Bootstrap Methods in 2D Integrable Quantum Field Theories

Thesis submitted for the degree
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Contents

1 Introduction

The bootstrap method was one of the first developments towards the description of quantized systems. It consists in a deductive scheme to the calculation of the scattering matrix, by requiring that the S–matrix elements satisfy some general properties that ought to be valid in a quantized theory. The advantage of this approach is that it is independent of whether or not a lagrangian formulation for the theory exists.

The original formulation considered general four dimensional models. The hope that the bootstrap method could provide a means for resolving or classify these theories was not fulfilled. Other methods as functional integration and the perturbation expansion took its place and became the main tools in the examination of quantum field theories.

Meanwhile one has understood that in that case bootstrap methods have been applied to models where they were unable to work efficiently. This thesis treats with two dimensional integrable field theories, and as we will explain in detail, in this case one can use the bootstrap principle to determine the exact quantum structure of a theory.

The interest in two dimensional systems is twofold. On the one hand, they can be seen as a playground of field theory, since in this case calculations are much simpler. It is in this context that one can learn how interacting field theories behave, not approximately as one can study them by perturbative methods or numerical simulations, but exactly by calculating analytic expressions for the quantum correlation-functions.

On the other hand, many of the field theories we will discuss are important as continuum limits of statistical field theories. One well known example are conformal field theories, which describe critical points of a statistical system, and predict various critical exponents and other universal finite-size scaling amplitudes. They have been resolved, that is the exact quantum correlation-functions are known (they are given as solutions of differential equations) and the operator content is classified.

Massive integrable field theories, which are the topic of this thesis, describe off-critical regions in the space of coupling constants. Unfortunately, for these theories less is known. Here is where the bootstrap method comes to work. The framework in which we will describe massive theories is the Lehmann-Symanzik-Zimmermann (LSZ) formalism. It uses asymptotic states as a basis of the particle space and an S–matrix, which connects them. Having an integrable field theory, additional constraints appear and allow to determine the exact particle spectrum and the full S–matrix, once an initial input is given. We will discuss this method in detail in part II.

The S–matrix provides only on–shell information. In order to understand the full quantum structure of a theory, one needs to determine also the off–shell properties. This is a subject of current interest. Still it is not possible to give a simple expression for the correlation functions as in the massless case. But, through the form-factor bootstrap approach, correlation-functions can be written as an infinite sum over form–factor contributions. Also it should be possible to determine the operator content of a theory. We will describe these methods and current results in part III.

Bootstrap methods give a way to understand the exact quantum structure of two dimensional integrable massive field theories. Nevertheless, they are not the only approach to reach this goal. Already for massless theories one has seen that the underlying algebraic structure was the key in the solution of the models. Nowadays many properties of conformal field theories are deduced from representation theory of the Virasoro algebra, which is the relevant symmetry algebra. For massive theories the algebraic approach becomes rather involved and is still in its beginnings. Even if many objects in the massive theory can be encoded into a (quantum) group theoretic language, it is not yet fully understood how to construct them from the principle in terms of the off–critical symmetry algebra.

In this stage of investigation it is important to have an alternative method which does not rely on the underlying symmetry. This role is played by the bootstrap method. It is based on the principles of quantum field theory rather than on representation theoretic arguments. Therefore calculations can be done explicitly and many results can be obtained without knowing the exact structure of the off-critical symmetry algebra.

In this thesis, I want to describe how the bootstrap principle works explicitly and which results can be obtained.

Part I

Basic Elements of the Bootstrap Approach

The bootstrap approach is supposed to be a method to determine the exact quantum structure of integrable massive two dimensional field theories. In order to discuss to which extend it can provide a means for such an ambitious goal, we need to introduce some fundamental concepts. This will be done in this part of the thesis.

First we will discuss two dimensional integrable models in general, review some results on conformal field theory, describe perturbations of them, and give a realization in terms of affine Toda field theories. Then we will concentrate on massive theories and introduce some fundamental properties of the bootstrap method. Since the subject is rather vast, we will analyze only simple systems in order not to obscure the physical concepts with technical complications. The techniques to treat with more complicated systems are introduced later on when needed.

2 Integrable Massive Two Dimensional Field Theories

The most well known examples of integrable field theories are probably the conformal models in two dimensions (see for example [22, 41, 56, 57]). Since they are also the best explored, we want to discuss some of their basic properties. They provide an example of a theory where the full quantum structure has been determined. In the following section we discuss how this has been achieved.

2.1 Conformal field theory

Conformal field theories (CFT) describe the critical points corresponding to a second order phase transition. This implies scaling invariance, and since we restrict ourselves to two dimensional systems, also conformal invariance.

Conformal transformations in general are defined as coordinate transformations such that the angle between intersecting curves at the intersection point is preserved. In two dimensions this means that $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$ with

$$\partial_{\bar{z}} f(z) = 0 \ , \qquad \partial_z \bar{f}(\bar{z}) = 0 \ .$$

In fact it is the group of analytic coordinate transformations. Expand

$$f(z) = z + \sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1} \quad , \quad \bar{f}(\bar{z}) = \bar{z} + \sum_{n=-\infty}^{\infty} \bar{\epsilon}_n \bar{z}^{n+1} \quad .$$

These transformations are generated by

$$l_n = z^{n+1}\partial_z \quad \text{and} \quad \bar{l}_n = \bar{z}^{n+1}\partial_{\bar{z}} \quad ,$$

which satisfy the classical commutation relations

$$\begin{gathered} [l_m, l_n] = (n-m) l_{m+n} \ , \ \left[\bar{l}_m, \bar{l}_n\right] = (n-m)\bar{l}_{m+n} \ \text{and} \\ \left[\bar{l}_m, l_n\right] = 0 \ . \end{gathered} \tag{2.1}$$

Quantizing the theory this symmetry algebra turns into a couple of Virasoro algebras,

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad , \\ \left[\bar{L}_m, \bar{L}_n\right] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad , \\ \left[L_m, \bar{L}_n\right] &= 0 \quad . \end{aligned} \tag{2.2}$$

The generators L_n and $\bar{L}_n$ are the Laurant coefficients of the stress-energy tensor,

$$T_{zz} \equiv T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \ , \qquad T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{z^{n+2}} \quad ,$$

$$T_{z\bar{z}} = \frac{1}{4}\Theta = 0 \tag{2.3}$$

The anomaly c in (2.2) appears also in the operator product expansion (OPE) of the stress energy tensor, namely

$$T(z_1)T(z_2) = \frac{c}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{(z_1 - z_2)} + \dots \tag{2.4}$$

For a certain range of the central charge, $0 < c < 1$, the theories can be classified. They are called *minimal models* and are determined by [12, 52, 60]

$$c = 1 - \frac{6(p-q)^2}{pq} \quad , \quad p,q \text{ positive relative prime integers} \tag{2.5}$$

$$h_{r,s} = \frac{(sp-rq)^2-(p-q)^2}{4pq} \quad , \quad 1 \leq s \leq q-1 \ , \ 1 \leq r \leq p-1 \quad . \tag{2.6}$$

Every minimal model is defined by a central charge labeled by two co-prime integers p, q, and denoted as $\mathcal{M}_{p,q}$. The operator content of each theory is specified by the respective $h_{r,s}$, being the conformal dimensions of the so–called *primary fields*. Equation (2.6) shows that minimal models contain only a finite number of primary fields. All other fields of the theory are contained in a Verma module structure. They are obtained by applying $L_{-k_1}L_{-k_2}\dots L_{-k_n}$ to a primary field Φ, and are called *descendent fields.*

A reduction of the space of states is obtained through the so-called *null-states* [12, 11]. They are specific linear combinations of descendent operators, which have zero norm. Therefore they can be eliminated from the space of states. These null-relations can be used to determine the quantum correlation functions. One can define an operator $\mathcal{L}_n$ to give the action of L_n on the correlator of primary fields as

$$\langle \Phi_1 \dots \Phi_n L_{-k} \Phi \rangle = \mathcal{L}_{-k} \langle \Phi_1 \dots \Phi_n \Phi \rangle \quad . \tag{2.7}$$

Through the Ward identity these can be determined in terms of differential operators. In this way a null relation turns into a differential equation for the correlator, which in principle can be solved to compute all correlators of these theories. Explicit expressions have been obtained through the Coulomb gas approach [42, 41], which defines the primary fields in terms of vertex operators of a bosonic model with a charge at infinity.

We pause here to emphasize the physical importance of (2.5) and (2.6). The representation theory of the Virasoro algebra in principle allows us to describe the possible scaling dimensions of fields of two dimensional conformal field theories, and thereby the possible critical indices of two dimensional systems at their second order phase transition [22, 121]. In the case of systems with $0 < c < 1$ this has turned out to give a complete classification of possible two dimensional critical behaviour. For example the two point function of primary fields can be determined as

$$\langle 0|\Phi_h(z)\Phi_{h'}(0)|0\rangle = \frac{C_{h,h'}}{z^{2h+2h'}}\delta_{h,h'} \quad , \tag{2.8}$$

where $C_{h,h'}$ is a normalization constant. In confrontation consider now the Ising model. The correlator of the order parameter σ has at criticality the behaviour

$$\langle \sigma_r \sigma_0 \rangle \sim \frac{1}{|r|^{\frac{1}{4}}} \quad .$$

Comparing with (2.8) one can assign the values $h_\sigma = \bar{h}_\sigma = \frac{1}{16}$ to this field [22]. But this in the above classification (2.6) corresponds to the field with weight $h_{1,2}$ of the CFT $\mathcal{M}_{3,4}$. Also the other operators of the Ising model can be brought in correspondence to those of the theory $\mathcal{M}_{3,4}$. This renders the identification of the critical Ising model with the model $\mathcal{M}_{3,4}$ complete. Similar other systems have been identified , e.g. the models $\mathcal{M}_{4,5}$, $\mathcal{M}_{5,6}$, $\mathcal{M}_{6,7}$ with $c = \frac{7}{11}, \frac{4}{5}, \frac{6}{7}$ were identified with the tricritical Ising model, 3-state Potts model and tricritical 3-state Potts model respectively. Actually all minimal models have been found to be critical points of statistical models [3, 59].

The above examples have a peculiarity: they are all of the type $\mathcal{M}_{p,p+1}$. The reason for that is the following: In order to select our models we did not include unitarity as a constraint on our theory. Unitarity restricts c and h to be positive, and only models of the type $\mathcal{M}_{p,p+1}$ survive [22, 56]. The problematic feature of a non unitary theory is that it contains correlators which diverge for large distances (compare (2.8)), since the theory contains at least one weight h smaller than zero. These models are of interest in statistical mechanics. An example is provided by the theory $\mathcal{M}_{2,5}$, which describes [23] the Yang–Lee singularity, *i.e.* the critical behaviour of an antiferromagnetic system in an imaginary magnetic field.

The CFT posses an infinite number of degrees of freedom and because they are integrable we expect also an infinite number of conserved quantities. A first one is provided by the stress–energy tensor $T(z)$, since $\partial_{\bar{z}}T(z) = 0$ because of conformal invariance. It is a descendant operator and lies in the Verma module of the identity $T(z) = L_{-2}I$. Actually an infinite number of conserved currents can be constructed out of the descendents of the module of the identity operator. The difficulty lies in the problem to select a linear independent set, eliminating derivative operators, which are obtained applying $L_{-1} \sim \partial_z$. The first independent conserved quantities are given in table 2.1.

spins	0	1	2	3	4	5	6
basis vectors	I	$\emptyset$	$T_2 = L_{-2}I$	$\emptyset$	$T_4 = L_{-2}^2 I$	$\emptyset$	$T_6^{(1)} = L_{-2}^3 I$ $T_6^{(2)} = L_{-3}^2 I$

Table 2.1: The first linear independent conserved quantities in CFT

For conformal systems defined on a torus there exists a general formula to calculate the degeneracy structure of the descendent fields in a Verma module corresponding to a primary field at every level k. Defining the theory on a torus corresponds to taking periodic boundary conditions in both directions of the lattice in a statistical system. On the torus the partition function takes a simple form [22]

$$Z(q, \bar{q}) = q^{-\frac{c}{24}} \bar{q}^{-\frac{c}{24}} tr\left(q^{L_0} \bar{q}^{L_0}\right) \ , \tag{2.9}$$

where q is a parameter depending on the geometry of the torus. This can be decomposed as

$$Z(q, \bar{q}) = \sum_{h, \bar{h}} N_{h, \bar{h}} \chi_h(q) \chi_{\bar{h}}(\bar{q}) \ , \tag{2.10}$$

where one defines the character χ_h as

$$\chi_h(q) \equiv q^{-\frac{c}{24}} tr_{\,h} q^{L_0} = q^{-\frac{c}{24}+h} \sum_{n=0}^{\infty} d_h(n) q^n \ , \tag{2.11}$$

$d_h(n)$ is the degeneracy of states in a representation at level n. The number of *independent* states at this level n is given by $d_0(n) - d_0(n-1)$. The quantities $N_{h,\bar{h}}$ are fixed by modular invariance. This means that one requires the partition function to be invariant under modular transformations, which characterize special base changes on the torus. For a given central charge there can exist more than one theory satisfying these requirements. They are classified in the ADE series [21].

If there were no null states the degeneracy $d_h(n)$ in (2.11) would be given by $d_h(n) = P(n) = \prod_{k=1}^{\infty}(1 - n^k)$, the number of partitions of the integer n, because all descendent states would be independent. For the minimal models one needs to eliminate all the null-states and the sub-modules of descendents they create. The expression for the character expansion in this case is given [98] by

$$\chi_{r,s}(c) = q^{-\frac{c}{24}} \frac{1}{P(q)} \sum_{k=-\infty}^{\infty} \left[q^{[2kpp'+rp-sp']^2-(p-p')^2]/4pp'} - (s \leftrightarrow -s) \right] \quad , \tag{2.12}$$

corresponding to the operator $h_{r,s}$ of the model $\mathcal{M}_{p,p'}$.

2.2 Extended conformal symmetries

The minimal models above are not the only conformal field theories with a reduced space of states, which allow an explicit calculation of correlation functions and exhibit a Verma module structure. There exist extensions of the Virasoro algebra providing an enlarged symmetry algebra, as for example supersymmetry, current algebra or W-symmetry. We want to discuss here only the W-algebras, because of their interrelation with Toda field theories (see section 2.4).

W-algebras are non-linear extensions of the Virasoro algebra, related to some finite Lie algebra (see e.g. [15, 14, 49]). We will consider only the $SL(n)$ models, which in the following will be called W_n–algebras. The algebra is easiest defined through the OPE of the fields $T(z)$ and $W^{(k)}$ with $k = 3, \ldots, n$. They are given by (2.4) and

$$\begin{aligned} T(z)W^{(k)}(z') &= \frac{kW^{(k)}(z')}{(z-z')^2} + \frac{\partial W^{(k)}(z')}{z-z'} \quad , \\ W^{(k)}(z)W^{(l)}(z') &= \sum_{r=0}^{k+l-1} : \frac{\mathcal{P}(W^{(r)}(z'))}{(z-z')^{k+l-r}} : \quad , \end{aligned} \tag{2.13}$$

where $: \ldots :$ denotes normal ordering and $\mathcal{P}(W^{(r)})$ is a differential polynomial of $W^{(r)}$. As the pure Virasoro algebra, these models allow a reduction of the space of states in the range $0 < c < n-1$, and the corresponding central charge and dimensions of the primary fields are given by [14]

$$\begin{aligned} c &= (n-1)\left(1 - \frac{n(n+1)(p-q)^2}{p.q}\right) \quad p, q > n-1 \quad , \\ h_{\mathrm{rs}} &= \frac{1}{2pq}\left(\sum_{i=1}^{n-1}(pr_i - qs_i)\vec{\omega}_i\right)^2 - \frac{n(n^2-1)(p-q)^2}{24pq} \quad . \end{aligned} \tag{2.14}$$

Herein, $\vec{\omega}_i$, $i = 1, \ldots n-1$ are the fundamental weights of $SL(n)$, normalized as $\vec{\omega}_i.\vec{\omega}_j = \frac{i}{n}(n-j)$ for $i < j$. The limits on the labels r_i and s_i are that $\sum r_i < q$ and $\sum s_i < p$. As in the pure conformal case (which corresponds to W_2), unitary theories are obtained for

$q = p+1$. W–algebras are related to the ultraviolet limit of the affine Toda field theories, to be discussed in section 2.4.

2.3 Perturbations of conformal field theories

One way to define an off-critical theory, is to perturb a conformal field theory by some relevant field $\Phi(x)$,

$$H_p = H_{CFT} + \lambda \int \Phi(x) d^2x \quad . \tag{2.15}$$

For certain fields $\Phi(x)$ the integrability of the CFT can be preserved under this perturbation. Nevertheless, the quantities T_s^α (see table 2.1) which defined an infinite set of conserved quantities in CFT, cannot serve any more as integrals of motion. This because their existence relied on conformal invariance which is destroyed by the perturbation. As a consequence, the energy momentum tensor is not any more traceless. At lowest order we have

$$\partial_{\bar{z}} T = [H_p, T(z)] \quad , \quad H_p = \lambda \int \Phi(x) d^2x \quad . \tag{2.16}$$

But since the field Φ is a well–defined object in CFT, it is possible to evaluate the commutator in (2.16) [45, 115], giving

$$\partial_{\bar{z}} T = \partial_z(\lambda(1-h)\Phi) \quad . \tag{2.17}$$

This can be symbolically written as $L_{-2}I = L_{-1}\Phi$. The same technique can be applied in order to construct a deformed current based on $T_4 = L_{-2}^2 I$. By dimensional analysis one finds that $\partial_{\bar{z}} L_{-2}^2 I$ must be an element of $\mathbb{P}_3$, where we define $\mathbb{P}_s$ as the space of descendents of Φ at level s. That is, in general we obtain that

$$\partial_{\bar{z}} L_{-2}^2 I = (A L_{-1}^3 + B L_{-2} L_{-1} + C L_{-3})\Phi \tag{2.18}$$

with some coefficients A, B, C. In order to obtain a conservation law, we want the right hand side to be proportional to L_{-1}, that is of derivative form. Assume there exists a null state at level 3. Then L_{-3} can be expressed in terms of the other two elements, and we have obtained a conservation law at this level.

This analysis can be generalized to higher levels, but soon becomes rather involved. In [115] a method was developed which proves the existence of a conservation law, without determining its precise form. It is called the *counting argument.* For, let $\mathbb{L}_s$ be the space of descendents and $\widehat{\mathbb{L}}_s$ be the space of *independent* descendents of I at level s. Then by dimensional counting $\partial_z \mathbb{L}_{s+1}$ must take values in the space $\mathbb{P}_s$ (as before in the example). $\mathbb{P}_s$ and $\widehat{\mathbb{P}}_s$ are constructed analogously to $\mathbb{L}_s$ and $\widehat{\mathbb{L}}_s$ but using the primary field Φ instead of I. If the dimension of $\widehat{\mathbb{L}}_{s+1}$ is larger than that of $\widehat{\mathbb{P}}_s$, one can find a quantity which maps onto $L_{-1}\Phi_{s-1}$ and therefore defines a conserved current. The respective dimensions of the modules $\widehat{\mathbb{P}}_s$ and $\widehat{\mathbb{L}}_{s+1}$ can be computed using the character expansion (2.12). For example for the perturbations of the type $\Phi_{1,3}$ one obtains

s	1	2	3	4	5	6	7
$dim\ (\widehat{\mathbb{L}}_{s+1})$	1	0	1	0	2	0	3
$dim\ (\widehat{\mathbb{P}}_s)$	0	1	0	2	1	3	2

(2.19)

Using the above argumentation one reads off the existence of conserved charges at level $s = 1, 3, 5, 7, \ldots$.

Doing a similar analysis for other operators $\Phi_{r,s}$ one finds that only *three* of them are selected to have conserved quantities for any minimal model: $\Phi_{1,3}$, $\Phi_{1,2}$ and $\Phi_{2,1}$. For the series $\Phi_{1,2}$ for example, one finds the conserved spins $s = 1, 5, 7, 11 \ldots$.

2.4 Toda field theory

In order to get a lagrangian description of perturbations of CFT, one uses a realization in terms of Toda field theory. Let's analyze the lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{\beta^2}\sum_{j=1}^{r} e^{\beta\alpha_j.\phi} \quad . \tag{2.20}$$

This action is built intrinsically over a Lie algebra, using the following notation: Let r be the rank of a finite Lie algebra $\mathcal{G}$ and let ϕ be an r–component scalar field. The exponential interaction is determined by the α_i, which are defined to be the positive simple roots of $\mathcal{G}$. The equations of motion in light cone coordinates are

$$\partial_+\partial_-\phi_j = -\frac{1}{\beta}\sum_{i=1}^{r}\alpha_i e^{\beta\alpha_i.\phi} \quad ,$$

or redefining $\varphi = \alpha_i\phi$

$$\partial_+\partial_-\varphi_j = -\frac{1}{\beta}\sum_{i=1}^{r} a_{ij} e^{\beta\varphi_j} \quad , \tag{2.21}$$

a_{ij} being the Cartan matrix of $\mathcal{G}$, $a_{ij} = \frac{\langle \alpha_i . \alpha_j \rangle}{\langle \alpha_j . \alpha_j \rangle}$. The energy momentum tensor in light cone coordinates takes the form

$$T_{\pm\pm} = -\frac{1}{2}\partial_\pm \phi \partial_\pm \phi + \frac{1}{\beta}\rho \partial_\pm^2 \phi \ , \qquad T_{\pm,\mp} = 0 \ .$$

Herein $\rho = \sum_i \lambda_i$, where the λ_i denote the fundamental weights of the algebra. Since it is traceless, we have a classically conformal invariant theory. This property survives after quantization. Computing the conformal algebra generated by the quantized energy momentum tensor [58], one finds that the central charge is related to the coupling constant β as

$$c = r + 12\rho^2(\frac{1}{\beta} + \beta)^2 = r\left(1 + h(h+1)(\frac{1}{\beta} + \beta)^2\right) \ , \qquad (2.22)$$

where h is the dual Coxeter number of $\mathcal{G}$ [34]. We will mainly examine A_n Toda theories, for which $h = n + 1$. The symmetries of these models are described by the W_n algebras discussed in section 2.2 [8, 15].

Comparing (2.14) and (2.22) one realizes that it is impossible to obtain realizations of perturbations of minimal models or their W–extended versions, as long as the coupling constant β is chosen real. The situation changes if one performs an analytic continuation to imaginary coupling constant $\gamma = i\beta$ [58]. The simplest algebra is A_1, for which $h = 2$. Parameterizing $\gamma^2 = \frac{p}{q}$, the central charge (2.22) becomes

$$c = 1 - \frac{6(p-q)^2}{pq} \ ,$$

which are the values for the minimal models (2.5). In this case the Toda field equations become

$$\partial^2 \phi = -\frac{2}{\beta} e^{\beta\phi} \ ,$$

which is the Liouville equation. Using the algebras A_n we obtain the values of the central charge for the W_n algebras (2.14).

Toda field theories can be built also over affine Kac-Moody algebras $\hat{\mathcal{G}}$ instead of a finite Lie algebra. As long as the full set of generators of the Kac-Moody algebra is included in the construction, one obtains again a conformal invariant model [4, 7]. But a consistent theory can be defined also by including only the 0^{th} root. In this case one obtains a set of massive models, usually denoted as affine Toda field theories (ATFT). The action is given by (2.20), but now including also the 0^{th} root, *i.e.* one adds a term

$$\delta V(\phi) = \frac{\lambda}{\beta^2} e^{\beta\alpha_0 . \phi} \quad , \tag{2.23}$$

to the lagrangian (2.20). The new minimum of the potential $\phi^{(0)}$ is finite and satisfies

$$\sum_i \alpha_i e^{\beta\alpha_i . \phi^{(0)}} = \lambda \alpha_0 e^{\beta\alpha_0 \phi^{(0)}} \quad .$$

In order to understand the structure of the excitations above the vacuum of this theory one analyzes the potential

$$V = \frac{1}{\beta^2} \sum_{i=1}^{r} e^{\beta\alpha_i(\phi^{(0)}+\phi)} + \frac{\lambda}{\beta^2} e^{\beta\alpha_0(\phi^{(0)}+\phi)} \quad .$$

Since α_0 can be expressed in terms of the other roots $\alpha_0 = -\sum_{n=1}^{r} n_i\alpha_i$, one can rewrite V as

$$V = \frac{k^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta\alpha_i . \phi} \quad ,$$

where the constant k is given by $k^2 = \lambda e^{\beta\alpha_0\phi^{(0)}}$. Expanding around the minimum $\phi = 0$ one obtains

$$V(\phi) = \frac{k^2}{\beta^2} \sum_{i=0}^{r} n_i + \frac{1}{2}(M^2)^{ab}\phi^a\phi^b + c^{abc}\phi^a\phi^b\phi^c + \dots \quad , \tag{2.24}$$

where summation over the indices a,b,c is intended. The second order term defines the mass matrix

$$(M^2)^{ab} = \sum_{i=0}^{r} n_i \alpha_i^a \alpha_i^b \quad , \tag{2.25}$$

and three point couplings c^{abc} are given by the third order term

$$c^{abc} = k^2\beta \sum_i n_i \alpha_i^a \alpha_i^b \alpha_i^c \quad . \tag{2.26}$$

It has been shown [93] that as well the conformal invariant as the perturbed Toda field theories are classically integrable.

The simplest example is given by the perturbations of the Liouville model, corresponding to A_1. In this case we have two possibilities to obtain an affine Toda field theory. One possibility is to choose $A_1^{(1)}$, with the Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. The corresponding equation of motion is $\partial^2\varphi = \frac{2}{\beta}\sinh\beta\varphi$, the so-called sinh–Gordon equation.

Its potential has a single vacuum. Similar as for the conformal TFT one can perform an analytic continuation $\gamma = i\beta$ and the equation of motion becomes the Sine-Gordon equation

$$\partial^2\varphi = \frac{2}{\gamma}\sin\gamma\varphi \quad . \tag{2.27}$$

The structure of the potential has now completely changed. Instead of an unique vacuum the Sine-Gordon potential exhibits an infinite number of degenerate vacua, and the solutions of (2.27) are of soliton and breather type [31].

The second possibility to construct an affine Lie algebra from the finite algebra A_1, is to choose $A_2^{(2)}$, with the Cartan matrix $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$. The corresponding equation of motion is the so–called Bullogh-Dodd equation

$$\partial^2\varphi = \frac{1}{\beta}(e^{-2\beta\varphi} - 2e^{\beta\varphi}) \quad . \tag{2.28}$$

Similar as the Sine-Gordon equation it admits soliton solutions if an analytic continuation in the coupling constant is performed.

The identification of the Liouville model with minimal models goes across the Coulomb gas approach (for example see [42]). As we already mentioned, this approach describes a massless scalar field embedded in a space with a charge placed at infinity, whose value is $-2\ \alpha_0$, where α_0 is a rational number. The primary fields are given by the vertex-operators

$$V_{r,s} = e^{i\alpha_{r,s}\phi(x)} \quad ,$$

with

$$\alpha_{r,s} = \frac{[(1-r)\alpha_+ + (1-s)\alpha_-]}{2} \tag{2.29}$$

and r, s positive integers and $\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2+1}$. The central charge is given by $c = 1 - 24\alpha_0^2$. The weights of these vertex operators are those given in (2.6). Therefore the fields $V_{r,s}$ are seen as a realization of the primary fields of the minimal models $\Phi_{r,s}$. Taking the classical limit which corresponds to take $c \to \infty$ one would like to identify the corresponding classical objects. To do this we use the relation

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \quad .$$

which is obtained by inverting the definitions of c and α_{pm} above. We see that any operator $V_{r,s}$ with $r > 1$ will explode in the limit $c \to \infty$. Therefore only operators of the kind $V_{1,n}$ can be seen in a classical analog of the theory. These operators have classical dimensions $-\frac{n-1}{2}$, and can be identified with the field

$$e^{-i\frac{n-1}{2}\beta\phi} \tag{2.30}$$

in the Liouville theory. They are precisely of the form (2.23). And indeed, perturbing by these fields, one finds for $n = 3$ the Sine-Gordon lagrangian and for $n = 2$ the Izergin-Korepin model [62].

The above analysis shows that the two perturbations $\Phi_{1,2}$ and $\Phi_{1,3}$ can both be realized as classical lagrangian field theories. This is not so for perturbations by the operator $\Phi_{2,1}$, which does not have a classical analogue. However, note that interchanging $\Phi_{1,2}$ and $\Phi_{2,1}$ corresponds to interchanging α_+ and α_- in (2.29), or changing the role of r and s. Therefore one conjectures [101], that those perturbations are also connected to the $A_2^{(2)}$ lagrangian.

We briefly summarize the principle results concerning the case of real coupling Toda field theory. They have been analyzed in great detail and the masses and couplings for the respective algebras have been calculated [18, 29]. Many results can be put in a unified form [40, 53] using a group theoretic notion. For simplicity we specialize to the simply-laced case. This also implies that we consider only non-twisted algebras.

The masses are given by diagonalizing the mass-matrix (2.25). Since this matrix is proportional to the incidence-matrix of the corresponding Dynkin diagram, the masses are given by

$$m_i = \beta x_i \alpha_i^2 \sin\frac{\pi}{h} \quad , \tag{2.31}$$

that is, they are proportional to the components of the Perron-Frobenius eigenvector x_i of the incidence matrix. h again denotes the Coxeter number of the corresponding finite dimensional simple Lie algebra $\mathcal{G}$.

The three point couplings (2.26) are given by the so-called *area-rule*, which reads as

$$c_{ijk} = \sigma_{ijk}\frac{4\beta}{\sqrt{h}}\Delta_{ijk} \quad , \tag{2.32}$$

where Δ_{ijk} is the area of the triangle whose sides are formed by the masses m_i, m_j and m_k and $\sigma_{ijk} = \pm 1$ is a sign factor. Going to higher order couplings $n \geq 3$ one finds that they can be completely determined in terms of the masses and the three–point couplings.

3 The S–Matrix Approach

In the last section we have discussed massive field theories which are integrable both classically as quantistically. Therefore one expects that they are also resolvable, that is that the full quantum structure can be determined, especially the correlation functions. Unfortunately up to now no straight forward method exists in order to achieve this goal. Nevertheless several methods have been proposed all of which are still under development. One approach is the bootstrap method. As a first step one evaluates the on–shell structure of the theory, that is the mass spectrum and the S–matrix. As a second step one can construct the form–factors and then finally the correlation functions.

The S matrix is introduced into quantum field theory through the Lehmann-Simanzik-Zimmermann formulation [44, 61]. Suppose we have a massive quantum field theory. There are two sets of creation and annihilation operators

$$a^{\epsilon}_{in}(\beta), a^{\star}_{in,\epsilon}(\beta) \quad , \quad a^{\epsilon}_{out}(\beta), a^{\star}_{out,\epsilon}(\beta) \quad , \tag{3.1}$$

which describe the physical particles asymptotically for $t \to -\infty, t \to \infty$. In this region they are supposed to be free. They satisfy canonical commutation relations. The momentum p_μ is parameterized in terms of the rapidity β, defined as

$$p_0(\beta) = m \cosh \beta \quad , \quad p_1(\beta) = m \sinh \beta \quad . \tag{3.2}$$

One also assumes that there exists a common vacuum for these operators

$$a^{\epsilon}_{in}(\beta)|0\rangle = a^{\epsilon}_{out}(\beta)|0\rangle = 0 \quad . \tag{3.3}$$

These asymptotic states should be connected by a unitary transformation, which is the *S–matrix*,

$$a^{\star}_{in,\epsilon_1}(\beta_1)\ldots a^{\star}_{in,\epsilon_n}(\beta_n)|0\rangle =$$

$$= \sum_m S^{\epsilon'_1,\ldots,\epsilon'_m}_{\epsilon_1,\ldots,\epsilon_n}(\beta'_1 \ldots \beta'_m|\beta_1 \ldots \beta_n) a^\star_{out,\epsilon'_1}(\beta'_1) a^\star_{out,\epsilon'_m}(\beta'_m)|0\rangle \quad . \tag{3.4}$$

Further one requires locality, unitarity and crossing symmetry in the theory. These properties restrict the form of the components of $S(\beta'_1 \ldots \beta'_m|\beta_1 \ldots \beta_n)$.

For two dimensional integrable systems, which provide an infinite number of conservation laws the requirements for a consistent scattering theory become severe constraints on the S–matrix elements. One can determine the form of the eigenvalues of the conservation laws. The first of them are $I_1 = p_0 + p_1$ and $I_{-1} = p_0 - p_1$ being the integrals of $T(z)$ and $\bar{T}(\bar{z})$, with eigenvalues Me^β and $Me^{-\beta}$ respectively, where M denotes the mass. Higher conservation laws I_s have eigenvalues of the form $M_s e^{\beta s}$, $M_s e^{-\beta s}$, M_s being a constant depending on the spin s. Due to locality of $I_{\pm s}$, their eigenvalues on multi-particle states are the sums of one particle eigenvalues. The S–matrix must commute with I_s, I_{-s} and therefore eigenvalues of "in" and "out" states coincide. So if we have the rapidities $\{\beta'_1 \ldots \beta'_n\}$ after scattering, they should satisfy the infinite set of equations

$$\sum_{j=1}^{n} e^{\pm s\beta_j} M_s = \sum_{j=1}^{m} e^{\pm s\beta'_j} M_s \quad ,$$

for any conserved spin s. The only solution consistent with analyticity, is $n = m$ and even more, $\{\beta'_1 \ldots \beta'_n\} = \{\beta_1 \ldots \beta_n\}$. Therefore the scattering is purely elastic.

Since the theory contains only massive particles, the interaction is expected to be short range. This implies that if one splits the interaction region into domains such that

$$|x_a - x_b| \overset{\sim}{<} R \ , \quad |x_a - x_i| \gg R \ , \quad |x_b - x_i| \gg R \ , \ |x_i - x_j| \gg R \ , \quad i,j = 1,\ldots,n \quad ,$$

one can describe in each region the process as a two particle scattering while the other particles behave approximately as free ones. Because the conservation laws must hold after every interaction processes, the n-particle scattering matrix reduces to a product of $n(n-1)/2$ two particle scattering one [116]. This property is called *factorized scattering.*

Factorized scattering implies that for calculating the full S–matrix it is sufficient to know all two–particle S–matrix elements. They can be in general represented as

$$\langle 0|a^{\epsilon'_1}_{out}(\beta'_1) a^{\epsilon'_2}_{out}(\beta'_2) a^\star_{in,\epsilon_1}(\beta_1) a^\star_{in,\epsilon_2}(\beta_2)|0\rangle = \delta(\beta_1 - \beta'_1)\delta(\beta_2 - \beta'_2) S^{\epsilon'_1\epsilon'_2}_{\epsilon_1\epsilon_2}(\beta_i - \beta_j) \quad , \tag{3.5}$$

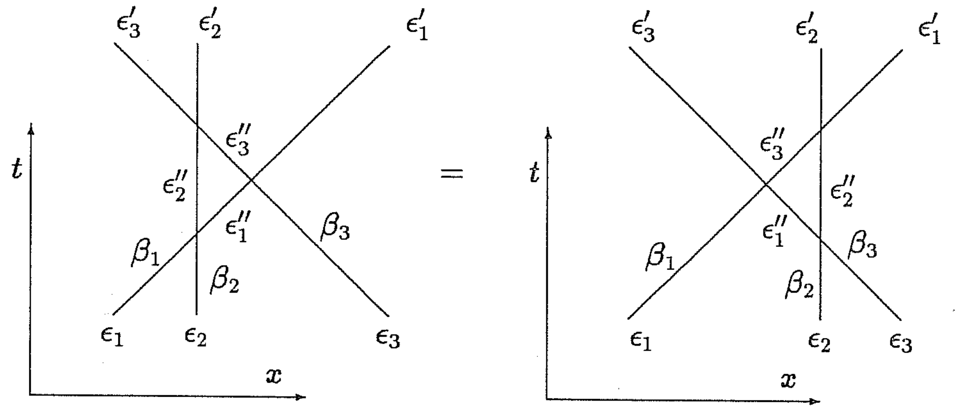
with $\beta_2 > \beta_1$, $\beta'_2 > \beta'_1$ and the ϵ_i labeling possible internal degrees of freedom. Because of

Lorenz invariance the elements $S^{\epsilon_1'\epsilon_2'}_{\epsilon_1\epsilon_2}(\beta_1-\beta_2)$ depend only on the difference of rapidities, $\beta_{ij}\equiv\beta_i-\beta_j$.

An important consequence of factorized scattering is the Yang–Baxter equation. For, consider a three particle scattering. This can be decomposed in two different ways into two particle scattering processes, which should be equivalent. The result is the Yang–Baxter equation,

$$\begin{aligned} &\sum_{\epsilon_1'''\epsilon_2'''\epsilon_3'''} S^{\epsilon_1'\epsilon_2'}_{\epsilon_1\epsilon_2}(\beta_1-\beta_2)S^{\epsilon_1''\epsilon_3'}_{\epsilon_1\epsilon_3'''}(\beta_1-\beta_3)S^{\epsilon_2''\epsilon_3''}_{\epsilon_2\epsilon_3}(\beta_2-\beta_3) \\ =&\sum_{\epsilon_1'''\epsilon_2'''\epsilon_3'''} S^{\epsilon_2'\epsilon_3'}_{\epsilon_2''\epsilon_3''}(\beta_2-\beta_3)S^{\epsilon_1'\epsilon_3''}_{\epsilon_1''\epsilon_3}(\beta_1-\beta_3)S^{\epsilon_1''\epsilon_2''}_{\epsilon_1\epsilon_2}(\beta_1-\beta_2) \quad , \end{aligned} \tag{3.6}$$

which we have depicted in figure 3.1.



(3.7)

Figure 3.1: Graphical representation of the Yang–Baxter equation

For the description of the space of physical states it is useful to introduce the so–called Zamolodchikov–Faddeev operators. Their existence is an axiom in this approach. One assumes the existence of a physical vacuum $|0\rangle$ which is annihilated by operators $Z^{\epsilon}(\beta)$,

$$Z^{\epsilon}(\beta)|0\rangle = 0 \quad .$$

The physical states are created by

$$|Z_{\epsilon_1}(\beta_1)\ldots Z_{\epsilon_n}(\beta_n)\rangle = Z^{*}_{\epsilon_1}(\beta_1)\ldots Z^{*}_{\epsilon_n}(\beta_n)|0\rangle \quad . \tag{3.8}$$

The importance of these operators lies in their commutation relations, which are governed by the S-matrix,

$$Z^{\epsilon_1}(\beta_1)Z^{\epsilon_2}(\beta_2) \;=\; S^{\epsilon_1\epsilon_2}_{\epsilon_1'\epsilon_2'}(\beta_1-\beta_2)Z^{\epsilon_2'}(\beta_2)Z^{\epsilon_1'}(\beta_1) \quad ,$$

$$
\begin{aligned}
Z^{\star}_{\epsilon_1}(\beta_1)Z^{\star}_{\epsilon_2}(\beta_2) &= S^{\epsilon_1'\epsilon_2'}_{\epsilon_1\epsilon_2}(\beta_1-\beta_2)Z^{\star}_{\epsilon_2'}(\beta_2)Z^{\star}_{\epsilon_1'}(\beta_1) \quad , \\
Z^{\epsilon_1}(\beta_1)Z^{\star}_{\epsilon_2}(\beta_2) &= Z^{\star}_{\epsilon_2'}(\beta_2)S^{\epsilon_1\epsilon_2'}_{\epsilon_1'\epsilon_2}(\beta_1-\beta_2)Z^{\epsilon_1'}(\beta_1) + 2\pi\delta^{\epsilon_1}_{\epsilon_2}\delta(\beta_1-\beta_2) \quad .
\end{aligned} \tag{3.9}
$$

Not all of the states (3.8) are independent. By exchanging the ordering of the elements in a state (3.8) one acquires a factor (*i.e.* an S-matrix element), and hence exchanged states are linearly dependent. For this one introduces ordered bases, which can be identified with the 'in' and 'out' states:

$$
\begin{gathered}
|Z_{\epsilon_1}(\beta_1)\dots Z_{\epsilon_n}(\beta_n)\rangle_{in} = |Z_{\epsilon_{i_n}}(\beta_{i_n})\dots Z_{\epsilon_{i_1}}(\beta_{i_1})\rangle \\
|Z_{\epsilon_1}(\beta_1)\dots Z_{\epsilon_n})(\beta_n)\rangle_{out} = |Z_{\epsilon_{i_1}}(\beta_{i_1}\dots Z_{\epsilon_{i_n}}(\beta_{i_n})\rangle \\
\text{where} \quad \beta_{i_1} < \beta_{i_2} < \dots < \beta_{i_n} \quad .
\end{gathered} \tag{3.10}
$$

Another important property of these operators is their behaviour under the Poincare group. They transform under Lorentz transformations $L(\alpha)$ and the translations T_y as

$$
\begin{aligned}
U_L Z_\epsilon(\beta)U_L^{-1} &= Z_\epsilon(\beta+\alpha) \\
U_{T_y} Z_\epsilon(\beta)U_{T_y}^{-1} &= e^{ip_\mu(\beta)y^\mu} Z_\epsilon(\beta)
\end{aligned} \tag{3.11}
$$

In this approach one need to assume the existence of the Zamolodchikov–Faddeev operators and their properties. For some models they can be constructed explicitly in terms of the underlying symmetry algebra [108].

3.1 Analytic structure

So far we have examined only the constraints on the S–matrix amplitudes which derive from integrability. Further requirements derive from general S–matrix properties as unitarity and crossing symmetry.

Recall the analytic structure of the S–matrix in terms of the invariant energy squared

$$
s = 4p^\mu p_\mu = 4m^2 \sinh(\frac{\beta_{12}}{2}) \quad . \tag{3.12}
$$

The matrix $S^{\epsilon_1'\epsilon_2'}_{\epsilon_1\epsilon_2}$ is analytic in the complex s-plane, with two cuts along the real axes $s \leq 0$ and $s \geq 4m^2$. This is pictured in fig.3.2, wherein dots indicate possible poles, which correspond to bound states. The cut for $s \leq 0$ corresponds to the threshold in the variable $t = (p_1 - p_4)^2$, which is kept fixed. This Riemann surface is composed of two

Figure 3.2: Cuts and poles of the S matrix in the s plane

sheets and the two cuts can be replaced by a single one going from $s = 0$ to $s = 4m^2$. The transformation to the rapidity variable β,

$$\beta = \ln\left(\frac{s - 2m^2 + \sqrt{s(s-4m^2)}}{2m^2}\right) \quad ,$$

which is the inverse of (3.12), transforms the physical sheet of the s-plane into the strip $0 \leq \mathrm{Im}\,\theta \leq \pi$. The edges of the right and left cuts of the physical sheet get mapped into the axes $\mathrm{Im}\,\theta = 0$ and $\mathrm{Im}\,\theta = \pi$ respectively. The axes $\mathrm{Im}\,\theta = l\pi$, $l = -1, \pm 2 \ldots$ correspond to the edges of cuts of the other complex s plane sheets (see fig.3.3).

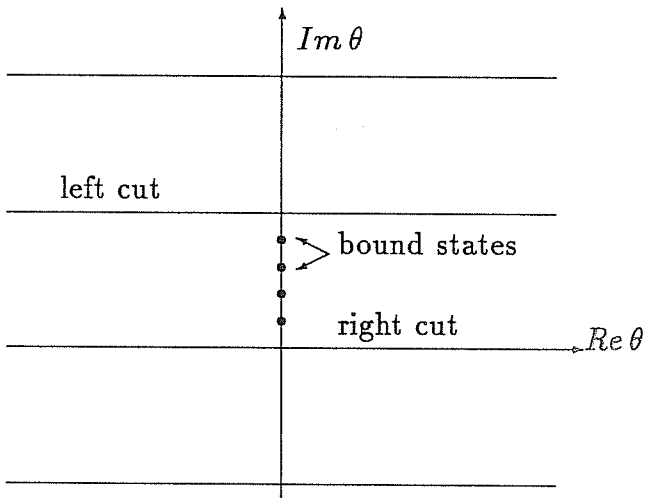


Figure 3.3: Analytic structure of the S matrix in the θ-plane

If the S–matrix is real–analytic, *i.e.* $S^\dagger(\beta) = S(-\beta)$, the unitarity requirement can be written as

$$S^{\epsilon_1'\epsilon_2'}_{\epsilon_1\epsilon_2}(\beta) S^{\epsilon_1\epsilon_2}_{\epsilon_1''\epsilon_2''}(-\beta) = \delta^{\epsilon_1'}_{\epsilon_1''}\delta^{\epsilon_2'}_{\epsilon_2''} \quad . \tag{3.13}$$

Finally let us determine the implications of crossing symmetry. Regarding both s and t as complex variables, one can reach the region $t \geq 4m^2$, $s \leq 0$ by analytic continuation, which describes the 'crossed' scattering process. The corresponding transformation is

$s \to 4m^2 - s$ or in terms of the rapidity, $\beta \to i\pi - \beta$. This yields the crossing symmetry relation

$$S_{\epsilon_1 \epsilon_2}^{\epsilon_1' \epsilon_2'}(\beta) = c_{\epsilon_1 \epsilon_1''} S_{\epsilon_1''' \epsilon_2}^{\epsilon_1'' \epsilon_2'}(i\pi - \beta) c^{\epsilon_1' \epsilon_1'''} \quad , \tag{3.14}$$

where c is the matrix of charge conjugation, with the property $c^2 = 1$.

3.2 Bound states and the bootstrap principle

A remarkable property of two dimensional integrable scattering theories is the bootstrap equation. It allows to determine the full scattering matrix once the S–matrix element for the fundamental particle is given.

Let us assume that our system is composed of only self–conjugate particles, that is particles without any internal symmetry. In this case the S–matrix can be written as S_{ab} a,b indicating the kind of particles which scatter. Since we require spatial reflection symmetry we obtain that $S_{ab}(\theta) = S_{ba}(\theta)$, *i.e.* the order of the indices is irrelevant.

Assume that the S–matrix element S_{ab} exhibits a pole $\theta = iu_{ab}^c$, *i.e.* the particles a and b form some bound state c at this value of the rapidity. Expressing the energy in this channel in terms of the masses and the rapidity one finds the identity

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos u_{ab}^c \quad . \tag{3.15}$$

This relation allows to determine the mass of the bound state c.

Because of crossing symmetry one expects also a pole at $\theta = i\bar{u}_{ab}^c$ in the S– matrix element S_{ab}, where $\bar{u}_{ab}^c \equiv \pi - u_{ab}^c$. Further, if the S–matrix gives rise to the bound–state process $a, b \to c$, then by crossing symmetry also the processes $b, c \to a$ and $c, a \to b$ exist. The poles of the respective S–matrix elements are determined by the masses of the particles. A particular useful observation is to note that the relation (3.15) corresponds to the cosine law for a triangle. This implies that the fusion rapidities are given by the angles formed by the triangle with sides of length m_a, m_b and m_c (compare fig. 3.4).

Having determined the mass of the bound state, the next step is to determine the S–matrix elements involving this new particle. This is possible because in the bootstrap approach the bound states are themselves identified with particles appearing as asymptotic states. Assume we know the scattering amplitudes S_{ad} and S_{bd}. Then the

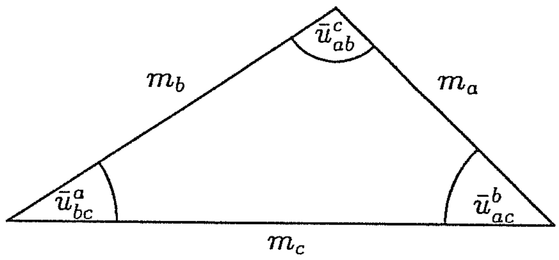


Figure 3.4: The mass triangle with the fusion angles

scattering amplitude S_{cd}, c being the particle corresponding to the pole in the amplitude S_{ab}, is given by

$$S_{cd}(\theta) = S_{ad}(\theta + i\bar{u}_{ac}^{b}) S_{bd}(\theta - i\bar{u}_{bc}^{a}) \quad , \tag{3.16}$$

which is called the *bootstrap equation.* This relation is shown in figure 3.5.

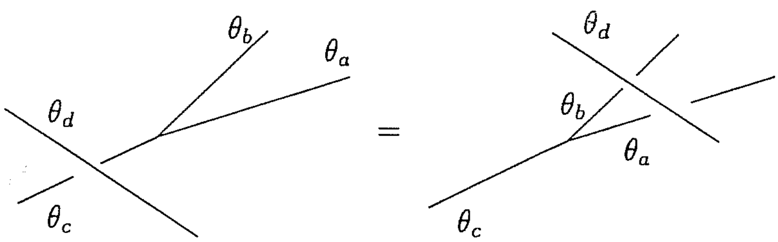


Figure 3.5: Graphical representation of the bootstrap equation

In principle this allows to start from a given amplitude S_{ab} and calculate all amplitudes of the particles appearing in our theory. For scalar particles the unitarity and crossing relations read as

$$S_{ab}(\theta) = S_{ab}(i\pi - \theta) \quad , \qquad S_{ab}(\theta) S_{ab}(-\theta) = 1 \quad . \tag{3.17}$$

There exists a general solution to these equations, namely

$$S(\theta) = \prod_x \frac{\tanh(\frac{\theta}{2} + i\pi x)}{\tanh(\frac{\theta}{2} - i\pi x)} \equiv \prod_x f_x(\theta) \quad . \tag{3.18}$$

This means that without any input from a specific model, the constraints deriving from integrability and the analytic structure have restricted the possible form of the S–matrix to that given in (3.18). Therefore it is possible to construct consistent S–matrices by classifying all possible combinations of the type (3.18) which fulfill the axioms of the bootstrap method. This analysis will be pursued in section 5.1.

In order to investigate systems with internal degrees of freedom, one needs to understand the degeneracy structure of the S–matrix elements. We postpone the discussion on the bootstrap equations in this case to chapter 6, in which S–matrices for degenerate particles will be discussed in detail.

4 Form–Factors

The complete description of a quantum theory is given by the knowledge of the whole set of correlation functions

$$\langle \mathcal{O}_{i_1}(x_1)\ldots\mathcal{O}_{i_n}(x_n)\rangle \equiv \langle 0|T(\mathcal{O}_{i_1}(x_1)\ldots\mathcal{O}_{i_n}(x_n))|0\rangle \quad . \tag{4.1}$$

For integrable two dimensional massive models a very efficient approach exists for the computation of these functions. Take for example the two point function of some local operator $\mathcal{O}$ and expand it as

$$\begin{aligned}\langle\mathcal{O}(x)\mathcal{O}(0)\rangle = \sum\int \frac{d\beta_1\ldots d\beta_n}{n!(2\pi)^n}\langle 0|\mathcal{O}(x)|Z_{\epsilon_1}(\beta_1),\ldots,Z_{\epsilon_n}(\beta_n)\rangle_{in}\times \\ {}^{in}\langle Z_{\epsilon_n}(\beta_n),\ldots,Z_{\epsilon_1}(\beta_1)|\mathcal{O}(0)|0\rangle \quad ,\end{aligned} \tag{4.2}$$

where the intermediate states are given by the bases of asymptotic states (see section 3). The matrix elements in this expansion can be calculated in closed form through the *form factor bootstrap method* which we will discuss in this chapter. The metyhod is based on the fact [104, 105] that the on–shell factorized scattering data are sufficient for the construction of the matrix–elements in the expansion (4.2).

As a first step let us reduce the set of independent matrix elements and define the form–factors. In the expansion of the n–point function, we encounter the general matrix element,

$$_{out}\langle Z_{\epsilon'_m}(\beta'_m),\ldots,Z_{\epsilon'_1}(\beta'_1)|\mathcal{O}(x)|Z_{\epsilon_1}(\beta_1),\ldots,Z_{\epsilon_n}(\beta_n)\rangle_{in} \quad . \tag{4.3}$$

This can be reduced to the corresponding matrix element of $\mathcal{O}(0)$. Since translations act on the operator $\mathcal{O}$ as $\mathcal{O}$ as $U_{T_y}\mathcal{O}(x)U_{T_y}^{-1} = \mathcal{O}(x+y)$, one obtains

$$_{out}\langle Z_{\epsilon'_m}(\beta'_m),\ldots,Z_{\epsilon'_1}(\beta'_1)|\mathcal{O}(x)|Z_{\epsilon_1}(\beta_1),\ldots,Z_{\epsilon_n}(\beta_n)\rangle_{in} =$$

$$\exp\left\{i\left(\sum_{i=m+1}^{n} p_\mu(\beta_i)-\sum_{i=1}^{m} p_\mu(\beta_i)\right)x^\mu\right\}\,{}_{out}\langle Z_{\epsilon'_m}(\beta'_m),\ldots,Z_{\epsilon'_1}(\beta'_1)|\mathcal{O}(0)|Z_{\epsilon_1}(\beta_1),\ldots,Z_{\epsilon_n}(\beta_n)\rangle_{in} \tag{4.4}$$

Form-factors are defined as

$$F^{\epsilon'_m\ldots\epsilon'_1}_{\epsilon_1\ldots\epsilon_n}(\beta'_m,\ldots,\beta'_1|\beta_1,\ldots,\beta_n)\equiv$$

$$\langle Z_{\epsilon'_m}(\beta'_m),\ldots,Z_{\epsilon'_1}(\beta'_1)|\mathcal{O}(0)|Z_{\epsilon_1}(\beta_1),\ldots,Z_{\epsilon_n}(\beta_n)\rangle \quad , \tag{4.5}$$

i.e. as the analytic continuation of the matrix elements (4.3) to all values of β_i. They are graphically represented in *fig*.4.1. The analytic continuation is an important step, since

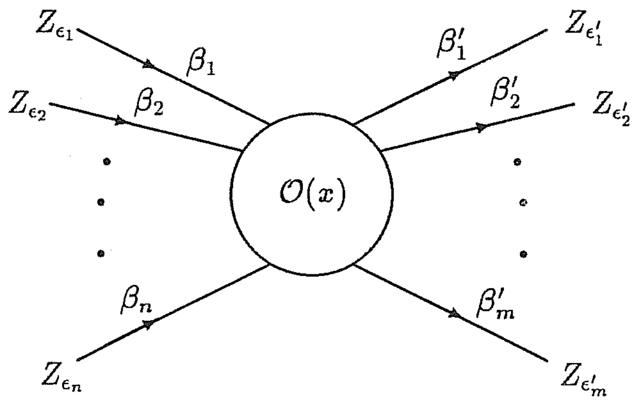


Figure 4.1: General matrix elements of the operator $\mathcal{O}(x)$ between asymptotic states

it allows the use of the properties of the Faddeev-Zamolodchikov operators, introduced in section (3).

For example, using crossing symmetry one can reduce the set of independent form-factors to those containing particles only on the right (see fig.4.2) by

$$F^{\epsilon'_m\ldots\epsilon'_1}_{\epsilon_1\ldots\epsilon_n}(\beta'_m,\ldots,\beta'_1|\beta_1,\ldots,\beta_n)=F_{\epsilon_1\ldots\epsilon_n\epsilon'_m\ldots\epsilon'_1}(\beta_1,\ldots,\beta_n,\beta'_m-i\pi,\ldots,\beta'_1-i\pi) \quad . \tag{4.6}$$

4.1 Form–factor equations

Let us discuss the properties that the form–factors must satisfy. Heuristically these equations are derived from the properties of the Faddeev- Zamolodchikov operators together

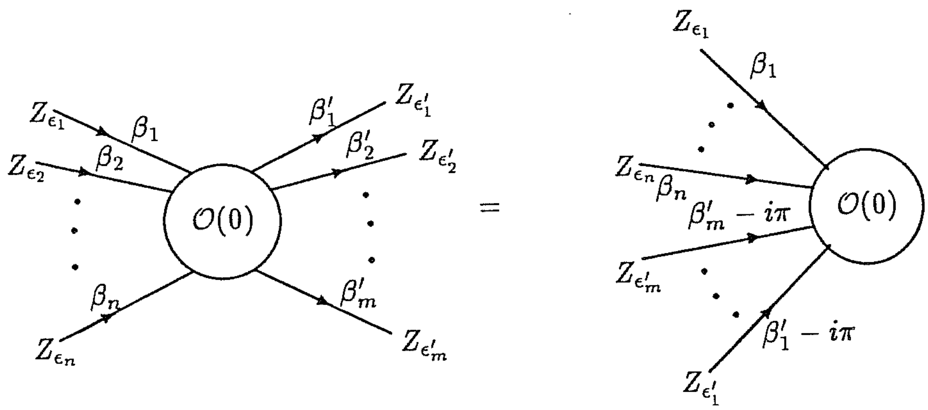


Figure 4.2: Relation of general matrix elements to the form–factors

with crossing symmetry and CPT invariance [65, 107]. As we will see the only information needed is the factorized scattering data.

For simplicity, we discuss these equations in the case of scalar self-conjugate asymptotic particle states. A consequence of the commutation relations (3.9) is the symmetry property,

$$F_{\epsilon_1\ldots\epsilon_i\epsilon_{i+1}\ldots\epsilon_n}(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots\beta_n) = S_{\epsilon_i\epsilon_{i+1}}(\beta_i-\beta_{i+1})\,F_{\epsilon_1\ldots\epsilon_{i+1}\epsilon_i\ldots\epsilon_n}(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots\beta_n) \quad . \tag{4.7}$$

Figure 1.3 shows this situation: exchanging to particles they scatter and give the S–matrix element in (4.7).

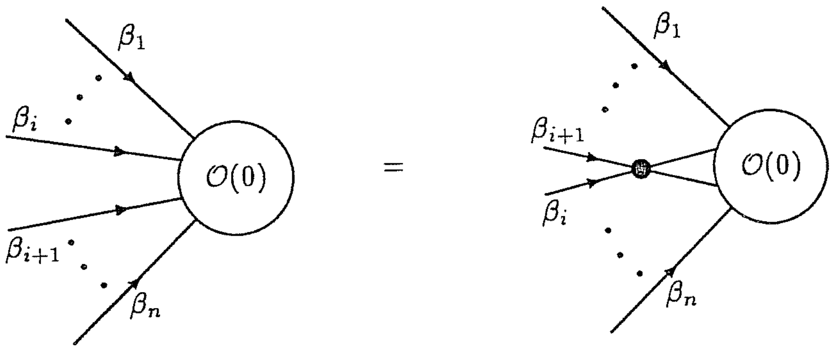


Figure 4.3: The symmetry property arises by crossing two asymptotic states. The large dot on the right hand side stands for an S–matrix element

Consider the analytic continuation $\beta_1 \longrightarrow \beta_1 + 2\pi i$, which from the kinematical point of view brings back to the initial configuration, but changes the ordering of the particles in the function $F_{\epsilon_1\ldots\epsilon_n}(\beta_1,\ldots,\beta_n)$. The result is shown in figure 4.4. One sees, that the analytic continuation can be related to the original form–factor in an alternative way by scattering all other particles. The consequence for the form–factor is the constraint equation

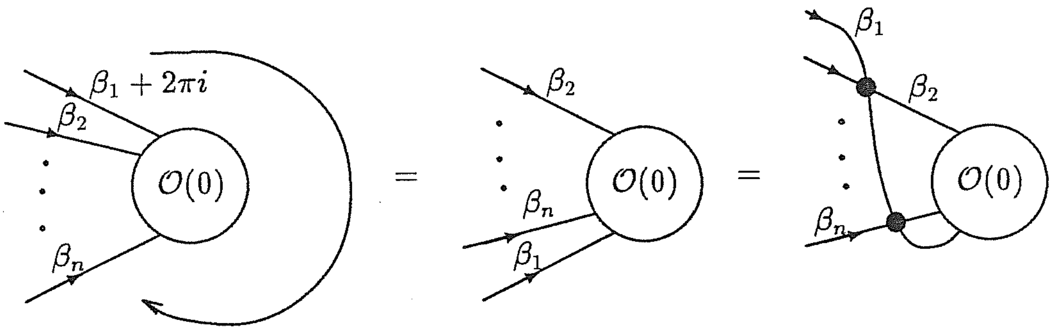


Figure 4.4: Analytic continuation of the form–factor by $\beta_1 \longrightarrow \beta_1 + 2\pi i$

$$F_{\epsilon_1\epsilon_2\ldots\epsilon_n}(\beta_1+2\pi i,\beta_2,\ldots,\beta_n) = F_{\epsilon_2\ldots\epsilon_n\epsilon_1}(\beta_2,\ldots,\beta_n,\beta_1) =$$
$$S_{\epsilon_1\epsilon_2}S_{\epsilon_1\epsilon_3}\ldots S_{\epsilon_1\epsilon_n}\; F_{\epsilon_1\epsilon_2\ldots\epsilon_n}(\beta_1,\beta_2,\ldots,\beta_n) \quad . \tag{4.8}$$

So we have obtained the first two form-factor equations, which are also known as "Watson's equations" [111].

A further constraint is imposed by relativistic invariance. Assume that the operator $\mathcal{O}$ has spin s. Then

$$F_{\epsilon_1\ldots\epsilon_n}(\beta_1+\Lambda,\ldots,\beta_n+\Lambda) = e^{s\Lambda}F_{\epsilon_1\ldots\epsilon_n}(\beta_1,\ldots,\beta_n) \quad . \tag{4.9}$$

The form-factor equations discussed so far cannect from-factors corresponding to a fixed particle number. n and m particle form–factors are independent of each other up to now. The final two constraint equations have a recursive structure and link form–factors of *different* particle numbers. Their originate from the pole–structure of the form–factors.

If particles A_i, A_j form a bound state A_k, the corresponding two-particle scattering amplitude exhibits a pole (see section 3.2), with the residue

$$-i \lim_{\beta'\to iu_{ij}^k}(\beta - iu_{ij}^k)S_{ij}(\beta) = (\Gamma_{ij}^k)^2 \quad ; \tag{4.10}$$

Γ_{ij}^{k} is the three–particle on–shell vertex (see fig.4.5).

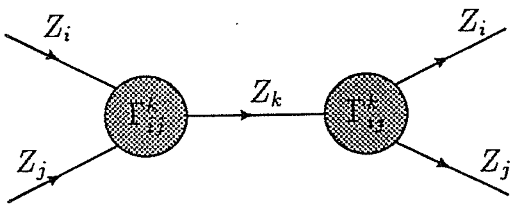


Figure 4.5: Residue of a bound state pole of the scattering amplitude

Corresponding to this bound state the form–factor exhibits a pole at the point $\beta_i - \beta_j = i\bar{u}_{ij}^{k}$ with the residue (see fig. 4.6)

$$-i \lim_{\beta' \to \beta} (\beta' - \beta) F_{ij\epsilon_1 \ldots \epsilon_n}(\beta' + i\bar{u}_{ik}^{j}, \beta - \bar{u}_{jk}^{i}, \beta_1, \ldots, \beta_{n-1}) =$$
$$= \Gamma_{ij}^{k} F_{k\epsilon_1 \ldots \epsilon_n}(\beta, \beta_1, \ldots, \beta_{n-1}) \quad . \tag{4.11}$$

All other binding rules can be derived with the help of equation (4.7).

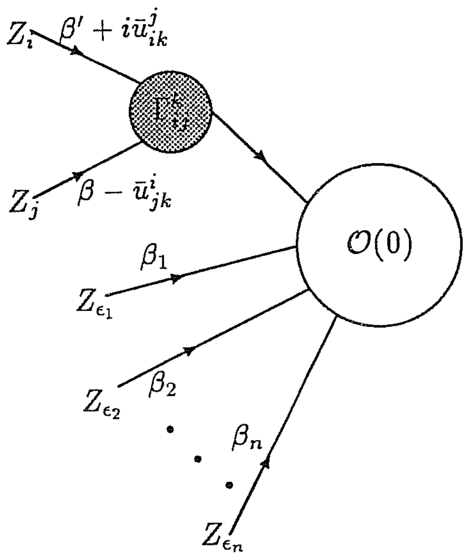


Figure 4.6: Recursion relation of the form factor corresponding to a bound state pole

The other type of poles has a kinematical origin and corresponds to zero–angle scattering. They appear in the form-factors where particles of the same kind are involved.

Their origin can be understood by the following argument [113]. Examine the form–factor

$$F_{\epsilon\epsilon\epsilon_1\ldots\epsilon_n}(\beta' + i\pi, \beta, \beta_1, \ldots, \beta_n) \quad , \tag{4.12}$$

which we have depicted in figure 4.7 a). By (4.8) the same graph can be represented by the expression

$$F_{\epsilon\epsilon_1\ldots\epsilon_n\epsilon}(\beta, \beta_1, \ldots, \beta_n, \beta' - i\pi) \quad . \tag{4.13}$$

Now examine these two expressions in the limit where the rapidities β and β' coincide. The two limit processes describe two different kinematical regions and are shown in figure 4.7 b). Their difference for $\delta \to 0$

$$F_{\epsilon\epsilon\epsilon_1\ldots\epsilon_n}(\beta' + i\pi - i\delta, \beta, \beta_1, \ldots, \beta_n) - F_{\epsilon\epsilon_1\ldots\epsilon_n\epsilon}(\beta, \beta_1, \ldots, \beta_n, \beta' - i\pi + i\delta)$$
$$\stackrel{\beta'\to\beta}{\longrightarrow} \left(1 - \prod_{i=1}^{n} S_{\epsilon\epsilon_i}(\beta - \beta_i)\right) F_{\epsilon_1\ldots\epsilon_n}(\beta_1, \ldots, \beta_n) \tag{4.14}$$

indicates a simple pole of (4.12) at $\beta' = \beta$ which gives rise to the residue–equation

$$-i \lim_{\beta'\to\beta} (\beta' - \beta) F_{\epsilon\epsilon\epsilon_1\ldots\epsilon_n}(\beta' + i\pi, \beta, \beta_1, \ldots, \beta_n) =$$
$$\left(1 - \prod_{i=1}^{n} S_{\epsilon\epsilon_i}(\beta - \beta_i)\right) F_{\epsilon_1\ldots\epsilon_n}(\beta_1, \ldots, \beta_n) \quad . \tag{4.15}$$

In [104, 105] it was shown that the operators defined by the matrix elements (4.3) satisfy proper locality relations provided that the form–factors satisfy the properties (4.6)-(4.9) and the residue equations (4.15) and (4.11).

4.2 Parameterization of the n–particle form–factor

In order to find solutions for the above discussed equations one needs to find a convenient parametrization of the form–factors. A solution process which has shown to be very useful [54, 65, 119], is to start with the calculation of the two–particle form–factor and then to parameterize the n–particle form–factor in terms of it. Let us discuss these steps in more detail.

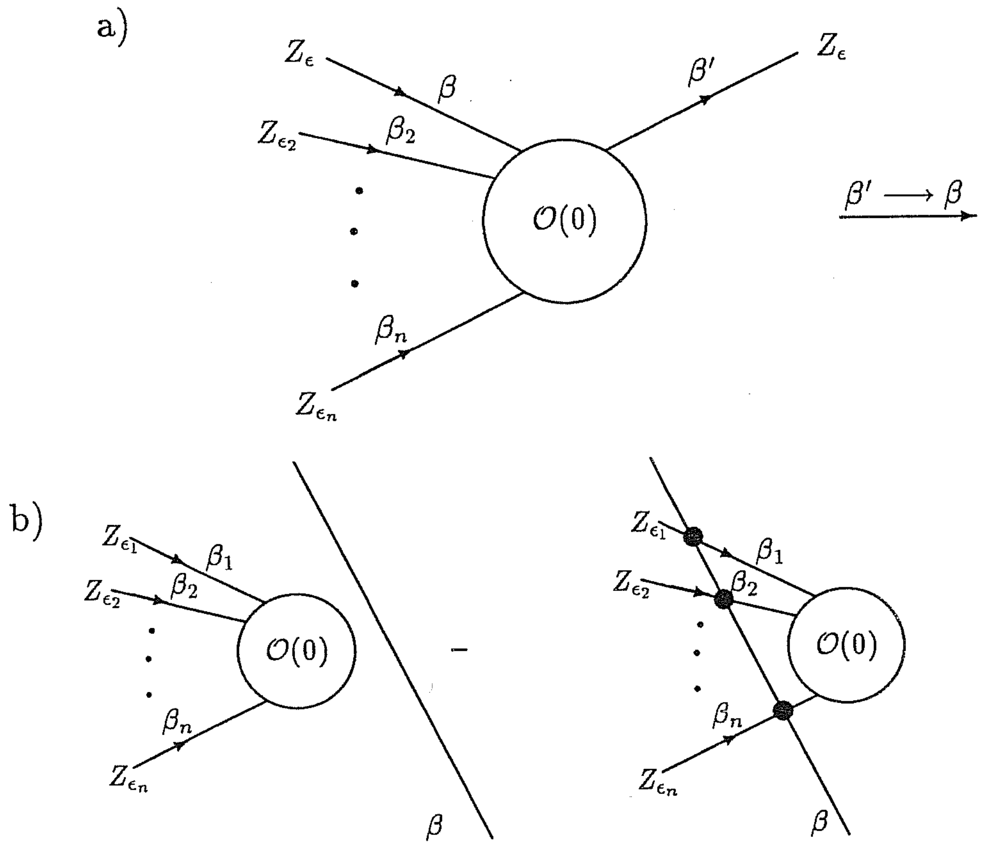


Figure 4.7: Recursion relation for the kinematical poles

The Watson's equations for $n = 2$ read as

$$F_{ab}(\theta) = S_{ab}(\theta)F_{ab}(-\theta) \quad , \quad F_{ab}(i\pi - \theta) = F_{ab}(i\pi + \theta) \quad . \tag{4.16}$$

This set of equations can be solved with the help of the following observation [65]. If the S–matrix element S_{ab} can be written in an integral representation of the form

$$S_{ab}(\theta) = \exp\left\{\int_0^\infty \frac{dx}{x} f(x) \sinh\left(\frac{x\theta}{i\pi}\right)\right\} \quad , \tag{4.17}$$

then a solution of (4.16) is given by

$$F_{ab}(\theta) = \exp\left\{\int_0^\infty \frac{dx}{x} f(x) \frac{\sin^2\left(\frac{x(i\pi-\theta)}{2\pi}\right)}{\sinh x}\right\} \quad . \tag{4.18}$$

Note that multiplying the expression (4.18) by an arbitrary function of $\cosh\beta$ we find another solution of equations (4.16). In order to determine the final form of $F_{ab}(\theta)$ it is necessary to consider a specific theory and to know the physical nature of the operator $\mathcal{O}$. In chapter 8 we will discuss extensively this problem analyzing a specific theory, the Sinh-Gordon model.

In order to select one specific solution we define the *minimal two particle form–factor* F_{ab}^{min} as the solution of equations (4.16) with the additional property that it is analytic in $0 < \text{Im}\beta < \pi$ and has no zeros in the same range.

For the discussion of the structure of the n–particle form–factors, we choose for simplicity the form–factor $F_{1,\dots,1}$ where the indices indicate that we discuss the form–factor corresponding to the fundamental particle of the considered theory. This is just a technical simplification since other form–factors can be treated in a similar way. In the following we drop the indices referring to the form–factor as F_n. In general the form–factor can be parameterized as

$$F_n(\theta_1, \dots, \theta_n) = K_n(\theta_1, \dots, \theta_n) \prod_{i<j} F^{min}(\theta_{ij}) \tag{4.19}$$

Then the function K_n needs to satisfy Watson's equations (4.7) and (4.8) with an S–matrix factor $S = 1$. Therefore it is a completely symmetric, $2\pi i$–periodic function of β_i. It must contain all expected kinematical and bound state poles. Finally it will contain the information on the operator $\mathcal{O}$.

Since we know the possible scattering processes we can split the function K_n further in order to determine the pole structure. The kinematical poles are expected at the rapidity values $\theta_i \to \theta_j + i\pi$. These poles can be generated by the completely symmetric function $\prod_{i<j}(\cosh \frac{1}{2}\beta_{ij})^{-1}$, satisfying the periodicity requirement. Assume further that the S–matrix element S_{11} exhibits poles at $\theta = i\alpha_l\pi$ for $l = 1, \ldots, p$. Then according to (4.11) also the form–factor exhibit poles which can be generated by the function

$$\prod_{i<j} \frac{1}{\sinh \frac{1}{2}(\theta_{ij} - i\alpha_l\pi) \sinh \frac{1}{2}(\theta_{ij} + i\alpha_l\pi)} \quad ,$$

for each pole. The final parameterization of the n–particle form–factor reads as

$$F_n(\theta_1, \ldots, \theta_n) = Q_n(\theta_1, \ldots, \theta_n) \prod_{i<j} \frac{F^{min}(\theta_{ij})}{\cosh \frac{1}{2}\beta_{ij} \prod_{l=1}^{p} \sinh \frac{1}{2}(\theta_{ij} - i\alpha_l\pi) \sinh \frac{1}{2}(\theta_{ij} + i\alpha_l\pi)} \quad . \tag{4.20}$$

Q_n is now a symmetric function free of singularities. A similar parameterization is possible also for the other form–factors $F_{\epsilon_1 \ldots \epsilon_n}$, but is more complicated since the terms in the product $\prod_{i<j}$ will depend on the corresponding indices.

The form factor equations have been reduced through this parameterization to a set of coupled recursive relations for Q_n. It has been shown [25] that the most general form for Q_n is

$$Q_n(\beta_1, \ldots, \beta_n) = \frac{1}{(\prod_{i=1}^{n} e^{\beta_i})^N} \mathcal{P}(e^{\beta_1}, \ldots, e^{\beta_n}) \quad ,$$

with $\mathcal{P}$ a polynomial. In this way one obtains polynomial recursive equations which in certain cases can be solved explicitly.

In this chapter we discussed only the defining equations for the form– factors and some of their general properties. This is due to the fact that the behaviour of the form–factors will depend on the chosen theory and on the operator under investigation. In chapter 8 we will solve explicitely the form–factor equations for the Sinh–Gordon model and discuss some physical implications.

Part II

S-Matrix Bootstrap

"Bootstrap"–method means literally the determination of the S–matrix by itself, without additional information. In fact, in chapter 3 we have discussed the equations corresponding to to crossing invariance, unitarity, integrability and bound states, which allow the explicit determination of the full S–matrix. In this part we want to discuss some applications of this method, show to what extend it is self-contained and use it to determine the on–shell scattering data of several models of perturbed conformal field theory.

The S–matrix takes a particular simple form if the spectrum consists only of scalar particles. We devote chapter 5 to this class of models. For that case explicit calculations are rather simple and one can analyze in detail the consequences of the bootstrap method.

Unfortunately the set of diagonal S-matrices is rather limited. For this in general one needs also to consider the degeneracy structure of the particles. For example most perturbed minimal models fall into this class of non–diagonal S–matrices. In chapter 6 we discuss the possible description of such systems. We will study how the bootstrap equations change and how they can be applied in this more general case.

5 Diagonal S-Matrices

Even though the diagonal S–matrices are a rather limited set, they are an ideal playground in order to examine the bootstrap–ideas. In section 5.1 we present our results [74] on an axiomatic bootstrap approach. This means that we take the constraints on the S–matrix as a set of axioms and try to classify all consistent solutions of it.

In section 5.2 we review some results on Toda field theories. Real coupled theories exhibit a spectrum of scalar particles (see section 2.4) and the S–matrices can be written in a unified form using a group theoretic language. In order to describe S–matrices of imaginary coupled theories we introduce the basic notions of quantum group. We discuss in which cases these theories become scalar S–matrices through the mechanism of quantum group reduction.

One class of such quantum group reduced models are the perturbed minimal models $\mathcal{M}_{2,2n+3} + \Phi_{1,2}$. In section 5.3 we give our construction of the S–matrices for these theories [72, 71] and discuss their features. We show that in order to explain their analytic structure the bootstrap principle has to be generalized.

Finally in section 5.4 we discuss a peculiar perturbed minimal model $\mathcal{M}_{2,9} + \Phi_{1,4}$ whose S–matrix we have constructed in [71]. This perturbation is especially interesting, since it does not fall into the class of general perturbations $\Phi_{1,2}, \Phi_{2,1}$ and $\Phi_{1,3}$.

5.1 Axiomatic bootstrap approach

We saw in section 3.2, that a particular simple class of S–matrices is given by those massive integrable systems which are parity invariant and have no degeneracy in the mass spectrum. In these cases the general solution of unitarity and crossing invariance (3.17) conditions for the two-body scattering is given by

$$S_{ab}(\theta) = \prod_{x \in X_{ab}} \frac{\tanh \frac{1}{2}(\theta + i\pi x)}{\tanh \frac{1}{2}(\theta - i\pi x)} = \prod_{x \in X_{ab}} f_x(\theta) \quad . \tag{5.1}$$

The sets of positive $\{x\}$'s or $\{1 - x\}$'s in (5.1) are related to the position of the poles in the scattering amplitudes S_{ab}, *i.e.* they signal possible bound states in this channel.

An interesting open problem is the classification of *all* consistent sets X_{ab} satisfying the bootstrap equations. Such a classification would give also a list of all possible consitent integrable massive field theories with a spectrum consisting of scalar particles. Further interesting conjectures are related to this. There is a hope that all consistent S-matrices can be related to some Toda theories. No counter examples are known at this time. But at the other hand it is not even possible to prove, that all o all consistent bootstrap systems correspond to some integrable field theory. A general answer to these questions is not available at this time, but as we will show, a first step in this direction cn be done by classifying the consistent bootstrap systems,

Let us analyze systems of the form (5.1). Starting from the scattering amplitude S_{11} of the lightest particle, one can compute the S–matrices of the bound states with higher mass using iteratively the bootstrap equation (3.16). But not all initial S_{11} give rise to a closed bootstrap process. Furthermore, since in principle we have the possibility to choose for each function f_x the pole at $\theta = i\pi x$ or that at $\theta = i\pi(1-x)$ in order to continue the bootstrap, at each step of the process we can have ramification points. The natural structure associated to eqs. (3.16) is that of a schematic tree with the node of each set of branches representing an S-matrix reached at that stage of iteration and the branches from each node are the possible new singularities one can use to continue the process (fig.5.1).

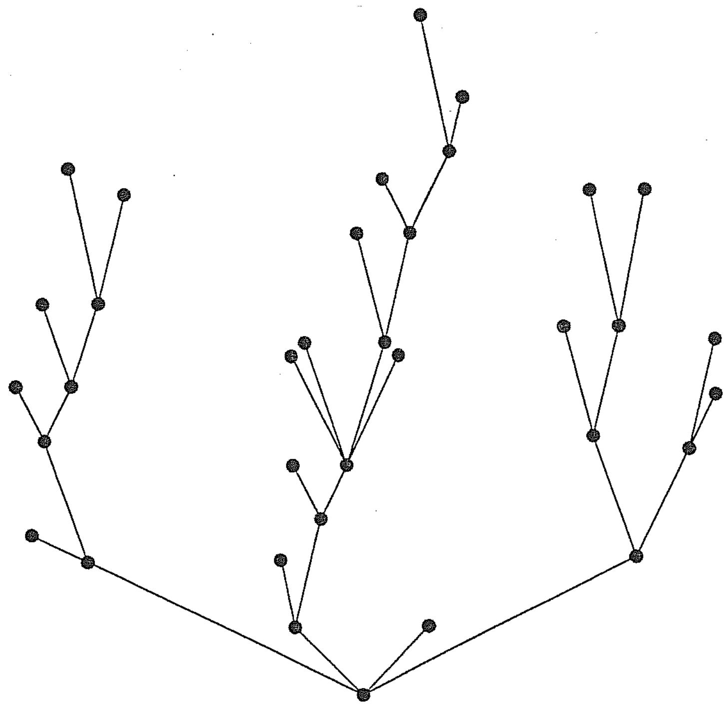


Figure 5.1: The "bootstrap tree"

The problem is to select S_{11} and then, out of all possible trees arising from it, only those ones, which give rise to a consistent set of S-matrices. The precise requirements of consistency will be formulated below. One can easily convince oneself that many initial choices for S_{11} can be immediately discarded because the result of applying eqs. (3.16) will not be a function of the form (5.1), *i.e.* eqs. (3.16) map outside the space $\mathcal{F}$. But it remains to investigate those ones which give rise to a set of functions belonging to the functional space $\mathcal{F}$.

One of the fundamental questions is whether the bootstrap tree closes or will grow to include an infinite number of particles. For instance, it seems possible to generate easily an infinite path in the following way. Let us take $S_{11} = f_{\frac{1}{3}-\epsilon}$ $(\epsilon > 0)$ and choose the singularity at $\theta = i\pi(\frac{1}{3} - \epsilon)$ to start with. This gives a bound state A_2 with mass equal to $m_2 = 2\cos\pi(\frac{1}{6} - \frac{\epsilon}{2})m_1$ and, using (3.16), $S_{22} = f_{\frac{1}{3}+2\epsilon}(f_{\frac{1}{3}-\epsilon})^2$. Then, choosing as a new singularity the pole at $\theta = i\pi(\frac{1}{3} + 2\epsilon)$ we find a new bound state A_3 with mass $m_3 = 2\cos\pi(\frac{1}{6} + \epsilon)\ \ m_2$. Proceeding in this way and taking ϵ arbitrarily small, it seems that we have generated a bootstrap with infinitely many particles, a subset of those with the unbounded masses $m_n \sim 3^{(n-1)/2}m_1$. Actually this is not a consistent system (see below) and the aim of this paper is to select only those paths in the bootstrap tree which give rise to a consistent set of S-matrices. After the discussion of the requirements a consistent set of S-matrices has to satisfy, we analyze the bootstrap tree generated by $S_{11} = f_x$, *i.e.* with only one singularity. We conclude that the only consistent solution is obtained when $x = 2\pi/(2n+1)$, $n = 1, 2, \ldots$. The spectrum is that of the breathers of the sine-Gordon model, $m_k = 2m\sin k\pi/(2n+1)$, $k = 1, \ldots n$. For all other values of x, the S-matrices obtained by the bootstrap equations (3.16) present higher order poles not explained only referring to the particle content of the theory or they are not compatible with the existence of higher conserved charges. In the first case the theory is not complete whereas in the latter case we are not allowed to use elasticity and factorization to solve the scattering problem. Later on we present our investigation of the bootstrap system defined by a S_{11} with higher number of f_x terms and we comment our results.

The bootstrap systems are severely constrained by a set of consistency equations and by the requirement that the higher order singularities find explanation in terms of multiple re-scattering processes. In this section we discuss the role of the consistency equations and in the next one we recall the basic features of the higher order poles. Let us assume that in the theories under consideration there exist a set of integrals of motion Q_s where s indicates the spin. The operators Q_s act diagonally on the asymptotic states

$$Q_s \mid A_{a_1}(\theta_{a_1})\ldots A_{a_n}(\theta_{a_n}) > \ = \sum_{i=1}^{n} \gamma_s^{a_i}\ \ e^{s\theta_{a_i}} \mid A_{a_1}(\theta_{a_1})\ldots A_{a_n}(\theta_{a_n}) > \quad . \tag{5.2}$$

The eigenvalues γ_s^a are constants which depend on particle identity and they satisfy a linear system of consistency equations which involve the resonance angles u_{ab}^c and the spins s [115]

$$\gamma_s^a \ e^{is\overline{u}_{ac}^b} + \gamma_s^b \ e^{is\overline{u}_{bc}^a} = \gamma_s^c \quad . \tag{5.3}$$

These homogeneous equations are always satisfied by $\gamma_s^a = 0$ $(\forall a, s)$ but in this case (5.2) implies the absence of any operator Q_s. In this case we cannot any longer use the factorization property of the S-matrix to solve the scattering problem. Suppose that there are nontrivial solutions of (5.3). Normalizing the nonzero eigenvalues of the lightest particle A_1 to 1, it is easy to show by induction that all other eigenvalues are real and eqs. (5.3) can be written in terms of two equations

$$\gamma_s^a = \frac{\sin(s\overline{u}_{bc}^a)}{\sin(s\overline{u}_{ac}^b)}\gamma_s^b \tag{5.4}$$

and

$$(\gamma_s^c)^2 = (\gamma_s^a)^2 + (\gamma_s^b)^2 + 2\gamma_s^a\gamma_s^b\cos(su_{ab}^c) \quad . \tag{5.5}$$

Eq. (5.4) is particularly useful because in order to have a non zero value for γ_s^a and γ_s^b the above ratio of sines should be *independent* of any bound state A_c appearing in the channel $\mid A_a A_b >$. Therefore, knowing the resonance angle of one bound state in this channel we can use this equation (a) to correctly identify the location of the other ones or (b) to prove that it is not possible to have higher order conserved charges compatible with the bootstrap. Simple examples clarify the above consideration. Consider two bootstrap systems defined by the following S_{11} functions:

1.

$$S_{11} = -f_{\frac{1}{9}} f_{\frac{5}{9}} \quad . \tag{5.6}$$

We identify the poles at $\theta = i\pi/9$ and $i5\pi/9$ with two new bound states m_2 and m_3. Applying (3.16) we can compute

$$S_{12} = f_{\frac{1}{6}} f_{\frac{17}{18}} f_{\frac{11}{18}} f_{\frac{1}{2}} \quad . \tag{5.7}$$

In this amplitude the pole at $\theta = i17\pi/18$ corresponds to the particle A_1. This angle is u_{12}^1 and therefore fixes the ratios (5.4) and from table 5.1 we see that we can identify the poles at $\theta = i\pi/6$ and $\theta = i11\pi/18$ as due to other bound states. The conserved spins are $s = 1, 5, 7, 9, 11, 13, 17$, mod (18).

2.

$$S_{11} = -f_{\frac{1}{5}} f_{\frac{1}{7}} \quad . \tag{5.8}$$

Let us apply the bootstrap eqs. (3.16) using for instance $u_{11}^2 = \pi/7$. We obtain

$$S_{12} = f_{\frac{3}{14}} f_{\frac{1}{14}} f_{\frac{19}{70}} f_{\frac{9}{70}} \quad . \tag{5.9}$$

In this amplitude the pole at $\theta = i13\pi/14$ is due to the particle A_1 and this angle fixes the ratios γ_1^s/γ_2^s. But from table 5.2 we see that there is no other pole in this amplitude which gives the same value of these ratios for $s = 1, 3, \ldots 35$. Hence the bootstrap system defined by (5.8) is not supported by the existence of higher additional charges and is not a consistent model. The same result is obtained starting by any other possible u_{11}^2.

s	$\frac{\gamma^2}{\gamma^1}$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	$\frac{11\pi}{18}$	$\frac{7\pi}{18}$
1	1.970	1.970	1.970	1.970	1.970
3	1.732	1.732	0.303	−0.866	0.647
5	1.286	1.286	−1.177	1.286	−0.920
7	0.684	0.684	2.159	0.684	−0.989
9	0	0	−1.219	0	−2.888
11	−0.684	−0.684	0.850	−0.684	−0.258
13	−1.286	−1.286	0.333	−1.286	1.040
15	−1.732	−1.732	10.190	0.866	0.068
17	−1.970	−1.970	−0.507	−1.970	3.603

Table 5.1: Ratio of γ_s^2/γ_s^1, s=1,...17, calculated for the poles occurring in the amplitude S_{12} for the Bootstrap starting with $S_{11} = -f_{\frac{1}{9}} f_{\frac{5}{9}}$. The first column contains the ratios for the identified pole u_{12}^1.

In order to have a consistent set of elastic S-matrices, one has also to analyze the additional constraints related to the higher order singularities introduced by the bootstrap equations. The basic idea is due to Coleman and Thun [31] and has been generalized in [18, 29]. In two dimensions, a box diagram corresponding to multiparticle scattering

s	$\frac{\gamma^2}{\gamma^1}$	$\frac{3\pi}{14}$	$\frac{11\pi}{14}$	$\frac{19\pi}{70}$	$\frac{51\pi}{70}$	$\frac{9\pi}{70}$	$\frac{61\pi}{70}$
1	1.950	1.950	1.950	1.950	1.950	1.950	1.950
3	1.564	1.564	−0.344	1.325	−0.841	1.812	0.836
5	0.868	0.868	−0.705	0.284	1.204	1.547	−0.66
7	0	0	−3.343	−0.821	0.450	1.171	−1.41
9	−0.868	−0.868	0.906	−1.582	0.142	0.710	−4.28
11	−1.564	−1.564	−0.296	0.087	−1.086	0.195	−1.08
13	−1.950	−1.950	6.053	−1.857	−1.438	−0.340	0.534
15	−1.950	−1.950	−1.122	−0.855	0.961	−0.862	1.184
17	−1.564	−1.564	0.905	0.289	−2.275	−1.340	−5.405
19	−0.868	−0.868	−1.064	1.225	−1.569	−1.765	1.498
21	0	0	0.840	1.441	0.659	−2.194	−0.423
23	0.868	0.868	−1.039	3.016	−0.792	−11.529	−1.069
25	1.564	1.564	2.395	1.458	0.078	−1.548	1.136
27	1.950	1.950	−0.002	0.281	−0.416	−1.596	−2.353
29	1.950	1.950	0.731	−0.798	1.841	−1.381	0.315
31	1.564	1.564	−2.052	−1.388	1.527	−1.023	1.013
33	0.868	0.868	−1.410	0.088	−1.009	−0.568	−0.429
35	0	0	−0.118	−2.271	2.518	−0.052	5.248

Table 5.2: Ratio of γ_s^2/γ_s^1, s=1,...35, calculated for the poles occurring in the amplitude S_{12} calculated for the Bootstrap starting with $S_{11} = -f_{\frac{1}{5}} f_{\frac{1}{7}}$. The first column contains the ratios for the identified pole u_{12}^1.

is singular if it can be drawn as a geometrical figure with all internal and external lines on-shell (fig.5.2).

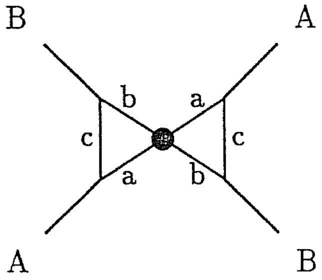


Figure 5.2: Double scattering process responsible for higher order pole singularities in the S-matrix

This is equivalent to evaluating the discontinuity of this graph by the Cutkosky rules: the point interactions correspond to S-matrix elements and the lines to the on-mass shell propagators. The higher order poles are located at

$$\theta_{AB} = 2\pi - u^a_{Ac} - u^b_{Bc} \quad . \tag{5.10}$$

If S_{ab} is regular at this value of the rapidity, we obtain a double pole, otherwise if S_{ab} is itself singular at θ_{AB}, we get a higher order singularity. Of course this explanation only works if we can actually draw such a graph, *i.e.* if

$$u^A_{ac} + u^B_{bc} < \pi \quad . \tag{5.11}$$

If this condition does not hold, it is not possible to explain the appearance of higher order poles in an S-matrix theory.

In particular, as was noticed in [18], the scattering amplitude S_{11} of the lightest particle cannot have higher order poles because the resonance angle of two heavy particles with the lightest one is greater than $2\pi/3$ and therefore it is impossible to draw a figure like *fig*.5.2 with the particle A_1 on all four external legs and the internal ones on-shell.

Let us consider a bootstrap system with

$$S_{11} = f_x(\theta) \quad . \tag{5.12}$$

Our approach consists in applying eqs.(3.16) as far as there are singularities in the functions S_{ab} identifiable as bound states. We prove [74] that there is only one possible way to implement the bootstrap which satisfies the consistency equations (5.4). We will find that the spectrum is given by

$$m_k = 2m \sin \frac{kx\pi}{2} \quad , \tag{5.13}$$

where m is an arbitrary normalization constant. Moreover, if we also require a consistent explanation of the higher order poles, we have to put the mass of the lightest particle A_{2n+1} produced by the bootstrap equal to zero, and decouple this particle from the massive sector of the theory. This is equivalent to have the following quantization condition for x

$$m_{2n+1} = 0, \qquad \Longrightarrow x = \frac{2}{2n+1} \quad . \tag{5.14}$$

In order to get familiar how this works, let us first study the cases when x is close to 2/3.

a) $x > 2/3$. In this case the singularity at $\theta = i\pi x$ corresponds to a bound state A_2 with mass m_2 less than m_1

$$\frac{m_2}{m_1} = 2\cos\frac{x\pi}{2} \quad . \tag{5.15}$$

and therefore contradicts the assumption that A_1 was the lightest particle. This fact alone is not necessarily a drawback since we were aware that the bootstrap allows computing scattering amplitudes choosing any arbitrary particle as starting point. Therefore the only possible interpretation would be that our initial identification of the lightest particle was wrong. But the real difficulty comes when we compute

$$S_{22} = f_{2x}(f_x)^2 \quad . \tag{5.16}$$

because we see that in this amplitude (which is now that of the lightest particle) appears a double pole which cannot occur. Hence there is no consistent set of S-matrices starting from $S_{11} = f_x$ when $x > 2/3$.

b) x slightly less than 2/3, $x = (2/3 - \epsilon)$, $(\epsilon \to 0)$. In this case the bootstrap produces three bound states with masses (fig.5.3)

$$m_k = 2m\sin\frac{kx\pi}{2}, \qquad k = 1,2,3 \tag{5.17}$$

and S-matrices

$$S_{11} = f_x \qquad S_{12} = f_{\frac{3x}{2}} f_{\frac{x}{2}} \qquad S_{13} = f_x f_{2x}$$
$$S_{22} = f_{2x}(f_x)^2 \quad S_{23} = f_{\frac{5x}{2}} f_{\frac{x}{2}} (f_{\frac{3x}{2}})^2 \quad S_{33} = f_{3x}(f_x f_{2x})^2 \tag{5.18}$$

As in case a), we obtain a particle A_3 with mass m_3 less than m_1. S_{33} contains as well

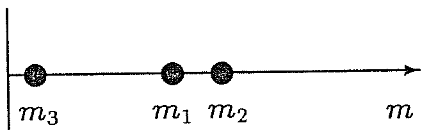


Figure 5.3: Mass spectrum generated by $S_{11} = f_x$ with x close to $2\pi/3$

unwanted double poles. The only way to make this system consistent is to push $m_3 \to 0$ and correspondingly decouple A_3 from the rest of the theory. In this limit all S-matrices involving A_3 go to the identity and the other particle state A_2 becomes identical to A_1. The initial three-particle system collapses to that one with only one particle state and S-matrix

$$S_{11} = f_{\frac{2}{3}}(\theta) \quad . \tag{5.19}$$

This corresponds to the S-matrix of the Yang–Lee model [24], *i.e.* $\mathcal{M}_{2,5} + \Phi_{1,3}$.

Let us consider now the general case and let us prove that there exists only one path in the bootstrap tree which satisfies the consistency equations. The proof is by induction [74]. Starting with $S_{11} = f_x$, we obtain a new bound state whose mass can be written as

$$\frac{m_2}{m_1} = 2\cos\frac{x\pi}{2} = \frac{2m\sin x}{2m\sin\frac{x}{2}} \tag{5.20}$$

where m is an arbitrary mass scale. We can compute S_{12} by applying (3.16)

$$S_{12} = S_{11}(\theta - i\frac{x\pi}{2}) S_{11}(\theta + i\frac{x\pi}{2}) = f_{\frac{3x}{2}} f_{\frac{x}{2}} \quad . \tag{5.21}$$

We get a function with four singularities: those at $\theta = \frac{1}{2}i\pi x$ and $\theta = (1 - x/2)\pi$, from the $f_{\frac{x}{2}}$ term, and those at $\theta = \frac{3}{2}i\pi x$ and $\theta = i(1 - \frac{3}{2}x)\pi$ from the $f_{\frac{3x}{2}}$ term. Among these, that one at $\theta = i(1 - x/2)\pi$ corresponds to the bound state A_1. Therefore we have correctly identified this angle as the resonance angle due to a bound state. We can now apply (5.4) in order to decide which of the two poles in $f_{\frac{3x}{2}}$ corresponds to a new bound state A_3. The answer turns out to be that one at $\theta = \frac{3}{2}i\pi x$. This means that we cannot use the other singularity at $\theta = (1 - \frac{3}{2}x)\pi$ to implement further the bootstrap if we require a non zero solution of the consistency equations but we are obliged to follow the path defined by the resonance angle $u_{12}^3 = \frac{3}{2}\pi x$. The mass of the new particle is

$$\frac{m_3}{m_1} = \frac{2m \sin\frac{3\pi x}{2}}{2m \sin\frac{\pi x}{2}} \quad . \tag{5.22}$$

We can compute

$$S_{13} = S_{12}(\theta - i\overline{u}_{23}^1)S_{11}(\theta + i\overline{u}_{13}^2) = f_x f_{2x} \quad . \tag{5.23}$$

Repeating here the same reasoning of before, we can identify the pole at $\theta = i(1-x)\pi$ as u_{13}^2 and in this way fix the ratio of the conserved quantities γ_1/γ_3 in (5.4). The singularity due to a new bound state A_4 is that at $\theta = iu_{13}^4 = i2\pi x$. The mass of this new bound state is

$$\frac{m_4}{m_1} = \frac{2m \sin(2\pi x)}{2m \sin\frac{\pi x}{2}} \quad . \tag{5.24}$$

The process can be continued up to the particle A_{2n+1} where n is defined by

$$\frac{2}{2n+3} < x \leq \frac{2}{2n+1} \tag{5.25}$$

and has to be completed by the computation of the remaining S-matrices. The mass spectrum is given by

$$m_k = 2m \sin\frac{kx\pi}{2}, \qquad k = 1, 2, \ldots 2n+1 \quad . \tag{5.26}$$

The particle A_{2n+1} is the lightest one and the corresponding S-matrix element has a plethora of double poles. We can get a consistent set of S-matrices only if we put $m_{2n+1} = 0$ and decouple this particle from the theory. In this limit the remaining $2n$ particles become identical in couple and we end up with a n-particle system with generic S-matrix [51, 102]

$$S_{ab} = f_{\frac{|a-b|}{2n+1}} f_{\frac{a+b}{2n+1}} \prod_{k=1}^{min(a,b)-1} \left(f_{\frac{|a-b|+2k}{2n+1}}\right)^2 \quad . \tag{5.27}$$

$(a, b = 1, 2, \ldots n)$. All double poles have now an explanation in terms of multiscattering processes and the conserved spins are all odd numbers but multiple of $2n+1$

$$s = 1, 3, \ldots, 2n-1, 2n+3, \ldots, 4n+1 \pmod{4n+2} \quad . \tag{5.28}$$

The price to be paid is that these S-matrices are not *one-particle unitary* and indeed correspond to the $\Phi_{1,3}$ deformation of the non-unitary minimal models $\mathcal{M}_{2,2n+3}$ [24, 51, 102].

The problem to find consistent sets of S-matrices starting with a S_{11} with more than one singularity gets more complicated. In order to investigate the bootstrap tree generated by a generic S_{11} we have designed a computer algorithm which spans all possible paths and selects only those ones which satisfy our requirements of consistency. With the help of this algorithm, which includes also one-particle unitarity as an additional requirement, we found [74] analyzing a large set of initial S-matrix elements ($\sim 10^9$) with two to five factors f_x the only two systems:

$$S_{11} = -f_{\frac{1}{9}} f_{\frac{5}{9}} \quad , \tag{5.29}$$

which corresponds to the minimal S-matrix of the E_7 system [50, 29] and

$$S_{11} = f_{\frac{1}{3}} f_{\frac{2}{5}} f_{\frac{1}{15}} \quad , \tag{5.30}$$

which corresponds to the minimal S-matrix of the E_8 system [115, 50]. In the case of S_{11} with four and five functions f_x, in the range we analyzed, the bootstrap does not give rise to any consistent system.

Let us summerize the results of this section. We have presented our investigations on the classification of the non-degenerate S-matrices in two dimensions. Our analysis is based on the bootstrap equations (3.16) and consistency requirements alone. In the case of S_{11} with one function, $S_{11} = f_x$, the only consistent bootstrap system is obtained when $x = 2/(2n+1)$. The cases of consistent S-matrices with higher number of singularities fall in the E_7 (two f_x functions in S_{11}) and E_8 (three f_x functions in S_{11}) systems. We have not found any example which is not related to the Dynkin diagrams and therefore to Toda field theories. This gives additional support to the hypothesis that the only consistent sets of non-degenerate S-matrices reduce to those constructed on the Lie algebras [29, 89].

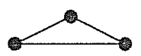
5.2 S-matrices for ATFT

In section 2.4 we have analyzed the classical properties of ATFT. We saw, that the models behave extremely different, depending on whether the coupling constant is chosen real or imaginary. Much more is known for real coupling theories. They exhibit a spectrum of only scalar particles and this reduces the possible form of the S–matrices to (5.1). For all the real coupled ATFT the S-matrices have been conjectured. For the simply–laced theories one can write down the S–matrices and their pole–structure in a unified way, relating them to the properties of the root system of the corresponding Lie–algebra. For non–simply laced algebras such a unified scheme is still missing and the S–matrices have a more complicated structure.

On the other hand our main interest are imaginary coupled ATFT. All of them have a soliton–spectrum and therefore also the S–matrix elements of the fundamental particle are degenerate. But they can have scalar bound states, called breathers. For rational values of the coupling constant the model can be reduced in order to describe deformed minimal models. For certain special values the solitons drop out of the physical spectrum under this reduction mechanism and only scalar particles remain. In this way we again obtain S-matrices of a purely diagonal structure.

5.2.1 Real coupled ATFT

For real coupled ATFT the particle spectrum consists of only scalar particles. Let us first discuss the simply–laced case. A special feature allows a simple determination of the S–matrix. This is that the ratios of the masses remain unchanged after quantization [5, 18, 29]. But because of eq. (3.15) the masses determine the possible fusion angles (see fig.3.4), and therefore the poles of the S–matrix. Requiring further the existence of conservation laws it is possible to construct the S–matrix.

Before discussing the general case, let us analyze a simple example. Consider the algebra $A_2^{(1)}$. The corresponding Dynkin–diagram is , and the masses of the particles (2.31) are $m_1 = 2m \sin\frac{\pi}{3}$ and $m_2 = 2m\sin\frac{2\pi}{3}$, m_2 being the antiparticle of m_1. Now the fusion angle between the particles is $u_{11}^2 = \frac{2\pi}{3}$, which must correspond to a pole of the S–matrix. Therefore we can conjecture the S–matrix as $S_{11} = f_{\frac{2}{3}}$. It turns out that this is not the full S–matrix but only it's *minimal part.* The additional factors depend on the coupling constant and do not introduce additional poles. They are determined by perturbative calculations [18, 29]. The full S–matrix finally takes the form

$$S_{11} = f_{\frac{2}{3}} f_b f_{-\frac{2}{3+b}} \quad ,$$

where b is the coupling constant depending factor given by

$$b(\beta) = \frac{\beta^2}{2\pi h}\left(1 + \frac{\beta^2}{4\pi}\right)^{-1} \quad . \tag{5.31}$$

The above analysis is only schematic and incomplete. The conserved charges, bootstrap equations, higher pole structure, *etc.* must be checked, in order to confirm that the proposed S–matrix is consistent. But generalizing the above method all simply–laced S–matrices have been constructed in a case by case study [18, 29]. Using the underlying algebra they can be expressed in a unified way. Take the Dynkin–diagram of the corresponding finite dimensional Lie–algebra of the ATFT which you want to examine and color the nodes alternatingly. Associate to this bicoloring a value $c(i) = \pm 1$. Further introduce the following building blocks for the S–matrix:

$$\{x\}_\theta = \frac{[x]_\theta}{[x]_{-\theta}} \quad , \quad [x]_\theta = \frac{< x+1 >_\theta < x-1 >_\theta}{< x+1-b >_\theta < x-1+b >_\theta}$$

$$\text{and} \qquad < x >_\theta = \sinh \frac{1}{2}(\theta + \frac{i\pi x}{h}) \quad .$$

Then the S–matrix can be written in general as [40]

$$S_{ij}(\theta) = \prod_{q=1}^{h} \left\{ 2q - \frac{c(i)+c(j)}{2} \right\}_\theta^{-\frac{1}{2}\lambda_i \gamma^q \gamma_j} \quad , \tag{5.32}$$

where γ_i are the simple roots and λ_j denote the fundamental weights of the algebra.

This expression satisfies the bootstrap rules and all consistency requirements demanded for a scattering matrix [53]. Further it is consistent with perturbative calculations [19].

Non-simply laced theories are less explored. The fundamental problem is that the masses renormalize, that is perturbative calculations show that their ratios change with the coupling constant. Therefore in general it is also impossible to find a solution to the S–matrix constraint equations which is independent of the coupling constant. In other words no *minimal S–matrix* exists for these theories. Nevertheless recently the S–matrices have been proposed [37], and checked to be consistent [35].

5.2.2 Imaginary coupled ATFT

Because of their soliton spectrum, imaginary coupled ATFT are much more difficult to describe. In section 3 we saw that the main requirement is that the S–matrix must satisfy the Yang–Baxter equation (3.6) additionally to unitarity and crossing constraints. Now a large class of Yang–Baxter algebras is given by the affine quantum groups. It is an obvious guess, that the S–matrix of an ATFT corresponding to an algebra $\hat{\mathcal{G}}$ might have the same degeneracy structure as the R–matrix of the quantum–group $U_q(\hat{\mathcal{G}})$, that is

$$S_{\epsilon_1\epsilon_2}^{\epsilon_1'\epsilon_2'}(\beta) = S_0(\beta) R_{\epsilon_1\epsilon_2}^{\epsilon_1'\epsilon_2'}(\beta) \quad .$$

The factor $S_0(\beta)$ is necessary in order to guarantee unitarity and crossing symmetry. This ansatz was confirmed in the case of the Sine–Gordon model. There the S–matrix was computed by different means [116] and identified with the R–matrix of $U_q(A_1^{(1)})$.

Since we will treat with these kind of models in following sections, we want to take the opportunity to introduce the basic notions on quantum groups. Since we do not have the space for a complete exposition, we refer the reader to the literature [43, 48, 66, 83, 110].

The algebra $U_q(\mathcal{G})$ can be seen as a deformation of a Lie algebra, in which a complex *deformation parameter* q has been introduced. The deformed commutation relations are given by

$$\begin{gathered}
[H_i, H_j] = 0 \ , \qquad k_i k_j = k_j k_i \ , \\
\left[H_i, X_j^{\pm}\right] = \pm a_{ij} X_j^{+} \ , \qquad k_i X_j^{\pm} = q^{\pm\frac{a_{ij}}{2}} X_j^{\pm} k_i \ , \\
[X_i^+, X_j^-] = \delta_{ij} [H]_q \ , \\
\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_q \left(X_i^{\pm}\right)^{1-a_{ij}-n} X_j^{\pm} \left(X_i^{\pm}\right)^n = 0 \quad i \neq j \quad ,
\end{gathered} \tag{5.33}$$

with

$$[n]_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}} \qquad (\overset{q\to 1}{\longrightarrow} n),$$

$$\left[\begin{array}{c} n \\ m \end{array}\right]_q \equiv \frac{[n]_q[n-1]_q \dots [n-m+1]_q}{[m]_q[m-1]_q \dots [1]_q} \quad ,$$

a_{ij} is the Cartan matrix of the corresponding simple Lie Algebra, and with the abbreviations

$$k_j = q_j^{\frac{H_j}{2}} \quad , \quad q_j = q^{(r_j, r_j)} \quad .$$

In the limit $q \to 1$, the commutation relations (5.33) go over into those of the Lie algebra $\mathcal{G}$. This algebra can be given a Hopf algebra structure by introducing the so-called co-product

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \quad , \quad \Delta(X_i^{\pm}) = X_i^{\pm} \otimes q^{\frac{H_i}{2}} + q^{-\frac{H_i}{2}} \otimes X_i^{\pm} \ , \tag{5.34}$$

The co-product must satisfy certain consistency rules. They are expressed in terms of an element R, which is an invertible element of the algebra and satisfies

$$\begin{aligned} (\Delta \otimes id)R &= R_{13}R_{23} \ , \\ (id \otimes \Delta)R &= R_{13}R_{12} \ , \\ (\sigma \circ \Delta)h &= R(\Delta h)R^{-1}, \ \forall h \in \mathcal{G} \quad . \end{aligned} \tag{5.35}$$

(σ is the permutation map of two spaces). The R-matrix is an object acting in the tensor-space, i.e. $R = \sum R^{(1)} \otimes R^{(2)}$. The first two equations in (5.35) give as a consequence the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad . \tag{5.36}$$

For example the R-matrix for $U_q(sl(2))$ is given by [66]

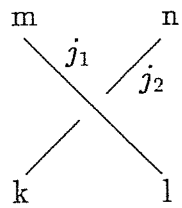
$$R = q^{\frac{H \otimes H}{2}} \sum \frac{(1-q^{-2})^n}{[n]_q!} (q^{\frac{H}{2}} X^+ \otimes q^{-\frac{H}{2}} X^-)^n q^{\frac{n(n-1)}{2}} \quad , \tag{5.37}$$

or explicitly for the fundamental representation

$$R = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \quad . \tag{5.38}$$

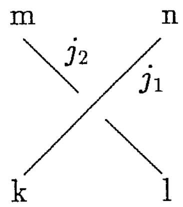
If we represent the R–matrix as

$$\left(R^{j_1 j_2}\right)^{kl}_{mn} \longrightarrow$$



, (5.39)

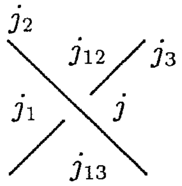
$$\left(\left(R^{j_1 j_2}\right)^{-1}\right)^{kl}_{mn} \longrightarrow$$



, (5.40)

we can draw now a similar picture for the Yang–Baxter equation as in (3.1).

There is another important representation of quantum groups, which we will need. It is the so-called shadow-world representation. In that case the R-matrix is substituted by the quantum $6j$-symbols,



$$\longrightarrow \quad (-1)^{j_{13}+j_{12}-j-j_1}\, q^{c_j+c_{j_1}-c_{j_{13}}-c_{j_{12}}} \quad \times$$

$$\left\{\begin{matrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{matrix}\right\}_q \quad . \tag{5.41}$$

They are given as [66]

$$\begin{aligned}
\left\{\begin{matrix} a & b & e \\ d & c & f \end{matrix}\right\}_q &= \sqrt{[2e+1][2f+1]}\ \ (-1)^{c+d+2e-a-b} \ \times \\
&\Delta(abe)\Delta(acf)\Delta(dce)\Delta(dbf)\ \ \textstyle\sum_z (-1)^z [z+1]! \ \times \\
&([z-a-b-e]![z-a-c-f]![z-b-d-f]![z-d-c-e]! \ \times \\
&[a+b+c+d-z]![a+d+e+f-z]![b+c+e+f-z]!)^{-1}
\end{aligned} \tag{5.42}$$

wherein we use the conventions that $[0]! = 1$ and the sum runs only over z such that no factor $[x]$ is less than zero. Further,

$$\Delta(abc) = \left(\frac{[-a+b+c]![a-b+c]![a+b-c]!}{[a+b+c+1]!}\right)^{\frac{1}{2}} \quad .$$

Note that now the indices are placed in the spaces between the lines. The transition to this representation has a correspondence in statistical models. It is the analogue of the change from vertex-models to solid on solid (SOS) models.

Finally let us discuss affine quantum groups. Their R-matrices contain an additional parameter, called the *spectral parameter*. Again we just give two basic examples. For the fundamental representation of $A_N^{(1)}$, the R-matrix becomes

$$R(x,q) = q^{\frac{1}{2}} x R_{12} - q^{-\frac{1}{2}} x^{-1} R_{21}^{-1} \quad . \tag{5.43}$$

Or for example the R matrix for $A_2^{(2)}$ in the fundamental representation can be written as

$$R(x) = (x^{-1} - 1) q^3 R_{12}(q) + (1 - x) q^{-3} R_{21}^{-1}(q) + q^{-5}(q^4 - 1)(q^6 + 1) I \quad , \tag{5.44}$$

with R_{12} being the R matrix of the spin 1 representation of $U_q(sl(2))$. These spectral depending R–matrices satisfy now the Yang- Baxter equation (3.6), and can therefore be used to construct S-matrices for degenerate particles.

5.2.3 Application to Sine-Gordon theory

For the Sine-Gordon model the mapping into the S–matrix is [13, 97, 102]

$$S(\beta) = S_0(\beta) R\left(x = e^{\frac{\pi\beta}{\xi}}, q = -e^{-\frac{i\pi^2}{\xi}}\right) \quad , \tag{5.45}$$

where $R(x,q)$ is given by (5.43). The scalar factor S_0 is obtained by requiring unitarity (3.13) and crossing symmetry (3.14). Since the R–matrix (5.43) satisfies

$$R(x,q) R(x^{-1},q) = (x^{-1}q - xq^{-1})(qx - x^{-1}q^{-1}) \quad . \tag{5.46}$$

We require

$$S_0(x) S_0(x^{-1}) = \frac{1}{(x^{-1}q - xq^{-1})(qx - x^{-1}q^{-1})} \quad .$$

$R(x,q)$ is already crossing invariant, and therefore it must be also S_0 ,

$$S_0(x) = S_0(-\frac{1}{xq}) \quad .$$

The minimal solution to these equations is

$$S_0(\beta) = \frac{1}{\sinh \frac{\pi}{\xi}(\beta - i\pi)} \times$$

$$\times \prod_{k=0}^{\infty} \frac{\Gamma(\frac{2k\pi}{\xi} + 1 + \frac{i\beta}{\xi})\Gamma(\frac{2k\pi}{\xi} + \frac{\pi}{\xi} - \frac{i\beta}{\xi})\Gamma(\frac{2k\pi}{\xi} + \frac{\pi}{\xi} + 1 - \frac{i\beta}{\xi})\Gamma(\frac{2k\pi}{\xi} + \frac{2\pi}{\xi} + \frac{i\beta}{\xi})}{\Gamma(\frac{2k\pi}{\xi} + 1 - \frac{i\beta}{\xi})\Gamma(\frac{2k\pi}{\xi} + \frac{\pi}{\xi} + \frac{i\beta}{\xi})\Gamma(\frac{2k\pi}{\xi} + \frac{\pi}{\xi} + 1 + \frac{i\beta}{\xi})\Gamma(\frac{2k\pi}{\xi} + \frac{2\pi}{\xi} - \frac{i\beta}{\xi})} \quad , \tag{5.47}$$

which has an integral representation [102]

$$S_0(\beta) = \frac{1}{\sinh \frac{\pi}{\xi}(\beta - i\pi)} \exp\left(-i \int_0^{\infty} \frac{\sin k\beta \sinh(\frac{\pi - \xi}{2})k}{k \cosh \frac{\pi k}{2} \sinh \frac{\xi k}{2}} dk\right) \quad . \tag{5.48}$$

Let us analyze the bound states of this fundamental particle. Analyzing the poles of S_0 we find that they are located at

$$\beta = i\pi - in\xi \quad , \quad n \geq 0 \ ,$$
$$\beta = in\xi \quad , \quad n \geq 0 \ ,$$

in the physical strip. Choosing $\beta = i\pi - in\xi$ as s channel pole we find that the R matrix degenerates into a 1 dimensional projector at these points. This can be interpreted as teh creation of scalar particles at those rapidity values. The S–matrix of the lightest one is

$$S_{b_1 b_1} = \frac{\sinh(\frac{\beta}{2} - \frac{i\xi}{2} + \frac{i\pi}{2})\sinh(\frac{\beta}{2} + \frac{i\xi}{2})}{\sinh(\frac{\beta}{2} + \frac{i\xi}{2} - \frac{i\pi}{2})\sinh(\frac{\beta}{2} - \frac{i\xi}{2})} = f_{\xi} \ . \tag{5.49}$$

Recall that this was exactly the type of S–matrices we investigated in section 5.1, just that there we had *no* underlying solitons and the parameter ξ turned out to be quantized. In order to make contact with these results we need to analyze the theory for rational coupling constants.

Let us change to the shadow world representation. For that we substitute in

$$R(x,q) = xq^{\frac{1}{2}} R_{12} - q^{-\frac{1}{2}} x^{-1} R_{21}^{-1} \ ,$$

the R matrix by the shadow-world objects

$$\left(R_{\frac{1}{2}\frac{1}{2}}\right)_{cd}^{ab} \longrightarrow (-1)^{a+c-b-d} \, q^{c_d + c_b - c_a - c_c} \left\{ \begin{array}{ccc} \frac{1}{2} & d & c \\ \frac{1}{2} & b & a \end{array} \right\}_q \ , \tag{5.50}$$

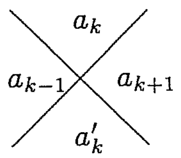
and similarly for $R^{-1}_{\frac{1}{2}\frac{1}{2}}$.

This corresponds to a base change also for the particles. We do not have solitons as a basis any more, but the new objects can be interpreted as kinks interpolating between different vacua. The bases is written as

$$|\beta_1, j_1|a_1|\beta_2, j_2|a_2|\ldots|a_{n-1}|\beta_n k_n\rangle \quad .$$

β_i are again the rapidities, j_i are the $U_q(sl(2))$ spins which also automatically distinguish breathers from kinks and a_i are the values assigned to the dual lattice. Then we can interpret the S-matrix as scattering of kinks. The kink-kink amplitudes can be pictured as

$$S\left(\beta\left|\begin{array}{cc} a_{k-1} & a_k \\ a_{k+1} & a'_k \end{array}\right|\right) =$$



In the case that the parameter q appearing in the quantum group becomes a root of unity the representations can be restricted [2, 95]. Let $\frac{\xi}{\pi} = \frac{r}{s-r}$. Then q is a root of unity, $q^r = 1$ and because of periodicity, the values of the spin can be limited to $\frac{1}{2}(r-2)$; this allows to limit the values a_k by $j_{max} = \frac{r}{2} - 1$. Further, from the shadow–world representation one has the restriction

$$|a_k - a_{k+1}| = \frac{1}{2} \quad , \tag{5.51}$$

which indicates that the R–matrix is related to the fundamental representation, where the values of the spin can be only $\pm\frac{1}{2}$.

Physically this restriction can be interpreted as a limitation on the kinks, which can only connect certain vacua. So in some way the other vacua are effectively decoupled from the theory. This idea was formulated in [46], showing that in the sine-Gordon theory there exists a BRST symmetry which decouples the higher soliton sectors. The analysis shows that the S–matrix (5.45) at the values $\xi = \frac{r}{s-r}\pi$ describes the $\Phi_{1,3}$ perturbations of the minimal models $\mathcal{M}_{r,s}$.

Let us look at $\Phi_{1,3}$ perturbations of $\mathcal{M}_{2,2n+3}$. The scattering matrices become particularly simple, since $\xi = \frac{2\pi}{2n+1}$. Because of the restriction $a_k \leq \frac{r-2}{2}$, we see that the solitons completely decouple from the theory and only breathers survive. The resulting S–matrices are exactly those found in (5.27), where now the result from that section is explained from an algebraic point of view.

These are not the only scalar S–matrices which derive from the restricted Sine-Gordon model. Take $\xi = \frac{3}{s-3}\pi$, corresponding to the Φ_{13}-perturbed models $\mathcal{M}_{3,s}$. The restriction allows only two vacua and therefore only *one* possible kink which behaves like a scalar particle. The simplest case corresponds to $s = 4$. In physical terms we are describing the perturbations of $\mathcal{M}_{3,4}$, the critical Ising model, in the thermal direction. The complicated expression for the general S–matrix reduces in this restriction to $S = -1$.

Also the other S–matrices of this series can be described in terms of only scalar particles [28, 97], though their soliton structure is reflected in the crossing relation, which reads $S(i\pi - \beta) = -S(\beta)$.

5.2.4 S–Matrices for restrictions of the Izergin Korepin model

Similar to the previous section we can describe $\Phi_{1,2}$ perturbations of minimal models. We know from section 2.4 that they correspond to the quantum group $A_2^{(2)}$. Therefore the natural ansatz [101] is to take

$$S_{12}(\beta) = S_0(\beta) R_{12}(x = e^{\frac{2\pi}{\xi}\beta}, q = e^{\frac{\pi^2 i}{\gamma}}) \tag{5.52}$$

with the R–matrix (5.44) corresponding to the $A_2^{(2)}$ quantum group. The parameter $\gamma = \frac{r}{s}$ corresponds to the model $\mathcal{M}_{r,s}$ which is perturbed in the $\Phi_{1,2}$ direction, and

$$\xi = \frac{2}{3}\frac{\pi\gamma}{2\pi - \gamma} \quad .$$

These parameters are chosen in order to match the R–matrix consistently with the corresponding field theory.

The parameter S_0 is given by

$$S_0(\beta) = \frac{1}{\pi^2}\Gamma\left(\frac{\pi + i\beta}{\xi}\right)\Gamma\left(\frac{\xi - \pi - i\beta}{\xi}\right)\Gamma\left(\frac{\frac{2}{3}\pi + i\beta}{\xi}\right)\Gamma\left(\frac{\xi - \frac{2}{3}\pi - i\beta}{\xi}\right) \quad \Xi(\beta) \quad , \tag{5.53}$$

where $\Xi(\beta)$ is the following infinite product

$$\begin{aligned} \Xi(\beta) \quad = \quad & \prod_{k=0}^{\infty} \left\{ \frac{\Gamma(\frac{\pi}{\xi} + \frac{2k\pi - i\beta}{\xi})\Gamma(\frac{2\pi}{\xi} + \frac{2k\pi + i\beta}{\xi})}{\Gamma(\frac{\pi}{\xi} + \frac{2k\pi + i\beta}{\xi})\Gamma(\frac{2\pi}{\xi} + \frac{2k\pi - i\beta}{\xi})} \frac{\Gamma(1 + \frac{2k\pi + i\beta}{\xi})\Gamma(\frac{\xi + \pi}{\xi} + \frac{2k\pi - i\beta}{\xi})}{\Gamma(1 + \frac{2k\pi - i\beta}{\xi})\Gamma(\frac{\xi + \pi}{\xi} + \frac{2k\pi + i\beta}{\xi})} \right. \\ \times \quad & \left. \frac{\Gamma(\frac{\pi}{3\xi} + \frac{2k\pi + i\beta}{\xi})\Gamma(\frac{4\pi}{3\xi} + \frac{2k\pi - i\beta}{\xi})}{\Gamma(\frac{\pi}{3\xi} + \frac{2k\pi - i\beta}{\xi})\Gamma(\frac{4\pi}{3\xi} + \frac{2k\pi + i\beta}{\xi})} \frac{\Gamma(\frac{2\pi + 3\xi}{3\xi} + \frac{2k\pi - i\beta}{\xi})\Gamma(\frac{5\pi + 3\xi}{3\xi} + \frac{2k\pi + i\beta}{\xi})}{\Gamma(\frac{2\pi + 3\xi}{3\xi} + \frac{2k\pi + i\beta}{\xi})\Gamma(\frac{5\pi + 3\xi}{3\xi} + \frac{2k\pi - i\beta}{\xi})} \right\} \end{aligned} \tag{5.54}$$

which again has an integral representation

$$S_0(\beta) = \left(\sinh\frac{\pi}{\xi}(\beta - i\pi)\sinh\frac{\pi}{\xi}\left(\beta - \frac{2\pi i}{3}\right)\right)^{-1} \times \exp\left(-2i\int_0^\infty \frac{dx}{x}\frac{\sin\beta x \sinh\frac{\pi x}{3}\cosh\left(\frac{\pi}{6}-\frac{\xi}{2}\right)x}{\cosh\frac{\pi x}{2}\sinh\frac{\xi x}{2}}\right) \quad . \tag{5.55}$$

In this form the S–matrix does not satisfy unitarity: even if $S(\beta)S(-\beta) = 1$, we do not necessarily have unitarity, simply because for $|q| = 1$ and $x \in \mathbb{R}$, which is a physically interesting situation, one has that $R^*_{12}(x) \neq R_{21}(x^{-1})$, so in that case the S matrix is not unitary. The poles of S_0 are located at

$$\beta = \begin{cases} i\pi - i\xi m, & i\xi m, \quad m > 0 \quad , \\ \frac{2\pi i}{3} - i\xi m, & \frac{\pi i}{3} + i\xi m, \quad m \geq 0 \quad . \end{cases} \tag{5.56}$$

Note that the first line corresponds exactly to the sine-Gordon model pole structure. Also here the poles $i\pi - i\xi m$ correspond to the creation of breathers. The S–matrix of the lightest one is [101]

$$S_{b_1,b_1}(\beta) = f_{\frac{2}{3}}(\beta) f_{\frac{\xi}{\pi}}(\beta) f_{\frac{\xi}{\pi}-\frac{1}{3}}(\beta) \quad . \tag{5.57}$$

Now consider the second set of poles in (5.56). For the poles $\frac{2\pi i}{3} - i\xi m$, we find that the R matrix degenerates into a 3-dimensional projector at these points. Hence, these poles can be interpreted as that at those rapidity values higher kinks are formed.

The hope is that the RSOS restriction of the R-matrix yields S-matrices which have a sensible physical interpretation, *i.e.* satisfy unitarity. For that we change to the shadow–world representation and take $q^r = 1$. The RSOS states of the reduced model

$$|\,\beta_1, j_1, k_1, |\,a_1\,| \quad \beta_2, j_2, k_2, \ldots |\,a_{n-1}\,| \quad \beta_n, j_n, k_n > \tag{5.58}$$

are now characterized by their rapidity β_i, by their type k (which distinguishes the kinks from the breathers), by their $U_q(sl(2))$ spin j and by the variables a_i characterizing the dual lattice, constrained by the limitations

$$a_i \leq \frac{r-2}{2} \quad , \qquad |\,a_k - 1\,| \;\leq\; a_{k+1} \;\leq\; \min(a_k + 1, r - 3 - a_k) \quad . \tag{5.59}$$

The S-matrix of these RSOS states is given by replacing in (5.44) the R–matrix by the $6j$-symbols [101]

$$
\begin{aligned}
&S\left(\beta_k-\beta_{k+1}\left|\begin{array}{cc} a_{k-1} & a_k \\ a_{k+1} & a'_k\end{array}\right.\right) = \\
&\frac{i}{4}\,S_0(\beta_k-\beta_{k+1})\left[\left\{\begin{array}{ccc} 1 & a_{k-1} & a_k \\ 1 & a_{k+1} & a'_k\end{array}\right\}_q\right. \\
&\times\left(\left(\exp\left(\frac{2\pi}{\xi}(\beta_{k+1}-\beta_k)\right)-1\right)\; q^{c_{a_{k+1}}+c_{a_{k-1}}-c_{a_k}-c_{a'_k}+3}(-1)^{\nu}\right. \\
&-\left.\left(\exp\left(-\frac{2\pi}{\xi}(\beta_{k+1}-\beta_k)\right)-1\right)\; q^{-(c_{a_{k+1}}+c_{a_{k-1}}-c_{a_k}-c_{a'_k}+3)}(-1)^{-\nu}\right) \\
&\left.\quad+\,q^{-5}(q^6+1)(q^4-1)\delta_{a_k,a'_k}\right]\,.
\end{aligned}
\tag{5.60}
$$

Herein c_a are given as $c_a = a(a+1)$, $\nu = a_k + a'_k - a_{k+1} - a_{k-1}$ and the $6j$-symbols are those of (5.42).

Again let us discuss the cases when this model reduces to only scalar particles. First there are the theories $\mathcal{M}_{2,2n+1}$, corresponding to $\xi = \frac{\pi}{3n}$, which will be described in the next section, and contain only breather states. From the restriction (5.59) we see that also in the theories $\mathcal{M}_{3,s}$ all kink states drop out of the spectrum, and only breathers remain. The simplest model is $\mathcal{M}_{3,4} + \Phi_{1,2}$. Its S–matrix contains 8 particles and was found originally in a completly different way [115]. It corresponds to the minimal part of the S–matrix of the E_8 ATFT. The particle content and S–matrices of the theories $\mathcal{M}_{3,s}$, $s > 4$ are still unexplored.

For the series $\mathcal{M}_{4,s}$ there are two vacua, but only one kink state, which again behaves as a scalar particle. Also in these models only the first member of the series, $\mathcal{M}_{4,5}$ which corresponds to the tricritical Ising model, is known. Its S–matrix is described by the minimal part of the E_7 ATFT [29].

For the other models $\mathcal{M}_{r,s}$, $r > 4$ kinks are present. The only exception is the model $\mathcal{M}_{6,7}$ where an additional symmetry allows a reduction to a scalar S–matrix [101].

As discussed in section 2.4, $\Phi_{2,1}$-perturbed models can be obtained by an interchange of the parameters r and s labeling the conformal theory $\mathcal{M}_{r,s}$. That is, the S–matrix of $\mathcal{M}_{r,s} + \Phi_{2,1}$ is given by the formal expression for the S–matrix of $\mathcal{M}_{s,r} + \Phi_{1,2}$. The only two examples of reductions to scalar S–matries are the Ising model $\mathcal{M}_{3,4}$ with the S-matrix $S = -1$, and the three state Potts model $\mathcal{M}_{5,6}$ [101].

5.3 S-Matrices of the $\Phi_{1,2}$ perturbed minimal models $\mathcal{M}_{2,2n+1}$

The Kac table of the minimal models $\mathcal{M}_{2,2n+1}$ extends along one row. The counting argument and an explicit computation for the $\Phi_{1,2}$ perturbed models show the existence of a conserved current with spin $s = 5$ but not that one with spin $s = 3$ [115]. The fusion rules of these CFT do not have any internal symmetry. These two facts together allow the possibility to have the "ϕ^3"- property in the S-matrices of the $\Phi_{1,2}$ perturbed models.

From the analysis made by Smirnov, we know that in these models the kinks play the role of quarks, in the sense that they form bound states which can occur as asymptotic states but themselves they cannot [103]. How many breathers are in the spectrum? We claim that for the models $\mathcal{M}_{2,2n+1}$, their number is $(n-1)$. The reason is the following. For these models, $\xi = \frac{\pi}{3n}$ and a very special situation happens at these values (see appendix). The $(n-1)$ poles between $i\pi$ and $\frac{2\pi i}{3}$ (and the crossing ones), which are those of the breathers b_i $(i = 1, 2, \ldots n-1)$, are now third order poles whereas all other poles relative to the kinks become fourth order poles (see figure 5.3). According to the

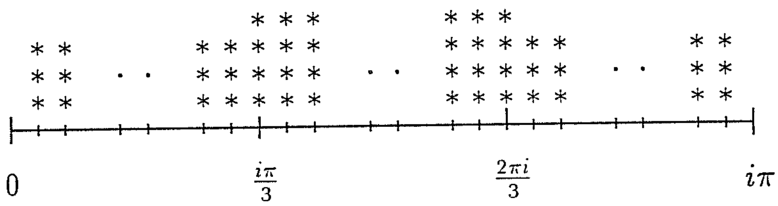


Figure 5.4: Pole-structure of the kink-kink S matrix

interpretation of the odd and even order poles put forward in [18, 29], this precludes the possibility of creating higher kinks. Therefore, all breathers (which, generally, are bound states of kinks) are just the $(n-1)$'s relative to the third order poles in the amplitude of the fundamental kink[1]. This conclusion is further supported by the analysis of the

[1]Although S_{12} is not really sensible of a physical interpretation, as we discuss in the next section, its

S-matrices of the breathers b_i with the fundamental kink: in these amplitudes there appears only the pole of the fundamental kink and no other singularities.

We now construct the S-matrix in the sector of the $(n-1)$ breathers b_i. Using (5.57) one finds for the fundamental particle b_1

$$S_{b_1 b_1}(\beta) = f_{\frac{1}{3n}}(\beta) f_{\frac{2}{3}}(\beta) f_{-\frac{n-1}{3n}}(\beta) \ . \tag{5.61}$$

We identify as the physical poles $u_{11}^1 = \frac{2\pi}{3}$ and $u_{11}^2 = \frac{\pi}{3n}$. The first one is interpreted as a bound state corresponding to the fusion $b_1 b_1 \to b_1 \to b_1 b_1$. This means that this S-matrix has the "ϕ^3"- property and therefore we cannot have a spin $s = 3$ current in the set of conserved quantities [115]. The second pole we assign to the breather b_2. Its mass is given by

$$\frac{m_2}{m_1} = \frac{\sin \frac{2\pi}{6n}}{\sin \frac{\pi}{6n}} \ . \tag{5.62}$$

Using the bootstrap equations (3.16), we can compute the amplitude $S_{b_1 b_2}$

$$S_{b_1 b_2}(\beta) = f_{\frac{3}{6n}}(\beta) f_{\frac{1}{6n}}(\beta) f_{\frac{2n+1}{6n}}(\beta) f_{\frac{-2n+3}{6n}}(\beta) \ . \tag{5.63}$$

Herein the pole $u_{12}^1 = \frac{5\pi}{6n}$ corresponds to the particle 1. Only if n is larger than 3, we get a new particle b_3 at the pole $u_{12}^3 = \frac{\pi}{2n}$. Otherwise, for $n = 3$, this factor cancels with the zero $f_{\frac{-2n+3}{6n}}(\beta)$. Treating similarly the scattering of b_1 and b_3, a new bound state b_4 appears and so on. By induction we obtain the whole sequence S_{b_1, b_k} $(k = 1, 2, \ldots < n-1)$

$$S_{b_1 b_k} = f_{\frac{k+1}{6n}} f_{\frac{k-1}{6n}} f_{\frac{2n+k-1}{6n}} f_{\frac{-2n+k+1}{6n}} \ . \tag{5.64}$$

The remaining scattering amplitudes are obtained again by induction, applying the bootstrap equations. Finally the general S-matrix S_{b_p, b_k} (for $k \geq p = 1, 2, \ldots n-1$) is given by [72, 71]

$$\begin{aligned} S_{b_p b_k} &= f_{\frac{k+p}{6n}} \left(f_{\frac{k+p-2}{6n}} \ldots f_{\frac{k-p+2}{6n}} \right)^2 f_{\frac{k-p}{6n}} \\ &\quad \times f_{\frac{2n+k+p-2}{6n}} \ldots f_{\frac{2n+k-p+2}{6n}} f_{\frac{2n+k-p}{6n}} \\ &\quad \times f_{\frac{-2n+k-p+2}{6n}} \ldots f_{\frac{-2n+k+p-2}{6n}} f_{\frac{-2n+k+p}{6n}} \ . \end{aligned} \tag{5.65}$$

analytic properties enter the structure of singularities in the RSOS physical sector.

The exact mass spectrum is

$$m_k = \sin\frac{k\pi}{6n} \quad , \qquad k = 1, 2, \ldots n-1 \ . \tag{5.66}$$

Postponing the discussion on the analytic structure of these S–matrix elements to the next section, here we make some general comments. First of all, notice that the first line of factors in (5.65) corresponds exactly to the structure of the S matrices found for the $\Phi_{1,3}$ deformation of these models [24, 51, 89]. Further, these are the poles identified as physical ones, and therefore also the mass-spectrum has the identical structure as in the $\Phi_{1,3}$ case. Secondly, the number of poles in the physical sheet given by the functions in the second line of (5.65), coincides with the number of zeros given by the functions of the third lines. Therefore, for what concerns the computation of the effective central charge in the ultraviolet regime of these scattering theories, the matrix N_{ab} which enters the thermodynamical Bethe ansatz (TBA) coincides with that of the deformation $\Phi_{1,3}$ (see section 7.1). Then it is not surprising to find also in this case the correct values

$$c(0) = \frac{2(n-1)}{2n+1} \quad . \tag{5.67}$$

In [71] the TBA and the Truncation space approach have been applied. The truncation spectrum is in good agreement with the S matrices. Also the scaling region of the system was examined in the TBA, and the exponents of the theory calculated. We will analyze one example in section 7.1. All results obtained confirm that the above S matrices describe perturbations of the models $\mathcal{M}_{2,2n+1}$ in the direction $\Phi_{1,2}$.

5.3.1 Pole structure of the S-Matrix

Looking at equation (5.65), it seems that the bootstrap program has not been carried out to the end. In fact, there are still poles in the S–matrix which have not been identified as particles. This, because above we only analyzed those poles which give rise to the breathers, corresponding to the first line of the general amplitude (5.65). Besides the argument we already gave for the truncation of the spectrum to the $(n-1)$ breathers only, the unphysical origin of these remaining poles also shows up in the conservation laws. If interpreted as singularities due to new particles, these spurious poles would not be consistent with conserved quantities of higher spin [115], and therefore the entire theory would be spoiled.

The domain of analyticity of an elastic S–matrix consists of a two-sheet Riemann surface with square-root singularities at the threshold points of the s and u channel, respectively at $(m_1+m_2)^2$ and at $(m_1-m_2)^2$. The mapping

$$\beta = \ln\left(\frac{s-m_1^2-m_2^2+\sqrt{((s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}}{2m_1m_2}\right) \tag{5.68}$$

transforms the physical sheet of the s plane into the strip $0 \leq \text{Im } \beta \leq \pi$. The second sheet is mapped into the strip $-\pi \leq \text{Im } \beta \leq 0$, and both repeat with period 2π. In order to understand the origin of the spurious poles in the S–matrices (5.65), it is better to interpret the singularities in a function f_{-x} not as zeros on the physical strip but as poles on the second sheet of the Riemann surface. Concerning this point, let us observe the following facts. The expression of the mass of a bound state A_c in a scattering state $\mid A_a A_b\rangle$ is an even function of the resonance angle u_{ab}^c

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos u_{ab}^c \quad . \tag{5.69}$$

Hence, reversing the sign of u_{ab}^c, the value of m_c does not change. Moreover, suppose we have given a closed bootstrap system with a generic S matrix of the form[2]

$$S_{ab}(\beta) = \prod_{x_i} f_{x_i}(\beta) \quad , \tag{5.70}$$

where all $x_i > 0$. Let us change all factors f_{x_i} into f_{-x_i}. If we now apply the bootstrap equation to the zeros instead of the poles, we again end up with a closed system with the same spectrum as the original one.

Hence, if one has an S-matrix with poles only in one sheet, the interpretation is the usual one. All odd-order poles must correspond to bound states. These, according to the bootstrap-principle, have to be followed and must give rise to conserved quantities of higher spin. The interesting situation though occurs, when poles appear in both sheets of the Riemann surface. For special values of their positions, it may happen that, through

[2]For simplicity, we consider here the case of purely elastic diagonal S–matrices. The argument given in the text can be easily generalized to the other cases.

$u_{11}^2 = \frac{2\pi}{5}$	$u_{12}^1 = \frac{4\pi}{5}$ $u_{12}^2 = \frac{3\pi}{5}$
	$u_{22}^3 = \frac{4\pi}{5}$

Table 5.3: Resonance angles of the $A_4^{(2)}$ model.

the bootstrap, they overlap each other and produce spurious poles. In these cases it is also possible that expected particles disappear from the spectrum and reappear as zeros [3]. In order to understand this mechanism better, let us consider a particularly simple and illustrative example, *i.e.* the second model of the $A_{2n}^{(2)}$ affine Toda field theories [18, 29]. The whole set of S–matrices of this system is given by

$$\begin{aligned} S_{11} &= f_{-b} f_{\frac{2}{5}} f_{b-\frac{2}{5}} \\ S_{12} &= f_{\frac{3}{5}} f_{b-\frac{3}{5}} f_{\frac{4}{5}} f_{b-\frac{4}{5}} \\ S_{22} &= f_{-b} f_{\frac{4}{5}} f_{b-\frac{4}{5}} f_{\frac{2}{5}} f_{b-\frac{2}{5}} f_{\frac{2}{5}} f_{-b-\frac{2}{5}} \ . \end{aligned} \tag{5.71}$$

The poles corresponding to the bound states are given by the b-independent terms (which are the minimal S–matrices). The values are in the Table 5.3. The mass spectrum is

$$m_1 = M \ , \qquad m_2 = 2M \ \cos\frac{\pi}{5} \ . \tag{5.72}$$

The remaining functions in (5.71) introduce zeros on the physical sheet. For a finite value of the coupling constant g, the terms containing $b(g)$, given by eq. (5.31), do not modify the spectrum. But, changing g, the zeros move around and at the self–dual point they overlap with the poles, producing the following set of S–matrices

$$S_{11} = f_{\frac{2}{5}} (f_{-\frac{1}{5}})^2 \ , \quad S_{12} = f_{-\frac{3}{5}} f_{\frac{4}{5}} \ , \quad S_{22} = f_{-\frac{4}{5}} \ . \tag{5.73}$$

If we retained the usual interpretation of the bound states as poles in the physical strip of the amplitudes, we would conclude, that in the above system the bound state A_2 has

[3] These arguments were recently generalized in [35] in order to explain the simgularity structure of non–simply laced real coupled ATFT. There the method was called "generalized bootstrap principle"

disappeared from the amplitude S_{12} as well as A_1 from S_{22}. Actually, as result of the collision of the zeros with the poles, we see that these particles have been moved onto the second sheet.

The same pattern is established for all other models of the affine Toda field theories $A_{2n}^{(2)}$ [35]. On the other hand, using the analysis made in [18, 29], it is possible to see that the S–matrices of the affine Toda field theories of the simply laced algebras (ADE) do not show this overlapping behaviour. A natural interpretation of the peculiar features of the series $A_{2n}^{(2)}$ comes from its group origin. This series is obtained as a folding of the simply-laced models $A_{2n}^{(1)}$ under the Z_2 automorphism of their Dynkin diagram. This folding projects the $2n$ fields of the original theory onto a n-dimensional subspace. n particles of the original $A_{2n}^{(1)}$ theories rearrange themselves as the new particles of the reduced models $A_{2n}^{(2)}$ (and then they appear as poles in the physical strip[4]), but we may think of the other n's of the initial model as particles living on the second sheet of the Riemann surface of the latter one.

A similar mechanism is responsible for the spurious poles in the S-matrix (5.65) of the $\Phi_{1,2}$ deformation of the $\mathcal{M}_{2,2n+1}$ models [72, 71]. The only difference is that the locations of the zeros are now not adjustable parameters, but they are fixed from the beginning. The first one occurs in the amplitude $S_{b_1b_1}$ through the term $f_{-\frac{n-1}{3n}}$. If we calculate the mass of the particle corresponding to it, we find $m_x = \frac{1}{2} - \sin(\frac{n-2}{6n}\pi)$. But we also get the same mass at the spurious pole in $S_{b_1b_2}$, namely at $u_{12}^x = \frac{4n-1}{6n}\pi$. Therefore this singularity can be interpreted as a zero which through the action of the bootstrap appears as a pole in the physical sheet.

The whole analysis of the analytic structure of the S–matrices is based on some basic steps which we clarify through the first non trivial model of our system[5], that one corresponding to $\mathcal{M}_{2,7}$. This model has two physical particles with S-matrices given by

$$S_{b_1b_1} = f_{\frac{1}{9}} f_{\frac{2}{3}} f_{-\frac{2}{9}} \quad , \quad S_{b_1b_2} = f_{\frac{1}{18}} f_{\frac{7}{18}} \quad , \quad S_{b_2b_2} = f_{\frac{2}{3}} f_{\frac{1}{9}} f_{\frac{5}{9}} \quad . \tag{5.74}$$

[4]Here we also remind that the $A_{2n}^{(2)}$ theories are the only non-simply laced theories which seem to be consistent without inclusion of other fields in the Lagrangian. For instance, the one loop corrections do not spoil the classical mass ratios [18, 29].

[5]Notice that for $n = 2$ we have the Yang-Lee model, in which holds the identification $\Phi_{1,2} \equiv \Phi_{1,3}$. Therefore, the S-matrix for this system reduces to that one discussed in [24].

Now follow the above interpretation and calculate the S-matrix relative to the pole at $\beta = -i\frac{2\pi}{9}$ on the second sheet of the Riemann surface of $S_{b_1b_1}$. Denoting this spurious particle by a_1, we have

$$S_{b_1a_1} = (f_{\frac{2}{9}})^2 f_{\frac{5}{9}} f_{-\frac{1}{9}} f_{-\frac{1}{3}} \quad , \quad S_{b_2a_1} = (f_{\frac{1}{6}})^2 f_{\frac{1}{2}} f_{\frac{5}{18}} f_{-\frac{1}{18}} \quad , \quad S_{a_1a_1} = (f_{\frac{2}{3}})^3 (f_{\frac{1}{9}})^2 (f_{-\frac{2}{9}})^2 \ . \tag{5.75}$$

In the amplitude $S_{b_1a_1}$ we can easily identify the particles b_1 (relative to $u_{b_1a_1}^{b_1} = \frac{10\pi}{9}$) and b_2 (with $u_{b_1a_1}^{b_2} = \frac{5\pi}{9}$). The pole at $\beta = -\frac{\pi}{3}$ on the second sheet gives rise to a new spurious particle a_2 and turns up as a pole in the physical sheet of the amplitude $S_{b_2b_2}$, i.e. that one at $u_{b_2b_2}^{a_2} = \frac{5\pi}{9}$.

Still we have not finished the analysis of this model. There remains one spurious pole in the amplitude $S_{b_2b_2}$ to be explained. This is the singularity at $\beta = i\frac{\pi}{9}$. But, looking at the general amplitude, we see that it arose from a cancellation of a zero with a double pole coming from the multi-scattering graph of the figure 5.5. This is exactly the same

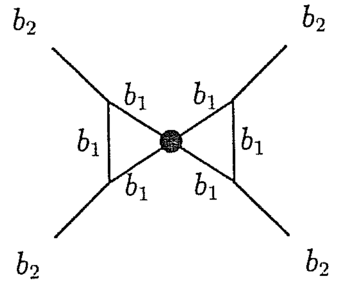


Figure 5.5: Multiscattering process responsible for higher order pole singularities in the S-matrix

mechanism we encountered in the example of ATFT $A_{2n}^{(2)}$, i.e. fine tuning the value of the coupling constant to a special value, one can boost a pole from one sheet of the Riemann surface into the other.

The analysis in the case of $\Phi_{1,2}$ perturbations is much more involved than in the $\Phi_{1,3}$ case but the basic mechanisms explained in the example above, successfully applied to all the models defined by (5.65).

5.4 $\Phi_{1,4}$ deformation of $\mathcal{M}_{2,9}$

Up to now, we have discussed only the general integrable perturbations $\Phi_{1,3}$, $\Phi_{1,2}$ and $\Phi_{2,1}$. These are integrable for all minimal models. It might though happen, that for certain specific models also other additional perturbations turn out to be integrable. These can be easily determined by the counting argument.

In [85], M. Martins pointed out the connection between the models $\mathcal{M}_{2,9}+\Phi_{1,4}$ and $\mathcal{M}_{8,9}+\Phi_{1,2}$. Basically, the argument relies on the identification of the fields $\Phi_{1,k}$ with the classical exponential operators appearing in the Liouville lagrangian (see section 2.4). This analysis implies, that the field $\Phi_{1,2}(\gamma)$ is related to the field $\Phi_{1,5}(\tilde{\gamma})$, provided that $\gamma = 4\tilde{\gamma}$ [6]. In the model $\mathcal{M}_{2,9}$ we have $\Phi_{1,4}\equiv\Phi_{1,5}$. Hence, it should be possible to recover the $\Phi_{1,4}$ deformation of this model using the analysis of the $\Phi_{1,2}$ deformation of the unitary model $\mathcal{M}_{8,9}$. The above observation also makes the origin of the integrability of the $\Phi_{1,4}$ deformation less mysterious, which is usually prevented by counting argument and null-vector considerations.

In [103] it was conjectured, that for the model $\mathcal{M}_{8,9}$, the spectrum consists of four particles: two kinks with the masses

$$M \quad , \qquad 2M\cos\frac{\pi}{15} \quad , \tag{5.76}$$

and two breathers with the masses

$$2M\sin\frac{4\pi}{15} \quad , \qquad 4M\sin\frac{4\pi}{15}\cos\frac{3\pi}{5} \quad . \tag{5.77}$$

The S–matrices of the fundamental kinks and of the fundamental breather are given by eqs. (5.60) and (5.57) respectively. If we would like to construct the S-matrix proposed by Martins from the massive theory $\mathcal{M}_{8,9}+\Phi_{1,2}$, we need to restrict the space of states such that it contains only particles with scalar behaviour and not kink-like. Hence we have to find a combination of the kink amplitudes which give rise to the S–matrix of the fundamental particle of the ones given below (5.80). The combination of the kink

[6] It is also necessary to make a corresponding rescaling in the exponential term of the Liouville action.

S–matrices we are looking for is [71]

$$S(\beta) = S\left(\beta\left|\begin{matrix}1 & 1\\ 1 & 1\end{matrix}\right.\right) + S\left(\beta\left|\begin{matrix}1 & 1\\ 2 & 2\end{matrix}\right.\right) = S_0(\beta)\sinh\frac{\pi}{\xi}(\beta + i\pi)\sinh\frac{\pi}{\xi}\left(\beta + \frac{2\pi i}{3}\right) \quad . \tag{5.78}$$

Simplifying the expression (5.55) for S_0, i.e. with $\xi = \frac{8\pi}{15}$, one obtains

$$S_0(\beta) = \frac{f_{\frac{2}{3}}(\beta) f_{\frac{2}{15}}(\beta) f_{\frac{7}{3}}(\beta) f_{-\frac{1}{15}}(\beta) f_{-\frac{2}{5}}(\beta)}{\sinh\frac{\pi}{\xi}(\beta + i\pi)\sinh\frac{\pi}{\xi}(\beta + \frac{2\pi i}{3})} \quad . \tag{5.79}$$

Using the above expression (5.79), the expression (5.78) reduces to the S–matrix proposed by Martins for the fundamental particle of $\mathcal{M}_{2,9} + \Phi_{1,4}$ [85]. It satisfies the usual requirement of unitarity and it is a crossing symmetric function. Therefore it is breather-like and a restriction to a subspace of scalar particles is possible. Now we want to analyze the bootstrap system which comes from (5.78).

The bootstrap closes with four particles and we have clculated the full S-matrix in [71]. In order to easily identify the appearance of the physical bound states, we have introduced a compact notation: a factor ${}^a f_x$ in a S–matrix element S_{bc}-matrix means that this pole gives rise to the physical particle a through $u_{bc}^a = x\pi$. The S–matrix is given by

$$\begin{aligned}
S_{11} &= {}^2f_{\frac{7}{15}}\ {}^3f_{\frac{2}{15}}\ {}^1f_{\frac{2}{3}} f_{-\frac{1}{15}} f_{-\frac{2}{5}} & S_{12} &= {}^1f_{\frac{23}{30}}\ {}^3f_{\frac{13}{30}}\\
S_{13} &= {}^1f_{\frac{14}{15}}\ {}^2f_{\frac{11}{15}}\ {}^4f_{\frac{1}{5}} f_{-\frac{2}{15}} f_{-\frac{1}{3}} (f_{\frac{2}{5}})^2 & S_{14} &= {}^3f_{\frac{13}{15}} f_{\frac{7}{15}} (f_{\frac{2}{3}})^2\\
\\
S_{22} &= {}^2f_{\frac{2}{3}} f_{\frac{7}{15}}\ {}^4f_{\frac{1}{5}} & S_{23} &= {}^1f_{\frac{5}{6}} f_{\frac{1}{2}} f_{\frac{3}{10}} f_{\frac{11}{30}}\\
S_{24} &= f_{\frac{7}{30}} f_{\frac{3}{10}}\ {}^2f_{\frac{9}{10}} f_{\frac{11}{30}} (f_{\frac{13}{30}})^2 & &\\
\\
S_{33} &= ({}^3f_{\frac{2}{3}})^3\ (f_{\frac{2}{15}})^2\ (f_{\frac{7}{15}})^2\ f_{-\frac{1}{15}} f_{-\frac{2}{5}} & S_{34} &= {}^1f_{\frac{1}{15}} f_{\frac{1}{5}} f_{\frac{7}{15}} (f_{\frac{4}{15}})^2\ (f_{\frac{2}{5}})^3\\
\\
S_{44} &= ({}^4f_{\frac{2}{3}})^3\ (f_{\frac{7}{15}})^3\ f_{\frac{4}{15}} f_{\frac{2}{15}} (f_{\frac{1}{5}})^2\ f_{\frac{2}{5}} \quad . & &
\end{aligned} \tag{5.80}$$

The particles A_1 and A_3 correspond respectively to the fundamental and the higher kink and their masses coincide with those given in (5.76). The other two particles A_2 and A_4 correspond, on the contrary, to the two breathers present in the $\mathcal{M}_{8,9} + \Phi_{1,2}$ model and their masses coincide with those in (5.77).

As before, we realize that in the S-matrix of the fundamental particle poles appear in both sheets of the Riemann-surface. Hence, we expect spurious poles along the bootstrap procedure, which appear indeed. The same mechanism we already applied to the previous systems works successfully also here. We calculate for the zero's the corresponding fusion-angles and masses, and see that these "spurious" particles also appear on the physical sheet of the Riemann-surface. For example, consider the zero $u_{11}^x = -\frac{1}{15}$. This turns out to be exactly the "spurious" particle appearing in S_{22} at $u_{22}^x = \frac{8}{15}\pi$. Other singularities in (5.80) can be analyzed similarly.

A non trivial check of our conclusions has already been done using the truncation method and, actually, this was the way how the S–matrix of the fundamental particle has been conjectured [85]. A further check using the thermodynamic Bethe ansatz (TBA) will be presented in section 7.1.

6 Degenerate S–matrices

The set of diagonal S–matrices is rather limited. In fact, many physically interesting models exhibit a particle spectrum containing also degenerate particles. Especially most of the perturbed minimal models fall into this class. The main complication which arises for these models is that additionally to the dynamical structure one needs to treat also with the degeneracy structure of the particles.

In section 5.2.2 we gave a short introduction of how S–matrices of this type can be constructed with the use of a quantum group R–matrix. We also discussed that in order to describe perturbed minimal models, one needs to consider the restricted shadow–world representation (also called IRF representation). In section 6.1 we give an alternative, though closely related description of such models based on the notion of graph–state models.

This construction will be applied to $\Phi_{1,2}$ and $\Phi_{2,1}$ perturbed minimal models in section 6.2. We present there our results [70] on the physical applications of this formalism. Finally in section 6.3 we carry out the bootstrap for the unitary minimal models $\mathcal{M}_{p,p+1}$ perturbed by the operator $\Phi_{1,2}$. We obtain explicitly the higher kink S–matrix elements, which is a merit of the graph–state formulation since it simplifies considerably the calculations.

In section 6.4 we discuss scattering theories related to the hard square lattice model (HSLM) which we analyzed in [69]. We discuss these theories in great detail, since the HSLM geometry is the simplest non–trivial geometry for describing S–matrices with kink excitations.

One specific model of this class is described in section 6.5. It is the model $\mathcal{M}_{4,5}+\Phi_{2,1}$, which corresponds to the tricritical Ising model perturbed by the subleading magnetization operator. We describe our investigation of the S–matrix of this model [32, 33, 91] which is a rather subtle question since two alternative proposals exist.

6.1 Scattering amplitudes based on graph-state models

A necessary requirement for the construction of an S–matrix for degenerate particles is that the amplitudes satisfy the Yang–Baxter equation (3.6). A very simple approach is to use a transfer-matrix of a critical integrable statistical mechanical model and try to interpret it as a scattering theory by adjusting unitarity and the crossing-relation.

In order to describe perturbed minimal models the statistical models we will use are the so–called graph–state models [9, 36, 38]. Let us review their properties: They can be parameterized in general as

$$\omega(a,b,c,d;u) = \delta_{ac}\sin(\lambda-u) + \delta_{bd}\sin(u)\frac{\psi(a)^{\frac{1}{2}}\psi(c)^{\frac{1}{2}}}{\psi(b)} \quad . \tag{6.1}$$

The restrictions on the values a, b, c, d, describing a plaquette of a square lattice, are defined by a two-dimensional graph, where a link between two sites means that these two values can occur as neighbouring sites on the lattice. The Boltzmann weights (6.1) satisfy the following properties:

$$\begin{aligned} &\textstyle\sum_c \ \omega(b,d,c,a;u)\ \omega(a,c,f,g;u+v)\ \omega(c,d,e,f;v) = \\ &\textstyle\sum_c \ \omega(a,b,c,g;v)\ \omega(b,d,e,c;u+v)\ \omega(c,e,f,g;u) \quad ; \end{aligned} \tag{6.2}$$

$$\sum_e \omega(e,c,d,a;-u)\ \omega(b,c,e,a;u) = \delta_{bd}\ \rho(u)\rho(-u) \quad ; \tag{6.3}$$

$$\omega(a,b,c,d;u) = \frac{\psi(a)^{\frac{1}{2}}\,\psi(c)^{\frac{1}{2}}}{\psi(d)^{\frac{1}{2}}\,\psi(b)^{\frac{1}{2}}}\ \omega(d,a,b,c;\lambda-u) \quad ; \tag{6.4}$$

$$\omega(a,b,c,d;u) = \omega(a,d,c,b;u) = \omega(c,b,a,d;u) \quad . \tag{6.5}$$

The *crossing point* λ in (6.1) and the so called crossing parameters $\psi(a)$ are determined [36] by the eigenvalue equation

$$\sum_{b\sim a}\psi(b) = \Lambda\psi(a) \quad \text{with} \quad \Lambda = 2\cos(\lambda) \quad . \tag{6.6}$$

In this equation the symbol $\sim$ means that the sum runs over states admissible from the geometry of the model, *i.e.* those fixed by the corresponding incidence matrix. The elegance of these models lies in the fact that the only information needed to construct them is the associated graph.

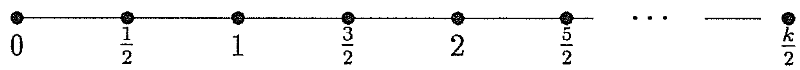
We have presented this construction here only in a schematic way in order to summarize the principle features. In the following we will describe it in detail and generalize it, in order to construct and analyze the scattering amplitudes of perturbed minimal models.

6.1.1 Temperly–Lieb algebras from incidence graphs

The construction above applies to general two-dimensional graphs. We will now concentrate ourselves on the specific graphs describing perturbed minimal models. These graphs are usually picturized fusion algebras of some Wess-Zumino-Witten (WZW) model [38, 55]. For our purpose, describing perturbations of conformal field theory, we use graphs based on the fusion-rules of $SU(2)$ WZW-models, which read as

$$\phi_{j_1} \times \phi_{j_2} = \sum_{j_3=|j_1-j_2|}^{\min(j_1+j_2,k-j_1-j_2)} \phi_{j_3} \quad . \tag{6.7}$$

For the fundamental representation (spin j=$\frac{1}{2}$) they coincide with the graph of the A_{k+1} Dynkin-diagrams, where k denotes the level of the underlying Kac-Moody algebra,

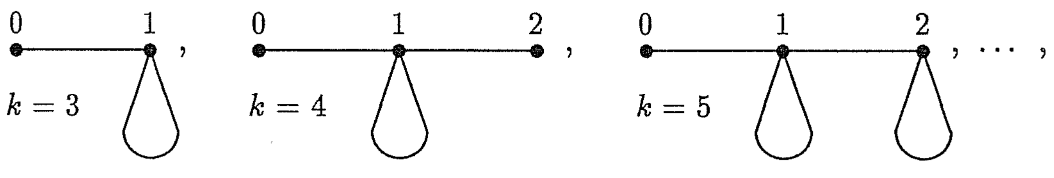


(6.8)

This coincides with the restrictions imposed on the scattering theories for $\Phi_{1,3}$ perturbed models (5.51).

We know that the $\Phi_{1,2}$ perturbed models are constructed on the spin $j = 1$ representation, which also allows the coupling of a site to itself. Such models are usually constructed by the fusion-procedure. Since the corresponding S–matrix is built on the *fundamental* representation of the algebra $A_2^{(2)}$ such a method cannot succeed in our case. Recently there has been an approach to construct so-called 'dilute' models [99]. One uses the *same* incidence graph (6.8), but generalizes the ansatz (6.1) for the amplitudes.

On the other hand, the fusion rules (6.7) suggest to consider different incidence diagrams:



(6.9)

where we have indicated also the corresponding level of the Kac-Moody algebra.

Given a graph, one can find a representation of the Temperly-Lieb algebra (TLA),

$$E_i E_j = E_j E_i \quad \text{for } |i-j| \geq 2 \ , \tag{6.10}$$

$$E_i E_{i\pm1} E_i = E_i \ , \quad E_i^2 = \sigma(j)^{\frac{1}{2}} E_i \ , \tag{6.11}$$

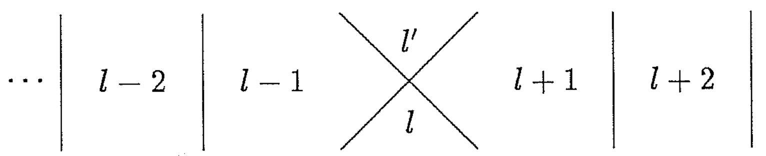
by diagonalizing the incidence matrix of the diagram. The generators E_i act in an n-particle space,

$$E_i = 1_1 \otimes 1_2 \otimes \ldots \otimes 1_{i-1} \otimes E \otimes 1_{i+1} \otimes \ldots \otimes 1_n \quad , \tag{6.12}$$

or

$$\langle l_1', l_2', \ldots, l_n' | E_i | l_1, \ldots, l_n \rangle = \prod_{j \neq i} \delta_{l_j, l_j'} E^{l,l'}_{l-1,l+1} \quad . \tag{6.13}$$

For graph-state models this action can be visualized as



$$\ldots \sim E^{l,l'}_{l-1,l+1} \quad , \tag{6.14}$$

and the generators are matrices in the indices l and l'. Let us define the parameter $\lambda \equiv \frac{\pi}{k+2}$. Then the largest eigenvalue, corresponding to the Perron-Frobenius eigenvector[7] for the case (6.8) is $\sigma(\frac{1}{2}) = 2\cos\lambda$ whereas for the case (6.9) it is $\sigma(1) = 1 + 2\cos 2\lambda$. This becomes more similar introducing a notion borrowed from the quantum-group language. Let us define $q = e^{i\lambda}$ and the quantum-symbol $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. Then we find that $\sigma(\frac{1}{2}) = [\frac{1}{2}]$ and $\sigma(1) = [1]$.

Also the eigenvectors have the same structure. They are $\psi(a) = [2a+1]$, where the numbers a take the values of the labels on the nodes of the corresponding diagram, that is half-integers for (6.8) and integers for (6.9) respectively. The indices a are restricted in both cases by the bound $a \leq \frac{k}{2}$. Finally, the generators (6.14) are constructed out of the eigenvectors [36] as

$$E^{ac}_{bd} = \frac{[2a+1]^{\frac{1}{2}}[2c+1]^{\frac{1}{2}}}{[2b+1]^{\frac{1}{2}}[2d+1]^{\frac{1}{2}}} \delta_{bd} \quad . \tag{6.15}$$

[7]The other eigenvalues lead in general to imaginary Boltzmann weights

Note that the amplitudes (6.1) can now be constructed as

$$\omega(a,b,c,d;u) = \sin(\lambda - u)\,1 + \sin u\, E_i \qquad (6.16)$$

Here we encounter an obstacle in our construction. The graph–state amplitudes (6.1) describe a critical statistical model only if the largest eigenvalue Λ_{max} of the adjacency matrix satisfies $\Lambda_{max} \leq 2$. Above this limit the model undergoes a first order phase coexistence [10], and cannot be used to construct a scattering matrix for a field theory. The graphs with $\Lambda_{max} \leq 2$ can be classified [94, 96] and coincide with the Dynkin diagrams of the simply-laced affine and finite Lie-Algebras. The graphs (6.8) are in this class, but the graphs (6.9) in general not. Therefore the ansatz (6.1) can not be used for that type of graphs. We will need to generalize the concept of the graph–state models in order to describe also the $\Phi_{1,2}$ scattering amplitudes.

6.1.2 Braid group

In order to extend the above construction we recall how the amplitudes (6.1) are obtained from the TLA. The first step is to find a braid-group representation, that is elements b_i satisfying

$$\begin{aligned} b_i b_j &= b_j b_i \quad ,\text{for } \ |i-j| \geq 2 \ , \\ b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1} \quad . \end{aligned} \qquad (6.17)$$

For the graphs (6.8), which lead to $\Phi_{1,3}$ perturbed models, one defines the braid-group generators by the linear transformation

$$b_k = 1 - e^{i\lambda} E_k \quad . \qquad (6.18)$$

In that way one obtains a Hecke algebra, that is a braid group satisfying an additional quadratic relation, which in our case reads

$$(b_k - 1)(b_k + e^{2i\lambda}) = 0 \quad . \qquad (6.19)$$

The existence of this relation is a consequence of the TLA algebra.

Let us turn to the graphs (6.9) now. As discussed, the Hecke algebra construction is of no use, since it will not lead to a critical lattice model. We are therefore looking for an algebra which is related to the spin 1 representation. The natural choice [36] is to construct a representation of the BWM algebra [16], which is defined by the relations

$$
\begin{aligned}
&g_i g_j = g_j g_i \ , \quad \text{for } |i-j| \geq 2 \ , \\
&g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \ ; \\
\\
&e_i e_j = e_j e_i \ , \quad \text{for } |i-j| \geq 2 \ , \\
&e_i e_{i\pm1} e_i = e_i \ , \quad e_i^2 = (m^{-1}(l+l^{-1})-1)e_i \ ; \\
\\
&g_i + g_i^{-1} = m(1+e_i) \ , \quad g_i^2 = m(g_i + l^{-1}e_i) - 1 \ , \\
&g_{i+1} g_i e_{i\pm1} = e_i g_{i\pm1} g_i = e_i e_{i\pm1} \ , \\
&g_{i\pm1} e_i g_{i\pm1} = g_i^{-1} e_{i\pm1} g_i^{-1} \ , \quad e_{i\pm1} e_i e_{i\pm1} = g_i^{-1} e_{i\pm1} \ , \\
&e_{i\pm1} e_i g_{i\pm1} = e_{i\pm1} g_i^{-1} \ , \quad g_i e_i = e_i g_i = l^{-1} e_i \ , \quad e_i g_{i\pm1} e_i = l e_i
\end{aligned}
\tag{6.20}
$$

with $e_i = -E_i$ and $g_i = -ib_i$. The parameters appearing in the algebra are $m = -i(q^2 - q^{-2})$ and $l = iq^4$. This algebra implies a third order relation for the braid group generators [36], which in our notation reads as

$$
(b_i - q^{-2})(b_i + q^2)(b_i + q^{-4}) = 0 \quad . \tag{6.21}
$$

Not all of the relations in (6.20) are independent [80], but in order to clearly see the relation of braid group and Temperly-Lieb algebra we listed them anyway.

It is an easy task to construct the Hecke algebra given the TLA algebra, since the transformation (6.18) between the generators is linear. This is not so for the BWM algebra, since the relation between b_i and e_i is a quadratic one, and therefore it is not straight forward to construct braid group generators from the TLA ones. This is the reason, why in general it is unusual to talk about graph-state models based on the BWM algebra.

We have seen that the TLA expressions can be written in a unified form for the spin $\frac{1}{2}$ and the spin 1 representations. Therefore we conjecture that this is also the case for the braid group generators. We construct the expression for the spin $\frac{1}{2}$ case, and verify that it fulfills the BWM algebra for $j=1$. The unified expression we have found is

$$b^{ac}_{bd} = q^{(c_d-c_c-c_a+c_b)}(-1)^{(b+d-a-c)} \times \left\{ \begin{array}{ccc} j & b & a \\ j & d & c \end{array} \right\}_q \times (-1)^{(a-c)\frac{1}{j}} \quad . \qquad (6.22)$$

These generators (6.22) satisfy a further property: the crossing symmetry,

$$b^{ac}_{bd} = \left(b^{bd}_{ac}\right)^{-1} \left(\frac{[2a+1][2c+1]}{[2b+1][2d+1]}\right)^{\frac{1}{2}} \quad . \qquad (6.23)$$

Here we see the advantage of our construction. As in the Hecke case, also here the crossing parameters are built out of the eigenvectors of the corresponding graph.

6.1.3 Introducing the spectral parameter

In [63] it was shown that given a representation of the braid group which factors either through the Hecke algebra or through the BWM algebra, one can introduce a spectral parameter x with a mechanism called *universal baxterization.* In the Hecke algebra case one finds

$$R_i(x) = q^{-1}xb_i - qx^{-1}b_i^{-1} \quad , \qquad (6.24)$$

which is equivalent to (6.1). For the BWM case the spectral parameter depending solution is written as

$$R_i(x) = (x^{-1}-1)kg_i + m(k+k^{-1}) + (x-1)k^{-1}g_i^{-1} \quad , \qquad (6.25)$$

with $k=q^3$.

Also the amplitudes (6.25) are constructed on the base of a graph. Even though they are related to the BWM algebra rather then to the Hecke algebra, they share still all their properties (6.2)-(6.5), as completeness

$$\begin{aligned} &\sum_e R^{ae}_{bd}(x) R^{ec}_{bd}(x^{-1}) = \\ &= \delta_{ac} \times (x^{-1}q^3 + xq^{-3})(x^{-1}q^2 - xq^{-2})(xq^3 + x^{-1}q^{-3})(xq^2 - x^{-1}q^{-2}) \quad , \qquad (6.26) \end{aligned}$$

the crossing relation

$$R^{ac}_{bd}(x) = \left(\frac{[2a+1][2c+1]}{[2b+1][2d+1]}\right)^{\frac{1}{2}} R^{bd}_{ac}(-x^{-1}q^6) \quad , \qquad (6.27)$$

and PCT symmetry

$$R^{ac}_{bd}(x) = R^{ac}_{db}(x) = R^{ca}_{bd}(x) \quad . \tag{6.28}$$

Also in the BWM case the crossing parameters are directly obtained from the Perron-Frobenius eigenvector, but of the graph (6.9).

Finally we mention the so-called *symmetry-breaking transformations* [36], which leave untouched the Yang-Baxter equation and the completeness-relation, but can change the parameters appearing in the crossing-relation. They are given by

$$R^{ac}_{bd}(e^u) \to \tilde{R}^{ac}_{bd}(e^u) = \alpha_{abcd}(u).\beta_{abcd}.\gamma_{abcd} \times R^{ac}_{bd}(e^u) \quad , \tag{6.29}$$

with

$$\begin{aligned} \alpha_{abcd}(u) &= e^{[-p(a)+p(b)-p(c)+p(d)]u} \quad , \\ \beta_{abcd} &= \frac{p'(a)}{p'(c)} \quad , \\ \gamma_{abcd} &= e^{\omega[(b-c)(c-d)-(a-d)(b-a)]} \quad . \end{aligned}$$

Herein $p(.)$ and $p'(.)$ are arbitrary functions, and ω an arbitrary parameter.

The so constructed graph state models (6.25) coincide for the BWM case with the scattering amplitudes conjectured by Smirnov (5.52), if we apply a symmetry breaking transformation of the type $R^{ac}_{bd}(x) \to (-1)^{(a-c)} R^{ac}_{bd}(x)$.

6.2 Application to scattering theories

Now we want to apply this mathematical formalism to the problem of scattering theories describing deformations of conformal field theories. Having now at hand a graph-state formulation also for Smirnov's amplitudes, we obtain a unified description of $\Phi_{1,3}$, $\Phi_{1,2}$ and $\Phi_{2,1}$ perturbations. The expressions for braid group and TLA are the same if related to the corresponding graph. The difference is, that $\Phi_{1,3}$ models are related to the Hecke algebra, whereas $\Phi_{1,2}$ and $\Phi_{2,1}$ models are related to the BWM structure.

From now on, we will concentrate mainly onto the R-matrix built on the BWM-algebra (6.25). The $\Phi_{1,3}$ perturbed models have been described in an algebraic way in [13, 97, 102].

6.2.1 Interpretation of the R-matrix as consistent scattering amplitudes

In order to identify the corresponding scattering theory one needs to relate the spectral parameter x to the rapidity variable β. This causes that a whole series of scattering theories get related to the same R-matrix. Let us explain this mechanism for the R-matrix (6.25).

Since we have in mind the scattering theories (5.52), we make the ansatz $x = e^{\frac{2\pi\beta}{\xi}}$. Crossing symmetry in scattering theories is usually implied in a strict way, *i.e.* $S(\beta)^{ac}_{bd} = S(i\pi-\beta)^{bd}_{ac}$. For the R-matrix we have the relation (6.27), which includes also the crossing factors given by the Perron-Frobenius eigenvector of the incidence diagram. They can be eliminated by a gauge transformation, using (6.29). On the other hand one needs to be aware, that in scattering theory there is another constraint determining the crossing factors. These are the residues, since for a scattering theory describing a unitary field theory the signs of the residues are fixed. One can show, that the residues, using the gauge of the amplitudes (5.52) have the right sign.

Second, in order to achieve crossing symmetry the transformation $\beta \longrightarrow i\pi - \beta$ must correspond to $x \longrightarrow -x^{-1}q^6$. This implies that

$$\frac{2\pi^2 i}{\xi} = i\pi + \frac{6i\pi}{r} + 2n\pi i \quad , \tag{6.30}$$

from which we find the relation for the parameter $\xi = \frac{2\pi r}{\pm 6+2nr-3r}$, with $n \in \mathbb{Z}$ and with $r \equiv k+2$. But in order to implement the symmetry of the diagram (6.9) dynamically we need a bound state at the pole $\beta = \frac{2i\pi}{3}$. This requires that at this point the R-matrix must degenerate into a 3-dimensional projector. In the appendix we have collected some information on the projectors and the necessary $6j$-symbols. We need that

$$e^{\frac{2\pi\beta}{\xi}}\Big|_{\beta=\frac{2\pi i}{3}} = q^4 \quad .$$

This condition eliminates part of the possible values for the parameter ξ, leaving $\xi = \frac{2\pi r}{\pm 6+6nr-3r}$.

This condition is equivalent to an idea proposed by Zamolodchikov [117] who constructed a factorizable scattering theory for the tricritical Ising model perturbed by the subleading magnetization ($\mathcal{M}_{4,5} + \Phi_{2,1}$). He required that one of the amplitudes needs to become zero at the pole at $\frac{2\pi i}{3}$. In our formulation this amplitude corresponds to R^{11}_{00}, since a kink interpolating the vacuum 0 to the vacuum 0 does not exist (see the graph (6.9), which has no tadpole at the node 0). This condition is automatically fulfilled if the amplitudes degenerate into a three-dimensional projector at this point.

Let us show that this is the case: in the previous paragraph the R-matrix was given in terms of $6j$-symbols. Since this R-matrix is an affinization of a quantum group in the shadow-world representation [66], we can also express the projectors as $6j$-symbols, that is

$$P^{ac;j}_{bd} = \left\{ \begin{matrix} 1 & 1 & j \\ b & d & a \end{matrix} \right\} \left\{ \begin{matrix} 1 & 1 & j \\ b & d & c \end{matrix} \right\} \quad . \tag{6.31}$$

The exact relation for the $3d$-projector is

$$R^{ac}_{bd}(x = q^4, q) = [2][4]P^{ac;1}_{bd} \quad . \tag{6.32}$$

From the expressions given in the appendix, we easily compute the residues at the pole. The general amplitude needed in order to verify the Zamolodchikov condition is

$$R^{l+1,l+1}_{ll}(q^4, q) = (q^2 - q^{-2})[2]\frac{[2l][2l+3]}{[2l+2][2l+1]} \quad ,$$

which becomes zero for $l = 0$.

As a last ingredient for a physical scattering theory one needs unitarity, that is $S(\beta)S^{\star}(\beta) = 1$. Since the elements of the R-matrix are real[8], R satisfies also real analyticity, *i.e.* $S^{\star}(\beta) = S(-\beta)$. Additionally we have the completeness property (6.26), and therefore the R-matrix multiplied by a scalar factor S_0, which eliminates the terms on the right hand side of (6.26) is unitary. But this factor coincides with (5.52) with the corresponding parameter ξ.

Confronting the resulting theories with (5.52), we find that all of the perturbed conformal scattering theories $\mathcal{M}_{r,mr\pm1} + \Phi_{1,2}$ correspond to the same R-matrix (6.25), once the parameter r is fixed.The parameter m encodes this arbitrariness, and takes the values $m = 1, 2, \ldots$. The 'formal' theory $\mathcal{M}_{r,r-1} + \Phi_{1,2}$ should be interpreted as the scattering theory for $\mathcal{M}_{r-1,r} + \Phi_{2,1}$. For all of these theories, even those describing non-unitary field

[8]This is related to the fact, that we used the highest eigenvalue in diagonalizing the incidence matrix of the diagrams (6.9), whose corresponding eigenvector is the Frobenius-Perron eigenvector.

theories, the scattering matrix of the fundamental particle is unitary, that is $SS^{\star} = 1$. Since through the bootstrap this property is preserved also for other particles, *all* of these models are consistent scattering theories. This is a larger set of unitary scattering theories as was realized before.

Let us discuss the consequences of the above calculations. We found that *one* R-matrix corresponds to *many* different scattering theories, according to how one relates the rapidity variable to the spectral parameter. In the past there was the believe that there is a unique way to find a physical scattering theory given an R-matrix and fixing the truncation level corresponding to the incidence diagram. It was determined by the principle of "minimality". This principle was commonly used in order to eliminate ambiguities deriving from the fact that the factor S_0 can not be derived uniquely, but has always an ambiguity of so-called CDD-factors. Minimality says, that the physical scattering theory corresponding to a given R-matrix is that one, which introduces the minimal number of poles and zeros in the physical strip. We see now, that this is not a fundamental principle. We find that the theories belonging to one R-matrix depend on how the spectral parameter is related to the rapidity variable, and the S-matrix of the fundamental particle differ from each other by CDD-factors. These factors of course usually introduce further poles in the physical strip, and therefore generate a completely different physical scattering theory. Analyzing the allowed parameters ξ we find that the theory with the minimal number of poles and zeros corresponds to a deformed *unitary* conformal theory. This fact was explicitly discussed for scattering theories of perturbed minimal models $\mathcal{M}_{5,n}$ in [69].

6.2.2 Ultraviolet limit

An important information is obtained from the ultraviolet limit of the theory. This, because the amplitudes (5.52) are supposed to describe deformations of minimal models. In the ultraviolet limit one should be able to determine the corresponding conformal field theory. Usually one needs to perform the thermodynamic Bethe ansatz in order to get information as the central charge and the dimension of the perturbing relevant operator. We want to discuss here possibilities to read off information of the conformal field theory directly from the S-matrix.

For $\beta \to \infty$ the S-matrix becomes again proportional to the braid-group generators (6.22), but with the gauge-transformation, that is

$$b^{ac}_{bd} = S_0(\beta \to \infty)\, q^{(c_d - c_c - c_a + c_b)}(-1)^{(b+d-a-c)} \times \left\{ \begin{matrix} j & b & a \\ j & d & c \end{matrix} \right\}_q \quad . \tag{6.33}$$

This expression is valid also for $\Phi_{1,3}$ perturbations, which correspond to the spin $j = \frac{1}{2}$. One notices that these expressions are proportional to the braiding matrices of conformal blocks of the WZW-models [1]. This is an important fact, since it supports the hope that also deformed minimal models can be described by similar algebraic structures as the original CFT.

Now we use the results from section 6.1. Let us view these braid-group generators as matrices in the indices a and c. Since they satisfy (for spin $j = 1$) a third order relation (6.21), there can be only 3 independent eigenvalues. This is strongly related to the fact that it is built on a BWM algebra. The same fact holds also for the corresponding R-matrices, whose non-diagonal components are given by the braid group generators. By means of diagonalization one finds that the eigenvalues correspond to the amplitudes $S^{11}_{00}(\beta)$, $S^{11}_{01}(\beta)$ and $S^{11}_{02}(\beta)$ which define three independent phase-shifts. We have calculated their asymptotic behaviour, and found that they behave as

$$\begin{aligned} \lim_{\beta\to\infty} S^{11}_{00}(\beta) &= e^{2i\pi h_{3,1}} \quad , \\ \lim_{\beta\to\infty} S^{11}_{01}(\beta) &= e^{i\pi h_{3,1}} \quad , \\ \lim_{\beta\to\infty} S^{11}_{00}(\beta) &= e^{i\pi(2h_{3,1} - h_{5,1})} \quad , \end{aligned} \tag{6.34}$$

where $h_{3,1}$ and $h_{5,1}$ are the anomalous dimensions of the corresponding fields of the original conformal field theory. These are exactly the dimensions appearing in the operator product expansion of $\Psi \equiv \Phi_{3,1}$ of the original minimal model $\mathcal{M}_{r,mr\pm1}$:

$$\Psi(z)\Psi(0) = \frac{1}{z^{2h_{3,1}}} \; 1 + \frac{C_{\Psi,\Psi,\Psi}}{z^{h_{3,1}}} \; \Psi(0) + \frac{C_{\Psi,\Psi,\Phi_{5,1}}}{z^{2h_{3,1}-h_{5,1}}} \; \Phi_{5,1}(0) + \dots \tag{6.35}$$

Of course if one considers the series of theories $\mathcal{M}_{r-1,r} + \Phi_{2,1}$ one finds that the corresponding field is $\Phi_{1,3}$ instead of $\Phi_{3,1}$,

$$\Psi(z)\Psi(0) = \frac{1}{z^{2h_{1,3}}} \; 1 + \frac{C_{\Psi,\Psi,\Psi}}{z^{h_{1,3}}} \; \Psi(0) + \frac{C_{\Psi,\Psi,\Phi_{1,5}}}{z^{2h_{1,3}-h_{1,5}}} \; \Phi_{5,1}(0) + \dots \tag{6.36}$$

This correspondence for $\Phi_{2,1}$ perturbed unitary theories has been found in [33].

Similar one can analyze the asymptotic phase-shifts of the $\Phi_{1,3}$- perturbed models. They satisfy a second order relation (6.19) and therefore the braid group generators as well as the R-matrix (6.24) have only two eigenvalues. They correspond to the amplitudes $S_{00}^{\frac{1}{2}\frac{1}{2}}(\beta)$ and $S_{01}^{\frac{1}{2}\frac{1}{2}}(\beta)$. It is not surprising that their asymptotic phase-shifts determine the dimensions of the OPE of the field $\Phi_{2,1}$,

$$S_{00}^{\frac{1}{2}\frac{1}{2}}(\beta\to\infty) = e^{2i\pi h_{2,1}} \quad , \quad S_{01}^{\frac{1}{2}\frac{1}{2}}(\beta\to\infty) = e^{i\pi(2h_{2,1}-h_{3,1})} \quad . \tag{6.37}$$

Note that for deriving these results not only the algebraic structure (braid-group) was important, but also the *dynamical* information. This is contained in the factor S_0. The phase-shifts in the ultra violet limit can therefore be used to put a further constraint on the CDD-ambiguity arising in the determination of S_0.

6.2.3 Bootstrap equations

For the use in the next section, let us discuss the Bootstrap equations in this context. For degenerate particles in the IRF description they were developed in [27]. The equations are

$$S_{ad}^{bd} f_{abc} = \sum_g f_{egd} S_{ag}^{eb}(\theta + i\bar{u})\, S_{db}^{cg}(\theta - i\bar{u}) \quad . \tag{6.38}$$

The constants f can be obtained from the scattering matrices [67] as

$$\mathrm{Res}_{\theta=iu} S_{bd}^{ac} = i f_{bad} f_{bcd} \quad , \tag{6.39}$$

where u is the corresponding S-matrix pole. It is useful to exploit the quantum-group symmetry in order to reformulate the above equations, since the above definition of the constants f leads to a system of quadratic equations to solve and therefore leaves an ambiguity of a sign.

Since the S-matrix for $\Phi_{1,2}$ perturbed minimal models is proportional to the $A_2^{(2)}$ quantum group R-matrix, one can also rewrite the bootstrap-equations in terms of the pentagon-identity. This determines the constants f as $6j$-symbols. Or more explicitly,

$$f_{abc} = \left\{ \begin{array}{ccc} 1 & 1 & j \\ a & c & b \end{array} \right\} \quad , \tag{6.40}$$

where the spin j corresponds to the projector, into which the S-matrix degenerates at the pole. This correspondence can also be seen from the form of the projectors (6.31).

6.3 Bootstrap for the models $\mathcal{M}_{p,p+1} + \Phi_{1,2}$

The unitary minimal series perturbed by the operator $\Phi_{1,2}$ was analyzed in [101]. He established the spectrum of all theories except $\mathcal{M}_{5,6}$ and wrote down the S–matrix of the fundamental kink as well as that one of the fundamental breather. We apply now the bootstrap-equations in the IRF formulation in order to write down the *complete explicit* S-matrix of kinks and their bound-states, the breathers. As a byproduct we also establish the particle content of the model $\mathcal{M}_{5,6}$, which contains two kinks and four breathers.

One of the main advantages of the IRF-representation is that the bootstrap equations become rather simple. They reduce to a set of scalar equations. In the usual vector representation, the matrix structure of the particles has to be taken in account. This makes explicit calculations very involved. In our formulation, in principle it is sufficient to consider only one particular amplitude, in order to carry out the bootstrap, since the bootstrap equations have become a set of *scalar* equations. This allows us to calculate higher kink S-matrices explicitly. They are needed in order to close the bootstrap, since if they exhibit further poles, they have to be interpreted as further particles and must be included into the spectrum. Let us use the abbreviations

$$(x)^{\pm} = \frac{\Gamma(\frac{2k\pi}{\xi} + x \pm \frac{i\beta}{\xi})}{\Gamma(\frac{2k\pi}{\xi} + x \mp \frac{i\beta}{\xi})} \quad , \tag{6.41}$$

and

$$\langle x \rangle = \frac{\tanh(\frac{\theta}{2} + i\pi x)}{\tanh(\frac{\theta}{2} - i\pi x)} \quad . \tag{6.42}$$

Then the S-matrices of the kinks are

$$S_{K_1,K_1}(\beta) = \left(\sinh\frac{\pi}{\xi}(\beta - i\pi)\sinh\frac{\pi}{\xi}(\beta - \frac{2\pi i}{3})\right)^{-1} \times \quad ,$$

$$\prod_{k=0}^{\infty}(\frac{\pi}{\xi})^{-}(\frac{2\pi}{\xi})^{+}(1)^{+}(1+\frac{\pi}{\xi})^{-}(\frac{\pi}{3\xi})^{+}(\frac{4\pi}{3\xi})^{-}(1+\frac{2\pi}{3\xi})^{-}(1+\frac{5\pi}{3\xi})^{+} \times R^{ac}_{bd} \tag{6.43}$$

$$S_{K_1,K_2}(\beta) \quad = \quad \left(\cosh\frac{\pi}{\xi}(\beta - i\pi)\cosh\frac{\pi}{\xi}(\beta - \frac{2\pi i}{3})\right)^{-1}$$

$$\prod_{k=0}^{\infty}(\frac{1}{2}+\frac{\pi}{\xi})^-(\frac{1}{2}+\frac{2\pi}{\xi})^+(\frac{1}{2})^+(\frac{1}{2}+\frac{\pi}{\xi})^- \times$$
$$(\frac{1}{2}+\frac{\pi}{3\xi})^+(\frac{1}{2}+\frac{4\pi}{3\xi})^-(\frac{1}{2}+\frac{2\pi}{3\xi})^-(\frac{1}{2}+\frac{5\pi}{3\xi})^+ \times$$
$$\tilde{R}^{ac}_{bd} \times \langle\frac{2i\pi}{3}-\frac{\xi}{2}\rangle\langle\frac{2\pi}{3}+\frac{\xi}{2}\rangle \quad , \tag{6.44}$$

where $\tilde{R}$ is the R-matrix with a spectral parameter shifted by a phase-factor of $\frac{\pi}{2}$. Finally,

$$S_{K_2,K_2}(\beta) = S_{K_1,K_1}(\beta)\langle\frac{2i\pi}{3}-\xi\rangle\langle\frac{2\pi}{3}\rangle^2\langle\xi\rangle \quad . \tag{6.45}$$

The analytic structure is exhibited in figures 6.1 and 6.2.

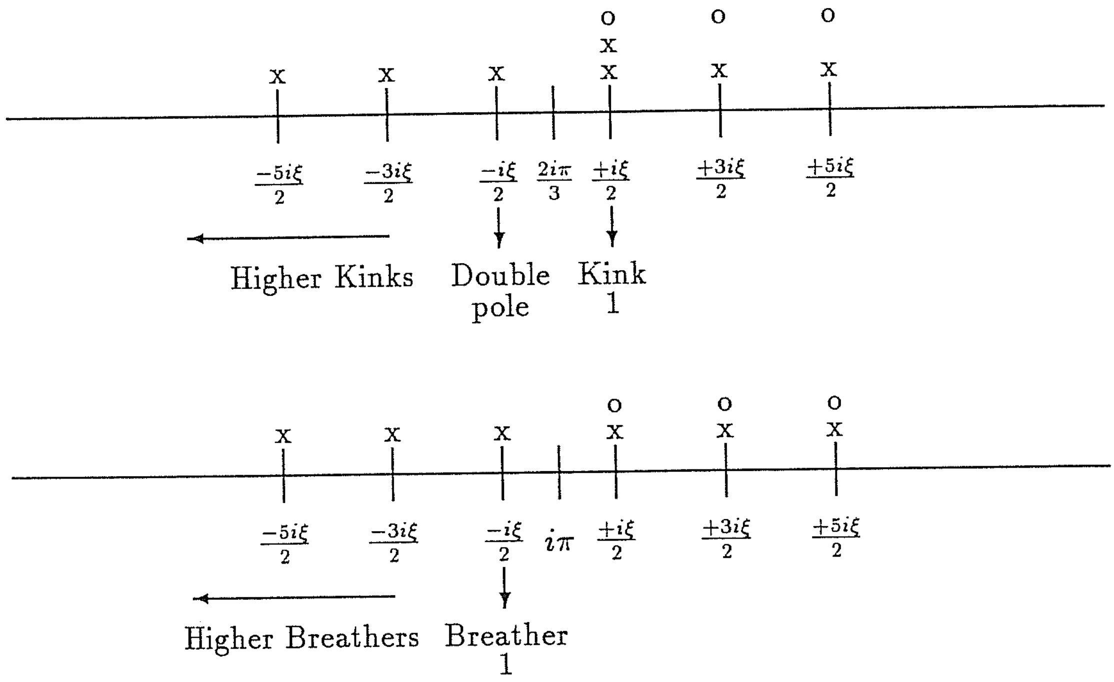


Figure 6.1: Pole structure of the S-matrix S_{K_1,K_2}

In both cases we showed only the direct channel poles, the crossed ones being in a one to one correspondence. The double poles in the kink-kink S-matrices can all be explained in terms of elementary scattering processes [31]. They are exhibited in figure 6.3.

The S-matrix elements involving the breathers are the following:

$$S_{K_1,B_1}(\beta) = \langle\frac{\pi}{2}-\frac{\xi}{2}\rangle_{K_1}\langle\frac{5\pi}{6}-\frac{\xi}{2}\rangle_{K_2} \quad ,$$
$$S_{K_1,B_2}(\beta) = \langle\frac{2\pi}{3}\rangle^2\langle\frac{2\pi}{3}-\xi\rangle_{K_2}\langle\xi\rangle \quad ,$$

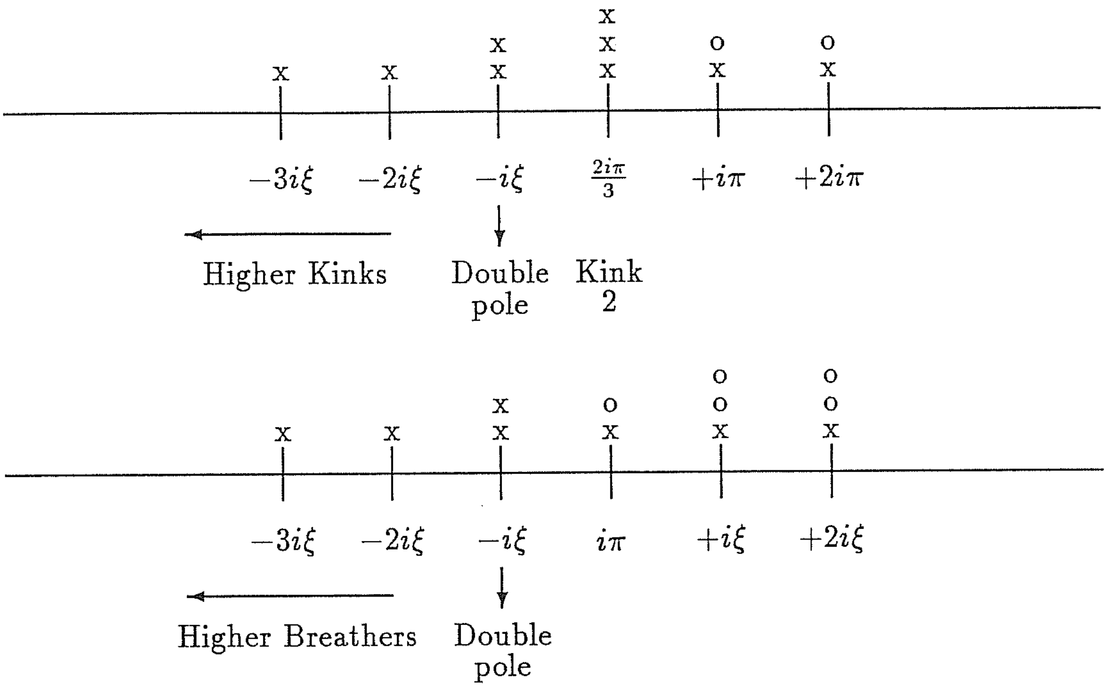


Figure 6.2: Pole structure of the S-matrix S_{K_2,K_2}

$$\begin{aligned}
S_{K_2,B_1}(\beta) &= \langle\frac{\pi}{2}\rangle\langle\frac{\pi}{6}\rangle_{K_1}\langle\frac{\pi}{6}+\xi\rangle\langle-\frac{\pi}{6}+\xi\rangle \quad , \\
S_{K_2,B_2}(\beta) &= \langle\frac{\pi}{3}+\frac{\xi}{2}\rangle^3\langle\pi-\frac{\xi}{2}\rangle\langle\frac{\pi}{3}-\frac{\xi}{2}\rangle_{K_1}\langle\pi-\frac{3\xi}{2}\rangle\langle-\frac{\pi}{3}+\frac{3\xi}{2}\rangle \\
S_{B_1,B_1}(\beta) &= \langle\xi\rangle\langle\frac{2\pi}{3}\rangle_{B_1}\langle-\frac{\pi}{3}+\xi\rangle_{B_2} \quad , \\
S_{B_1,B_2}(\beta) &= \langle\frac{\pi}{2}-\frac{\xi}{2}\rangle^2\langle-\frac{\pi}{6}+\frac{3\xi}{2}\rangle\langle-\frac{\pi}{2}+\frac{3\xi}{2}\rangle\langle\frac{\pi}{2}+\frac{\xi}{2}\rangle\langle-\frac{\pi}{6}+\frac{\xi}{2}\rangle_{B_1} \quad , \\
S_{B_2,B_2}(\beta) &= (\langle\frac{2\pi}{3}\rangle^3)_{B_2}\langle\xi\rangle^3\langle-\frac{\pi}{3}+\xi\rangle^2\langle\frac{\pi}{3}+\xi\rangle\langle-\frac{\pi}{3}+2\xi\rangle\langle-\frac{2\pi}{3}+2\xi\rangle \quad .
\end{aligned} \tag{6.46}$$

The lower indices indicate the bound state corresponding to that pole.

The model $\mathcal{M}_{5,6}$ exhibits two more breathers, even though no more kinks are generated. This can be seen from fig 6.1 and 6.2. Since $\xi \leq \frac{\pi}{2}$ new breathers are created. But since $\xi \geq \frac{\pi}{3}$ no new kink poles come into the physical strip. The third breather with the mass

$$M_3 = 4m \sin\left(\frac{5}{21}\pi\right) \sin\left(\frac{3}{7}\pi\right)$$

is a bound state of K_1 and K_2 at the pole $u_{K_1,K_2}^{B_3} = \frac{2}{7}\pi$. The heaviest breather is a bound state of K_2 with itself at rapidity $u_{K_2,K_2}^{B_4} = \frac{1}{21}\pi$ and has mass

$$M_4 = 4m \cos\left(\frac{2\pi}{21}\right) \cos\left(\frac{\pi}{42}\right) \quad .$$

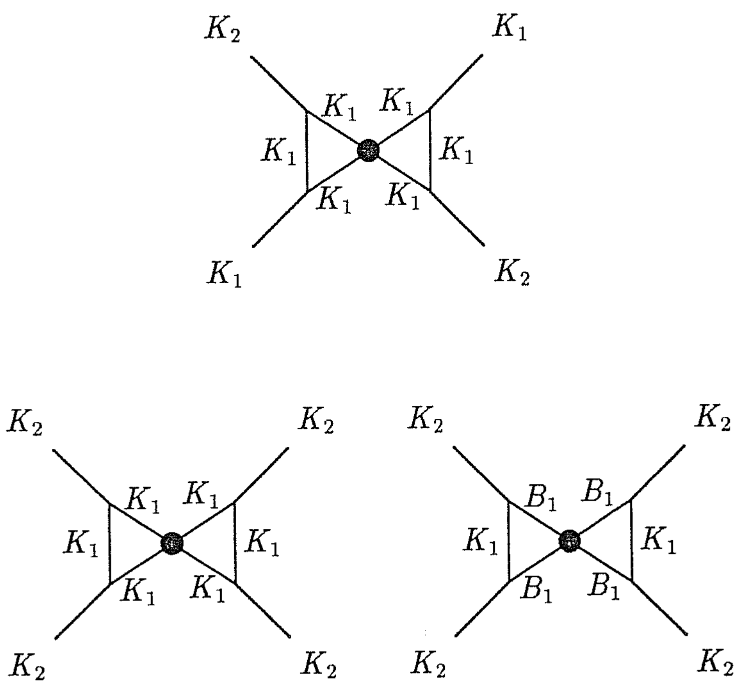


Figure 6.3: Multi scattering processes responsible for higher order poles in the S-matrices S_{K_1,K_2} and S_{K_2,K_2}.

In [86] the truncation method was performed for this model, and all but the heaviest particle were conjectured. This is not surprising since for heavy particles the finiteness of the basis in the truncated Hilbert space causes rather big systematical errors. The remaining breather part of the S-matrix of this model is

$$
\begin{aligned}
S_{13} &= \langle \tfrac{31}{42} \rangle_2 \langle \tfrac{3}{14} \rangle_4 \langle \tfrac{13}{14} \rangle_1 \langle \tfrac{11}{42} \rangle \langle \tfrac{19}{42} \rangle \langle \tfrac{17}{42} \rangle^2 \quad , \qquad S_{14} = \langle \tfrac{6}{7} \rangle_3 \langle \tfrac{2}{3} \rangle^2 \langle \tfrac{10}{21} \rangle^2 \langle \tfrac{4}{21} \rangle \langle \tfrac{2}{7} \rangle \langle \tfrac{5}{21} \rangle \quad , \\
S_{23} &= \langle \tfrac{5}{6} \rangle_1 \langle \tfrac{5}{14} \rangle^2 \langle \tfrac{13}{42} \rangle^2 \langle \tfrac{1}{2} \rangle \langle \tfrac{3}{14} \rangle \langle \tfrac{19}{42} \rangle \quad , \qquad S_{24} = \langle \tfrac{19}{21} \rangle_2 \langle \tfrac{2}{7} \rangle^2 \langle \tfrac{5}{21} \rangle \langle \tfrac{8}{21} \rangle^2 \langle \tfrac{3}{7} \rangle^3 \langle \tfrac{1}{7} \rangle \langle \tfrac{10}{21} \rangle \quad , \\
S_{33} &= \langle \tfrac{2}{3} \rangle_3^3 \langle \tfrac{1}{7} \rangle^2 \langle \tfrac{10}{21} \rangle^3 \langle \tfrac{2}{7} \rangle \langle \tfrac{8}{21} \rangle \langle \tfrac{4}{21} \rangle \quad , \\
S_{34} &= \langle \tfrac{13}{14} \rangle_1 \langle \tfrac{17}{42} \rangle^4 \langle \tfrac{3}{14} \rangle^2 \langle \tfrac{19}{42} \rangle^3 \langle \tfrac{11}{42} \rangle^3 \langle \tfrac{5}{14} \rangle \langle \tfrac{5}{42} \rangle \langle \tfrac{5}{21} \rangle \quad , \\
S_{44} &= \langle \tfrac{2}{3} \rangle_4^5 \langle \tfrac{2}{7} \rangle^2 \langle \tfrac{1}{7} \rangle^2 \langle \tfrac{10}{21} \rangle^5 \langle \tfrac{8}{21} \rangle^3 \langle \tfrac{4}{21} \rangle^3 \langle \tfrac{3}{7} \rangle \langle \tfrac{1}{21} \rangle \langle \tfrac{5}{21} \rangle \quad .
\end{aligned}
\tag{6.47}
$$

Herein the indices of the S-matrix elements correspond to breathers. This is the complete breather-part of this S-matrix.

A final confirmation of these S-matrices is expected from the thermodynamic Bethe ansatz. It involves higher level Bethe ansatz techniques, and gets rather complicated since there the spectrum consists of *two* degenerate particles.

Let us summarize the results of this section: We have analyzed the IRF structure which lies under $\Phi_{1,2}$ and $\Phi_{2,1}$-perturbed conformal field theories. As the $\Phi_{1,3}$-perturbed models, they can be built as graph state models, but using an BWM-algebra instead of a Hecke-algebra as underlying structure. Using this formulation we have given a unified way of describing scattering amplitudes of deformed minimal models in the $\Phi_{1,3}$, $\Phi_{1,2}$ and $\Phi_{2,1}$ directions. These exceed in general the integrable perturbations (apart of some special perturbations for single models, see *e.g.* section 5.4).

We have obtained the complete set of unitary scattering theories, which are $\mathcal{M}_{r,mr\pm 1} + \Phi_{1,2}$ and $\mathcal{M}_{r-1,r} + \Phi_{2,1}$. We have analyzed the ultraviolet limit, which is determined by the braid group generators of the BWM algebra. They are proportional to the braiding matrices of the conformal blocks of the corresponding WZW model $SU(2)$. Moreover, diagonalizing the phase-shifts in the UV-limit one finds that the resulting eigenvalues describe exactly the dimensions appearing in the OPE of $\Phi_{3,1}$ of the original minimal model $\mathcal{M}_{r,mr\pm1}$ ($\Phi_{1,3}$ for $\Phi_{2,1}$ perturbed models). The crossing parameters are determined from the Frobenius Perron eigenvector of the incidence graph.

Having an explicit expression of the residues in form of $6j$ symbols, we have rewritten the bootstrap-equations in a form which allows explicit calculations in a simple way. We then used that to calculate the S-matrix elements involving the higher kink, which appears in the unitary series $\mathcal{M}_{r,r+1} + \Phi_{1,2}$. A non-trivial degeneracy structure persists for the models $r \geq 5$. We find the whole S-matrices involving kinks and breathers of these theories. The S-matrix elements among kinks exhibit double poles which can all be described by elementary scattering processes of the lightest kink and the lightest breather. As a byproduct we also determine the spectrum of the model $\mathcal{M}_{5,6} + \Phi_{1,2}$ which was not known before.

We have calculated these S-matrices *explicitly*. They are the essential input, if one wants to determine the off-shell structure of these field theoretical models. One possibility is the thermodynamic Bethe ansatz. But, as already mentioned, it seems rather difficult to be carried out, because of the complicated spectrum. The other important approach is to calculate the form-factors, since they allow to get an expression for the correlation-functions. Since now the complete S-matrix is known, it should not be difficult to calculate at least the two and three particle form factors.

6.4 Scattering Theories related to the HSLM model

We will apply the formalism of the last section to a specific geometry, the Hard Square Lattice Model (HSLM) [10] in the critical regime, which means choosing a trigonometric parameterization of the Boltzmann weights. We will find that every consistent configuration describes the geometry of a $\Phi_{1,3}$ perturbation of the series $\mathcal{M}_{5,n}$. A more restricted set describes the $\Phi_{1,2}$ and $\Phi_{2,1}$ perturbations of these models.

The HSLM in the critical regime can be formulated as a graph state model with a graph consisting of two sites, labelled by 0 and 1, of which only one can couple to itself (figure 6.4) [36]. The associated incidence matrix is given by

$$I = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad . \tag{6.48}$$

The eigenvalue equation (6.6) has two possible solutions. The first one is given by $\lambda = \frac{\pi}{5}$

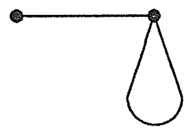


Figure 6.4: Graph defining the geometry of the HSLM

and the associated eigenvector is the Frobenius-Perron eigenvector of the incidence matrix (6.48). This solution corresponds to the usual parameterization of the HSLM where the crossing parameters $\psi(a)$ are all positive. The second solution is given by $\lambda = \frac{3\pi}{5}$. The properties (6.2) - (6.5) for the corresponding Boltzmann weights (6.1) are equally fulfilled but now some of them are imaginary due to the fact that the crossing parameters $\psi(a)$ take also negative values.

We want to interpret the above Boltzmann weights as possible scattering amplitudes of a $(1+1)$ dimensional relativistic QFT, presenting a set of degenerate vacua associated to the sites of the graph. These are labelled by the indices a, b, c and d in (6.1) and the scattering processes occur in terms of kink configurations K_{ab}

$$|K_{bc}K_{cd}\rangle = S_{bd}^{ac}|K_{ba}K_{ad}\rangle \quad , \tag{6.49}$$

wherein the kinks K_{ab} are meant to connect the degenerate vacua of the corresponding soliton theory. In order to describe a physical scattering theory, one requires a complete set of asymptotic states, linked by the S–matrix which has to be unitary,

$$S\, S^{\dagger} = 1 \quad , \tag{6.50}$$

and crossing symmetric

$$S_{bd}^{ac}(i\pi - \beta) = S_{ac}^{db} \quad . \tag{6.51}$$

One can release this second requirement introducing a non-trivial charge conjugation (see *e.g.* [92]) which, in terms of the structure of the kink vacua, means that one introduces an asymmetric basis, because the coupling constants of the theory g_{abc} will not any more be invariant under the rotation of the indices. As long as this asymmetry arises from a basis change one can equally describe the scattering theory in this asymmetric basis.

The usual ansatz for the S–matrix is to define

$$S_{bd}^{ac}(\beta) = S_0(\beta)\ \ \omega(a, b, c, d; \pm\frac{i\beta}{\xi}) \quad , \tag{6.52}$$

where S_0 is determined by the unitarity condition (see below). From the point of view of the initial IRF model, S_0 simply multiplies the Boltzmann weights by an overall factor. In terms of scattering theory though, it is with this function that one introduces the dynamics. This, because the Boltzmann weights (6.1) are entire functions, and possible bound states are introduced only through the poles of S_0.

We start our discussion with the first solution $\lambda = \frac{\pi}{5}$ of eq. (6.6). Implementing the condition of crossing symmetry (6.4) through the correspondence $\mp\frac{\pi}{\xi} = \frac{\pi}{5} + 2n\pi$, n being integer (we require ξ positive), we obtain the following possible values of ξ

$$\xi = \frac{5}{10n \mp 1} \quad . \tag{6.53}$$

By a basis change in the space spanned by the asymptotic states, *i.e.* a symmetry breaking transformation (6.29), it is always possible to implement crossing symmetry with a trivial charge conjugation operator. This transformation acts on the amplitudes as

$$S_{bd}^{ac} \longrightarrow \left(\frac{\psi(a)\psi(c)}{\psi(b)\psi(d)}\right)^{-\frac{\beta}{2\pi i}} S_{bd}^{ac} \quad . \tag{6.54}$$

However, it introduces the unpleasant feature of an oscillating behaviour into the S-matrix, and therefore represents an inconvenient basis to compare the S–matrix with an underlying CFT reached in the ultraviolet limit (for a discussion of this point see ref. [32, 33]).

Unitarity links the Hilbert spaces of "in" and "out" states through the S–matrix. Whenever the amplitudes are real, *i.e.* the crossing parameters are positive, the unitarity equations are equivalent to

$$\sum_e S^{ed}_{ca}(\beta) S^{be}_{ca}(-\beta) = \delta_{bd} \quad . \tag{6.55}$$

Therefore, the unitarity equations (6.55) are fulfilled, if one chooses S_0 to cancel the factor $\rho(u)\rho(-u)$ in (6.3).

In the case $\lambda = \frac{\pi}{5}$, $\rho(u) = \sin(\frac{\pi}{5} - u)$ and $S_0(\beta)$ has to satisfy

$$S_0(\lambda - u) = S_0(u) \quad \text{and} \quad S_0(u)S_0(-u) = \frac{1}{\sin(\frac{\pi}{5} - u)\sin(\frac{\pi}{5} + u)} \quad . \tag{6.56}$$

The simplest solution of this system is given by

$$S_0^{(1)} = \frac{\sin(\frac{3\pi}{5} - u)}{\sin(\frac{\pi}{5} - u)\sin(\frac{3\pi}{5} + u)} \quad . \tag{6.57}$$

Notice that this expression introduces poles and zeros into the physical strip. Any other solution differs by CDD-factors [89, 116], that is factors of the kind f_x (compare eq. (3.18)). As we will show, these CDD-factors give rise to two *different* possible series of models.

Our parameterization, $u = \pm\frac{i\beta}{\xi}$, was chosen in order to facilitate the comparison with RSOS restrictions of the sine-Gordon S–matrix [13, 97]. From the values (6.53), this would correspond to $\Phi_{1,3}$ perturbations of conformal field theories $\mathcal{M}_{5,10n\pm4}$, ξ being related to the renormalized sine-Gordon coupling γ through $\xi = \frac{\gamma^2}{8}$. To show that this is really the case, we need to discuss the nature of the vacua and to compare their analytic structure.

The $\Phi_{1,3}$ perturbations of the minimal models $\mathcal{M}_{5,10n\pm4}$, belong to a class of theories known as RSOS(4), that is they present *four* possible degenerate vacua differing consecutively by $\frac{1}{2}$, *i.e.* $\{0, \frac{1}{2}, 1, \frac{3}{2}\}$. On the other hand, the S-matrix with $S_0(\beta)$ given in (6.57) was constructed with only two vacuum states. Therefore, the above mentioned identification seems problematic. However, due to a Z_2 symmetry of the RSOS(4) model, their amplitudes [13, 97] satisfy

$$S_{bd}^{ac}(\beta) = S_{\frac{3}{2}-b,\frac{3}{2}-d}^{\frac{3}{2}-a,\frac{3}{2}-c}(\beta) \quad , \tag{6.58}$$

that is, the amplitudes effectively reduce to a geometry described by two vacua but defined on the graph of the HSLM. This is due to the diagrammatic identity [3, 36] shown in figure 6.5 .

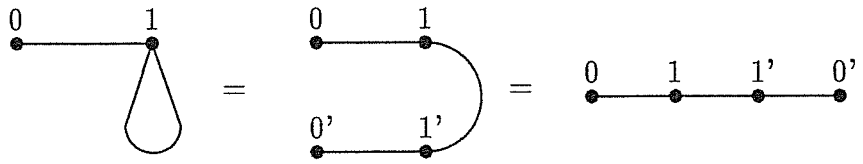


Figure 6.5: Diagrammatic identity relating RSOS(4) and the HSLM

It remains now to analyze the analytic structure of the two scattering theories we are comparing. The $S_0(\beta)$ factor corresponding to RSOS(4) for the two series of the parameter ξ are given by

$$\begin{aligned} \xi &= \frac{5}{10n+1} \quad \longrightarrow \quad S_0^{(SG)} = S_0^{(1)} \prod_{m=0}^{2n-1} [m\xi][-m\xi - \frac{\xi}{5}] \prod_{m=0}^{n-1} [m\xi + \frac{2\xi}{5}] \quad , \\ \xi &= \frac{5}{10n-1} \quad \longrightarrow \quad S_0^{(SG)} = S_0^{(1)} \prod_{m=0}^{2n} [m\xi][-m\xi + \frac{\xi}{5}] \prod_{m=1}^{n} [m\xi - \frac{2\xi}{5}] \quad . \end{aligned} \tag{6.59}$$

As expected, they differ from (6.57) only by CDD-factors. They cancel out all poles and zeros in the minimal expression of $S_0^{(1)}$ and introduce the required poles in order to create the Sine-Gordon breathers. We conclude that for every possible value of ξ given in (6.53), we find one system describing a $\Phi_{1,3}$ perturbation of a minimal model.

On the other hand we know that the HSLM naturally arises in $\Phi_{1,2}$ and $\Phi_{2,1}$ perturbations through the RSOS reduction of the Izergin-Korepin model. They have an intrinsic "Hard Hexagon" geometry, in that also the dynamics exhibits the structure of the graph of fig. 6.4. Let us explain this point some more in detail. In the theory there exist only three fundamental kinks K_{01}, K_{10} and K_{11}. What we mean by the dynamical structure is that the model exhibits a ϕ^3 interaction whose structure is determined by the kinks, respecting the geometry of the graph in fig. 6.4. That is, in the scattering process S_{00}^{11} we can not have a pole at $\beta = \frac{2i\pi}{3}$, which would correspond to a bound state K_{00}. This one can see stretching the amplitudes as shown in fig.6.6. On the other hand we require a pole

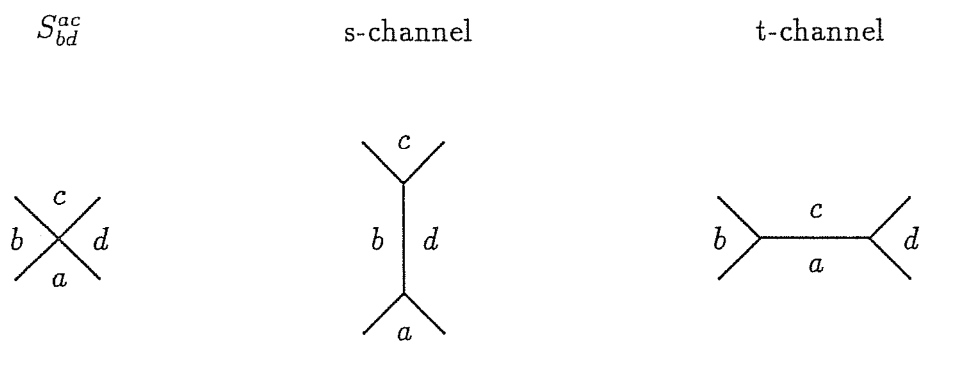


Figure 6.6: Intermediate states in the s- and t-channel of the scattering amplitudes

in the crossed channel $\beta = \frac{i\pi}{3}$ since this corresponds to the bound state K_{11}. Therefore we derive two conditions we need:

1. The function S_0 must exhibit a pole at $\beta = \frac{2i\pi}{3}$ and at $\beta = \frac{i\pi}{3}$. With this self-interaction the geometry of the model gets implemented dynamically. Note that such a pole is forbidden for the RSOS(4) model. In that case, even though the Boltzmann weights of the amplitudes take the form of those of the HSLM the vacua carry a different "coloring" which forbids the ϕ^3 interaction.

2. The pole at $\beta = \frac{2i\pi}{3}$ has to be cancelled by a zero appearing in the Boltzmann weight of amplitude S^{11}_{00}. Then, because of crossing symmetry, the pole $\beta = \frac{i\pi}{3}$ in S^{00}_{11} gets automatically cancelled.

The first requirement we can always fulfill, since we are free to add CDD factors [9]. Considering the second requirement, we find $\xi = \frac{5}{30n\pm 9}$ as the consistent set of parameters ξ. We compare this with S–matrices arising from the RSOS restriction of the Izergin-Korepin model, in which the spectral parameter is identified as $u = \pm\frac{2\pi i}{\xi}$. We find that the amplitudes correspond exactly to those of the models $\mathcal{M}_{5,5n+1} + \Phi_{1,2}$, with $n = 1, 2, \ldots$ for $u = -\frac{2i\beta}{\xi}$ and to $\mathcal{M}_{5,5n+4} + \Phi_{1,2}$ with $n = 1, 2, \ldots$ and the model $\mathcal{M}_{4,5} + \Phi_{2,1}$ with $u = +\frac{2i\beta}{\xi}$. We again checked that the expression given by Smirnov [101] differs from (6.57) only by CDD factors, and guarantees the appearance of poles at $\beta = \frac{2\pi i}{3}$ and at $\beta = \frac{\pi i}{3}$.

[9]This freedom can be restricted by the bootstrap equations [27].

Let us turn to the sector $\lambda = \frac{3\pi}{5}$. The obvious problem lies here in the fact, that some of the amplitudes are imaginary. This causes that real analyticity is violated, and the relation (6.3) does not coincide any more with the unitarity requirement (6.55). It is not clear yet, which kind of physics such systems describe. For example for the model $\mathcal{M}_{3,5}$ two contrary explanations are given. On the one hand, in [90] it was observed that scattering matrices of this kind describe well defined *massive* systems. On the other hand in [87] the same system was shown to correspond to a *massless* one. Since both authors rely on the same technique (the truncation space approach), one needs to wait until other methods determine the nature of this scattering theory.

Let us observe some facts about these models. Using a *symmetry breaking transformation* one can render the amplitudes real. Explicitly this can be done by transforming the amplitudes into

$$S_{bd}^{ac}(\beta) = S_0(\beta)\ \ e^{\frac{i\pi}{2}(a-c)}\ \ \omega(a,b,c,d;\frac{i\beta}{\xi}) \quad . \tag{6.60}$$

This though, causes an explicit breaking of parity, and does not cure the problem of unitarity.

The interpretation of such a system as a massive model is clearly difficult. Since the unitarity of the S–matrix is one of the basic axioms of the LSZ formalism, it is not easy to see how to drop it. A possibility to save unitarity might be to define a new star operation which guarantees that $S^{\dagger}(\beta) \equiv S(-\beta)$. With this assumption these S–matrices could describe consistently the scattering of the kinks, and would be able to be interpreted as the physical content of integrable field theories.

If the system is interpreted as a massless theory the situation is less clear. The unitarity requirement in this case amounts from investigating the axioms of scattering theory in the limit where the masses of the particles go to zero. Even if the resulting equations have been applied successfully in order to describe massless flows, from an axiomatic point of view this procedure is not correct. In this sense for massless models the 'S-matrix' might be seen just as an algebraic structure, which describes the symmetry of the theory. Therefore it is not clear whether real analyticity is a fundamental requirement in this case. Certainly the completeness relation is necessary, since it derives from the quantum group symmetry of the model.

Keeping these problems in mind, we can repeat the analysis of the preceeding section. Instead of presenting the calculation, which is done straightforward, we will just state the results. Implementing the crossing symmetry gives again a quantization of the parameter ξ,

$$\xi = \frac{5}{10n \pm 3} \quad . \tag{6.61}$$

The factor S_0 is determined as before, in order to fulfill (6.55) and we find that the resulting theories correspond to the expression calculated from the quantum group reduction of perturbations of minimal models, but now of the series $\mathcal{M}_{5,10n\pm2}$ perturbed by $\Phi_{1,3}$. Again we look also for a dynamical implementation of the symmetry of the model, and find S–matrices resulting from the Izergin-Korepin model, that is $\mathcal{M}_{5,5n+2}$, $\mathcal{M}_{5,5n+3}$ perturbed by $\Phi_{1,2}$, and $\mathcal{M}_{3,5}$ perturbed by $\Phi_{2,1}$. The amplitudes of these models again differ from the ones of the $\Phi_{1,3}$ perturbations only by CDD factors.

We found that all possible HSLM models lie in the class of $\Phi_{1,3}$ perturbations of the models $\mathcal{M}_{5,10n+k}$, $k = 2,4,6,8$, with the folding described above. A more restricted set (by the condition of dynamical symmetry), gives *all* $\Phi_{1,2}$ and $\Phi_{2,1}$ perturbations of the series $\mathcal{M}_{5,n}$. It is interesting to understand, why we did not find the $\Phi_{1,3}$ perturbations of the models $\mathcal{M}_{5,10n+k}$, with $k = 1,3,7,9$. The reason is that they are not in the usual class of IRF models, since it turns out that their Boltzmann weights do not satisfy eq.(6.4), but

$$\omega(a,b,c,d;u) = -\frac{\psi(a)^{\frac{1}{2}}\,\psi(c)^{\frac{1}{2}}}{\psi(d)^{\frac{1}{2}}\,\psi(b)^{\frac{1}{2}}}\ \omega(d,a,b,c;\lambda' - u) \quad . \tag{6.62}$$

Still their amplitudes are given by (6.1), but the crossing point is shifted, *i.e.* $\lambda' = \lambda + \pi$. Then properties (6.2),(6.3) and (6.5) are again fulfilled. As before one can identify all possible geometries with those of $\Phi_{1,3}$ perturbations, now of the models $\mathcal{M}_{5,10n+k}$, $k = 1,3,7,9$. The minus sign in the crossing relation gets fixed by the requirement that the function S_0 satisfies

$$S_0(\lambda' - u) = -S_0(u) \quad \text{and} \quad S_0(u)S_0(-u) = \frac{1}{\sin(\frac{\pi}{5} - u)\sin(\frac{\pi}{5} + u)} \quad . \tag{6.63}$$

One again finds a "simplest" solution similar to (6.57), which differs from the expression for the RSOS(4) models [97] only by CDD factors. Note that also for this series one could formulate models with self-interaction. One example is the S–matrix given in [117], which we will discuss in detail in section 6.5.2. Even though they are seen to represent consistent S–matrices, they do not correspond to models arising from the quantum group reduction of either the Sine-Gordon, nor the Izergin-Korepin model.

Summarizing, we have shown that due to crossing symmetry a quantization of the parameter ξ, which relates the rapidity with the spectral parameter of the HSLM, occurs. We found that to every parameter, which gives rise to a consistent configuration, there exists one model which can be identified as a model $\mathcal{M}_{5,n}$ perturbed by the operator $\Phi_{1,3}$. Imposing the condition of "dynamical symmetry", *i.e.* the property of a ϕ^3 interaction, selects a subset of the above configurations. These include all the $\Phi_{1,2}$ and $\Phi_{2,1}$ perturbations, again of models $\mathcal{M}_{5,n}$ and $\mathcal{M}_{n,5}$ respectively, but the set of consistent models is larger.

6.5 The TIM Perturbed with the Subleading Magnetization Operator

The model $\mathcal{M}_{4,5}$ perturbed by $\Phi_{2,1}$ clearly falls in the class discussed above. Nevertheless it is of a special interest, because there exist two *different* proposals for the scattering matrix. This is why we will discuss it here from a more physical point of view, and present the proposals for the S-matrices explicitly. In section 7.3 we will show, how one can definitely decide which of the two conjectures is the right one.

The Ising model with vacancies is described at its critical point by the unitary minimal CFT $\mathcal{M}_{4,5}$. Its central charge is $c = \frac{7}{10}$ and it has six primary fields appearing in the Kac-table (table 6.1), four of them relevant. It represents the universality class ϕ^6 of the Landau-Ginzburg theory

$$\int \mathcal{D}\phi \;\; e^{-\int (\nabla\phi)^2 + \lambda_6\phi^6 + \lambda_4\phi^4 + \lambda_3\phi^3 + \lambda_2\phi^2 + \lambda_1\phi) \;\; d^2r} \tag{6.64}$$

at the tricritical point $\lambda_i = 0, \quad i = 1, \ldots, 4$. The primary fields in the Kac-table can be identified with normal ordered Landau-Ginzburg fields. They are collected in table 6.2.

A perturbation of this critical point with the field $\Phi_{2,1}$ with anomalous dimension $(h, \bar{h}) = (\frac{7}{16}, \frac{7}{16})$, which is identified with the subleading magnetization, drives the TIM into a massive regime. The spectrum using the TCSA was computed first in [77]. The

$\frac{3}{2}$	$\frac{7}{16}$	0
$\frac{3}{5}$	$\frac{3}{80}$	$\frac{1}{10}$
$\frac{1}{10}$	$\frac{3}{80}$	$\frac{3}{5}$
0	$\frac{7}{16}$	$\frac{3}{2}$

Table 6.1: Kac-table of the minimal model $\mathcal{M}_{4,5}$

lowest energy levels for the calculation with *periodic* boundary conditions are reproduced in figure 6.8. One reads off that the ground state is double degenerate. This corresponds to the Landau-Ginzburg picture, in which the potential exhibits two asymmetric degenerate vacua (see fig. 6.7).

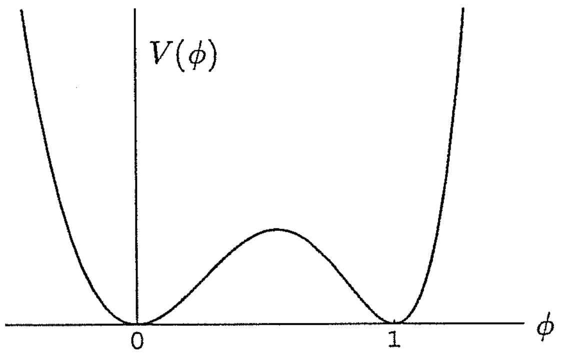


Figure 6.7: Landau-Ginzburg potential for the subleading magnetic perturbation of TIM.

This asymmetry can be understood from the fact that the subleading magnetization explicitly breaks the Z_2 symmetry of the theory, since the operator is odd under these transformations (see table 6.2). For this reason the theory can exhibit the "ϕ^3-property", *i.e.*, the absence of a conserved current of spin 3 and therefore the possibility to form a bound state through the process $AA \rightarrow A \rightarrow AA$. This picture is confirmed by the counting argument, which shows the existence of conserved currents with spins $s = (1, 5, 7, 11, 13)$ [77, 115].

Looking again at fig. 6.8, we see above the ground state a single excitation at mass m, below the threshold at mass $2m$. This feature is explained qualitatively by the asymmetry of the Landau-Ginzburg potential, since if the potential was symmetric, also the bound

Field	Identification	Z_2-symmetry
identity	$1 \equiv \Phi_{0,0}$	even
leading energy	$\epsilon \equiv \Phi_{\frac{1}{10},\frac{1}{10}} =: \phi^2 :$	even
subleading energy	$\epsilon' \equiv \Phi_{\frac{6}{10},\frac{6}{10}} =: \phi^4 :$	even
irrelevant field	$\epsilon'' \equiv \Phi_{\frac{3}{2},\frac{3}{2}} =: \phi^6 :$	even
leading magnetization	$\sigma \equiv \Phi_{\frac{3}{80},\frac{3}{80}} =: \phi :$	odd
subleading magnetization	$\sigma' \equiv \Phi_{\frac{7}{16},\frac{7}{16}} =: \phi^3 :$	odd

Table 6.2: Landau-Ginzburg identification of the primary fields of the model $\mathcal{M}_{4,5}$

state should be double degenerate. Moreover, the fact that this bound state has the same mass m as the fundamental excitations was explained by Zamolodchikov [117] as a consequence of elastic scattering.

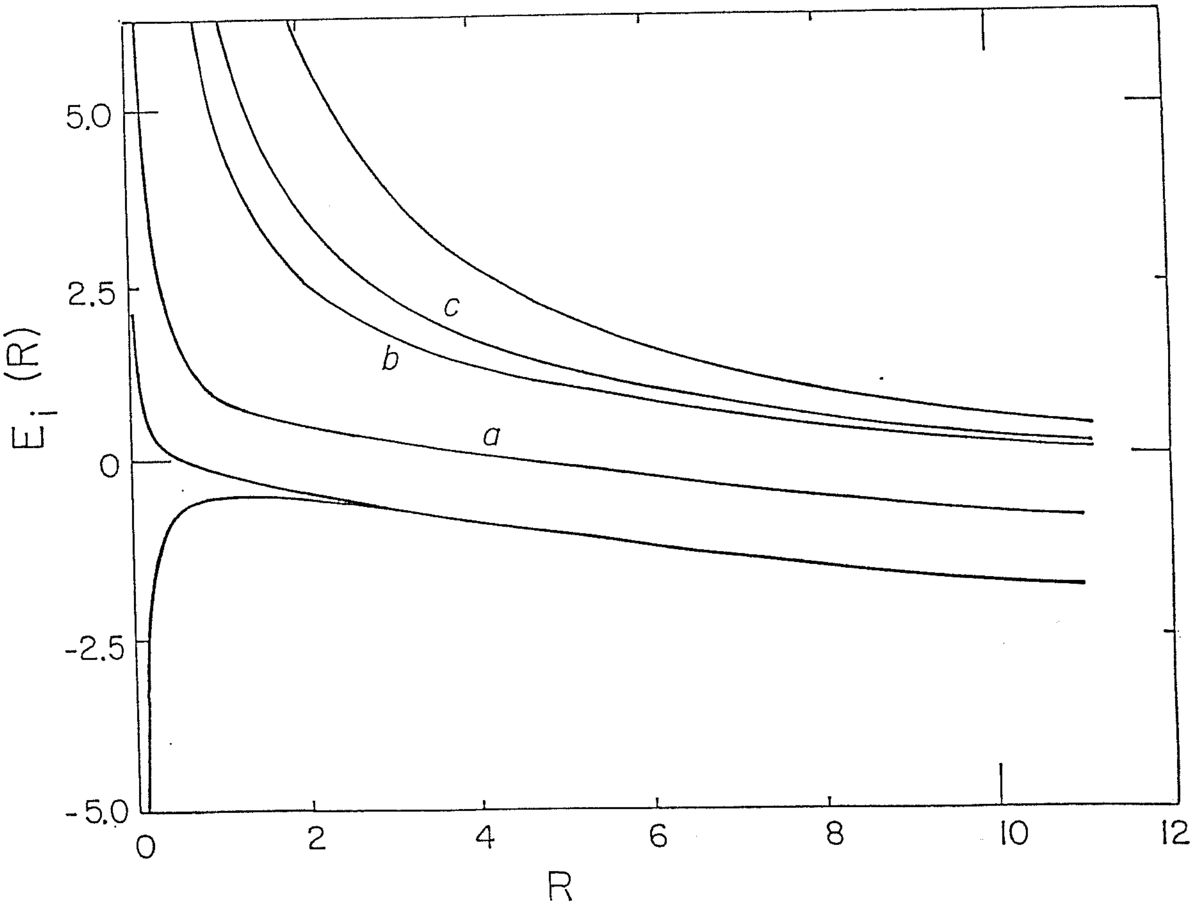


Figure 6.8: First energy levels for the subleading magnetic perturbation of TIM with periodic boundary conditions on the strip.

This information is sufficient to read off some general features the S-matrix must possess. First of all, the fundamental S–matrix must be degenerate, giving rise to two independent kink-configurations. Secondly their S-matrix elements must exhibit a pole for a value of the rapidity $\beta = \frac{2\pi i}{3}$, so as to generate a *single* bound state with the same mass m, and no further bound states below the threshold. In the following we will describe two different conjectures for this S-matrix, both satisfying the above requirements.

6.5.1 The S-Matrices using Smirnov's approach

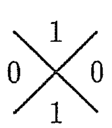
Using equation (5.52) we can calculate the amplitudes for the S-matrix conjectured by Smirnov. They have been computed in [32]: the parameters in (5.52) are $r = 4$ and $\xi = \frac{10\pi}{9}$. From eq. (5.59), the only possible values of a_i are 0 and 1 and the one-particle states are the vectors: $\mid K_{01}\rangle$, $\mid K_{10}\rangle$ and $\mid K_{11}\rangle$. All of them have the same mass m. Notice that the state $\mid K_{00}\rangle$ is not allowed. A basis for the two-particle asymptotic states is

$$\mid K_{01}K_{10}\rangle, \quad \mid K_{01}K_{11}\rangle, \quad \mid K_{11}K_{11}\rangle, \quad \mid K_{11}K_{10}\rangle, \quad \mid K_{10}K_{01}\rangle \ . \tag{6.65}$$

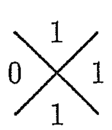
The scattering processes are

$$\begin{aligned}
\mid K_{01}(\beta_1)K_{10}(\beta_2)\rangle &= S_{00}^{11}(\beta_1-\beta_2)\mid K_{01}(\beta_2)K_{10}(\beta_1)\rangle \ , \\
\mid K_{01}(\beta_1)K_{11}(\beta_2)\rangle &= S_{01}^{11}(\beta_1-\beta_2)\mid K_{01}(\beta_2)K_{11}(\beta_1)\rangle \ , \\
\mid K_{11}(\beta_1)K_{10}(\beta_2)\rangle &= S_{10}^{11}(\beta_1-\beta_2)\mid K_{11}(\beta_2)K_{10}(\beta_1)\rangle \ , \\
\mid K_{11}(\beta_1)K_{11}(\beta_2)\rangle &= S_{11}^{11}(\beta_1-\beta_2)\mid K_{11}(\beta_2)K_{11}(\beta_1)\rangle + S_{11}^{10}(\beta_1-\beta_2)\mid K_{10}(\beta_2)K_{01}(\beta_1)\rangle \ , \\
\mid K_{10}(\beta_1)K_{01}(\beta_2)\rangle &= S_{11}^{00}(\beta_1-\beta_2)\mid K_{10}(\beta_2)K_{01}(\beta_1)\rangle + S_{11}^{10}(\beta_1-\beta_2)\mid K_{11}(\beta_2)K_{11}(\beta_1)\rangle \ .
\end{aligned} \tag{6.66}$$

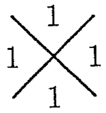
Explicitly, the above amplitudes are given by



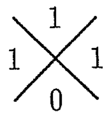
$$= S_{00}^{11}(\beta) = \frac{i}{2}\ S_0(\beta)\ \sinh\left(\frac{9}{5}\beta - i\frac{\pi}{5}\right) \quad ,$$



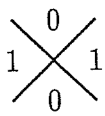
$$= S_{01}^{11}(\beta) = -\frac{i}{2}\ S_0(\beta)\ \sinh\left(\frac{9}{5}\beta + i\frac{\pi}{5}\right) \quad ,$$



$$= S_{11}^{11}(\beta) = \frac{i}{2}\, S_0(\beta)\, \frac{\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{2\pi}{5}\right)}\, \sinh\left(\frac{9}{5}\beta - i\frac{2\pi}{5}\right) \quad ,(6.67)$$



$$= S_{11}^{01}(\beta) = -\frac{i}{2}\, S_0(\beta) \left(\frac{\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{2\pi}{5}\right)}\right)^{\frac{1}{2}} \sinh\left(\frac{9}{5}\beta\right) \quad ,$$



$$= S_{11}^{00}(\beta) = -\frac{i}{2}\, S_0(\beta)\, \frac{\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{2\pi}{5}\right)}\, \sinh\left(\frac{9}{5}\beta + i\frac{2\pi}{5}\right) \quad .$$

The function $S_0(\beta)$ is given by

$$\begin{aligned} S_0(\beta) &= -\left(\sinh\frac{9}{10}(\beta - i\pi)\sinh\frac{9}{10}\left(\beta - \frac{2\pi i}{3}\right)\right)^{-1} \\ &\quad \times w\left(\beta, -\frac{1}{5}\right) w\left(\beta, +\frac{1}{10}\right) w\left(\beta, \frac{3}{10}\right) \\ &\quad \times t\left(\beta, \frac{2}{9}\right) t\left(\beta, -\frac{8}{9}\right) t\left(\beta, \frac{7}{9}\right) t\left(\beta, -\frac{1}{9}\right) \quad , \end{aligned} \tag{6.68}$$

where

$$w(\beta, x) = \frac{\sinh\left(\frac{9}{10}\beta + i\pi x\right)}{\sinh\left(\frac{9}{10}\beta - i\pi x\right)} \quad ;$$

$$t(\beta, x) = \frac{\sinh\ \frac{1}{2}(\beta + i\pi x)}{\sinh\ \frac{1}{2}(\beta - i\pi x)} \quad .$$

The S-matrix satisfies of course the unitarity relations and also the factorization equations. The explicit crossing relation is given by

$$S_{bd}^{ac}(\beta) = (-1)^{a-c+d-b}\left(\frac{[2a+1][2c+1]}{[2b+1][2d+1]}\right)^{\frac{1}{2}} S_{ac}^{db}(i\pi - \beta) \quad . \tag{6.69}$$

Let us discuss now some features of this S-matrix. The amplitudes (6.67) are periodic along the imaginary axis of β with period $10\pi i$. In figure 6.9 we show the analytic structure of the S-matrix. Note that the only poles giving rise to a bound state are located at $\beta = \frac{2\pi i}{3}$, and at $\beta = \frac{\pi i}{3}$, since all other poles are overlapped by zeros, and do not give rise to a singularity of the S-matrix. The direct channel corresponds to the pole

Figure 6.9: Pole structure of $S_0(\beta)$: $*$ are the locations of the poles and o the positions of the zeroes.

at $\beta = \frac{2\pi i}{3}$. Since we require the model to describe a unitary field theory, we expect the residues at this pole to be imaginary positive. They are given by

$$
\begin{aligned}
r_1 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}} \quad S_{00}^{11}(\beta) = 0 \;; \\
r_2 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}} \quad S_{01}^{11}(\beta) = i \left(\frac{s\left(\frac{2}{5}\right)}{s\left(\frac{1}{5}\right)}\right)^2 \omega \;; \\
r_3 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}} \quad S_{11}^{11}(\beta) = i \;\omega \;; \\
r_4 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}} \quad S_{11}^{01}(\beta) = i \left(\frac{s\left(\frac{2}{5}\right)}{s\left(\frac{1}{5}\right)}\right)^{\frac{1}{2}} \omega \;; \\
r_5 &= \mathrm{Res}_{\beta=\frac{2\pi i}{3}} \quad S_{11}^{00}(\beta) = i \,\frac{s\left(\frac{2}{5}\right)}{s\left(\frac{1}{5}\right)} \;\omega \;;
\end{aligned}
\tag{6.70}
$$

where we use the abbreviation $s(x) = \sin(\pi x)$ and

$$
\omega = \frac{5}{9}\,\frac{s\left(\frac{1}{5}\right) s\left(\frac{1}{10}\right) s\left(\frac{4}{9}\right) s\left(\frac{1}{9}\right) s^2\left(\frac{5}{18}\right)}{s\left(\frac{3}{10}\right) s\left(\frac{1}{18}\right) s\left(\frac{7}{18}\right) s^2\left(\frac{2}{9}\right)} \quad . \tag{6.71}
$$

Their numerical values are collected in Table 6.3. Indeed the residues are positive, apart the one of the amplitude S_{00}^{11} in which an additional zero cancels the pole, and therefore the residue becomes zero. Let us discuss other properties of the scattering theory under consideration. For real values of β, the amplitudes $S_{00}^{11}(\beta)$ and $S_{01}^{11}(\beta)$ are numbers of modulus 1. It is therefore convenient to define the following phase shifts

$$
\begin{aligned}
S_{00}^{11}(\beta) &\equiv e^{2i\delta_0(\beta)} \;; \\
S_{01}^{11}(\beta) &\equiv e^{2i\delta_1(\beta)} \;.
\end{aligned}
\tag{6.72}
$$

	Res at $\frac{2\pi i}{3}$	Res at $\frac{i\pi}{3}$
$S_{00}^{11}(\beta)$	0	-0.957340 i
$S_{01}^{11}(\beta)$	0.957340 i	0.591669 i
$S_{11}^{11}(\beta)$	0.365671 i	-0.365671 i
$S_{11}^{01}(\beta)$	0.465141 i	0.752614 i
$S_{11}^{00}(\beta)$	0.591669 i	0

Table 6.3: The residues on the poles $\frac{2\pi i}{3}$ and $\frac{i\pi}{3}$ of the RSOS S-matrix (2.15).

The non-diagonal sector of the scattering processes is characterized by the 2×2 symmetric S-matrix

$$\begin{pmatrix} S_{11}^{11}(\beta) & S_{11}^{01}(\beta) \\ S_{11}^{01}(\beta) & S_{11}^{00}(\beta) \end{pmatrix} . \tag{6.73}$$

We can define the corresponding phase shifts by diagonalizing the matrix (6.73). The eigenvalues turn out to be [32, 33] the same functions as in (6.72)

$$\begin{pmatrix} e^{2i\delta_0(\beta)} & 0 \\ 0 & e^{2i\delta_1(\beta)} \end{pmatrix} . \tag{6.74}$$

A basis of eigenvectors is given by

$$\mid \phi_i(\beta_1)\phi_i(\beta_2)\rangle \;=\; \sum_{j=0}^{1} U_{ij} \mid K_{1j}(\beta_1)K_{j1}(\beta_2)\rangle \quad , \qquad i = 0,1 \quad , \tag{6.75}$$

where U is an unitary matrix which does not depend on β

$$U \;=\; \frac{1}{\sqrt{1+a^2}} \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix} . \tag{6.76}$$

The asymptotic behaviour of the phase shifts is the following:

$$\begin{aligned} \lim_{\beta\to\infty} \; e^{2i\delta_0(\beta)} &= e^{\frac{6\pi i}{5}} \; ; \\ \lim_{\beta\to\infty} \; e^{2i\delta_1(\beta)} &= e^{\frac{3\pi i}{5}} \; . \end{aligned} \tag{6.77}$$

We can use this nontrivial asymptotic values of the phase-shifts in order to define generalized bilinear commutation relations for the "kinks" ϕ_0 and ϕ_1 [109, 64, 103]

$$\phi_i(t,x)\phi_j(t,y) = \phi_j(t,y)\phi_i(t,x) \;\; e^{2\pi i s_{ij}\epsilon(x-y)} \; . \tag{6.78}$$

The generalized "spin" s_{ij} is a parameter related to the asymptotic behaviour of the S-matrix. A consistent assignment is given by

$$\begin{aligned} s_{00} &= \frac{3}{5} = \frac{\delta_0(\infty)}{\pi} \quad ; \\ s_{01} &= 0 \quad ; \\ s_{11} &= \frac{3}{10} = \frac{\delta_1(\infty)}{\pi} \quad . \end{aligned} \tag{6.79}$$

Notice that the previous monodromy properties are those of the chiral field $\Psi = \Phi_{\frac{6}{10},0}$ of the original CFT of the TIM. This field occupies the position $(1,3)$ in the Kac-table of the model. The operator product expansion of Ψ with itself reads

$$\Psi(z)\Psi(0) = \frac{1}{z^{\frac{6}{5}}}\ 1 + \frac{C_{\Psi,\Psi,\Psi}}{z^{\frac{3}{5}}}\ \Psi(0) + \ldots \tag{6.80}$$

where $C_{\Psi,\Psi,\Psi}$ is the structure constant of the OPE algebra. Moving z around the origin, $z \to e^{2\pi i}z$, the phase acquired from the first term on the right hand side of (6.80) comes from the conformal dimension of the operator Ψ itself. In contrast, the phase obtained from the second term is due to the insertion of an additional operator Ψ. A similar structure appears in the scattering processes of the "kinks" ϕ_i: in the amplitude of the kink ϕ_0 there is no bound state in the s-channel (corresponding to the "identity term" in (6.80)) whereas in the amplitude of ϕ_1 a kink can be created as a bound state for $\beta = \frac{2\pi i}{3}$ (corresponding to the "Ψ term" in (6.80)). In the ultraviolet limit, the fields ϕ_i should give rise to the operator $\Psi(z)$, similarly to the case analyzed in [101]. Note that a rigorous proof of this statement would require the analysis of the form factors.

This fact is a particular case of a general situation of the RSOS S-matrices coming from Smirnov's reduction, discussed in section 6.2.2. In the previous case of TIM, with $r = 4$, only *two* independent phase-shifts could appear, since the last channel in (6.36) could not be opened because $\Phi_{1,5}$ does not appear in the Kac-table of the primary fields of the original CFT, and the singular part of the OPE stops after the two first terms.

Now let us finally turn to the spectrum. We first analyze the bound state structure. In the amplitude S_{00}^{11} there is no bound state in the direct channel but only the singularity coming from the state $\mid K_{11}\rangle$ exchanged in the t-channel. This is easily seen from fig. 6.6, where we stretch the original amplitudes along the vertical direction (s-channel) and along the horizontal one (t-channel). Since the state $| K_{00}\rangle$ is not physical, the residue in the direct channel is zero. In the amplitude S_{01}^{11} we have the bound state $| K_{01}\rangle$ in the direct channel and the singularity due to $| K_{11}\rangle$ in the crossed channel. In S_{11}^{11}, the state $| K_{11}\rangle$ appears as a bound state in both channels. In S_{11}^{01} the situation is reversed with respect to that of S_{01}^{11}, as it should be from the crossing symmetry property (6.69): the state $| K_{11}\rangle$ appears in the t-channel and $| K_{01}\rangle$ in the direct channel. Finally, in S_{11}^{00} there is the bound state $| K_{11}\rangle$ in the direct channel but the residue on the t-channel pole is zero, again because $| K_{00}\rangle$ is unphysical. This situation is, of course, that obtained by applying crossing to S_{00}^{11}.

Let us compare this picture with the data coming from the TCSA. The one-particle line a of fig.6.8 corresponds to the state $| K_{11}\rangle$. This energy level is not doubly degenerate because the state $| K_{00}\rangle$ is forbidden by the RSOS selection rules, eq. (5.59). With periodic boundary conditions, the kink states $| K_{01}\rangle$ and $| K_{10}\rangle$ are projected out and $| K_{11}\rangle$ is the only one-particle state that can appear in the spectrum. In order to determine the pattern of the energy levels obtained from TCSA and to relate the scattering processes to the data of the original unperturbed CFT (along the line suggested in [76]), we would need a higher-level Bethe ansatz technique. This is because our actual situation deals with kink-like excitations in contrast to that of ref. [76] which considers only diagonal, breather-like S–matrices. The Bethe-ansatz technique gets quite complicated in the case of a S-matrix with kink excitations, and has been carried out only for particular examples [120], all of them describing $\Phi_{1,3}$ perturbations of minimal models.

6.5.2 The Approach of Zamolodchikov

The problem of finding a theoretical explanation for the energy levels of the $\Phi_{2,1}$ perturbed TIM was first discussed in [117]. In his notation, the one particle states are given by

$$| K_{+}\rangle, \quad | K_{-}\rangle, \quad | B\rangle \ , \tag{6.81}$$

which we can identify with our $| K_{10}\rangle$, $| K_{01}\rangle$ and $| K_{11}\rangle$, respectively. The two-particle amplitudes of the scattering processes were defined in [117] to be

$$| B(\beta_1)B(\beta_2)\rangle \ = \ a(\beta_1-\beta_2) \,|\, B(\beta_2)B(\beta_1)\rangle + b(\beta_1-\beta_2) \,|\, K_{+}(\beta_2)K_{-}(\beta_1)\rangle \quad ,$$

$$
\begin{aligned}
\mid K_-(\beta_1)B(\beta_2)\rangle &= c(\beta_1-\beta_2)\mid K_-(\beta_2)B(\beta_1)\rangle \quad , \\
\mid B(\beta_1)K_+(\beta_2)\rangle &= c(\beta_1-\beta_2)\mid B(\beta_2)K_+(\beta_1)\rangle \quad , \\
\mid K_-(\beta_1)K_+(\beta_2)\rangle &= d(\beta_1-\beta_2)\mid K_-(\beta_2)K_+(\beta_1)\rangle \quad , \\
\mid K_+(\beta_1)K_-(\beta_2)\rangle &= e(\beta_1-\beta_2)\mid K_+(\beta_2)K_-(\beta_1)\rangle + b(\beta_1-\beta_2)\mid B(\beta_2)B(\beta_1)\rangle \quad .
\end{aligned}
\tag{6.82}
$$

They are in correspondence with those of (6.66) if we make the assignments

$$
\begin{aligned}
a(\beta) &\rightarrow S_{11}^{11}(\beta) \;; \\
b(\beta) &\rightarrow S_{11}^{10}(\beta) \;; \\
c(\beta) &\rightarrow S_{01}^{11}(\beta) \;; \\
d(\beta) &\rightarrow S_{00}^{11}(\beta) \;; \\
e(\beta) &\rightarrow S_{11}^{00}(\beta) \;.
\end{aligned}
\tag{6.83}
$$

In order to solve the Yang-Baxter equations which ensure the factorization of the scattering processes, Zamolodchikov noticed that the above amplitudes coincide with the definitions of the Boltzmann weights of the "Hard Square Lattice Gas". Therefore, he borrowed Baxter's solution [10] in the case where it reduces to a trigonometric form

$$
\begin{aligned}
a(\beta) &= \frac{\sin\left(\frac{2\pi}{5}+\lambda\beta\right)}{\sin\left(\frac{2\pi}{5}\right)}\; R(\beta) \quad ; \\
b(\beta) &= e^{\delta\beta}\,\frac{\sin(\lambda\beta)}{\left[\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{\pi}{5}\right)\right]^{\frac{1}{2}}}\; R(\beta) \quad ; \\
c(\beta) &= e^{-\delta\beta}\,\frac{\sin\left(\frac{\pi}{5}-\lambda\beta\right)}{\sin\left(\frac{\pi}{5}\right)}\; R(\beta) \quad ; \\
d(\beta) &= e^{-2\delta\beta}\,\frac{\sin\left(\frac{\pi}{5}+\lambda\beta\right)}{\sin\left(\frac{\pi}{5}\right)}\; R(\beta) \quad ; \\
e(\beta) &= e^{2\delta\beta}\,\frac{\sin\left(\frac{2\pi}{5}-\lambda\beta\right)}{\sin\left(\frac{2\pi}{5}\right)}\; R(\beta) \quad .
\end{aligned}
\tag{6.84}
$$

Here δ and λ are arbitrary parameters and $R(\beta)$ is an arbitrary function. In order to fix completely the amplitudes the following requirements were imposed:

1. the unitarity conditions, eqs.(3.13);

2. the absence of a pole in the direct channel of the amplitude $d(\beta)$;

3. crossing symmetry, implemented in the following form

$$\begin{aligned} a(\beta) &= a(i\pi-\beta) \; ; \\ b(\beta) &= c(i\pi-\beta) \; ; \\ d(\beta) &= e(i\pi-\beta) \; . \end{aligned} \tag{6.85}$$

The final form of the S–matrices is given by

$$\begin{aligned} a(\beta) &= e^{-2i\pi\delta}\, \frac{\sin\left(\frac{2\pi-6i\beta}{5}\right)\sin\left(\frac{3\pi+6i\beta}{5}\right)}{\sin\left(\frac{\pi-6i\beta}{5}\right)\sin\left(\frac{2\pi+6i\beta}{5}\right)} \; ; \\ b(\beta) &= e^{-\delta(i\pi-\beta)}\, \frac{\sin\left(\frac{6i\beta}{5}\right)\sin\left(\frac{3\pi+6i\beta}{5}\right)}{\sin\left(\frac{\pi-6i\beta}{5}\right)\sin\left(\frac{2\pi+6i\beta}{5}\right)} \; ; \\ c(\beta) &= e^{-\delta\beta}\, \frac{\sin\left(\frac{\pi+6i\beta}{5}\right)\sin\left(\frac{3\pi+6i\beta}{5}\right)}{\sin\left(\frac{\pi-6i\beta}{5}\right)\sin\left(\frac{2\pi+6i\beta}{5}\right)} \; ; \\ d(\beta) &= e^{-2\delta\beta}\, \frac{\sin\left(\frac{3\pi+6i\beta}{5}\right)}{\sin\left(\frac{2\pi+6i\beta}{5}\right)} \; ; \\ e(\beta) &= e^{-2\delta(i\pi-\beta)}\, \frac{\sin\left(\frac{3\pi+6i\beta}{5}\right)}{\sin\left(\frac{\pi-6i\beta}{5}\right)} \; . \end{aligned} \tag{6.86}$$

Herein δ is an imaginary number satisfying

$$e^{-2\pi i\delta} = \frac{s\left(\frac{1}{5}\right)}{s\left(\frac{2}{5}\right)} \; . \tag{6.87}$$

All amplitudes but $d(\beta)$ have a simple pole at $\beta = \frac{2\pi i}{3}$. Their residues are given by

$$\begin{aligned} r_1 &= \text{Res}_{\beta=\frac{2\pi i}{3}} \quad a(\beta) = i\,\frac{5}{6}\,\frac{\left(s\left(\frac{1}{5}\right)\right)^3}{\left(s\left(\frac{2}{5}\right)\right)^2} \; ; \\ r_2 &= \text{Res}_{\beta=\frac{2\pi i}{3}} \quad b(\beta) = -i\,\frac{5}{6}\,\frac{\left(s\left(\frac{1}{5}\right)\right)^{\frac{13}{6}}}{\left(s\left(\frac{2}{5}\right)\right)^{\frac{7}{6}}} \; ; \\ r_3 &= \text{Res}_{\beta=\frac{2\pi i}{3}} \quad c(\beta) = i\,\frac{5}{6}\,\frac{\left(s\left(\frac{1}{5}\right)\right)^{\frac{4}{3}}}{\left(s\left(\frac{2}{5}\right)\right)^{\frac{1}{3}}} \; ; \\ r_4 &= \text{Res}_{\beta=\frac{2\pi i}{3}} \quad d(\beta) = 0 \; ; \end{aligned} \tag{6.88}$$

	Res at $\frac{2\pi i}{3}$	Res at $\frac{i\pi}{3}$
$S_{11}^{11}(\beta)$	0.187095 i	-0.187095 i
$S_{11}^{10}(\beta)$	-0.279395 i	-0.417229 i
$S_{10}^{11}(\beta)$	0.417229 i	0.279395 i
$S_{00}^{11}(\beta)$	0	-0.417229 i
$S_{11}^{00}(\beta)$	0.417229 i	0

Table 6.4: The residues on the poles $\frac{2\pi i}{3}$ and $\frac{i\pi}{3}$ of the Zamolodchikov S-matrix (2.36).

$$\tau_5 \;=\; \mathrm{Res}_{\beta=\frac{2\pi i}{3}} \quad e(\beta) = i\;\frac{5}{6}\;\frac{\left(s\left(\frac{1}{5}\right)\right)^{\frac{4}{3}}}{\left(s\left(\frac{2}{5}\right)\right)^{\frac{1}{3}}} \quad .$$

Their numerical values are collected in Table 6.4.

In the asymptotic limit $\beta \to \infty$, all amplitudes but $a(\beta)$ have an oscillating behaviour

$$\begin{aligned} a(\beta) &\sim e^{-2\pi i\delta} \quad ; \\ b(\beta) &\sim e^{-\delta i\pi}\; e^{\delta\beta}\; e^{\frac{3\pi i}{5}} \quad ; \\ c(\beta) &\sim e^{-\delta\beta}\; e^{-\frac{3\pi i}{5}} \quad ; \\ d(\beta) &\sim e^{-2\delta\beta}\; e^{-\frac{i\pi}{5}} \quad ; \\ e(\beta) &\sim e^{-2\delta i\pi}\; e^{2\delta\beta}\; e^{\frac{-4\pi i}{5}} \quad . \end{aligned} \tag{6.89}$$

This of course is due to the gauge transformation performed in order to insure strict crossing symmetry.

As for Smirnov's amplitudes we tried to relate the asymptotic behaviour to the OPE of some conformal field of the model $\mathcal{M}_{4,5}$. But even not taking in account these oscillatory factors we were not able to establish such a link. Up to now the link of the asymptotic phase shifts to the CFT lies on a purely observational basis, *i.e.*, no rigorous theoretical explanation is available for this fact. Therefore we would like to stress, that the absence of such a link for Zamolodchikov's proposal does not imply the inconsistency of his conjecture.

6.5.3 Appendix to section 6

Here we collected some necessary information on the projectors. The R-matrix (6.25) can be written in terms of projectors as

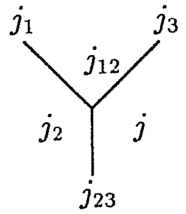
$$\begin{aligned} R(x) &= (q^2x^{-\frac{1}{2}} - x^{\frac{1}{2}}q^{-2})(q^{-3}x^{-\frac{1}{2}} + q^3x^{\frac{1}{2}})P_0 + (q^2x^{\frac{1}{2}} - x^{-\frac{1}{2}}q^{-2})(q^{-3}x^{\frac{1}{2}} + q^3x^{-\frac{1}{2}})P_1 \\ &+ (q^2x^{-\frac{1}{2}} - x^{\frac{1}{2}}q^{-2})(q^{-3}x^{\frac{1}{2}} + q^3x^{-\frac{1}{2}})P_2 \quad . \end{aligned} \tag{6.90}$$

We look for points where $R(x) \sim P_i$, and therefore the other terms must vanish. We find:

$$\begin{array}{llll} P_0 \text{ vanishs if}: & x = q^4 & \text{or} & x = iq^{-6} \ , \\ P_1 & x = q^{-4} & & x = iq^6 \ , \\ P_2 & x = q^4 & & x = iq^6 \ . \end{array} \tag{6.91}$$

This means that $R \sim P_0$ at $x = iq^6$ and $R \sim P_1$ at $x = q^4$. Note that R never becomes proportional to P_2, and therefore we can not form bound states of spin 2 in an hypothetical S-matrix based on the R-matrix (6.25).

We give now the $6j$ symbols which are necessary to carry out the bootstrap. Note that they correspond to the fusion coefficient, graphically displayed as



$$\longrightarrow \left\{ \begin{matrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{12} \end{matrix} \right\}_q \ . \tag{6.92}$$

The only non-zero fusion coefficients for spin 1 are

$$\left\{ \begin{matrix} 1 & 1 & 1 \\ l & l & l \end{matrix} \right\}_q = f_{l,l,l} = (q^{2l+1} + q^{-2l-1}) \left(\frac{[2]}{[4][2l][2l+2]} \right)^{\frac{1}{2}} \ ,$$

$$\left\{ \begin{matrix} 1 & 1 & 1 \\ l & l-1 & l \end{matrix} \right\}_q = f_{l,l,l-1} = \left(\frac{[2][2l+2]}{[4][2l]} \right)^{\frac{1}{2}} \ ,$$

$$\left\{ \begin{matrix} 1 & 1 & 1 \\ l & l+1 & l \end{matrix} \right\}_q = f_{l,l,l+1} = -\left(\frac{[2][2l]}{[4][2l+2]} \right)^{\frac{1}{2}} \ ,$$

$$\left\{ \begin{matrix} 1 & 1 & 1 \\ l & l & l-1 \end{matrix} \right\}_q = f_{l,l-1,l} = -\left(\frac{[2l-1][2][2l+2]}{[2l+3][4][2l]} \right)^{\frac{1}{2}} \ ,$$

$$\left\{\begin{array}{ccc} 1 & 1 & 1 \\ l & l & l+1 \end{array}\right\}_q = f_{l,l+1,l} = \left(\frac{[2l+3][2][2l]}{[2l+1][4][2l+2]}\right)^{\frac{1}{2}} .$$

The $6j$-symbols for spin 0, that is the fusion coefficients for the breathers, are

$$\left\{\begin{array}{ccc} 1 & 1 & 0 \\ l & l & l \end{array}\right\}_q = -\left(\frac{1}{[3]}\right)^{\frac{1}{2}} ,$$

$$\left\{\begin{array}{ccc} 1 & 1 & 0 \\ l & l & l+1 \end{array}\right\}_q = \left(\frac{[2l+3]}{[3][2l+1]}\right)^{\frac{1}{2}} ,$$

$$\left\{\begin{array}{ccc} 1 & 1 & 0 \\ l & l+1 & l \end{array}\right\}_q = \left(\frac{[2l-1]}{[3][2l+1]}\right)^{\frac{1}{2}} .$$

7 Confirmation of Bootstrap Results

In the last two chapters we have shown how S–matrices are constructed. The methods we discussed were not based on a straightforward calculation, but rather on symmetry arguments which were used to *conjecture* the exact form. Especially in section 6.5 we have discussed two proposals for an S–matrix for the same field theory.

A way in order to calculate the S–matrix would be to perform the Bethe ansatz for the corresponding field theory. In this way one could *derive* the S–matrix exactly. Unfortunately such calculations are very involved and have been applied to few models. The methods we have described can be considered as a "short–cut", using symmetry considerations and general arguments from quantum field theory in order to give a conjecture for the exact S–matrix.

If the S–matrix is constructed in this way, it should be confirmed by independent methods in order to establish its link with the underlying field theory. For this purpose numerous different methods have been proposed and the subject has grown enormously during the last years. we will sketch in this chapter the most important methods. That is, we will just describe the basic ideas and give some simple examples of their application linked to the theories we have discussed in the previous chapters. Also we will show how these methods can be used in order to decide between the two conjectured S–matrices described in section 6.5.

We should note here that all methods we will describe in this chapter have a more general scope as to confirm bootstrap results. A complete description of them would be separate topic of research. For that we limit ourselves to discuss them in the connection of the bootstrap-principle and especially we want to show how they can be used to confirm conjectured S–matrices.

7.1 The TBA equations

Consider an integrable massive scattering theory on a cylinder. This implies factorized scattering, and so one can assume that the wave function of the particles is well described by a free wave function in the intermediate region of two scattering. Take the ansatz

$$\psi(x_1 \ldots x_n) = e^{i\sum p_j x_j} \sum_P A(P)\Theta(x_P) \quad ;$$

$A(P)$ are coefficients of the momenta whose ordering is specified by

$$\Theta(x_P) = \begin{cases} 1 & \text{if} \quad x_{p_1} < \ldots < x_{p_n} \\ 0 & \quad \text{otherwise} \end{cases} \quad .$$

Let the permutation P differ from P' by the exchange of the indices k and j. Then

$$A(P') = S_{kj}(\beta_k - \beta_j)A(P) \quad . \tag{7.1}$$

We impose antiperiodic boundary conditions for our wave functions, which provides that two particles cannot have equal momenta, leading to the condition

$$A(k, p_2, \ldots, p_n) = -e^{ip_k L} A(p_2, \ldots, p_n, k) \quad , \tag{7.2}$$

L being the length of the strip on which we consider the theory. Permuting one particle through all others and comparing (7.1) and (7.2), one finds

$$e^{iLm_k \sinh\beta_k} \prod_{j\neq k} S_{kj}(\beta_k - \beta_j) = -1 \quad \text{for} \quad k = 1, 2, \ldots n \quad . \tag{7.3}$$

We introduce the phase $\delta_{kj}(\beta_k - \beta_j) \equiv -i \ln S_{kj}(\beta_k - \beta_j)$. In terms of these the equations become

$$Lm_k \sinh\beta_k + \sum_{j\neq k} \delta_{kj}(\beta_k - \beta_j) = 2\pi n_k \quad \text{for } k = 1, 2, \ldots, n \quad , \tag{7.4}$$

n_k being some integers. These coupled transcendental equations for the rapidities are called the Bethe ansatz equations. One tries to solve these equations in the thermodynamic limit introducing densities of rapidities for each particle species and transferring the equations into integral equations. That is, let $\rho_1^{(a)}(\beta) = \frac{n}{\Delta\beta}$, where we assume that there are n particles in the small interval $\Delta\beta$, be the particle density and $\rho^{(a)}(\beta) = \frac{n_k}{\Delta\beta}$ be the level density corresponding to the particle a, then (7.4) become

$$m_a L \cosh\beta + \sum_{b=1}^{n} \int_{-\infty}^{\infty} \varphi_{ab}(\beta - \beta')\rho_1^{(a)}(\beta')d\beta' = 2\pi\rho^{(a)} \quad , \tag{7.5}$$

with $\varphi_{ab}(\beta) = \frac{d}{d\theta}\delta_{ab}(\beta)$. In order to compute the ground state energy one needs to minimize the free energy

$$RLf(\rho,\rho_1) = RH_B(\rho_1) + S(\rho,\rho_1) \quad , \tag{7.6}$$

where $H_B = \sum_a m_a \int \cosh\beta \rho_1^{(a)} d\beta$ and S denotes the entropy. The extremum condition for a fermionic system[10] takes the form [118]

$$-rM_a \cosh\beta + \epsilon_a(\beta) = \sum_{b=1}^{n} \int_{-\infty}^{\infty} \varphi_{ab}(\beta-\beta') \log(1+e^{-\epsilon_b(\beta)}) \frac{d\beta'}{2\pi} \quad , \tag{7.7}$$

where we introduced the so-called pseudo-density $e^{-\epsilon_a} \equiv \frac{\rho_1^{(a)}}{\rho^{(a)}-\rho_1^{(a)}}$, the scaling length $r = Rm_1$ and the rescaled masses $M_a = \frac{m_a}{m_1}$; m_1 is the lightest particle mass. These coupled integral equations are called the TBA equations. The extremal free energy depends only on the ratios $\frac{\rho_1^{(a)}}{\rho^{(a)}}$ and is given by

$$f(R) = -\frac{r}{2\pi} \sum_{a=1}^{n} M_a \int_{-\infty}^{\infty} \cosh\beta \log(1+e^{-\epsilon_a(\beta)}) d\beta \quad . \tag{7.8}$$

One can extract several physical quantities from the solution of the TBA-equations ([67, 76, 85, 88, 112, 114, 118]). Since very little is known about non-critical systems, one tries to examine the equations in the ultraviolet limit, which corresponds to $r \longrightarrow 0$, where the underlying field theory should become a CFT. The central charge is related to the vacuum bulk energy, and is given by

$$c(r) = \frac{3r}{\pi^2} \sum_{a=1}^{n} M_a \int_{-\infty}^{\infty} \cosh\beta \log(1+e^{-\epsilon_a(\beta)}) d\beta \quad . \tag{7.9}$$

Having calculated the central charge one would like to extract the conformal dimension of the perturbing operator. For small r, one expects that $f(r)$ reproduces the behaviour predicted by conformal perturbation theory. For the applications we have in mind this expnsion reads as

$$c(r) = c(0) - \frac{3f_0}{\pi} r^2 + \sum_{k=1}^{\infty} f_k r^{yk} \quad . \tag{7.10}$$

$c(0)$ is not really the central charge, but the so-called *effective* central charge, $c(0) = c - 24h_{min}$ where h_{min} is the lowest dimension of the CFT. This distinction is of course important in non-unitary theories, which contain also fields with negative dimensions.

[10] We use the fermionic TBA equations since in diagonal scattering up to now they turned out to be the relevant ones, see e.g.[118] for the general theory

The exponent y is related to the perturbing field by $y = 2(1-h)$ if the theory is unitary and by $y = 4(1-h)$ if it is non-unitary. The coefficients f_k are related to correlation functions of the CFT [67, 88, 118], and provide an ultimate important check of the theory.

A simple check which in many cases can be performed analytically, is the calculation of the central charge $c(0)$. The equations (7.7) take the form

$$\epsilon_a = -\sum_{b=1}^{N} N_{ab} \ln(1 + e^{-\epsilon_b}) \ . \tag{7.11}$$

The matrix N_{ab} in (7.11) is given as

$$N_{ab} = \int_{-\infty}^{\infty} \frac{d\beta}{2\pi} \varphi_{ab}(\beta) = \frac{1}{2\pi}(\delta_{ab}(\infty) - \delta_{ab}(-\infty)) \ . \tag{7.12}$$

The effective central charge $c(0)$ can be obtained by integration

$$c(0) = \sum_{a=1}^{n} \frac{6}{\pi^2} \int_0^{\infty} dx \frac{x + \frac{\epsilon_a}{2}}{e^{x+\epsilon_a} + 1} \ . \tag{7.13}$$

The matrix N_{ab} is very simple to determine. Since we deal with diagonal S matrices we know that the general form is $S_{ab}(\beta) = \prod_i f_{x_i}(\beta)$. Analyzing (7.12) for this situation one sees that $\varphi_{ab}(\beta) = \sum_i \varphi[f_{x_i}](\beta)$ so that one can sum up individual contributions coming from the single factors $f_{x_i}(\beta)$. Then $N_{ab} = \sum N[f_{x_i}]$ which is $N_{ab} = \sum \mathrm{sgn}(x_i)$ for $-1 < x \leq 1$.

As an example we consider the theories $\mathcal{M}_{2,2n+3}$, discussed in section 5.3. The solution of (7.11) can be obtained in closed form. One finds [67]

$$e^{\epsilon_a}(\mathcal{M}_{2,2n+3}) = \frac{\sin(\frac{a\pi}{2n+3}) \sin(\frac{(a+2)\pi}{2n+3})}{\sin^2(\frac{\pi}{2n+3})} \ . \tag{7.14}$$

To do the summation over the integral one uses the identity

$$\int_0^{\infty} dx \frac{x + \frac{\epsilon}{2}}{e^{x+\epsilon} + 1} = L(\frac{1}{1 + e^{\epsilon}}) \ , \tag{7.15}$$

where $L(x)$ is the so called Roger's dilogarithmic function, given by

$$L(x) = -\frac{1}{2} \int_0^{x} dy [\frac{\ln y}{1-y} + \frac{\ln(1-y)}{y}] \ . \tag{7.16}$$

There exist sum-rules for these functions which allow one to calculate $c(0)$ exactly [79]. One finds that $c(0) = \frac{2n}{n+3}$ which is the right result.

Also the scattering theory $\mathcal{M}_{2,9}$ perturbed with $\Phi_{1,4}$ (see section 5.4) can be confirmed with the exact calculation of the central charge [71]. Again the ultraviolet limit is taken solving Eqs. (7.7), with the S matrices given in (5.80), for $R \to 0$. In this limit the scaling function is written in terms of the Rogers dilogarithmic function

$$c(0) = -12 \sum_{i=1}^{4} L(x_i); \tag{7.17}$$

where

$$\begin{aligned} x_1^{-1} &= 4\cos^2\left(\frac{\pi}{18}\right) \\ x_2^{-1} &= \left(2\cos\left(\frac{2\pi}{9}\right)+1\right)^2 \\ x_3^{-1} &= 4\cos^2\left(\frac{\pi}{18}\right)\left(2\cos\left(\frac{\pi}{9}\right)+1\right) \\ x_4^{-1} &= 64\cos^2\left(\frac{2\pi}{9}\right)\cos^4\left(\frac{\pi}{9}\right) \ . \end{aligned} \tag{7.18}$$

Using the properties of the Rogers dilogarithmic function we can exactly perform the sum of Eq. (7.17) and our final result is $c = \frac{2}{3}$, which agrees with the corresponding conformal field theory (see Eq.(5.67)).

It is very important to perform the TBA also for finite r, since only in this way the scaling region can be explored. In [17] we have designed an algorithm which performs these computations, and becomes powerful especially when one investigates the region $r \to 0$. Here we give just one simple example of such a calculation. Let us extract the dimension of the perturbing operator for the conjectured scattering theory of $\mathcal{M}_{2,7}+\Phi_{1,2}$ (see section 5.3). The output data of our program is given in table 7.1 and is clearly compatible with theoretical result. The TBA becomes even more powerful if combined with the Truncation Space Approach (see section 7.2). There many quantities can be "measured" and then be confronted with TBA results [67, 76, 85, 112, 114, 118]. In that way also some of the coefficients f_i in eq.(7.10) can be checked. If a thorough analysis of this kind is performed for a scattering theory (as has been done for example for the theories $\mathcal{M}_{2,2n+3}+\Phi_{1,2}$ in [71]), the scattering theory is given a very strong support. Such an examination can substitute the difficult Bethe ansatz calculations which one would need to do otherwise in order to prove the correctness of a conjectured S–matrix.

```
computed cexact =     .57142857D+00
r=    .100000000D-01  central charge=   .571388072E+00
r=    .200000000D-01  central charge=   .571269945E+00
r=    .300000000D-01  central charge=   .571077712E+00
r=    .400000000D-01  central charge=   .570814111E+00
r=    .500000000D-01  central charge=   .570481500E+00

error in extrapolation -.1112E-05
estimated exponent        .257142827D+01
theoretical exponent      .257142857D+01
estimated dimension of the corresponding operator
for a     unitary theory: DELTA=    .357143D+00
for a non-unitary theory: DELTA=   -.285714D+00
 total cpu time (secs)    .176E+05
```

Table 7.1: Output of the program tba [17] for the conjectured S–matrix for the theory $\mathcal{M}_{2,7} + \Phi 1, 2$.

7.2 The truncation conformal space approach

The TCSA allows us to study the crossover from massless to massive behaviour in a theory with the space coordinate compactified on a circle of radius $R/2\pi$. The method consists in truncating the infinite-dimensional Hilbert space of the CFT up to a level Λ in the Verma modules, diagonalizing the Hamiltonian

$$H_p(\lambda, R) = \frac{2\pi}{R} \left(L_0 + \bar{L}_0 - \frac{c}{12}\right) + \mu \int_0^R \Phi_{r,s}(x)\ dx \quad . \tag{7.19}$$

An efficient algorithm has been developed for performing such a computation [78]. In our case, the truncation Λ is fixed at level 5 in the Verma modules. The parameter μ in (7.19) is a dimensionful coupling constant, related to the mass scale of the perturbed theory

$$[\mu] = m^{2-2h_{r,s}} \quad . \tag{7.20}$$

In the following we fix the mass scale putting $\mu = 1$.

In a finite geometry the spectrum of the hamiltonian (7.19) is discrete. The energy eigenvalues take the scaling form

$$E_i(R) = \frac{1}{R} F_i(\rho) \quad , \tag{7.21}$$

with the scaling variable $\rho = \frac{R}{\xi}$. The correlation length ξ is defined as the Compton wave length of the lightest particle in the thermodynamic limit, $\xi = \frac{1}{m}$. In the ultraviolet regime ($\rho \ll 1$), the spectrum is dominated by the conformal part of the Hamiltonian, and behaves as

$$E_i(R) \simeq \frac{2\pi}{R}\left(2h_i - \frac{c}{12}\right) \quad , \quad mR \ll 1 \quad , \tag{7.22}$$

whereas in the infrared region ($\rho \gg 1$) it is characterized by a set of stable particles. There the scaling functions F_i become

$$F_i \sim \frac{1}{2\pi}\left[\epsilon_0 (\frac{R}{\xi})^2 + \frac{M_i}{m}\frac{R}{\xi}\right] \quad , \tag{7.23}$$

so that

$$E_i(R) \simeq \epsilon_0 m^2 R + M_i \quad , \quad mR \gg 1 \quad , \tag{7.24}$$

where M_i is the mass gap of the ith level. However, the above infrared asymptotic behaviour holds only in the ideal situation when the truncation parameter Λ goes to infinity. In practice, for finite Λ, the linear behaviour of eq. (7.24) is realized only within a finite region of the R axis. The large R behaviour is dictated by truncation effects. In order to find the physical regions, we make use of the parameter (introduced in ref. [77])

$$\rho_i(R) = \frac{d \ \log E_i(R)}{d \ \log R} \quad . \tag{7.25}$$

The parameter ρ_i lies between the values $\rho_i = -1$ (in the ultraviolet region) and $\rho_i = 1$ (in the infrared one). In the limit of large R (the truncation-dominated regime), $\rho_i = 1 - 2h$.

The "window" in R where the linear infrared behaviour holds depends upon the perturbing field and, for the case of operators with anomalous dimension $h \geq \frac{1}{2}$, it can be completely shrunk away. This phenomenon is related to the divergencies which appear in a perturbative expansion of the Hamiltonian (7.19), which must be renormalized. Under these circumstances, it is more convenient to consider the differences of energies, which are not renormalized.

As an example we will discuss the model $\mathcal{M}_{4,5} + \Phi_{2,1}$, which we discussed in section 6.5. It is of special interest, since for this perturbation two different proposals for a scattering matrix exist.

In the case of the subleading magnetic perturbation of TIM, the anomalous dimension of $\Phi_{2,1}$ is $h = \frac{7}{16}$, which is near $\frac{1}{2}$. Looking at fig. 6.8, we see that the onset of the infrared region of the two lowest levels is around $R \sim 2$ and persists only for few units in R. It is well known in statistical mechanics, that the largest two eigenvalues of the transfer matrix of two-dimensional ferromagnetic systems have an exponential energy splitting for large R [10]. As was shown in [77] this is also the case in TCSA. That is, in the region of the on-set of the infra-red behaviour, the ground state levels approach each other exponentially [77]

$$E_1 - E_0 \sim e^{-mR} \quad . \tag{7.26}$$

But this allows us to extract the mass "experimentally" by measuring the splitting of the first two lines. We find

$$m = 0.98 \pm 0.02 \quad . \tag{7.27}$$

From fig. 7.1, we see that for the third level, that of one-particle state, the ultraviolet behaviour extends till $R \sim 0.5$. The crossover region is in the interval $0.5 \leq R \leq 2$.

Beyond this interval, the infrared regime begins but the "window" of infrared behaviour is quite narrow, in the neighbourhood of $R \sim 3$. Considering the differences of energies with respect to those of the degenerate ground states (fig. 7.2) one can also read off the mass-gap and see that it is consistent with the value extracted from the exponential approach of the two lowest levels. In fig. 7.2, the third line defines the threshold, with a mass-gap $2m$.

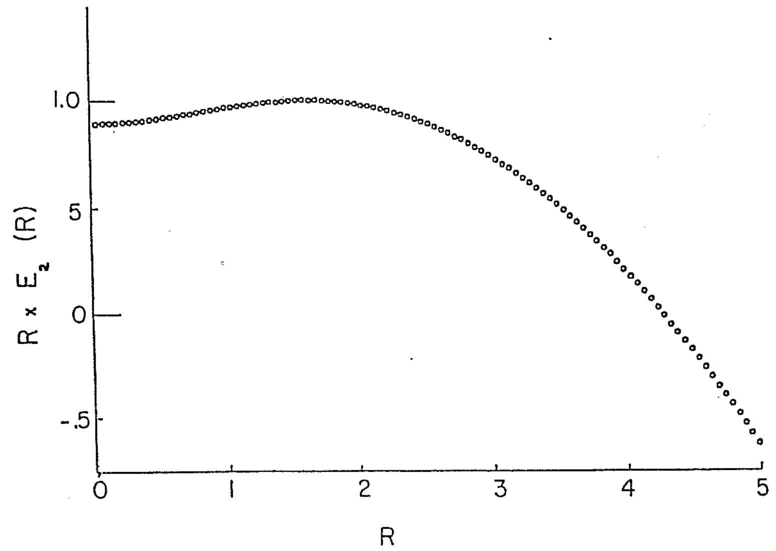


Figure 7.1: Energy level which corresponds to the one–particle state in the infrared region

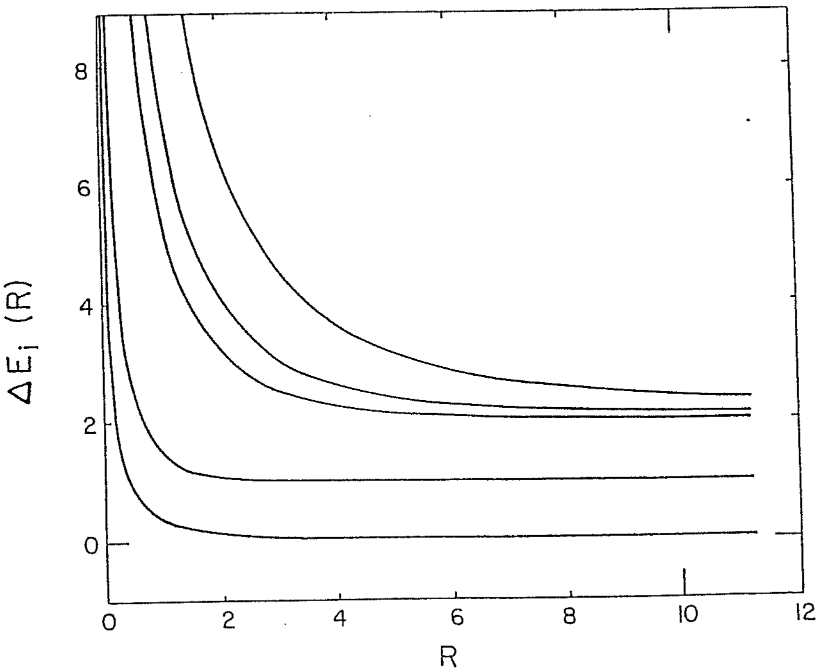


Figure 7.2: The lowest energy differences with respect to the ground state

7.3 Finite-size effects

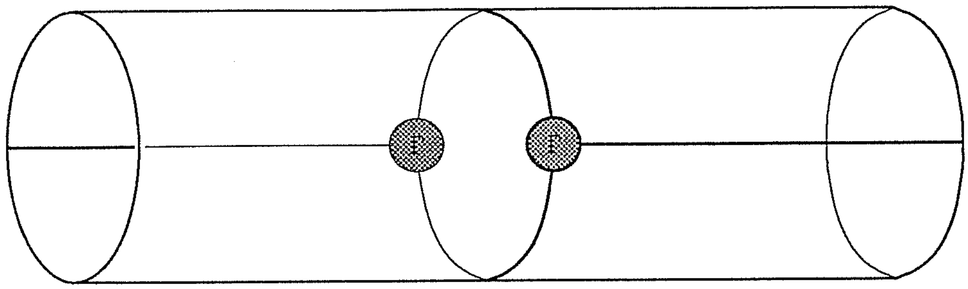
We would like to decide between the two proposals for the scattering matrix for $\mathcal{M}_{4,5} + \Phi_{2,1}$. One possibility would be to consider the TBA and compare its results with the truncation data. This is unfortunately very difficult, since we deal with a degenerate particle and the diagonalization of the transfer-matrix has not been carried out yet for this model. Therefore we have to think of other possibilities.

In the ideal situation, when the truncation level $\Lambda \to \infty$, the crossover between the intermediate region ($mR \sim 1$) and the infrared one ($mR \gg 1$) is controlled by off-mass shell effects and has an exponential behaviour. The computation of these finite-size corrections has been put performed by Lüscher [81], and we refer the reader to this reference for a detailed discussion of the subject. Rigorously speaking, this analysis is valid for the case of only one vacuum in the theory, but the degeneracy of the ground state gives only subleading contribution (see below), so that we can use Lüsher's results, at least at leading order. The idea is to consider perturbative corrections to the propagator, which is known exactly for the infinite volume theory. The leading corrections come from topologically non-trivial diagrams, which wind around the cylinder exactly once. The analysis is independent of the details of the interaction, and the above statement is very general.

Since these corrections involve the scattering matrix it is exactly the method we looked for. It gives a prediction for the spectrum which we can check from the TCSA data. Let us discuss it in more detail.

The diagrams contributing to the corrections are shown in fig.7.3. Note that the only state which can propagate is K_{11} since K_{00} is forbidden from the HSLM geometry, and K_{01} and K_{10} from the boundary conditions.

The diagrams can be computed from the data extracted from the S-matrix. The first correction involves the on-mass-shell three-particle vertex Γ, which is extracted from the residue at $\beta = \frac{2\pi i}{3}$ of the amplitudes $S_{11}^{11}(\beta)$ for the Smirnov and Zamolodchikov S-matrix respectively. The second correction comes from an integral over the momentum of the intermediate virtual particle, interacting via the S-matrix $S_{11}^{11}(\beta)$. The final result leading correction:



sub–leading correction:

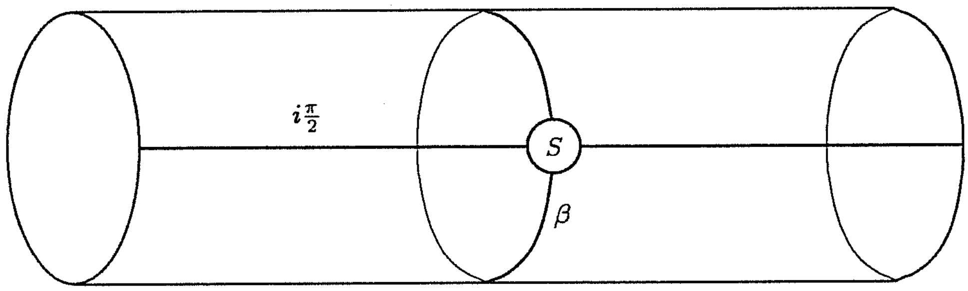


Figure 7.3: Finite volume off-mass-shell corrections to one-particle energy

becomes

$$\Delta E(R) \equiv E_2(R) - E_0(R) = m + i\,\frac{\sqrt{3}m}{2}\Gamma^2 \exp\left(-\frac{\sqrt{3}mR}{2}\right) \tag{7.28}$$
$$-m\int_{-\infty}^{\infty}\frac{d\beta}{2\pi}\; e^{-mR\cosh\beta}\cosh\beta\left(S\left(\beta+\frac{i\pi}{2}\right)-1\right) \quad .$$

In this analysis we have to be aware that the ground state of our potential is degenerate, therefore (7.28) is correct up to further subleading corrections of the form $\mathcal{O}(e^{-mR})$. The problem of comparing the S-matrix with the truncation data, using this approach, has been addressed to in [32]. There we adopted the following procedure: first we computed numerically the integral on the intermediate particles in both cases of RSOS and Zamolodchikov's S-matrix and we subtracted it from the numerical data obtained from the TCSA. After this subtraction, we made a fit of the data with a function of the form

$$G(R) = A + Be^{-\frac{\sqrt{3}}{2}mR} + Ce^{-mR} \quad . \tag{7.29}$$

The first term should correspond to the mass term. The coefficient of the second one is the quantity we need in order to extract the residue of the S-matrix at $\beta = \frac{2\pi i}{3}$

$$\frac{2}{\sqrt{3}m} \quad B = i \ \text{Res}_{\beta=\frac{2\pi i}{3}} \quad S_{11}^{11}(\beta) \quad . \tag{7.30}$$

The third term is a subleading one, which takes into account: a) the asymptotic exponential approach of the lowest levels of our TCSA data to the (unknown) theoretical vacuum energy $E_0(R)$; b) the possible subleading corrections to (7.28), arising from tunneling processes. These tunneling processes might not be strictly proportional to e^{-mR}, but in the region in which we are measuring, a term of the order $R^{\alpha}e^{-mR}$ will behave as e^{-mR}, since the exponential decay will overwhelm the polynomial behaviour.

In the case of RSOS S-matrix, the best fit gives the following values

$$\begin{aligned} A &= 0.97 \pm 0.02 \ ; \\ B &= -0.29 \pm 0.02 \ ; \\ C &= -0.36 \pm 0.02 \ . \end{aligned} \tag{7.31}$$

The corresponding curve is drawn in fig.7.4, together with the data obtained from TCSA. The mass term agrees with our previous calculation (eq. (7.27)). The second term gives for the residue at $\beta = \frac{2\pi i}{3}$ the value 0.34 ± 0.02. This is consistent with that of the RSOS S-matrix. In our fit procedure, the value of the residue we extracted through (7.30) is stable with respect to small variation of the mass value. Increasing (decreasing) m, B increases (decreases) as well, in such a way that the residue takes the same value (within the numerical errors). This a pleasant situation because it allows an iterative procedure to find the best fit of the data: one can start with a trial value for m (let's say $m = 1$) and plug it into (7.29). From the A-term which comes out from the fit, one gets a new determination of the mass m that can be again inserted into (7.29) and so on. Continued iteration does not affect significantly the value we extract for the residue, but converges to an accurate measurement of the mass. The values in (7.31) were obtained in this way.

With Zamolodchikov's S-matrix the best fit of the data (with the same iterative procedure as before) gives the result

$$A = 0.96 \pm 0.02 \ ;$$

$$B = -1.10 \pm 0.02 \; ; \qquad (7.32)$$
$$C = 1.14 \pm 0.02 \; .$$

The residue extracted from these data (1.29 ± 0.01) is not consistent with that one of the amplitude $S_{11}^{11}(\beta)$. The situation does not improve even if we *fix* the coefficient of $e^{-\frac{\sqrt{3}m}{2}}$ to be that one predicted by Table 7.2, namely $B = -0.158$ and leave as free parameters for a best fit A and C. In this case, our best determination of A and C were $A = 0.965$ and $C = -0.046$. The curve is plotted in fig.7.4 together with the data obtained from TCSA. It shows clearly that the S-matrix derived from Smirnov's approach is consistent with the truncation data. In [68] the the same question is addressed but with different numerical methods which are based also on TCSA. The authors confirm our investigation and draw the same conclusions.

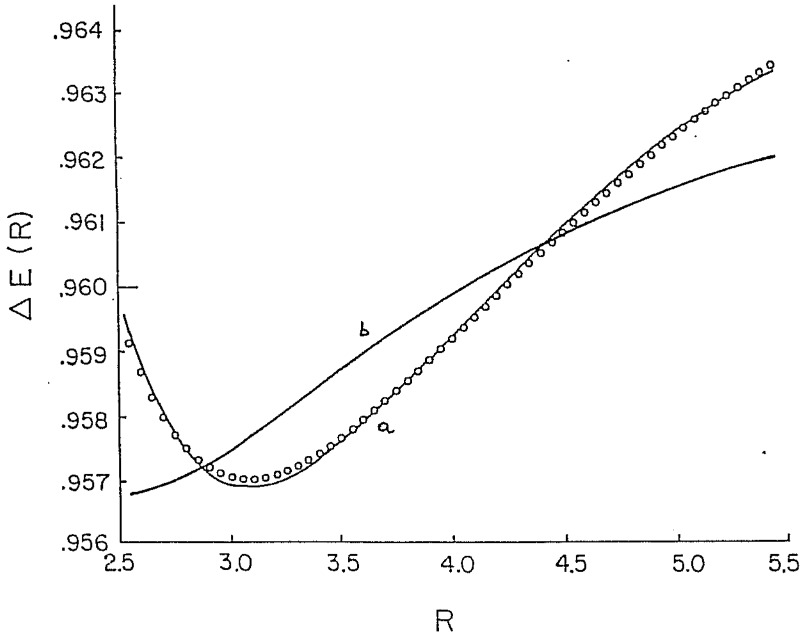


Figure 7.4: Energy difference of the one-particle state with respect to the double degenerate ground state, $\Delta E(R) \equiv E_2(R) - E_0(R)$, compared with off-mass-shell corrections. The dots are the numerical data obtained from TCSA, a *is the curve for the RSOS S-matrix, with A, B and C given in (3.11),* b *is the curve for Zamolodchikov's S-matrix with the B term equal to the theoretical value and A and C coming from a best fit.*

R	$\Delta E(R)$	$I_1(R)$	$I_2(R)$
2.5	0.95913	-0.0625322	-0.0174576
2.55	0.958681	-0.058955	-0.0164808
2.6	0.9583	-0.0555943	-0.0155615
2.65	0.957983	-0.0524359	-0.0146961
2.7	0.957722	-0.0494667	-0.0138811
2.75	0.957511	-0.0466744	-0.0131135
2.8	0.957345	-0.0440477	-0.0123902
2.85	0.957221	-0.0415761	-0.0117085
2.9	0.957132	-0.0392498	-0.011066
2.95	0.957076	-0.0370596	-0.0104601
3.	0.957048	-0.034997	-0.00988871
3.05	0.957046	-0.0330542	-0.00934973
3.1	0.957068	-0.0312238	-0.00884121
3.15	0.95711	-0.0294988	-0.00836133
3.2	0.95717	-0.0278729	-0.00790841
3.25	0.957247	-0.02634	-0.00748085
3.3	0.957337	-0.0248945	-0.00707716
3.35	0.95744	-0.0235313	-0.00669595
3.4	0.957553	-0.0222453	-0.0063359
3.45	0.957676	-0.021032	-0.00599579
3.5	0.957807	-0.019887	-0.00567447
3.55	0.957945	-0.0188064	-0.00537086
3.6	0.958089	-0.0177864	-0.00508394
3.65	0.958239	-0.0168233	-0.00481276
3.7	0.958393	-0.015914	-0.00455642
3.75	0.958549	-0.0150553	-0.00431408

Table 7.2: Difference of energy $\Delta E \equiv E_2 - E_0$ compared with the finite volume off-mass-shell corrections. In the first column, the values of R. In the second column, the numerical data obtained from TCSA. In the third and fourth columns, the numerical values of the integral $\int_{-\infty}^{\infty} \frac{d\beta}{2\pi}\ e^{-mR\cosh\beta}\cosh\beta\left(S\left(\beta+\frac{i\pi}{2}\right)-1\right)$ for the RSOS S-matrix and for the Zamolodchikov's S-matrix, respectively. The integral is computed for a value of m, obtained self-consistently from the best fit of the data.

R	$\Delta E(R)$	$I_1(R)$	$I_2(R)$
3.8	0.958709	-0.0142442	-0.00408495
3.85	0.95887	-0.013478	-0.00386828
3.9	0.959033	-0.0127541	-0.00366337
3.95	0.959198	-0.0120701	-0.00346957
4.	0.959363	-0.0114238	-0.00328625
4.05	0.959528	-0.0108129	-0.00311283
4.1	0.959693	-0.0102355	-0.00294875
4.15	0.959857	-0.00968962	-0.0027935
4.2	0.960021	-0.00917357	-0.00264659
4.25	0.960183	-0.00868562	-0.00250756
4.3	0.960346	-0.00822421	-0.00237597
4.35	0.960505	-0.00778784	-0.00225142
4.4	0.960663	-0.00737512	-0.00213352
4.45	0.96082	-0.00698473	-0.00202191
4.5	0.960973	-0.00661542	-0.00191624
4.55	0.961128	-0.00626603	-0.00181618
4.6	0.961279	-0.00593545	-0.00172144
4.65	0.961426	-0.00562265	-0.00163172
4.7	0.96158	-0.00532664	-0.00154676
4.75	0.961718	-0.0050465	-0.00146629
4.8	0.961862	-0.00478135	-0.00139007
4.85	0.962	-0.00453039	-0.00131787
4.9	0.962143	-0.00429282	-0.00124948
4.95	0.962279	-0.00406792	-0.00118469
5.	0.962408	-0.003855	-0.00112331
5.05	0.962541	-0.0036534	-0.00106515
5.1	0.962667	-0.00346252	-0.00101005
5.15	0.962795	-0.00328177	-0.000957836
5.2	0.962916	-0.00311059	-0.000908356
5.25	0.963039	-0.00294848	-0.000861466
5.3	0.963154	-0.00279494	-0.000817027
5.35	0.96328	-0.00264952	-0.00077491
5.4	0.963387	-0.0025117	-0.00073499

Table 7.2: - continued

Part III

Form-Factor Bootstrap

In the last part we have described extensively the S–matrix bootstrap method for various models. The S–matrix describes the on–shell properties of a massive quantum theory and therefore determines completely its infrared properties. Also in the ultraviolet regime a complete description of the theory is given, since it is determined by some conformal field theory.

One would like to understand the full quantum structure of a theory, that is to be able to study it over the whole range of distances and to describe the cross–over between ultraviolet and infrared behaviour. For that it is necessary to calculate the off-shell correlators. A complete description of the theory is given if we know the whole set of its correlation functions.

In the bootstrap approach this goal can be achieved through the calculation of form–factors. In section 4 we have already described the basic principles of this method and discussed the form–factor axioms. Here we will examine a specific example in order to show the method at work. We will be able to determine the operator-content of the theory, and give a general expression for the form–factors. All these results are obtained by solving the constraint equations for the form–factors without any additional information about the theory, apart from the on–shell data. In this sense one can consider the results as a pure merit of the bootstrap principle.

8 Form-Factors in the Sinh-Gordon Model

In order to study the form–factor bootstrap method we choose a specific model, the Sinh–Gordon model. This because as we saw in chapter 4, the bootstrap axioms depend on the nature of the considered theory. Especially one needs the on–shell data of the theory, the set of asymptotic states and the full S–matrix.

We choose the Sinh–Gordon model for several reasons. First of all it has only one particle and no bound states, which simplifies enourmosly the form–factor axioms. Further its S–matrix is of a rather simple form. On the other hand the model is highly non–trivial as a quantum field theory and exhibits a coupling constant dependence. In the following section 8.1 we will discuss its properties in some more detail.

In section 8.2 we discuss the form–factor equations for this specific model and choose a convenient parametrization for the form–factors. In section 8.3 we present our results [73] on the solution of these equations. We obtain the general solution in a closed form and also determine the operator content of the theory. In section 8.4 we discuss the cluster property of the form–factors and identify the form–factors for the exponential operators $e^{\alpha\phi}$ where ϕ is the field of the Sinh–Gordon action. Finally in section 8.5 we discuss the construction of correlation functions. We give some examples of their calculation and confirm our identification of the exponential operators by calculating their anomaolous dimensions in the scaling limit.

8.1 General properties of the Sinh-Gordon model

The Sinh-Gordon theory is defined by the action

$$\mathcal{S} = \int d^2x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{g^2}\cosh g\phi(x)\right] . \tag{8.1}$$

It is the simplest example of an affine Toda field theory, namely the real coupled $A_1^{(1)}$ theory. It possesses a Z_2 symmetry $\phi \to -\phi$. Though many features of ATFT have already been discussed in section 2.4 and 5.2, we will summarize the principle properties here for the special case of the Sinh-Gordon theory.

There are numerous alternative viewpoints for the Sinh-Gordon model. First, it can be regarded either as a perturbation of the free massless conformal action by means of the relevant operator $\cosh g\phi(x)$. Alternatively, it can be considered as a perturbation of the conformal Liouville action

$$\mathcal{S} = \int d^2x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \lambda e^{g\phi}\right] \quad , \tag{8.2}$$

by means of the relevant operator $e^{-g\phi}$ or as a conformal affine A_1-Toda theory [7] in which the conformal symmetry is broken by setting the free field to zero.

In a perturbative approach to the quantum field theory defined by the action (8.1), the only ultraviolet divergencies which occur in any order in g come from tadpole graphs and can be removed by a normal ordering prescription with respect to an arbitrary mass scale M. All other Feynman graphs are convergent and give rise to finite wave function and mass renormalisation. The coupling constant g does not renormalise.

An essential feature of the Sinh-Gordon theory is its integrability, which in the classical case can be established by means of the inverse scattering method [47]. This corresponds to an infinity series of conservation laws

$$\partial_z T_{s+1} = \partial_{\bar{z}} \Theta_{s-1} . \tag{8.3}$$

The corresponding charges Q_s are given by

$$Q_s = \oint [T_{s+1}\, dz + \Theta_{s-1}\, d\bar{z}] . \tag{8.4}$$

The integer-valued index s which labels the integrals of motion is the spin of the operators. Non-trivial conservation laws are obtained for odd values of s

$$s = 1, 3, 5, 7, \ldots \tag{8.5}$$

In analogy to the Sine-Gordon theory [100], an infinite set of conserved charges Q_s with spin s given in (8.5) also exists for the quantized version of the Sinh-Gordon theory. They are diagonalised by the asymptotic states with eigenvalues given by

$$Q_s\, |\beta_1, \ldots, \beta_n> = \chi_s \sum_{i=1}^{n} e^{s\beta_i}\, |\beta_1, \ldots, \beta_n> , \tag{8.6}$$

where χ_s is the normalization constant of the charge Q_s. The existence of these higher integrals of motion precludes the possibility of production processes and hence guarantees that the n-particle scattering amplitudes are purely elastic and factorized into $n(n-1)/2$ two-particle S-matrices. The exact expression for the Sinh-Gordon theory is given by [5]

$$S(\beta, B) = \frac{\tanh \frac{1}{2}(\beta - i\frac{\pi B}{2})}{\tanh \frac{1}{2}(\beta + i\frac{\pi B}{2})} , \tag{8.7}$$

where B is the following function of the coupling constant g

$$B(g) = \frac{2g^2}{8\pi + g^2} . \tag{8.8}$$

corresponding to eq.(5.31). For real values of g the S-matrix has no poles in the physical sheet and hence there are no bound states, whereas two zeros are present at the crossing symmetric positions

$$\beta = \begin{cases} i\frac{\pi B}{2} \\ i\frac{\pi(2-B)}{2} \end{cases} \tag{8.9}$$

An interesting feature of the S-matrix is its invariance under the map [18, 29]

$$B \to 2 - B \tag{8.10}$$

i.e. under the *strong-weak* coupling constant duality

$$g \to \frac{8\pi}{g}. \tag{8.11}$$

This duality is a property shared by the unperturbed conformal Liouville theory (8.2) [84] and it is quite remarkable that it survives even when the conformal symmetry is broken.

8.2 Form factors of the Sinh-Gordon model

We will consider matrix elements between in-states and out-states of hermitian local spinless scalar operators $\mathcal{O}(x)$. Since we have only one non degenerate particle the form-factors do not carry any indices and we can simplify the notation used in chapter 4 and denote them by

$$F_n^{\mathcal{O}}(\beta_1, \beta_2, \ldots, \beta_n) = \langle 0 \mid \mathcal{O}(0,0) \mid Z(\beta_1), Z(\beta_2), \ldots, Z(\beta_n)\rangle_{in}. \tag{8.12}$$

For local scalar operators $\mathcal{O}(x)$, relativistic invariance implies that the form–factors F_n are functions of the difference of the rapidities β_{ij}

$$F_n(\beta_1, \beta_2, \ldots, \beta_n) = F_n(\beta_{12}, \beta_{13}, \ldots, \beta_{ij}, \ldots) \quad, i < j \ . \tag{8.13}$$

Let us discuss the axioms for the form-factors (see section 4) for this special case. Except for the poles corresponding to the one-particle bound states in all sub-channels, we expect the form–factors F_n to be analytic inside the strip $0 < \text{Im}\beta_{ij} < 2\pi$. The form–factors of a hermitian local scalar operator $\mathcal{O}(x)$ satisfy a set of equations (see section 4), known as Watson's equations [111], which for our case assume the simple form

- $$F_n(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots,\beta_n) = F_n(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots,\beta_n)S(\beta_i-\beta_{i+1}) \;, \qquad (8.14)$$

- $$F_n(\beta_1+2\pi i,\ldots,\beta_{n-1},\beta_n) = F_n(\beta_2,\ldots,\beta_n,\beta_1) = \prod_{i=2}^{n} S(\beta_i-\beta_1)F_n(\beta_1,\ldots,\beta_n) \;. \qquad (8.15)$$

Since we do not have poles in the S–matrix, we cannot form any bound states. This implies that for this case the eq.(4.11) does not give any constraint. The only recursive equation is given by that for the kinematical poles

- $$-i\lim_{\tilde{\beta}\to\beta}(\tilde{\beta}-\beta)F_{n+2}(\tilde{\beta}+i\pi,\beta,\beta_1,\beta_2,\ldots,\beta_n) = \left(1-\prod_{i=1}^{n} S(\beta-\beta_i)\right) F_n(\beta_1,\ldots,\beta_n). \qquad (8.16)$$

The general solution of Watson's equations (8.14),(8.15) can always be brought into the form [65]

$$F_n(\beta_1,\ldots,\beta_n) = K_n(\beta_1,\ldots,\beta_n)\prod_{i<j} F_{\rm min}(\beta_{ij}) \;, \qquad (8.17)$$

where $F_{\rm min}(\beta)$ satisfies (8.14) and (8.15) for the case $n=2$, is analytic in $0 \leq \text{Im}\ \beta \leq \pi$ and has no zeros in $0 < \text{Im}\ \beta < \pi$. These requirements uniquely determine this function, up to a normalization constant. The remaining factors K_n then satisfy Watson's equations with $S_2 = 1$, which implies that they are completely symmetric, $2\pi i$-periodic functions of the β_i. They must contain all the physical poles expected in the form–factor under consideration and must satisfy a correct asymptotic behaviour for large value of β_i. Both requirements depend on the nature of the operator $\mathcal{O}$.

Let us notice that one condition on the asymptotic behaviour of the form–factors is dictated by relativistic invariance. In fact, a simultaneous shift in the rapidity variables results in

$$F_n^{\mathcal{O}}(\beta_1+\Lambda,\beta_2+\Lambda,\ldots,\beta_n+\Lambda) \;=\; F_n^{\mathcal{O}}(\beta_1,\beta_2,\ldots,\beta_n) \;, \qquad (8.18)$$

For form–factors of an operator $\mathcal{O}(x)$ of spin s, the previous equation generalizes to

$$F_n^{\mathcal{O}}(\beta_1+\Lambda,\beta_2+\Lambda,\ldots,\beta_n+\Lambda) \;=\; e^{s\Lambda}\, F_n^{\mathcal{O}}(\beta_1,\beta_2,\ldots,\beta_n) \;, \qquad (8.19)$$

Secondly, in order to have a power-law bounded ultraviolet behaviour of the two-point function of the operator $\mathcal{O}(x)$ (which is the case we will consider), we have to require that the form- factors behave asymptotically at most as $\exp(k\beta_i)$ in the limit $\beta_i \to \infty$, with k being a constant independent of i. This means that, once we extract from K_n the denominator which gives rise to the poles, the remaining part has to be a symmetric function of the variables $x_i \equiv e^{\beta_i}$, with a finite number of terms, i.e. a symmetric polynomial in the x_i's. It is convenient to introduce a basis in this functional space given by the *elementary symmetric polynomials* $\sigma_k^{(n)}(x_1, \ldots, x_n)$ which are generated by [82]

$$\prod_{i=1}^{n}(x + x_i) = \sum_{k=0}^{n} x^{n-k}\, \sigma_k^{(n)}(x_1, x_2, \ldots, x_n). \tag{8.20}$$

Conventionally the $\sigma_k^{(n)}$ with $k > n$ and with $n < 0$ are zero. The explicit expressions for the other cases are

$$\begin{aligned} &\sigma_0 = 1 \quad , \\ &\sigma_1 = x_1 + x_2 + \ldots + x_n \quad , \\ &\sigma_2 = x_1x_2 + x_1x_3 + \ldots x_{n-1}x_n \quad , \\ &\vdots \qquad \vdots \\ &\sigma_n = x_1x_2\ldots x_n \quad . \end{aligned} \tag{8.21}$$

The $\sigma_k^{(n)}$ are linear in each variable x_i and their total degree is k. For later convenience we state one more of their properties: they satisfy the recursive equation

$$\sigma_k^{(n+2)}(-x, x, x_1, \ldots, x_n) = \sigma_k^{(n)}(x_1, x_2, \ldots, x_n) - x^2\sigma_{k-2}^{(n)}(x_1, x_2, \ldots, x_n) \ . \tag{8.22}$$

The Z_2 symmetry of the model is realized by a map σ, whose effect on the elementary field of the theory is $\sigma(\phi) = -\phi$. According to this symmetry we can label the operators by their Z_2 parity. The simplest representative of the odd sector is given by the (renormalised) field $\phi(x)$ itself. It creates a one-particle state from the vacuum. Our normalization is fixed to be (see sect. 4.3)

$$F_1^{\phi}(\beta) = < 0 \mid \phi(0) \mid \beta >_{\text{in}} = \frac{1}{\sqrt{2}} \ . \tag{8.23}$$

For the even sector, an important operator is given by the energy-momentum tensor

$$T_{\mu\nu}(x) = 2\pi\,(: \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}(x) \ :) \tag{8.24}$$

where :: denotes the usual normal ordering prescription with respect an arbitrary mass scale M. Its trace $T_\mu^\mu(x) = \Theta(x)$ is a spinless operator whose normalization is fixed in terms of its two-particle form–factor

$$F_2^\Theta(\beta_{12} = i\pi) = {}_{\text{out}} < \beta_1 \mid \Theta(0) \mid \beta_2 >_{\text{in}} = 2\pi m^2 \;, \tag{8.25}$$

where m is the physical mass.

An essential step for the computation the form–factors is the determination of $F_{\text{min}}(\beta)$, introduced in (8.17). It satisfies the equations

$$\begin{aligned} F_{\text{min}}(\beta) &= F_{\text{min}}(-\beta)\, S_2(\beta) \;, \\ F_{\text{min}}(i\pi - \beta) &= F_{\text{min}}(i\pi + \beta) \;. \end{aligned} \tag{8.26}$$

As shown in [65], the easiest way to compute $F_{\text{min}}(\beta)$ (up to a normalization $\mathcal{N}$) is to exploit an integral representation of the S-matrix. For the Sinh-Gordon theory we have

$$F_{\text{min}}(\beta, B) = \mathcal{N} \exp\left[8 \int_0^\infty \frac{dx}{x} \frac{\sinh\left(\frac{xB}{4}\right) \sinh\left(\frac{x}{2}(1-\frac{B}{2})\right) \sinh\frac{x}{2}}{\sinh^2 x} \sin^2\left(\frac{x\hat{\beta}}{2\pi}\right)\right] \;. \tag{8.27}$$

We choose our normalization to be

$$\mathcal{N} = \exp\left[-4 \int_0^\infty \frac{dx}{x} \frac{\sinh\left(\frac{xB}{4}\right) \sinh\left(\frac{x}{2}(1-\frac{B}{2})\right) \sinh\frac{x}{2}}{\sinh^2 x}\right] \;. \tag{8.28}$$

The analytic structure of $F_{\text{min}}(\beta, B)$ can be easily read from its infinite product representation in terms of Γ functions

$$F_{\text{min}}(\beta, B) = \prod_{k=0}^{\infty} \left| \frac{\Gamma\left(k+\frac{3}{2}+\frac{i\hat{\beta}}{2\pi}\right)\Gamma\left(k+\frac{1}{2}+\frac{B}{4}+\frac{i\hat{\beta}}{2\pi}\right)\Gamma\left(k+1-\frac{B}{4}+\frac{i\hat{\beta}}{2\pi}\right)}{\Gamma\left(k+\frac{1}{2}+\frac{i\hat{\beta}}{2\pi}\right)\Gamma\left(k+\frac{3}{2}-\frac{B}{4}+\frac{i\hat{\beta}}{2\pi}\right)\Gamma\left(k+1+\frac{B}{4}+\frac{i\hat{\beta}}{2\pi}\right)} \right|^2 \tag{8.29}$$

$F_{\text{min}}(\beta, B)$ has a simple zero at the threshold $\beta = 0$ since $S(0) = -1$ and its asymptotic behaviour is given by

$$\lim_{\beta\to\infty} F_{\text{min}}(\beta, B) = 1 \quad . \tag{8.30}$$

It satisfies the functional equation

$$F_{\text{min}}(i\pi + \beta, B) F_{\text{min}}(\beta, B) = \frac{\sinh\beta}{\sinh\beta + \sinh\frac{i\pi B}{2}} \quad , \tag{8.31}$$

which we will use in the next section in order to find a convenient form for the recursive equations of the form–factors.

Since the Sinh-Gordon theory has no bound states, the only poles which appear in any form–factor $F_n(\beta_1,\ldots,\beta_n)$ are those occurring in every three-body channel. Additional poles in the n-body intermediate channel are excluded by the elasticity of the scattering theory. Using the identity

$$(p_1+p_2+p_3)^2-m^2 \;=\; 8m^2\cosh\frac{1}{2}\beta_{12}\cosh\frac{1}{2}\beta_{13}\cosh\frac{1}{2}\beta_{23}\;, \tag{8.32}$$

all possible three-particle poles are taken into account by the following parameterization of the function

$$K_n(\beta_1,\ldots,\beta_n) \;=\; \frac{Q'_n(\beta_1,\ldots,\beta_n)}{\prod\limits_{i<j}\cosh\frac{1}{2}\beta_{ij}}\;, \tag{8.33}$$

where Q'_n is free of any singularity. The second equation in (8.14) implies that Q'_n is $2\pi i$-periodic (anti-periodic) when n is an odd (even) integer. Hence, with a re-definition of Q'_n into Q_n, the general parameterization of the form–factor $F_n(\beta_1,\ldots\beta_n)$ is chosen to be

$$F_n(\beta_1,\ldots,\beta_n) \;=\; H_n\, Q_n(x_1,\ldots,x_n)\prod_{i<j}\frac{F_{\rm min}(\beta_{ij})}{x_i+x_j} \tag{8.34}$$

where $x_i = e^{\beta_i}$ and H_n is a normalization constant. The denominator in (8.34) may be written more concisely as $\det\Sigma$ where the entries of the $(n-1)\times(n-1)$-matrix Σ are given by $\Sigma_{ij}=\sigma^{(n)}_{2i-j}(x_1,\ldots,x_n)$.

The normalization constants for the form–factors of odd and even operators are conveniently chosen to be

$$H_{2n+1}=H_1\mu^{2n}\quad,\quad H_{2n}=H_2\mu^{2n-2} \tag{8.35}$$

with

$$\mu\equiv\left(\frac{4\sin(\pi B/2)}{F_{min}(i\pi,B)}\right)^{\frac{1}{2}} \tag{8.36}$$

where H_1 and H_2 are the initial conditions, fixed by the nature of the operator.

The functions $Q_n(x_1,\ldots,x_n)$ are symmetric polynomials in the variables x_i. As consequence of eq. (8.18), for form–factors of spinless operators the total degree should be $n(n-1)/2$ in order to match the total degree of the denominator in (8.34). The order of the degree of Q_n in each variable x_i is fixed by the nature and by the asymptotic behaviour of the operator $\mathcal{O}$ which is considered.

Applying the parameterization (8.34), together with the identity (8.31), the recursive equations (8.16) take on the form

$$(-)^n\, Q_{n+2}(-x,x,x_1,\ldots,x_n) \;=\; x D_n(x,x_1,x_2,\ldots,x_n)\, Q_n(x_1,x_2,\ldots,x_n) \qquad (8.37)$$

where we have introduced the function

$$D_n = \frac{-i}{4\sin(\pi B/2)}\left(\prod_{i=1}^{n}\left[(x+\omega x_i)(x-\omega^{-1}x_i)\right] - \prod_{i=1}^{n}\left[(x-\omega x_i)(x+\omega^{-1}x_i)\right]\right) \qquad (8.38)$$

with $\omega = \exp(i\pi B/2)$. Using the generating function (8.20) of the symmetric polynomials, the function D_n can be expressed as

$$D_n(x,x_1,\ldots,x_n) = \sum_{k=1}^{n}\;\sum_{m=1,odd}^{k} [m]\, x^{2(n-k)+m}\sigma_k^{(n)}\sigma_{k-m}^{(n)}(-1)^{k+1} \quad . \qquad (8.39)$$

We have introduced the symbol $[n]$ defined by

$$[n] \equiv \frac{\sin(n\frac{B}{2})}{\sin\frac{B}{2}} \qquad (8.40)$$

8.3 Solution of the recursive equations

In this section, we discuss the most general solution of the recursive equations (8.37) in the space of symmetric polynomials $\mathcal{P}$ of total degree $n(n-1)/2$. Any independent solution defines the matrix elements of a local scalar operator in the theory. A clarification is in order here. In the following we will only consider the form–factors of *irreducible operators.* Their form–factors have the distinguished property that they cannot be further factorized in terms of the elementary symmetric polynomial $\sigma_k^{(n)}$. We will not consider for instance form–factors of *derivative scalar operators* of the kind $(\partial\bar{\partial})^k\mathcal{O}$. The reason for this is that they can be simply obtained by multiplying the form–factor $F_n^{\mathcal{O}}$ by the term $(\frac{\sigma_{n-1}\sigma_1}{\sigma_n})^k$.

8.3.1 General solution for Q_3 and Q_4

To understand the structure of the linear space of the form–factors, it is worth considering the first polynomials $Q_1, .., Q_4$.

By relativistic invariance, Q_1 has to be a constant which we denote by $A_1^{(1)}$. For the same reason Q_2 is proportional to σ_1, since this is the only symmetric polynomial of degree 1, and we define $Q_2 = A_1^{(2)}\sigma_1$.

At level 3, the most general symmetric polynomial of total degree 3 is given by

$$\mathcal{P}_3 = A_1^{(3)}\sigma_3 + A_2^{(3)}\sigma_1\sigma_2 + A_3^{(3)}\sigma_1^3 \,, \tag{8.41}$$

and in order to be identified as a form–factor of the theory it should satisfy the recursive equation

$$-\mathcal{P}_3(-x,x,x_1) = xD_1(x)Q_1(x_1) = -x^2\sigma_1^{(1)}A_1^1 \,. \tag{8.42}$$

The solution of (8.42) gives rise to the most general form–factor Q_3

$$Q_3 = (A_1^{(1)} - A_2^{(3)})\sigma_3 + A_2^{(3)}\sigma_1\sigma_2 \,. \tag{8.43}$$

Given $A_1^{(1)}$, the linear space of form–factors at level 3 is a one-dimensional manifold parameterized by $A_2^{(3)}$, for instance.

An analogous result holds for Q_4. Starting with the most general symmetric polynomial of total degree 6

$$\begin{aligned}\mathcal{P}_4 &= A_1^{(4)}\sigma_4\sigma_2 + A_2^{(4)}\sigma_3\sigma_2\sigma_1 + A_3^{(4)}\sigma_4\sigma_1^2 + A_4^{(4)}\sigma_3^2 + A_5^{(4)}\sigma_3\sigma_1^3 \\ &+ A_6^{(4)}\sigma_2^3 + A_7^{(4)}\sigma_2^2\sigma_1^2 + A_8^{(4)}\sigma_2\sigma_1^4 + A_9^{(4)}\sigma_1^6 \,,\end{aligned} \tag{8.44}$$

and imposing the recursive equation (8.37) with initial condition $A_1^{(2)}$, the final form of Q_4 is given by

$$Q_4 = A_2^{(4)}\sigma_3\sigma_2\sigma_1 + A_3^{(4)}(\sigma_4\sigma_1^2 + \sigma_3^2) \quad , \tag{8.45}$$

with

$$A_3^{(4)} + A_2^{(4)} = A_1^{(2)} \,.$$

Also in this case the linear space of form–factors at level 4 is a one-dimensional manifold.

It is interesting to note that both solutions can be written as a sum of determinants[11]

$$\begin{aligned} Q_3 &= A_1^{(3)} \begin{Vmatrix} 0 & \sigma_3 \\ -1 & 0 \end{Vmatrix} + A_2^{(3)} \begin{Vmatrix} \sigma_1 & [2]\sigma_3 \\ 0 & \sigma_2 \end{Vmatrix} \,, \\ Q_4 &= A_1^{(4)} \begin{Vmatrix} \sigma_1 & [2]\sigma_3 & 0 \\ 0 & \sigma_2 & [2]\sigma_4 \\ 0 & 0 & \sigma_3 \end{Vmatrix} + A_2^{(4)} \begin{Vmatrix} [2]\sigma_1 & [3]\sigma_3 & 0 \\ 1 & [2]\sigma_2 & [3]\sigma_4 \\ 0 & \sigma_1 & [2]\sigma_3 \end{Vmatrix} \,. \end{aligned}$$

[11] We use the notation $||A|| \equiv \det A$.

Let us discuss some properties of $Q_1, \ldots, Q_4$. First of all at any step of the recursion process a new free parameter enters the solution. Secondly, the partial degree[12] of Q_n ($n = 1, \ldots, 4$) does not exceed $n-1$. Hence, all these form–factors will be at the most constant for $\beta_i \to \infty$. Finally, seeking solutions of the form–factor equations which vanish as $\beta_i \to \infty$, there is only a *unique* class of polynomials, i.e. Q_2 and Q_4 identically zero and $Q_3 \sim \sigma_3$. In [54] it has been shown that these are the first form-factors of the elementary field ϕ.

8.3.2 Properties of the general solution Q_n

Important properties of the polynomials Q_n can be easily obtained by analyzing the recursive equations (8.37). As a first step let us show that the partial degree of the polynomials Q_n satisfies

$$\deg(Q_n) \leq n-1 \; . \tag{8.46}$$

We have seen above that this property is true for $Q_1,..,Q_4$. To prove that this persists for higher polynomials, the cases (a) $Q_n \neq 0$ and (b) $Q_n = 0$ have to be considered separately.

- In the case (a) the proof is done by induction. Let us assume $\deg(Q_n) \leq n-1$. Since D_n is bilinear in $\sigma^{(n)}$ (see eq. (8.39)), the partial degree of $Q_{n+2}(-x, x, x_1, \ldots x_n)$ in the variables $x_1, \ldots x_n$ is smaller or equal to $n+1$. But the partial degree of $Q_{n+2}(x_1, x_2, \ldots x_{n+2})$ is equal to $Q_{n+2}(-x, x, x_1 \ldots, x_n)$, therefore the partial degree of Q_{n+2} must be less or equal to $n+1$.

- In the case (b), the space of the solutions is given by the kernel of eq. (8.37), i.e.

$$Q_{n+2}(-x, x, \ldots, x_{n+2}) = 0 \; . \tag{8.47}$$

In the space of polynomials $\mathcal{P}$ of total degree $\frac{(n+2)(n+1)}{2}$, there is only one solution of this equation, i.e.

$$Q_{n+2} = \prod_{i<j}^{n+2} (x_i + x_j) \; . \tag{8.48}$$

[12]Note that the partial degree in each x_i determines the asymptotic behaviour of the form–factor and is fixed by the total number of $\sigma^{(n)}$'s in each term of the sum.

This polynomial has partial degree $n+1$ and coincides with the denominator of eq. (8.34).

We have therefore shown that the partial degree of Q_n must be less or equal to $(n-1)$ for any spinless irreducible operator. A first consequence of this statement is that the form–factors of such operators cannot diverge in the limit $\beta_i \rightarrow \infty$. Secondly, it is now easy to understand the appearance of one additional parameter at each step of the recursion process. This is simply because the total dimension of the space of the polynomials Q_n is given by the dimension of the space of the polynomial Q_{n-2}, summed with the dimension of the kernel. Since the kernel is a one-dimensional manifold, the dimension of the space of solutions increases exactly by one at each step of the recursion. With the initial conditions $\dim(Q_1) = \dim(Q_2) = 1$, we finally obtain

$$\dim(Q_{2n-1}) = \dim(Q_{2n}) = n \quad . \tag{8.49}$$

Therefore the most general form–factor at level n of irreducible scalar operators span a finite linear space which can be expressed in terms of a basis Q_n^k

$$\begin{aligned} Q_{2n}(A_1^{(2n)}, \ldots, A_n^{(2n)}) &= \sum_{p=1}^{n} A_p^{(2n)} Q_{2n}^p \qquad (8.50) \\ Q_{2n-1}(A_1^{(2n-1)}, \ldots, A_n^{(2n-1)}) &= \sum_{p=1}^{n} A_p^{(2n-1)} Q_{2n-1}^p \ . \end{aligned}$$

Each of these polynomials defines the matrix elements of a quantum operator of the Sinh-Gordon model. Note that the dimension of the linear space of the form–factors grows exactly as the number of powers of the elementary field $\phi(x)$, i.e. ϕ^k, $k < n$. Therefore we expect to identify the tower structure of the form–factors with the space of the matrix elements of the composite operators ϕ^k.

8.3.3 Elementary solutions

An interesting class of solutions of the recursive equations (8.37) from which we can extract a basis for the form–factor space is given by[13]

$$Q_n(k) = ||M_{ij}(k)|| \quad , \tag{8.51}$$

[13]For simplicity we suppress the dependance of $Q_n(k)$ on the variables x_i.

where $M_{ij}(k)$ is an $(n-1)\times(n-1)$ matrix with entries

$$M_{ij}(k) = \sigma_{2i-j}\,[i-j+k] \quad . \tag{8.52}$$

These polynomials, which we call *elementary solutions*, depend on an arbitrary integer k and satisfy

$$Q_n(k) \;=\; (-1)^{n+1} Q_n(-k) \quad . \tag{8.53}$$

Although all $Q_n(k)$ are solutions of (8.37) not all of them can be linearly independent. The reason is that the dimension of space of the solutions at level $N=2n$ (or $N=2n-1$) is n at most. The first non trivial $Q_n(k)$ are given by the determinant

$$Q_3(k) \;=\; \left\| \begin{array}{cc} [k]\sigma_1 & [k+1]\sigma_3 \\ [\text{k-1}] & [k]\sigma_2 \end{array} \right\| \quad . \tag{8.54}$$

Using the trigonometrical identity $[n]^2 - [n-1][n+1] = 1$, it is easy to see that they satisfy eq. (8.37) (with $A_0^1 = 1$) for any integer value of k.

The whole set[14] of form–factors of the elementary field $\phi(x)$ and the trace of the energy-momentum tensor $\Theta(x)$ can be easily expressed in terms of the $Q_n(k)$. In fact the form–factors of the field $\phi(x)$ are given by $Q_n(0)$. Note that the form–factors of $\phi(x)$ are automatically zero for even n, in agreement with the Z_2 parity of the model. On the contrary, for odd n they vanish asymptotically when $\beta_i \to \infty$, as follows from the LSZ reduction formula [54]. Concerning the form–factors of $\Theta(x)$, these are given by the even polynomials $Q_{2n}(1)$, which tend to a constant when $\beta_i \to \infty$.

We conclude that the most general solution can be written as a linear combination of the elementary polynomials $Q_n(k)$. Only the first few of them are linearly independent, and we can write the general solution as

$$\begin{aligned} Q_{2n}(A_1^{(2n)},\ldots,A_n^{(2n)}) &= \sum_{p=1}^{n} A_p^{(2n)} Q_{2n}(p) \\ Q_{2n-1}(A_1^{(2n-1)},\ldots,A_n^{(2n-1)}) &= \sum_{p=1}^{n} A_p^{(2n-1)} Q_{2n-1}(p) \quad . \end{aligned} \tag{8.55}$$

This is the same as eq. (8.50), but now with the elementary solutions as base-vectors. It is interesting to note that that the polynomials contain only terms appearing in the determinant $|\sigma_{2i-j}|$.

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[14]The first representatives of them were computed in [54].

Consider an n-particle form-factor given through the polynomial given by eq. (8.55). The coefficients in the sum will automatically determine all form-factors $i < n$, because of the recursion-relations (8.37).

Usually, though one is taking an opposite approach, that is one fixes the 1-particle form-factor and wants to determine the structure of the more particle form-factors. In this case, the general structure of the coefficients A_k^n is not known. The first ones can be obtained through the recursive equations (8.37) starting by fixing the constant A_0^1 and finding the relations between A_0^3 and A_1^3, then determining the relations among A_i^5 and so on. Here we give the result of such a procedure:

$$\begin{array}{cccc} & & & A_0^1 \\ & & A_1^2 & \\ & & A_1^3 & A_0^3 = A_0^1 - A_1^3 \\ A_2^4 & A_1^4 = A_1^2 - [2]A_2^4 & & \\ A_2^5 & A_1^5 = A_1^3 - ([1]-[3])A_2^5 & & A_0^5 = A_0^3 + [3]A_2^5 \end{array} \tag{8.56}$$

8.3.4 A special class of operators $\mathcal{O}^n$

We consider a special class of solutions, which will allow us to understand better the space of local operators. We define our operators $\mathcal{O}^n$ through their n-particle form-factor, given as

$$F_n^{\mathcal{O}^n} = H_n \prod_{i<j}^{n} F^{min}$$

This is obtained using the special solution to our recursive equations $Q_n = |\sigma_{2i-j}|$, which cancels the denominator $\prod(x_i + x_j)$ in the form-factor. As a first example let us analyze the operator $\mathcal{O}^3$: We want $Q_3 = \sigma_1\sigma_2 - \sigma_3$. From (8.43) we see that this automatically requires $A_0^1 = 0$.

Now we can easily generalize this result to general n. Again require the condition that $F_n^{\mathcal{O}^n} = H_n \prod F^{min}$, that is that the corresponding $Q_n = \det \sigma_{2i-j}^{(n)}$. Then $Q_m = 0$ for $m < n$.

To show this consider

$$Q_n(-x, x, x_1, \ldots x_n) = \left|\sigma_{2i-j}^{(n-2)} - x^2\sigma_{2i-j-2}^{(n-2)}\right| \tag{8.57}$$

Now adding $-x^2$ times the column i to the column $i+1$ we find

$$Q_n(-x, x, x_1, \dots x_n) = \left|\sigma_{2i-j}^{(n-2)}\right| \tag{8.58}$$

which is identical 0, since the last column is 0. Since D_n is not zero we conclude that $Q_{n-2} = 0$. This further implies in general that all $F_m^n = 0$ for $m < n$.

We want to find the coefficients $A_k^n(\mathcal{O}^m)$ which give us these polynomials, in order to perform the base-transformation. Up to now we have only well-defined $F_n^{\mathcal{O}^n}$. For this form-factor the polynomial Q_n is given by the following coefficients:

$$A_i^n = 0 \quad , \; for \; i < n$$

and

$$\begin{aligned} A^n_{\frac{n-1}{2}} \equiv q_1(\frac{n-1}{2}) &= \left([\frac{n-1}{2}]\prod_{i=1}^{n-2}[i]\right)^{-1} \qquad for \;\; n \; odd \\ A^n_{\frac{n}{2}} \equiv q_2(\frac{n}{2}) &= \left((2\cos(\pi B/2))^{(\frac{n}{2}-1)(mod\,2)}\prod_{i=1}^{n-1}[i]\right)^{-1} \qquad for \;\; n \; even \end{aligned} \tag{8.59}$$

In order to construct form-factors $F_k^{\mathcal{O}^n}$, with $k > n$, we simply use the linearity of our original base and the recursive equations. Therefore we can choose in general the coefficients to be

$$A_k^n(\mathcal{O}^m) = \delta_{k,m} q_i(k) \tag{8.60}$$

In fig. 8.1 we have exhibited the structure of these operators. A similar diagram can be drawn for the operators with even index. This clarifies the structure of the space of states. At every level we have one linearly independent solution more, which we have now constructed explicitly.

8.4 Cluster-property and exponential operators

We can associate a quantum operator $\Psi_k(x)$ to any elementary solution $Q_n(k)$ $(k \neq 0)$. To identify such operators it is interesting to analyze the cluster property of their form–factors. By cluster transformation we generally mean the behaviour of a form–factor under the shift of a subset of the rapidities, i.e.

$$F_n^{\mathcal{O}_a}(\beta_1 + \Delta, \dots, \beta_m + \Delta, \beta_{m+1}, \dots, \beta_n) \; . \tag{8.61}$$

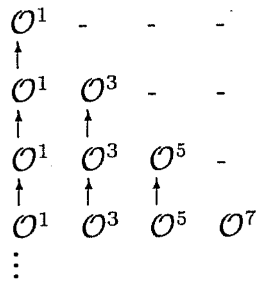

Figure 8.1: Tower like structure of the operators $\mathcal{O}^n$

Taking the limit $\Lambda \to \infty$, $F_n^{\mathcal{O}_a}$ can be decomposed into two functions of m and $(n-m)$ variables respectively. It is easy to prove that both functions satisfy all the set of axioms for the form–factors. Therefore they can be considered as form–factors of some operators $\mathcal{O}_b$ and $\mathcal{O}_c$

$$\lim_{\Lambda\to\infty} F_n^{\mathcal{O}_a}(\beta_1+\Delta,\ldots,\beta_m+\Delta,\beta_{m+1},\ldots,\beta_n) = F_m^{\mathcal{O}_b}(\beta_1,\ldots,\beta_m)F_{n-m}^{\mathcal{O}_c}(\beta_{m+1},\ldots,\beta_n) \tag{8.62}$$

Shortly,

$$\mathcal{O}_a \to \mathcal{O}_b \times \mathcal{O}_c \ .$$

We will prove that the operators Ψ_k are mapped onto themselves under the cluster transformation, i.e.

$$\Psi_k \to \Psi_k \times \Psi_k \ . \tag{8.63}$$

To this aim, let us introduce some notations.

It is useful to define the cluster-operator $\mathcal{C}_m$ (acting on the symmetric functions) by means of

$$\mathcal{C}_m\left(f(x_1,\ldots,x_n)\right) \equiv f(x_1e^{\Delta},x_2e^{\Delta},\ldots,x_me^{\Delta},x_{m+1},\ldots,x_n) \ \ m<n \ . \tag{8.64}$$

For example,

$$\mathcal{C}_1(\sigma_1^{(n)}) = e^{\Delta}x_1+x_2\ldots+x_n = e^{\Delta}\sigma_1^1+\hat{\sigma}_1^{(n-1)} \quad , \tag{8.65}$$

where we have defined

$$\hat{\sigma}_i^{(n-k)} \equiv \sigma_i^{(n-k)}(x_{n-k+1},x_{n-k+2},\ldots,x_n) \quad .$$

One easily finds that

$$\mathcal{C}_m(\sigma_k^{(n)}) = \sum_{i=1}^{k} \sigma_{k-i}^{(m)} e^{(k-i)\Delta} \hat{\sigma}_i^{(n-m)} \quad . \tag{8.66}$$

Since the cluster properties are fixed by the leading term of this sum, we have

$$\begin{aligned} \mathcal{C}_m(\sigma_k^{(n)}) &\sim \sigma_m^{(m)} e^{m\Delta} \hat{\sigma}_{k-m}^{(n-m)} \quad m \leq k \quad , \\ \mathcal{C}_m(\sigma_k^{(n)}) &\sim \sigma_k^{(m)} e^{k\Delta} \qquad\qquad m \geq k \quad . \end{aligned} \tag{8.67}$$

Now let us consider separately the cluster property of each term entering their parametrization

$$F_n^k(\beta_1, \ldots, \beta_n) = H_n^k \, Q_n(k) \prod_{i<j}^{n} \frac{F_{\rm min}(\beta_{ij})}{(x_i + x_j)} \quad . \tag{8.68}$$

From eq. (8.30), we have

$$\prod_{i<j}^{n} F_{\rm min}(\beta_{ij}) \longrightarrow \prod_{i<j}^{m} F_{\rm min}(\beta_{ij}) \prod_{i<j=m+1}^{n} F_{\rm min}(\beta_{ij}) \quad . \tag{8.69}$$

Using eq. (8.67), the cluster property of the elementary solution $Q_n(k)$ is given by

$$\mathcal{C}_m(Q_n(k)) \sim h(n,m) \, [k] \, Q_m(k) \, Q_{n-m}(k) \quad , \tag{8.70}$$

where

$$h(n,m) = e^{\Delta m (n - \frac{m+1}{2})} \, (\sigma_m^{(m)})^{n-m} \quad .$$

Concerning the denominator in (8.68), we write it initially as

$$\prod_{i<j}^{n} (x_i + x_j) = ||\sigma_{2i-j}^{(n)}||^{\langle n-1 \rangle} \quad , \tag{8.71}$$

where the index $\langle n-1 \rangle$ indicates the dimension of the matrix. Applying again eq. (8.67), we obtain

$$\mathcal{C}_m \left(||\sigma_{2i-j}^{(n)}||^{\langle n-1 \rangle} \right) \sim h(n,m) \, ||\sigma_{2i-j}^{(m)}||^{\langle m-1 \rangle} \times ||\hat{\sigma}_{2i-j}^{(n-m)}||^{\langle n-m-1 \rangle} \quad . \tag{8.72}$$

Choosing the normalization constants H_1^k and H_2^k as

$$H_1^k = \mu[k] \quad , \quad H_2^k = \mu^2 \, [k] \quad , \tag{8.73}$$

and using eqs. (8.69), (8.70) and (8.72), we conclude that the form–factors of Ψ_k are mapped onto themselves under the cluster transformation. Since this is a distinguished property of exponential operators [107], it is natural to identify the operators Ψ_k with the fields $e^{kg\phi}$.

8.5 Calculation of correlation functions

The form–factor approach was developed in order to calculate correlation functions. Even if we have now analytic expressions for the form–factors, we still need to do the integrals and the infinite sum in (4.2). In general it is not clear how to rewrite this in a useful form, and to possibly extract a simple expression or a differential equation for the correlation functions. Such results were only obtained for the Ising model [6, 113].

In this sense one can criticize the form–factor approach. However by numerical investigations it has been found that the form–factor series converges extremely fast [54, 119], and therefore provides a quantitative good approximation for the correlation functions. This fact was explained in [26] with the following argument.

At threshold ($\beta \to 0$) the minimal two particle form-factor $F_{min}(\beta)$ goes linearly to zero since $S(0) = -1$. Now consider for example the four particle contribution to the two point function. If the form–factor was constant, the contribution would be

$$\int \prod_1^4 d\beta_i e^{-Mr\sum_1^4 \cosh\beta_i} \sim \frac{e^{-4Mr}}{r^2} \quad .$$

In momentum space this corresponds to a branch point at $q^2 = -16M^2$ of the form $\ln(q^2 + 16M^2)$. But since we know that F_{min} vanishes at threshold, an additional factor $\prod_{i<j} \beta_{ij}^2$ appears in the integral, which leads to a behaviour of the form $\frac{e^{4Mr}}{r^8}$. Its effect is that the branch point gets softened to the form $(q^2 + 16M^2)^3$.

The same suppression effect exists for all higher contributions. The m–particle branch cut would be, on grounds of phase–space alone of the form $(q^2+m^2M^2)^{m/4-1}$, but actually gets softened to $(q^2 + m^2M^2)^{m^2/4-1}$. This means that one expects for the whole range of the coupling constant, that the correlation function is approximated by the lowest contributions of the form–factor series.

Let us give here one example of such a calculation in the Sinh–Gordon model. The ultraviolet behaviour of the Sinh–Gordon model is governed by a free massless theory with $c = 1$. In the infrared regime on the other hand it is massive and therefore $c = 0$. The change of the central charge going from short to long distances is dictated by the c-theorem and expressed in terms of the spectral representation of the two–point function of the trace of the stress-energy tensor [20] as

$$\Delta c = \int_0^\infty d\mu c_1(\mu) \quad , \tag{8.74}$$

with

$$c_1(\mu) = \frac{6}{\pi^2\mu^3} Im G(p^2 = -\mu^2) \quad ,$$
$$G(p^2) = \int d^2x e^{-ipx} \langle \Theta(x)\Theta(0)\rangle_{conn} \quad .$$

The two–point correlation function of Θ can as usual be expressed in terms of the form–factors, giving

$$c_1(\mu) = \frac{12}{\mu^3}\sum_{n=1}^{\infty}\frac{1}{(2n)!}\int\frac{d\beta_1\ldots d\beta_{2n}}{(2\pi)^{2n}}|F_{2n}^{\Theta}(\beta_1,\ldots\beta_{2n})|^2$$
$$\times\,\delta(\sum_i m\sinh\beta_i)\delta(\sum_i m\cosh\beta_i - \mu) \quad . \tag{8.75}$$

For the Sinh-Gordon theory $\Delta c = 1$ and the convergence of this series has been studied in [54]. It was shown that the sum rule is saturated by the two-particle contribution.

Let us now analyze the anomalous dimensions of the operators Ψ_k which we conjectured in section 8.4 to be related to the exponential fields $e^{kg\phi}$ [73]. Also this involves the computation of the correlation functions, since in order to extract the scaling behaviour we need to analyze

$$G_k(x) = \langle\Psi_k(x)\,\Psi_k(0)\rangle = \tag{8.76}$$
$$= \sum_{n=0}^{\infty}\int\frac{d\beta_1\ldots d\beta_n}{n!(2\pi)^n}F_n^{\Psi_k}(\beta_1\ldots\beta_n)F_n^{\Psi_k}(\beta_n\ldots\beta_1)\exp\left(-mr\sum_{i=1}^{n}\cosh\beta_i\right) \quad .$$

Let us present the method of how this can be done [102]. In the ultraviolet limit we expect this correlation function to have the scaling behaviour

$$G_k(x) \overset{mr\to 0}{\sim} (mr)^{-4h_k} \quad . \tag{8.77}$$

Since the kth term of the series (8.77) behaves as $(\ln mr)^k$ when $mr \to 0$ a straightforward investigation of the series is not efficient in order to extract the scaling dimensions.

Let us rewrite the series (8.77) in the following form,

$$G_k(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{-\infty}^{\infty} H_n(\beta_1,\ldots,\beta_n)e^{-mr\sum_j\cosh\beta_j}d\beta_1,\ldots,d\beta_n \quad , \tag{8.78}$$

where

$$H_n = \frac{(-1)^n}{(2\pi)^n} F_n^{\Psi_k}(\beta_1 \dots \beta_n) F_n^{\Psi_k}(\beta_n \dots \beta_1) \tag{8.79}$$

This function has now exactly the form of the grand partition function of a gas [39] with the coordinate dependent fugacity $z_i(\beta) = \frac{1}{2\pi} e^{-mr\cosh\beta_i}$. Using this observation one can use the standard expansion techniques used in statistical mechanics.

As a first step observe that because of (8.62) also the quantities H_n fulfill the cluster–property

$$H_n(\beta_1, \dots, \beta_l, \beta_{l+1} + \Lambda, \dots, \beta_n + \Lambda) \overset{\Lambda\to\infty}{\longrightarrow} H_l(\beta_1, \dots, \beta_l) H_{n-l}(\beta_{l+1} \dots, \beta_n) + O(e^{-\Lambda}) \ , \tag{8.80}$$

This allows us to apply a cluster–expansion (see e.g. [39]) by expanding the logarithm of $G_k(x)$ as

$$\ln G_k(x) = -\sum_{n=1}^{\infty} \int \tilde{H}_k(\beta_1, \dots, \beta_n) e^{-mr\sum_j \cosh\beta_j} d\beta_1, \dots, d\beta_n \quad , \tag{8.81}$$

with the coefficients

$$\tilde{H}_n(\beta_1, \dots, \beta_n) = \sum_{q=1}^{n} \sum_{n=n_1+\dots+n_q} \frac{(-1)^{k+q}}{q} \frac{1}{k_1! k_2! \dots k_q!}$$

$$\times H_{n_1}(\beta_1, \dots, \beta_{n_1}) H_{n_2}(\beta_{n_1+1}, \dots, \beta_{n_1+n_2}) \dots H_{n_q}(\beta_{n_1+\dots n_{q-1}+1}, \dots, \beta_n) \tag{8.82}$$

$\tilde{H}_n$ is a symmetric function of its arguments. Because of (8.80) it has the property

$$\tilde{H}_n((\beta_1, \dots, \beta_l, \beta_{l+1} + \Lambda, \dots, \beta_n + \Lambda) \overset{\Lambda\to\infty}{\longrightarrow} O(e^{-\Lambda}) \ , \ \ l \neq 0, \ \ l \neq k \tag{8.83}$$

This is why the integrals in (8.81) behave as

$$\int \tilde{H}_n(\beta_1, \dots, \beta_n) e^{-mr\sum \cosh\beta_j} d\beta_1 \dots d\beta_n \overset{mr\to 0}{\sim} 2\ln(mr) \int \tilde{H}_n(0, \beta_2, \dots, \beta_n) d\beta_2 \dots d\beta_n \quad . \tag{8.84}$$

This integral is convergent due to (8.83). It follows that one can write

$$\ln G_k(x) \sim -4\ln(mr) h_k \tag{8.85}$$

where

$$h_k = \frac{1}{2} \sum_{k=1}^{\infty} \int d\beta_2 \dots d\beta_n \tilde{H}_n(0, \beta_2, \dots, \beta_n) \quad . \tag{8.86}$$

With this formula at hand it is a simple calculation to extract the scaling dimension of the operators Ψ_k at lowest order. We find $h_k(g) = -k^2 g^2/8\pi$. This coincides with the anomalous dimensions of the exponential operators $e^{k\, g\, \phi(x)}$ calculated in the free massless bosonic theory which governs the ultraviolet limit. This confirms the identification of these operators with the exponentials, as we have conjectured in section 8.4.

Conclusions

In this thesis we have presented our results on the application of the bootstrap principle in two dimensional massive integrable field theories. We have applied this method at verious levels. We started out with the investigation of the on–shell data, *i.e.* the S–matrix, then turned to the calculation of the form–factors and finally constructed correlation functions.

The space we have dedicated to the various subjects reflects the situation in those fields of research. While the S–matrix bootstrap is well understood and has been widely applied, the form–factor bootstrap method is still at the beginnings of its development. Only for few models the complete set of form–factors has been determined in a closed form.

The final goal of the bootstrap method is to determine the off-shell correlation functions. Even though one has a closed expression given by the form–factor expansion which gives an extraordinary good quantitative approximation, the goal is to express them in some form which allow analytical calculations. It is a general hope that the correlators should satisfy some differential equation as it is the case in the few known examples of massive models and also in conformal field theory. To achieve this goal is a topic of future research.

Let us resume the major new results which have been obtained in this thesis. We started out by investigating S–matrices containing only scalar particles. Since in this case the form of the single S–matrix elements is constraint severely, it is possible to use an axiomatic approach in order to classify all consistent bootstrap systems. We have shown how the method works, introducing the concept of the 'bootstrap tree'. For a simple class of models we have also obtained the result, that the only consistent S-matrices originate from some kind of affine Toda field theory.

After reviewing some results on S–matrices for affine Toda theories we concentrated on a specific series of models: $\mathcal{M}_{2,2n+3} + \Phi_{1,2}$. These models have also a spectrum of only scalar particles, but the S–matrix element of the fundamental particle exhibits zeros in the physical strip besides the poles which generate the bound states. This causes some pathologies in the S–matrix as 'spurious' poles which cannot be explained by the usual bootstrap mechanism. We showed how the bootstrap principle has to be enlarged for these cases and how then the S–matrices become consistent. Finally we discuss the model $\mathcal{M}_{2,9} + \Phi_{1,4}$, which exhibits similar features.

In order to describe degenerate particles one needs to take care of the internal symmetry in addition to the dynamical structure prescribed by the bootstrap equations. We discussed a fruitful approach for perturbed minimal models in section 6. It was based on the idea to formulate the S–matrices in the language of graph–state models. This gives a very simple interpretation for the crossing symmetry, ultraviolet limit and the bootstrap equations. We have shown some specific applications of these methods. Using the bootstrap equations we have calculated the full S–matrix for the models $\mathcal{M}_{r,r+1} + \Phi_{1,2}$ for $r > 4$, which is the case when they contain degenerate particles. Also we have shown that the bootstrap closes on those particles.

As a specific example we have discussed the hard square lattice model. It is one of the simplest geometries for scattering theories exhibiting non–trivial degenerate particles. We have classified all consistent scattering theories in this geometry. Finally we have focused our attention on a model of particular interest in this class. It is the tricritical Ising model perturbed by the subleading magnetization operator ($\mathcal{M}_{4,5} + \Phi_{2,1}$). For this specific theory two different proposals for the scattering theory exist.

This ambiguity lead us to review the methods which can be used to confirm a proposed S–matrix. We sketched in simple terms the TBA and TCSA, applying them to models discussed before. We then concentrated specifically on the problem to decide, which of the two scattering matrices for $\mathcal{M}_{4,5} + \Phi_{2,1}$ is the right one. Using the theory of finite-size corrections we find the consistent one.

In order to describe the off–shell properties of a massive integrable field theory it is useful to calculate the form-factors. These can be used to get a quantitative approximation for the correlation-function. In the last part of the thesis we showed that they can be also used to determine the operator content of the field theory. We have carried out a detailed analysis for the Sinh-Gordon model, and showed how the local operators of the theory are organized in a tower like structure.

In the field of the application of the bootstrap method there are still many open quesions. It is my believe that especially the off-critical properties of the theory deserve more attention. As shown in chapter 8 the form–factor bootstrap approach has turned out to be a very powerful means in order to classify the operator content of a theory and calculate the full set of form–factors. I hope to be able to dedicate my future research to the generalization of this method.

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