ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Thesis submitted for the degree of Doctor Philosophiae

ROTATIONS ON A RIEMANNIAN MANIFOLD

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Academic Year 1992/93

TRIESTE

SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
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TRIESTE
Strada Costiera 11

Il presente lavoro, svolto sotto la direzione del Prof. Lieven Vanhecke, costituisce la tesi presentata dal dott. Lorenzo Nicolodi per l' ottenimento dell' attestato di ricerca postuniversitaria denominato "Doctor Philosophiae", settore di Analisi Funzionale e Applicazioni, presso la S.I.S.S.A. di Trieste. Ai sensi del Decreto del Ministro della Pubblica Istruzione n. 419 del 24/04/1987, tale diploma è equipollente al titolo di Dottore di Ricerca in Matematica.

Acknowledgments

I would like to express my gratitude to Professor Lieven Vanhecke, who not only had the patience of following me in the work, teaching me a lot of mathematics, but who also sustained me by his warmth and enthusiasm during my two-year stay at the Katholieke Universiteit Leuven.

Everyone at the Department of Mathematics of the K.U. Leuven has been nice and helpful in different ways. I would like to mention, in particular, the members and the visitors of the Section of Geometry with whom I had the privilege to relate more closely: E. Boeckx, P. Bueken, M. Djorić, E. García–Río, O. Kowalski, Ph. Tondeur and P. Vanhaecke. I have also benefitted from useful discussions with several people on various occasions.

I wish to thank here Professor Franco Tricerri, whose confidence in me has been a great source of encouragement all along.

Moreover, I express my appreciation to the S.I.S.S.A., Trieste, and in particular to Professor Graziano Gentili as responsible for the Section of Geometry. Finally, thanks are extended to the Consiglio Nazionale delle Ricerche for the financial support during my stay in Leuven (grant n. 203.01.50).

This work is dedicated to my parents.

Contents

Introduction

In this work we shall be concerned with the study of (local) rotations on a Riemannian manifold. Our aim is to investigate the interplay between the properties of these local diffeomorphisms and the geometry of the manifold, focusing especially on the Riemannian curvature. This study originated from the general problem of determining to what extent the properties of a family of geometrically defined objects on a Riemannian manifold (M, g) influence the geometry of (M, g) (cf. [CV1],[GV1],[V2],[V4]).

The development of the theory of rotations initiated with the study of geodesic reflections (geodesic symmetries) and was then extended to the study of reflections with respect to curves and submanifolds. Here we shall continue this theory considering the more general notion of rotations around points, curves and submanifolds of a Riemannian manifold.

Rotations will be defined as a special class of local diffeomorphisms that generalize in a natural way the corresponding concepts in Euclidean space. As rotations in Euclidean space have properties that are directly related to the specific nature of Euclidean geometry, one can expect that rotations on a Riemannian manifold with special curvature will have particular properties. On the other hand, if one supposes that rotations have special properties, one may ask how this reflects in the curvature of the ambient space and in the extrinsic and intrinsic geometry of the submanifold.

Besides dealing with volume–preserving, holomorphic and symplectic reflections and rotations, throughout the work special emphasis is given to the relationship between isometric and harmonic rotations around points, curves and submanifolds.

To illustrate the nature of the problem we are interested in, we briefly recall some significant results from the theory of reflections in various contexts (cf. [Bu],[V4] for a more extended survey).

Local geodesic reflections are local generalizations to arbitrary Riemannian manifolds of reflections with respect to a point in Euclidean space. These local diffeomorphisms may be used to define special classes of Riemannian manifolds. To start with, a classical result by E. Cartan asserts that all (local) geodesic reflections (geodesic symmetries) on a Riemannian manifold (M, g) are isometric if and only if the Riemann curvature tensor R is parallel with respect to the Levi Civita connection, i.e., (M, g) is (locally) symmetric. Furthermore, it is clear that isometric reflections are harmonic maps and hence locally symmetric spaces have harmonic geodesic reflections. On the other hand, it was proved in [DVV] that the converse also holds. So, locally symmetric spaces may be defined as Riemannian manifolds with either isometric or harmonic geodesic reflections. Alternative characterizations of locally symmetric spaces have also been given in terms of the invariance, under local geodesic reflections, of operators related to the extrinsic and intrinsic geometry of small geodesic spheres, e.g., the shape operator and the Ricci operator. For this we refer to Chapters II and III (see also [VW2],[DV1]).

In the framework of almost Hermitian geometry the consideration of almost Hermitian manifolds with symplectic (with respect to the Kähler form) geodesic reflections was initiated by Jacob in [Ja]. Using the known classification of compact homogeneous Kähler manifolds, he gave, in a special case, an affirmative answer to a conjecture of S. Kobayashi stating that such manifolds with symplectic geodesic reflections are Hermitian symmetric spaces. This conjecture has been considered from a different and much more general point of view by K. Sekigawa and L. Vanhecke in [SV1] where it is proved that locally Hermitian symmetric manifolds may be characterized as almost Hermitian manifolds with symplectic or holomorphic geodesic reflections (cf. also Chapter 2). So, they provided a complete positive answer to Kobayashi conjecture.

Symplectic geodesic reflections lead to consider another important class of reflections. In fact, it is clear that symplectic geodesic reflections are volume–preserving. The class of manifolds with volume–preserving (up to sign) local geodesic reflections were first studied by D'Atri and Nickerson in [DN1],[DN2]. This class contains, for example, the locally symmetric spaces, the harmonic spaces, the generalized Heisenberg groups, the naturally reductive and the commutative spaces. Besides these examples, all known examples of Riemannian manifolds with volume–preserving geodesic reflections are locally homogeneous spaces, i.e., the pseudogroup of local isometries acts transitively on these manifolds. Up to now it is an open problem whether the manifolds with volume–preserving local geodesic reflections are locally homogeneous. For more details, references and further results we refer to [Bu],[K2],[V3],[V4].

We already mentioned that the notion of geodesic reflections can be extended and one may consider (local) reflections with respect to curves and submanifolds of (M, g). By doing this, one obtains results illustrating the relation which exists between the geometry of (M, g), the geometry of the submanifolds and the geometry of tubular neighborhoods about them. Again, the properties of these reflections have been used to define some particular classes of manifolds. Geodesic reflections with respect to geodesics were first introduced by L. Vanhecke and T.J. Willmore in [VW2] and were there used to characterize locally symmetric spaces and spaces of constant curvature. As a matter of fact, the authors proved that a manifold is locally symmetric if and only if the geodesic reflections with respect to all geodesics are volume–preserving and that a Riemannian manifold is a space of constant curvature if and only if the local geodesic symmetries with respect to all geodesics are isometries. As concerns spaces of constant curvature it was then explicitly proved in [VV-A] that the isometric property can be replaced by the harmonic one. This result may also be seen as a consequence of the general result proved in [DGV] according to which, for analytic data, the geodesic reflection with respect to a submanifold B of a Riemannian manifold (M, g) is harmonic if and only if it is isometric. For a survey we refer to [V4] and to Chapters III and IV for a discussion of other properties of geodesic reflections.

Local reflections with respect to curves and submanifolds have also been shown to be useful for studying almost contact geometry. When the manifold is an almost contact metric manifold, it is natural to consider geodesic reflections with respect to the integral curves of the characteristic vector field of the manifold, but also reflections with respect to other naturally defined curves play an important role. For instance, on a Sasakian manifold the integral curves of the characteristic vector field are geodesics and having either isometric or harmonic reflections in these curves is a characteristic property of the (locally) φ-symmetric manifolds. For an extensive study of reflections and rotations in the context of almost contact geometry and for a complete and update bibliography we refer to [Bu].

The characteristic vector field of a Sasakian manifold generates a Riemannian foliation with geodesic one-dimensional leaves. These manifolds are just special cases of Riemannian foliations. Reflections with respect to the leaves of a Riemannian foliation have been considered in [ToV4]. The particular case of a Riemannian flow generated by a unit Killing vector field was investigated by several other authors. For a systematic account on this topic we refer to [GD].

The research presented in this work starts with a study of point rotations. These are generalizations to arbitrary Riemannian manifolds of point rotations in ordinary Euclidean space and in turn generalize the already considered local geodesic reflections. A rotation s_m around a point m on a Riemannian manifold is the local diffeomorphism defined, in a sufficiently small neighborhood of m, by

$s_m = \exp_m \circ S \circ \exp_m^{-1}$, where S is a $(1,1)$-tensor field on the manifold (M, g) that preserves the metric g. S is called a *rotation field* and s_m a (local) *S-rotation.* In this way we obtain a field $s : m \mapsto s_m$ of S-rotations such that $ds_{m|m} = S_m$.

Several examples of rotations occur in differential geometry. Let S be a rotation field on (M, g) such that $I - S$ is non-singular (I denotes the Kronecker tensor field) and suppose that, for each $m \in M$, s_m is a local isometry which preserves S. Then (M, g) together with the field s of S-rotations is called a *Riemannian locally s-regular manifold.* This class of manifolds was first introduced in [GL] (cf. [K1] and Chapter 2 for more details and references). When S is of finite order, i.e., $S^k = I$ for some $k \in \mathbb{N}$, (M, g) is called a *locally k-symmetric space* and we obtain the locally symmetric spaces when $k = 2$. The rotation field S associated to a Riemannian locally s-regular manifold is such that both the covariant derivatives ∇S and $\nabla^2 S$, taken with respect to the Levi Civita connection, are S-invariant. In [SV3] it is proved that a rotation field S with these properties induces a locally s-regular structure on (M, g) if and only if, for each $m \in M$, the S-rotation s_m is harmonic. Alternative geometric characterizations of the class of Riemannian locally s-regular manifolds in terms of the s-invariance of the shape operators and of the Ricci operators of small geodesic spheres may be found, respectively, in [LV1] and [DV1]. See also Chapter 2.

Concerning almost Hermitian geometry there is a very natural class of rotations to be considered. Let (M, g, J) be an almost Hermitian manifold; then we may consider the field j of J-rotations that is defined by $j : m \mapsto j_m = \exp_m \circ J_m \circ \exp_m^{-1}$, for all $m \in M$. Again, one may expect that there will be a relation between the properties of these local diffeomorphisms and the geometry of (M, g, J). One of our purposes in this work is to investigate this. We shall concentrate on aspects about Riemannian and almost Hermitian geometry and consider isometric, harmonic, symplectic and holomorphic J-rotations.

In Chapter 2 it is proved, for example, that on an almost Hermitian manifold each J-rotation is harmonic if and only if it is isometric. Moreover, the only Kähler manifolds all of whose J-rotations around all points are either isometric or harmonic are the locally Hermitian symmetric spaces (cf. [NV1]). Furthermore, as these particular rotations induce global diffeomorphisms on the (small) geodesic spheres of (M, g), we are interested here in the study of the invariance, under these diffeomorphisms, of some operators and naturally defined functions connected to the extrinsic and intrinsic geometry of (small) geodesic spheres. All this leads again to new characterizations of some special classes of almost Hermitian manifolds, in particular of Hermitian symmetric spaces (cf. [NV1]).

In Chapter 3 and Chapter 4 we continue our research by introducing the notions of rotations around curves and submanifolds (cf. [NV3],[NV4]). Our aim is to treat analogous problems as those treated for reflections. More specifically, we shall deal with harmonic rotations and focus as before on the relation between harmonic and isometric rotations. There are several examples of rotations. Every orientation–preserving element f of the isotropy subgroup at some point p of an odd–dimensional homogeneous Riemannian manifold is a rotation around the geodesic through p given by $\gamma(t) = \exp_m(tv)$, where v is a unit tangent vector corresponding to the eigenvalue $+1$ of $df_{|p}$. Another example was provided by J. Inoguti (cf. [In]) in the context of K-contact geometry. The author introduced the so-called φ-rotation around a flow line of the characteristic vector field on a K-contact manifold and proved that within the class of K-contact manifolds the locally φ-symmetric Sasakian manifolds are characterized by having isometric φ-rotations. Also in this situation one finds that the harmonic property is equivalent to the isometric one (cf. [BuV]).

It will be clear from the detailed discussion that, in general, the problems for rotations are more complicated than for reflections and this will be reflected in restrictions on the class of ambient spaces or the class of submanifolds considered. Among other results, we prove that harmonic and isometric rotations around a totally geodesic submanifold with flat normal connection coincide when the ambient space is a locally symmetric Einstein manifold. From this we also obtain that a rotation around a geodesic in a locally symmetric Einstein manifold is harmonic if and only if it is an isometry (cf. [NV3],[NV4]). The curve and submanifold cases are treated separately in Chapter 3 and 4 to better underline the role played by the geometry of the submanifold, namely by the second fundamental form and the normal connection of the submanifold.

The method employed to prove our results, as well as to prove most of the results we refer to throughout the work, is based on the power series expansions of suitable geometric quantities connected with the specific problems. For this it is necessary to consider the theory of power series expansions of tensor fields in normal coordinates and, more generally, in Fermi coordinates. Although there are general procedures (cf. [Gr4],[GV2],[GV3]), we shall use here Jacobi vector fields to provide the needed power series expansions (cf. [V4]).

The preparatory material and the basic technical tools we shall need in the course of the work are presented in Chapter 1.

The original contributions of this thesis are essentially those contained in the items [NV1],[NV2],[NV3],[NV4] of the bibliography, except that alternative proofs have been provided on some occasions and some new results have been added throughout.

Chapter 1

Preliminaries on Riemannian manifolds

In this chapter we review the basic notions regarding Riemannian manifolds and provide the set-up for studying the geometry of normal neighborhoods and of tubular neighborhoods about curves and submanifolds. In the course of the exposition we shall use standard material as presented in [Be2],[Go],[GKM],[KN],[ON] and [V4]. So, we shall not always specify the precise references.

1.1 Exponential map and Jacobi fields

Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold (henceforth, if not otherwise stated, all objects will be assumed C^∞). Let TM denote the tangent bundle, g the Riemannian metric and ∇ the corresponding Levi Civita connection, which is the unique torsion-free connection for which g is parallel, i.e., $\nabla g = 0$. This is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \end{aligned}$$

for any X, Y, $Z \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the set of vector fields on M. The *Riemann curvature tensor* R of ∇ is given, using the following sign convention, by

$$R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z \tag{1.1}$$

for all X, Y, $Z \in \mathfrak{X}(M)$.

The *Ricci tensor* ρ at a point m in M is defined by

$$\rho_{XY} = \rho(X,Y) = \mathrm{tr}(Z \mapsto R_{XZ}Y)$$

with $X, Y, Z \in T_m M$. Therefore, if $\{E_1, \ldots, E_n\}$ is an orthonormal frame at m, $\rho_{XY} = \sum_{i=1}^{n} g(R_{XE_i}Y, E_i)$. We recall that a Riemannian manifold (M, g) is an *Einstein manifold* provided $\rho = cg$ for some constant c.

If N is an arbitrary manifold and $f : N \to M$ a differentiable map, $df : TN \to TM$ denotes the differential of f. ∇ naturally extends to a covariant derivative for vector fields along f. For any vector field A on N and any vector field Y along f the covariant derivative $\nabla_A Y$ is uniquely determined by requiring the chain rule $\nabla_v(X \circ f) = \nabla_{df(v)} X$ for any tangent vector $v \in TN$ and any vector field X on M. As a consequence we obtain for the Levi Civita connection :

$$\nabla_A df(B) - \nabla_B df(A) - df([A, B]) = 0, \tag{1.2}$$

$$R_{df(A)df(B)} Y = \nabla_A \nabla_B Y - \nabla_B \nabla_A Y - \nabla_{[A,B]} Y, \tag{1.3}$$

where A, B are vector fields on N and Y a vector field along f.

For a curve $\gamma : I = [a, b] \to M$ the parameter vector field on I with respect to the parameter t will be denoted by $\frac{d}{dt}$ and the tangent vector $\dot\gamma(t) = (d\gamma(\frac{d}{dt}))(t)$ of γ at t is also denoted by $(\frac{d}{dt}\gamma)(t)$. The covariant derivative $\nabla_{\frac{d}{dt}} Y$ for a vector field Y along γ will also be abbreviated by Y'.

A curve γ is said to be a *geodesic* if it satisfies the second order non-linear differential equation (the *geodesic equation*)

$$\dot\gamma' = 0.$$

It is worth noting that the geodesics with respect to the Levi Civita connection are solutions of a natural variational problem in Riemannian geometry. They are precisely the stationary points (with respect to fixed boundary values $\gamma(a)$, $\gamma(b)$) of the energy functional

$$E(\gamma) = \frac{1}{2} \int_a^b g(\dot\gamma(t), \dot\gamma(t)) dt.$$

Stationary points of the length functional

$$L(\gamma) = \frac{1}{2} \int_a^b g(\dot\gamma(t), \dot\gamma(t))^{\frac{1}{2}} dt$$

also give rise to geodesics, but possibly only after reparametrization, as they need not be parametrized by (an affine transformation of) arc length, i.e., need not satisfy $g(\dot\gamma(t), \dot\gamma(t)) = \text{const}$.

The *exponential map* $\exp : TM \to M$ is determined by the initial value problem for geodesics, that is, if $v \in T_mM$ then $\exp(v) = \gamma_v(1)$, where γ_v is the geodesic with initial conditions $\gamma_v(0) = m$ and $\dot\gamma_v(0) = v$. The exponential map at m, denoted by $\exp_m$, is the restriction of exp to the tangent space T_mM. exp as well as $\exp_m$ are smooth and moreover the differential of $\exp_m$ at the origin of T_mM is the identity map. Therefore $\exp_m$ is a diffeomorphism in a neighborhood of the zero vector of T_mM.

A *Jacobi field* $Y(t)$ along a geodesic γ is a vector field along γ that is a solution of the second order linear differential equation (the *Jacobi equation*)

$$Y''(t) + R_{\dot\gamma(t)Y(t)}\dot\gamma(t) = 0. \tag{1.4}$$

Any Y is uniquely determined by its initial conditions of order zero and one.

For a geodesic γ the operator R_γ along γ defined on $\{\dot\gamma(t)\}^\perp \subset T_{\gamma(t)}M$ by

$$R_\gamma : X \mapsto R_{\dot\gamma(t)X}\dot\gamma(t)$$

is the *Jacobi operator* along γ. As a consequence of the symmetries of R we have that R_γ is self-adjoint. For each $m \in M$ and each $v \in T_mM$ one also defines the Jacobi operator as the self-adjoint endomorphism R_v of $\{v\}^\perp \subset T_mM$ by $R_v(w) = R_{vw}v$, for all $w \in \{v\}^\perp$. The Jacobi operator has been shown to be a useful tool to describe the curvature along geodesics in a Riemannian manifold (for some applications see for example [BV]).

Jacobi fields along a geodesic arise naturally as variational vector fields of one-parameter families of geodesics. In fact if V is a geodesic variation of γ, i.e.,

$$V : I \times (-\epsilon, \epsilon) \to M, (t, s) \mapsto V(t, s)$$

is differentiable and $V(t, 0) = \gamma(t)$ and $t \mapsto V(t, s)$ is a geodesic for all $s \in (-\epsilon, \epsilon)$, then $Y(t) = (\frac{\partial}{\partial s}V)(t, 0)$ is a Jacobi field along γ since (1.2), $[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = 0$ and (1.3) yield

$$\begin{aligned} Y''(t) &= (\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} V)(t,0) = (\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} V)(t,0) \\ &= (\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} V)(t,0) - (\nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} V)(t,0) \\ &= -R_{\dot\gamma(t)Y(t)}\dot\gamma(t). \end{aligned}$$

Therefore the Jacobi equation is the linearization of the geodesic equation along γ and it contains the infinitesimal information about geodesics and the curvature along geodesics. Conversely, any solution of the Jacobi equation (1.4) can be obtained as a variational vector field of a suitable geodesic variation of the geodesic γ (cf. [KN]).

Note that V can be written in the following way. If α is the curve $\alpha(s) = V(0,s)$ and ξ the vector field along α given by $\xi(s) = (\frac{\partial}{\partial t}V)(0,s)$ then $V(t,s) = \exp_{\alpha(s)} t\xi(s)$ and $Y(t)$ is a Jacobi field along the geodesic $\gamma(t) = \exp_{\alpha(0)} t\xi(0)$. Observe that $\xi(0)$ is the initial velocity vector of the geodesic γ and that $Y'(t) = (\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} V)(t,0)$. The initial conditions of $Y(t)$ in terms of α and ξ are $Y(0) = (\frac{d}{ds}\alpha)(0)$ and $Y'(t) = \dot\xi(0)$.

The preceding discussion indicates a general method to generate Jacobi vector fields with prescribed initial values. We state this as a lemma.

LEMMA 1.1. *Let $\gamma : I \to M, t \mapsto \gamma(t)$ be a unit speed geodesic with $\gamma(0) = m$, and let $v, w \in T_mM$. Then for any curve $\alpha : (-\epsilon, \epsilon) \to M, s \mapsto \alpha(s)$ with $\alpha(0) = m$ and $\dot\alpha(0) = v$ and any vector field ξ along α with $\xi(0) = \dot\gamma(0)$ and $\xi'(0) = w$,*

$$V(t,s) = \exp_{\alpha(s)} t\xi(s)$$

defines, on a suitable open subset of $\mathbb{R}^2$, a geodesic variation of γ and the vector field $Y(t) = (\frac{\partial}{\partial s}V)(t,0)$ is the unique Jacobi field along γ with $Y(0) = v$, $Y'(0) = w$ and $Y'(t) = (\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} V)(t,0)$.

REMARK. Among the variations with the properties as in the Lemma 1.1 there is a natural one to be considered. Let γ and α be as in the lemma and let X, W denote the ∇-parallel vector fields along α with $X(0) = \dot\gamma(0)$ and $W(0) = w$. Then α and $\xi(s) = X(s) + sW(s)$ give rise to a variation V which generates the Jacobi field Y with $Y(0) = v$, $Y'(0) = w$.

For example, the Jacobi field with initial conditions $Y(0) = 0$, $Y'(0) = w$ along the geodesic $\exp_m(tu)$ is obtained from the variation $V(t,s) = \exp_m(t(u+sw))$. Here $\alpha(s)$ is the constant curve $\alpha(s) \equiv m$, $\xi(s) = u + sw$ and

$$Y(t) = d\exp_{m|tu} tw.$$

This shows that in a neighborhood of the zero vector of T_mM *the differential of the restriction* $\exp_{|T_mM} = \exp_m$ *is determined by Jacobi vector fields on* M *with these initial conditions.*

More generally, the differential of exp is completely determined by Jacobi fields. In fact, any tangent vector u to TM can be written as the tangent vector $u = (\frac{d}{ds}\xi)(0)$ of a curve $s \mapsto \xi(s)$ in TM. ξ is a vector field along the base curve $\alpha(s) = \pi \circ \xi(s)$, where $\pi : TM \to M$ denotes the projection of TM. Then, if V and Y are defined as above we have

$$d\exp_{\alpha(s)|\xi(0)}(u) = \frac{d}{ds}(\exp_{\alpha(s)} \circ \xi(s))(0) = (\frac{d}{ds}V)(1,0) = Y(1).$$

1.2 Normal neighborhoods, normal coordinates and power series expansions

Let $m \in M$, then an open set U is said to be a *normal neighborhood* of m if it is diffeomorphic via $\exp_m$ to an open starshaped neighborhood of the zero vector in T_mM. On a normal neighborhood U of m one can introduce *normal (geodesic) coordinates* $(x^1, \ldots, x^n)$ at m defined by

$$x^i(\exp_m(\sum_{j=1}^n a^j e_j) = a^i, \qquad i = 1, \ldots, n,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space T_mM at m. Clearly, $\frac{\partial}{\partial x^i}(m) = e_i$, $i = 1, \ldots, n$ and moreover

$$g_{ij}(m) = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})(m) = \delta_{ij},$$

$$\frac{\partial g_{ij}}{\partial x^k}(m) = 0 \iff \Gamma^k_{ij}(m) = 0 \iff (\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j})(m) = 0,$$

for all $i, j, k = 1, \ldots, n$.

We recall that the Christoffel symbols Γ_{ij}^k are defined by $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x^k}$ and are expressed in terms of the metric g by the following formula

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^n g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right),$$

where g^{kl} is the (k, l) entry of the inverse matrix of (g_{ij}).

Next, let γ be a geodesic parametrized by arc length r such that $\gamma(0) = m$, $\dot{\gamma}(0) = u = e_1$ and extend the basis $\{e_1, \ldots, e_n\}$ to a basis $\{E_1(r), \ldots, E_n(r)\}$ by parallel translation along γ. Now, by definition of normal coordinates we have for sufficiently small r

$$\frac{\partial}{\partial x^a}(\gamma(r)) = d\exp_{m|ru}(e_a) \qquad a = 2, \ldots, n.$$

According to Lemma 1.1 and the successive remark, this implies that the vector fields along γ given by

$$Y_a(r) = r \frac{\partial}{\partial x^a}(\gamma(r)) \qquad a = 2, \ldots, n. \tag{1.5}$$

are the unique Jacobi vector fields with initial conditions

$$Y_a(0) = 0, \qquad Y_a'(0) = e_a, \qquad a = 2, \ldots, n. \tag{1.6}$$

In this way we get a description along γ of the normal coordinate vector fields $\frac{\partial}{\partial x^i}$'s in terms of Jacobi fields, which are easier to handle as they are solutions of the Jacobi equation with prescribed initial conditions. Next, put

$$Y_a(r) = (AE_a)(r), \qquad a = 2, \ldots, n. \tag{1.7}$$

Then $r \mapsto A(r)$ is an endomorphism-valued function. Each $A(r)$ is an endomorphism of the space $\{\dot{\gamma}(r)\}^\perp$ and these spaces may be identified via the parallel translation along γ with respect to ∇ using the parallel basis $\{E_1, \ldots, E_n\}$. Actually, $A(r)$ can be thought of as an endomorphism of $\{\dot{\gamma}(0)\}^\perp$ once we assume that for $X \in \{\dot{\gamma}(0)\}^{\perp}$, $A(r)X = \tau_r^{-1}A(r)\tau_r X$, where τ_r denotes the parallel translation along γ.

Using (1.7) in (1.4) and the initial conditions (1.6) we obtain the following endomorphism-valued Jacobi equation

$$A'' + R_\gamma \circ A = 0, \tag{1.8}$$

with initial conditions

$$A(0) = 0, \qquad A'(0) = Id_{|u^\perp}. \tag{1.9}$$

For R_γ we make the same identification as for $A(r)$.

By the Jacobi equation (1.8) and the initial conditions (1.9) we get then

$$\begin{aligned} A(r) = rI - \frac{r^3}{6}R_u - \frac{r^4}{12}R'_u + \frac{r^5}{5!}(-3R''_u + R_u \circ R_u) \\ + \frac{r^6}{6!}(-4R'''_u + 4R'_u \circ R_u + 2R_u \circ R'_u) + O(r^7), \end{aligned} \tag{1.10}$$

where $R_u^{(k)} = (\nabla^k_{u\ldots u}R)_u.u$ and $O(r^7)$ means terms of order seven or more in r.

We present now some applications of the preceding formulae to study the geometry in normal neighborhoods.

We start by determining the components $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ of the metric tensor g with respect to normal coordinates about m. For general methods to write down power series expansions of covariant tensor fields we refer to [Gr4],[Gr7], [GV1],[Pe],[Gh]. The fact that

$$\frac{\partial}{\partial x^1}(\gamma(r)) = \frac{\partial}{\partial r}(\gamma(r)),$$

and the Gauss Lemma (cf. [CE]) yield at $p = \exp_m(ru)$

$$g_{11}(p) = 1, \qquad g_{1a}(p) = 0, \qquad a = 2, \ldots, n. \tag{1.11}$$

Moreover, we have

$$g_{ab}(p) = \frac{1}{r^2} g(r\frac{\partial}{\partial x^a}, r\frac{\partial}{\partial x^b})(\gamma(r)) = \frac{1}{r^2} g(AE_a, AE_b)(r). \tag{1.12}$$

Therefore we obtain

$$\begin{aligned} g_{ab}(p) = \delta_{ab} &- \frac{r^2}{3} R_{uaub}(m) - \frac{r^3}{6} (\nabla_u R)_{uaub}(m) \\ &+ \frac{r^4}{180} \{-9(\nabla^2_{uu} R)_{uaub} + 8 \sum_{i=1}^{n} R_{uaui} R_{ubui}\}(m) \\ &+ \frac{r^5}{90} \{-(\nabla^3_{uuu} R)_{uaub} + 2 \sum_{i=1}^{n} ((\nabla_u R)_{uaui} R_{ubui} \\ &+ (\nabla_u R)_{ubui} R_{uaui})\}(m) \\ &+ \frac{r^6}{7!} \{-10(\nabla^4_{uuuu} R)_{uaub} \\ &+ 34 \sum_{i=1}^{n} ((\nabla^2_{uu} R)_{uaui} R_{ubui} + (\nabla^2_{uu} R)_{ubui} R_{uaui}) \\ &+ 55 \sum_{i=1}^{n} (\nabla_u R)_{uaui} (\nabla_u R)_{ubui} \\ &- 16 \sum_{i,j=1}^{n} R_{uaui} R_{ubuj} R_{uiuj}\} + O(r^7), \end{aligned} \tag{1.13}$$

where $a, b = 2, \ldots, n$ and R_{uaub} stands for $R_{ue_a ue_b} = g(R_{ue_a} u, e_b)$. By using (1.10), (1.13) and the relations $g_{ik} g^{kj} = \delta_{ij}$ we get the expansions for the contravariant components of g:

$$g^{11}(p) = 1, \qquad g^{1a}(p) = 0, \tag{1.14}$$

$$\begin{aligned} g^{ab} = \delta_{ab} &+ \frac{r^2}{3} R_{uaub}(m) + \frac{r^3}{6} (\nabla_u R)_{uaub}(m) \\ &+ \frac{r^4}{60} \{3(\nabla^2_{uu} R)_{uaub} + 4 \sum_{i=1}^{n} R_{uaui} R_{ubui}\}(m) \\ &+ O(r^5). \end{aligned} \tag{1.15}$$

As a further application we describe how the endomorphism A intervenes in the computation of functions and operators related to the geometry of geodesic spheres in a Riemannian manifold (cf. [V4],[VW1]).

Let $G_m(\overline{r}) = \exp_m S_m(\overline{r})$, where $S_m(\overline{r})$ denotes the sphere centered at the origin of T_mM and radius $\overline{r}$, $\overline{r}$ sufficiently small. It is a hypersurface of (M,g). The extrinsic geometry of $G_m(\overline{r})$ is described by the *shape operator* T_m, defined at $p = \exp_m(\overline{r}u)$ by

$$T_m(p)X = (\nabla_X \frac{\partial}{\partial r})(p), \qquad X \in T_pG_m(\overline{r}), \tag{1.16}$$

where $\frac{\partial}{\partial r}$ denotes the outward unit normal vector field along $G_m(\overline{r})$.

As the Jacobi vector fields Y_a, $a = 2, \ldots, n$, are tangent to the geodesic sphere with center m, (1.16) leads to

$$T_m(p)Y_a = \nabla_{Y_a}\frac{\partial}{\partial r} = \nabla_{\frac{\partial}{\partial r}} Y_a = Y_a', \tag{1.17}$$

due to the fact that, as $\Gamma(r,s) = \exp_m(r\cos(s)u + r\sin(s)e_a)$ is a diffeomorphism of a neighborhood of $(\overline{r}, 0)$ in $\mathbb{R}^2$ onto a piece of surface through p in M, on this surface we have $[\frac{\partial}{\partial r}, \frac{\partial}{\partial s}] = 0$ and then $\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial r} = \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial s}$ by the formula (1.2).

Therefore, at $p = \exp_m(ru)$,

$$T_m(p) = (A'A^{-1})(r). \tag{1.18}$$

So, the *mean curvature* h_m of the geodesic sphere $G_m(r)$ is given by

$$h_m := \mathrm{tr}T_m(p) = \mathrm{tr}(A'A^{-1})(r) = \frac{(\det A)'}{\det A}(r). \tag{1.19}$$

Finally, let

$$\theta_m(p) = (\det(g_{ij}))^{\frac{1}{2}}(p), \tag{1.20}$$

where m is the center of the normal coordinate system $(x_1, \ldots, x_n)$ and $p = \exp_m(ru)$ as above. θ_m is the *volume density function* of $\exp_m$ and by (1.11) and (1.12) we have

$$\theta_m(p) = \frac{1}{r^{n-1}}(\det A)(r). \tag{1.21}$$

Taking into account (1.19) and (1.21) we then obtain

$$h_m(p) = \frac{n-1}{r} + \left(\frac{\theta'_m}{\theta_m}\right)(p). \tag{1.22}$$

1.3 Tubular Neighborhoods, Fermi coordinates and power series expansions

Let (M, g) be an n-dimensional Riemannian manifold as above and let B be a connected topologically embedded submanifold of dimension q. The case $q = 1$ will be dealt with in detail later. In what follows we consider B as a Riemannian manifold with the induced metric and normal bundle $\nu(M)$. First, we briefly recall some useful formulae from the theory of submanifolds ([C1]).

1.3.1 Basic equations for submanifolds

$\mathfrak{X}(B)$ will denote the set of tangent vector fields of B and $\overline{\mathfrak{X}}(B)$ the set of tangent vector fields of M along B. Then

$$\overline{\mathfrak{X}}(B) = \mathfrak{X}(B) \oplus \mathfrak{X}^{\perp}(B),$$

where $\mathfrak{X}^{\perp}(B)$ consists of all vector fields normal to B. $\mathcal{F}(B)$ denotes the algebra of real-valued C^{∞} functions on B.

Let ∇ denote the Levi Civita connection of M and consider the covariant derivative $\nabla_X U$ of a vector field U along B in the direction X, $X \in \mathfrak{X}(B)$. The connection ∇ induces one on B by taking U tangent to B and projecting $\nabla_X U$ onto the tangent bundle TB. This projection agrees with the intrinsically defined Levi Civita connection $\widetilde{\nabla}$ of the induced metric on B. We have the *Gauss formula*

$$\nabla_X Y = \widetilde{\nabla}_X Y + T(X, Y), \tag{1.23}$$

$X, Y \in \mathfrak{X}(B)$, where $T : \mathfrak{X}(B) \times \mathfrak{X}(B) \to \mathfrak{X}^{\perp}(B)$ is a symmetric $\mathcal{F}(B)$-bilinear map, the so-called *second fundamental form* of B in M. The *mean curvature vector field*

H of B in M is defined by

$$H = \sum_{i=1}^{q} T(E_i, E_i), \tag{1.24}$$

where $\{E_1, \ldots, E_q\}$ is an orthonormal frame field tangent to B. A submanifold is *minimal* if $H = 0$, and it is *totally geodesic* if $T = 0$.

There is also an induced metric connection on the normal bundle, the *normal connection* $\nabla^{\perp}$. It is given by projecting $\nabla_X N$ onto $\nu(B)$, where now N is a section of $\nu(B)$, that is $N \in \mathfrak{X}^{\perp}(B)$ and $X \in \mathfrak{X}(B)$. We have the *Weingarten formula*

$$\nabla_X N = \nabla^{\perp}_X N + T(N)X, \tag{1.25}$$

where $T(N)$ is related to T by $g(T(N)X, Y) = -g(T(X,Y), N)$, $X, Y \in \mathfrak{X}(B)$ (note that $g(Y, N) = 0$ implies $0 = g(\nabla_X Y, N) + g(Y, \nabla_X N) = g(T(X,Y), N) + g(Y, T(N)X)$.

If $\widetilde{R}$ and R are the curvature tensors of $\widetilde{\nabla}$ and ∇, respectively, the tangential component of $R_{XY}Z$ is given in terms of $\widetilde{R}$ and T by the *Gauss equation*

$$g(R_{XY}Z, W) = g(\widetilde{R}_{XY}Z, W) + g(T(X,Z), T(Y,W)) - g(T(X,W), T(Y,Z)), \tag{1.26}$$

where $X, Y, Z, W \in \mathfrak{X}(B)$.

On the other hand, by taking the normal component of $R_{XY}Z$ we have the *Codazzi equation*

$$(R_{XY}Z)^{\perp} = (\overline{\nabla}_X T)(Y, Z) - (\overline{\nabla}_Y T)(X, Z), \tag{1.27}$$

where $(\overline{\nabla}_X T)(Y, Z) = \nabla^{\perp}_X (T(Y, Z)) - T(\widetilde{\nabla}_X Y, Z) - T(Y, \widetilde{\nabla}_X Z)$.

The *normal curvature tensor* $R^{\perp}$ is defined by

$$R^{\perp}_{XY} N = \nabla^{\perp}_{[X,Y]} N - [\nabla^{\perp}_X, \nabla^{\perp}_Y] N,$$

where $X, Y \in \mathfrak{X}(B)$ and $N \in \mathfrak{X}^{\perp}(B)$. $R^{\perp}$ verifies the *Ricci equation* given by

$$g(R^{\perp}_{XY} U, V) = g(R_{XY} U, V) + g(T(U)X, T(V)Y) - g(T(V)X, T(U)Y) \tag{1.28}$$

for X, Y tangent vector fields of B and U, V normal vector fields along B.

The normal curvature tensor represents an obstruction for the local existence of normal parallel sections (i.e., parallel with respect to the normal connection) of the normal bundle as it is stated in the following lemma (see for example [C1]).

LEMMA 1.2. *The following are equivalent:*

(1) $R^{\perp}$ *is identically zero.*

(2) *Let q be any point in B. Then any vector $N_{|q}$ in $\nu_q(B)$ may be uniquely extended to a local $\nabla^{\perp}$-parallel section of $\nu(B)$.*

1.3.2 Tubular neighborhoods and the normal exponential map

The exponential map of ν, also called normal exponential map, and denoted by $\exp_\nu$, is the restriction to $\nu(B)$ of the exponential map on TM and is given explicitly by

$$\exp_\nu((m,v)) = \exp_m(v)$$

for $m \in B$ and $v \in \nu_m(B)$. $\exp_\nu$ is defined on an open domain containing the zero section of $\nu(B)$ that we shall always suppose sufficiently small in order to have a diffeomorphic $\exp_\nu$ (cf. [GKM],[ON]).

The differential of the normal exponential map is described by the so-called *B-Jacobi vector fields.* A Jacobi field Y along a geodesic $\gamma : [0,b] \to M, r \mapsto \gamma(r)$ normal to B is a B-Jacobi field if it comes from a variation V of γ through normal geodesics, that is to say (cf. Section 1.1)

$$Y(r) = \frac{d}{ds}\exp_\nu r\xi(s)_{|s=0},$$

where $\xi(s)$ is a normal vector field along the curve $\alpha(s) = \pi \circ \xi(s)$ and $\xi(0) = \dot\gamma(0)$. Here $\pi : \nu(B) \to B$ is the projection of the normal bundle. Moreover, the method described in Lemma 1.1 to generate Jacobi fields can be adapted with slight changes to the specific case of B-Jacobi fields. In particular we have that B-Jacobi fields are characterized by their initial conditions, i.e., Y is a B-Jacobi field along γ if and only if

$$Y(0) \in T_mB \qquad \text{and} \qquad Y'(0) - T(\dot\gamma(0))Y(0) \quad \text{is orthogonal to} \quad B.$$

For $r > 0$ we let

$$\widetilde{U}_B(r) = \{u \in \nu_m(B) : m \in B, \|u\| < r\}.$$

If $\exp_\nu$ maps $\widetilde{U}_B(r)$ diffeomorphically onto the open subset

$$U_B(r) = \exp_\nu(\widetilde{U}_B(r)),$$

then $U_B(r)$ is called a *tubular neighborhood* of B.

$U_B(r)$ can also be described as follows:

$$\begin{aligned} U_B(r) &= \{m \in M : \text{there exists a geodesic } \gamma \text{ with length} \\ &\qquad L(\gamma) < r \text{ from } m \text{ to } B \text{ meeting } B \text{ orthogonally}\} \\ &= \bigcup_{m\in B} \{\exp_m(v) : v \in \nu_m(B),\ \|v\| < r\}. \end{aligned}$$

We also denote by $P_B(r)$ the *tube* of radius r about B given by

$$P_B(r) = \{m \in U_B(r) : d(m, B) = r\},$$

where $d(\ ,\ )$ denotes the distance induced by the metric. It is a smooth hypersurface.

There is a natural parametrization for tubular neighborhoods about B in terms of $\exp_\nu$ when a parametrization of B is known. We describe it in what follows.

1.3.3 Fermi coordinates

Let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame field of (M, g) along B in a neighborhood $U \subset B$ of a point $m \in B$. We choose $E_1, \ldots, E_q$ to be tangent vector fields and $E_{q+1}, \ldots, E_n$ orthonormal sections of $\nu(B)$. If $(y^1, \ldots, y^q)$ is a system of coordinates for B on U such that $\frac{\partial}{\partial x^i}(m) = E_i(m)$, $i = 1, \ldots, q$, then the *Fermi coordinates* $x^1, \ldots, x^n$ relative to m, $(y^1, \ldots, y^n)$ and the frame field $E_{q+1}, \ldots, E_n$ are given by (cf. [Gr6],[Gr7],[GKM],[V4])

$$x^i(\exp_\nu \sum_{\alpha=q+1}^{n} t^\alpha E_\alpha(b)) = y^i(b), \qquad i = 1, \ldots, q$$

$$x^a(\exp_\nu \sum_{\alpha=q+1}^{n} t^\alpha E_\alpha(b)) = t^\alpha, \qquad \alpha = q+1, \ldots, n,$$

where $b \in U$ and the t^α's are small enough, in accordance with the hypothesis made above for $\exp_\nu$.

Now fix a normal unit vector u at m and consider the geodesic normal to B given by $\gamma(r) = \exp_m(ru)$. We shall specialize the frame field $\{E_1, \ldots, E_m\}$ in such a way that $E_n(m) = u$. Next, let $\{e_1(r), \ldots, e_n(r)\}$ be the frame field along $\gamma(r)$ obtained by parallel translation of $\{E_1(m), \ldots, E_n(m)\}$ with respect to the Levi Civita connection ∇ of (M, g).

Again, by definition of Fermi coordinates and taking into account the discussion made above about B-Jacobi vector fields, it is easily seen that the vector fields

$$Y_i(r) = \frac{\partial}{\partial x^i}(\gamma(r)), \qquad Y_a(r) = r\frac{\partial}{\partial x^a}(\gamma(r)), \tag{1.29}$$

$i = 1, \ldots, q$; $a = q+1, \ldots, n-1$, are Jacobi vector fields along γ with initial conditions

$$\begin{cases} Y_i(0) = E_i(m), & Y_i'(0) = \nabla_{\dot\gamma(0)}\frac{\partial}{\partial x^i} = (\nabla_{E_i}E_n)(m), \quad i = 1, \ldots, q, \\ Y_a(0) = 0, & Y_a'(0) = E_a(m), \qquad\qquad\qquad\qquad a = q+1, \ldots, n. \end{cases} \tag{1.30}$$

Then the endomorphism-valued function $r \mapsto D_u(r)$ defined by

$$Y_\alpha(r) = D_u(r)e_\alpha(r), \qquad \alpha = 1, \ldots, n-1, \tag{1.31}$$

satisfies the endomorphism-valued Jacobi equation

$$D_u'' + R_\gamma \circ D_u = 0. \tag{1.32}$$

Each $D_u(r)$ is an endomorphism of the space $\{\dot\gamma(r)\}^\perp$ and as we mentioned above, these spaces may be identified via the parallel translation along γ by using the parallel basis $\{e_A(r)\}$. Using the Gauss and Weingarten formulae introduced above the initial conditions for $D_u(r)$ with respect to the basis $\{E_1(m), \ldots, E_{n-1}(m)\}$ are given in matrix form by

$$D_u(0) = \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix}, \qquad D_u'(0) = \begin{pmatrix} T(u) & 0 \\ -{}^t\bot(u) & I_{n-q-1}, \end{pmatrix} \tag{1.33}$$

where

$$T(u)_{ij} = g(T(u)E_i, E_j)(m), \qquad \perp (u)_{ia} = g(\nabla^{\perp}_{E_i} E_a, E_n)(m);$$

I_q and I_{n-q-1} are the identity matrices of order q and $n-q-1$, respectively.

REMARK. Note that the local orthonormal frame $\{E_{q+1}, \dots, E_n\}$ can always be chosen to be parallel with respect to the normal connection at any single point in U. Moreover, according to Lemma 1.2, it can be taken parallel on U if and only if the normal connection is flat.

Again, as an application of the preceding formulae we compute the covariant and contravariant components of the metric tensor with respect to Fermi coordinates (see also [V4],[NV4]). Let

$$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), \qquad g_{ia} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^a}), \qquad g_{ab} = g(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}),$$

$i, j = 1, \dots, q$; $a, b = q+1, \dots, n$ and $p = \exp_m(ru)$. Then the generalized Gauss Lemma (cf. [Gr7]) and $\frac{\partial}{\partial r}(\gamma(r)) = \frac{\partial}{\partial x^n}(\gamma(r))$ imply

$$g_{in} = g_{an} = 0, \qquad g_{nn} = 1.$$

Next, using (1.29) we get

$$\text{(1.34)} \qquad \begin{cases} g_{ij}(p) = g(D_u(r)e_i, D_u(r)e_j), \\ g_{ia}(p) = \dfrac{1}{r} g(D_u(r)e_i, D_u(r)e_a), \\ g_{ab}(p) = \dfrac{1}{r^2} g(D_u(r)e_a, D_u(r)e_b). \end{cases}$$

The use of the Jacobi equation (1.32) and the initial conditions (1.33) for D_u yield the following power series expansions (by abuse of notation we will also denote by the same symbol the linear map corresponding to the matrix ${}^t\bot(u)$):

$$
\begin{aligned}
g_{ij}(p) = {} & g(E_i, E_j)(m) + 2r g(T(u)E_i, E_j)(m) \\
& + r^2\{-g(R_u E_i, E_j) + g(T(u)^2 E_i, E_j) \\
& \quad + g({}^t\bot(u)E_i, {}^t\bot(u)E_j)\}(m) \\
& - \frac{r^3}{3}\{g(R'_u E_i, E_j) + 2g(R_u E_i, T(u)E_j) \\
& \quad + 2g(R_u E_j, T(u)E_i) - 2g(R_u E_i, {}^t\bot(u)E_j) \\
& \quad - 2g(R_u E_j, {}^t\bot(u)E_i)\}(m) \\
& - \frac{r^4}{12}\{g(R''_u E_i, E_j) - 4g(R_u^2 E_i, E_j) \\
& \quad + 3g(R'_u E_i, T(u)E_j) \\
& \quad + 3g(R'(u)E_j, T(u)E_i) \\
& \quad + 4g(R_u T(u)E_i, T(u)E_j) \\
& \quad - 3g(R'_u E_i, {}^t\bot(u)E_j) - 3g(R'_u E_j, {}^t\bot(u)E_i) \\
& \quad + 4g(R_u {}^t\bot(u)E_i, {}^t\bot(u)E_j) \\
& \quad - 4g(R_u {}^t\bot(u)E_j, T(u)E_i) \\
& \quad - 4g(R_u T(u)E_j, {}^t\bot(u)E_i)\}(m) + O(r^5),
\end{aligned}
\tag{1.35}
$$

$$
\begin{aligned}
g_{ia}(p) = {} & -r g({}^t\bot(u)E_i, E_a)(m) - \frac{2r^2}{3} g(R_u E_i, E_a)(m) \\
& - \frac{r^3}{12}\{3g(R'_u E_i, E_a) + 4g(R_u T(u)E_i, E_a) \\
& \quad - 4g(R_u {}^t\bot(u)E_i, E_a)\}(m) \\
& - \frac{r^4}{30}\{2g(R''_u E_i, E_a) - 4g(R_u^2 E_i, E_a) \\
& \quad + 5g(T(u)E_i, R'_u E_a) \\
& \quad - 5g({}^t\bot(u)E_i, R'_u E_a)\}(m) + O(r^5),
\end{aligned}
\tag{1.36}
$$

$$
\begin{aligned}
(1.37) \qquad g_{ab}(p) = g(E_a, E_b)(m) &- \frac{r^2}{3} g(R_u E_a, E_b)(m) \\
&- \frac{r^3}{6} g(R'_u E_a, E_b)(m) \\
&+ \frac{r^4}{180} \{8g(R_u^2 E_a, E_b) - 9g(R''_u E_a, E_b)\}(m) \\
&+ O(r^5).
\end{aligned}
$$

Furthermore, by solving the equations

$$
\begin{cases}
\delta_{ij} = g_{ik} g^{kj} + g_{ia} g^{aj}, \\
0 = g_{ik} g^{ka} + g_{ib} g^{ba}, \\
\delta_{ab} = g_{ak} g^{kb} + g_{ac} g^{cb},
\end{cases}
$$

with $k, i, j = 1, \ldots, q$ and $a, b, c = q+1, \ldots, n$, we obtain

$$
\begin{aligned}
(1.38) \qquad g^{ij}(p) = g(E_i, E_j)(m) &- 2rg(T(u)E_i, E_j)(m) \\
&+ r^2 \{3g(T(u)^2 E_i, E_j) + g(R_u E_i, E_j)\}(m) \\
&+ \frac{r^3}{3} \{g(R'_u E_i, E_j) - 4g(T(u)E_i, R_u E_j) \\
&\quad - 4g(R_u E_i, T(u)E_j) \\
&\quad - 12g(T(u)^3 E_i, E_j)\}(m) + O(r^4),
\end{aligned}
$$

$$
\begin{aligned}
(1.39) \qquad g^{ia}(p) = rg({}^t\bot(u)E_i, E_a)(m) &+ \frac{2r^2}{3} \{g(R_u E_i, E_a) \\
&- 3\sum_k g(T(u)E_i, E_k) g({}^t\bot(u)E_k, E_a)\}(m) \\
\frac{r^3}{4} \{g(R'(u)E_i, E_a) &- 4g(R_u T(u)E_i, E_a) \\
&+ 4\sum_k g(R_u E_i, E_k) g({}^t\bot(u)E_k, E_a) \\
&+ 12\sum_k g(T(u)^2 E_i, E_k) g({}^t\bot(u)E_k, E_a)\}(m) \\
+ O(t^4),
\end{aligned}
$$

$$
\begin{aligned}
(1.40) \qquad g^{ab}(p) = {} & g(E_a, E_b)(m) + \frac{r^2}{3}\{g(R_u E_a, E_b) \\
& +3\sum_k g({}^t\bot(u)E_k, E_a)g({}^t\bot(u)E_k, E_b)\}(m) \\
& +\frac{r^3}{6}\{g(R'_u E_a, E_b) \\
& -12\sum_{k,l} g(T(u)E_k, E_l)g({}^t\bot(u)E_k, E_a)g({}^t\bot(u)E_l, E_b) \\
& +4\sum_k g(R_u E_k, E_a)g({}^t\bot(u)E_k, E_b) \\
& +4\sum_k g(R_u E_k, E_b)g({}^t\bot(u)E_k, E_a)\}(m) + 0(r^4).
\end{aligned}
$$

1.3.4 The curve case

We specialize now the preceding formulae to the case of a (unit speed) embedded curve $\sigma : [a, b] \mapsto M$. As the normal connection of $\nu(\sigma)$ is obviously flat, it is not restrictive to work with normal vector fields along σ which are parallel with respect to the normal connection of $\nu(\sigma)$. This considerably simplifies the treatment.

Let $\{E_1(t), \dots, E_n(t)\}$ be an orthonormal frame along σ, such that $E_1(t) = \dot{\sigma}(t)$ and $E_2, \dots, E_n$ are normal vector fields along σ which are parallel with respect to the normal connection. Then, on a tubular neighborhood U_σ of σ, the Fermi coordinates $(x_1, \dots, x_n)$ with respect to $\sigma(t_o)$, $t_o \in [a, b]$, and the frame field $\{E_2, \dots, E_n\}$ are defined by

$$
\begin{aligned}
x^1(\exp_{\sigma(t)} \sum_{j=2}^n t^j E_{j\,|\sigma(t)}) &= t - t_o \\
x^i(\exp_{\sigma(t)} \sum_{j=2}^n t^j E_{j\,|\sigma(t)}) &= t^i, \qquad i = 2, \dots, n.
\end{aligned}
$$

For $p \in U_\sigma$ we have $p = \exp_{\sigma(t)} v$, where $v = \sum_{i=2}^n x^i E_i(t) = ru$, $\|u\| = 1$ and $r^2 = \sum_{i=2}^n (x^i)^2$.

In general σ is not a geodesic and we put

$$\kappa_u = g(\ddot{\sigma}, u),$$

where $\ddot{\sigma}(t) = (\nabla_{\frac{\partial}{\partial t}}\dot{\sigma})(t)$ is the *(mean) curvature vector* of σ normal to σ at t. If u is $\nabla^{\perp}$–parallel along σ, we have

$$(\nabla_{\frac{\partial}{\partial t}}u)(\sigma(t)) = g((\nabla_{\frac{\partial}{\partial t}}u)(t), \dot{\sigma}(t))\dot{\sigma}(t),$$

Therefore, since $g(u, \dot{\sigma}) = 0$,

$$(\nabla_{\frac{\partial}{\partial t}}u)(\sigma(t)) = -g(u, \ddot{\sigma}(t))\dot{\sigma}(t) = -\kappa_u\dot{\sigma}(t). \tag{1.41}$$

Thus if we assume that $E_n(t) = \dot{\gamma}(0)$, where $\gamma : r \mapsto \exp_{\sigma(t)}(ru)$, (1.41) reads in Fermi coordinates

$$(\nabla_{\frac{\partial}{\partial x^1}}u)(\sigma(t)) = -\kappa_u\dot{\sigma}(t).$$

Hence the initial conditions for $D_u(r)$ are (cf. (1.30))

$$D_u(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-2} \end{pmatrix}, \qquad D'_u(0) = \begin{pmatrix} -\kappa_u & 0 \\ 0 & I_{n-2} \end{pmatrix}.$$

Proceeding as in the general case we get for $m = \sigma(t)$ and $p = \exp_{\sigma(t)}(ru)$ the following power series expansions for the components of the metric with respect to Fermi coordinates:

$$\begin{aligned} g_{11}(p) = 1 &- 2r\kappa_u(m) + r^2(\kappa_u^2 - R_{1u1u})(m) \\ &- \frac{r^3}{3}\{(\nabla_u R)_{1u1u} - 4\kappa_u R_{1u1u}\}(m) \\ &- \frac{r^4}{12}\{(\nabla^2_{uu}R)_{1u1u} - 4\sum_{c=2}^{n} R^2_{1ucu} \\ &\quad - 6\kappa_u(\nabla_u R)_{1u1u} + 4\kappa_u^2 R_{1u1u}\}(m) + O(r^5), \end{aligned} \tag{1.42}$$

$$
\begin{aligned}
g_{1a}(p) = & -\frac{2r^2}{3}R_{1uau}(m) \\
& -\frac{r^3}{12}\{3(\nabla_u R)_{1uau} - 4\kappa_u R_{1uau}\}(m) \\
& -\frac{r^4}{30}\{2(\nabla^2_{uu} R)_{1uau} - 4R_{1u1u}R_{1uau} \\
& \qquad -4\sum_{c=2}^{n} R_{1ucu}R_{aucu} - 5\kappa_u(\nabla_u R)_{1uau}\}(m) \\
& + O(r^5),
\end{aligned}
\tag{1.43}
$$

$$
\begin{aligned}
g_{ab}(p) = \delta_{ab} - \frac{r^2}{3}R_{uaub}(m) - \frac{r^3}{6}(\nabla_u R)_{uaub}(m) \\
- \frac{r^4}{180}\{9(\nabla^2_{uu} R)_{uaub} - 8R_{1uau}R_{1ubu} \\
- 8\sum_{c=2}^{n} R_{uauc}R_{ubuc}\}(m) + O(r^5),
\end{aligned}
\tag{1.44}
$$

$$
\begin{aligned}
g^{11}(p) = & \, 1 + 2r\kappa_u(m) + r^2(R_{1u1u} + 3\kappa_u^2)(m) \\
& \frac{r^3}{3}\{(\nabla_u R)_{1u1u} + 8\kappa_u R_{1u1u} + 12\kappa_u^3\}(m) \\
& \frac{r^4}{36}\{3(\nabla^2_{uu} R)_{1u1u} + 24R^2_{1u1u} + 4\sum_{c=2}^{n} R^2_{1ucu} \\
& \quad + 30\kappa_u(\nabla_u R)_{1u1u} + 180\kappa_u^2 R_{1u1u} + 180\kappa_u^4\}(m) \\
& + O(r^5),
\end{aligned}
\tag{1.45}
$$

$$
\begin{aligned}
g^{1a}(p) = & \frac{2r^2}{3}R_{1uau}(m) + \frac{r^3}{4}\{(\nabla_u R)_{1uau} + \kappa_u R_{1uau}\}(m) \\
& \frac{r^4}{45}\{3(\nabla^2_{uu} R)_{1uau} + 24R_{1u1u}R_{1uau} \\
& \quad + 4\sum_{c=2}^{n} R_{1ucu}R_{aucu} + 15\kappa_u(\nabla_u R)_{1uau} \\
& \quad + 60\kappa_u^2 R_{1uau}\}(m) + O(r^5),
\end{aligned}
\tag{1.46}
$$

$$(1.47)\qquad \begin{aligned} g^{ab}(p) = \delta_{ab} + \frac{r^2}{3}R_{uaub}(m) + \frac{r^3}{6}(\nabla_u R)_{uaub}(m) \\ \frac{r^4}{60}\{3(\nabla^2_{uu}R)_{uaub} + 24R_{1uau}R_{1ubu} \\ + 4\sum_{c=2}^{n} R_{uauc}R_{ubuc}\}(m) + O(r^5). \end{aligned}$$

1.4 Almost Hermitian geometry

1.4.1 Almost Hermitian manifolds

An *almost complex manifold* is a smooth manifold M equipped with a $(1,1)$-tensor field J wich satisfies $J^2 = -I$, where I denotes the identity map. M is necessarily even-dimensional and orientable. Note that J gives TM a structure of complex vector bundle over M by posing $(a+ib)X = aX + bJX$, $X \in \mathfrak{X}(M)$, a, b real numbers. The complexification $T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ of TM splits into conjugate subbundles under the action of J,

$$T_{\mathbb{C}}M = T^{(1,0)}(M) \oplus T^{(0,1)}(M),$$

where $J_{|T^{(1,0)}(M)} = i$ and $T^{(0,1)}(M)$ is the complex conjugate of $T^{(1,0)}(M)$.

$T^{(1,0)}(M)$ is generated by elements of the form $U = \frac{1}{2}(X - iJX)$, X tangent vector and the mapping $X \mapsto U$ defines a $\mathbb{C}$–linear isomorphism between TM and $T^{(1,0)}(M)$.

A map $f : M \to N$ between almost complex manifolds (M, J^M) and (N, J^N) is said to be *holomorphic* if its differential $df : TM \to TN$ commutes with the respective almost complex structures, i.e.,

$$df \circ J^M = J^N \circ df,$$

and *anti-holomorphic* if it anti-commutes, i.e., $df \circ J^M = -J^N \circ df$.

A *Hermitian metric* on an almost complex manifold M is a Riemannian metric g such that

$$g(JX, JY) = g(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$. Every almost complex manifold admits such a metric, for example by taking $g(X,Y) = h(X,Y) + h(JX, JY)$ with h an arbitrary metric. An almost complex manifold with an Hermitian metric is called an *almost Hermitian manifold* .

Let (M, g, J) be an almost Hermitian manifold and N a submanifold of M. N is said to be a *holomorphic* submanifold of M if $JT_m(N) = T_m(N)$ for all $m \in N$ and N is a *totally real* submanifold if for all $m \in N$ we have $JT_m(N) \subset T_m^{\perp}(N)$, where $T_m^{\perp}(N)$ denotes the normal space (with respect to g) to T_mN in T_mM.

In almost Hermitian geometry two additional tensors defined in terms of J play a special role. The first is the *Kähler form* Ω defined for $X, Y \in \mathfrak{X}(M)$ by

$$\Omega(X, Y) = g(X, JY).$$

Notice that Ω is skew-symmetric and defines a differential two-form. Moreover it is non-degenerate as bilinear form on each tangent space.

The second is the so-called Nijenhuis tensor N defined by

$$N(X, Y) = [X, Y] - [JX, JY] + J[JX, Y] + J[X, JY].$$

The condition $N = 0$ means that the almost complex structure is *integrable* (cf. [KN],vol.II,[Go],[NN]), that is, locally there exist coordinates $(z^1, \ldots, z^m)$, $2m = \dim_{\mathbb{R}} M$, $z^k = x^k + iy^k$, $k = 1, \ldots, m$, for which

$$J(\frac{\partial}{\partial x^k}) = \frac{\partial}{\partial y^k}.$$

Since any two such systems of coordinates are related by a biholomorphic change of variables, the integrability condition is equivalent to M being a *complex manifold.*

We describe now some types of almost Hermitian manifolds obtained by assuming additional conditions on J and Ω. In what follows ∇ will denote the Levi Civita connection of the Hermitian metric g.

DEFINITION. An almost Hermitian manifold (M, g, J) is (cf. [Gr1],[V1])

(1) *Hermitian* if $N = 0$ or, equivalently, $(\nabla_X J)Y - (\nabla_{JX} J)JY = 0$;
(2) *Kähler* if $\nabla J = 0$; (In this case $\nabla\Omega = 0$ and so $d\Omega = 0$. Moreover J is necessarily integrable.)
(3) *Nearly-Kähler* if $(\nabla_X J)Y + (\nabla_Y J)X = 0$;
(4) *Quasi-Kähler* if $(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$;
(5) *Almost-Kähler* if Ω is closed, i.e., $d\Omega = 0$;
(6) *Semi-Kähler* if $\delta\Omega = 0$, where δ denotes the codifferential.

A remarkable feature of Kähler manifolds is that the Riemann curvature tensor satisfies the so-called *Kähler identity,* i.e., (cf. [KN], vol. II)

$$R_{XY} \circ J = J \circ R_{XY}, \qquad X, Y \in \mathfrak{X}(M).$$

Kähler and almost-Kähler manifolds are examples of a broader class of manifolds which we will treat now.

A smooth manifold M equipped with a closed 2-form Ω which is non-degenerate as a bilinear form on each tangent space is called *symplectic* . (M, Ω) is necessarily even-dimensional and orientable. Among other restrictions, most of them of topological kind, a further restriction on a smooth manifold to be symplectic is that *it must admit an almost-Kähler structure.* To see this we pick any Riemannian metric g on a symplectic manifold (M, Ω). Then Ω can be represented with respect to g by a skew-symmetric endomorphism A on TM, i.e., $\Omega(X, Y) = g(X, AY)$. $-A^2$ is then positive definite so that it has a positive square root B. Set $J = AB^{-1}$. Since A and B commute, $J^2 = -I$, while $\Omega(JX, JY) = g(AB^{-1}X, AAB^{-1}Y) = -g(AB^{-1}X, BY) = -g(AX, Y) = \Omega(X, Y)$. On the other hand, $\Omega(JX, X) = g(AB^{-1}X, AX) = g(BX, X)$, which is strictly positive for $X \neq 0$. This says that the modified metric $g'(X, Y) = g(X, BY)$ gives M the structure of an almost-Kähler manifold with Kähler 2-form Ω.

If (M, Ω) is symplectic, then a *symplectic diffeomorphism* of M is a diffeomorphism $f : M \to M$ which preserves Ω, i.e.,

$$f^*\Omega = \Omega.$$

For example the group of holomorphic isometries of a Kähler manifold M consists of symplectic diffeomorphisms of (M, Ω).

However, we emphasize that our use of the word symplectic in the framework of Hermitian geometry does not imply necessarily that the Kähler form Ω is closed (cf. [SV1],[V3]).

1.4.2 Holomorphic normal coordinates

Let (M, g, J) be a Kähler manifold with respect to the Hermitian metric g. g extends by complex linearity to a complex bilinear form g on $T_{\mathbb{C}}M$ (this form cannot be positive definite) and determines a positive definite Hermitian scalar product h on $T_{\mathbb{C}}M$, whose components with respect to complex coordinates $(z^1, \ldots, z^m)$ are defined by

$$h_{\alpha\beta} = h(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}) = g(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z^\beta}}) = g_{\alpha\overline{\beta}}, \qquad \alpha, \beta = 1, \ldots, m.$$

Then h induces by restriction a positive definite Hermitian scalar product on the holomorphic tangent bundle $TM \simeq T^{(1,0)}(M)$.

This Hermitian scalar product h is (locally) expressed in the form

$$h = \sum_{\alpha,\beta} h_{\alpha\beta} dz^\alpha \otimes \overline{dz^\beta} \qquad (h_{\alpha\beta} = \overline{h_{\beta\alpha}}).$$

The corresponding Riemannian metric g is then

$$g = 2Re(\sum_{\alpha,\beta} h_{\alpha\beta} dz^\alpha \otimes \overline{dz^\beta}).$$

In this setting the following are equivalent (cf. [Go],[KN],vol.II):

(1) (M, g) is Kähler.

(2) At each point m there are complex coordinates $(z^1, \ldots, z^m)$ such that

$$h_{\alpha\beta}(m) = \delta_{\alpha\beta}, \qquad \frac{\partial h_{\alpha\beta}}{\partial z^\gamma}(m) = 0 = \frac{\partial h_{\alpha\beta}}{\partial \overline{z^\delta}}(m) \tag{1.48}$$

for all $\alpha, \beta, \gamma, \delta = 1, \ldots, m$.

(3) For each point m there exists a system of complex coordinates $(z^1, \ldots, z^m)$ centered at m such that h coincides with the flat metric $\sum_\alpha dz^\alpha \otimes \overline{dz^\alpha}$ up to the order 1 at m, i.e.,

$$h_{\alpha\beta}(z, \overline{z}) = \delta_{\alpha\beta} + O(2), \tag{1.49}$$

where $O(2)$ means terms of order two or more in the z^α and $\overline{z^\alpha}$.

Complex coordinates as in (2) or (3) are called *holomorphic normal coordinates.*

REMARK. If $z^k = x^k + iy^k$, $k = 1, \ldots, m$, are holomorphic normal coordinates, it holds that

$$(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial y^l})(m) = 0, \qquad (\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l})(m) = 0, \qquad (\nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^l})(m) = 0,$$

$$g(\tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial x^l})(m) = \delta_{kl}, \quad g(\tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial y^l})(m) = \delta_{kl}, \quad g(\tfrac{\partial}{\partial y^k}, \tfrac{\partial}{\partial y^l})(m) = \delta_{kl},$$

for $k, l = 1, \ldots, m$. Therefore, the real and imaginary parts of normal holomorphic coordinates at a given point m match Riemann normal coordinates at m up to the first order. In other words $\exp_m$ is holomorphic to second order at m in normal holomorphic coordinates. The request for $\exp_m$ to be holomorphic in normal holomorphic coordinates forces M to be a flat Kähler manifold. In fact, Riemann normal coordinates $(x^1, \ldots, x^{2m})$ on U about m are real and imaginary parts of normal holomorphic coordinates if and only if

$$J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^{m+i}}, \qquad i = 1, \ldots, m,$$

on U or, equivalently, if for $p = \exp_m(ru) \in U$, $u \in T_m M$,

$$J_p \circ d\exp_{m|ru}(e_i) = d\exp_{m|ru}(Je_i). \tag{1.50}$$

Now, using Jacobi fields as indicated in Section 1.1 and taking into account that M is Kähler ($\nabla J = 0$), we write down power series expansions about m for both sides of (1.50). The first necessary condition we get is

$$R_u \circ J_m - J_m \circ R_u = 0,$$

which yields

$$R_{uxuy} - R_{uJxuJy} = 0, \qquad u, x, y \in T_m M.$$

Putting $u = x = y$ we obtain $R_{uJuuJu} = 0$ and hence $R = 0$ (see [KN],vol.II, p.166).

1.5 Harmonic maps

Let $\phi : M \to N$ be a smooth map between two Riemannian manifolds with metrics g and h, respectively. The differential $d\phi : TM \to TN$ can be interpreted as a homomorphism from the tangent bundle TM of M to the pull-back $\phi^{-1}TN$ of the tangent bundle of N, i.e., $d\phi$ can be considered as a section of the bundle $T^*M \otimes \phi^{-1}TN$. Let ∇ denote both the Levi Civita connection on TM and the pull-back connection on $\phi^{-1}TN$. With the due identifications, the covariant derivative of $d\phi$ is then defined by

$$(\nabla_X d\phi)Y = \nabla_X(d\phi Y) - d\phi(\nabla_X Y), \qquad X, Y \in \mathfrak{X}(M).$$

Since ∇ is torsion-free

$$(\nabla_X d\phi)(Y) - (\nabla_Y d\phi)X = [d\phi X, d\phi Y] - d\phi[X, Y] = 0.$$

Therefore $\nabla d\phi$ is a section of the bundle $S^2T^*M \otimes \phi^{-1}TN$, i.e., is a symmetric bilinear form on TM with values in $\phi^{-1}TN$ and is called the *second fundamental form* of ϕ. The trace of $\nabla d\phi$ taken with respect to the metric g is called the *tension field* of ϕ and is denoted by $\tau(\phi)$. The map ϕ is *harmonic* if $\tau(\phi) = 0$ (cf. [ES],[EL]).

Let $U \subset M$ be a domain with coordinates $(x^1, \ldots, x^m)$ and $V \subset N$ a domain with coordinates $(y^1, \ldots, y^n)$ such that $\phi(U) \subset V$ and suppose ϕ is locally represented by $y^\alpha = \phi^\alpha(x^1, \ldots, x^m)$, $\alpha = 1, \ldots, n$. Then we have

$$\nabla(d\phi)^\gamma_{ij} = \frac{\partial^2 \phi^\gamma}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij} \frac{\partial \phi^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j}, \tag{1.50}$$

where ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbols of (M, g) and (N, h), respectively. Hence ϕ is harmonic if and only if

$$\tau^\gamma(\phi) = g^{ij}(\nabla(d\phi))^\gamma_{ij} = 0, \qquad \gamma = 1, \ldots, n. \tag{1.51}$$

Harmonic maps occur in many different situations.

For example, if $N = \mathbb{R}$, τ is just the Laplace-Beltrami operator on (M, g) and harmonic maps are just harmonic functions.

If $M = [a, b] \subset \mathbb{R}$ then a map from M into N is harmonic if and only if it is a geodesic. But as we saw in Section 1.1, the geodesics are precisely the critical points of the energy functional. In general, it can be seen that harmonic maps arise as solutions of a variational problem and an arbitrary map ϕ is harmonic if and only if it corresponds to a critical point of the *energy functional*

$$E(\phi) = \frac{1}{2}\int_M \|d\phi\|^2$$

where

$$\|d\phi\|^2 = g^{jk}\frac{\partial\phi^r}{\partial x^j}\frac{\partial\phi^s}{\partial x^k}h_{rs}$$

and $\tau(\phi) = 0$ is the associated Euler-Lagrange equation.

If ϕ is a Riemannian immersion, i.e., $\phi^*h = g$, then $\nabla d\phi$ is the classical second fundamental form of the submanifold $\phi(M)$ in N and ϕ is harmonic if and only if it is a minimal immersion.

If ϕ is a holomorphic map between open sets $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$, where both $\mathbb{C}^m$ and $\mathbb{C}^n$ are considered with the flat metric, then each component ϕ^α of ϕ, $\alpha = 1, \ldots, n$, satisfies the Cauchy-Riemann equations

$$\frac{\partial\phi^\alpha}{\partial\overline{z^\beta}} = 0, \qquad \beta = 1, \ldots, m.$$

Therefore ϕ is harmonic since each component satisfies the Laplace equation

$$\sum_{\beta=1}^{m}\frac{\partial^2\phi^\alpha}{\partial z^\beta\partial\overline{z^\beta}} = 0.$$

If (M, g) and (N, h) are Kähler manifolds and ϕ is holomorphic, we can choose holomorphic normal coordinates at $p \in M$ and $\phi(p) \in N$ (cf. Section 1.4). For such a choice of coordinates $\tau(\phi)(p)$ reduces to the usual Laplacian and then, by arguing as above, ϕ is harmonic.

More generally, one can give conditions on almost Hermitian manifolds M, N in order that a holomorphic map $\phi : M \to N$ should necessarily be harmonic. To this purpose we state the following theorem by Lichnerowicz [Li].

THEOREM 1.3. *Let $\phi : M \to N$ be either a holomorphic or an anti-holomorphic map of almost Hermitian manifolds, where M is semi-Kähler and N is quasi-Kähler. Then ϕ is harmonic.*

Chapter 2

Rotations around points

In this chapter we present some aspects of the theory of rotations around points on a Riemannian manifold (M, g). A rotation s_m around a point $m \in M$ is the local diffeomorphism defined by $s_m = \exp_m \circ S \circ \exp_m^{-1}$, where S is a given $(1,1)$-tensor field on M which preserves the metric g (the *rotation field*). The concept of rotations extends in a natural way the corresponding notion for ordinary Euclidean space. In the same way as rotations in Euclidean space have properties that are directly related to Euclidean geometry, we can expect that in a Riemannian manifold there should be a strong relation between rotations and the curvature properties of M. Therefore, it is likely that on a specific Riemannian manifold obtained by some restriction on the curvature, the rotations will have particular properties and, on the other hand, if the rotations have special properties one may ask how these reflect in the curvature of the ambient space.

After introducing the notion of a rotation around a point we discuss different examples of rotations such as reflections, S-regular rotations and J-rotations. We concentrate on aspects about Riemannian, Hermitian and symplectic geometry and consider in particular isometric, symplectic, holomorphic and volume-preserving rotations. Moreover, we study these properties of rotations in relation with the extrinsic and intrinsic geometry of the (small) geodesic spheres of a Riemannian manifold. In this way, we again obtain information on the curvature of the ambient space.

2.1 Definitions and examples

Let (M, g) be a Riemannian manifold and ∇ its Levi Civita connection. Furthermore, let S be a $(1,1)$-tensor field on M which preserves g, i.e.,

$$g(SX, SY) = g(X, Y)$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. S is said to be a *rotation field* on M. For any rotation field S we define on a sufficiently small geodesic ball B_m with center $m \in M$ a local diffeomorphism s_m, which fixes m, by

$$s_m = \exp_m \circ S_m \circ \exp_m^{-1}.$$

We call s_m a *(local) S-rotation* . In this way we obtain a field $s : m \mapsto s_m$ of S-rotations such that

$$ds_{m|m} = S_m.$$

If $S - I$ is non-singular (I denotes the Kronecker tensor field), i.e., S_m has no fixed tangent vector $X \in T_mM$ for each $m \in M$, then m is an isolated fixed point of s_m and we say that S is a *free rotation field* and s_m a *free rotation.* In the literature, S and s_m have also been called a *symmetry tensor field* and a *symmetry,* respectively (cf. [GL],[K1],[LV2]).

For a system of normal coordinates $(x^1, \ldots, x^n)$ on B_m centered at m and such that $\frac{\partial}{\partial x^i} = e_i$, $i = 1, \ldots, n$, we have the following analytic expression for s_m:

$$x^i \circ s_m = S^i_j(m) x^j,$$

where $S^i_j(m)$ are the components of S at m with respect to the chosen basis. They form an orthogonal matrix.

The following are some well-known examples of free rotations that occur in differential geometry.

EXAMPLE 1. For $S = -I$, s_m defines the local *geodesic reflection* or *geodesic symmetry* with respect to m. See [V3],[V4] for applications and additional references.

EXAMPLE 2. Let $\mathcal{F}(M)$ be the ring of all real-valued smooth functions on M. For all non-negative integers p, q let $\mathcal{T}^p_q(M)$ be the module over $\mathcal{F}(M)$ of all smooth tensor fields with contravariant and covariant orders p and q, respectively. Then, for a rotation field S we say that any $T \in \mathcal{T}^p_q(M)$ is S-invariant if for all $\omega_1, \dots, \omega_p \in \mathcal{T}^0_1(M)$ and all $X_1, \dots, X_q \in \mathcal{T}^1_0(M)$

$$T(\omega_1 S, \dots, \omega_p S, X_1, \dots, X_q) = T(\omega_1, \dots, \omega_p, SX_1, \dots, SX_q),$$

where $(\omega S)X = \omega(SX)$ for $\omega \in \mathcal{T}^0_1(M)$ and $X \in \mathcal{T}^1_0(M)$.

In particular, a free rotation field S is said to be *regular* if ∇S and $\nabla^2 S$ are S-invariant.

Next, let S be a free rotation field and suppose that for each $m \in M$, s_m is a local isometry. Moreover, let S be s-invariant, that is

$$S(ds_m X) = ds_m(SX)$$

for each vector field X defined on some neighborhood of m. (M, g) together with the field s of S-rotations is called a Riemannian *locally s-regular manifold* .

If in addition $S^k = I$ for some $k \in \mathbb{N}_+$, $k \neq 1$, (M, g) together with S is called a *k-symmetric space* (see [GL],[K1]). Note that Riemannian locally s-regular manifolds are necessarily locally homogeneous and hence analytic.

For $k = 2$ $(S = -I)$ we obtain the classical locally symmetric spaces. For $k = 3$ we have the locally 3-symmetric spaces. Such manifolds are almost Hermitian manifolds (M, g, J), where the almost complex structure is determined canonically by the field s of S-rotations as follows:

$$S_m = ds_{m|m} = -\frac{1}{2} I_m + \frac{\sqrt{3}}{2} J_m.$$

With respect to this almost complex structure the locally 3-symmetric spaces are quasi-Kähler manifolds and in the connected, simply connected, complete case they are nearly-Kähler if and only if they are naturally reductive homogeneous spaces. For more details about locally 3-symmetric manifolds we refer to [Gr3],[K1],[TV3].

Any locally s-regular (in particular, any k-symmetric) Riemannian manifold can be defined either by local isometries as above or equivalently by the following tensor conditions on S and on the Riemannian curvature tensor ([GL],[K1]).

LEMMA 2.1. *Let S be a regular rotation field on (M, g) such that R and ∇R are S-invariant. Then $\nabla^2 R$ is S-invariant and hence all the covariant derivatives of R are S-invariant. Moreover, (M, g) is a locally s-regular manifold with respect to S.*

EXAMPLE 3. On an almost Hermitian manifold (M, g, J), the almost complex structure provides a natural free rotation field and hence a field of J-rotations.

So far we have considered only examples of free rotations. In the next are considered non-free rotations.

EXAMPLE 4. Let (M, g) be a Riemannian manifold and P a $(1,1)$-tensor field on M such that $P^2 = I$ and $g(PX, PY) = g(X, Y)$, $X, Y \in \mathfrak{X}(M)$. P is said to be an *almost product structure* on M and (M, g, P) is called an *almost product manifold* (cf. [Gr2]). The tensor fields $V = \frac{I+P}{2}$ and $H = \frac{I-P}{2}$ verify $V^2 = H^2 = I$ and determine complementary distributions on M, neither of which is in general integrable.

2.2 Isometric rotations

First, we note that any isometry f fixing a point m is a rotation when restricted to a suitable neighborhood of m. Its rotation field is $df_{|m}$. We want to determine necessary and sufficient conditions for an S-rotation to be an isometry. In the analytic case we obtain the following criterion.

PROPOSITION 2.2. *Let (M, g) be an analytic Riemannian manifold and $m \in M$. Then the S-rotation s_m is an isometry if and only if*

$$(2.1) \qquad (\nabla^k_{X\dots X} R)_{XYXY} = (\nabla^k_{SX\dots SX} R)_{SXSYSXSY}, \qquad k = 0, 1, 2, \dots$$

for each $X, Y \in T_m M$, X orthogonal to Y.

We refer to [KV3] for a proof. According to Proposition 2.2, if S satisfies (2.1) we have as a consequence that the conditions

$$(2.2) \qquad (\nabla^k_{X_1\dots X_k} R)_{XYZV} = (\nabla^k_{SX_1\dots SX_k} R)_{SXSYSZSV}, \qquad k = 0, 1, 2, \dots$$

$X_1 \dots X_k, X, Y, Z, V \in T_m M$, hold. The conditions (2.2) are the classical conditions for the existence of a local isometry in the analytic situation ([KN],vol.I). Thus, in the analytic case, (2.1) implies (2.2). From the algebraic point of view this implication has a non-trivial meaning and leads to the following (cf. [KV3])

PROBLEM. Do the conditions (2.1) imply the conditions (2.2) for any k ?

For $k = 0$ this is the standard fact that the sectional curvature uniquely determines the Riemann curvature tensor. For $k = 1$ the implication is also true due to (see [VW2],[Gr5],[CV1] and Lemma 2.24 in [Be1])

LEMMA 2.3. *In a Riemannian manifold the following are equivalent:*

(1) $\nabla R = 0$;
(2) $(\nabla_X R)_{XYXY} = 0$;
(3) $(\nabla_V R)_{WXYZ} + (\nabla_X R)_{WZYV} + (\nabla_Z R)_{WVYX} = 0$.

It is worth noting that the proofs of Lemma 2.3 given in [VW2],[Gr5] and [CV1] are of algebraic kind, while the proof provided in [Be1] is based on the use of Jacobi vector fields and the geodesic reflections.

Now we specialize Proposition 2.2 to the different examples considered above. In the case of geodesic reflections (2.1) yields $(\nabla_X R)_{XYXY} = 0$ and then for Lemma 2.3 this is equivalent to $\nabla R = 0$. Hence, if (2.1) holds for each m, we get the well-known fact that *(M, g) is locally symmetric if and only if the geodesic reflections s_m are isometries for each $m \in M$.*

For a regular rotation field S on (M, g), Proposition 2.2 and Lemma 2.1 yield ([NV2])

PROPOSITION 2.4. *Let (M, g) be a Riemannian manifold and S a regular rotation field. Then all S-rotations s_m are isometries if and only if*

$$R_{XYXY} = R_{SXSYSXSY}, \tag{2.3}$$

$$(\nabla_X R)_{XYXY} = (\nabla_{SX} R)_{SXSYSXSY}, \tag{2.4}$$

for each $X, Y \in T_m M$, X orthogonal to Y and all $m \in M$. In particular, if (2.3) and (2.4) hold, then (M, g) is an s-regular manifold with respect to S, and conversely.

Finally, for J-rotations we have ([NV1])

PROPOSITION 2.5. *Let (M, g, J) be an almost Hermitian manifold. Then each J-rotation j_m is an isometry if and only if (M, g, J) is locally symmetric and*

$$R_{XYXY} = R_{JXJYJXJY} \tag{2.5}$$

for all vector fields X, Y *on* M.

As an alternative we present here a direct proof that makes use of Jacobi fields.

PROOF. First, let j_m be an isometry. Since $dj_{m|m} = J_m$, we have

$$R_{J_m X J_m Y J_m X J_m Y} = R_{XYXY},$$

where $X, Y \in T_m M$. Hence (2.5) holds. Moreover, since j_m^2 is the geodesic reflection centered at m, these reflections are isometric and then, as is well-known, (M, g) is locally symmetric (cf. [He]).

Conversely, let $\widetilde{B}_m$ be the ball about the origin in $T_m M$ corresponding to B_m by the exponential map. Let $p \in B_m$ and $v \in T_p M$. Then there exists a unique $u \in \widetilde{B}_m$ and also a $w \in T_u(T_m M) \simeq T_m M$ such that $d\exp_{m|u}(w) = v$. Here $v = Y(1)$, where Y is the unique Jacobi field along the geodesic $\gamma : t \mapsto \exp_m(tu)$ with initial conditions $Y(0) = 0, Y'(0) = w$ (cf. Chapter 1). So $g(v, v) = g(Y(1), Y(1))$. Moreover the definition of j_m yields $dj_{m|p}(v) = d\exp_m(J_m w)$ and hence $g(dj_{m|p}(v), dj_{m|p}(v)) = g(\overline{Y}(1), \overline{Y}(1))$, where $\overline{Y}(1)$ is the unique Jacobi field along $\overline{\gamma} : t \mapsto \exp_m(tJ_m u)$, $\overline{\gamma}(0) = m$, with initial conditions $\overline{Y}(0) = 0, \overline{Y}'(0) = J_m w$. Now, let $(E_1, \ldots, E_n)$ be the orthonormal frame field along γ obtained by parallel translation of $(e_1, \ldots, e_n)$ and $(\overline{E}_1, \ldots, \overline{E}_n)$ that obtained by parallel translation of $(Je_1, \ldots, Je_n)$ along $\overline{\gamma}$. Put $Y(t) = \sum Y_i(t) E_i$, $\overline{Y}(t) = \sum \overline{Y}_i(t) \overline{E}_i$. As (M, g) is locally symmetric and, from (2.5), R is J-invariant, both $Y_i(t)$ and $\overline{Y}_i(t)$ satisfy the same system of linear differential equations with the same initial values. Hence we get $g(\overline{Y}(1), \overline{Y}(1)) = g(Y(1), Y(1))$ and then j_m is an isometry.

An almost Hermitian manifold (M, g, J) is said to belong to the class $\mathcal{AH}_3$ if and only if R is J-invariant ([V1]). Any Kähler manifold belongs to this class and in particular (2.5) is automatically satisfied.

COROLLARY 2.6. *A Kähler manifold is locally Hermitian symmetric if and only if each* J*-rotation is an isometry.*

REMARK. Note that there are several examples of non-Kählerian symmetric almost Hermitian manifold of the class $\mathcal{AH}_3$. S^6 with a nearly-Kähler structure is perhaps the most well-known one. Flat non-Kählerian manifolds also provide examples. Some of them are constructed in [TV1],[TV2]. For other classes of examples we refer to [V1].

2.3 Harmonic and isometric rotations

We now deal with harmonic rotations and study their relationship with isometric rotations.

First, it is clear that an isometric reflection s_m is a harmonic map, i.e., $\tau(s_m) = 0$. Hence all locally symmetric spaces have harmonic geodesic reflections. Actually, these spaces are the only Riemannian manifolds with this property according to the following characterization proved in [DVV].

PROPOSITION 2.7. *A Riemannian manifold is locally symmetric if and only if all geodesic reflections are harmonic.*

For a regular rotation field S we have ([SV3])

PROPOSITION 2.8. *Let (M, g) be a Riemannian manifold and S a regular rotation field. Then all s_m are harmonic if and only if they are isometric, that is, if and only if (M, g) is a locally s-regular manifold with respect to S.*

The study of the relationship between harmonic and isometric rotations becomes much more difficult when we drop the condition of regularity on the rotation field S. The best result we can state is (cf. [NV2])

PROPOSITION 2.9. *Let (M, g) be a locally symmetric Riemannian manifold and S a rotation field on (M, g). Then any s_m is harmonic if and only if it is an isometry.*

Since, by hypothesis, the manifold is locally symmetric, to prove Proposition 2.9 we have just to show that if s_m is harmonic then the curvature tensor R is S-ivariant at m. This can be done by following a method similar to the one used in [SV3] to prove Proposition 2.8. We will illustrate this method in proving the next result concerning J-rotations on an almost Hermitian manifold ([NV1]).

PROPOSITION 2.10. *All J-rotation on an almost Hermitian manifold are harmonic if and only if they are isometric.*

PROOF. If j_m is isometric it is also harmonic. Conversely, suppose $\tau(j_m) = 0$. In a system of normal coordinates $(x^1, \ldots, x^n)$ this is equivalent to (cf. (1.50))

$$\tau(j_m)^k(p) = g^{ij}(p)\{-\Gamma^l_{ij}(p)J^k_l(m) + \Gamma^k_{ab}(j_m(p))J^a_i(m)J^l_j(m)\} = 0, \tag{2.6}$$

where $p = \exp_m(ru)$, u unit vector in T_mM. Put

$$-\Gamma^l_{ij}(p)J^k_l(m) + \Gamma^k_{ab}(j_m(p))J^a_i(m)J^l_j(m) = \sum_{t=1}^{5} \alpha^k_{tij}(m,u)r^t + O(r^6).$$

Using (2.6) and the expression for g^{ij} (cf. (1.14), (1.15)) we get that j_m is harmonic if and only if

(A) $\sum_i \alpha^k_{1ii}(m,u) = 0;$

(B) $\sum_i \alpha^k_{2ii}(m,u) = 0;$

(C) $\sum_i \alpha^k_{3ii}(m,u) + \frac{1}{3}\sum_{i,j} \alpha^k_{1ij}(m,u)R_{uiuj}(m) = 0;$

(D) $\sum_i \alpha^k_{4ii}(m,u) + \frac{1}{3}\sum_{i,j} \alpha^k_{2ij}(m,u)R_{uiuj}(m)$

$$+\frac{1}{6}\sum_{i,j}\alpha^k_{1ij}(m,u)(\nabla_u R)_{uiuj}(m) = 0;$$

(E) $\sum_i \alpha^k_{5ii}(m,u) + \frac{1}{3}\sum_{i,j} \alpha^k_{3ij}(m,u)R_{uiuj}(m) + \frac{1}{6}\sum_{i,j}\alpha^k_{2ij}(m,u)(\nabla_u R)_{uiuj}(m)$

$$+\frac{1}{60}\sum_{i,j}\alpha^k_{1ij}(m,u)\{3(\nabla^2_{uu}R)_{uiuj} + 4\sum_s R_{uius}R_{ujus}\}(m) = 0.$$

To compute α^k_{tij} for $t = 1, \ldots, 5$ we need to know the Christoffel symbols. A straightforward but long computation shows that

$$\Gamma^k_{ij}(p) = \frac{1}{2}\sum_{l=1}^{n} g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right) = \sum_{l=1}^{5} \beta^l_{ijk}(m,u)r^l + O(r^6),$$

where

$$\beta^1_{ijk}(m,u) = -\frac{1}{3}\{R_{uijk} + R_{ujik}\},$$

$$\beta^2_{ijk}(m,u) = -\frac{1}{12}\{(\nabla_j R)_{uiuk} + (\nabla_i R)_{ujuk} - (\nabla_k R)_{uiuj} - 2(\nabla_u R)_{ujki} - 2(\nabla_u R)_{uikj}\},$$

$$\begin{aligned}\beta^3_{ijk}(m,u) = &-\frac{1}{40}(\nabla^2_{ju}R)_{uiuk} - \frac{1}{40}(\nabla^2_{uj}R)_{uiuk} - \frac{1}{20}(\nabla^2_{uu}R)_{uijk} \\ &- \frac{1}{40}(\nabla^2_{iu}R)_{ujuk} - \frac{1}{40}(\nabla^2_{ui}R)_{ujuk} - \frac{1}{20}(\nabla^2_{uu}R)_{ujik} \\ &+ \frac{1}{40}(\nabla^2_{ku}R)_{uiuj} + \frac{1}{40}(\nabla^2_{uk}R)_{uiuj} - \frac{4}{45}\sum_s R_{uijs}R_{ukus} \\ &- \frac{4}{45}\sum_s R_{ujis}R_{ukus} + \frac{1}{15}\sum_s R_{jkus}R_{uius} + \frac{1}{15}\sum_s R_{ikus}R_{ujus},\end{aligned}$$

$$\begin{aligned}\beta^4_{ijk}(m,u) = &\frac{1}{180}\{-(\nabla^3_{juu}R)_{uiuk} - (\nabla^3_{uju}R)_{uiuk} - (\nabla^3_{uuj}R)_{uiuk} - (\nabla^3_{iuu}R)_{ujuk} \\ &- (\nabla^3_{uiu}R)_{ujuk} - (\nabla^3_{uui}R)_{ujuk} + (\nabla^3_{kuu}R)_{uiuj} + (\nabla^3_{uku}R)_{uiuj} \\ &+ (\nabla^3_{uuk}R)_{uiuj} - 2(\nabla^3_{uuu}R)_{uijk} - 2(\nabla^3_{uuu}R)_{ujik}\} \\ &+ \frac{1}{90}\{-\frac{3}{2}\sum_s(\nabla_j R)_{uius}R_{ukus} - 4\sum_s(\nabla_u R)_{uijs}R_{ukus} \\ &+ \sum_s(\nabla_j R)_{ukus}R_{uius} - 4\sum_s(\nabla_u R)_{ukus}R_{uijs} \\ &+ \sum_s(\nabla_i R)_{ujus}R_{ukus} - \frac{13}{2}\sum_s(\nabla_u R)_{ujis}R_{ukus} \\ &+ \sum_s(\nabla_i R)_{ukus}R_{ujus} - 4\sum_s(\nabla_u R)_{ukus}R_{ujis} \\ &- \sum_s(\nabla_k R)_{uius}R_{ujus} - \sum_s(\nabla_k R)_{ujus}R_{uius} \\ &+ 3\sum_s(\nabla_u R)_{uius}R_{jkus} + 3\sum_s(\nabla_u R)_{jkus}R_{uius} \\ &+ 3\sum_s(\nabla_u R)_{ujus}R_{ikus} + 3\sum_s(\nabla_u R)_{ikus}R_{ujus}\},\end{aligned}$$

$$
\begin{aligned}
\beta^5_{ijk}(m,u) = \frac{1}{2(7!)}\{ & -10(\nabla^4_{juuu}R)_{uiuk} - 10(\nabla^4_{ujuu}R)_{uiuk} - 10(\nabla^4_{uuju}R)_{uiuk} \\
& - 10(\nabla^4_{uuuj}R)_{uiuk} - 20(\nabla^4_{uuuu}R)_{uijk} - 10(\nabla^4_{iuuu}R)_{ujuk} \\
& - 10(\nabla^4_{uiuu}R)_{ujuk} - 10(\nabla^4_{uuiu}R)_{ujuk} - 10(\nabla^4_{uuui}R)_{ujuk} \\
& - 20(\nabla^4_{uuuu}R)_{ujik} + 10(\nabla^4_{kuuu}R)_{uiuj} + 10(\nabla^4_{ukuu}R)_{uiuj} \\
& + 10(\nabla^4_{uuku}R)_{uiuj} + 10(\nabla^4_{uuuk}R)_{uiuj} + 34\sum_s(\nabla^2_{ju}R)_{uius}R_{ukus} \\
& + 34\sum_s(\nabla^2_{uj}R)_{uius}R_{ukus} + 34\sum_s(\nabla^2_{uu}R)_{uijs}R_{ukus} \\
& + 132\sum_s(\nabla^2_{uu}R)_{uius}R_{jkus} + 34\sum_s(\nabla^2_{iu}R)_{ujus}R_{ukus} \\
& + 34\sum_s(\nabla^2_{ui}R)_{ujus}R_{ukus} + 34\sum_s(\nabla^2_{uu}R)_{ujis}R_{ukus} \\
& + 132\sum_s(\nabla^2_{uu}R)_{ujus}R_{ikus} - 34\sum_s(\nabla^2_{ku}R)_{uius}R_{ujus} \\
& - 34\sum_s(\nabla^2_{uk}R)_{uius}R_{ujus} - 132\sum_s(\nabla^2_{uu}R)_{kius}R_{ujus} \\
& + 34\sum_s(\nabla^2_{ju}R)_{ukus}R_{uius} + 34\sum_s(\nabla^2_{uj}R)_{ukus}R_{uius} \\
& + 132\sum_s(\nabla^2_{uu}R)_{jkus}R_{uius} + 34\sum_s(\nabla^2_{uu}R)_{ukus}R_{uijs} \\
& + 34\sum_s(\nabla^2_{iu}R)_{ukus}R_{ujus} + 34\sum_s(\nabla^2_{ui}R)_{ukus}R_{ujus} \\
& + 34\sum_s(\nabla^2_{uu}R)_{ukus}R_{ujis} - 34\sum_s(\nabla^2_{ku}R)_{ujus}R_{uius} \\
& - 34\sum_s(\nabla^2_{uk}R)_{ujus}R_{uius} + 55\sum_s(\nabla_j R)_{uius}(\nabla_u R)_{ukus} \\
& + 55\sum_s(\nabla_u R)_{uijs}(\nabla_u R)_{ukus} + 55\sum_s(\nabla_u R)_{uius}(\nabla_j R)_{ukus} \\
& + 55\sum_s(\nabla_u R)_{uius}(\nabla_u R)_{jkus} + 110\sum_s(\nabla_u R)_{uius}(\nabla_u R)_{ukjs}
\end{aligned}
$$

$$
\begin{aligned}
&+55\sum_s(\nabla_i R)_{ujus}(\nabla_u R)_{ukus}+55\sum_s(\nabla_u R)_{ujis}(\nabla_u R)_{ukus}\\
&+55\sum_s(\nabla_u R)_{ujus}(\nabla_i R)_{ukus}+110\sum_s(\nabla_u R)_{ujus}(\nabla_u R)_{ikus}\\
&-55\sum_s(\nabla_k R)_{uius}(\nabla_u R)_{ujus}-55\sum_s(\nabla_u R)_{kius}(\nabla_u R)_{ujus}\\
&-55\sum_s(\nabla_u R)_{uius}(\nabla_k R)_{ujus}\\
&-16\sum_{a,b}R_{uija}R_{ukub}R_{uaub}-16\sum_{a,b}R_{uiua}R_{jkub}R_{uaub}\\
&-16\sum_{a,b}R_{uiua}R_{ukjb}R_{uaub}-16\sum_{a,b}R_{uiua}R_{ukub}R_{jaub}\\
&-16\sum_{a,b}R_{uiua}R_{ukub}R_{uajb}-16\sum_{a,b}R_{ujia}R_{ukub}R_{uaub}\\
&-16\sum_{a,b}R_{ujua}R_{ikub}R_{uaub}-16\sum_{a,b}R_{ujua}R_{ukib}R_{uaub}\\
&-16\sum_{a,b}R_{ujua}R_{ukub}R_{iaub}-16\sum_{a,b}R_{ujua}R_{ukub}R_{uaib}\\
&+16\sum_{a,b}R_{kiua}R_{ujub}R_{uaub}+16\sum_{a,b}R_{uika}R_{ujub}R_{uaub}\\
&+16\sum_{a,b}R_{uiua}R_{kjub}R_{uaub}+16\sum_{a,b}R_{ukua}R_{ujib}R_{uaub}\\
&+16\sum_{a,b}R_{ukua}R_{ujub}R_{iaub}+16\sum_{a,b}R_{ukua}R_{ujub}R_{uaib}\}\\
&+\frac{1}{360}\sum_l R_{ukul}\{-3(\nabla^2_{ju}R)_{uiul}-3(\nabla^2_{uj}R)_{uiul}-6(\nabla^2_{uu}R)_{uijl}\\
&-3(\nabla^2_{iu}R)_{ujul}-3(\nabla^2_{ui}R)_{ujul}-6(\nabla^2_{uu}R)_{ujil}+3(\nabla^2_{lu}R)_{uiuj}\\
&+3(\nabla^2_{ul}R)_{uiuj}+\frac{8}{3}\sum_s R_{uijs}R_{ulus}+\frac{8}{3}\sum_s R_{ujis}R_{ulus}\\
&+8\sum_s R_{uius}R_{jlus}+8\sum_s R_{ujus}R_{ilus}\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{72}\sum_l(\nabla_u R)_{ukul}\{(\nabla_j R)_{uiul}+(\nabla_i R)_{ujul}\\
&-(\nabla_l R)_{uiuj}-2(\nabla_u R)_{liuj}-2(\nabla_u R)_{uilj}\}\\
&-\frac{1}{180}\sum_l\{R_{uijl}+R_{ujil}\}\{3(\nabla^2_{uu}R)_{ukul}\\
&+4\sum_s R_{ukus}R_{ulus}\}.
\end{aligned}
$$

According to the explicit expressions for the Christoffel symbols we obtain the following consequences from the conditions (A),...,(E).

From (A) we obtain for the Ricci tensor ρ at m

$$\rho_{uk}-\rho_{JuJk}=0,$$

where, as we have already mentioned, k stands for $\frac{\partial}{\partial x^k}(m)$, $k=1,\dots,n$. Since $\frac{\partial}{\partial x^k}(m)$ and u are arbitrary this implies that the Ricci tensor ρ is *J-invariant.*

From (B) we get that $\nabla\rho$ is J-invariant and hence $\nabla\rho=0$. Using the preceding facts about ρ and $\nabla\rho$, the condition (C) yields

$$5\sum_{i,j}R_{uiuj}(R_{uiuj}-R_{JuJiJuJj})-3\sum_{i,j}(R^2_{uiuj}-R^2_{JuJiJuJj})=0. \tag{2.7}$$

To derive useful results from (2.7) we identify T_mM with the n-dimensional Euclidean space $\mathbb{E}^n$ via the orthonormal basis $\{\frac{\partial}{\partial x^k}(m)\}$, $k=1,\dots,n$ and consider the left hand side of (2.7) as a function on the unit sphere $S^{n-1}(1)$. Integrating (2.7) on S^{n-1} using the formulae given in Lemma 4.10 (see also [CV1],[Gr7]), we obtain

$$\sum_{a,b,i,j=1}^n R_{aibj}(R_{aibj}-R_{JaJiJbJj})=0$$

and hence also

$$\sum_{a,b,i,j=1}^n R_{JaJiJbJj}(R_{JaJiJbJj}-R_{aibj})=0.$$

The sum of these two conditions yields

$$\sum_{a,b,i,j=1}^{n} (R_{aibj} - R_{JaJiJbJj})^2 = 0,$$

from which follows that $R_{aibj} = R_{JaJiJbJj}$ and then that R is *J-invariant* or, with the terminology introduced above, M is an $\mathcal{AH}_3$-manifold.

Taking into account the results from conditions (A), (B) and (C), the condition (D) yields

$$\sum_{ij} R_{uiuj}((\nabla_u R)_{uiuj} - (\nabla_{Ju} R)_{JuJiJuJj}) = 0.$$

We will not use this condition to prove our result. After a long computation, in which we use the fact that R is J-invariant, we obtain from the condition (E)

$$\begin{aligned} &4\sum_{ij} R_{uiuj}((\nabla^2_{uu} R)_{uiuj} - (\nabla^2_{JuJu} R)_{JuJiJuJj}) \\ &+ 22\sum_{ij}[((\nabla_u R)_{uiuj})^2 - ((\nabla_{Ju} R)_{JuJiJuJj})^2] \\ &- 70\sum_{ij}(\nabla_u R)_{uiuj}((\nabla_u R)_{uiuj} - (\nabla_{Ju} R)_{JuJiJuJj}) = 0. \end{aligned}$$

Integrating again over the unit sphere, we obtain

$$\sum_{a,b,c,i,j} (\nabla_a R)_{bicj}((\nabla_a R)_{bicj} - (\nabla_{Ja} R)_{JbJiJcJj}) = 0$$

and hence

$$(\nabla_a R)_{bicj} = (\nabla_{Ja} R)_{JbJiJcJj},$$

which means that ∇R is J-invariant and then that (M, g, J) is locally symmetric. This and the fact that M is an $\mathcal{AH}_3$-manifold imply, according to Proposition 2.5, that j_m is an isometry.

COROLLARY 2.11. *An almost Hermitian manifold (M, g, J) is a locally symmetric manifold of class $\mathcal{AH}_3$ if and only if each J-rotation is harmonic.*

REMARK. Instead of integrating over $S^{n-1}(1)$ we may proceed as follows. As above, we identify T_mM with the n-dimensional Euclidean space $\mathbb{E}^n$. The left hand side of (2.7) may be regarded, after multiplication with r^4, as a homogeneous polynomial of order 4. Next, let Δ denote the Laplacian of $\mathbb{E}^n$. Then, taking twice the Laplacian of (2.7) we get the same result as with the integration process. A third method for reducing expressions like (2.7) is the linearization and summation procedure illustrated in Lemma 2.28.

2.4 Symplectic and holomorphic rotations

In this section we deal with holomorphic and symplectic rotations in the context of Hermitian geometry, namely with rotations that preserve, respectively, the almost complex structure J and the Kähler form Ω of an almost Hermitian manifold (M, g, J) (cf. Section 1.4).

It is well-known that all geodesic reflections on a Hermitian symmetric space preserve both g and J (see [He]) and hence they are symplectic. Moreover, the following holds (cf. [SV1]).

PROPOSITION 2.12. *An almost Hermitian manifold (M, g, J) is locally isometric to a Hermitian symmetric space if and only if the geodesic reflections (geodesic symmetries) are either symplectic or holomorphic.*

The proof is obtained using power series expansions for Ω and J, respectively. First one gets that the manifold is Kähler and once in the Kähler situation the following result is crucial to obtain the proof (cf. [Gr3],[SV1]).

LEMMA 2.13. *A Kähler manifold (M, g, J) is locally symmetric if and only if $(\nabla_X R)_{XJXXJX} = 0$ for all tangent vector fields X.*

We also refer to [V3] for a discussion of this criterion.

For J-rotations we are able to prove [NV1]

PROPOSITION 2.14. *An almost Hermitian manifold (M, g, J) is locally isometric to a Hermitian symmetric space if and only if each J-rotation is either*

(1) *holomorphic, or*
(2) *symplectic.*

PROOF. ($\Rightarrow$) From Corollary 2.6 each j_m is an isometry and then, as soon as j_m is holomorphic (respectively, symplectic) we get easily that j_m is also symplectic (respectively, holomorphic). Therefore, we are left to prove that in our hypotheses j_m is either holomorphic or symplectic.

Consider $\widetilde{J} = dj_m^{-1} \circ J \circ dj_m$. It is a tensor field of type $(1,1)$ which clearly satisfies $\widetilde{J}^2 = -I$. Moreover, it is a parallel tensor field due to the fact that j_m is an affine transformation with respect to the Levi Civita connection and that M is Kähler. Since $\widetilde{J}$ and J have the same value at m we have $\widetilde{J} = J$, that is j_m is holomorphic.

($\Leftarrow$) Let j_m be holomorphic (respectively, symplectic) for each $m \in M$. Then each geodesic reflection j_m^2 is also holomorphic (respectively, symplectic). According to Proposition 2.12 we conclude that (M, g, J) is a locally symmetric Kähler manifold.

Next, we consider S-rotations with S regular. In this case the known results can be summarized as follows (cf. [LV2]).

PROPOSITION 2.15. *Let (M, g, J) be an almost Hermitian manifold and S a regular rotation field on M. Then, for each $m \in M$, the following are equivalent:*

(1) *s_m is isometric and $J, \nabla J$ are S-invariant;*
(2) *s_m is isometric and $\Omega, \nabla\Omega$ are S-invariant;*
(3) *s_m is holomorphic;*
(4) *s_m is symplectic.*

COROLLARY 2.16. *Let (M, g, J, s) be an almost Hermitian manifold with a local s-regular structure. If J and ∇J are S-invariant, then each (local) rotation s_m, $m \in M$, is holomorphic and symplectic.*

Conversely, suppose (M, g) admits an almost Hermitian structure J and a rotation field S for which the corresponding rotations are holomorphic (respectively, symplectic). Then J and ∇J (respectively, Ω and $\nabla\Omega$) are S-invariant. Moreover, if S is regular then each s_m is a local isometry and hence (M, g) is a locally s-regular manifold with respect to S.

REMARK. This generalizes the above characterization of locally Hermitian symmetric spaces (cf. Proposition 2.12) to the class of s-regular almost Hermitian manifolds.

Dropping the hypothesis of regularity on S we can prove

PROPOSITION 2.17. *Let (M, g, J) be a locally Hermitian symmetric space and S a rotation field on M. The following statements are equivalent for each $m \in M$:*

(1) *s_m is isometric and $SJ = JS$;*
(2) *s_m is holomorphic;*
(3) *s_m is symplectic.*

PROOF. (3) $\Rightarrow$ (1) For a system of normal geodesic coordinates $(x^1, \dots, x^n)$ centered at m we have, as indicated in Chapter 1, the following power series expansion for $\Omega_{ij} = \Omega(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, $i, j = 1, \dots, n$:

$$\Omega_{ij}(\exp_m(ru)) = \alpha_{0ij} + r\alpha_{1ij} + \frac{r^2}{2}\alpha_{2ij} + \frac{r^3}{6}\alpha_{3ij} + O(r^4),$$

where

$$(2.8) \qquad \begin{cases} \alpha_{0ij} = \Omega_{ij}(m), \\ \alpha_{1ij} = 0, \\ \alpha_{2ij} = -\frac{1}{3}\{\sum_t R_{uiut}\Omega_{tj} + \sum_t R_{ujut}\Omega_{it}\}(m). \end{cases}$$

Now, if s_m is symplectic we have

$$(2.9) \qquad \Omega_{ij}(\exp_m(ru)) = S_i^a(m)S_j^b(m)\Omega_{ab}(\exp_m(rSu)).$$

By equating the α_{2ij} in (2.9) we get

$$(2.10) \qquad R_{XYXJZ} - R_{XJYXZ} = R_{SXSYSXSJZ} - R_{SXSJYSXSZ}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Observe that the symplectic property yields at once that

$$(2.11) \qquad SJ = JS,$$

a condition that has already been used implicitly to write (2.10). Next, define

$$(2.12) \qquad T_{XYZW} = R_{XYZW} - R_{SXSYSZSW}.$$

T verifies the algebraic conditions of an algebraic curvature tensor and moreover the Kähler identity for R and (2.11) imply that T satisfies the Kähler identity too,

$$T_{XYJZJW} = T_{XYZW}.$$

Furthermore, (2.10) implies $T_{XJXXJX} = 0$ and hence $T = 0$ (see [KN],vol.II,p.166). Thus R is S-invariant. The S-invariance of R and a Jacobi field argument as presented in the proof of Proposition 2.5 yield that s_m is an isometry.

(1) $\Rightarrow$ (3) If $JS = SJ$, then Ω is S-invariant and at any point $p \in M$, Ω is invariant under the action of $ds_{p|p} = S_p$. Since Ω is also parallel, it is uniquely determined by its value at one point, and since s_p is isometric and then preserves parallelism locally, it follows that Ω is invariant under the action of s_p and hence s_p is symplectic.

Analogously we can prove that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) As usual, let $(x^1, \ldots, x^n)$ be a system of normal coordinates centered at m. First we note that, if J^i_j denote the components of the almost complex structure J with respect to $(x^1, \ldots, x^n)$, then

$$-J^i_j = \sum_{k=1}^{n} \Omega_{jk} g^{ki}, \qquad i, j = 1, \ldots, n.$$

From this, the previous expansion for $\Omega_{ij}(\exp_m(ru))$ (cf. (2.8)) and the expansion for g^{ij} (cf. (1.14), (1.15)) we have

$$J^i_j(\exp_m(ru)) = \gamma^i_{0j} + r\gamma^i_{1j} + \frac{r^2}{2}\gamma^i_{2j} + O(r^3),$$

where

$$\begin{cases} \gamma^i_{0j} = J^i_j(m), \\ \gamma^i_{1j} = 0, \\ \gamma^i_{2j} = -\frac{1}{3}\{\sum_t R_{ujut}J^t_i + \sum_t R_{uiut}J^t_j\}(m). \end{cases}$$

We now compare this power series with that for $J^i_j(\exp_m(rS_m u))$ and following the same procedure as in the proof of (3) $\Rightarrow$ (1) we obtain the required result.

2.5 Rotations and the geometry of geodesic spheres

It is clear that rotations in general induce global diffeomorphisms on the (small) geodesic spheres. We are interested here in the study of the invariance under these diffeomorphisms of some operators and naturally defined functions related to the intrinsic and extrinsic geometry of the geodesic spheres. This leads again to new characterizations of special classes of manifolds, e.g., locally symmetric spaces, locally s-regular manifolds, Hermitian symmetric spaces, and to new criteria for rotations to have certain properties.

We will present here results concerning the shape operator (extrinsic geometry) and the Ricci operator (intrinsic geometry) of geodesic spheres. For $p \in G_m(r)$ we denote by $T_m(p)$ the shape operator at p of the geodesic sphere with center m and by $\widetilde{Q}_m(p)$ the Ricci operator of the geodesic sphere $G_m(r)$ at p. We recall that the Ricci operator is the symmetric (1,1)-tensor associated to the Ricci tensor $\widetilde{\rho}_{XY} = \operatorname{tr}(Z \mapsto \widetilde{R}_{XZ}Y)$ of the geodesic sphere $G_m(r)$. $\widetilde{R}$ denotes the curvature tensor of $G_m(r)$ and X, Y, Z are tangent vectors to $G_m(r)$ at p (cf. Section 1.3). Using the Gauss equation for the hypersurface $G_m(r)$ (cf. (1.26)) we obtain the following power series expansion for the Ricci operator $\widetilde{Q}_m(p)$ at p (see for example [CV1]):

$$\begin{aligned}\widetilde{Q}_m(p) =& \frac{n-2}{r^2}I + \{Q - \rho(u,.)u - \frac{1}{3}\rho(u,u)I - \frac{n}{3}R_u\}(m)\\ &+ r\{\nabla_u Q - (\nabla_u\rho)(u,.)u - \frac{1}{4}(\nabla_u\rho)(u,u)I - \frac{n+1}{4}R'_u\}(m) + O(r^2),\end{aligned}$$

where Q denotes the Ricci operator of the ambient space (M, g).

In [VW2] was given a criterion for isometric reflections, or equivalently, for a manifold to be locally symmetric, by using the shape operator of small geodesic spheres.

PROPOSITION 2.18. *Let (M, g) be a Riemannian manifold. Then (M, g) is locally symmetric if and only if*

$$T_{\exp_m(ru)}(m) = T_{\exp_m(-ru)}(m), \tag{2.13}$$

for all u and small r.

This result was then generalized to the broader class of locally s-regular manifolds in [LV1]:

PROPOSITION 2.19. *Let S be a regular rotation field on (M, g). For each $m \in M$ choose a geodesic ball $B_m(r)$ which is a normal neighborhood of each of its points. Suppose that for each unit vector $u \in T_mM$ there exists a positive r_o such that if $0 < \sigma < r_o$ and $p = \exp_m(\sigma u)$ then*

$$T_{s_m(p)} \circ S_m = S_m \circ T_p, \tag{2.14}$$

where each side of this equation is considered as a linear map of $\{u\}^\perp$ and s_m is the local rotation determined by S. Then (M, g) is a locally s-regular manifold. Conversely, any locally s-regular manifold has the above property.

REMARK. A criterion for S-rotations when S is not necessarily regular may be obtained analogously within the class of locally symmetric spaces.

As concerns J-rotations we have the following characterization of locally Hermitian symmetric spaces [NV1].

PROPOSITION 2.20. *Let (M, g) be a Kähler manifold. Then it is locally Hermitian symmetric if and only if*

$$T_{j_m(p)}(m) \circ J_m = J_m \circ T_p(m) \tag{2.15}$$

for all $p = \exp_m(ru)$ and all sufficiently small radii r.

PROOF. For a locally Hermitian symmetric space, (2.15) follows from Corollary 2.6.

Conversely, suppose (2.15) holds. Then we also have

$$T_{j_m^2(p)}(m) \circ J_m = J_m \circ T_{j_m(p)}(p) \tag{2.16}$$

and consequently (2.15), (2.16) yield

$$J_m \circ T_p(m) = J_m \circ T_{j_m^2(p)}(m)$$

and hence $T_p(m) = T_{j_m^2(p)}(m)$. But this is just (2.13) which is a characteristic property of locally symmetric spaces.

REMARK. Instead of (2.15) it is sufficient to suppose that (2.15) holds when applied to the vector $J_m u$ which is tangent to the geodesic sphere $G_p(r)$ at m. So, the necessary and sufficient condition becomes

$$T_p(m)J_m u = J_m T_{j_m(p)}(m)u.$$

Another characterization is the following [NV1].

PROPOSITION 2.21. *Let (M, g, J) be a Kähler manifold. Then it is locally Hermitian symmetric if and only if the shape operator T_m is locally j_m-invariant, i.e.,*

$$T_m \circ dj_m = dj_m \circ T_m \tag{2.17}$$

for all $m \in M$ and all sufficiently small radii r.

PROOF. First, let (M, g, J) be locally Hermitian symmetric. Then j_m is an isometry (cf. Corollary 2.6) and hence (2.17) holds.

Conversely, suppose (2.17) holds. Then we have

$$dj_{m|p} \circ T_m(p) = T_m(j_m(p)) \circ dj_{m|p} \tag{2.18}$$

and also

$$dj_{m|j_m(p)} \circ T_m(j_m(p)) = T_m(j_m^2(p)) \circ dj_{m|j_m(p)}. \tag{2.19}$$

From (2.18) and (2.19) we get $dj_m^2 \circ T_m = T_m \circ dj_m^2$. This means that T_m is preserved by the geodesic reflection j_m^2. But this is characteristic for locally symmetric spaces as can be deduced from the following characterization of locally s-regular symmetric manifolds (cf. [LV1]).

PROPOSITION 2.22. *Let S be a regular rotation field on (M, g) and suppose for each $m \in M$ the shape operator is locally s_m-invariant, i.e., $T_m \circ ds_m = ds_m \circ T_m$. Then (M, g) is a locally s-regular manifold associated to S.*

Conversely, any locally s-regular manifold has the above property.

There is a great similarity between the properties of the shape operators and the Ricci operators of small geodesic spheres. We present here the analogues of the preceding results for the Ricci operator.

The following was proved in [DV1].

PROPOSITION 2.23. *Let S be a regular rotation field on (M,g), where* $\dim M >$ 3. *Then (M,g) is a locally s-regular manifold with associated rotation field S if and only if*

$$\widetilde{Q}_{s_m(p)}(m) \circ S_m = S_m \circ \widetilde{Q}_p(m) \tag{2.20}$$

for all m and all $p = \exp_m(ru)$ with r sufficiently small.

When $S = -I$ this provides another characterization of locally symmetric spaces.

COROLLARY 2.24. *A Riemannian manifold M of dimension greater than* 3 *is locally symmetric if and only if*

$$\widetilde{Q}_{\exp_m(ru)}(m) = \widetilde{Q}_{\exp_m(-ru)}(m)$$

for all $m \in M$, all unit vector $u \in T_mM$ and all sufficiently small r.

For J-rotations we get ([NV1])

PROPOSITION 2.25. *A Kähler manifold (M,g,J) of dimension greater than or equal to* 4 *is locally Hermitian symmetric if and only if*

$$\widetilde{Q}_m \circ dj_m = dj_m \circ \widetilde{Q}_m$$

or, equivalently,

$$\widetilde{Q}_{j_m(p)}(m) \circ J_m = J_m \circ \widetilde{Q}_p(m)$$

for all $m \in M$, all $p = \exp_m(ru)$ and all sufficiently small r.

Next, we note that the local diffeomorphism j_m induces in a natural way a global diffeomorphism on sufficiently small geodesic spheres with center m. We still denote it by j_m. We are now interested in the invariance of the function

$$p \mapsto \kappa_m(p) = g(T_m(p)J_p\gamma'(r), J_p\gamma'(r)).$$

$\kappa_m(p)$ is the curvature at p of the geodesic of $G_m(r)$ tangent to $J_p\gamma'(r)$. Note that $\gamma'(r) = \frac{\partial}{\partial r}\big|_p$. Then we get (cf. [NV1])

PROPOSITION 2.26. *A Kähler manifold (M, g, J) is locally Hermitian symmetric if and only if for each $m \in M$*

$$j_m^* \kappa_m = \kappa_m.$$

The proof is similar to the preceding ones in this section. Finally, we consider the function

$$p \mapsto \overline{\kappa}_m(p) = \kappa_p(m) = g(T_p(m) J_m u, J_m u).$$

In a similar way as for Proposition 2.26 we get ([NV1])

PROPOSITION 2.27. *A Kähler manifold (M, g, J) is locally Hermitian symmetric if and only if for each $m \in M$*

$$j_m^* \overline{\kappa}_m = \overline{\kappa}_m.$$

REMARK. a) All the proofs of the results concerning J-rotations rely on Proposition 2.5 and on the known results about geodesic reflections (cf. [NV1]). One may also give direct proofs by using power series expansions. We will give a short indication of this method. First, we need the following *linearization result.*

LEMMA 2.28. *Let (M, g, J) be a Kähler manifold with $\dim M = n$ and suppose there exist real numbers a and b, $b \neq 0$, such that for all tangent vectors X we have*

$$a \nabla_X \rho_{XX} g(X, X) + b \nabla_X R_{XJXXJX} = 0, \tag{2.21}$$

where $\{(n+4)a + 6b\}\{(n+6)a + 12b\} \neq 0$. Then

$$\nabla_X R_{XJXXJX} = 0. \tag{2.22}$$

PROOF. We indicate the two main steps in the proof. First, we replace X by $\alpha X + \beta Y$ in (2.21) and write down the coefficient of $\alpha^3 \beta^2$. Using the Kähler and the first Bianchi identities, we obtain

$$\begin{aligned} a\{&(\nabla_X \rho)_{XX} g(Y, Y) + 4(\nabla_X \rho)_{XY} g(X, Y) + (\nabla_X \rho)_{YY} g(X, X) \\ &+ 2(\nabla_Y \rho)_{XY} g(X, X) + 2(\nabla_Y \rho)_{XX} g(X, Y)\} \\ &+ b\{10(\nabla_X R)_{XJYXJY} + 2(\nabla_X R)_{XYXY} + 4(\nabla_{JX} R)_{JXJYXJY}\} = 0. \end{aligned}$$

Now, put $Y = e_i, i = 1, \ldots, n$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis and sum over i. This yields

$$\{(n+6)a + 12b\}\nabla_X \rho_{XX} + 2a\nabla_X \tau g(X, X) = 0, \tag{2.23}$$

where τ denotes the scalar curvature. Then, by a similar linearization and summation procedure applied to (2.23) we get

$$\{(n+4)a + 6b\}\nabla_X \tau = 0, \tag{2.24}$$

So, (2.22) follows from (2.21), (2.23) and (2.24).

Finally, one proceeds as follows. Using normal coordinates and Jacobi vector fields one may write down power series expansions for the operators and functions needed (cf. Chapter 1). Then the conditions in the statements lead quickly to a relation of the form (2.21) and hence the results follow from Lemma 2.13 and Lemma 2.28. We note that for Proposition 2.14 the power series expansions lead first to $\nabla J = 0$.

b) For the study of the invariance under a field of rotations of other functions and geometric objects related to the extrinsic and intrinsic geometry of geodesic spheres we refer to [DV1],[LV1],[NV1].

Chapter 3

Rotations around curves

In this chapter we present the study of local rotations around a smooth embedded curve $\sigma : [a, b] \to M$ in a Riemannian manifold (M, g). These transformations are local diffeomorphisms that generalize in a natural way rotations around a straight line in ordinary Euclidean space. They are determined by means of a field of endomorphisms along the curve that for each $m \in \sigma$ fix the tangent vectors to σ and when restricted to the fibres of the normal bundle of σ behave as linear isometries.

Reflections with respect to a curve provide a class of examples of such rotations. We refer to [V4] for further details about this study. When σ is a constant curve, we obtain the rotations around a point. We study problems similar to those for rotations around a point. The main purpose is to study *harmonic* rotations and to compare them with *isometric* rotations. In Section 3.1 we define the notion of rotation and derive in the analytic case a set of necessary and sufficient conditions for a rotation to be isometric. This will be used later in the chapter when we will consider harmonic rotations in relationship with isometric rotations and we need a criterion to recognize isometric rotations. In particular, we show that for free rotations these two concepts coincide for locally symmetric Einstein spaces. It is not known to us whether this result can be extended to general Riemannian manifolds.

3.1 Rotations and isometries in tubular neighborhoods

Let f be an isometry of (M, g) whose (totally geodesic) fixed point set $F(f)$ has positive dimension and let σ be a curve as above contained in $F(f)$. Then we have

LEMMA 3.1. *On a sufficiently small tubular neighborhood U_σ of σ the isometry f can be expressed in the form*

$$f = \exp_\sigma \circ df_{|\sigma} \circ \exp_\sigma^{-1}. \tag{3.1}$$

PROOF. For each point $p \in U_\sigma$ there exists a unique geodesic $\gamma : [0,1] \to M$ of minimal length such that $p = \gamma(1)$ and $\sigma(t) = \gamma(0)$ for some $t \in [a,b]$. Furthermore, $\dot\gamma(0) = \exp_{\sigma(t)}^{-1}(p)$. The curve $f \circ \gamma$ is also a geodesic emanating from the same point $\sigma(t)$ and with initial velocity $df_{|\sigma(t)}(\dot\gamma(0))$. Hence

$$f(p) = f(\gamma(1)) = \exp_{\gamma(t)}(df_{|\sigma(t)}\dot\gamma(0)).$$

REMARK. There are several examples of Riemannian manifolds endowed with isometries as described above. For example, let (M,g) be a homogeneous Riemannian manifold and let K be the isotropy group at some point of M. Since the linear isotropy representation of K in T_pM is faithful the isotropy group at p can be identified with a subgroup of $O(T_pM)$, the linear isotropy group at p. Now, if we suppose that dim$M$ is odd, then any orientation-preserving element $df_{|p}$ of the linear isotropy group admits the eigenvalue 1. Let v be a unit tangent vector corresponding to this eigenvalue and consider the geodesic through p given by $\exp_p(tv)$. Then $f(\exp_p(tv))$ is also a geodesic with the same initial conditions as those of $\exp_p(tv)$ and hence $f(\exp_p(tv)) = \exp_p(tv)$.

Motivated by these considerations, we turn to the definition of a rotation.

DEFINITIONS. Let $S(t)$ be a field of linear endomorphisms

$$S(t) : T_{\sigma(t)}M \to T_{\sigma(t)}M$$

along the curve σ such that $S(t)$ restricted to $T_{\sigma(t)}\sigma$ is the identity map and on each fibre $\nu(\sigma)_{\sigma(t)}$ of the normal bundle $\nu(\sigma)$, $S(t)$ is a linear isometry, that is

$$S(t)\dot\sigma(t) = \dot\sigma(t), \qquad g(S(t)X, S(t)Y) = g(X,Y)$$

for all $X, Y \in \nu(\sigma)_{\sigma(t)}$. Then $S(t)$ is said to be a *rotation field along* σ. In what follows we shall use the same notation $S(t)$ to denote the operator on $T_{\sigma(t)}M$ as well as its restriction to the fibre of $\nu(\sigma)$ at $\sigma(t)$.

Now, let U_σ be a tubular neighborhood of σ with sufficiently small radius. Then, the local diffeomorphism s_σ defined by

$$s_\sigma = \exp_\sigma \circ S \circ \exp_\sigma^{-1}$$

is called a *(local) S-rotation around σ*. Moreover, if $S - I$ is non-singular in the normal bundle, we say that s_σ is a *free S-rotation* .

For $S = -I$, s_σ defines the *reflection with respect to σ* (cf. [VW2],[V4]). Note that we have

$$s_\sigma : U_\sigma \to U_\sigma : \exp_\sigma(\sigma(t), v) \mapsto \exp_\sigma(\sigma(t), S(t)v).$$

Furthermore, σ is contained in the fixed point set of s_σ.

The analytic expression of s_σ follows easily by using Fermi coordinates (cf. Chapter 1):

$$x^1 \circ s_\sigma = x^1, \qquad x^i \circ s_\sigma = S^i_j(t) x^j, \tag{3.2}$$

where $S^i_j(t)$ are the components of $S(t)$ at $\sigma(t)$ with respect to the basis $\{E_2, \ldots, E_n\}$ defined in Section 1.3. Moreover, we have

$$ds_{\sigma|\sigma(t)} = S(t)$$

for all $t \in [a, b]$.

From the expression (3.2) it is clear that the study of S-rotations is different and somewhat more complicated than that of rotations around a point due to the special role played by the x^1-coordinate.

REMARK. Note that S is parallel along σ if and only if S is parallel with respect to $\nabla^\perp$ and $S\ddot{\sigma} = \ddot{\sigma}$. In this case it follows that each higher order covariant derivative of σ is also an eigenvector of S with eigenvalue $+1$, that is,

$$S\sigma^{(k)} = \sigma^{(k)}, \qquad k \in \mathbb{N}_o.$$

So, once a parallel rotation field S is given, we have restrictions on σ. For example, if S defines a reflection, i.e., $S = -I$ in $\nu(\sigma)$, then $\ddot{\sigma} = 0$ and hence σ is a geodesic. The same holds when S is a free rotation field.

Note that Lemma 3.1 yields that each isometry f is a rotation around any curve σ contained in its fixed point set and its rotation field is $df_{|\sigma}$. As may be checked directly, this rotation field is parallel. We stress the fact that the isometric rotations around σ are exactly the isometries which have a (totally geodesic) fixed point set of positive dimension containing σ and this is the only relation between the curve and the isometry.

Now, we will look for conditions under which a rotation field S along σ defines an *isometric rotation* . This criterion will be used later in this chapter.

PROPOSITION 3.2. *Let $\sigma : [a,b] \to M$ be an embedded curve in a Riemannian manifold (M,g) and suppose that the S-rotation s_σ is an isometry. Then*

$$(1)\ S \text{ is parallel along } \sigma; \tag{3.3}$$

$$(2)\ (\nabla^k_{u\ldots u}R)_{uxuy} = (\nabla^k_{Su\ldots Su}R)_{SuSxSuSy}, \tag{3.4}$$

for all $u \in \nu(\sigma)_{\sigma(t)}$, all $x, y \in T_{\sigma(t)}M$, all $t \in [a,b]$ and all $k \in \mathbb{N}$.

Conversely, if (M,g) is analytic and S is a rotation field along σ such that (1) *and* (2) *hold, then the corresponding S-rotation is an isometry.*

PROOF. First, let s_σ be an isometry. Then $ds_{\sigma|\sigma}$ is parallel along σ and since $ds_{\sigma|\sigma} = S$, S is parallel. Then (3.4) follows since any isometry preserves the curvature tensor and its covariant derivatives.

To prove the converse one may use the power series expansions for the components of the metric tensor with respect to Fermi coordinates (cf. (1.42), (1.43), (1.44)). Then it is not difficult to see that the coefficients in the expansions only depend on the subset

$$\{(\nabla^k_{u\ldots u}R)_{u\cdot u}, u \in \nu(\sigma)_{\sigma(t)}, k \in \mathbb{N}\}$$

of the set of all covariant derivatives of the curvature tensor R and on the (mean) curvature vector $\ddot{\sigma}$ of σ. Then the conditions (3.4) and $S\ddot{\sigma} = \ddot{\sigma}$, which follows at once from $S\dot{\sigma} = \dot{\sigma}$ since S is parallel, yield that s_σ is an isometry. This finishes the proof.

For the special case of reflections with respect to σ and for analytic data Proposition 3.2 reads as follows ([CV2]):

COROLLARY 3.3. *Let (M, g) be a Riemannian manifold and $\sigma : [a, b] \to M$ an embedded curve in M. Then the reflection s_σ is an isometry if and only if*

$$\begin{aligned} &(1) && \sigma \text{ is a geodesic;} \\ &(2) && R_u^{(2k)} v \text{ is normal to } \sigma, \\ &&& R_u^{(2k+1)} v \text{ is tangent to } \sigma, \\ &&& R_u^{(2k+1)} \dot{\sigma} \text{ is normal to } \sigma \end{aligned}$$

for all vectors u, v normal to σ and all $k \in \mathbb{N}$, where $R_u^{(l)} = (\nabla^l_{u \ldots u} R)_{u \cdot} u$.

The criterion given in Proposition 3.2 becomes considerably simpler for locally symmetric spaces.

COROLLARY 3.4. *Let (M, g) be a locally symmetric Riemannian manifold and σ a curve as above. Then the S-rotation s_σ is an isometry if and only if*

$$S \text{ is parallel along } \sigma; \tag{3.5}$$

$$R_{uxuy} = R_{SuSxSuSy} \tag{3.6}$$

for all $u \in \nu(\sigma)_{\sigma(t)}$, $x, y \in T_{\sigma(t)}$ and all $t \in [a, b]$.

Moreover, for real, complex and quaternionic space forms, one knows the explicit form of the curvature tensor (see for example [Be2],[V3]). We consider now these spaces case by case.

I. Let (M, g) be a space of constant curvature c. Since

$$R_{XYZW} = c\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\},$$

$X, Y, Z, W \in \mathfrak{X}(M)$, (3.6) is always satisfied. This leads to

COROLLARY 3.5. *Let (M, g) be a space of constant curvature c. Then the S-rotation s_σ is an isometry if and only if the rotation field S is parallel along σ.*

For reflections we have ([CV2],[ToV2])

COROLLARY 3.6. *Let (M, g) be a space of constant curvature c. Then a reflection in a curve σ is an isometry if and only if σ is a geodesic.*

To see this, suppose first that s_σ is an isometry. Then the result follows from the remark above. Conversely, suppose σ is a geodesic. Then $S\dot{\sigma} = \dot{\sigma}$ gives $S'\dot{\sigma} = 0$. Moreover, for $u \in \nu(\sigma)_{\sigma(t)}$ we have $Su = -u$. Hence (note that $\nabla_{\frac{\partial}{\partial t}} u$ is perpendicular to σ since σ is a geodesic)

$$(\nabla_{\frac{\partial}{\partial t}} S)u = -S(\dot{u}) - \dot{u} = \dot{u} - \dot{u} = 0,$$

that is S is parallel.

II. Let (M, g, J) be a Kähler manifold of constant holomorphic sectional curvature $c \neq 0$. Then we have

$$\begin{aligned} R_{XYZW} =& \frac{c}{4}\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + g(JX,Z)g(JY,W) \\ &- g(JX,W)g(JY,Z) + 2g(JX,Y)g(JZ,W)\}, \end{aligned}$$

$X, Y, Z, W \in \mathfrak{X}(M)$. From this we easily derive

COROLLARY 3.7. *Let (M, g, J) be a Kähler manifold of constant holomorphic sectional curvature $c \neq 0$. Then the S-rotation s_σ is an isometry if and only if S is parallel along σ and*

$$SJ\dot{\sigma} = J\dot{\sigma} \quad \textit{and} \quad SJu = JSu, \textit{ or} \tag{3.7}$$

$$SJ\dot{\sigma} = -J\dot{\sigma} \quad \textit{and} \quad SJu = -JSu \tag{3.8}$$

for all u orthogonal to σ.

III. Let (M, g) be a quaternionic Kähler manifold of constant quaternionic sectional curvature $c \neq 0$. In this case the Riemann curvature tensor has the form

$$\begin{aligned} R_{XYZW} =& \frac{c}{4}\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + \sum_{\alpha=1}^{3}[g(J_\alpha X,Z)g(J_\alpha Y,W) \\ &- g(J_\alpha X,W)g(J_\alpha Y,Z) + 2g(J_\alpha X,Y)g(J_\alpha Z,W)]\}, \end{aligned}$$

where (J_1, J_2, J_3) is a set of defining almost complex structures.

From this one derives

COROLLARY 3.8. *Let (M, g) be a quaternionic Kähler manifold of constant quaternionic sectional curvature $c \neq 0$. Then the S-rotation s_σ is an isometry if and only if S is parallel and*

$$SJ_\alpha = \sum_{\beta=1}^{3} a_{\alpha\beta} J_\beta S, \qquad \alpha = 1, 2, 3, \tag{3.9}$$

where $A = (a_{\alpha\beta}) \in SO(3)$ and $a_{\alpha\beta}$ are functions of t.

As we pointed out above, when s_σ is a reflection, S is *parallel if and only if σ is a geodesic.* In this case we derive, from(3.7) and (3.8) (respectively, (3.9)), that s_σ can never be an isometry except when $\dim M = 2$ (respectively, dimM = 4) in which cases (M, g) has constant curvature and then, according to Corollary 3.6, s_σ is always an isometry.

Note that this last property is characteristic for real space forms. Indeed, we have (cf. [VW2],[V4]):

PROPOSITION 3.9. *A Riemannian manifold is a space of constant curvature if and only if the reflections in all geodesics are isometries.*

For locally symmetric spaces and reflections, (3.6) reduces to

$$R_{uvu\dot{\sigma}} = 0$$

for all u, v orthogonal to σ. In [CV2] it is given a nice geometrical interpretation of this condition relating to the existence of totally geodesic submanifolds in locally symmetric spaces. This aspect will be dealt with in the next Chapter, Section IV.1, Proposition 4.3, to which we refer.

Using the theory by Chen and Nagano about (M_+, M_-)–totally geodesic submanifolds in symmetric spaces (cf. [C2]) it is known that Riemannian manifolds of constant sectional curvature $c \neq 0$ are the only irreducible symmetric spaces which admit totally geodesic hypersurfaces. Hence by applying Proposition 4.3 we have the following result ([V4]).

PROPOSITION 3.10. *Let (M, g) be a locally irreducible symmetric space. Then (M, g) is a space of constant curvature if it admits a curve σ such that the reflection s_σ with respect to σ is an isometry.*

3.2 Harmonic and isometric rotations around curves

Now we focus on harmonic rotations around a curve $\sigma : [a, b] \to M$ with emphasis on their relationship with isometric rotations. The earlier results for reflections were given in [VV-A].

PROPOSITION 3.11. *Let σ be a topologically embedded curve in a Riemannian manifold M. If the local reflection in σ is harmonic, then σ is a geodesic.*

Furthermore, we have a characterization of space forms in terms of harmonic reflections with respect to geodesics.

PROPOSITION 3.12. *Let (M, g) be a connected Riemannian manifold. Then (M, g) is a space of constant curvature if and only if the local reflections with respect to all geodesics are harmonic.*

Concerning the problem of the relationship between harmonic and isometric reflection we can state, for analytic data, the following special case of Proposition 4.5 ([DGV]), that indeed clarifies why Proposition 3.9 is equivalent to Proposition 3.12.

PROPOSITION 3.13. *The reflection s_σ with respect to an embedded curve σ of a Riemannian manifold (M, g) is harmonic if and only if it is an isometry.*

For an S-rotation s_σ we aim to prove the following results [NV3].

PROPOSITION 3.14. *Let $\sigma : [a, b] \to M$ be a smooth embedded curve in a Riemannian manifold M and s_σ an S-rotation around σ. If s_σ is harmonic, then S is parallel along σ. Moreover, if s_σ is a free S-rotation, then σ is a geodesic.*

PROPOSITION 3.15. *Let s_σ be a free rotation on a locally symmetric space such that the Ricci tensor is S-invariant. Then s_σ is harmonic if and only if it is isometric.*

As consequences we get the following corollaries.

COROLLARY 3.16. *A free rotation s_σ on a locally symmetric Einstein space is harmonic if and only if it is an isometry.*

COROLLARY 3.17. *A rotation around a geodesic in a locally symmetric Einstein space is harmonic if and only if it is an isometry.*

We use Fermi coordinates to express the fact that the tension field τ of s_σ vanishes. From (1.51) we get that s_σ is harmonic if and only if

$$(3.10)\qquad \begin{aligned} \tau^c(s_\sigma)(p) &= \{g^{11}(\nabla ds_\sigma)^c_{11} + 2g^{1a}(\nabla ds_\sigma)^c_{1a} + g^{ab}(\nabla ds_\sigma)^c_{ab}\}(p) = 0, \\ \tau^1(s_\sigma)(p) &= \{g^{11}(\nabla ds_\sigma)^1_{11} + 2g^{1a}(\nabla ds_\sigma)^1_{1a} + g^{ab}(\nabla ds_\sigma)^1_{ab}\}(p) = 0, \end{aligned}$$

with $a, b, c = 2, \ldots, n$, $p = \exp_{\sigma(t)}(ru)$, $\|u\| = 1$ and where

$$\begin{aligned} (\nabla ds_\sigma)^1_{11}(p) = &-\Gamma^1_{11}(p) + \Gamma^1_{\alpha\beta}(s_\sigma(p))\dot S^\alpha_\delta \dot S^\beta_\mu x^\delta x^\mu + \Gamma^1_{\alpha 1}(s_\sigma(p))\dot S^\alpha_\delta x^\delta \\ &+ \Gamma^1_{1\beta}(s_\sigma(p))\dot S^\beta_\mu x^\mu + \Gamma^1_{11}(s_\sigma(p)), \end{aligned}$$

$$(\nabla ds_\sigma)^1_{1a}(p) = -\Gamma^1_{1a}(p) + \Gamma^1_{\alpha\beta}(s_\sigma(p))\dot S^\alpha_\gamma S^\beta_a x^\gamma + \Gamma^1_{1\beta}(s_\sigma(p))S^\beta_a,$$

$$(\nabla ds_\sigma)^1_{ab}(p) = -\Gamma^1_{ab}(p) + \Gamma^1_{\alpha\beta}(s_\sigma(p))S^\alpha_a S^\beta_b,$$

$$(3.12)\qquad \begin{aligned} (\nabla ds_\sigma)^c_{11}(p) = &\ddot S^c_\gamma x^\gamma - \Gamma^1_{11}(p)\dot S^c_\delta x^\delta - \Gamma^k_{11}(p)S^c_k \\ &+ \Gamma^c_{\alpha\beta}(s_\sigma(p)\dot S^\alpha_\mu \dot S^\beta_\nu x^\mu x^\nu + \Gamma^c_{\alpha 1}(s_\sigma(p))\dot S^\alpha_\mu x^\mu \\ &+ \Gamma^c_{1\beta}(s_\sigma(p))\dot S^\beta_\nu x^\nu + \Gamma^c_{11}(s_\sigma(p)), \end{aligned}$$

$$\begin{aligned} (\nabla ds_\sigma)^c_{1a}(p) = &\dot S^c_a - \Gamma^k_{1a}(p)S^c_k - \Gamma^1_{1a}(p)\dot S^c_\mu x^\mu \\ &+ \Gamma^c_{\alpha\beta}(s_\sigma(p))\dot S^\alpha_\mu S^\beta_a x^\mu + \Gamma^c_{1\beta}(s_\sigma(p))S^\beta_a, \end{aligned}$$

$$(\nabla ds_\sigma)^c_{ab}(p) = -\Gamma^k_{ab}(p)S^c_k - \Gamma^1_{ab}(p)\dot S^c_\mu x^\mu + \Gamma^c_{\alpha\beta}(s_\sigma(p))S^\alpha_a S^\beta_b.$$

Next, we put

$$\tau^c(s_\sigma)(p) = \sum_{t=0}^{3} A_t^c r^t + O(r^4), \qquad c = 2,\dots,n, \tag{3.13}$$

$$\tau^1(s_\sigma)(p) = \sum_{t=0}^{3} A_t^1 r^t + O(r^4). \tag{3.14}$$

Then (3.10) and (3.11) give the following necessary conditions for a harmonic rotation s_σ:

$$A_t^c = 0, \qquad A_t^1 = 0, \qquad t = 0,1,2,3, \qquad c = 2,\dots,n. \tag{3.15}$$

In order to write down the expressions for the Christoffel symbols, we set

$$g_{AB}(p) = \sum_{l=0}^{3} \alpha_{AB}^C(m) r^C + O(r^4),$$
$$g^{AB}(p) = \sum_{l=0}^{3} \beta_{AB}^C(m) r^C + O(r^4),$$
$$\frac{\partial g_{AB}}{\partial x^C}(p) = \sum_{l=0}^{3} \gamma_{ABC}^l(m) + O(r^4)$$

for $A, B = 1,\dots,n$, $m = \sigma(t)$ (see Chapter 1, formulae (1.42)–(1.47)). We have for $a, b, c = 2,\dots,n$

$$\Gamma_{11}^1(p) = \frac{r}{2}\gamma_{111}^1(m) + \frac{r^2}{2}\{\gamma_{111}^2 + \beta_{11}^1\gamma_{111}^1 - \sum_{a=2}^{n}\beta_{1a}^2\gamma_{11a}^0\}(m) + O(r^3),$$

$$\Gamma_{1a}^1(p) = \frac{1}{2}\gamma_{11a}^0 + \frac{r}{2}\{\gamma_{11a}^1 + \beta_{11}^1\gamma_{11a}^0\}(m)$$
$$+ \frac{r^2}{2}\{\gamma_{11a}^2 + \beta_{11}^1\gamma_{11a}^1 + \beta_{11}^2\gamma_{11a}^0\}(m) + O(r^3),$$

$$\Gamma_{ab}^1(p) = \frac{r}{2}\{\gamma_{1ab}^1 + \gamma_{1ba}^1\}(m) + \frac{r^2}{2}\{\gamma_{1ab}^2 + \gamma_{1ba}^2 - \gamma_{ab1}^2$$
$$+ \beta_{11}^1\gamma_{1ab}^1 + \beta_{11}^1\gamma_{1ba}^1\}(m) + O(r^3),$$

$$
\begin{aligned}
\Gamma^c_{11}(p) = &- \frac{1}{2}\gamma^0_{11c}(m) - \frac{r}{2}\gamma^1_{11c}(m) \\
&+ \frac{r^2}{2}\{2\gamma^2_{1c1} - \gamma^2_{11c} - \sum_{a=2}^{n} \beta^2_{ca}\gamma^0_{11a}\}(m) \\
&+ \frac{r^3}{2}\{2\gamma^3_{1c1} - \gamma^3_{11c} - \sum_{a=2}^{n} \beta^2_{cd}\gamma^1_{11d} \\
&- \sum_{a=2}^{n} \beta^3_{ca}\gamma^0_{11a} + \beta^2_{c1}\gamma^1_{111} + \beta^1_{c1}\gamma^2_{111}\}(m) + O(r^4),
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^c_{1a}(p) = &\frac{r}{2}\{\gamma^1_{1ca} - \gamma^1_{1ac}\}(m) \\
&\frac{r^2}{2}\{\gamma^2_{1ca} + \gamma^2_{ac1} - \gamma^2_{1ac} \\
&+ \beta^2_{c1}\gamma^0_{11a}\}(m) + O(r^3),
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^c_{ab}(p) = &\frac{r}{2}\{\gamma^1_{acb} + \gamma^1_{bca} - \gamma^1_{abc}\}(m) \\
&+ \frac{r^2}{2}\{\gamma^2_{acb} + \gamma^2_{bca} - \gamma^2_{abc}\}(m) \\
&+ \frac{r^3}{2}\{\gamma^3_{acb} + \gamma^3_{bca} - \gamma^3_{abc} + \beta^2_{1c}\gamma^1_{a1b} \\
&+ \beta^2_{1c}\gamma^1_{b1a} + \sum_{d=2}^{n} \beta^2_{cd}\gamma^1_{adb} \\
&+ \sum_{d=2}^{n} \beta^2_{cd}\gamma^1_{bda} - \sum_{d=2}^{n} \beta^2_{cd}\gamma^1_{abd}\}(m) + O(r^4).
\end{aligned}
$$

Hence, using the expressions for the Christoffel symbols, the expressions for the components of the metric g in terms of Fermi coordinates (cf. (1.42)–(1.47)), (3.10), (3.11) and (3.12), we compute the desired coefficients A^c_t and A^1_t, $c = 2, \ldots, n$. We omit the lengthy but straightforward computations.

PROOF OF PROPOSITION 3.14. Using $A_0^c = 0$, $c = 2, \dots, n$, we obtain

$$g(\ddot{\sigma}, E_c) - g(\ddot{\sigma}, SE_c) = 0, \qquad c = 2, \dots, n.$$

This yields

$$({}^tS - I)\ddot{\sigma} = k\dot{\sigma}$$

for some real k. On the other hand, $({}^tS - I)\ddot{\sigma}$ is orthogonal to $\dot{\sigma}$ and hence we have

$$S\ddot{\sigma} = \ddot{\sigma}. \tag{3.16}$$

Next, from conditions $A_1^c = 0$, $c = 2, \dots, n$, we obtain, taking into account (3.16),

$$g(\ddot{S}u, E_c) - (R_{1u1S^{-1}c} - R_{1Su1c}) - \frac{2}{3}\sum_{a=2}^{n}(R_{uaS^{-1}ca} - R_{SuSacSa}) = 0 \tag{3.17}$$

or, equivalently,

$$g(\ddot{S}u, SE_c) - (R_{1u1c} - R_{1Su1Sc}) - \frac{2}{3}\sum_{a=2}^{n}(R_{uaca} - R_{SuSaScSa}) = 0, \tag{3.18}$$

where Sc and $S^{-1}c$ denote the vectors $(S\frac{\partial}{\partial x^c})(\sigma(t))$ and $(S^{-1}\frac{\partial}{\partial x^c})(\sigma(t))$, respectively, for $c = 2, \dots, n$. Now, put $E_c = u$ in (3.18). Then we obtain

$$g(\ddot{S}u, Su) - (R_{1u1u} - R_{1Su1Su}) - \frac{2}{3}\sum_{a=2}^{n}(R_{uaua} - R_{SuSaSuSa}) = 0. \tag{3.19}$$

Since $\|u\| = 1$ we have $g(\dot{S}u, Su) = 0$. Differentiating once again we get

$$\begin{aligned} 0 &= g(\ddot{S}u, Su) + g(\dot{S}\dot{u}, Su) + g(\dot{S}u, \dot{S}u) + g(\dot{S}u, S\dot{u}) \\ &= g(\ddot{S}u, Su) + g(\dot{S}u, \dot{S}u) + g(({}^tS\dot{S} + {}^t\dot{S}S)u, \dot{u}) \\ &= g(\ddot{S}u, Su) + g(\dot{S}u, \dot{S}u) \end{aligned} \tag{3.20}$$

because ${}^tS\dot{S} +^t \dot{S}S = 0$ on normal vectors to σ and $({}^tS)^{\cdot} =^t (\dot{S})$. Using this in (3.19), then putting $u = E_c$ and summing up with respect to $c = 2, \ldots, n$, we get with $\dot{S}\dot{\sigma} = 0$,

$$\|\dot{S}\|^2 + \frac{2}{3}\sum_{a,c=2}^{n}(R_{caca} - R_{ScSaScSa}) = 0.$$

This implies

$$\nabla_{\frac{\partial}{\partial t}} S = 0, \qquad \text{i.e., } S \text{ is parallel.}$$

Finally, it is clear from (3.16) that, if s_σ is free, then $\ddot{\sigma} = 0$ and this finishes the proof.

PROOF OF PROPOSITION 3.15. Since any isometry is harmonic we only have to prove the direct part of the proposition. So, let s_σ be harmonic. According to Proposition 3.2 we have to prove

$$R_{1u1u} = R_{1Su1Su}, \qquad R_{1uau} = R_{1SuSaSu}, \qquad R_{aubu} = R_{SaSuSbSu},$$

$a, b = 2, \ldots, n$.

From (3.19) and the fact that S is parallel we get

$$R_{1u1u} - R_{1Su1Su} + \frac{2}{3}\sum_{a=2}^{n}(R_{uaua} - R_{SuSaSuSa}) = 0 \tag{3.21}$$

and since ρ is S-invariant this yields at once

$$R_{1u1u} = R_{1Su1Su} \tag{3.22}$$

for all $u \in \nu(\sigma)_{\sigma(t)}$.

Further, we consider the conditions $A_3^c = 0$. As s_σ is free, σ is a geodesic. By putting $S^{-1}c = u$, these conditions become

(3.23)

$$-60\sum_{a=2}^{n} R_{1uau}(R_{1uau} - R_{1SuSaSu}) + 36\sum_{a=2}^{n}(R_{1uau}^2 - R_{1SuSaSu}^2)$$
$$+6\sum_{a,b=2}^{n}(R_{uaub}^2 - R_{SuSaSuSb}^2) - 10\sum_{a,b=2}^{n} R_{uaub}(R_{uaub} - R_{SuSaSuSb}) = 0.$$

To handle (3.23) we integrate it over the unit sphere $S^{n-2}(1)$ in $\nu(\sigma)_{\sigma(t)}$. For this technique we refer to Lemma 4.10 in the next chapter (cf. [CV1],[GV1],[Gr7]).

First note that the integrals of

$$\sum_{a=2}^{n}(R_{1uau}^2 - R_{1SuSaSu}^2) \qquad \text{and} \qquad \sum_{a=2}^{n}(R_{uaub}^2 - R_{SuSaSuSb}^2)$$

are zero. Next, put

$$A = \sum_{a=2}^{n} R_{1uau}(R_{1uau} - R_{1SuSaSu}),$$
$$B = \sum_{a,b=2}^{n} R_{uaub}(R_{uaub} - R_{SuSaSuSb})$$

and let

$$c_{n-2} = \frac{(n-1)\pi^{\frac{n-1}{2}}}{(\frac{n-1}{2})!}$$

denote the volume of the unit sphere in the Euclidean space $\mathbb{E}^{n-1}$. Then we have

$$\begin{aligned}
\int_{S^{n-2}(1)} A du &= \frac{c_{n-2}}{(n-1)(n+1)} \sum_{a,i,j=2}^{n} \{R_{1iai}R_{1jaj} + R_{1iaj}R_{1iaj} \\
&\quad + R_{1iaj}R_{1jai} - R_{1iai}R_{1SjSaSj} - R_{1iaj}R_{1SiSaSj} \\
&\quad - R_{1iaj}R_{1SjSaSi}\} \\
&= \frac{c_{n-2}}{(n-1)(n+1)}\{\sum_{\alpha=1}^{n}\sum_{a,j=2}^{n}(R_{1\alpha aj}^2 - R_{1\alpha aj}R_{1S\alpha SaSj}) \\
&\quad + \sum_{a=2}^{n}\sum_{\alpha\beta=1}^{n}(R_{1\alpha a\beta}R_{1\beta a\alpha} - R_{1\alpha a\beta}R_{1S\beta SaS\alpha})\} \\
&= \frac{c_{n-2}}{(n-1)(n+1)}\{\sum_{\alpha,\beta,\gamma=1}^{n}(R_{1\alpha\beta\gamma}^2 + R_{1\alpha\beta\gamma}R_{1\gamma\beta\alpha} \\
&\quad - R_{1\alpha\beta\gamma}R_{1S\alpha S\beta S\gamma} - R_{1\alpha\beta\gamma}R_{1S\gamma S\beta S\alpha}) \\
&\quad - 3\sum_{\alpha,\beta=1}^{n}(R_{1\alpha 1\beta}^2 - R_{1\alpha 1\beta}R_{1S\alpha 1S\beta})\}.
\end{aligned}$$

Now we use the following identities (see for example [GV1])

$$\sum_{\alpha,\beta,\gamma=1}^{n} R_{1\alpha\beta\gamma}R_{1\gamma\beta\alpha} = \frac{1}{2}\sum_{\alpha,\beta,\gamma=1}^{n} R^2_{1\alpha\beta\gamma}, \tag{3.24}$$

$$\sum_{\alpha,\beta,\gamma=1}^{n} R_{1\alpha\beta\gamma}R_{1S\gamma S\beta S\alpha} = \frac{1}{2}\sum_{\alpha,\beta,\gamma=1}^{n} R_{1\alpha\beta\gamma}R_{1S\alpha S\beta S\gamma} \tag{3.25}$$

to obtain

$$\int_{S^{n-2}(1)} Adu = \frac{3c_{n-2}}{2(n-1)(n+1)}\sum_{\alpha,\beta,\gamma=1}^{n} R_{1\alpha\beta\gamma}(R_{1\alpha\beta\gamma} - R_{1S\alpha S\beta S\gamma}). \tag{3.26}$$

By the same procedure we compute the integral of B:

$$\int_{S^{n-2}(1)} Bdu = \frac{c_{n-2}}{(n-1)(n+1)}\sum_{a,b=2}^{n}\{(\rho_{ab} - R_{1a1b})^2 + \frac{3}{2}\sum_{i,j=2}^{n} R^2_{iajb}$$
$$- (\rho_{ab} - R_{1a1b})(\rho_{SaSb} - R_{1Sa1Sb}) + \frac{3}{2}\sum_{i,j=2}^{n} R_{iajb}R_{SiSaSjSb}\}.$$

Taking into account (3.22) and the S-invariance of ρ we obtain

$$\int_{S^{n-2}(1)} Bdu = \frac{3c_{n-2}}{2(n-1)(n+1)}\sum_{a,b,i,j=2}^{n} R_{iajb}(R_{iajb} - R_{SiSaSjSb}). \tag{3.27}$$

Finally, (3.21), (3.26) and (3.27) yield

$$6\sum_{\alpha,\beta,\gamma=1}^{n} R_{1\alpha\beta\gamma}(R_{1\alpha\beta\gamma} - R_{1S\alpha S\beta S\gamma}) + \sum_{a,b,i,j=2}^{n} R_{iajb}(R_{iajb} - R_{SiSaSjSb}) = 0$$

and hence also

$$6\sum_{\alpha,\beta,\gamma=1}^{n}(R_{1\alpha\beta\gamma}-R_{1S\alpha S\beta S\gamma})^2+\sum_{a,b,i,j=2}^{n}(R_{iajb}-R_{SiSaSjSb})^2=0.$$

This gives

$$\begin{aligned}R_{1\alpha\beta\gamma}&=R_{1S\alpha S\beta S\gamma}, &\quad \alpha,\beta,\gamma&=1,\ldots,n,\\ R_{iajb}&=R_{SiSaSjSb}, &\quad i,j,a,b&=2,\ldots,n.\end{aligned}$$

Thus the required result is proved.

REMARK. In the preceding propositions we restricted to the case of locally symmetric spaces with S-invariant Ricci tensor. In the more general case the conditions $A_t^c=0$, $A_t^1=0$ become rather complicated and involved. Up to now we do not know if these results can be extended to the case of general Riemannian manifolds.

3.3 Survey of related results

Let θ_σ be the *volume density function* of $\exp_\sigma$, where σ is a curve as above. θ_σ is defined with respect to Fermi coordinates along σ by

$$\theta_\sigma(p)=(\det(g_{ij}(p)))^{\frac{1}{2}}. \tag{3.28}$$

Using (1.34) and (3.28) we have, with the notations and symbols introduced in Chapter 1, Section 1.3,

$$\theta_\sigma(p)=\frac{(\det D_u)(r)}{r^{n-2}}. \tag{3.29}$$

Consider now the shape operator T_σ of the tube $P_\sigma(r)$ at $p=\exp_\sigma(ru)$. Proceeding as in the case of the geodesic spheres (cf. Section 1.2) it is not difficult to prove that T_σ takes the form

$$T_\sigma(p)=(D_u'D_u^{-1})(r) \tag{3.30}$$

and therefore the *mean curvature* $h_\sigma(p)$ of $P_\sigma(r)$ is given by

$$h_\sigma(p)=\operatorname{tr}(D_u'D_u^{-1})(r)=\frac{(\det D_u)'}{\det D_u}(r)=\frac{n-2}{r}+(\frac{\theta_\sigma'}{\theta_\sigma})(p). \tag{3.31}$$

Using (3.31) and the expansion for the mean curvature of the tube $P_\sigma(r)$ we obtain the following power series expansion for the volume density function (cf. [CV1],[V4]):

$$\theta_\sigma(\exp_m(ru)) = 1 - \kappa_u r - \frac{1}{6}(\rho_{uu} + 2R_{1u1u})r^2 + O(r^3). \tag{3.32}$$

An S-rotation s_σ around a curve σ is *volume-preserving* (up to sign) if and only if (cf. [V4])

$$\theta_\sigma(s_\sigma(p)) = \theta_\sigma(p). \tag{3.33}$$

Volume–preserving reflections with respect to curves have been considered (see [V4] for more details) and may be used to characterize special classes of manifolds. For example, it was proved by L. Vanhecke and T.J. Willmore in [VW2] that a Riemannian manifold is locally symmetric if and only if the reflections with respect to all geodesics are volume–preserving.

Note that if the reflection with respect to a curve σ is volume–preserving then σ is a geodesic. This is just a special case of the fact that if a free S-rotation s_σ around a curve σ is volume–preserving then σ is a geodesic. In fact the first condition we get from (3.32) and (3.33) is

$$g(\ddot{\sigma}, u) = g(\ddot{\sigma}, Su)$$

and therefore if S is free $\ddot{\sigma}$ must be zero, i.e., σ is a geodesic. Apart from this, very little is known about volume–preserving rotations around curves.

For example, we can mention that for a space of constant curvature $c \neq 0$, due to the fact that the Jacobi equation can be solved explicitely, the volume density function for a geodesic is given by (cf. [GV3],[V4])

$$\theta_\sigma(\exp_m(ru)) = (\cos\sqrt{c}r)(\frac{\sin\sqrt{c}r}{\sqrt{c}r})^{n-2}$$

and therefore any rotation around a geodesic σ is always volume–preserving.

Also in the case of a space with constant holomorphic curvature $c \neq 0$, the volume density function for a geodesic σ is given explicitly (cf. [GV3]) by

$$\theta_\sigma(\exp_m(ru)) = (\frac{2}{\sqrt{c}r}\sin\frac{\sqrt{c}}{2}r)^{2n-2}(\cos^2\frac{\sqrt{c}}{2}r - a^2\sin^2\frac{\sqrt{c}}{2}r),$$

where $a = g(\dot{\sigma}(0), Ju)$. Therefore, we must have

$$g(\dot{\sigma}, Ju)g(\dot{\sigma}, Ju) = g(\dot{\sigma}, JSu)g(\dot{\sigma}, JSu)$$

and hence $SJ\dot{\sigma} = \pm J\dot{\sigma}$. So, a rotation s_σ around a geodesic σ is volume–preserving if and only if $SJ\dot{\sigma} = \pm J\dot{\sigma}$.

Reflections and rotations with respect to curves have been used extensively in *(almost) contact geometry* . There it is natural to consider reflections with respect to integral curves of the characteristic vector field but also reflections with respect to other curves play an important role. In [In], the author considers special rotations around ξ-geodesics on a Sasakian manifold, the so-called *(local) φ-rotations* . For this kind of arguments the most recent and extensive reference is the doctoral dissertation by P. Bueken [Bu] to which we refer the interested reader.

In several cases, for example in the framework of K-contact geometry itself, the manifolds are just special cases of Riemannian foliations. Reflections with respect to the leaves of a Riemannian foliation have been considered in [ToV4]. In that paper the special case of a Riemannian flow generated by a unit Killing vector field was also treated. This study was then continued and developed by other authors in subsequent works. For these problems we refer to [GD].

Chapter 4

Rotations around submanifolds

In this chapter we continue our study by considering the notion of a rotation around a submanifold.

4.1 Isometries and rotations in tubular neighborhoods

Let f be an isometry of (M, g) and suppose it has a fixed point set of positive dimension. Let B be a (totally geodesic) connected component of this fixed point set. Then, on a sufficiently small tubular neighborhood of B, f can be represented as

$$f = \exp_\nu \circ df_{|B} \circ \exp_\nu^{-1},$$

where $df_{|B}$ is the differential map of f calculated along B. So, $df_{|B}$ is a (1,1)-tensor field along B which is a linear isometry on each fibre of the normal bundle of B and which is the identity map on the vectors tangent to B.

With this example in mind we will now introduce the notion of rotation around a submanifold ([NV4]).

DEFINITIONS. A (1,1)-tensor field S along B, that is, an $\mathcal{F}(B)$-linear map

$$S : \overline{\mathfrak{X}}(B) \to \overline{\mathfrak{X}}(B),$$

is said to be a *rotation field along B* if

$$S(\mathfrak{X}^\perp(B)) \subseteq \mathfrak{X}^\perp(B), \qquad S_{|\mathfrak{X}(B)} = I_{\mathfrak{X}(B)}$$

and

$$g(SU, SV) = g(U, V)$$

for all $U, V \in \mathfrak{X}^{\perp}(B)$.

On a sufficiently small tubular neighborhood of B, the local diffeomorphism defined by

$$s_B = \exp_\nu \circ S \circ \exp_\nu^{-1}$$

is said to be a *(local) S-rotation around the submanifold B*. If $S - I$ is non-singular on the normal bundle, s_B is said to be a *free S-rotation*.

Note that for $S = -I$, s_B defines the (local) reflection with respect to B. Furthermore, we have

$$s_B : \exp_\nu(m, v) \mapsto \exp_\nu(m, Sv)$$

and B is contained in the fixed point set of s_B. Finally, the analytic expression of s_B in terms of Fermi coordinates is

$$\begin{cases} x^i \circ s_B = x^i, & i = 1, \ldots, q, \\ x^a \circ s_B = S^a_b(x^1, \ldots, x^q)x^b, & a, b = q+1, \ldots, n, \end{cases}$$

where S^a_b are the components of S with respect to the basis $\{E_{q+1}, \ldots, E_n\}$ introduced above (cf. Chapter 1).

The covariant differential of S along B is the $\mathcal{F}(B)$-linear function $\nabla S : \mathfrak{X}(B) \times \overline{\mathfrak{X}}(B) \to \overline{\mathfrak{X}}(B)$ defined by

$$(\nabla S)(X, V) = (\nabla_X S)V = \nabla_X(SV) - S\nabla_X V.$$

Then it is easy to see that $\nabla S = 0$ is equivalent to the two following statements:

(1) S preserves the second fundamental form of B, that is, $ST(X, Y) = T(X, Y)$;

(2) S preserves the normal connection of B, that is $S\nabla^{\perp}_X U = \nabla^{\perp}_X(SU)$ (or, equivalently, $(\nabla^{\perp}_X S)U = 0$),

for $X, Y \in \mathfrak{X}(B)$ and $U \in \mathfrak{X}^{\perp}(B)$. If S is *free*, $\nabla S = 0$ *implies that B is totally geodesic.* In particular, when S determines the reflection with respect to B, then $\nabla S = 0$ is equivalent to the fact that B is totally geodesic since $\nabla^{\perp}_X S = 0$ is automatically satisfied.

From the remarks made above at the beginning of the section it follows that an isometry f of (M, g) is a rotation around the (totally geodesic) connected components of its fixed point set (which was supposed to have a positive dimension). Its rotation field is the differential map of f along B. It is easy to see that this field is parallel along B. Now, we will derive a criterion for a rotation to be isometric.

PROPOSITION 4.1. *Let B be a submanifold of (M, g) as specified above and let s_B be an S-rotation around B. Then, if s_B is an isometry,*

$$(1)\ \nabla S = 0 \quad along \quad B;$$
$$(2)\ (\nabla^k_{u\dots u}R)_{uxuy} = (\nabla^k_{Su\dots Su}R)_{SuSxSuSy},$$

for all normal vectors u, all tangent vectors x, y of M and all $k \in \mathbb{N}$.

The converse also holds for analytic data.

PROOF. If $s_B = \exp_\nu \circ S \circ \exp_\nu^{-1}$ is an isometry, then $s_B = \exp_\nu \circ ds_{B|B} \circ \exp_\nu^{-1}$ and hence $S = ds_{B|B}$ is parallel along B. Moreover, since any isometry preserves the curvature tensor and its covariant derivatives, we have (2).

Conversely, given (1) and (2), we have to prove that $s_B^* g = g$. Using (1) this reduces to

$$g_{ij}(p) = g_{ij}(s_B(p)),\ g_{ia}(p) = g_{i\alpha}(s_B(p))S^\alpha_a(m),\ g_{ab}(p) = g_{\beta\gamma}(s_B(p))S^\beta_a(m)S^\gamma_b(m),$$

for $i, j = 1, \dots, q$; $a, b, \alpha, \beta, \gamma = q+1, \dots, n$ where $p = \exp_m(ru)$. As we explained in Section 1.3, the components of the metric tensor with respect to Fermi coordinates are given in terms of the operator D_u. The Jacobi equation (1.32) yields

$$D_u^{l+2}(0) = -\sum_{k=0}^{l} \binom{l}{k} R^{(l-k)}(0) D_u^{(k)}(0), \qquad l \in \mathbb{N}.$$

Then, the Taylor expansion of $D_u(r)$ together with the initial conditions,

$$T(Su) = ST(u) = T(u), \qquad S\nabla^\perp u = \nabla^\perp(Su),$$

and (2) yield the required result.

This proposition generalizes the result obtained in [CV2] for reflections, i.e., when $S = -I$. In this case the proposition reads as follows.

PROPOSITION 4.2. *Let (M, g) be an analytic Riemannian manifold and B a submanifold. Then the reflection s_B is a local isometry if and only if*

(1) *B is totally geodesic;*

(2) *$(\nabla^{2k}_{u\dots u}R)_{uv}u$ is normal to B,*
$(\nabla^{2k+1}_{u\dots u}R)_{uv}u$ is tangent to B, and
$(\nabla^{2k+1}_{u\dots u}R)_{ux}u$ is normal to B

for all normal vectors u, v of B, any tangent vector x of B and all $k \in \mathbb{N}$.

When (M, g) is locally symmetric (2) of the above proposition reduces to

$$R_{uv}u \text{ is normal to } B$$

for all u, v normal to B. As we already mentioned in Chapter 3, Section 3.1, there are two useful geometric interpretations of this condition for a totally geodesic submanifold B in a locally symmetric space (cf. [CV2]).

PROPOSITION 4.3. *Let B be a totally geodesic submanifold in a locally symmetric space (M, g). Then the following statements are equivalent:*

(1) *$R_{uv}u$ is normal to B for all u, v normal to B;*
(2) *Through each $m \in B$ there exists a totally geodesic submanifold $\overline{B}$ such that $T_m\overline{B} = \nu(B)_m$;*
(3) *B is curvature-adapted to M, i.e., $R_u(T_mB) \subset T_mB$ and $R_u \circ T(u) = T(u) \circ R_u$ for each $m \in B$ and every (unit) normal vector u at m.*

For the notion of a *curvature-adapted* submanifold we refer to [BV] (see also [Gr7] for the notion of *compatible* submanifold), where one can also find further applications. It is clear that every submanifold of a space of constant curvature is curvature-adapted. Moreover, in spaces of constant holomorphic sectional curvature the holomorphic submanifolds provide examples of curvature-adapted submanifolds. Other examples of curvature-adapted submanifolds are the geodesic spheres in $\mathfrak{P}$-spaces (cf. [BV]).

In [CO] it was pointed out that a totally geodesic submanifold B in a Kähler manifold of constant holomorphic sectional curvature $c \neq 0$ is either holomorphic or totally real with $\dim B = \frac{1}{2}\dim M$. Then from the preceding discussion the following holds true ([CV2]).

COROLLARY 4.4. *(a) Let (M, g) be a space of constant curvature. Then the reflection s_B is an isometry if and only if B is totally geodesic.*

(b) Let (M, g) be a Kähler manifold of constant holomorphic sectional curvature $c \neq 0$. Then the reflection s_B is an isometry if and only if either B is a holomorphic totally geodesic submanifold or a totally real totally geodesic submanifold of dimension $\frac{1}{2}\dim M$.

4.2 Harmonic and isometric rotations

In this section we concentrate on harmonic S-rotations in relation with isometric S-rotations around a submanifold of a Riemannian manifold.

The relationship between isometric and harmonic reflections for analytic data is completely clarified by the following result [DGV].

PROPOSITION 4.5. *Let s_B denote the reflection with respect to a submanifold B in a Riemannian manifold (M, g). Then s_B is harmonic if and only if it is an isometry.*

For S-rotations the problem is not solved in general. Nevertheless, we have [NV4]:

PROPOSITION 4.6. *Let B be a submanifold of (M, g) as specified above and let s_B be an S-rotation around B. If s_B is harmonic, then S preserves the mean curvature vector field of B and if s_B is a free rotation, then B is a minimal submanifold. Moreover, if B is totally geodesic with flat normal connection and s_B a harmonic rotation, then $\nabla S = 0$ along B.*

PROPOSITION 4.7. *Let (M, g) be a locally symmetric space with S-invariant Ricci tensor and let B be a totally geodesic submanifold with flat normal connection and s_B an S-rotation around B. Then s_B is harmonic if and only if it is an isometry.*

This gives at once

COROLLARY 4.8. *Let (M, g) be a locally symmetric Einstein space and let B be a totally geodesic submanifold with flat normal connection and s_B a rotation around B. Then s_B is harmonic if and only if s_B is isometric.*

In the next part of the section we will prove these results. Using Fermi coordinates and with the notations and symbols introduced in Section 1.5 we first state

LEMMA 4.9. *An S-rotation s_B around the submanifold B is harmonic if and only if for all $m \in B$*

$$(4.1) \quad \begin{aligned} \tau^c(s_B)(p) &= \{g^{ij}(\nabla ds_B)^c_{ij} + 2g^{ia}(\nabla ds_B)^c_{ia} + g^{ab}(\nabla ds_B)^c_{ab}\}(p) = 0, \\ \tau^k(s_B)(p) &= \{g^{ij}(\nabla ds_B)^k_{ij} + 2g^{ia}(\nabla ds_B)^k_{ia} + g^{ab}(\nabla ds_B)^k_{ab}\}(p) = 0, \end{aligned}$$

for $i,j,k=1,\dots,q; a,b,c=q+1,\dots,n$, *where* $p=\exp_m(ru)$, $u\in\nu(B)_m$, $\|u\|=1$ *and*

$$
\begin{aligned}
(\nabla ds_B)^k_{ij}(p) &= -\Gamma^k_{ij}(p)+\Gamma^k_{ij}(s_B(p))+\Gamma^k_{i\beta}(s_B(p))\frac{\partial S^\beta_\mu}{\partial x^j}x^\mu \\
&\quad +\Gamma^k_{\alpha j}(s_B(p))\frac{\partial S^\alpha_\delta}{\partial x^i}x^\delta+\Gamma^k_{\alpha\beta}(s_B(p))\frac{\partial S^\alpha_\delta}{\partial x^i}\frac{\partial S^\beta_\mu}{\partial x^j}x^\delta x^\mu, \\
(\nabla ds_B)^k_{ia}(p) &= -\Gamma^k_{ia}(p)+\Gamma^k_{i\beta}(s_B(p))S^\beta_a+\Gamma^k_{\alpha\beta}(s_B(p))\frac{\partial S^\alpha_\delta}{\partial x^i}S^\beta_a x^\delta, \\
(\nabla ds_B)^k_{ab}(p) &= -\Gamma^k_{ab}(p)+\Gamma^k_{\alpha\beta}(s_B(p))S^\alpha_a S^\beta_b, \\
(\nabla ds_B)^c_{ij}(p) &= \frac{\partial^2 S^c_\alpha}{\partial x^i\partial x^j}x^\alpha-\Gamma^l_{ij}(p)\frac{\partial S^c_\delta}{\partial x^l}x^\delta-\Gamma^\alpha_{ij}(p)S^c_\alpha \\
&\quad +\Gamma^c_{ij}(s_B(p))+\Gamma^c_{i\beta}(s_B(p))\frac{\partial S^\beta_\delta}{\partial x^j}x^\delta \\
&\quad +\Gamma^c_{\alpha j}(s_B(p))\frac{\partial S^\alpha_\mu}{\partial x^i}x^\mu+\Gamma^c_{\alpha\beta}(s_B(p))\frac{\partial S^\alpha_\delta}{\partial x^i}\frac{\partial S^\beta_\mu}{\partial x^j}x^\delta x^\mu, \\
(\nabla ds_B)^c_{ia}(p) &= \frac{\partial S^c_a}{\partial x^i}-\Gamma^l_{ia}(p)\frac{\partial S^c_\delta}{\partial x^l}x^\delta-\Gamma^\alpha_{ia}(p)S^c_\alpha \\
&\quad +\Gamma^c_{i\beta}(s_B(p))S^\beta_\alpha+\Gamma^c_{\alpha\beta}(s_B(p))\frac{\partial S^\alpha_\delta}{\partial x^i}S^\beta_a x^\delta, \\
(\nabla ds_B)^c_{ab}(p) &= -\Gamma^l_{ab}(p)\frac{\partial S^c_\delta}{\partial x^l}x^\delta-\Gamma^\alpha_{ab}S^c_\alpha+\Gamma^c_{\alpha\beta}(s_B(p))S^\alpha_a S^\beta_b,
\end{aligned}
\tag{4.2}
$$

where S^α_β *and its partial derivatives are evaluated at* m.

Further, put

$$
\begin{aligned}
\tau(s_B)^c &= \sum_{k=0}^{3} A^c_k t^k + O(t^4), \qquad c=q+1,\dots,n,\\
\tau(s_B)^i &= \sum_{k=0}^{3} A^i_k t^k + O(t^4), \qquad i=1,\dots,q.
\end{aligned}
$$

From (4.1) it follows that if s_B is a harmonic rotation then we have

$$A_k^c = 0, \qquad A_k^i = 0$$

for $k = 0,1,2,3; c = q+1,\dots,n$ and $i = 1,\dots,q$. To compute A_k^c and A_k^i we need to know the Christoffel symbols. To this end, we put

$$\begin{aligned}
g_{AB}(p) &= \sum_{l=0}^{3} \alpha_{AB}^{C}(m) r^C + O(r^4), \\
g^{AB}(p) &= \sum_{l=0}^{3} \beta_{AB}^{C}(m) r^C + O(r^4), \\
\frac{\partial g_{AB}}{\partial x^C}(p) &= \sum_{l=0}^{3} \gamma_{ABC}^{l}(m) + O(r4)
\end{aligned}$$

for $A, B = 1,\dots,n$ (see Chapter 1, formulae (1.35)–(1.40)). We have, for $i,j,k = 1,\dots,q; a,b,c = q+1,\dots,n$,

$$\begin{aligned}
\Gamma_{ij}^{k}(p) = {} & \frac{1}{2}\{\gamma_{ikj}^{0} + \gamma_{jki}^{0} - \gamma_{ijk}^{0}\}(m) \\
& + \frac{r}{2}\{\gamma_{ikj}^{1} + \gamma_{jki}^{1} - \gamma_{ijk}^{1} \\
& - \sum_{d=q+1}^{n} \beta_{kd}^{1}\gamma_{ijd}^{0}\}(m) \\
& + \frac{r^2}{2}\{\gamma_{ikj}^{2} + \gamma_{jki}^{2} - \gamma_{ijk}^{2} \\
& + \sum_{l=1}^{q} \beta_{kl}^{1}\gamma_{ilj}^{1} + \sum_{l=1}^{q} \beta_{kl}^{1}\gamma_{jli}^{1} - \sum_{l=1}^{q} \beta_{kl}^{1}\gamma_{ijl}^{1} \\
& - \sum_{d=q+1}^{n} \beta_{kd}^{2}\gamma_{ijd}^{0} + \sum_{d=q+1}^{n} \beta_{kd}^{1}\gamma_{idj}^{1} \\
& + \sum_{d=q+1}^{n} \beta_{kd}^{1}\gamma_{jdi}^{1} - \sum_{d=q+1}^{n} \beta_{kd}^{1}\gamma_{idj}^{1}\}(m) + O(r^3),
\end{aligned}$$

$$
\begin{aligned}
\Gamma^k_{ia}(p) = {} & \frac{1}{2}\gamma^0_{ika}(m) \\
& + \frac{r}{2}\{\gamma^1_{ika} + \gamma^1_{aki} - \gamma^1_{iak} + \sum_{l=1}^{q} \beta^1_{kl}\gamma^0_{ila} \\
& + \sum_{d=q+1}^{n} \beta^1_{kd}\gamma^0_{ida} - \sum_{d=q+1}^{n} \beta^1_{kd}\gamma^0_{iad}\}(m) \\
& + \frac{r^2}{2}\{\gamma^2_{ika} + \gamma^2_{aki} - \gamma^2_{iak} \\
& \sum_{l=1}^{q} \beta^1_{kl}\gamma^1_{ila} + \sum_{l=q}^{n} \beta^1_{kl}\gamma^1_{ali} - \sum_{l=1}^{q} \beta^1_{kl}\gamma^1_{ial} \\
& + \sum_{l=1}^{q} \beta^2_{kl}\gamma^0_{ila} + \sum_{d=q+1}^{n} \beta^1_{kd}\gamma^1_{ida} + \sum_{d=q+1}^{n} \beta^1_{kd}\gamma^1_{adi} \\
& - \sum_{d=q+1}^{n} \beta^2_{kd}\gamma^0_{ida} - \sum_{d=q+1}^{n} \beta^2_{kd}\gamma^0_{iad}\}(m) + O(r^3),
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^k_{ab}(p) = {} & \frac{1}{2}\{\gamma^0_{akb} + \gamma^0_{bka}\}(m) + \frac{r}{2}\{\gamma^1_{akb} \\
& \gamma^1_{bka} + \sum_{l=1}^{q} \beta^1_{kl}\gamma^0_{alb} + \sum_{l=1}^{q} \beta^1_{kl}\gamma^0_{bla}\}(m) \\
& + \frac{r^2}{2}\{\gamma^2_{akb} + \gamma^2_{bka} - \gamma^2_{abl} + \sum_{l=1}^{q} \beta^1_{kl}\gamma^1_{alb} \\
& + \sum_{l=1}^{q} \beta^1_{kl}\gamma^1_{bla} + \sum_{l=1}^{q} \beta^2_{kl}\gamma^0_{alb} + \sum_{l=1}^{q} \beta^2_{kl}\gamma^0_{bla} \\
& + \sum_{d=q+1}^{n} \beta^1_{kd}\gamma^1_{adb} + \sum_{d=q+1}^{n} \beta^1_{kd}\gamma^1_{bda} \\
& - \sum_{d=q+1}^{n} \beta^1_{kd}\gamma^1_{abd}\}(m) + O(r^3),
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^c_{ij}(p) = & -\frac{1}{2}\gamma^0_{ijc}(m) + \frac{r}{2}\{\gamma^1_{icj} + \gamma^1_{jci} \\
& - \gamma^1_{ijc}\}(m) + \frac{r^2}{2}\{\sum_{l=1}^{q} \beta^1_{cl}\gamma^1_{ilj} + \sum_{l=1}^{q} \beta^1_{cl}\gamma^1_{jli} \\
& - \sum_{l=1}^{q} \beta^1_{cl}\gamma^1_{ijl} + \gamma^2_{icj} + \gamma^2_{jci} - \gamma^2_{ijc} \\
& - \sum_{d=q+1}^{n} \beta^2_{cd}\gamma^0_{ijd}\}(m) \\
& + \frac{r^3}{2}\{\gamma^3_{icj} + \gamma^3_{jci} - \gamma^3_{ijc} + \sum_{d=q+1}^{n} \beta^2_{cd}\gamma^1_{idj} \\
& + \sum_{d=q+1}^{n} \beta^2_{cd}\gamma^1_{jdi} - \sum_{d=q+1}^{n} \beta^2_{cd}\gamma^1_{ijd} - \sum_{d=q+1}^{n} \beta^3_{cd}\gamma^0_{ijd} \\
& + \sum_{l=1}^{q} \beta^2_{cl}\gamma^1_{ilj} + \sum_{l=1}^{q} \beta^2_{cl}\gamma^1_{jli} - \sum_{l=1}^{q} \beta^2_{cl}\gamma^1_{ijl} \\
& + \sum_{l=1}^{q} \beta^1_{cl}\gamma^2_{ilj} + \sum_{l=1}^{q} \beta^1_{cl}\gamma^1_{jli} - \sum_{l=1}^{q} \beta^1_{cl}\gamma^2_{ijl}\}(m) + O(r^4),
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^c_{ia}(p) = & \frac{1}{2}\{\gamma^0_{ica} - \gamma^0_{iac}\}(m) \\
& + \frac{r}{2}\{\sum_{l=1}^{q} \beta^1_{cl}\gamma^0_{ila} + \gamma^1_{ica} - \gamma^1_{iac}\}(m) \\
& + \frac{r^2}{2}\{\sum_{l=1}^{q} \beta^1_{cl}\gamma^1_{ila} + \sum_{l=1}^{q} \beta^1_{cl}\gamma^1_{ali} - \sum_{l=1}^{q} \beta^1_{cl}\gamma^1_{ial} \\
& + \sum_{l=1}^{q} \beta^2_{cl}\gamma^0_{ila} + \gamma^2_{ica} + \gamma^2_{aci} \\
& - \gamma^2_{iac} + \sum_{d=q+1}^{n} \beta^2_{cd}\gamma^0_{ida}\}(m) + O(r^3),
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^c_{ab}(p) = \frac{r}{2}\{&\sum_{l=1}^{q}\beta^1_{cl}\gamma^0_{alb} + \sum_{l=1}^{q}\beta^1_{cl}\gamma^0_{bla} \\
&+ \gamma^1_{acb} + \gamma^1_{bca} - \gamma^1_{abc}\}(m) \\
&+ \frac{r^2}{2}\{\sum_{l=1}^{q}\beta^1_{cl}\gamma^1_{alb} + \sum_{l=1}^{q}\beta^1_{cl}\gamma^1_{bla} + \sum_{l=1}^{q}\beta^2_{cl}\gamma^0_{alb} \\
&+ \sum_{l=1}^{q}\beta^2_{cl}\gamma^0_{bla} + \gamma^2_{acb} + \gamma^2_{bca} - \gamma^2_{abc}\}(m) \\
&+ \frac{r^3}{2}\{\sum_{l=1}^{q}\beta^1_{cl}\gamma^2_{alb} + \sum_{l=1}^{q}\beta^1_{cl}\gamma^2_{bla} - \sum_{l=1}^{q}\beta^1_{cl}\gamma^2_{abl} \\
&+ \sum_{l=1}^{q}\beta^2_{cl}\gamma^1_{alb} + \sum_{l=1}^{q}\beta^2_{cl}\gamma^1_{bla} + \sum_{l=1}^{q}\beta^3_{cl}\gamma^0_{alb} \\
&+ \sum_{l=1}^{q}\beta^3_{cl}\gamma^0_{bla} + \gamma^3_{acb} + \gamma^3_{bca} \\
&- \gamma^3_{abc} + \sum_{d=q+1}^{n}\beta^2_{cd}\gamma^1_{adb} + \sum_{d=q+1}^{n}\beta^2_{cd}\gamma^1_{bda} \\
&- \sum_{d=q+1}^{n}\beta^2_{cd}\gamma^1_{abd}\}(m) + O(r^4).
\end{aligned}
$$

Therefore, using (4.1), (4.2), the expressions for g^{AB} and for the Christoffel symbols we are able to compute the coefficients A^c_k and A^i_k. We delete the lengthy but straightforward computations.

PROOF OF PROPOSITION 4.6. First, from $A^c_0 = 0, c = q+1, \ldots, n$ we get

$$
S\sum_{i=1}^{q} T_{E_i}E_i = \sum_{i=1}^{q} T_{E_i}E_i.
$$

This is equivalent to $SH = H$, where H is the mean curvature vector. So, if s_B is a free rotation, we get $H = 0$ and hence B is a minimal submanifold.

Next, suppose that B is totally geodesic and that $\nabla^\perp$ is flat. Then the frame field $\{E_{q+1},\ldots,E_n\}$ can be taken parallel with respect to the normal connection in a neighborhood of $m \in B$ (cf. Section 1.3). Under these hypotheses, the conditions $A_i^c = 0$ yield

$$\sum_{i=1}^{q} g((\nabla^{\perp^2}_{E_iE_i}S)u, SE_c) - \sum_{i=1}^{q}(R_{uici} - R_{SuiSci}) - \frac{2}{3}\sum_{a=q+1}^{n}(R_{caua} - R_{ScSaSuSa}) = 0, \tag{4.3}$$

where as before $R_{ABCD} = R_{E_AE_BE_CE_D}$, $A,B,C,D = 1,\ldots,n$. Now, for a normal unit vector field u we have $g((\nabla^\perp_{E_i}S)u, Su) = 0$ and then, differentiating again and using the identity ${}^tSS = I$, we get

$$g((\nabla^{\perp^2}_{E_iE_i}S)u, Su) = -g((\nabla^\perp_{E_i}S)u, (\nabla^\perp_{E_i}S)u).$$

Next, we substitute this in (4.3), replace u by E_c and sum with respect to $c = q+1,\ldots,n$. This gives

$$\sum_{i=1}^{q}\sum_{c=q+1}^{n} g((\nabla^\perp_{E_i}S)E_c, (\nabla^\perp_{E_i}S)E_c) + \sum_{i=1}^{q}\sum_{c=q+1}^{n}(R_{cici} - R_{SciSci}) + \frac{2}{3}\sum_{a,c=q+1}^{n}(R_{caca} - R_{ScSaScSa}) = 0,$$

and therefore

$$\sum_{i=1}^{q}\sum_{c=q+1}^{n} g((\nabla^\perp_{E_i}S)E_c, (\nabla^\perp_{E_i}S)E_c) = 0.$$

As $\nabla^\perp : \mathfrak{X}(B)\times\mathfrak{X}^\perp(B) \to \mathfrak{X}^\perp(B)$ is $\mathcal{F}(B)$-linear, we obtain $\nabla^\perp S = 0$ along B. So, since B is totally geodesic, $\nabla S = 0$ along B.

Proof of Proposition 4.7. Since any isometry is harmonic we only have to prove the converse. So, let s_B be a harmonic rotation around B. According to Proposition 4.1 and Proposition 4.6 we have to prove that

$$R_{iuju} = R_{iSujSu}, \qquad R_{iuau} = R_{iSuSaSu}, \qquad R_{aubu} = R_{SaSuSbSu}$$

for $i, j = 1, \dots, q; a, b = q + 1, \dots, n$.

First, since $\nabla S = 0$ and since the Ricci tensor is S-invariant, (4.3) yields, by taking $E_c = u$,

$$\sum_{i=1}^{q} (R_{uiui} - R_{SuiSui}) = 0 \tag{4.4}$$

for all $u \in \nu(B)_m$.

Next, we consider the conditions $A_3^c = 0, c = q + 1, \dots, n$. Then, after a lengthy computation, we obtain

$$\begin{aligned} &30 \sum_{i,j=1}^{q} (R_{uiuj}^2 - R_{SuiSuj}^2) - 45 \sum_{i,j=1}^{q} R_{uiuj}(R_{uiuj} - R_{SuiSuj}) \\ &- 60 \sum_{i=1}^{q} \sum_{a=q+1}^{n} R_{uiua}(R_{uiua} - R_{SuiSuSa}) \\ &+ 36 \sum_{i=1}^{q} \sum_{a=q+1}^{n} (R_{iuau}^2 - R_{iSuSaSu}^2) \\ &+ 6 \sum_{a,b=q+1}^{n} (R_{uaub}^2 - R_{SuaSub}^2) \\ &- 10 \sum_{a,b=q+1}^{n} R_{uaub}(R_{uaub} - R_{SuSaSuSb}) = 0. \end{aligned} \tag{4.5}$$

The left hand side of (4.5) may be considered as a function on the unit sphere $S^{n-q-1}(1)$ in $\nu(B)_m$. We shall integrate this function over this sphere. To this purpose we recall the following auxiliary lemma (cf. [CV1],[Gr7],[GV1],[GV2]).

LEMMA 4.10. *Let* $u = \sum_{a=q+1}^{n} u_a E_a$ *be an orthonormal decomposition of the unit vector* $u \in \nu(B)_m$ *with respect to an orthonormal basis* $\{E_{q+1}, \dots, E_n\}$ *of*

$\nu(B)_m$. *Then we have*

$$\int_{S^{n-q-1}(1)} u_a d\mu = 0, \qquad \int_{S^{n-q-1}(1)} u_a u_b u_c d\mu = 0$$

$$\int_{S^{n-q-1}(1)} u_a u_b d\mu = c_{n-q-1}\frac{1}{n-1}\delta_{ab},$$

$$\int_{S^{n-q-1}(1)} u_a^2 u_b^2 d\mu = \frac{1}{3}\int_{S^{n-q-1}(1)} u_a^4 d\mu = c_{n-q-1}\frac{1}{(n-q)(n-q+2)}, \quad (a \neq b),$$

$$\int_{S^{n-q-1}(1)} u_a u_b u_c u_d d\mu = 0 \text{ whenever at least three indices are different,}$$

$$\int_{S^{n-q-1}(1)} u_a^3 u_b d\mu = 0, \qquad a \neq b,$$

where $a, b, c, d = q+1, \ldots, n$; $d\mu$ *denotes the volume element of* $S^{n-q-1}(1)$ *and*

$$c_{n-q-1} = \frac{(n-q)\pi^{\frac{n-q}{2}}}{(\frac{n-q}{2})!}$$

is the volume of a unit sphere in the Euclidean space $\mathbb{E}^{n-q}$.

As a consequence of this lemma one gets at once that the integrals of

$$\sum_{i,j=1}^{q}(R_{uiuj}^2 - R_{SuiSuj}^2), \qquad \sum_{i=1}^{q}\sum_{a=q+1}^{n}(R_{iuau}^2 - R_{iSuSaSu}^2)$$

and

$$\sum_{a,b=q+1}^{n}(R_{uaub}^2 - R_{SuaSub}^2)$$

vanish. Next, put

$$A = \sum_{i,j=1}^{q} R_{uiuj}(R_{uiuj} - R_{SuiSuj}),$$

$$B = \sum_{i=1}^{q} \sum_{a=q+1}^{n} R_{uiua}(R_{uiua} - R_{SuiSuSa}),$$

$$C = \sum_{a,b=q+1}^{n} R_{uaub}(R_{uaub} - R_{SuSaSuSb}).$$

Then, by integration, we get first

$$\int_{S^{n-q-1}(1)} A d\mu = \frac{c_{n-q-1}}{(n-q)(n-q+2)} \sum_{i,j=1}^{q} \sum_{a,b=q+1}^{n} \{R_{aibj}R_{aibj} + R_{aibj}R_{biaj} - R_{aibj}R_{SaiSbj} - R_{aibj}R_{SbiSaj}\}.$$

Now, we use the first Bianchi identity and the Ricci equation for the submanifold B (cf. (1.28)) in order to obtain

$$\int_{S^{n-q-1}(1)} A d\mu = \frac{c_{n-q-1}}{(n-q)(n-q+2)} \sum_{i,j=1}^{q} \sum_{a,b=q+1}^{n} (R_{aibj} - R_{SaiSbj})^2.$$

Next, we have

$$\int_{S^{n-q-1}(1)} B d\mu = \frac{c_{n-q-1}}{(n-q)(n-q+2)} \sum_{i=1}^{q} \sum_{a,b,c=q+1}^{n} \{R_{biba}R_{cica} + R_{bica}R_{bica} + R_{bica}R_{ciba} - R_{biba}R_{SciScSa} - R_{bica}R_{SbiScSa} - R_{bica}R_{SciSbSa}\}.$$

Observe that

$$\sum_{c=q+1}^{n} (R_{biba}R_{cica} - R_{biba}R_{SciScSa}) = R_{biba}(\rho_{ia} - \sum_{j=1}^{q} R_{jija} - \rho_{iSa} + \sum_{j=1}^{q} R_{jijSa})$$
$$= \sum_{j=1}^{q} R_{biba}(R_{jijSa} - R_{jija})$$

since ρ is S-invariant. As the submanifold is totally geodesic, $R_{E_j E_i} E_j$ is tangent to B and then the last expression vanishes. Furthermore, because of the curvature identities

$$\sum_{a,b,c} R_{ibca} R_{icba} = \frac{1}{2} \sum_{a,b,c} R_{ibca} R_{ibca},$$

$$\sum_{a,b,c} R_{ibca} R_{iScSbSa} = \frac{1}{2} \sum_{a,b,c} R_{ibca} R_{iSbScSa},$$

we eventually obtain

$$\int_{S^{n-q-1}(1)} B d\mu = \frac{3c_{n-q-1}}{2(n-q)(n-q+2)} \sum_{i=1}^{q} \sum_{a,b,c=q+1}^{n} R_{ibca}(R_{ibca} - R_{iSbScSa}).$$

For the integral of C we get

$$\int_{S^{n-q-1}(1)} C d\mu = \frac{c_{n-q-1}}{(n-q)(n-q+2)} \sum_{a,b,\alpha,\beta=q+1}^{n} \{R_{\alpha a \alpha b} R_{\beta a \beta b} + R_{\alpha a \beta b} R_{\alpha a \beta b}$$
$$+ R_{\alpha a \beta b} R_{\beta a \alpha b} - R_{\alpha a \alpha b} R_{S\beta S a S\beta S b} - R_{\alpha a \beta b} R_{S\alpha S a S\beta S b} - R_{\alpha a \beta b} R_{S\beta S a S\alpha S b}\}.$$

Proceeding in the same way as above we obtain

$$\int_{S^{n-q-1}(1)} C d\mu = \frac{c_{n-q-1}}{(n-q)(n-q+2)} \sum_{a,b,\alpha,\beta=q+1}^{n} \{R_{\alpha a \alpha b} R_{\beta a \beta b} - R_{\alpha a \alpha b} R_{S\beta S a S\beta S b}$$
$$+ \frac{3}{2} R_{\alpha a \beta b}(R_{\alpha a \beta b} - R_{S\alpha S a S\beta S b})\}.$$

Now, the S-invariance of ρ yields

$$\sum_{a,b,\alpha,\beta=q+1}^{n} R_{\alpha a \alpha b}(R_{\beta a \beta b} - R_{S\beta S a S\beta S b}) = -\sum_{i=1}^{q} \sum_{a,b,\alpha=q+1}^{n} R_{\alpha a \alpha b}(R_{iaib} - R_{iSaiSb})$$

and this vanishes because of (4.4). So, we get

$$\int_{S^{n-q-1}(1)} C d\mu = \frac{3c_{n-q-1}}{2(n-q)(n-q+2)} \sum_{a,b,\alpha,\beta=q+1}^{n} R_{\alpha a\beta b}(R_{\alpha a\beta b} - R_{S\alpha Sa S\beta Sb})$$
$$= \frac{3c_{n-q-1}}{4(n-q)(n-q+2)} \sum_{a,b,\alpha,\beta=q+1}^{n} (R_{\alpha a\beta b} - R_{S\alpha Sa S\beta Sb})^2.$$

From all these computations, (4.5) integrated over $S^{n-q-1}(1)$ takes the form

$$6\sum_{i,j=1}^{q}\sum_{a,b=q+1}^{n}(R_{aibj} - R_{SaiSbj})^2 + 6\sum_{i=1}^{q}\sum_{a,b,c=q+1}^{n}(R_{iabc} - R_{iSaSbSc})^2$$
$$+ \sum_{a,b,\alpha,\beta=q+1}^{n}(R_{\alpha a\beta b} - R_{S\alpha Sa S\beta Sb})^2 = 0,$$

and this is equivalent to

$$R_{aibj} = R_{SaiSbj}, \qquad R_{iabc} = R_{iSaSbSc}, \qquad R_{\alpha a\beta b} = R_{S\alpha Sa S\beta Sb}$$

for $i, j = 1, \ldots, q; a, b, c, \alpha, \beta = q+1, \ldots, n$, from which the required result follows.

4.3 Holomorphic and symplectic rotations and related questions

In this section we consider holomorphic and symplectic free rotations around a submanifold B of an almost Hermitian manifold (M, g, J). We shall put emphasis on the influence that such rotations have on the extrinsic geometry of the submanifold.

This study was initiated by B.Y. Chen and L. Vanhecke for the case of reflections in [CV2] and continued in [CV3].

First of all, note that if a free S-rotation s_B is either holomorphic or symplectic, then we obtain the condition

$$SJ = JS,$$

and this implies at once that the submanifold is holomorphic. Analogously, if s_B is anti-holomorphic, then the submanifold B is totally real.

The consequences on the second fundamental form of B induced by the holomorphic and symplectic properties of the reflection with respect to B are summarized in the following ([CV2])

PROPOSITION 4.11. *Let (M, g, J) be an almost Hermitian manifold and B a submanifold. If the reflection with respect to B is holomorphic (respectively, symplectic) then*

(1) *B is a holomorphic submanifold;*
(2) *the second fundamental form T of B in M satisfies $T(X,Y) - T(JX, JY) = 0$ (respectively, $T(X,Y) + T(JX, JY) = 0$);*
(3) *$(\nabla_u J)X$ is normal to B*

for all vectors X, Y tangent to B and all u normal to B.

For a generic free rotation s_B there is an analogous result.

PROPOSITION 4.12. *Let (M, g, J) be an almost Hermitian manifold and B a submanifold. If the free S-rotation s_B is holomorphic (respectively, symplectic) then*

(1) *B is a holomorphic submanifold;*
(2) *the second fundamental form T of B in M satisfies $T(X,Y) - T(JX, JY) = 0$ (respectively, $T(X,Y) + T(JX, JY) = 0$);*
(3) *$(\nabla_u J)X$ is normal to B*

for all vectors X, Y tangent to B and all u normal to B.

The proof of this proposition is obtained in the same way as that of Proposition 4.11 by computing power series expansions of the Kähler form and the almost complex structure in terms of Fermi coordinates along the submanifold. We sketch the proof for a symplectic rotation. Let $p = \exp_m(u)$, r sufficiently small, $u \in \nu(B)_m$, $\|u\| = 1$, $m \in B$. Then s_B is symplectic if and only if

$$\Omega(ds_B(\frac{\partial}{\partial x^A}), ds_B(\frac{\partial}{\partial x^B}))(p) = \Omega(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B})(p), \qquad A, B = 1, \ldots, n.$$

In particular,

$$\Omega(ds_B(\frac{\partial}{\partial x^i}), ds_B(\frac{\partial}{\partial x^j}))(p) = \Omega(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})(p), \qquad i, j = 1, \ldots, q. \tag{4.6}$$

or, equivalently,

$$\Omega(\frac{\partial}{\partial x^i} + \sum_{\alpha=q+1}^{n} \frac{\partial(S^\alpha_\beta)}{\partial x^i} x^\beta \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^j} + \sum_{\alpha=q+1}^{n} \frac{\partial(S^\gamma_\delta)}{\partial x^j} x^\delta \frac{\partial}{\partial x^\gamma}) = \Omega(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}).$$

Using the initial conditions for $D_u(r)$ (cf. Section 1.3) we have

$$\begin{aligned}\Omega_{ij}(p) = {} & g(E_i, JE_j)(m) + r\{g(T(u)E_i, JE_j) + g(E_i, JT(u)E_j) \\ & + g(E_i, J'(u)E_j)\}(m) + O(r^2),\end{aligned}$$

$$\begin{aligned}\Omega_{ia}(p) = {} & g(E_i, JE_a)(m) + r\{g(T(u)E_i, JE_a) - g({}^t\bot(u)E_i, JE_a) \\ & + g(E_i, J'E_a)\}(m) + O(r^2),\end{aligned}$$

$$\begin{aligned}\Omega_{ab}(p) = {} & g(E_a, JE_b)(m) + rg(E_a, J'E_b)(m) \\ & + \frac{r^2}{6}\{3g(E_a, J''E_b) - g(E_a, JR_uE_b) - g(R_uE_a, JE_b)\}(m) + O(r^3),\end{aligned}$$

where $i, j = 1, \ldots, q; a, b = q+1, \ldots, n$ and $J'(u)$ stands for $\nabla_u J$.

Comparing the first order terms in (4.6) we obtain

$$\begin{aligned}& g(u, T(E_i, JE_j)) - g(u, T(JE_i, E_j)) + g(E_i, J'(u)E_j) = \\ & \quad g(Su, T(E_i, JE_j)) - g(Su, T(JE_i, E_j)) + g(E_i, J'(Su)E_j)).\end{aligned}$$

Using the fact that B is holomorphic and replacing E_i with JE_i we get

$$\begin{aligned}& g(u, T(JE_i, JE_j)) + g(u, T(E_i, E_j)) + g(JE_i, J'(u)E_j) = \\ & \quad g(u, {}^tST(JE_i, JE_j)) + g(u, {}^tST(E_i, E_j)) + g(JE_i, J'(Su)E_j)).\end{aligned}$$

Then, for all X, Y tangent to B we have

$$\begin{aligned}& g(u, T(JX, JY) + T(X, Y) - {}^tS(T(JX, JY) + T(X, Y))) = \\ & \quad - g(JX, J'(u)Y - J'(Su)Y).\end{aligned}$$

Observe now that the right-hand side of the preceding formula is skew-symmetric and the left-hand side is symmetric. This implies

$${}^tS(T(JX, JY) + T(X, Y)) = T(JX, JY) + T(X, Y)$$
$$\text{and } J'(u)Y - J'(Su)Y \text{ is normal to } B$$

and thus, since S is free,

$$T(JX,JY)+T(X,Y)=0$$

and (3).

From the relation $T(X,Y)+T(JX,JY)=0$ for the second fundamental form we have easily that a symplectic free rotation implies that the submanifold B is *minimal* .

This result can also be obtained as a consequence of the fact that any symplectic rotation preserves the volume.

A free rotation s_B is *volume-preserving* (up to sign) if and only if

$$\theta_B(\exp_m(ru) = \theta_B(\exp_m(rSu)) \tag{4.7}$$

for $u \in \nu(B)_m$, $m \in B$ and r sufficiently small, where θ_B denotes the *volume density function* of $\exp_{\nu(B)}$ defined, in a sufficiently small tubular neighborhood of B, by

$$\theta_B = (\det(g_{AB}))^{\frac{1}{2}}, \qquad A, B = 1, \ldots, n. \tag{4.8}$$

As has been proved in [VW2] for the curve case and in [KV2] in general (see also [V4]), the following important formula holds

$$\theta_B(p) = r^q \theta_m(p) \det\{T(u) + B_u(r)\},$$

where $\theta_m(p)$ denotes the volume density function of $\exp_m$ in (M,g) (cf. Section 1.2), $T(u)$ is the shape operator of B with respect to u and

$$B_u(r)_{ij} = g(T_p(m)E_i, E_j)(m), \qquad i,j = 1, \ldots, q = \dim B.$$

Here $T_p(m)$ is the shape operator at m of the geodesic sphere $G_p(r)$ with center p and radius r.

The remarkable feature of this formula is that θ_B is completely represented by quantities relating only to the extrinsic geometry of B and the geodesic spheres of the ambient space that are tangent to B.

Taking into account the power series expansions of θ_m and B_u (cf. Chapter 1, and [CV1]) we obtain

$$\theta_B(\exp_m(ru)) = 1 + r\ \mathrm{tr}T(u) + O(r^2).$$

Thus, if s_B is volume-preserving (up to sign) then

$$\mathrm{tr}T(u) = \mathrm{tr}T(Su),$$

that is S preserves the mean curvature vector of B. Since S is free we then obtain that B is a minimal submanifold.

Volume-preserving reflections with respect to submanifolds have been investigated in [ToV1] (see also [V4]) to which we refer for further details. Other problems connected with the study of reflections with respect to submanifolds in relation with the extrinsic and intrinsic geometry of tubular neighborhoods can be found in [ToV2],[ToV3].

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