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Nonregular Representations of CCR Algebras

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Contents

INTRODUCTION

Many models of Quantum Field Theory, Many Body Theory and Quantum Statistical Mechanics are formulated and sometimes solved in terms of variables or fields which formally satisfy the Canonical Commutation Relations (CCR), but which actually cannot be represented as operators in a Hilbert space because of their bad infrared behaviour. Tipical examples are:

1. the infinite quantum harmonic lattice in thermal equilibrium and the Free Bose gas, both in space dimensions $d \leq 2$; in the first case the singular variable is the local deplacement from the equilibrium positions and in the second case is the local particle density [AM,BR];
2. the electrons in a periodic potential (Bloch electrons), which are a prototype model in the anlysis of θ-angle superselection structures [AM,J];
3. the massless quantum electrodynamics in two space–time dimensions (QED_2 or Schwinger model) [LS], where the singular field variables are the charged fields;
4. the Stückelberg-Kibble model in 1+1 and 2+1 space–time dimensions, where the singular variable is the analogue of the (phase of the) Higgs field;
5. the massless scalar field in two space–time dimensions, which plays an important rôle as a basic or building block field in the solution of several two dimensional problems [W,SW,DFZ];
6. the $U(1)$ current algebra on the circle, where the singular variables are the charged fields [BMT];
7. quite generally statistical models involving unbounded variables with a flat distribution (required by symmetry properties).

The strategies usually adopted to discuss the above models fall essentially in two categories:

a. relax positivity and represent the singular fields as operators on an indefinite metric space [W,MPS2];
b. use a restricted set of field variables (the ones which can be represented as operators in a Hilbert space) and describe the remaining degrees of freedom by others methods, like e.g. as morphisms of the regular field algebra [SW,BMT].

Both strategies have their own advantages and disadvantages. The aim of this thesis is to discuss an alternative approach based on the representation of the Weyl exponentials of the singular variables in a positive metric Hilbert space.

This will allow to keep the canonical structure (in Weyl form) also for the singular variables and to clarify the relation between infrared singularities and the so arising *nonregular representations* of the CCR.

The mathematical treatment of these latter can be done in a rather compact way and it yields a convenient and usable framework for discussing explicit models. We suppose to be given a CCR algebra $\mathcal{A}$, and a regular representation π of a subalgebra $\mathcal{A}_0$ of $\mathcal{A}$. Our results can be summarized as follows:

1. a *nonregular* representation extending π to $\mathcal{A}$ always exist (*Sect I.2*);
2. nonregular representations of a CCR algebra $\mathcal{A}$ are determined by the representations of the subalgebra $\mathcal{A}_0$ if the latter is a *maximal domain of regularity* (see *Sect. I.3*); in the various models mentioned above $\mathcal{A}_0$ can be interpreted as the algebra of observables, the current algebra etc.
3. nonregular representations of a CCR algebra $\mathcal{A}$ decompose into inequivalent representations of the maximal regular subalgebra $\mathcal{A}_0$ (*Sect. I.4*);
4. irreducible nonregular representations are characterized (*Section I.3* and *Appendix A*).

These structure properties prove very useful for a rigorous mathematical treatment of the infrared singular variables or fields which naturally enter in the Hamiltonian formulation of the above models (*Sect. II*). One also gets insight on the representations of field algebras $\mathcal{A}$ charged under a gauge–like group when the observable algebra $\mathcal{A}_0$ is a CCR algebra. In particular one has (*Sect. I.4*):

5. "charged" field algebras $\mathcal{A}$ can be obtained as CCR extensions (called *extended CCR algebras*) of the algebra $\mathcal{A}_0$, briefly "core" subalgebra; furthermore an extended CCR algebra uniquely determines a "gauge group" $\mathcal{G}$ leaving its core subalgebra pointwise invariant;
6. vacuum representations of extended CCR algebras decompose into inequivalent representations labelled by charges which annihilate the vacuum: more precisely, if the subgroup of $\mathcal{G}$ leaving the vacuum invariant is non–trivial, then the corresponding vacuum representation of the extended CCR algebra $\mathcal{A}$ is nonregular.

Moreover, nonregular representations of extended CCR algebras allow for a full solution of the bosonization problem in 1+1 dimensions (*Appendix B*); in particular:

7. fermionic degrees of freedom may be obtained in terms of bosonic CCR algebras,

8. local fermion fields can be proved to exist as *ultrastrong limits* of bosonic (Weyl) operators of an extended CCR algebra.

Previous treatments of fermionic bosonization were done with reference to specific models and in terms of correlation functions (i.e. in a given representation); along this direction the best result seems that in [CR], where local fermion fields are obtained as strong limits on a dense set of vectors in given representation. An algebraic formula which constructs a local fermion field (at a given time) in terms of canonical boson operators seems to be lacking in the literature, apart from the very suggestive Mandelstam–Skyrme formula [M,Sk], whose mathematical meaning is however not clear. The result 8. mentioned above provides such an algebraic fermion bosonization and it is made possible by the use of nonregular representations of CCR algebras [AMS1].

It is worthwile to remark that the mathematical framework discussed above for the description of infrared singular fields should prove useful in the constructive approach to Many Body or QFT problems which involve severe infrared problems (like Coulomb systems, gauge theories etc.). In fact, by introducing an infrared cutoff (where necessary), the models may be formulated in terms of canonical algebras, involving variables associated to the charged fields, which may require nonregular representations. In this sense, the above approach may be regarded as a concrete realization of the DHR approach [DHR,DR]. The latter is based on the analysis of the representations of the observable algebras $\mathcal{A}_{obs}$ and describes their charged state representations in terms of the vacuum representation and of charged morphisms of $\mathcal{A}_{obs}$. Here, we exploit the canonical structure of the variables which enter in the definition of the Hamiltonian, of the equation of motion etc; the mathematical problems connected with infrared singular variables are resolved by the use of nonregular representations. The construction of the charged morphisms is now directly obtained in terms of the fields which enter in the definition of the model (and/or may be obtained as CCR extensions of the observable algebra), so that the charged state representations of the observable algebra are provided by the vacuum representation of the extended CCR algebra.

The general framework discussed in this paper also provides a useful and relatively simple tratment of quantum fields on a circle, a problem which has recently attracted much interest in connection with string theory and with representations of Kač–Moody and Virasoro algebras [FK,F,Se]. The quantization of such systems leads to nonregular representations of CCR algebras; in this way one has a systematic treatment where infrared problems do not arise and the field algebra is *simple.*

The structures discussed in Section I.4 naturally lead to the occurrence of superselection rules also for quantum mechanical systems with a finite number of degrees of freedom (i.e. to the breakdown of Von Neumann uniqueness): the mechanism is that the nonregular factorial representation of the *simple* Weyl algebra of canonical variables decomposes into inequivalent factorial representations of its regular observable subalgebra. The resulting structure may be regarded as an alternative route to the solution of the problem considered by Landsman [L] through a different approach. His starting point is the classification of the representations of a *non simple* C^**–algebra*, which plays the rôle of the algebra of observables, whereas our strategy is to construct a simple algebra of canonical variables and to identify its subalgebra of observables.

The *use* of nonregular representations of CCR algebras in mathematical physics is not new.

Plane waves have been recognised to define nonregular representations of the CCR algebra (finite number of degrees of freedom) in [FVW], where a uniqueness theorem is proved for this class of representations.

Perhaps the best known example of nonregular state is the tracial state (infinite temperature KMS state). It has a crucial rôle in the proof of the existence (and uniqueness in the nondegenerate case) of the C^*–structure for the CCR algebra over the generic symplectic space [Sl,BR,P]. Central (and hence nonregular) states have been introduced in [MSTV] in order to characterize the various C^*–structures which are allowed when the symplectic form of the CCR is degenerate.

Buchholz and Fredenhagen have argued that nonregular representations naturally arise in the construction of charged states and explicitely implement their idea in the case of the algebra of the electromagnetic field [BF]. They use a very particular type of nonregular ground state and point out in this special case some among the main structure properties we'll show in Section I to hold in full generality.

Following the ideas in [BF], Nill [N1,N2] has recently introduced nonregular ground states in the analysis of Chern–Simons theory (in the canonical formulation) and has investigated the resulting superselection structure of the theory. He presented also a first attempt of analysis of nonregular states into a functional integral language.

Hurst and Grundling [Gr,GH,H] have studied the theory of quantum constrainted systems introducing the concept of Dirac state. This kind of state shares a property which is very similar to our major structure assumption (Condition A in Section I.3) in the analysis of nonregular representations of CCR algebras. However, no further attention is paid there to this aspect.

As it is explained in Section I.4, the appearence of nonregular representations in gauge theories, and more generally in constrainted quantum systems, is not fortuitous: a nonregular representation of a CCR algebra always gives rise to a "gauge group" leaving the regularly represented subalgebra pointwise invariant.

Narnhofer and Thirring have met with nonregular representations of CCR algebras both in the analysis of Gauge Field Theory models [NT1] and in the construction of equilibrium (KMS) states for simple quantum hamiltonians on $\mathbb{R}^2$ or on the 2–torus [NT2].

This thesis and the works from which it arises [AMS1,AMS2,AMS3] offer the first treatment of the mathematical structure of nonregular representations of CCR algebras, togheter with a clear discussion of the relevant models.

NONREGULAR REPRESENTATIONS OF WEYL ALGEBRAS

1. CCR algebras

Our aim is to study nonregular representations of Canonical Commutation Relations (CCR) algebras. In order to make clear the environment in which we move and to fix notations, we recall basic facts about CCR algebras and their regular representations.

Let be given a symplectic space (V,σ), i.e., V be a real linear space and σ be a symplectic form on it (σ is *not* supposed to be nondegenerate). More generally [Sl], V may be an abelian group and the form σ is substituted by a bicharacter $b(\cdot,\cdot)$ on V.

A CCR algebra $\mathcal{A}(V,\sigma)$ is the *-algebra generated by elements $\delta(F)$, $F \in V$; the product and the involution in the algebra are defined by

$$\begin{aligned} \delta(F)\delta(G) &= \delta(F+G)e^{-\frac{i}{2}\sigma(F,G)} \\ \delta(F)^* &= \delta(-F) \qquad\qquad F,G \in V. \end{aligned} \tag{1.1}$$

These equations imply $\delta(0) = \mathbb{1}$ and $\delta(F)^{-1} = \delta(F)^*$.

$\mathcal{A}(V,\sigma)$ can be completed to a C^*-algebra [Ma], the C^* norm being unique exactly when the form σ is nondegenerate [Sl,MSTV,BR,P]. In this case $\mathcal{A}(V,\sigma)$ is simple. A state ω on $\mathcal{A}(V,\sigma)$ is a positive and normalized linear functional on it. We denote by $E_{\mathcal{A}(V,\sigma)}$ the set of states over the CCR algebra $\mathcal{A}(V,\sigma)$. In the GNS representation $(\pi_\omega,\mathcal{H}_\omega,\psi_\omega)$ of $\mathcal{A}(V,\sigma)$ induced by a state ω, the elements $\delta(F)$, $F \in V$, are represented by unitary operators.

A representation π of $\mathcal{A}(V,\sigma)$ is said to be *regular* if the map $\lambda \in \mathbb{R} \longmapsto \pi(\delta(\lambda F))$ is strongly continuous for every $F \in V$. A state ω on $\mathcal{A}(V,\sigma)$ is said to be *regular* if the associated GNS representation π_ω of $\mathcal{A}(V,\sigma)$ is regular. The regularity of ω is equivalent to the fact that $\lambda \in \mathbb{R} \longmapsto \omega(\delta(\lambda F))$ is a continuous mapping for every $F \in V$.

If the representation π is regular then there are selfadjoint operators $\Phi(F)$ on $\mathcal{H}_\omega$ such that $\pi(\delta(\lambda F)) = e^{i\lambda\Phi(F)}$ for every $\lambda \in \mathbb{R}$, $F \in V$. The operators $\Phi(F)$ satisfy the unbounded form of the Canonical Commutation Relations.

Most of the models we study are quasi free: they are identified [MV,P,BR] by a state on $\mathcal{A}(V,\sigma)$ (usually the ground state or an equilibrium state) of the form

$$\omega_q(\delta(F)) = e^{-\frac{1}{4}q(F)} \qquad \forall F \in V,$$

where $\mathcal{A}(V,\sigma)$ is the CCR algebra describing the kinematics of every single model, and $q(\cdot)$ is a quadratic form on V which arises from an inner product. The form $q(\cdot)$ satisfies

$$|\sigma(F,G)|^2 \leq q(F)q(G) \qquad \forall F,G \in V. \tag{1.2}$$

This condition is equivalent to the positivity of the linear functional ω_q [MV,P].

2. Flat nonregular states on CCR algebras

We introduce and study in this section a particular class of nonregular representations of Weyl algebras. We suppose to this end to have a CCR algebra $\mathcal{A}(V_0,\sigma_0)$ and a state ω on it. A symplectic space (V,σ) is given such that V_0 is a linear subspace of V, and that $\sigma_{|V_0\times V_0} = \sigma_0$. In this case we say that (V_0,σ_0) is a *symplectic subspace* of (V,σ). $\mathcal{A}(V_0,\sigma_0)$ is then a *-subalgebra of $\mathcal{A}(V,\sigma)$. Where confusions do not arise, we'll use the same symbol σ to indicate a symplectic form on V or its restriction to subspaces.

We prove in Prop. 2.1 below that ω can *always* be extended to a *nonregular state* Ω on $\mathcal{A}(V,\sigma)$. In the proofs the fact that (V_0,σ_0) is a symplectic subspace of (V,σ) is crucial. Indeed, two different proofs of this proposition are given. The first proof generalizes an argument by Buchholz and Fredenhagen [BF] and displays the "charged sectors" structure which will be further analyzed in Section 4. The second proof uses the fact that one can always construct an abelian (and hence amenable) group of automorphisms of $\mathcal{A}(V,\sigma)$, whose action leaves $\mathcal{A}(V_0,\sigma_0)$ pointwise invariant. The existence of this group follows *solely* from the fact that (V_0,σ_0) is a symplectic subspace of (V,σ). We denote by $E_\omega \subset E_{\mathcal{A}(V,\sigma)}$ the set of states extending $\omega \in E_{\mathcal{A}(V_0,\sigma_0)}$ to $\mathcal{A}$.

Proposition 2.1

Let $\mathcal{A}(V,\sigma)$ be a CCR algebra. Let $\mathcal{A}(V_0,\sigma_0)$ be a CCR subalgebra of $\mathcal{A}(V,\sigma)$ and ω be a state on $\mathcal{A}(V_0,\sigma_0)$. Then the linear functional on $\mathcal{A}(V,\sigma)$ defined by

$$\Omega(\delta(F)) = \begin{cases} \omega(\delta(F)) & F \in V_0 \\ 0 & F \in V\backslash V_0. \end{cases} \tag{2.1}$$

is positive and hence is a state on $\mathcal{A}(V,\sigma)$.

First proof. It is enough to show that

$$\Omega((\sum_{i=1}^{N}\mu_i\delta(F_i))^*(\sum_{i=1}^{N}\mu_i\delta(F_i)))\geq 0 \qquad F_i\in V,\ N<+\infty$$

for the state Ω as in (2.1). One easily calculates that

$$\begin{aligned}&\Omega((\sum_{i=1}^{N}\mu_i\delta(F_i))^*(\sum_{i=1}^{N}\mu_i\delta(F_i)))=\\&=\sum_{F_i-F_j\in V_0}\overline{\mu}_i\mu_j e^{\frac{i}{2}\sigma(F_i,F_j)}\omega(\delta(F_j-F_i)).\end{aligned} \tag{2.2}$$

Let's partition the set $\{F_i\}_{i=1\cdots N}$ into equivalence classes mod V_0. Let G^k, $k=1\cdots M\leq N$ be a representative from each class. One notices that, for suitable G^k's,

$$\begin{aligned}\omega(\delta(F_j-F_i))&\equiv\omega(\delta(F_j-G^k-(F_i-G^k)))\\ \frac{i}{2}\sigma(F_i,F_j)&=\frac{i}{4}\{\sigma(2F_i-G^k,G^k)-\sigma(2F_j-G^k,G^k)\}+\\&\quad+\frac{i}{2}\sigma(F_i-G^k,F_j-G^k).\end{aligned}$$

It follows that, if we call R^k the equivalence class identified by G^k, I_k the set of the indices $i=1\cdots N$ such that $F_i\in R^k$, and we set

$$\alpha_{ik}:=\mu_i e^{\frac{i}{4}\sigma(2F_i-G^k,G^k)},\qquad u_{ik}:=F_i-G^k:$$

$$\begin{aligned}(2.2)&=\sum_{k=1}^{M}\sum_{F_i,F_j\in R^k}\overline{\alpha}_{ik}\alpha_{jk}e^{\frac{i}{2}\sigma(F_i-G^k,F_j-G^k)}\omega(\delta(F_j-G^k-(F_i-G^k)))\equiv\\&\equiv\sum_{k=1}^{M}\sum_{u_{ik},u_{jk}\in V_0}\overline{\alpha}_{ik}\alpha_{jk}e^{\frac{i}{2}\sigma(u_{ik},u_{jk})}\omega(\delta(u_{jk}-u_{ik}))=\\&=\sum_{k=1}^{M}\sum_{i,j\in I_k}\overline{\alpha}_i\alpha_j e^{\frac{i}{2}\sigma(u_i,u_j)}\omega(\delta(u_j-u_i))=\\&=\sum_{k=1}^{M}\omega((\sum_{i\in I_k}\alpha_i\delta(u_i))^*(\sum_{j\in I_k}\alpha_j\delta(u_j)))\geq 0\end{aligned}$$

by the positivity of ω.

It is obvious that $\Omega(\mathbb{1})=\Omega(\delta(0))=\omega(\delta(0))=1$. Hence Ω is a normalized, positive linear functional on $\mathcal{A}(V,\sigma)$.

Second proof. This proof is organized as follows.

a) We construct an *amenable* group $\mathcal{G}_0$ of automorphisms of $\mathcal{A}(V,\sigma)$ leaving $\mathcal{A}(V_0,\sigma_0)$ pointwise invariant.
b) For every $\omega \in E_{\mathcal{A}(V_0,\sigma_0)}$, we show that the functional Ω as in eq. (2.1) is a $\mathcal{G}_0$-*invariant state* on $\mathcal{A}(V,\sigma)$.

a) Consider the set

$$\nu_0 := \{\phi \in V_{\mathbb{R}}^* : \ \phi(F) = 0 \quad \forall F \in V_0\}.$$

It is an abelian group since $\nu_0 \simeq (V/V_0)^*$.
We then define the following abelian group of automorphisms of $\mathcal{A}(V,\sigma)$ (symmetry group)

$$\mathcal{G}_0 := \{\alpha_\phi \in Aut\mathcal{A}(V,\sigma),\ \alpha_\phi(\delta(F)) = e^{i\phi(F)}\delta(F), \quad \phi \in \nu_0\}.$$

$\mathcal{G}_0$ is isomorphic to ν_0.
The action of $\mathcal{G}_0$ on $\mathcal{A}(V,\sigma)$ leaves $\mathcal{A}(V_0,\sigma_0)$ pointwise invariant.
Since $\mathcal{G}_0$ is abelian it is amenable (see th. 1.2.1 in [G]); let η be an invariant mean on it.

b) Take now $\omega \in E_{\mathcal{A}(V_0,\sigma_0)}$, and let $\hat{\omega} \in E_\omega$. Since $\hat{\omega}(\alpha_\phi(\delta(F)))$ is continuous and bounded as a function of $\phi \in \mathcal{G}_0$, one may define

$$\eta(\hat{\omega}(\alpha_\phi(\delta(F)))) \qquad \forall F \in V$$

and where ϕ runs over $\mathcal{G}_0$. It is well known (see for instance the Lemma on pag. 172 in [E]) that this mapping defines a state on $\mathcal{A}(V,\sigma)$, which we call $\eta\hat{\omega}$:

$$\eta\hat{\omega}(\delta(F)) := \eta(\hat{\omega}(\alpha_\phi(\delta(F)))) \qquad \forall F \in V.$$

We show that $\Omega = \eta\hat{\omega}$.
Notice first that, since $\mathcal{A}(V_0,\sigma_0)$ is left pointwise invariant by the action of $\mathcal{G}_0$ on $\mathcal{A}(V,\sigma)$, one has $\eta\hat{\omega} = \hat{\omega} = \omega$ on $\mathcal{A}(V_0,\sigma_0)$.
Moreover, $\eta\hat{\omega}$ is $\mathcal{G}_0$-invariant and so, for every $\tilde{\phi} \in \nu_0$,

$$\eta\hat{\omega}(\alpha_{\tilde{\phi}}(\delta(F))) \equiv \eta(\hat{\omega}(\alpha_{\phi+\tilde{\phi}}(\delta(F)))) = \eta\hat{\omega}(\delta(F)) \qquad \forall F \in V.$$

But

$$\eta\hat{\omega}(\alpha_{\tilde{\phi}}(\delta(F))) = e^{i\tilde{\phi}(F)}\eta\hat{\omega}(\delta(F)) \qquad \forall F \in V$$

since $e^{i\tilde{\phi}(F)}$ is a c-number and $\eta\hat{\omega}$ is linear.

Fixed anyhow $F \in V \backslash V_0$, there is $\psi \in \nu_0$ such that $\psi(F) \neq 0$, and one can always set $\psi(F) \neq 2n\pi$, $n \in \mathbb{Z}$, by rescaling the functional ψ.
It follows that

$$\eta\hat{\omega}(\delta(F)) = 0 \qquad \forall F \in V \backslash V_0.$$

Hence $\Omega = \eta\hat{\omega}$ and Ω is a state on $\mathcal{A}(V, \sigma)$. *q.e.d.*

One has then $(\alpha_\phi)^*\Omega = \Omega$ for every $\phi \in \nu_0$ and so α_ϕ is implemented in π_Ω by a unitary operator U_ϕ.

Remark. If one singles out any nontrivial additive subgroup $\mathcal{G}$ of $\mathcal{G}_0$, the above proposition shows that a $\mathcal{G}$-invariant state extending ω to $\mathcal{A}(V, \sigma)$ always exists.
As an example, which will be relevant in the following, fix $G \in V$ and consider the sets

$$\nu_G := \{\phi \in \nu_0 : \ \phi(G) = 2k\pi, \ k \in \mathbb{Z}\}$$

and

$$\mathcal{G}_G := \{\alpha^\phi \in \mathcal{G}_0 : \ \phi \in \nu_G\}.$$

$\mathcal{G}_G$ is clearly an additive subgroup of $\mathcal{G}_0$. By the above proof, we are able to determine the form of the $\mathcal{G}_G$-invariant states extending ω to $\mathcal{A}(V, \sigma)$: it results

$$\Omega(\delta(F)) = \begin{cases} \omega(\delta(F)) & F \in V_0 \\ 0 & F \in V \backslash \{V_0 + \{nG\}\} \quad n \in \mathbb{Z}. \end{cases}$$

Nothing can be said about the value of $\Omega(\delta(G))$.

Remark. The above proof implies that, given ω, Ω is the *unique* $\mathcal{G}_0$-invariant state extending ω to $\mathcal{A}(V, \sigma)$. Ω is then an extremal $\mathcal{G}_0$-invariant state if ω is pure (since $\mathcal{G}_0 = \{\mathbb{1}\}$ on $\mathcal{A}(V_0, \sigma_0)$).

The states whose existence is proved in Prop. 2.1 and in the above remark are a particular class of nonregular states on CCR algebras. In the following, they will be referred to as *flat* nonregular states (we will often skip the word "flat" or "nonregular").

We now prove a sort of converse of Prop. 2.1: we suppose indeed to be given a CCR algebra $\mathcal{A}(V, \sigma)$ and a (possibly nonregular) representation π of it. We show that the set $\mathcal{R}_V^\pi$ of $F \in V$, such that $\lambda \in \mathbb{R} \longmapsto \pi(\delta(\lambda F))$ is strongly continuous, is a linear subspace of V. Hence the $\delta(\cdot)$ indexed by the elements of this set give rise in a natural way to a CCR subalgebra $\mathcal{A}_\pi$ (it depends on π!) of $\mathcal{A}(V, \sigma)$, which we call the *regular subalgebra* of $\mathcal{A}(V, \sigma)$ in representation π.

In particular, if π is the GNS representation induced by a (possibly nonregular) state Ω, $\pi \equiv \pi_\Omega$, the set $\mathcal{R}^\pi_{V}$ coincides with the set

$$\mathcal{R}^\Omega_{V} := \{F \in V : \ \lambda \in \mathbb{R} \longmapsto \Omega(\delta(\lambda F)) \in C^0\}.$$

Proposition 2.2

Let $\mathcal{A}(V,\sigma)$ be a CCR algebra and π be a representation of it. Then

$$\mathcal{R}^\pi_{V} := \{F \in V : \ \lambda \in \mathbb{R} \longmapsto \pi(\delta(\lambda F)) \text{ is strongly continuous}\}$$

is a linear subspace of V and so $(\mathcal{R}^\pi_{V}, \sigma)$ is a symplectic subspace of (V,σ).

Proof. It is obvious that $\mathcal{R}^\pi_{V}$ is a cone in V: $F \in \mathcal{R}^\pi_{V}$ implies $\alpha F \in \mathcal{R}^\pi_{V}$ for every $\alpha \in \mathbb{R}$.

If $F \in \mathcal{R}^\pi_{V}$, then $\pi(\delta(\lambda F))$ is a unitary group which is strongly continuous in $\lambda \in \mathbb{R}$.
Let $F, G \in \mathcal{R}^\pi_{V}$. One easily calculates that

$$\pi(\delta(\lambda(F+G))) = \pi(\delta(\lambda F))\pi(\delta(\lambda G))e^{\frac{i}{2}\lambda^2\sigma(F,G)}. \tag{2.3}$$

The phase factor is clearly continuous in $\lambda \in \mathbb{R}$.
Moreover, since both $\pi(\delta(\lambda F))$ and $\pi(\delta(\lambda G))$ belong to the unit ball in $\mathcal{B}(\mathcal{H}_\Omega)$, since they are strongly continuous in $\lambda \in \mathbb{R}$ and since the strong operator topology is jointly continuous on the unit ball in $\mathcal{B}(\mathcal{H}_\Omega)$, it follows that the left-hand side in (2.3) is strongly continuous in $\lambda \in \mathbb{R}$.
Hence $F + G \in \mathcal{R}^\pi_{V}$ whenever $F, G \in \mathcal{R}^\pi_{V}$.
In this way we have proved that, fixed a representation π of $\mathcal{A}(V,\sigma)$, $(\mathcal{R}^\pi_{V}, \sigma)$ is a symplectic subspace of (V,σ). *q.e.d.*

In the following, since we always refer to GNS representations of $\mathcal{A}(V,\sigma)$, the algebra $\mathcal{A}(\mathcal{R}^\Omega_{V}, \sigma) \equiv \mathcal{A}_\Omega$ will be referred to as the *regular subalgebra* of $\mathcal{A}(V,\sigma)$ in representation π_Ω.

Remark. In the quasifree case, $\Omega_q(\delta(F)) = 0$ is equivalent to $q(F) = +\infty$. This condition on q may be formalized in a precise way as follows.

Definition 2.3

We call *generalized quadratic form* over a symplectic space (V,σ) a map $q : V \longrightarrow \mathbb{R}^+ \cup \{+\infty\}$ such that the following properties hold, $\forall \lambda \in \mathbb{R}\backslash\{0\}, \forall F, G \in V$

A.

$$q(\lambda F) = \lambda^2 q(F) \qquad q(F) \geq 0 \qquad q(0) = 0$$

$$q(F+G)^{1/2} \leq q(F)^{1/2} + q(G)^{1/2}$$

$$q(F+G) + q(F-G) = 2q(F) + 2q(G)$$

(with the obvious convention when $q(F)$ or $q(G)$ equals infinity). The following possibilities are allowed:

$$\{F \in V : q(F) = +\infty\} \neq \emptyset$$
$$V_q^0 := \{F \in V : q(F) = 0\} \neq \{0\}.$$

B.

$$|\sigma(F,G)|^2 \leq q(F)q(G) \qquad \forall F, G \in V : q(F) < +\infty, q(G) < +\infty.$$

As a consequence of Prop. 2.1 above, one has

Lemma 2.4

Let (V,σ) be a symplectic space, $q(\cdot)$ a generalized quadratic form on it; $\mathcal{A}(V,\sigma)$ the CCR algebra associated to (V,σ). Then the linear functional on $\mathcal{A}(V,\sigma)$ defined by

$$\Omega_q(\delta(F)) = \begin{cases} \exp(-\frac{1}{4}q(F)) & q(F) < +\infty \\ 0 & q(F) = +\infty. \end{cases} \tag{2.4}$$

is positive and hence is a state on $\mathcal{A}(V,\sigma)$ (generalized quasifree state).

Remark. One easily shows that $\mathcal{R}_V^{\Omega_q} = V_q$, where

$$V_q := \{F \in V : \ q(F) < +\infty\} \tag{2.5}$$

is a linear subspace of V, and the restriction of Ω_q to $\mathcal{A}(V_q,\sigma)$ is a quasifree state.

3. Minimally nonregular flat states

By the above propositions, the following definitions do make sense:

Definition 3.1

Let $\mathcal{A}(V_0,\sigma_0)$ be a CCR subalgebra of $\mathcal{A}(V,\sigma)$. Then a regular state ω on $\mathcal{A}(V_0,\sigma_0)$ has $\mathcal{A}(V_0,\sigma_0)$ as its *maximal domain of regularity* in $\mathcal{A}(V,\sigma)$ if there is *no regular* extension of ω to a larger CCR subalgebra $\mathcal{A}(V_1,\sigma_1)$ of $\mathcal{A}(V,\sigma)$, $V_0 \not\subseteq V_1 \subset V$.

- This definition makes sense by Prop. 2.1, saying that *there are non*regular extensions of ω to $\mathcal{A}(V,\sigma)$.

Definition 3.2

A nonregular state Ω on a CCR algebra $\mathcal{A}(V,\sigma)$ is *minimally nonregular* if the regular subalgebra $\mathcal{A}_\Omega$ of $\mathcal{A}(V,\sigma)$ is its maximal domain of regularity.

- A nonregular state on the generic CCR algebra always exists: take $V_0 = \{0\}$ in Prop. 2.1. As it is easily seen, this state induces the type II_1 representation first introduced by Slawny [Sl].

Our goal is to characterize minimally nonregular *flat* states. They are relevant since in some sense they are determined by their restriction to the regular subalgebra. We suppose to have a CCR algebra $\mathcal{A}(V,\sigma)$ and a subalgebra $\mathcal{A}(V_0,\sigma_0)$ of it. A *regular* state ω on $\mathcal{A}(V_0,\sigma_0)$ is given. We want to study the nonregular extensions of ω to $\mathcal{A}(V,\sigma)$. To this end we do the following, crucial, assumption (as usual, we take $\Omega \in E_\omega$):

Condition A

There is $\{F_n\}_{n\in\mathbb{N}} \in V_0$*, with* $\lim_{n\to+\infty}\sigma(F_n,\cdot) \in V^*_{\mathbb{R}}$*, such that*

$$\lim_{n\to+\infty}\omega(\delta(F_n)) = 1 \tag{3.1}$$

Remark. There are cases in which $\{F_n\}_{n\in\mathbb{N}}$ in Condition A is a constant sequence: $\exists F \in V_0$, $F \neq 0$, such that $\omega(\delta(F)) = 1$. It follows from the subsequent Lemma 3.3 that, if $\Omega \in E_\omega$,

$$\pi_\Omega(\delta(F)) \in \mathcal{Z}_\Omega(\mathcal{A}(\mathcal{R}^\Omega_V,\sigma)).$$

A situation of this type occurs, for instance, in the quantization of systems with constraints (like gauge theories). First class constraints are identified precisely by the above condition (see the works by Grundling and Hurst [GH,Gr,H]), the states complying with it being referred to as "Dirac states".

Remark. In the quasifree case, Condition A above is replaced by the following, which involves only the form q associated to the state ω_q.

Condition A$_q$

There is $\{F_n\}_{n\in\mathbb{N}} \in V_0$*, with* $\lim_{n\to+\infty}\sigma(F_n,\cdot) \in V^*_{\mathbb{R}}$*, such that*

$$\lim_{n\to+\infty} q(F_n) = 0 \tag{3.2}$$

The existence of *nontrivial* (i.e., $\exists G \in V : \lim_{n\to+\infty}\sigma(F_n,G) \neq 0$) sequences complying with Condition A forces any state $\Omega \in E_\omega$ to be nonregular: we will see this later. In the two following lemmas, we show that $\lim_{n\to+\infty}\pi_\Omega(\delta(F_n))$ exists in the strong operator topology and that it belongs to the centre of (the von Neumann algebra associated in π_Ω to) the regular subalgebra $\mathcal{A}_\Omega$.

Lemma 3.3

Let $\mathcal{A}(V,\sigma)$ be a CCR algebra. Let $\mathcal{A}(V_0,\sigma_0)$ be a CCR subalgebra of $\mathcal{A}(V,\sigma)$ and ω be a regular state on $\mathcal{A}(V_0,\sigma_0)$. Let $\Omega \in E_\omega$. Then, for every $\{F_n\}_{n\in\mathbb{N}} \in V_0$ which satisfies Condition A and for every $G \in \mathcal{R}^{\Omega}_{V}$, one has

$$\lim_{n\to+\infty} \sigma(F_n, G) = 0.$$

Proof. Let Ω be any extension of ω to $\mathcal{A}(V,\sigma)$: clearly $V_0 \subseteq \mathcal{R}^{\Omega}_{V}$. One notices then that (3.1) implies $s - \lim_{n\to+\infty} \pi_\Omega(\delta(F_n))\psi_\Omega = \psi_\Omega$. Indeed, one has

$$\|(\pi_\Omega(\delta(F_n)) - \mathbb{1})\psi_\Omega\|^2 = 2 - 2Re\{\omega(\delta(F_n))\} \longrightarrow 0$$

as $n \to +\infty$, if (3.1) is true.
Hence

$$\lim_{n\to+\infty} \Omega(\delta(-F_n)\delta(G)\delta(F_n)) = \Omega(\delta(G)) \qquad \forall G \in V. \tag{3.3}$$

On the other hand, one explicitely calculates that

$$\lim_{n\to+\infty} \Omega(\delta(-F_n)\delta(G)\delta(F_n)) = \Omega(\delta(G)) \lim_{n\to+\infty} e^{i\sigma(F_n,G)} \qquad \forall G \in V, \tag{3.4}$$

where $\lim_{n\to+\infty} \sigma(F_n, G)$ exists for every $G \in V$.
Now, if $G \in \mathcal{R}^{\Omega}_{V}$, then there is an interval $I_G \subset \mathbb{R}$ such that, for every $\mu \in I_G$, $\Omega(\delta(\mu G)) \neq 0$. Choose anyhow $G \in \mathcal{R}^{\Omega}_{V}$. We show that $\lim_{n\to+\infty} \sigma(F_n, G) = 0$. Indeed, $\lim_{n\to+\infty} \sigma(F_n, \mu G)$ exists for every $\mu \in I_G$ and it is always possible, by suitably choosing $\tilde{\mu} \in I_G$, to have $\lim_{n\to+\infty} \sigma(F_n, \tilde{\mu} G) \neq 2n\pi$, $n \in \mathbb{Z}\backslash\{0\}$.
Comparing (3.3) and (3.4) with $G \longrightarrow \tilde{\mu} G$ one has, since $\Omega(\delta(\tilde{\mu} G)) \neq 0$,

$$\tilde{\mu} \lim_{n\to+\infty} \sigma(F_n, G) = \lim_{n\to+\infty} \sigma(F_n, \tilde{\mu} G) = 0 \qquad \forall G \in \mathcal{R}^{\Omega}_{V}.$$

Since $\tilde{\mu} \neq 0$ the thesis follows. *q.e.d.*

As a consequence of this lemma we have the following result.

Lemma 3.4

Let $\mathcal{A}(V,\sigma)$ be a CCR algebra. Let $\mathcal{A}(V_0,\sigma_0)$ be a CCR subalgebra of $\mathcal{A}(V,\sigma)$ and ω be a regular state on $\mathcal{A}(V_0,\sigma_0)$. Let $\Omega \in E_\omega$. Then, for every $\{F_n\}_{n\in\mathbb{N}} \in V_0$ satisfying Condition A, $U \equiv \lim_{n\to+\infty} \pi_\Omega(\delta(F_n))$ exists in the strong operator topology in $\mathcal{H}_\Omega$; U is a unitary operator and

$$U \in \mathcal{Z}_\Omega(\mathcal{A}(\mathcal{R}^{\Omega}_{V}, \sigma)).$$

Proof. The first statement follows from the fact that the sequence $\pi_\Omega(\delta(F_n))$ is uniformly bounded, the existence of $s - \lim_{n\to+\infty} \pi_\Omega(\delta(F_n))\psi_\Omega$ and from the standard observation that

$$\lim_{n\to+\infty} \|(\pi_\Omega(\delta(F_n)) - \pi_\Omega(\delta(F_m)))\pi_\Omega(\delta(G))\psi_\Omega\| =$$
$$\lim_{n\to+\infty} \|(\pi_\Omega(\delta(F_n))e^{-i\sigma(F_n,G)} - \pi_\Omega(\delta(F_m))e^{-i\sigma(F_m,G)})\psi_\Omega\| \quad \forall G \in V.$$

(recall that $\lim_{n\to+\infty} \sigma(F_n, \cdot) \in V^*_{\mathbb{R}}$).
The limit operator U is unitary as a strong limit of unitary operators with strongly convergent adjoints.
U is in $\pi_\Omega(\mathcal{A}(V_0,\sigma_0))'' \subseteq \pi_\Omega(\mathcal{A}(\mathcal{R}^\Omega_{\Upsilon},\sigma))''$ since it is a strong limit of elements in $\pi_\Omega(\mathcal{A}(V_0,\sigma_0))$; it is in $\pi_\Omega(\mathcal{A}(\mathcal{R}^\Omega_{\Upsilon},\sigma))'$ by Lemma 3.3. *q.e.d.*
It will be subsequently shown that the operators U in this Lemma are strictly correlated to the operators U_ϕ introduced just after Prop. 2.1.

Lemma 3.3 implies that, if for some $G \in V$ and $\{F_n\}_{n\in\mathbb{N}} \in V_0$ satisfying Condition A one has $\lim_{n\to+\infty} \sigma(F_n, G) = \alpha \neq 0$, then $G \notin \mathcal{R}^\Omega_{\Upsilon}$ and so $\delta(G)$ is not regularly represented in π_Ω, for every $\Omega \in E_\omega$.

Starting from this observation, a more detailed analysis proceeds as follows. Let's consider the set of functionals identified by Condition A and define

$$\Phi_{\sigma\omega} := \{\varphi \in V^*_{\mathbb{R}} : \varphi = \lim_{n\to+\infty} \sigma(F_n, \cdot),\ \{F_n\}_{n\in\mathbb{N}} \in V_0 \text{ satisfying Condition A}\}.$$

It is very important to stress that, given $\mathcal{A}(V,\sigma)$ and $\mathcal{A}(V_0,\sigma_0)$, $\Phi_{\sigma\omega}$ is *completely determined* by the regular state ω. $\Phi_{\sigma\omega}$ is not empty since it contains at least the null functional; by Lemma 3.3 it is a subset of ν_0. Set then

$$V^0_{\sigma\omega} := \{G \in V : \forall\varphi \in \Phi_{\sigma\omega}\ \varphi(G) = 0\}.$$

$V^0_{\sigma\omega}$ is a real linear subspace of V. It follows from Lemma 3.3 that $V^0_{\sigma\omega}$ contains V_0 (and $\mathcal{R}^\Omega_{\Upsilon}$, $\forall\Omega \in E_\omega$) and that

Lemma 3.5

If $\exists G \in V$ such that $G \notin V^0_{\sigma\omega}$, then no $\Omega \in E_\omega$ is a regular state. Moreover, if $V_0 = V^0_{\sigma\omega}$ then ω has $\mathcal{A}(V_0,\sigma_0)$ as maximal domain of regularity in $\mathcal{A}(V,\sigma)$.

This means that the domain of regularity of ω can (possibly) be enlarged up to the CCR algebra $\mathcal{A}(V^0_{\sigma\omega},\sigma)$ and nothing can be said about the form of the (possibly regular) extensions of ω to this algebra. No state in $E_\omega \cap E_{\mathcal{A}(V^0_{\sigma\omega},\sigma)}$ can be extended to a regular state on $\mathcal{A}(V,\sigma)$. In order to study these nonregular extension we introduce

$$V_{\sigma\omega} := \{G \in V : \ \forall\varphi \in \Phi_{\sigma\omega} \ \exists m \in \mathbb{Z} : \ \varphi(G) = 2m\pi\}.$$

Remark. Clearly $V^0_{\sigma\omega} \subseteq V_{\sigma\omega}$. In the quasifree case one always has that $V^0_{\sigma\omega} = V_{\sigma\omega}$ and so $V_{\sigma\omega}$ is a linear space (see below the proof of Corollary 3.8).

It is easily seen that $V_{\sigma\omega}$ is a closed additive subgroup of V: it can be written as the sum of a vector space, $V^0_{\sigma\omega}$, and of a discrete subgroup. It follows that it is well defined [Sl] the CCR algebra

$$\mathcal{A}_{obs} := \mathcal{A}(V_{\sigma\omega},\sigma) = \{\delta(F) \in \mathcal{A}(V,\sigma) : \ F \in V_{\sigma\omega}\}.$$

Remark. It results from the definition of $\mathcal{A}_{obs}$ that

$$\lim_{n\to+\infty} \pi_\Omega(\delta(F_n)) \in \mathcal{Z}_\Omega(\mathcal{A}_{obs}),$$

where $\{F_n\}_{n\in\mathbb{N}}$ satisfies Condition A and $\Omega \in E_\omega$.

By the same argument as in the proof of Lemma 3.3 (i.e. comparing equations (3.3) and (3.4)) it is immediate that

Lemma 3.6

For every $\Omega \in E_\omega$, one has

$$\Omega(\delta(F)) = 0 \qquad F \notin V_{\sigma\omega} :$$

the states in $E_\omega \cap E_{\mathcal{A}_{obs}}$ extend in unique way to states in E_ω.

It is now easy to give a condition which provides us with a *unique* extension $\Omega \in E_\omega$; Ω is then minimally nonregular, with $\mathcal{A}_\Omega = \mathcal{A}(V_0,\sigma_0)$.

Proposition 3.7

Let $\mathcal{A}(V,\sigma)$ be a CCR algebra. Let $\mathcal{A}(V_0,\sigma_0)$ be a CCR subalgebra of $\mathcal{A}(V,\sigma)$ and ω be a regular state on $\mathcal{A}(V_0,\sigma_0)$. Consider the two statements

i. $\forall G \in V\backslash V_0 \ \exists\{F_n\}_{n\in\mathbb{N}} \in V_0$ satisfying Condition A such that

$$\lim_{n\to+\infty} e^{i\sigma(F_n,G)} = e^{i\alpha} \neq 1.$$

ii. ω *admits a unique extension* Ω *to* $\mathcal{A}(V,\sigma)$, *namely the flat nonregular state given by eq. (2.1).*

It follows that i. $\Rightarrow$ *ii. and in particular* Ω *is minimally nonregular. Moreover, if* ω *is pure so is* Ω .

Remark. In condition *i.*, we may substitute $V \backslash V_0$ with V/V_0.

In the quasifree case, one proves the equivalence between a strengthened version of *i.* and *ii.* above. We see this in the following

Corollary 3.8

Let $\mathcal{A}(V,\sigma)$ *be a CCR algebra. Let* $\mathcal{A}(V_0,\sigma_0)$ *be a CCR subalgebra of* $\mathcal{A}(V,\sigma)$ *and* ω_q *be a regular quasifree state on* $\mathcal{A}(V_0,\sigma_0)$. *Then the following two statements are equivalent*

i'. $\forall G \in V \backslash V_0 \ \exists \{F_n\}_{n\in\mathbb{N}} \in V_0$ *satisfying Condition* A_q *such that*

$$\lim_{n\to+\infty} \sigma(F_n, G) = \alpha \neq 0$$

ii. ω *admits a unique extension* Ω *to* $\mathcal{A}(V,\sigma)$, *namely the flat nonregular state given by eq. (2.1).*

Proof. To prove that *i'.* $\Rightarrow$ *ii.* one shows that *i'.* here and *i.* in Prop. 3.7 are equivalent for quasifree states. Indeed, note that $\lim_{n\to+\infty} q(F_n) = 0$ is equivalent to $\lim_{n\to+\infty} \omega_q(\delta(F_n)) = 1$.
Moreover, $\lim_{n\to+\infty} q(\lambda F_n) = \lambda^2 \lim_{n\to+\infty} q(F_n) \quad \forall \lambda \in \mathbb{R}$. Hence, if the sequence $\{F_n\}_{n\in\mathbb{N}}$ complies with Condition A_q, so does $\{\lambda F_n\}_{n\in\mathbb{N}}$ for every $\lambda \in \mathbb{R}$. It follows that, if $\alpha \neq 0$ above, one can always choose the sequence in *i.'* in such a way that $\alpha \neq 2n\pi \quad \forall n \in \mathbb{Z}$. One concludes that, for quasifree states, *i.* and *i.'* are equivalent and then Prop. 3.7 implies that *i'.* $\Rightarrow$ *ii.*
We now prove that *ii.* $\Rightarrow$ *i'.*
We notice that the latter is equivalent to the fact that, for any $G \in V \backslash V_0$, $\sigma(G,\cdot)$ is a linear functional on V_0 which is not bounded with respect to the inner product $[\cdot,\cdot]_q$ associated to q.
Suppose now that *i.'* is false. Then, for some $G \in V \backslash V_0$, $\sigma(G,\cdot)$ is a bounded linear functional on V_0. As a consequence, $\sigma(G,F) = 0$ if $q(F) = 0$ and

$$Q(G) \equiv \sup_{\substack{F \in V_0 \\ q(F) \neq 0}} \frac{|\sigma(G,F)|^2}{q(F)} < +\infty.$$

Then q can be extended as a finite Hilbert quadratic form to $V_0 + \mathrm{Span}[G]$ by

$$[G, F]_q = 0 \quad \forall F \in V_0, \qquad q(G) = Q(G).$$

In this way one gets a quasifree state which extends ω_q to $\mathcal{A}(V_0 + \mathrm{Span}[G], \sigma)$, and so *ii.* is false. *q.e.d.*

Remark. As Corollary 3.8 shows, for quasifree states condition *i.* is *equivalent* to the fact that Ω is minimally nonregular.

As an aside, we characterize pure flat nonregular states.

Lemma 3.9

Let $\mathcal{A}(V, \sigma)$ be a CCR algebra and Ω be a flat nonregular state on it. If Ω is pure then the restriction ω of it to $\mathcal{A}(\mathcal{R}_V^\Omega, \sigma)$ is pure.

Proof. Suppose that ω is not pure and hence decomposable:

$$\omega = \lambda\omega_1 + (1 - \lambda)\omega_2 \qquad \lambda \in (0, 1);\ \omega_i \in E_{\mathcal{A}_\Omega},\ i = 1, 2.$$

By Prop. 2.1, the state ω_i, $i = 1, 2$ can be extended to a *flat* nonregular state on $\mathcal{A}(V, \sigma)$:

$$\Omega_i(\delta(F)) = \begin{cases} \omega_i(\delta(F)) & F \in \mathcal{R}_V^\Omega \\ 0 & F \in V \backslash \mathcal{R}_V^\Omega. \end{cases}$$

Then Ω can be decomposed as follows:

$$\Omega = \lambda\Omega_1 + (1 - \lambda)\Omega_2 \qquad \Omega_i \in E_{\mathcal{A}(V,\sigma)},\ i = 1, 2$$

and hence it is not pure. *q.e.d.*

It then immediately follows that

Proposition 3.10

Let Ω be a flat nonregular state on $\mathcal{A}(V, \sigma)$ whose restriction ω to $\mathcal{A}_\Omega$ satisfies i. in Prop. 3.7. Then Ω is pure iff ω is.

The fact that Ω is minimally nonregular is essential for the above proposition to hold. This is shown in the following

Lemma 3.11

Let Ω be a flat nonregular state on $\mathcal{A}(V, \sigma)$ which is not minimally nonregular. Then Ω is not pure.

Proof. If Ω is not minimally nonregular, then $\exists G_0 \in V$, $\exists \tilde{\Omega} \in E_\omega$ such that $\lambda \in \mathbb{R} \longmapsto \tilde{\Omega}(\delta(\lambda G_0))$ is continuous and in particular $\tilde{\Omega}(\delta(G_0)) \neq 0$. By Prop. 2.1 we can always choose $\tilde{\Omega}$ so that $\tilde{\Omega}(\delta(F)) = 0$ for every $F \in V \backslash \{\mathcal{R}_V^\Omega + \mathrm{Span}[G_0]\}$. For every $F \in V$, we decompose

$$F = F_\Omega + \lambda(F) G_0 + F',$$

where $F_\Omega \in \mathcal{R}_V^\Omega$. Since $\lambda(F)$ is a real linear functional on V, it is well defined the state $\tilde{\Omega}^\alpha$ on $\mathcal{A}(V,\sigma)$ given by

$$\tilde{\Omega}^\alpha(\delta(F)) = e^{i\alpha\lambda(F)} \tilde{\Omega}(\delta(F)) \qquad \forall F \in V,\ \alpha \in \mathbb{R}.$$

One easily verifies that

$$\Omega = w^* - \lim_{n \to +\infty} \frac{1}{2n\pi} \int_{-n\pi}^{n\pi} d\alpha \tilde{\Omega}^\alpha.$$

We now exhibit a nontrivial decomposition of Ω.

To this end, take $I_{a,0} = [-a, a]$, with $0 < a < \pi$, and let $I_{a,n}$ be its translate by $2n\pi$, $n \in \mathbb{Z}$. Set then $I_a := \bigcup_n \{I_{a,n}\}$ and define

$$\Omega_1 := w^* - \lim_{n \to +\infty} \frac{1}{2na} \int_{I_a \cap [-n\pi, n\pi]} d\alpha \tilde{\Omega}^\alpha$$

$$\Omega_2 := w^* - \lim_{n \to +\infty} \frac{1}{2n(\pi - a)} \int_{\{\mathbb{R} \backslash I_a\} \cap [-n\pi, n\pi]} d\alpha \tilde{\Omega}^\alpha.$$

They are states on $\mathcal{A}(V,\sigma)$ as w^*-limits of sequences of states; furthermore the decomposition (which is nontrivial since $\Omega_1(\delta(G_0)) = \frac{\sin a}{a} \tilde{\Omega}(\delta(G_0))$ and $\Omega_2(\delta(G_0)) = -\frac{\sin a}{\pi - a} \tilde{\Omega}(\delta(G_0))$ as it is easily verified)

$$\Omega = \frac{a}{\pi} \Omega_1 + \frac{(\pi - a)}{\pi} \Omega_2$$

holds so that Ω is not pure. *q.e.d.*

Remark. A characterization of pure and primary *generalized quasifree states* is given in Appendix A.

4. Structure of the representations induced by minimally nonregular flat states

In the following proposition we analyze the GNS representation π_Ω of $\mathcal{A}(V,\sigma)$ in terms of representations of $\mathcal{A}_{obs} = \mathcal{A}(V_{\sigma\omega}, \sigma)$. We denote by f an element in $V \backslash V_{\sigma\omega}$ taken as a representative of the equivalence class in $V/V_{\sigma\omega}$.

Proposition 4.1

Let $\mathcal{A}(V,\sigma)$ be a CCR algebra. Let $\mathcal{A}(V_0,\sigma_0)$ be a CCR subalgebra of $\mathcal{A}(V,\sigma)$ and ω be a regular state on $\mathcal{A}(V_0,\sigma_0)$. Let $\Omega \in E_\omega$ and $\tilde{\omega} \equiv \Omega_{|\mathcal{A}_{obs}}$. Then π_Ω decomposes into the direct sum of inequivalent representations of $\mathcal{A}_{obs}$, labelled by the equivalence classes $f \in V/V_{\sigma\omega}$:

$$\pi_\Omega = \bigoplus_{f\in V/V_{\sigma\omega}} \pi_{\tilde{\omega}_f}, \qquad \mathcal{H}_\Omega = \bigoplus_{f\in V/V_{\sigma\omega}} \mathcal{H}_{\tilde{\omega}_f}$$

where $\tilde{\omega}_f := \tilde{\omega}\circ\rho_f$, $\rho_f \in Aut\mathcal{A}_{obs} : \ \delta(G) \longmapsto e^{i\sigma(f,G)}\delta(G) \quad \forall G \in V$. Moreover every $\pi_{\tilde{\omega}_f}$ is irreducible (factorial) iff $\pi_{\tilde{\omega}}$ is.

Proof. The last statement follows directly from the fact that $\rho_f \in Aut\mathcal{A}_{obs}$.
We then notice that, for every $F \in V_{\sigma\omega}$, ρ_F is implemented in $\pi_{\tilde{\omega}}$ by $\pi_{\tilde{\omega}}(\delta(F)) \in \mathcal{A}_{obs}$; by definition of GNS construction, one actually has $\mathcal{H}_{\tilde{\omega}_F} \equiv \mathcal{H}_{\tilde{\omega}}$. On the other hand, let $G \in V\backslash V_{\sigma\omega}$. Then by the definition of $V_{\sigma\omega}$ there is $\{F_n\}_{n\in\mathbb{N}} \in V_0$ such that

$$\begin{aligned}&\tilde{\omega}_G(\delta(F_n)) = \tilde{\omega}\circ\rho_G(\delta(F_n)) = \\ =&e^{i\sigma(G,F_n)}\tilde{\omega}(\delta(F_n)) \to e^{i\alpha}, \qquad e^{i\alpha} \neq 1\end{aligned}$$

which implies, since $|e^{i\alpha}| = 1$, that there exists

$$s - \lim_{n\to+\infty} \pi_{\tilde{\omega}_G}(\delta(F_n))\psi_{\tilde{\omega}_G} = e^{i\alpha}\psi_{\tilde{\omega}_G}, \quad e^{i\alpha} \neq 1. \tag{4.1}$$

If $\pi_{\tilde{\omega}} \simeq \pi_{\tilde{\omega}_G}$, and we call T the unitary intertwiner between the two representations, then one should have $\psi_{\tilde{\omega}_G} = T\psi_{\tilde{\omega}}$ and

$$T(s - \lim_{n\to+\infty} \pi_{\tilde{\omega}}(\delta(F_n))\psi_{\tilde{\omega}}) = s - \lim_{n\to+\infty} \pi_{\tilde{\omega}_G}(\delta(F_n))\psi_{\tilde{\omega}_G}$$

at the same time, which is not true whenever (4.1) is true.
Hence $\pi_{\tilde{\omega}} \not\simeq \pi_{\tilde{\omega}_G}$ if $G \in V\backslash V_{\sigma\omega}$.
We then argue from these results that $\pi_{\tilde{\omega}_G}$ and $\pi_{\tilde{\omega}_F}$ are equivalent iff $G - F \in V_{\sigma\omega}$ (since both ρ_F and ρ_G are invertible), i.e. iff $G \simeq F$ mod $V_{\sigma\omega}$, and the result follows from the definition of GNS construction. *q.e.d.*

We want to give a deeper insight into the structure displayed by Prop. 4.1. We consider a regular state ω on $\mathcal{A}(V_0,\sigma_0)$ and an extension Ω of it to $\mathcal{A}(V,\sigma)$. If ω satisfies condition *i.* in Prop. 3.7, then Ω is minimally nonregular and $\mathcal{A}_\Omega = \mathcal{A}(V_0,\sigma_0)$.

Referring to the construction in the proof of Prop. 2.1, we show that $\mathcal{A}(V,\sigma)$ identifies a gauge group $\mathcal{G}_0$, and that it may be interpreted as a field algebra. In the case in which condition *i.* in Prop. 3.7 holds, we analyze the decomposition of π_Ω into inequivalent representations of $\mathcal{A}(V_0, \sigma_0)$ and study the structure of the space of the "gauge charges".

The gauge group

As in the proof of Prop. 2.1, we consider the set

$$\nu_0 := \{\phi \in V_{\mathbb{R}}^* : \ \phi(F) = 0 \quad \forall F \in V_0\}.$$

It is an abelian group (actually a real vector space) and $\nu_0 \simeq (V/V_0)^*$.
The gauge group $\mathcal{G}_0$ is then defined as the abelian group of automorphisms of $\mathcal{A}(V, \sigma)$

$$\mathcal{G}_0 := \{\alpha_\phi \in Aut\mathcal{A}(V, \sigma), \ \alpha_\phi(\delta(F)) = e^{i\phi(F)}\delta(F), \ \phi \in \nu_0\}.$$

$\mathcal{G}_0$ is isomorphic to ν_0.
The action of $\mathcal{G}_0$ on $\mathcal{A}(V, \sigma)$ leaves $\mathcal{A}(V_0, \sigma_0)$ pointwise invariant.
Recall that $\mathcal{G}_0$ is amenable, since it is abelian.

We now come to the representation π_Ω of $\mathcal{A}(V, \sigma)$. Recall that $V_0 \subseteq \mathcal{R}_V^\Omega$. One then shows that

Proposition 4.2

The subgroup $\mathcal{G}_\Omega$ of $\mathcal{G}_0$ given by

$$\mathcal{G}_\Omega = \{\alpha_\phi \in \mathcal{G}_0 : \ \phi(F) = 2m\pi, \ m \in \mathbb{Z}, \ \forall F \in V_{\sigma\omega}\}$$

leaves Ω invariant. Every one-parameter subgroup $\alpha_{\lambda\phi}$ of $\mathcal{G}_\Omega$, ϕ fixed, is implemented in π_Ω by a strongly continuous one-parameter group of unitary operators. Their generators define the gauge charges; *they annihilate the vacuum ψ_Ω.*
$\mathcal{G}_\Omega$ is nontrivial iff Ω defines a nonregular representation of $\mathcal{A}(V, \sigma)$.

Proof. The last statement is obvious.
Next, let's define the additive subgroup of ν_0

$$\nu_{\sigma\omega} := \{\phi \in \nu_0 : \ \phi(F) = 2m\pi, \ m \in \mathbb{Z}, \quad \forall F \in V_{\sigma\omega}\}.$$

If $\phi \in \nu_{\sigma\omega}$, then $\alpha_\phi = \mathbb{1}$ on $\mathcal{A}_{obs}$ by the definition of $\mathcal{A}_{obs}$; if $G \in V \backslash V_{\sigma\omega}$, one has

$$\Omega(\alpha_\phi(\delta(G))) = e^{i\phi(G)}\Omega(\delta(G)) = 0 = \Omega(\delta(G))$$

by Prop. 3.6. It follows

$$(\alpha_\phi)^* \Omega = \Omega.$$

The first statement is proved.

Let then $\alpha_{\lambda\phi}$ be any one-parameter subgroup of $\mathcal{G}_\Omega$ given by

$$\alpha_{\lambda\phi}(\delta(G)) = e^{i\lambda\phi(G)}\delta(G) \qquad \forall F \in V.$$

Subgroups of this type are for instance generated by those $\phi \in \nu_\Omega$, where

$$\nu_\Omega := \{\phi \in \nu_{\sigma\omega} : \ \phi(F) = 0 \ \forall F \in V_{\sigma\omega}\}.$$

Since $(\alpha_{\lambda\phi})^*\Omega = \Omega$, $\alpha_{\lambda\phi}$ is implemented, for every $\lambda \in \mathbb{R}$, by a unitary operator $U_\phi(\lambda)$ in π_Ω and

$$\|(U_\phi(\lambda) - \mathbb{1})\pi_\Omega(\delta(G))\psi_\Omega\| = |e^{i\lambda\phi(G)} - 1| \longrightarrow 0 \qquad \forall G \in V$$

as $\lambda \to 0$, i.e. strong continuity holds. The existence of the gauge charges as generators of the groups $U_\phi(\lambda)$ follows from Stone's theorem. If $U_\phi(\lambda) = e^{i\lambda Q_\phi}$, Q_ϕ a gauge charge, then $U_\phi(\lambda)\psi_\Omega = \psi_\Omega$ implies $Q_\phi\psi_\Omega = 0$. *q.e.d.*

Notice that the set of gauge charges Q_Ω is a linear space.

Additional information are obtained if $V_0 = \mathcal{R}_V^\Omega$ and ω satisfies condition *i.* in Prop. 3.7.

Proposition 4.3

If ω satisfies condition i. in Prop. 3.7, then

a. $\mathcal{G}_\Omega = \mathcal{G}_0$

b. π_Ω *decomposes into the direct sum of inequivalent representations of* $\mathcal{A}(\mathcal{R}_V^\Omega, \sigma)$, *labelled by the gauge charges*

$$\pi_\Omega = \bigoplus_{f \in V/\mathcal{R}_V^\Omega} \pi_{\omega_f}$$

where $\omega_f = \omega \circ \rho_f$ *as in Prop. 4.1.*

c. The "charged fields" $\pi_\Omega(\delta(f))$, $f \in V/\mathcal{R}_V^\Omega$, *act as intertwiners between the inequivalent representations of* $\mathcal{A}(\mathcal{R}_V^\Omega, \sigma)$.

Proof. a. follows from $V_0 = \mathcal{R}_V^\Omega$.

b. and *c.* are easy consequences of Prop. 4.1. *q.e.d.*

We now look at the structure of the set of the gauge charges: we suppose again that $V_{\sigma\omega} = \mathcal{R}_V^\Omega \not\subseteq V$.

It is clear from the definition of ν_Ω that

$$U_\phi(\lambda) \in \pi_\Omega(\mathcal{A}(\mathcal{R}_V^\Omega, \sigma))' \qquad \forall \phi \in \nu_\Omega,\ \lambda \in \mathbb{R}.$$

One may say that the implementers of the gauge symmetries belong to (and so the gauge charges are affiliated to) the commutant of the algebra $\mathcal{A}_\Omega$. We are now looking for conditions implying that the gauge charges (or at least some among them) are actually *central* charges. This amounts to have

$$U_\phi(\lambda) \in \mathcal{Z}_\Omega(\mathcal{A}(\mathcal{R}_V^\Omega, \sigma)) \qquad \text{for some } \phi \in \nu_\Omega,\ \forall \lambda \in \mathbb{R}.$$

From this point of view, Lemmas 3.3 and 3.4 are useful.
Indeed, let $\{F_n\}_{n\in\mathbb{N}} \in V_0$ comply with Condition A. Then $\lim_{n\to+\infty} \sigma(F_n, \cdot) \in V_{\mathbb{R}}^*$ and actually $\lim_{n\to+\infty} \sigma(F_n, \cdot) \in \nu_\Omega$, by Lemma 3.3.
On the other hand, there exists $\lim_{n\to+\infty} \pi_\Omega(\delta(F_n))$ in the strong operator topology in $\mathcal{H}_\Omega$ and it belongs to $\mathcal{Z}_\Omega(\mathcal{A}(\mathcal{R}_V^\Omega, \sigma))$; the limit operator U is unitary (Lemma 3.4).
Since $\mathcal{R}_V^\Omega \neq V$, U is not the identity operator: indeed there is $G \in V$ such that $\lim_{n\to+\infty} \sigma(F_n, G) \neq 0$ (we then set $\lim_{n\to+\infty} \sigma(F_n, \cdot) \equiv \phi_G$). Then, for every $\lambda \in \mathbb{R}$, $\lambda\phi_G$ belongs to ν_Ω and is nontrivial. By Prop. 4.2, the one-parameter subgroup $\alpha_{\lambda\phi_G}$ of $\mathcal{G}_\Omega$ is implemented by a strongly continuous one-parameter group of unitaries $U_{\phi_G}(\lambda)$.
If $U_{\phi_G}(\lambda) \equiv e^{i\lambda Q_{\phi_G}}$, then $U = U_{\phi_G}(\lambda = 1) = e^{iQ_{\phi_G}}$ with Q_{ϕ_G} selfadjoint and affiliated to the centre of the observable algebra.
In the following Lemma we see that, if condition *i.* in Prop. 3.7 is true for ω, then every gauge charge is a linear combination of charges of this type.

Lemma 4.4

Let Ω be a nonregular state on $\mathcal{A}(V, \sigma)$ whose restriction ω to $\mathcal{A}_\Omega$ satisfies i. in Prop. 3.7. Then the set $\Phi_{\sigma\omega}$ actually spans the space $\nu_0 = \nu_\Omega$.

Proof. Every map $\lim_{n\to+\infty} \sigma(F_n, \cdot)$, $\{F_n\}_{n\in\mathbb{N}} \in \mathcal{R}_V^\Omega$ complying with Condition A, is a true linear functional on the whole V. By Lemma 3.3, $\Phi_{\sigma\omega}$ is contained in ν_0.
Since $\nu_0^* \equiv (V/\mathcal{R}_V^\Omega)^{**} = V/\mathcal{R}_V^\Omega$ as algebraic duals of linear spaces, and since ν_0 is a closed linear subspace of V^*, if $\mathrm{Span}\{\phi \in \Phi_{\sigma\omega}\} \not\subseteq \nu_0$, there is $G \in V/\mathcal{R}_V^\Omega$ such that $G(\phi) \equiv \phi(G) = 0$ for every $\phi \in \Phi_{\sigma\omega}$, that is
$\exists G \in V/\mathcal{R}_V^\Omega$ such that, $\forall \{F_n\}_{n\in\mathbb{N}} \in V_0$ complying with Condition A, one has

$$\lim_{n\to+\infty} \sigma(F_n, G) = 0$$

which contradicts *i.* in Prop. 3.7. *q.e.d.*

Remark. The above reasoning is more trasparent if the existence of $\{F_n\}_{n\in\mathbb{N}} \in \mathcal{R}^{\Omega}_{V}$ such that $\lim_{n\to+\infty}\omega(\delta(F_n)) = 1$, implies $\lim_{n\to+\infty}\omega(\delta(\lambda F_n)) = 1$ for every $\lambda \in \mathbb{R}$.

This is true for instance in the quasifree case, where $\lim_{n\to+\infty}\omega_q(\delta(F_n)) = 1$ is equivalent to $\lim_{n\to+\infty} q(F_n) = 0$ and then

$$\lim_{n\to+\infty} q(\lambda F_n) = \lambda^2 \lim_{n\to+\infty} q(F_n) = 0.$$

This fact implies indeed that $\lambda\phi \in \Phi_{\sigma\omega}$ for every $\lambda \in \mathbb{R}$ whenever $\phi \in \Phi_{\sigma\omega}$. Since $q(F+G)^{\frac{1}{2}} \leq q(F)^{\frac{1}{2}} + q(G)^{\frac{1}{2}}$ $\forall F, G \in \mathcal{R}^{\Omega}_{V}$, $\phi, \psi \in \Phi_{\sigma\omega}$ entails $\phi + \psi \in \Phi_{\sigma\omega}$. It follows that in the hypotesis of the above Lemma one actually has $\Phi_{\sigma\omega} = \nu_0 = \nu_\Omega$.

EXAMPLES

1. The harmonic oscillator in the limit of zero frequency or of infinite temperature.

The model is defined by the quantum Hamiltonian

$$H = \frac{1}{2m}(\hat{p}^2 + m^2\nu^2\hat{x}^2).$$

The Weyl algebra describing the kinematics is generated by the operator

$$W(\underline{u}) := \exp\left(i(u_1\hat{x} + u_2\hat{p})\right)$$

with $\underline{u} :=< u_1, u_2 >\in \mathbb{R}^2$. The symplectic form is

$$\sigma(\underline{u}, \underline{v}) := u_1 v_2 - v_1 u_2 \tag{1.1}$$

which is nondegenerate on $\mathbb{R}^2 \times \mathbb{R}^2$.
The equilibrium states at inverse temperature $\beta = \frac{1}{T}$ are the quasifree states defined by

$$\omega_\beta(W(\underline{u})) = \exp\left(-\frac{1}{4}\{\frac{u_1^2}{m\nu} + m\nu u_2^2\}\coth\frac{\nu\beta}{2}\right). \tag{1.2}$$

The zero temperature ($\beta \to 0$) state in the limit $\nu \to 0$ (free particle limit) is a generalized quasifree state defined by

$$\begin{aligned} \omega_0(W(< 0, u_2 >)) &= 1 \qquad \forall u_2 \in \mathbb{R} \\ \omega_0(W(< u_1, 0 >)) &= 0 \qquad \forall u_1 \in \mathbb{R}\ u_1 \neq 0, \end{aligned}$$

corresponding to $q(0, u_2) = 0$, $q(u_1, 0) = +\infty$, if $u_1 \neq 0$. Furthermore

$$V_q = \{\underline{u} \in \mathbb{R}^2 : \underline{u} =< 0, u_2 >\}$$

is the maximal domain of regularity for ω_0 (if not, there would be a regular and quasifree extension and this would contradict the nondegeneracy of σ).

For non zero temperature states the $\nu \to 0$ limit also defines a generalized quasifree state given by (see also [NT2])

$$\begin{aligned} \omega_\beta^0(W(< u_1, 0 >)) &= 0 \qquad u_1 \neq 0 \\ \omega_\beta^0(W(< 0, u_2 >)) &= \exp\left(-\frac{m}{2\beta}u_2^2\right). \end{aligned} \tag{1.3}$$

It is worthwile to remark that, in an equilibrium state at non–zero temperature, the Maxwell distribution of the velocity of a free quantum particle only arises in the infinite volume limit and it requires a uniform distribution of the position so that the use of nonregular representations is necessary. For fixed ν and $\beta \to 0$ the state ω_β converges on the Weyl algebra to the (nonregular) central state $\hat{\omega}(\delta(F)) = 0$, $\forall F \neq 0$.

2. Quantum particle on a circle.

The observable algebra $\mathcal{A}_{obs}$ is generated by the Weyl operators

$$W(n,v) = \exp i(n\varphi + vL)$$

over the additive group $V_0 = \{n,v\}$, $v \in \mathbb{R}$, $n \in \mathbb{N}$, with the usual symplectic form $\sigma((n_1,v_1),(n_2,v_2)) = n_1v_2 - n_2v_1$. The extended algebra $\mathcal{A} = \mathcal{A}(V,\sigma)$, $V = \mathbb{R} \times \mathbb{R}$, which includes the singular variables $\exp i\alpha\varphi$, $\alpha \in \mathbb{R}$, defines a gauge group $\mathcal{G}$ which is the group of rotations of $2k\pi$, $k \in \mathbb{N}$. In fact, the gauge group is defined by the group of real functionals Φ on V, vanishing, modulo 2π, on V_0:

$$\Phi(\alpha,\beta) = \Phi(\alpha + n, 0) \mod 2\pi \quad ,$$

and therefore

$$\Phi(\alpha,\beta) = 2m\pi\alpha \ , \quad m \in \mathbb{N} \ . \tag{2.1}$$

Hence, the action of the gauge group on $\mathcal{A}$ is given by

$$W(\alpha,\beta) \to \exp i2\pi m\alpha \ W(\alpha,\beta) \tag{2.2}$$

i.e., it coincides with rotations of angle $\theta = 2m\pi$. The gauge group is unbroken in the representation of $\mathcal{A}_{obs}$ defined by the generalized quasi-free state (the ground state of a rotator) given by

$$\begin{aligned} \Omega(W(n,v)) &= 0 \qquad n \neq 0 \\ &= 1 \qquad n = 0. \end{aligned} \tag{2.3}$$

The charged fields $W(\theta,0)$, $\theta \in [0,1)$, are singular variables and intertwine between inequivalent representations of $\mathcal{A}_{obs}$. They define a one parameter group of automorphisms β^θ, $\theta \in [0,1)$, of $\mathcal{A}_{obs}$:

$$\beta^\theta(W(n,v)) = \exp i\theta v \ W(n,v)$$

which is broken; the states $\Omega_\theta \equiv (\beta^\theta)^*\Omega$ correspond to wave functions with boundary conditions $\psi(0) = e^{i2\pi\theta}\psi(2\pi)$ and in the representations defined by them the spectrum of L is $\{\theta + m\,,\ m \in \mathbb{N}\}$. The automorphisms β^θ have therefore the meaning of "boosts" on the angular momentum L.

The occurrence of nonregular representations displayed by the above simple examples is actually a general fact whenever one deals with states which are homogeneous with respect to one field variable. In the functional integral language this occurs when the functional measure is homogeneous along one direction in field configuration space.

It should not be a surprise that such a phenomenon already occurs at the level of quantum mechanical systems with a finite number of degrees of freedom. The point is that Von Neumann theorem [BR] on the unitary equivalence of the representations of the CCR algebras is evaded by the lack of continuity of the Weyl group, a feature which is sometimes required by the physics of the problem.

3. Quantum harmonic lattice.

We consider a finite volume Bravais lattice in d dimensions: let $\mathbf{R}$ be the vector identifying the single lattice site, $\mathbf{K}$ the vector identifying the sites of the reciprocal lattice, v the volume of the primitive cell. The kinematics is described by the canonical coordinates $q(\mathbf{R})$ and $p(\mathbf{R})$: we fix here the origin in the centre of mass of the lattice, hence the variable $q(\mathbf{R})$ represents the deplacement from the mean value of the position referred to the origin. One has, forgetting about domain problems, the CCR

$$[q_\mu(\mathbf{R}), p_\nu(\mathbf{R}')] = i\delta_{\mu\nu}\delta_{\mathbf{R}\mathbf{R}'} \qquad \mu,\nu = 1,\cdots,d.$$

The dynamics is given, supposing one has only a nearest-neighbor interaction, by the quantum hamiltonian

$$H = \frac{1}{2m}\sum_{\mathbf{R}} p(\mathbf{R})^2 + \frac{1}{2}\sum_{<\mathbf{R}\mathbf{R}'>,\mu,\nu} q_\mu(\mathbf{R})D_{\mu\nu}(\mathbf{R}-\mathbf{R}')q_\nu(\mathbf{R}). \tag{3.1}$$

In order to construct a CCR algebra that describes the infinite lattice, we denote by V the set of lattice functions with values in $\mathbb{R}^d \times \mathbb{R}^d$ such that, if $\underline{\alpha}(\mathbf{R}) := <\alpha_1(\mathbf{R}), \alpha_2(\mathbf{R})> \in V$, it holds

$$\sum_{\mathbf{R}} \alpha_1(\mathbf{R})^2 < +\infty \qquad \sum_{\mathbf{R}} \alpha_2(\mathbf{R})^2 < +\infty.$$

Let then σ be the nondegenerate symplectic form on V defined by

$$\sigma(\underline{\alpha},\underline{\beta}) = \sum_{\mathbf{R}} \alpha_1(\mathbf{R})\cdot\beta_2(\mathbf{R}) - \sum_{\mathbf{R}} \alpha_2(\mathbf{R})\cdot\beta_1(\mathbf{R}).$$

V is a real linear space and hence our infinite lattice is described by the CCR *-algebra $\mathcal{A}(V,\sigma)$.

There are subalgebras of $\mathcal{A}(V,\sigma)$ which are better suited to describe every finite subsystem: we think of the infinite lattice as imbedded in $\mathbb{R}^d$ and let $\Lambda_N \subset \mathbb{R}_d$ be open, bounded and containing in its interior N sites of the lattice. We denote with V_N the real linear space of maps from Λ_N with values in $\mathbb{R}^d \times \mathbb{R}^d$. The restriction of σ to $V_N \times V_N$ is still nondegenerate: hence $\mathcal{A}(V_N,\sigma)$ is a *-subalgebra of $\mathcal{A}(V,\sigma)$. Lasty, $\bigcup_{\Lambda_N} V_N$ is a subspace of V and the * subalgebra associated to it, we call it $\mathcal{A}$, contains the whole information relative to finite subsystems. If $\delta(\underline{\alpha})$ is a generator of $\mathcal{A}(V,\sigma)$ contained in $\mathcal{A}$, then it exists a Λ_N such that $\underline{\alpha} \in V_N$. Such a generator admits an obvious representation as Weyl operator

$$W(\underline{\alpha}) = \exp\,(i \sum_{\mathbf{R}\in\Lambda_N} \{q(\mathbf{R})\cdot\alpha_1(\mathbf{R}) + p(\mathbf{R})\cdot\alpha_2(\mathbf{R})\}).$$

We identify $\mathcal{A}(V_N,\sigma)$ with this representation.

The Gibbs canonical equilibrium state at inverse temperature $\beta \in \mathbb{R}^+$ in a finite volume $\Lambda_N \in \mathbb{R}^d$ is the quasifree state given by

$$\omega^{\beta}_{\Lambda_N}(\delta(\underline{\alpha})) = \exp\,(-\frac{1}{4N}\sum_{k,s}\{\frac{|\tilde{\alpha}_1(\underline{k})\cdot\underline{\epsilon}_s(\underline{k})|^2}{m\omega_s(\underline{k})}+ \\ + m\omega_s(\underline{k})|\tilde{\alpha_2}(\underline{k})\cdot\underline{\epsilon}_s(\underline{k})|^2\}\coth\frac{\beta\omega_s(\underline{k})}{2}) \qquad (3.2)$$

where $\underline{\tilde{\alpha}}(\underline{k}) := \sum_{\mathbf{R}\in\lambda_N}\exp\,(-i\underline{k}\cdot\underline{\mathbf{R}})\underline{\alpha}(\mathbf{R})$ and $\delta(\underline{\alpha}) \in \mathcal{A}(V_N,\sigma)$.

Polarization vectors $\underline{\epsilon}_s(\underline{k})$ and normal frequencies $\omega_s(\underline{k})$ (with wave vectors $\underline{k}$ restricted to the first Brillouin zone) are obtained as solutions of the eigenvalue problem

$$D_{\mu\nu}(\underline{k})\epsilon_\nu(\underline{k}) = m\omega(\underline{k})^2\epsilon^\mu(\underline{k})$$

where

$$D_{\mu\nu}(\underline{k}) := \sum_{\mathbf{R}} D_{\mu\nu}(\mathbf{R})\exp\,(i\underline{k}\cdot\mathbf{R})$$

with the sum restricted to nearest neighbors of the origin.

It is absolutely essential to notice that

$$\omega(\underline{k}) \approx |\underline{k}| \qquad (3.3)$$

in the range of small $|\underline{k}|$ (see for instance chap. 23 in [AM]). Furthermore, since we are using coordinates which are referred to to centre of mass, there are no zero modes.

It is then easy to go to the thermodynamical limit. Let indeed $\delta(\underline{\alpha}) \in \mathcal{A}$ and let $\Lambda'_N \nearrow \mathbb{R}^d$ in the sense that it eventually contains every bounded $\Lambda \subset \mathbb{R}^d$. It is then immediate that

$$\lim_{\Lambda'_N \nearrow \mathbb{R}^d} \omega^\beta_{\Lambda'_N}(\delta(\underline{\alpha})) = \omega^\beta(\delta(\underline{\alpha}))$$

where ω^β is the g.q.s. on $\mathcal{A}$ defined by

$$\omega^\beta(\delta(\underline{\alpha})) = \exp\Big(- \frac{v}{4} \int \frac{d^d k}{(2\pi)^d} \sum_{s=1}^{d} \{ \frac{|\tilde{\alpha}_1(\underline{k}) \cdot \underline{\epsilon}_s(\underline{k})|^2}{m\omega_s(\underline{k})} + \\ + m\omega_s(\underline{k}) |\tilde{\alpha}_2(\underline{k}) \cdot \underline{\epsilon}_s(\underline{k})|^2 \} \coth \frac{\beta \omega_s(\underline{k})}{2} \Big). \tag{3.4}$$

Remembering equation (3.3) we have the following results, since the sums in (3.2) approximate the integral in (3.4)

if $d \leq 2$ and if $\sum_{\mathbf{R}} \alpha_1(\mathbf{R}) = \tilde{\alpha}_1(0) \neq \underline{0}$ then $\omega^\beta(\delta(\underline{\alpha})) = 0$ for every $\beta \in \mathbb{R}^+$.
if $d = 1$ and if $\tilde{\alpha}_1(0) \neq \underline{0}$ then also $\lim_{\beta \to +\infty} \omega^\beta(\delta(\underline{\alpha})) = 0$.

Moreover, since $\underline{\alpha}(\mathbf{R}) \in \bigcup_{\Lambda_N} V_N$, it follows that $\tilde{\underline{\alpha}}(\underline{k})$ is analytic in $\underline{k}$. Hence the above conditions are also necessary for the expectation value to be zero.

Thus one has a nonregular representation. The physical meaning is that in this case all the variables $q_\mu(\mathbf{R})$ have a uniform distribution in the infinite volume limit and hence their mean does not exist (*crystals do not exist* in $d = 1, 2$)

4. Bloch electrons and θ angle structure.

The framework discussed in Section I allows to clarify the occurrence of a structure of sectors labelled by an angle θ in the case of electrons in a periodic potential $V(x) = V(x + a)$, and its relation with the θ vacua structure of QCD.

The analogy has been stressed by Jackiw [J] and the purpose of this paragraph is to offer a mathematically rigorous version which does not rely on the semiclassical approximation: the non-normalizable Bloch wave functions are treated here as non regular states over the CCR algebra generated by the Weyl operators; the analog of the large gauge transformations of QCD, the chiral transformations and their breaking will emerge clearly and fit naturally in the framework discussed above.

The CCR algebra $\mathcal{A}(V,\sigma)$ is generated by the Weyl operators $W(\alpha,\beta)$, with $\alpha,\beta \in \mathbb{R}$, and the Bloch states Ω_k^n are defined by the Bloch wave functions [AM] (we set $a = 1$ in the following)

$$\psi_k^n(x) = e^{ikx} v_k^n(x), \quad v_k^n(x) = v_k^n(x+1), \quad k \in [0, 2\pi).$$

The ground state ($n = k = 0$) is thus given by [DB]

$$\Omega_0^0(W(\alpha,\beta)) = 0 \quad \text{if} \quad \alpha \neq 2\pi n, \ n \in \mathbb{N}$$

$$\Omega_0^0(W(2\pi n,\beta) = e^{i\pi n\beta} \int_0^1 \bar{v}_0^0(x) v_0^0(x+\beta) e^{i2\pi nx}\, dx.$$

The Weyl operators $W(\alpha, 0)$, which generate the boosts on $\mathcal{A}(V,\sigma)$, are represented non regularly by Ω_0^0; all the states Ω_k^n defined by the Bloch wave functions are represented by vectors in the GNS representation space defined by Ω_0^0. The nonregularity of the representation is forced by the invariance of Ω_0^0 under the non compact discrete group of lattice translations. The regular subalgebra $\mathcal{A}_0$ is generated by the Weyl operators $W(0,\beta) = \exp i\beta p$.

The group of discrete (lattice) translations $T_n = \exp inp$, $n \in \mathbb{N}$ plays the rôle of an unbroken gauge group $\mathcal{G}$; they are the analog of the large gauge transformations of QCD. The subalgebra of $\mathcal{A}$ which is left pointwise invariant under such (gauge) group may be identified with the observable algebra $\mathcal{A}_{obs}$; it is generated by $\exp i\beta p$ and $\exp i2\pi mx$ with $\beta \in \mathbb{R}$, $m \in \mathbb{N}$.

The GNS representation space $\mathcal{H}$ given by the state Ω_0^0 over $\mathcal{A}$ decomposes into disjoint irreducible representations of $\mathcal{A}_{obs}$, labelled by an angle $\theta \in [0, 2\pi)$

$$\mathcal{H} = \oplus_{\theta \in [0,2\pi)} \mathcal{H}_\theta, \quad \mathcal{H}_\theta \equiv W(\theta, 0)\mathcal{H}_0 \tag{4.1}$$

$$\mathcal{H}_0 = \overline{\mathcal{A}_{obs}\Psi_0}, \tag{4.2}$$

where Ψ_0 is the vector in $\mathcal{H}$ which corresponds to the state Ω_0^0.

In fact, the spaces $\mathcal{H}_\theta$ are orthogonal, since $\forall \Psi = A\Psi_0$, $\Psi' = B\Psi_0$ $A, B \in \mathcal{A}_{obs}$

$$(W(\theta,0)\Psi\, ,\, W(\theta',0)\Psi') = (\Psi_0\, ,\, W(-\theta+\theta',0)C\Psi_0) \tag{4.3}$$

where C is a finite linear combination of Weyl operators $W(2\pi n_i, \beta_i)$, so that the right side vanishes when $\theta \neq \theta'$ (because then $-\theta + \theta' + 2\pi n_i$ cannot be an integer times 2π). Furthermore, $D \equiv \mathcal{A}(V,\sigma)\Psi_0$ is dense in $\mathcal{H}$ and each vector of D

can be written as a linear combination of vectors of the form $W(\theta_i, 0)A_i\Psi_0$ with $A_i \in \mathcal{A}_{obs}$, and therefore

$$\mathcal{H} = \overline{D} = \oplus_\theta \mathcal{H}_\theta.$$

Finally, $\mathcal{A}_{obs}$ has a non trivial center $\mathcal{Z}_{obs}$, generated by the group $\mathcal{G}$ of lattice translations $T_n = \exp inp$, $n \in \mathbb{N}$, and Ω_0^0 is a pure state over $\mathcal{A}_{obs}$. In fact, $\mathcal{G}$ is unbroken, since $\Omega_0^0(T_n) = 1$ implies $T_n\Psi_0 = \Psi_0$; therefore

$$\Omega_\theta(T_m) \equiv \Omega_0^0(W(-\theta, 0)T_m W(\theta, 0)) = e^{im\theta},$$

which implies

$$T_m\Psi_\theta = e^{im\theta}\Psi_\theta \quad (\,\Psi_\theta \equiv W(\theta, 0)\Psi_0\,). \tag{4.4}$$

Hence, $\mathcal{Z}_{obs}$ is non trivial and the representations of $\mathcal{A}_{obs}$ given by the states Ω_θ are inequivalent; they are irreducible since all the states Ω_θ are pure as a consequence of purity of Ω_0^0.

The group of charged automorphisms of $\mathcal{A}_{obs}$ is generated by $\exp i\alpha x$, $\alpha \in [0, 2\pi)$, and the following equations are satisfied:

$$e^{i\alpha x}\Psi_\theta = \Psi_{\theta+\alpha} \tag{4.5}$$

$$T_n e^{i\alpha x} T_n^{-1} = a^{i\alpha(x+n)}. \tag{4.6}$$

They are the analog of the chiral transformations in QCD and are necessarily broken in each irreducible representation of $\mathcal{A}_{obs}$ given by the vectors Ψ_θ. As clarified by the present analysis,the crucial point is that the algebra of observables has a non trivial center which is not left pointwise invariant under the "chiral" transformations (no use has been made of the semiclassical approximation and of the tunnelling mechanism). We recover in this way the mechanism of chiral symmetry breaking obtained previously by different approaches [LS,KS,MPS1,MS1,MS3].

It may be remarked that in the case of Bloch electrons the analog of the chiral transformations have a simple action on the vectors Ψ_θ and on the corresponding sectors (equation (4.5)), but not on the "ground state vectors" $\Psi_\theta^B \in \mathcal{H}_\theta$, defined by the Bloch wave function $\psi_\theta^0(x) = \exp i\theta x\; v_\theta^0(x)$ (because $v_\theta^0(x) \neq v_{\theta+\alpha}^0(x)$).

As far as the analog of the n-vacua is concerned, one may investigate in general the representations of $\mathcal{A}_{obs}$ defined by wave functions $\psi(x) \in L^2$, i.e., by states with some localization in x, e.g., wave functions $\psi_n(x)$ localized around the minima x_n of the periodic potential. In this case one necessarily gets reducible representations of $\mathcal{A}_{obs}$. In fact, the operators $T_n \in \mathcal{Z}_{obs}$ cannot be represented by c-numbers t_n (a necessary condition for irreducibility), since, by the unitarity of T_n one must then have $|t_n| = 1$, and, on the other side, $T_n\psi(x) = \psi(x+n)$ implies $|(\psi, T_n\psi)| < 1$. For the Wannier wave functions ψ_W one actually gets $(\psi_W, T_n\psi_W) = 0$.

The above argument applies to any representation of $\mathcal{A}_{obs}$ in which the "chiral" transformations $\exp i\alpha x$ are implemented by strongly continuous unitary operators, so that the "chiral" charge is well defined; in fact, in this case the wave functions are in L^2 and one necessarily gets reducibility.

In conclusion, one is led to use the representations defined by the vectors Ψ_θ (or by the Bloch states) quite generally in order to have irreducible representations of $\mathcal{A}_{obs}$ (in QFT this is equivalent to the validity of the cluster property) and this automatically implies the breaking of the "chiral" transformations. The so emerging θ angle structure is merely a consequence of the structure of $\mathcal{A}_{obs}$ (in particular its non trivial center).

5. Free Bose gas.

We suppose that the gas is confined in a volume $\Lambda \in \mathbb{R}^d$. It is described by the CCR C^*-algebra $\overline{\mathcal{A}(L^2(\Lambda), \sigma)} \equiv \mathcal{A}_\Lambda$, where

$$\sigma(f,g) = Im(f,g)_{L^2} \qquad \forall f,g \in L^2(\Lambda). \tag{5.1}$$

The dynamics acts at the one particle level and defines on $\mathcal{A}_\Lambda$ the one-parameter group of *-automorphisms α^t :

$$\alpha^t(\delta(f)) = \delta(e^{itH_\Lambda} f) \qquad \forall f \in L^2(\Lambda)$$

where H_Λ is the selfadjoint extension of the Laplacian $-\triangle$ on $L^2(\Lambda)$ corresponding to *Dirichlet* boundary conditions on $\partial\Lambda$.

The Gibbs grancanonical factor with chemical potential $\mu \in \mathbb{R}$ and inverse temperature $\beta \in \mathbb{R}^+$ is the gauge invariant quasifree state

$$\omega_\Lambda^{\mu\beta}(\delta(f)) := \exp\{-\frac{(f, (\mathbb{1} + ze^{-\beta H_\Lambda})(\mathbb{1} - ze^{-\beta H_\Lambda})^{-1} f)}{4}\} \tag{5.2}$$

for every $f \in L^2(\Lambda)$, with $z = e^{\mu\beta}$. This state is regular.

For $d \geq 3$, in the thermodynamical limit there are two distinct regimes. In the first one there is a single phase, characterized by high temperature and low density (corresponding to $z < 1$ for Dirichlet boundary conditions). In the second one (low temperature and high density, corresponding to $z = 1$) a finite fraction of particles occupy the lowest energy level (Bose-Einstein condensation). There is a multiplicity of phases, overyone of them characterized by its own value of the particle density, all values in the interval $[\rho_c(\beta), +\infty[$ being allowed.

$\rho_c(\beta)$ is the critical value of the particle density (local particle number per unit volume: it is independent of the shape of Λ, for these boundary conditions [BR]):

$$\rho_c(\beta) = (2\pi)^{-d} \int d^d p e^{-\beta p^2} (1 - e^{-\beta p^2})^{-1}. \tag{5.3}$$

This is the value, at $z = 1$, of the density as a function of β and z

$$\rho(\beta, z) = (2\pi)^{-d} \int d^d p e^{-\beta p^2} (1 - z e^{-\beta p^2})^{-1}. \tag{5.4}$$

The integral defining $\rho_c(\beta)$ is divergent in $d \leq 2$: it is obtained as the (divergent) limit of well defined Riemann sums giving the value of the local particle number per unit volume when the system is confined in Λ. The critical density goes to infinity but nonregular representations appear, as shows the following

Proposition (5.2.31 in [BR]) *With the above notations,*

$$\omega^{\mu\beta}(A) = \lim_{\Lambda' \nearrow \mathbb{R}^d} \omega^{\mu\beta}_{\Lambda'}(A)$$

exist for $z = 1$ and for every $\beta \in \mathbb{R}^+$, $A \in \overline{\bigcup_\Lambda \mathcal{A}_\Lambda}$ when $\Lambda' \nearrow \mathbb{R}^d$ in the sense that eventually contains every $\Lambda \subseteq \mathbb{R}^d$. The limit state is the gauge invariant quasifree state defined by

$$\omega^{\mu\beta}(\delta(f)) := e^{-\frac{\|f\|}{4}} \exp\{-\frac{1}{(2\pi)^d} \int d^d p |\tilde{f}(p)|^2 e^{-\beta p^2} (1 - e^{-\beta p^2})^{-1}\} \tag{5.5}$$

for every $f \in \bigcup_\Lambda L^2(\Lambda)$. In particular, $\omega^{\mu\beta}(\delta(f)) = 0$ if $d = 1, 2$ and $\int d^d x f(x) \neq 0$.

We obtain hence, in $d \leq 2$, a g.q.s. with

$$V_q = \{f \in \bigcup_\Lambda L^2(\Lambda) : \tilde{f}(0) = 0\} \tag{5.6}$$

and $\tilde{f}(0)$ is well defined since $\tilde{f}(p)$ is analytic.

Remark. It is very important to notice that $\omega^{\mu\beta}$ is not locally normal, thanks to the discontinuity of $\omega^{\mu\beta}(\delta(\lambda f))$ in $\lambda = 0$ for every $f \in L^2(\mathcal{O})$ with $\tilde{f}(0) \neq 0$, *chosen anyhow* an open set $\mathcal{O} \in \mathbb{R}^d$. In this model, this property is confirmed by the nonexistence of the local particle number discussed above.

6. Massless scalar field in $1+1$ dimensions.

The Weyl algebra can be thought of as formally generated by the exponentials of the canonical "time zero" fields $\phi(f_1)$, $\pi(f_2)$, $f_1, f_2 \in \mathcal{S}_{\mathrm{real}}(\mathbb{R}) \equiv \mathcal{S}$ and the CCR are described by the standard symplectic form

$$\sigma(F, G) \equiv \int (f_1 g_2 - f_2 g_1)\, dx$$

over the (real) vector space $V := \mathcal{S} \times \mathcal{S}$. Since σ is nondegenerate (V, σ) identifies a unique C^*-algebra $\mathcal{A}(V, \sigma)$. The time evolution which characterizes the model is described by a one–parameter group of automorphisms of $\mathcal{A}(V, \sigma)$.

It is well known [W] that, for infrared reasons, the vacuum state ω_0 for the massless field time evolution does not define a regular representation of the Weyl algebra $\mathcal{A}(V, \sigma)$. The state ω_0 is in fact well defined on the observable algebra

$$\mathcal{A}_0 \equiv \mathcal{A}(V_0, \sigma_0),\ V_0 \equiv \partial\mathcal{S} \times \mathcal{S},\ \partial\mathcal{S} \equiv \{f = \partial g\ ,\ g \in \mathcal{S}\},\ \sigma_0 \equiv \sigma_{|V_0 \times V_0}$$

where it is given by

$$\omega_0(W(F)) = \exp(-q(F)/4)\quad , \tag{6.1}$$

$$q_0(F) = \int \frac{1}{\omega(p)} |\tilde{f}_1(p)|^2 + \omega(p)|\tilde{f}_2(p)|^2\, dp\quad ,\quad \omega(p) \equiv |p|\ ; \tag{6.2}$$

it has a unique extension (as a positive linear functional) to $\mathcal{A}(V, \sigma)$, call it Ω_0, which is of the form (6.1), with

$$q(F) = +\infty\ \text{ for }\ F \notin V_0. \tag{6.3}$$

This is a consequence of Corollary I.3.8 and of the following (see also [SW])

Lemma 6.1

Fixed anyhow $G \in V \backslash V_0$, $\sigma(\cdot, G)$ is an unbounded linear functional on V_0, when this is equipped with the inner product induced by q_0.

Proof. We argue by contradiction. Indeed, if, fixed anyhow $G \in V \backslash V_0$, $\sigma(\cdot, G)$ is bounded, we can extend it by continuity to $\overline{V_0}^{q_0}$. This space contains elements of the form $F_n = <0, f_n>$, with

$$\tilde{f}_n(p) = \begin{cases} |p|^{-1+\frac{1}{n}} & |p| < \delta \\ 0 & |p| \geq \delta. \end{cases}$$

It is easy to see, for instance using mollifiers, that the continuous extensions of σ and q to $\overline{V_0}^{q_0}$ are still given by the standard integral expressions. It is then immediate that

$$q_0(F_n) = \int \omega(p)|\tilde{f}_n(p)|^2 dp = \int_{-\delta}^{\delta} |p|^{-1+\frac{1}{n}} dp = n\delta^{\frac{2}{n}}.$$

If $G \in V \backslash V_0$, then $\epsilon > 0$, $b > 0$ exist such that

$$|Re\tilde{g}_1(p)| > b \qquad if\ |p| < \epsilon.$$

With $\epsilon = \delta$ we obtain

$$|\sigma(F_n, G)| = |\int_{-\delta}^{\delta} |p|^{-1+\frac{1}{n}} \tilde{g}_1(p) dp| = \int_{-\delta}^{\delta} |p|^{-1+\frac{1}{n}} |Re\tilde{g}_1(p)| dp > 2bn\delta^{\frac{1}{n}}.$$

It follows that, for every $G \in V \backslash V_0$,

$$\frac{|\sigma(F_n, G)|^2}{q(F_n)} > 4b^2 n$$

and $\sigma(\cdot, G)$ is not bounded. *q.e.d.*

The gauge group $\mathcal{G}$ is the one-parameter group of automorphisms β^λ corresponding to $\varphi \to \varphi + \lambda$, $\lambda \in \mathbb{R}$. The Weyl operators $W(f_1, 0) \equiv \exp i\varphi(f_1)$, $f_1 \notin \partial\mathcal{S}$ define charged fields. The gauge group $\mathcal{G}$ is *unbroken* in the ground state representation of $\mathcal{A}(\mathcal{S} \times \mathcal{S}, \sigma)$ defined by Ω_0.

Call $U(\lambda)$ the unitary implementer of β^λ and Q the associated generator. We can give a precise identification of them. Indeed, let $\{F_n\}_{n \in \mathbb{N}} :=< 0, f_n >$ with

$$\tilde{f}_n(p) = \begin{cases} \frac{1}{2n}|p|^{-1+\frac{1}{n}} & |p| < \frac{1}{n} \\ 0 & |p| > \frac{1}{n}. \end{cases} \tag{6.4}$$

For n fixed, $F_n \in \overline{V_0}^{q_0}$ and so $\pi_{\omega_0}(\delta(F_n))$ is an unitary operator in $\pi_{\omega_0}(\mathcal{A}(\overline{V_0}^{q_0}, \sigma_0))$, which is included in $\pi_{\omega_0}(\mathcal{A}(V_0, \sigma_0))''$. Indeed, if $F_n \in \overline{V_0}^{q_0}$, there is a sequence $\{F_{n,m}\} \in V_0$ approximating it in the q norm and hence σ-weakly with respect to V_0. It follows that $\pi_{\omega_0}(\delta(F_n))$ is unitary as a strong limit of a sequence of unitary operators with strongly convergent adjoints.

By the same argument, $s - \lim_{n \to +\infty} \pi_{\omega_0}(\delta(F_n))$ exists and it is an unitary operator in $\pi_{\omega_0}(\mathcal{A}(\overline{V_0}^{q_0}, \sigma_0))'' = \pi_{\omega_0}(\mathcal{A}(V_0, \sigma_0))''$.

Moreover, notice that $\mathcal{S}' - \lim_{n \to +\infty} \tilde{f}_n(p) = \delta(p)$, and that $\lim_{n \to +\infty} q(F_n) = 0$.

Hence

$$U(\lambda) = s - \lim_{n\to+\infty} \pi_{\omega_0}(\delta(\lambda F_n)). \tag{6.5}$$

It follows from this that

$$U(\lambda) \in \pi_{\omega_0}(\mathcal{A}(V_0,\sigma_0))' \cap \pi_{\omega_0}(\mathcal{A}(V_0,\sigma_0))''. \tag{6.6}$$

If we fix $n < +\infty$, there exists the Stone's generator $\Phi_{\omega_0}(F_n)$ of $\pi_{\omega_0}(\delta(\lambda F_n)), \quad \lambda \in \mathbb{R}$.

$\{\Phi_{\omega_0}(F_n)\}$ is a sequence of selfadjoint operators on $\mathcal{H}_{\omega_0}$. It follows then from (6.5) that

$$\lim_{n\to+\infty} \Phi_{\omega_0}(F_n) \longrightarrow Q$$

in the strong resolvent sense [RS].

Since $(\mathcal{S}\times\mathcal{S})/(\partial\mathcal{S}\times\mathcal{S})$ is isomorphic to $\mathbb{R}$, the Hilbert space $\mathcal{H}_{\Omega_0}$ decomposes into disjoint representations of $\mathcal{A}_0$ labelled by the gauge charges $\alpha \in \mathbb{R}$

$$\mathcal{H}_{\Omega_0} = \oplus_{\alpha\in\mathbb{R}}\mathcal{H}_\alpha \tag{6.7}$$

where $\mathcal{H}_\alpha = \overline{\mathcal{A}_0\Psi_\alpha}$, $\Psi_\alpha = W(f,0)\Psi_0$, $\int f(x)\,dx = \alpha$, and Ψ_0 is the vector corresponding to Ω_0.

Similarly, equilibrium (KMS) states over $\mathcal{A}_0$ are given by the regular and quasifree states

$$\omega_\beta(\delta(F)) := \exp(-\frac{1}{4}q_\beta(F)) \qquad \forall F \in V_0.$$

Here q_β is related to q by the standard factor $\coth(\frac{\omega(p)\beta}{2})$, $\beta = \frac{1}{T}$, which yields the fulfillment of the KMS condition [RST].

Furthermore, ω_β has an extension to $\mathcal{A}(\mathcal{S}\times\mathcal{S},\sigma)$ given by

$$\Omega_\beta(\delta(F)) = \begin{cases} \omega_\beta(\delta(F)) & \forall F \in V_0 \\ 0 & \text{otherwise.} \end{cases} \tag{6.8}$$

The state Ω_β *i.* is invariant under space translations *ii.* is primary and *iii.* satisfies the KMS condition. Actually is the unique extension of ω_β with the properties *i.* and *ii.*. In fact, any extension $\hat{\Omega}_\beta$ with such properties must satisfy the cluster property

$$\lim_{a\to+\infty} \hat{\Omega}_\beta(e^{i[\phi(f)-\phi(f_a)]}) = |\hat{\Omega}_\beta(e^{i\phi(f)})|^2$$

where f is real and $f_a(x) = f(x-a)$. Since $f - f_a \in \partial\mathcal{S}$ the left hand side can be computed on ω_β and the limit $a \to +\infty$ vanishes. Hence $\hat{\Omega}_\beta = \Omega_\beta$

In conclusion, ground state or equilibrium state representations of the CCR algebra of the massless scalar field in $1+1$ dimensions exist compatibly with positivity but they are nonregular.

7. Current algebra in $1+1$ dimensions and its extensions.

The (observable) current algebra is generated by the Weyl operators

$$W(f_1, f_2) = \exp i(j_0(f_1) + j_1(f_2)) \qquad f_1 \, , \, f_2 \in \mathcal{S}$$

and it is isomorphic to the CCR algebra $\mathcal{A}_0 \equiv \mathcal{A}(\partial\mathcal{S} \times \mathcal{S}, \sigma)$, through the relations

$$j_0(x) = -\frac{1}{\sqrt{\pi}}\partial_1\varphi(x) \; , \;\; j_1(x) = -j_0^5(x) = -\frac{1}{\sqrt{\pi}}\pi(x) \; , \tag{7.1}$$

with $\varphi(x)$ a pseudo-scalar field in $1+1$ dimensions and $\pi(x)$ its canonically conjugated momentum.

One can then look for a canonical extension of $\mathcal{A}_0$ which may be interpreted as a field algebra charged under vector and axial charges, i.e., under the automorphisms generated by the local charges

$$Q_R \equiv j_0(f_R) \quad , \quad Q_R^5 \equiv j_0^5(f_R) = -j_1(f_R) \tag{7.2}$$

where $f_R(x) = f(x/R)$, $f \in \mathcal{D}(\mathbb{R})$, $f(x) = 1/\sqrt{\pi}$ if $|x| \leq 1$, $f(x) = 0$ if $|x| \geq 1+\epsilon$. The most natural extension is provided by $\mathcal{A} \equiv \mathcal{A}(\mathcal{S} \times \partial^{-1}\mathcal{S}, \sigma_1)$ where

$$\partial^{-1}\mathcal{S} \equiv \{f \in C^\infty(\mathbb{R}) , \; \partial f \in \mathcal{S}(\mathbb{R})\} \; , \tag{7.3}$$

$$\sigma_1(F, G) \equiv \int (f_1 g_2 - f_2 g_1) \, dx \; .$$

The gauge group $\mathcal{G}$ is generated by the one parameter groups β^λ , β_5^μ, $\lambda, \mu \in \mathbb{R}$, corresponding to the transformations

$$\varphi \to \varphi \quad , \quad \partial^{-1}\pi \to \partial^{-1}\pi + \lambda \; , \tag{7.4}$$

$$\varphi \to \varphi + \mu \quad , \quad \partial^{-1}\pi \to \partial^{-1}\pi \; ; \tag{7.5}$$

$\mathcal{G}$ is therefore isomorphic to $\mathbb{R} \times \mathbb{R}$.

The identification of the ground state representation of such CCR algebras depends on the time evolution automorphism , i.e. on the specific models.

The state ω_0 defined by (6.1) and (6.2) defines a ground state representation of $\mathcal{A}_0$ (for the massless time evolution); as in the previous section, it has a unique extension to $\mathcal{A}$, which defines a nonregular representation of $\mathcal{A}$. The gauge group $\mathcal{G}$ is unbroken in this representation and one gets a decomposition into charged sectors labelled by the vector and axial charges. Again, the charged fields act as intertwiners between inequivalent representations of $\mathcal{A}_0$.

8. The Schwinger model.

The Schwinger model in the bosonized form in the Coulomb gauge may be defined by the time zero current algebra $\mathcal{A}_0 \equiv \mathcal{A}(\partial\mathcal{S} \times \mathcal{S}, \sigma)$ and the following equations of motion

$$\frac{d}{dt} j_0(x) = -(1/\sqrt{\pi})\, \partial_1 \pi(x)$$

$$\frac{d}{dt} j_1(x) = -(1/\sqrt{\pi}) \frac{d}{dt} \pi(x) = \partial_1 j_0 - (e^2/\pi)\, \partial_1^{-1} j_0(x) \tag{8.1}$$

(For a more general discussion see [MS1,MS3,MS5]; here for simplicity we consider the dynamics with no variables at infinity).

One is then led to consider the algebra $\mathcal{A}(\mathcal{S} \times \mathcal{S}, \sigma)$, which can be regarded as the observable algebra $\mathcal{A}_{obs}$ (stable under time evolution !), and the dynamics is then equivalent to that of a massive scalar field, with mass $m^2 = e^2/\pi$. It is important to stress that the chiral transformations (7.5) (which act trivially on the time zero current algebra) define a non-trivial group, isomorphic to $\mathbb{R}$, of automorphisms of the observable algebra $\mathcal{A}_{obs}$.

We consider the canonical extension of $\mathcal{A}_{obs}$ identified by the addition of bosonized fermion field variables at time zero (actually, smeared fermion *fields* belong to the strong closure of this extension [AMS]), namely the "field algebra" $\mathcal{A} = \mathcal{A}(V, \sigma)$ with

$$V = \{F = (f_1, f_2) \in \mathcal{S} \times \partial^{-1}\mathcal{S}\ ,\ (f_2(\infty) - f_2(-\infty)) = n\sqrt{\pi}\,,\, n \in \mathbb{N}\}. \tag{8.2}$$

The gauge group is now the $U(1)$ group β^λ defined by

$$\beta^\lambda W(f_1, f_2) = e^{i\lambda(f_2(\infty) - f_2(-\infty))/\sqrt{\pi}}\, W(f_1, f_2) \tag{8.3}$$

The ground state on $\mathcal{A}_{obs}$ is defined by the quadratic form q_m given by equation (6.2), with $\omega(p) = (p^2 + m^2)^{1/2} \equiv \omega_m(p)$. It defines a regular quasi free representation of $\mathcal{A}_0$ and also a regular quasi free representation of $\mathcal{A}_{obs}$.

The ground state $\omega_{(m)}$ has a unique extension Ω_m from $\mathcal{A}_{obs}$ to $\mathcal{A}$, since all $F \in V/V_0$, $V_0 = \mathcal{S} \times \mathcal{S}$, define unbounded linear functionals on V_0. The chiral automorphisms β_5^μ are broken in π_{Ω_m}, whereas the gauge group β^λ is unbroken.

The GNS representation of the field algebra $\mathcal{A}$ defined by Ω_m decomposes into a direct sum of irreducible inequivalent representations of $\mathcal{A}_{obs}$ labelled by the gauge charge $q = n \in \mathbb{N}$. Such representations are not stable under time evolution, so that the time translations are not implementable and the energy is not defined in the charged sectors. Actually, the charged state Ω_F defined by

$$\Omega_F(\cdot) = \Omega_m(W(-F) \cdot W(F)) \quad , \quad F \notin V_0$$

is mapped into a one-parameter family of states Ω_F^t on $\mathcal{A}_{obs}$, of the form $\Omega_F^t = \Omega_{F^t}$ with

$$
\begin{aligned}
F^t = (f_1^t, f_2^t) \ \in \partial^{-1}\mathcal{S} \times \partial^{-1}\mathcal{S} \quad , \\
\tilde{f}_1^t(k) = \cos\omega_m t\, \tilde{f}_1(k) + \omega_m \sin\omega_m t\, \tilde{f}_2(k) \\
\tilde{f}_2^t(k) = -\omega_m^{-1} \sin\omega_m t\, \tilde{f}_1(k) + cos\omega_m t\, \tilde{f}_2(k)
\end{aligned}
\tag{8.4}
$$

(the symplectic form σ extends naturally from V to $V^{ext} \equiv \partial^{-1}\mathcal{S} \times \partial^{-1}\mathcal{S}$, and the state Ω_m has again a unique extension to $\mathcal{A}^{ext} \equiv \mathcal{A}(V^{ext}, \sigma^{ext})$).

The above time evolution of states can be viewed as the result of a time evolution automorphism α^t of the extended field algebra $\mathcal{A}^{ext}$, defined by $\alpha^t(W(F)) = W(F^{-t})$ and by equations (8.4). In the representation of $\mathcal{A}^{ext}$ defined by the unique extension of Ω_m, the time evolution is described by a unitary group which is not strongly continuous (except on the vacuum sector), so that the energy is not defined in the charged sectors.

The above treatment of the Schwinger model displays a general characterization of confinement, by which charged sectors exist but the corresponding states are not physically observable since they do not have finite energy (for a similar mechanism in massless QED_4 and in QED_3 see [MS2].

9. Current algebra on the circle.

Representations of the $U(1)\times U(1)$ current algebra on the circle have played an important rôle in connection with the classification of the representations of Kač-Moody and Virasoro algebras [FK,F,Se]. The classification of the positive energy representations of the $U(1) \times U(1)$ current algebra on the circle has been given in [BMT] by implementing the DHR strategy. Here, we offer a canonical approach which provides the charged morphisms as Weyl operators which intertwine between the vacuum representation and the charged sectors.

It is worthwhile to remark that the treatment in terms of extended Weyl algebras can also be done on the real line, without space compactification, and is therefore also suited for models with breaking of conformal symmetry.

In two dimensional conformal quantum field theory models, the $U(1) \times U(1)$ current algebra is generated by conserved chiral currents which have "left/right moving" components living on a circle, $J_\pm(z)$, $|z| = 1$. The Weyl exponentials

$$
W(u) = \exp iJ(u) \quad , \quad J(u) \equiv \int \frac{dz}{2\pi i}\, J(z)u(z) \quad , \tag{9.1}
$$

$J = J_+$ or J_-, $u \in \mathcal{S}(S^1)$, the space of C^∞ real functions defined on the circle $|z| = 1$, generate a CCR algebra $\mathcal{A}_0 \equiv \mathcal{A}(\mathcal{S}(S^1), \sigma)$, where the symplectic form σ is given by

$$\sigma(u,v) = [J(u)\,,\, J(v)] = \int \frac{dz}{2\pi i}\, u'(z)v(z) \tag{9.2}$$

Such an algebra has a non trivial center generated by $J_{n=0} = J(1) \equiv Q$ where J_n are defined (with equation (9.2) extended to complex functions) by

$$J_n = J(z^n) \quad , \quad J_n^* = J_{-n} \tag{9.3}$$

and satisfy

$$[J_n\,,\, J_m] = m\,\delta_{n+m,0} \quad . \tag{9.4}$$

The "conformal Hamiltonian" generates the following automorphism on $\mathcal{A}_0$:

$$\alpha_t(W(u)) = W(u_t) \quad , \quad u_t(z) = u(e^{-it}z) \tag{9.5}$$

The various QFT models with a $U(1) \times U(1)$ current algebra can then be recovered on the basis of the classification of all the positive energy representations of $\mathcal{A}_0$, and in [BMT] this is obtained according to the philosophy of Doplicher, Haag and Roberts, by means of localized morphisms of $\mathcal{A}_0$.

These representations can also be obtained by introducing a canonical extension of $\mathcal{A}_0$; in this way one recovers a (charged) field algebra $\mathcal{A}$, a gauge group $\mathcal{G}$ and the decomposition of the quasi free representations of $\mathcal{A}$ into inequivalent positive energy representations of $\mathcal{A}_0$. More precisely, one considers the canonical extension of $\mathcal{A}_0$ given by $\mathcal{A}(V, \sigma)$, where

$$V = \{\mathcal{S}(S^1) + \lambda X\,,\ \lambda \in \mathbb{R}\} \tag{9.6}$$

and X is characterized by

$$\sigma(X,u) = \int \frac{dz}{2\pi i}\,\frac{u(z)}{z} = (1/2\pi)\int_0^{2\pi} u(e^{i\theta})\,d\theta\ , \quad \forall u \in \mathcal{S}(S^1) \tag{9.7}$$

(and, of course, $\sigma(X,X) = 0$).
This extension of σ to V is nondegenerate (remember that σ is degenerate on $\mathcal{S}(S^1) \times \mathcal{S}(S^1)$). Notice also that

$$e^{iQ}W(\lambda X)e^{-iQ} = e^{i\lambda}W(\lambda X) \tag{9.8}$$

and so $W(X)$ has unit "charge". X is uniquely characterized by the above formulas *up to a constant.* Hence $W(X)$ is uniquely determined *in* $\mathcal{A}$ up to

$\eta(X)\exp(i\lambda(X)Q)$, where η is a phase factor.

The new element X plays an essential rôle in the construction of charged fields which implement the local automorphisms ("charge shifts") of $\mathcal{A}_0$ given by

$$\gamma_\rho(W(u)) = \exp(i\int \frac{dz}{2\pi i}\,\rho(z)u(z))\; W(u) \quad , \quad z\rho(z) \in \mathcal{S}(S^1) \quad . \tag{9.9}$$

In fact, if $\int \rho(z)\,dz = 0$, γ_ρ is implementable by an element of $\mathcal{A}_0$ (by equation (9.2) and the existence of a regular primitive of ρ). On the other hand, the generic function $\rho(z)$ on the circle with $z\rho(z) \in \mathcal{S}(S^1)$ can be written as

$$\rho(z) = i\frac{d}{dz}\rho_1(z) + \frac{a_\rho}{z} \quad , \tag{9.10}$$

with $a_\rho \in \mathbb{R}$ and $\rho_1 \in \mathcal{S}(S^1)$. The corresponding automorphism γ_ρ, equation (9.9), is implemented by $W(F_\rho)$ with

$$F_\rho = a_\rho X + \rho_1 \quad , \tag{9.11}$$

which is an element of the extended space V. Clearly, a_ρ is the "charge" carried by γ_ρ

$$a_\rho = \int \frac{dz}{2\pi i}\,\rho(z)$$

and the rôle of X is to implement the singular part of ρ, namely a_ρ/z. ρ_1 is determined up to an additive constant, and this corresponds to the multiplication of $W(F_\rho)$ (equation (9.11)) by an element of the center of $\mathcal{A}_0$.

The canonical extension of the current algebra also allows for a very simple derivation of the *fusion rule* for localized automorphisms. To this purpose, we consider, as in [BMT], the automorphisms γ_ρ with ρ localized in subintervals of the circle which do not include the "point at infinity", e.g. $\theta = \pi$, corresponding to $x = \pm\infty$ in the compactification of the line. equation (9.10) can be integrated in the complement of the support of ρ and gives *locally*

$$\rho_1(z) = a_\rho(i\log z + c_\rho) \quad ;$$

fixing the cut in the definition of $\log z$ at $\theta = \pi$, c_ρ takes constant values $c_\rho^\pm$ for $\theta > \pi$ and $\theta < \pi$, and continuity of ρ_1 (which belongs to $\mathcal{S}(S^1)$) implies $c_\rho^+ - c_\rho^- = 2\pi$. Since ρ_1 is defined up to a constant, we can impose the condition

$$a_\rho c_\rho^+ = -i\int \frac{dz}{2\pi i}\,\rho(z)\log z \quad .$$

This uniquely fixes the implementers $W(F_\rho)$, and then the Weyl relations in $\mathcal{A}(V)$ immediately give the "fusion rule" for charge shifts γ_ρ, γ_σ localized in disjoint intervals:

$$W(F_\rho)W(F_\sigma) = W(F_{\rho+\sigma})e^{\pm i\pi a_\rho a_\sigma} \quad ,$$

where the $\pm$ signs correspond to the cases in which ρ is localized after/before σ, in the sense of increasing θ, in the punctured circle (with $\theta = \pi$ removed).

We now turn to the gauge group $\mathcal{G}$ associated to $\mathcal{A}$ as a canonical extension of $\mathcal{A}_0$. It is clear from the definition that

$$\mathcal{G} = \{\alpha_\phi(W(F)) = e^{i\phi(F)}W(F),\ F \in V,\ \phi \in V'_{\text{real}} :\ \phi(u) = 0 \quad \forall u \in \mathcal{S}(S^1)\}$$

and that every $\alpha_\phi \in \mathcal{G}$ is of the form $\alpha_{\lambda\tilde{\phi}}$ where $\tilde{\phi}$ is fixed and $\lambda \in \mathbb{R}$. Since $\phi(u + \lambda X) = \lambda\phi(X)$ for every $u \in \mathcal{S}(S^1)$, one can choose $\tilde{\phi}$ such that $\tilde{\phi}(X) = 1$ and this amounts to fix $\tilde{\phi}(\cdot) = \sigma(\cdot, 1)$. $\mathcal{G}$ is a group of *inner* automorphisms of $\mathcal{A}$: the automorphism $\alpha_{\lambda\tilde{\phi}}$, $\lambda \in \mathbb{R}$, is implemented by the central element $W(\lambda)$ (=exp($i\lambda Q$) in every representation in which the center of $\mathcal{A}_0$ is regularly represented).

As a further point, we know from the analysis in [BMT] how to characterize the sets of (α^t,β)-KMS and ground states for the dynamics (9.5) above.

The factor (α^t,β)-KMS states on $\mathcal{A}_0$ are defined by

$$\omega_\beta^g(W(u)) := \exp\{ig\tilde{u}_0 - \frac{1}{2}\sum_{n=1}^{\infty} n \coth\frac{n\beta}{2}\ |\tilde{u}_n|^2\} \qquad \forall u \in \mathcal{S}(S^1),$$

where $g \in \mathbb{R}$ and $\tilde{u}_n := \int \frac{dz}{2\pi i}u(z)z^{n-1}$

The pure ground states on $\mathcal{A}_0$ are defined by

$$\omega_0^g(W(u)) := \exp\{ig\tilde{u}_0 - \frac{1}{2}\sum_{n=1}^{\infty} n|\tilde{u}_n|^2\} \qquad \forall u \in \mathcal{S}(S^1).$$

We will call in the following π_g the representation determined by ω_0^g, and call q_β and q_0 the g.q.f. associated to the above states, respectively.

We are now faced to the problem of extending the dynamics in (9.5) to the algebra $\mathcal{A}$: we require to this end that $\mathcal{A}$ is stable under the action of the extended dynamics (which we call $\tilde{\alpha}^t$). Using the fact that $\mathcal{A}$ is the smallest $*$-algebra containing $\mathcal{A}_0$ and elements which are Q-charged, Equation (9.8) above and the fact that

$$\tilde{\alpha}^t(e^{i\lambda Q}) = \alpha^t(e^{i\lambda Q}) = e^{i\lambda Q} \qquad \forall\lambda \in \mathbb{R},$$

one obtains that the most general extension of the dynamics to $\mathcal{A}$ is determined by the following time evolution for $W(X)$ (η is a phase factor):

$$\tilde{\alpha}^t(W(gX)) = \eta(g,t)e^{i\lambda(g,t)Q}W(gX) \qquad \forall g \in \mathbb{R}.$$

The following results are stated for the choice $\eta(g,t) \equiv 1$, $\lambda(g,t) \equiv 0$, but they hold for all values of these parameters.

It is easily shown that both ω_β and ω_0 (we set here $g = 0$) have $\mathcal{A}_0$ as a maximal domain of regularity in $\mathcal{A}$. Indeed, one has $q_0(1) = q_\beta(1) = 0$ and $\sigma(X,1) = 1$. Hence $\sigma(X,\cdot)$ is not q–bounded on $\mathcal{S}(S^1)$ and by Corollary I.3.8 we have the result. As a consequence, the unique extensions Ω_β and Ω_0 of ω_β and ω_0 to $\mathcal{A}$ are a factor $(\tilde{\alpha}^t, \beta)$-KMS state and a pure ground state, respectively. Hence, we can recover the decomposition into sectors given in [BMT] (we use for the sake of simplicity only the pure states here):

Proposition 9.1

The gauge group $\mathcal{G}$ is unbroken in π_{Ω_0}.

The representation π_{Ω_0} decomposes into a direct sum of inequivalent irreducible representations of $\mathcal{A}_0$, labelled by the value of the gauge charge:

$$\pi_{\Omega_0} = \bigoplus_{g \in \mathbb{R}} \pi_g .$$

The charged fields $W(gX)$, $g \in \mathbb{R}$, act as intertwiners between the inequivalent representations of $\mathcal{A}_0$.

10. The Stückelberg-Kibble model.

We discuss here an application to a quantum field model which is relevant (see e.g. [MS4,MS2]) for understanding the Higgs and confinement mechanisms in different space–time dimensions (see also [NT1]).

The Stückelberg–Kibble (S–K) model is defined as an approximation of the abelian Higgs–Kibble model, by freezing the modulus of the Higgs field $\chi = |\chi| \exp i\varphi$ to $|\chi| = 1$ [MS2,MS4]. The observable algebra is generated by the current $j_\mu = \partial_\mu \varphi + eA_\mu$ and by the electromagnetic field $F_{\mu\nu}$. In the Coulomb gauge, such a field algebra can be realized in terms of a CCR algebra

$$j_0 = \pi \quad , \quad j_i = \partial_i \varphi \quad ,$$

$$F_{i0} = \partial_i A_0 \quad , \quad A_0 = \triangle^{-1}\pi \quad , \quad F_{ij} = \partial_i A_j - \partial_j A_i \quad , \tag{10.1}$$

with φ and π canonical fields. In more than $1+1$ dimensions one has a non-trivial transverse algebra generated by F_{ij}, but it decouples from the other fields and it will not be considered for simplicity. Hence, the observable algebra at $t = 0$ can be identified with the CCR algebra

$$\mathcal{A}_{obs} = \mathcal{A}(\partial S \times \partial \triangle^{-1} S, \sigma) \quad ,$$

where $\triangle^{-1} S$ is the space of C^∞ functions, f, bounded by polynomials, with $\triangle f \in \mathcal{S}$, $\triangle$ the Laplace operator; $\partial \triangle^{-1} S$ denotes the space of (partial) derivatives of functions in $\triangle^{-1} S$.

The dynamics is defined by the following infrared cutoff Hamiltonians

$$H_L = \frac{1}{2} \int d^s x \, [\, (\partial \varphi)^2 + \pi^2 \,] + \frac{e^2}{2} \int d^s x \, d^s y \, \pi(x) \pi(y) U_L(x-y) \tag{10.2}$$

where $U_L(x) \equiv f_L(x) V(x)$ with $\triangle V(x) = -\delta(x)$, and $f_L(x) = f(x/L)$, $f \in \mathcal{D}(\mathbb{R}^s)$, $f(x) = 1$ for $|x| < 1$, $f(x) = 0$ for $|x| > 1 + \epsilon$.

The removal of the infrared cutoff can be done by taking strong limits, in a class of representations, of the (infrared cutoff) time evolution of $\mathcal{A}_{obs}$; one obtains a one–parameter group α^t, $t \in \mathbb{R}$, of automorphisms of $\mathcal{A}_{obs}$, as in [MS4].

The charged field algebra $\mathcal{A}$ of the S–K model can be obtained as a canonical extension of $\mathcal{A}_{obs}$, with $V_0 \equiv \partial S \times \partial \triangle^{-1} S \subset V \equiv \mathcal{S} \times \partial \triangle^{-1} S$, i.e., $\mathcal{A} = \mathcal{A}(\mathcal{S} \times \partial \triangle^{-1} S, \sigma)$, with the natural extension of σ.

The gauge group is isomorphic to $\mathbb{R}$ and is defined by $\varphi \to \varphi + \lambda$, $\partial \triangle^{-1} \pi \to \partial \triangle^{-1} \pi$; it is a $U(1)$ group on the "compact fields" $\exp i\varphi(f)$, $\int f \, d^s x = m \in \mathbb{Z}$, which are the original variables of the S–K model.

The ground state Ω_0 on $\mathcal{A}_{obs}$ is given by the quadratic form (6.2) with ω_m replaced by $p^2/\omega_m(p)$, $\omega_m(p) \equiv \sqrt{p^2 + 4\pi e^2}$. The state Ω_0 has a unique extension Ω from $\mathcal{A}_{obs}$ to $\mathcal{A}$ (since all $F \in V/V_0$ define unbounded functionals on V_0).

The representation π_Ω defined by Ω on $\mathcal{A}$ is non regular in $1+1$ and $2+1$ dimensions, regular in $3+1$. In $1+1$ and $2+1$ dimensions the gauge group is therefore unbroken in π_Ω, and correspondingly the Hilbert space $\mathcal{H}_\Omega$ decomposes into irreducible inequivalent representations of $\mathcal{A}_{obs}$ labelled by the gauge charge:

$$\mathcal{H}_\Omega = \oplus_{\alpha \in \mathbb{R}} \mathcal{H}_\alpha \quad ;$$

in $3+1$ dimensions the gauge group is spontaneously broken and there are no charged sectors.

The charged sectors in $1+1$ and $2+1$ dimensions are not stable under the time evolution, and the energy cannot be defined, except on the vacuum sector, i.e. *confinement* takes place as *spontaneous breaking of the time translations on the observable algebra, in the charged sectors.* Under the time evolution each charged sector is mapped into a one-parameter family of inequivalent representations and this can be viewed as the result of a time evolution of a larger field algebra $\mathcal{A}^{ext} = \mathcal{A}(\mathcal{S} \times \triangle^{-1}\mathcal{S})$ (which gives rise to the larger gauge group $\mathbb{R} \times \mathbb{R}$), and the (unique) extension of Ω to $\mathcal{A}^{ext}$ defines a Hilbert space in which the time translations are implementable by unitary operators which are not strongly continuous, except on the vacuum sector. Charged states are therefore excluded in different space dimensions by two different mechanisms, one (confinement) involving infinite energies, the other (Higgs) resulting from the fact that charged fields applied to the vacuum give rise to states in the vacuum sector (for related results see [MS2]).

APPENDIX A

EXTREMAL GENERALIZED QUASIFREE STATES

1. Primary states

We recall the standard characterization of primary quasifree states (prop. 11 and th. 3 in [MV]).

Let $\mathcal{A}(V,\sigma)$ be a CCR *-algebra (σ is assumed to be nondegenerate). Let ω_q be the quasifree state associated to the finite and nondegenerate generalized quadratic form $q : V \to \mathbb{R}^+$. Then ω_q is primary iff the continuous extension of σ to $\overline{V}^q$ is nondgenerate.

To extend this result to generalized quasifree states we introduce the notion of σq–topology.

Definition A.1

Let (V,σ) be a symplectic space and q be a generalized quadratic form on it. We call σq-topology the locally convex topology defined by the neighborhood basis at the origin $I_{\epsilon,\epsilon_i,G_i}$ where, for every finite set $\{G_i\} \in V$ and with $\epsilon, \epsilon_i \in \mathbb{R}$,

$$I_{\epsilon,\epsilon_i,G_i} = \{F \in V : q(F) < \epsilon, \quad |\sigma(F,G_i)| < \epsilon_i\}. \tag{A.1}$$

Remark. Notice that we may take the G_i's to be in $V \backslash V_q$. Indeed, if $G_i \in V_q$ for some i, then the second condition in $(A.1)$ is implied by the first, with $\epsilon_i = \epsilon q(G_i)$, by the positivity condition (I.1.2). Hence, if $V = V_q$, this topology reduces to the strong topology on V induced by the q norm.

If both σ and q are degenerate, there may be elements $F \in V$ such that $q(F) = 0$ and $\sigma(F,\cdot) \equiv 0$ on V. Hence the σq-topology is not Haussdorff. Therefore we introduce

Definition A.2

We call $\overline{V}^{\sigma q}$ the space obtained from V adjoining to it all limit points of σq-Cauchy nets and then going to the quotient with respect to the nets that admits the origin as a limit point.

Remark. If $V = V_q$, $\overline{V}^{\sigma q}$ coincides with $\overline{V}^{q}$, the Hilbert space canonically obtained from $(V, [\cdot,\cdot]_q)$ by completion and quotient over the zero sequences.

As a first result we have

Lemma A.3

Given a decomposition $V = V_q + V'$, then $\overline{V}^{\sigma q} = \overline{V_q}^{\sigma q} + V'$. Furthermore, $q(\cdot)$ and $\sigma(\cdot,\cdot)$ are continuous (σ is jointly continuous) in the σq-topology and they have then a unique continuous extension to $\overline{V}^{\sigma q}$.

Proof. The first statement is an immediate consequence of the definition of $\overline{V}^{\sigma q}$. It follows that, if the net $\{F_\alpha\} \in V$ is σq-Cauchy, one can write $F_\alpha = G_\alpha + H$, $\{G_\alpha\} \in V$ a σq-convergent net and $H \in V'$, fixed.
The continuity of $q(\cdot)$ follows at once. The joint continuity of $\sigma(\cdot,\cdot)$ follows from the preceding decomposition, from the σ-weak and the q-strong convergence of $\{G_\alpha\}$ combined with the standard positivity condition, true on pairs of vectors in V_q. *q.e.d.*

As far as the characterization of primary generalized quasifree state is concerned, we state the following

Conjecture A.4

Let (V,σ) be a symplectic space, q be a generalized quadratic form on it and ω_q the associated generalized quasifree state. Then ω_q is primary iff the continuous extension of σ to $\overline{V}^{\sigma q}$ is nondegenerate.

We have been able to prove the "only if" part of this conjecture. The proof is a simple generalization of th. 3 proof in [MV]. Notice that we cover also the case, not treated in [MV], in which q is finite but degenerate and σ is degenerate.

Proposition A.5

Let (V,σ) be a symplectic space and q be a generalized quadratic form on it. If the continuous extension of σ to $\overline{V}^{\sigma q}$ is degenerate, then ω_q is not primary.

Proof. If σ is degenerate on $\overline{V}^{\sigma q}$, let $F_0 \neq 0$ in $\overline{V}^{\sigma q}$ such that $\sigma(F_0,\cdot) \equiv 0$ on $\overline{V}^{\sigma q}$ and let $\{F_\alpha\}$ be a net in V σq-convergent to F_0. This means that $\{F_\alpha\}$ converges to F_0 q–strongly and σ–weakly.
As a consequence, $\lim_\alpha \pi_{\omega_q}(\delta(F_\alpha))$ exists in the strong operator topology and hence defines an element U in $\pi_{\omega_q}(\mathcal{A}(V,\sigma))''$. Since σ is degenerate on F_0 and $\{F_\alpha\}$ is σq convergent to F_0, U is in the centre $\pi_{\omega_q}(\mathcal{A}(V,\sigma))' \cap \pi_{\omega_q}(\mathcal{A}(V,\sigma))''$. Notice then that U is unitary: it is a strong limit of unitary operators with, by the definition of

adjoint in Weyl algebras, strongly convergent adjoints. If ω_q is primary we should have

$$U = \lambda \mathbb{1} \qquad |\lambda| = 1.$$

But $F_0 \neq 0$ in $\overline{V}^{\sigma q}$ and since σ is degenerate on F_0, necessarily $q(F_0) \neq 0$ and so, by the continuity of $q(\cdot)$,

$$1 = |\lambda| = |(\psi_{\omega_q}, U\psi_{\omega_q})| = \lim_\alpha \exp\left(-\frac{1}{4}q(F_\alpha)\right) = \exp\left(-\frac{1}{4}q(F_0)\right) < 1.$$

Hence ω_q is not primary. *q.e.d.*

We suppose now to be given a primary generalized quasifree state ω_q on $\mathcal{A}(V,\sigma)$; we want to characterize its primary extension to $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$. It turns out that there is a unique, up to phases, primary extension (see also ch. 5.4.3 in [BR]).

Proposition A.6

*Let $\mathcal{A}(V,\sigma)$ be a CCR *-algebra, ω_q be a* primary *generalized quasifree state on it. Then primary states exist extending ω_q to $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$ and they are all of the form*

$$\Omega_\phi(\delta(F)) := \exp(i\phi(F))\exp\left(-\frac{1}{4}q_e(F)\right) \qquad \forall F \in \overline{V}^{\sigma q} \tag{A.2}$$

with $q_e(\cdot)$ the unique continuous extension of $q(\cdot)$ to $\overline{V}^{\sigma q}$ and $\phi(\cdot)$ a real additive functional on $\overline{V}^{\sigma q}$. Furthermore, $\mathcal{A}(V,\sigma)$ is strongly dense in $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$ in any representation defined by the states Ω_ϕ.

Proof. We first show that every state as in $(A.2)$ is primary.

Let $\Omega_0(\delta(F)) := \exp(-\frac{1}{4}q_e(F)) \quad \forall F \in \overline{V}^{\sigma q}$. Then the generalized quasifree state Ω_0 on $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$ is primary iff ω_q is primary on $\mathcal{A}(V,\sigma)$. Indeed, the definition of $\overline{V}^{\sigma q}$ implies that $\pi_{\omega_q}(\mathcal{A}(V,\sigma))$ is strongly dense in $\pi_{\Omega_0}(\mathcal{A}(\overline{V}^{\sigma q},\sigma))$. In particular the centers of the associated Von Neumann algebras coincide and this proves our statement.

Since $\delta(\cdot) \to \exp(i\phi(\cdot))\delta(\cdot)$ defines an automorphism of $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$, every state Ω_ϕ as in $(A.2)$ is primary.

Let, conversely, Ω be a primary state extending ω_q to $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$. We want to show that $\Omega = \Omega_\phi$ for some real additive functional ϕ. Let then $F \in \overline{V}^{\sigma q}$. If $\{F_\alpha\} \in V$ is a net σq-convergent to F, it is clear from $(A.1)$ that, for every $G \in \overline{V}^{\sigma q}$,

$$\lim_\alpha [\delta(F_\alpha - F), \delta(G)] = 0 \tag{A.3}$$

in the C^*-norm sense (the unique C^*-norm on $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$: ω_q is primary and hence the continuous extension of σ to $\overline{V}^{\sigma q}$ is nondegenerate!).

We then have

$$s-\lim_{\alpha}\pi_{\Omega}(\delta(F_{\alpha}-F))=s-\lim_{\alpha}\pi_{\Omega}(\delta(F_{\alpha}))\pi_{\Omega}(\delta(-F)).$$

Since π_{Ω} is a factor it follows from $(A.3)$ that

$$s-\lim_{\alpha}\pi(\delta(F_{\alpha}))=\exp\left(-i\phi(F)\right)\pi_{\Omega}(\delta(F))$$

where $\phi(F)\in\mathbb{R}$, since the limit operator is unitary. As a consequence, fixed anyhow $F\in\overline{V}^{\sigma q}$,

$$\begin{aligned}\Omega(\delta(F)) &= \exp\left(i\phi(F)\right)\Omega(s-\lim_{\alpha}\pi_{\Omega}(\delta(F_{\alpha})))=\\ &=\exp\left(i\phi(F)\right)\lim_{\alpha}\omega_q(\delta(F_{\alpha}))=\exp\left(i\phi(F)\right)\lim_{\alpha}\exp\left(-\frac{1}{4}q(F_{\alpha})\right)=\\ &=\exp\left(i\phi(F)\right)\exp\left(-\frac{1}{4}q_e(F)\right).\end{aligned}$$

This shows that $\phi(F)$ is independent of the net used to approximate F. The additivity of $\phi(\cdot)$ comes from the fact that $\exp\left(i\phi(\cdot)\right)$ is obtained as a strong limit and that the strong operator topology is jointly continuous on the uniformly bounded sets. By the above construction it is also clear that $\mathcal{A}(V,\sigma)$ is strongly dense in $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$ in all representations defined by the states Ω_{ϕ}. *q.e.d.*

2. Pure states

Pure (Fock) regular quasifree states are characterized as follows.

Let $\mathcal{Q}$ be the set of quasifree states ω_q on the CCR *-algebra $\mathcal{A}(V,\sigma)$, where q is a finite generalized quadratic form over the nondegenerate symplectic space (V,σ) and the continuous extension of σ to $\overline{V}^q$ is nondegenerate.

Then $D_q\in\mathcal{B}(\overline{V}^q)$ exists such that $D_q^{\dagger}=-D_q$, $D_q^{\dagger}D_q\leq\mathbb{1}$ and $\sigma(\cdot,\cdot)=[\cdot,D_q\cdot]_q$ on $\overline{V}^q$. The polar decomposition $D_q=J|D_q|$ gives $J^2=-\mathbb{1}$ and $J^{\dagger}=-J$.

One shows that $A_q=-D_q^{-1}$ exists with a dense domain and that it is a normal operator. Furthermore, one has that $A_q=J|A_q|$ with $|A_q|=|D_q|^{-1}\geq\mathbb{1}$; $A_q^{\dagger}A_q\geq\mathbb{1}$.

A quasifree state $\omega_q\in\mathcal{Q}$ is pure iff $A_q^{\dagger}A_q=\mathbb{1}$.

Our generalization is based on the following

Definition A.7

A generalized quadratic form q on a symplectic space (V,σ) is said *minimal* on (V,σ) if there is no generalized quadratic form $q' \neq q$ on (V,σ) such that

$$q'(F) \leq q(F) \qquad \forall F \in V. \tag{A.4}$$

The existence of minimal generalized quadratic forms on the symplectic space (V,σ) is a direct consequence of Zorn's lemma.

Remark. It is immediate that if q is minimal then $q_{|V_q} \equiv \hat{q}$ is minimal on (V_q,σ_q), $\sigma_q \equiv \sigma_{|V_q\times V_q}$ and that this is its maximal domain of regularity in (V,σ).
It is simple to check that a finite generalized quadratic form is minimal on (V,σ) iff the continuous extension of σ to $\overline{V}^q$ is nondegenerate and $A_q^\dagger A_q = \mathbb{1}$.

Then we have

Proposition A.8

Let q be a generalized quadratic form over the symplectic space (V,σ). Then the generalized quasifree state ω_q on $\mathcal{A}(V,\sigma)$ is pure iff q is minimal.

Proof. **1.** q minimal implies ω_q pure.

By Proposition I.3.8 it is enough to show that the restriction of ω_q to its maximal domain of regularity $\mathcal{A}(V_q,\sigma_q)$ is pure. To this end we note that $\ker\sigma_q = \ker\hat{q}$.

Indeed, finiteness of $\hat{q}$ and positivity condition (I.1.2) imply that $\ker\hat{q} \subset \ker\sigma_q$. Let then $F_0 \in \ker\sigma_q$, that is such that $\sigma_q(F_0,G) = 0 \quad \forall G \in V_q$. Setting $\sigma_q(\cdot,\cdot) = [\cdot, D_q\cdot]_q$ we have that $F_0 \in \ker D_{\hat{q}} = \ker|D_q|$. We define

$$\hat{q}_{min}(G) \equiv [G,|D_{\hat{q}}|G]_{\hat{q}} \qquad \forall G \in V_q, \tag{A.5}$$

and it is easy to verify that $\hat{q}_{min}$ complies with Definition I.2.3. Since $|D_q| \leq \mathbb{1}$, one has

$$\hat{q}_{min}(G) \leq \hat{q}(G) \qquad \forall G \in V_q.$$

But $F_0 \in \ker|D_q|$ implies $\hat{q}_{min}(F_0) = 0$ and $\hat{q}$ is minimal: so $\hat{q}(F_0) = \hat{q}_{min}(F_0) = 0$.
Hence $F_0 \in \ker\hat{q}$ and $\ker\sigma_q \subset \ker\hat{q}$.
We conclude that $\ker\sigma_q = \ker\hat{q}$.

It follows from this that

$$\pi_{\omega_{\hat{q}}}(\delta(F_0)) = \mathbb{1} \qquad \forall F_0 \in \ker\hat{q}. \tag{A.6}$$

Schwartz's inequality and $\ker\sigma_q = \ker\hat{q}$ imply that $\hat{q}$ and σ_q are well defined on the quotient space $V/\ker\hat{q}$. $(V/\ker\hat{q},\sigma_q)$ is thus a nondegenerate symplectic space and $\hat{q}$ a finite nondegenerate generalized quadratic form on it; so $\mathcal{A}(V/\ker\hat{q}, \sigma_q)$ is a well defined CCR *-algebra and $\hat{q}$ induces a generalized quasifree state ω'_q on it. But then from $(A.6)$ it follows that

$$\pi_{\omega_{\hat{q}}}(\mathcal{A}(V,\sigma)) \simeq \pi_{\omega'_{\hat{q}}}(\mathcal{A}(V/\ker\hat{q},\sigma_q)). \tag{A.7}$$

One can verify that $\hat{q}$ is minimal on $(V/\ker\hat{q}, \sigma_q)$ and that the continuous extension of σ_q to $\overline{V/\ker\hat{q}}^q$ is nondegenerate. By the standard argument [MV], ω'_q is pure and $\pi_{\omega'_{\hat{q}}}(\mathcal{A}(V/\ker\hat{q},\sigma)$ is irreducible. Using $(A.7)$ one concludes that so is also $\pi_{\omega_{\hat{q}}}(\mathcal{A}(V,\sigma))$, i.e. ω_q is pure.

2. ω_q pure implies q minimal.

It is convenient to distinguish two cases depending on whether (V_q, σ_q) is the maximal domain of regularity of q or not. In the first case q non minimal implies $q_{|V_q} \equiv \hat{q}$ non minimal. $\hat{q}$ is a finite generalized quadratic form and the associated operator D_q is such that $|D_q| \leq (1-\epsilon)\mathbb{1}$, for some $\epsilon > 0$. Hence there is a vector $H_P \in \overline{V_q}^q$ with spectral support relative to $|D_q|$ below $1-\epsilon$. Define then on $V_q \times V_q$

$$q_H(\cdot) := \hat{q}(\cdot) - \epsilon\hat{q}(P_H\cdot)$$

where P_H projects on H. It is easily verified that $q_H(\cdot)$ satisfies the positivity condition (I.1.2) on (V,σ) and that

$$q_H(F) \leq \hat{q}(F) \qquad \forall F \in V.$$

Let ω_α be the generalized quasifree state on $\mathcal{A}(V,\sigma)$ defined by

$$\omega_\alpha(\delta(F)) = \begin{cases} \exp(i\alpha\sqrt{\epsilon}[H,F]_{\hat{q}})\exp(-\frac{1}{4}q_H(F)) & \forall F \in V_q \\ 0 & \text{otherwise} \end{cases}$$

with $\alpha \in \mathbb{R}$. A trivial computation shows that

$$\omega_q = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} d\alpha \exp(-\alpha^2)\omega_\alpha$$

so that ω_q is not pure.

In the second case, the very same proof as in Lemma I.3.11 applies and so ω_q is not pure. *q.e.d.*

Remark. The above Proposition generalizes the standard result [MV] on the characterization of pure quasifree states in two respects:

i. σ is allowed to be degenerate on V.

ii. The quasifree states may be not regular.

APPENDIX B

ALGEBRAIC FERMION BOSONIZATION

It is well known [CR,CRW,DFZ,St,SW] that the fermion bosonization involves essentially two steps: i) the construction of anticommuting variables out of bosonic ones; ii) the construction of *local* Fermi fields in terms of Bose fields.

The first step amounts to constructing *infrared singular* fields out of a bosonic algebra (roughly the reason is that one has to smear bosonic fields with test functions which do not vanish at infinity). For a review of the extensive literature see [St,CR]. The point that we want to make is that such construction fits very naturally in the framework of non-regular representations of CCR algebras and actually appears very simple in such an approach. The crucial ingredient is the use of canonical extensions of CCR algebras and of their non-regular representations.

The second step is essentially an ultraviolet problem and, as we will see, the above framework allows for an improvement with respect to the literature on the subject, namely the construction of local fermi fields as *ultrastrong limits* of bosonic variables in all the representations which are locally Fock with respect to the ground state representation of the massless scalar field, rather than *strong limits on a dense set of states* of specific bosonic models [St,CR,CRW]. Our results will be established at the level of canonical variables, with no reference to the dynamics; the boson-fermion correspondence will thus emerge at the kinematical level, starting from the $U(1) \times U(1)$ current algebra $\mathcal{A}_0$ defined as the canonical Weyl algebra over the symplectic space $(\partial\mathcal{S} \times \mathcal{S},\, \sigma)$, where

$$\mathcal{S} \equiv \{f \in \mathcal{S}_{real}(\mathbb{R})\}, \qquad \partial\mathcal{S} \equiv \{f = \partial g \ \text{ for some } g \in \mathcal{S}\}$$

and for $F = (f_1, f_2)$ and $G = (g_1, g_2)$ in $\partial\mathcal{S} \times \mathcal{S}$

$$\sigma(F, G) \equiv \int dx\, (f_1 g_2 - f_2 g_1) \,.$$

One then introduces a canonical extension $\mathcal{A}$ of $\mathcal{A}_0$, $\mathcal{A} = \mathcal{A}(\mathcal{S} \times \partial^{-1}\mathcal{S},\, \sigma)$ where

$$\partial^{-1}\mathcal{S} \equiv \{f \in C^\infty(\mathbb{R}),\ \partial f \in \mathcal{S}\}\ ;$$

such an algebra can be interpreted as a charged field algebra associated to $\mathcal{A}_0$; it has a C^* norm, which is unique since σ is not degenerate, and therefore it can be represented as a C^* algebra of operators in a Hilbert space. The extension of a state from $\mathcal{A}_0$ to $\mathcal{A}$ defines a non-regular representation of $\mathcal{A}$ if a $U(1)$ subgroup is unbroken.

1. Anticommuting variables and fermionic algebras

The first point is to show that the field algebra $\mathcal{A}(\mathcal{S} \times \partial^{-1}\mathcal{S}, \sigma)$ contains anticommuting variables. To make the concept of anticommutativity precise we introduce the following notion of *localization* :
Given a bounded interval I of the real line, we say that $F = (f_1, f_2) \in \mathcal{S} \times \partial^{-1}\mathcal{S}$ is localized in I if

$$\mathrm{supp}(F) \equiv \mathrm{supp}(f_1) \cup \mathrm{supp}(\partial f_2) \subset I \quad . \tag{B.1}$$

The corresponding elements $W(F)$ of the Weyl algebra will also be said to be localized in I.

Definition B.1
A subset K of localized elements of $\mathcal{S} \times \partial^{-1}\mathcal{S}$ is said to describe fermionic degrees of freedom, or briefly is called a *fermionic subset*, if the corresponding set $\mathcal{F}_K$ of Weyl operators $W(F)$, $F \in K$
i) is invariant under the space translations α_x (i.e. $F \in K \Rightarrow F_x \in K$, with $F_x(y) \equiv F(y - x)$, $\forall x \in \mathbb{R}$),
ii) is invariant under the adjoint operation (i.e. $F \in K \Rightarrow -F \in K$),
iii) Weyl operators which are localized in disjoint intervals anticommute.

In order to construct a fermionic subset K we start with a localized $F \in \mathcal{S} \times \partial^{-1}\mathcal{S}$ and add to it $-F$ and all the functions obtained from them by space translations, so that i) and ii) hold. To satisfy iii) we note that:

Lemma B.2
Given a localized element $F = (f_1, f_2) \in \mathcal{S} \times \partial^{-1}\mathcal{S}$, the space translation invariant set

$$\mathcal{W}_F \equiv \{\alpha_x(W(F)) \ , \ x \in \mathbb{R}\} \tag{B.2}$$

satisfies iii) iff $\exists n \in \mathbb{Z}$ such that

$$(f_2(\infty) - f_2(-\infty)) \int f_1(x)\, dx = (2n + 1)\,\pi. \tag{B.3}$$

Proof. It suffices to consider the anticommutator

$$\{W(F), \alpha_x(W(F))\} = W(F+F_x)\, 2\cos(\sigma(F,F_x)/2) \tag{B.4}$$

for $\mp x$ large enough. In the two cases one has respectively

$$\sigma(F,F_x) = \pm(f_2(\infty) - f_2(-\infty)) \int f_1(y)\, dy \quad , \tag{B.5}$$

so that the r.h.s. of eq.$(B.4)$ vanishes iff eq.$(B.3)$ holds for some $n \in \mathbb{Z}$. *q.e.d.*

In conclusion, if eq.$(B.3)$ holds, the set $\mathcal{W}_F \cup \mathcal{W}_{-F}$ satisfies i), ii), iii). It describes one fermion degree of freedom. To find *maximal* sets of fermionic operators one is led by Lemma B.2 to consider the following two "charges", defined, $\forall F \in \mathcal{S} \times \partial^{-1}\mathcal{S}$, by

$$q_1(F) \equiv \int f_1(x)\, dx \ , \qquad q_2(F) \equiv f_2(\infty) - f_2(-\infty) \ . \tag{B.6}$$

Let us first fix $q_1(F) = \pm q_1 \neq 0$ (both signs are required by condition ii) above). Now, if F and G are elements of a fermionic subset K, with $q_1(F) = q_1$, $q_1(G) = \epsilon q_1$, $\epsilon = \pm 1$, then, by Lemma B.2, $\exists n_F, n_G \in \mathbb{Z}$ such that

$$q_2(F) = \frac{(2n_F+1)\pi}{q_1} \quad , \quad q_2(G) = \epsilon \frac{(2n_G+1)\pi}{q_1} \quad . \tag{B.7}$$

Furthermore, the anticommutativity condition iii) between F and G_x for sufficiently large $|x|$ gives

$$q_1(g_2(-\infty) - \epsilon f_2(\infty)) = (2k+1)\,\pi \qquad \text{for some } k \in \mathbb{Z} \ ,$$

$$q_1(g_2(\infty) - \epsilon f_2(-\infty)) = (2k'+1)\,\pi \qquad \text{for some } k' \in \mathbb{Z} \ , \tag{B.8}$$

which imply (by subtracting and summing)

$$k' - k = \epsilon(n_F + n_G + 1) \ , \tag{B.9}$$

$$g_2(-\infty) = \epsilon f_2(-\infty) + \frac{(2k + 2\epsilon n_F + 1 + \epsilon)\pi}{q_1} \quad . \tag{B.10}$$

In conclusion, a maximal fermionic subset with charge $\pm q_1$ is characterized by

$$q_1(F) = \epsilon\, q_1 \ , \quad \epsilon = \pm 1 \ ,$$

$$q_2(F) = \epsilon \frac{(2n_F + 1)\pi}{q_1} \ , \quad n_F \in \mathbb{Z} \qquad (B.11)$$

$$f_2(-\infty) = \epsilon c + \frac{2m\pi}{q_1} \ , \quad m \in \mathbb{Z}$$

where c is a (fixed) real number. Such a subset will be denoted by K_{c,q_1}.

Given c and q, the Weyl operators $W(F)$, $F \in K_{c,q}$ generate, through finite sums and products, a $*$-algebra $\mathcal{F}_{c,q}$ which will be called a *fermionic algebra.* Such an algebra is finitely generated by the set of i) all Weyl operators $W(F)$ with F localized, $q_1(F) = \epsilon q$, $\epsilon = \pm 1$, $q_2(F) = \pm \epsilon\pi/q$, $f_2(-\infty) = \epsilon c$, (fermionic operators with charge $\epsilon q/\sqrt{\pi}$ and chiral charge $\pm\epsilon\sqrt{\pi}/q$) and ii) the operator $W_\infty \equiv W(F_\infty)$, $F_\infty \equiv (0, 2\pi/q) \in \mathcal{S} \times \partial^{-1}\mathcal{S}$.

Proposition B.3

The $*$-algebra $\mathcal{F}_{c,q}$ has a non trivial center $Z_{c,q}$ generated by W_∞.

Proof. Clearly W_∞ commutes with the above set of generators of $\mathcal{F}_{c,q}$ and therefore it belongs to $Z_{c,q}$. To prove the converse, we first remark that $\mathcal{F}_{c,q}$ is finitely generated by the $W(F)$ with F belonging to the additive group $\mathcal{G}_{c,q}$ whose elements H are localized and characterized by

$$\int h_1(x)\, dx = mq \ ,$$

$$h_2(\infty) - h_2(-\infty) = (2N + m)\pi/q \ , \qquad (B.12)$$

$$h_2(-\infty) = mc + \frac{2M\pi}{q} \ , \quad m, N, M \in \mathbb{Z} \ .$$

Now, if $A \equiv \sum \lambda_i W(H_i)$, $H_i \in \mathcal{G}_{c,q}$, does not belong to the algebra generated by W_∞ then, for some index j , $h_{j,1} \neq 0$ or $h_{j,2} \neq$ const. In either case, one can find a $G \in \mathcal{G}_{c,q}$ such that $\sigma(G, H_j) \neq 2n\pi$. On the other hand, $A \in Z_{c,q}$ implies

$$0 = [W(G), A] = \sum \lambda_i W(G + H_i)\ 2i \sin(\sigma(G, H_i)/2)$$

and therefore $\lambda_i \sin(\sigma(G, H_i)/2) = 0\ \forall i$, in contrast with the above condition for $i = j$. *q.e.d.*

It is worthwhile to remark that in the labelling of $\mathcal{F}_{c,q}$ only the charge q is important, since a change of the constant c (from c to c', say) can be obtained through the following automorphism of the field algebra $\mathcal{A}$, which leaves the current algebra $\mathcal{A}_0$ pointwise invariant:

$$F = (f_1, f_2) \ \rightarrow \ F' = (f_1, f_2 + \frac{c' - c}{q} \int f_1(x)\, dx\,) \quad . \qquad (B.13)$$

Thus, without loss of generality, we can choose $c = 0$; then the algebra $\mathcal{F}_q \equiv \mathcal{F}_{0,q}/Z_{0,q}$ is generated by the current algebra $\mathcal{A}_0$ (with test functions of compact support) and by the Weyl operators $W(F), f_1 = \epsilon\, q\, \partial f$, $f_2 = \pm(\epsilon\pi/q)\, f$, where f is a fixed element of $\partial^{-1}S$ with $\int \partial f(x)\, dx = 1$, $f(-\infty) = 0$. They are the CCR counterpart of the charged morphisms of Streater and Wilde [SW]; here they are realized as operators belonging to a canonical extension of the current algebra $\mathcal{A}_0$.

The algebras $\mathcal{F}_{c,q}$ are maximal in the sense that any extension of the fermionic subset $K_{c,q}$ to a larger fermionic subset gives rise to the same fermionic algebra $\mathcal{F}_{c,q}$. In fact, if G belongs to a fermionic subset which extends $K_{c,q}$, then in the same way as in the proof of Proposition B.3 one finds that

$$q_1(G) = (2l+1)\, q\ , \quad l \in \mathbb{Z}\ ,$$

$$g_2(-\infty) = (2l+1)\, c + 2M\pi/q\ , \quad M \in \mathbb{Z}\ ,$$

$$q_2(G) = (2N+2l+1)\, \pi/q\ , \quad N \in \mathbb{Z}\ ,$$

so that, by eq.$(B.12)$, G belongs to the additive group $\mathcal{G}_{c,q}$ which characterizes $\mathcal{F}_{c,q}$.

2. Local Fermi fields as ultrastrong limits of Bose fields.

The above construction of fermionic algebras solves the infrared part of the fermion bosonization, namely the construction of anticommuting fields out of (infrared) extended Bose fields. In this section we will discuss the ultraviolet part, namely the construction of *local* Fermi fields, with the following features:

i) the local Fermi fields will be obtained as ultrastrong limits of bosonic variables, a possibility which was regarded as unlikely in the literature [St,CR]; since the operator product is ultrastrongly continous, an immediate advantage of this result is that all the algebraic operations commute with such limit.

ii) the construction is done at the canonical level, without reference to specific dynamical models and, in this sense, it can be regarded as a rigorous version of heuristic algebraic formulas of fermion bosonization, like Mandelstam's formula [M,Sk].

In order to approximate local Fermi fields, we consider the following Weyl operators: let $\rho(x) \in \mathcal{D}(\mathbb{R})$, with support contained in $[-1/2, 1/2]$, and

$$\rho(x) \geq 0\ , \quad \sqrt{2\pi}\tilde{\rho}(0) \equiv \int \rho(x)\, dx = \sqrt{\pi}\ .$$

Put $\rho^{(\epsilon)}(x) \equiv 1/\epsilon\ \rho(x/\epsilon)$, $\epsilon > 0$, and define (with $\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ for $x \leq 0$)

$$F_{\pm}^{(\epsilon)} = (\pm\rho^{(\epsilon)}, \theta * \rho^{(\epsilon)}) \in \mathcal{S} \times \partial^{-1}\mathcal{S} \tag{B.14}$$

The following discussion makes use of the representation π_{ω_0} of $\mathcal{A}_0$ defined, through the GNS construction, by the state ω_0 (the vacuum state of the massless scalar field in $1+1$ dimensions):

$$\omega_0(W(G)) = \exp(-||G||_0^2/4) \quad , \quad ||G||_0^2 = \int dk\, [\frac{1}{|k|}|\tilde{g}_1(k)|^2 + |k||\tilde{g}_2(k)|^2]$$

As we have seen in Section II.6, the state ω_0 has a unique extension Ω_0 to $\mathcal{A}$, which defines a non regular representation of $\mathcal{A}$; the corresponding GNS vectors will be denoted by Ψ_{ω_0} and Ψ_{Ω_0}. The non regularity of Ω_0 on $\mathcal{A}$ is equivalent to the absence of spontaneous breaking of the charge and the chiral charge in the representation defined by the massless ground state. In the proof of Theorem B.4 only the fact that Ω_0 extends ω_0 enters.

Theorem B.4

Let ω be any state on the current algebra $\mathcal{A}_0$, which is locally normal with respect to ω_0, and let ω_{ext} denote an extension of ω to the field algebra $\mathcal{A}$, and $(\pi_{\omega_{ext}}\,,\mathcal{H}_{ext}\,,\Psi_{\omega_{ext}})$ the corresponding GNS representation; then for any $g \in \mathcal{D}(\mathbb{R})$ the operators

$$\psi_{\pm}^{(\epsilon)}(g) = \frac{C}{\epsilon^{1/2}} \int \pi_{\omega_{ext}}(\alpha_y(W(F_{\pm}^{(\epsilon)})))g(y)\, dy \tag{B.15}$$

with C a suitable constant, converge in the ultrastrong operator topology on $\mathcal{B}(\mathcal{H}_{ext})$ as $\epsilon \to 0^+$. The limits define right/left handed fermions $\psi_{R/L}(g)$ which satisfy the canonical anticommutation relations.

Remark. Without loss of generality, it suffices to establish the Theorem for g real; the result for complex g follows by linearity.

Proof. **Step 1:** $||\psi_{\pm}^{(\epsilon)}(g)||$ is bounded uniformly in $\epsilon > 0$, and therefore existence of the limit in the ultrastrong topology defined by $\mathcal{H}_{ext}$ is implied by strong convergence on a dense subspace of $\mathcal{H}_{ext}$, e.g. the linear space $\mathcal{A}\Psi_{\omega_{ext}}$.

We start by remarking that, for every fixed $\epsilon > 0$, $\alpha_x(W(F_{\pm}^{(\epsilon)}))$ is ultrastrongly continuous in x in $\pi_{\omega_{ext}}$; in fact, since $F_x^{(\epsilon)} - F^{(\epsilon)} \in \partial\mathcal{S} \times \mathcal{S}$, by the local normality of ω (as a state on $\mathcal{A}_0$) w.r.t. ω_0, this follows from the ultrastrong continuity of $\alpha_x(W(F_{\pm}^{(\epsilon)}))$ in π_{ω_0}, which follows from $||F_x - F||_0 \to 0$, as $x \to 0$, $\forall F \in \mathcal{S}\times\partial^{-1}\mathcal{S}$.

Then, the integral of eq.$(B.15)$ is well defined as a Bochner integral and therefore $\psi_{\pm}^{(\epsilon)}(g) \in \mathcal{B}(\mathcal{H}_{ext})\ \forall \epsilon > 0$. Furthermore (dropping the $\pm$ in the following)

$$||\psi^{(\epsilon)}(g)||^2 \leq ||\{\psi^{(\epsilon)}(g)^\dagger, \psi^{(\epsilon)}(g)\}||$$

$$\leq \int_{|x-y|<\epsilon} dx\, dy\, |g(x)|\, |g(y)|\, ||\{\psi^{(\epsilon)}(x)^\dagger, \psi^{(\epsilon)}(y)\}||$$

$$\leq \frac{2C^2}{\epsilon} \int_{|x-y|<\epsilon} dx\, dy\, |g(x)|\, |g(y)| \ \rightarrow\ 4\, C^2\, ||g||_{L^2}^2 \ .$$

Step 2: For any GNS representation π_Ω of $\mathcal{A}$, strong convergence of the vector $\psi^{(\epsilon)}(g)\Psi_\Omega$ implies strong convergence of $\psi^{(\epsilon)}(g)\mathcal{A}\Psi_\Omega$, as $\epsilon \rightarrow 0^+$.

In fact, $\forall G \in \mathcal{S} \times \partial^{-1}\mathcal{S}$, $\sigma^\epsilon(x) \equiv \sigma(G, F_x^{(\epsilon)})$ and its limit when $\epsilon \rightarrow 0^+$ (which exists uniformly in x) are C^∞ functions of x, so that the behaviour of

$$||(\psi^{(\epsilon')}(g) - \psi^{(\epsilon)}(g))W(G)\Psi_\Omega|| = ||(\psi^{(\epsilon')}(e^{i\sigma^{\epsilon'}} g) - \psi^{(\epsilon)}(e^{i\sigma^\epsilon} g))\Psi_\Omega||$$

for $\epsilon, \epsilon' \rightarrow 0^+$ is governed by the strong limit of $\psi^{(\epsilon)}(e^{i\sigma^0} g)\Psi_\Omega$, which exists by assumption, and by the limit of

$$||\psi^{(\epsilon)}((e^{i\sigma^\epsilon} - e^{i\sigma^0})g)\Psi_\Omega|| \leq ||\psi^{(\epsilon)}((e^{i\sigma^\epsilon} - e^{i\sigma^0})g)|| \ ,$$

which vanishes as $\epsilon \rightarrow 0^+$ by the estimate given in step 1.

Step 3: $\psi^{(\epsilon)}(g)\Psi_{\Omega_0}$ converges strongly as $\epsilon \rightarrow 0^+$.

Since, by the properties of ω_0, for g real

$$\omega_0(\psi^{(\epsilon')}(g)^\dagger\, \psi^{(\epsilon)}(g)) = \omega_0(\psi^{(\epsilon')}(g)\, \psi^{(\epsilon)}(g)^\dagger) \ ,$$

$\psi^{(\epsilon)}(g)\Psi_{\Omega_0}$ is a Cauchy sequence in the strong topology provided

$$\lim_{\epsilon \rightarrow 0} \omega_0(\{\psi^{(\epsilon)}(g)^\dagger, \psi^{(\epsilon)}(g)\}) = \lim_{\epsilon \rightarrow 0} \lim_{\epsilon' \rightarrow 0} \omega_0(\{\psi^{(\epsilon)}(g)^\dagger, \psi^{(\epsilon')}(g)\}) \ . \qquad (B.16)$$

An explicit calculation gives

$$\omega_0(\{\psi^{(\epsilon)}(g)^\dagger, \psi^{(\epsilon')}(g)\}) = \frac{C^2}{(\epsilon\epsilon')^{1/2}} \int dx\, dy\, g(x)\, g(y)\, 2\, Re\ \exp R_{\epsilon,\epsilon'}(x-y) \quad (B.17)$$

where

$$R_{\epsilon,\epsilon'}(x-y) = -1/2 \int |\tilde{\rho}(\epsilon k) - \tilde{\rho}(\epsilon' k)e^{ik(x-y)}|^2\, dk/|k| +$$

$$+i\int sink(x-y)\,\tilde\rho(\epsilon k)\,\tilde\rho(\epsilon' k)\,dk/k \qquad (B.18)$$

Furthermore, at least as a limit in $\mathcal{S}'$,

$$\lim_{\epsilon'\to 0}(C^2/(\epsilon\epsilon')^{1/2})\exp R_{\epsilon,\epsilon'}(x-y) = (C^2/\epsilon)\ \exp R_{\epsilon,0}(x-y)$$

where

$$R_{\epsilon,0}(x-y) = 2\int_0^\infty (\tilde\rho(0)\,\tilde\rho(k)e^{ik(x-y)/\epsilon} - \tilde\rho(k)^2)\,dk/k \qquad (B.19)$$

(note that $(\epsilon\epsilon')^{-1/2}\exp R_{\epsilon,\epsilon'}$ are continuous functions, polynomially bounded uniformly in ϵ'). On the other hand,

$$\omega_0(\{\psi^{(\epsilon)}(g)^\dagger, \psi^{(\epsilon)}(g)\}) = (C^2/\epsilon)\int dx\,dy\,g(x)\,g(y)\,2\,Re\ \exp R_\epsilon(x-y) \quad (B.20)$$

where

$$R_\epsilon(x) = 2\int_0^\infty \tilde\rho(k)^2(e^{ikx/\epsilon}-1)\,dk/k \qquad (B.21)$$

Now, the distributions $Re\ \exp R_\epsilon(x)$ and $Re\ \exp R_{\epsilon,0}(x)$ are positive and have supports which shrink to the origin in the limit $\epsilon \longrightarrow 0^+$. It is therefore enough to control the integrals $1/\epsilon \int dx\ \exp R_\epsilon(x)$ and similarly for $R_{\epsilon,0}$. By partial integration, such integrals can be written as

$$\int dx \exp[-2\int_0^\infty dk \log k\,\frac{d}{dk}(e^{ikx}\tilde\rho(k)^2)\ +\ C'\,] \qquad (B.22)$$

and

$$\int dx \exp[-2\int_0^\infty dk \log k\,\frac{d}{dk}(e^{ikx}\tilde\rho(0)\tilde\rho(k))\ +\ C'\,]\quad , \qquad (B.23)$$

$$C'\ =\ 2\int_0^\infty dk\ \log k\,\frac{d}{dk}\tilde\rho(k)^2\quad . \qquad (B.24)$$

By a scale transformation $x = \lambda^{-1}\,y,\ k = \lambda\,q$, the above integrals, e.g. eq.$(B.22)$, become (neglecting the common factor $\exp C'$)

$$\lambda^{-1}\int dy \exp[-2\int_0^\infty dq \log q\,\frac{d}{dq}(e^{iqy}\tilde\rho(\lambda q)^2) + 2\tilde\rho(0)^2\log\lambda\ ]\ =$$

$$=\lim_{R\to\infty}\int_{-R}^{R} dy\ \exp[-2\int_{-\infty}^\infty dx\,[\log(-i(x+y)+\epsilon)-\gamma\,]\,\frac{1}{\lambda}\,(\rho*\rho)(\frac{x}{\lambda})]$$

with γ the Euler constant $\gamma = \int_0^\infty \exp(-x)\log x\,dx$; we have used $\tilde\rho(0)^2\ = 1/2$ and the fact that $\log(-iz+\epsilon)-\gamma$ is the Fourier transform of $-d/dq(\theta(q)\log q)$. The integral in y can be evaluated in the complex y plane where the contour can be closed in the upper half plane, since $\log(-i(x+y)+\epsilon)$ is analytic there and the x integral is limited to $|x| < \lambda$ by the support properties of ρ; since $\tilde\rho(0)^2\ = 1/2$, the integrand behaves like $1/y$ for $Im\,y$ positive and $|y|$ large. The result depends only on the value of $\tilde\rho(k)^2$ at the origin and it coincides with the integral of eq.$(B.23)$ since $\tilde\rho(0)\tilde\rho(k)$ and $\tilde\rho(k)^2$ coincide there. The value of both integrals is πe^γ.

Step 4: As a byproduct of Step 3, one has that $(1/\epsilon)\, Re\ \exp R_\epsilon(x)$ converges to $\delta(x)$, apart from a constant; the expectation on ω_0 of the anticommutators $\{\psi^{(\epsilon)}(x)^\dagger_\pm\,, \psi^{(\epsilon)}(y)_\pm\}$ (with the same index $+$ or $-$) converges therefore to a δ function provided the constant C in eq.(B.15) is chosen as

$$C\ =\ (\exp(-\gamma - C')\,/2\pi)^{1/2}\quad .$$

Step 5: The conclusions hold for any state ω locally normal (as a state on $\mathcal{A}_0$) with respect to ω_0; in fact, the strong convergence of $\psi^{(\epsilon)}(g)$ in the GNS representation space of $\mathcal{A}$ defined by any extension of ω is equivalent to the strong convergence of $W(-F)\psi^{(\epsilon)}(g)$, with F defined by taking $\epsilon = 1$ in eq.(B.14); the latter operators belong to $\mathcal{A}_0$ and, as in Step 2, it is enough to study strong convergence in the GNS space defined by $\mathcal{A}_0$ and ω (the "vacuum sector").

Now, for fixed y, since $\rho(x)$ has compact support, there is a finite interval I such that, $\forall\epsilon \in (0,1]$, $W(-F)\psi^{(\epsilon)}(y)$ belongs to $\mathcal{A}_0(I)$, the algebra generated by Weyl operators $W(G)$, $G \in \partial\mathcal{S} \times \mathcal{S}$ with support in I. Hence,

$$\pi_\omega(W(-F)\psi^{(\epsilon)}(g)) \in \pi_\omega(\mathcal{A}_0(I))''\quad ,$$

with a suitable I, for g of compact support.

As a consequence, $W(-F)\psi^{(\epsilon)}(g)$ is ultrastrongly convergent in all the representations of $\mathcal{A}_0$ which define the same ultrastrong topology as π_{ω_0} on the local algebras $\mathcal{A}_0(I)$, and therefore in all the representations defined by states ω on $\mathcal{A}_0$ which are locally normal with respect to ω_0.

Step 6: The limit field operators $\psi(g)$ satisfy the CAR's.

In fact, thanks to the strong convergence established in Step 3, and to the result of Step 4, we only have to show that the anticommutators considered in Step 4 converge to c-numbers, and that those of the form $\{\psi^{(\epsilon)}(g)\,, \psi^{(\epsilon)}(h)\}$ or $\{\psi^{(\epsilon)}(g)^\dagger\,, \psi^{(\epsilon)}(h)^\dagger\}$ converge to zero, in the space $\mathcal{H}_{ext}$.

The latter point follows by reducing the analysis, as in Step 5, to the strong convergence to zero of operators of the form, e.g.,

$$B^{(\epsilon)} = W(-F_1)W(-F_2)\,\{\psi^{(\epsilon)}(g)\,, \psi^{(\epsilon)}(h)\},$$

with F_1 and F_2 so chosen that $B^{(\epsilon)}$ belongs to the current algebra localized in a fixed interval.

As in Steps 2 and 5, this strong limit in $\mathcal{H}_{ext}$ vanishes iff it vanishes in the representation of $\mathcal{A}_0$ defined by ω_0; this vanishes as a consequence of the vanishing of the weak limit, which follows from the support properties of the anticommutators and the divergence of the quadratic form which defines ω_0, in the limit $\epsilon \to 0^+$. The same analysis applies to the anticommutators of $\psi_+^\dagger$ with ψ_-.

In order to show that the limits of the anticommutators $\{\psi^{(\epsilon)}(g)_\pm^\dagger\,,\psi^{(\epsilon)}(h)_\pm\}$ are c-numbers, we show that they belong to the center of the strong closure of local current algebras $\mathcal{A}_0(I)$ and that the representation of these algebras in $\mathcal{H}_{ext}$ is factorial.

The last point follows from the quasi-equivalence of the representation of these algebras in $\mathcal{H}_{ext}$ to their GNS representation defined by ω_0, as shown by the following argument: any $W(F)$, with $F \in \mathcal{S} \times \partial^{-1}\mathcal{S}$ can be written in the form $\exp i\alpha\, W(F')W(F_{I+\epsilon})$ with $F_{I+\epsilon}$ localized in a finite interval slightly larger than I and F' with support disjoint from I; from this it follows immediately that, for bounded nets in $\mathcal{A}_0(I)$, strong convergence in π_ω implies strong convergence in $\mathcal{H}_{ext}$, and strong convergence to 0 in π_ω implies strong convergence to 0 in $\mathcal{H}_{ext}$ (the converse statements are obvious); therefore the Von Neumann algebras generated by the representations of a local current algebra $\mathcal{A}_0(I)$ in the GNS space $\mathcal{H}_{\pi_\omega}$ and in $\mathcal{H}_{ext}$ are isomorphic; the local normality of ω with respect to ω_0, which defines factorial representations of $\mathcal{A}_0(I)$, implies the result.

Therefore, we only have to show that the above operators, in the representation π_{ω_0}, commute with all the (suitably localized) Weyl operators in $\mathcal{A}(\partial\mathcal{S}\times\mathcal{S},\sigma)$.

Using the Weyl relations, it is enough to show that $\forall G \in \partial\mathcal{S} \times \mathcal{S}$

$$\lim_{\epsilon\to 0+} \omega_0([\{\psi^{(\epsilon)}(g)^\dagger\,,\psi^{(\epsilon)}(h)\}\,,W(G)]) = 0 \quad ,$$

i.e.

$$\lim_{\epsilon\to 0+} \omega_0(W(G)\{\psi^{(\epsilon)}(e^{i\sigma^\epsilon}g)^\dagger\,,\psi^{(\epsilon)}(e^{-i\sigma^\epsilon}h)\}) = \lim_{\epsilon\to 0+} \omega_0(W(G)\{\psi^{(\epsilon)}(g)^\dagger\,,\psi^{(\epsilon)}(h)\}) \; ,$$

which follows from

$$\lim_{\epsilon\to 0+} \omega_0(\{\psi^{(\epsilon)}(x)^\dagger\,,\psi^{(\epsilon)}(y)\}) = \delta(x-y) \; ,$$

the definition given by eq.(B.15), and the properties of the Weyl product. *q.e.d.*

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