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Asymptotic Dirichlet Problems

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"Doctor Philosophiæ"*

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TRIESTE

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Introduction

The aim of this thesis is the study of Dirichlet problems in domains with a “fragmented” boundary. More precisely, we are given a relatively compact open set $D \subset M$ (with sufficiently smooth boundary ∂D), M being \mathbf{R}^d or a Riemannian manifold, and a sequence (E_h) of closed subsets of D (with smooth boundary); we want to study, for every $f \in L^2(D)$, the asymptotic behavior of the sequence (u_h) of the solutions to the Dirichlet problems:

$$(P_h) \quad \begin{cases} -\Delta u_h = f, & \text{in } D \setminus E_h \\ u_h \in H_0^1(D \setminus E_h). \end{cases}$$

If we let $h \rightarrow +\infty$, it is not evident what could be the possible limit of the sequence (u_h) (should it exist).

Assume that, for each $h \in \mathbf{N}$, the closed set E_h is the disjoint union of h balls of radius $r_h > 0$ contained in the unit cube of \mathbf{R}^d ; then it has been proved by J. Rauch & M. Taylor in [R-T] that the behavior of the sequence (u_h) is linked with the behavior of the sequence

$$\alpha_h \stackrel{\text{def}}{=} \begin{cases} hr_h^{d-2}, & \text{if } d \geq 3, \\ h/|\log r_h|, & \text{if } d = 2; \end{cases}$$

more precisely, we have the following relation

$$\alpha_h \longrightarrow \begin{cases} +\infty \\ c \in [0, +\infty[\end{cases} \implies u_h \longrightarrow \begin{cases} 0 \\ u_c \end{cases}$$

where u_c is the solution to the Dirichlet problem

$$(P_c) \quad \begin{cases} -\Delta u_c + cu_c = f, & \text{in } D \\ u_c \in H_0^1(D). \end{cases}$$

We remark that the convergence of the sequence (u_h) to u_c is proved in [R-T, §6] using probabilistic methods.

Yet in a probabilistic setting, G.C. Papanicolaou & S.R.S. Varadhan studied in [P-V] the associated problem of diffusion, also when the centers of the balls are randomly distributed in D .

In an analytical framework, this kind of problems has been studied by D. Cioranescu & F. Murat in [C-M, §§1 and 2], arranging periodically the “holes” E_h in \mathbf{R}^d , $d \geq 2$.

Problems as (P_h) have been also studied by E.Ya. Hruslov, A.V. Marchenko using the orthogonal projection method in [Hru1], and by a capacity method in [Hru2], [Ma-Hru1]; we mention also the papers by E.Ya. Hruslov & A.V. Marchenko, D. Cioranescu & J. Saint Jean Paulin in [Ma-Hru1], [C-SJP1], and [C-SJP2] for other results on this argument. In a Riemannian framework, problems similar to (P_h) have been studied by I. Chavel & E.A. Feldman in [Ch, Chapter IX], [Ch-F2]; using probabilistic methods, they prove, among other results, the convergence of the eigenvalues. We mention, in this framework, several papers of S. Ozawa [O1]–[O5] in which, using the Green’s function method, the convergence of the spectrum is analyzed. Still in a Riemannian framework, G. Dal Maso, R. Gulliver & U. Mosco in [DM-G-M] studied similar problems, also when an increasing number of handles is attached to the manifold. We mention also the papers of P. Bérard, G. Besson, S. Gallot, I. Chavel, G. Courtois, E.A. Feldman [Bes], [Be-Ga], [Cou], [Ch-F1], [Ch-F3] in which these authors studied (with different methods) the case of a Riemannian manifold with a submanifold of codimension greater than or equal to 2 excised, with particular attention to the convergence of the eigenvalues.

Again in \mathbf{R}^d , the (sequence of) problems (P_h) may be included in the framework of the Relaxed Dirichlet problems, introduced by G. Dal Maso & U. Mosco in [DM-M1], [DM-M2]. A typical Relaxed Dirichlet problem can be written as

$$(RDP) \quad \begin{cases} -\Delta u + \mu u = f, & \text{in } D \\ u \in H_0^1(D), \end{cases}$$

where μ is a (positive, Borel) measure which belongs to a suitable family $\mathcal{M}_0(D)$ of measures (cf. Notation below). We point out that the problem (RDP) has to be interpreted in the *weak sense*: we say that the function u is a (weak) solution to (RDP) if $u \in H_0^1(D) \cap L^2(D, \mu)$ and it satisfies

$$(WF) \quad \int_D \nabla u \cdot \nabla v \, dx + \int_D uv \, d\mu = \int_D fv \, dx,$$

for every $v \in H_0^1(D) \cap L^2(D, \mu)$.

In this framework we can consider the sequence of problems

$$(RDP_h) \quad \begin{cases} -\Delta u_h + \mu_h u_h = f, & \text{in } D \\ u_h \in H_0^1(D), \end{cases}$$

where (μ_h) is a given sequence of measures in $\mathcal{M}_0(D)$. Given a sequence (E_h) as above, we can define a sequence (μ_h) of measures, denoted by (∞_{E_h}) (cf. Notation below), such that the problem (RDP_h) , for each $h \in \mathbb{N}$, reduces to (P_h) ; cf. [DM-M1].

Associated with (RDP) , there is an energy functional

$$F_\mu(v) = \int_D |\nabla v|^2 dx + \int_D v^2 d\mu - 2 \int_D f v dx,$$

for $v \in H_0^1(D) \cap L^2(D, \mu)$.

It is possible to prove (cf. [DM-M2, Theorem 2.4]) that the solution to (RDP) (exists, is unique and) may be characterized in variational term as the minimum point of the functional $F_\mu(\cdot)$ on $H_0^1(D)$. So we are led to consider the sequence of functionals (F_{μ_h}) , for $(\mu_h) \subset \mathcal{M}_0(D)$, and its Γ -limit: it is not difficult to prove, indeed, that the solutions of (RDP_h) converge to the solution of (RDP) , whenever (F_{μ_h}) Γ -converges to F_μ .

Following [DM-M1], we may introduce in the space $\mathcal{M}_0(D)$ a notion of convergence, called γ -convergence, in the following way: we say that a sequence (μ_h) γ -converges to a measure $\mu \in \mathcal{M}_0(D)$ if and only if the sequence of the corresponding functionals (F_{μ_h}) Γ -converges to F_μ . For $D \subset \mathbb{R}^d$ it turns out that the space $\mathcal{M}_0(D)$, equipped with the topology induced by the γ -convergence, is metrizable and sequentially compact; we refer to [DM-M1], [DM-M2] for more details.

With these motivations, we define on a Riemannian manifold M a similar space $\mathcal{M}_0(M)$; given a sequence $(\mu_h) \subset \mathcal{M}_0(M)$, we introduce the sequence of the energy functionals $F_{\mu_h} : L^2(M) \rightarrow [0, +\infty]$, defined by

$$F_{\mu_h}(v) = \begin{cases} \int_M [|\nabla v|_g^2 + v^2] dV_g + \int_M v^2 d\mu_h, & \text{if } v \in H_0^1(M) \\ +\infty, & \text{otherwise in } L^2(M), \end{cases}$$

and study the Γ -limit of this sequence. A general result in Γ -convergence says that any sequence of functionals defined on a separable, metric space, admits a subsequence which Γ -converges to a lower semicontinuous functional F . Considering

the sequence (F_{μ_h}) , we then know that there is a functional F which is the Γ -limit of (a sub-sequence of) (F_{μ_h}) . Hence our interest shifts to a possible representation of F on $H_0^1(M)$ as

$$(\Gamma) \quad F(v) = \int_M [|\nabla v|_g + v^2] dV_g + \int_M v^2 d\mu,$$

for a suitable measure $\mu \in \mathcal{M}_0(M)$.

In the second section of the first chapter we prove that there exists a measure $\mu \in \mathcal{M}_0(M)$ such that (Γ) holds. This integral representation is a consequence of an abstract result concerning the representation of bilinear forms defined on a Riesz space. To prove this abstract result, we used an extension to bilinear forms of the construction of the Daniell integral. We point out that this method of representing the limit functional was first used by G. Buttazzo, G. Dal Maso & U. Mosco in [Bu-DM-M1] in a slight different context.

Let E be a Borel set and, for a measure $\mu \in \mathcal{M}_0(M)$, let us introduce $\mu^E(\cdot) \stackrel{\text{def}}{=} \mu(E \cap \cdot)$, the restriction of the measure μ to E . In the third section of the first chapter we prove the continuity of this restriction operator under the γ -convergence, *i.e.* if (μ_h) is a sequence in $\mathcal{M}_0(M)$ which γ -converges to $\mu \in \mathcal{M}_0(M)$, then we have that the sequence of the restriction (μ_h^E) γ -converges to μ^E , not for every Borel set E , but for “sufficiently many” sets E , in a sense made precise in Proposition 3.9 & Theorem 3.10.

In the third section we also show that, under suitable assumptions, the measure μ occurring in $(\Gamma\text{-lim})$ is absolutely continuous w.r.t. a given Radon measure $\nu \in \mathcal{M}_0(M)$ (cf. Theorem 3.14). An ingredient in the proof is the fact that a sequence $(\mu_h) \subset \mathcal{M}_0(M)$ γ -converges to $\mu \in \mathcal{M}_0(M)$ if and only if the sequence of the corresponding μ -capacities, $\text{cap}_{\mu_h}(\cdot)$, (cf. Notation below) converges *weakly* (in a sense introduced by E. De Giorgi & G. Letta in [De G-L]) to $\text{cap}_\mu(\cdot)$; cf. Proposition 3.8 in this respect. We must notice that in the case $M = \mathbf{R}^d$, Theorem 3.14 has been proved by G. Buttazzo, G. Dal Maso & U. Mosco in [Bu-DM-M2] and the Proposition 3.8 has been proved by G. Dal Maso in [DM2].

In the fourth section we show that the γ -convergence of measures is equivalent to the strong convergence in $L^2(M)$ of the resolvent operators. This is sufficient to establish a convergence result for the spectrum.

The second chapter is devoted to illustrate what we could call “the derivation method”: it consists in an application of the Theorem 3.14 in the first chapter

plus a (sort of) super-additivity result for the harmonic capacity (cf. Lemma 4.2). It can be said that this method reduces the proof of the convergence of the problems (P_h) to the study of the asymptotic behavior of the harmonic capacity of the holes E_h . We use this method to give an explicit example that shows how to approximate the problem (RDP) by means of a suitable sequence of problems (P_h) ; in this framework we also mention the papers by A. Braides, G. Dal Maso & A. Malusa [Br-Ma], [DM-Ma].

More precisely, in this chapter we are given a self similar fractal K , as introduced by J.E. Hutchinson in [Hu], of (Hausdorff) dimension α and its associated Hausdorff measure restricted to K , $H_{|K}^\alpha$; we exploit the self similar structure of K in order to build up the “holes” E_h by an iterating procedure. We show that the sequence of the solutions (u_h) of (P_h) converges to the solution u of (RDP) with

$$(Frac) \quad \mu = c_1 H_{|K}^\alpha.$$

In the first section of this chapter we introduce the family of self similar fractals, and some notation we shall need. In the second section the construction of the sequence (E_h) is carried out, through an iterative procedure, starting from a “model set” E . The constant c_1 in (Frac) is explicitly given in term only of the measure $H^\alpha(K)$ and the (harmonic) capacity of E . We conclude this section, stating the main result (Theorem 3.1) and presenting a pictorial example. The third section is devoted to the proof of the main result; a crucial step is a (sort of) “superadditivity” result for the capacity of very fragmented sets: we remark that in general the capacity, as set function, is not superadditive.

It is interesting to note that in the “trivial” case of the Lebesgue measure (or of the $d - 1$ -dimensional Hausdorff measure restricted to an hyperplane) the usual homogenization technique can be seen as a particular case of this method. Our construction applies to fractals of dimension larger than $d - 2$: this limitation is natural, at least in our framework, since each Borel set E , having finite Hausdorff measure of dimension less than or equal to $d - 2$, has capacity zero (cf. [Z, Theorem 2.6.16]).

In the last chapter we are interested in the asymptotic behavior of the Dirichlet problems on domains in a Riemannian manifold M , with randomly distributed small holes. More precisely, given $f \in L^2(M)$, we consider the sequence of

problems

$$(R_h) \quad \begin{cases} -\Delta u_h + u_h = f, & \text{in } M \setminus F_h \\ u_h \in H_0^1(M \setminus F_h), \end{cases}$$

where (F_h) is a sequence of *random sets* contained in M (cf. Definition 3.4). Note that in this framework each solution u_h is a random variable. In particular, we are interested in finding conditions which assure that the sequence of solutions (u_h) of (R_h) converges in probability, as $h \rightarrow +\infty$, to the solution of the deterministic Relaxed Dirichlet Problem

$$(D) \quad \begin{cases} -\Delta u + u + \mu u = f, & \text{in } D \\ u \in H_0^1(D), \end{cases}$$

where μ is a “deterministic” measure of $\mathcal{M}_0(M)$, that can be explicitly computed in term of the distribution law of the sets F_h .

We take as “random holes” the following family of closed sets: for each $h \in \mathbb{N}$

$$F_h \stackrel{\text{def}}{=} \bigcup_{i \in I_h} B_{r_h}(x_h^i),$$

where $I_h = \{1, 2, \dots, h\}$, $r_h > 0$, $x_h^i \in M$, for every $i \in I_h$. The conditions we look for are given in terms of appropriate assumptions on the family of independent, identically distributed random variables $(x_h^i)_{i \in I_h}$, distributed, for each $h \in \mathbb{N}$, with law β , on the “size” of the sequence of the radii (r_h) , and on a suitable assumption on the measure $\beta \in \mathcal{M}_0(M)$ (cf. Assumptions 3.5). We use also a result proved by M. Balzano in [Ba1, Theorem 3.1] and which can be also applied in our Riemannian framework. We prove that under qualitative assumptions, that are weaker than in [Ba1] (even in \mathbf{R}^d , which was the framework in [Ba1]), we have still the convergence in probability of (u_h) to u .

We mention (besides the aforesaid paper of G. Papanicolaou & S.R.S. Varadhan [P-V]) also J.R. Baxter, R.V. Chacon, M. Kac, S. Ozawa in [B-Cha-J], [B-J], [K], [O6], [O7], where problems similar to (R_h) are studied, with different techniques (Brownian motion methods the first three papers, and by Green function method in the latter two). The fluctuations around the solution of the limit problem have been investigated *e.g.* by R. Figari, E. Orlandi & S. Teta in [Fi-Or-T].

In the first section of this chapter we introduce some notation; in the second we introduce (besides the meta-harmonic capacity) also the harmonic capacity, and we show some preliminary results on the harmonic capacity of concentric balls in Riemannian manifold. In the third section we introduce the problem, our assumptions, and prove our main result (Theorem 3.6). The method of our proof follows the lines of that in [Ba1]. We use in particular the fact that the measure ν , arising as the γ -limit of the sequence (μ_h) , can be characterized as the least superadditive set function greater than or equal to $\text{cap}_\nu(\cdot)$; moreover we need also to analyse the asymptotic behavior of the expectations and the covariances of certain random variables (cf. Theorem 3.3). Then the proof of the Theorem 3.6 follows essentially from a “superadditive argument” for the harmonic capacity.

This thesis is a collection of papers: more precisely, the first chapter is taken from [N], the second chapter is taken from a joint paper with A. Braides, [Br-N], and the third is also a joint paper with M. Balzano, [Ba-N].

1. Asymptotic Behavior of Dirichlet Problems on a Riemannian Manifold

Section 1. Preliminaries & main notation

This section is devoted to a series of definitions, notation and preliminary results, which will be used in the sequel. For the definition of a Riemannian manifold we refer *e.g.* to books [Au], [Ch], [G-H-L].

Let M be an *oriented, connected* Riemannian manifold of class C^3 , with dimension $\dim M \stackrel{\text{def}}{=} d \geq 2$; let g be its metric tensor. Associated to g there is the *Laplace-Beltrami operator* Δ , acting on real valued functions defined on M .

We recall that M is a metric space and the distance $d(x, y)$ between any two points $x, y \in M$ is given by the infimum of all piecewise C^1 curves joining x and y (see *e.g.* [G-H-La, Proposition 2.91]); the diameter of any set $E \subset M$ is denoted by $\text{diam } E \stackrel{\text{def}}{=} \sup\{d(x, y) : x, y \in E\}$. We say that E is bounded if $\text{diam } E < +\infty$. By $B_\rho(x)$ we denote the *open geodesic ball* of center $x \in M$ and radius ρ .

We derive, from our regularity assumption on M , the following property.

Property 1. The metric components $(g_{ij})_{i=1}^d$ belong to $C^2(M)$ and, for every relatively compact open set $A \subset M$, there exists $\kappa > 0$ such that:

$$(1.1) \quad \kappa^{-1} \sum_{i=1}^d (\xi^i)^2 \leq g_{ij}(x) \xi^j \xi^i \leq \kappa \sum_{i=1}^d (\xi^i)^2$$

for all $x \in A$, and for all $\xi \in \mathbf{R}^d$.

We shall use the Einstein convention over repeated indices, viz. $a_i b^i \stackrel{\text{def}}{=} \sum_i a_i b^i$.

By $\langle \cdot, \cdot \rangle_g$ we mean the scalar product in $T_x M$, the tangent space of M at x , and by $|\cdot|_g$ we mean the norm induced by this scalar product. If the subscript g is omitted, it is meant that the metric considered is the standard Euclidean metric in \mathbf{R}^d .

By $\mathcal{B}(M)$ we indicate the class of the Borel sets of M .

The Lebesgue integral of a measurable function $f : M \rightarrow \mathbf{R}$ can be expressed locally as

$$\int_M f \, dV_g = \int_{\phi(U)} (f \sqrt{\det(g)}) \circ \phi^{-1}(x) \, dx,$$

where (U, ϕ) is a local chart of M .

For more details about the Lebesgue integral on a Riemannian manifold, we refer to the book of Aubin [Au, pp.29–30]. The measure of a Borel set E of M will be denoted by $V(E)$; sometimes with a little abuse of language, we speak of $V(E)$ as the Lebesgue measure of E .

We define $L^2(M)$ as the Hilbert space of all (equivalence classes of) measurable functions $f : M \rightarrow \mathbf{R}$ for which the integral of f^2 is finite; its scalar product is

$$(f, h) \stackrel{\text{def}}{=} \int_M fh \, dV_g$$

and the associated norm is given by

$$\|f\| \stackrel{\text{def}}{=} \int_M f^2 \, dV_g.$$

From now on, the term *Borel* (resp. *Radon*) measure will always mean *positive Borel* (resp. *positive Radon*) measure. We say that a Borel measure μ is a Radon measure if $\mu(K) < \infty$ for every compact subset K of M . Given a Borel measure μ on M , we denote by $L^p(M, \mu)$ (resp. $L^p_{\text{loc}}(M, \mu)$) all (equivalence classes of) measurable functions defined on M whose p -th power is integrable (resp. locally integrable) on M , for $p \geq 1$.

Let X be a vector field; we say that X is *measurable* if $X \circ \eta^{-1} : \eta(U) \rightarrow \mathbf{R}$ is a measurable function for every local chart (U, η) of M . Following [Ch], let us define

$$\mathcal{L}^2(M) = \left\{ X : \int_M |X|_g^2 \, dV_g < +\infty, X \text{ Borel function on } M \right\}.$$

Given any two continuous vector fields X and Y , we define the inner product on $\mathcal{L}^2(M)$ by

$$(X, Y)_{\mathcal{L}^2(M)} = \int_M \langle X, Y \rangle_g \, dV_g.$$

Given a function $f \in L^2(M)$, we say that $Y \in \mathcal{L}^2(M)$ is a weak derivative of f if

$$(Y, X) = -(f, \text{div} X),$$

For all C^1 vector field X with compact support on M .

Since there exists at most one weak derivative Y , we write in analogy with the smooth case $Y = \nabla f$; in local coordinates $\nabla f = (g^{1j}D_j f, \dots, g^{dj}D_j f)$, where $D_i f = \frac{\partial f}{\partial x^i}$. Here and after g^{ij} will denote the inverse matrix of g_{ij} .

The Sobolev space $H^1(M)$ consists of those measurable functions $f \in L^2(M)$ possessing weak derivative.

We remark that $H^1(M)$ inherits the scalar product

$$(f, h)_{H^1(M)} = (f, h)_{L^2(M)} + (\nabla f, \nabla h)_{L^2(M)},$$

so that $H^1(M)$ is a Hilbert space.

With $H_0^1(M)$, we shall indicate the closure of $C_c^\infty(M)$ with respect to the norm induced by the scalar product in $H^1(M)$.

In a Riemannian manifold we can define in a natural way the following functional

$$(1.1) \quad \Psi_M(u) = \int_M [|\nabla u|_g^2 + u^2] dV_g, \text{ for } u \in H^1(M),$$

which, in local coordinates, becomes

$$\Psi_M(u) = \int_M (g^{ij}(x)D_i u D_j u + u^2) \sqrt{|g|} dx;$$

we shall use both the notation (cf. Remark 1.1 below), and write

$$\int_M (g^{ij}(x)D_i u D_j u + u^2) dV_g = \int_M [|\nabla u|_g^2 + u^2] dV_g.$$

We define the set function $c(A)$, called the meta-harmonic capacity of A , for $A \in \mathcal{B}(M)$, as

$$c(A) = \inf \{ \Psi_M(u) : u \in H^1(M), u \geq 1 \text{ in an open neighbourhood of } A \}.$$

1.1 Remark. The notion of capacity is intrinsic, that is, it does not depend on the choice of the coordinates. We say that a property $P(x)$ holds *quasi everywhere* (q.e.) if this property holds for all $x \in M$ except for a set Z , with $c(Z) = 0$.

Since each function in $H^1(\mathbf{R}^d)$ (has a representative which) is defined up to a set of capacity zero (see [Z]), this property continues to hold for the functions in $H^1(M)$, because of the Property 1.

1.2 Definition. We define $\mathcal{M}_0(M)$ as the family of all Borel measures on M that vanish on all sets of capacity zero.

1.3 Example. Let A and B two subsets of M , $B \subset \mathcal{B}(M)$. Define

$$\infty_A(B) = \begin{cases} 0, & \text{if } c(A \cap B) = 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then $\infty_A(\cdot)$ belongs to $\mathcal{M}_0(M)$.

1.4 Example. The volume form V_g belongs to $\mathcal{M}_0(M)$.

Now let us consider $\mu \in \mathcal{M}_0(M)$; since every function $u \in H^1(M)$ defined up to a set of capacity zero, the following functional

$$\Phi_M(u) = \int_M u^2 d\mu$$

is well defined on $H^1(M)$ for every $\mu \in \mathcal{M}_0(M)$ (possibly $\Phi_M(u) = +\infty$).

1.5 Notation. Besides Φ_M and Ψ_M , we shall introduce two functionals Φ and Ψ which take into account the boundary condition " $u = 0$ on ∂M " for both Ψ_M and Φ_M : we define $\Phi(u) = \Phi_M(u)$ and $\Psi(u) = \Psi_M(u)$ if $u \in H_0^1(M)$ while $\Phi(u) = \Psi_M(u) = +\infty$ otherwise in $L^2(M)$.

1.6 Definition. We say that a set $A \subset M$ is *quasi open* (resp. *quasi closed*, *quasi compact*) if for every $\varepsilon > 0$, there exists an open set (resp. closed, compact) U , such that

$$c(U \Delta A) < \varepsilon,$$

where Δ here denotes the symmetric difference between two sets. A set A is quasi open if and only if A^c is quasi closed, where A^c is the complementation w.r.t. M ; moreover countable union (or finite intersection) of quasi open sets is still quasi open.

A function $f : M \rightarrow \mathbf{R}$ is said to be *quasi continuous* in M if, for every $\varepsilon > 0$ there exists a set $E \subset M$, with $c(M \setminus E) < \varepsilon$, such that the restriction $f|_E : E \rightarrow \mathbf{R}$ is continuous.

Let us introduce the following set function, called μ -capacity: $\text{cap}_\mu : \mathcal{B}(M) \rightarrow [0, +\infty]$ defined by

$$(1.2) \quad \text{cap}_\mu(A) = \inf \left\{ \Psi_M(u) + \int_M (u-1)^2 d\mu : u \in H_0^1(M) \right\}.$$

It has been proved (see [DM-M1, Example 5.4]) that in general this set function is not a Choquet capacity (cf. [Cho, Definition 1.1]), since cap_μ may be not continuous along decreasing sequence of compact sets.

Let \mathcal{E} be a family of subsets contained in M ; following [De G-L] we say that \mathcal{E} is *dense* in $\mathcal{P}(M)$ (class of all subsets of M) if for every pair (K, V) , K compact, V open, $K \subset V$, there exists $E \in \mathcal{E}$, such that

$$K \subset E \subset V.$$

We say that \mathcal{E} is *rich* in $\mathcal{P}(M)$ if for every chain $(E_t)_{t \in T}$ in $\mathcal{P}(M)$, the set

$$\{t \in T : E_t \notin \mathcal{E}\}$$

is at most countable.

By *chain* we mean a family of subsets of M such that T is a non-empty open interval of \mathbf{R} , \overline{E}_t is compact for every $t \in T$ and $\overline{E}_s \subset \text{int}(E_t)$ for every $s < t$, $t, s \in T$

1.7 Proposition. *Every rich family is dense.*

Proof. For the proof, we refer to [DM2, Proposition 4.8]. This proof depends essentially on the Urysohn Lemma, which holds true in any normal topological space, such as a differentiable manifold. \square

1.8 Lemma. *Let $\alpha : \mathcal{P}(M) \rightarrow [0, +\infty]$ be an increasing function. Let $\mathcal{E}(\alpha)$ be the family of all subsets of M such that \overline{E} is compact in M and $\alpha(\text{int}E) = \alpha(\overline{E})$. Then $\mathcal{E}(\alpha)$ is rich in $\mathcal{P}(M)$.*

Proof. For the proof, we refer to [De G-L, Proposition 4.7]. \square

We now show that on a differentiable manifold it is possible to construct suitable coverings.

Let $(W_i)_{i \in I}$ be an open cover of a (paracompact) differential manifold M , $I \subset \mathbf{N}$. For every W_i there exists an open set V_i such that $V_i \subset\subset W_i$ and $(V_i)_{i \in I}$ is an open cover. Let $\phi_i \in C_c^\infty(W_i)$ such that $\phi_i = 1$ on V_i and $0 \leq \phi_i \leq 1$ on W_i ; define, for all $i \in I$ and for every $\rho \in (0, 1)$

$$U_i(\rho) = \{x \in W_i : \phi_i(x) > \rho\}.$$

We have that, for every $i \in I$,

$$V_i \subset\subset U_i(\rho) \subset\subset W_i;$$

hence the family $(U_i(\rho))_{i \in I}$ is an open cover of M .

1.9 Lemma. *Under the above assumptions, for every Borel measure σ there exists a family of pairwise disjoint open sets $\mathcal{U} = (U_i)_{i \in I}$ such that $U_i \subset\subset W_i$ for every $i \in I$,*

$$\bigcup_{i \in I} U_i = M \setminus \left[\bigcup_{i \in I} \partial U_i \right]$$

and $\sigma(\bigcup_{i \in I} \partial U_i) = 0$.

Proof. Let $U_i(\rho)$ as above; let us consider the function $f_i(\rho) = \sigma(U_i(\rho))$ for every $i \in I$. This function is positive and decreasing on $(0, 1)$, so it has, at most, a countable set of discontinuity points (ρ_h^i) , $i \in I$, $h \in \mathbf{N}$; let us assume that ρ is a continuity point for each f_i , $i \in I$ and consider

$$U_1 = U_1(\rho) \quad \text{and} \quad U_i = U_i(\rho) \setminus [\overline{U_1 \cup \dots \cup U_{i-1}}]$$

for every $i \geq 2$. Each U_i is open and $U_i \cap U_j = \emptyset$ for every $i \neq j$. Let us suppose that $x \in M \setminus \bigcup_{i \in I} \partial U_i$; since $(U_i(\rho))$ is an open cover, then $x \in U_k(\rho)$, for some $k \in I$. By definition of the U_i 's, this implies that $x \in U_j$ for one index $j \in I$; hence $x \in \bigcup_{i \in I} U_i$. The other inclusion is immediate.

Since ρ is a continuity point for each f_i , $i \in I$, we have that $\sigma(\partial U_i(\rho)) = 0$ for every $i \in I$. Now it is easy to prove the last assertion: in fact $\partial U_1 = \partial U_1(\rho)$ and for every $i \in I$

$$\partial U_{i+1} = \partial [U_{i+1}(\rho) \setminus [\overline{U_1 \cup \dots \cup U_i}]] \subset \partial U_{i+1}(\rho) \cup \partial U_i \dots \partial U_1.$$

This yields, by induction, that $\sigma(\partial U_i) = 0$, for every $i \in I$; hence $\sigma(\bigcup_{i \in I} \partial U_i) \leq \sum_{i \in I} \sigma(\partial U_i) = 0$. The proof is complete. \square

1.10 Lemma. *Under the assumptions of the above Lemma and for a finite I , for every $\varepsilon > 0$ and for every Borel measure σ , there exists an open cover $(U_i)_{i \in I}$ of M such that $U_i \subset\subset W_i$ and*

$$\sigma(U_i \cap U_j) < \varepsilon$$

for every $i, j \in I, i \neq j$.

Proof. Let $f_i(\rho) = \sigma(U_i(\rho)), i \in I$, as in the above lemma. Let us assume that $\bar{\rho}$ is a continuity point for each $f_i, i \in I$. For a given $\varepsilon_i > 0$ (which may depend on i), if ρ is sufficiently close to $\bar{\rho}, \rho < \bar{\rho}$, we have

$$\sigma(\partial U_i(\bar{\rho})) \leq \sigma(U_i(\rho) \setminus U_i(\bar{\rho})) = \sigma(U_i(\rho)) - \sigma(U_i(\bar{\rho})) < \varepsilon_i/2.$$

This implies, in particular, that $\sigma(\partial U_i(\bar{\rho})) = 0$ for every continuity point $\bar{\rho}$ for f_i , for every $i \in I$. Let $0 < \rho < \bar{\rho}$ and define

$$U_1 = U_1(\rho), \quad U_i = U_i(\rho) \setminus \left[\overline{U_1(\bar{\rho})} \cup \dots \cup \overline{U_{i-1}(\bar{\rho})} \right]$$

for every $i \geq 2$. It is easily seen that $(U_i)_{i \in I}$ is an open cover of M ; if $U_i \cap U_j \neq \emptyset$, then we have that

$$U_i \cap U_j \subset \left[U_i(\rho) \setminus \overline{U_i(\bar{\rho})} \right] \cup \left[U_j(\rho) \setminus \overline{U_j(\bar{\rho})} \right].$$

Given $\varepsilon > \max\{\varepsilon_i, \varepsilon_j\} > 0$, we have that $\sigma(U_i \cap U_j) < \varepsilon$ if we take ρ sufficiently close to $\bar{\rho}$, since $\bar{\rho}$ is a continuity point for both f_i and f_j . The proof is complete.

□

Section 2. A representation theorem

As we said in the Introduction, this section is devoted to the representation of the Γ -limit functional F ; the main result is the Theorem 2.3 which is stated in the first part of the section. The second part is devoted to a generalization of the Daniell's construction of the integral for the bilinear forms. In the third part, we prove Theorem 2.3 using the abstract result of the second part.

2.1 Definition. Let be X a metric space, $(F_h)_h$ a sequence of functionals $F_h : X \rightarrow [0, +\infty]$ and let $F : X \rightarrow [0, +\infty]$.

We say that the sequence $(F_h)_h$ Γ -converges to F in X if and only if the following conditions (a) and (b) hold true:

(a) for every sequence $(u_h)_h$ in X converging to some $u \in X$ as $h \rightarrow +\infty$, we have

$$F(u) \leq \liminf_{h \rightarrow +\infty} F_h(u_h);$$

(b) for every $u \in X$ there exists a sequence $(u_h)_h$ such that

$$F(u) \geq \limsup_{h \rightarrow +\infty} F_h(u_h).$$

The following compactness result holds (cf. [DG-F, Proposition 3.1]).

2.2 Proposition. *Assume that X is a separable metric space. For every sequence $(F_h)_h$ of functionals there exists a subsequence $(F_{h_k})_k$ which Γ -converges in X to a lower-semicontinuous functional as $k \rightarrow +\infty$.*

Let $(\mu_h)_h$ be a sequence of measures belonging to $\mathcal{M}_0(M)$ and for every $h \in \mathbb{N}$, let us consider the functional $F_h : L^2(M) \rightarrow [0, +\infty]$, defined by

$$F_h(u) = \begin{cases} \Psi(u) + \Phi_h(u), & \text{if } u \in H_0^1(M), \\ +\infty, & \text{otherwise in } L^2(M) \end{cases}$$

where

$$(2.0) \quad \Phi_h(u) = \int_M u^2 d\mu_h$$

and

$$\Psi(u) = \int_M [|\nabla u|_g^2 + u^2] dV_g.$$

We stress that F_h takes into account the boundary condition " $u = 0$ on ∂M ".

By the previous proposition, a subsequence of $(F_h)_h$ Γ -converges in $L^2(M)$ to a functional $F : L^2(M) \rightarrow [0, +\infty]$. The following result provides an integral representation of the limit functional F .

2.3 Theorem. *Suppose that $F_h \xrightarrow{\Gamma} F$ in $L^2(M)$ as $h \rightarrow +\infty$; then there exists a measure $\mu \in \mathcal{M}_0(M)$, such that*

$$F(u) = \int_M [|\nabla u|_g^2 + u^2] dV_g + \int_M u^2 d\mu, \quad \forall u \in H_0^1(M),$$

while $F(u) = +\infty$ if $u \notin H_0^1(M)$.

2.4 Remark. To prove the theorem we define the functional $\Phi : H_0^1(M) \rightarrow [0, +\infty]$ by

$$(2.1) \quad \Phi(u) = F(u) - \Psi(u);$$

we have to show that

$$\Phi(u) = \int_M u^2 d\mu \quad \forall u \in H_0^1(M)$$

for a suitable measure $\mu \in \mathcal{M}_0(M)$. This will be performed through various steps.

2.5 Definition. Before giving the properties of the functional Φ , we want to make precise what we mean by *extended valued quadratic functional* $F : X \rightarrow [0, +\infty]$, acting on any real vector space X . We say that F is a *quadratic functional* if it satisfies the following conditions

$$F(0) = 0, \quad F(u) \geq 0, \quad F(tu) = t^2 F(u), \quad \forall t \in \mathbf{R}$$

$$F(u + v) + F(u - v) = 2[F(u) + F(v)].$$

Since the functional F may admit the value $+\infty$, we shall follow hereafter the following (usual) convention about the algebraic properties of $\overline{\mathbf{R}} : 0 \cdot \infty = 0, +\infty + t = +\infty$ and $-\infty + t = -\infty$ for every $t \in \mathbf{R}$.

2.6 Remark. It is possible to prove that $F : X \rightarrow [0, +\infty]$ is a quadratic functional on X if and only if there exist a linear subspace Y of X and a bilinear form $B : Y \times Y \rightarrow \mathbf{R}$ such that

$$F(x) = \begin{cases} B(x, x) & \text{if } x \in Y \\ +\infty & \text{if } x \notin Y. \end{cases}$$

It is clear that in this case

$$Y = \{x \in X : F(x) < +\infty\}$$

and $B(x, y) = \frac{1}{2} [F(x + y) - F(x) - F(y)]$ (polarization identity), for every $x, y \in Y$. The proof of this fact is similar to the one given in [Y, Chapter I, p. 39].

2.7 Theorem. Let $u, v \in H_0^1(M)$ and let Φ be the functional defined by (2.1).

Then:

- (i) if $0 \leq u \leq v$ a.e. on M , then $\Phi(u) \leq \Phi(v)$;
- (ii) $\Phi(|u|) \leq \Phi(u)$;
- (iii) $\Phi(u + v) \leq \Phi(u) + \Phi(v)$, if $u \wedge v = 0$ a.e. on M ;
- (iv) $\Phi(u) = \lim_h \Phi(u_h)$, for every increasing sequence of positive functions $(u_h)_h$ such that $u_h \rightarrow u$ q.e. on M ;
- (v) $\Phi(\cdot)$ is a real extended quadratic functional.

Proof. (i) Let $(v_h)_h$ be a sequences in $H_0^1(M)$ converging to v , such that

$$\Psi(v) + \Phi(v) = \lim_{h \rightarrow +\infty} [\Psi(v_h) + \Phi_h(v_h)].$$

Since $v \geq 0$, it is not restrictive to take $v_h \geq 0$. It is easy to see that for every h

$$\Psi(u \wedge v_h) + \Psi(u \vee v_h) = \Psi(u) + \Psi(v_h)$$

$$\Phi_h(u \wedge v_h) \leq \Phi_h(v_h)$$

and $u \wedge v_h$ tends to u , while $u \vee v_h$ tends to v ; by Γ -convergence we have

$$\Psi(u) + \Phi(u) \leq \liminf_h [\Psi(u \wedge v_h) + \Phi_h(u \wedge v_h)],$$

and by the lower semicontinuity of Ψ

$$\Psi(v) \leq \liminf_h [\Psi(u \vee v_h)].$$

This two relations together give

$$\begin{aligned} \Psi(u) + \Psi(v) + \Phi(u) &\leq \liminf_h [\Psi(u \wedge v_h) + \Phi_h(u \wedge v_h)] + \\ &\quad + \liminf_h \Psi(u \vee v_h) \leq \\ &\leq \liminf_h [\Psi(u) + \Phi_h(v_h) + \Psi(v_h)] = \\ &= \Psi(u) + \Phi(v) + \Psi(v). \end{aligned}$$

This implies that $\Phi(u) \leq \Phi(v)$ and so (i) is proved.

(ii) Let u be a function in $H_0^1(M)$ and let $(u_h)_h$ be a sequence in $H_0^1(M)$ converging to u in $L^2(M)$, such that

$$\Psi(u) + \Phi(u) = \lim_h [\Psi(u_h) + \Phi_h(u_h)];$$

since $|u_h| \rightarrow |u|$ in $L^2(M)$, $\Phi_h(|u|) = \Phi_h(u)$ and $\Psi(|u|) = \Psi(u)$,

$$\begin{aligned} \Psi(|u|) + \Phi(|u|) &\leq \liminf_h [\Psi(|u_h|) + \Phi_h(|u_h|)] \leq \\ &\leq \liminf_h [\Psi(u_h) + \Phi_h(u_h)] = \\ &= \Psi(u) + \Phi(u) < +\infty. \end{aligned}$$

This gives $\Phi(|u|) \leq \Phi(u)$, since $\Psi(|u|) = \Psi(u)$ for every $u \in H_0^1(M)$.

(iii) By definition of Γ -convergence, there exist two sequences $(u_h)_h$ and $(v_h)_h$ of non-negative functions converging in $L^2(M)$ respectively to u and v , such that

$$\Psi(u) + \Phi(u) = \lim_h [\Psi(u_h) + \Phi_h(u_h)]$$

$$\Psi(v) + \Phi(v) = \lim_h [\Psi(v_h) + \Phi_h(v_h)].$$

Since $u_h \vee v_h$ converges in $L^2(M)$ to $u \vee v = u + v$ as $h \rightarrow +\infty$, we have

$$\begin{aligned} \Psi(u + v) + \Phi(u + v) &\leq \liminf_h [\Psi(u_h \vee v_h) + \Phi_h(u_h \vee v_h)] \leq \\ &\leq \liminf_h [\Psi(u_h) + \Psi(v_h) + \Phi_h(u_h) + \Phi_h(v_h)] = \\ &= \Psi(u) + \Psi(v) + \Phi(u) + \Phi(v). \end{aligned}$$

Since $\Psi(u + v) = \Psi(u) + \Psi(v)$, and $u \wedge v = 0$, we have proved that $\Phi(u + v) \leq \Phi(u) + \Phi(v)$.

(iv) As $u_h \geq 0$, it is possible to prove that there exists a sequence $(v_h)_h$ in $H_0^1(M)$ such that $0 \leq v_h \leq u_h$ a.e. and v_h converges to u strongly in $H_0^1(M)$. In the case $M = \mathbf{R}^d$ the proof is given in [DM1, Lemma 1.6]; the general case can be obtained by a partition of unity subordinate to a given atlas. Since the functional Φ , by definition, is the difference between the limit functional F , which is lower semicontinuous, and the functional Ψ , which is continuous w.r.t. the strong topology of $H_0^1(M)$, we get that Φ is lower semicontinuous on the strong topology of $H_0^1(M)$, hence we have, using also (i),

$$\Phi(u) \leq \liminf_h \Phi(v_h) \leq \liminf_h \Phi(u_h).$$

On the other hand, we have from (i) $\Phi(u_h) \leq \Phi(u)$ for every h , hence $\limsup_h \Phi(u_h) \leq \Phi(u)$, and the conclusion follows.

(v) From Proposition V in [S] we have that for every $u, v \in H_0^1(M)$, $t \in \mathbf{R}$, the limit functional F is quadratic, according to the Definition 2.5; hence we get that the functional Φ itself satisfies the same conditions above, since the functional Ψ is quadratic. The proof of Theorem 2.4 is now complete. \square

In the remainder of this section, we give an integral representation of the limit functional Φ , occurring in the Theorem 2.3, by means of a measure μ belonging to the class $\mathcal{M}_0(M)$. The methods we use do not depend mostly on the “concrete” spaces which enter in Theorem 2.2, rather on the structure of such spaces. So we suppose that a *real valued* quadratic functional G is given on a Riesz space \mathcal{L} and we assume that G satisfies the Hypotheses 2.10 below. We consider also the bilinear form β associated to this functional; by means of a Daniell’s type extension result adapted to our situation, the bilinear form β is extended to $\widehat{\mathcal{L}} \times \widehat{\mathcal{L}}$, where $\widehat{\mathcal{L}}$ is the monotone class generated by \mathcal{L} , and the measure μ is constructed by means of this extension. At this point, we turn to our “concrete” functional Φ and we give the required representation.

2.8 Definition. Let \mathcal{L} be a (real) vector space of (real) functions defined on an arbitrary set Ω . We say that \mathcal{L} is a *Riesz space* if $|f| \in \mathcal{L}$ whenever $f \in \mathcal{L}$.

If \mathcal{L} is a Riesz space, we define $\mathcal{L}^+ = \{f \in \mathcal{L} : f \geq 0\}$, where the order is defined pointwise.

2.9 Remark. Since $f^+ = |f| - f$, we have that \mathcal{L} is a Riesz space if and only if it contains f^+ (or f^-) for any $f \in \mathcal{L}$. This implies that a Riesz space is closed under the operations \vee and \wedge , defined by $(f \wedge g)(x) = \min\{f(x), g(x)\}$ and $(f \vee g)(x) = \max\{f(x), g(x)\}$. Therefore $f \in \mathcal{L}^+$ if and only if $f^- = 0$.

2.10 Hypotheses. Let $G : \mathcal{L} \rightarrow [0, +\infty[$ be a *quadratic* functional, according to Definition 2.4, which satisfies the following properties:

- (i) if $u, v \in \mathcal{L}$ and $0 \leq u \leq v$, then $G(u) \leq G(v)$;
- (ii) $G(|u|) \leq G(u)$;
- (iii) $G(u + v) \leq G(u) + G(v)$, if $u, v \in \mathcal{L}$ such that $u \wedge v = 0$;
- (iv) $G(u) = \lim_h G(u_h)$, for every increasing sequence in \mathcal{L}^+ of positive functions $(u_h)_h$ converging pointwise to u .
- (v) $G(\cdot)$ is a real quadratic functional with *finite values*.

2.11 Definition. A monotone class \mathcal{S} on a set Ω is a class of real valued functions defined on Ω such that:

- (i) if $(u_h)_h$ is an increasing sequence in \mathcal{S} having a majorant in \mathcal{S} , then $u = \sup_h u_h \in \mathcal{S}$;
- (ii) if $(u_h)_h$ is a decreasing sequence having a minorant in \mathcal{S} , then $u = \inf_h u_h \in \mathcal{S}$.

Let \mathcal{L} be a Riesz space; the monotone class generated by \mathcal{L} (i.e. the smallest monotone class generated by \mathcal{L}) is still a Riesz space which will be denoted by $\widehat{\mathcal{L}}$. Let us define for $f, g \in \mathcal{L}$

$$\beta(f, g) = \frac{1}{2}[G(f + g) - G(f) - G(g)].$$

From Remark 2.5 it follows that β is a bilinear form. Since $\beta(f, f) = G(f)$, β is the bilinear form associated to the quadratic functional G . Observe that β is *symmetric*.

2.12 Definition. Let \mathcal{L} be a Riesz space. We say that a bilinear form β defined on $\mathcal{L} \times \mathcal{L}$ is :

- *positive* if

$$\beta(u, v) \geq 0, \quad \text{for every } u, v \geq 0, u, v \in \mathcal{L};$$

- *local* if, given $u, v \in \mathcal{L}$ with $|u| \wedge |v| = 0$, we have $\beta(u, v) = 0$;

- *continuous on monotone sequences* if, given $u, v \in \mathcal{L}$, we have

$$\beta(u, v) = \lim_{h \rightarrow +\infty} \beta(u, v_h),$$

where $(v_h)_h$ is an increasing sequence in \mathcal{L} such that $v = \sup_{h \in \mathbb{N}} v_h$.

The following three propositions show that the functional G , satisfying Hypotheses 2.10, fulfills the three properties listed in Definition 2.12.

2.13 Proposition. *Assume Hypotheses 2.10 and let us suppose that $u, v \geq 0$; then $\beta(u, v) \geq 0$.*

Proof. Since $\beta(\cdot, \cdot)$ is the bilinear form associated to G , we have

$$\beta(u, v) = \lim_{t \rightarrow 0^+} \frac{G(u + tv) - G(u)}{2t},$$

since $u + tv \geq u$; then the proof follows from the property (i) of Hypotheses 2.10.

□

2.14 Proposition. *Assume Hypotheses 2.10; if $f, g \in \mathcal{L}$ such that $|f| \wedge |g| = 0$, then $\beta(f, g) = 0$.*

Proof. It is not restrictive to assume that both f, g are non-negative; as β is positive, we have the inequality $\beta(f, g) \geq 0$. To prove the opposite inequality, by definition of β (Remark 2.6), we have to prove that

$$G(u + v) \leq G(u) + G(v);$$

this follows from the property (iii) of Hypotheses 2.10. □

2.15 Proposition. *Assume Hypotheses 2.10, let $f, g \in \mathcal{L}$, $f \geq 0$ and let $(g_h)_h$ be an increasing sequence in \mathcal{L} such that $g = \sup_{h \in \mathbb{N}} g_h$. Then*

$$\beta(f, g) = \lim_{h \rightarrow +\infty} \beta(f, g_h).$$

Proof. It is not restrictive to assume $g \geq 0$ and $g_h \geq 0$. By the Schwarz's inequality

$$|\beta(f, g_h - g)|^2 \leq \beta(f, f)\beta(g_h - g, g_h - g) = G(f)G(g_h - g).$$

From Hypotheses 2.10-(v), we have

$$G(g_h - g) = 2G(g) + 2G(g_h) - G(g_h + g);$$

taking Hypotheses 2.10-(iv) into account, we have

$$\lim_{h \rightarrow +\infty} G(g_h - g) = 4G(g) - G(2g) = 0$$

and the conclusion follows. \square

Definition 2.16—Condition D_1 . We say that a functional $I : \mathcal{L} \rightarrow \mathbf{R}$ satisfies the conditions (D_1) if

- (1) I is linear, i.e. $I(af + bg) = aI(f) + bI(g)$, $\forall a, b \in \mathbf{R}$ and $\forall f, g \in \mathcal{L}$;
- (2) I is increasing, i.e. $I(f) \geq 0$, for every $f \in \mathcal{L}^+$;
- (3) I is continuous on monotone sequences, i.e. if $f \in \mathcal{L}^+$, $f_h \searrow 0$ as $h \rightarrow +\infty$, then $\lim_h I(f_h) = 0$.

A functional I satisfying (D_1) is called a *Daniell integral*.

Definition 2.17—Condition D_2 . We say that a functional $\beta : \mathcal{L} \times \mathcal{L} \rightarrow \mathbf{R}$ satisfies the Daniell's conditions (D_2) if for every fixed \bar{u}, \bar{v} contained in \mathcal{L}^+ , the functionals $\beta(\bar{u}, \cdot)$ and $\beta(\cdot, \bar{v})$ satisfy conditions (D_1), that is

- (i) $\beta(\bar{u}, \cdot)$ and $\beta(\cdot, \bar{v})$ are linear, i.e. $\beta(\bar{u}, av_1 + bv_2) = a\beta(\bar{u}, v_1) + b\beta(\bar{u}, v_2)$ and $\beta(au_1 + bu_2, \bar{v}) = a\beta(u_1, \bar{v}) + b\beta(u_2, \bar{v})$, for every $a, b \in \mathbf{R}$ and for u_1, u_2, v_1, v_2 in \mathcal{L} ;
- (ii) $\beta(\bar{u}, \cdot)$ and $\beta(\cdot, \bar{v})$ are increasing, i.e. $\beta(\bar{u}, v) \geq 0$ and $\beta(u, \bar{v}) \geq 0$, for every u, v contained in \mathcal{L}^+ ;
- (iii) $\beta(\bar{u}, \cdot)$ and $\beta(\cdot, \bar{v})$ are continuous on monotone sequences, i.e. if $(u_h), (v_h)$ are two sequences contained in \mathcal{L} with $u_h \searrow 0$ and $v_h \searrow 0$ as h goes to ∞ then $\lim_{h \rightarrow +\infty} \beta(\bar{u}, v_h) = 0$ and $\lim_{h \rightarrow +\infty} \beta(u_h, \bar{v}) = 0$.

The following result is classical in the theory of Daniell's integral (see e.g. [C-W-S, Chapter III]).

2.18 Proposition. Let \mathcal{L} be a Riesz space and let $\widehat{\mathcal{L}}$ be the monotone class generated by \mathcal{L} . Let I_0 be a linear form satisfying the Daniell's condition (D_1) above; then there exists an unique positive linear form

$$I : \widehat{\mathcal{L}} \rightarrow \mathbf{R}$$

still satisfying the conditions (D_1) , such that $I = I_0$ on \mathcal{L} .

It is possible to prove a similar extension result for bilinear form.

2.19 Theorem. *Suppose that β is a bilinear form satisfying the conditions (D_2) above; then there exists a unique bilinear form*

$$\widehat{\beta} : \widehat{\mathcal{L}} \times \widehat{\mathcal{L}} \longrightarrow \mathbf{R}$$

which still satisfies (D_2) and extends β .

Proof. For every $v \in \mathcal{L}^+$, $\beta(\cdot, v)$ may be extended to a form $\widetilde{\beta}(\cdot, v)$, defined on $\widehat{\mathcal{L}}$, which still satisfies (D_1) . Set for every $v \in \mathcal{L}$, $v = v^+ - v^-$,

$$\widetilde{\beta}(\cdot, v) = \widetilde{\beta}(\cdot, v^+) - \widetilde{\beta}(\cdot, v^-).$$

Hence we have

$$\widetilde{\beta} : \widehat{\mathcal{L}} \times \mathcal{L} \longrightarrow \mathbf{R}.$$

For every $u \in \widehat{\mathcal{L}}$, $u \geq 0$, $\widetilde{\beta}(u, \cdot)$ still continues to satisfy (1) and (2) of (D_1) . In order to prove condition (3), we first remark that, as a consequence of the definition of the monotone class, each element of $\widehat{\mathcal{L}}$ is between two elements of \mathcal{L} ; with this remark, it is easy to realize that the continuity of $\widetilde{\beta}(u, \cdot)$ holds true.

Therefore the form $\widetilde{\beta}(u, \cdot)$ satisfies (D_1) for every $u \in \widehat{\mathcal{L}}^+$; by Theorem 2.18 there exists a unique form $\widehat{\beta}(u, \cdot)$ defined on $\widehat{\mathcal{L}}$ that extends $\widetilde{\beta}(u, \cdot)$ and such that $\widehat{\beta}(u, \cdot)$ satisfies (D_1) on $\widehat{\mathcal{L}}^+$. For every $u \in \widehat{\mathcal{L}}$ we define

$$\widehat{\beta}(u, \cdot) = \widehat{\beta}(u^+, \cdot) - \widehat{\beta}(u^-, \cdot).$$

As before, it is possible to prove that $\widehat{\beta}$ still satisfies the Daniell's conditions (D_1) w.r.t. the first variable. Therefore $\widehat{\beta}$ satisfies (D_2) . Let us prove the uniqueness of the bilinear form $\widehat{\beta}$. Suppose that there exist two extensions of bilinear form β , let us call them β_1 and β_2 , both satisfying the Daniell's conditions (D_2) ; let us consider for a fixed $u \in \mathcal{L}$,

$$\mathcal{A} = \left\{ v \in \widehat{\mathcal{L}} : \beta_1(u, v) = \beta_2(u, v) \right\}.$$

Since $\beta_1|_{\mathcal{L} \times \mathcal{L}} = \beta_2|_{\mathcal{L} \times \mathcal{L}} = \beta$, \mathcal{A} contains \mathcal{L} ; moreover \mathcal{A} is a monotone class, since β_1 and β_2 both satisfy conditions (D_2) . So \mathcal{A} is a monotone class containing

\mathcal{L} and contained in $\widehat{\mathcal{L}}$, hence $\mathcal{A} = \widehat{\mathcal{L}}$, by the minimality of $\widehat{\mathcal{L}}$. This shows that $\beta_1(u, v) = \beta_2(u, v)$ for every $u \in \mathcal{L}$ and for every $v \in \widehat{\mathcal{L}}$.

Now fix $v \in \widehat{\mathcal{L}}$ and consider

$$\mathcal{B} = \left\{ u \in \widehat{\mathcal{L}} : \beta_1(u, v) = \beta_2(u, v) \right\}.$$

Arguing as in the previous step, it may be proved that \mathcal{B} coincides with $\widehat{\mathcal{L}}$ and $\beta_1(u, v) = \beta_2(u, v)$ for every $u \in \widehat{\mathcal{L}}$ and for every $v \in \mathcal{L}$.

This concludes the proof. \square

2.20 Remark. We stress that if β is *local* or *symmetric*, then the same property holds for $\widehat{\beta}$.

2.21 Remark. We recall that a family \mathcal{E} of subsets of Ω is called δ -ring, if the following three conditions are satisfied:

- (1) $\emptyset \in \mathcal{E}$;
- (2) if A and B are in \mathcal{E} , then $A \cup B$ and $A \setminus B$ belongs to \mathcal{E} ;
- (3) if (E_h) is a sequence of elements of \mathcal{E} , then $\bigcap_{h \in \mathbb{N}} E_h$ is in \mathcal{E} .

We say that a family \mathcal{F} of subsets of Ω is called a σ -ring if it is a δ -ring and

- (4) if $(F_h)_h$ is a sequence of elements in \mathcal{F} , then $\bigcup_h F_h$ is in \mathcal{F} .

If \mathcal{E} is a δ -ring, we indicate by \mathcal{E}_σ the σ -ring generated by \mathcal{E} ; it may be proved that \mathcal{E}_σ consists of all unions of increasing sequences from \mathcal{E} (cf. e.g. [C-W-S, Proposition 2.1.11]).

A set function μ defined on a δ -ring \mathcal{E} is called *measure* on \mathcal{E} if the following conditions are satisfied:

- (5) $\mu(\bigcup_h A_h) = \sum_h \mu(A_h)$, for any finite or countable family of disjoint elements of \mathcal{E} whose union is in \mathcal{E} ;
- (6) $\mu(A) \geq 0$, for every $A \in \mathcal{E}$.

2.22 Theorem. Let $\widehat{\mathcal{L}}$ be the monotone class generated by \mathcal{L} and let $\beta : \mathcal{L} \times \mathcal{L} \rightarrow \mathbf{R}$ be a bilinear form which is local, positive, symmetric and continuous on monotone sequences; let us suppose that $\widehat{\mathcal{L}}$ satisfies the Stone condition:

$$f \in \widehat{\mathcal{L}} \Rightarrow f \wedge 1 \in \widehat{\mathcal{L}}.$$

Denote by \mathcal{E} the class $\mathcal{E} = \{E \subset \Omega : 1_E \in \widehat{\mathcal{L}}\}$ and by μ the set function $\mu : \mathcal{E} \rightarrow [0, +\infty]$ defined by $\mu(E) = \widehat{\beta}(1_E, 1_E)$, where $\widehat{\beta}$ is the extension of β given by Theorem 2.19. Then \mathcal{E} is a δ -ring, μ is a measure on \mathcal{E} , $\widehat{\mathcal{L}}$ is a subset of $L^2(\Omega, \mathcal{E}, \mu)$ and

$$(2.2) \quad \widehat{\beta}(f, g) = \int_{\Omega} fg d\mu,$$

for every $f, g \in \widehat{\mathcal{L}}$.

Proof. The proof of the theorem is achieved through six steps.

Step 1. $u \in \widehat{\mathcal{L}}, t > 0 \Rightarrow \{u > t\} \subset \mathcal{E}$. In fact for every $t > 0$ we have

$$h [(u - t)^+] \wedge 1 = [h(u - u \wedge t) \wedge 1] \in \widehat{\mathcal{L}}$$

by the Stone condition. Since $1_{\{u > t\}}$ is the limit of the increasing sequence $h [(u - t)^+] \wedge 1$ which is bounded from the above by u/t , we get $1_{\{u > t\}} \in \widehat{\mathcal{L}}$.

Step 2. $E \in \mathcal{E}, u \in \widehat{\mathcal{L}} \Rightarrow u1_E \in \widehat{\mathcal{L}}$. It is not restrictive to assume $u \geq 0$; in this case, $u1_E$ is the limit of the increasing sequence $(u \wedge h1_E)$. Since $u \wedge h1_E \in \widehat{\mathcal{L}}$, which is bounded from the above by u for every h , we get $u1_E \in \widehat{\mathcal{L}}$.

Step 3. $E, F \in \mathcal{E}, E \subset F$ implies

$$(2.3) \quad \widehat{\beta}(1_E, 1_F) = \widehat{\beta}(1_E, 1_E) = \mu(E)$$

This is a consequence of the local property of $\widehat{\beta}$, see Proposition 2.14 and Remark 2.20. Moreover this property implies also

$$\widehat{\beta}(1_E, 1_{F \setminus E}) = 0, \quad \forall E, F \in \mathcal{E}.$$

Step 4. μ is a measure on the δ -ring \mathcal{E} . \mathcal{E} is a δ -ring because $\widehat{\mathcal{L}}$ is a monotone Riesz space. The fact that μ is a measure, follows from conditions (D_2) and from the local property of β : in fact, taking Step 3 into account, we have that

- the *finite additivity* of μ comes from the linearity of β ;
- the *countable additivity* follows from the continuity of β along monotone sequences.

Step 5. Let $u \in \widehat{\mathcal{L}}, E \in \mathcal{E}$; then

$$\widehat{\beta}(u, 1_E) = \int_{\Omega} u1_E d\mu = \int_E u d\mu.$$

Assume, for a moment, that u is a positive step function

$$u = \sum_{i=1}^n a_i 1_{A_i};$$

then, by Step 3 and by the local property of $\widehat{\beta}$, we can also suppose that $A_i \subset E$. By the linearity,

$$\widehat{\beta}(u, 1_E) = \sum_{i=1}^n a_i \beta(1_{A_i}, 1_E) = \sum_{i=1}^n a_i \mu(A_i) = \sum_{i=1}^n a_i \mu(A_i \cap E).$$

If $u \in \widehat{\mathcal{L}}$, $u \geq 0$, there exists a sequence of step function $\phi_h \in \widehat{\mathcal{L}}$ (cf. Step 1) such that $0 \leq \phi_n \nearrow u$; so the conclusion follows from the continuity of β along monotone sequences.

Step 6. If $u, v \in \widehat{\mathcal{L}}$, then

$$\widehat{\beta}(u, v) = \int_{\Omega} u v d\mu.$$

We may assume that both $u, v \geq 0$. The equality above is true for every v which is a step function. The proof is then achieved by a monotone approximation argument, as above. \square

Now we take up again to our concrete functional Φ defined by (2.1). In order to apply the abstract part of this section, we introduce H_{Φ} as the class of all quasi-continuous Borel functions $f : M \rightarrow \mathbf{R}$ for which there exists a function $u \in H_0^1(M)$ such that $u = f$ q.e. on M and $\Phi(u) < +\infty$. Since every function $u \in H_0^1(M)$ has a quasi-continuous representative which is unique up to a set of capacity zero (see [Z]), we can define G on H_{Φ} by setting $G(f) = \Phi(u)$.

2.23 Remark. We stress that $G(f) = G(g)$ if $f, g \in H_{\Phi}$ and $f = g$ up to a set of capacity zero.

2.24 Remark. From Theorem 2.7-(v), we have that H_{Φ} is a vector space of real functions defined on M . With this definition, G is a *finite real valued* quadratic functional on H_{Φ} .

2.25 Remark. From Theorem 2.7-(ii), we find that H_Φ is a Riesz space.

We define the bilinear form

$$(2.3) \quad B(f, g) = \frac{1}{2}[G(f + g) - G(f) - G(g)]$$

which results to be *local, positive and continuous on monotone sequences*, as it has been proved in the Propositions 2.13, 2.14, 2.15 above. We apply Theorem 2.19, which assures the existence of the extension \widehat{B} , enjoying of the same properties of B on the monotone class $\widehat{H_\Phi}$ generated by H_Φ . Using Theorem 2.22 we represent \widehat{B} by means of a measure μ . This measure gives, in turn, the required representation of the limit functional Φ , as we prove in the following proposition.

2.26 Proposition. *Let $\Phi : H_0^1(M) \longrightarrow [0, +\infty]$ be a functional satisfying conditions (i), ..., (v) of Theorem 2.7. Then there exists a measure $\mu \in \mathcal{M}_0$ such that*

$$(2.4) \quad \Phi(u) = \int_M u^2 d\mu$$

for every $u \in H_0^1(M)$.

Proof. Let \mathcal{E}_σ be the σ -ring generated by \mathcal{E}_Φ . The measure μ of Theorem 2.22 can be extended to a measure defined on the Borel σ -field on M , which we still denote by μ , such that $\mu(A) = +\infty$ whenever A is not in \mathcal{E}_σ ; from Remark 2.28 below, it follows that μ is in \mathcal{M}_0 .

From Theorem 2.19 and from the formula $G(f) = B(f, f)$, we have that

$$(2.5) \quad G(f) = \int_M f^2 d\mu$$

for every $f \in H_\Phi$. By definition of G , this implies that (2.4) holds for any $u \in H_0^1(M)$ with $\Phi(u) < +\infty$.

To complete the proof let u be a function in $H_0^1(M)$ with $\Phi(u) = +\infty$ and, in order to get a contradiction, let us suppose that

$$\int_M u^2 d\mu < +\infty.$$

For every $\varepsilon > 0$

$$\mu(\{|u| > \varepsilon\}) \leq \varepsilon^{-2} \int_M u^2 d\mu < +\infty,$$

so $A = \{|u| > \varepsilon\} \in \mathcal{E}_\sigma$; by a monotone class argument, there exists an increasing sequence $(g_h)_h$ in H_Φ , such that $g_h \rightarrow +\infty$ on A and let

$$f_h = g_h \wedge (|u| - \varepsilon)^+;$$

then $f_h \in H_\Phi$ and $\lim_h f_h = (|u| - \varepsilon)^+$, so that (cf. Theorem 2.7-(iv))

$$\Phi(|u| - \varepsilon) = \lim_h \Phi(f_h) = \lim_h \int_M f_h^2 d\mu = \int_M (|u| - \varepsilon)^+{}^2 d\mu.$$

Since $(|u| - \varepsilon)^+$, as ε goes to zero, converges to $|u|$, by Theorem 2.7-(iv) we have

$$\Phi(|u|) = \int_M |u|^2 d\mu < +\infty.$$

So by theorem 2.7-(i) and (v)

$$\Phi(u) \leq 2[\Phi(u^+) + \Phi(u^-)] \leq 4\Phi(|u|) < +\infty$$

and we get the contradiction. \square

2.27 Remark. From Remark 2.6 it is clear that $F(u) = +\infty$ if $u \notin H_0^1(M)$, where F is the Γ -limit functional in Theorem 2.3.

Proof of Theorem 2.3. The functional Φ defined by (2.1) satisfies the conditions (i), ..., (v) of Theorem 2.7, so it is enough to apply Proposition 2.26 and take into account the above remark. The proof of Theorem 2.3 is then complete. \square

2.28 Remark. The measure μ is absolutely continuous with respect to capacity, that is $\mu(E) = 0$ if E is a Borel set of capacity zero. In fact, by definition, $E \in \mathcal{E}_\Phi$ if and only if $1_E \in \widehat{H}_\Phi$; if E has capacity zero, then 1_E coincides q.e. with the function identically zero. Hence $1_E \in \widehat{H}_\Phi$ and (cf. Remark 2.23)

$$\mu(E) = B(1_E, 1_E) = G(1_E) = G(0) = 0.$$

2.29 Remark. It may happen that the δ -ring \mathcal{E}_Φ is not closed under the complement operation, hence in general A^c does not belong to \mathcal{E}_Φ if $A \in \mathcal{E}_\Phi$, as the following example shows.

2.29 Example. Let us consider Ω a bounded open set of \mathbf{R}^d and let C be a closed subset of Ω ; let

$$\mu = \infty_C.$$

In this situation H_Φ consists of $H_0^1(\Omega)$ functions that are zero on C , up to a set of capacity zero, and this property still continues to hold for functions in \widehat{H}_Φ . By definition, $E \in \mathcal{E}$ if and only if $1_E \in \widehat{H}_\Phi$, i.e. $1_E = 0$ q.e. on C ; so the property to be zero on C is true if $E \subset \Omega \setminus C$, then there is no hope to have $E^c \in \mathcal{E}$.

Section 3. Continuity of the restriction operator

In this section we face the problem of the continuity of the restriction operator μ^E w.r.t. the γ -convergence. This kind of result was firstly tackled in the euclidean case in [DM2].

We introduce also the notion of *image measure* (see [Dell-Me] for more details) which allows us to extend a previous result of [Bu-DM-M2, Theorem 5.2] to Riemannian manifolds (Theorem 3.14). Moreover the equivalence stated in Proposition 3.8, which concerns the equivalence between the γ -convergence of (μ_h) to μ and the convergence of the corresponding μ -capacities, is essential in proving Theorem 3.14, as it was in the Euclidean case. For the definition of the μ -capacity, we refer to Section 1 in this chapter.

For every open submanifold D in M and for every $\nu \in \mathcal{M}_0$, we shall denote by $F_\nu^D : L^2(D) \rightarrow [0, +\infty]$ the functional defined by

$$(3.0) \quad F_\nu^D(u) = \begin{cases} \int_D [|\nabla u|^2 + u^2] dV_g + \int_D u^2 d\nu, & \text{if } u \in H_0^1(D), \\ +\infty, & \text{otherwise in } L^2(D). \end{cases}$$

Note that $u = 0$ on ∂D .

For ease of notation, when $D = M$, we shall set $F_\nu^M = F_\nu$.

3.1 Definition. With the notations of section 2, we say that the sequence $(\mu_h)_h$ of measures in $\mathcal{M}_0(M)$ γ -converges to $\mu \in \mathcal{M}_0(M)$ if the sequence of the corresponding functionals $(F_{\mu_h})_h$ Γ -converges in $L^2(M)$ to the functional F_μ .

We have the following result

3.2 Proposition. *Let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(M)$ and let $\mu \in \mathcal{M}_0(M)$. The following are equivalent:*

- (a) $(\mu_h)_h$ γ -converges to μ ;
- (b) $(F_{\mu_h}^D)_h$ Γ -converges to F_μ^D in $L^2(D)$, for every open submanifold D in M .

3.3 Remark. For every open submanifold D in M , let us define

$$(3.1) \quad F_+^D(u) = \inf \left\{ \limsup_h F_{\mu_h}^D(u_h) : u_h \rightarrow u \text{ in } L^2(D) \right\},$$

and

$$(3.2) \quad F_-^D(u) = \inf \left\{ \liminf_h F_{\mu_h}^D(u_h) : u_h \rightarrow u \text{ in } L^2(D) \right\}.$$

If $D = M$ we denote the corresponding functionals by F_+ and F_- . By definition we have $F_+^D(u) \geq F_-^D(u)$ for every $u \in L^2(D)$. By a diagonal argument, it is easy to see that the infima in (3.1) and (3.2) are achieved by suitable sequences; moreover F_+^D and F_-^D are lower semicontinuous on $L^2(D)$ [DeG-F, Proposition 1.8].

It is easy to realize that F_μ^D is the Γ -limit in $L^2(D)$ of $F_{\mu_h}^D$ if and only if $F_\mu^D = F_+^D = F_-^D$ on $L^2(D)$; therefore the inequalities

$$F_+^D(u) \leq F_\mu^D(u) \leq F_-^D(u) \quad \text{on } H_0^1(D),$$

are equivalent to the Γ -convergence in $L^2(D)$ of $F_{\mu_h}^D$ to F_μ^D .

3.4 Remark. Let D be an open submanifold of M . If $u \in H_0^1(D)$, we can extend it to the whole manifold M by putting $u = 0$ outside D ; so we get $u \in H_0^1(M)$ and this extension is still denoted by u .

Proof of the Proposition 3.2. (b) \Rightarrow (a) It is trivial, since we can always take $D = M$.

(a) \Rightarrow (b) Let us assume (a), which is equivalent to suppose $F_+ = F_\mu = F_-$ on $L^2(M)$. Let us prove that

$$F_\mu^D \leq F_-^D \quad \text{on } H_0^1(D).$$

Let $u \in H_0^1(D)$ with $F_-^D(u) < +\infty$. By (3.2) there exists a sequence $(u_h)_h$ converging to u in $L^2(D)$ such that

$$\liminf_h F_{\mu_h}^D(u_h) = F_-^D(u);$$

since $F_-^D(u)$ is finite, we may assume that, up to a subsequence, $u_h \in H_0^1(D)$; hence $u_h \in H_0^1(M)$ and $F_{\mu_h}^D(u_h) = F_{\mu_h}(u_h)$, so that

$$F_-^D(u) = \liminf_h F_{\mu_h}^D(u_h) = \liminf_h F_{\mu_h}(u_h) \geq F_-(u) = F_\mu(u) = F_\mu^D(u).$$

It remains to prove that

$$F_+^D \leq F_\mu^D.$$

Let $u \in H_0^1(D)$ and such that $F_\mu^D(u) < +\infty$; it is not restrictive to assume that the support of u is compact. This is possible since for every $u \in H_0^1(D)$ there exists a sequence $v_h \in H_0^1(D)$, with $\text{supp } v_h$ compact in D for every h , such that $v_h \rightarrow u$ strongly in $H_0^1(D)$ and $v_h^2 \nearrow u^2$ q.e. in D ; therefore $F_\mu^D(v_h) \rightarrow F_\mu^D(u)$. Now extend the function u by putting $u = 0$ outside D , so that $u \in H_0^1(M)$ and $F_+(u) = F_\mu(u) = F_-(u)$. By (3.1) there exists a sequence $(u_h)_h$ converging to u in $L^2(M)$ such that

$$F_+(u) = F_\mu^D(u) = \limsup_h F_{\mu_h}(u_h) < +\infty.$$

This yields that $u_h \in H_0^1(M)$ for h large enough and

$$\limsup_h \int_M [|\nabla u|_g^2 + u^2] dV_g \leq \limsup_h F_{\mu_h}(u_h) < +\infty$$

so that $(u_h)_h$ converges weakly to u in $H_0^1(M)$. Let ζ be a smooth function, with compact support in D such that $\zeta = 1$ on $\text{supp } u$; then ζu_h belongs to $H_0^1(D)$ and ζu_h converges strongly in $L^2(D)$ to $\zeta u = u$ (by Rellich's theorem); hence

$$\begin{aligned} F_+^D(u) &\leq \limsup_h F_{\mu_h}^D(\zeta u_h) = \\ &= \limsup_h \left\{ \int_D [|\nabla u_h|_g^2 \zeta^2 + 2\zeta u_h \langle \nabla u_h, \nabla \zeta \rangle_g + u_h^2 |\nabla \zeta|_g^2] dV_g + \int_D \zeta^2 u_h^2 d\mu_h \right\} \leq \\ &\leq \limsup_h F_{\mu_h}(u_h) + 2 \int_D [\zeta u \langle \nabla u, \nabla \zeta \rangle_g + u^2 |\nabla \zeta|_g^2] dV_g = F_\mu(u), \end{aligned}$$

where in the last equality we have used the fact that $\nabla \zeta = 0$ on $\text{supp } u$. The proof of the proposition is then complete. \square

In order to examine the asymptotic behavior of the cap_{μ_h} , we need the following Proposition 3.6, and Proposition 3.7. Before entering into the details, we define the following functional which is slight different from Ψ .

3.5 Definition. Let N be a submanifold of M and let $\Psi_N : L^2(D) \rightarrow [0, +\infty]$ be the functional defined by

$$\Psi_N(u) = \begin{cases} \int_D [|\nabla u|_g^2 + u^2] dV_g & \text{if } u \in H^1(N); \\ +\infty, & \text{otherwise in } L^2(N). \end{cases}$$

We want to stress that Ψ_N does not take into account any boundary conditions, unlike Ψ defined by (2.0). Moreover Ψ_M and Ψ coincide on $H_0^1(M)$.

3.6 Proposition. Let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(M)$ which γ -converges to $\mu \in \mathcal{M}_0(M)$. Let A be an open set in M and let N be an open submanifold of M such that $A \subset N \subset M$. Then

$$(3.4) \quad \Psi_N(u) + \int_A u^2 d\mu \leq \liminf_h \left[\Psi_N(u_h) + \int_A u_h^2 d\mu_h \right],$$

for every $u \in H^1(N)$ and for every $u_h \in H^1(N)$ converging to u weakly in $L^2(N)$.

Proof. First of all we remark that it is not restrictive to assume that the lower limit at the right hand side of (3.4) is a finite limit; hence u_h is a bounded sequence in $H^1(N)$ converging to u weakly in $H^1(N)$. Now let us prove first the case of A contained in a single chart (U, ω) with coordinate system (x_1, \dots, x_d) . Let K be a compact set, $K \subset A$, and consider $\tau \in C_c^\infty(A)$, $0 \leq \tau \leq 1$, $\tau = 1$ on K . Since $u_h \in H^1(N)$, $\tau u_h \in H_0^1(N)$, $\text{supp}(\tau u_h)$ is in A and τu_h converges to τu strongly in $L^2(N)$, so by condition (a) of Γ -convergence (cf. Definition 2.1) we have

$$\Psi_M(\tau u) + \int_M (\tau u)^2 d\mu \leq \liminf_h \left[\Psi_M(\tau u_h) + \int_M (\tau u_h)^2 d\mu_h \right],$$

that in local coordinates, using the summation convention, it reads

$$\begin{aligned} & \int_\Omega [g^{ij} D_j(\tau u) D_i(\tau u) + (u\tau)^2] b(x) dx + \int_M (\tau u)^2 d\mu \leq \\ & \leq \liminf_h \left[\int_\Omega [a_{ij} D_j(\tau u_h) D_i(\tau u_h) + (u_h \tau)^2] b(x) dx + \int_M (\tau u_h)^2 d\mu \right] \end{aligned}$$

where $b(x) = \sqrt{|g(x)|}$ is a function in $L^\infty(\Omega)$, $\Omega = \omega(U)$ is a bounded open set in \mathbf{R}^d and D_i are the distributional derivatives (in \mathbf{R}^d). We have, expliciting the computations,

$$\begin{aligned} & \int_{\Omega} [g^{ij} D_j \tau D_i \tau] u^2 b(x) dx + 2 \int_{\Omega} [g^{ij} D_j \tau D_i u] u \tau b(x) dx + \\ & + \int_{\Omega} [g^{ij} D_j u D_i u + u^2] \tau^2 b(x) dx + \int_M \tau^2 u^2 d\mu \leq \\ & \leq \liminf_h \left[\int_{\Omega} (g^{ij} D_j \tau D_i \tau) u_h^2 b(x) dx + 2 \int_{\Omega} (g^{ij} D_j \tau D_i u_h) u_h \tau b(x) dx + \right. \\ & \left. + \int_{\Omega} (g^{ij} D_j u_h D_i u_h) \tau^2 b(x) dx + \int_{\Omega} (u_h \tau)^2 b dx + \int_M (u_h \tau)^2 d\mu \right]. \end{aligned}$$

Since u_h converges weakly to u in $H^1(N)$, the first and the second term on the right hand side tend, as $h \rightarrow +\infty$, to the first and to the second on the left hand side and $\int_{\Omega} \tau^2 u_h^2 b dx \rightarrow \int_{\Omega} \tau^2 u^2 b dx$, hence

$$(3.5) \quad \begin{aligned} & \int_{\Omega} [g^{ij} D_j u D_i u] \tau^2 b(x) dx + \int_M u^2 \tau^2 d\mu \leq \\ & \leq \liminf_h \left[\int_{\Omega} [g^{ij} D_j u_h D_i u_h] b(x) dx + \int_M \tau^2 u_h^2 d\mu \right] \end{aligned}$$

By lower semicontinuity w.r.t. the weak topology of $H^1(N)$, we have also

$$(3.6) \quad \begin{aligned} & \int_{\Omega} (g^{ij} D_j u D_i u) (1 - \tau^2) b dx \leq \\ & \leq \liminf_h \int_{\Omega} (g^{ij} D_j u_h D_i u_h) (1 - \tau^2) b dx; \end{aligned}$$

adding (3.5) and (3.6), we obtain

$$\begin{aligned} & \int_N [|\nabla u|_g^2 + u^2] dV_g + \int_K u^2 d\mu \leq \\ & \leq \liminf_h \left[\int_N [|\nabla u_h|_g^2 + u_h^2] dV_g + \int_A u_h^2 d\mu_h \right] \end{aligned}$$

and taking $K \nearrow A$, we have the desired result.

If we have not A contained in a single local chart, then we may consider $A \cap U_i$ where $\mathcal{U} = (U_i)_i$ is a family of open set given in Lemma 1.9, where the Borel measure σ is defined by $\sigma(B) = \int_B [|\nabla u|_g^2 + u^2] dV_g + \int_B u^2 d\mu$, for every $B \in \mathcal{B}(N)$. We apply the above argument to $A \cap U_i$ and we get the assertion of the proposition, since

$$\begin{aligned} & \int_N [|\nabla u|_g^2 + u^2] dV_g + \int_A u^2 d\mu = \\ & = \sum_i \left[\int_{U_i} [|\nabla u|_g^2 + u^2] dV_g + \int_{A \cap U_i} u^2 d\mu \right], \end{aligned}$$

and

$$\begin{aligned} & \sum_i \left[\liminf_h \int_{U_i} [|\nabla u_h|_g^2 + u_h^2] dV_g + \int_{A \cap U_i} u_h^2 d\mu_h \right] \leq \\ & \leq \liminf_h \left[\sum_i \int_{U_i} [|\nabla u_h|_g^2 + u_h^2] dV_g + \int_{A \cap U_i} u_h^2 d\mu_h \right] \leq \\ & \leq \liminf_h \left[\int_N [|\nabla u_h|_g^2 + u_h^2] dV_g + \int_A u_h^2 d\mu_h \right]. \end{aligned}$$

The proof is complete. \square

3.7 Proposition. *Let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(M)$ which γ -converges to $\mu \in \mathcal{M}_0(M)$. Let N be an open submanifold of M , K a compact set and A an open set such that $K \subset A \subset N$. Then for every $u \in H^1(N)$ there exists a sequence $(u_h)_h$ in $H^1(N)$ such that u_h converges strongly in $L^2(N)$ to u , $u - u_h \in H_0^1(N)$ and*

$$(3.7) \quad \Psi_N(u) + \int_A u^2 d\mu \geq \limsup_h \left[\Psi_N(u_h) + \int_K u_h^2 d\mu_h \right].$$

Proof. We may assume that $u \in L^2(N, \mu)$; by a diagonal argument it is enough to show that for every $\varepsilon > 0$ there exists a sequence $(u_h)_h$ in $H_0^1(N)$ such that u_h converges to u strongly in $L^2(N)$, $u_h - u \in H_0^1(N)$ and

$$\Psi_N(u) + \int_A u^2 d\mu + \varepsilon \geq \limsup_h \left[\Psi_N(u_h) + \int_K u_h^2 d\mu_h \right].$$

It is not restrictive to assume that A is a relatively compact open set.

Let ε be given and let W be an open set of N such that

$$K \subset W \subset \overline{W} \subset A$$

and $\Psi_{\overline{W} \setminus K}(u) < \varepsilon$. We suppose, at first, that A is contained in a local chart (U, ω) . Let $\zeta \in C_c^\infty(A)$, $\zeta = 1$ on (a neighborhood of) \overline{W} and $0 \leq \zeta \leq 1$ on A ; define $v = u\zeta$ that belongs to $H_0^1(N) \cap L^2(N, \mu)$. From Definition 2.1-(b), there exists a sequence $(v_h)_h$ in $H_0^1(M)$ converging to u strongly in $L^2(M)$ and such that

$$\Psi_M(v) + \int_M v^2 d\mu \geq \limsup_h \left[\Psi_M(v_h) + \int_M v_h^2 d\mu_h \right].$$

Set $E = M \setminus \overline{W}$; then we have

$$\begin{aligned} & \Psi_{\overline{W}}(u) + \int_{\overline{W}} u^2 d\mu + \Psi_E(v) + \int_E v^2 d\mu \geq \\ & \geq \limsup_h \left[\Psi_{\overline{W}}(v_h) + \int_{\overline{W}} v_h^2 d\mu_h \right] + \liminf_h \left[\Psi_E(v_h) + \int_E v_h^2 d\mu_h \right]. \end{aligned}$$

By Proposition 3.6 we have that

$$\Psi_E(v) + \int_E v^2 d\mu \leq \liminf_h \left[\Psi_E(v_h) + \int_E v_h^2 d\mu_h \right]$$

and, by definition, $v \in L^2(E, \mu)$, so

$$(3.8) \quad \Psi_{\overline{W}}(u) + \int_{\overline{W}} u^2 d\mu \geq \limsup_h \left[\Psi_{\overline{W}}(v_h) + \int_{\overline{W}} v_h^2 d\mu_h \right].$$

Let $\xi \in C_c^\infty(W)$, $\xi = 1$ in a neighborhood of K , $0 \leq \xi \leq 1$. Define

$$u_h = \xi v_h + (1 - \xi)u;$$

then

$$u_h = v_h \quad \text{in a neighbourhood of } K$$

and converges strongly in $L^2(N)$ to u . Writing $|\nabla u_h|_g^2$ in local coordinates, we have, for every $\varepsilon \in (0, 1)$

$$(3.9) \quad \begin{aligned} & g^{ij} D_j u_h D_i u_h \leq \left[\frac{\xi}{1 - \varepsilon} \right] [g^{ij} D_j v_h D_i v_h] + \\ & \left[\frac{1 - \xi}{1 - \varepsilon} \right] [g^{ij} D_j u D_i u] + \left[\frac{v_h - u}{\varepsilon} \right] [g^{ij} D_j \xi D_i \xi]. \end{aligned}$$

Since v_h converges to u strongly in $L^2(W)$, using (3.8) and (3.9), we get

$$\begin{aligned}
& \limsup_h \left[\Psi_N(u_h) + \int_K u_h^2 d\mu_h \right] \leq \\
& \leq \frac{1}{1-\varepsilon} \limsup_h \left[\Psi_W(v_h) + \int_W v_h^2 d\mu_h \right] + \frac{1}{1-\varepsilon} \Psi_{\overline{W} \setminus K}(u) \leq \\
& \leq \frac{1}{1-\varepsilon} \left[\Psi_{\overline{W}}(u) + \int_{\overline{W}} u^2 d\mu + \Psi_{\overline{W} \setminus K}(u) \right] \leq \\
& \leq \frac{1}{1-\varepsilon} \left[\Psi_N(u) + \int_W u^2 d\mu + \varepsilon \right].
\end{aligned}$$

The proof is then achieved in the case that A is contained in a single chart. If this does not happen, we consider an atlas $(W_i)_{i \in I}$, where I is a finite set of indexes; now, for a given $\varepsilon > 0$, we apply the argument of the previous step to $A \cap U_i$, where $\mathcal{U} = (U_i)_i$ is the open cover given in Lemma 1.10 and the measure considered is $\sigma(B) = \int_B [|\nabla u|_g^2 + u^2] dV_g + \int_B u^2 d\mu$ for every B Borel set in M . We apply to $A \cap U_i$ the above arguments and after a summation, we get the proof of the proposition. In fact,

$$\begin{aligned}
& \sum_i \left[\Psi_{N \cap U_i}(u) + \int_{A \cap U_i} u^2 d\mu \right] = \\
& = \sum_i \limsup_h \left[\Psi_{N \cap U_i}(u_h) + \int_{K \cap U_i} u_h^2 d\mu_h \right] \geq \\
& \geq \limsup_h \left[\sum_i \left(\Psi_{N \cap U_i}(u_h) + \int_{K \cap U_i} u_h^2 d\mu_h \right) \right] \geq \\
& \geq \limsup_h \left[\Psi_N(u_h) + \int_K u_h^2 d\mu_h \right],
\end{aligned}$$

while

$$\sum_i \left[\Psi_{N \cap U_i}(u) + \int_{A \cap U_i} u^2 d\mu \right] = \Psi_N(u) + \int_A u^2 d\mu + \varepsilon$$

since $\sigma(U_i \cap U_j) < \varepsilon$. □

Using similar methods to those in [DM2], the following results may be proved using Propositions 3.6 and 3.7 above.

We recall that $\mathcal{P}(M)$ stands for the class of all subsets of M .

3.8 Proposition. For a given $\mu \in \mathcal{M}_0(M)$, consider the set function cap_μ . Then the following are equivalent:

- (a) $(\mu_h)_h$ is a sequence in $\mathcal{M}_0(M)$ which γ -converges to μ in $\mathcal{M}_0(M)$;
 (b) the inequalities

$$\text{cap}_\mu(K) \leq \liminf_h \text{cap}_{\mu_h}(U)$$

$$\text{cap}_\mu(U) \geq \limsup_h \text{cap}_{\mu_h}(K)$$

hold true for every K, U compact and open sets respectively in M , with $K \subset U$;

- (c) for every open set $U \subset M$

$$\begin{aligned} \text{cap}_\mu(U) &= \sup \left\{ \liminf_h \text{cap}_{\mu_h}(K) : K \subset U, K \text{ compact} \right\} = \\ &= \sup \left\{ \limsup_h \text{cap}_{\mu_h}(K) : K \subset U, K \text{ compact} \right\}; \end{aligned}$$

- (d) the family of all $E \subset M$ such that

$$\text{cap}_\mu(E) = \lim_h \text{cap}_{\mu_h}(E)$$

is dense in $\mathcal{P}(M)$.

3.9 Proposition. For every $\mu \in \mathcal{M}_0(M)$ let \mathcal{H} be the family of all subsets E of M such that

$$\text{cap}_\mu(V \cap \overline{E}) = \text{cap}_\mu(V \cap E)$$

for every open set $V \subset M$. Then \mathcal{H} is rich in $\mathcal{P}(M)$ and $(\mu_h^E)_h$ γ -converges to μ^E for every $E \in \mathcal{H}$ and for every $(\mu_h)_h$ γ -converging to μ in $\mathcal{M}_0(M)$.

Finally we give the main result of this section.

3.10 Theorem. Let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(M)$. The following are equivalent:

- (a) $(\mu_h)_h$ γ -converges to a measure $\mu \in \mathcal{M}_0(M)$;
 (b) the sequence of functionals $F_{\mu_h}^D$, defined by (3.0), Γ -converges in $L^2(D)$ to the functional $F_{\mu^D}^D$, for every open submanifold D of M and for every Borel set $E \in \mathcal{H}$, $E \subset D$.

Proof. (b) \Rightarrow (a) It follows by taking $D = E = M$.

(a) \Rightarrow (b) The γ -convergence of μ_h to μ implies that $F_{\mu_h}^D$ Γ -converges in $L^2(D)$ to F_{μ}^D and this is equivalent to say that μ_h^D γ -converges to μ . The same argument of Proposition 3.9 applies to μ_h^D , so μ_h^E γ -converges to μ^E , for every $E \in \mathcal{H}$, $E \subset D$. This concludes the proof. \square

In this last part we introduce the notion of image measure and prove the Theorem 3.14.

3.11 Definition. Let $(\Omega, \mathcal{F}, \lambda)$ be a measure space and let (E, \mathcal{E}) be a measurable space; let $f : \Omega \rightarrow E$ be a \mathcal{F} -measurable function. The image measure of λ under f , denoted by $f(\lambda)$, is the measure μ on (E, \mathcal{E}) defined by

$$\mu(A) = \lambda(f^{-1}(A)), \quad \forall A \in \mathcal{E}.$$

Let g be a \mathcal{E} -measurable mapping from E into a measurable space (G, \mathcal{G}) . We have the equation

$$g(f(\lambda)) = (g \circ f)(\lambda).$$

3.12 Remark. With the notations above, let η be a real valued \mathcal{E} -measurable function on E ; η is μ -integrable if and only if $\eta \circ f$ is λ -integrable and

$$(3.10) \quad \int_E \eta \, d\mu = \int_{\Omega} (\eta \circ f) \, d\lambda.$$

3.13 Remark. Let $\mu \in \mathcal{M}_0(M)$, let (U, f) be a local chart and let $L = -\sum_{i,j=1}^d D_i(\sqrt{|g|} g^{ij} D_j)$ the Laplace-Beltrami operator read through (U, f) . For every $A \subset B \subset f(U)$, where $A \in \mathcal{B}(\mathbb{R}^d)$, B open set, let us define

$$\text{cap}_{\mu}^L(A, B) = \min_{u \in H_0^1(f^{-1}(B))} \left\{ \int_B (g^{ij} D_j u D_i u \sqrt{|g|}) \circ f^{-1} \, dx + \int_A (u - 1)^2 \circ f^{-1} \, d\tilde{\mu} \right\},$$

where $\tilde{\mu} = f(\mu)$; then one immediately realizes that $\text{cap}_{\mu}^L(A, B)$ is anything else but the set function $\text{cap}_{\mu}(f^{-1}(A), f^{-1}(B))$ read through (U, f) .

3.14 Theorem. *Let $\bar{N} = N \cup \partial N$ be a d -dimensional compact Riemannian manifold with boundary, let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(N)$, let ν be a Radon measure in $\mathcal{M}_0(N)$ and let $h : N \rightarrow \mathbb{R}$ be a nonnegative function. Assume that*

- for q.e. $x \in N$

$$\begin{aligned} h(x) &= \liminf_{r \rightarrow 0} \liminf_{h \rightarrow \infty} \frac{\text{cap}_{\mu_h}(B_r(x), B_{2r}(x))}{\nu(B_r(x))} \\ &= \liminf_{r \rightarrow 0} \limsup_{h \rightarrow \infty} \frac{\text{cap}_{\mu_h}(B_r(x), B_{2r}(x))}{\nu(B_r(x))}; \end{aligned}$$

- $h \in L^1_{loc}(N, \nu)$;
- $h(x) < +\infty$ q.e. on N .

Then μ_h γ -converges to $\mu = f\nu$.

Proof. By compactness of $\mathcal{M}_0(N)$ w.r.t the γ -convergence, we know that, up to a subsequence, $\mu_h \xrightarrow{\gamma} \mu$, with $\mu \in \mathcal{M}_0(N)$. Let (U, f) be a local chart of N ; let us introduce the image measures $\tilde{\mu}_h = f(\mu_h)$, $\tilde{\mu} = f(\mu)$ of μ_h , μ respectively under f in \mathbb{R}^d . From formula (1) it is easy to realize that the image measures $\tilde{\mu}_h$, $\tilde{\mu}$ belongs to $\mathcal{M}_0(\mathbb{R}^d)$ and $\tilde{\mu}_h \xrightarrow{\gamma} \tilde{\mu}$: to realize this fact it is sufficient to write the functionals F_{μ_h} and F_μ on the coordinate chart (U, f) . So if we set $\tilde{\nu} = f(\nu)$ and define $\tilde{h} = h \circ f^{-1}$, we have

- for q.e. $y \in f(U)$

$$\begin{aligned} \tilde{h}(y) &= \liminf_{r \rightarrow 0} \liminf_{h \rightarrow \infty} \frac{\text{cap}_{\tilde{\mu}_h}(f(B_r(y)), f(B_{2r}(y))))}{\tilde{\nu}(f(B_r(y)))} \\ &= \liminf_{r \rightarrow 0} \limsup_{h \rightarrow \infty} \frac{\text{cap}_{\tilde{\mu}_h}(f(B_r(y)), f(B_{2r}(y))))}{\tilde{\nu}(f(B_r(y)))}; \end{aligned}$$

- $\tilde{h} \in L^1_{loc}(\mathbb{R}^d, \tilde{\nu})$
- $\tilde{h}(y) < +\infty$ q.e. on \mathbb{R}^d .

We can apply now the result in [Bu-DM-M2, Theorem 5.2] and we get $\tilde{\mu} = \tilde{h}\tilde{\nu}$. Now let us prove that $\mu = h\nu$ on U . Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ any function in $L^1_{loc}(\mathbb{R}^d, \tilde{\mu})$; we have

$$\int_{\mathbb{R}^d} \phi d\tilde{\mu} = \int_{\mathbb{R}^d} \phi \tilde{h} d\tilde{\nu}.$$

Using the formula (3.10) above, the left and the right member of the equality above are respectively equal to

$$\int_U \phi \circ f d\mu = \int_U (\phi \circ f)(\tilde{h} \circ f) d\nu,$$

hence we get $\mu = h\nu$, where $h = \bar{h} \circ f$.

Now we prove that this procedure may be extended to the whole N . Given $\mu \in \mathcal{M}_0(N)$ and $\epsilon > 0$, there exists a cover of N $\{(U_i, f_i)\}$ such that $\mu(U_i \cap U_j) < \epsilon$ (cf. Lemma 1.10); let, moreover, (α_i) the partition of unity associated to this cover. For any μ -integrable function $g : N \rightarrow \mathbb{R}$, we have

$$\int_N g(\omega)\mu(d\omega) = \sum_i \int_{V_i} \alpha_i(\omega)g(\omega)\mu(d\omega),$$

where $V_i = \text{supp } \alpha_i \subset U_i$. Where $V_i \cap V_j \neq \emptyset$, we have $\mu(V_i \cap V_j) < \epsilon$. So, up to an arbitrary $\epsilon > 0$, the above procedure can be extended to the whole manifold. \square

Section 4. Convergence of the solutions

As we said in the Introduction, we study in this section the convergence of the solutions of Dirichlet problems

$$(4.1) \quad \begin{cases} -\Delta_g u_h + \mu_h u_h + u_h = f & \text{in } M, \\ u_h \in H_0^1(M), \end{cases}$$

where $\mu_h \in \mathcal{M}_0(M)$. We show also a result concerning the convergence of the eigenvalues $(\sigma_h^{(i)})_{i \in \mathbb{N}}$, $h \in \mathbb{N}$, of the problem

$$(4.2) \quad \begin{cases} -\Delta_g u_h + \mu_h u_h + \lambda u_h = f & \text{in } M, \\ u_h \in H_0^1(M). \end{cases}$$

Let us consider, for an arbitrary $\mu \in \mathcal{M}_0(M)$ and for a given $f \in L^2(M)$, the following Dirichlet problem

$$(4.3) \quad \begin{cases} -\Delta_g u + \mu u + \lambda u = f & \text{in } M \\ u \in H_0^1(M), \end{cases}$$

where Δ_g is the Laplace-Beltrami operator and $\lambda > 0$.

4.1 Definition. We say that u is a (weak) solution to the problem (4.3), if $u \in H_0^1(M) \cap L^2(M, \mu)$ and

$$\int_M \langle \nabla u, \nabla z \rangle_g dV_g + \int_M uz d\mu + \lambda \int_M uz dV_g = \int_M fz dV_g$$

for every $z \in H_0^1(M) \cap L^2(M, \mu)$.

4.2 Remark. It may be proved, using variational methods similar to those in [B-DM-M], that for every $\lambda > 0$ and for every $f \in L^2(M)$, there exists a unique solution u to (4.3); moreover u may be characterized as the minimum point of the following problem

$$\min_{v \in H_0^1(M)} \left[F_\lambda(v) - \int_M fv dV_g \right],$$

where $F_\lambda : H_0^1(M) \rightarrow [0 + \infty]$ is the functional defined by

$$F_\lambda(v) = \int_M [|\nabla v|_g^2 + \lambda v^2] dV_g + \int_M v^2 d\mu.$$

4.3 Definition. Let $\mu \in \mathcal{M}_0(M)$; then the resolvent operator for the Dirichlet problem (4.3) is defined as

$$(4).4 \quad R_\lambda^\mu : L^2(M) \longrightarrow L^2(M)$$

that associates to every $f \in L^2(M)$ the solution u to the problem (4.3).

By definition, it follows that $R_\lambda^\mu f \in H_0^1(M) \cap L^2(M, \mu)$; since $R_\lambda^\mu f$ is the solution of (4.3), then the following estimate holds true

$$\|R_\lambda^\mu f\|_{L^2(M)} \leq (1/\lambda) \|f\|_{L^2(M)}$$

for every $f \in L^2(M)$. Moreover R_λ^μ is a positive operator.

4.4 Remark. In the space $\mathcal{M}_0(M)$ it is possible to introduce an equivalence relation \simeq in this way: we say that μ is equivalent to ν ($\mu \simeq \nu$ in symbol), if and only if

$$\int_M w^2 d\mu = \int_M w^2 d\nu$$

for every $w \in H_0^1(M)$. It is possible to prove that \simeq is actually an equivalence relation; cf. [DM2], [DM-M1], [B-DM-M] for more details about this topic.

4.5 Definition. We say that σ is an eigenvalue and u is an eigenfunction of the quadratic form

$$F_0(v) = \int_M |\nabla v|_g^2 dV_g + \int_M v^2 d\mu$$

if $u \in H_0^1(M) \cap L^2(M, \mu)$, $u \neq 0$ and the following equation is satisfied:

$$(4.5) \quad \int_M \langle \nabla u, \nabla z \rangle_g dV_g + \int_M uz d\mu = \sigma \int_M uz dV_g,$$

for every $z \in H_0^1(M) \cap L^2(M, \mu)$.

Given $\lambda > 0$, it follows from the definition that σ is an eigenvalue of F_0 if and only if $1/(\sigma + \lambda)$ is a proper value of the resolvent operator R_λ^μ .

The following proposition is a consequence of an abstract result in Γ -convergence (cf. [A, Theorem 3.26]).

4.6 Proposition. *Let (μ_h) be a sequence in $\mathcal{M}_0(M)$ and let $\lambda > 0$; then μ_h γ -converges to μ if and only if $R_\lambda^{\mu_h}$ converges strongly in $L^2(M)$ to R_λ^μ .*

We may state the following result.

4.7 Theorem. *Let (μ_h) be a sequence in $\mathcal{M}_0(M)$; then the following are equivalent:*

- (μ_h) γ -converges to μ ;
- the sequence (u_h) of the solutions of the problems (4.2) converges strongly in $L^2(M)$ to the solution u of the problem (4.3).

Proof. Let $f \in L^2(M)$ be given; by definition $R_\lambda^{\mu_h} f = u_h$, with $\lambda = 1$, is the solution of the problem (4.2). Then the equivalence follows from Proposition 4.6.

□

Before going on, we have to know if F_0 has actually infinitely many eigenvalues. The following result, taken from [DM-G-M], gives an answer in this direction.

4.8 Proposition. *If μ is not equivalent to ∞_M , then $\text{Ker}(R_\lambda^\mu)$ has infinite codimension.*

Proof. Since μ is not equivalent to ∞_M , then there exists $w \in H_0^1(M)$ such that

$$\int_M w^2 d\mu \neq \int_M w^2 d\infty_M,$$

hence w is not identically zero in M ; moreover $w \in L^2(M, \mu)$. Therefore the set $A = \{x \in M : w(x) \neq 0\}$ has positive measure. Let $\phi \in C_c^\infty(M)$, $0 \leq \phi \leq 1$ on M , and take as test function in (4.5) $v = w\phi \in L^2(M, \mu)$. If $f \in \text{Ker}(R_\lambda^\mu)$, then we have $f = 0$ a.e. on A ; so every function which vanishes on $M \setminus A$ is orthogonal to $\text{Ker}(R_\lambda^\mu)$. This implies that $\text{codim Ker}(R_\lambda^\mu) = \infty$. On the other hand, if μ is equivalent to ∞_M , then the resolvent operator is identically zero as an operator on $L^2(M)$. This means that it has the only proper value zero; in this case we say that $\sigma = +\infty$ is the only eigenvalue of F_0 with infinite multiplicity. \square

Let $\mu \in \mathcal{M}_0(M)$; since R_λ^μ , for every μ not equivalent to ∞_M , is a positive, compact operator and its kernel has infinite codimension, it has infinitely many positive proper values. Therefore F_0 has infinitely many eigenvalues which can be rearranged in an increasing sequence:

$$0 \leq \sigma^{(1)} \leq \sigma^{(2)} \leq \dots \rightarrow +\infty,$$

where each eigenvalue is repeated according to its multiplicity.

Theorem 4.7, together with Proposition 4.8, allow us to state the following result about the convergence of the eigenvalues.

4.9 Theorem. *If $(\mu_h)_h$ is a sequence in $\mathcal{M}_0(M)$ which γ -converges to a measure $\mu \in \mathcal{M}_0(M)$, then*

$$\sigma_h^{(i)} \longrightarrow \sigma^{(i)}, \text{ as } h \rightarrow +\infty,$$

where $\sigma_h^{(i)}$ is the i -th eigenvalue of the problem (4.2) and $\sigma^{(i)}$ is the i -th eigenvalue of the problem (4.3) (counted according to their multiplicities).

Proof. If μ is not equivalent to ∞_M , then from Proposition 4.6 it follows that the resolvent operator $R_\lambda^{\mu_h}$ converges strongly to the resolvent operator R_λ^μ in $L^2(M)$. This strong convergence is sufficient to give the convergence of the eigenvalues (see [D-S, Lemma XI.9.5]). \square

4.10 Remark. In the pathological case $\mu \simeq \infty_M$, we use the convention $\sigma^{(i)} = +\infty$ for every $i \in \mathbb{N}$; using arguments similar to those in [DM-G-M], it is still possible to prove that $\sigma_h^{(i)} \rightarrow +\infty$, for every $i \in \mathbb{N}$.

2. Fractal Relaxed Dirichlet Problems

1. Notation. Fractals, Self Similar Sets.

We shall denote by $\#(E)$ the cardinality of the set E if E is finite, and $+\infty$ otherwise. If μ is a set function defined on the subsets of a set X and $E \subset X$, we shall indicate by $\mu|_E(A) = \mu(A \cap E)$ for every $A \subset X$. Moreover if $f : X \rightarrow Y$ is a function, then we define $f_{\#}\mu(E) = \mu(f^{-1}(E))$.

We shall denote by $B_r(x)$ the open ball in \mathbb{R}^d with center x and radius r . In all that follows we shall assume $d \geq 2$. If E is a measurable subset of \mathbb{R}^n , and $r > 0$ we shall denote by $\mathcal{H}^r(E)$ its r -dimensional Hausdorff measure, defined as

$$\mathcal{H}^r(E) = \sup_{\varepsilon > 0} \inf \left\{ \sum_{i \in I} m(r) (\text{diam}(E_i))^r : \right. \\ \left. : E_i \text{ measurable, } E \subset \bigcup_{i \in I} E_i, \text{diam}(E_i) \leq \varepsilon \right\},$$

where

$$m(r) = 2^{-r} \frac{\Gamma(1/2)^r}{\Gamma((r/2) + 1)}$$

and Γ is Euler's function. By $\omega_{d-1} = \mathcal{H}^{d-1}(\partial B_1(0))$ we denote the $d - 1$ -dimensional measure of the unit sphere.

The letter c will denote a strictly positive constant whose value may vary from line to line, depending only on the fixed parameters of the problem. If $F(p)$, $G(p)$ are two quantities depending on a parameter p , then by $F(p) \approx G(p)$ as $p \rightarrow \bar{p}$ we mean that the ratio F/G approaches 1 when $p \rightarrow \bar{p}$.

Self Similar Sets.

We introduce now the class of (strictly) self similar fractals, as studied by Hutchinson in [Hu].

We shall denote with $\mathcal{S} = \{S_1, \dots, S_N\}$ a finite family of similitudes on \mathbb{R}^n with common ratio $\rho < 1$. The *dimension of similitude* of \mathcal{S} will be the number $\alpha = -\log_{\rho} N$ (i.e. $N\rho^{\alpha} = 1$).

We define the following sets of indices: $C(N) = \{1, \dots, N\}^{\mathbb{N}}$, and $C_p(N) = \{1, \dots, N\}^p$, for every $p \in \mathbb{N}$. If $(i) = (i_1, i_2, \dots) \in C(N)$, then $(i)_p = (i_1, \dots, i_p) \in C_p(N)$ will denote the 'projection' of (i) on $C_p(N)$.

For every $\beta = (\beta_1, \dots, \beta_p) \in C_p(N)$ we define the similitude $S_\beta = S_{\beta_p} \circ \dots \circ S_{\beta_1}$, and $s_\beta \in \mathbb{R}^d$ the unique fixed point of S_β . Moreover, for every $(i) \in C(N)$ we set

$$(1.1) \quad s_{(i)} = \lim_{p \rightarrow +\infty} s_{(i)_p},$$

and

$$(1.2) \quad K_S = \{s_{(i)} : (i) \in C(N)\}.$$

The *coordinate map* $\pi : C(N) \rightarrow K_S$, $(i) \mapsto s_{(i)}$ is continuous if we consider the product topology on $C(N)$.

We will say that a set $E \subset \mathbb{R}^d$ is *S-invariant* if we have $\bigcup_{i=1}^N S_i(E) = E$.

1.1 Theorem. (Hutchinson [Hu]) *The set K_S is the unique closed bounded S-invariant set.*

If ν is a regular Borel measure with bounded support and finite mass, we define the measure

$$(1.3) \quad S\nu = \sum_{i=1}^N \rho^\alpha S_{i\#}\nu = \sum_{i=1}^N \frac{1}{N} S_{i\#}\nu;$$

$$i.e. \quad S\nu(E) = \sum_{i=1}^N \rho^\alpha \nu(S_i^{-1}(E)).$$

On $C(N)$ we shall consider the product measure induced by the probability measure with mass $\frac{1}{N}$ on each $i \in \{1, \dots, N\}$, and we shall denote it by τ .

1.2 Theorem. (Hutchinson [Hu]) *The measure $\nu_S = \pi_{\#}(\tau)$ is the unique regular Borel measure verifying:*

- i) ν_S has compact support (the support of ν_S is in fact K_S);
- ii) ν_S is a probability measure (i.e. $\nu_S(\mathbb{R}^d) = 1$);
- iii) ν_S is S-invariant (i.e. $S\nu = \nu$).

We will say that a set $J \subset \mathbf{R}^d$ is *self similar* if there exists a family of similitudes \mathcal{S} as above such that

- i) J is \mathcal{S} -invariant;
- ii) J has Hausdorff dimension $k \geq 0$, $\mathcal{H}^k(J) > 0$, and $\mathcal{H}^k(S_i(J) \cap S_j(J)) = 0$ if $i \neq j$.

We will say that the family of similitudes \mathcal{S} satisfies the *open set condition* (see [8]), and that $K_{\mathcal{S}}$ is a *self similar fractal*, if there exists a bounded open set $\mathcal{O} \subset \mathbf{R}^d$ such that $\mathcal{H}^\alpha(K_{\mathcal{S}} \setminus \mathcal{O}) = 0$, $S_i(\mathcal{O}) \subset \mathcal{O}$ for every $i \in \{1, \dots, N\}$, and $S_i(\mathcal{O}) \cap S_j(\mathcal{O}) = \emptyset$ if $i \neq j$. For a possible choice of \mathcal{S} and of the set \mathcal{O} satisfying the open set condition for the Von Koch curve we refer to [Br-D'A].

With this definition of self similar fractals, we recover most of the well-known fractals obtained by an iteration construction, such as the Von Koch curve, the Sierpinski gasket etc. (cf. [M], [F], [Hu] for more details and examples). Let us remark that in this framework are included also “trivial” fractals such as the cube $[0, 1]^n$ in \mathbf{R}^d or the k -dimensional cube $[0, 1]^k$ in \mathbf{R}^d , $k < n$.

The following theorem is the fundamental result for self similar fractals.

1.3 Theorem. (Hutchinson [Hu]) *If the family of similitudes \mathcal{S} verifies the open set condition, then $K_{\mathcal{S}}$ is a self similar set, the Hausdorff dimension of $K_{\mathcal{S}}$ is α , $0 < \mathcal{H}^\alpha(K_{\mathcal{S}}) < +\infty$, and*

$$(1.4) \quad \nu_{\mathcal{S}} = \frac{1}{\mathcal{H}^\alpha(K_{\mathcal{S}})} \mathcal{H}^\alpha|_{K_{\mathcal{S}}}.$$

Capacity.

The main tool for the proof of our results will be a compactness and derivation lemma (Lemma 1.4 below). Before stating it, we shall need to define the notions of *capacity*.

Let F be an open subset of \mathbf{R}^d , and E a Borel subset of F ; the capacity of E with respect to F is

$$(1.5) \quad \text{cap}(E, F) = \inf \left\{ \int_F |Du|^2 \, dx : u \geq 1 \text{ on an open neighbourhood of } E, u \in H_0^1(F) \right\}.$$

We say that a property $P(x)$ holds for *quasi every* $x \in E$ (or quasi-everywhere in E) if

$$\text{cap}(\{x \in E : P(x) \text{ is not verified}\}, F) = 0.$$

Note that the property of being of capacity zero is independent of the set F .

It can be proven (see [Cho]) that for every Borel subset E of F , there exists a function $u \in H_0^1(F)$ such that $u \in H_0^1(F)$, $u \geq 1$ quasi everywhere on E , and

$$\text{cap}(E, F) = \int_F |Du|^2 dx;$$

this function will be called the *capacitary potential* of E with respect to F .

For example, if we consider two concentric balls $B_r(x) \subset B_R(x)$, $0 < r < R$, then we have

$$\text{cap}(B_r(x), B_R(x)) = (d-2)\omega_{d-1} \frac{(rR)^{d-2}}{R^{d-2} - r^{d-2}} \quad \text{for } d \geq 3$$

and

$$\text{cap}(B_r(x), B_R(x)) = 2\pi \left(\log \frac{R}{r} \right)^{-1} \quad \text{for } d = 2.$$

Notice that from the definition of capacity, we have $\text{cap}(tE, tF) = t^{d-2} \text{cap}(E, F)$ for any real number $t > 0$.

1.4 Lemma. *Let (E_h) be a sequence of closed subsets of \mathbf{R}^d and let ν be a Radon measure such that $\nu(B) = 0$ on sets B of capacity zero. Let us suppose that the following hypotheses are satisfied:*

(i) *for every $x \in \mathbf{R}^d$ and for every $T > 0$ we have*

$$(1.6) \quad \liminf_{t \rightarrow 0^+} \limsup_{k \rightarrow +\infty} \frac{\text{cap}(B_t(x) \cap E_k, B_T(x))}{\nu(B_t(x))} \\ = \liminf_{t \rightarrow 0^+} \liminf_{k \rightarrow +\infty} \frac{\text{cap}(B_t(x) \cap E_k, B_T(x))}{\nu(B_t(x))} = \varphi(x);$$

(ii) $\varphi(x) < +\infty$ for q.e. $x \in \mathbf{R}^d$;

(iii) $\varphi \in L^1(\mathbf{R}^d, \nu)$.

Let us define the measure $\mu = \varphi\nu$. Then, for every bounded open set Ω of \mathbb{R}^d , and for every $f \in L^2(\Omega)$, the solutions (u_h) to the Dirichlet problem

$$(1.7) \quad \begin{cases} -\Delta u_h = f & \text{in } \Omega \setminus E_h \\ u_h \in H_0^1(\Omega \setminus E_h) \end{cases}$$

converge in $L^2(\Omega)$, as $h \rightarrow +\infty$, to the weak solution u of the Dirichlet problem

$$(1.8) \quad \begin{cases} -\Delta u + u\mu = f & \text{in } \Omega \\ u \in H_0^1(\Omega) \cap L^1(\Omega, \mu), \end{cases}$$

i.e., the unique function $u \in H_0^1(\Omega) \cap L^1(\Omega, \mu)$ such that

$$\int_{\Omega} Du Dv dx + \int_{\Omega} uv d\mu = \int_{\Omega} f v dx$$

for every $v \in C^\infty(\Omega)$ with compact support in Ω (cf. [DM-M1, Proposition 3.8]).

Proof. The existence of a measure $\tilde{\mu}$ and of a function u which satisfy (1.8) has been proven in the more general framework of the so-called Relaxed Dirichlet Problem (cf. Propositions 4.9 and 4.10 in [DM-M1]). In [Bu-DM-M2, Theorem 5.2] it is proven that $\tilde{\mu} = \mu = \varphi\nu$. \square

2. The Main Result

From now on we shall consider as fixed a family of similitudes \mathcal{S} verifying the open set condition, and the corresponding set $K = K_{\mathcal{S}}$, as defined in Section 1.

Let us fix $x_0 \in \mathcal{O}$, and let us define

$$(2.1) \quad R = \frac{1}{2} \text{dist}(x_0, \partial\mathcal{O}).$$

Let $c_0 > 0$; for every $p \in \mathbb{N}$ we shall set

$$(2.2) \quad R_p = R\rho^p,$$

and

$$(2.3) \quad \rho_p = \begin{cases} c_0(R\rho^p)^{\alpha/(d-2)} = c_0(R_p)^{\alpha/(d-2)} & \text{if } d \geq 3, \\ (R\rho^p) \exp\left(-\frac{1}{c_0}(R\rho^p)^{-\alpha}\right) = (R_p) \exp\left(-\frac{1}{c_0}(R_p)^{-\alpha}\right) & \text{if } d = 2. \end{cases}$$

Moreover let us fix a set $E \subset B_1(0)$, with finite capacity with respect to $B_1(0)$, and define for every $\beta \in C_p(N)$

$$(2.4) \quad x_\beta = S_\beta(x_0), \quad B_\beta^p = x_\beta + \rho_p E,$$

and

$$(2.5) \quad B_p = \bigcup \{\overline{B_\beta^p} : \beta \in C_p(N)\}.$$

Note that we have $|x_\gamma - x_\beta| > 2R_p$ if $\beta, \gamma \in C_p(N)$, and $\beta \neq \gamma$; moreover, if $\alpha > d - 2$, we have for large p

$$(2.6) \quad \text{dist}(B_\gamma^p, B_\beta^p) > R_p.$$

We briefly illustrate the construction in (2.1)–(2.6) with some figures in section 4.

We are now in a position to state the main result.

2.1 Theorem. *Let K be a self similar fractal of Hausdorff dimension $\alpha > d - 2$, R defined by (2.1), $c_0 > 0$, $E \subset B_1(0)$ a set with finite capacity with respect to $B_1(0)$, and let us set*

$$(2.7) \quad c_1 = \begin{cases} c_0^{d-2} R^\alpha \frac{1}{\mathcal{H}^\alpha(K)} \text{cap}(E, \mathbf{R}^d) & \text{if } d \geq 3 \\ c_0 R^\alpha \frac{1}{\mathcal{H}^\alpha(K)} \lim_{t \rightarrow +\infty} (\log t \text{cap}(E, B_t(0))) & \text{if } d = 2. \end{cases}$$

Let \mathcal{E}_p be constructed as in (2.1)–(2.6), let Ω be a bounded open subset of \mathbf{R}^d , and $f \in L^2(\Omega)$. Then, the weak solutions u_p to the Dirichlet problem

$$(2.8) \quad \begin{cases} -\Delta u_p = f & \text{in } \Omega \setminus B_p \\ u_p \in H_0^1(\Omega \setminus B_p) \end{cases}$$

($p \in \mathbb{N}$) converge in $L^2(\Omega)$, as $p \rightarrow +\infty$, to the weak solution u of the Dirichlet problem

$$(2.9) \quad \begin{cases} -\Delta u + c_1 u \mathcal{H}^\alpha|_{K \cap \Omega} = f & \text{in } \Omega \\ u \in H_0^1(\Omega), \end{cases}$$

i.e., the unique function $u \in H_0^1(\Omega)$ such that

$$(2.10) \quad \int_\Omega Du Dv dx + c_1 \int_{K \cap \Omega} u v d\mathcal{H}^\alpha = \int_\Omega f v dx$$

for every $v \in H_0^1(\Omega)$.

2.2 Remark. 1) If we take $E = B_1(0)$, the constant c_1 may be written as

$$c_1 = \begin{cases} \frac{c_0^{d-2} R^\alpha}{\mathcal{H}^\alpha(K)} \text{cap}(B_1(0), \mathbf{R}^d) & \text{if } d \geq 3 \\ \frac{c_0 R^\alpha}{\mathcal{H}^\alpha(K)} \lim_{t \rightarrow +\infty} \left(\log t \text{cap}(B_1(0), B_t(0)) \right) & \text{if } d = 2, \end{cases}$$

i.e. we have

$$c_1 = \begin{cases} c_0^{d-2} \frac{(d-2)\omega_{d-1} R^\alpha}{\mathcal{H}^\alpha(K)} & \text{if } d \geq 3 \\ c_0 \frac{2\pi R^\alpha}{\mathcal{H}^\alpha(K)} & \text{if } d = 2. \end{cases}$$

2) As a particular case of Theorem 2.1, we get the results of D. Cioranescu & F. Murat [C-M, §2], where they obtain in the limit the Lebesgue measure. In fact, we can consider Ω as a subset of an d -dimensional cube $Q = [-T, T]^d$. The cube Q itself can be seen as a self similar set, by choosing the 2^d similitudes which carry it into 2^d sub-cubes of side length T . In this case, the procedure of iteration coincides with the usual homogenization technique. In the same way, we can obtain as a limit the $(d-1)$ -dimensional Hausdorff measure restricted to an $(d-1)$ -hyperplane, as in [C-M].

3) It will be clear from the proof of Theorem 2.1 that the same conclusion holds true when we consider more general sets B_p obtained as in (2.5) where the $B_p^\beta \subset B_{R_p}$ are not necessarily similar to each other; it suffices that

$$\text{cap}(B_p^\beta, B_{R_p}) \approx c_1 \rho^{\alpha p} \mathcal{H}^\alpha(K)$$

uniformly in β as $p \rightarrow +\infty$.

4) If $\text{cap}(B_p^\beta, B_{R_p}(x_\beta)) \rho^{-\alpha p}$ tends to $+\infty$ as $p \rightarrow +\infty$, uniformly in β , then the limit problem (2.9) becomes

$$\begin{cases} -\Delta u = f & \text{in } \Omega \setminus K \\ u \in H_0^1(\Omega \setminus K). \end{cases}$$

5) Theorem 2.1 may be interpreted in term of Γ -convergence of the energy functionals related to problems (2.8), (3,9) (cf. the recent book [DM] for an introduction to the theory of Γ -convergence). In fact, let $\mathcal{F}_p : H_0^1(\Omega) \rightarrow [0, +\infty]$ be defined as

$$\mathcal{F}_p(u) = \begin{cases} \int_\Omega |Du|^2 dx & \text{if } u|_{\Omega \setminus B_p} \in H_0^1(\Omega \setminus B_p) \\ +\infty & \text{otherwise;} \end{cases}$$

then the sequence (\mathcal{F}_p) Γ -converges with respect to the $L^2(\Omega)$ -topology, as $p \rightarrow +\infty$, to the functional $\mathcal{F} : H_0^1(\Omega) \rightarrow [0, +\infty]$ defined by $\mathcal{F}(u) = \int_{\Omega \setminus K} |Du|^2 dx + \int_{K \cap \Omega} u^2 d\mathcal{H}^\alpha$ (see Proposition 4.10 in [DM-M1]).

3. Proof of the main result

We begin by proving two simple results regarding the structure of self similar fractals.

3.1 Proposition. *Let $K = K_S$ be a self similar fractal. Let V be a bounded open subset of \mathbf{R}^d such that $\mathcal{H}^\alpha(\partial V \cap K_S) = 0$. Then*

$$(3.1) \quad \mathcal{H}^\alpha(V \cap K_S) = \lim_{p \rightarrow +\infty} \rho^{p\alpha} \mathcal{H}^\alpha(K_S) \#\{\beta \in C_p(N) : S_\beta(\mathcal{O}) \cap V \neq \emptyset\},$$

and

$$(3.2) \quad \mathcal{H}^\alpha(V \cap K_S) = \lim_{p \rightarrow +\infty} \rho^{p\alpha} \mathcal{H}^\alpha(K_S) \#\{\beta \in C_p(N) : B_\beta^p \cap V \neq \emptyset\}.$$

Proof. Let $I = \{\beta \in C_p(N) : S_\beta(\mathcal{O}) \cap V \neq \emptyset\}$, and let us define

$$V_q = \{x \in \mathbf{R}^d : \text{dist}(x, V) < \rho^q \text{diam} \mathcal{O}\}.$$

We have

$$\mathcal{H}^\alpha(V \cap K_S) = \mathcal{H}^\alpha(\overline{V} \cap K_S) = \lim_{q \rightarrow +\infty} \mathcal{H}^\alpha(V_q \cap K_S).$$

If $p \geq q$, then $V_q \supset \bigcup \{S_\beta(\mathcal{O}) : \beta \in I\}$ so that

$$\mathcal{H}^\alpha(V_q \cap K_S) \geq \sum_{\beta \in I} \mathcal{H}^\alpha(K_S \cap S_\beta(\mathcal{O})) = \rho^{p\alpha} \mathcal{H}^\alpha(K_S) \#(I).$$

Passing to the limit first as $p \rightarrow +\infty$, and then as $q \rightarrow +\infty$ we have

$$\begin{aligned} \lim_{q \rightarrow +\infty} \mathcal{H}^\alpha(V_q \cap K_S) &\geq \limsup_{p \rightarrow +\infty} \rho^{\alpha p} \mathcal{H}^\alpha(K_S) \#(I) \\ &\geq \liminf_{p \rightarrow +\infty} \rho^{\alpha p} \mathcal{H}^\alpha(K_S) \#(I) \geq \mathcal{H}^\alpha(V \cap K_S), \end{aligned}$$

and hence the proof of (3.1) is achieved; in the same way we can prove (3.2). \square

3.2 Proposition. *Let \mathcal{S} satisfy the open set condition. For every $R > 0$ there exists $M_R > 0$ such that, for every $p \in \mathbb{N}$, and for every $\beta \in C_p(N)$, we have*

$$(3.3) \quad \#\{\gamma \in C_p(N) : \text{dist}(S_\beta(\mathcal{O}), S_\gamma(\mathcal{O})) < R\rho^p\} \leq M_R.$$

Proof. Fixed $R > 0$, let us consider the set

$$D_{p,\beta}^R = \{x \in \mathbf{R}^d : 0 < \text{dist}(x, S_\beta(\mathcal{O})) < \text{diam}S_\beta(\mathcal{O}) + R\rho^p (= \rho^p(\text{diam}\mathcal{O} + R))\}.$$

We have $|D_{p,\beta}^R| = c_R(\rho^p)^d$ (with c_R a constant depending only on R and \mathcal{O}). Let us consider

$$N_{p,\beta}^R = \{\gamma \in C_p(N) : \text{dist}(S_\beta(\mathcal{O}), S_\gamma(\mathcal{O})) < R\rho^p\}.$$

Since \mathcal{O} is open, it contains a ball of radius R_0 , and hence each $S_\gamma(\mathcal{O})$ contains a ball of radius $R_0\rho^p$. We have then

$$|D_{p,\beta}^R| \geq N_{p,\beta}^R(R_0)^d |B_1(0)|\rho^{pd},$$

so that $N_{p,\beta}^R(R_0)^d |B_1(0)| \leq c_R$. We can take then $M_R = c_R(R_0)^{-d} |B_1(0)|^{-1}$. \square

We can proceed now in the proof of Theorem 2.1. Let us first notice that, if $\alpha > d - 2$, the measure $\mathcal{H}^\alpha|_K$ belongs to $H_{\text{loc}}^{-1}(\mathbf{R}^d)$ (and hence it is zero on all sets of capacity zero). In fact, by [Z, Th 4.7.5] it is sufficient to remark that for all $x \in K$

$$\int_0^1 \frac{\mathcal{H}^\alpha|_K(B_r(x))}{r^{d-2}} \frac{dr}{r} \leq c \int_0^1 \frac{dr}{r^{d-2-\alpha+1}} < +\infty.$$

The next step will be to estimate the capacity in (1.6) in order to calculate the limits therein. The estimate from above follows from the (strong) sub-additivity of the capacity, while the estimate from below is proven using the following result.

3.3 Lemma. *Let V be a bounded open subset of \mathbf{R}^d . There exists a constant c depending only on d, R and α such that if we define*

$$(3.4) \quad \delta = \delta(V) = \begin{cases} c(\text{diam}V)^{\alpha-d+2} & \text{if } d \geq 3 \\ c(\text{diam}V)^\alpha |\log(\text{diam}V)| & \text{if } d = 2 \end{cases}$$

and we have $\delta < 1$, then for every $T > 2\text{diam}(V)$ and for every $x \in V$,

$$(3.5) \quad \begin{aligned} & \text{cap}(V \cap \mathcal{B}_p, B_T(x)) \\ & \geq (1 - \delta)^2 \#\{\beta \in C_p(N) : B_\beta^p \cap V \neq \emptyset\} \text{cap}(\rho_p E, B_{R_p}(0)). \end{aligned}$$

for sufficiently large p .

Proof. Let us define $I = \{\beta \in C_p(N) : B_\beta^p \cap V \neq \emptyset\}$. We claim that, if the capacitary potential u of $V \cap \mathcal{B}_p$ with respect to $B_T(x)$ satisfies $u \leq \delta$ on $\partial B_{R_p}(x_\beta)$ for every $\beta \in I$, then the proof is achieved. In fact, let us assume that $u \leq \delta$ on $\partial B_{R_p}(x_\beta)$ for every $\beta \in I$; let us define $v = \frac{1}{(1-\delta)}(u - \delta)^+$. By the definition of the capacitary potential, it is easy to see that $v \in H_0^1(B_T(x))$, $v \geq 1$ q.e on $V \cap \mathcal{B}_p$ and $v = 0$ q.e on $\partial B_{R_p}(x_\beta)$, for every $\beta \in I$, hence we have

$$\text{cap}(B_\beta^p, B_{R_p}(x_\beta)) \leq \int_{B_{R_p}(x_\beta)} |Dv|^2 dx$$

and therefore

$$(3.6) \quad \int_{B_T(x)} |Dv|^2 dx \geq \sum_{\beta \in I} \int_{B_{R_p}(x_\beta)} |Dv|^2 dx \geq \sum_{\beta \in I} \text{cap}(B_\beta^p, B_{R_p}(x_\beta)).$$

By definition of v , we have also

$$(3.7) \quad \begin{aligned} \int_{B_T(x)} |Dv|^2 dx &= \frac{1}{(1-\delta)^2} \int_{B_T(x)} |D(u - \delta)^+|^2 dx \\ &\leq \frac{1}{(1-\delta)^2} \int_{B_T(x)} |Du|^2 dx = \frac{\text{cap}(V \cap \mathcal{B}_p, B_T(x))}{(1-\delta)^2}. \end{aligned}$$

We obtain the assertion by (3.6) and (3.7).

Now it remains to prove that $u \leq \delta$ on $\partial B_{R_p}(x_\beta)$ for every $\beta \in I$. We start with the case $d = 2$.

For every $\beta \in I$ consider the function

$$(3.8) \quad u_\beta(x) = \left(\log \frac{\rho_p}{2T}\right)^{-1} \log \frac{|x - x_\beta|}{2T}$$

which is the solution to

$$(3.9) \quad \begin{cases} -\Delta u_\beta = 0 & \text{in } B_{2T}(x_\beta) \\ u_\beta = 1 & \text{on } \partial B_{\rho_p}(x_\beta) \\ u_\beta = 0 & \text{on } \partial B_{2T}(x_\beta), \end{cases}$$

and define

$$(3.10) \quad z(x) = \sum_{\beta \in I} u_{\beta}(x);$$

observe that z is superharmonic on \mathbb{R}^2 , as sum of superharmonic functions. Since, for every $\beta \in I$, we have $B_T(x) \subset\subset B_{2T}(x_{\beta})$, it follows that $u_{\beta}(y) \geq 0$ for every $y \in \partial B_T(x)$ and for every $\beta \in I$. Therefore $z \geq 0$ on $\partial B_T(x)$ and $z \geq 1$ on $\mathcal{E}_p \cap V$; hence $z \geq u$ in $B_T(x)$, since the capacitary potential u is characterized to be smaller than any other positive superharmonic function which is greater than or equal to 1 on $\mathcal{E}_p \cap V$. To achieve the proof it is sufficient, by the maximum principle, to consider a fixed $y \in \partial B_{R_p}(x_{\beta})$ and prove that $z(y) \leq \delta$.

By definition of z , we have

$$z(y) = \sum_{\beta \in I} \left(\log \frac{\rho_p}{2T} \right)^{-1} \log \frac{|y - x_{\beta}|}{2T}.$$

Let $I_q = \{\beta \in I : R\rho^{p-q} \leq |x_{\beta} - y| < R\rho^{p-q-1}\}$; then

$$(3.11) \quad z(y) \leq \sum_{q=0}^{\bar{q}} \#(I_q) \left(\log \frac{\rho_p}{2T} \right)^{-1} \log \left(\frac{R\rho^{p-q}}{2T} \right),$$

where $\bar{q} = \bar{q}(h, p) = p - \left\lceil \log_{\rho} \left(\frac{1}{R} \text{diam} V \right) \right\rceil$. Now we have

$$I_q \subset \{\beta \in I : |x_{\beta} - y| < R\rho^{p-q-1}\},$$

so by Proposition 3.2 we get

$$\#(I_q) \leq N^{q+1} \#\{\gamma \in C_{p-q-1}(N) : \text{dist}(S_{\bar{\beta}}(\mathcal{O}), S_{\gamma}(\mathcal{O})) < R\rho^{p-q-1}\} \leq MN^{q+1},$$

where $\bar{\beta} \in C_{p-q-1}(N)$ is determined by $x_{\beta} \in S_{\bar{\beta}}(\mathcal{O})$. Recalling that $N = \rho^{-\alpha}$, we obtain

$$(3.12) \quad z(y) \leq \frac{MN}{\log \left(\frac{2T}{\rho_p} \right)} \sum_{q=0}^{\bar{q}} \log \left(\frac{2T\rho^{q-p}}{R} \right) \rho^{-q\alpha}.$$

We observe that the function

$$x \mapsto \rho^{-x\alpha} \log \left(\frac{2T}{R} \rho^{x-p} \right) = N^x \left(\log \left(\frac{2T}{R} \rho^{-p} \right) + x \log \rho \right)$$

is increasing in $(0, \bar{q} + 1)$ for p large enough. Therefore we can estimate the sum at the right hand side by an integration as follows

$$\begin{aligned}
\sum_{q=0}^{\bar{q}} \log\left(\frac{2T}{R} \rho^{q-p}\right) \rho^{-q\alpha} &\leq \int_0^{\bar{q}+1} \log\left(\frac{2T}{R} \rho^{x-p}\right) \rho^{-x\alpha} dx \\
(3.13) \qquad \qquad \qquad &= \left(\log 1/\rho\right)^{-1} \int_1^{\rho^{-\bar{q}-1}} \log\left(\frac{2T}{R} \rho^{-p} y^{-1}\right) y^{\alpha-1} dy \\
&\leq c \rho^{-p\alpha} (\text{diam} V)^\alpha \left| \log(\text{diam} V) \right|.
\end{aligned}$$

From (3.12), we obtain (recalling the definition of ρ_p)

$$\begin{aligned}
z(y) &\leq c \rho^{-p\alpha} (\text{diam} V)^\alpha \left| \log(\text{diam} V) \right| \frac{MN}{\log\left(\frac{T}{\rho_p}\right)} \\
(3.14) \qquad \qquad \qquad &\leq c \frac{\rho^{-p\alpha} (\text{diam} V)^\alpha \left| \log(\text{diam} V) \right|}{c_0^{-1} (R\rho^p)^{-\alpha} - \log\left(\frac{R\rho^p}{T}\right)} \leq c (\text{diam} V)^\alpha \left| \log(\text{diam} V) \right|.
\end{aligned}$$

We can take then $\delta = c (\text{diam} V)^\alpha \left| \log(\text{diam} V) \right|$.

In the case $d \geq 3$ we may proceed in the same way as in the case $d = 2$, using in (3.8) the functions

$$u_\beta(x) = \frac{\rho_p^{d-2}}{T^{d-2} - \rho_p^{d-2}} \left(\left(\frac{|x - x_\beta|}{T} \right)^{2-d} - 1 \right),$$

which verify (3.9), and defining the function z as in (3.10). We get then, as in (3.11) and (3.12),

$$\begin{aligned}
z(y) &\leq \sum_{q=0}^{\bar{q}} \#(I_q) \frac{\rho_p^{d-2}}{T^{d-2} - \rho_p^{d-2}} \left(\frac{R\rho^{p-q}}{T} \right)^{2-d} \\
&\leq c \rho_p^{d-2} \sum_{q=0}^{\bar{q}} \#(I_q) \rho^{(p-q)(2-d)} \leq c \rho^{p(\alpha-d+2)} \sum_{q=0}^{\bar{q}} N^q \rho^{(n-d)q} \\
&= c \rho^{p(\alpha-d+2)} \sum_{q=0}^{\bar{q}} \rho^{(n-d-\alpha)q} \leq c \rho^{p(\alpha-d+2)} (\rho^{d-2-\alpha})^{\bar{q}+1} \leq c (\text{diam} V)^{\alpha-d+2},
\end{aligned}$$

and we achieve the proof of the Lemma. \square

We can now conclude the proof of Theorem 2.1. It will suffice to compute the function φ as defined in Lemma 1.4, and show that $\varphi(x) \equiv c_1$ q.e. on K .

We choose $\nu = \mathcal{H}^\alpha|_K$, $E_p = \mathcal{B}_p$, and $T > 0$ a fixed real number, and compute then φ for every $x \in K$ using (1.6). Notice that $\mathcal{H}^\alpha(B_t(x) \cap K) > 0$ for every $t > 0$, and we have $\mathcal{H}^\alpha(\partial B_t(x) \cap K) = 0$, except for at most a countable number of t . Let $I = \{\beta \in \mathbb{C}_p(N) : B_\beta^p \cap B_t(x) \neq \emptyset\}$. From (3.2) we obtain

$$(3.15) \quad \mathcal{H}^\alpha(B_t(x) \cap K) = \lim_{p \rightarrow +\infty} \rho^{p\alpha} \mathcal{H}^\alpha(K) \#(I).$$

Recall that, from the elementary properties of the capacity (cf. [Cho]), we have

$$\begin{aligned} \text{cap}(B_t(x) \cap \mathcal{B}_p, B_T(x)) &\leq \sum_{\beta \in I} \text{cap}(B_\beta^p, B_{R_p}(x_\beta)) \\ &= \#(I) \text{cap}(\rho_p E, B_{R_p}(0)), \end{aligned}$$

hence by (3.15) we get

$$\liminf_{t \rightarrow 0} \limsup_{p \rightarrow +\infty} \frac{\text{cap}(B_t(x) \cap \mathcal{B}_p, B_T(x))}{\mathcal{H}^\alpha(B_t(x) \cap K)} \leq \lim_{p \rightarrow +\infty} \frac{\text{cap}(\rho_p E, B_{R_p}(0))}{\rho^{p\alpha} \mathcal{H}^\alpha(K)}.$$

By using Lemma 3.3 and (3.2), we obtain also

$$\begin{aligned} &\liminf_{t \rightarrow 0} \liminf_{p \rightarrow +\infty} \frac{\text{cap}(B_t(x) \cap \mathcal{B}_p, B_T(x))}{\mathcal{H}^\alpha(B_t(x) \cap K)} \\ &\geq \liminf_{t \rightarrow 0} \liminf_{p \rightarrow +\infty} \frac{\text{cap}(\rho_p E, B_{R_p}(0)) (1 - \delta(B_t(x)))^2}{\mathcal{H}^\alpha(K) \rho^{p\alpha}} \\ &= \lim_{p \rightarrow +\infty} \frac{\rho^{-\alpha p}}{\mathcal{H}^\alpha(K)} \text{cap}(\rho_p E, B_{R_p}(0)). \end{aligned}$$

Hence the hypotheses of Lemma 1.4 are established.

If $d \geq 3$, we have

$$\begin{aligned} \varphi(x) &= \lim_{p \rightarrow +\infty} \frac{\rho^{-\alpha p}}{\mathcal{H}^\alpha(K)} \rho_p^{d-2} \text{cap}(E, \rho_p^{-1} B_{R_p}(0)) \\ &= \frac{c_0^{d-2} R^\alpha}{\mathcal{H}^\alpha(K)} \text{cap}(E, \mathbf{R}^d) = c_1; \end{aligned}$$

in the case $d = 2$ we have

$$\varphi(x) = \lim_{p \rightarrow +\infty} \frac{\rho^{-\alpha p}}{\mathcal{H}^\alpha(K)} \text{cap}(E, \rho_p^{-1} B_{R_p}(0)) = c_1.$$

Finally, we remark that since $\mathcal{H}^\alpha|_K \in H^{-1}(\Omega)$, we have that $L^1(\Omega, \mathcal{H}^\alpha|_K) \supset H_0^1(\Omega)$; therefore formula (2.10) is valid for test functions in $H_0^1(\Omega)$. \square

4. Figures

We take as \mathcal{O} the larger hexagon in Fig. 1 and S_1, \dots, S_7 the seven similitudes with $\rho = 1/3$ which carry \mathcal{O} into the smaller hexagons. Fig. 2 shows a possible choice of x_0 and R .

In Fig. 3 it is shown \mathcal{B}_5 with $E = B_1(0)$ for a proper choice of c_0 .

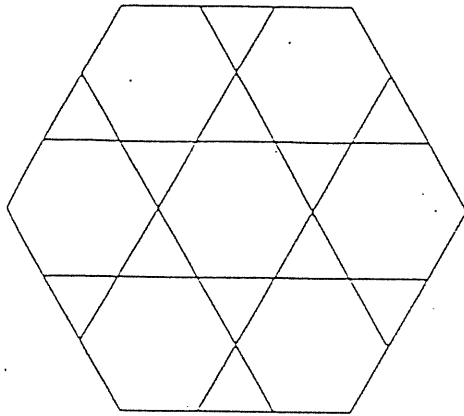


Figure 1

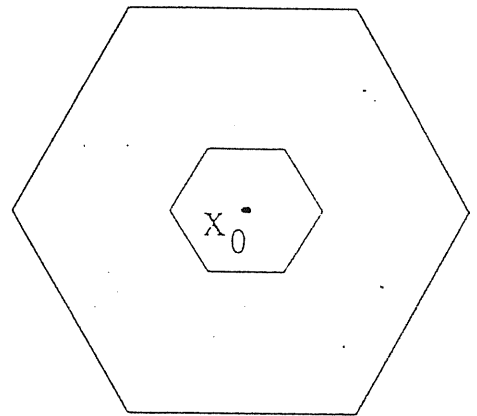


Figure 2

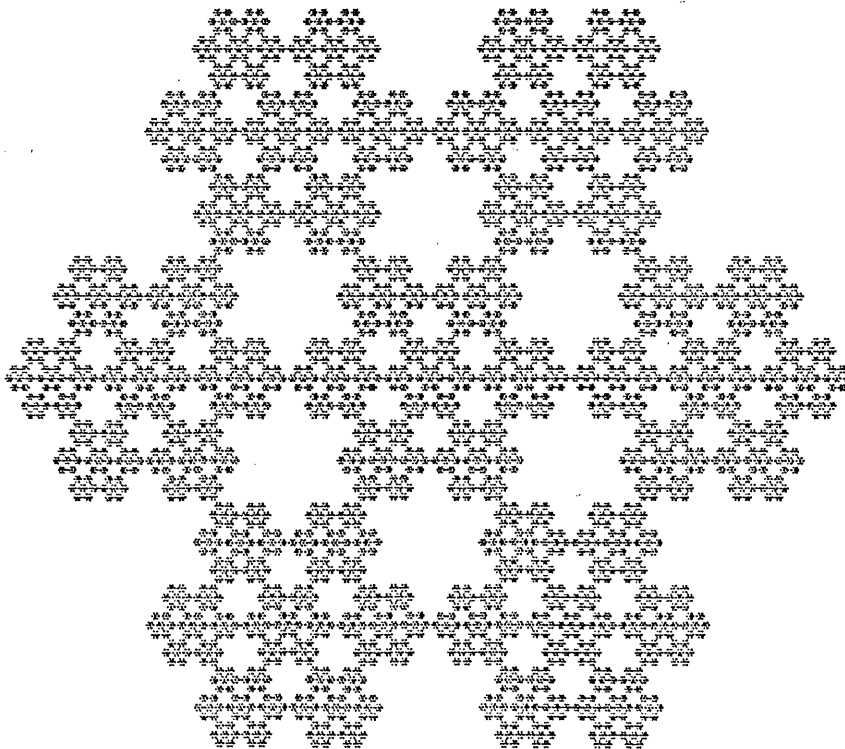


Figure 3

3. Dirichlet Problems on a Riemannian Manifold with Random Holes

1. Notation & Preliminaries

We recall that we are given an *oriented, connected* Riemannian manifold M of class C^3 , with $\dim M \stackrel{\text{def}}{=} d \geq 2$.

By $B_\rho(x)$ we denote the *open geodesic ball* of center $x \in M$ and radius ρ .

We recall here Property 1, introduced in the first section of the first chapter; we also introduce a second property, concerning the sectional curvature of M . Both these properties follow from the regularity assumption on M .

Property 1. The metric components $(g_{ij})_{i=1}^d$ belong to $C^2(M)$ and, for every relatively compact open set $A \subset M$, there exists $\kappa > 0$ such that:

$$(1.1) \quad \kappa^{-1} \sum_{i=1}^d (\xi^i)^2 \leq g_{ij}(x) \xi^j \xi^i \leq \kappa \sum_{i=1}^d (\xi^i)^2$$

for all $x \in A$, and for all $\xi \in \mathbf{R}^d$.

Let $x \in M$ and let $R : M_x \times M_x \times M_x \rightarrow M_x$ be the (Riemann) curvature tensor; given two linear independent vector fields $\xi, \eta \in M_x$, we introduce $K : M_x \times M_x \rightarrow \mathbf{R}$ defined by

$$K(\xi, \eta) = \frac{\langle R(\xi, \eta)\xi, \eta \rangle}{\left[\langle \xi, \xi \rangle \langle \eta, \eta \rangle - (\langle \xi, \eta \rangle)^2 \right]}$$

$K(\xi, \eta)$ is the *sectional curvature* of the 2-dimensional plane determined by ξ, η ; $\langle \cdot, \cdot \rangle$ denotes the scalar product in M_x induced by the metric tensor g .

Property 2. For each $x \in M$ there exist $\lambda^+(x)$, $\lambda^-(x)$, which are bounded on compact sets, such that

$$\lambda^-(x) \leq K(\xi, \eta) \leq \lambda^+(x),$$

for all 2-dimensional plane determined by ξ, η , as ξ, η vary in M_x .

In the following we shall also consider \mathbf{R}^d , always equipped with the euclidean metric, given by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, i, j = 1, \dots, d, \end{cases}$$

and we denote the open ball of center $x \in \mathbf{R}^d$ and radius $r > 0$ by $B_r^e(x)$.

We let ω_d denote the $(d - 1)$ -dimensional Hausdorff measure of $\{y \in \mathbf{R}^d : \sum_{i=1}^d (y^i)^2 = 1\}$; cf. the first section in the second chapter. The letter c will denote a strictly positive constant whose value may vary from line to line.

2. Capacities on manifold

2.1 Definition. Let $A \subset M$ be a relatively compact open set and let E be a Borel subset of A . The *harmonic capacity* of E w.r.t. A is

$$(2.1) \quad \text{cap}(E, A) = \inf_{u \in H_0^1(A)} \left\{ \int_A g^{ij} \partial_j u \partial_i u \, dV_g, u \geq 1 \text{ a.e. on a neighborhood of } E \right\}.$$

Let now E be a Borel subset of M . The *meta-harmonic capacity* of E is defined as

$$(2.2) \quad c(E) = \inf_{u \in H^1(M)} \left\{ \int_M (g^{ij} \partial_j u \partial_i u^2 + u^2) \, dV_g, u \geq 1 \text{ a.e. on a neighborhood of } E \right\}.$$

2.2 Remark. Let A be a relatively compact open set of M ; it can be proven that the harmonic capacity is a set functions which satisfies the following properties:

- (a) if $E_1 \subset E_2$ are two Borel sets, then $\text{cap}(E_1, A) \leq \text{cap}(E_2, A)$;
 (b) if (E_h) is an increasing sequence of Borel sets of A and $E = \bigcup_{h \in \mathbb{N}} E_h \subset A$, then

$$\text{cap}\left(\bigcup_{h \in \mathbb{N}} E_h, A\right) = \sup_{h \in \mathbb{N}} \text{cap}(E_h, A);$$

- (c) if (K_h) is a decreasing sequence of compact sets contained in A and $K = \bigcap_{h \in \mathbb{N}} K_h$, then $\text{cap}\left(\bigcap_{h \in \mathbb{N}} K_h, A\right) = \inf_{h \in \mathbb{N}} \text{cap}(K_h, A)$;
 (d) if E_1, E_2 are two Borel sets of A , then $\text{cap}(E_1 \cup E_2, A) \leq \text{cap}(E_1, A) + \text{cap}(E_2, A)$.

Analogous properties to (a), (b), (c), (d) hold true for the meta-harmonic capacity; cf. Proposition 1.4 in [DM1].

2.3 Remark. Using standard variational methods, such those in [K-S, Chapter II, §6], it is possible to prove that the infimum both in (2.1) and in (2.2) is attained. We will call the (unique) function $u_E \in H_0^1(A)$ which realizes the minimum in (2.1) the *capacitary potential* of E w.r.t. A . From Definition 2.1 it follows that

$$\text{cap}(E, A) \leq c(E) \leq \text{cap}(E, A) + \int_A (u_E)^2 dV_g.$$

This implies that $c(E) = 0$ if and only if $\text{cap}(E, A) = 0$.

In the sequel we shall need the following result.

2.4 Proposition. Let $p \in M$ and consider $B_r(p) \subset B_R(p)$, with $0 < r < R$. Denoting by v_r the capacitary potential of $B_r(p)$ w.r.t. $B_R(p)$, we have

$$(2.3) \quad \lim_{r \rightarrow 0} \frac{1}{\text{cap}(B_r(p), B_R(p))} \int_{B_R(p)} (v_r)^2 dV_g = 0.$$

Proof. We prove (2.3) first in \mathbb{R}^d . The harmonic capacity (in \mathbb{R}^d) of $B_r^e(p)$ w.r.t. $B_R^e(p)$ is

$$\text{cap}(B_r^e(p), B_R^e(p)) = \min_{v \in H_0^1(B_R^e(p))} \left\{ \int_{B_R^e(p)} \sum_{i=1}^d |\partial_i v|^2 dy : v \geq 1 \text{ on } B_r^e(p) \right\},$$

and let w_r be the capacity potential of $B_r^e(p)$ w.r.t. $B_R^e(p)$. It is easy to see that

$$w_r(y) = \begin{cases} \frac{r^{d-2}}{R^{d-2} - r^{d-2}} \left(\frac{R^{d-2}}{|y|^{d-2}} - 1 \right) & \text{if } y \in B_R^e(p) \setminus \overline{B_r^e(x)} \\ 1 & \text{if } y \in \overline{B_r^e(p)}, \end{cases}$$

for $d \geq 3$, while

$$w_r(y) = \begin{cases} \left(\log \frac{R}{r} \right)^{-1} (\log R - \log |y|) & \text{if } y \in B_R^e(p) \setminus \overline{B_r^e(x)} \\ 1 & \text{if } x \in \overline{B_r^e(p)}, \end{cases}$$

for $d = 2$, and

$$\text{cap}(B_r^e(p), B_R^e(p)) = \begin{cases} \omega_d(d-2) \frac{r^{d-2}}{1 - (r/R)^{d-2}}, & \text{if } d \geq 3 \\ 2\pi \left(\log \frac{R}{r} \right)^{-1}, & \text{if } d = 2. \end{cases}$$

Now an easy computation gives

$$(2.4) \quad \lim_{\rho \rightarrow 0} \frac{1}{\text{cap}(B_r^e(p), B_R^e(p))} \int_{B_R^e(p)} w_\rho^2 dV_g = 0.$$

Now we are given $B_r(p) \subset B_R(p)$ in a Riemannian manifold M ; it is not restrictive to assume that $B_R(p)$ is contained in a local chart (U, ϕ) . The expression of the Laplace-Beltrami operator in local coordinates is

$$L = \partial_i (a^{ij}(y) \partial_j),$$

where $a^{ij}(y) = a^{ji}(y) = \sqrt{\det g} g^{ij}(y)$; moreover there exists $\kappa > 0$ such that

$$(2.5) \quad \kappa^{-1} \sum_{i=1}^d (\xi_i)^2 \leq a^{ij}(y) \xi_i \xi_j \leq \kappa \sum_{i=1}^d (\xi_i)^2$$

for every $\xi \in \mathbf{R}^d$ and for every $y \in \mathcal{O} \stackrel{\text{def}}{=} \phi(U) \subset \mathbf{R}^d$. Let $x = \phi(p)$ and consider, for every $0 < \rho < \sigma$,

$$(2.6) \quad \text{cap}_L(B_\rho^e(x), B_\sigma^e(x)) \stackrel{\text{def}}{=} \min_{u \in H_0^1(B_\sigma^e(x))} \left\{ \int_{B_\sigma^e(x)} a^{ij} \partial_i u \partial_j u : u \geq 1 \text{ a.e. on } B_\rho^e(x) \right\},$$

and let $u_L(\cdot)$ be the function which realizes the minimum in (2.6). From (2.5) it follows that

$$(2.7) \quad \kappa^{-1} \text{cap}(B_\rho^e(x), B_\sigma^e(x)) \leq \text{cap}_L(B_\rho^e(x), B_\sigma^e(x)) \leq \kappa \text{cap}(B_\rho^e(x), B_\sigma^e(x));$$

moreover from Corollary 7.1 in [L-S-W] there exists a constant $c = c(\kappa, \sigma, \mathcal{O})$ such that

$$(2.8) \quad c^{-1} w_\rho(y) \leq u_L(y) \leq c w_\rho(y)$$

for every $y \in B_\sigma^e(x)$. From (2.7) and (2.8) we get

$$\frac{(u_L(y))^2}{\text{cap}_L(B_\rho^e(x), B_\sigma^e(x))} \leq \kappa c^2 \frac{(w_\rho(y))^2}{\text{cap}_H(B_\rho^e(x), B_\sigma^e(x))}$$

and using (2.4) we have

$$\lim_{\rho \rightarrow 0} \frac{1}{\text{cap}_L(B_\rho^e(x), B_\sigma^e(x))} \int_{B_\sigma^e(x)} (u_L(y))^2 dy = 0.$$

Noticing that

$$\text{cap}(B_r(p), B_R(p)) = \text{cap}_L(\varphi(B_r(p)), \varphi(B_R(p))),$$

to get the result it is enough to show that

$$\lim_{r \rightarrow 0} \frac{1}{\text{cap}_L(\varphi(B_r(p)), \varphi(B_R(p)))} \int_{B_R(p)} v_r^2 dV_g = 0.$$

Let ρ be such that $0 < \rho \leq \text{Lip}(\varphi, B_R(p))r$ so we have

$$B_\rho^e(x) \subset \varphi(B_r(p))$$

and let $0 < \sigma$ be such that

$$\varphi(B_R(p)) \subset B_\sigma^e(x).$$

Then on one hand we have

$$\text{cap}_L(\varphi(B_r(p)), \varphi(B_R(p))) \geq \text{cap}_L(B_\rho^e(x), B_\sigma^e(x)),$$

while on the other hand we have from the maximum principle that the capacity potential v_r is less than or equal to u_L . So

$$\frac{1}{\text{cap}_L(\varphi(B_r(p)), \varphi(B_R(p)))} \int_{B_R(p)} v_r^2 dV \leq \frac{1}{\text{cap}_L(B_\rho^e(x), B_\sigma^e(x))} \int_{B_\sigma^e(x)} u_L^2 dy.$$

Now we pass to the limit as $r \rightarrow 0$ (hence as $\rho \rightarrow 0$) and we get the result. \square

For the capacity potential u_E which minimizes the functional in (2.1) we give a *representation formula* in Proposition 2.7 below by means of the *Green function* of the Laplace-Beltrami operator. This result can be proven adapting to our case the methods developed in [L-S-W, §§ 5& 6]. Before stating the result, we introduce and give some properties of the Green function for Δ which are needed in the following.

2.5 Definition. ([Au, Definition 4.11]) Let $\bar{W} = W \cup \partial W$ be a compact manifold with boundary ∂W . The Green function $g_W(x, y)$ of the Laplace-Beltrami operator, with Dirichlet boundary condition on ∂W , is the function which satisfies, for $x, y \in W$,

$$\Delta_{(y)}g_W(x, y) = \delta_x \quad \text{for } x, y \in W$$

in the sense of distribution and which vanishes for $x, y \in \partial W$; here δ_x is the Dirac mass at x . The subscript “(y)” indicates here that the Laplace-Beltrami operator acts on $y \mapsto g_W(x, y)$.

The properties we shall need are listed in the following proposition.

2.6 Proposition. Let $\bar{W} = W \cup \partial W$ be an oriented compact manifold with boundary ∂W . There exists $g_W(x, y)$, the Green function of the Laplace-Beltrami operator, which has the following properties:

- (i) $g_W(x, y) > 0$ for $x, y \in W$;
- (ii) $g_W(x, y) = g_W(y, x)$;
- (iii) we have the following estimate

$$(2.9) \quad g_W(x, y) < \begin{cases} Cr^{2-d}, & \text{for } d > 2, \\ C(|\ln r| + 1), & \text{for } d = 2, \end{cases}$$

where $r = d(x, y)$ and C is a constant which depends on the distance of x to the boundary.

Proof. See Theorem 4.17-(a), (e), (c) in [Au]. □

2.7 Proposition. *Let A be a relatively compact open set in M , let $E \subset A$ be a compact set and let u_E be the capacitary potential of E w.r.t. A . There exists a Radon measure μ_E for which the integral*

$$\int_A g_A(x, y) \mu_E(dy)$$

exists, it is finite almost everywhere (w.r.t. the Lebesgue measure on A) and

$$u_E(x) = \int_A g_A(x, y) \mu_E(dy).$$

The measure μ_E is called the capacitary distribution of E and μ_E vanishes on all Borel sets having harmonic capacity zero. Moreover the measure μ_E is supported on ∂E . \square

Let

$$c(r, R) = \begin{cases} \omega_d(d-2) \frac{r^{d-2}}{1 - (r/R)^{d-2}}, & \text{if } d \geq 3 \\ 2\pi \left(\log \frac{R}{r}\right)^{-1}, & \text{if } d = 2; \end{cases}$$

we recall that in the particular case of $M = \mathbf{R}^d$, we have

$$c(r, R) = \text{cap}(B_r^e(x), B_R^e(x)).$$

The following result concerns an estimate on the harmonic capacity of two concentric balls in M in term of $c(r, R)$.

2.8 Lemma. *For each $x \in M$ there exist $\lambda'(x) \leq \lambda''(x)$, which are bounded on compact sets, such that*

$$(2.10) \quad \begin{aligned} (1 + \lambda'(x)R^2 + O(R^3))c(r, R) &\leq \text{cap}(B_r(x), B_R(x)) \\ &\leq (1 + \lambda''(x)R^2 + O(R^3))c(r, R), \end{aligned}$$

for any $0 < r < R < \bar{R}$, where \bar{R} may depend on x .

Proof. Let us denote by \mathcal{O}_x the largest open set in M_x such that for any $\xi \in \mathcal{O}_x$ the geodesic

$$\gamma_\xi(t) = \exp(t\xi)$$

minimizes the distance from x to $\gamma_\xi(t)$, for any $x \in [0, 1]$. We set $\exp_x(\mathcal{O}_x) = D_x$, and we have that $\exp : \mathcal{O}_x \rightarrow D_x$ is a diffeomorphism. Let

$$\bar{R} \stackrel{\text{def}}{=} \sup\{\rho > 0 : \overline{B(x, \rho)} \subset D_x\}.$$

Note that for any $0 < R < \bar{R}$ the ball $B(x, R) = \exp(\mathcal{D}_R(x))$, where we have set $\mathcal{D}_R(x) \stackrel{\text{def}}{=} \{\xi \in M_x : \langle \xi, \xi \rangle_g < R\}$ ([Ch, p.65]). Let R be the Riemann curvature tensor, and Ric be the Ricci tensor,

$$\text{Ric}(\cdot, \cdot) = \sum_{i=1}^d \langle R(\cdot, e_i)\cdot, e_i \rangle_g$$

(cf. [Ch, p. 60]); let $(e_i)_{i=1}^d$ be an orthonormal basis in M_x . Using *Riemann normal coordinates* on D_x it is possible to show ([Ch, p. 318]) that for every $\xi \in \mathcal{O}_x$ we have

$$(2.11) \quad \begin{aligned} g^{ij}(y)|_{y=\exp_x \xi} &= \delta^{ij} + 1/3 \langle R(\xi, e_i)\xi, e_j \rangle_g + O(|\xi|_g^3) \\ \sqrt{\det g}(y)|_{y=\exp_x \xi} &= 1 - 1/6 \text{Ric}(\xi, \xi) + O(|\xi|_g^3); \end{aligned}$$

as the map

$$(\zeta, \sigma) \mapsto \langle R(\xi, \zeta)\xi, \sigma \rangle_g$$

is symmetric ([Ch, p. 59]), we may assume that $\langle R(\xi, e_i)\xi, e_j \rangle_g = 0$, for every $j \neq i$. Note that we have

$$\mathcal{D}_R(x) = \{\xi \in M_x : \langle \xi, \xi \rangle_g < R\} = \{\xi = (\xi^1 e_1, \dots, \xi^d e_d) : \sum_{i=1}^d (\xi^i)^2 < R^2\},$$

since

$$|\xi|_g^2 = \langle \xi, \xi \rangle_g = \xi^i \xi^k \langle e_k, e_i \rangle_g = \xi^i \xi^k \delta_{ik} = \sum_{i=1}^d (\xi^i)^2.$$

We want to estimate $\langle R(\xi, e_i)\xi, e_j \rangle$ and $\text{Ric}(\xi, \xi)$ in term of $|\xi|^2$. We recall that

$$\langle R(\xi, e_i)\xi, e_i \rangle = K(\xi, e_i) \left[|\xi|_g^2 - (\xi^i)^2 \right],$$

and from the property 2 we have

$$\lambda^-(x) \leq K(\xi, e_i) \leq \lambda^+(x).$$

Assume, for simplicity, that λ^-, λ^+ satisfy $0 \leq |\lambda^-(x)| \leq \lambda^+(x)$. Then

$$|\langle R(\xi, e_i)\xi, e_i \rangle_g| \leq \lambda^+(x)|\xi|_g^2;$$

moreover

$$(2.12) \quad \begin{aligned} |\text{Ric}(\xi, \xi)| &= \left| \sum_{i=1}^d \langle R(\xi, e_i)\xi, e_i \rangle_g \right| = \left| \sum_{i=1}^d K(\xi, e_i) \left[|\xi|_g^2 - (\xi^i)^2 \right] \right| \\ &\leq \lambda^+(x) \left| \sum_{i=1}^d \left[|\xi|_g^2 - (\xi^i)^2 \right] \right| = (d-1)\lambda^+(x)|\xi|_g^2. \end{aligned}$$

Let $u \in C^1(B_R(x))$; from the first formula in (2.11) we have

$$\sum_{i=1}^d |\partial_i u|^2 \left[1 - \lambda^+ R^2 + O(R^3) \right] \leq g^{ij} \partial_j u \partial_i u \leq \sum_{i=1}^d |\partial_i u|^2 \left[1 + \lambda^+ R^2 + O(R^3) \right].$$

Integrating on $B_R(x)$

$$(2.13) \quad \begin{aligned} \left[1 - \lambda^+ R^2 + O(R^3) \right] \int_{B_R(x)} \sum_{i=1}^d |\partial_i u|^2 dV_g &\leq \int_{B_R(x)} g^{ij} \partial_j u \partial_i u dV_g \\ &\leq \left[1 + \lambda^+ R^2 + O(R^3) \right] \int_{B_R(x)} \sum_{i=1}^d |\partial_i u|^2 dV_g. \end{aligned}$$

As

$$\int_{B_R(x)} \sum_{i=1}^d |\partial_i u|^2 dV_g = \int_{\mathcal{D}_R(x)} \left(\left[\sum_{i=1}^d |\partial_i u|^2 \sqrt{\det g} \right] \circ \exp_x(y) \right) dy,$$

we obtain from the second formula in (2.11), and from (2.12),

$$(2.14) \quad \begin{aligned} \left[1 - (d-1)\lambda^+(x)R^2 + O(R^3) \right] \int_{\mathcal{D}_R(x)} \sum_{i=1}^d |\partial_i u|^2 dy &\leq \int_{B_R(x)} \sum_{i=1}^d |\partial_i u|^2 dV_g \\ &\leq \left[1 + (d-1)\lambda^+(x)R^2 + O(R^3) \right] \int_{\mathcal{D}_R(x)} \sum_{i=1}^d |\partial_i u|^2 dy. \end{aligned}$$

Now from (2.13) and (2.14) we have for every $u \in C^1(B_R(x))$

$$\begin{aligned} \left[1 - d\lambda^+(x)R^2 + O(R^3) \right] \int_{\mathcal{D}_R(x)} \sum_{i=1}^d |\partial_i u|^2 dy &\leq \int_{B_R(x)} g^{ij} \partial_j u \partial_i u dV_g \\ &\leq \left[1 + d\lambda^+(x)R^2 + O(R^3) \right] \int_{\mathcal{D}_R(x)} \sum_{i=1}^d |\partial_i u|^2 dy, \end{aligned}$$

and from this we can conclude. \square

Let us indicate by \mathcal{B} the σ -algebra of all Borel subsets contained in M , by \mathcal{U} the family of all relatively compact open subsets of M and by \mathcal{K} the family of all compact subset of M .

2.9 Definition. We indicate by \mathcal{M}_0^* the class of all Borel measures μ on M such that

- (-) $\mu(B) = 0$ whenever $c(B) = 0$;
- (-) $\mu(B) = \inf\{\mu(A) : A \text{ quasi open}, B \subset A\}$, for every $B \in \mathcal{B}$.

We recall that a Borel set $B \subset M$ is said to be *quasi open* (resp. *quasi closed*) if for every $\varepsilon > 0$ there exists an open set (resp. a closed set) U such that

$$c(B \Delta U) < \varepsilon;$$

here Δ denotes the symmetric difference between B and U . Moreover B is quasi open if and only if B^c is quasi closed; the countable union or the finite intersection of quasi open sets is still quasi open.

The measure $\mu(B) = \int_B f dV$, for $f \in L^1(M)$, belongs to \mathcal{M}_0^* , as well the singular measure

$$(2.15) \quad \infty_E(B) = \begin{cases} 0 & \text{if } c(E \cap B) = 0 \\ +\infty & \text{if } c(E \cap B) > 0, \end{cases}$$

where E is a (quasi) closed subset of M .

2.10 Definition. Let $\mu \in \mathcal{M}_0^*$. For every $B \in \mathcal{B}$, we define the μ -capacity of B as

$$\text{cap}_\mu(B) = \inf \left\{ \int_M (g^{ij} \partial_j u \partial_i u + u^2) dV_g + \int_B (u - 1)^2 d\mu : u \in H^1(M) \right\}.$$

2.11 Remark. If $\mu = \infty_F$, for a (quasi) closed set F , then $\text{cap}_{\infty_F}(B) = c(B \cap F)$.

2.12 Proposition. For every $\mu \in \mathcal{M}_0^*$ the set function $\text{cap}_\mu(\cdot)$ satisfies the following properties:

- (a) $\text{cap}_\mu(\emptyset) = 0$;
- (b) if $B_1, B_2 \in \mathcal{B}$, $B_1 \subset B_2$, then $\text{cap}_\mu(B_1) \leq \text{cap}_\mu(B_2)$;
- (c) if (B_n) is an increasing sequence in \mathcal{B} , $B = \bigcup_n B_n$, then $\text{cap}_\mu(B) = \sup_n \text{cap}_\mu(B_n)$;
- (d) if (B_n) is a sequence in \mathcal{B} , $B \subset \bigcup_n B_n$, then $\text{cap}_\mu(B) \leq \sum_n \text{cap}_\mu(B_n)$;
- (e) $\text{cap}_\mu(B_1 \cup B_2) + \text{cap}_\mu(B_1 \cap B_2) \leq \text{cap}_\mu(B_1) + \text{cap}_\mu(B_2)$, for all $B_1, B_2 \in \mathcal{B}$;
- (f) $\text{cap}_\mu(B) \leq \text{cap}(B)$, for every $B \in \mathcal{B}$;
- (g) $\text{cap}_\mu(B) \leq \mu(B)$, for every $B \in \mathcal{B}$;
- (h) $\text{cap}_\mu(K) = \inf\{\text{cap}_\mu(U) : K \subset U, U \in \mathcal{U}\}$, for every $K \in \mathcal{K}$;
- (i) $\text{cap}_\mu(B) = \sup\{\text{cap}_\mu(K) : K \subset B, K \in \mathcal{K}\}$, for every $B \in \mathcal{B}$.

Proof. All these properties can be proven as in [DM2, §§2, 3], so we refer to this paper for the proof. \square

We may associate to every $\mu \in \mathcal{M}_0^*$ the functional $F_\mu : L^2(M) \rightarrow [0, +\infty]$ defined by

$$F_\mu(v) = \begin{cases} \int_M (g^{ij} \partial_j v \partial_i v + u^2) dV_g + \int_M u^2 d\mu - 2 \int_M f v dV_g, & \text{if } u \in H_0^1(M), \\ +\infty, & \text{otherwise.} \end{cases}$$

We recall that each function $u \in H^1(M)$ is defined up to a set of capacity zero; since the measure μ does not charge (Borel) sets of capacity zero, it follows that the functional above is well defined and F_μ is lower semicontinuous w.r.t. the strong topology of $L^2(M)$.

2.13 Definition. Let (μ_h) be a sequence in \mathcal{M}_0^* and let $\mu \in \mathcal{M}_0^*$. We say that (μ_h) γ -converges to μ if the following conditions are satisfied:

- (a) for every $u \in H_0^1(M)$ and for every sequence (u_h) in $H_0^1(M)$ converging to u in $L^2(M)$ we have

$$F_\mu(u) \leq \liminf_{h \rightarrow +\infty} F_{\mu_h}(u_h);$$

- (b) for every $u \in H_0^1(M)$ there exists a sequence (u_h) in $H_0^1(M)$ such that u_h converges to u in $L^2(M)$ and

$$F_\mu(u) \geq \limsup_{h \rightarrow +\infty} F_{\mu_h}(u_h).$$

2.14 Remark. It can be proven that there exists a unique metrizable topology τ_γ on \mathcal{M}_0^* which induces the γ -convergence. All topological notions we shall consider are relative to τ_γ w.r.t. which \mathcal{M}_0^* is also *metrizable and compact*; see [DM-M, §4] for more details. \square

We recall here the Proposition 3.8 in the first Chapter.

2.15 Proposition. *Let (μ_h) be a sequence in \mathcal{M}_0^* and let $\mu \in \mathcal{M}_0^*$. Then (μ_h) γ -converges to μ if and only if*

$$(a) \text{ cap}_\mu(U) \leq \liminf_{h \rightarrow +\infty} \text{cap}_{\mu_h}(U),$$

$$(b) \text{ cap}_\mu(K) \geq \limsup_{h \rightarrow +\infty} \text{cap}_{\mu_h}(K)$$

are satisfied for every $K \in \mathcal{K}$ and for every $U \in \mathcal{U}$. \square

2.16 Remark. By the above proposition it follows that a sub-base for the topology τ_γ is given by $\{\mu \in \mathcal{M}_0^* : \text{cap}_\mu(U) > t\}$, $\{\mu \in \mathcal{M}_0^* : \text{cap}_\mu(K) < s\}$, for $t, s > 0$, $K \in \mathcal{K}$ and $U \in \mathcal{U}$. We therefore may speak about open and closed sets in \mathcal{M}_0^* , hence about Borel sets whose family we denote by $\mathcal{B}(\mathcal{M}_0^*)$. \square

From the next proposition we get some useful measurability properties of the μ -capacity. By $\mathcal{B}(\mathcal{M}_0^*)$ we denote the Borel σ -field of \mathcal{M}_0^* .

2.17 Proposition. *The family $\mathcal{B}(\mathcal{M}_0^*)$ is the smallest σ -algebra for which the function $\text{cap}_\mu(U) : \mathcal{M}_0^* \rightarrow [0, +\infty]$ is measurable for every $U \in \mathcal{U}$ (resp. the function $\text{cap}_\mu(K) : \mathcal{M}_0^* \rightarrow [0, +\infty]$ is measurable for every $K \in \mathcal{K}$).*

Proof. The proof of these results can be obtained adapting the proofs of Proposition 2.3 and Proposition 2.4 in [B]. \square

From the previous proposition we have the following consequence.

2.18 Corollary. *Let (Λ, Σ, P) be a measure space and let $m : \Lambda \rightarrow \mathcal{M}_0^*$ be a function. The following statements are equivalent:*

- (i) m is $\Sigma/\mathcal{B}(\mathcal{M}_0^*)$ -measurable;
- (ii) $\text{cap}_{m(\cdot)}(U)$ is Σ -measurable, for every $U \in \mathcal{U}$;
- (iii) $\text{cap}_{m(\cdot)}(K)$ is Σ -measurable, for every $K \in \mathcal{K}$. \square

2.19 Lemma. *Let A be a relatively compact open set; for every compact set $K \subset A \subset M$, and for every $R > 0$, the real-valued function, defined on $M \times \cdots \times M$ (p -times) by*

$$(x_1, \dots, x_p) \mapsto \text{cap} \left(\bigcup_{i=1}^p \overline{B_R(x_i)} \cap K, A \right)$$

is upper semicontinuous in $M \times \cdots \times M$.

Proof. Adapt the proof of Lemma 3.1 in [B]. □

3. Dirichlet Problems with Random Holes

Let (Ω, Σ, P) be a probability space. We shall denote by \mathbb{E} and Cov respectively the *expectation* and the *covariance* of a random variable w.r.t. the measure P .

3.1 Definition. A measurable function $m : \Omega \rightarrow \mathcal{M}_0^*$ will be called a *random measure*.

We recall that necessary and sufficient conditions for the measurability of the function $m : \Omega \rightarrow \mathcal{M}_0^*$ are given in Corollary 2.18.

Let $\mu \in \mathcal{M}_0^*$. Let us consider the following Dirichlet problem, formally written for every $f \in L^2(M)$ as

$$(3.1) \quad \begin{cases} -\Delta_g u + \lambda u + \mu u = f & \text{in } M \\ u \in H_0^1(M). \end{cases}$$

We say that $u \in H_0^1(M) \cap L^2(M, \mu)$ is a *weak solution* of (3.1) if

$$\int_M (\langle \nabla u, \nabla v \rangle_g + \lambda uv) dV_g + \int_M uv d\mu = \int_M fv dV_g$$

for any $v \in H_0^1(M) \cap L^2(M, \mu)$.

Let $\mu \in \mathcal{M}_0^*$; the *resolvent operator* for the Dirichlet problem (3.1)

$$R_\lambda^\mu : L^2(M) \longrightarrow L^2(M)$$

is defined as the operator that associates to every $f \in L^2(M)$ the unique solution u to (3.1). Observe that R_λ^μ is a *positive* and *linear* operator.

In the sequel we are interested in sequences of Dirichlet problems such as

$$(3.2) \quad \begin{cases} -\Delta_g u + \lambda u + m_h u = f & \text{in } M \\ u \in H_0^1(M), \end{cases}$$

where (m_h) is a sequence of random measures. In particular we want also to study the asymptotic behavior as $h \rightarrow +\infty$ of the resolvent operators $R_\lambda^{m_h}$ associated to the random measures m_h . The following theorem gives an answer in this sense; its proof can be obtained adapting the arguments used in [B, Theorem 4.1].

We recall that \mathcal{U} denotes the family of all relatively compact open sets in M .

3.2 Definition. Let us define the following set functions:

$$\begin{aligned} \alpha'(U) &= \liminf_{h \rightarrow +\infty} \mathbb{E}[\text{cap}_{m_h(\cdot)}(U)] \\ \alpha''(U) &= \limsup_{h \rightarrow +\infty} \mathbb{E}[\text{cap}_{m_h(\cdot)}(U)] \end{aligned}$$

for every $U \in \mathcal{U}$. Next consider the *inner regularization* of α' and α'' defined for every $U \in \mathcal{U}$ by

$$\begin{aligned} \overline{\alpha'}(U) &= \sup\{\alpha'(V) : V \in \mathcal{U}, \overline{U} \subset V\}, \\ \overline{\alpha''}(U) &= \sup\{\alpha''(V) : V \in \mathcal{U}, \overline{U} \subset V\}. \end{aligned}$$

Then extend $\overline{\alpha'}$ and $\overline{\alpha''}$ to arbitrary Borel sets $B \in \mathcal{B}$ by

$$(3.3) \quad \overline{\alpha'}(B) = \inf\{\overline{\alpha'}(U) : V \in \mathcal{U}, B \subset U\},$$

$$(3.4) \quad \overline{\alpha''}(B) = \inf\{\overline{\alpha''}(U) : V \in \mathcal{U}, B \subset U\}.$$

Finally denote by ν' , ν'' the least superadditive set functions defined on \mathcal{B} greater than or equal to $\overline{\alpha'}$ and $\overline{\alpha''}$ respectively.

3.3 Theorem. Let (m_h) be a sequence of random measures. Let α' and α'' be defined as in Definition 3.2 above; let ν' and ν'' be the least superadditive set functions on \mathcal{B} greater than or equal to $\overline{\alpha'}$ and $\overline{\alpha''}$ respectively. Assume that

(i) $\nu'(B) = \nu''(B)$ and denote by $\nu(B)$ their common value, for every $B \in \mathcal{B}$;

(ii) there exist $\varepsilon > 0$, a continuous increasing function $\xi : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ with $\xi(0,0) = 0$ and a Radon measure β on \mathcal{B} such that

$$\limsup_{h \rightarrow +\infty} \text{Cov}(\text{cap}_{m_h(\cdot)}(U), \text{cap}_{m_h(\cdot)}(V)) \leq \xi(\text{diam}(U), \text{diam}(V))\beta(U)\beta(V)$$

for every pair $U, V \in \mathcal{U}$ such that $\bar{U} \cap \bar{V} = \emptyset$ and $\text{diam}(U) < \varepsilon$, $\text{diam}(V) < \varepsilon$.

Then, for every $\lambda > 0$, $R_\lambda^{m_h}$ converges strongly in probability to R_λ^ν , that is

$$\lim_{h \rightarrow +\infty} P\{\omega \in \Omega : \|R_\lambda^{m_h}(\omega)(f) - R_\lambda^\nu(f)\|_{L^2(M)} > \eta\} = 0$$

for every $\eta > 0$ and for any $f \in L^2(M)$. □

From now on, we shall consider a particular class of random measures, which are related to Dirichlet problems with random holes.

Let us denote by \mathcal{C} the family of closed sets contained in A .

3.4 Definition. A function $F : \Omega \rightarrow \mathcal{C}$ is called a *random set* if the function $m : \Omega \rightarrow \mathcal{M}_0^*$ defined by $m(\omega) = \infty_{F(\omega)}$ for each $\omega \in \Omega$ is Σ -measurable, where $\infty_{F(\omega)}$ is the singular measure defined in (2.15).

Let $F : \Omega \rightarrow \mathcal{C}$ be a function; using the notation introduced in §2, it follows from Corollary 2.18 that the following are equivalent:

- a) F is a random set;
- b) $\text{cap}_{F(\cdot)}(U)$ is Σ -measurable for every $U \in \mathcal{U}$;
- c) $\text{cap}_{F(\cdot)}(K)$ is Σ -measurable for every $K \in \mathcal{K}$.

Let (F_h) be a sequence of random sets and let $(m_h(\omega))$ be the sequence of random measures so defined

$$m_h(\omega) = \infty_{F_h(\omega)} \quad \text{for each } \omega \in \Omega.$$

Let $f \in L^2(M)$ and $\lambda > 0$ a parameter. We shall consider the weak solutions u_h of the following Dirichlet problems on random domains

$$(3.5) \quad \begin{cases} -\Delta_g u_h + \lambda u = f & \text{in } M \setminus F_h \\ u \in H_0^1(M \setminus F_h). \end{cases}$$

As in [DM-M], it can be shown that the above Dirichlet problem can be written using the measures ∞_{F_h} as

$$(3.6) \quad \begin{cases} -\Delta_g u_h + \lambda u + \infty_{F_h} u_h = f & \text{in } M \\ u \in H_0^1(M); \end{cases}$$

the resolvent operator is defined as

$$R_\lambda^{\infty_{F_h}}(f) = \begin{cases} R_\lambda^h(f), & \text{on } M \setminus F_h \\ 0, & \text{on } F_h, \end{cases}$$

where R_λ^h is the resolvent operator associated to (3.5).

Let \mathcal{R} be the class of Radon measures on M (\mathcal{R} class of measures having compact support whenever $d = 2$). For $\beta \in \mathcal{R}$, we define

$$\mathcal{E}(\beta, A) = \begin{cases} \iint_{A \times A} \frac{\beta(dx)\beta(dy)}{d(x,y)^{d-2}}, & \text{if } d \geq 3, \\ \iint_{A \times A} \log \frac{2e}{d(x,y)} \beta(dx)\beta(dy), & \text{if } d = 2, \end{cases}$$

for each relatively compact open set $A \subset M$. In analogy with the euclidean case, we call $\mathcal{E}(\beta, A)$ the *energy* of β on A .

We say that $\beta \in \mathcal{R}$ has *finite energy* if

$$\sup_{A \in \mathcal{U}} \mathcal{E}(\beta, A) < +\infty,$$

where \mathcal{U} is the class of all relatively compact open subsets of M .

3.5 Assumptions. Let us assume the following hypotheses:

- (i₁) let β be a probability law on M of finite energy;
- (i₂) for every $h \in \mathbf{N}$ we set $I_h = \{1, \dots, h\}$ and we consider h measurable functions $x_h^i : \Omega \rightarrow M$, $i \in I_h$, such that $(x_h^i)_{i \in I_h}$ is a family of independent, identically distributed random variables with probability distribution β , viz.

$$P(x_i \in B) = \beta(B), i \in I_h,$$

for every Borel set $B \subset M$;

(i₃) let (r_h) be a sequence of strictly positive numbers such that

$$]0, +\infty[\ni \ell \stackrel{\text{def}}{=} \begin{cases} \lim_{h \rightarrow +\infty} h r_h^{d-2}, & \text{if } d \geq 3, \\ \lim_{h \rightarrow +\infty} h \left(\ln \frac{1}{r_h} \right)^{-1}, & \text{if } d = 2. \end{cases}$$

From now on, we shall consider the sequence of closed set (E_h) , given by

$$(3.7) \quad E_h \stackrel{\text{def}}{=} \bigcup_{i \in I_h} \overline{B_{r_h}(x_h^i)}.$$

By Lemma 2.19 it follows that the sets E_h are actually random sets, according to Definition 3.4.

We will prove the following result.

3.6 Theorem. *Let (E_h) be the sequence of random sets, as defined by (3.7). Assume the general hypotheses (i_1) , (i_2) , (i_3) . Then for any $\psi \in L^2(M)$ and for every $\varepsilon > 0$*

$$\lim_{h \rightarrow +\infty} P\{\omega \in \Omega : \|R_\lambda^{\infty E_h(\omega)}(\omega)\psi - R_\lambda(\omega)\psi\|_{L^2(M)} > \varepsilon\} = 0$$

where R_λ is the resolvent operator associated with the measure

$$\nu = \begin{cases} \omega_d(d-2)\ell\beta, & \text{if } d \geq 3, \\ 2\pi\ell\beta, & \text{if } d = 2 \end{cases}$$

We set for ease of notation

$$\omega(d) = \begin{cases} \omega_d(d-2), & \text{if } d \geq 3, \\ 2\pi, & \text{if } d = 2. \end{cases}$$

By Theorem 3.3, the above result is an immediate consequence of the following proposition.

3.7 Proposition. *Let (E_h) be the sequence of random sets defined in (3.7). Let α', α'' be the set functions as in Definition 3.2. Then if general hypotheses (i_1) , (i_2) , (i_3) hold, we have*

(t₁) $\nu'(B) = \nu''(B) = \omega(d)\ell\beta(B)$, for every $B \in \mathcal{B}$;

(t₂) there exist a constant $\varepsilon > 0$, an increasing continuous function $\xi : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ with $\xi(0,0) = 0$ and a Radon measure β_1 such that

$$\limsup_{h \rightarrow +\infty} |\text{Cov}[c(E_h(\cdot) \cap U), c(E_h(\cdot) \cap V)]| \leq \xi(\text{diam } U, \text{diam } V)\beta_1(U)\beta_1(V)$$

for any $U, V \in \mathcal{U}$ such that $\bar{U} \cap \bar{V} = \emptyset$ with $\text{diam}(U) < \varepsilon$, $\text{diam}(V) < \varepsilon$.

Notation. Let us consider, for $d \geq 2$,

$$R_h = \left(\frac{1}{h}\right)^{1/d}.$$

By 1_Z we shall denote the *characteristic function* of the set Z .

3.8 Remark. From the assumption (i_3) it follows that

$$\lim_{h \rightarrow +\infty} \frac{r_h}{R_h} = 0, \quad d \geq 2.$$

We have indeed, for $d \geq 3$,

$$\left(\frac{r_h}{R_h}\right)^{d-2} = h^{\frac{d-2}{d}} r_h^{d-2} = h r_h^{d-2} h^{-\frac{2}{d}}$$

which goes to zero, as $h \rightarrow +\infty$; for $d = 2$ we have the asymptotic estimate

$$\frac{r_h/R_h}{\exp(-h)} \rightarrow 1,$$

as $h \rightarrow +\infty$.

To achieve the proof of the Proposition 3.7 we need some more notation.

Let A be a relatively compact open subset of M and let $(x_i)_{i \in I}$ be a finite family of independent, identically distributed random variables, with values in M , and with distribution given by

$$P(x_i \in B) = \beta(B), \forall B \in \mathcal{B},$$

where β is the probability measure on M introduced in the Assumption 3.5- i_1 . Let δ , R and r be positive numbers, $0 < \delta < 1$, $0 < r < 2r < R < 1$; for every open set $A \subset\subset A'$ ($A \subset\subset A'$ and $\text{diam } A < 1$ when $d = 2$), let us introduce the following random sets of indices:

$$I(A) = \{i \in I : B_R(x_i) \subset A, d(x_i, x_j) \geq 4R, \forall j \in I, j \neq i\},$$

$$I_\delta(A) = \begin{cases} \left\{ i \in I(A) : \left(\frac{r}{R}\right)^{d-2} + \sum_{\substack{j \in I(A) \\ j \neq i}} \frac{r^{d-2}}{(d(x_i, x_j) - 2R)^{d-2}} \leq \delta \right\}, & d \geq 3 \\ \left\{ i \in I(A) : \left(\log \frac{R}{r}\right)^{-1} \left[\log \frac{2e}{R} + \sum_{\substack{j \in I(A) \\ j \neq i}} \log \frac{2e}{(d(x_i, x_j) - 2R)} \right] \leq \delta \right\}, & d = 2, \end{cases}$$

and $J_\delta(A) \stackrel{\text{def}}{=} I(A) \setminus I_\delta(A)$.

Loosely speaking, the (random) set $I(A)$ gives a “separating condition” among the x_i 's ($2R$ could play the role of “separating radius”), while if $i \in I_\delta(A)$ then we have an upper bound on the potential at x_i .

In the following lemma we prove a sort of “superadditive” result for the harmonic capacity of $\bigcup_{i \in I_\delta(A)} B_r(x_i)$ w.r.t. A , and we estimate the expectation of the number of $J_\delta(A)$. This lemma will be essential in the proof of the Proposition 3.7.

3.9 Lemma. *Let δ , r , and R be positive numbers, $0 < \delta < 1$, $0 < r < 2r < R < 1$; let A, A' be relatively compact open sets, $A \subset A' \subset M$. With the notation introduced above, for every $C > 2\omega(d)$ and for every compact set $K \subset M$, there exists $R_K > 0$ such that if $R < R_K$ and $A \subset K$, then we have*

$$(i) \quad (1 - \delta)^2 \sum_{i \in I_{\delta/C}(A)} \text{cap}(B_r(x_i), B_R(x_i)) \leq \text{cap}\left(\bigcup_{i \in I_{\delta/C}(A)} B_r(x_i), A\right),$$

and

$$(ii) \quad \mathbb{E}[\#(J_{\delta/C}(A))] \leq \frac{C}{\delta} \left[\#(I) \left(\frac{r}{R}\right)^{d-2} + 2^{d-2} r^{d-2} \mathcal{E}(\beta, A) (\#(I))^2 \right].$$

Proof. Let $C > 2\omega(d)$ and let u be the capacitary potential of $\bigcup_{i \in I_{\delta/C}(A)} B_r(x_i)$ w.r.t. A ; if $u < \delta$ on $\partial B_R(x_i)$, for each $i \in I_{\delta/C}(A)$, we claim that the proof is achieved. Let us introduce indeed $v = \frac{(u-\delta)^+}{(1-\delta)}$; the function $v \in H_0^1(A)$, $v \geq 1$ q.e. on $\bigcup_{i \in I_{\delta/C}(A)} B_r(x_i)$ and $v = 0$ on $\partial B_R(x_i)$, for every $i \in I_{\delta/C}(A)$. We then have

$$\text{cap}(B_r(x_i), B_R(x_i)) \leq \int_{B_R(x_i)} |\nabla v|^2 dV_g$$

for every $i \in I_{\delta/C}(A)$. Hence

$$\int_A |\nabla v|^2 dV_g \geq \sum_{i \in I_{\delta/C}(A)} \int_{B_R(x_i)} |\nabla v|^2 dV_g \geq \sum_{i \in I_{\delta/C}(A)} \text{cap}(B_r(x_i), B_r(x_i)).$$

On the other hand we have also

$$\begin{aligned} \int_A |\nabla v|^2 dV_g &= \frac{1}{(1-\delta)^2} \int_A |\nabla(u-\delta)^+|^2 dV_g \leq \frac{1}{(1-\delta)^2} \int_A |\nabla u|^2 dV_g \\ &= \frac{1}{(1-\delta)^2} \text{cap}\left(\bigcup_{i \in I_{\delta/C}(A)} B_R(x_i), A\right); \end{aligned}$$

therefore we have

$$\text{cap}\left(\bigcup_{i \in I_{\delta/C}(A)} B_R(x_i), A\right) \geq (1-\delta)^2 \sum_{i \in I_{\delta/C}(A)} \text{cap}(B_r(x_i), B_r(x_i)).$$

Now we verify that $u \leq \delta$, on $\partial B_R(x_i)$, for each $i \in I_{\delta/C}(A)$. Consider first the case $d \geq 3$. Let A' be an open set such that $A \subset\subset A' \subset\subset M$; let

$$u_i(x) = \int_{A'} g(x, y) \mu_i(dy),$$

be the capacitary potential of $B_r(x_i)$ w.r.t. A' ; $g(\cdot, \cdot)$ is the Green function (with Dirichlet boundary condition) on A' and μ_i is the capacitary distribution of $B_r(x_i)$ w.r.t. A' (see Proposition 2.7). Define

$$z(x) := \sum_{i \in I_{\delta/C}(A)} u_i(x), \quad \forall x \in A'.$$

Adapting a classical comparison result ([KS, Chapter 6, Sect. 7]) to our case, we find that for each $x \in A$

$$u(x) \leq z(x).$$

Let $y \in \partial B_R(x_i)$, for a fixed $i \in I_{\delta/C}(A)$; we have

$$z(y) = \sum_{i \in I_{\delta/C}(A)} u_i(y) = \int_{A'} g(y, \xi) \mu_i(d\xi) + \sum_{\substack{j \in I_{\delta/C}(A) \\ j \neq i}} \int_{A'} g(y, \zeta) \mu_j(d\zeta).$$

We recall (see Proposition 2.6-(iii)) that

$$g(y, x) \leq c(d(y, x))^{d-2},$$

where $c > 0$ is uniform for $x, y \in A$. For $\xi \in \partial B_r(x_i)$ we have $d(y, \xi) \geq R - r$, while for $\zeta \in \partial B_r(x_j)$, $j \neq i$, we have

$$\begin{aligned} d(x_i, x_j) &\leq d(x_i, y) + d(y, \zeta) + d(\zeta, x_j) \\ &= R + d(y, \zeta) + r, \end{aligned}$$

hence

$$d(y, \zeta) \geq d(x_i, x_j) - 2R.$$

Then

$$z(y) \leq \left(\frac{1}{R-r}\right)^{d-2} \text{cap}(B_r(x_i), A') + \sum_{\substack{j \in I_{\delta/C}(A) \\ j \neq i}} \frac{\text{cap}(B_r(x_j), A')}{(d(x_i, x_j) - 2R)^{d-2}}.$$

As $2r \leq R$, we have $R - r \geq (1/2)R$, hence

$$\begin{aligned} z(y) &\leq \left(\frac{2}{R}\right)^{d-2} \text{cap}(B_r(x_i), A') + 2^{d-2} \sum_{\substack{j \in I_{\delta/C}(A) \\ j \neq i}} \frac{\text{cap}(B_r(x_j), A')}{(d(x_i, x_j) - 2R)^{d-2}} \\ &\leq \left(\frac{2}{R}\right)^{d-2} \text{cap}(B_r(x_i), B_R(x_i)) + 2^{d-2} \sum_{\substack{j \in I(A) \\ j \neq i}} \frac{\text{cap}(B_r(x_j), B_R(x_j))}{(d(x_i, x_j) - 2R)^{d-2}}, \end{aligned}$$

since $\text{cap}(B_r(x_k), A') \leq \text{cap}(B_r(x_k), B_R(x_k))$, for every $k \in I_{\delta/C}(A)$, and $I_{\delta/C}(A) \subset I(A)$.

Now we recall (cf. Lemma 2.8) that there exists a constant $\lambda > 0$ such that for every $x \in A$

$$\text{cap}(B_r(x), B_R(x)) \leq \omega(d)(1 + \lambda R^2 + O(R^3)) \frac{r^{d-2}}{1 - (r/R)^{d-2}},$$

hence

$$z(y) \leq C \left(\frac{r}{R}\right)^{d-2} + C \sum_{\substack{j \in I(A) \\ j \neq i}} \frac{r^{d-2}}{(d(x_j, x_i) - 2R)^{d-2}},$$

which is less than δ , by definition of $I_{\delta/C}(A)$.

The proof of (i) in the case $d = 2$ is similar: we consider, as in the previous case,

$$u_i(x) = \int_{A'} g(x, \xi) \mu_i(d\xi),$$

and for each $x \in A'$ we define

$$z(x) := \sum_{i \in J_\delta(A)} u_i(x).$$

Applying the same comparison principle we have $u(x) \leq z(x)$. Using the estimate for the Green function in (2.9) (recall that $\text{diam } A < 1$),

$$g(x, \xi) \leq c \log \frac{e}{d(x, \xi)},$$

where c is uniform for $x, \xi \in A$, we find that

$$z(y) \leq c \left(\log \frac{R}{r} \right)^{-1} \left(\log \frac{2e}{R} + \sum_{\substack{j \in I(A) \\ j \neq i}} \log \frac{2e}{d(x_i, x_j) - 2R} \right),$$

which is less than δ , by definition of $J_{\delta/C}(A)$.

We now prove the inequality in (ii). As before, we consider first the case $d \geq 3$. First of all we note that

$$J_{\delta/C}(A) = \left\{ i \in I : \left(\frac{r}{R} \right)^{d-2} + \sum_{\substack{j \in I(A) \\ j \neq i}} \frac{r^{d-2}}{(d(x_i, x_j) - 2R)^{d-2}} > \frac{\delta}{C} \right\}.$$

Therefore

$$\#(J_{\delta/C}(A)) < \frac{C}{\delta} \sum_{i \in J_\delta} \left[\left(\frac{r}{R} \right)^{d-2} + \sum_{\substack{j \in I(A) \\ j \neq i}} \frac{r^{d-2}}{(d(x_i, x_j) - 2R)^{d-2}} \right],$$

hence

$$\begin{aligned} \mathbb{E}[\#(J_{\delta/C}(A))] &< \frac{C}{\delta} \left(\#(I) \left(\frac{r}{R} \right)^{d-2} + \mathbb{E} \left[\sum_{\substack{i, j \in I(A) \\ j \neq i}} \frac{r^{d-2}}{(d(x_i, x_j) - 2R)^{d-2}} \right] \right) \\ &< \frac{C}{\delta} \left(\#(I) \left(\frac{r}{R} \right)^{d-2} + 2^{d-2} r^{d-2} \mathbb{E} \left[\sum_{\substack{i, j \in I(A) \\ j \neq i}} \frac{1}{d(x_i, x_j)^{d-2}} \right] \right) \\ &= \frac{C}{\delta} \left(\#(I) \left(\frac{r}{R} \right)^{d-2} + 2^{d-2} r^{d-2} \sum_{\substack{i, j \in I \\ j \neq i}} \mathbb{E} \left[\frac{\mathbf{1}_A(x_i) \mathbf{1}_A(x_j)}{d(x_i, x_j)^{d-2}} \right] \right) \\ &= \frac{C}{\delta} \left(\#(I) \left(\frac{r}{R} \right)^{d-2} + 2^{d-2} r^{d-2} \sum_{\substack{i, j \in I \\ j \neq i}} \int \int_{A \times A} \frac{\beta(dx) \beta(dy)}{(d(x, y))^{d-2}} \right) \\ &\leq \frac{C}{\delta} \left(\#(I) \left(\frac{r}{R} \right)^{d-2} + 2^{d-2} r^{d-2} \mathcal{E}(\beta, A) (\#(I))^2 \right). \end{aligned}$$

The case $d = 2$ is proven with similar computations. \square

3.10 Remark. Since the measure β has finite energy, it does not charge sets of capacity zero; this implies that the measure σ , defined on the Borel family of $M \times M$ by

$$\sigma(E) \stackrel{\text{def}}{=} \int \int_E \frac{\beta(dx)\beta(dy)}{(d(x,y))^{d-2}}, \quad d \geq 3,$$

does not charge singletons. This property holds also in the case $d = 2$.

Now consider a non-atomic, finite measure σ on a separable, metric space X . For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every A with $\text{diam } A < \delta$ we have $\sigma(A) < \varepsilon$. In fact, suppose for the moment that the measure has support in a compact subset of X and assume, by contradiction, that there exists $\varepsilon_o > 0$ such that for every $h \in \mathbb{N}$ there exists A_h with $\text{diam}(A_h) < 1/h$ and $\sigma(A_h) \geq \varepsilon_o$. Let $x_h \in A_h$; then $x_h \rightarrow x \in X$ and for h sufficiently large we have

$$A_h \subset B_r(x).$$

Then $\sigma(A_h) \geq \varepsilon_o$ implies $\sigma(B_r(x)) \geq \varepsilon_o$. If we let $r \rightarrow 0$, we have $B_r(x) \rightarrow \{x\}$, hence $\sigma(\{x\}) \geq \varepsilon_o > 0$, but we have a contradiction, since the measure σ is non-atomic. If the finite measure σ is not supported on a compact set, given $\varepsilon > 0$, there exists a compact set K such that $\sigma(X \setminus K) < \varepsilon$; now we repeat the argument above in the compact set K .

Proof of the Proposition 3.7. For simplicity we assume $d \geq 3$; the case $d = 2$ is proved in the same way. Let $C > 2\omega(d)$.

Let A be a relatively compact open subset of M , with $\text{diam } A < 1$; we recall that $c(\infty_{E_h}, A) = c(A \cap E_h)$ (cf. Remark 2.11), and that $I_h = \{1, \dots, h\}$, $h \in \mathbb{N}$; let us introduce

$$I_h(A) = \{i \in I_h : B_{R_h}(x_h^i) \subset A, |x_h^i - x_h^j| \geq 2R_h, \forall j \in I, j \neq i\},$$

$$I_{\frac{\delta}{C}, h}(A) = \left\{ i \in I_h(A) : \left(\frac{r_h}{R_h} \right)^{d-2} + \sum_{\substack{j \in I(A) \\ j \neq i}} \frac{r_h^{d-2}}{(|x_h^i - x_h^j| - R_h)^{d-2}} \leq \frac{\delta}{C} \right\}$$

and set $J_{\frac{\delta}{C}, h}(A) = I_h(A) \setminus I_{\frac{\delta}{C}, h}(A)$. Denote by E_h' the random set

$$E_h' \stackrel{\text{def}}{=} \bigcup_{i \in I_{\frac{\delta}{C}, h}(A)} \overline{B_{r_h}(x_h^i)}.$$

Note that $c(E_h \cap A) \geq c(E'_h \cap A) \geq \text{cap}(E'_h \cap A, A)$, where the last inequality follows from Remark 2.3. We apply Lemma 3.9-(i) and find that

$$\begin{aligned}
\text{cap}(E'_h \cap A, A) &\geq (1 - \delta)^2 \sum_{i \in I_{\frac{\delta}{c}, h}(A)} \text{cap}(B_{r_h}(x_h^i), B_{R_h}(x_h^i)) \\
(3.13) \qquad &\geq \omega(d) \left[1 + \lambda_1(R_h)^2\right] \left[1 - \delta\right]^2 \times \\
&\quad \times \left[\#(I_h(A)) - \#(J_{\frac{\delta}{c}, h}(A))\right] \frac{r_h^{d-2}}{1 - (r_h/R_h)^{d-2}}.
\end{aligned}$$

From the first line to the second one we have applied Lemma 2.8; $\lambda_1 = \inf_{z \in A} \lambda'(z)$, where $\lambda'(\cdot)$ is the function occurring at the left hand side of (2.10); we recall that $\lambda'(\cdot)$ is bounded on compact sets.

We introduce

$$A'_h \stackrel{\text{def}}{=} \{x \in M : d(x, A) < R_h\},$$

and $K_h(A'_h) \stackrel{\text{def}}{=} \{i \in I_h : B_{R_h}(x_h^i) \subset A'_h\}$; notice that $E_h \cap A \subset \bigcup_{i \in K_h(A'_h)} B_{r_h}(x_h^i)$.

We have

$$\begin{aligned}
c(E_h \cap A) &\leq c\left(\bigcup_{i \in K_h(A'_h)} B_{r_h}(x_h^i) \cap A'_h\right) \leq \sum_{i \in K_h(A'_h)} c(B_{r_h}(x_h^i)) \\
(3.14) \qquad &\leq \sum_{i \in K_h(A'_h)} \left[\text{cap}(B_{r_h}(x_h^i), B_{R_h}(x_h^i)) + \int_{B_{R_h}(x_h^i)} (u_h^i)^2 dV_g \right] \\
&\leq \omega(d) \left[1 + \lambda_2(R_h)^2\right] \left[\#(K_h(A'_h))\right] \frac{r_h^{d-2}}{1 - (r_h/R_h)^{d-2}} \\
&\quad + \omega(d)h \int_{B_{R_h}(x_h^i)} (u_h^i)^2 dV_g.
\end{aligned}$$

In the second inequality of the first line we have used the subadditive property of the meta-harmonic capacity (cf. Remark 2.2-(d)); in the second line we have applied the inequality stated in Remark 2.3; u_h^i is the capacitary potential of $B_{r_h}(x_h^i)$ w.r.t. $B_{R_h}(x_h^i)$ and finally we have used Lemma 2.8 in the last line; $\lambda_2 = \sup_{z \in A} \lambda''(z)$, where $\lambda''(\cdot)$ is the function occurring at the right hand side of (2.10); we recall that $\lambda''(\cdot)$ is bounded on compact sets.

Incidentally we show that for every $\omega \in \Omega$ we have

$$\lim_{h \rightarrow +\infty} h \int_{B_{R_h}(x_h^i)} (u_h^i)^2 dV_g = 0.$$

Consider indeed the case $d \geq 3$. We have from Lemma 2.8

$$\text{cap}(B_{r_h}(x_h^i), B_{R_h}(x_h^i)) \leq \omega(d)(1 + \lambda_2(R_h)^2 + O(R_h)^3) \frac{r_h^{d-2}}{1 - (r_h/R_h)^{d-2}};$$

hence, there is a constant $c > 0$ such that for every $\omega \in \Omega$, we have

$$h \int_{B_{R_h}(x_h^i)} (u_h^i)^2 dV_g \leq \frac{c}{\text{cap}(B_{r_h}(x_h^i), B_{R_h}(x_h^i))} \int_{B_{R_h}(x_h^i)} (u_h^i)^2 dV_g$$

and

$$\lim_{h \rightarrow +\infty} \frac{1}{\text{cap}(B_{r_h}(x_h^i), B_{R_h}(x_h^i))} \int_{B_{R_h}(x_h^i)} (u_h^i)^2 dV_g = 0,$$

as follows from Proposition 2.4. The case $d = 2$ is shown in a similar manner.

By the Law of Large Numbers we have, for the random variables $\#(I_h(A))$ and $\#(K_h(A'_h))$,

$$(3.15) \quad \beta(A) = \begin{cases} \lim_{h \rightarrow +\infty} \frac{\mathbb{E}[\#(I_h(A))]}{h}, \\ \lim_{h \rightarrow +\infty} \frac{\mathbb{E}[\#(K_h(A'_h))]}{h}, \end{cases}$$

for every $A \in \mathcal{U}$ with $\beta(\partial A) = 0$.

Therefore from (3.13), (3.14), (3.15), Lemma 3.9 and Remark 3.8 we get

$$(3.16) \quad \overline{\alpha}''(B) \leq \omega(d)\ell\beta(B),$$

for every $B \in \mathcal{B}$ and

$$(3.17) \quad \overline{\alpha}'(B') \geq \omega(d)\ell\beta(B')(1 - \delta)^2 - C\omega(d)\frac{2^{d-2}\ell^2}{\delta}\mathcal{E}(\beta, B')$$

for every $B' \in \mathcal{B}$, with $\text{diam } B' < 1$. From (3.16) we have

$$\nu''(B) \leq \omega(d)\ell\beta(B)$$

for every $B \in \mathcal{B}$. To achieve the proof of (t₁), we have to prove that

$$(3.18) \quad \nu'(B) \geq \omega(d)\ell\beta(B)$$

for each $B \in \mathcal{B}$. Let us fix $B \in \mathcal{B}$; for arbitrary $0 < \eta < 1$, take a Borel partition $(B_k)_{k \in K}$ of B with $\text{diam } B_k < \eta$. Since ν' is superadditive, we have

$$\begin{aligned} \nu'(B) &\geq \sum_{k \in K} \nu'(B_k) \geq \omega(d)(1 - \delta)^2 \ell \beta(B) - C\omega(d) \sum_{k \in K} \frac{2^{d-2} \ell^2}{\delta} \mathcal{E}(\beta, B_k) \\ &= \omega(d)(1 - \delta)^2 \ell \beta(B) - C\omega(d) \sum_{k \in K} \frac{2^{d-2} \ell^2}{\delta} \int \int_{B_k \times B_k} \frac{\beta(dx)\beta(dy)}{|x - y|^{d-2}} \\ &\geq \omega(d)(1 - \delta)^2 \ell \beta(B) - C\omega(d) \frac{2^{d-2} \ell^2}{\delta} \int \int_{D_\eta} \frac{\beta(dx)\beta(dy)}{|x - y|^{d-2}} \\ &= \omega(d)(1 - \delta)^2 \ell \beta(B) - C\omega(d) \frac{2^{d-2} \ell^2}{\delta} \mathcal{E}(\beta, D_\eta), \end{aligned}$$

where $D_\eta = \{(x, y) \in M \times M : d(x, y) < \eta\}$, so that $B_k \times B_k \subset D_\eta$, for every $k \in K$; notice that $\text{diam } D_\eta < \eta$. Since β is a measure of finite energy, and taking into account of Remark 3.10, we find that

$$\lim_{\eta \rightarrow 0} \mathcal{E}(\beta, D_\eta) = 0,$$

and we get (3.18); letting $\delta \rightarrow 0$ we have $\nu'(B) \geq \omega(d)\ell\beta(B)$; as $\nu''(B) \leq \omega(d)\ell\beta(B)$, and $\nu''(\cdot) \geq \nu'(\cdot)$, we get $\nu(B) = \omega(d)\ell\beta(B)$, which proves (t₁).

Proof of (t₂). First of all we observe that by the Strong Law of Large Numbers we have

$$(3.20) \quad \lim_{h \rightarrow +\infty} \frac{\#(I_h(U))}{h} = \beta(U), \quad \text{for a.e. } \omega \in \Omega$$

and

$$(3.21) \quad \lim_{h \rightarrow +\infty} \frac{\#(I_h(U))}{h} = \beta(U), \quad \text{in } L^1(\Omega)$$

for any $U \in \mathcal{U}$. Actually we have, for any $U \in \mathcal{U}$,

$$(3.22) \quad \lim_{h \rightarrow +\infty} \frac{\#(I_h(U))}{h} = \beta(U), \quad \text{in } L^2(\Omega)$$

since $\frac{\#(I_h(U))}{h}$ is an equibounded sequence of random variables.

By (3.20), (3.21), (3.22) we have

$$\begin{aligned} (3.23) \quad &\liminf_{h \rightarrow +\infty} \mathbb{E}[c(E_h(\cdot) \cap U)c(E_h(\cdot) \cap V)] \\ &\geq C\omega(d)\ell(1 - \delta)^2(1 - \delta)^2 \limsup_{h \rightarrow +\infty} \left\{ \mathbb{E} \left[\frac{\#(I_h(U))}{h} \frac{\#(I_h(V))}{h} \right] \right. \\ &\quad \left. - \mathbb{E} \left[\frac{\#(I_h(U))}{h} \frac{\#(J_{\delta,h}(V))}{h} \right] - \mathbb{E} \left[\frac{\#(I_h(V))}{h} \frac{\#(J_{\delta,h}(U))}{h} \right] \right\} \end{aligned}$$

for any pair $U, V \in \mathcal{U}$ with diameter less than 1 and $\bar{U} \cap \bar{V} = \emptyset$. From (3.22) we obtain

$$(3.24) \quad \lim_{h \rightarrow +\infty} \mathbb{E} \left[\frac{\#(I_h(U))}{h} \frac{\#(I_h(V))}{h} \right] = \beta(U)\beta(V);$$

moreover by Lemma 3.9 and (3.20)

$$(3.25) \quad \limsup_{h \rightarrow +\infty} \mathbb{E} \left[\frac{\#(I_h(U))}{h} \frac{\#(J_{\delta,h}(V))}{h} \right] \leq C\omega(d)\beta(U) \frac{2^{d-2}}{\delta} \ell^2 \mathcal{E}(\beta, V)$$

and

$$(3.26) \quad \limsup_{h \rightarrow +\infty} \mathbb{E} \left[\frac{\#(I_h(V))}{h} \frac{\#(J_{\delta,h}(U))}{h} \right] \leq C\beta(V) \frac{2^{d-2}}{\delta} \omega(d) \ell^2 \mathcal{E}(\beta, U)$$

for any $U, V \in \mathcal{U}$. Then by (3.23), (3.24), (3.25), (3.26)

$$(3.27) \quad \liminf_{h \rightarrow +\infty} \mathbb{E}[c(E_h(\cdot) \cap U)c(E_h(\cdot) \cap V)] \geq \omega(d)\ell(1-4\delta) \times \\ \left[\beta(U)\beta(V) - C\omega(d)\beta(U) \frac{2^{d-2}}{\delta} \ell^2 \mathcal{E}(\beta, V) - C\omega(d)\beta(V) \frac{2^{d-2}}{\delta} \ell^2 \mathcal{E}(\beta, U) \right]$$

for every $U, V \in \mathcal{U}$ with diameter less than 1 and $\bar{U} \cap \bar{V} = \emptyset$.

By (3.14) and (3.22), we also deduce

$$(3.28) \quad \limsup_{h \rightarrow +\infty} \mathbb{E}[c(E_h(\cdot) \cap U)c(E_h(\cdot) \cap V)] \leq \omega(d)\ell^2 \beta(U)\beta(V)$$

for any $U, V \in \mathcal{U}$ with $\beta(\partial U) = \beta(\partial V) = 0$.

Estimates similar to (3.27) and (3.28) for the upper and lower limit of the sequence $\mathbb{E}[c(E_h(\cdot) \cap U)]\mathbb{E}[c(E_h(\cdot) \cap V)]$ can be obtained in the same way. Therefore we can deduce, for any $\delta > 0$, and for every $U, V \in \mathcal{U}$ with diameter less than 1 and $\bar{U} \cap \bar{V} = \emptyset$,

$$(3.29) \quad \limsup_{h \rightarrow +\infty} |\text{Cov}[c(E_h(\cdot) \cap U), c(E_h(\cdot) \cap V)]| \leq \omega(d)\ell^2 \beta(U)\beta(V) - \omega(d)\ell^2 [1-4\delta] \times \\ \times \left[\beta(U)\beta(V) - \omega(d)\beta(U) \frac{2^{d-2}}{\delta} \ell \mathcal{E}(\beta, V) - \omega(d)\beta(V) \frac{2^{d-2}}{\delta} \ell \mathcal{E}(\beta, U) \right] \\ \leq c\delta \left[\tilde{\beta}(U)\mathcal{E}(\beta, V) + \tilde{\beta}(V)\mathcal{E}(\beta, U) + \delta^2 \tilde{\beta}(U)\tilde{\beta}(V) \right] \\ \leq c\delta \left[\tilde{\beta}(U)\mathcal{E}(\beta, V) + \tilde{\beta}(V)\mathcal{E}(\beta, U) + \tilde{\beta}(U)\tilde{\beta}(V) \right],$$

where $\tilde{\beta}(\cdot)$ is the Radon measure: $\tilde{\beta}(\cdot) = (1/\delta)\beta(\cdot)$ and $c = \omega(d) \max\{C, 4\ell^2, 2^{d-2}\ell\}$. Take now $\delta = \max\{\text{diam } U, \text{diam } V\}$ and we have

$$(3.30) \quad \limsup_{h \rightarrow +\infty} |\text{Cov}[c(E_h(\cdot) \cap U), c(E_h(\cdot) \cap V)]| \\ \leq c\ell^2\delta \left[\tilde{\beta}(U)\mathcal{E}(\beta, V) + \tilde{\beta}(V)\mathcal{E}(\beta, U) + \tilde{\beta}(U)\tilde{\beta}(V) \right].$$

Taking in (3.30)

$$\beta_1(U) \stackrel{\text{def}}{=} \tilde{\beta}(U) + \mathcal{E}(\beta, U),$$

for every $U \in \mathcal{U}$ and $\xi(x, y) \stackrel{\text{def}}{=} \max\{x, y\}$, the item (t₂) is proved, and hence the proof of the proposition is accomplished. \square

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