac{1}{2} (|W_1|^2 + |W_2|^2) \qquad (3.45c)
\end{aligned}$$

So, H_1 and H_5 must be harmonic in the transverse directions and the usual relation between S and W_i must be fulfilled.

One can check that the supersymmetry conditions reduce to (3.17) provided H is replaced by the product $H_1 H_5$, except in the i directions (i.e.the ones longitudinal to the D5-brane but transverse to the D1-brane), where the condition reads:

$$\partial_i \epsilon + H_5^{-1/2} W_5 (i\sigma_2) \gamma_{\underline{i}} \epsilon = 0 \qquad (3.46)$$

We stress that such equations were obtained assuming the spinor had definite chirality with respect to the worldvolume of both D-branes. As in the case of the D1-brane, apparently such conditions are too restrictive to allow for any supersymmetry.

3.6 Summary and conclusions

We have found a solution describing a fully localized D3-brane on a pp-wave background. Such solution turns out to be supersymmetric, and it preserves, apparently 1/8 supersymmetries. We have also shown how to generalize our results to the case of other Dp-branes. Our conclusions are summarized in table (3.6). Finally we have studied the D1/D5-brane system, which turns out to be non supersymmetric.

	D1	D3	D5	D7
$\Gamma_{wv}\epsilon = \epsilon$	0 susy	1/8 susy	0 susy	1/4 susy
$\Gamma_{wv}\epsilon = -\epsilon$	0 susy	1/8 susy	1/4 susy	1/4 susy

Table 3.1: Summary of the results for Dp-branes.

As a possible further development, it could be interesting to study the gauge theories living on the worldvolume of such D-branes, or to study the backgrounds obtained from ours after some dualities. In this way other Dp and Dp/Dp' solutions are expected.

Chapter 4

Non local charges on AdS and PP-waves.

In spite of the high amount of symmetry of $AdS_5 \times S^5$, due to the presence of RR fluxes, it is not known how to quantize string theory in this background, however, one might hope that the theory could be exactly solvable. In [9, 10] it was shown that the world-sheet CFT contains, at the classical level, an infinite set of non local charges of the type arising in integrable models [1]. On the other hand there has been lot of progress in the understanding of integrability on the gauge theory side, since [6] and subsequent works, and one expects that both lines of development will play a fundamental role in fully understanding the AdS/CFT duality.

The aim of this chapter is to study the structure of the infinite set of non-local conserved charges found in [10] for the sigma-model describing strings on pp-waves. [2]

In the first two sections we review the construction of non-local charges for the Green-Schwarz superstring on $AdS_5 \times S^5$. We recall the results found in [10] that will be useful in the following and then show that the Green-Schwarz superstring on pp-waves possesses charges with the same structure. In section 4.3 we show how to write the explicit form of these charges in the light cone gauge and we do it for the first non trivial orders. In section 4.4 we write the explicit form for the first non trivial charges for $AdS_5 \times S^5$

[1] In [44] this set of charges was shown to exist also in the pure spinor formalism of strings in $AdS_5 \times S^5$ [45]. Further it was shown that such charges are BRST invariant in the pure spinor formalism and κ-symmetric in the Green-Schwarz formulation [46].

[2] This chapter is based on [47]

and show that its Penrose limit coincides with the charges previously found. We also check that the semi classical value of the AdS charge for a rotating string on S^5 (dual of a BMN state) coincides with that of the pp-wave charge when applied to the same BMN state. Finally we end with a summary and some conclusions.

4.1 Flat connections in coset sigma models

Consider the non linear sigma model with Lagrangian $L \sim Tr(\partial_i g^{-1} \partial^i g)$, where the field $g(x)$ takes values in the group G. The global symmetry is $G \times G$, left and right multiplication. In the following we will focus on the conserved current corresponding to left multiplication:

$$j_i = -(\partial_i g) g^{-1} \tag{4.1}$$

This current takes values in the Lie algebra $\mathcal{G}$. Writing the current as a one-form one sees that

$$d * j = 0, \qquad dj + j \wedge j = 0 \tag{4.2}$$

Thus the current can be regarded as a flat gauge connection. Actually taking the following linear combination

$$a = j\frac{1}{2}(1 \pm \cosh\lambda) + *j\frac{1}{2}\sinh\lambda \tag{4.3}$$

One can easily check that $da + a \wedge a = 0$, so one obtains two one-parameter families of flat connections. As we will see in the next section, this allows for the construction of an infinite number of conserved charges.

Now let us consider G/H coset models, where we identify $g(x) \equiv g(x)h(x)$. Left multiplication by G is still a global symmetry. To construct the action define

$$J = g^{-1} j g = -g^{-1} \partial g \tag{4.4}$$

This is invariant under left multiplication. Further decompose J according to the decomposition of the Lie algebra $\mathcal{G} = \mathcal{H} \oplus \mathcal{K}$:

$$J = H + K \tag{4.5}$$

Then H transforms as a connection under $\mathcal{H}$-gauge transformation whereas K transform covariantly. It is easy to see that $k = gKg^{-1}$ is gauge invariant, the Lagrangian is then $L \sim Tr(k_i k^i) = Tr(K_i K^i)$.

We will use capital letters X to denote currents that are conjugated by right multiplication, generally corresponding to some decomposition under representations of $\mathcal{H}$, then $x = gXg^{-1}$ is conjugated by left multiplication. We will focus on the $\mathcal{H}$-gauge invariants, *i.e.* the x other than h. Notice however that the x do not have simple decompositions under the Lie algebra, to use such decompositions we must refer back to the X. Note also that

$$dx = g(dX)g^{-1} - j \wedge x - x \wedge j \tag{4.6}$$

The construction of flat connections can be extended provided the coset is a symmetric space, that is, in addition to $[\mathcal{H}, \mathcal{H}] \subseteq \mathcal{H}$ and $[\mathcal{H}, \mathcal{K}] \subseteq \mathcal{K}$, which follow from the subgroup structure, we have $[\mathcal{K}, \mathcal{K}] \subseteq \mathcal{H}$ as well. To see this, note that $dJ = J \wedge J$, and decompose both sides under $G = \mathcal{H} \oplus \mathcal{K}$:

$$\begin{aligned} dH &= H \wedge H + K \wedge K \ , \\ dK &= H \wedge K + K \wedge H \ . \end{aligned} \tag{4.7}$$

If the coset were not a symmetric space, then $K \wedge K$ would be a sum of two pieces, one of which is in $\mathcal{H}$ and the other in $\mathcal{K}$.[3] Transforming to the x forms, we have

$$\begin{aligned} dh &= -k \wedge h - h \wedge k \ , \\ dk &= -2k \wedge k \ . \end{aligned} \tag{4.8}$$

The gauge invariant k is also the Noether current for the global symmetry, $d{*}k = 0$. The current $2k$ is both flat and conserved, and so can be used to construct two families of flat connections precisely as above.

4.2 Nonlocal charges in $AdS_5 \times S^5$

The Green-Schwarz superstring on $AdS_5 \times S^5$ can be considered as a non-linear sigma model where the field $g(x)$ takes values in the coset superspace [48, 49, 50]:

[3] When $\mathcal{K}$ is a subalgebra, $K \wedge K$ contributes only to dK, and it is again possible to construct flat connections.

$$\frac{G_{AdS}}{H_{AdS}} = \frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}, \tag{4.9}$$

whose bosonic part is

$$\frac{SO(4,2)}{SO(4,1)} \times \frac{SO(6)}{SO(5)} = AdS_5 \times S^5. \tag{4.10}$$

The bosonic generators of G_{AdS} are the translations P^a and rotations J^{ab}, with $a, b = 0, ..., 4$, (generators of $SO(4,2)$) and translations $P^{a'}$ and rotations $J^{a'b'}$, with $a', b' = 0, ..., 4$, (generators of $SO(6)$). The fermionic generators are 32 spinors $Q^{\alpha\alpha',I}$ with $\alpha, \alpha' = 1, ..., 4$ and $I = 1, 2$. H_{AdS} is the stability subgroup of G_{AdS}, generated by the rotations J^{ab} and $J^{a'b'}$.

Then, it follows that the Lie algebra of $PSU(2,2|4)$ can be decomposed in the following way:

$$\mathcal{G}_{AdS} = \mathcal{H}_{AdS} + \mathcal{P} + \mathcal{Q}_1 + \mathcal{Q}_2, \tag{4.11}$$

with $\mathcal{H}_{AdS}$ the Lie algebra of H_{AdS}, $\mathcal{P}$ the algebra of the translations, and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ two copies of the $(\mathbf{4},\mathbf{4})$ representation of $\mathcal{H}_{AdS}$.

We focus on the current

$$J = -g^{-1}\partial g = H + P + Q_1 + Q_2. \tag{4.12}$$

Using the relation $dJ = J \wedge J$ and the $\mathcal{Z}_4$ grading respected by the algebra, with the following charges

$$\mathcal{H}: 0, \quad \mathcal{P}: 2, \quad \mathcal{Q}_1: 1, \quad \mathcal{Q}_2: 3 \tag{4.13}$$

we can find equations for dH, dP, dQ_1 and dQ_2:

$$\begin{aligned}
dH &= H\wedge H + P\wedge P + Q_1\wedge Q_2 + Q_2\wedge Q_1 \ , \\
dP &= H\wedge P + P\wedge H + Q_1\wedge Q_1 + Q_2\wedge Q_2 \ , \\
dQ_1 &= H\wedge Q_1 + Q_1\wedge H + P\wedge Q_2 + Q_2\wedge P \ , \\
dQ_1 &= H\wedge Q_2 + Q_2\wedge H + P\wedge Q_1 + Q_1\wedge P \ .
\end{aligned} \tag{4.14}$$

In terms of the lower-case currents they read:

$$
\begin{aligned}
dh &= -h\wedge h + p\wedge p - h\wedge p - p\wedge h - h\wedge q - q\wedge h + \tfrac{1}{2}(q\wedge q - q'\wedge q') \ , \\
dp &= -2p\wedge p - p\wedge q - q\wedge p + \tfrac{1}{2}(q\wedge q + q'\wedge q') \ , \\
dq &= -2q\wedge q \ , \\
dq' &= -2p\wedge q' - 2q'\wedge p - q\wedge q' - q'\wedge q \ ,
\end{aligned}
\tag{4.15}
$$

where we have defined $q = q_1 + q_2$ and $q' = q_1 - q_2$. This can be supplemented with the equations of motion [51]

$$
\begin{aligned}
d{*p} &= p\wedge {*q} + {*q}\wedge p + \tfrac{1}{2}(q\wedge q' + q'\wedge q) \ , \\
0 &= p\wedge({*q} - q') + ({*q} - q')\wedge p \ , \\
0 &= p\wedge(q - {*q'}) + (q - {*q'})\wedge p \ .
\end{aligned}
\tag{4.16}
$$

Next, we define

$$
a = \alpha p + \beta {*p} + \gamma q + \delta q', \tag{4.17}
$$

then, by requiring a to be a flat connection, *i.e.* $da + a\wedge a = 0$, we find two one-parameter families of solutions, given by

$$
\begin{aligned}
\alpha &= -2\sinh^2\lambda \ , \\
\beta &= \mp 2\sinh\lambda\cosh\lambda \ , \\
\gamma &= 1 \pm \cosh\lambda \ , \\
\delta &= \sinh\lambda \ .
\end{aligned}
\tag{4.18}
$$

Given a flat connection, the following equation

$$
dU = -aU, \tag{4.19}
$$

is integrable. On a simply connected space, and with initial condition $U(x_0, x_0) = 1$, then

$$
U(x, x_0) = \mathcal{P}\exp\left(-\int_{x_0}^{x} a\right) \tag{4.20}
$$

With $\mathcal{P}$ the path ordering in the Lie algebra. This allows the construction of an infinite number of conserved charges, given by

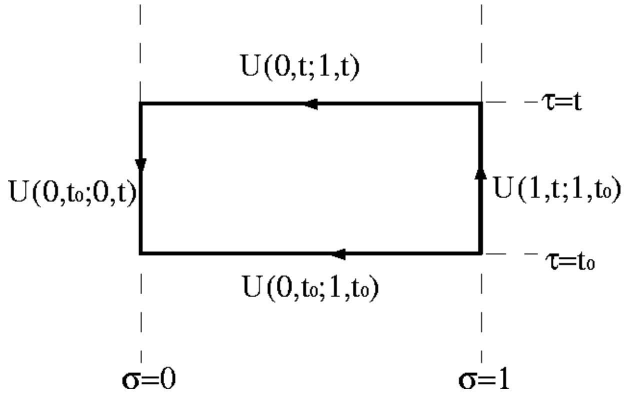

Figure 4.1: Evolution operators on the strip.

$$Q^{a\pm}(t) = U^{a\pm}(\infty, t; -\infty, t). \tag{4.21}$$

For a a flat connection, in particular 4.17, this charge can be shown to be conserved for an appropriate falloff of the fields at infinity.

As we are interested in closed string theory, the world-sheet satisfies periodic boundary conditions. Considering the product of four Wilson lines that form a closed path in the strip (and don't enclose any singularity) we have (see figure (4.1))

$$U^{a\pm}(0, t_0; 0, t)U^{a\pm}(0, t; 1, t)U^{a\pm}(1, t; 1, t_0) = U^{a\pm}(0, t_0; 1, t_0), \tag{4.22}$$

from this equation it is easy to see that the following quantity

$$Q^{a\pm}(t) = \langle U^{a\pm}(0, t; 1, t)\rangle, \tag{4.23}$$

is conserved. Here the spatial coordinate σ is restricted to the $[0, 1]$ interval and we have assumed periodic boundary conditions, so that $U^{a\pm}(0, t_0; 0, t) = (U^{a\pm}(1, t; 1, t_0))^{-1}$. With $\langle\mathcal{O}\rangle$ we denote some invariant cyclic operator. For the case of $AdS_5 \times S_5$, whose isometry algebra is semi-simple, the operator $\langle\rangle$ can simply be taken to be the trace.[4]

[4]More precisely, when the algebra is semi-simple the trace is equivalent to the more general bilinear with the required properties we could take. This point will became more clear when considering the PP-wave algebra, that is non-semi-simple.

A convenient way to see the infinite set of charges is by Taylor expanding in the parameter λ, for instance

$$Q^{a_-}(t) = 1 + \sum_{n=1} \lambda^n Q_n. \qquad (4.24)$$

Writing $a_- = \lambda a^{(1)} + \lambda^2 a^{(2)} + ...$, we have for the first order charges

$$Q_1 = \int_0^1 d\sigma a_1^{(1)}(\sigma), \qquad (4.25)$$

$$Q_2 = \int_0^1 d\sigma a_1^{(2)}(\sigma) + \int_0^1 d\sigma \int_0^\sigma d\sigma' a_1^{(1)}(\sigma) a_1^{(1)}(\sigma'), \qquad (4.26)$$

and so on.

4.3 Nonlocal charges on PP-Waves

In this subsection we will argue that these non-local charges exist also for the case of the pp-wave and have the same form as in (4.23).

Let us consider the Green-Schwarz superstring on a pp-wave RR background

$$ds^2 = -2dx^+dx^- - x_I^2 dx^{+2} + dx_I^2 \qquad (4.27)$$

$$F^{-i_1 \ldots i_4} = 2\epsilon^{i_1 \ldots i_4}, \qquad F^{-i'_1 \ldots i'_4} = 2\epsilon^{i'_1 \ldots i'_4} \qquad (4.28)$$

$I = 1, \ldots 8$, describes the eight flat directions of the pp-wave, $i, j = 1, \ldots 4$ and $i', j' = 5, \ldots 8$. and x^+ is taken to be the light-cone evolution parameter.

The Green-Schwarz superstring on a pp-wave background can be regarded as a non linear sigma model on a coset superspace [51]

$$\frac{G}{H} \qquad (4.29)$$

The transformation group G is spanned by the following generators: The even (bosonic) part of the superalgebra includes ten "translation" generators P^μ, $SO(4)$ rotation generators J^{ij}, $i, j = 1, \ldots, 4$, $SO'(4)$ rotation generators $J^{i'j'}$, $i', j' = 5, \ldots, 8$ and eight rotation generators in the (x^-, x^I) plane J^{+I}.

The odd (fermionic) part of the superalgebra consists of the complex 16-component spinor Q_α, $\alpha = 1, \dots 16$. The stability group H, is generated by J^{ij}, $J^{i'j'}$ and J^{+I}. The algebra between the relevant generators is given in the appendix A.

Then, the Lie algebra $\mathcal{G}$ can be decomposed as follows

$$\mathcal{G} = \mathcal{H} + \mathcal{P} + \mathcal{Q}_1 + \mathcal{Q}_2 \; . \tag{4.30}$$

Hence the current J can be decomposed as in (5.27). The algebra in this case also respects a $\mathcal{Z}_4$ grading that together with the condition $dJ = J \wedge J$ leads to equations of the exact form of (4.16).

For simplicity, from now on we will deal explicitly with the bosonic fields, however the discussion can be carried out for the fermionic fields as well. The equation of motion reads [51]:

$$d * p = 0, \tag{4.31}$$

which is the same as (4.17). We can then construct the same set of nonlocal conserved charges (4.23), from the connection

$$a = \alpha p + \beta{*}p, \tag{4.32}$$

with α and β given by (4.18). In terms of Cartan 1-forms

$$p = p_\mu L^\mu, \qquad p_\mu = g P_\mu g^{-1}, \tag{4.33}$$

with L^μ 1-forms on the two dimensional world-sheet and P_μ generators of G. We know that string theory on this background is exactly solvable in the light cone gauge, so it is interesting to ask what is the form of these charges when we choose such a gauge.

4.4 Explicit form of the charges on the PP-wave

In the light cone gauge, the Cartan 1–forms become

$$L^+ = dx^+, \qquad L^I = dx^I, \qquad L^- = dx^- - \frac{1}{2} x_I^2 dx^+, \tag{4.34}$$

with x^μ two dimensional fields on the world-sheet, depending in general on the world-sheet coordinates (τ, σ) with metric g_{ab}. Further we fix

$$\sqrt{g}g^{ab} = \eta^{ab}, \qquad x^+(\tau,\sigma) = \tau, \qquad p^+ = 1. \tag{4.35}$$

With this choice, the flat connection takes the form

$$a = \alpha p + \beta(*p) = \pm 2\lambda(*p) - 2\lambda^2 p \pm \frac{4}{3}\lambda^3(*p) + ..., \tag{4.36}$$

with

$$p = p_+ d\tau + p_I dx^I + p_-(dx^- - \frac{1}{2}x_I^2 d\tau). \tag{4.37}$$

where $p_+ = p^-$ and $p_- = p^+$. It is important to notice that p is written in terms of lower-case generators, of the form $p^\mu = gP^\mu g^{-1}$, and hence depends on the world-sheet coordinates through g. Choosing a coset representative of the form $g(x^\mu) = e^{\tau P^-}e^{x^I P^I}$ we can express the lower-case generators in terms of upper-case generators as follows

$$p^I = \cos\tau P^I - \sin\tau J^{+I}, \tag{4.38}$$

$$p^- = P^- - \frac{|x|^2}{2}P^+ + x^I\cos\tau J^{+I} + x^I\sin\tau P^I, \tag{4.39}$$

$$p^+ = P^+. \tag{4.40}$$

where we have defined $|x|^2 = Sum_I x^I x^I$. Writing the charges in terms of upper-case generators, we see their explicit dependence on the world-sheet coordinates, then we can construct these order by order by Taylor expanding in the parameter λ.

4.4.1 First order charge

To first order in λ we obtain the charge

$$Q_1 = \left\langle \int_0^1 d\sigma(P^- + A(\sigma)P^+ + B^I(\sigma)P^I + C^I(\sigma)J^{+I}) \right\rangle, \tag{4.41}$$

with

$$A(\sigma) = \partial_\tau x^- - |x|^2 \tag{4.42}$$

$$B^I(\sigma) = \sin\tau x^I + \cos\tau\partial_\tau x^I \tag{4.43}$$

$$C^I(\sigma) = \cos\tau x^I - \sin\tau\partial_\tau x^I \tag{4.44}$$

where we are not showing the σ dependence of the world-sheet fields. By using (A.14) we can write $A(\sigma)$ purely in terms of the fields x^I

$$A(\sigma) = -\frac{1}{2}(\partial_\tau x^I \partial_\tau x^I + \partial_\sigma x^I \partial_\sigma x^I + x^I x^I). \tag{4.45}$$

where sum over I is understood. By using the equations of motion, we can see that the coefficient of every generator is a classically conserved charge. So we have the following conserved quantities

$$Q_T = 1, \qquad Q_A = \int_0^1 d\sigma A(\sigma), \qquad Q_{B^I} = \int_0^1 d\sigma B^I(\sigma), \qquad Q_{C^I} = \int_0^1 d\sigma C^I(\sigma). \tag{4.46}$$

Since every coefficient is conserved, this implies the conservation of Q_1, for every invariant $\langle\rangle$ we choose.

It is interesting to notice that if we plug the mode expansion of the fields x^I (see appendix A) we find the following expressions for the charges

$$Q_A = \frac{1}{2}\left(p_0^I p_0^I + x_0^I x_0^I\right) + \sum_{n\neq 0}\left(\alpha_n^{I1}\alpha_{-n}^{I1} + \alpha_n^{I2}\alpha_{-n}^{I2}\right), \tag{4.47}$$

$$Q_{B^I} = p_0^I, \qquad Q_{C^I} = x_0^I. \tag{4.48}$$

The classical Poisson-Dirac brackets among these charges are given by

$$\{Q_{B^I}, Q_{C^J}\}_{PB} = \delta^{IJ} Q_T, \qquad \{Q_{B^I}, Q_A\}_{PB} = Q_{C^I}, \qquad \{Q_A, Q_{C^I}\}_{PB} = Q_{B^I}, \tag{4.49}$$

whereas, of course, Q_T has zero Poisson-Dirac bracket with every operator. Note that this is the same algebra satisfied by the bosonic generators of the pp-wave algebra.

From (4.47) we see that this set of charges represent the constants of motion p_0^I, x_0^I and the Hamiltonian, that are, of course, the quantities associated with the symmetries of the pp-wave.

4.4.2 Second order charge

For the second order charge we have [5]

[5] Q_2 has in general a single integral term, proportional to $\int p_1$, however, this is 0 in this case, since p_1 is a total derivative in the spatial coordinate.

$$Q_2 = \left\langle \int_0^1 d\sigma \int_0^\sigma d\sigma' \mathcal{A}(\sigma)\ \mathcal{A}(\sigma') \right\rangle \tag{4.50}$$
$$\mathcal{A}(\sigma) = P^- + A(\sigma)P^+ + B^I(\sigma)P^I + C^I(\sigma)J^{+I}$$

In order to study Q_2 let us introduce the following notation:

$$Q_{\mathcal{A}\mathcal{B}} = \int_0^1 d\sigma \int_0^\sigma d\sigma' \mathcal{A}(\sigma)\mathcal{B}(\sigma'), \tag{4.51}$$

where $\mathcal{A}, \mathcal{B}$ can take the values $1, A, B^I$ or C^I. Then we find the following conserved quantities

$$\begin{aligned} & & & Q_{11}, \quad Q_{AA}, \quad Q_{BB}, \quad Q_{CC} \\ Q_{\{1,A\}} = Q_{1A} + Q_{A1}, & \quad Q_{\{1,B\}} = Q_{1B} + Q_{B1}, & \quad Q_{\{1,C\}} = Q_{1C} + Q_{C1}, \\ Q_{\{A,B\}} = Q_{BA} + Q_{AB}, & \quad Q_{\{A,C\}} = Q_{AC} + Q_{CA}, & \quad Q_{\{B,C\}} = Q_{BC} + Q_{CB}, \\ & Q_a = \frac{1}{2} Q_{[C^I,B^I]} \langle P^+ \rangle + Q_{[B^I,1]} \langle J^{+I} \rangle + Q_{[1,C^I]} \langle P^I \rangle. \end{aligned}$$

where we have introduced the notation $Q_{[\mathcal{A},\mathcal{B}]} = Q_{\mathcal{A}\mathcal{B}} - Q_{\mathcal{B}\mathcal{A}}$. The charges in the first three lines, can be written as product of the charges appearing in Q_1, in fact

$$\begin{aligned} Q_{\{\mathcal{A},\mathcal{B}\}} = \int_0^1 d\sigma \int_0^\sigma d\sigma' \mathcal{A}(\sigma)\mathcal{B}(\sigma') + \int_0^1 d\sigma \int_0^\sigma d\sigma' \mathcal{B}(\sigma)\mathcal{A}(\sigma') = \\ = \int_0^1 d\sigma \int_0^1 d\sigma' \mathcal{A}(\sigma)\mathcal{B}(\sigma') = Q_{\mathcal{A}} Q_{\mathcal{B}}, \end{aligned} \tag{4.52}$$

where we have used that $\mathcal{A}$ and $\mathcal{B}$ commute. As for the last charge, note that since the invariant operator is cyclic, then $\langle T^A T^B \rangle - \langle T^B T^A \rangle = \langle [T^A, T^B] \rangle = 0$ and it becomes trivial.

In order to get some non-trivial conserved charge we should go to higher order.

4.4.3 Third and higher order charges

In this subsection we try to determine the general structure of the higher order charges and give an algorithm to write them explicitly.

As seen in the previous section, the charge of order n will have a local contribution (only one integral) plus a bi-local (two integrals), etc, up to a n-local contribution, given schematically by a sum of terms of the form

$$Q_{A_1...A_N} = \int_0^1 d\sigma_1 \mathcal{A}_1 \int_0^{\sigma_1} d\sigma_2 \mathcal{A}_2 ... \int_0^{\sigma_{n-1}} d\sigma_n \mathcal{A}_n, \tag{4.53}$$

times the corresponding product of generators, plus all the permutations.

In order to study these combinations, let us notice that we can write an arbitrary permutation in the following form

$$\begin{aligned} Q_{A_{I_1}...A_{I_n}} &= \int_0^1 d\sigma_1 \mathcal{A}_1 \int_{\sigma_1^i}^{\sigma_1^{i+1}} d\sigma_2 \mathcal{A}_2 ... \int_{\sigma_{n-1}^j}^{\sigma_{n-1}^{j+1}} d\sigma_n \mathcal{A}_n \\ Q_{A_{I_1}...A_{I_p} B A_{I_{p+1}},...A_{I_n}} &= \int_0^1 d\sigma_1 \mathcal{A}_1 \int_{\sigma_1^i}^{\sigma_1^{i+1}} d\sigma_2 \mathcal{A}_2 ... \int_{\sigma_{n-1}^j}^{\sigma_{n-1}^{j+1}} d\sigma_n \mathcal{A}_n \int_{\sigma_n^{n-p}}^{\sigma_n^{n-p+1}} d\sigma_{n+1} \mathcal{B}, \end{aligned} \tag{4.54}$$

with σ_m^i taking the values $0, \sigma_1, ..., \sigma_m, 1$ in increasing order. In other words, we express the permutations by interchanging the intervals of integration instead of the order of the $\mathcal{A}_i$.

We can give a precise recursive relation giving the integral of any permutation if we complement (4.54) with the following relation

$$\int_0^1 d\sigma \mathcal{B}(\sigma) \int_0^{\sigma} d\sigma_1 \mathcal{A}(\sigma_1) = \int_0^1 d\sigma \mathcal{A}(\sigma) \int_{\sigma}^1 d\sigma_1 \mathcal{B}(\sigma_1). \tag{4.55}$$

For such a n-local integral, we have $n!$ permutations, however, not all of them are independent. For instance, with the recursive relation given here, it is easy to see that the completely symmetric sum of all the permutations is just the product of the local integrals

$$Q_{A_1...A_n} + \text{permutations} = Q_{A_1} ... Q_{A_n}. \tag{4.56}$$

More generally, one can prove that

$$Q_{A_1...A_n B} + Q_{A_1...A_{n_1} B A_n} + ... + Q_{B A_1...A_n} = Q_B Q_{A_1...A_n}, \tag{4.57}$$

from where (4.56) as well as other relations can be shown. By using (4.57) together with the commutation relations among the generators one can write an arbitrary order charge in a way in which lower order contributions are explicit. For instance, for the third order charge we have [6]

$$\begin{aligned} Q_3 = Q_{ABC}\left\langle P^A P^B P^C\right\rangle &= \frac{1}{6}\left(Q_{ABC}\left\langle P^A P^B P^C\right\rangle + \text{permutations}\right) = \\ &= \frac{1}{12} Q_A Q_B Q_C \left\langle P^A\{P^B, P^C\}\right\rangle + \frac{1}{12} Q_A Q_{[B,C]}\left\langle P^A [P^B, P^C]\right\rangle + \\ &\qquad + \frac{1}{6} Q_{A[B,C]}\left\langle P^A[P^B, P^C]\right\rangle = \\ &= \frac{1}{12} Q_A Q_B Q_C \left\langle P^A\{P^B, P^C\}\right\rangle + \frac{1}{4} Q_A Q_{[B,C]}\left\langle P^A[P^B, P^C]\right\rangle . \end{aligned} \tag{4.58}$$

As happened for Q_2, the first contribution of the right-hand side of (4.58) is conserved, independently of the choice of $\langle\rangle$, since it is the product of conserved charges. Let us focus on the nontrivial piece

$$Q_3^{NT} = Q_A Q_{[B,C]}\left\langle P^A, [P^B, P^C]\right\rangle = Q_A Q_{[B,C]} f_D^{BC}\left\langle P^A P^D\right\rangle . \tag{4.59}$$

At this point we need to give an expression for $\left\langle P^A P^D\right\rangle$, that we will call Ω^{AD}. In our case P^A can take the "values" P^I, J^{+I}, P^- and P^+. If we take $\Omega^{AB} = Tr(P^A P^B)$, then we obtain

$$\Omega^{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and as can be easily seen, we obtain trivial conserved charges. The fact that Ω^{AB} is degenerate is due to the fact that the pp-wave algebra is non semi-simple. Fortunately, the most general Ω^{AB} with the required properties has been given for the algebra under consideration [52][53] [7]

$$\Omega^{AB} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & b' & k \\ 0 & 0 & k & 0 \end{pmatrix} \tag{4.60}$$

[6] There will be also a local term, that is proportional to Q_1, and a bi-local term, whose contribution vanish for the case under consideration.

[7] For the product of $P^I P^J$, or $J^{+I} J^{+J}$, we take the invariant to be proportional to δ^{IJ}.

With this choice one can see that Q_3^{NT} is non trivial and conserved:

$$
\begin{aligned}
Q_3^{NT} &= Q_{[C^I,B^I]}Q_1 + Q_{[B^I,1]}Q_{C^I} + Q_{[1,C^I]}Q_{B^I} = \\
= -2\int_0^1 d\sigma \int_0^1 d\sigma' \sigma x^I(\sigma)\partial_\tau x^I(\sigma') &+ 2\int_0^1 d\sigma \int_0^1 d\sigma' \sigma \partial_\tau x^I(\sigma) x^I(\sigma') + \\
+ \left(\int_0^1 d\sigma \int_0^\sigma d\sigma' x^I(\sigma)\partial_\tau x^I(\sigma') \right. &- \left. \int_0^1 d\sigma \int_0^\sigma d\sigma' \partial_\tau x^I(\sigma) x^I(\sigma')\right).
\end{aligned} \tag{4.61}
$$

Up to an overall factor k, note that the charge does not depend on b'. Expanding in modes one has

$$
Q_3^{NT} = \sum_{n\neq 0} \frac{2}{w_n k_n}\left(\alpha_n^{I1}\alpha_{-n}^{I1} - \alpha_n^{I2}\alpha_{-n}^{I2}\right). \tag{4.62}
$$

In order to evaluate higher order charges, one should give an expression for higher order invariants. In general they will have contribution from lower order invariants, plus some independent piece. We stress that in general, for a given order charge, there will be terms conserved by themselves, for instance, at fourth order we find

$$
Q_4^I = \int_0^1 d\sigma x^I(\sigma)\int_0^\sigma d\sigma' x^I(\sigma') + \int_0^1 d\sigma \partial_\tau x^I(\sigma)\int_0^\sigma d\sigma' \partial_\tau x^I(\sigma'), \tag{4.63}
$$

plus some other complicated contributions. Note that even if the complete charge at a given order is a Casimir of the group, there will be components that are conserved by themselves and need not to be associated to Casimirs. Plugging the oscillator expressions for the fields x^I in (4.63) we obtain

$$
Q_4^I = \frac{1}{2}(x_0^I x_0^I + p_0^I p_0^I) - i\sum_m \frac{k_n}{w_n^2}(\alpha_n^1\alpha_{-n}^1 - \alpha_n^2\alpha_{-n}^2). \tag{4.64}
$$

From the oscillator expressions for Q_3^{NT} and Q_4^I it is evident that they will have vanishing classical Poisson-Dirac brackets between them, so we see that they will not generate any new charge. This is different to the situation of sigma models with boundary conditions at infinity, where all the infinite set of classically conserved non-local charges is generated by the first non-local charges.

As we will see in the next section, there is another procedure to recover such charges, that simply consist in taking the Penrose limit of the corresponding charges on $AdS_5 \times S^5$. In this case the algebra is semi-simple, and so we can take the trace as invariant operator.

4.5 $AdS_5 \times S^5$ charges and their Penrose limit

In the previous section we have given the explicit form for the infinite set of non-local classically conserved charges for the pp-wave by studying directly string theory on such background. It is expected that such charges are the Penrose limit of the charges for the $AdS_5 \times S^5$. In this section we will prove that this is the case for the first non-local charges.

4.5.1 Explicit form of the Charges on $AdS_5 \times S_5$

As before, the first order charge can be written as (the trace is understood)

$$Q^1_{AdS} = \int (L_a p^a + L_{a'} p^{a'}), \tag{4.65}$$

with the Cartan 1-forms given by [54]

$$L^a_{\mathrm{AdS}} = dy^a + \left(\frac{\sinh y}{y} - 1\right) dy^b \boldsymbol{\Upsilon}_b{}^a \tag{4.66}$$

$$L^{a'}_{\mathrm{AdS}} = dy^{a'} + \left(\frac{\sin y'}{y'} - 1\right) dy^{b'} \boldsymbol{\Upsilon}_{b'}{}^{a'}, \tag{4.67}$$

with

$$\begin{aligned} y = \sqrt{y^2} = \sqrt{y^a y_a}, &\qquad y' = \sqrt{y'^2} = \sqrt{y^{a'} y_{a'}}, \\ \boldsymbol{\Upsilon}^b_a = \delta_a{}^b - \frac{y_a y^b}{y^2} \quad , &\qquad \boldsymbol{\Upsilon}^{b'}_{a'} \;=\; \delta_{a'}{}^{b'} - \frac{y_{a'} y^{b'}}{y'^2} \qquad . \end{aligned} \tag{4.68}$$

In this coordinates the bosonic Lagrangian (equivalently the metric) reads:

$$\mathcal{L} = (dy)^2 + (\sinh y)^2 d\Omega_4^2 + (dy')^2 + (\sin y')^2 d\Omega_4'^2 \tag{4.69}$$

where the 4-sphere metrics are given by

$$d\Omega_4^2 = \frac{(dy^a)(dy^a) - (dy)^2}{y^2}, \tag{4.70}$$

and similar for $d\Omega_4'^2$. As before, in order to show the explicit dependence on the world-sheet coordinates it is convenient to express lower-case generators in terms of upper-case generators, for the present case we obtain

$$p^a = \cosh y P^a + \left(\frac{1-\cosh y}{y^2}\right) y^a y^b P^b - \frac{\sinh y}{y} y^b J^{ab}, \tag{4.71}$$

$$p^{a'} = \cos y' P^{a'} + \left(\frac{1-\cos y'}{y'^2}\right) y^{a'} y^{b'} P^{b'} + \frac{\sin y'}{y'} y^{b'} J^{a'b'}. \tag{4.72}$$

Now in terms of upper-case generators

$$Q^1_{AdS} = \int (C_a P^a + C_{ab} J^{ab} + C'_a P^{a'} + C_{a'b'} J^{a'b'}), \tag{4.73}$$

with

$$C_a = \cosh y L_a + \frac{1-\cosh y}{y^2} y^b dy^b y^a, \quad C_{ab} = -\frac{\sinh y}{y}(L^a y^b - L^b y^a), \tag{4.74}$$

$$C_{a'} = \cos y' L_{a'} + \frac{1-\cos y'}{y'^2} y^{b'} dy^{b'} y^{a'}, \quad C_{a'b'} = \frac{\sin y'}{y'}(L^{a'} y^{b'} - L^{b'} y^{a'}) \tag{4.75}$$

So the charges $\int_0^1 d\sigma C_{a,0}$, etc, are conserved quantities, and they represent the isometries of background. As it is well know, by performing the Penrose limit, the isometries of $AdS_5 \times S_5$ map into the isometries of the pp-waves, so the Penrose limit of this first order charges are the first order charges found previously.

In order to find higher order charges we should worry about the invariant $<>$. The algebra of $AdS_5 \times S_5$ is semi-simple, so we can simply take the trace of product of upper-case operators. Again the second order charge will be trivial and we should go to the third order.

Let us write the upper-case generators as T^a, then $[T^a, T^b] = f^c_{ab} T^c$. Next, let us choose a representation of the algebra in which $Tr(T^a T^b) \propto \delta_{ab}$ then the non trivial third order charge becomes

$$Q^3_{AdS} = f^a_{bc}\left(\int_0^1 C_a\right)\left(\int_0^1 d\sigma C_b(\sigma)\int_0^\sigma d\sigma' C_c(\sigma')\right), \tag{4.76}$$

where now C_a is the coefficient of T^a in the first order charge, etc.

4.5.2 Penrose limit

The Penrose limit of the Cartan 1-forms was taken in [54], to which we refer the reader for details, here we repeat basically their analysis. [8] We define $y^\mp = y_\pm = (y^9 \pm y^0)/\sqrt{2}$ and then perform the rescaling

$$y^- \to \Omega^2 y^-, \quad y^+ \to y^+, \quad y^{\hat{i}} \to \Omega y^{\hat{i}}. \tag{4.77}$$

The Cartan 1-forms should also be rescaled as $L^- \to \Omega^2 L^-$, $L^+ \to L^+$ and $L^{\hat{i}} \to \Omega L^{\hat{i}}$.

Finally by performing the following change of coordinates

$$x^{\hat{i}} = \frac{\sin y^+}{y^+} y^{\hat{i}}, \tag{4.78}$$

$$x^+ = y^+, \tag{4.79}$$

$$x^- = y^- + \frac{y^{\hat{i}} y^{\hat{i}}}{2y^+}\left(1 - \frac{\sin 2y^+}{2y^+}\right), \tag{4.80}$$

we obtain the Cartan 1 forms used in the previous section for the pp-wave as $\Omega \to 0$ (see (4.34)).

$$L^- = dx^- - \frac{1}{2} x^{\hat{i}} x^{\hat{i}} dx^+, \qquad L^+ = dx^+, \qquad L^{\hat{i}} = dx^{\hat{i}}. \tag{4.81}$$

In order to show that Q^3_{AdS} maps into Q^{NT}_3 we need also to show that the $AdS_5 \times S_5$ algebra goes to the pp-wave algebra, with the correct structure constants f^A_{BC}. This was done in [55], by performing the following rescaling

$$P^+ \to \frac{1}{\Omega^2} P^+, \qquad P^{\hat{i}} \to \frac{1}{\Omega} P^{\hat{i}}, \qquad P^{\hat{i}}_* \to \frac{1}{\Omega} P^{\hat{i}}_*, \tag{4.82}$$

[8] Our conventions interchange + and - with respect to the conventions used in [54].

where P_*^a are the boost generator, and then taking the $\Omega \to 0$ limit. Notice that these rescalings corresponds to rescalings on the coordinates.

So we see that Q^3_{AdS} maps into Q_3^{NT}. In order to contruct the charges for $AdS_5 \times S^5$ we can take the trace of products of operators as invariant form without lost of generality, since the algebra is semi-simple. It is interesting to notice that the Penrose limit of such charges is equivalent to consider the charges on the pp-wave but using now the non-degenerate invariant (4.60), as done in the previous section.

4.5.3 An explicit check

As an explicit check that the Penrose limit of the charges of $AdS_5 \times S^5$ are the charges on the pp-wave we can consider the following exercise.

Since we have the expression for Q_3^{NT} in terms of the mode expansion of the coordinates x^I, we can compute its value when applied to BMN operators [9]:

$$Q_3^{NT}\alpha^\dagger_{n_1}\alpha^\dagger_{n_2}...\alpha^\dagger_{n_L}|0> \sim \left(\frac{1}{n_1}+\frac{1}{n_2}+...+\frac{1}{n_L}\right)\alpha^\dagger_{n_1}\alpha^\dagger_{n_2}...\alpha^\dagger_{n_L}|0>, \quad (4.83)$$

where we have taken the classical limit, *i. e.* $\sqrt{1+n^2} \approx 1$ [10]. On the other hand, on $AdS_5 \times S_5$, the dual of the BMN operators are believed to be rotating strings on an equator of S^5 with very large angular momentum. So it is interesting to check the value of Q^3_{AdS} when we plug in its expression in the semi classical solution corresponding to such a rotating string. This was done for the Hamiltonian in [56].

In the following we will focus on the S^5 part, the analysis for the AdS_5 is very much the same. First, let us change coordinates to the one used by [4]. In such coordinates, the relevant part of the metric turns out to be

$$ds^2 = d\phi^2 \cos^2\theta + d\theta^2 + \sin^2\theta d\Omega_3^2, \quad (4.84)$$

with $d\Omega_3^2$ the metric of a 3-sphere parametrized by angles α, β and γ. Performing this change of coordinates and setting $\alpha = \beta = \gamma = 0$ for simplicity

[9] For future convenience, we use the notation used in [4], where $n > 0$ for left movers and $n < 0$ for right movers.

[10] More explicitly, reintroducing the dimensionful parameters $\sqrt{m^2 + \frac{n^2}{(\alpha' p^+)^2}} \approx m$.

we can rewrite the coefficients $C_{a'}$ in terms of the new coordinates, as we focus on the S^5 only we suppress the primes henceforth,

$$C_{J^{45}} = -C_{J^{54}} = \cos\phi d\theta + \sin\theta\cos\theta\sin\phi d\phi, \tag{4.85}$$

$$C_{P^4} = \sin\phi d\theta - \sin\theta\cos\theta\cos\phi d\phi, \tag{4.86}$$

$$C_{P^5} = -\cos^2\theta d\phi. \tag{4.87}$$

We will consider a string rotating in the equator defined by $\theta = 0$, that is, we will consider $\phi = J\tau$ and small oscillations around $\theta = 0$. The Lagrangian becomes

$$\mathcal{L} = \cos^2\theta(d\phi)^2 + (d\theta)^2 \approx J^2(1+\theta^2) + (d\theta)^2 \tag{4.88}$$

By using the approximation of small perturbation and the explicit form of ϕ we obtain

$$\begin{aligned} j^A \equiv C_{J^{45}} = -C_{J^{54}} &= \cos(J\tau)\, d\theta + J\sin(J\tau)\,\theta d\tau, \\ j^B \equiv C_{P^4} &= \sin(J\tau)\, d\theta - J\cos(J\tau)\,\theta d\tau, \\ j^C \equiv C_{P^5} &= -Jd\tau. \end{aligned} \tag{4.89}$$

By using the approximate equation of motion

$$(\partial_\sigma^2 - \partial_\tau^2)\theta - J^2\theta = 0, \tag{4.90}$$

it is easy to see that the quantities $Q^A = \int_0^1 j_0^A$, etc, are conserved. Let us assume the perturbation to be of the form [11]

$$\theta(\tau,\sigma) = \sum_{i=1}^{K} \frac{A_i}{\omega_{n_i}} \left(e^{i(\omega_{n_i}\tau + 2\pi n_i\sigma)} + e^{-i(\omega_{n_i}\tau + 2\pi n_i\sigma)}\right) \tag{4.91}$$

with $\omega_n = \sqrt{J^2 + 4\pi n^2}$. Then we find

[11] The presence of 0-modes will not change the final result.

$$Q^3_{AdS} = Q^A Q^{[B,C]} + Q^C Q^{[A,B]} + Q^B Q^{[C,A]} =$$
$$= J(\int_0^1 \partial_\tau \theta \int_0^\sigma \theta - \int_0^1 \theta \int_0^\sigma \partial_\tau \theta) = \sum_{i=1}^{K} \frac{J^2 A_i^2}{n_i \sqrt{J^2 + 4\pi n_i^2}}, \tag{4.92}$$

that coincides with (4.83) for large J. In fact, one can notice that the form of (4.89) coincides with that of the coefficients B^I, C^I and 1, of the pp-wave, from which Q_3^{NT} is built.

Of course the very same method can be used to compute these charges for other string states on $AdS_5 \times S^5$, as done in [56].

4.6 Summary and conclusions

We have shown that the closed superstring on the maximally susy pp-wave possesses an infinite tower of non-local classically conserved charges. We have then shown that they coincide with the Penrose limit of the charges present for $AdS_5 \times S^5$.

In order to construct these charges in closed string theory, one must impose periodic boundary conditions on the world-sheet coordinates and then take some invariant of the group elements appearing in the charge. As a consequence it is not clear whether it is possible to generate all the tower of non-local currents by repeated Poisson Dirac brackets of the first non-local currents (or some finite number of them), as opposed to what happens when one considers the uncompactified sigma model. Indeed, from the first order charges explicitly obtained in this paper it is not possible to obtain more non-local conserved quantities.

On the other hand, when one considers closed string theory on pp-waves, which has a non semi-simple algebra, the naive invariant, *i.e.* the trace, turns out to be degenerate, and one should look for a non degenerate invariant in order to obtain non-trivial charges. The non degenerate bilinear invariant for the algebra under consideration was found in [52][53] and we use it in order to compute the first non trivial non-local conserved charge. Remarkably, this charge coincides with the Penrose limit of the first non trivial non-local conserved charge for $AdS_5 \times S^5$, whose algebra is semi-simple and we can use the trace as non-degenerate invariant.

There are many possible further directions to pursue. The super-Yangian algebra was explicitly studied in [57], one obvious generalization of the present work is to consider the Penrose limit of the super-Yangian algebra. The question about the gauge dual of such charges is interesting. As the AdS/CFT correspondence is more precise in the pp-wave limit (as the string spectrum can be exactly computed there) it may be simpler to study the dual of the charges in this limit.

Even though string theory is exactly solvable on pp-waves (in the ligth cone gauge), the non-local charges described in this chapter could provide a clue about the role played by these currents in the full $AdS_5 \times S^5$ background.

Chapter 5

PP-waves against free SYM

As already mentioned, in [28] and [29] the spectrum of strings on flat space, organized in AdS representations, along with its KK descendants was found to match that of free $\mathcal{N} = 4$ SYM. Besides the spectrum one would like to have a precise matching between the generators of the symmetry algebra on the two sides, having the Hamiltonian on the string side would allow to compute conformal dimensions, for instance. On the other hand one would like to have a systematic way to compute correlation functions on the string theory side and compare those with that of free $\mathcal{N} = 4$ SYM. In view of eq. (2.27) and the above discussion it is natural to try to mimic [28] and [29] but using strings on pp-waves as our starting point.

In this chapter we report some attempts in finding a correspondence between strings on pp-waves (with some additional restrictions to be described below) and free $\mathcal{N} = 4$ SYM.

We begin by assigning to each string oscillator Δ and J quantum numbers, this allows us to construct the string spectrum (by applying oscillators to a tower of vacua $|L\rangle$ with $L = 0, 1, 2, ...$) and read the quantum numbers of the gauge dual to each state. In section 5.2 we match the string spectrum to that of free $\mathcal{N} = 4$ SYM. In order to have a precise correspondence some restrictions must be imposed on the "level" of the vacua, depending on the kind of oscillators acting on them, in this way we find a matching up to level one in string theory and for all the string states with two (bosonic) oscillators. In section 5.3 we develop a discrete SFT formalism in order to compute three-point correlations functions in string theory and in section 5.4 we compare these results with those obtained from gauge theory .

On the way we find many difficulties and the matching is not precise (but almost!) so the general picture turns out to be not completely satisfactory. We end this chapter by discussing some possible sources for such mismatch.

5.1 Identification of generators in the two sides of the correspondence

The aim of this section is to find a correspondence between the generators of strings on pp-waves and that of free $\mathcal{N}=4$ SYM. In particular we will find Δ and J assignments for every string oscillator.

5.1.1 Identification of the conformal and Kaluza-Klein generators

In the following we show that the creation and annihilation operators of the zero modes $a_0^{\mu\dagger}, a_0^{\mu}$, with $\mu = 1,2,3,4$, of the plane wave string theory can be associated with the conformal generators K^μ and P^μ of the conformal field theory. We also identify the Kaluza-Klein generators on the S^5 P_i, P_i^* with $i = 5,6,7,8$, with the creation and annihilation operators $a_0^{i\dagger}$ and a_0^i of the plane wave string theory. In order to do that we use the fact that the plane wave algebra is a contraction of the $AdS_5 \times S^5$ algebra [55] and keep track of the generators during the contraction. The relevant part of the Euclidean AdS_5 algebra is given by

$$[M_{-1,0}, M_{-1,\mu}] = M_{0,\mu}, \quad [M_{-1,0}, M_{0,\mu}] = M_{-1,\mu} \tag{5.1}$$

Here $M_{-1,0}$ refers to the Cartan generator Δ, to diagonalize the action of Δ we consider the linear combinations $P_\mu = M_{-1,\mu} + M_{0,\mu}$ and $K_\mu = M_{-1,\mu} - M_{0,\mu}$, then the algebra reduces to

$$[\Delta, P_\mu] = P_\mu, \quad [\Delta, K_\mu] = -K_\mu \tag{5.2}$$

The relevant part of the $SO(6)$ algebra is given by

$$[R_{-1,0}, R_{-1,i}] = -R_{0,i}, \quad [R_{-1,0}, R_{0,i}] = R_{-1,i} \tag{5.3}$$

As before, we rename the Cartan generator $R_{-1,0}$ to iJ and diagonalize its action by considering the linear combinations $P_i = R_{-1,i} + iR_{0,i}$ and $P_i^* = R_{-1,i} - iR_{0,i}$. We obtain the simple relations

$$[J, P_i] = P_i, \quad [J, P_i^*] = -P_i^* \tag{5.4}$$

Note that the symmetry generators of S^5 and AdS_5 are decoupled. The plane wave algebra is obtained by considering the following linear combinations

$$P^+ = \frac{1}{2}(\Delta + J), \quad P^- = \Delta - J \tag{5.5}$$

and then scaling the generators according to

$$\begin{aligned} P^+ \to \frac{1}{\Omega^2}P^+, \quad K_\mu \to \frac{1}{\Omega}K_\mu, \quad P_\mu \to \frac{1}{\Omega}P_\mu, \\ P_i \to \frac{1}{\Omega}P_i, \quad P_i^* \to \frac{1}{\Omega}P_i^* \end{aligned} \tag{5.6}$$

In terms of these re-scaled operators the commutation relations (5.2) and (5.4) read

$$\begin{aligned} [P^+, P_\mu] = \frac{\Omega}{2}P_\mu, \quad [P^+, K_\mu] = -\frac{\Omega}{2}K_\mu, \\ [P^+, P_i] = \frac{\Omega}{2}P_i, \quad [P^+, P_i^*] = -\frac{\Omega}{2}P_i^*, \\ [P^-, P_\mu] = P_\mu, \quad [P^-, K_\mu] = -K_\mu, \\ [P^-, P_i] = -P_i, \quad [P^-, P_i^*] = P_i^*, \end{aligned} \tag{5.7}$$

The plane wave algebra is then obtained by taking $\Omega \to 0$. P^+ thus becomes a central element.

Consider the following global charges of the light cone string theory on the plane wave background

$$P^I = p_0^I, \quad J^{+I} = -ix_0^I, \quad I = 1, 2, \ldots 8 \tag{5.8}$$

Here x_0^I and p_0^I are the zero modes of the position and the momentum on the world sheet. Consider the linear combination $a_0^{I\dagger} = p_0 - ix_0^I$ and $a_0^I = p_0 + ix_0^I$. Their commutation relations with P^- are given by [35]

$$[P^-, a_0^{I\dagger}] = a_0^{I\dagger}, \quad [P^-, a^I] = -a_0^I \tag{5.9}$$

Comparing these relations with the 3rd and 4th line of (5.7), (5.2) and (5.4) we can make the following identifications and read out the Δ and J quantum number of the oscillators under consideration

$$\begin{aligned} P_\mu \to a_0^{\mu\dagger}, \quad \Delta = 1, J = 0, \\ K_\mu \to a_0^\mu, \quad \Delta = -1, J = 0, \\ P_i \to a_0^i, \quad \Delta = 0, J = 1, \\ P_i^* \to a_0^{i\dagger}, \quad \Delta = 0, J = -1. \end{aligned} \tag{5.10}$$

5.1.2 Identification of the superconformal generators

The commutation relations of the $\mathcal{N}=4$ supercharges with Δ and J are

$$\begin{aligned}
[\Delta, Q^+] &= \frac{1}{2}Q^+, \quad [J, Q^+] = \frac{1}{2}Q^+, \\
[\Delta, Q^-] &= \frac{1}{2}Q^-, \quad [J, Q^-] = -\frac{1}{2}Q^-, \\
[\Delta, S^+] &= -\frac{1}{2}S^+, \quad [J, S^+] = \frac{1}{2}S^+, \\
[\Delta, S^-] &= -\frac{1}{2}S^-, \quad [J, S^-] = -\frac{1}{2}S^-,
\end{aligned} \tag{5.11}$$

where $Q^\pm$ and $S^\pm$ are the supersymmetry and the superconformal charges. They are 16-dimensional Weyl spinors, the superscripts indicate their R-charges J. We identify the realization of these charges in the light cone plane wave string theory in the $m \to \infty$ limit. Only the dynamical charges of the light cone string theory have m dependence, they are given by [35] [1]

$$\begin{aligned}
Q^{-1} &= 2p_0^I\bar\gamma^I\theta_0^1 - 2mx_0^I\bar\gamma^I\Pi\theta_0^2 + \sum_{n=1}^{\infty}\left(2\sqrt{\omega_n}c_n a_n^{I\dagger}\bar\gamma^I\eta_n + \frac{im}{\sqrt{\omega_n}c_n}\tilde a_n^{I\dagger}\bar\gamma^I\Pi\tilde\eta_n + \text{h.c.}\right) \\
Q^{-2} &= 2p_0^I\bar\gamma^I\theta_0^2 + 2mx_0^I\bar\gamma^I\Pi\theta_0^2 + \sum_{n=1}^{\infty}\left(2\sqrt{\omega_n}c_n \tilde a_n^{I\dagger}\bar\gamma^I\tilde\eta_n - \frac{im}{\sqrt{\omega_n}c_n}a_n^{I\dagger}\bar\gamma^I\Pi\eta_n + \text{h.c.}\right),
\end{aligned} \tag{5.12}$$

where

$$\omega_n = \sqrt{(2\pi n)^2 + m^2}, \qquad c_n = \frac{1}{\sqrt{1 + (\frac{\omega_n - 2\pi n}{m})^2}} \tag{5.13}$$

We see that in the $m \to \infty$ limit, $\omega_n \to m$ and $c_n \to 1/\sqrt{2}$. Plugging this limit in the dynamical charges (5.12) and considering the following linear

[1] In the conventions of [35] $m = 2\pi\alpha' p^+ \mu$.

combinations we obtain

$$\begin{aligned} Q^+ = Q^{-1} + i\bar{\Pi}Q^{-2} &= 2\sqrt{2m}\Big(a_0^{\mu\dagger}\bar{\gamma}^\mu(\theta_R + \bar{\theta}_L) + a_0^i\bar{\gamma}^i(\theta_L + \bar{\theta}_R) \\ &+ \sum_{n=1}^{\infty}\left(a_n^{\mu\dagger}\bar{\gamma}^\mu\eta_n + a_n^i\bar{\gamma}^i\eta_n^\dagger + i\tilde{a}_n^{\mu\dagger}\bar{\gamma}^\mu\Pi\tilde{\eta}_n - i\tilde{a}_n^i\bar{\gamma}^i\Pi\tilde{\eta}_n^\dagger\right)\Big) \\ S^- = Q^{-1} - i\bar{\Pi}Q^{-2} &= 2\sqrt{2m}\Big(a_0^{i\dagger}\bar{\gamma}^i(\theta_R + \bar{\theta}_L) + a_0^\mu\bar{\gamma}^\mu(\theta_L + \bar{\theta}_R) \\ &+ \sum_{n=1}^{\infty}\left(a_n^{i\dagger}\bar{\gamma}^i\eta_n + a_n^\mu\bar{\gamma}^\mu\eta_n^\dagger + i\tilde{a}_n^{i\dagger}\bar{\gamma}^i\Pi\tilde{\eta}_n - i\tilde{a}_n^\mu\bar{\gamma}^\mu\Pi\tilde{\eta}_n^\dagger\right)\Big) \end{aligned} \tag{5.14}$$

Fermionic oscillators are denoted by θ (which is a complex Weyl spinor) and we use the conventions of [35]

$$\theta_0 = \frac{1}{\sqrt{2}}(\theta_0^1 + i\theta_0^2), \qquad \bar{\theta}_0 = \frac{1}{\sqrt{2}}(\theta_0^1 - i\theta_0^2) \tag{5.15}$$

$$\theta_{-n}^I \equiv \frac{1}{\sqrt{2}}\eta_n^{I\dagger}, \qquad \theta_n^I \equiv \frac{1}{\sqrt{2}}\eta_n^I, \qquad n = 1, 2, 3, ... \tag{5.16}$$

$$\theta_R = \frac{1+\Pi}{\sqrt{2}}\theta_0, \qquad \theta_L = \frac{1-\Pi}{\sqrt{2}}\theta_0. \tag{5.17}$$

We have defined the proper linear combinations in preparation for their Δ and J assignment. The fermionic vacuum is defined by $\theta_R|0\rangle = \bar{\theta}_L|0\rangle = 0$. The linearly realized supercharges are independent of m and can also be grouped into the following linear combinations

$$\begin{aligned} S^+ &= Q^{+1} + i\Pi Q^{+2} = \theta_R + \bar{\theta}_L, \\ Q^- &= Q^{+1} - i\Pi Q^{+2} = \theta_L + \bar{\theta}_R, \end{aligned} \tag{5.18}$$

Let us now assign the Δ and J charges to the above generators. The dynamical generators commute with P^-, therefore they both have $\Delta = J$. We also have the following commutation relations

$$\begin{aligned} [P_\mu, S^-] = [a_0^{\mu\dagger}, S^-] = \bar{\gamma}^\mu Q^-, &\qquad [P_\mu, Q^+] = 0, \\ [K_\mu, Q^+] = [a_0^\mu, Q^+] = \bar{\gamma}^\mu S^+, &\qquad [K_\mu, S^-] = 0 \\ [P_-, Q^+] &= [P_-, S^-] = 0, \\ [P_-, Q^-] = Q^-, &\qquad [P_-, S^+] = -S^- \end{aligned} \tag{5.19}$$

Inspection of the commutation relations suggests the following assignment of Δ and J for the charges:

$$\begin{aligned} Q^+ &: \quad \Delta = \frac{1}{2}, \quad J = \frac{1}{2} \\ S^- &: \quad \Delta = -\frac{1}{2}, \quad J = -\frac{1}{2} \\ Q^- &: \quad \Delta = \frac{1}{2}, \quad J = -\frac{1}{2} \\ S^+ &: \quad \Delta = -\frac{1}{2}, \quad J = \frac{1}{2} \end{aligned} \tag{5.20}$$

5.1.3 Assignment of Δ and J to string oscillators

It is not straightforward to assign Δ and J charges to the string oscillators. Inspired by [29] we rely on the extrapolation of the the BMN formula (2.23) for $g_{YM} = 0$, which is given by

$$\Delta - J = \nu \tag{5.21}$$

where ν is the number of excited oscillators. Therefore for a given oscillator, the difference between Δ and J is one. In addition we use (5.20) in order to infer the Δ and J quantum numbers of each string oscillator. We propose the following assignments

$$\begin{aligned} a_n^{i\dagger} &: \quad \Delta = n, \quad J = n-1, \\ a_n^{i} &: \quad \Delta = -n, \quad J = -(n-1), \\ a_n^{\mu\dagger} &: \quad \Delta = n+1, \quad J = n, \\ a_n^{\mu} &: \quad \Delta = -(n+1), \quad J = -n, \\ \eta_n^{\dagger} &: \quad \Delta = n+1/2, \quad J = n-1/2, \\ \eta_n &: \quad \Delta = -(n+1/2), \quad J = -n+1/2, \\ \theta_L + \bar{\theta}_R &: \quad \Delta = 1/2, \quad J = -1/2, \\ \theta_R + \bar{\theta}_L &: \quad \Delta = -1/2, \quad J = +1/2, \end{aligned} \tag{5.22}$$

Note that substituting these charge assignments in the expressions for the supersymmetry generators in (5.14) and (5.18) we obtain the charge assignments in (5.20). Also, the identification of the charges for the bosonic zero modes in (5.10) agree with the more general assignment for the bosonic oscillators in (5.22).

5.2 Organization of the plane wave spectrum into Yang Mills states

It would seem that now that we have the individual assignments of Δ and J for each string oscillator it should be easy to construct the spectrum and match it with the free Yang-Mills spectrum. However it turns out that we need to put certain bounds on the J quantum number of the vacuum depending on the kind of oscillators acting on it, so as to agree with the Yang Mills spectrum for finite J. We will see this in the various examples below.

5.2.1 Supergravity modes

All supergravity states are generated by the action of various symmetry generators on the highest weight state (HWS) $\mathrm{Tr}(Z^J)$. In the plane wave string theory we identify this state with the vacuum with J units of R-charge[2].

$$\mathrm{Tr}(Z^J) \leftrightarrow |J; \Delta = J\rangle \equiv |J\rangle \tag{5.23}$$

To fill in the full $SO(6)$ content of this HWS representation we can act on (**??**) with the J lowering operator on the gauge theory side. This is given by

$$J^{i\bar{Z}} = \phi^i \frac{\partial}{\partial Z} - \bar{Z} \frac{\partial}{\partial \phi^i} \tag{5.24}$$

As we saw in the previous section, on the string theory side this symmetry is generated by the operator $a_0^{i\dagger}$. On the gauge theory we can keep acting by $J^{i\bar{Z}}$ till we obtain $\mathrm{Tr}\bar{Z}^J$. Note that after J actions of $J^{i\bar{Z}}$, the J quantum number turns negative and the number of Φ^i's start decreasing till one is left with all $\bar{Z}$'s. To mimic this feature in the string theory we fill out the complete $SO(6)$ representation in the following way. Start from $|J\rangle$ and act with $a_0^{i\dagger}$'s J times. Acting with these operators lowers the J quantum number till zero increasing the number of ϕ^i. Then for the remaining states start from the lowest weight state $|-J\rangle$ and act by the operator a_0^i $J-1$

[2]Notice that the R-charge J is not necesarily large.

times. The action of these operators increases the J quantum number and increases the number of ϕ^i. Thus on the string side we constrain the number of $a_0^{i\dagger}$ acting on $|J\rangle$ or the number of a_0^i on $|-J\rangle$. This can also be thought of as bounds on J given certain number of $a_0^{i\dagger}$, in fact the J quantum number of the vacuum is always greater than or equal to the number of $a_0^{i\dagger}$ acting on it. A similar condition applies for the lowest weight state $|-J\rangle$. Note that if one works in the sector of very large J and very small number of impurities (BMN limit), these constraints are immaterial but obviously realized.

As a simple check of imposing this constraint in order to get the right number of states in string theory, we evaluate the string partition function and compare it with the partition function of all the $SO(6)$ traceless symmetric tensors on the gauge theory. On the string side the partition function is given by

$$\mathcal{Z} = \sum_{l=0}^{\infty}\sum_{j=0}^{l}(x^l q^j x^{-j}) + \sum_{l=0}^{\infty}\sum_{j=0}^{l-1} x^{-l}q^j x^j \tag{5.25}$$

Powers of q count the number of oscillators a_0^i, either creation or annihilation, acting on the corresponding vacuum and powers of x count the J quantum number. The first sum counts the states with $a_0^{i\dagger}$ acting on $|J\rangle$ and the second sum counts the states with a_0^i acting on $|-J\rangle$. Summing both terms in the partition function we obtain

$$\mathcal{Z} = \frac{1}{1-x}\frac{1}{1-q} + \frac{1/x}{1-1/x}\frac{1}{1-q} \tag{5.26}$$

Note that though we had a constrained sum in (5.25), in the partition function the sum becomes free and one gets a product of the partition functions of the q oscillators and a oscillator which creates the Z or the $\bar{Z}$ states. Now let us examine the partition function of all traceless symmetric tensors in the gauge theory. We need to know the decomposition of a $SO(6)$ traceless symmetric tensor into $SO(4)$ symmetric tensors including the trace and the J quantum number of the $SO(2)$. This is given by

$$[0,n,0] = [n]^0 \oplus [n-1]^{\pm 1} \oplus [n-2]^{\pm 2} \oplus \ldots \oplus [0]^{\pm n} \tag{5.27}$$

here $[n]$ refers to a rank n symmetric tensor of $SO(4)$ which includes the trace and the superscripts indicate the J quantum number. The partition function of all totally symmetric traceless tensors is given by

$$\mathcal{Z} = \sum_{l=0}^{\infty}\sum_{j=0}^{l} q^{l-j}x^j + \sum_{l=0}^{\infty}\sum_{j=1}^{l} q^{l-j}x^{-j} \tag{5.28}$$

Here q keeps track of the rank of the $SO(4)$ tensor while x counts the J charge. The first term is for the term in (5.27) with $J = 0$ and positive J, while the second term is for J negative. Again the sum can be performed and it gives

$$\mathcal{Z} = \frac{1}{1-x}\frac{1}{1-q} + \frac{1/x}{1-1/x}\frac{1}{1-q} \tag{5.29}$$

which is the same as the string theory counting of the supergravity states.

5.2.2 Level one states and the Konishi multiplet

In order to simplify the search of string states which correspond to Yang Mills operators it is convenient to look for superconformal primaries. Since we know what the annihilation operators $S^{\pm}$ correspond to in string theory, we can construct these states and look for their duals. S^{+} is just the zero mode annihilation operator for the fermions, so obviously it annihilates the vacuum. From the structure of S^{-} in (5.14) we see that creation operators of bosonic oscillators in the i directions come together with fermionic annihilation operators and vice-versa for the μ directions. Thus the following states are superconformal primaries at level one

$$a_1^{(i\dagger}\tilde{a}_1^{j)\dagger}|J\rangle, \quad a_1^{[i\dagger}\tilde{a}_1^{j]\dagger}|J\rangle, \quad a_1^{i\dagger}\tilde{a}_1^{i\dagger}|J\rangle \tag{5.30}$$

These are states in the symmetric traceless (**9**), anti-symmetric (**6**) and scalar (**1**) representations of $SO(4)$. From the assignments (5.22) we see that the oscillator part of all of the above operators have $\Delta = 2$ and $J = 0$. Thus the degeneracy can be broken only if the level of the vacuum, J, is different for these states. The information we have about the string theory is not enough to predict what the allowed values of J are, in fact it seems all values of J are allowed. However, as we have seen for the case of the supergravity modes, there will be bounds on J coming from the requirement that these string states match up with states of the Yang-Mills. We assume that they are all HWS of $SO(6)$. Consider the $SO(6)$ scalar on the Yang-Mills side $\mathrm{Tr}(\phi^I\phi^I)$, where $I = 4,5,6,7,8,9$, it is a dimension 2 operator, a superconformal primary and not part of the 1/2-BPS multiplet.[3] Thus the scalar in (5.30) with $|J>=|0>$ has the right quantum numbers to be identified with the Yang-Mills scalar $\mathrm{Tr}(\phi^I\phi^I)$.

[3]Half-BPS SYM states are dual to supergravity states.

The obvious choice for the operator to be identified with the anti-symmetric representation of $SO(4)$ is $\mathrm{Tr}(\phi^{[I}\phi^{J]})$, but this is zero because of the cyclicity of the trace. The next candidates come from operators with three scalar fields, the operator $\mathrm{Tr}\phi^I\phi^J\phi^K$ such that I, J is anti-symmetric and K is symmetric with respect to I or J is also zero due to the trace. Thus the only candidate is the completely antisymmetry representation of $SO(6)$ $\mathrm{Tr}(\phi^{[I}\phi^J\phi^{K]})$, whose HWS with respect to $SO(2)_J$ is $\mathrm{Tr}(\phi^{[i}\phi^{j]}Z)$. Thus the lower bound for J for the antisymmetric representation in (5.30) is $|J>=|1>$. We see by this choice the quantum number $\Delta = 3$ also agrees on the string side.

For the state in the symmetric traceless representation of $SO(4)$ in (5.30) and $\Delta \geq 2$, the obvious candidate is the symmetric traceless representation $\mathrm{Tr}(\phi^{(I}\phi^{J)})$, however this is ruled out as it is dual to a supergravity state. The only non zero length 3 operator with at least two symmetric indexes is the completely symmetric traceless representation which is also a supergravity state. Among the length 4 operators the non zero operator which has the right symmetry properties is $\mathrm{Tr}(\phi^I\phi^J\phi^K\phi^L)$ such that two indexes say I, K and J, L are symmetric but I, J and K, L are anti-symmetric among them. This is the $(2, 0, 2)$ representation of $SU(4)$. The HWS with respect to $SO(2)_J$ is given by

$$Tr(\phi^{(i}Z\phi^{j)}Z - \phi^{(i}\phi^{j)}Z^2) \tag{5.31}$$

Thus the symmetric representation of $SO(4)$ in (5.30) has at least $|J>=|2>$. In conclusion we have the following identifications

$$\begin{aligned} a_1^{(i\dagger}\tilde{a}_1^{j)\dagger}|J\rangle \leftrightarrow \mathrm{Tr}(\phi^{(i}Z\phi^{j)}Z - \phi^{(i}\phi^{j)}Z^2) : &\qquad [2,0,2] = \mathbf{84} \\ a_1^{[i\dagger}\tilde{a}_1^{j]\dagger}|J\rangle \leftrightarrow \mathrm{Tr}(\phi^{[i}\phi^{j]}Z) : &\qquad [2,0,0]+[0,0,2] = \mathbf{10}+\mathbf{10}*, \\ a_1^{i\dagger}\tilde{a}_1^{i\dagger}|J\rangle, \leftrightarrow \mathrm{Tr}(\phi^I\phi^I) : &\qquad [0,0,0] = \mathbf{1} \end{aligned} \tag{5.32}$$

The last column denotes the $SU(4)$ representation of which the states are highest weight of. All of these states are superconformal primaries, in fact they are the ground states of the short multiplets in which the long Konishi multiplet decomposes into [28] [4]

$$\mathcal{A}^2_{[000](00)} \to \mathcal{CC}^{1,1}_{[000](00)} + \mathcal{BC}^{\frac{1}{4},\frac{3}{4}}_{[200](00)} + \mathcal{CB}^{\frac{3}{4},\frac{1}{4}}_{[002](00)} + \mathcal{BB}^{\frac{1}{4},\frac{1}{4}}_{[202](00)} \tag{5.33}$$

[4] The long supermultiplet $\mathcal{A}^{\Delta_0,(j_1,j_2)}_{[k,p,q]}$ is obtained by the unconstrained action of the sixteen supercharges on the ground state $[k,p,q]_{(j_1,j_2)}$ of conformal dimension Δ_0. For some particular states some combinations of the charges may annihilate them, and we obtain short or semi-short multiplets, see [28] for the details.

With $[k, p, q]$ denoting an $SU(4)$ representation in terms of Dynkin labels and (j_1, j_2) denoting an $SO(4)$ representation in terms of spins. The "bottom" component of $\mathcal{CC}$ has $\Delta = 2$ and is the Konishi scalar $\mathrm{Tr}(\Phi^I \Phi^I)$. It is a semi-short multiplet of dimension $2^8 \times 5$. The "bottom" component of the multiplet $\mathcal{BC}$ along with $\mathcal{CB}$ is $\mathrm{Tr}(\Phi^{[I}\Phi^J\Phi^{K]})$, in the completely antisymmetric representation. The two multiplets combined have dimension $2(10 \times 2^{11} - 256 \times 37)$. Finally, the bottom component of the multiplet $\mathcal{BB}$ is the length 4 state $\mathrm{Tr}(\phi^{\{I}\phi^{(J\}}\phi^{\{K)}\phi^{L\}})$, and it has dimension $10 \times 2^{12} + 256 \times 5$. Note that adding the dimensions of all the sub multiplets gives 2^{16} which is the expected dimension of a long multiplet.

Though we have only written down the highest weight states in (5.30) we obtain the full $SO(6)$ representation by the action of $a_0^{i\dagger}$ on the highest weight states and a_0^i on the lowest weight states. The procedure is similar to the gravity modes discussed in the previous subsection.

5.2.3 Generalization to two oscillators states

In the following we generalize the results obtained in the previous section for string states of the form $a_n^{i\dagger}\tilde{a}_n^{j\dagger}|J\rangle$. The basic strategy is to count the number of inequivalent states in SYM with given quantum numbers and fix the bounds on $|J\rangle$ in order to have the same number of states (for every representation) in the string spectrum.

A generic state $a_n^{i\dagger}\tilde{a}_n^{j\dagger}|L\rangle$ has quantum numbers $\Delta = 2n + L$ and $J = 2n + L - 2$ and transform in the $SO(4)$ representations $\mathbf{1}, \mathbf{6}$ and $\mathbf{9}$. So we need to count inequivalent SYM states (taking into account the cyclicity of the trace) with a given conformal dimension Δ and R-charge $J = \Delta - 2$, *i.e.* states of the form $TrZ^l\phi^i Z^{L-l-2}\phi^j$, transforming in a definite $SO(4)$ representation.

In order to do that we can use a particular case of Polya's theory [28, 60, 61]. According to this technique, the inequivalent necklaces of length n, built from beads of p different types $a_1, a_2, ..., a_p$ are given by

$$P_n(a_1, a_2, ..., a_p) \equiv \frac{1}{n}\sum_{d|n} \varphi(d)(a_1^d + a_2^d + ... + a_p^d)^{n/d} \tag{5.34}$$

where the sum runs over the divisors of n and $\varphi(d)$ is the Euler's totient function, denoting the number of numbers relatively prime to d, smaller than

d, with $\varphi(1) \equiv 1$. For instance, for $n = 6$ and two different kind of beads a and b the different necklaces are given by

$$\begin{aligned} P_6(a,b) &= \frac{1}{6}[(a+b)^6 + (a^2+b^2)^3 + 2(a^3+b^3)^2 + 2(a^6+b^6)] \\ &= a^6 + a^5 b + 3a^4 b^2 + 4a^3 b^3 + 3a^2 b^4 + ab^5 + b^6 \end{aligned} \tag{5.35}$$

Where a^6 denotes the necklace with 6 beads of type a, etc. If we are interested just in the total number of inequivalent necklaces then we should compute $P_n(1,1,...,1)$.

Hence, the quantity of inequivalent states of length L composed by the fields Z and Φ^i, $i = 1,...,4$ is given by $P_L(Z, \Phi^1, \Phi^2, \Phi^3, \Phi^4)$. As we are interested in keeping just the information of the R-charge, we need to compute $P_L(Z,1,1,1,1)$, then the coefficient of Z^{L-2} in such polynomial equals the quantity of inequivalent states with $\Delta = L$ and $J = L - 2$.

In order to know how these states distribute among the $SO(4)$ representations $\mathbf{1}, \mathbf{6}$ and $\mathbf{9}$ it is enough to note that the representations $\mathbf{1}$ and $\mathbf{9}$ appear the same quantity of times at each level, and the quantity of singlets is given by $P_L(Z,1)$. In table (5.2.3) we can see the number of inequivalent states and the representations in which they appear for any given length.

Length=Δ	states	Representations
2	10	$\mathbf{1}+\mathbf{9}$
3	16	$\mathbf{1}+\mathbf{6}+\mathbf{9}$
4	26	$2.\mathbf{1}+\mathbf{6}+2.\mathbf{9}$
5	32	$2.\mathbf{1}+2.\mathbf{6}+2.\mathbf{9}$
6	42	$3.\mathbf{1}+2.\mathbf{6}+3.\mathbf{9}$
2n	$16n-6$	$n.\mathbf{1}+(n-1)\mathbf{6}+n.\mathbf{9}$
2n+1	$16n$	$n.\mathbf{1}+n.\mathbf{6}+n.\mathbf{9}$

From the analysis of the supergravity spectrum, section 5.2.1, we know that at every length L there is a state transforming in the $\mathbf{9}$ representation, dual to the supergravity state $a_0^{(i\dagger}a_0^{j)\dagger}|L>$

It is now straightforward to fix the bounds for arbitrary level for every representation, so as to match string states with those of free SYM. Actually one can see that we have the same bounds as those at level one. In summary, the following string states

$$a_n^{i\dagger}\tilde{a}_n^{i\dagger}(|0\rangle + |1\rangle + |2\rangle + ...), \tag{5.36}$$
$$a_n^{[i\dagger}\tilde{a}_n^{j]\dagger}(|1\rangle + |2\rangle + |3\rangle + ...) \tag{5.37}$$
$$a_n^{(i\dagger}\tilde{a}_n^{j)\dagger}(|2\rangle + |3\rangle + |4\rangle + ...) \tag{5.38}$$

reproduce exactly the same quantum numbers, representations and multiplicities, when summing over n, of the corresponding YM states (table (5.2.3)).

As the string level grows there is increasing ambiguity in fixing the precise dictionary between string and gauge theory states, since the conformal dimension and R-charge of the state $a_n^{i\dagger}\tilde{a}_n^{j\dagger}|L\rangle$ depends on the combination $2n + L$ and not n and L separately. We know, however, that for large R-charge there is a precise dictionary, see (2.19), and ours should reduce to that for large J. This assumption together with the fact that the duals of the string states should form an orthogonal basis (also for small J) will help us in order to conjecture the precise dictionary, as will be done in section 5.4.

Once we have the precise dictionary between the spectrum on the two theories the most natural test is to compute correlation functions on both sides of the correspondence. This is straightforward in free YM as it involves simply Wick contractions (apart from the position dependence). In order to compute correlation functions in string theory we develop a discretized light-cone SFT formalism, on PP-waves, on the limit of large mass, *i.e.* $\mu \to \infty$.

In this picture the string i is composed by N_i bits, which are non interacting in the limit considered, and string interactions are viewed as splitting of the bits $N_3 \to N_1 + N_2 = N_3$

5.3 Discrete light cone string field theory

In this section we compute the 3-string vertex $|V_3\rangle$ in discrete light cone SFT on pp-waves, for the process in which one string, 3, splits into two, 1 and 2 (or two strings, 1 and 2, join into 3). This vertex depends (on a way to be described in the next sections) on the overlapping matrices $X^r_{m,n}$ which express the Fourier basis of the string r in terms of the Fourier basis of the string 3. [5]

[5] String field theory for the closed type IIB string theory was developed in [58] for flat space-time, for a introduction to light cone SFT on pp-waves see[59].

We start by setting up notations and computing the various overlaps mentioned above. We focus on one of the bosonic coordinates. The string action is given by (we are not showing the τ dependence)

$$S = \frac{1}{2N} \int d\tau \sum_{s=-(N-1)/2}^{(N+1)/2} \left(\dot{X}^2(s) - ((X(s) - X(s-1))^2 - m^2 X^2(s) \right) \tag{5.39}$$

here we have scaled the world sheet coordinates to $X \to \sqrt{2\pi\alpha'}X$. The mass of the plane wave is given by $m = \mu\sqrt{\alpha'}2\pi$. Note that with these conventions, the world sheet, as well as the target space coordinates are dimensionless. μ has dimensions of mass. We have discretized the σ direction of the light cone string by N bits. In (5.39) N is taken to be odd for convenience, and $X(s+N) = X(s)$. Solving the equations of motion we obtain

$$X(s,\tau) = \cos m\tau x_0 + \frac{1}{m}\sin m\tau p_0 + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{1}{\omega_n} \left(e^{-i(\omega_n\tau - \frac{2\pi n s}{N})}\alpha_n + e^{-i(\omega_n\tau + \frac{2\pi n s}{N})}\tilde{\alpha}_n \right) \tag{5.40}$$

here

$$\omega_n = \sqrt{m^2 + 4\sin^2\frac{\pi n}{N}}, \quad \text{for } n > 0 \qquad \omega_n = -\sqrt{m^2 + 4\sin^2\frac{\pi n}{N}}, \quad \text{for } n < 0 \tag{5.41}$$

The summation over n in (5.40) runs from $-\frac{N-1}{2}$ to $\frac{N-1}{2}$. Similarly $P = \frac{\dot{X}}{N}$, is given by

$$\begin{aligned} P(s,\tau) &= \frac{1}{N}\Big[-m\sin m\tau x_0 + \cos m\tau p_0 \\ &+ \frac{1}{\sqrt{2}} \Big(\sum_{n\neq 0} e^{-i(\omega_n\tau - \frac{2\pi n s}{N})}\alpha_n + e^{-i(\omega_n\tau + \frac{2\pi n s}{N})}\tilde{\alpha}_n \Big)\Big] \end{aligned} \tag{5.42}$$

The commutation relations among the oscillators are given by

$$[\alpha_n, \alpha_m] = \omega_n \delta_{n+m,0} \ , \quad [\tilde{\alpha}_n, \tilde{\alpha}_m] = \omega_n \delta_{n+m,0} \ , \quad [x_0, p_0] = i. \tag{5.43}$$

It is convenient to redefine the operators according to

$$\begin{aligned} a_n &= \frac{\alpha_n}{\sqrt{\omega_n}}, \qquad a_n^\dagger = \frac{\alpha_{-n}}{\sqrt{|\omega_n|}} \\ a_{-n} &= \frac{\tilde{\alpha}_n}{\sqrt{\omega_n}}, \qquad a_{-n}^\dagger = \frac{\tilde{\alpha}_{-n}}{\sqrt{|\omega_n|}} \end{aligned} \tag{5.44}$$

where $n > 0$. After the change of normalizations, the commutation relations read

$$[a_n^\dagger, a_m] = \delta_{m,n} \tag{5.45}$$

The overlapping matrices are obtained by matching the momentum modes before and after the string splits. Let us suppose the string splits at worldsheet time $\tau = 0$, and let us rewrite the modes in a convenient form at $\tau = 0$. The worldsheet coordinate is given by

$$X(s) = x_0 + \frac{i}{\sqrt{2}} \sum_{n=1}^{\frac{N-1}{2}} \frac{1}{\sqrt{|\omega_n|}} \Big((a_n - a_n^\dagger + a_{-n} - a_{-n}^\dagger) \cos \frac{2\pi n s}{N} \tag{5.46}$$
$$+\, i(a_n + a_n^\dagger - a_{-n} - a_{-n}^\dagger) \sin \frac{2\pi n s}{N} \Big)$$

performing the following linear transformations

$$\frac{a_n + a_{-n}}{\sqrt{2}} \to a_n, \quad \frac{a_n^\dagger + a_{-n}^\dagger}{\sqrt{2}} \to a_n^\dagger, \tag{5.47}$$
$$\frac{a_n - a_{-n}}{\sqrt{2}} \to -ia_{-n}, \quad \frac{a_n^\dagger - a_{-n}^\dagger}{\sqrt{2}} \to ia_{-n}^\dagger$$

and substituting in (5.46) we obtain

$$\begin{aligned} X(s) &= x_0 + i \sum_{n=1}^{\frac{N-1}{2}} \frac{1}{\sqrt{|\omega_n|}} \left((a_n - a_n^\dagger) \cos \frac{2\pi n s}{N} + (a_{-|n|} - a_{-|n|}^\dagger) \sin \frac{2\pi n s}{N} \right) \\ &= x_0 + \sqrt{2} \sum_{n=1}^{\frac{N-1}{2}} \left(x_{|n|} \cos \frac{2\pi n s}{N} + x_{-|n|} \sin \frac{2\pi n s}{N} \right) \end{aligned} \tag{5.48}$$

where we have defined

$$a_n = \frac{1}{\sqrt{2}} \left(\frac{p_n}{\sqrt{|\omega_n|}} - i\sqrt{|\omega_n|} x_n \right), \quad a_n^\dagger = \frac{1}{\sqrt{2}} \left(\frac{p_n}{\sqrt{|\omega_n|}} + i\sqrt{|\omega_n|} x_n \right), \tag{5.49}$$

Going through the same change of variables for the mode expansion of P given in (5.42) we obtain

$$P(s) = \frac{1}{N} \left(p_0 + \sqrt{2} \sum_{n=1}^{\frac{N-1}{2}} \left(p_n \cos \frac{2\pi n s}{N} + p_{-|n|} \sin \frac{2\pi n s}{N} \right) \right) \tag{5.50}$$

For N even we take the positions of the bits to be at 1/2 integer points. This is so as to have a symmetric distribution of bits about the origin, therefore the expansion of the momentum mode for N even is given by

$$P(s) = \frac{1}{N}\left[p_0 + \sqrt{2}\sum_{n=1}^{\frac{N}{2}}\left(p_n \cos\frac{2\pi n(s+1/2)}{N} + p_{-|n|}\sin\frac{2\pi n(s+1/2)}{N}\right)\right] \tag{5.51}$$

The inverse relations are given by

$$p_0 = \sum_{s=-\frac{N}{2}}^{\frac{N}{2}-1} P(s), \tag{5.52}$$

$$p_n = \sqrt{2}\sum_{s=-\frac{N}{2}}^{\frac{N}{2}-1} P(s)\cos\frac{2\pi n}{N}(s+1/2),$$

$$p_{-|n|} = \sqrt{2}\sum_{s=-\frac{N}{2}}^{\frac{N}{2}-1} P(s)\sin\frac{2\pi n}{N}(s+1/2)$$

With similar inversion relations for N odd.

5.3.1 Calculation of the overlaps

Requiring momentum conservation across the point where the string splits we obtain

$$P^{(3)}(s) + P^{(1)}(s) + P^{(2)}(s) = 0 \tag{5.53}$$

$P^{(1)}(s)$ is defined on the points lying between $-(N^1-1)/2$ and $(N^1-1)/2$, while $P^{(2)}$ is defined on the points from $(N^1-1)/2+1$ to $(N^3-1)/2$ and $-(N^1-1)/2-1$ to $-(N^3-1)/2$. Here we are discussing the case when an odd number of bits of the third string splits into an odd number of bits of the first string and an even number of bits of the second string. The overlapping matrix X^1_{mn} is just the coefficient of $p^{(3)}_n$ in the expansion of $p^{(1)}_m$ in Fourier modes of $P^{(3)}(s)$. Assuming $m, n > 0$ we obtain

$$
\begin{aligned}
X^1_{mn} &= \frac{2}{N_1} \sum_{s=-\frac{N_1-1}{2}}^{\frac{N_1-1}{2}} \cos\frac{2\pi ns}{N_1} \cos\frac{2\pi ms}{N_3} = \qquad (5.54) \\
&= \frac{1}{N_1}\left(\frac{\sin\pi(\frac{n}{N_1}+\frac{m}{N_3})N_1}{\sin\pi(\frac{n}{N_1}+\frac{m}{N_3})} + \frac{\sin\pi(\frac{n}{N_1}-\frac{m}{N_3})N_1}{\sin\pi(\frac{n}{N_1}-\frac{m}{N_3})} \right) \\
X^1_{m0} &= \frac{\sqrt{2}}{N_1} \sum_{s=-\frac{N_1-1}{2}}^{\frac{N_1-1}{2}} \cos\frac{2\pi ms}{N_3}, \\
&= \frac{\sqrt{2}}{N_1}\left(\frac{\sin\pi(\frac{m}{N_3})N_1}{\sin\pi(\frac{m}{N_3})} \right), \\
X^1_{0n} &= 0, \\
X^1_{00} &= 1, \\
X^1_{0-n} = X^1_{m-n} &= \frac{2}{N_1} \sum_{s=-\frac{N_1-1}{2}}^{\frac{N_1-1}{2}} \sin\frac{2\pi ns}{N_1} \cos\frac{2\pi ms}{N_3} = 0, \\
X^1_{-m0} = X^1_{-mn} &= \frac{2}{N_1} \sum_{s=-\frac{N_1-1}{2}}^{\frac{N_1}{2}} \cos\frac{2\pi ns}{N_1} \sin\frac{2\pi ms}{N_3} = 0, \\
X^1_{-m-n} &= \frac{2}{N_1} \sum_{s=-\frac{N_1-1}{2}}^{\frac{N_1-1}{2}} \sin\frac{2\pi ns}{N_1} \sin\frac{2\pi ms}{N_3} \\
&= \frac{1}{N_1}\left(\frac{\sin\pi(\frac{n}{N_1}+\frac{m}{N_3})N_1}{\sin\pi(\frac{n}{N_1}+\frac{m}{N_3})} - \frac{\sin\pi(\frac{n}{N_1}-\frac{m}{N_3})N_1}{\sin\pi(\frac{n}{N_1}-\frac{m}{N_3})} \right)
\end{aligned}
$$

The overlap function of the second string with the third string, denoted by X^2_{mn}, is the same as X^1_{mn}, after replacement of N_1 by N_2 in the above expressions and multiplication by $(-1)^{n+m+1}$. To show this let us explicitly calculate the overlap function of the second string with the third string for a particular case. As the number of bits of the second string is even for the case under consideration, they are defined on half integer lattice points. The different overlaps, for $n > 0, m > 0$, are given by

$$\begin{aligned} X^2_{mn} &= \frac{2}{N_2}\left(\sum_{t=0}^{\frac{N_2}{2}-1} \cos(\frac{2\pi n}{N_2}(t+\frac{1}{2}))\cos(\frac{2\pi m}{N_3}(t+\frac{N_1-1}{2}+1))\right. \\ &+ \left.\sum_{t=-1}^{-\frac{N_2}{2}} \cos(\frac{2\pi n}{N_2}(t+\frac{1}{2}))\cos(\frac{2\pi m}{N_3}(t-\frac{N_1-1}{2}))\right), \\ &= \frac{1}{N_2}\left(\frac{\sin\frac{\pi m}{N_3}N_1}{\sin\pi(\frac{n}{N_2}+\frac{m}{N_3})} + \frac{\sin\pi\frac{(-m)}{N_3}N_1}{\sin\pi(\frac{n}{N_2}-\frac{m}{N_3})}\right) \\ &= (-1)^{n+m+1}\frac{1}{N_2}\left(\frac{\sin\pi(\frac{n}{N_2}+\frac{m}{N_3})N_2}{\sin\pi(\frac{n}{N_2}+\frac{m}{N_3})} + \frac{\sin\pi(\frac{n}{N_2}-\frac{m}{N_3})N_2}{\sin\pi(\frac{n}{N_2}-\frac{m}{N_3})}\right) \end{aligned} \quad (5.55)$$

Note that in the last step we have used $N_3 = N_1 + N_2$. As for the overlapping of the 3rd string with itself we have obviously $X^3_{mn} = \delta_{mn}$.

5.3.2 Construction of the vertex in the $\mu \to \infty$ limit

The SFT three-vertex is given by [22][6]

$$|V_3\rangle = \exp\left(\frac{1}{2}\sum_{r,s=1}^{3} a^{\dagger T}_{(r)} N^{rs} a^{\dagger}_{(s)}\right)|0\rangle \quad (5.56)$$

where the Neumann functions are given by

$$N^{rs} = \delta^{rs} - 2C^{1/2}_{(r)} X^{(r)T}\Gamma^{-1}X^{(s)}C^{1/2}_{(s)} \quad (5.57)$$

with

$$\Gamma = \sum_{r=1}^{3} X^{(r)} C_{(r)} X^{(r)T} \quad (5.58)$$

and

$$C_{(r)kk'} = N_r\sqrt{m^2 + 4\sin^2\frac{\pi m}{N_r}}\delta_{kk'} \quad (5.59)$$

[6]Note that the oscillator basis used here differs from that of the previous section, the relevant change of basis will be given when computing actual correlation functions, done in the next section.

In the limit under consideration, $m \to \infty$, $C_{(r)kk'}$ reduces to $N_r m \delta_{kk'}$, and hence Γ reduces to $m \sum_{r=1}^{3} N_r X^{(r)} X^{(r)T}$. In this limit Γ can be shown to be proportional to the identity. Let us examine in detail

$$\Gamma_{kk'} = m \left(N_3 \delta_{kk'} + N_1 (X^{(1)} X^{(1)T})_{kk'} + N_2 (X^{(2)} X^{(2)T})_{kk'} \right) \tag{5.60}$$

For the case $k, k' > 0$. Consider the second term in the above expression, use (5.54) and write write

$$\begin{aligned} N_1 \sum_{n=0}^{\frac{N_1-1}{2}} X^{(1)}_{kn} X^{(1)}_{k'n} &= \frac{4}{N_1} \sum_{s,t=-\frac{N_1-1}{2}}^{\frac{N_1-1}{2}} \sum_{n=1}^{\frac{N_1-1}{2}} \cos \frac{2\pi ns}{N_1} \cos \frac{2\pi ks}{N_3} \cos \frac{2\pi nt}{N_1} \cos \frac{2\pi k't}{N_3} \\ &+ \frac{2}{N_1} \sum_{s,t=-\frac{N_1-1}{2}}^{\frac{N_1-1}{2}} \cos \frac{2\pi ks}{N_3} \cos \frac{2\pi k't}{N_3} = \qquad (5.61) \\ &= \frac{2}{N_1} \sum_{s,t=-\frac{N_1-1}{2}}^{\frac{N_1-1}{2}} \sum_{n=-\frac{N_1-1}{2}}^{\frac{N_1-1}{2}} \cos \frac{2\pi ns}{N_1} \cos \frac{2\pi ks}{N_3} \cos \frac{2\pi nt}{N_1} \cos \frac{2\pi k't}{N_3} \end{aligned}$$

Summing over n we obtain the delta function $\frac{N_1}{2}(\delta(s-t) + \delta(s+t))$. Thus the sum reduces to

$$N_1 \sum_{n=0}^{\frac{N_1-1}{2}} X^{(1)}_{kn} X^{(1)}_{k'n} = 2 \sum_{t=-\frac{N_1-1}{2}}^{\frac{N_1-1}{2}} \cos \frac{2\pi kt}{N_3} \cos \frac{2\pi k't}{N_3} \tag{5.62}$$

Now let's examine the last term in (5.60), again using (5.55) we obtain

$$\begin{aligned} N_2 \sum_{n=0}^{\frac{N_2}{2}} X^{(2)}_{kn} X^{(2)}_{k'n} &= 2 \left(\sum_{t=0}^{N_2/2-1} \cos \frac{2\pi k}{N_3}(t + \frac{N_1-1}{2} + 1) \cos \frac{2\pi k'}{N_3}(t + \frac{N_1-1}{2} + 1) \right. \\ &+ \left. \sum_{t=-1}^{-N_2/2} \cos \frac{2\pi k}{N_3}(t - \frac{N_1-1}{2}) \cos \frac{2\pi k'}{N_3}(t - \frac{N_1-1}{2}) \right) = \qquad (5.63) \\ &= 2 \sum_{t=\frac{N_1+1}{2}}^{\frac{N_3-1}{2}} \cos \frac{2\pi kt}{N_3} \cos \frac{2\pi k't}{N_3} + 2 \sum_{t=-\frac{N_3-1}{2}}^{-\frac{N_1+1}{2}} \cos \frac{2\pi kt}{N_3} \cos \frac{2\pi k't}{N_3} \end{aligned}$$

Note that here we have used $N_3 = N_2 + N_1$. Adding the three terms of (5.60) we obtain

$$\Gamma_{kk'} = mN_3\delta_{kk'} + 2m \sum_{-\frac{N_3-1}{2}}^{\frac{N_3-1}{2}} \cos\frac{2\pi ks}{N_3}\cos\frac{2\pi k's}{N_3} = 2mN_3\delta_{kk'} \tag{5.64}$$

Though we have discussed the case $k, k' > 0$ we can repeat the same line of arguments for other ranges of k and arrive at the same conclusions. Thus we have the result that in the $\mu \to \infty$ limit Γ is proportional to the identity. If we consider a bit non conserving process, by going through the same algebra we will obtain a doubling of terms when the bits overlap. Therefore we cannot use the result for the final sum in (5.64), thus for such cases $\Gamma_{k,k'}$ will not be proportional to $\delta_{kk'}$.

Now that we have all the ingredients of $|V_3\rangle$ we can proceed and compute three point correlation functions.

5.4 Correlation Functions

In this section we compute correlation functions among various operators to be defined below and compare gauge and string theory results.

5.4.1 Yang Mills correlation functions

Definition of the states

Let us begin by defining the following operators [7]

$$O^J_{vac} = \frac{1}{\sqrt{J}} Tr[Z^J] \leftrightarrow |J\rangle \tag{5.65}$$

$$O^J_i = \frac{1}{\sqrt{J+1}} Tr[\phi^i Z^J] \leftrightarrow a_0^{i\dagger}|J+1\rangle \tag{5.66}$$

For a generic two-oscillators string state in the symmetric representation we propose the following YM dual

[7] A normalization factor depending on the rank of the gauge group is not included.

$$a_n^{(i\dagger}\tilde{a}_n^{j)\dagger}|J-2n+2\rangle \leftrightarrow \mathcal{O}^J_{(i,j),n} = N_n \sum_{l=0}^{J} Tr[\phi^i Z^l \phi^j Z^{J-l}] \cos\frac{\pi n(2l+1)}{J+1} \quad (5.67)$$

Obviously the dictionary reduces to that of BMN for large J. As seen in section 5.2.3, the number of string states is given by the conditions $J-2n+2 \geq 2$ and $n \geq 0$, *i.e.* we have $\frac{J+2}{2}$ states for J even and $\frac{J+1}{2}$ for J odd. Inequivalent states on the gauge theory are easily accounted for by noticing the following identifications

$$\mathcal{O}^J_{(i,j),n} \equiv \mathcal{O}^J_{(i,j),-n} \equiv -\mathcal{O}^J_{(i,j),J+1+n} \quad (5.68)$$

so we can restrict ourselves to the range $0 \leq n < \frac{J+1}{2}$, and hence we have the same quantity of states in both sides. Furthermore one can check that for the particular case $n=1$ and $J=2$ the dictionary is equivalent to (5.32), as expected. In order to fix the normalization constant and check that such states form an orthonormal basis we compute

$$< \bar{\mathcal{O}}^J_{(1,2),m}\mathcal{O}^J_{(1,2),n} >= N_n^2 \sum_{l=0}^{J} \cos\frac{\pi m(2l+1)}{J+1}\cos\frac{\pi n(2l+1)}{J+1} = N_n^2(\frac{J+1}{2})\delta_{m,n} \quad (5.69)$$

Except for $m=n=0$ where we obtain twice that result. So we choose $N_0 = \frac{1}{\sqrt{J+1}}$ and $N_{n>0} = \frac{\sqrt{2}}{\sqrt{J+1}}$.

A similar analysis can be repeated for the antisymmetric representation, in this case

$$a_n^{[i\dagger}\tilde{a}_{-n}^{j]\dagger}|J-2n+2\rangle \leftrightarrow \mathcal{O}^J_{[i,j],n} = \frac{\sqrt{2}}{\sqrt{J+2}} \sum_{l=0}^{J} Tr[\phi^i Z^l \phi^j Z^{J-l}] \sin\frac{\pi n(2l+2)}{J+2} \quad (5.70)$$

Again, the particular case $n=1$ and $J=1$ agrees with (5.32). For completeness we give the YM dual of the singlet representation

$$a_n^{i\dagger}\tilde{a}_n^{i\dagger}|J-2n+2\rangle \leftrightarrow \frac{1}{\sqrt{J+3}}\left(\frac{1}{2}\sum_{l=0}^{J} Tr[\phi^i Z^l \phi^j Z^{J-l}]\cos\frac{\pi n(2l+3)}{J+3} - 2\cos\frac{\pi n}{J+3}Tr[\bar{Z}Z^{J+1}]\right) \quad (5.71)$$

Again in the case of $m = n = 0$ the normalization factor has an additional $1/\sqrt{2}$ factor. The presence of the term with $\bar{Z}$ does not affect the counting of inequivalent states but is required for orthonormality. As before, (5.32) is a particular case of 5.71, with $n = 1$ and $J = 0$.

Notice that our dictionary coincides with that of [62], obtained from diagonalizing the one loop anomalous dimension matrix for two impurities operators.

Correlation functions

Let us begin by computing [8]

$$\langle \bar{\mathcal{O}}_1^{J_1} \bar{\mathcal{O}}_2^{J_2} \mathcal{O}^J_{(i,j);n} \rangle = N \sum_{l=0}^{J} \langle Tr[\phi^1 \bar{Z}^{J_1}] Tr[\phi^2 \bar{Z}^{J_2}] Tr[\phi^1 Z^l \phi^2 Z^{J-l}] \rangle f(l) \quad (5.72)$$

With $J = J_1 + J_2$ and $f(l) = \cos \frac{\pi n(2l+1)}{J+1}$. In order to proceed we need to compute the correlation function on the r.h.s of (5.72). As we are considering large N free SYM such correlation function is just given by the different Wick contractions, on a planar way, among the fields under consideration. It is not hard to see that

$$< Tr[\phi^1 \bar{Z}^{J_1}] Tr[\phi^2 \bar{Z}^{J_2}] Tr[\phi^1 Z^l \phi^2 Z^{J-l}] >= \begin{cases} l+1 & \text{for } l < J_2 \\ J_2 + 1 & \text{for } J_2 \leq l < J_1 \\ J_1 + J_2 - l + 1 & \text{for } J_1 \leq l \end{cases}$$

where we have supposed $J_1 \geq J_2$ without lost of generality. By plugging this into (5.72) we obtain

$$< \bar{\mathcal{O}}_1^{J_1} \bar{\mathcal{O}}_2^{J_2} \mathcal{O}^J_{(i,j);n} >= \frac{K}{2} (-1)^n \frac{\sin \frac{n\pi(J_1+1)}{J+1} \sin \frac{n\pi(J_2+1)}{J+1}}{(\sin \frac{n\pi}{J+1})^2} \quad (5.73)$$

With K some factor. In the same way it is easy to compute

[8] It is a well know result that the position dependence of three-point correlation functions of local operators is fixed in conformal field theories, for instance, for three scalar primaries operators we have $\langle O^1(x_1) O^2(x_2) O^3(x_3) \rangle = \frac{f^{1,2,3}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3} |x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1} |x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}}$, throughout this theses we will be interested in the structure constants $f^{1,2,3}$.

$$< \bar{\mathcal{O}}_1^{J_1} \bar{\mathcal{O}}_2^{J_2} \mathcal{O}^J_{[i,j];n} >= 0 \tag{5.74}$$

Finally, we can compute

$$< \bar{\mathcal{O}}^{J_1}_{vac} \bar{\mathcal{O}}^{J_2}_{(1,2);n} \mathcal{O}^J_{(1,2);m} >= \frac{N}{2} \frac{\sin(\frac{(J_2+1)\pi m}{J+1}) \sin(\frac{(J_2+1)\pi m}{J+1} + \pi(\frac{m}{J+1} - \frac{n}{J_2+1}))}{(\sin \pi(\frac{m}{J+1} - \frac{n}{J_2+1}))^2} + (n \to -n) \tag{5.75}$$

5.4.2 String computations

Consider the process of two supergravity states joinning into a string state

$$a_0^{i\dagger}|N_1\rangle + a_0^{j\dagger}|N_2\rangle \to a_n^{i\dagger}\tilde{a}_n^{j\dagger}|N_3\rangle \tag{5.76}$$

N_1, N_2 and N_3 refer to the number of bits in each string. The relevant overlap is given by

$$C_{0,0,n} = \left(\langle N_1|a_0^{(1)i} \otimes \langle N_2|a_0^{(2)j} \otimes \langle N_3|a_n^{(3)i}\tilde{a}_n^{(3)j}\right)|V_3\rangle \tag{5.77}$$

As already mentioned, in order to use the results of the previous section we should change our basis, this is simply done by

$$\begin{aligned} a_n &\to \frac{1}{\sqrt{2}}(a_{|n|} - i a_{-|n|}), \qquad \tilde{a}_n \to \frac{1}{\sqrt{2}}(a_{|n|} + i a_{-|n|}), \\ a_n^\dagger &\to \frac{1}{\sqrt{2}}(a^\dagger_{|n|} + i a^\dagger_{-|n|}), \qquad \tilde{a}_n^\dagger \to \frac{1}{\sqrt{2}}(a^\dagger_{|n|} - i a^\dagger_{-|n|}), \end{aligned} \tag{5.78}$$

Substituting this in (5.77) and using (5.54), (5.55) and (5.57) we obtain

$$C_{0,0,n} = \frac{1}{2} N_{n0}^{(31)i} N_{n0}^{(32)j} = \frac{1}{2} X_{n0}^{(1)} X_{n0}^{(2)} = \frac{\sin\frac{\pi n N_1}{N_3} \sin\frac{\pi n N_2}{N_3}}{(\sin\frac{\pi n}{N_3})^2} \tag{5.79}$$

Up to an overall normalization. Let us now look at the following two impurity process

$$|N_1\rangle + a_n^{i\dagger}\tilde{a}_n^{j\dagger}|N_2\rangle \to a_m^{i\dagger}\tilde{a}_m^{j\dagger}|N_3\rangle \tag{5.80}$$

Again performing the required change of basis in (5.78) we get the following contribution from the Neumann functions to the overlap

$$\left(\langle N_1| \otimes \langle N_2|a_n^{(2)i}\tilde{a}_n^{(2)j} \otimes \langle N_3|a_m^{(3)i}\tilde{a}_m^{(3)j}\right)|V_3\rangle = \tag{5.81}$$
$$N_{nm}^{(31)i}N_{nm}^{(31)j} - N_{-n-m}^{(31)i}N_{nm}^{(31)j} - N_{nm}^{(31)i}N_{-n-m}^{(31)j} + N_{-n-m}^{(31)i}N_{-n-m}^{(31)j} =$$
$$\left(N_{nm}^{(31)i} - N_{-n-m}^{(31)i}\right)\left(N_{nm}^{(31)j} - N_{-n-m}^{(31)j}\right) = \left(\frac{\sin\pi(\frac{n}{N_2} - \frac{m}{N_3})N_2}{\sin\pi(\frac{n}{N_2} - \frac{m}{N_3})}\right)^2$$

Up to an overall factor. Again we have used (5.54) and (5.55) and (5.57).

5.4.3 Comparison of the results

The SYM correlation function (5.73) is to be compared with the string theory correlation function (5.79). First of all we notice the same structure on both results, in particular the appearance of a term like $\sin\frac{\pi n}{N}$ in both denominators, not present in the usual BMN limit, where $n \ll N$ and hence $sin\frac{\pi n}{N} \approx \frac{n}{N}$. Actually both correlation functions agree if

$$\frac{N_1}{N_3} = \frac{J_1+1}{J+1}, \qquad \frac{N_2}{N_3} = \frac{J_2+1}{J+1}, \qquad N_3 = J+1. \tag{5.82}$$

However this is not compatible with the condition $N_1 + N_2 = N_3$. It seems as if we should “shift” the number of bits of some of the strings, by one.

Again, the same structure can be seen in the denominators of (5.75) and (5.81) and again in the numerators there is a relative phase $\pi(\frac{n}{J+1} - \frac{m}{J_2+1})$ between the two results.

Notice that the mismatch between string and gauge theory results vanish when the number of bits, or J, is very large.

5.5 Conclusions

In this chapter we have reported some attempts at finding a correspondence between strings on pp-waves (with some additional restrictions) and free $\mathcal{N}=4$ SYM.

We have found a precise match of the spectrum for string states with two oscillators, as well as for the supergravity states. There are, however, two problems one should overcame in this picture. The first has to do with the fact that there is no algorithm that allows one to find the allowed vacua when more than two oscillators are acting on them, on the other hand, the strategy to look for superconformal primaries on both sides of the correspondence is rather laborious already at level 2. [9] The second problem is that we don't know how to obtain the dual of a generic SYM state containing arbitrary number of fields Z and $\bar{Z}$, for instance $TrZ^3\bar{Z}^3$. On the positive side the KK descendants have an explicit string realization.

We have also developed a method for computing correlation functions in string theory, by using a discretized version of light-cone SFT, but there is a small mismatch between these and the corresponding SYM correlations functions. However, the fact that the correlations functions are "almost" working and have the same structure, make us think of seriously taking some kind of bit description for free $\mathcal{N} = 4$ SYM.

Another issue is that even if we have obtained an identification of the gauge theory symmetry generators in terms of string theory operators, the operators on the two sides satisfy different algebras!. A related issue is the following, let us suppose we want to work on position space, then (if the bit picture is right) there should be string oscillators such that when they act on a string of $Z's$ (the vacuum) they insert some field, let's say ϕ^i, at a given position. However, such oscillator cannot act at the same position twice! *i.e.* we should, in some way, forbid two oscillators (of this kind) to act at the same place. We don't know how these constraints will reflect on the string theory we are considering, in momentum space, but it is expected that, before getting a precise matching of states and correlation functions, one should implement this. Note that when the number of $Z's$ is very large compared with the number of impurities, these constraints are irrelevant.

[9] The matching between string and gauge states was seen to hold up to level two, including fermionic primaries.

Chapter 6

Oscillator approach to weakly coupled SYM

One of the problems of the approach followed in the previous chapter was the fact that we did not have the full $AdS_5 \times S^5$ algebra on the string side. Further, if one works in position space, one should impose some conditions on the string oscillators, since one cannot act twice on the same site with the oscillators "inserting" ϕ^i, for instance.

An oscillator description of such algebra was developed in [63], where the supergravity spectrum was computed. In this chapter we use such oscillators as string oscillators acting on position space and show how the difficulties of the previous approach are overcome.

In section 6.1 we give a brief introduction to the oscillator construction (following [63, 64, 65]) and show the dictionary between oscillator states and SYM letters. In section 6.2 we compute the single and multi-bit partition function and show that it agrees with that of free $\mathcal{N} = 4SYM$. In section 6.3 we derive the three-string vertex, by symmetry considerations, in oscillator variables and show that, for some simple examples, the correlations functions computed with such vertex agree with those of free $\mathcal{N} = 4$ SYM. In section 6.4 we consider the first order corrections in the 't Hooft coupling in the gauge theory and try to include such correction on the string theory side. In string field theory corrections to two and three points functions are fixed once the corrections to the Hamiltonian are given and in particular there is a precise relation between quantum corrections to two and three point functions. We show that such relation is actually satisfied for gauge theory correlation functions of operators in the $SO(6)$ sector. We end with some discussions of the results obtained in this chapter.

6.1 The oscillator construction

The symmetry group of type IIB superstring theory on $AdS_5 \times S^5$ is $(P)SU(2,2|4)$, whose even (bosonic) subgroup is $SU(2,2) \times SU(4) \times U(1)$. This coincides with the symmetry group (superconformal group plus isometries) of $\mathcal{N}=4$ SYM in four dimensions. In the following we show how the oscillator method can be used in order to construct the spectrum and generators of such gauge theory.

$SU(2,2)$ is the covering group of the conformal group $SO(2,4)$, we shall denote its two $SU(2)$ subgroups by $SU(2)_L$ and $SU(2)_R$. Let us consider a set of bosonic oscillators a_i, b_r and $a^i = (a_i)^\dagger, b^r = (b_r)^\dagger$, satisfying the canonical commutation relations

$$[a_i, a^j] = \delta_i^j, \qquad [b_r, b^s] = \delta_r^s \tag{6.1}$$

with $i, r = 1, 2$ an index in the fundamental representation of $SU(2)_L$ and $SU(2)_R$ respectively. The $SU(2,2)$ generators are then given by the following bilinears

$$A_{ir} = a_i b_r, \qquad A^{ir} = a^i b^r \tag{6.2}$$

$$L^i_j = a^i a_j - \frac{1}{2}\delta^i_j a^l a_l, \qquad R^r_q = b^r b_q - \frac{1}{2}\delta^r_q b^l b_l \tag{6.3}$$

$$E = \frac{1}{2}(a^i a_i + b^r b_r) + 1 = \frac{1}{2}(N_a + N_b) + 1 \tag{6.4}$$

with A_{ir}, A^{ir} the non-compact generators, L^i_j and R^r_q the generators of the $SU(2)_L$ and $SU(2)_R$ respectively and E the $U(1)$ generator in the maximal compact subgroup of $SU(2,2)$. $SU(4)$ also contains two $SU(2)$ subgroups, denoted with $SU(2)_{k_1}$ and $SU(2)_{k_2}$. We can consider a set of fermionic oscillators α_γ, β_μ and $\alpha^\gamma = (\alpha_\gamma)^\dagger, \beta^\mu = (\beta_\mu)^\dagger$, satisfying the canonical anti-commutation relations

$$\{\alpha_\gamma, \alpha^\delta\} = \delta_\gamma^\delta \qquad \{\beta_\mu, \beta^\nu\} = \delta_\mu^\nu \tag{6.5}$$

with $\gamma, \mu = 1, 2$ and index in the fundamental of $SU(2)_{k_1}$ and $SU(2)_{k_2}$ respectively. As for the case of $SU(2,2)$ we can express the generators of $SU(4)$ in terms of bilinears

$$J_{\gamma\mu} = \alpha_\gamma \beta_\mu, \qquad J^{\gamma\mu} = \alpha^\gamma \beta^\mu \tag{6.6}$$

$$M^\gamma_\delta = \alpha^\gamma \alpha_\delta - \frac{1}{2}\delta^\gamma_\delta N_\alpha, \qquad S^\mu_\nu = \beta^\mu \beta_\nu - \frac{1}{2}\delta^\mu_\nu N_\beta \tag{6.7}$$

$$J = 1 - \frac{1}{2}(N_\alpha + N_\beta) \tag{6.8}$$

We can generate the spectrum of states by defining the Fock vacuum

$$a_i|0>= b_r|0>= \alpha_\gamma|0>= \beta_\mu|0>= 0 \tag{6.9}$$

and then by acting on it with the creation operators. In addition to E and J we can define the following $U(1)$ charges

$$C = \frac{1}{2}(N_a + N_\alpha - N_b - N_\beta) \tag{6.10}$$

$$B = N_\alpha - N_\beta \tag{6.11}$$

C is the generator of the $U(1)$ in the bosonic subgroup of $AdS_5 \times S^5$ and it commutes with the rest of the generators, one can mode it out and obtain the $PSU(2,2|4)$ algebra. Rather remarkably by considering only states with zero charge C we obtain precisely the letters of $\mathcal{N} = 4$ SYM, with the following identifications

$$|0\rangle \equiv |Z\rangle \leftrightarrow Z \tag{6.12}$$

$$\alpha^\gamma \beta^\mu|0\rangle \equiv |\phi^{\gamma\mu}\rangle \leftrightarrow \phi^{\gamma\mu} = \phi^i(\sigma^i)^{\gamma\mu} \tag{6.13}$$

$$\alpha^1\alpha^2\beta^1\beta^2|0\rangle \equiv |\bar{Z}\rangle \leftrightarrow \bar{Z} \tag{6.14}$$

$$a^i\beta^\mu|0\rangle \leftrightarrow \tilde{\lambda}^{+1/2,i\mu}, \qquad b^r\alpha^\gamma|0\rangle \leftrightarrow \lambda^{+1/2,r\gamma} \tag{6.15}$$

$$a^i\beta^1\beta^2\alpha^\gamma|0\rangle \leftrightarrow \tilde{\lambda}^{-1/2,i\gamma} \qquad b^r\alpha^1\alpha^2\beta^\mu|0\rangle \leftrightarrow \lambda^{-1/2,r\mu} \tag{6.16}$$

$$a^ia^j\alpha^1\alpha^2|0\rangle \leftrightarrow F^{ij} \qquad b^ib^j\beta^1\beta^2|0\rangle \leftrightarrow \tilde{F}^{ij} \tag{6.17}$$

$$a^ib^j \leftrightarrow (\sigma^\mu)^{i,j}\partial_\mu \tag{6.18}$$

In the fermionic states the superscript denotes R-charge and tilde denotes negative "chirality" (see discussion below). It is easy to see that with these identifications we have the right index structure and degrees of freedom, moreover the quantum number E can be identified with the conformal dimension Δ, J can be identified with the R-charge and B with the so called "bonus" symmetry (see [66]) or hypercharge. It is now straightforward to construct the dual of single trace gauge invariant operators of a given length L. In order to do that we consider the Fock vacuum of level L, as the tensor product of L vacua, living at different sites:

$$|L> \equiv |0>_1 \otimes |0>_2 \otimes ... \otimes |0>_L \tag{6.19}$$

then add an additional index $s = 1, ..., L$ to the oscillators described above, labeling the site at which they act, and now act with these oscillators on the Fock vacuum (oscillators acting on different sites will commute or anticommute). So for instance [1]

$$\alpha^\gamma_{(1)}\beta^\mu_{(1)}\alpha^\delta_{(l+2)}\beta^\nu_{(l+2)}|L> \leftrightarrow Tr[\phi^{\gamma\mu}Z^l\phi^{\delta\nu}Z^{L-l-2}] \tag{6.20}$$

Due to the cyclicity of the trace one has to impose periodic boundary conditions, *i.e.* $|s_{L+1}> = |s_1>$, etc. Notice that the charge C is zero locally, *i.e.* at every site. The superconformal charges are represented as bilinears in the oscillators as well

$$Q^{-\ i\mu} = a^i\beta^\mu, \qquad Q^{-\ r\gamma} = b^r\alpha^\gamma \tag{6.21}$$

$$Q^{+\ i}_\gamma = a^i\alpha_\gamma \qquad Q^{+\ r}_\mu = b^r\beta_\mu \tag{6.22}$$

$$S^+_{i\mu} = a_i\beta_\mu, \qquad S^+_{r\gamma} = b_r\alpha_\gamma \tag{6.23}$$

$$S^{-\gamma}_{\ i} = a_i\alpha^\gamma \qquad S^{-\mu}_{\ r} = b_r\beta^\mu \tag{6.24}$$

The generators of the various groups as well as the supercharges act on a given state of length L in a natural way, *i.e.* $\mathcal{J} = \sum_{i=1}^L \mathcal{J}(i)$ with $\mathcal{J}(i)$ acting on the site i [2]. The full generators satisfy the following commutation relations

[1] A more precise dictionary will be given when computing correlation functions.

[2] one has to be careful when $\mathcal{J}(i)$ is fermionic since it will acquire a sign ± 1 when passing through the sites $1, ..., i-1$

$$[A_{ir}, A^{jq}] = \delta^q_r L^j_i + \delta^j_i R^q_r + \delta^j_i \delta^q_r E \tag{6.25}$$

$$[J_{\gamma\mu}, J^{\delta\nu}] = \delta^\nu_\mu M^\delta_\gamma + \delta^\delta_\gamma S^\nu_\mu - \delta^\nu_\mu \delta^\delta_\gamma J \tag{6.26}$$

$$\{Q^{-\ i\mu}, S^+_{j\nu}\} = \delta^\mu_\nu L^i_j - \delta^i_j S^\mu_\nu + \frac{1}{2}\delta^i_j \delta^\mu_\nu (E + J + C) \tag{6.27}$$

$$\{Q^{-\ r\gamma}, S^+_{q\delta}\} = \delta^\gamma_\delta R^r_q - \delta^r_q M^\gamma_\delta + \frac{1}{2}\delta^r_q \delta^\gamma_\delta (E + J - C) \tag{6.28}$$

$$\{Q^{+\ i}_\gamma, S^{-\delta}_{\ \ j}\} = \delta^\gamma_\delta L^i_j + \delta^i_j M^\delta_\gamma + \frac{1}{2}\delta^i_j \delta^\delta_\gamma (E - J + C) \tag{6.29}$$

$$\{Q^{+\ r}_\mu, S^{-\nu}_{\ \ q}\} = \delta^\nu_\mu R^r_q + \delta^r_q S^\nu_\mu + \frac{1}{2}\delta^r_q \delta^\nu_\mu (E - J - C) \tag{6.30}$$

It is clear that the generator B never appears in the l.h.s. of a (anti)commutator: it acts as an external automorphism.

Note that our notion of bit is very different to that of [25], see discussion in 2.3.1. The bit we are considering corresponds to the singleton representation of $SU(2,2|4)$ and not to the full supergravity spectrum.

Diffrently from the approach followed in the previous chapter, notice that now we have, by construction, the full AdS algebra. Furthermore, the fermionic nature of the oscillators α and β ensures that one cannot apply them more than a definite number of times on the same site. On the other hand one should be able to take arbitrary number of derivatives of SYM fields, and this is ensured by the bosonic nature of the $SU(2,2)$ oscillators.

6.2 Partition function

6.2.1 One bit partition function

In the following we will compute the partition function for a single bit. In order to do that we should simply compute

$$\mathcal{Z} = \sum_{S_{phys}} < S|e^{-\tau H}|S > \tag{6.31}$$

with $H = E - J = \frac{1}{2}(N_a + N_b + N_\alpha + N_\beta)$. The sum runs over physical states, *i.e.* those with zero central charge C, and as a consequence the oscillators do not act freely. The $C = 0$ condition can be imposed by inserting $e^{i\mu C}$, treating the oscillators as free and at the end integrating over μ.

$$\mathcal{Z}(q) = \int d\mu \sum_S < S|e^{-\tau H} e^{-i\mu C}|S >= \tag{6.32}$$
$$= \oint_{|x|=1} \frac{dx}{x} \left(\frac{1}{1-qx}\right)^2 \left(\frac{1}{1-qx^{-1}}\right)^2 (1+qx)^2(1+qx^{-1})^2$$

where we have defined $e^{-\tau} \equiv q$, $e^{-i\mu} \equiv x$. The contour integral has a simple pole at $x = 0$ and a double pole at $x = q$ and can be easily evaluated

$$\mathcal{Z}(q) = 1 + \frac{16q^2(1+q^2)}{(1-q^2)^3} = 1 + 16(q^2 + 4q^4 + 9q^6 + 16q^8 + ...) = 1 + 16\sum_{l=1}^{\infty} l^2 q^{2l} \tag{6.33}$$

The coefficient of q^{2l} gives the number of single bit states with $\Delta - J = l$. In order to compare our counting with that of $\mathcal{N} = 4$ SYM let us have a closer look at the single letter states. In the table (6.2.1) we can see the different single letter fields of $\mathcal{N} = 4$ SYM, together with their quantum numbers (Δ and J) and the number of physical states

Field	Δ	J	$\Delta - J$	Number
$\partial^{\mu_1}...\partial^{\mu_k} Z$	$k+1$	1	k	$(k+1)^2$
$\partial^{\mu_1}...\partial^{\mu_k} \phi^i$	$k+1$	0	k+1	$4(k+1)^2$
$\partial^{\mu_1}...\partial^{\mu_k} \bar{Z}$	$k+1$	-1	k+2	$(k+1)^2$
$\partial^{\mu_1}...\partial^{\mu_k} \lambda^+$	k+3/2	1/2	k+1	$4(k+2)(k+1)$
$\partial^{\mu_1}...\partial^{\mu_k} \lambda^-$	k+3/2	-1/2	k+2	$4(k+2)(k+1)$
$\partial^{\mu_1}...\partial^{\mu_k} F^{ij}, \bar{F}^{ij}$	k+2	0	k+2	$2(k+3)(k+1)$

In order to count physical states one should take into account the equations of motion. Let us consider a massless scalar field ϕ satisfying $\partial^\mu \partial_\mu \phi = 0$, the quantity of physical fields is the number of states of the form $\partial^{\mu_1}...\partial^{\mu_k}\phi$, *i.e.* $\frac{(k+3)(k+2)(k+1)}{6}$ minus the number of states of the form $\partial^{\mu_1}...\partial^{\mu_{k-2}}\partial^\mu\partial_\mu\phi$, *i.e.* $\frac{(k+1)(k)(k-1)}{6}$, so we obtain $(k+1)^2$ in accordance with the table 6.2.1. A similar counting can be carried out for the other fields.

The sum of all the contributions for a given $\Delta - J = k$ can be seen to be $16k^2$ in perfect agreement with (6.33). It is remarkable that the fields of $\mathcal{N} = 4$ SYM at a given level $\Delta - J$ are just multiples of perfect squares!

In the same way we can compute the partition function giving other quantum numbers, for instance, the one bit partition function for Δ turns out to be

$$\mathcal{Z}_{\Delta}(q) = \frac{2q^2(3+q)}{(1+q)^3} = 6q^2 - 16q^3 + 30q^4 - ... \tag{6.34}$$

Now the coefficient of q^l gives the quantity of one letter fields with conformal dimension $l/2$. Odd powers of q, with negative multiplicities, correspond to fermionic states.

6.2.2 Multi-bit partition function

Now that we have the single bit partition function $\mathcal{Z}(q_2)$ we can consider the partition function for states of n bits. The only subtlety arises from the fact that operators in SYM are cyclically symmetric, and we should impose this condition on the partition function. Let us define g as the generator of the cyclic group, *i.e.*

$$g|s_1 > |s_2 > ...|s_n >= |s_2 > ...|s_n > |s_1 > \tag{6.35}$$

One can compute the result of the insertion of g^m into the multi-bit analog of (6.32)

$$\int d\mu \sum_{S_1,...,S_n} < S_n|... < S_1|e^{-\tau H}e^{-i\mu C}g^m|S_1 > ...|S_n >= \left(\mathcal{Z}(q_2^{\frac{n}{(m,n)}})\right)^{(m,n)} \tag{6.36}$$

With (m, N) we denote the largest common divisor of m and n. Then cyclically invariant states can be kept by inserting a projector $P = \frac{1}{n}(1+g+g^2+...+g^{n-1})$.

More generically we can consider the following set of projectors $P_{w^l} = \frac{1}{n}(1 + w^l g + w^{2l} g^2 + ... + w^{(n-1)l} g^{n-1})$ with $w = e^{\frac{2\pi i}{n}}$, they can be seen to satisfy

$$P_{w^l}P_{w^{l'}} = \delta_{l,l'}P_{w^l}, \qquad \sum_{l=0}^{N-1} P_{w^l} = 1, \qquad gP_{w^l} = w^{-l}P_{w^l} \tag{6.37}$$

Next we can insert $\sum_{l=0}^{n-1} q_1^l P_{w^l}$, with $q_1^N = 1$, then, by summing over q_1 only the term proportional to q_1^0 is kept, *i.e.* the cyclically invariant term. The final result for the partition function for n bits is then

$$\mathcal{Z}^n(q_1, q_2) = \frac{1}{n} \sum_{l=0}^{n-1} q_1^l \left(\sum_{m=1}^{n} \left(\mathcal{Z}(q_2^{\frac{n}{(m,n)}}) \right)^{(m,n)} w^{lm} \right) \tag{6.38}$$

Let us focus for a moment on the cyclically invariant piece

$$\frac{1}{n} \sum_{m=1}^{n} \left(\mathcal{Z}(q_2^{\frac{n}{(m,n)}}) \right)^{(m,n)} = \sum_{d|n} \frac{\varphi(d)}{n} \mathcal{Z}(q_2^d)^{\frac{n}{d}} \tag{6.39}$$

The sum runs over the divisor of n and $\varphi(d)$ denotes the number of m, such that $(m, n) = \frac{n}{d}$. Since $(m, n) = \frac{n}{d}$ then $m = \frac{n}{d}a$ with $(a, d) = 1$, on the other hand, since $m \leq n$ then $a < d$ (unless $d = 1$) so $\varphi(d)$ is given by the number of coprimes with d and smaller than d, with $\varphi(1) = 1$, but this is nothing but the Euler's totient function.

The result we have obtained agrees with what is expected from Polya's theory! (see section 5.2.3). One may as well compute the terms proportional to q_1^l

$$\sum_{d|N} c(d, l) \frac{1}{N} \mathcal{Z}(q_2^d)^{\frac{N}{d}} \tag{6.40}$$

with $c(d, l)$ the so called Ramanujan's sum.

$$c(d, l) = \sum_{a=1,(a,d)=1}^{d} e^{\frac{2\pi i l a}{d}} \tag{6.41}$$

The complete partition function is then the sum over the partial partition functions for fixed numbers of bits

$$\mathcal{Z}(q_2) = \sum_{n=2}^{\infty} \mathcal{Z}^n(q_1, q_2) = \sum_{n=2}^{\infty} \sum_{l=0}^{n-1} q_1^l \sum_{d|n} \frac{c(d, l)}{n} \mathcal{Z}(q_2^d)^{\frac{n}{d}} \tag{6.42}$$

Since on the r.h.s of q_1 depends on the term in the sum I am looking at (as it is a Nth root of one) we are not allowed to write a "stringy" partition function, on which $q_1 \approx e^{2\pi i \sigma}$.

Since there are infinite number of states with a given $\Delta - J$ (for instance TrZ^J has $\Delta - J = 0$ for every J) every term in (6.2.2) is divergent and the full partition function is not very useful. One can consider, instead, the partition function encoding information on Δ

$$\mathcal{Z}_\Delta = \sum_{n=2}^{\infty} \sum_{d|n} \frac{\varphi(d)}{n} \mathcal{Z}_\Delta(q^d)^{\frac{n}{d}} = 21q^4 - 96q^5 + 376q^6 - 1344q^7 + \ldots \qquad (6.43)$$

which agrees with the free SYM partition function computed in [29]. As already mentioned, in the oscillator construction it is very easy to include the information on the other quantum numbers, just by inserting the corresponding operators in (6.33).

6.3 String overlap in the oscillator construction

In this section we derive the three string vertex by using conservation laws.

Let us first define inner products and selection rules for the states in the oscillator construction. We normalize $\langle 0|0\rangle = 1$. The dictionary with YM states implies that as $|0\rangle$ has $J_{\rm YM} = 1$, the state $\langle 0|$ has $J_{\rm YM} = -1$, however the operator $J = -\frac{1}{2}(N_\alpha + N_\beta) + 1$ still has eigenvalue $J = 1$ when acting on the state $\langle 0|$. Therefore, the conservation law for the inner product on the oscillator construction is $J_{\rm in} = J_{\rm out}$, unlike the case of YM where $J_1 + J_2 = 0$. This point has to be kept in mind for all the processes we will discuss. For instance for a three string process the conservation law is $J^{(1)} + J^{(2)} = J^{(3)}$, instead of $J^{(1)} + J^{(2)} + J^{(3)} = 0$, which is what one would have expected from intuition in YM.

Consider the "ket" state $\alpha^\dagger \sigma^i \beta^\dagger |0\rangle$, where $\sigma^i = (1, i\vec{\sigma})$, its "bra" is given by $\langle 0|\beta \bar{\sigma}^i \alpha$. Here $\bar{\sigma}^i = (1, -i\vec{\sigma})$, the inner product of these two states is 8.

In the next subsection we will construct the 3 vertex in terms of a state in the 3-Hilbert space of strings. Since the conservation laws in our conventions might be unfamiliar we will construct the two point function as a state in the 2-Hilbert space of strings and derive the conservation laws for this case. Consider the 2 point function $\langle 0|\mathcal{O}_1^\dagger \mathcal{O}_2|0\rangle$ where $\mathcal{O}_1$ and $\mathcal{O}_2$ are 2 operators written in terms of oscillators and are normal ordered. For any generator $\mathcal{J}$

which annihilates the vacuum $|0\rangle$ and $\langle 0|$ we obtain the following conservation law

$$\langle 0|[\mathcal{J}, \mathcal{O}_1^\dagger]\mathcal{O}_2|0\rangle + \langle 0|\mathcal{O}_1^\dagger[\mathcal{J}, \mathcal{O}_2]|0\rangle = 0 \tag{6.44}$$

A similar analysis also holds for the generators for which $|0\rangle$ and $\langle 0|$ are eigenvectors with the same eigenvalue, for instance the J charge and Δ. Thus the following generators satisfy the above conservation law (as explained in section 6.1 $\mathcal{J} = \sum_s \mathcal{J}_{(s)}$)

$$\begin{aligned}
J = -\frac{1}{2}(N_\alpha + N_\beta) + 1, \qquad & \Delta = \frac{1}{2}(N_a + N_b) + 1, \\
M^+ = \alpha_1^\dagger \alpha_2, \quad M^- = \alpha_2^\dagger \alpha_1, \quad & M^3 = \frac{1}{2}(\alpha_1^\dagger \alpha_1 - \alpha_2^\dagger \alpha_2), \\
S^+ = \beta_1^\dagger \beta_2, \quad S^- = \beta_2^\dagger \beta_1, \quad & S^3 = \frac{1}{2}(\beta_1^\dagger \beta_1 - \beta_2^\dagger \beta_2), \\
, L^+ = a_1^\dagger a_2, \quad L^- = a_2^\dagger a_1, \quad & L^3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2), \\
R^+ = b_1^\dagger b_2, \quad R^- = b_2^\dagger b_1, \quad & R^3 = \frac{1}{2}(b_1^\dagger b_1 - b_2^\dagger b_2), \\
& Q_{ij}^{+a} = a_i^\dagger \alpha_j, \quad S_{ij}^{-a} = \alpha_i^\dagger a_j, \\
& Q_{ij}^{+b} = b_i^\dagger \beta_j, \quad S_{ij}^{-b} = \beta_i^\dagger b_j,
\end{aligned} \tag{6.45}$$

$SU(2)$ indices are contracted with the invariant Levi-Civita tensor, *i.e.* $\alpha_1^\dagger \alpha_2 = \alpha_\gamma^\dagger \epsilon^{\gamma\delta} \alpha_\delta = \alpha^1 \alpha_2 - \alpha^2 \alpha_1$. Note that all these generators commute with the Hamiltonian $H = \Delta - J$. The 2-point function of the physical states can be written in terms of a state in the 2-Hilbert space as

$$\langle 0|\mathcal{O}_1^\dagger \mathcal{O}_2|0\rangle = \left(\langle 0|\mathcal{O}_1^\dagger \otimes \langle 0|\mathcal{O}_2^\dagger\right)|V_2\rangle \tag{6.46}$$

where $|V_2\rangle$ is given by

$$\begin{aligned}
&|V_2\rangle = \\
&\exp\left[\sum_{s=0}^{l}(\alpha_{(s)}^{(1)\dagger}\alpha_{(s)}^{(2)\dagger} + \beta_{(s)}^{(1)\dagger}\beta_{(s)}^{(2)\dagger} + a_{(s)}^{(1)\dagger}a_{(s)}^{(2)\dagger} + b_{(s)}^{(1)\dagger}b_{(s)}^{(2)\dagger})\right]|0\rangle^{(1)} \otimes |0\rangle^{(2)}
\end{aligned} \tag{6.47}$$

Again, contraction of $SU(2)$ indices is done by means of the by the invariant tensor ϵ^{12}. Only such an invariant reproduces the two point function $\langle 0|\beta\bar{\sigma}^i\alpha\alpha^\dagger\sigma^j\beta^\dagger|0\rangle = 2\delta^{ij}$

Let us now write the conservation laws in (6.44) as an operator equation on the 2-vertex $|V_2\rangle$, we obtain

$$^{(1)}\langle 0|[\mathcal{J}, \mathcal{O}_1^\dagger] \otimes {}^{(2)}\langle 0|\mathcal{O}_2^\dagger + {}^{(1)}\langle 0|\mathcal{O}_1^\dagger \otimes {}^{(2)}\langle 0|[\mathcal{J}, \mathcal{O}_2^\dagger] \ \ |V_2\rangle = 0 \tag{6.48}$$

for any two operators. This implies the following operator equation on $|V_2\rangle$.

$$\mathcal{J}^{(1)}|V_2\rangle = \mathcal{J}^{\dagger(2)}|V_2\rangle \tag{6.49}$$

for generators which do not carry any $SU(2)$ indices, if G carries $SU(2)$ indices, a careful analysis shows that $G^\dagger$ also involves change of the $SU(2)$ labels by the ϵ tensor.

One can think of $|V_2\rangle$ as a solution to the operator equations corresponding to the conservation laws (6.49). The solution is given by (6.47). The generators which do not annihilate the vacuum also give rise to operator equations for $|V_2\rangle$ for instance Q^-_{ij}, S^+_{kl} anti-commute to generators which either annihilate the vacuum or have a definite eigen value on the vacuum. Thus, they give rise to operator equations of the following type for $|V_2\rangle$

$$[Q^-_{ij}, S^+_{kl}]^{(1)}|V_2\rangle = [Q^{-1}_{ij}, S^+_{kl}]^{(2)\dagger}|V_2\rangle \tag{6.50}$$

Now we proceed to compute the three string vertex, as a simple generalization of the two string vertex 6.47.

6.3.1 Length conserving processes

In a length conserving process, 2 strings of length $l^{(1)}$ and $l^{(2)}$ join together to form a string of length $l^{(3)} = l^{(1)} + l^{(2)}$, see figure (6.1,A) . The interaction vertex for these strings is given by the delta function overlap, bits of the string 1 and 2 are put in one to one correspondence with the bits of string 3. The $l^{(1)}$ bits of the first string overlap with the first $l^{(1)}$ bits of string 3 and $l^{(2)}$ bits of the second string overlap with the next $l^{(2)}$ bits of the third string. Thus we write the interaction vertex of three string states

$$\langle 0|\mathcal{O}^{\dagger(1)}\mathcal{O}^{\dagger(2)}\mathcal{O}^{(3)}|0\rangle = \left(\langle 0|\mathcal{O}^{\dagger(1)} \otimes \langle 0|\mathcal{O}^{\dagger(2)} \otimes \langle 0|\mathcal{O}^{\dagger(3)}\right)|V_3\rangle \tag{6.51}$$

The overlap of these bits are determined by the scalar product discussed in the previous section. Going through the same analysis as in the previous section for the two strings overlap we find that the three vertex $|V_3\rangle$ satisfies the following conservation laws

$$(\mathcal{J}^{(1)} + \mathcal{J}^{(2)})|V_3\rangle = \mathcal{J}^{(3)\dagger}|V_3\rangle \tag{6.52}$$

where $\mathcal{J}^{(1)}$, $\mathcal{J}^{(2)}$, $\mathcal{J}^{(3)}$ is one of the various charges listed in (6.45), corresponding to the strings 1, 2 and 3 respectively. Thus the construction of the three vertex is naturally obtained by generalizing the two vertex in (6.47). The vertex which satisfies all these conservation laws is given by

$$\begin{aligned} |V_3\rangle &= \exp\left[\sum_{s=0}^{l^{(1)}-1}(\alpha_s^{(1)\dagger}\alpha_s^{(3)\dagger}+\beta_s^{(1)\dagger}\beta_s^{(3)\dagger}+a_s^{(1)\dagger}a_s^{(3)\dagger}+b_s^{(1)\dagger}b_s^{(3)\dagger})\right. \\ &+ \left.\sum_{s=l^{(1)}}^{l^{(3)}-1}(\alpha_s^{(2)\dagger}\alpha_s^{(3)\dagger}+\beta_s^{(2)\dagger}\beta_s^{(3)\dagger}+a_s^{(2)\dagger}a_s^{(3)\dagger}+b_s^{(2)\dagger}b_s^{(3)\dagger})\right] \end{aligned} \tag{6.53}$$

Note that to obtain the conservation laws in (6.52) the contraction structure in the bilinears is performed by the $SU(2)$ invariant tensor ϵ_{ij}. We will verify that this reproduces the structure constants for the three point functions.

6.3.2 Length non-conserving process

Consider a process in which two strings of length $l^{(1)}$ and $l^{(2)}$ join together to form a string of length $l^{(3)}$, with $l^{(1)}+l^{(2)}-2l=l^{(3)}$, here $2l$ refers to the length "loss" in the interaction, see figure (6.1,B).

The length violation is always even as l bits of the string 1 overlaps with l bits of the string 2, thus $l^{(3)}$ is shorter by $2l$ bits. We will choose $l^{(3)}$ to be such that $l^{(3)}\geq l^{(2)}\geq l^{(1)}\geq l$, The interaction vertex is constructed by the overlap of the $l^{(1)}-l$ bits of the first string with the first $l^{(1)}-l$ bits of the third string, the remaining l bits of the first string overlap with the first l bits of the second string. Finally, the remaining $l^{(2)}-l$ bits of the second string overlap with the bits left over of the third string. In terms of correlation function, this vertex is represented as

$$\langle 0|\mathcal{O}^{(1)\dagger}_{l^{(1)}-l}\mathcal{O}^{(1)\dagger}_{l}\mathcal{O}^{(2)\dagger}_{l^{(2)}-l}\mathcal{O}^{(2)}_{l}\mathcal{O}^{(3)}|0\rangle = \left(\langle 0|\mathcal{O}^{(1)\dagger}_{l^{(1)}-l}\mathcal{O}^{(1)\dagger}_{l}\otimes\langle 0|\mathcal{O}^{(2)\dagger}_{l^{(2)}-l}\mathcal{O}^{(2)\dagger}_{l}\otimes\langle 0|\mathcal{O}^{(3)\dagger}\right)|V_3\rangle \tag{6.54}$$

Here the subscript l refers to those bits in string one and two which overlap with one another. The conservation laws on $|V_3\rangle$ derived from the above correlation function are the following

$$\begin{aligned} \mathcal{J}^{(1)}_{l^{(1)}-l}+\mathcal{J}^{(2)}_{l^{(2)}-l} &= \mathcal{J}^{(3)\dagger}, \\ \mathcal{J}^{(1)}_{l} &= \mathcal{J}^{(2)\dagger}_{l} \end{aligned} \tag{6.55}$$

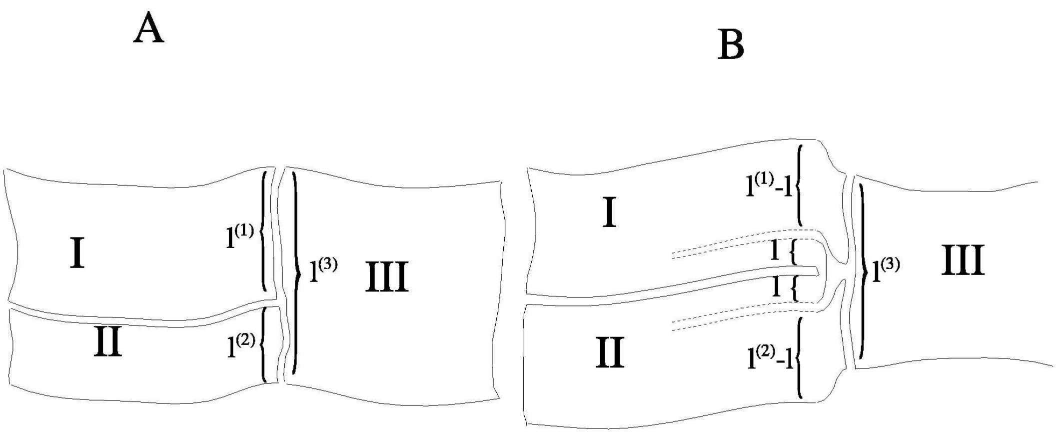

Figure 6.1: Joining of strings 1 and 2 into string 3, for a length conserving process (A) and a length non conserving process (B). Remember that every strip represent a closed string world-sheet, so we have periodic boundary conditions.

The three vertex satisfying these conservation laws is given by

$$
\begin{aligned}
|V_3\rangle &= \exp\left[N^{(13)} + N^{(23)} + N^{(12)}\right], \qquad (6.56)\\
N^{(13)} &= \sum_{s=0}^{l^{(1)}-1-l} \alpha(s)^{(1)}\alpha(s)^{(3)} + \beta(s)^{(1)}\beta(s)^{(3)} + a(s)^{(1)}a(s)^{(3)} + b(s)^{(1)}b(s)^{(3)},\\
N^{(23)} &= \sum_{s=l^{(1)}-l}^{l^{(3)}-1} \alpha(s+l)^{(2)}\alpha(s)^{(3)} + \beta(s+l)^{(2)}\beta(s)^{(3)} + a(s+l)^{(2)}a(s)^{(3)} + b(s+l)^{(2)}b(s)^{(3)},\\
N^{(12)} &= \sum_{s=l^{(1)}-l}^{l^{(3)}-1} \alpha(s)^{(1)}\alpha(s)^{(2)} + \beta(s)^{(1)}\beta(s)^{(2)} + a(s)^{(1)}a(s)^{(2)} + b(s)^{(1)}b(s)^{(2)},
\end{aligned}
$$

6.3.3 Examples

In the following we consider some examples of various processes to test the vertex (6.53). Consider the following Yang-Mill operators

$$O^{(1)} = \frac{1}{\sqrt{N^{J_1+1}}}\mathrm{Tr}(\phi^i Z^{J_1}), \quad O^{(2)} = \frac{1}{\sqrt{N^{J_2+1}}}\mathrm{Tr}(\phi^j Z^{J_2}), \tag{6.57}$$
$$O^{(3)} = \frac{1}{\sqrt{N^{J_1+J_1+2}}}\mathrm{Tr}(\phi^i \bar{Z}^J \phi^j \bar{Z}^{J_1+J_1-J}), \quad i \neq j, J_1 < J_2.$$

The normalizations are chosen such that in the large N limit the leading term in their two point function is canonically normalized. The two point function of $\phi^i_{ab}(x_1)\phi^j_{a'b'}(x_2)$ is given by $\delta^{ij}\delta_{aa'}\delta_{bb'}/|x_1 - x_2|^2$. The three point functions of these operators in the large N limit is given by [3]

$$\begin{aligned}
\langle \mathcal{O}^{(1)}(x_1)\mathcal{O}^{(2)}(x_2)\mathcal{O}^{(3)}(x_3)\rangle &= \frac{J+1}{N|x_2-x_3|^{2(J_2+1)}|x_3-x_1|^{2(J_1+1)}}, \quad J < J_1 \qquad (6.58)\\
&= \frac{J_1+1}{N|x_2-x_3|^{2(J_2+1)}|x_3-x_1|^{2(J_1+1)}}, \quad J_1 \leq J \leq J_2\\
&= \frac{J_1+J_2-J+1}{N|x_2-x_3|^{2(J_2+1)}|x_3-x_1|^{2(J_1+1)}}, \quad J > J_2
\end{aligned}$$

The multiplicity occurring in the structure constants are due to the cyclicity of the trace, see section 5.4. To evaluate this correlation function using the string vertex, we first set up the dictionary for the states.

$$^{1}\langle J_1+1| \textstyle\sum_{s=0}^{J_1} \alpha(s)\bar{\sigma}^i\beta(s) \rightarrow \mathcal{O}^{(1)}, \tag{6.59}$$
$$\textstyle\sum_{t=0}^{J_1+J_2+1} \alpha^\dagger(t)\sigma^i\beta^\dagger(t)\alpha^\dagger(t+J+1)\sigma^j\beta^\dagger(t+J+1)|J_1+J_2+2\rangle^{(3)} \rightarrow \mathcal{O}^{(3)}$$
$$^{(2)}\langle J_2+1| \textstyle\sum_{u=J_1+1}^{J_1+J_2+1} \alpha(u)\bar{\sigma}^i\beta(u) \rightarrow \mathcal{O}^{(2)}$$

Evaluating the vertex on these states we obtain

$$\begin{aligned}
\langle \mathcal{O}^{(1)}| \otimes \langle \mathcal{O}^{(2)}| \otimes \langle \mathcal{O}^{(3)}| \; |V_3\rangle &= \sum_{t=0}^{J_1+J_2+1}\sum_{u=J_1+1}^{J_1+J_2+1}\sum_{s=0}^{J_1} \delta(s,t)\delta(u,t+J+1)\\
&= J+1, \quad J < J_1, \qquad (6.60)\\
&= J_1+1, \quad J_1 \leq J \leq J_2,\\
&= J_1+J_2-J+1 \quad J > J_2
\end{aligned}$$

[3] As in the previous chapter we will be interested in the structure constant of such correlation functions.

Up to an overall factor, in perfect agreement with (6.58).

As an example of length non conserving process, consider the following operators

$$\mathcal{O}_1 = \frac{1}{\sqrt{N^{J_1+2}}} Tr(\phi^j Z^l \phi^i Z^{J_1-l}) \tag{6.61}$$

$$\mathcal{O}_2 = \frac{1}{\sqrt{N^{J_2+2}}} Tr(\phi^i Z^m \phi^k Z^{J_2-m}) \tag{6.62}$$

$$\mathcal{O}_3 = \frac{1}{\sqrt{N^{J_3+2}}} Tr(\phi^j \bar{Z}^n \phi^k \bar{Z}^{J_3-n}) \tag{6.63}$$

With $i \neq j \neq k$ and $J_1 + J_2 = J_3$. The correlation function of these three operators is given by (up to the position dependence)

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \frac{\delta_{l+m,n}}{N} \tag{6.64}$$

For this process the length violation is two units. The corresponding string states are

$$\begin{aligned}
\sum_{s=0}^{J_1+1} \alpha^\dagger(s)\sigma^i\beta^\dagger(s)\alpha^\dagger(s+l+1)\sigma^j\beta^\dagger(s+l+1)|J_1+2\rangle^{(1)} &= |\mathcal{O}^{(1)}\rangle \\
\sum_{t=J_1+1}^{J_3+2} \alpha^\dagger(t)\sigma^i\beta^\dagger(t)\alpha^\dagger(t+m+1)\sigma^j\beta^\dagger(t+m+1)|J_2+2\rangle^{(2)} &= |\mathcal{O}^{(2)}\rangle \\
\sum_{u=0}^{J_3+1} \alpha^\dagger(u)\sigma^i\beta^\dagger(u)\alpha^\dagger(u+n+1)\sigma^j\beta^\dagger(u+n+1)|J_3+2\rangle^{(3)} &= |\mathcal{O}^{(3)}\rangle
\end{aligned}$$

Evaluating the length non conserving vertex (6.56) with $l = 1$ we obtain

$$\langle \mathcal{O}^{(1)}| \otimes \langle \mathcal{O}^{(2)}| \otimes \langle \mathcal{O}^{(3)}| \ \ |V_3\rangle = \sum_{u=0}^{J_3+1}\sum_{s=0}^{J_1+1}\sum_{t=J_1+1}^{J_3+2} \delta_{s,u}\delta_{s+l+1,J_1+1}\delta_{t,J_1+1} = \delta_{m+l,n}$$

In perfect agreement with (6.64). Let us notice that all the SYM correlations functions have a factor $\frac{1}{N}$, so the string coupling constant should be proportional to $\frac{1}{N}$ and the three vertex should include a normalization factor $\frac{1}{N}$,as expected.

6.4 Quantum corrections

In the previous section we have seen that SFT two and three point correlation functions agree with those of free $\mathcal{N}=4$ SYM. In this section we include the first order corrections in the 't Hooft coupling λ for the gauge theory correlations functions and then try to obtain them from the string theory side. We will focus on single trace operators composed only by scalars without derivatives, *i.e.* we restrict ourselves to the $SO(6)$ sector. Operators of this kind do not mix, at one loop, with other kinds of operators, so it is consistent to consider such subsector by itself.

As shown in [6] these operators can be naturally mapped into states of an $SO(6)$ spin chain and remarkably the one loop anomalous dimension matrix is given by the integrable Hamiltonian of such spin chain. Since then a huge industry has been developed in order to compute anomalous dimensions in $\mathcal{N}=4$ SYM [7, 8, 67] . Quantum corrections to three point functions have been computed in [68] for general operators in the $SO(6)$ sector.

In SFT n-points correlations functions are determined by the Hamiltonian. In the previous sections we have seen that the free Hamiltonian actually reproduces two and three point correlation functions of the free gauge theory. At the moment we don't know how to introduce quantum corrections to the string Hamiltonian, but it is interesting to consider the first order corrections in the 't Hooft coupling to the Hamiltonian and to see whether SFT, with such "interaction" Hamiltonian as an input, reproduces the first order corrections to two and three points gauge theory correlations functions. Remarkably, the answer turns out to be positive for gauge operators in the $SO(6)$ sector.

In the following we give a brief review of how one loop quantum corrections to two and three point functions are computed in gauge theory and then compare such results with those of SFT.

6.4.1 One loop corrections to two and three gauge theory correlation functions

Given two bare scalar conformal primary operators $\mathcal{O}^B$ with free scaling dimension Δ_0 the two point function to first order in λ takes the form

$$\langle \bar{\mathcal{O}}^B_\alpha(x_1)\mathcal{O}^B_\beta(x_2)\rangle = \frac{g_{\alpha\rho}}{|x_{12}|^{2\Delta_0}}(\delta^\rho_\beta - g^{\rho\sigma}h_{\sigma\beta}\log(x_{12}^2\Lambda^2)) \tag{6.65}$$

$(g^{-1}h)^\alpha_\beta$ is the anomalous dimension matrix. By a unitary transformation we can diagonalize the matrices: $U^\dagger g U = \delta_{\alpha\beta} A_\alpha$ and $U^\dagger (g^{-1}h) U = \delta^\alpha_\beta \gamma_\alpha$, with A_α the normalization and γ_α the anomalous dimension. Decomposing the normalization as $A_\alpha = N_\alpha^2(1 + 2a_\alpha\lambda + \mathcal{O}(\lambda^2))$, where a_α is a scheme dependent constant, the two point function for a orthogonalized operator $\mathcal{O}_\alpha$ becomes

$$\langle \bar{\mathcal{O}}_\alpha(x_1)\mathcal{O}_\alpha(x_2)\rangle = \frac{N_\alpha^2(1+2a_\alpha\lambda)}{|x_{12}|^{2\Delta_0}}(1 - \gamma_\alpha Log|x_1 2\Lambda|^2). \tag{6.66}$$

The two point function of renormalized primary operators $\mathcal{O}_\alpha^R = \mathcal{O}_\alpha(1 - a_\alpha\lambda + \gamma_\alpha Log|\Lambda/\mu|)$ takes the form

$$\langle \bar{\mathcal{O}}^R_\alpha(x_1)\mathcal{O}^R_\alpha(x_2)\rangle = \frac{N_\alpha^2}{|x_{12}|^{2\Delta_0}|x_{12}\mu|^{2\gamma_\alpha}}. \tag{6.67}$$

As required by conformal invariance. Single trace gauge invariant operators composed just by scalars fields can be put in one to one correspondence with states of an $SO(6)$ spin chain

$$Tr(\phi^{i_1}\phi^{i_2}...\phi^{i_L}) \leftrightarrow |\phi^{i_1}\rangle...|\phi^{i_L}\rangle \tag{6.68}$$

Very remarkably, the anomalous dimensions of such operators are given by the integrable Hamiltonian of the $SO(6)$ spin chain [6]:

$$H = \frac{\lambda}{16\pi^2}\sum_{i=1}^{L} H_{i,i+1}, \qquad H_{i,i+1} = K_{i,i+1} + 2I_{i,i+1} - 2P_{i,i+1} \tag{6.69}$$

$H_{i,i+1}$ is a two body operator, acting on the sites i and $i+1$, *i. e.* the spin chain Hamiltonian describes a nearest neighbor interaction. K, I and P are called trace, identity and permutation operators and their action on the fields under consideration is given by

$$K\ |\phi^I\rangle|\phi^J\rangle = \delta^{IJ}|\phi^K\rangle|\phi^K\rangle \tag{6.70}$$
$$I\ |\phi^I\rangle|\phi^J\rangle = |\phi^I\rangle|\phi^J\rangle \tag{6.71}$$
$$P\ |\phi^I\rangle|\phi^J\rangle = |\phi^J\rangle|\phi^I\rangle \tag{6.72}$$

As two simple examples consider the length two operator in the completely symmetric traceless representation and the Konishi scalar

$$Tr(\phi^{(I}\phi^{J)}) \leftrightarrow |\phi^{(I}\rangle|\phi^{J)}\rangle = |\phi^I\rangle|\phi^J\rangle + |\phi^J\rangle|\phi^I\rangle - \frac{1}{3}\delta^{IJ}|\phi^K\rangle|\phi^K\rangle \quad (6.73)$$
$$Tr(\phi^I\phi^I) \leftrightarrow |\phi^I\rangle|\phi^I\rangle \quad (6.74)$$

It is easy to see that

$$H_{12}|\phi^{(I}\rangle|\phi^{J)}\rangle = 0 \quad (6.75)$$
$$H_{12}|\phi^I\rangle|\phi^I\rangle = 6|\phi^I\rangle|\phi^I\rangle \Rightarrow H|\phi^I\rangle|\phi^I\rangle = 12(\frac{\lambda}{16\pi^2})|\phi^I\rangle|\phi^I\rangle \quad (6.76)$$

The state in the symmetric traceless representation is a BPS state and its anomalous dimension is zero, as expected, as for the Konishi scalar, its anomalous dimension (6.76) coincides with that computed in [69].

Conformal invariance constrains the three-point function for renormalized primary operators to be

$$\langle \mathcal{O}^R_\alpha(x_1)\mathcal{O}^R_\beta(x_2)\mathcal{O}^R_\rho(x_3)\rangle = \frac{N_\alpha N_\beta N_\rho c_{\alpha\beta\rho}}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\rho}|x_{13}|^{\Delta_\alpha-\Delta_\beta+\Delta_\rho}|x_{23}|^{\Delta_\alpha+\Delta_\beta+\Delta_\rho}|\mu|^{\gamma_\alpha+\gamma_\beta+\gamma_\rho}} \quad (6.77)$$

where $c_{\alpha\beta\rho}$ is the OPE coefficient. We are interested in finding the one-loop correction to the structure constant. Therefore, we decompose

$$c_{\alpha\beta\rho} = c^0_{\alpha\beta\rho}(1 + \lambda c^1_{\alpha\beta\rho} + \mathcal{O}(\lambda^2)) \quad (6.78)$$

Where we have supposed $c^0_{\alpha\beta\rho} \neq 0$. In computing the quantum corrections to three point functions two classes of Feynman diagrams contribute, pair-wise contributions, arising already in two point functions and three-wise contributions, acting on the three operators, see figure 6.2.

In [68] the quantum corrections for the three point functions of $SO(6)$ operators was found to be

$$\lambda c^1_{\alpha\beta\rho} = -\frac{1}{2}(\gamma_\alpha + \gamma_\beta + \gamma_\rho) - \frac{\lambda}{16\pi^2}(b_{12} + b_{23} + b_{31}) \quad (6.79)$$

Where b_{ij} denote the pair-wise contributions among the operators i and j, or, in another words, the sandwich of H in (6.69) among the overlapping between the operators i and j. Notice the interesting result that $c^1_{\alpha\beta\rho}$ can be entirely expressed in terms of pairwise quantities!.

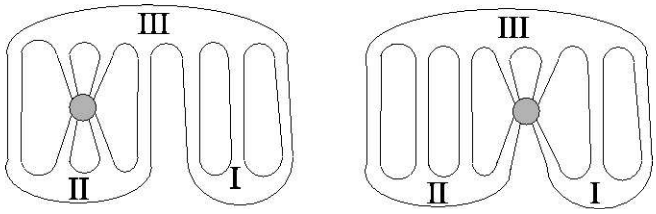

Figure 6.2: Pair-wise contributions (left) appear already in two point functions, the fat vertex denotes all kinds of contributions, quartic interactions, gluon exchange and self energy. On the right we see genuine three-wise interactions.

6.4.2 Comparison with string field theory

From 6.47 and 6.56 we see we can write the SFT vertices (at zeroth order in the 't Hooft coupling) as

$$|V_2\rangle^0 = e^{H^0_{12}}|0\rangle^{(1)}|0\rangle^{(2)} \tag{6.80}$$
$$|V_3\rangle^0 = e^{H^0_{12}+H^0_{23}+H^0_{31}}|0\rangle^{(1)}|0\rangle^{(2)}|0\rangle^{(3)} \tag{6.81}$$

With H^0_{ij} the "Hamiltonian" action on the strings i and j, etc, *i.e.*:

$$H^0_{ij} = \sum_s \alpha^{(i)}_{(s)}\alpha^{(j)}_{(s)} + \beta^{(i)}_{(s)}\beta^{(j)}_{(s)} + a^{(i)}_{(s)}a^{(j)}_{(s)} + b^{(i)}_{(s)}b^{(j)}_{(s)} \tag{6.82}$$

The sum runs over the sites in which the strings under consideration overlap. When focusing on the $SO(6)$ sector it is enough to consider just the fermionic oscillators α and β, but the following discussion is general.

As already mentioned, at the moment we don't know how to introduce corrections in the 't Hooft coupling to the string Hamiltonian, but we may suppose that there is an interaction Hamiltonian such that up to order one in λ

$$H_{ij} = H^0_{ij} + H^1_{ij} \tag{6.83}$$

H^1_{ij} is the anomalous dimension Hamiltonian (6.69) (properly expressed in terms of oscillators) acting on the strings i and j, however we don't need its explicit form in the following discussion [4]. Then, with this new Hamiltonian the two and three vertices will get corrected, for instance

$$\begin{aligned} |V_2\rangle &= |V_2\rangle^0 + |V_2\rangle^1 + \mathcal{O}(\lambda^2) = e^{H^0_{12}+H^1_{12}}|0\rangle^{(1)}|0\rangle^{(2)} = \\ &= (1 + H^1_{12} + \mathcal{O}(\lambda^2))e^{H^0_{12}}|0\rangle^{(1)}|0\rangle^{(2)} \\ |V_2\rangle^1 &= H^1_{12}|V_2\rangle^0 \end{aligned} \tag{6.84}$$

In the same way it is easy to check that

$$|V_3\rangle^1 = (H^1_{12} + H^1_{23} + H^1_{31})|V_3\rangle^0 \tag{6.85}$$

Given three string states $|\alpha\rangle$, $|\beta\rangle$ and $|\rho\rangle$, we have

$$\begin{aligned} \langle\alpha|\langle\alpha|V_2\rangle^1 &= \gamma_\alpha, \quad \langle\beta|\langle\beta|V_2\rangle^1 = \gamma_\beta, \quad \langle\rho|\langle\rho|V_2\rangle^1 = \gamma_\rho \\ \langle\alpha|\langle\beta|V_2\rangle^1 &\equiv -\frac{\lambda}{16\pi^2}b_{12}, \quad \langle\beta|\langle\rho|V_2\rangle^1 \equiv -\frac{\lambda}{16\pi^2}b_{23}, \quad \langle\rho|\langle\alpha|V_2\rangle^1 \equiv -\frac{\lambda}{16\pi^2}b_{31} \end{aligned}$$

Then it is straightforward to compute the corrections to three point functions for the normalized string states $|\tilde{s}\rangle = \frac{|s\rangle}{\sqrt{1+\lambda_s}}$

$$\langle\tilde{\alpha}|\langle\tilde{\beta}|\langle\tilde{\rho}||V_3\rangle^1 = -\frac{1}{2}(\gamma_\alpha + \gamma_\beta + \gamma_\rho) - \frac{\lambda}{16\pi^2}(b_{12} + b_{23} + b_{31}) \tag{6.86}$$

In perfect agreement with (6.79)!. Many comments are in order, first of all notice that the SFT answer is completely general and it should apply also for the full SYM spectrum and not just to the $SO(6)$ subsector. Second, notice that the SFT answer is very easy to get and to understand, corrections to n-point functions are fixed once the world-sheet Hamiltonian corrections are given, in another words, once the norm of the states is known up to that order. On the other hand, in the gauge theory getting (6.79) is rather non trivial and it becomes even more complicated for operators which don't belong to the $SO(6)$ sector.

[4] As the string Hamiltonian is $\Delta - J$ and we expect J not to receive any quantum corrections, then the quantum corrections to the Hamiltonian are given by the quantum corrections to the conformal dimensions.

6.5 Conclusions

In this chapter we have shown how the spectrum, together with all the generators, of free $\mathcal{N} = 4$ SYM can be obtained from the oscillators construction. In order to get the precise algebra and spectrum, the nature (bosonic or fermionic) of the oscillators seems to play a fundamental role. Notice that the notion of bits arising from the oscillator construction is very different to that of [25], as there a single bit behaves basically as a superparticle on AdS, with the full supergravity spectrum and not only the doubleton representation.

The partition function of the oscillator construction was shown to agree with that of free $\mathcal{N} = 4$ SYM, though we were not able to write it as a string theory partition function.

Thinking of these oscillators as string oscillators acting on position space we have obtained the three strings vertex, which allowed us to compute correlations functions on the string theory. We have tested, with success, some simple correlations functions for length conserving and length non conserving processes.

In SFT once the norm of the states is known (up to some given order in the 't Hooft coupling) two and three point correlation functions are given, this implies, in particular, that quantum corrections to gauge theory three points correlation functions can be expressed in terms of pair wise quantities, appearing already in two point correlation functions. This was shown explicitly to be the case for gauge operators in the $SO(6)$ sector, where SFT and gauge theory answer are in perfect agreement.

The next step would be to test the SFT "prediction" for gauge operators not belonging to the $SO(6)$ sector [70]. An interesting sector is given by considering operators of the form [8] $Tr(D_\mu^k\phi...D_\mu^{k'}\phi)$, *i.e.* covariant derivatives on one complex direction acting on a fixed kind of scalar field. Such subsector doesn't mix with others at one loop, further its spectrum at one loop contains a rich structure of Harmonic numbers and we have a richer structure of primaries and descendants as well (for the $SO(6)$ sector all the operators are primaries). On the string theory side the relevant oscillators are bosonic, consistent with the fact that for a given length there are infinite number of gauge invariant operators. We should also mention that the gauge theory computation seems to be rather non trivial, since there are many diagrams one should add, even for a small length and number of derivatives, so we find such computation interesting by itself.

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Appendix A

Notations

In most of the theses we use $a_n, \tilde{a}_n$ for bosonic annihilation oscillators, left and right respectively, with $n > 0$ and $a_n^\dagger, \tilde{a}_n^\dagger$ for the corresponding creation oscillators.

In chapter 3 we have use α_n^1, α_n^2 for bosonic left and right oscillators, with $n > 0$ for annihilation and $n < 0$ for creation oscillators.

Any possible changes of such conventions during the text are clearly indicated.

The commutation relations between the bosonic generators of the $AdS_5 \times S^5$ algebra are

$$\left[P^a, P^b\right] = J^{ab} \qquad \left[P^{a'}, P^{b'}\right] = -J^{a'b'}, \tag{A.1}$$

$$\left[P^a, J^{bc}\right] = \eta^{ab}P^c - \eta^{ac}P^b, \qquad \left[P^{a'}, J^{b'c'}\right] = \eta^{a'b'}P^{c'} - \eta^{a'c'}P^{b'}, \tag{A.2}$$

$$\left[J^{ab}, J^{cd}\right] = \eta^{bc}J^{ad} + 3 \text{ terms} \qquad \left[J^{a'b'}, J^{c'd'}\right] = \eta^{b'c'}J^{a'd'} + 3 \text{ terms}, \tag{A.3}$$

with $a = 0, ..., 4$, so(4,1) vector indices, in the tangent space of AdS_5, $a' = 0, ..., 4$ so(5) vector indices in the tangent space of S^5, $\eta^{ab} = diag(-++++)$ and $\eta^{a'b'} = diag(+++++)$.

The commutation relations between the bosonic generators of the pp-wave algebra are

$$[P^-, P^I] = -J^{+I}, \tag{A.4}$$

$$[P^I, J^{+J}] = -\delta^{IJ}P^+, \qquad [P^-, J^{+I}] = P^I, \tag{A.5}$$

with $I = 1, ..., 8$. These generators admit the following field representation

$$P^+ = 1, \qquad P^I = -\int_0^1 d\sigma(\cos\tau\partial_\tau x^I + \sin\tau x^I), \tag{A.6}$$

$$J^{+I} = \int_0^1 d\sigma(\sin\tau\partial_\tau x^I - \cos\tau x^I). \tag{A.7}$$

The two dimensional fields $x^I(\tau, \sigma)$ satisfy the following equations of motion and periodicity conditions:

$$(-\partial_\tau^2 + \partial_\sigma^2)x^I - x^I = 0 \tag{A.8}$$

$$x^I(\tau, 0) = x^I(\tau, 1), \qquad \partial_\sigma x^I(\tau, 0) = \partial_\sigma x^I(\tau, 1) \tag{A.9}$$

Such equations admit as solution

$$x^I(\sigma, \tau) = \cos\tau\, x_0^I + \sin\tau\, p_0^I + \mathrm{i}\sum_{n\neq 0} \frac{1}{\omega_n} e^{-\mathrm{i}\omega_n\tau}(e^{\mathrm{i}k_n\sigma}\alpha_n^{1I} + e^{-\mathrm{i}k_n\sigma}\alpha_n^{2I}), \tag{A.10}$$

with the frequencies defined by

$$\omega_n = \sqrt{k_n^2 + 1}, \qquad n > 0; \qquad \omega_n = -\sqrt{k_n^2 + 1}, \qquad n < 0, \tag{A.11}$$

$$k_n = 2\pi n, \qquad n = \pm 1, \pm 2, ... \tag{A.12}$$

The coordinate x^- can be expressed in terms of x^I by the following constraints

$$\partial_\sigma x^- = -\partial_\tau x^I \partial_\sigma x^I, \tag{A.13}$$

$$\partial_\tau x^- = \frac{1}{2}(-\partial_\tau x^I \partial_\tau x^I - \partial_\sigma x^I \partial_\sigma x^I + x^I x^I). \tag{A.14}$$

Bibliography

[1] G. 't Hooft, "A Planar Diagram Theory For Strong Interactions," Nucl. Phys. B **72** (1974) 461.

[2] J. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity" Adv. Theor. Math. Phys. **2** (1998) 231 [arXiv:hep-th/9711200].

[3] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** 253 (1998), [arXiv:hep-th/9802150].

[4] D. Berenstein, J. Maldacena and H. Nastase, *Strings in flat space and pp–wave from N=4 super–Yang–Mills*, [arXiv:hep-th/0202021].

[5] R.R. Metsaev, *Type IIB Green–Schwarz superstring in plane wave Ramond–Ramond background* Nucl. Phys. **B 625** (2002) 70, [arXiv:hep-th/0112044].

[6] J. A. Minahan and K. Zarembo, "The Bethe-ansatz for N = 4 super Yang-Mills," JHEP **0303** (2003) 013 [arXiv:hep-th/0212208].

[7] N. Beisert, C. Kristjansen and M. Staudacher, "The dilatation operator of N = 4 super Yang-Mills theory," Nucl. Phys. B **664** (2003) 131 [arXiv:hep-th/0303060].

[8] N. Beisert, "The complete one-loop dilatation operator of N = 4 super Yang-Mills theory," Nucl. Phys. B **676** (2004) 3 [arXiv:hep-th/0307015].

[9] G. Mandal, N. V. Suryanarayana and S. R. Wadia, "Aspects of semi-classical strings in AdS(5)," Phys. Lett. B **543** (2002) 81 [arXiv:hep-th/0206103].

[10] I. Bena, J. Polchinski and R. Roiban, "Hidden symmetries of the AdS(5) x S**5 superstring" arXiv:hep-th/0305116.

[11] L. Dolan, C. R. Nappi and E. Witten, "A relation between approaches to integrability in superconformal Yang-Mills theory," arXiv:hep-th/0308089.

[12] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, Phys. Rep. **323** (2000) 183 [arXiv:hep-th/9905111].

[13] J. L. Petersen, "Introduction to the Maldacena conjecture on AdS/CFT," Int. J. Mod. Phys. A **14** (1999) 3597 [arXiv:hep-th/9902131].

[14] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS/CFT correspondence," arXiv:hep-th/0201253.

[15] M. Bertolini, "Four lectures on the gauge-gravity correspondence," Int. J. Mod. Phys. A **18** (2003) 5647 [arXiv:hep-th/0303160].

[16] J. M. Maldacena, "TASI 2003 lectures on AdS/CFT," arXiv:hep-th/0309246.

[17] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, JHEP **0201** (2002) 255, [arXiv:hep-th/0110242].

[18] A. Pankiewicz, "Strings in plane wave backgrounds," Fortsch. Phys. **51** (2003) 1139 [arXiv:hep-th/0307027].

[19] J. C. Plefka, "Lectures on the plane-wave string / gauge theory duality," Fortsch. Phys. **52** (2004) 264 [arXiv:hep-th/0307101].

[20] R. Russo and A. Tanzini, "The duality between IIB string theory on pp-wave and N = 4 SYM: A status report," Class. Quant. Grav. **21** (2004) S1265 [arXiv:hep-th/0401155].

[21] R. Penrose, *Any spacetime has a plane wave as a limit*, in *Differential Geometry and Relativity*, pp.271-75, Reidel, Dordrecht, 1976.

[22] M. Spradlin and A. Volovich, "Superstring interactions in a pp-wave background," Phys. Rev. D **66** (2002) 086004 [arXiv:hep-th/0204146].

[23] B. Sundborg, "Stringy gravity, interacting tensionless strings and massless higher spins," Nucl. Phys. Proc. Suppl. **102** (2001) 113 [arXiv:hep-th/0103247].

[24] C. B. Thorn, "Fields in the language of string: Divergences and renormalization," arXiv:hep-th/0311026.

[25] A. Dhar, G. Mandal and S. R. Wadia, "String bits in small radius AdS and weakly coupled N = 4 super Yang-Mills theory. I," arXiv:hep-th/0304062.

[26] R. Gopakumar, "From free fields to AdS," Phys. Rev. D **70** (2004) 025009 [arXiv:hep-th/0308184].

[27] R. Gopakumar, "From free fields to AdS. II," Phys. Rev. D **70** (2004) 025010 [arXiv:hep-th/0402063].

[28] M. Bianchi, J. F. Morales and H. Samtleben, JHEP **0307**, 062 (2003) [arXiv:hep-th/0305052].

[29] N. Beisert, M. Bianchi, J. F. Morales and H. Samtleben, JHEP **0402**, 001 (2004) [arXiv:hep-th/0310292].

[30] P. Bain, P. Meessen and M. Zamaklar, *Supergravity solutions for D-branes in Hpp-wave backgrounds* [arXiv:hep-th/0205106].

[31] A. Dabholkar and S. Parvizi, *Dp Branes in PP-wave Background*, [arXiv:hep-th/0203231].

[32] K. Skenderis and M. Taylor, *Branes in AdS and pp-wave spacetimes*, [arXiv:hep-th/0204054]

[33] A. Kumar, R. R. Nayak and S. Siwach, "D-brane solutions in pp-wave background," Phys. Lett. B **541** (2002) 183 [arXiv:hep-th/0204025].

[34] L. F. Alday and M. Cirafici, "An example of localized D-branes solution on pp-wave backgrounds," JHEP **0305** (2003) 006 [arXiv:hep-th/0301253].

[35] R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond–Ramond background*, [arXiv:hep-th/0202109].

[36] J. Maldacena, L. Maoz, *Strings on pp-waves and massive two dimensional field theories*, [arXiv:hep-th/0207284]

[37] I. Bakas and J. Sonnenschein, *On integrable models from pp-wave string backgrounds*, [arXiv:hep-th/0211257].

[38] M. Cvetic, H. Lu, C. N. Pope, K. S. Stelle, *Linearly-realised Worldsheet Supersymmetry in pp-wave Backgrounds*, [arXiv:hep-th/0209193]

[39] O. Bergman, M.R. Gaberdiel, M.B. Green, *D-brane interactions in type IIB plane-wave background*, [arXiv:hep-th/0205183]

[40] K.S. Stelle, *Lectures on Supergravity p-branes*, [arXiv:hep-th/9701088].

[41] A. Biswas, A. Kumar and K. L. Panigrahi, *p–p' Branes in PP-wave Background*, [arXiv:hep-th/0208042]

[42] E. Gava, A.B. Hammou, J.F. Morales and K.S. Narain, *AdS/CFT correspondence and D1/D5 systems in theories with 16 supercharges*, [arXiv:hep-th/0102043], E. Gava, A.B. Hammou, J.F. Morales and K.S. Narain *The D1/D5 System in $\mathcal{N}=4$ String Theories*, [arXiv:hep-th/0012118].

[43] J.G. Russo and A.A. Tseytlin, *On solvable models of type IIB Superstring in NS–NS and R–R plane wave backgrounds*, [arXiv:hep-th/0202179].

[44] B. C. Vallilo, "Flat currents in the classical AdS(5) x S**5 pure spinor superstring," arXiv:hep-th/0307018.

[45] N. Berkovits, "Super-Poincare covariant quantization of the superstring," JHEP **0004** (2000) 018 [arXiv:hep-th/0001035].

[46] N. Berkovits, "BRST cohomology and nonlocal conserved charges," arXiv:hep-th/0409159.

[47] L. F. Alday, "Nonlocal charges on AdS(5) x S**5 and pp-waves," JHEP **0312** (2003) 033 [arXiv:hep-th/0310146].

[48] R. R. Metsaev and A. A. Tseytlin, "Type IIB superstring action in AdS(5) x S(5) background," Nucl. Phys. B **533**, 109 (1998) [arXiv:hep-th/9805028].

[49] R. Kallosh, J. Rahmfeld and A. Rajaraman, "Near horizon superspace," JHEP **9809**, 002 (1998) [arXiv:hep-th/9805217].

[50] R. Roiban and W. Siegel, "Superstrings on AdS(5) x S(5) supertwistor space," JHEP **0011**, 024 (2000) [arXiv:hep-th/0010104].

[51] R. R. Metsaev, "Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background," Nucl. Phys. B **625** (2002) 70 [arXiv:hep-th/0112044].

[52] D. Cangemi and R. Jackiw, "Gauge invariant formulations of lineal gravity," Phys. Rev. Lett. **69** (1992) 233 [arXiv:hep-th/9203056].

[53] Chiara R. Nappi and Edward Witten, "A WZW model based on a non-semi-simple group", [arXiv:hep-th/9310112].

[54] M. Hatsuda, K. Kamimura and M. Sakaguchi, "Gauge invariant action for superstring in Ramond-Ramond plane-wave background," Nucl. Phys. B **644** (2002) 40 [arXiv:hep-th/0207157].

[55] M. Hatsuda, K. Kamimura and M. Sakaguchi, "From super-AdS(5) x S**5 algebra to super-pp-wave algebra," Nucl. Phys. B **632** (2002) 114 [arXiv:hep-th/0202190].

[56] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "A semi-classical limit of the gauge/string correspondence," Nucl. Phys. B **636** (2002) 99 [arXiv:hep-th/0204051].

[57] M. Hatsuda and K. Yoshida, "Classical integrability and super Yangian of superstring on AdS(5) x S**5," arXiv:hep-th/0407044.

[58] M. B. Green, J. H. Schwarz and L. Brink, "Superfield Theory Of Type Ii Superstrings," Nucl. Phys. B **219** (1983) 437.

[59] M. Spradlin and A. Volovich, "Light-cone string field theory in a plane wave," arXiv:hep-th/0310033.

[60] G. Polya and R. Read, "Combinatorial enumeration of groups, graphs and chimical groups", Springer Verlag, New York 1987.

[61] A. M. Polyakov, "Gauge fields and space-time," Int. J. Mod. Phys. A **17S1** (2002) 119 [arXiv:hep-th/0110196].

[62] N. Beisert, "BMN operators and superconformal symmetry," Nucl. Phys. B **659** (2003) 79 [arXiv:hep-th/0211032].

[63] M. Gunaydin and N. Marcus, "The Spectrum Of The S**5 Compactification Of The Chiral N=2, D = 10 Supergravity And The Unitary Supermultiplets Of U(2,2/4)," Class. Quant. Grav. **2** (1985) L11.

[64] M. Gunaydin, D. Minic and M. Zagermann, "Novel supermultiplets of SU(2,2|4) and the AdS(5)/CFT(4) duality," Nucl. Phys. B **544** (1999) 737 [arXiv:hep-th/9810226].

[65] S. Fernando, M. Gunaydin and O. Pavlyk, "Spectra of PP-wave limits of M-/superstring theory on AdS(p) x S(q) spaces," JHEP **0210** (2002) 007 [arXiv:hep-th/0207175].

[66] K. A. Intriligator and W. Skiba, "Bonus symmetry and the operator product expansion of N = 4 super-Yang-Mills," Nucl. Phys. B **559** (1999) 165 [arXiv:hep-th/9905020].

[67] N. Beisert and M. Staudacher, "The N = 4 SYM integrable super spin chain," Nucl. Phys. B **670** (2003) 439 [arXiv:hep-th/0307042].

[68] K. Okuyama and L. S. Tseng, "Three-point functions in N = 4 SYM theory at one-loop," JHEP **0408** (2004) 055 [arXiv:hep-th/0404190].

[69] D. Anselmi, "The N = 4 quantum conformal algebra," Nucl. Phys. B **541** (1999) 369 [arXiv:hep-th/9809192].

[70] L. Alday,J. David,E. Gava,K. Narain, "Work in progress"