Scuola Internazionale Superiore di Studi Avanzati - Trieste

Algebraic and Geometrical aspects of rational Gaudin Models

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Chapter 1

Introduction

Rational Gaudin models were introduced in 1976 by Gaudin [25]. They describe a lattice of N spins interacting through long range forces that depend on a set of $2N$ parameters $\epsilon_1, \ldots, \epsilon_N, \eta_1, \ldots, \eta_N$. The corresponding Hamiltonian reads:

$$H = \sum_{i \neq j}^{N} \frac{\eta_i - \eta_j}{\epsilon_i - \epsilon_j} \vec{\sigma_i} \cdot \vec{\sigma_j} \tag{1.0.1}$$

The Hamiltonians (1.0.2) belong to the tensor product of N "spin 1/2" representations of $sl(2)$: $\vec{\sigma_j}$ denotes the vector $(\sigma_j^1, \sigma_j^2, \sigma_j^3)$ where σ_j^k refer to the k–th Pauli matrix acting on the j–th spin. Gaudin proved that, for any choice of the parameters $\{\epsilon_i, \eta_i\}_{i=1}^{N}$, the N operators $H_1, \ldots, H_N$

$$H_i = \sum_{j \neq i} \frac{1}{\epsilon_i - \epsilon_j} \vec{\sigma_i} \cdot \vec{\sigma_j} \tag{1.0.2}$$

commute among themselves and the Hamiltonian is the linear combination

$$H = \sum_{i=1}^{N} \eta_i H_i \tag{1.0.3}$$

Using the so called Bethe Ansatz, Gaudin showed that the problem of simultaneous diagonalization of the Hamiltonians H_i can be reduced to the problem of solving a system of algebraic equations called "Bethe equations". Gaudin showed also that the Hamiltonians can be generalized to the tensor product of N finite dimensional representations $\rho_i, i = 1, \ldots, N$ of an arbitrary (semisimple) Lie algebra $\mathfrak{g}$. Such generalized Hamiltonians can be described in terms of the Lax matrix introduced in the works of Sklyanin [55] and Jurčo [33]:

$$L(\lambda) = \sum_{i=1}^{N} \frac{A_i^q}{\lambda - \epsilon_i} \tag{1.0.4}$$

where ϵ_i are again arbitrary parameters and A_i^q are suitable $\rho_i(\mathfrak{g})$–valued matrices to be defined later, the label q (for quantum) denotes the fact that the entries of the matrices A_i^q are operators. Due to the existence of an r–matrix for the Lax matrix (1.0.4) the quadratic spectral invariants of $L(\lambda)$

$$\mathrm{Tr}(\mathrm{res}_{\lambda=\epsilon_i} L(\lambda)^2) = \sum_{j\neq i} \mathrm{Tr}\left(\frac{A_i^q A_j^q}{\epsilon_i - \epsilon_j}\right) \tag{1.0.5}$$

define a family of commuting observables that reduces to (1.0.2) in the spin 1/2 case.

In [33], Jurčo used the algebraic Bethe Ansatz to diagonalize the observables (1.0.5) in the case of $\mathfrak{g}$ being one of the simple classical Lie algebras. In the subsequent papers [34, 35] he showed as some interesting models of non–linear quantum optics could be related to particular representations of the Gaudin Hamiltonians (1.0.5). In this way he could use the algebraic Bethe Ansatz for the Gaudin models as a unified approach to a large variety of models of quantum optics.

The interest of Sklyanin and his coworkers in Gaudin models was more oriented toward the theory of separation of variables (SoV) both in the classical and quantum case [55, 56, 57]. In particular the Gaudin system was used as a simple model on which one could investigate the intertwining relations between the inverse scattering method (in both its classical and quantum version) and the theory of SoV. Following Sklyanin, this topic was investigated by many other authors (see e.g. [40, 36, 1, 51, 31, 27]).

The classical version of Gaudin models can be obtained replacing the generators of the Lie algebra $\rho_i(\mathfrak{g})$ in the matrices A_i^q with the corresponding coordinate functions on the Lie–Poisson manifold associated with $\mathfrak{g}$. We will denote the matrices we obtain after this substitution by A_i^c. Classical Hamiltonian systems associated with Lax matrices of the form (1.0.4) were widely studied e.g. by Reyman and Semenov-Tian-Shansky in the context of a general group–theoretic approach to classical integrable systems [48, 50]. In particular they proved that if one add a constant matrix σ with simple spectrum to the classical version of the Lax matrix (1.0.4)

$$L(\lambda) = \sum_{i=1}^{N} \frac{A_i^c}{\lambda - \epsilon_i} + \sigma \tag{1.0.6}$$

then $L(\lambda)$ defines a completely integrable system whenever $\mathfrak{g}$ is one of the classical Lie algebras A_n, B_n, C_n, D_n.

We would like to mention the fact that recently a certain interest in Gaudin models arose in the theory of condensed matter physics. In fact , it has been noticed [52, 53, 3] that the BCS model (describing superconductivity in metals) and the $sl(2)$ Gaudin model are very closely related. In particular, this relation allowed Amico and Osterloh [3] to translate the results of Sklyanin on correlation functions of the $sl(2)$ Gaudin model [58] to the BCS model, obtaining the exact correlation functions in the canonical ensemble.

The aim of this thesis is to study algebraic and geometrical aspects of classical Gaudin Models, with a view towards quantization. At first will frame the classical rational Gaudin models inside the so called bihamiltonian scheme. We will show that there exist two compatible Poisson tensor P and Q such that the spectral invariants of the Lax matrix (1.0.6) can be seen as elements of the Gel'fand–Zakharevich (GZ) sequences [29] associated with P and Q. P will be just the standard product structure associated with the Lie–Poisson manifold $\mathcal{P}_{\mathfrak{g}}^{\otimes N}$. The vector field associated by the Poisson tensor P to a function f can be written in terms of the matrices A_i^c as

$$\dot{A}_i^c = \left[A_i^c, \frac{\partial f}{\partial A_i^c}\right] \qquad i = 1, \dots, N$$

So we will write the Poisson tensor P in the symbolic form:

$$P = \begin{pmatrix} [A_1^c, .] & 0 & \dots & 0 \\ 0 & [A_2^c, .] & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & [A_N^c, .] \end{pmatrix} \tag{1.0.7}$$

We find the second Poisson tensor Q using the following argument: we define the change of coordinates $(A_1^c, \dots, A_N^c) \to (B_0, \dots, B_{N-1})$ induced by putting the rational Lax matrix (1.0.6) in polynomial form:

$$\left(\prod_{i=1}^{N}(\lambda - \epsilon_i)\right)\left(\sum_{i=1}^{N}\frac{A_i^c}{\lambda - \epsilon_i} + \sigma\right) \equiv \lambda^N \sigma + \sum_{i=0}^{N-1} B_i \lambda^i \tag{1.0.8}$$

On the space of polynomial pencils of matrices a family of mutually compatible Poisson brackets are defined [50, 42]. They will be termed, for the sake of brevity, RSTS tensors. This family can be described by saying that there is a map from degree N polynomials in the variable λ to the set of Poisson structures on the manifold of polynomial Lax matrices of the form (1.0.8) which sends the monomials $\lambda^0, \dots, \lambda^N$ into $N+1$ fundamental Poisson brackets, $\Pi_l, l = 0, \dots, N$. In the "coordinates" $B_0, \dots, B_{N-1}, \sigma$, the diagonal Poisson tensor P (1.0.7) is given by the sum

$$P = \sum_{l=0}^{N}(-1)^{N-l-1} s_{N-l}(\epsilon_1, \dots, \epsilon_N)\Pi_l, \tag{1.0.9}$$

where the s_i's are the elementary symmetric polynomials in the ϵ_i's, Since the Poisson tensors Π_l form a $N+1$-parameter family of compatible Poisson tensors, we can choose as a second Poisson tensor a suitable linear combination of them to have a bihamiltonian structure on $\mathcal{P}_{\mathfrak{g}}^{\otimes N}$. We choose the Poisson tensor

$$Q = \sum_{l=0}^{N-1} (-1)^{N-l} s_{N-l-1}(\epsilon_1, \ldots, \epsilon_N)\Pi_l, \tag{1.0.10}$$

This choice is motivated by the fact that finding the GZ sequences associated with the bihamiltonian pencil $Q - \lambda P$ turns out to be quite easy. We show that in the case of a generic matrix σ (i.e. with simple spectrum), the elements of such GZ sequences coincide with the spectral invariants of the Lax matrix (1.0.6); but "small differences" arise when this is not the case. In fact, if the spectrum of σ is not simple, then the spectral invariants of the Lax matrix $L(\lambda)$ do not provide all the involutive integrals of motion. We show that if the spectrum of σ degenerates then the rank of the Poisson tensor Q drops and the additional Casimirs of Q provide the missing Hamiltonians.

Afterwards, we concentrate on a particular case of rational Gaudin model, i.e., on the case when $\sigma = 0$ and $\eta_j = \epsilon_j \; \forall j$. In such case the Gaudin Hamiltonian becomes parameter independent:

$$H_s^{c,q} = \sum_{i,j=1}^{N} \mathrm{Tr}\left(A_i^{c,q} A_j^{c,q}\right) \tag{1.0.11}$$

and describes a (respectively classical and quantum) superintegrable Hamiltonian system (as argued in [30]). Thanks to the superintegrability property, it is possible to introduce a new set of parameter–independent observables

$$I_m^{c,q} = \sum_{i,j=1}^{m} \mathrm{Tr}\left(A_i^{c,q} A_j^{c,q}\right) \qquad m = 2, \ldots, N-1 \tag{1.0.12}$$

that commute among themselves and with the Hamiltonian (1.0.11) both in the classical and in the quantum case. This new set of integrals was introduced for the first time by Calogero [13] in the classical case and by Calogero and Van Diejen [14] in the quantum case, even if the connection with Gaudin models was not noticed. Such connection was unveiled in two subsequent papers by Ballesteros and Ragnisco [8] and Karimipour [38]. In particular, Ballesteros and Ragnisco showed that the quantities

$$H^{c,q}(m,n) = \mathrm{Tr}\left(\left(\sum_{i=1}^{m} A_i^{c,q}\right)^n\right) \tag{1.0.13}$$

commute among them for any value of n and m and with the total generators

$$\phi_\tau^{c,q} = \mathrm{Tr}\left(A_i^{c,q}\tau\right) \tag{1.0.14}$$

both in the classical and in the quantum case. This statement was proved using the standard Hopf–algebra structure defined respectively on the algebra of polynomials on the dual of a Lie algebra $Pol(\mathfrak{g}^*)$ in the classical case, and on the Universal Enveloping algebra $U(\mathfrak{g})$ in the quantum case.

In the $su(2)$ quantum case it was shown [45, 46], that the set of observables (1.0.11,1.0.12) can be simultaneously diagonalized by using just representation theory, without any need of solving Bethe equations. However, in general, the number of Hamiltonians (1.0.13) is equal to the number of degrees of freedom only if the rank of $\mathfrak{g}$ is 1.

It is interesting to remark that, in a completely independent framework, namely investigating a geometrical problem on the moduli spaces of polygons in $\mathbb{R}^3$, Kapovich and Millson [37] obtained the same set of integrals (1.0.11), (1.0.12) in the $sl(2)$. Their work was later generalized by Flaschka and Millson [24] that showed that if one consider a highly degenerated submanifold of $\mathcal{P}^{\otimes N}_{u(m)}$, then the integrals (1.0.13) and (1.0.14) are a sufficient number to provide complete integrability.

Dealing with the classical $sl(2)$ case, Kapovich and Millson integrated the system and found explicitly the action–angle variables. Hence, the set of observables (1.0.11,1.0.12) appear much easier to cope with than the corresponding set of observables (1.0.5). Nevertheless, the problem of completing the set of integrals (1.0.13) to a completely integrable system for algebras $\mathfrak{g}$ of rank higher than one was solved only in the very special case of Flaschka and Millson.

In the following we will work mainly with the classical rational Gaudin models, so we will write A_i for A_i^c dropping the label c. We will solve the problem of completing the set of integrals (1.0.13) in the classical case using bihamiltonian techniques; we will construct a further Poisson tensor R compatible with P but *not* with Q [1], whose GZ sequences reproduce, in the $sl(2)$ case, the Hamiltonians (1.0.11,1.0.12). The expression of R in terms of the matrices A_i is given by

$$R = \begin{pmatrix} 0 & [A_1,\cdot] & \cdots & [A_1,\cdot] \\ [A_1,\cdot] & [A_2 - A_1,\cdot] & \cdots & [A_2,\cdot] \\ \vdots & \vdots & \ddots & \vdots \\ [A_1,\cdot] & [A_2,\cdot] & \cdots & [(N-1)A_N - \sum_{i=1}^{N-1} A_i,\cdot] \end{pmatrix}. \tag{1.0.15}$$

We prove that, for $\mathfrak{g}$ a classical Lie algebra, the GZ sequences associated with P and R define a completely integrable system, that contains among its Hamiltonians the functions (1.0.13). The elements of the GZ sequences can be seen

[1] This proves that R, differently from Q, does not belong to the RSTS family of Poisson tensors associated with P.

as spectral invariants of the $N-1$ "Lax" matrices

$$L_i(\lambda) = (\lambda - i + 2)A_i + \sum_{j=1}^{i-1} A_j \qquad i = 2, \ldots, N \tag{1.0.16}$$

The functions (1.0.14) appear, also in this case, as Casimirs of R with very peculiar GZ sequences; in fact, they satisfy the equation:

$$Rd\phi_\tau = (N-1)Pd\phi_\tau$$

so that we can imagine these sequences as closed loops. An Abelian subalgebra of the algebra spanned by the functions (1.0.14) can be obtained introducing a N–th Lax matrix

$$L_{N+1}(\lambda) = \sigma + \frac{\sum_{i=1}^{N} A_i}{\lambda} \tag{1.0.17}$$

with σ a constant matrix with simple spectrum. We prove that the spectral invariants of the Lax matrices $L_i(\lambda)$, $i = 2, \ldots, N+1$ define an integrable system whenever $\mathfrak{g}$ is a classical Lie algebra.

After solving the problem of integrability we pass to the problem of separability of the Hamiltonians coming from the Lax matrices (1.0.16). We apply recently developed bihamiltonian techniques [20, 21] to solve this problem in the $\mathfrak{g} = sl(r)$ case. We find a set of separation coordinates *alternative* to the "standard" one found by Sklyanin [55, 56] and the "Montreal group" [1], based on the standard Lax representation for the homogeneous Gaudin model (that is, the Lax matrix (1.0.6) with $\sigma = 0$). This should not be regarded as a surprise, since it is well known that super-integrability is related with the existence of different sets of separation coordinates. In this set of coordinates the separation coordinates live on spectral curves whose genus is independent of the number of particles, while, in the standard picture, the genus of the spectral curve grows linearly with it. We adopt the following strategy: first we took a complete set of Casimirs for the Poisson tensor P:

$$C_i^{(m)} = \mathrm{Tr}(A_i^m) \quad i = 1, \ldots, N \quad m = 2, \ldots, r \tag{1.0.18}$$

and defined a corresponding set of $N(r-1)$ vector fields $Z_i^{(m)}$, $(i = 1, \ldots, N;\ m = 2, \ldots r)$ such that

- $\left(Z_i^{(m)} C_j^{(n)}\right) = \delta_{ij}\delta_{mn}$
- the functions invariants along the vector fields $Z_i^{(m)}$ form a Poisson subalgebra with respect to both Poisson tensors P and R.

Thanks to the above properties, we have that bivector $\widetilde{R}$ defined by

$$\widetilde{R} = R - \sum_{i=1}^{N} \sum_{m=2}^{r} X_i^{(m)} \wedge Z_i^{(m)}$$

where $X_i^{(m)} = PdC_i^{(m)}$, is still a Poisson tensor compatible with the Poisson tensor P and, moreover, can be restricted to the same symplectic leaf $\mathcal{S}$ of P. In fact, by construction:

$$\widetilde{R}dC_i^{(m)} = 0 \qquad \forall i, m$$

In this way, we can construct a Nijenhuis tensor $\mathcal{N}$ on $\mathcal{S}$:

$$N = \widetilde{R}|_{\mathcal{S}}(P|_{\mathcal{S}})^{-1}$$

Where $\widetilde{R}|_{\mathcal{S}}$ and $(P|_{\mathcal{S}})$ denote respectively the restrictions of the Poisson tensor $\widetilde{R}$ and P to the symplectic leaf $\mathcal{S}$. From the bihamiltonian theory we know that the eigenvalues of $\mathcal{N}$ are at least double and that the corresponding eigenspaces play a crucial role in SoV. In fact if we are able to find pairs of functions ρ_i, σ_i (called *Stäeckel* functions) such that their differentials form a complete set of eigenvectors for the adjoint of $\mathcal{N}$:

$$\mathcal{N}^* d\rho_i = \lambda_i d\rho_i \qquad \mathcal{N}^* d\sigma_i = \lambda_i d\sigma_i$$

then the functions ρ_i, σ_i satisfies commutation relations of the form

$$\{\rho_i, \sigma_j\} = f_i(\rho_i, \sigma_i)\delta_{ij}$$

for some set of functions f_i, and separates the Hamiltonians of the GZ sequences associated with $R - \lambda P$. A transformation of coordinates $\rho_i, \sigma_i \to \mu_i, \sigma_i$ such that the differentials of the new functions span again the λ_i eigenspace and have canonical Poisson brackets can be always found.

As a consequence of the vanishing of the Nijenhuis torsion of $\mathcal{N}$, the eigenvalues λ_i always satisfy

$$\mathcal{N}^* d\lambda_i = \lambda_i d\lambda_i$$

so that half of the separation coordinates can be found algebraically.

There are some special cases when even the other half of the separation coordinates can be found algebraically. This happens if we succeed in finding a complete set of polynomial Casimirs $H_a^{(l)}(\lambda)$ of the Poisson pencil $R - \lambda P$ such that it holds:

$$\text{Lie}_{Z_i}^{(m)} \text{Lie}_{Z_j}^{(n)} (H_a^{(l)}(\lambda)) = 0 \tag{1.0.19}$$

In such a case we say that the bihamiltonian manifold $\mathcal{P}_{\mathfrak{g}}^{\otimes N}, R - \lambda P$ admits an *affine* structure.

Luckily, this is exactly our case: in fact choosing the complete set of polynomial Casimirs $H_a^{(l)}(\lambda)$ as the coefficients of the expansion in powers of μ of the characteristic polynomial:

$$\det(\mu - L_a(\lambda)) = \mu^r + \sum_{l=2}^{r} H_a^{(l)}(\lambda)\mu^{r-l}$$

the equations (1.0.19) are satisfied. Applying the algebraic recipe to find Stäeckel functions on affine bihamiltonian manifolds to our case, we find that they can be characterized as solutions of the systems of equations (the index a being kept fixed)

$$\widetilde{M}_a^{kr}(\lambda, \xi) = 0 \qquad k = 1, \ldots, r \tag{1.0.20}$$

where $\widetilde{M}_a(\lambda, \xi)$ denotes the classical adjoint of the matrix

$$M_a(\lambda, \xi) = \xi \mathbb{I} - L_a(\lambda)$$

We would like to underline the striking similarities of equations (1.0.20) with the equations occurring in the Lax theory of SoV [56, 1, 15].

Finally we show by explicit computation that the Stäeckel functions defined by equations (1.0.20) satisfy the Poisson relations

$$\{\lambda_a^i, \xi_a^i\} = (\lambda_a^i - a + 2)(\lambda_a^i - a + 1)$$

so that separation variables are given by λ_a^i and

$$\mu_a^i = \frac{\xi_a^i}{(\lambda_a^i - a + 2)(\lambda_a^i - a + 1)}$$

In this thesis we deal mainly with classical Hamiltonian systems; however in the last chapter of it we take in consideration some quantum aspects related to the set of observables (1.0.13). We said that the commutativity of (1.0.13) relies on a common algebraic structure of Lie–Poisson and Universal Enveloping algebras. As a consequence, the classical Hamiltonians

$$H^c(m, n) = \mathrm{Tr}\left(\left(\sum_{i=1}^{m} A_i^c\right)^n\right) \tag{1.0.21}$$

can be quantized simply replacing the matrices A_i^c with their quantum counterparts A_i^q. We showed that we get an integrable system if we consider the classical Hamiltonians $H^c(n,m)$ toghether with the other spectral invariants of the N Lax matrices

$$L_i^c(\lambda) = (\lambda - i + 2)A_i^c + \sum_{j=1}^{i-1} A_j^c \qquad i = 2,\dots,N \tag{1.0.22}$$

$$L_{N+1}^c(\lambda) = \sigma + \frac{\sum_{i=1}^N A_i^c}{\lambda} \tag{1.0.23}$$

It is then natural to wonder if also these further spectral invariants can be quantized in a simple way as well. At the moment, we have only a partial answer to this question. In particular, we show that if one replaces the matrices A_i^c with their quantum counterparts A_i^q in the Lax matrices (1.0.16), then the resulting quantum Lax matrices (we rescaled the spectral parameter λ):

$$L_i^q(\lambda) = \lambda A_i^q + \sum_{j=1}^{i-1} A_j^q \qquad i = 2,\dots,N$$

satisfy:

$$\left[\mathrm{Tr}\left(L_i^q(\lambda)^m\right), \mathrm{Tr}\left(L_j^q(\lambda)^n\right)\right] = 0 \qquad i \neq j$$

In this way we prove that the spectral invariants of different Lax matrices commute. However, at the moment, we succeeded in proving the commutativity of the spectral invariants of a given Lax matrix $L_i^q(\lambda)$ among them only until cubic terms:

$$\left[\mathrm{Tr}\left(L_i^q(\lambda)^m\right), \mathrm{Tr}\left(L_i^q(\lambda)^n\right)\right] = 0 \qquad m,n = 1,2,3$$

Outline of the thesis

- In chapter 2 we give an essential review of the bihamiltonian theory of SoV, just illustrating the main theorems. Special emphasis is put on the theory of SoV for GZ manifolds admitting an affine structure.

  We give also a brief survey on Lie–Poisson manifolds and on Poisson product structure with the aim of introducing concepts, notations and conventions to be widely used in subsequent chapters.

- In chapter 3 we recall the history of rational Gaudin models and we give a detailed description of the results previously obtained on the set of parameter independent integrals (1.0.11,1.0.12) by Ballesteros and Ragnisco on one side and by Flaschka, Kapovich and Millson on the other.

- In chapter 4 we introduce a bihamiltonian structure for the inhomogeneous Gaudin model using the results of Reyman and Semenov–Tian–Shansky. We show that the Lax and the bihamiltonian approach are essentially equivalent. We comment on the differences arising when the spectrum of the matrix σ in (1.0.6) is not simple.

- In chapter 5 we concentrate on the set of parameter independent integrals. We introduce a further Poisson tensor R that is compatible with P but not with Q. We show that the Hamiltonians of the GZ sequences associated with P and R can be seen as the spectral invariants of N Lax matrices. We prove that such invariants define a completely integrable Hamiltonian system on $\mathcal{P}_{\mathfrak{g}}^{\otimes N}$ for $\mathfrak{g}$ belonging to the class of classical Lie algebras.

- In chapter 6 we apply the machinery of bihamiltonian theory of Sov to separate the set of parameter independent Hamiltonians just obtained in chapter 5 for the case $\mathfrak{g} = sl(r)$. We show that the separation coordinates are given by particular points on the spectral curves associated with our Lax matrices. We have N Lax matrices and they have all the same genus g so that in our case the genus of the spectral curves is independent from the number of "particles" N. We give explicit examples of the SoV procedure in the cases $\mathfrak{g} = sl(2)$ and $\mathfrak{g} = sl(3)$.

- In the last chapter we give some preliminary results on the quantization of the integrals we obtained by the GZ sequences associated with the Poisson tensors P and R.

- Finally, in appendices we give the proofs of two lemmas that were too cumbersome to be placed in the text.

Chapter 2

Mathematical background

2.1 Bihamiltonian manifolds and integrability

A classical Hamiltonian system is a triple $(M, \{\cdot,\cdot\}, H)$ where M is a differentiable manifold, H is a differentiable function on M and $\{\cdot,\cdot\}$ denotes the Poisson bracket. The Poisson bracket is a composition law on differentiable functions on M that is bilinear, skewsymmetric and satisfies the Leibnitz rule and the Jacobi identity. A manifold M endowed with a Poisson bracket is called a "Poisson manifold" and if the Poisson bracket is non–degenerate a "symplectic manifold".

We can always associate with a Poisson bracket a Poisson tensor

$$P : T^*M \to TM$$

through the equality:

$$\{f, g\} = \langle df, Pdg \rangle$$

If $\dim(M) = m$ and $y^1, \ldots, y^m$ is a set of (local) coordinates of M we can write the Poisson tensor in the form:

$$P = P^{\alpha\beta}(y^1, \ldots, y^m)\frac{\partial}{\partial y^\alpha} \otimes \frac{\partial}{\partial y^\beta} \qquad \alpha, \beta = 1, \ldots, m$$

where $P^{\alpha\beta}(y^1, \ldots, y^m) \in C^\infty(M)$. It follows that if f and g are two C^∞ functions on M, their Poisson bracket is also a C^∞ function. Hence, the Poisson bracket defines on $C^\infty(M)$ an algebra structure that together with the functions' pointwise multiplication makes $C^\infty(M)$ a Poisson algebra.

Definition 2.1.1 *A "Poisson algebra" is a vector space A endowed with two bilinear operations: a "multiplication" $m : A \otimes A \to A$ and a "Poisson bracket"*

$\{\cdot,\cdot\}: A\otimes A \to A$ *satisfying for any* $a,b,c\in A$*:*

$$m(m(a,b),c) = m(a,m(b,c)) \qquad \text{(Associativity)} \qquad (2.1.1)$$
$$m(a,b) = m(b,a) \qquad \text{(Commutativity)} \qquad (2.1.2)$$
$$\{a,b\} = -\{b,a\} \qquad \text{(Skewsymmetry)} \qquad (2.1.3)$$
$$\{a,\{b,c\}\} + \{b,\{c,a\}\} + \{c,\{a,b\}\} = 0 \qquad \text{(Jacobi identity)} \qquad (2.1.4)$$
$$\{m(a,b),c\} = m(a,\{b,c\}) + m(b,\{a,c\}) \qquad \text{(Leibnitz rule)} \qquad (2.1.5)$$

On a Poisson manifold we can associate with any differentiable function f a vector field X_f defined as

$$X_f(g) = \{g,f\}$$

Using the Poisson tensor the vector field X_f can be written in the equivalent form

$$X_f = Pdf$$

In this way to our Hamiltonian system is naturally associated a dynamical system on M defined by the vector field X_H. In general, solving an Hamiltonian system is not at all an easy task and the main result is the celebrated Arnold-Liouville theorem [4]:

Theorem 2.1.2 *Let M be a $2n$ dimensional symplectic manifold and let $F_1,\dots,F_n$ be a family of n involutive functions*

$$\{F_i,F_j\} = 0 \qquad i,j = 1,\dots,n$$

and let us consider the level set

$$M_f = \{x:\ F_i(x) = f_i,\ i = 1,\dots,n\}$$

then:

1. *M_f is a regular invariant manifold under the flux associated with any function F_i.*

2. *If M_f is compact and connected, then it is diffeomorphic to an n–dimensional torus.*

3. *A point on M_f evolve along a quasi–periodic motion under the flux associated with any function F_i.*

4. *the Hamiltonian systems associated with any function F_i can be solved by quadratures*

So, when one has an Hamiltonian system, it is natural to search for functions in involution with the Hamiltonian and among themselves. One way to perform such a task is to frame the Hamiltonian system inside the bi–Hamiltonian scheme. An Hamiltonian system is said to be "bi–Hamiltonian" if on the Poisson manifold M it is possible to define a second Poisson bracket $\{\cdot,\cdot\}_1$ *compatible* with the original one (that from now on we will denote with $\{\cdot,\cdot\}_0$) in such a way that the Hamiltonian vector field X_H can also be written as:

$$X_H(G) = \{H_1, G\}_1$$

for a suitable function H_1. Two Poisson brackets are said to be compatible if the Poisson pencil

$$\{f,g\}_\lambda = \{f,g\}_1 - \lambda\{f,g\}_0$$

verifies the Jacobi identity for any value of the real parameter λ. We will call P_λ the Poisson tensor associated with $\{\cdot,\cdot\}_\lambda$.

Associated with the two Poisson brackets we will have two Poisson tensors P_0 and P_1. If one of them is invertible (let us say P_0), then we can build up a "Nijenhuis" tensor on M:

$$\mathcal{N} = P_1(P_0)^{-1} \tag{2.1.6}$$

i.e. a $(1,1)$ tensor with vanishing torsion:

$$T(\mathcal{N})(x,y) = [\mathcal{N}x,\mathcal{N}y] - \mathcal{N}[\mathcal{N}x,y] - \mathcal{N}[x,\mathcal{N}y] + \mathcal{N}^2[x,y] = 0 \quad x,y \in TM$$

The relevance of the Nijenhuis tensor is given by the following theorem:

Theorem 2.1.3 *The functions*

$$I_k = \frac{1}{k}\mathrm{Tr}(\mathcal{N}^k)$$

satisfy the relations:

$$\mathcal{N}^* dI_k = dI_{k+1} \tag{2.1.7}$$

where $\mathcal{N}^*$ is the tensor dual to $\mathcal{N}$.

From the above theorem it easily follows that the functions I_k form an involutive family with respect to both brackets:

$$\{I_j, I_k\}_{0,1} = 0 \qquad j,k = 1,\ldots,n$$

In fact, equation (2.1.7) implies:

$$P_1 dI_k = P_0 dI_{k+1}$$

so that we have (assuming $k > j$):

$$\{I_j, I_k\}_0 = \langle dI_j, P_0 dI_k\rangle = \langle dI_j, P_1 dI_{k-1}\rangle = \{I_j, I_{k-1}\}_1 =$$
$$-\langle dI_{k-1}, P_1 dI_j\rangle = -\langle dI_{k-1}, P_0 dI_{j+1}\rangle = \{I_{j+1}, I_{k-1}\}_1$$

one can continue to raise one index and to lower the other until they match, then because of the skewsimmetry of Poisson tensors, the bracket vanishes.

If P_0 is invertible, we can define a symplectic two form $\omega : TM \to T^*M$ in the following way:

$$\{f, g\}_0 = \omega(X_f, X_g) \tag{2.1.8}$$

With the definitions (2.1.6) and (2.1.8) the Nijenhuis tensor $\mathcal{N}$ and the symplectic tensor ω are related by the equations:

$$\omega(\mathcal{N}x, y) = \omega(x, \mathcal{N}y) \tag{2.1.9}$$
$$L_{\mathcal{N}x}(\omega)y - L_{\mathcal{N}y}(\omega)x + \omega\mathcal{N}[x, y] + d\omega(x, y) = 0 \quad \forall x, y \in T^*M \tag{2.1.10}$$

where L_X denotes the Lie derivative with respect to the vector field X.

Definition 2.1.4 *An "$\omega\mathcal{N}$ manifold" is a differential manifold M endowed with a symplectic tensor ω and with a Nijenhuis tensor $\mathcal{N}$ fulfilling the coupling conditions (2.1.9),(2.1.10).*

So we have that a bihamiltonian manifold where one of the two Poisson tensor is invertible has a natural structure of $\omega\mathcal{N}$ manifold.

We have seen that, when we have a bihamiltonian manifold and one of the two Poisson tensor is not degenerated, we can easily find a family of functions in involution. Unfortunately, the Hamiltonian systems possessing this geometric structure are very few, so that the above described approach cannot be applied in the majority of cases; in fact, if both Poisson structures are degenerated it is impossible to define the Nijenhuis tensor $\mathcal{N}$. In such a case one can resort to the anchored Lenard scheme introduced by Gel'fand and Zakharevich [29]. Their approach is based on the fact that if the Poisson brackets $\{,\}_1$ and $\{,\}_0$ are both degenerated, then, for any fixed value $\lambda = \lambda^0$, the Poisson bracket $\{,\}_1 - \lambda^0\{,\}_0$ will be degenerated as well. From this fact it follows, under some additional hypotheses, the existence of a set of Casimir functions $C_{t,\lambda}$, ($t = 1, \dots, r$ for some r) for the Poisson bracket $\{\cdot,\cdot\}_\lambda$. The functions $C_{t,\lambda}$ are called "Casimir of the pencil". This statement is made precise in the following proposition.

Proposition 2.1.5 *Consider a bihamiltonian manifold M and a point $m_0 \in M$. Fix $\lambda^0 \in \mathbb{R}$. Suppose that the corank of the bracket $\{,\}_1 - \lambda\{,\}_0$ at m is r for any m near m_0 and any λ near λ^0. Then there is a neighborhood $U \times \mathcal{U}$ of $(m_0, \lambda_0) \in M \times \mathbb{R}$ and r families $C_{t,\lambda}$, $1 \le t \le r$, $\lambda \in \mathcal{U}$, of functions on U such that*

1. *for any given t $C_{t,\lambda}$ is a Casimir of the pencil $\{,\}_1 - \lambda\{,\}_0$ on U and*

2. *for any given $\lambda \in \mathcal{U}$ the functions $C_{t,\lambda}$, $1 \le t \le r$, are independent.*

The idea of the anchored Lenard scheme is to put $\lambda_0 = 0$ and write a formal series in λ for a Casimir C_λ defined near λ^0:

$$C_\lambda = H_0 + \lambda H_1 + \lambda^2 H_2 + \dots$$

The condition that C_λ is a Casimir function of the Poisson pencil can be written explicitly as:

$$\{H_0 + \lambda H_1 + \lambda^2 H_2 + \dots, f\}_1 - \lambda\{H_0 + \lambda H_1 + \lambda^2 H_2 + \dots, f\}_0 = 0$$

equating the coefficients of the various power in λ one easily gets:

$$\{H_i, f\}_1 = \{H_{i+1}, f\}_0 \qquad \forall f \in C^\infty(M) \tag{2.1.11}$$

$$\{H_0, f\}_0 = 0 \qquad \forall f \in C^\infty(M) \tag{2.1.12}$$

Definition 2.1.6 *Consider a formal series $H_0 + \lambda H_1 + \lambda^2 H_2 + \dots$ in λ with H_i being functions on M. Call it a "formal λ-family" on M if the sequence H_k satisfies the recurrence relation (2.1.11). Call this formal λ-family "anchored" if H_0 is a Casimir for $\{,\}_0$.*

We have just seen that equation (2.1.11) implies the involutivity of the elements H_k with respect to both Poisson brackets. Now we show that thanks to equation (2.1.12) elements of different *anchored* formal λ-family Poisson commute as well.

Proposition 2.1.7 *Given two* anchored *formal λ-families $H_0 + \lambda H_1 + \lambda^2 H_2 + \dots$ and $H'_0 + \lambda H'_1 + \lambda^2 H'_2 + \dots$ one has $\{H_i, H'_j\}_0 = \{H_i, H'_j\}_1 = 0$ for any i and j.*

Proof. Put $H_i = H'_i = 0$ for $i < 0$. In this way we can apply (2.1.11) for $i < 0$ too. For any i, j we have:

$$\{H_i, H'_j\}_0 = \{H_{i-1}, H'_j\}_1 = -\{H'_j, H_{i-1}\}_1 = \{-H'_{j+1}, H_{i-1}\}_0 = \{H_{i-1}, H'_{j+1}\}_0$$

Repeating this process i times and using (2.1.12) we get:

$$\{H_i, H'_j\}_0 = \{H_0, H'_{j+i}\}_0 = 0$$

Moreover

$$\{H_i, H'_j\}_1 = \{H_{i+1}, H'_j\}_0 = 0$$

□

This last proposition allows us to define the "Anchored Lenard scheme" to find large families of involutive functions on a bihamiltonian manifold.

Proposition 2.1.8 *Consider a maximal collection $H_0^{(1)}, \dots, H_0^{(r)}$ of independent Casimir functions for $\{,\}_0$ near $m_0 \in M$. Let $H_i^{(t)}, t = 1, \dots, r, i \geq 0$ be solutions to recursion relations (2.1.11) with $H_0^{(t)}$ as the initial data. Then there are numbers $k_1, \dots, k_r$ such that the collection $\{H_i^{(t)}\}$, $1 \leq t \leq r$, $0 \leq i \leq k_t$, is functionally independent, and all functions $H_i^{(t)}$, $1 \leq t \leq r$, $i > k_t$, depend functionally on this collection.*

The proof of this proposition can be found in [28].

Summarizing, until now we have the following picture: if in a point $m_0 \in M$ and for λ belonging to a neighborhood $\mathcal{U}$ of $\lambda = 0$ the Poisson pencil has constant corank r then one can define r Casimir of the pencil $\{C_{t,\lambda}\}_{t=1}^r$ in some neighborhood $U \times \mathcal{U}$ of $(m_0, 0)$. Now consider a maximal collection $H_0^{(1)}, \dots, H_0^{(r)}$ of independent Casimir functions for $\{,\}_0$ near $m_0 \in M$, and consider the equations

$$\{H_i^t, f\}_1 = \{H_{i+1}^t, f\}_0 \qquad \forall f \in C^\infty(M) \quad t = 1, \dots, r \tag{2.1.13}$$

as recursive equations for the functions $H_1^t, H_2^t, \dots$. Then the existence of the Casimirs $\{C_{t,\lambda}\}_{t=1}^r$ guarantees the solvability of equations (2.1.13), so that equations (2.1.13) can be solved on the same neighborhood $U \times \mathcal{U}$ of $(m_0, 0)$ on which the Casimirs are defined. Finally, proposition (2.1.8) assures us that we have to solve only a *finite* number of equations.

So we can think of the anchored Lenard scheme as a process consisting of two steps: first one finds a maximal independent collection of Casimir functions for the bracket $\{,\}_0$, then one solves recurrence relations (2.1.13) with these functions as initial data until new functions start to depend upon the old ones.

2.2 Separation of variables

Even if one has established the integrability of a given Hamiltonian system, the problem of finding explicitly its solutions is far from being solved. One of the oldest and possibly, most effective methods to solve a Hamiltonian dynamical system is the method of separation of variables (SoV). Recently, investigations on the relations between SoV and the inverse scattering method (also in its quantum formulation) had the effect of strongly reviving the interest in this field (see [57]). Following this renewed interest the issue of separation of variables in bihamiltonian systems has been thoroughly investigated [17, 21, 12, 32]. In this section we present the main results on this subject following [21]. We begin with the definition of separable systems.

Definition 2.2.1 *An n-tuple $(H_1, \ldots, H_n)$ of functionally independent Hamiltonians will be said to be separable in a set of canonical coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ if there exist n relations, called separation relations, of the form:*

$$\phi_i(q_i, p_i, H_1, \ldots, H_n) = 0\,, \quad i = 1, \ldots, n\,, \quad \textit{with} \det\left[\frac{\partial \phi_i}{\partial H_j}\right] \neq 0 \qquad (2.2.1)$$

The reason for this definition is that the stationary Hamilton-Jacobi equations for the Hamiltonians H_i can be collectively solved by the additively separated complete integral

$$W(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} W_i(q_i; \alpha_1, \ldots, \alpha_n) \qquad (2.2.2)$$

where the W_i are found by quadratures as the solutions of ordinary differential equations.

2.2.1 Separation of variables on ωN manifolds

Let M be an $\omega\mathcal{N}$ manifold.

Definition 2.2.2 *A set of local coordinates (x_i, y_i) on M is called a set of "Darboux-Nijenhuis (DN) coordinates" if they are canonical with respect to the symplectic form ω,*

$$\omega = \sum_{i=1}^{n} dy_i \wedge dx_i$$

and are such that the recursion operator $\mathcal{N}$ takes the diagonal form

$$\mathcal{N} = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i} \otimes dx_i + \frac{\partial}{\partial y_i} \otimes dy_i\right) \qquad (2.2.3)$$

This means that the only nonzero Poisson brackets are:

$$\{x_i, y_j\}_0 = \delta_{ij} \qquad \{x_i, y_j\}_1 = \lambda_i \delta_{ij}$$

The assumption contained in (2.2.3) that the eigenvalues λ_i of $\mathcal{N}$ are at least double is not restrictive, since its eigenspaces have even dimension, equal to the dimension of the kernel of $P_0 - \lambda P_1$.

Definition 2.2.3 *A $2n$-dimensional $\omega\mathcal{N}$ manifold M is said to be "semisimple" if its recursion operator $\mathcal{N}$ has, generically, n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. It is called "regular" if the eigenvalues of $\mathcal{N}$ are functionally independent on M.*

The definition of semisimple $\omega\mathcal{N}$ manifold follows from the fact that it can be shown [28, 43, 59] that every point of a semisimple $\omega\mathcal{N}$ manifold has a neighborhood where DN coordinates can be found, while that of regular $\omega\mathcal{N}$ manifold is motivated by the fact that if M is regular then half of the DN coordinate are provided by the eigenvalues of the recursion operator $\mathcal{N}$. Indeed, as a consequence of the vanishing of the Nijenhuis torsion of $\mathcal{N}$, the eigenvalues λ_i always satisfy

$$\mathcal{N}^* d\lambda_i = \lambda_i d\lambda_i$$

The remaining half of DN coordinates can always be found by quadratures [21] (in some cases, they happen to be found algebrically):

Proposition 2.2.4 *In a neighborhood of a point of a regular $\omega\mathcal{N}$ manifold it is possible to find by quadratures n functions $\mu_1, \dots, \mu_n$ that, along with the eigenvalues $\lambda_1, \dots, \lambda_n$ form a set of DN coordinates.*

Such coordinates will be called a set of "special Darboux–Nijenhuis coordinates".

Having introduced the concept of DN coordinates we can state the main theorem on separability of bihamiltonian systems defined on $\omega\mathcal{N}$ manifolds:

Theorem 2.2.5 *Let M be a semisimple $\omega\mathcal{N}$ manifold and let $(H_1, \dots, H_n)$ be a set of n functionally independent Hamiltonians on M. Then the following statements are equivalent:*

1. *The Hamiltonians $(H_1, \dots, H_n)$ are separable in DN coordinates;*
2. *The tangent distribution to the foliation defined by $(H_1, \dots, H_n)$ is Lagrangian with respect to ω and invariant with respect to $\mathcal{N}$;*
3. *The functions $(H_1, \dots, H_n)$ are in bi-involution, i.e., $\{H_i, H_j\}_0 = 0$ and $\{H_i, H_j\}_1 = 0$ for all i, j.*

Theorem (2.2.5) give us a simple criterion to establish separability of an integrable Hamiltonian system $(H_1, \dots, H_n)$ defined on a $\omega\mathcal{N}$ manifold M; moreover, if the system is separable and M is regular then half of the separation coordinates are provided by the eigenvalues of $\mathcal{N}$. However, finding the second half of the separation coordinates is not a trivial task and until now there is not any algorithmic procedure. Such a procedure can be exhibited in the case of Gel'fand–Zakharevich systems admitting an affine structure as we will show later.

As we have seen, the eigenspaces of $\mathcal{N}^*$ plays a relevant role for the separation of variables. This leads us to introduce the following definition:

Definition 2.2.6 *A function f on an $\omega\mathcal{N}$ manifold is said to be a* Stäckel function *(relative to the eigenvalue λ_i of $\mathcal{N}$) if*

$$\mathcal{N}^* df = \lambda_i df \tag{2.2.4}$$

The following property of Stäckel functions, which also explains their name, will be very useful later.

Proposition 2.2.7 *Let M be a semisimple $\omega\mathcal{N}$ manifold. A function f on M is a Stäckel function relative to λ_i if and only if, in any system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of DN coordinates, f depends only on x_i and y_i.*

Proof. Clearly, if $f = f(x_i, y_i)$ then $\mathcal{N}^* df = \lambda_i df$. Conversely, if (2.2.4) holds, then df belongs to the λ_i–eigenspace of $\mathcal{N}^*$, so that df is a linear combination of dx_i and dy_i and therefore f depends only on x_i and y_i □

Using proposition 2.2.7 and the implicit function theorem is not difficult to show that

Proposition 2.2.8 *Let M be a semisimple ωN manifold. If ρ and σ are two Stäckel function relative to the same eigenvalue of $\mathcal{N}$ λ_i, then their Poisson bracket has the form:*

$$\{\rho, \sigma\}_0 = f(\rho, \sigma) \tag{2.2.5}$$

Suppose that on a semisimple $\omega\mathcal{N}$ manifold M of dimension $2n$ we are able to find n pairs of functions $\rho_i, \sigma_i,\ i = 1, \ldots, n$ satisfying:

$$\mathcal{N}^* d\rho_i = \lambda_i d\rho_i \qquad \mathcal{N}^* d\sigma_i = \lambda_i d\sigma_i$$

then by proposition (2.2.8) the only non zero Poisson brackets will be of the form:

$$\{\rho_i, \sigma_j\}_0 = f_i(\rho_i, \sigma_i)\delta_{ij} \tag{2.2.6}$$

for some collection of functions f_i. Suppose that on a point m_0 of M all the functions f_i are different from zero. Then there will be a neighborhood U of m_0 where this will be still true. For $m \in U$, equations (2.2.6) entail the functional independence of the functions ρ_i, σ_i, so that they can be considered as local coordinates on M.

Let $(H_1, \ldots, H_n)$ be a set of functionally independent Hamiltonians in bi–involution. Then by theorem 2.2.5 they are separable in some set of DN coordinates $x_i, y_i,\ i = 1, \ldots, n$. Then, for $m \in U$ it will exist a change of variables from the coordinates x_i, y_i to the coordinates ρ_i, σ_i so that separation relations of the form (2.2.1) hold for the variables ρ_i, σ_i as well. However we cannot say that the system is separable also in the variables ρ_i, σ_i because we are missing canonicity. This problem can be solved replacing, for example, the functions σ_i with the new functions

$$\mu_i(m) = \int_{\rho_i(m_0)}^{\rho_i(m)} \frac{1}{f_i(\lambda_i, x)} dx \qquad m \in U$$

It is clear that the new variables ρ_i, μ_i will have canonical Poisson brackets.

2.2.2 Separation of variables on affine Gel'fand–Zakharevich manifolds

So far, the theory of separability for bihamiltonian systems has been developed for the special class defined on $\omega\mathcal{N}$ manifolds introduced in the previous section. In the case of bihamiltonian systems of Gel'fand Zakharevich type one has to pursue (when possible) the following strategy:

1. consider a symplectic leaf $\mathcal{S}$ of one of the two Poisson tensors (let us say P_0)
2. take the natural restriction of P_0 to $\mathcal{S}$: $P_0|_{\mathcal{S}}$
3. define a "deformation" $\widetilde{P}_1$ of the second Poisson tensor P_1 in such a way that $\widetilde{P}_1$ is still compatible with P_0 and, moreover, can be restricted to the same symplectic leaf $\mathcal{S}$ of P_0
4. define the restriction $\widetilde{P}_1|_{\mathcal{S}}$ of $\widetilde{P}_1$ to $\mathcal{S}$
5. use the bihamiltonian theory of Sov on the $\omega\mathcal{N}$ manifold defined by $\mathcal{S}$ with the compatible Poisson tensors $P_0|_{\mathcal{S}}$ and $\widetilde{P}_1|_{\mathcal{S}}$.

First of all we illustrate the reduction procedure.

To concoct out of P_1 the suitable deformation $\widetilde{P}_1$, one can adopt the following strategy:

First one fixes a complete set $C_1, \dots, C_k$ of Casimirs of P_0, and considers the first vector fields of the Lenard chains associated with C_i, i.e.,

$$X_a = P_1 dC_a, a = 1, \dots, k.$$

Then one considers a distribution $\mathcal{Z}$ that is transversal to the symplectic leaves of P_0. For any basis $W_1, \dots, W_k$ in $\mathcal{Z}$, thanks the transversality condition, the matrix

$$[G_0]^i_j = \mathrm{Lie}_{W_j}(C_i) \tag{2.2.7}$$

is nonsingular (say on an open set $U \subset M$). So, the tensor defined by

$$\widetilde{P}_1 = P_1 - \sum_{i,j} X_i \wedge [G_0^{-1}]_{i,j} W_j \tag{2.2.8}$$

is well defined and restricts to the generic symplectic leaf $\mathcal{S}$ of P, since, by construction, $\widetilde{P}_1 dC_j = 0$, $j = 1, \ldots, k$. Notice that, if we define a new basis in $\mathcal{Z}$ by

$$Z_i = \sum_j [G_0^{-1}]^i_j W_j \text{ so that } \mathrm{Lie}_{Z_i}(C_j) = \delta_{i,j}, \tag{2.2.9}$$

the expression of the deformed tensor $\widetilde{P}_1$ simplifies to

$$\widetilde{P}_1 = P_1 - \sum_i X_i \wedge Z_i. \tag{2.2.10}$$

We will call a basis of $\mathcal{Z}$ satisfying (2.2.9) a *normalized basis for the transversal distribution.*

The proof of the following Proposition can be found in [20]

Proposition 2.2.9 *Let $(M, P_1 - \lambda P_0)$ be a $2n+k$ bihamiltonian manifold with corank(P_0) $= k$, and suppose that there exists a distribution $\mathcal{Z} \subset TM$ of dimension k, s.t.:*

1. *$\mathcal{Z}$ intersect transversally the symplectic foliation of P_0.*
2. *the space of functions invariant under $\mathcal{Z}$ is a Poisson subalgebra for the whole pencil $(M, P_1 - \lambda P_0)$.*

Then, if $\widetilde{P}_1$ is the deformation of P_1 defined by (2.2.8), $\widetilde{P}_1 - \lambda P_0$ is still a Poisson pencil on M, and its restriction endows the generic symplectic leaves of P_0 with the structure of a ωN manifold.

□

Having described the reduction procedure we are going to use, we have to define affine Gel'fand–Zakharevich manifolds.

Definition 2.2.10 *We say that a bihamiltonian manifold $(M, P_1 - \lambda P_0)$, endowed with a transversal distribution $\mathcal{Z}$ satisfying the assumptions of Proposition 2.2.9 "admits an affine structure" if it is possible to choose a complete set of Casimir of P_0, and a corresponding basis of* normalized flat generators *Z_b, $b = 1, \ldots$, corank (P_0) in $\mathcal{Z}$ such that, for every Casimir of the Poisson pencil $H^a(\lambda)$ and every b, c one has, in addition to Equation (2.2.9)*

$$Lie_{Z_b} Lie_{Z_c}(H^a(\lambda)) = 0. \tag{2.2.11}$$

The notion of affine structure for a bihamiltonian manifold was clarified in [21] in connection with the problem of the Stäckel separability of bihamiltonian systems. For the purposes of the present thesis, we remark that an affine Poisson pencil satisfies some special properties, to be illustrated in the following.

Let $(M, P_1 - \lambda P_0)$ be a corank k affine bihamiltonian manifold, and let Z_a, $a = 1, \ldots, k$ be a set of flat generators for the transversal distribution $\mathcal{Z}$. Let us consider the polynomial Casimirs

$$H^{(a)}(\lambda) = \lambda^{n_a} H_0^a + \cdots + H_{n_a}^a, \qquad a = 1, \ldots, k \tag{2.2.12}$$

and their deformations along the flat generators, that is, the k^2 polynomials

$$D_b^a(\lambda) = \mathrm{Lie}_{Z_b} H^{(a)}(\lambda) = \lambda^{n_a} \delta_a^b - D_{b,1}^a \lambda^{n_a - 1} - \cdots - D_{b,n_a}^a. \tag{2.2.13}$$

The polynomials $D_b^a(\lambda)$ are invariant along $\mathcal{Z}$, so that they can be considered as functions on the generic symplectic leaves of P_0. They satisfy remarkable differential relations. Indeed it holds:

Proposition 2.2.11 *The actions of $\widetilde{P}_1$ and P_0 on the deformation of the Casimirs of the pencil are related by the following formula:*

$$\widetilde{P}_1 dD_b^a(\lambda) = \lambda P_0 dD_b^a(\lambda) + \sum_{c=1}^{k} D_c^a(\lambda) P_0 dD_{b,1}^c. \tag{2.2.14}$$

Proof. We consider the characteristic property of a Casimir of the Poisson pencil,

$$P_\lambda dH^{(a)}(\lambda) = 0$$

and derive it w.r.t. Z_b. We get:

$$\mathrm{Lie}_{Z_b}(P_\lambda)\, dH^{(a)}(\lambda) + P_\lambda dD_b^a(\lambda) = 0. \tag{2.2.15}$$

Since $\mathrm{Lie}_{Z_b}(P_\lambda) = \sum_c [Z_b, X_c^1] \wedge Z_c$ with $X_c^1 = P_1 dH_0^c = P_0 dH_1^c$, whence $[Z_b, X_c^1] = -P_0 dD_{b,1}^c$, we see that eq.(2.2.15) takes the form

$$P_\lambda dD_b^a(\lambda) - \sum_c D_c^a(\lambda) P_0 dD_{b,1}^c - \sum_c \langle [Z_b, X_c^1], dH^{(a)}(\lambda)\rangle \cdot Z_c = 0. \tag{2.2.16}$$

Finally, noticing that $\{H_1^c, H^{(a)}(\lambda)\}_0 = 0$ and using the hypothesis of affinity of the GZ manifold, we see that the latter equation yields

$$P_\lambda dD_b^a(\lambda) - \sum_c D_c^a(\lambda) P_0 dD_{b,1}^c - \sum_c (X_c^1 \wedge Z_c) \cdot (dD_b^a(\lambda)) = 0, \tag{2.2.17}$$

which, in view of (2.2.10), yields the statement.

□

In the next proposition we show that for affine Gel'fand–Zakharevich manifolds we can use the matrix $D(\lambda)$ to find a set of coordinates (λ_i, ρ_i) such that, when restricted to the symplectic leaves $\mathcal{S}$ of P_0, they satisfies:

$$\mathcal{N}^* d\lambda_i = \lambda_i d\lambda_i \qquad \mathcal{N}^* d\rho_i = \lambda_i d\rho_i$$

where the tensor $\mathcal{N}$ is defined as $\mathcal{N} = \widetilde{P}_1|_S (P_0|_S)^{-1}$. As we argued at the end of the previous section, such a property almost solve the SoV problem for the Hamiltonians $(H_0^a, \dots, H_{n_a}^a)_{a=1}^k$.

Proposition 2.2.12 *Let $D(\lambda)$ be a $k \times k$ polynomial matrix of the form*

$$D_b^a(\lambda) = \lambda^{n_a} \delta_a^b - \lambda^{n_a - 1} D_{b,1}^a - \dots - D_{b,n_a}^a, \; a, b = 1, \dots, k, \qquad (2.2.18)$$

where the $D_{b,p}^a$ are smooth independent functions on a $\omega\mathcal{N}$ manifold M, satisfying equation (2.2.14) and let us denote with $\widetilde{D}(\lambda)$ its classical adjoint matrix. Then:

1. *The determinant $\Delta(\lambda) = \det D(\lambda)$ is a monic polynomial of degree $\nu = \sum_a n_a$*

$$\Delta(\lambda) = \lambda^\nu - \lambda^{\nu-1}\Delta_1 - \dots - \Delta_\nu$$

   and satisfies

$$P_1 d\Delta(\lambda) = \lambda P_0 d\Delta(\lambda) + \Delta(\lambda) P_0 d\Delta_1. \qquad (2.2.19)$$

2. *The roots λ_i of $\Delta(\lambda)$ satisfy*

$$P_1 d\lambda_i = \lambda_i P_0 \, d\lambda_i,$$

3. *Any ratio $\rho(\lambda) := \widetilde{D}_b^a(\lambda)/\widetilde{D}_c^a(\lambda)$ (provided they are not identically vanishing or divergent) of elements belonging to the same row "a" of $\widetilde{D}(\lambda)$, evaluated at the roots λ_i of $\Delta(\lambda)$ satisfy the equation*

$$\widetilde{P}_1 d\rho(\lambda_i) = \lambda_i P_0 d\rho(\lambda_i). \qquad (2.2.20)$$

Proof. The meaning of the power expansion (2.2.18) is that the (a, a) entry of D_b^a is a monic degree n_a polynomial, while all other entries in the a-th row are of degree not exceeding $n_a - 1$. From this fact it follows:

$$\Delta(\lambda) = \prod_{a=1}^k D_a^a(\lambda) + O(\lambda^{\nu-2}), \text{ whence } \Delta_1 = \sum_{a=1}^k D_{a,1}^a. \qquad (2.2.21)$$

We multiply the matrix equation (2.2.14) say, on the left, by the classical adjoint $\widetilde{D}$ of D, to get

$$\sum_c \widetilde{P}_1 \widetilde{D}^a_c(\lambda) dD^c_b(\lambda) = \sum_c \lambda P_0 \widetilde{D}^a_c(\lambda) dD^c_b(\lambda) + \sum_{c,d} \widetilde{D}^a_c(\lambda) D^c_d(\lambda) P_0 dD^d_{b,1}. \tag{2.2.22}$$

Recalling that $\sum_c \widetilde{D}^a_c(\lambda) D^c_d(\lambda) = \delta^a_d \Delta(\lambda)$, equation (2.2.22) becomes:

$$\sum_c \widetilde{P}_1 \widetilde{D}^a_c(\lambda) dD^c_b(\lambda) = \sum_c \lambda P_0 \widetilde{D}^a_c(\lambda) dD^c_b(\lambda) + \Delta(\lambda) P_0 dD^a_{b,1} \tag{2.2.23}$$

Now taking the trace in equation (2.2.23) and using the identity $\mathrm{Tr}(\widetilde{D}(\lambda) dD(\lambda)) = d\Delta(\lambda)$, and the equation (2.2.21) we get the proof of the first item.

To prove item number 2, we first notice that

$$d(\Delta(\lambda_i)) = d\Delta(\lambda)\Big|_{\lambda=\lambda_i} + \frac{\partial \Delta(\lambda)}{\partial \lambda}\Big|_{\lambda=\lambda_i} d\lambda_i, \tag{2.2.24}$$

where $d\Delta(\lambda) = -\sum_{j=1}^{\nu} \lambda^{\nu-j} d\Delta_j$. Taking into account the relation (2.2.19), we get

$$(P_1 - \lambda P_0) d\Delta(\lambda_i) = \Delta(\lambda_i) P_0 d\Delta_1 = 0 \tag{2.2.25}$$

and substituting equation (2.2.24) in the first member of equation (2.2.25)

$$0 = (P_1 - \lambda P_0) d\Delta(\lambda_i) = (P_1 - \lambda P_0) d\Delta(\lambda)\Big|_{\lambda=\lambda_i} + \frac{\partial \Delta(\lambda)}{\partial \lambda}\Big|_{\lambda=\lambda_i} (P_1 - \lambda_i P_0) d\lambda_i, \tag{2.2.26}$$

Using again equation (2.2.19) we get

$$\Delta(\lambda_i) P_0 d\Delta_1 + \frac{\partial \Delta(\lambda)}{\partial \lambda}\Big|_{\lambda=\lambda_i} (P_1 - \lambda_i P_0) d\lambda_i = 0 \tag{2.2.27}$$

which implies the assertion, since

$$\Delta(\lambda_i) = 0, \text{ while } \frac{\partial \Delta(\lambda)}{\partial \lambda}\Big|_{\lambda=\lambda_i} \neq 0,$$

thanks to the fact that being the coefficients $\Delta_{a,i}$ functionally independent, the roots wil be generically simple.

The proof of the third assertion is basically contained in the proof of Proposition 8.4 of [21]. We limit ourselves to sketch it. By using the relations (2.2.14) and (2.2.19), together with the defining relation $\widetilde{D}^a_c(\lambda) D^c_d(\lambda) = \delta^a_d \Delta(\lambda)$, one arrives at the matrix equation

$$\sum_c \left(\widetilde{P}_1 d\widetilde{D}^a_c - \lambda P_0 d\widetilde{D}^a_c\right) D^c_b = \Delta(\lambda) P_0 \left(d\Delta_1 \delta^a_b - dD^a_{1,b}\right) \tag{2.2.28}$$

Taking once again into account that $\Delta(\lambda)$ has simple eigenvalues, we see that each "row" $(\widetilde{P}_1 d\widetilde{D}^a_c - \lambda P_0 d\widetilde{D}^a_c)$, evaluated at $\lambda = \lambda_i$ must be proportional to the corresponding row of $\widetilde{D}$, that is, there must exist vector fields $X^{(i)}_c$ in TM such that

$$(\widetilde{P}_1 d\widetilde{D}^a_c - \lambda P_0 d\widetilde{D}^a_c)|_{\lambda=\lambda_i} = X^{(i)}_c \otimes \widetilde{D}^a_c|_{\lambda=\lambda_i}, \quad a = 1, \ldots, k.$$

If we consider the ratios of elements of the same row, as in the statement number 3, we see that the correponding vector $X^{(i)}_c$ must vanish, whence the thesis.

□

We give some comments on the content of proposition 2.2.12. Point number 1 tells us that the number of solutions λ_i to the equation $\Delta(\lambda_i) = 0$ is exactly equal to the number of Hamiltonians functions one obtains by the polynomial Casimirs (2.2.12). Point number 2 state that the restriction to the symplectic leaves of P_0 of the λ_i are the eigenvalues of the Nijenhuis tensor $\mathcal{N}$. Finally point number 3 give us a method to complete the set λ_i with another set of Stäckel functions.

2.3 Lie–Poisson brackets

In the present thesis, we will apply these and other ideas to *linear* Poisson brackets. Such Poisson brackets can always be seen as Lie–Poisson brackets defined on C^∞ functions on the dual of a Lie algebra $\mathfrak{g}$. We give a short review of Lie–Poisson brackets also in order to fix notations and conventions to be used later on.

Let us denote with $\mathfrak{g}$ a Lie algebra, with $\mathfrak{g}^*$ its dual and with $\langle \cdot, \cdot \rangle$ a non degenerate pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. It is well known that it is possible to endow the space $\mathcal{F}(\mathfrak{g}^*)$ of differentiable functions on $\mathfrak{g}^*$ with a Poisson bracket called "Lie–Poisson" bracket defined in the following way. The Lie–Poisson bracket of two functions $f, g \in \mathcal{F}(\mathfrak{g}^*)$ is given in the point $A \in g^*$ by:

$$\{f, g\}(A) = \langle A, [df|_A, dg|_A] \rangle \tag{2.3.1}$$

Since $df, dg \in \mathfrak{g}^{**} \simeq \mathfrak{g}$ the commutator in (2.3.1) is defined.

We give an explicit coordinate representation of the Poisson bracket (2.3.1). Let M be the dimension of $\mathfrak{g}$, let $X^1, \ldots, X^M$ be a set of generators for $\mathfrak{g}$ and $C^{\alpha\beta}_\gamma$ be the associated structure constants:

$$[X^\alpha, X^\beta] = C^{\alpha\beta}_\gamma X^\gamma$$

Using the pairing $\langle \cdot, \cdot \rangle$ we can identify the dual basis to X^α $\{X_\alpha\}_{\alpha=1}^M$ in $\mathfrak{g}^*$:

$$\langle X_\alpha, X^\beta \rangle = \delta^\beta_\alpha$$

A generic vector of $\mathfrak{g}^*$ can then be written in the form (using Einstein's summation convention):

$$A = y^\alpha X_\alpha$$

where y^α are the coordinate functions on $\mathfrak{g}^*$. By definition the differential of a coordinate function y^α (in the point A) is the element in $\mathfrak{g}^{**}$ satisfying

$$\langle A, dy^\alpha \rangle = y^\alpha \Longrightarrow dy^\alpha = X^\alpha$$

hence, we have:

$$\{y^\alpha, y^\beta\}(A) = \langle A, [dy^\alpha, dy^\beta] \rangle = \langle A, [X^\alpha, X^\beta] \rangle = C_\gamma^{\alpha\beta} \langle A, X^\gamma \rangle = C_\gamma^{\alpha\beta} y^\gamma$$

It follows that the Poisson tensor P corresponding to the Lie–Poisson bracket (2.3.1) can be written in the y coordinates as:

$$P = C_\gamma^{\alpha\beta} y^\gamma \frac{\partial}{\partial y^\alpha} \otimes \frac{\partial}{\partial y^\beta} \tag{2.3.2}$$

To make explicit computations is convenient to work in the fundamental representation ρ of $\mathfrak{g}$ taking the trace as a non–degenerate bilinear form. On a simple Lie algebra $\mathfrak{g}$ all non–degenerate bilinear form are proportional; it follows that the matrix

$$g^{\alpha\beta} = \mathrm{Tr}(\rho(X^\alpha)\rho(X^\beta)) \tag{2.3.3}$$

is a multiple of the standard Cartan–Killing metric and consequently is invertible. The dual element to a generator X^α is defined by:

$$X_\beta = g_{\alpha\beta}\rho(X^\alpha) \tag{2.3.4}$$

Then the generic element of $\mathfrak{g}^*$ is given by the matrix

$$A = y^\beta X_\beta = g_{\alpha\beta}\rho(X^\alpha) y^\beta \tag{2.3.5}$$

Now our aim is to express the Lie–Poisson bracket (2.3.1) in matrix form so that we can use, in a certain sense, the matrix A as a single coordinate instead of using the coordinates $\{y^\alpha\}_{\alpha=1}^M$.

Lemma 1 *The Lie-Poisson bracket (2.3.1) associate with the matrix A the vector field*

$$\mathrm{Lie}_{X_F}(A) = \dot{A} = \left[A, \frac{\partial F}{\partial A}\right] \tag{2.3.6}$$

where $\frac{\partial F}{\partial A}$ is defined as the matrix satisfying:

$$\mathrm{Lie}_{X_F}(G) = \mathrm{Tr}\left(\dot{A}\frac{\partial G}{\partial A}\right) \tag{2.3.7}$$

Proof: First of all we find the expression of $\frac{\partial G}{\partial A}$ in the coordinates y^α. Using formula (2.3.2) we have:

$$\dot{A}_{lm} = \text{Lie}_{X_F}(A)_{lm} = \{A_{lm}, F\} = C_\gamma^{\alpha\beta} y^\gamma \frac{\partial}{\partial y^\alpha} \left(g_{\delta\epsilon} \rho(X^\delta)_{lm} y^\epsilon \right) \frac{\partial F}{\partial y^\beta} =$$
$$= C_\gamma^{\alpha\beta} y^\gamma g_{\delta\alpha} \rho(X^\delta)_{lm} \frac{\partial F}{\partial y^\beta}$$

Hence we can write in matrix form:

$$\dot{A} = C_\gamma^{\alpha\beta} y^\gamma g_{\delta\alpha} \rho(X^\delta) \frac{\partial F}{\partial y^\beta} \tag{2.3.8}$$

Using again equation (2.3.2) we have:

$$\text{Lie}_{X_F}(G) = \{G, F\} = -C_\gamma^{\alpha\beta} y^\gamma \frac{\partial F}{\partial y^\alpha} \frac{\partial G}{\partial y^\beta} \tag{2.3.9}$$

On the other hand using (2.3.8)

$$\text{Tr}\left(\dot{A} \frac{\partial G}{\partial A} \right) = C_\gamma^{\alpha\beta} y^\gamma g_{\delta,\alpha} \frac{\partial F}{\partial y^\beta} \text{Tr}\left(\rho(X^\delta) \frac{\partial G}{\partial A} \right) \tag{2.3.10}$$

Equality between equation (2.3.9) and (2.3.10) holds provided that

$$\frac{\partial G}{\partial A} = \frac{\partial G}{\partial y^\alpha} \rho(X^\alpha) \tag{2.3.11}$$

Now we must verify that equation (2.3.6) indeed holds. Using (2.3.11) we have:

$$\left[A, \frac{\partial F}{\partial A} \right] = -\left[g_{\alpha\delta} \rho(X^\alpha) y^\delta, \frac{\partial F}{\partial y^\beta} \rho(X^\beta) \right] = -g_{\alpha\delta} C_\gamma^{\alpha\beta} y^\delta \rho(X^\gamma) \frac{\partial F}{\partial y^\beta} \tag{2.3.12}$$

Since the trace is an *ad*–invariant bilinear form, the metric tensor g satisfies the equality:

$$-g_{\alpha\delta} C_\gamma^{\alpha\beta} = g_{\alpha\gamma} C_\delta^{\alpha\beta} \tag{2.3.13}$$

Substituting equation (2.3.13) inside (2.3.12) we see that formula (2.3.6) follows. □

2.3.1 Poisson brackets on direct product of Poisson algebras

If $(P, \{\cdot,\cdot\}_P)$ and $(Q, \{\cdot,\cdot\}_Q)$ are two Poisson algebras, then on the direct product $P \otimes Q$ is naturally defined a Poisson bracket given by:

$$\{a \otimes b, c \otimes d\}_{P \otimes Q} = \{a, c\}_P \otimes bd + ac \otimes \{b, d\}_Q \forall a, c \in P \quad \forall b, d \in Q \tag{2.3.14}$$

Let us call $\mathcal{P}_{\mathfrak{g}}$ the Poisson algebra defined on the dual of a Lie algebra $\mathfrak{g}$. From equation (2.3.14) it follows that we can define a Poisson bracket on the direct product of N–copies of the Lie–Poisson algebra associated with a simple Lie algebra $\mathfrak{g}$. Let us denote with

$$\mathcal{G} = \overbrace{\mathcal{P}_{\mathfrak{g}} \otimes \mathcal{P}_{\mathfrak{g}} \otimes \cdots \otimes \mathcal{P}_{\mathfrak{g}}}^{N} \tag{2.3.15}$$

the resulting Poisson algebra. If $C_{\gamma}^{\alpha\beta}$ are the structure constants of the Lie algebra $\mathfrak{g}$ then the Poisson tensor $P_{\mathcal{G}}$ on $\mathcal{G}$ is given by:

$$P_{\mathcal{G}} = \sum_{l=1}^{N} C_{\gamma}^{\alpha\beta} y_l^{\gamma} \frac{\partial}{\partial y_l^{\alpha}} \otimes \frac{\partial}{\partial y_l^{\beta}} \tag{2.3.16}$$

where y_l^{α} denotes the α–th coordinate on the l–th copy of $\mathcal{P}_{\mathfrak{g}}$. Defining the matrices

$$A_i = g_{\alpha\beta}\rho(X^{\alpha})y_i^{\beta} \quad i = 1, \ldots, N \tag{2.3.17}$$

where again ρ represent the fundamental representation of $\mathfrak{g}$, the Poisson tensor (2.3.16) turns out to be equivalent to the one defined by its Hamiltonian vector fields:

$$\dot{A}_i = \left[A_i, \frac{\partial F}{\partial A_i}\right] \tag{2.3.18}$$

where, if $Y = (Y_i, \ldots, Y_N)$ represents a tangent vector to M,the elements $\dfrac{\partial F}{\partial A_i} \in \rho(\mathfrak{g})$, $i = 1, \ldots, N$ are the elements defined by means of the expression of the Lie derivative of F w.r.t. Y as

$$\mathrm{Lie}_Y(F) = \sum_{i=1}^{N} \mathrm{Tr}\big(Y_i \cdot \frac{\partial F}{\partial A_i}\big). \tag{2.3.19}$$

We will often write the standard product Lie-Poisson tensor P (and other Poisson tensors) representing its action on the differential of a function by means of the matrix symbolic form:

$$\begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \vdots \\ \dot{A}_N \end{pmatrix} = \begin{pmatrix} [A_1, .] & 0 & \ldots & 0 \\ 0 & [A_2, .] & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & [A_N, .] \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial F}{\partial A_1} \\ \dfrac{\partial F}{\partial A_2} \\ \vdots \\ \dfrac{\partial F}{\partial A_N} \end{pmatrix}. \tag{2.3.20}$$

For this reason, we will term the standard Lie-Poisson tensor P the *diagonal* Poisson tensor.

2.4 Few remarks on finite dimensional quantum Hamiltonian systems

2.4.1 Quantum Hamiltonian systems and the problem of quantum integrability

A quantum Hamiltonian system is defined by an Hilbert space $\mathcal{H}$ and a set of self–adjoint operators $\{X^\alpha\}_{\alpha=1}^N$ on $\mathcal{H}$ (that determines the set of observables on $\mathcal{H}$) and a particular element of the set of observables $H(X^1, \ldots, X^N)$ that is called the *Hamiltonian* of the system. The set of observables is given by self–adjoint functions of the operators X^α. In this thesis we will work only with finite dimensional quantum systems, i.e., with the case in which the Hilbert space $\mathcal{H}$ is finite dimensional; many of the properties that we will state hold only in this framework. In such case, self–adjoint operators on $\mathcal{H}$ form an algebra with respect to two operations defined through operator composition: the anticommutator

$$A \circ B = \frac{1}{2}(AB + BA) \tag{2.4.1}$$

and the commutator

$$[A, B] = i\hbar\,(AB - BA) \tag{2.4.2}$$

In formula (2.4.2) $\hbar$ denotes a real parameter.

We notice that the anticommutator is a commutative non-associative product, while the commutator is an antysimmetric bilinear application satisfying the Leibnitz rule:

$$[A \circ B, C] = A \circ [B, C] + [A, C] \circ B \tag{2.4.3}$$

and Jacobi identity. We would like to stress the close similarity between the algebraic structure of the algebra of observables and that of Poisson algebras (see section 2.1).

Once given a quantum Hamiltonian system, the most interesting problem to solve from a physical point of view, is that of determining the spectrum and the eigenvectors of the Hamiltonian H. It may happens that the spectrum of H is degenerate; then diagonalizing H is not enough to give a complete description of the system and this aim can be reached only diagonalizing a larger number of observables. Since two observables can be simultaneously diagonalized only if they commute (in which case they are said to be *compatible*), the problem of finding families of commuting observables has a great importance in quantum physics. In particular a set of functionally independent[1] compatible

[1] For a definition of functional independence of compatible observables see e.g. [11]

observables is said to form a *complete set of compatible observables* if their common eigenvalues are non–degenerate.

The problem of finding families of compatible observables can be seen, in a certain sense, as the quantum analog of the problem of finding involutive families of functions in classical Hamiltonian systems. Thanks to the above-mentioned similarity between the algebraic structure of quantum and classical integrable systems , many notions and tools of the theory of classical integrable systems has been translated in the framework of quantum systems and viceversa (Lax matrices, r-matrix approach, etc.). Unfortunately, the notion itself of (Liouville) integrability is hardly generalizable to quantum systems. We recall that a classical Hamiltonian with N degrees of freedom is said to be integrable if it is possible to find N functionally independent Poisson commuting functions such that the Hamiltonian is functionally dependent on them. One could be tempted (as is done in many papers) to translate this notion in the most natural way, by saying that a quantum Hamiltonian system with N degrees of freedom [2] is integrable if it possesses N commuting functionally independent observables such that the Hamiltonian is functionally dependent on them. The trouble with this definition is that, while in the classical case the notion of functional independence (of a set of functions) is equivalent to a notion of linear independence (of their differentials), this does not happen in the quantum case. So, while in the classical case if $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ are two distinct sets of functionally independent functions and we denotes with F and G, respectively, the set of functions functionally dependent on them, then $F = G$ implies that $m = n$, this is nomore true in the quantum case. A typical example of such phenomenon is given by a theorem of Von Neumann, that state that if $A_1, \ldots, A_n$ is a set of functionally independent compatible observables , then there exist an observable A and n measurable functions f_i such that

$$A_i = f_i(A)$$

So, for quantum systems, the maximum number of functionally independent compatible observables depend on the choice of the observables and does not appear to be related to the number of degrees of freedom of the system.

On the other hand, in some cases (such as in the case of $su(2)$–rational Gaudin model) one is able to find a complete set of compatible observables whose number is exactly equal to the number of degrees of freedom and to diagonalize them simultaneously. Hence it appears natural to say that such systems are quantum integrable.

[2]The number f of degrees of freedom can be defined in the quantum case in complete analogy with the classical case, saying that if d is the number of operators X^α defining the algebra of observables and c is the dimension of the center of such algebra, then $f = \frac{d-c}{2}$.

2.4.2 Quantum systems and Universal Enveloping Algebras

If $\mathfrak{g}$ is a Lie algebra and it is possible to define a finite dimensional representation of it by means of self–adjoint operators (such as in the case of $\mathfrak{g} = su(2)$ using Pauli matrices), then on $\rho(\mathfrak{g})$ it is possible to define a finite dimensional quantum system simply taking as Hamiltonian a generic self–adjoint function of the generators of $\rho(\mathfrak{g})$; the Hilbert space of the system will be simply the vector space V defined by the representation ρ. If one is interested in algebraic properties (such as commutativity) of elements in $\rho(\mathfrak{g})$ it is more convenient to work with the Universal Enveloping Algebra (UEA) $U(\mathfrak{g})$ of $\mathfrak{g}$.

Given a Lie algebra $\mathfrak{g}$ its UEA $U(\mathfrak{g})$ is defined in the following way: one defines a formal associative product m on $\mathfrak{g}$ [3] that we denote by

$$m(X^\alpha, X^\beta) = X^\alpha X^\beta \qquad X^\alpha, X^\beta \in \mathfrak{g}$$

The associative algebra generated by the generators of $\mathfrak{g}$ through the formal product is usually denoted by $T(\mathfrak{g})$. Now $U(\mathfrak{g})$ is defined as the quotient of $T(\mathfrak{g})$ by the bilateral ideal $I(\mathfrak{g})$ generated by the elements:

$$X^\alpha X^\beta - X^\beta X^\alpha - C^{\alpha\beta}_\gamma X^\gamma \qquad X^\alpha, X^\beta, X^\gamma \in \mathfrak{g}$$

where $C^{\alpha\beta}_\gamma$ are the structure constants of the Lie algebra $\mathfrak{g}$. The Lie bracket of $\mathfrak{g}$ can be extended to all $U(\mathfrak{g})$ defining:

$$[X, Y] = XY - YX \qquad X, Y \in U(\mathfrak{g})$$

so that on $U(\mathfrak{g})$ two operations are defined: an associative product and a Lie bracket, the two structures being intertwined by the Leibnitz rule.

The fundamental property of UEA's, whereas their name, is the following. Given any representation ρ of $\mathfrak{g}$ there exists a map $\phi : U(\mathfrak{g}) \to \rho(\mathfrak{g})$ that is a morphism both with respect to the product and the commutator:

$$\begin{aligned} \phi(XY) &= \phi(X)\phi(Y) \\ \phi([X, Y]) &= [\phi(X), \phi(Y)] \end{aligned} \qquad X, Y \in U(\mathfrak{g})$$

So if two elements commutes at the level of the UEA $U(\mathfrak{g})$, they will commute in any representation of $\mathfrak{g}$.

Another interesting feature of UEA's is that, given two UEA's $U(\mathfrak{g}_1)$ and $U(\mathfrak{g}_2)$, one can naturally define a Lie bracket and an associative product on the

[3] We use a formal product in the definition of $U(\mathfrak{g})$ instead of the usual tensor product since in the following we will be interested in tensor products of UEA's, so we want to eliminate possible sources of confusion among the two operations.

tensor product $U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$ as:

$$(X \otimes Y)(Z \otimes W) = XZ \otimes YW \qquad X, Z \in U(\mathfrak{g}_1) \quad Y, W \in U(\mathfrak{g}_2)$$
$$[X \otimes Y, Z \otimes W] = [X, Z] \otimes YW + ZX \otimes [Y, W]$$

With the above notions of product and Lie bracket, also the Leibnitz rule is verified in $U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$, in particular it can be proven that it holds $U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2) \simeq U(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$.

2.4.3 Quantization of Lie–Poisson algebras

One of the more relevant notions originated from the similar algebraic structure of classical and quantum systems (see section 2.4.1) is that of *quantization* of a Poisson algebra. Informally speaking [4] a quantization of a Poisson algebra $\mathcal{P}$ is a map $Q_\hbar$ depending on a real parameter $\hbar$ that assign to any element in $\mathcal{P}$ a self–adjoint operator defined on a given Hilbert space $\mathcal{H}$, in such a way that $Q_\hbar$ is, up to $O(\hbar)$ terms, an homomorphism of the corresponding product and Lie bracket structures, i.e., it holds for any $f, g \in \mathcal{P}$:

$$\lim_{\hbar \to 0} \left(Q_\hbar(f) \circ Q_\hbar(g) - Q_\hbar(fg)\right) = 0 \tag{2.4.4}$$
$$\lim_{\hbar \to 0} \left([Q_\hbar(f), Q_\hbar(g)]_\hbar - i\hbar Q_\hbar(\{f, g\})\right) = 0 \tag{2.4.5}$$

Given a Lie algebra $\mathfrak{g}$ it does exist a quantization between the Poisson algebra of polynomials on $\mathfrak{g}^*$ with the usual Lie–Poisson structure $Pol(\mathfrak{g}^*)$ and (hermitean representations) of its UEA $U(\mathfrak{g})$.

Indeed, let $\{X^\alpha\}_{\alpha=1}^{\dim(\mathfrak{g})}$ be a set of generators of $\mathfrak{g}$ and $\{y^\alpha\}_{\alpha=1}^{\dim(\mathfrak{g})}$ the corresponding coordinate functions on the dual $\mathfrak{g}^*$ as in section 2.3. Let us rescale the commutator in $U(\mathfrak{g})$ by putting:

$$[X, Y]_\hbar = i\hbar\, [X, Y]$$

Then a map $Q_\hbar$ between $Pol(\mathfrak{g}^*)$ and $U(\mathfrak{g})$ satisfying (2.4.4) and (2.4.5) is defined by:

$$Q_\hbar(g_{i_1 \dots i_N} y^{i_1} \dots y^{i_N}) = \frac{1}{n(p)} \sum_{\{p\}} g_{p(i_1) \dots p(i_N)} X^{p(i_1)} \dots X^{p(i_N)}$$

where the sum is over all the permutations of the indices $i_1, \dots, i_N$ and $n(p)$ is the total number of such permutations. In other words, to quantize a polynomial one has to replace the coordinates y^α with the corresponding generators X^α and symmetrize.

[4]For a more formal definition see e.g. [41].

Property (2.4.4) follows immediately from the fact that the two terms $Q_\hbar(f) \circ Q_\hbar(g)$ and $Q_\hbar(fg)$ differ only for the ordering of the generators X^α. Using the fact that the structure constants are the same for the generators X^α and the coordinates y^α (see section 2.3):

$$[X^\alpha, X^\beta] = C^{\alpha\beta}_\gamma X^\gamma$$
$$\{y^\alpha, y^\beta\} = C^{\alpha\beta}_\gamma y^\gamma$$

and the Leibnitz rule, it's easy to show that the terms $[Q_\hbar(f), Q_\hbar(g)]_\hbar$ and $i\hbar Q_\hbar(\{f, g\})$ differ again only by ordering, so that also property (2.4.5) is satisfied.

Finally, since symmetric polynomials in the generators of $\mathfrak{g}$ span all the UEA $U(\mathfrak{g})$, we have that the image of $Pol(\mathfrak{g}^*)$ under $Q_\hbar$ is exactly $U(\mathfrak{g})$. In the following we will always pose $\hbar = 1$.

Chapter 3

A brief "history" of rational Gaudin models

Rational Gaudin models were introduced by M. Gaudin in 1976 [25] (see also [26]) as a class of exactly solvable quantum spin Hamiltonians. Gaudin showed that the spin operators

$$H_j = \sum_{l=1, l\neq j}^{N} \frac{\vec{\sigma}_j \cdot \vec{\sigma}_l}{\epsilon_j - \epsilon_l} \qquad j = 1, \dots, N \tag{3.0.1}$$

are mutually commuting and computed explicitly the eigenvalues and eigenvectors that diagonalize simultaneously the family (3.0.1). In equation (3.0.1) $\vec{\sigma_j}$ denotes the vector of matrices $(\sigma_j^1, \sigma_j^2, \sigma_j^3)$ where σ_j^k refer to the k-th Pauli matrix acting on the j-th spin:

$$\sigma_j^k = \overbrace{1 \otimes 1 \otimes \cdots \otimes 1}^{j-1} \otimes \sigma^k \otimes \overbrace{1 \otimes 1 \otimes \cdots \otimes 1}^{N-j} \qquad j = 1, \dots, N \qquad k = 1, 2, 3$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The N operators (3.0.1) are not all independent since it holds:

$$\sum_{j=1}^{N} H_j = 0$$

so that we can choose from (3.0.1) only $N-1$ independent operators. The simultaneous diagonalization of the operators (3.0.1) does not eliminate completely the degeneration of their spectra, so that to get a complete set of observables, one has to add to the family (3.0.1) an N-th commuting operator, given, for example, by the third component of the total spin:

$$S^3 = \sum_{j=1}^{N} \sigma_j^3$$

Out of the operators (3.0.1) Gaudin constructed the further operator

$$\mathcal{H} = \sum_{j<l} \frac{\eta_j - \eta_l}{\epsilon_j - \epsilon_l} \vec{\sigma_j} \cdot \vec{\sigma_l} = \sum_{j=1}^{N} \eta_j H_j \tag{3.0.2}$$

selecting it as Hamiltonian of the system. The reason of this choice is that while the H_j describes the interaction of a selected spin with the others, $\mathcal{H}$ describes the interaction of any spin with each other.

Gaudin himself pointed out that the system defined by the Hamiltonians (3.0.1) is amenable to various generalizations. First of all one can replace in (3.0.1) the Pauli matrices with the corresponding elements of an arbitrary $sl(2)$ representation; in fact Hamiltonians (3.0.1) can be seen as the images under a spin 1/2 representation of the elements of $U(sl(2))^{\otimes N}$ given by:

$$I_j = \sum_{l=1, l\neq j}^{N} \sum_{\alpha\beta=1}^{3} \frac{g_{\alpha,\beta} X_j^\alpha X_l^\beta}{\epsilon_j - \epsilon_l} \tag{3.0.3}$$

where X^1, X^2, X^3 denote the 3 $sl(2)$ generators and the metric $g^{\alpha\beta}$ is given by equation (2.3.3). Gaudin proved commutativity of the I_j at the level of the Universal Enveloping Algebra, showing in this way that the images of the I_j under any representation give a set of commuting operators.

Another straightforward generalization can be obtained substituting to the Lie algebra $sl(2)$ any semisimple Lie algebra $\mathfrak{g}$. The corresponding expression for the integrals I_j becomes:

$$\mathcal{I}_j = \sum_{l=1, l\neq j}^{N} \sum_{\alpha,\beta=1}^{\dim(\mathfrak{g})} \frac{g_{\alpha\beta} X_j^\alpha X_l^\beta}{\epsilon_j - \epsilon_l} \tag{3.0.4}$$

where $X^1, \ldots, X^{\dim(\mathfrak{g})}$ are a set of generators for the Lie algebra $\mathfrak{g}$. The commutativity of (3.0.4) can be again proved at the level of the Universal Enveloping Algebras, consequently giving us a set of $N-1$ commuting operators in any representation. The Gaudin Hamiltonian H_G is defined analogously to (3.0.2) as:

$$H_G = \sum_{j=1}^{N} \eta_j \mathcal{I}_j = \sum_{j,l=1, l\neq j}^{N} \sum_{\alpha,\beta=1}^{\dim(\mathfrak{g})} \frac{\eta_j - \eta_l}{\epsilon_j - \epsilon_l} g_{\alpha\beta} X_j^\alpha X_l^\beta \tag{3.0.5}$$

Later on, the rational Gaudin model was studied by Sklyanin [55] and Jurčo [33] from the point of view of the Quantum Inverse Scattering Method [39]. In particular, Sklyanin studied the $su(2)$ rational Gaudin model, diagonalizing the operators (3.0.1) by means of separation of variables and underlining the connections between his procedure and the functional Bethe Ansatz.

Jurčo noticed that the integrals (3.0.4) could be obtained, for any semisimple Lie algebra $\mathfrak{g}$, as spectral invariants of a Lax matrix of the following form:

$$L^q(\lambda) = \sum_{i=1}^{N} \frac{A_i^q}{\lambda - \epsilon_i} \tag{3.0.6}$$

$$A_i^q = \sum_{\alpha,\beta=1}^{\dim(\mathfrak{g})} g_{\alpha\beta}\rho(X^\alpha)X_i^\beta \tag{3.0.7}$$

where again ρ denotes the fundamental representation of $\mathfrak{g}$. Hence the matrix $L(\lambda)$ belong to the space $\rho(\mathfrak{g}) \otimes U(\mathfrak{g})^{\otimes N}$. The space $\rho(\mathfrak{g})$ is usually called the "auxiliary space". The integrals (3.0.4) can be expressed in term of the Lax matrix (3.0.6) as:

$$\mathcal{I}_j = \mathrm{res}_{\lambda=\epsilon_j}\mathrm{Tr}_0\left(L^q(\lambda)^2\right) \tag{3.0.8}$$

where Tr_0 denotes the trace taken only over the auxiliary space.

For the Lax matrix (3.0.6) it is possible to define an r matrix

$$r(\lambda) = \frac{1}{\lambda}\sum_{\alpha,\beta=1}^{\dim(\mathfrak{g})} g_{\alpha\beta}\rho(X^\alpha)\otimes\rho(X^\beta) \tag{3.0.9}$$

such that:

$$[L^q(\lambda)\otimes\mathbb{I}, \mathbb{I}\otimes L^q(\mu)] + [r(\lambda-\mu), L^q(\lambda)\otimes\mathbb{I} + \mathbb{I}\otimes L^q(\mu)] = 0 \tag{3.0.10}$$

where the commutator is taken on the auxiliary space.

The algebraic structure defined by equation (3.0.10) allowed Jurčo to use the Algebraic Bethe Ansatz to simultaneously diagonalize the set of operators (3.0.4) in all the cases when $\mathfrak{g}$ is a classical Lie algebra. Moreover, from equation (3.0.10) it is possible to give a proof of the commutativity of the quadratic spectral invariants (3.0.8) of $L^q(\lambda)$ [33] alternative to the one given by Gaudin [26]. We will comment on the relevance of the equation (3.0.10) for the existence of higher order (cubic, quartic and so on) "spectral invariants" of $L(\lambda)$ in the last chapter of this thesis.

3.0.4 Classical rational Gaudin models

We have seen that the Lax matrix (3.0.6) of the quantum rational Gaudin model (3.0.6) admits a (classical) r–matrix formulation (3.0.10). Let us now consider the Lie–Poisson manifold $\mathcal{P}_\mathfrak{g}$ associated with the Lie algebra $\mathfrak{g}$. If we perform the substitution $X_l^\alpha \mapsto y_l^\alpha$, where y_l^α denotes the coordinate dual to X_l^α, in the Hamiltonian (3.0.5) and in the Lax matrix (3.0.6), we obtain that the classical Hamiltonian

$$H_G^c = \sum_{j,l=1,l\neq j}^{N} \sum_{\alpha,\beta=1}^{\dim(\mathfrak{g})} \frac{\eta_j - \eta_l}{\epsilon_j - \epsilon_l} g_{\alpha\beta} y_j^\alpha y_l^\beta \tag{3.0.11}$$

and the new matrix

$$L^c(\lambda) = \sum_{i=1}^{N} \frac{A_i^c}{\lambda - \epsilon_i} \tag{3.0.12}$$

$$A_i^c = \sum_{\alpha,\beta=1}^{\dim(\mathfrak{g})} g_{\alpha\beta}\rho(X^\alpha) y_i^\beta \tag{3.0.13}$$

satisfy the following properties:

- The matrix $L^c(\lambda)$ admits a r–matrix formulation:

$$\sum_{ijkl}\{L_{ij}^c(\lambda), L_{kl}^c(\mu)\} e_{ij} \otimes e_{kl} = [L^c(\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes L^c(\mu), r(\lambda - \mu)] \tag{3.0.14}$$

  where $r(\lambda)$ is again given by equation (3.0.9).

- H_G^c belong to the family of spectral invariants of the matrix $L^c(\lambda)$.

$$H_G^c = \sum_{j=1}^{N} \eta_j \mathrm{res}_{\lambda=\epsilon_j} \mathrm{Tr}\left(L(\lambda)^2\right) \tag{3.0.15}$$

We would like to stress that equation (3.0.14) has deeper implications than its quantum analog (3.0.10). In fact, in the classical case equation (3.0.14) implies that *all* the spectral invariants of the Lax matrix $L^c(\lambda)$ form a family of involutive functions (see e.g. [5]). It is natural to call the Hamiltonian H_G^c the *classical version* of the Hamiltonian H_G. Consequently, the models associated with the Hamiltonian (3.0.11) are called *classical rational Gaudin models*.

Hamiltonian systems defined on direct products of Lie–Poisson manifolds with Lax matrices of the form (3.0.12) have been widely studied by Reyman and Semenov–Tian–Shansky [50]. From their results, it follows that the spectral invariants of the Lax matrix $L'(\lambda)$ are not enough to provide complete integrability of the model. To recover the missing integrals one has to notice that the spectral invariants of $L^c(\lambda)$ are invariant under the global action of the Lie group $\mathbf{G}$ associated with the Lie algebra $\mathfrak{g}$. In other words, if τ is an element of $\rho(\mathfrak{g})$, then

$$\phi_\tau = \mathrm{Tr}\left(\sum_{i=1}^{N} A_i \tau\right) \tag{3.0.16}$$

defines a function Poisson commuting with all the spectral invariants of $L^c(\lambda)$. Varying $\tau \in \mathfrak{g}$, the functions (3.0.16) span a Poisson algebra isomorphic to $\mathfrak{g}$. Such algebra is clearly not Abelian, so to get a complete family of involutive functions we must extract from the functions (3.0.16) a maximal Abelian subalgebra.

Instead of following this approach one can slightly modify the Lax matrix adding a constant term $\sigma \in \rho(\mathfrak{g})$ [50]:

$$L^c(\lambda, \sigma) = \sum_{i=1}^{N} \frac{A_i^c}{\lambda - \epsilon_i} + \sigma \tag{3.0.17}$$

and define a new Hamiltonian applying formula (3.0.15) to the new Lax matrix:

$$\begin{aligned} H_G^c(\sigma) &= \sum_{j=1}^{N} \eta_j \mathrm{res}_{\lambda=\epsilon_j} \mathrm{Tr}\left(L^c(\lambda,\sigma)^2\right) = \\ &= \sum_{j,l=1, l\neq j}^{N} \sum_{\alpha,\beta=1}^{\dim(\mathfrak{g})} \frac{\eta_j - \eta_l}{\epsilon_j - \epsilon_l} \mathrm{Tr}(A_i^c A_j^c) + \sum_{j=1}^{N} \eta_j \mathrm{Tr}(A_j^c \sigma) \end{aligned} \tag{3.0.18}$$

Now if we choose σ with simple spectrum we know from [50] that the spectral invariants of $L^c(\lambda, \sigma)$ defines a completely integrable system on $P_{\mathfrak{g}}^{\otimes N}$ for $\mathfrak{g}$ being one of the classical Lie algebras A_n, B_n, C_n, D_n.

A relevant member of the family of rational Gaudin Hamiltonians (3.0.18) is obtained when one put $\sigma = 0$, $\eta_j = \epsilon_j$, then H_G^c takes the very simple form:

$$H_s = \sum_{i<j} \mathrm{Tr}(A_i^c A_j^c) = \sum_{i<j} g_{\alpha,\beta} y_i^\alpha y_j^\beta \tag{3.0.19}$$

i.e. it becomes independent of the parameters η_i, ϵ_i. Hamiltonian (5.0.1) defines the so called "Homogeneous XXX Gaudin model". What distinguish the Homogeneous XXX Gaudin model from the other members of the family (3.0.18) is the fact that it is a *superintegrable* Hamiltonian system. There is a very simple argument proving the above claim [30]. Since the Hamiltonian (5.0.1) is parameter independent we can take two Lax matrices depending on two different sets of parameters:

$$L_1(\lambda) = \sum_{i=1}^{N} \frac{A_i^c}{\lambda - \epsilon_i} \tag{3.0.20}$$

$$L_2(\lambda) = \sum_{i=1}^{N} \frac{A_i^c}{\lambda - \xi_i} \tag{3.0.21}$$

This two Lax matrices have the following common family of spectral invariants:

$$\sum_{i=1}^{N} \mathrm{res}_{\lambda=\epsilon_i} \mathrm{Tr}(\lambda L_1^c(\lambda)^k) = \sum_{i=1}^{N} \mathrm{res}_{\lambda=\xi_i} \mathrm{Tr}(\lambda L_2^c(\lambda)^k)$$

that includes the Hamiltonian (5.0.1). For a generic choice of the parameters ϵ_i, ξ_i the other spectral invariants turn out to be independent. Hence the two Lax matrices $L_1(\lambda), L_2(\lambda)$ define two different sets of commuting observables having in common the Hamiltonian (5.0.1) (and possibly some other elements).

Thanks to the superintegrability of (5.0.1), in the $sl(2)$ case we can replace the family of integrals (3.0.3) with an alternative family of *parameter independent integrals* given by:

$$\tilde{I}_k = \sum_{i,j=1}^{k+1} g_{\alpha,\beta} y_i^\alpha y_j^\beta \qquad k = 1, \dots, N-1 \tag{3.0.22}$$

$$\tilde{I}_N = y_T^3 = \sum_{i=1}^{N} y_i^3 \tag{3.0.23}$$

This new family of integrals appeared for the first time in [13] even though in a disguised form. In fact Calogero considered the classical integrable Hamiltonian system defined on the standard symplectic space $\mathbb{R}^{2N}$

$$H_C = \sum_{j,k=1}^{N} p_j p_k \{\lambda + \mu \cos[\nu(q_j - q_k)]\} \tag{3.0.24}$$

together with the family of involutive integrals of motion:

$$h_m = \sum_{j,k=1}^{m} p_j p_k \{1 - \cos[\nu(q_j - q_k)]\} \qquad m = 1, \dots, N \tag{3.0.25}$$

The link between the family of integrals (3.0.24, 3.0.25) defined by Calogero and the integrals (3.0.22) of the Homogeneous XXX Gaudin model was clarified in an article by Ballesteros and Ragnisco[8], and in another one by Karimipour [38]. First of all one has to notice that the parameter ν can be seen as an overall multiplicative factor by means of the symplectic transformation

$$p_i \mapsto p_i' = \nu p_i \qquad q_i \mapsto \frac{q_i}{\nu}$$

so that in equations (3.0.24) and (3.0.25) one can put $\nu = 1$. Now if we choose the coordinates y_1, y_2, y_3 on $\mathcal{P}_{sl(2)}$ in such a way that the following Poisson relations hold:

$$\{y_3, y_1\} = -y_2 \quad \{y_3, y_2\} = y_1 \quad \{y_1, y_2\} = y_3$$

then the symplectic leaf of the Gaudin model defined by:

$$y_3^2 - y_1^2 - y_2^2 = 0$$

is isomorphic to the standard symplectic manifold and we can write the coordinates y_1, y_2, y_3 in the form:

$$y^1 = p\cos q \qquad y^2 = p\sin q \qquad y^3 = p \tag{3.0.26}$$

The integrals (3.0.3) rewritten in the (p, q) coordinates through (3.0.26) are exactly the integrals (3.0.25) defined by Calogero, while the Hamiltonian (3.0.24) is obtained using the same change of coordinates (3.0.26) from the function

$$\mu H_G + (\lambda - \mu)(y_T^3)^2$$

where y_T^3 is given by formula (3.0.23).

Starting from the study of the Hamiltonian system (3.0.24) Ballesteros and Ragnisco developed the so called *coproduct method* to construct integrable quantum and classical Hamiltonian systems. In the next section we will report briefly such method for the classical case.

The set of integrals (3.0.22) was introduced independently also by Kapovich and Millson in a very different context. In the last section of this chapter we will briefly recall their results and later generalizations of them by Flaschka and Millson [24].

3.1 Poisson–Hopf algebras and classical integrable systems

In this section we want to illustrate the so called "coproduct method" to construct integrable systems [8, 9, 6] and its connection with "bending flows" and the Homogeneous Gaudin model.

We recall some definitions from the theory of Hopf algebras.

Definition 3.1.1 *A Hopf algebra $\mathcal{A}$ is a linear space on a field K with the following linear applications*

- $m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ *multiplication*
- $\eta : K \longrightarrow \mathcal{A}$ *unity*
- $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ *coproduct*
- $\epsilon : \mathcal{A} \longrightarrow K$ *counity*
- $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ *antipode*

such that $\forall a, b, c \in \mathcal{A}$, $\forall \alpha \in K$:

1. $m(m(a, b), c) = m(a, m(b, c))$
2. $m(\eta(\alpha), a) = m(a, \eta(\alpha)) = \alpha a$
3. $(id \otimes \Delta)\Delta(a) = (\Delta \otimes id)\Delta(a)$
4. $(id \otimes \epsilon)\Delta(a) = a \otimes 1 \qquad (\epsilon \otimes id)\Delta(a) = 1 \otimes a$
5. $\Delta(m(a, b)) = m(\Delta(a), \Delta(b))$
6. $\epsilon(m(a, b)) = m(\epsilon(a), \epsilon(b))$
7. $m((\sigma \otimes id)\Delta(a)) = m((id \otimes \sigma)\Delta(a)) = \eta(\epsilon(a))$
8. $\sigma(m(a, b)) = m(\sigma(b), \sigma(a))$

where id *is the identity application in* $\mathcal{A}$.

Definition 3.1.2

- *A linear space* A *with a multiplication* m *and unity* η *satisfying properties* $1-2$ *is called an "associative algebra".*
- *A linear space* A *with a coproduct* Δ *and counity* ϵ *satisfying properties* $3-4$ *is called a "coassociative coalgebra".*
- *A linear space* A *with the operations* $(m, \eta, \Delta, \epsilon)$ *satisfying properties* $1-5$ *is called a "bialgebra".*

Having defined the notion of Hopf algebra we can define the main object we are interested in, i.e. what is a Poisson–Hopf algebra.

Definition 3.1.3 *$\mathcal{P}$ is a "Poisson–Hopf algebra" if $\mathcal{P}$ is both a Poisson and an Hopf algebra with the compatibility condition:*

$$\{\Delta(a), \Delta(b)\}_{\mathcal{P}\otimes\mathcal{P}} = \Delta(\{a, b\}_{\mathcal{P}}) \tag{3.1.1}$$

The Poisson bracket $\{\cdot, \cdot\}_{\mathcal{P}\otimes\mathcal{P}}$ is the one defined by (2.3.14). Our interest in Poisson–Hopf algebras is motivated by the fact that on a finitely generated Poisson algebra $\mathcal{P}$ it is possible to define a structure of Poisson–Hopf algebra. The multiplication m is the usual multiplication of functions, the identity η is defined as

$$\eta(\alpha) = \alpha \cdot 1$$

where 1 is the function identically equal to one, and the other maps are defined on the generators of $\mathcal{P}$ and the identity element as:

$$\Delta(y^\alpha) = y^\alpha \otimes 1 + 1 \otimes y^\alpha := y_1^\alpha + y_2^\alpha$$
$$\Delta(1) = 1 \otimes 1$$
$$\epsilon(y^\alpha) = 0$$
$$\epsilon(1) = 1$$
$$\sigma(y^\alpha) = -y^\alpha$$
$$\sigma(1) = 1$$

They can be extended to all $\mathcal{P}$ through properties 5, 6 and 8 of definition 3.1.1.

From Δ one can define recursively the linear applications

$$\Delta^{(i)} := (\Delta^{(2)} \otimes \overbrace{1 \otimes 1 \otimes \cdots \otimes 1}^{i-2}) \circ \Delta^{(i-1)}$$
$$\Delta^{(i)} : \mathcal{P} \longrightarrow \overbrace{\mathcal{P} \otimes \cdots \otimes \mathcal{P}}^{i} \qquad \Delta^{(2)} := \Delta$$

They are Poisson morphisms:

$$\{\Delta^{(i)}(a), \Delta^{(i)}(b)\}_{\mathcal{P}^{\otimes i}} = \Delta^{(i)}(\{a, b\}_{\mathcal{P}}) \tag{3.1.2}$$

The idea now is to construct an involutive family of functions in $\mathcal{P}^{\otimes N}$ applying the morphisms $\Delta^{(i)}$ to central elements in $\mathcal{P}$. In fact, we have the following theorem:

Theorem 3.1.4 *Let C_i be central elements in $\mathcal{P}$, then the functions*

$$K_{ij} = \Delta^{(j)}(C_i) \otimes \Delta^{(N-j)}(1) \tag{3.1.3}$$

are in involution on $\mathcal{P}^{\otimes N}$ and commutes with the $N-th$ coproduct of the generators:

$$\{K_{ij}, K_{lm}\} = 0 \tag{3.1.4}$$
$$\{K_{ij}, \Delta^{(N)}(y^\alpha)\} = 0 \tag{3.1.5}$$

Proof:

Let us denote with $\{y^\alpha\}_{\alpha=1}^M$ the coordinates in $\mathcal{P}$, and with $\Delta^{(i)}(\mathcal{P})$ the algebra generated by the elements $\{\Delta^{(i)}(y^\alpha)\}_{\alpha=1}^M$. Then we have the chain of inclusions:

$$\Delta^{(2)}(\mathcal{P}) \subset \mathcal{P} \otimes \mathcal{P} \qquad \Delta^{(i)}(\mathcal{P}) \subset \Delta^{(i-1)}(\mathcal{P}) \otimes \mathcal{P} \tag{3.1.6}$$

Using (3.1.6) and assuming $m \geq j$ we have:

$$\begin{aligned}
&\{K_{ij}, K_{lm}\} = \\
&= \{\Delta^{(j)}(C_i) \otimes \Delta^{(N-j)}(1), \Delta^{(m)}(C_l) \otimes \Delta^{(N-m)}(1)\} \in \\
&\in \{\Delta^{(j)}(C_i) \otimes \Delta^{(N-j)}(1), \Delta^{(j)}(\mathcal{P}) \otimes \mathcal{P}^{\otimes m-j} \otimes \Delta^{(N-m)}(1)\} = \\
&= \{\Delta^{(j)}(C_i), \Delta^{(j)}(\mathcal{P})\} \otimes \mathcal{P}^{\otimes m-j} \otimes \Delta^{(N-m)}(1) = 0
\end{aligned}$$

Equation (3.1.5) can be proved in the same way.

□

The independence of the functions (7.1.3) is assured if one starts from a set of algebraically independent Casimirs C_i. In fact by the homomorphism property of the coproduct it follows that the images of the Casimirs $\Delta^{(j)}(C_i)$ for a fixed j will be a set of algebraically independent functions on $\mathcal{P}^{\otimes j}$. On the other hand independence of sets corresponding to different values of j follows from the fact that they depend explicitly on different subsets of coordinates.

The simplest example on which one can apply the coproduct method is the Lie–Poisson algebra associated with the Lie algebra $sl(2)$. In fact on a Lie–Poisson algebra the polynomials in the natural coordinates on the dual form a finitely generated Poisson subalgebra. We have seen that for simple algebras we can arrange the generators of the Lie–Poisson algebra in a matrix (2.3.5), (2.3.17). The Casimir element of $\mathcal{P}_{sl(2)}$ can be written in term of the matrix A as:

$$C = \mathrm{Tr}(A^2) \tag{3.1.7}$$

Defining the coproduct of a matrix as $(\Delta(A))_{ij} = \Delta(A_{ij})$, we can write:

$$\Delta^{(i)}(C) = \mathrm{Tr}(\Delta^{(i)}(A^2)) = \mathrm{Tr}\left(\left(\sum_{k=1}^{i} A_k\right)^2\right) \tag{3.1.8}$$

From equation (3.1.8) we see that $\Delta^{(N)}(C) = 2H_G + \sum_{i=1}^{N} C_i$.

Applying the coproduct method to higher rank simple algebras, one has that the coproduct of the Casimirs elements K_{ij} are no more a sufficient number to assure complete integrability of the model (also taking into account the global invariance (3.1.5). We will exemplify this problem on $\mathcal{P}_{sl(r)}$. The Poisson algebra $\mathcal{P}_{sl(r)}$ has $r-1$ independent Casimirs C_i given by:

$$C_{i-1} = \mathrm{Tr}(A^i) \qquad i = 2, \ldots, r \tag{3.1.9}$$

whence it follows that the commuting functions K_{ij}, for any j, are simply the spectral invariants of the matrices $\sum_{k=1}^{j} A_k$. The dimension of the symplectic leaf is $r(r-1)$. So the number of degrees of freedom on the product manifold $\mathcal{P}_{sl(r)}^{\otimes N}$ is $Nr(r-1)/2$, while the K_{ij} define only $(N-1)(r-1)$ involutive functions. Using the global invariance (3.1.5) we can add to our set other $r(r-1)/2$ involutive functions ending with a total of $(N+r/2-1)(r-1)$. It follows that the coproduct method give us complete integrability only in the case $r=1$.

3.1.1 q-deformed integrable systems

The advantage of working in the Poisson–Hopf algebra setting is that one can naturally define "q-deformations" of such structures. Given a Poisson–Hopf algebra $\mathcal{P}$ one can define a Poisson–Hopf algebra $\mathcal{P}_q$ depending on a (complex) parameter q such that $\lim_{q\to 0}\mathcal{P}_q = \mathcal{P}$. Hence on $\mathcal{P}_q$ one can define deformations $(\{\cdot,\cdot\}_q, \Delta_q)$ of Poisson bracket and coproduct such that, for any value of the parameter q, Δ_q is still an homomorphism for $\{\cdot,\cdot\}_q$:

$$\{\Delta_q(a), \Delta_q(b)\} = \Delta_q(\{a,b\}) \qquad \forall a,b \in \mathcal{P}_q \tag{3.1.10}$$

We can again define inductively linear applications:

$$\Delta_q^{(i)} := (\Delta_q^{(2)} \otimes \overbrace{1 \otimes 1 \otimes \cdots \otimes 1}^{i-2}) \circ \Delta_q^{(i-1)}$$

that are Poisson morphisms:

$$\{\Delta_q^{(i)}(a), \Delta_q^{(i)}(b)\}_{\mathcal{P}_q^{\otimes i}} = \Delta^{(i)}(\{a,b\}_{\mathcal{P}_q}) \tag{3.1.11}$$

and then apply them to the central elements C_q^i of the Poisson–Hopf algebra $\mathcal{P}_q$ to construct the quantities:

$$K_{ij}^q := \Delta_q^{(j)}(C_i^q)$$

The proof of commutativity of such quantities is totally analogous to proof of theorem 3.1.4.

In this way we obtain a family of integrable systems depending on the deformation parameter q such that in the limit $q \to 0$ they give the integrable system one obtains applying the coproduct method to the undeformed Poisson–Hopf algebra. Plenty of examples of such systems can be found in [8], [9].

3.1.2 Extension to Poisson comodule algebras

For completeness we report a generalization of the coproduct method developed quite recently [7]. We have to introduce the notion of Poisson comodule algebras.

Definition 3.1.5

- *A (right) "coaction" of an Hopf algebra H on a vector space V is a linear map $\phi : V \longrightarrow V \otimes H$ such that*

$$(\phi \otimes id) \circ \phi = (id \otimes \Delta) \circ \phi$$

We will say that V is an "H-comodule".

- *If V is an algebra, we shall say that V is a "H–comodule algebra" if the coaction ϕ is a homomorphism with respect to the product on the algebra:*

$$\phi(ab) = \phi(a)\phi(b) \qquad \forall a, b \in V$$

- *Moreover, if V is also a Poisson algebra, then we will say that V is a "H–Poisson comodule algebra" if ϕ is an homomorphism also with respect to the Poisson bracket:*

$$\phi(\{a, b\}) = \{\phi(a), \phi(b)\} \qquad \forall a, b \in V$$

Now we can mimic the coproduct method replacing the coproduct with the coaction. Namely, we can define the chain of homomorphisms

$$\phi^{(i)} := (\phi^{(2)} \otimes \overbrace{1 \otimes 1 \otimes \cdots \otimes 1}^{i-2}) \circ \phi^{(i-1)}$$

Now, if C^i are central elements of the Poisson bracket in V, the elements $K_{ij} = \phi^{(j)}(C^i)$ are an involutive family and commutes with the image of any generator under the N–th coaction: $\{K_{ij}, \phi^{(N)}(y^\alpha)\}$. The proof of this fact is a simple extension of the proof of 3.1.4. Some examples of classical integrable systems one obtain with this method are given in [7].

Note that any Poisson–Hopf algebra is a H–comodule Poisson algebra with respect to itself, since the coaction map is given by the coproduct Δ.

3.2 Bending flows

The integrable systems defined by (3.0.22) was studied also by Kapovich and Millson [37] in a very different context, namely, the symplectic geometry of moduli spaces $M_{\mathbf{r}}$ of n–gons in three dimensional space with fixed side lengths $\mathbf{r} = (r_1, \ldots, r_n)$.

Consider the vector space $\mathcal{H}_2^0$ of 2×2 hermitean matrices with vanishing trace and identify it with the Lie algebra $\mathbf{u}_0(2)$ of skewhermitean matrices with vanishing trace through the linear map $\phi : \mathbf{u}_0(2) \to \mathcal{H}_2^0$ given by: $x \mapsto X = ix$. Using this identification and the trace as a non–degenerate bilinear form we can build the Lie–Poisson manifold associated with $\mathcal{H}_2^0$. As in section 2.3 we can introduce a matrix A such that the Lie–Poisson tensor is defined by the Hamiltonian vector fields:

$$\dot{A} = \left[A, \frac{\partial F}{\partial A}\right]$$

The Casimirs of this Poisson manifold are given by traces of powers of the matrix A. Since, in this case, A is a rank 1 matrix, we have only one Casimir function: $C = \mathrm{Tr}(A^2)$. We will consider the symplectic leaves $\mathrm{Tr}(A^2) = \frac{1}{2}r^2$ of this Poisson manifold with r a positive real number (the reason of this particular choice will be clear later). Once we have fixed r we can write the matrix A in the form:

$$A = r\mathbf{w} \otimes \mathbf{w}^* - \frac{1}{2} r I = \frac{1}{2} r \begin{pmatrix} |w_1|^2 - |w_2|^2 & 2w_1 w_2^* \\ 2w_1^* w_2 & |w_2|^2 - |w_1|^2 \end{pmatrix} \tag{3.2.1}$$

where $\mathbf{w} = (w_1, w_2)$ is a unimodular vector in $\mathbb{C}^2$.

Equation (3.2.1) defines a diffeomorphism between the $r-$*symplectic* leaf of $\mathcal{H}_2^{0*}$ and $\mathbb{CP}^2$. Such diffeomorphism is also a Poisson morphism if the symplectic structure on $\mathbb{CP}^2$ is given by $4r$ times the Fubini–Study one [24].

We can identify the matrix A defined in (3.2.1) with a vector in $\mathbb{R}^3$ of length r through the map

$$f : \mathbf{x} = (x_1, x_2, x_3) \to \hat{\mathbf{x}} = \frac{1}{2} \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix} \tag{3.2.2}$$

Let us denote with $N_{\mathbf{r}}$ the direct product of n symplectic leaves of $\mathcal{H}_2^{0*}$, where $\mathbf{r} = (r_1, \dots, r_n)$ are the values of the corresponding Casimirs. Associated with each symplectic leaves we will have a matrix A_i of the form (3.2.1) that we can think of as a vector in $\mathbb{R}^3$ of length r_i. Now let us consider the euclidean affine space $\mathbb{E}^3$ and let us fix a point $v_0 \in \mathbb{E}^3$. We associate with the matrices $A_1, \dots, A_n$ the broken line in $\mathbb{E}^3$ composed of the segments $A_i = \overrightarrow{v_{i-1}v_i}$. We will obtain a closed polygon if the matrices A_i satisfy:

$$\sum_{i=1}^{n} A_i = 0 \tag{3.2.3}$$

The momentum map associated with the action of $U(2)$ on $N_{\mathbf{r}}$ is given by:

$$\mu(A_1, \dots, A_n) = \sum_{i=1}^{n} A_i \tag{3.2.4}$$

Hence we can identify the space of closed n–gons with

$$\widetilde{M}_{\mathbf{r}} = \mu^{-1}(0) = \{\mathbf{A} \in N_{\mathbf{r}} \mid \sum_{i=1}^{n} A_i = 0\} \tag{3.2.5}$$

Finally to define the moduli space $M_{\mathbf{r}}$ of n–gons (with fixed sidelenghts $\mathbf{r}$) we have to quotient $\widetilde{M}_{\mathbf{r}}$ by the action of isometries. Translations can be ruled out fixing the starting point $v_0 \in \mathbb{E}^3$, while the rotations are given by the global action of $U(2)$, hence $M_{\mathbf{r}} = \widetilde{M}_{\mathbf{r}}/U(2)$.

Kapovich and Millson considered on the Poisson manifold $M_{\mathbf{r}}$ the set of Hamiltonian given by the length of the diagonals

$$\Delta^{(i)}(A) = \sum_{j=1}^{i} A_j \qquad i = 2, \ldots, n-2 \tag{3.2.6}$$

of the polygon $A_1, \ldots, A_n$. The length of the vectors (3.2.6) (as is clear from equation (3.2.2) is given by:

$$2\lambda_i = \sqrt{2\mathrm{Tr}\,(\Delta^{(i)}(A)^2)} \tag{3.2.7}$$

where with λ_i we denoted the eigenvalue of the matrix $\Delta^{(i)}(A)$. They showed that the λ_i give a complete set of involutive functions on $M_{\mathbf{r}}$ and gave a nice geometrical picture of the corresponding flows. The diagonal $\Delta(i)(A)$ divide the polygon in two parts and the flow associated with its eigenvalue λ_i rotate the first part of the polygon (the one containing the edges $A_1, \ldots, A_{i-1}$) around the diagonal at a constant speed. Motivated by this geometrical description they gave to the flows generated by the integrals (3.2.7) the name of *bending flows*. Moreover, they found explicitly action–angle variables. The action variables are exactly the eigenvalues λ_i, while the corresponding conjugate angles are given by the oriented dihedral angle between the two triangles spanned, respectively, by $\Delta^{(i-1)}(A), A_i, \Delta^{(i)}(A)$ and $\Delta^{(i)}(A), A_{i+1}, \Delta^{(i+1)}(A)$.

We remark that the integrals (3.2.7) differs by those one obtains applying the coproduct method to $M_{\mathbf{r}}$ only by a square root (see equation (3.1.8)).

3.2.1 Rank 1 generalization of bending flows

In [24] Flaschka and Millson proposed a generalization of the bending flows extending the results of Kapovich and Millson to the case where the matrices A_i are $m+1 \times m+1$ Hermitean matrices of rank one.

They denote with $\mathcal{H}_{m+1}$ the vector space of $m+1 \times m+1$ Hermitean matrices. As in the previous section they identify $\mathcal{H}_{m+1}$ with the Lie algebra $\mathbf{u}(m+1)$, so that they can associate with $\mathcal{H}_{m+1}$ a Lie–Poisson manifold. Fixed $r > 0$, they consider the $U(m+1)$–orbit through $\mathrm{diag}(r, 0, \ldots, 0)$ and denote it by $\mathcal{O}_r$. A matrix A belonging to $\mathcal{O}_r$ can be written in the form:

$$A = r\mathbf{w} \otimes \mathbf{w}^*$$

where $\mathbf{w} \in \mathbb{C}^{m+1}$ is a unit vector. Given a n-tuple of positive numbers $\mathbf{r} = (r_1, r_2, \ldots, r_n)$, they define $\widetilde{N}_{\mathbf{r}}$ to be the product symplectic manifold $\prod_{i=1}^{n} \mathcal{O}_{r_i}$.

The diagonal action of $U(m+1)$ on $\widetilde{N}_{\mathbf{r}}$ is Hamiltonian with momentum map μ given by:

$$\mu(A_1, \ldots, A_n) = \sum_{i=1}^{n} A_i$$

It follows that the preimage

$$\widetilde{M}_{\mathbf{r}} = \mu(\Lambda\mathbb{I}) = \{\mathbf{A} \in \widetilde{N}_{\mathbf{r}} \mid \sum_{i=1}^{n} A_i = \Lambda\mathbb{I}\}$$
$$\Lambda = \frac{1}{m+1}\sum_{i=1}^{n} r_i$$

is a Poisson manifold. By analogy with the case of Kapovich and Millson they call $\widetilde{M}_{\mathbf{r}}$ the space of closed n–gons, and the quotient $M_{\mathbf{r}} = \widetilde{M}_{\mathbf{r}}/U(m+1)$ the moduli space of n–gons (with side lengths $\mathbf{r}$).

Flaschka and Millson show that the eigenvalues of the diagonals

$$\Delta^{(i)}(A) = \sum_{j=1}^{i} A_j \qquad j = 2, \ldots, n-2$$

defines an integrable system on $M_{\mathbf{r}}$. Moreover they construct explicitly action–angle variables. The case $m = 1$ reproduces exactly the results of Kapovich and Millson.

We would like to stress that also the integrals defined by Flaschka and Millson are equivalent to those one obtain applying the coproduct method to the Poisson manifold $M_{\mathbf{r}}$. From the coproduct point of view we could say that Flaschka and Millson give a generalization of the coproduct method to suitable Poisson submanifolds of the Lie–Poisson manifold $\mathbf{u}(m+1)^{\otimes n}$.

Chapter 4

A Bihamiltonian Structure of the rational Gaudin model

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathcal{P}_\mathfrak{g}$ the associated Lie Poisson manifold. We have seen that the Lax matrix of the rational Gaudin model is given by:

$$L_r = \sum_{i=1}^{N} \frac{A_i}{\lambda - \epsilon_i} + \sigma \qquad (4.0.1)$$
$$A_i = \sum_{\alpha,\beta=1}^{\dim(\mathfrak{g})} g_{\alpha\beta}\rho(X^\alpha)y_i^\beta$$

A bihamiltonian structure for rational Gaudin models can be obtained using the following argument. Let us consider the map $\{A_i\} \longrightarrow \{B_i\}$ that sends the rational Lax matrix (4.0.1) to the polynomial Lax matrix

$$L_{pol} = \lambda^n\sigma + \sum_{i=0}^{N-1} B_i\lambda^i = \Big(\prod_{i=1}^{N}(\lambda - \epsilon_i)\Big) \cdot L_{rat} \qquad (4.0.2)$$

explicitly given by

$$B_l = (-1)^{N-l-1}\sum_{i=1}^{N} s_{N-l-1}(\epsilon_1,\ldots,\hat{\epsilon}_i,\ldots,\epsilon_N)\cdot A_i + \\ (-1)^{N-l}s_{N-l}(\epsilon_1,\ldots,\epsilon_N)\cdot\sigma, \qquad l = 0,\ldots,N-1, \qquad (4.0.3)$$

where $s_k(\epsilon_1,\ldots,\epsilon_N)$ denotes the k-th elementary symmetric polynomial in the variables $\epsilon_1,\ldots,\epsilon_N$.

On the space of polynomial pencils of matrices a family of mutually compatible Poisson brackets are defined [50, 42]. For the sake of brevity they will be termed RSTS tensors. In a nutshell, this family can be described by saying that there is a map from degree N polynomials in the variable λ to the set of Poisson structures on the manifold of polynomial Lax matrices of the form (4.0.2) which sends the monomials $\lambda^0, \ldots, \lambda^N$ into $N+1$ fundamental Poisson brackets, $\Pi_l, l = 0, \ldots, N$. In our case, the fundamental tensors Π_l can be represented by matrices having the following block–diagonal structure:

$$\Pi_l = \begin{pmatrix} C_l & 0 \\ 0 & D_l \end{pmatrix}, \tag{4.0.4}$$

with:

$$\begin{cases} (C_l)_{ij} = -[B_{i+j-l-1}, \cdot] & i,j = 1, \ldots, l \\ (D_l)_{ij} = [B_{i+j+l-1}, \cdot] & i,j = 1, \ldots, N-l \end{cases} \tag{4.0.5}$$
$$B_i = 0 \text{ if } i < 0 \text{ or } i > N, \qquad \text{and} \quad B_N = \sigma.$$

Lemma 2 *In the "coordinates" $B_0, \ldots, B_{N-1}, \sigma$, the diagonal Poisson tensor P (2.3.20) is given by the sum*

$$P = \sum_{l=0}^{N} (-1)^{N-l-1} s_{N-l}(\epsilon_1, \ldots, \epsilon_N)\Pi_l, \tag{4.0.6}$$

where the s_i's are the elementary symmetric polynomials in the ϵ_i's; that is, it is the tensor associated with the polynomial

$$p_N = \prod_{l=1}^{N} (\lambda - a_i).$$

This Lemma can be proved by means of a direct computation. For the reader's convenience, we collect its main steps in Appendix A.

Since the $N+1$ Poisson tensors (4.0.4) are compatible, we can choose as a second Poisson tensor a suitable linear combination of them to have a bihamiltonian structure on $\mathcal{P}_{\mathfrak{g}}^{\otimes N}$. Let

$$Q = \sum_{l=0}^{N-1} (-1)^{N-l} s_{N-l-1}(\epsilon_1, \ldots, \epsilon_N)\Pi_l, \tag{4.0.7}$$

be the tensor associated with the polynomial

$$p_{N-1} = \left(\frac{p_N}{\lambda}\right)_+ = \lambda^{N-1} - s_1\lambda^{N-2} + \cdots + (-1)^N s_{N-1}.$$

It turns out that all the integrals of motion that one can obtain from the spectral invariants of the Lax matrix (4.0.1) can be obtained by the GZ method applied to the pencil $Q - \lambda P$; in fact it holds (see, also, [50]):

Lemma 3 *All the vector fields associated with the spectral invariants of (4.0.1) are bihamiltonian with respect to the pair $Q - \lambda P$.*

Proof: It is convenient to work with the variables B_i. Let us define:

$$K_\alpha^{(i)} = \mathrm{Tr}\big(\mathrm{Res}_{\lambda=0}\big(\frac{(\sum_{j=1}^{N} B_j\lambda^j)^\alpha}{\lambda^i}\big)\big) \qquad (4.0.8)$$
$$i = 1, \ldots, \alpha N \qquad \alpha = 2, \ldots \mathrm{rk}(\mathfrak{g}).$$

For any fixed α, the αN functions (4.0.8) fulfill the relations [50]:

$$\Pi_i dK^{(j)} = \Pi_{i+k} dK^{(j+k)} = X^{(j-i)}. \qquad (4.0.9)$$

From (4.0.9) it follows that $X^{(i)} = 0$ if $i \leq 0$ or $i > N(\alpha-1)$; in fact, if $i \leq 0$ then $K^{(i)} = 0$ and $X^{(i)} = \Pi_0 dK^{(i)} = 0$, while if $i > N(\alpha - 1)$, then $K^{(N+i)} = const$ and $X^{(i)} = \Pi_N dK^{(N+i)} = 0$. Now let us set:

$$b_l = (-1)^{N-l+1} s_{N-l}(\epsilon_1, \ldots, \epsilon_N), \qquad (4.0.10)$$

we have:

$$PdK_\alpha^{(j)} = \sum_{l=0}^{N} b_l \Pi_l dK^{(j)} = \sum_{l=0}^{N} b_l X^{(j-l)}$$
$$QdK_\alpha^{(j)} = \sum_{l=1}^{N} b_l \Pi_{l-1} dK^{(j)} = \sum_{l=1}^{N} b_l X^{(j-l-1)}.$$

Then:

$$PdK_\alpha^{(j)} - QdK_\alpha^{(j+1)} = b_0 X_\alpha^{(j)}.$$

If one of the a_i is equal to zero, then $b_0 = \prod_{i=1}^{N} a_i = 0$ and the proof is concluded. Otherwise we need to find a function $F_\alpha^{(j)}$ such that

$$QdF_\alpha^{(j)} = b_0 X_\alpha^{(j)}$$

We proceed by induction. If $j = 1$, we have $b_0 X_\alpha^{(1)} = Q\frac{b_0}{b_1} dK_\alpha^{(1)}$, so that $F_\alpha^{(1)} = b_0/b_1 K_\alpha^{(1)}$. Now let $F_\alpha^{(i)}$ be such that: $b_0 X_\alpha^{(i)} = QdF_\alpha^{(i)} \quad i = 1, \ldots, j-1$. Then

$$Q\frac{b_0}{b_1} dK_\alpha^{(j)} = b_0 X_\alpha^{(j)} + \frac{b_0 b_2}{b_1} X_\alpha^{(j-1)} + \cdots + \frac{b_0 b_N}{b_1} X_\alpha^{(j-N+1)} \Longrightarrow$$
$$\Longrightarrow b_0 X_\alpha^{(j)} = Q\left(\frac{b_0}{b_1} dK_\alpha^{(j)} - \frac{b_2}{b_1} dF_\alpha^{(j-1)} - \cdots - \frac{b_N}{b_1} dF_\alpha^{(j-N+1)}\right).$$

So we have:

$$QdF_\alpha^{(j)} = b_0 X_\alpha^{(j)}, \quad \text{with } F_\alpha^{(j)} = \frac{b_0}{b_1} K_\alpha^{(j)} - \frac{1}{b_1} \sum_{i=1}^{N-1} b_{i+1} F_\alpha^{(j-i)} \qquad (4.0.11)$$

□

From the previous lemma it follows that we can solve the recursive equations

$$QdH_i = PdH_{i+1}$$

for any H_i that is a combination of the functions $K_\alpha^{(j)}$. So, to prove that we can define anchored Lenard sequences we show that the Casimirs of P can be written as combinations of the functions $K_\alpha^{(j)}$. We will discuss this point for the algebra A_n. For the other classical Lie algebras B_n, C_n, D_n the expression for the Casimirs in terms of the functions $K_{al}^{(j)}$ will be the same [10], but only a subset of them will be functionally independent. We have to distinguish two cases:

- if $b_0 \neq 0$, i.e. all the ϵ_i are different from zero, then a set of independent Casimirs of P for $\mathfrak{g} = A_n$ is given in terms by the following expressions:

$$C_{i,\alpha} = \sum_{j=1}^{\alpha N} \epsilon_i^j K_\alpha^{(j)} \qquad i = 1, \ldots, N \qquad \alpha = 2, \ldots, n+1. \qquad (4.0.12)$$

- if $b_0 = 0$ then only one among the ϵ_i, say ϵ_N, is zero. In this case equation (4.0.12) defines $n(N-1)$ independent Casimirs, instead of nN:

$$C_{i,\alpha} = \sum_{j=1}^{\alpha N} \epsilon_i^j K_\alpha^{(j)} \qquad i = 1, \ldots, N-1 \qquad \alpha = 2, \ldots, n+1. \qquad (4.0.13)$$

  The functions (4.0.13) turn out to be simultaneous Casimirs for both P and Q. The remaining n Casimirs of P (the rank of P is obviously the same in both cases) are given by

$$C_{N,\alpha} = K_\alpha^{(1)} \qquad \alpha = 2, \ldots, n+1. \qquad (4.0.14)$$

  In this case the GZ sequences take the very simple form:

$$PdK_\alpha^{(j)} = QdK_\alpha^{(j+1)} \qquad PdK_\alpha^{(1)} = 0 \qquad QdK^{(N(\alpha-1))} = 0$$

4.1 On the choice of σ

We have said that if σ is chosen to have simple spectrum the spectral invariants of (4.0.1) are a sufficient number in order to provide complete integrability for $\mathfrak{g}$ being one of the classical Lie algebras. If σ has not simple spectrum, then we are missing some integrals. These integrals can be recovered noticing that if τ belong to the commutant of σ, then the function:

$$F = Tr\left(\sum_{i=1}^{N} A_i \tau\right)$$

commutes with all the spectral invariants of L_r. In fact the vector field associated with F is given by:

$$\dot{A}_i = [A_i, \tau]$$

Hence:

$$\dot{L}_\tau = \frac{\dot{A}_i}{x - \epsilon_i} = \frac{[A_i, \tau]}{x - \epsilon_i} + [\sigma, \tau] = [L_r, \tau]$$

While this additional commuting flows are not intrinsic to the Lax approach, but must be inserted "a posteriori", they are naturally provided by the bihamiltonian approach. In fact, in the coordinates B_i, the last column of the Poisson tensor Q is given by:

$$\begin{pmatrix} b_1[\sigma, .] \\ b_2[\sigma, .] \\ \vdots \\ b_N[\sigma, .] \end{pmatrix} \tag{4.1.1}$$

So, if τ belong to the commutant of σ, the function $F_\tau = Tr\,(B_{N-1}\tau)$ is a Casimir for Q. Using the change of variables 4.0.3 we find that in the coordinates A_i the function F_τ is exactly $F_\tau = Tr\left(\sum_{i=1}^{N} A_i\tau\right)$. We have seen that P assign to F the vector field $\dot{A}_i = [A_i, \tau]$; from the expression (4.1.1) of the last column of the tensor Q it is obvious that the recurrence relation

$$PdF = QdG$$

cannot be solved if $\tau \neq \sigma$. So, for any possible choice of $\tau \neq \sigma$ in the stabilizer of σ we have an anchored Lenard chain starting from a Casimir of Q given by F_τ that stops after the first vector field X_τ. The elements in these kind of chains commute with the elements belonging to the "normal" Lenard anchored chains but not among them. What happens from the point of view of (2.1.5) is that Q and P have different ranks. So the point $\lambda = 0$ is a point where the corank of the Poisson tensor $Q - \lambda P$ drops and it cannot exist any neighborhood of $\lambda = 0$ such that the corank of $Q - \lambda P$ remains constant.

Chapter 5

The homogeneous case

As we have seen in chapter 3, the Hamiltonian

$$H_s = \sum_{i<j} \mathrm{Tr}(A_i A_j) = \sum_{i<j} g_{\alpha,\beta} X_i^\alpha X_j^\beta \tag{5.0.1}$$

defines a superintegrable Hamiltonian system. The two sets of involutive functions can be obtained as spectral invariants of the Lax matrices (3.0.20),(3.0.21), that depend on two different sets of parameters $\epsilon_1, \ldots, \epsilon_N$ and $\xi_1, \ldots, \xi_N$. Moreover, we have seen that for $\mathfrak{g} = sl(2)$, we can define a complete set of involutive functions that are parameter independent. If we consider a Lie algebra $\mathfrak{g}$ of higher rank, we have seen that we can obtain more parameter independent first integrals of (5.0.1) applying the coproduct method to the Poisson algebra $\mathcal{P}_\mathfrak{g}$, but they will provide complete integrability only on some special Poisson submanifolds of $\mathcal{P}_\mathfrak{g}$ as in the case of the bending flows.

In this chapter we will show that for $\mathfrak{g}$ being one of the classical Lie algebras A_n, B_n, C_n, D_n it is possible to define a set of parameter independent functions that give complete integrability of the Hamiltonian (5.0.1). They are obtained introducing a further Poisson structure R on the manifold $\mathcal{P}_\mathfrak{g}^{\otimes N}$. As we shall show it will be possible to combine it with the diagonal Poisson structure P of Eq. (2.3.20) to get a further *Poisson pencil*, not belonging to the RSTS family described in Section 4. The GZ method applied to the Poisson pencil $R - \lambda P$ will give rise to these new set of integrals. Since everything will be done in a Lie-algebraic setting, these results hold for a generic semisimple Lie algebra. We will show that we have complete integrability only for $\mathfrak{g}$ being A_n, B_n, C_n, D_n.

5.1 The additional bihamiltonian pencil

Let us consider the bivector R, defined, by means of the constructions outlined in Section 2.3 by the following matrix:

$$R = \begin{pmatrix} 0 & [A_1, \cdot] & \cdots & [A_1, \cdot] \\ [A_1, \cdot] & [A_2 - A_1, \cdot] & \cdots & [A_2, \cdot] \\ \vdots & \vdots & \ddots & \vdots \\ [A_1, \cdot] & [A_2, \cdot] & \cdots & [(N-1)A_N - \sum_{i=1}^{N-1} A_i, \cdot] \end{pmatrix}. \tag{5.1.1}$$

Proposition 5.1.1 *The bivector R defined by (5.1.1) is a Poisson bivector.*

Proof: Linearity and skewsymmetry are obvious, so we must prove only the Jacobi Identity. Also, we can limit ourselves to prove the assertions for the case of linear functions on $\mathcal{P}_{\mathfrak{g}}$. If F, G, H are such functions, identifying their differentials with the three N–tuples of matrices $\{\alpha_i\}, \{\beta_i\}, \{\gamma_i\}$, (e.g., $\dfrac{\partial F}{\partial A_i} = \alpha_i, \ldots$), the Poisson bracket is defined by:

$$\{F, G\}_R = \langle dF, RdG \rangle = \sum_{i,j,k} r_{ijk} \mathrm{Tr}\left(\alpha_i \left[A_k, \beta_j\right]\right) = \sum_{i,j,k} r_{ijk} \mathrm{Tr}\left(A_k \left[\beta_j, \alpha_i\right]\right),$$
$$r_{ijk} = (k-1)\delta_{ij}\delta_{jk} - \theta_{(i-k)}\delta_{ij} + \theta_{(j-i)}\delta_{ik} + \theta_{(i-j)}\delta_{jk},$$

where δ is the usual Kronecker symbol and

$$\theta_{(i)} = \begin{cases} 1 & \text{if } i > 0 \\ 0 & \text{if } i \leq 0 \end{cases}$$

The Jacobi identity reads

$$\{H, \{F, G\}_R\}_R + \{F, \{G, H\}_R\}_R + \{G, \{H, F\}_R\}_R =$$
$$\sum_{i,j,k,l,m} r_{ijk} r_{lmj} \left(\mathrm{Tr}\left(A_k \left[\left[\beta_m, \alpha_l\right], \gamma_i\right]\right) + \mathrm{Tr}\left(A_k \left[\left[\alpha_m, \gamma_l\right], \beta_i\right]\right) + \mathrm{Tr}\left(A_k \left[\left[\gamma_m, \beta_l\right], \alpha_i\right]\right)\right),$$

which, renaming the indexes, becomes

$$\sum_{i,j,k,l,m} r_{ijk} r_{lmj} \mathrm{Tr}\left(A_k \left[\left[\beta_m, \alpha_l\right], \gamma_i\right]\right) + r_{mjk} r_{ilj} \mathrm{Tr}\left(A_k \left[\left[\alpha_l, \gamma_i\right], \beta_m\right]\right) +$$
$$+ r_{ljk} r_{mij} \mathrm{Tr}\left(A_k \left[\left[\gamma_i, \beta_m\right], \alpha_l\right]\right).$$

A sufficient condition for the last expression to be zero is that for any k it holds:

$$\sum_j r_{ijk} r_{lmj} = \sum_j r_{mjk} r_{ilj} \tag{5.1.2}$$

In fact equation (5.1.2) implies that the quantity $\sum_j r_{ijk} r_{lmj}$ is invariant under cyclic permutations of the indexes i, l, m, hence if (5.1.2) holds we can write:

$$\begin{aligned} &\{H, \{F, G\}_R\}_R + \{F, \{G, H\}_R\}_R + \{G, \{H, F\}_R\}_R = \\ &= \sum_{i,j,k,l,m} r_{ijk} r_{lmj} \mathrm{Tr}\left(A_k \left(\left[\left[\beta_m, \alpha_l\right], \gamma_i\right] + \left[\left[\alpha_l, \gamma_i\right], \beta_m\right] + \left[\left[\gamma_i, \beta_m\right], \alpha_l\right]\right)\right), \end{aligned}$$

which vanishes thanks to the Jacobi identity in $\mathfrak{g}$.

Let us show that (5.1.2) does indeed hold in our case. By means of algebraic manipulations, namely cycling through i, l, m and renaming the indexes using the Kronecker's δ, we obtain:

$$\begin{aligned} &\sum_j r_{ijk} r_{lmj} - \sum_j r_{mjk} r_{ilj} = \\ &= \delta_{ik}\delta_{lm}[(l-i)\theta_{(l-i)} \sum_j \theta_{(j-i)}\theta_{(l-j)}] + \delta_{il}\delta_{mk}[(i-m)\theta_{(i-m)} + \sum_j \theta_{(j-m)}\theta_{(i-j)}] + \\ &+\delta_{lm}[\theta_{(i-k)}(\theta_{(l-i)} - \theta_{(l-k)}) + \theta_{(l-k)}\theta_{(i-l)}] + \delta_{il}[\theta_{(m-k)}(\theta_{(i-k)} - \theta_{(i-m)}) - \theta_{(i-k)}\theta_{(m-i)}] + \\ &+\delta_{ik}[\theta_{(l-i)}(\theta_{(m-l)} - \theta_{(m-i)}) + \theta_{(m-i)}\theta_{(l-m)}] + \delta_{mk}[\theta_{(l-m)}(\theta_{(i-m)} - \theta_{(i-l)}) - \theta_{(i-m)}\theta_{(l-i)}] \end{aligned}$$

Using the identities:

$$\begin{aligned} &\sum_j \theta_{(j-i)}\theta_{(l-j)} = (l-i-1)\theta_{(l-i)} \\ &\theta_{(i-k)}(\theta_{(l-i)} - \theta_{(l-k)}) + \theta_{(l-k)}\theta_{(i-l)} = -\theta_{(l-k)}\delta_{il} \end{aligned}$$

we see that

$$\sum_j r_{ijk} r_{lmj} - \sum_j r_{mjk} r_{ilj} = 0$$

□

Proposition 5.1.2 *The Poisson tensor R is compatible with the diagonal Poisson tensor P.*

Proof: We recall that

$$\{F, G\}_P = \sum_{i,j,k} \delta_{ij}\delta_{jk} \mathrm{Tr}\left(A_k \left[\frac{\partial G}{\partial A_j}, \frac{\partial F}{\partial A_i}\right]\right).$$

The characteristic condition for the compatibility of two Poisson tensors is

$$\{H, \{F, G\}_P\}_R + \{H, \{F, G\}_R\}_P + \text{cyclic permutations} = 0.$$

As in the proof of Proposition 5.1.1 above, one shows that a sufficient condition is that the quantity

$$s_{iklm} = \sum_j (r_{ijk}\delta_{lm}\delta_{mj} + \delta_{ij}\delta_{jk}r_{lmj})$$

be invariant under cyclic permutations of the indexes i, l, m for all k's. We have:

$$s_{iklm} = (k+i)\delta_{ik}\delta_{lm}\delta_{il} - \theta_{(i-k)}\delta_{lm}\delta_{il} + \theta_{(i-m)}\delta_{lm}\delta_{kl} + \theta_{(m-l)}\delta_{ik}\delta_{il} + \theta_{(l-m)}\delta_{ik}\delta_{im}$$

that manifestly satisfies this property.

□

Remark. By the previous proposition, we can endow, for every N, the space $\mathcal{P}_{\mathfrak{g}}^{\otimes N}$ with a bihamiltonian structure $P_\lambda = R - \lambda P$. A natural question arises, namely what is the connection with the RSTS family of Poisson structures discussed in Section 4. We do not have yet a complete answer to this point; however, as we will show in Appendix B, the new tensor R is *not* compatible with a generic element of the RSTS family (4.0.4).

5.2 The Lenard Chains

We construct the Lenard chains for the Poisson pencil $R - \lambda P$, using the GZ recipe discussed in Section 2.

Let us introduce $N-1$ Lax matrices:

$$L_a(\lambda) = (\lambda - (a-2))A_a + \sum_{k=1}^{a-1} A_k \qquad a = 2, \ldots, N \tag{5.2.1}$$

It holds:

Proposition 5.2.1 *Let F be a smooth function on M and let us consider the pencil of vector fields*

$$X_F^\lambda = P^\lambda dF := (R - \lambda P)dF$$

(we say that X_F^λ is Hamiltonian w.r.t. the pencil P_λ). Then, along X_F^λ, the matrices L_i of eq. (6.1.1) evolve according to a Lax equation,

$$Lie_{X_F^\lambda}(L_i) = [L_i(\lambda), M_i(\lambda)] \tag{5.2.2}$$

with

$$M_i(\lambda) = (i-\lambda)\frac{\partial F}{\partial A_{i+1}} + \sum_{k=i+2}^{N} \frac{\partial F}{\partial A_k}$$

Proof: Let us denote $\alpha_i = \frac{\partial F}{\partial A_i}$ The vector field X_F^λ is explicitly given by:

$$\mathrm{Lie}_{X_F^\lambda}(A_i) = (P^\lambda dF)_i = \sum_{j,k}(r_{ijk} - \lambda p_{ijk})\,[A_k, \alpha_k] =$$
$$= \sum_{j,k}((k-\lambda-1)\delta_{ij}\delta_{jk} - \theta(i-k)\delta_{ij} + \theta(j-i)\delta_{ik} + \theta(i-j)\delta_{jk})\,[A_k, \alpha_k] =$$
$$= \sum_{k=1}^{i-1}[A_k, \alpha_k - \alpha_i] + \left[A_i, (i-\lambda-1)\alpha_i + \sum_{k=i+1}^{N}\alpha_k\right]$$

Substituting in $L_i(\lambda)$ we get:

$$\begin{aligned}
\mathrm{Lie}_{X_F^\lambda}(L_i) &= (\lambda - i + 1)\left(\sum_{k=1}^{i}[A_k, \alpha_k - \alpha_{i+1}] + \left[A_{i+1}, (i-\lambda)\alpha_{i+1} + \sum_{k=i+2}^{N}\alpha_k\right]\right) + \\
&+ \sum_{j=1}^{i}\left(\sum_{k=1}^{i-1}[A_k, \alpha_k - \alpha_j] + \left[A_j, (j-\lambda-1)\alpha_j + \sum_{k=j+1}^{N}\alpha_k\right]\right) = \\
&= \left[\left((\lambda - i + 1)A_{i+1} + \sum_{j=1}^{i}A_j\right), \left((i-\lambda)\alpha_{i+1} + \sum_{k=i+2}^{N}\alpha_k\right)\right]
\end{aligned}$$

□

We can interpret this result by saying that we can associate with the homogeneous n-particle Gaudin system a set of n matrices depending on a parameter λ, satisfying a Lax equation along the "formal" (i.e., depending on the parameter λ) flows of vector fields that are Hamiltonian with respect to the pencil P_λ.

Proposition 5.2.2 *The coefficients $K_k^{(l)}$ of the expansion in powers of μ of the characteristic polynomial*

$$det(\mu - L_k(\lambda)) = \mu^N + \sum_{l=2}^{N} K_k^{(l)}(\lambda)\mu^{N-l}$$

are polynomial Casimirs of the pencil $P_\lambda = R - \lambda P$. Moreover, along any vector field X associated with any of the non–trivial coefficient of such polynomial Casimir, all matrices $L_k(\lambda)$ evolve according to Lax equations

$$\frac{d}{dt_X}L_k(\lambda) = X(L_k(\lambda)) = [L_k(\lambda), M_k(X)]$$

for suitable matrices $M_k(X)$.

Proof: Let us show that this is true for the equivalent set of spectral invariants $H_m^{(l)} = 1/(l+1)\mathrm{Tr}(L_m)^{l+1}$. This is the same of proving that, for any one–form α, that we can assume to be exact, $\alpha = dF$ we have

$$\langle \alpha, P_\lambda dH_m^{(l)} \rangle = 0.$$

Now, switching the action of the Poisson pencil on $\alpha = dF$ the LHS of this equation reads

$$\begin{aligned} L_{X_F^\lambda}(H_m^{(l)}) &= L_{X_F^\lambda}\left(\frac{1}{l+1}\mathrm{Tr}(L_m(\lambda))^{l+1}\right) \\ &= \frac{1}{l+1}\sum_{j=0}^{l-1}\mathrm{Tr}\left(L_m(\lambda)^j \cdot L_{X_F^\lambda}(L_m(\lambda)) \cdot L_m(\lambda)^{l-j}\right) \\ &= \mathrm{Tr}\left(L_m^l \cdot L_{X_F^\lambda}(L_m)\right) = \mathrm{Tr}\left(L_m^l(\lambda)[L_m(\lambda), M_m^F(\lambda)]\right) = 0 \end{aligned} \tag{5.2.3}$$

This proves the first assertion of the proposition, and, in particular, shows that all the vector fields X_m^{lj} associated (say, via P) with the coefficients of the expansion

$$K_m^{(l)} = \sum_{j=0}^{l} K_m^{lj}\lambda^j \tag{5.2.4}$$

are indeed bihamiltonian vector fields, i.e.:

$$\begin{aligned} X_m^{lj} = PdK_m^{lj-1} = RK_m^{lj} \qquad & j = 1, \ldots, l \\ RdK_m^{l0} = 0 \qquad PdK_m^{ll} = 0 & \end{aligned}$$

To prove the second statement we notice that $X_m^{lj} = RdK_m^{lj}$ can be written as a Hamiltonian vector field w.r.t. the pencil, considering the "truncated" polynomial $\left(\lambda^{-j}K^{(l)}(\lambda)\right)_+$,, where $(\cdot)_+$ denotes the nonnegative part of the expansion in λ. The assertion of the proposition follows from proposition 5.2.1.

□

5.2.1 Complete integrability for the classical Lie algebras

The results in section 5.2 have been obtained only requiring that does exist a metric $g^{\alpha\beta}$ on the Lie algebra $\mathfrak{g}$, hence they hold for any semisimple Lie algebra. In this section we prove that when $\mathfrak{g}$ is one of the classical Lie algebras, then the family of involutive functions obtained in section 5.2 (keeping into account also the global invariance) define a completely integrable system.

If $\mathfrak{g}$ is a semisimple Lie algebra then the dimension of the generic maximal symplectic leaves of the Poisson manifold $\mathcal{P}_\mathfrak{g}$ is given by $\dim(\mathfrak{g}) - \text{rank}(\mathfrak{g})$. So, the number of degrees of freedom for a Hamiltonian system defined on $\mathcal{P}_\mathfrak{g}^{\otimes N}$ is $\frac{N}{2}(\dim(\mathfrak{g}) - \text{rank}(\mathfrak{g}))$. From [48, 50] we know that if $\mathfrak{g}$ is one of the classical Lie algebras A_n, B_n, C_n, D_n, then each Lax matrix of the form (6.1.1) generically provide us with a set of $\frac{1}{2}(\dim(\mathfrak{g})+\text{rank}(\mathfrak{g}))$ independent functions. The spectral invariants of Lax matrices $L_a(\lambda)$ with consecutive indices intersects, in fact it holds:

$$\text{Tr}\left(\sum_{i=1}^{k+1} \text{res}_{\lambda=0}\frac{1}{\lambda^i}\left(L_a(\lambda)\right)^k\right) = \text{Tr}\left(\text{res}_{\lambda=0}\frac{1}{\lambda}\left(L_{a+1}(\lambda)\right)^k\right) =$$
$$= \text{Tr}\left(\sum_{i=1}^{a} A_i\right)^k \qquad a = 2, \ldots, N-1 \tag{5.2.5}$$

The index k in equation (5.2.5) runs over the so called *exponents* of the Lie algebra $\mathfrak{g}$ that are a given set of $\text{rank}(\mathfrak{g})$ integers. Moreover, it holds:

$$\text{Tr}\left(\text{res}_{\lambda=0}\frac{1}{\lambda}\left(L_2(\lambda)\right)^k\right) = \text{Tr}\left(A_1^k\right)$$

Such functions give us a set of $\text{rank}(\mathfrak{g})$ common Casimirs for the Poisson tensors P and R:

$$Pd\text{Tr}\left(A_1^k\right) = Rd\text{Tr}\left(A_1^k\right) = 0 \tag{5.2.6}$$

Summarizing, if we exclude from the counting the functions given by (5.2.5) and (5.2.6) then from each Lax matrix $L_a(\lambda)$ we obtain a set of $\frac{1}{2}(\dim(\mathfrak{g}) - \text{rank}(\mathfrak{g}))$ independent functions. Having excluded the functions (5.2.5) and (5.2.6) the spectral invariants of a Lax matrix $L_a(\lambda)$ depend on the first "a" coordinates, none excluded. So, spectral invariants of different Lax matrices depend on different sets of coordinates and are, therefore, functionally independent. Concluding, from the Lax matrices $L_a(\lambda)$, $a = 2, \ldots, N$ we obtain a total of $\frac{N-1}{2}(\dim(\mathfrak{g}) - \text{rank}(\mathfrak{g}))$ functionally independent involutive functions. To reach complete integrability we are missing $\frac{1}{2}(\dim(\mathfrak{g}) - \text{rank}(\mathfrak{g}))$ more integrals. To obtain such integrals we must resort again to the global invariance of the integrals defined by (5.2.4) under the global action of the Lie group G associated with the Lie algebra $\mathfrak{g}$.

As in chapter 4, if $\tau \in \mathfrak{g}$ then the functions

$$\phi_\tau = \text{Tr}\left(\sum_{i=1}^{N} A_i \tau\right) \tag{5.2.7}$$

Poisson commute with the spectral invariants of the Lax matrices $L_a(\lambda)$. Again these functions correspond to a decrease of the rank of the Poisson pencil $R - \lambda P$

for a particular value of λ. In fact, the functions (5.2.7) obey the particular "recurrence relations":

$$Rd\phi_\tau = (N-1)Pd\phi_\tau$$

so that for $\lambda = N-1$ the rank of the Poisson pencil $R-\lambda P$ degenerate abruptly. The Lenard "sequence" associated with the functions ϕ_τ is somewhat peculiar; indeed we can associate with each ϕ_τ a Lenard diagram which is a closed loop, to be compared with the usual ladder typical of iterable Hamiltonians.

One can extract from the algebra spanned by the functions (5.2.7) a maximal abelian subalgebra using the following trick. Fixed an element $\sigma \in \rho(\mathfrak{g})$ with simple spectrum, one can construct the Lax matrix:

$$L_{N+1}(\lambda) = \sigma + \frac{\sum_{i=1}^N A_i}{\lambda} \tag{5.2.8}$$

The matrix (5.2.8) admits a r-matrix formulation with respect to the ordinary product Poisson bracket $\{,\}_P$.

$$\sum_{ijkl}\{L_{N+1}(\lambda)_{ij}, L_{N+1}(\mu)_{kl}\}_P e_{ij} \otimes e_{kl} = [L_{N+1}(\lambda)\otimes \mathbb{I} + \mathbb{I}\otimes L_{N+1}(\mu), r(\lambda-\mu)] \tag{5.2.9}$$

where $r(\lambda)$ is again given by equation (3.0.9). We conclude that the spectral invariants of the Lax matrix $L_{N+1}(\lambda)$ form an involutive family [5]. Moreover, since they are functions only of the "global" variables $\sum_{i=1}^N A_i$ they are in involution also with the spectral invariants of the other Lax matrices $L_a(\lambda)$, $a = 2,\ldots,N$. Using again the results of ([48, 50]) we can state that for $\mathfrak{g}$ being one of the classical Lie algebras, the Lax matrix (5.2.8) generically provide us with a set of $\frac{1}{2}(\dim(\mathfrak{g}) + \mathrm{rank}(\mathfrak{g}))$ independent spectral invariants. However, it must be noticed that

$$\begin{aligned}\mathrm{Tr}\left(\sum_{i=1}^{k+1}\mathrm{res}_{\lambda=0}\frac{1}{\lambda^i}\left(L_N(\lambda)\right)^k\right) &= \mathrm{Tr}\left(\mathrm{res}_{\lambda=0}\lambda^{k-1}\left(L_{N+1}(\lambda)\right)^k\right) = \\ = \mathrm{Tr}\left(\sum_{i=1}^N A_i\right)^k \qquad & a = 2,\ldots,N-1\end{aligned} \tag{5.2.10}$$

So the rank$(\mathfrak{g})$ spectral invariants of $L_{N+1}(\lambda)$ (5.2.10) are in common with the Lax matrix $L_N(\lambda)$. The other ones, instead, give us a set of $\frac{1}{2}(\dim(\mathfrak{g}) - \mathrm{rank}(\mathfrak{g}))$ involutive functions that are functionally independent from the spectral invariants of the other Lax matrices. In fact they are the only functions (apart from those given by (5.2.10) which depend on the global variables $\sum_{i=1}^N A_i$.

In this way we have recovered the integrals we were missing to obtain complete integrability.

We end this Section with a comment concerning super–integrability of the model. To this end we remark that we have at our disposal two Poisson pencils to construct families of commuting integrals for the Gaudin (homogeneous) Hamiltonian H_G: the pencil $R - \lambda P$ and the pencil $Q - \lambda P$, described in Section 4. On the $N(r(r-1))$–dimensional generic symplectic leaves of P they give rise to two distinct $(N-1)(r(r-1))/2)$ families of integrals of the motion K_m^{lj} and $\widetilde{K}_m^{lj}$. Direct computations (which we performed for $r = 3, 4$ and $N \leq 6$) suggest that the number of functionally independent elements in the union of the two families be $(N-1)(r(r-1)) - (r-1)$. In other words, also taking into account the integrals coming from the global $SL(r)$ invariance of the model, we have super-integrability for the $sl(r)$ Gaudin model, that, however, is maximal only for the $sl(2)$ case.

Chapter 6

Separation of Variables for the sl(r) Homogeneous Gaudin Systems

In this chapter we will apply the results of section 2.2 and show how, for $\mathfrak{g} = sl(r)$, the bihamiltonian structure $P_\lambda = R - \lambda P$ associated with the parameter independent integrals of the Gaudin model provides a set of separation coordinates and relations for the H-J equations associated with H_G.

The first step is to show that P_λ induces a ωN manifold structure on the generic symplectic leaf $\mathcal{S}$ of P. We consider the distribution generated by the $N(r-1)$ vector fields

$$W_i^a := \frac{\partial}{\partial (A_i)_{r,a}}, \quad i = 1, \dots, N, \quad a = 1, \dots, r-1, \tag{6.0.1}$$

where $(A_i)_{r,a}$ denotes the $a-th$ entry of the last column of the A_i matrix.

Proposition 6.0.3 *The distribution $\mathcal{Z}$ defined by the vector fields W_i^a satisfies the hypotheses of Proposition 2.2.9.*

Proof. The key point is the following observation on the (ordinary) Lie-Poisson brackets on a single copy of $M = sl(r)$. The Poisson bracket of two functions F, G on M, is given by $\{F, G\} = \mathrm{Tr}(\frac{\partial F}{\partial A} \cdot [A, \frac{\partial G}{\partial A}]) = -\mathrm{Tr}(A \cdot [\frac{\partial F}{\partial A}, \frac{\partial G}{\partial A}])$. Let A_{ab} denote the a, b entry of A, and consider the family of $r-1$ vector fields on M defined by $W_a = \frac{\partial}{\partial A_{r,a}}$, $a = 1, \dots, r-1$, as well as the distribution $\mathcal{Z} = \langle W_a \rangle \subset TM$ defined by the W_a.

We notice that differentials of functions vanishing along $\mathcal{Z}$ admit a very simple matrix representation. Indeed W_i has a matrix representation given by the tensor product

$$\mathrm{Lie}_{W_i}(A) = |e_a\rangle \otimes \langle \epsilon_r|, a = 1, \dots, r-1,$$

where $|e_a\rangle$ is the standard basis of $\mathbb{C}^r$ (and $\langle \epsilon_a|$ the dual basis). So $\mathrm{Lie}_{W_i}(F) = 0$ iff $\left(\frac{\partial F}{\partial A}\right)_{r,a} = 0, a = 1, \ldots, r-1$, i.e., iff $\frac{\partial F}{\partial A}$ lies in the lower maximal parabolic subalgebra $\mathfrak{p}_-$ of $sl(r)$.

Let now W denote any element in $\mathcal{Z}$, and let F, G be functions such that $W(F) = W(G) = 0$, and let us compute $\mathrm{Lie}_W(\{F,G\})$. Thanks to the Leibniz property of the Lie derivative and the fact that W is a constant vector field we have that

$$\mathrm{Lie}_W(\{F,G\}) = -\mathrm{Tr}\left(\mathrm{Lie}_W(A)\cdot\left[\frac{\partial F}{\partial A}, \frac{\partial G}{\partial A}\right]\right) \tag{6.0.2}$$

which vanishes as well since $\mathfrak{p}_-$ is indeed a Lie subalgebra of $sl(r)$.

In the case of the N-particle $sl(r)$ Gaudin model, whose phase space is parametrized by n matrices A_i we consider the family of $N\cdot(r-1)$ vector fields defined in (6.0.1).

The distribution $\mathcal{Z}$ generated by these vector fields is generically transversal to the symplectic leaves of the Lie–Poisson product structure on $sl(r)^N$. We now prove that the space of functions vanishing along $\mathcal{Z}$ is a Poisson subalgebra for any *affine* Poisson tensor Q.

The brackets $\{F,G\}_Q = \langle dF, QdG\rangle$ are given by the multiple sum

$$\{F,G\}_Q = \sum_{i,j,k=1}^{N} \mathrm{Tr}\left(\frac{\partial F}{\partial A_i}\cdot\left(\sum_{k=1}^{N} c_{i,j}^k\left[A_k, \frac{\partial G}{\partial A_j}\right]\right) + d_{i,j}^k\left[\sigma_k, \frac{\partial G}{\partial A_j}\right]\right)$$

where σ_k denote constant matrices. Noticing that the differentials of functions F vanishing along $\mathcal{Z}$ are represented by n-tuples of matrices $dF = (\frac{\partial F}{\partial A_1}, \frac{\partial F}{\partial A_2}, \ldots, \frac{\partial F}{\partial A_n})$ with $\frac{\partial F}{\partial A_i} \in \mathfrak{p}_-$, $i = 1, \ldots, N$, we see that the Lie derivatives $W_a^i(\{F,G\}_Q)$ are given by multiple sums of terms like those of Eq. (6.0.2), and so vanish whenever $W_a^i(F) = W_a^i(G) = 0$.

□

To construct a set of flat generators Z_i^a for $\mathcal{Z}$, we can argue as follows. In the case of $M^1 = sl(n)$, we normalize the W_i^a with respect to the coefficients $C_1, \ldots, C_{r-1}$ of the characteristic polynomial of A. The normalization for the N sites case is done site by site. It is not difficult to realize that such normalized generators Z_a^i provide the GZ manifold $(M, R-\lambda P)$ with the structure of an affine GZ manifold, according to Definition 2.2.10.

Let us now consider the Lax matrices

$$L_a = (\lambda - a + 2)A_a + \sum_{b=1}^{a-1} A_b,\ a = 2, \ldots, N$$

define

$$M_a(\lambda, \xi) = \xi \mathbf{I} - L_a(\lambda),$$

and denote their classical adjoint with $\widetilde{M}_a(\lambda, \xi)$. The determinants of the matrices M_a define, thanks to Proposition 5.2.2, polynomial Casimirs $K_i^{(a)}(\lambda)$ for $R - \lambda P$, via:

$$\mathrm{Det}(M_a(\lambda, \xi)) = \xi^r + \sum_{i=1}^{r-1} K_i^{(a)}(\lambda)\xi^{r-i-1}. \tag{6.0.3}$$

We now list a few elementary but important facts following from the definitions of the Casimirs of the Poisson pencil $K^{(a)}(\lambda)$ and of the normalized transversal vector fields Z_i^a.

1. The $(N-1)(r-1) \times (N-1)(r-1)$ matrix of the deformations of the Casimirs w.r.t. the transversal vector fields has the following block form:

$$D_\beta^\alpha = \begin{bmatrix} D_2^2 & 0 & \cdots & \cdots & 0 \\ D_2^3 & D_3^3 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & 0 \\ D_2^k & D_3^k & \cdots & & D_k^k \end{bmatrix} \tag{6.0.4}$$

where the $(r-1)\times(r-1)$ matrices D_b^a are given by: $\left(D_b^a\right)_j^i = \mathrm{Lie}_{Z_i^b}(K_j^{(a)}(\lambda))$. The determinant $\mathcal{D}^a(\lambda)$ of D_a^a can be written as

$$\mathcal{D}^a(\lambda) = (\lambda - a + 2)^{r-1}\Delta^a(\lambda) \tag{6.0.5}$$

where $\Delta^a(\lambda)$ is a monic polynomial of degree $r(r-1)/2$. This follows from the observation that

$$\mathrm{Lie}_{Z_k^a}(\mathrm{Det}(M_a(\lambda, \xi)_a)) = -\mathrm{Tr}(\widetilde{M}_a(\lambda, \xi)\mathrm{Lie}_{Z_k^a}(L_a))$$

with $\mathrm{Lie}_{Z_k^a}(L_a) = (\lambda - a + 2)\mathrm{Lie}_{Z_k^a}(A_a)$ and the fact that $\mathcal{D}^a(\lambda) = \mathrm{Det}(D_a^a)$ is monic of degree $r(r+1)/2 - 1$.

2. Thanks to the lower diagonal block form of the matrix D of equation (6.0.4), its diagonal blocks D_a^a satisfy Proposition (2.2.11). So its determinant satisfy, according to Proposition 2.2.12

$$(\widetilde{R} - \lambda P)d\mathcal{D}^a(\lambda) = \mathcal{D}^a(\lambda)Pd\mathcal{D}_1^a, \tag{6.0.6}$$

and thanks to the factorization property (6.0.5), we have

$$(\widetilde{R} - \lambda P_0)d\Delta^a(\lambda) = \Delta^a(\lambda)P_0 d\Delta_1^a. \tag{6.0.7}$$

We now recall from Proposition 6.0.3 that the normalized generators Z_i^a and the "constant" ones W_i^a are related by the Jacobian matrix G of the Casimirs of P w.r.t. the W_j^a, i.e,

$$W_i^a = \sum_j G_{ij}^a Z_j^a, \text{ with } G_{ij}^a = \mathrm{Lie}_{W_j^a}(K_{i,0}^{(a)}) \; a = 2, \ldots, N$$

If we define matrices E^a by means of the relations

$$\mathrm{Lie}_{W_j^a} \mathrm{Det}(M_a(\xi, \lambda)) = \sum_{i=1}^{r-1} \xi^{r-1-i} E_{ij}^a \qquad (6.0.8)$$

we see that they are related with the previously introduced matrices D^a by means of $E^a = G^a \cdot D^a$. Since G is independent of λ, the zeroes of the determinant of E^a coincide with the zeroes of $\mathcal{D}^a$. Now, $\mathrm{Lie}_{W_j^a}\mathrm{Det}(M_a(\xi, \lambda))$ is nothing but the determinant of the minor of $M_a(\xi, \lambda)$ relative to the r, j entry, as it is clear from the very definition of W_j^a. So, we see that:

a) The roots of $\mathcal{D}^a$ are those values of λ for which the $r-1$ equations, defined for $a = 2, \ldots, N$,

$$\sum_{i=1}^{r-1} \xi^{r-1-i} \mathcal{D}_{ij}^a = 0 \qquad (6.0.9)$$

admit non-trivial solutions. In particular, the roots of $\Delta^a(\lambda)$ will define non trivial elements λ_i^a, $i = 1, \ldots, r(r-1)/2$.

b) The values ξ_i^a corresponding to the roots λ_i^a of Δ^a are given by suitably normalized elements of the adjoint matrix $\widetilde{D}^a$.

c) Since the $r(r-1)/2$ pairs $\{\xi_i^a, \lambda_i^a\}$, for fixed a, annihilate the $r-1$ minors of the matrix $M_a(\xi, \lambda)$ relative to the entries $(1, r), \ldots (r-1, r)$, they annihilate the minor relative to the (r, r) entry as well, hence they annihilate the last row of the adjoint matrix of $M_a(\xi, \lambda)$:

$$\widetilde{M}_a^{kr}(\xi, \lambda) = 0 \qquad \forall k$$

Taking into account the three above remarks and Proposition 2.2.12 we can state:

Proposition 6.0.4 *The $(n-1)r(r-1)/2$ pairs of functions $\{\xi_i^a, \lambda_i^a\}$ obtained by means of the construction outlined above satisfy:*

- *the Jacobi separation relations*

$$Det(M_a(\xi_i^a, \lambda_i^a)) = 0$$

- *the differential relations*

$$\widetilde{R}d\lambda_i^a = \lambda_i^a P_0 d\lambda_i^a, \quad \widetilde{R}d\xi_i^a = \lambda_i^a P_0 d\xi_i^a.$$

In particular, their brackets w.r.t. the canonical Lie–Poisson brackets are of the form:

$$\{\lambda_j^b, \xi_i^a\} = \delta_{ij}\delta^{ab}\varphi_{ij}^{ab}(\xi_i^a, \lambda_i^a). \tag{6.0.10}$$

□

To finish our job we have to:
i) Discuss about the coordinates associated with the global gauge invariance of the Gaudin Systems
ii) Explicitly construct, out of the coordinates found so far, a set of *canonical* separated coordinates.
Point i) can be solved as follows. We have seen that a maximal abelian family of functions associated with the global gauge invariance is given by the spectral invariants of the Lax matrix

$$L_{N+1}(\lambda) = \sigma + \frac{\sum_{i=1}^N A_i}{\lambda} \tag{6.0.11}$$

For separation of variables, a more convenient choice of the integrals associated with the global $SL(r)$ invariance can be done as follows [16]. We pick the $r-1$ independent elements $\phi_{h_1}, \ldots, \phi_{h_{r-1}}$ associated with, say, the standard Cartan subalgebra of $sl(r)$,

$$\phi_{h_i} = \mathrm{Tr}\left(h_i \sum_{i=1}^N A_i\right)$$

and the Gel'fand-Cetlyn invariants, that is, the Casimirs of the nested subalgebras

$$sl(2) \subset sl(3) \subset \cdots \subset sl(r), \tag{6.0.12}$$

under the map $sl(r)^N \to sl(r)$ sending the N-tuple $\{A_1, \ldots, A_N\}$ into the total sum, $A_{tot} = \sum_{i=1}^N A_i$. The advantage of working with this family of functions, instead with the one defined by the Lax matrix (6.0.11), is that it is invariant along the distribution $\mathcal{Z}$ defined by (6.0.1). Now, one notices that any function φ depending only on the "global" matrix variable $A_{tot} = \sum_{i=1}^n A_i$, which is invariant along the distribution $\mathcal{Z}$ satifies the differential relation

$$\mathcal{N}^* d\varphi = (N-1)d\varphi. \tag{6.0.13}$$

In particular, this family includes the mutually commuting Hamiltonians of Gel'fand–Cetlyn type just introduced. Indeed, the property (6.0.13) follows from the fact that such a function φ satisfies the relation

$$Rd\varphi = (N-1)Pd\varphi$$

with respect to the undeformed pencil, and from the property that φ commutes with all the Hamiltonians of the hierarchy. It is not difficult to show that one can find inside the ring of functions, a set of $r(r-1)/2$ canonical coordinates. Thanks to (6.0.13) they will have vanishing Poisson brackets with the Nijenhuis coordinates of Proposition 6.0.4.

The solution to point ii) above can be simply done by means of a direct computation of the Poisson brackets between ξ_i^a and λ_i^a. In particular, this computation will implicitly prove that these quantities are functionally independent.

Proposition 6.0.5 *Let* $M_a(\lambda,\xi) = \xi - (\lambda - a + 2)A_i + \sum_{k=1}^{a-1} A_k$, *and* $\widetilde{M}_a(\lambda,\xi)$ *its classical adjoint. A set of coordinates* (λ_a^i, ξ_a^i) *satisfying*

$$\widetilde{M}_a^{kj}(\lambda_a^i, \xi_a^i) = 0$$

for a fixed j *and any* k, *has Poisson brackets:*

$$\{\lambda_a^i, \xi_a^i\} = (\lambda_a^i - a + 2)(\lambda_a^i - a + 1) \qquad (6.0.14)$$

Proof. We follow almost verbatim the proof of Theorem 1.3 in [1] with few variants. We consider the basis in $\rho(sl(r))$ given in terms of the standard $gl(r)$ basis by:

$$e_{ij} \quad i \neq j = 1, \ldots, r \qquad e_{ii} - \frac{1}{r}\mathbb{I} \quad i = 1, \ldots, r-1$$

Let us take $j = 1$, then the points (λ_a^i, ξ_a^i) are determined by the conditions

$$\tilde{M}_a^{k1}(\lambda_a^i, \xi_a^i) = 0 \quad \forall k.$$

Generically, only two among these equations are independent, say

$$\tilde{M}_a^{11} = \tilde{M}_a^{21} = 0$$

This assures that, generically, the matrix

$$F_a^i := \begin{pmatrix} \frac{\partial \tilde{M}_a^{11}}{\partial \lambda} & \frac{\partial \tilde{M}_a^{11}}{\partial \xi} \\ \frac{\partial \tilde{M}_a^{21}}{\partial \lambda} & \frac{\partial \tilde{M}_a^{21}}{\partial \xi} \end{pmatrix} (\lambda_a^i, \xi_a^i) \qquad (6.0.15)$$

is invertible. Using the change of variables defined by F_a^i we can write:

$$\{\lambda_a^i, \xi_a^i\} = \frac{1}{\det F_a^i}\{\tilde{M}_a^{11}(\lambda_a^i, \xi_a^i), \tilde{M}_a^{21}(\lambda_a^i, \xi_a^i)\}. \qquad (6.0.16)$$

Using the derivation property of the bracket, we have:

$$\begin{aligned}
&\{\tilde{M}_a^{11}(\lambda,\xi), \tilde{M}_a^{21}(\lambda,\xi)\} = \\
&= \sum_{pqrs} \frac{\partial \tilde{M}_a^{11}(\lambda,\xi)}{\partial M_a^{pq}(\lambda,\xi)} \frac{\partial \tilde{M}_a^{21}(\lambda,\xi)}{\partial M_a^{rs}(\lambda,\xi)} \{M_a^{pq}(\lambda,\xi), M_a^{rs}(\lambda,\xi)\} \qquad (6.0.17)
\end{aligned}$$

The Poisson brackets of the entries of the matrix $M_a(\lambda,\xi)$ are given by:

$$\begin{aligned}
&\{M_a^{ij}(\lambda,\xi), M_a^{kl}(\sigma,\eta)\} = \\
&= \mathrm{tr}\left[\left((\lambda-a+2)(\sigma-a+2)A_a + \sum_{r=1}^{a-1} A_r\right)(e_{jk}\delta_{il} - e_{il}\delta_{jk})\right] = \\
&= \frac{1}{\lambda-\sigma}\left[(\lambda-a+1)(\sigma-a+2)(M_a^{jk}(\lambda,\xi)\delta_{il} - M_a^{il}(\lambda,\xi)\delta_{jk}) + \right. \\
&\left. + (\lambda-a+2)(\sigma-a+1)(M_a^{il}(\sigma,\eta)\delta_{jk} - M_a^{jk}(\sigma,\eta)\delta_{il})\right] \qquad (6.0.18)
\end{aligned}$$

and taking the limit $(\sigma,\eta)\to(\lambda,\xi)$ we finally obtain:

$$\begin{aligned}
&\{M_a^{ij}(\lambda,\xi), M_a^{kl}(\lambda,\xi)\} = \lim_{(\sigma,\eta)\to(\lambda,\xi)} \{M_{ij}^a(\lambda,\xi), M_{kl}^a(\sigma,\eta)\} = \\
&= (\lambda-a+1)(\lambda-a+2)\left(\frac{dM_a^{jk}(\lambda,\xi)}{d\lambda}\delta_{il} - \frac{dM_a^{il}(\lambda,\xi)}{d\lambda}\delta_{jk}\right). \quad (6.0.19)
\end{aligned}$$

Substituting (6.0.19) inside (6.0.17) we get

$$\begin{aligned}
&\{\tilde{M}_a^{11}(\lambda,\xi), \tilde{M}_a^{21}(\lambda,\xi)\} = \\
&= (\lambda-a+1)(\lambda-a+2)\sum_{prs}\left(\frac{\partial \tilde{M}_a^{11}}{\partial M_a^{pr}}\frac{\partial \tilde{M}_a^{21}}{\partial M_a^{rs}} - \frac{\partial \tilde{M}_a^{11}}{\partial M_a^{rs}}\frac{\partial \tilde{M}_a^{21}}{\partial M_a^{pr}}\right)\frac{dM_a^{ps}}{d\lambda}. \quad (6.0.20)
\end{aligned}$$

The proof that

$$\sum_{prs}\left(\frac{\partial \tilde{M}_a^{11}}{\partial M_a^{pr}}\frac{\partial \tilde{M}_a^{21}}{\partial M_a^{rs}} - \frac{\partial \tilde{M}_a^{11}}{\partial M_a^{rs}}\frac{\partial \tilde{M}_a^{21}}{\partial M_a^{pr}}\right)\frac{dM_a^{ps}}{d\lambda}\Bigg|_{(\lambda=\lambda_a^i,\xi=\xi_a^i)} = \det(F_a^i)$$

is given in [1]. Proposition 6.0.5 now follows from equation (6.0.16) and (6.0.20).

□

6.1 Examples

6.1.1 Separation of Variables in the sl(2) case

We consider now the N-particle $sl(2)$ case. The aim is to show that the Hamilton–Jacobi equations associated with the Hamiltonians

$$H_a = \mathrm{Tr}\big(A_i \cdot \sum_{j=1}^{a-1} A_j\big), \quad a = 2, \dots, N, \tag{6.1.1}$$

and, in particular, the H–J equations associated with the physical Hamiltonian $H_G = \frac{1}{2}\sum_{i=1}^{N} H_i$ are separable in a very "simple" set of coordinates.

We consider the manifold $M = \mathcal{P}_{sl(2)}^{\otimes N}$, endowed with the Poisson pencil $R - \lambda P$, explicitly parametrized in terms of the N matrices

$$A_i = \begin{bmatrix} h_i & f_i \\ e_i & -h_i \end{bmatrix}. \tag{6.1.2}$$

The generic symplectic leaf S of P is a $2N$ dimensional symplectic manifold, defined by the equations

$$C_i = \frac{1}{2}\mathrm{Tr} A_i^2 = h_i^2 + e_i f_i, \quad i = 1, \dots, N,$$

and can be (generically) endowed with the $2N$ coordinates $\{h_i, f_i\}_{i=1,\dots,N}$.

A set of normalized transverse vector fields are given in this case by:

$$Z_i = \frac{1}{f_i}\frac{\partial}{\partial e_i}, \tag{6.1.3}$$

According to the recipe given in 6, we obtain a set of $2N - 2$ Nijenhuis coordinates solving the system of equations:

$$\begin{cases} (\widetilde{M_a}(\xi,\lambda))_{12} = (\lambda - a + 2) f_a + \sum_{i=1}^{a-1} f_i = 0 \\ (\widetilde{M_a}(\xi,\lambda))_{22} = \xi - (\lambda - a + 2) h_a - \sum_{i=1}^{a-1} h_i = 0 \end{cases} \tag{6.1.4}$$

The solutions to (6.1.4) are given by the $2N - 2$ functions

$$\begin{aligned} \lambda_a &= -\frac{\sum_{k=1}^{a-1} f_k}{f_a} + (a-2), \qquad a = 2, \dots, N \\ \xi_a &= (\lambda_a - (a-2))\, h_a + \sum_{k=1}^{a-1} h_k, \qquad a = 2, \dots, N \end{aligned} \tag{6.1.5}$$

Using the explicit expressions of the Poisson brackets:

$$\{h_i, e_j\}_P = \delta_{ij} e_j, \quad \{h_i, f_j\}_P = -\delta_{ij} f_j, \quad \{e_i, f_j\}_P = 2\delta_{ij} h_j$$

$$\{h_i, e_j\}_R = \delta_{ij} \left[(i-1)e_i - \sum_{k=1}^{i-1} e_k \right] + \theta_{(i-j)} e_j + \theta_{(j-i)} e_i$$

$$\{h_i, e_j\}_R = -\delta_{ij} \left[(i-1)f_i - \sum_{k=1}^{i-1} f_k \right] - \theta_{(i-j)} f_j - \theta_{(j-i)} f_i$$

$$\{e_i, f_j\}_R = 2 \left\{ \delta_{ij} \left[(i-1)h_i - \sum_{k=1}^{i-1} h_k \right] + \theta_{(i-j)} h_j + \theta_{(j-i)} h_i \right\}.$$

it is not difficult to check that the coordinates λ_a, ξ_a are a set of Nijenhuis coordinates:

$$N^* d\lambda_a = \lambda_a d\lambda_a \qquad N^* d\xi_a = \lambda_a d\xi_a$$

Moreover, one can also verify that, as implied by proposition 6.0.5, the coordinates λ_a, ξ_a obey the Poisson brackets:

$$\{\lambda_a, \xi_b\} = \delta_{ab}(\lambda_a - (a-2))(\lambda_a - (a-1)), a = 2, \ldots, N. \tag{6.1.6}$$

Hence, one can choose

$$\phi_a = \frac{\xi_a}{(\lambda_a - (a-2))(\lambda_a - (a-1))} \quad a = 2, \ldots, N \tag{6.1.7}$$

which are canonically conjugated to the λ_a's.

One can complete the set $\lambda_a, \phi_a,\ a = 2, \ldots, N$ to a set of Darboux-Nijenhuis coordinates adding the two functions:

$$\lambda_1 = \sum_{i=1}^{N} f_i$$

$$\phi_1 = \frac{\sum_{i=1}^{N} h_i}{\sum_{i=1}^{N} f_i}$$

coming from the global $SL(2)$ invariance of the system.

6.1.2 Separation of variables in the sl(3) case

Now we consider the Poisson manifold $M = sl(3)^n$, endowed with the Poisson pencil $R - \lambda P$ and parametrized by the n matrices

$$A_i = \begin{pmatrix} h_{1,i} & e_{1,i} & e_{3,i} \\ f_{1,i} & h_{2,i} - h_{1,i} & e_{2,i} \\ f_{3,i} & f_{2,i} & -h_{2,i} \end{pmatrix} \quad i = 1, \ldots, n. \tag{6.1.8}$$

On this manifold the Poisson tensor P has $2n$ Casimirs:

$$C_2^i = \frac{1}{2}\mathrm{Tr}\big((A_i^2)\big), \quad C_3^i = \frac{1}{3}\mathrm{Tr}\big((A_i)^3\big) \quad i = 1, \dots, n. \tag{6.1.9}$$

The characteristic polynomials of the Lax matrices $L_a = (\lambda - a + 2)A_a + \sum_{b=1}^{a-1} A_b$ are expressed as

$$\Gamma(\mu, \lambda) = \mu^3 - \mu H_2^{(a)}(\lambda) - K_3^{(a)}(\lambda). \tag{6.1.10}$$

The transversal distribution $\mathcal{Z}$ is generated by the $2n$ flat generators:

$$\begin{aligned}
Z_i^2 &= \frac{1}{d}\left[(f_{3,i}(h_{1,i} - h_{2,i}) + f_{2,i}f_{1,i})\frac{\partial}{\partial e_{2,i}} + (f_{2,i}h_{1,i} - f_{3,i}e_{1,i})\frac{\partial}{\partial e_{3,i}}\right] \\
Z_i^3 &= \frac{1}{d}\left[f_{2,i}\frac{\partial}{\partial e_{3,i}} - f_{3,i}\frac{\partial}{\partial e_{2,i}}\right] \\
d &= f_{2,i}f_{3,i}(2h_{1,i} - h_{2,i}) + f_{2,i}^2 f_{1,i} - f_{3,i}^2 e_{1,i}
\end{aligned}$$

The symplectic leaves of P are generically parametrized by matrices A_i of the form:

$$\begin{pmatrix} h_{1,i} & e_{1,i} & \Phi_{3,i} \\ f_{1,i} & h_{2,i} - h_{1,i} & \Phi_{2,i} \\ f_{3,i} & f_{2,i} & -h_{2,i} \end{pmatrix}$$

where $\Phi_{2,i}$ and $\Phi_{3,i}$ are suitable functions of the coordinates $h_{1,i}, h_{2,i}, f_{1,i}, f_{2,i}, f_{3,i}, e_{1,i}$, parametrically depending on the Casimirs (6.1.9).

The coordinates (λ_a^i, ξ_a^i) can quite explicitly be found by means of the following steps:

We consider the matrix $M_a(\lambda, \xi) = \xi - L_a(\lambda)$ and its adjoint $\widetilde{M_a}(\lambda, \xi)$. We have to look for the common zeroes of the elements $\widetilde{M_a}(\lambda, \xi)_{3,1}$ and $\widetilde{M_a}(\lambda, \xi)_{3,2}$, that is, for the common zeroes of

$$\mathrm{Det}\begin{pmatrix} -L_a(\lambda)_{2,1} & \xi - L_a(\lambda)_{2,2} \\ -L_a(\lambda)_{3,1} & -L_a(\lambda)_{3,2} \end{pmatrix}, \qquad \mathrm{Det}\begin{pmatrix} \xi - L_a(\lambda)_{1,1} & -L_a(\lambda)_{1,2} \\ -L_a(\lambda)_{3,1} & -L_a(\lambda)_{3,2} \end{pmatrix}. \tag{6.1.11}$$

Taking into account the form of the vector fields Z_a^α and of the characteristic polynomial (6.1.10), we can identify the system (6.1.11) with

$$\begin{cases} \xi \mathrm{Lie}_{Z_a^2} H_2^{(a)} + \mathrm{Lie}_{Z_a^2} H_3^{(a)} = 0 \\ \xi \mathrm{Lie}_{Z_a^3} H_2^{(a)} + \mathrm{Lie}_{Z_a^3} H_3^{(a)} = 0 \end{cases}, \tag{6.1.12}$$

where

$$\mathrm{Det}(M_a(\xi,\lambda)) = \xi^3 - H_2^{(a)}\xi - H_3^{(a)}.$$

As we have noticed in Section 6 , we can factor out $(\lambda - a + 2)$ from each line of this system, and consider, in matrix form the equivalent system:

$$\langle \xi, 1| \begin{pmatrix} G^a_{2,2} & G^a_{2,3} \\ G^a_{3,2} & G^a_{3,2} \end{pmatrix}, \quad \mathrm{with} G^a_{\alpha,\beta} = \mathrm{Lie}_{Z^\alpha_a} H^{(a)}_\beta/(\lambda - a + 2). \qquad (6.1.13)$$

We notice that $G^a_{\alpha,\beta}$ are polynomials in λ of degree $\alpha - 1$, so that the three zeroes $\lambda_a^1, \lambda_a^2, \lambda_a^3$ represent the compatibility condition for the system (6.1.12); the corresponding coordinates $\xi_a^1, \xi_a^2, \xi_a^3$ are thus given by, e.g.,

$$\xi^\alpha_a = -G^a_{3,3}/G^a_{2,3}|_{\lambda=\lambda^\alpha_a}$$

We remark that our procedure for finding separation coordinates exactly matches the one introduced, in the framework of r-matrix theory, in [15].

To proceed in our analysis, it is useful to make the following identifications. Along with the Casimirs (6.1.9) we define:

$$B_a = \sum_{b=1}^{a-1} A_b,\ \mathsf{H}^a = \mathrm{Tr}(A_a B_a), \mathsf{K}^a = \mathrm{Tr}((A_a)^2 B_a),\ \mathsf{J}^a = \mathrm{Tr}(A_a (B_a)^2),\ a = 2, \dots, n. \qquad (6.1.14)$$

Also, it is convenient to shift the coordinates λ^i_a defining

$$\zeta^i_a = \lambda^i_a - a + 2, i = 1, 2, 3,\ \ a = 2, \dots, n. \qquad (6.1.15)$$

The coordinates ζ^i_a, ξ^i_a are still Nijenhuis coordinates and satisfy the separation relations

$$\mathrm{Det}(\xi^i_a - \zeta^i_a A_a - B_a) = 0 \qquad (6.1.16)$$

Since their Poisson brackets are

$$\{\xi^i_a, \zeta^j_b\} = \delta_{a,b}\delta^{i,j}\zeta^i_a(\zeta^i_a - 1), \qquad (6.1.17)$$

Darboux-Nijenhuis coordinates are given by the pairs $\zeta^i_a, \rho^i_a = \xi^i_a/\zeta^i_a(\zeta^i_a - 1)$. The explicit form of the separation relations (6.1.16), in terms of the coordinates ξ, ζ, the Casimirs, and the constants of the motion defined in (6.1.14) are:

$$\begin{aligned} \Gamma^a(\zeta_a, \xi_a) = \xi_a^3 - \xi_a\Big(C_2^a\zeta_a^2 + \mathsf{H}^a\zeta_a + \sum_{b=2}^{a-1}(C_2^b + \mathsf{H}^b) + C_2^1\Big) \\ - \Big(C_3^a\zeta_a^3 + \mathsf{K}^a\zeta_a^2 + \mathsf{J}^a\zeta_a + \sum_{b=2}^{a-1}(C_3^b + \mathsf{K}^b + \mathsf{J}^b) + C_3^1\Big) = 0. \end{aligned} \qquad (6.1.18)$$

The solution W of the (stationary) Hamilton-Jacobi equations of the $sl(3)$ Gaudin model can be expressed as:

$$W = \sum_{a=2}^{n} \left(\sum_{\alpha=1}^{3} \int^{P_a^\alpha} \frac{\xi d\zeta}{\zeta(\zeta-1)} \right) + \sum_{\alpha=1}^{3} H_t^\alpha q_t^a, \tag{6.1.19}$$

where P_a^α denotes the point $(\xi_a^\alpha, \zeta_a^\alpha) \in \Gamma^a$, and H_t^λ and q_t^α denote, respectively, a complete family of Gel'fand-Cetlyn Hamiltonians associated with the global $SL(3)$ invariance of the model, and their canonically conjugated variables.

To explicitly linearize the Hamiltonian flows, we have to compute the partial derivatives $\frac{\partial W}{\partial \mathsf{H}^a}, \frac{\partial W}{\partial \mathsf{K}^a}, \frac{\partial W}{\partial \mathsf{J}^a}$. Denoting with Γ_ξ^a the partial derivative w.r.t. ξ of the separation relations (6.1.18), we finally get:

$$\begin{aligned}
Q_{\mathsf{H}^a} &= \frac{\partial W}{\partial \mathsf{H}^a} = \sum_{\alpha=1}^{3} \left(\int^{P_a^\alpha} \frac{\xi d\zeta}{(\zeta-1)\Gamma_\xi^a} + \sum_{b=1}^{a-1} \int^{P_b^\alpha} \frac{\xi d\zeta}{\zeta(\zeta-1)\Gamma_\xi^b} \right), \\
Q_{\mathsf{K}^a} &= \frac{\partial W}{\partial \mathsf{K}^a} = \sum_{\alpha=1}^{3} \left(\int^{P_a^\alpha} \frac{\zeta d\zeta}{(\zeta-1)\Gamma_\xi^a} + \sum_{b=1}^{a-1} \int^{P_b^\alpha} \frac{d\zeta}{\zeta(\zeta-1)\Gamma_\xi^b} \right), \\
Q_{\mathsf{J}^a} &= \frac{\partial W}{\partial \mathsf{J}^a} = \sum_{\alpha=1}^{3} \left(\int^{P_a^\alpha} \frac{d\zeta}{(\zeta-1)\Gamma_\xi^a} + \sum_{b=1}^{a-1} \int^{P_b^\alpha} \frac{d\zeta}{\zeta(\zeta-1)\Gamma_\xi^b} \right).
\end{aligned} \tag{6.1.20}$$

In particular, we notice that the difference $Q_{\mathsf{K}^a} - Q_{\mathsf{J}^a}$ is given by the first kind Abelian integral on Γ^a

$$Q_{\mathsf{K}^a} - Q_{\mathsf{J}^a} = \sum_{\alpha=1}^{3} \left(\int^{P_a^\alpha} \frac{d\zeta}{\Gamma_\xi^a} \right).$$

In general, we see that the flows associated with the integrals $(\mathsf{H}^a, \mathsf{K}^a, \mathsf{J}^a)$, and hence with the Gaudin Hamiltonian

$$H_g = \sum_{a=2}^{n} \mathsf{H}^a,$$

linearize on the manifold $\mathcal{J}_{(3)}^2 \times \cdots \mathcal{J}_{(3)}^n$, where $\mathcal{J}_{(3)}^a$ is the threefold product of the Jacobian of the genus 1 curve $\Gamma^a(\xi, \zeta) = 0$, by means of the map (6.1.20) associated with the Abelian differentials (the last four being of the third kind)

$$\frac{d\zeta}{\Gamma_\xi^a}, \frac{\zeta d\zeta}{(\zeta-1)\Gamma_\xi^a}, \frac{d\zeta}{(\zeta-1)\Gamma_\xi^a}, \frac{d\zeta}{\zeta(\zeta-1)\Gamma_\xi^a}, \frac{\xi d\zeta}{\zeta(\zeta-1)\Gamma_\xi^a}. \tag{6.1.21}$$

The case of $sl(r)$, $r > 3$ can be treated analogously.

Chapter 7

Toward Quantization

We have seen that both in the classical case and in the quantum case, we can associate a Lax matrix to the rational Gaudin model

$$L^{c,q}(\lambda) = \sum_{i=1}^{N} \frac{A_i^{c,q}}{\lambda - \epsilon_i} + \sigma \tag{7.0.1}$$

where, if X_i^β, $i = 1, \dots, \dim(\mathfrak{g})$ are a set of generators for $\mathfrak{g}$, y_i^β, $i = 1, \dots, \dim(\mathfrak{g})$ are the corresponding coordinates on the associated Lie–Poisson manifold and ρ is a finite–dimensional representation of $\mathfrak{g}$, the matrices A_i^c and A_i^q are defined by:

$$A_i^c = g_{\alpha\beta}\rho(X^\alpha)y_i^\beta$$
$$A_i^q = g_{\alpha\beta}\rho(X^\alpha)X_i^\beta$$

Moreover, both in the classical and in the quantum case, the Lax matrix (7.0.1) admits an r–matrix formulation:

$$\{L^c(\lambda)_{ij}, L^c(\mu)_{kl}\}e_{ij} \otimes e_{kl} + [r(\lambda - \mu), L^c(\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes L^c(\mu)] = 0 \tag{7.0.2}$$
$$[L^q(\lambda) \otimes \mathbb{I}, \mathbb{I} \otimes L^q(\mu)] + [r(\lambda - \mu), L^q(\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes L^q(\mu)] \tag{7.0.3}$$

where $r(\lambda)$ is the same for the classical and quantum case:

$$r(\lambda) = \frac{1}{\lambda} g_{\alpha\beta}\rho(X^\alpha) \otimes \rho(X^\beta) \tag{7.0.4}$$

In the classical case equation (7.0.2) implies involutivity of all the spectral invariants of $L^c(\lambda)$ [5], while in the quantum case we have seen that Jurčo used equation (8.0.1) to prove only the commutativity of the quadratic observables [33, 54]:

$$H_i = \mathrm{res}_{\lambda=\epsilon_i} \mathrm{Tr}\left(L^q(\lambda)^2\right)$$

So, a natural question arises: does equation (8.0.1) implies also the commutativity of higher order operators in analogy with the classical case? An answer to this question was given by Feigin, Frenkel [22] (see also [23]); in fact, they proved that if equations (7.0.2) and (8.0.1) holds with an r–matrix of the form (7.0.4), then for any Lie algebra $\mathfrak{g}$ do exist quantum commuting operators that are deformations of the spectral invariants of the classical Lax matrix $L^c(\lambda)$. Unfortunately, their proof does not provide any explicit expression for these "higher Gaudin Hamiltonians" [23]. A way to obtain an explicit expression of such Hamiltonians in the case $\mathfrak{g} = gl(n)$ was suggested by Sklyanin in [57] but, due to the difficult computations needed, the problem of finding explicit formulas for arbitrary n is still unsolved.

On the other hand, in the classical case we have seen that, for $\sigma = 0$, some of the spectral invariants of $L^c(\lambda)$ define super–integrable Hamiltonian systems and consequently admit an alternative set of involutive functions. Such Hamiltonians are given by:

$$H^i_{SI} = \mathrm{Tr}\left(\left(\sum_{j=1}^{N} A^c_j\right)^i\right) \tag{7.0.5}$$

and can be seen as as the images of the Casimirs

$$C_i = \mathrm{Tr}\left((A^c)^i\right)$$

under the standard N–th coproduct:

$$H^i_{SI} = \Delta^{(N)}(C_i)$$
$$\Delta^{(m)}(A^c) = \sum_{j=1}^{m} (A^c_j)$$

In section 3 we have seen that, thanks to the homomorphism properties of the coproduct, one can construct the family of involutive functions

$$K_{ij} = \Delta^{(j)}(C_i) \otimes \Delta^{(N-j)}(1) \qquad j = 2, \ldots, N \tag{7.0.6}$$
$$K_{Ni} = H^i_{SI} \tag{7.0.7}$$

We showed as such functions define a completely integrable system only in very special cases. We faced the problem of completing the set of functions (7.1.4) to a completely integrable system in the case of $\mathfrak{g}$ being a classical Lie algebra. In such case a complete set of involutive functions is given by the spectral invariants of the N Lax matrices:

$$L^c_i(\lambda) = \lambda A^c_{i+1} + \Delta^i(A^c) \quad i = 1, \ldots, N-1 \qquad L^c_N(\lambda) = \sigma + \frac{\Delta^N(A^c)}{\lambda} \tag{7.0.8}$$

We will show in a moment that on UEA's it is possible to naturally define a structure of Hopf Algebra and consequently we have a natural definition of coproduct. Using the coproduct of $U(\mathfrak{g})$ one can easily "quantize" the family of functions (7.1.4) obtaining a family of commuting observables on $U(\mathfrak{g})^{\otimes N}$. So, using the coproduct approach, we obtain a family of compatible observables containing not only quadratic observables but also higher polynomials ones. A naturally arising question is if it is possible to "quantize" also the other integrals coming from the Lax matrices (7.0.8). We have some promising preliminary results on this question that we illustrate in the next section.

7.1 Coproduct method for Universal Enveloping Algebras

Since it is possible to define a structure of Hopf algebra also on UEA's, we can extend the coproduct method to quantum systems defined on $U(\mathfrak{g})^{\otimes N}$. We recall briefly how to define a Hopf algebra structure on $U(\mathfrak{g})$. As we have seen in chapter 2, a UEA is an associative algebra with unity so that to define an Hopf algebra structure on $U(\mathfrak{g})$ (see section 3) we are missing only the definitions of coproduct, counity and antipode. Such maps can be defined in total analogy with the case of Lie–Poisson algebras; Let us denote with $\{X^\alpha\}_{\alpha=1}^M$ a set of generators of $\mathfrak{g}$, then the action of the coproduct, the counity and the antipode on the generators of $\mathfrak{g}$ is given by:

$$\begin{aligned} &\Delta(X^\alpha) = X^\alpha \otimes 1 + 1 \otimes X^\alpha && \Delta(1) = 1 \otimes 1 \\ &\epsilon(X^\alpha) = 0 && \epsilon(1) = 1 \qquad \alpha = 1, \dots, M \\ &\sigma(X^\alpha) = -X^\alpha && \sigma(1) = 1 \end{aligned} \tag{7.1.1}$$

and can be extended to all $U(\mathfrak{g})$ imposing the properties of homomorphism with respect to the product for Δ and ϵ

$$\begin{aligned} \Delta(ab) &= \Delta(a)\Delta(b) \\ \epsilon(ab) &= \epsilon(a)\epsilon(b) \end{aligned} \qquad a, b \in U(\mathfrak{g})$$

and of antihomomorphism for σ:

$$\sigma(ab) = \sigma(b)\sigma(a) \quad a, b \in U(\mathfrak{g})$$

Since Δ is an homomorphism with respect to the product, it will be also an homomorphism with respect to the Lie bracket:

$$\Delta([a, b]) = \Delta(ab - ba) = \Delta(a)\Delta(b) - \Delta(b)\Delta(a) = [\Delta(a), \Delta(b)] \quad a, b \in U(\mathfrak{g}) \tag{7.1.2}$$

As in the classical case, we can define a chain of morphisms:

$$\Delta^{(i)} := (\Delta^{(2)} \otimes \overbrace{1 \otimes 1 \otimes \cdots \otimes 1}^{i-2}) \circ \Delta^{(i-1)}$$
$$\Delta^{(i)} : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{\otimes i} \qquad \Delta^{(2)} := \Delta$$
$$\left[\Delta^{(i)}(a), \Delta^{(i)}(b)\right] = \Delta^{(i)}([a,b]) \qquad a, b \in U(\mathfrak{g})$$

and we can state the "quantum" analog of theorem 3.1.4:

Theorem 7.1.1 *Let C_i be central elements in $U(\mathfrak{g})$, then the elements*

$$K_{ij} = \Delta^{(j)}(C_i) \otimes \Delta^{(N-j)}(1) \tag{7.1.3}$$

are commuting elements of $U(\mathfrak{g})^{\otimes N}$ and commutes with the $N-th$ coproduct of the generators:

$$[K_{ij}, K_{lm}] = 0$$
$$[K_{ij}, \Delta^{(N)}(X^\alpha)] = 0 \quad X^\alpha \in \mathfrak{g}$$

The proof of this theorem is exactly the same of theorem 3.1.4 provided one substitute $\mathcal{P}$ with $U(\mathfrak{g})$, Poisson brackets with commutators and the generators y^α of $\mathcal{P}$ with the generators X^α of $\mathfrak{g}$.

If $\mathfrak{g}$ is a classical Lie Algebra then, using the definition of coproduct given in (7.1.1), the set of commuting elements in $U(\mathfrak{g})^{\otimes N}$ one obtains using the coproduct method is given by:

$$K_{ij} = \mathrm{Tr}\left(\left(\sum_{l=1}^{j} A_l^q\right)^{m_i}\right) \qquad i = 1, \ldots, \mathrm{rk}(\mathfrak{g}) \quad j = 2, \ldots, N \tag{7.1.4}$$

where $m_1, \ldots, m_{\mathrm{rk}(\mathfrak{g})}$ denote a set of exponents of $\mathfrak{g}$ [10]. In this section we want to show that substituting A_i^c with A_i^q in the Lax matrices (7.0.8), we obtain N operator valued matrices

$$L_i^q = \lambda A_{i+1}^q + \Delta^i(A^q) \quad i = 1, \ldots, N-1 \qquad L_N^q = \sigma + \frac{\Delta^N(A^q)}{\lambda} \tag{7.1.5}$$

whose spectral invariants define a set of mutually commuting observables on $U(\mathfrak{g})^{\otimes N}$ containing the one defined by (7.1.4). From now on we will drop the label q for the Lax matrices (7.1.5), being always understood that we are dealing with the quantum case.

First of all we show that

Lemma 4 *It holds:*

$$\left[\mathrm{Tr}\left((L_i(\lambda))^m\right), \Delta^{(p)}(X^q)\right] = 0 \quad \text{for } p > i \quad X^q \in \mathfrak{g}$$

Proof: We recall that $\mathrm{Tr}(A_i^m)$ give us central elements of $U(\mathfrak{g})$ [10]. Hence, for $X^q \in \mathfrak{g}$ it holds:

$$[\mathrm{Tr}(A_i^m), X_i^q] = 0$$

Using the explicit expression of the matrices A_i^m this implies:

$$\begin{aligned}
&[\mathrm{Tr}\left(\left(A_i^m\right), X_i^q\right] = \\
&= \left[g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_m j_m} \mathrm{Tr}\left(\rho(X^{i_1}) \ldots \rho(X^{i_m})\right) X_i^{j_1} \ldots X_i^{j_m}, X_i^q\right] = \\
&= g_{i_1 j_1} \cdots g_{i_m j_m} g^{i_1 i_2 \ldots i_m} \left[X_i^{j_1} \ldots X_i^{j_m}, X_i^q\right] = \\
&= \sum_{r=1}^{m} g_{j_1 \ldots j_m} C_s^{j_r q} X_i^{j_1} \ldots X_i^{j_{r-1}} X_i^s X_i^{j_{r+1}} \ldots X_i^{j_m} = 0
\end{aligned}$$

Swapping names between j_r and s we get:

$$\sum_{r=1}^{m} g_{j_1 \ldots j_{r-1} s j_{r+1} \ldots j_m} C_{j_r}^{sq} X_i^{j_1} \ldots X_i^{j_m} = 0 \qquad (7.1.6)$$

Since it holds for any simple Lie algebra $\mathfrak{g}$, equation (7.1.6) enforces:

$$\sum_{r=1}^{m} g_{j_1 \ldots j_{r-1} s j_{r+1} \ldots j_m} C_{j_r}^{sq} = 0 \qquad (7.1.7)$$

for any choice of $j_1, \ldots, j_m$.

Now, using (7.1.7) and the commutation relations:

$$\begin{aligned}
&[X_{i+1}^j, \Delta^{(p)}(X^q)] = C_s^{jq} X_{i+1}^s \\
&[\Delta^{(i)}(X^j), \Delta^{(p)}(X^q)] = C_s^{jq} \Delta^{(i)}(X^s)
\end{aligned} \qquad p > i$$

we can easily prove the lemma:

$$\begin{aligned}
&\left[\mathrm{Tr}\left(\left(L_i(\lambda)\right)^m\right), \Delta^{(p)}(X^q)\right] = \left[\mathrm{Tr}\left(\left(\lambda A_{i+1} + \Delta^{(i)}(A)\right)^m\right), \Delta^{(p)}(X^q)\right] = \\
&= \left[g_{j_1 \ldots j_m} (\lambda X_{i+1}^{j_1} + \Delta^{(i)}(X^{j_1}) \ldots (\lambda X_{i+1}^{j_m} + \Delta^{(i)}(X^{j_m}), \Delta^{(p)}(X^q)\right] = \\
&\sum_{r=1}^{m} g_{j_1 \ldots j_{r-1} s j_{r+1} \ldots j_m} C_{j_r}^{sq} (\lambda X_{i+1}^{j_1} + \Delta^{(i)}(X^{j_1}) \ldots (\lambda X_{i+1}^{j_m} + \Delta^{(i)}(X^{j_m})) = 0
\end{aligned}$$

□

From lemma (4) it follows that the spectral invariants of a Lax matrix $L_i(\lambda)$ commute with those of a different Lax matrix $L_j(\mu)$:

$$[\mathrm{Tr}\left(\left(L_i(\lambda)\right)^m\right), \mathrm{Tr}\left(\left(L_j(\mu)\right)^n\right)] = 0 \qquad \text{if } i \neq j \quad \forall m, n \qquad (7.1.8)$$

In fact, assuming $j > i$, then

$$\mathrm{Tr}\left((L_j(\mu))^n\right) = g_{i_1\ldots i_n}\left((\mu X^{i_1}_{j+1} + \Delta^{(j)}(X^{i_1}))\ldots(\mu X^{i_n}_{j+1} + \Delta^{(j)}(X^{i_n}))\right) \quad (7.1.9)$$

but it holds

$$\left[\mathrm{Tr}\left((L_i(\lambda))^m\right), \Delta^{(j)}(X^q)\right] = 0$$
$$\left[\mathrm{Tr}\left((L_i(\lambda))^m\right), X^q_{j+1}\right] = 0$$

so that (7.1.8) is verified.

Let us define

$$H(i,l,m) := \mathrm{res}_{\lambda=0}\frac{1}{\lambda^{l+1}}\mathrm{Tr}\left(L_i(\lambda)^m\right) \qquad (i = 1,\ldots,N-1,\ l = 0,\ldots,m) \quad (7.1.10)$$

We have that

$$[H(i,l,m), H(j,p,q)] = 0 \qquad \text{if } i \neq j \quad (7.1.11)$$

The commuting observables coming from the coproduct are given by:

$$\mathrm{Tr}\left(\left(\sum_{k=1}^{i+1} A^q_k\right)^m\right) = \sum_{l=0}^{m} H(i,l,m) = H(i+1,0,m) \qquad i = 1,\ldots,N-1 \quad (7.1.12)$$

Consider equation (7.1.12) in the case $i = N-1$ as the definition of $H(N,0,m)$. From equation (7.1.12) it follows that the observables coming from the coproduct commute with *all* the spectral invariants of the matrices (7.0.1):

$$[H(i,0,m), H(j,p,q)] = 0 \quad i = 1,\ldots,N\ j = 1,\ldots,N-1 \quad (7.1.13)$$

In fact, we have seen that $[H(i,0,m), H(j,p,q)] = 0$ if $i \neq j$; if we have instead $i = j$ we can use equation (7.1.12) to write:

$$[H(i,0,m), H(i,p,q)] = \sum_{l=0}^{m}[H(i-1,l,m), H(i,p,q)] = 0 \qquad i > 1$$

If $i = 1$, then $H(1,0,m)$ is a Casimir and (7.1.13) is verified as well. Notice that

$$H(i,m,m) = \mathrm{Tr}\left(\left(A^q_{i+1}\right)^m\right)\ i = 1,\ldots,N-1 \quad (7.1.14)$$

are Casimirs in $U(\mathfrak{g})^{\otimes N}$, so that it holds also

$$[H(i,m,m), H(j,p,q)] = 0 \quad i = 1,\ldots,N\ j = 1,\ldots,N-1 \quad (7.1.15)$$

Thanks to the equations (7.1.11), (7.1.13) and (7.1.15), we can state commutativity of the spectral invariants of the Lax matrices (7.1.5) until cubic terms

Lemma 5 *It holds*

$$[\mathrm{Tr}\left((L_i(\lambda))^m\right), \mathrm{Tr}\left((L_i(\mu))^n\right)] = 0 \qquad \text{if} \quad m,n = 1,2,3 \tag{7.1.16}$$

Proof: We have to prove just the commutativity of the three quantities

$$H(i,1,2),\ H(i,1,3),\ H(i,2,3)$$

We have:

$$\begin{aligned}&[H(i,1,2), H(i,1,3)] = [H(i,0,2) + H(i,1,2) + H(i,2,2), H(i,1,3)] = \\ &[H(i+1,0,2), H(i,1,3)] = 0\end{aligned}$$

Equation $[H(i,1,2), H(i,2,3)] = 0$ can be proven exactly in the same way. Finally

$$\begin{aligned}&[H(i,1,3), H(i,2,3)] = [H(i,0,3) + H(i,1,3) + H(i,2,3) + H(i,3,3), H(i,2,3)] = \\ &= [H(i+1,0,3), H(i,2,3)] = 0\end{aligned}$$

□

7.2 An r–matrix formulation

Starting from a given Lax matrix $L_i(\lambda)$, we have shown explicitly how to construct linear, quadratic and cubic commuting observables. We prove the existence of higher order commuting observables in the following way. We introduce a set of Lax matrices $\widetilde{L}_i(\lambda)$ equivalent to the set (7.1.5) and we show that for this new set equation (8.0.1) holds with an r–matrix of the form (7.0.4). Then, thanks to the result of Feigin and Frenkel [22] the existence of higher order commuting observables is assured. However, we do not know if these new observables are still given by traces of powers of the matrices $\widetilde{L}_i(\lambda)$.

The matrices $\widetilde{L}_i(\lambda)$ are defined as:

$$\widetilde{L}_i(\lambda) = \frac{A_{i+1}}{\lambda - \epsilon_1} + \frac{\Delta^{(i)}(A)}{\lambda - \epsilon_2} \qquad i = 1,\dots,N-1 \tag{7.2.1}$$

$$\widetilde{L}_N(\lambda) = \sigma + \frac{\Delta^{(N)}(A)}{\lambda} \tag{7.2.2}$$

with ϵ_1 and ϵ_2 arbitrary parameters $\epsilon_1 \neq \epsilon_2$. It is easy to see that the spectral invariants of the matrices (7.2.1) coincide with those of $L_i(\lambda)$, $i = 1,\dots,N-1$:

$$\mathrm{Tr}\left(\mathrm{res}_{\lambda=0}\frac{1}{\lambda^m}(L_i(\lambda))^n\right) = \mathrm{Tr}\left(\mathrm{res}_{\lambda=\epsilon_1}(\epsilon_1-\epsilon_2)^{n-m+1}(\lambda-\epsilon_1)^{m-2}(\widetilde{L}_i(\lambda))^n\right)$$

$m = 2,\dots,n+1$

and

$$\mathrm{Tr}\left(\mathrm{res}_{\lambda=0}\frac{1}{\lambda}(L_i(\lambda))^n\right) = \mathrm{Tr}\left(\mathrm{res}_{\lambda=\epsilon_2}(\lambda-\epsilon_2)^{n-1}(\widetilde{L}_i(\lambda))^n\right)$$

Now we prove that the matrices $\widetilde{L}_i(\lambda)$ admits an r–matrix:

Lemma 6 *The matrices $\widetilde{L}_i(\lambda)$, $i = 1, \ldots, N$ satisfy the equations:*

$$\left[\widetilde{L}_i(\lambda)\otimes\mathbb{I}, \mathbb{I}\otimes\widetilde{L}_i(\mu)\right] + \left[r(\lambda-\mu), \widetilde{L}_i(\lambda)\otimes\mathbb{I} + \mathbb{I}\otimes\widetilde{L}_i(\mu)\right] = 0 \qquad (7.2.3)$$

where

$$r(\lambda) = \frac{1}{\lambda}g_{\alpha\beta}\rho(X^\alpha)\otimes\rho(X^\beta) \qquad (7.2.4)$$

Proof: This lemma is a plane consequence of the fact that $\widetilde{L}_1(\lambda)$ is exactly the Lax matrix of the two particles Gaudin model and consequently verifies (7.2.3). Using this fact and the homomorphism property of the coproduct one may argue that equation (7.2.3) is verified also for the other Lax matrices. However, for completeness, we report the explicit computation.

Consider $i = 1, \ldots, N-1$. Substituting the expressions of the matrices $\widetilde{L}_i(\lambda)$ we get:

$$\begin{aligned}
&\left[\widetilde{L}_i(\lambda)\otimes\mathbb{I}, \mathbb{I}\otimes\widetilde{L}_i(\mu)\right] = \\
&= g_{\alpha\beta}g_{\gamma\delta}\rho(X^\alpha)\otimes\rho(X^\gamma)\left[\frac{X^\beta_{i+1}}{\lambda-\epsilon_1} + \frac{\Delta^{(i)}(X^\beta)}{\lambda-\epsilon_2}, \frac{X^\delta_{i+1}}{\mu-\epsilon_1} + \frac{\Delta^{(i)}(X^\delta)}{\mu-\epsilon_2}\right] = \\
&= g_{\alpha\beta}g_{\gamma\delta}\rho(X^\alpha)\otimes\rho(X^\gamma)C^{\beta\delta}_\eta\left(\frac{X^\eta_{i+1}}{(\lambda-\epsilon_1)(\mu-\epsilon_1)} + \frac{\Delta^{(i)}(X^\eta)}{(\lambda-\epsilon_2)(\mu-\epsilon_2)}\right)
\end{aligned}$$

On the other hand, putting

$$f^\eta_i(\lambda) = \frac{X^\eta_{i+1}}{\lambda-\epsilon_1} + \frac{\Delta^{(i)}(X^\eta)}{\lambda-\epsilon_2}$$

we have:

$$\begin{aligned}
&\left[r(\lambda-\mu), \widetilde{L}(\lambda)\otimes\mathbb{I} + \mathbb{I}\otimes\widetilde{L}(\mu)\right] = \\
&= \frac{1}{\lambda-\mu}g_{\alpha\beta}g_{\gamma\eta}\left[\rho(X^\alpha)\otimes\rho(X^\beta), (\rho(X^\gamma)\otimes\mathbb{I})\,f^\eta_i(\lambda) + (\mathbb{I}\otimes\rho(X^\gamma))\,f^\eta_i(\mu)\right] = \\
&= \frac{1}{\lambda-\mu}g_{\alpha\beta}g_{\gamma\eta}\left\{C^{\alpha\gamma}_\delta\rho(X^\delta)\otimes\rho(X^\beta)f^\eta_i(\lambda) + C^{\beta\gamma}_\delta\rho(X^\alpha)\otimes\rho(X^\delta)f^\eta_i(\mu)\right\}
\end{aligned}$$

Using the identity

$$g_{\gamma\eta}C^{\alpha\gamma}_\delta = -g_{\gamma\delta}C^{\alpha\gamma}_\eta$$

and swapping the indexes $\gamma \leftrightarrow \beta, \alpha \leftrightarrow \delta$ in the first term and $\gamma \leftrightarrow \delta$ in the second we get:

$$\left[r(\lambda-\mu), \widetilde{L}(\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes \widetilde{L}(\mu)\right] =$$
$$= g_{\alpha\beta} g_{\gamma\delta} C_\eta^{\beta\delta} \rho(X^\alpha) \otimes \rho(X^\gamma) \left(\frac{f_i^\eta(\lambda) - f_i^\eta(\mu)}{\lambda - \mu} \right)$$

The lemma follows since it holds:

$$\frac{f_i^\eta(\lambda) - f_i^\eta(\mu)}{\lambda - \mu} = - \left(\frac{X_{i+1}^\eta}{(\lambda - \epsilon_1)(\mu - \epsilon_1)} + \frac{\Delta^{(i)}(X^\eta)}{(\lambda - \epsilon_2)(\mu - \epsilon_2)} \right)$$

The case $i = N$ can be proved by a similar computation.

□

Chapter 8

Open perspectives

We list the results obtained in this thesis together with the related open problems.

- We have established a Bihamiltonian formulation for the rational Gaudin model, it would be interesting to search also for a Bihamiltonian formulation of the trigonometric and elliptic models.

- We have derived a complete set of involutive parameter independent integrals for the $\mathfrak{g}$ Homogeneous Gaudin model when $\mathfrak{g}$ is a classical Lie algebra. We showed that such integrals can be seen as spectral invariants of N Lax matrices $\{L_i(\lambda)\}_{i=1}^N$ and that they generalize the Hamiltonians previously found by Ballesteros and Ragnisco [6, 8, 9] related to the coproduct approach. Since, as argued in section 3.1, the Hamiltonians defined by Ballesteros and Ragnisco can be "q–deformed", the possibility of extending this deformation procedure to all the spectral invariants of the Lax matrices $L_i(\lambda)$ could be investigated.

- We used the Bihamiltonian theory of SoV to separate the Homogeneous $sl(r)$ Gaudin model. It would be interesting to apply such procedure to the other classical Lie algebras as well. The main difficulty in solving this problem is to find a suitable set of transversal vector fields W_i for algebras different from $sl(r)$. It would be also interesting to apply the Bihamiltonian SoV procedure to the inhomogeneous models, so to compare analogies and differences with the Lax approach.

- In the last part of the thesis we faced the problem of quantizing the Hamiltonians coming from the Lax matrices $L_i(\lambda)$. We proved that if one introduces suitable quantizations $L_i^q(\lambda)$ of these Lax matrices, then the traces of the powers of the matrices $L_i^q(\lambda)$ commute until cubic terms. This allows us to quantize all the parameter independent Hamiltonians in the $\mathfrak{g} = su(3)$ case. It would be interesting to apply the techniques used in [45, 46] to diagonalize this family of operators.

- Finally, we have seen that in the quantum case we can use a r–matrix to describe the commutation rules of the entries of the Lax matrices $\widetilde{L}_i^q(\lambda)$

$$\begin{aligned}
\widetilde{L}_i^q(\lambda) &= \frac{A_{i+1}^q}{\lambda-\epsilon_1} + \frac{\Delta^{(i)}(A^q)}{\lambda-\epsilon_2} \qquad i=1,\dots,N-1 \\
\widetilde{L}_N^q(\lambda) &= \sigma + \frac{\Delta^{(N)}(A^q)}{\lambda} \\
[\widetilde{L}_i^q(\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes \widetilde{L}^q(\mu)] &+ [r(\lambda-\mu), \widetilde{L}^q(\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes \widetilde{L}^q(\mu)]
\end{aligned}$$

Such representation holds "mutatis mutandis" in the classical case:

$$\begin{aligned}
\widetilde{L}_i^c(\lambda) &= \frac{A_{i+1}^c}{\lambda-\epsilon_1} + \frac{\Delta^{(i)}(A^c)}{\lambda-\epsilon_2} \qquad i=1,\dots,N-1 \\
\widetilde{L}_N^c(\lambda) &= \sigma + \frac{\Delta^{(N)}(A^c)}{\lambda} \\
\{\widetilde{L}_i^c(\lambda)_{ab}, \widetilde{L}_i^c(\lambda)_{cd}\}e_{ab} \otimes e_{cd} &+ [r(\lambda-\mu), \widetilde{L}_i^c(\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes \widetilde{L}_i^c(\mu)] = 0
\end{aligned}$$

It would be very interesting to find a (possibly non–dynamical) global r–matrix (both for the classical and quantum case) for the "total" Lax matrix

$$L_T^{c,q}(\lambda) = \begin{pmatrix} \widetilde{L}_1^{c,q}(\lambda) & 0 & \dots & 0 \\ 0 & \widetilde{L}_2^{c,q}(\lambda) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \widetilde{L}_N^{c,q}(\lambda) \end{pmatrix}$$

Appendices

Appendix A

In this appendix we sketch the proof of Lemma 1. Actually we will prove the converse statement, i.e., that under the map (4.0.3) the Poisson tensor (4.0.6) is sent exactly in the diagonal Poisson tensor (2.3.20).

We denote with J the Jacobian of the transformation:

$$J_{ij} = \frac{\partial B_{i-1}}{\partial A_j} = (-1)^{N-i} s_{N-i}(\epsilon_1, \ldots, \hat{\epsilon_j}, \ldots, \epsilon_N). \tag{A.1}$$

Using the identity:

$$\sum_{j=1}^{N} x^{j-1} (-1)^{N-j} s_{N-j}(\epsilon_1, \ldots, \hat{\epsilon_k}, \ldots, \epsilon_N) = \prod_{l \neq k} (x - \epsilon_k) \tag{A.2}$$

the inverse matrix of A.1 is easily obtained:

$$(J^{-1})_{ij} = \frac{\epsilon_i^{j-1}}{\prod_{k \neq i} (\epsilon_i - \epsilon_k)}. \tag{A.3}$$

We have:

$$(J^{-1} P J^{-1})^t)_{in} = \frac{(-1)^N}{\prod_{m \neq i} (\epsilon_i - \epsilon_m) \prod_{p \neq n} (\epsilon_n - \epsilon_p)} (P_{in}^{(1)} - P_{in}^{(2)}) \tag{A.4}$$

with:

$$P_{in}^{(1)} = \sum_{r=0}^{N-1} \sum_{l=r+1}^{N} (-1)^l s_{N-l}(\epsilon_1, \ldots, \epsilon_N) \sum_{k=r+1}^{l} \epsilon_i^{r+l-k} \epsilon_n^{k-1} [B_r, \cdot] \tag{A.5}$$

$$P_{in}^{(2)} = \sum_{r=1}^{N} \sum_{l=0}^{r-1} (-1)^l s_{N-l}(\epsilon_1, \ldots, \epsilon_N) \sum_{k=l+1}^{r} \epsilon_i^{r+l-k} \epsilon_n^{k-1} [B_r, \cdot]. \tag{A.6}$$

Subtracting (A.5) and (A.6), by using induction and the identities

$$s_i(\epsilon_1,\ldots,\epsilon_{N+1}) = s_i(\epsilon_1,\ldots,\epsilon_N) + \epsilon_{N+1}s_{i-1}(\epsilon_1,\ldots,\epsilon_N) \tag{A.7}$$
$$s_i(\epsilon_1,\ldots,\epsilon_N) = 0 \text{ if } i<0 \text{ or } i>N$$
$$\sum_{l=0}^{N}(-1)^l s_{N-l}(\epsilon_1,\ldots,\epsilon_N)x^l = (-1)^N\prod_{i=1}^{N}(x-\epsilon_i) \tag{A.8}$$

one proves that the coefficient of B_r in formula (A.4) vanishes if $i \neq n$.

Now let us consider the diagonal terms.

$$P^{(1)}_{ii} - P^{(2)}_{ii} = \sum_{r=0}^{N}\left(\epsilon_i^r\sum_{l=0}^{N}(-1)^l s_{N-l}(\epsilon_1,\ldots,\epsilon_N)(l-r)\epsilon_i^{l-1}[B_r,\cdot]\right).$$

Since

$$\sum_{l=0}^{N}(-1)^l s_{N-l}(\epsilon_1,\ldots,\epsilon_N)\, l\, \epsilon_i^{l-1} = (-1)^N\frac{d}{dx}\left(\prod_j(x-\epsilon_j)\right)\Bigg|_{x=\epsilon_i} = (-1)^N\prod_{j\neq i}(\epsilon_i-\epsilon_j)$$

we get (using (4.0.3) and (A.2)) :

$$P^{(1)}_{ii} - P^{(2)}_{ii} = (-1)^N\prod_{j\neq i}(\epsilon_i-\epsilon_j)\sum_{r=0}^{N}\epsilon_i^r[B_r,\cdot] = (-1)^N\left(\prod_{j\neq i}(\epsilon_i-\epsilon_j)\right)^2[A_i,\cdot]$$

whence the assertion.

□

Appendix B

Lemma 7 *The Poisson tensors Q (4.0.7) and R (5.1.1) are not compatible for $N \geq 3$ for any choice of the parameters $\epsilon_1,\ldots,\epsilon_N$.*

Proof: The Poisson tensor Q can be written in the A coordinates as:

$$\{F,G\}_Q = \sum_{i,j,k} q_{ijk}\operatorname{Tr}\left(A_k\left[\frac{\partial G}{\partial A_j},\frac{\partial F}{\partial A_i}\right]\right),$$

with:

$$q_{ijk} = (-1)^N \left\{ \delta_{ij} \left(\xi_j \delta_{jk} + \beta_j \frac{(1-\delta_{jk})}{\eta_{jk}} \right) + \frac{(\beta_i \delta_{jk} - \beta_j \delta_{ik})}{\eta_{ji}} \right\}$$

$$\eta_{ij} = \begin{cases} \epsilon_i - \epsilon_j & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases} \qquad \beta_i = \prod_{k \neq i} \frac{\epsilon_k}{\eta_{ik}} \qquad \xi_i = \sum_{j \neq i} \frac{\beta_j}{\eta_{ji}}.$$

Using this expression is easy to evaluate the Schouten bracket of Q, R on the differentials of the functions

$$F = \mathrm{Tr}(A_1 h) \qquad G = \mathrm{Tr}(A_2 x) \qquad H = \mathrm{Tr}(A_2 h)$$

where with x and h we denoted two constant matrices satisfying $[h, x] = x$. We have:

$$[Q, R]_S(dF, dG, dH) =$$
$$= (-1)^N \left[\xi_2 - \xi_1 + \frac{(\beta_1 - \beta_2)}{\eta_{21}} + \beta_2 \sum_{j=3}^{N} \frac{1}{\eta_{2j}} \right] Tr(A_1 x) + \beta_1 \sum_{k=3}^{N} \frac{1}{\eta_{k1}} Tr(A_k x) \quad \text{(B.1)}$$

For expression (B.1) to be zero is necessary that $\beta_1 = 0$, i.e., we have to put to zero one among the constants $\epsilon_2, \ldots, \epsilon_N$. Let us suppose $\epsilon_k = 0$, $k \neq 1$, $k \neq 2$, then

$$[Q, R]_S(dF, dG, dH) = (-1)^N \left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1} \right) \mathrm{Tr}(A_1 x) \neq 0$$

because the constants ϵ_i must be distinct.

In the case $a_2 = 0$, we have instead:

$$[Q, R]_S(dF, dG, dH) = (-1)^N \mathrm{Tr}(A_1 x) \sum_{j=3}^{N} \frac{1}{\epsilon_j}. \qquad \text{(B.2)}$$

If $N = 3$ then (B.2) is different from zero, so we can assume $N > 3$. But if $\epsilon_2 = 0$ and $N > 3$ having defined:

$$F' = \mathrm{Tr}(A_1 h) \qquad G' = \mathrm{Tr}(A_2 x) \qquad H' = \mathrm{Tr}(A_3 h),$$

we have:

$$[Q, R]_S(dF', dG', dH') = (-1)^{N+1} \frac{1}{\epsilon_3} Tr(A_1 x) \neq 0$$

and the proof is concluded.

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