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Scuola Internazionale Superiore di Studi Avanzati - Trieste

Singularly perturbed elliptic problems

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Thesis submitted for the degree of *Doctor Philosophiae*
Academic Year 2003-2004

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In memoria del Prof. Vito Ranieri
mio professore al liceo...

I wish to thank my supervisor Prof. Antonio Ambrosetti for all his teachings. I am also grateful to Prof. Marino Badiale and Simone Secchi for their useful collaborations but not only. Thanks to Prof. Charles A. Stuart for having accepted to refer this thesis. I thank the Dipartimento di Matematica of the Università degli Studi di Bari and the Politecnico di Bari for the kind hospitality.

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Thanks...

Alessio Pomponio

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1 Introduction

In this thesis we present some nonlinear differential equations studied with a perturbation technique introduced by Ambrosetti and Badiale [3, 4].

In Part I we present briefly this method showing its main ideas. We use this technique to study, in Chapter 3

$$\begin{cases} -\Delta\psi - \lambda\psi = a(x)|\psi|^{p-1}\psi + b(x)|\psi|^{q-1}\psi & x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} \psi(x) = 0, \end{cases}$$

where $N \geq 1$, λ is a negative parameter, $1 < p < q \leq \frac{N+2}{N-2}$ if $N \geq 3$ (and $q < +\infty$ if $N = 1, 2$), $p < 1 + 4/N$ and with $a, b : \mathbb{R}^N \to \mathbb{R}$ satisfying suitable hypotheses. We find solutions bifurcating from the bottom of the essential spectrum of $-\Delta$. We will show that in some cases, this branch of solutions forms a smooth curve.

In Part II we focus our attention on the nonlinear Schrödinger equation. In Chapter 4 we present the main results obtained in the last twenty years, starting from the paper of Floer and Weinstein [36].

In Chapter 5 we study a generalization of the nonlinear Schrödinger equation, that is

$$\begin{cases} -\varepsilon^2 \operatorname{div}\big(J(x)\nabla u\big) + V(x)u = u^p & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases}$$

where $N \geq 3$, $p \in \left(1, \frac{N+2}{N-2}\right)$, $V : \mathbb{R}^N \to \mathbb{R}$, $J : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ are C^1 functions. Using the perturbation technique and the penalization method of del Pino and Felmer [31], we find solutions which concentrate near maxima and minima of an auxiliary functional depending on J and V.

In Chapter 6 we consider a nonlinear Schrödinger equation with critical Sobolev exponent:

$$\begin{cases} -\varepsilon^2 \Delta u + V(x) u = K(x) u^p + Q(x) u^\sigma & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases}$$

where $N \geq 3$, $1 < p < \sigma = \frac{N+2}{N-2}$, V, K and Q are C^2 function from $\mathbb{R}^N$ to $\mathbb{R}$. We will show that there exist solutions concentrating near the maximum and minimum points of an auxiliary functional which depends only on the potentials V, K and Q.

Part III is devoted to the singularly perturbed Neumann problem. In Chapter 7 we present some "biological" motivations of the study of this equation and some classical results.

In Chapter 8 and in Chapter 9, we consider a singularly perturbed Neumann problem in the presence of potentials. More precisely we study:

$$\begin{cases} -\varepsilon^2 \operatorname{div}(J(x) \nabla u) + V(x) u = u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain with external normal ν, $N \geq 3$, $1 < p < (N+2)/(N-2)$, $J \colon \mathbb{R}^N \to \mathbb{R}$ and $V \colon \mathbb{R}^N \to \mathbb{R}$ are C^2 functions. We show how the existence of solutions concentrating on the boundary $\partial\Omega$ and inside Ω is strictly linked to a suitable auxiliary functional depending on J and V and, in some cases, also on the mean curvature H of $\partial\Omega$.

Part I

The perturbation method

2 The perturbation method

In this chapter we give the main ideas and results of a variational method to study critical points of perturbed functionals and to deal with problems with lack of compactness. The method has been developed by Ambrosetti and Badiale in [3, 4], extending some previous results contained in [6, 7]. It allows us to treat problems in which a "small parameter" appears and permits to find existence and multiplicity results even in some situations where the concentration-compactness methods of P.L. Lions [53, 54, 55, 56] fails or requires heavy calculations. The key idea is to perform a suitable finite dimensional reduction and to find the solutions of the "perturbed" problem near the solutions of the "unperturbed" one.

We deal with a family of C^2 functionals $\{f_\varepsilon\}$, defined on a Hilbert space E, parametrized by $\varepsilon \geq 0$. We suppose that the "unperturbed" functional f_0 has a whole manifold of critical points. Under suitable assumptions, it is possible to prove the existence of critical points of each f_ε, for all ε sufficiently small. For the sake of completeness, we present below this technique in some details.

We want to find critical points of functionals of the form

$$f_\varepsilon(u) = \frac{1}{2}\|u\|^2 - F(u) + G(\varepsilon, u),$$

where $\|\cdot\|$ is the norm in E, $F : E \to \mathbb{R}$ and $G : \mathbb{R} \times E \to \mathbb{R}$. We need the following hypotheses:

($\mathbf{F_0}$) $F \in C^2$;
($\mathbf{G_0}$) G is continuous in $(\varepsilon, u) \in \mathbb{R} \times E$ and $G(0, u) = 0$ for all $u \in E$;
($\mathbf{G_1}$) G is of class C^2 with respect to $u \in E$.

We will use the notation $F'(u)$, respectively $G'(\varepsilon, u)$, to denote the functions defined by setting

$$(F'(u) \mid v) = DF(u)[v], \quad \forall\, v \in E,$$

and, respectively,

$$(G'(\varepsilon, u) \mid v) = D_u G(\varepsilon, u)[v], \quad \forall\, v \in E,$$

where $(\cdot \mid \cdot)$ is the scalar product in E. Similarly, $F''(u)$, resp. $G''(\varepsilon, u)$, denote the maps in $L(E, E)$ defined by

$$(F''(u)v \mid w) = D^2F(u)[v,w] \qquad (G''(\varepsilon,u)v \mid w) = D^2_{uu}G(\varepsilon,u)[v,w].$$

We also assume that F satisfies:

($\mathbf{F_1}$) there exists a d-dimensional C^2 manifold Z, $d \geq 1$, consisting of critical points of f_0, namely such that

$$z - F'(z) = 0, \quad \forall\, z \in Z.$$

Such a Z will be called a *critical manifold* of f_0.

Let T_zZ denote the tangent space to Z at z and I_E denote the Identity map in E. We further suppose:

($\mathbf{F_2}$) $F''(z)$ is compact $\forall\, z \in Z$;
($\mathbf{F_3}$) $T_zZ = \ker[I_E - F''(z)]$, $\forall\, z \in Z$.

We make the following further assumptions on G:

($\mathbf{G_2}$) the maps $(\varepsilon,u) \mapsto G'(\varepsilon,u)$, $(\varepsilon,u) \mapsto G''(\varepsilon,u)$ are continuous (as maps from $\mathbb{R} \times E$ to E, resp. to $L(E,E)$);
($\mathbf{G_3}$) there exist $\alpha > 0$ and a continuous function $\Gamma : Z \to \mathbb{R}$ such that, for all $z \in Z$,

$$\Gamma(z) = \lim_{\varepsilon \to 0} \frac{G(\varepsilon,z)}{\varepsilon^\alpha}$$

and

$$G'(\varepsilon,z) = o(\varepsilon^{\alpha/2}).$$

To avoid technicalities, we will assume $Z = \zeta(\mathbb{R}^d)$ where $\zeta \in C^2(\mathbb{R}^d, E)$. We will set $Z^r = \zeta(B_r)$, where $B_r = \{x \in \mathbb{R}^d : |x| \leq r\}$.

The main idea is to use the implicit function theorem to produce a good local deformation of Z in the normal direction (see Figure 1). More precisely, we have the following lemma (for the proof, see [3, 4]).

Lemma 2.1. *Given $R > 0$, there exist $\varepsilon_0 > 0$ and a smooth function*

$$w\colon (-\varepsilon_0, \varepsilon_0) \times Z^R \to E$$

such that:

(i) $w(0,z) = 0$ for all $z \in Z^R$;
(ii) $w(\varepsilon,z) \perp T_zZ$, for all $(\varepsilon,z) \in (-\varepsilon_0,\varepsilon_0) \times Z^R$;
(iii) $f'_\varepsilon(z + w(\varepsilon,z)) \in T_zZ$, for all $(\varepsilon,z) \in (-\varepsilon_0,\varepsilon_0) \times Z^R$;
(iv) $\|w\| = o(\varepsilon^{\alpha/2})$ as $\varepsilon \to 0$, uniformly in $z \in Z^R$;
(v) $\|D_zw(\varepsilon,z)\| \to 0$ as $\varepsilon \to 0$, uniformly in $z \in Z^R$.

Let us define

$$Z_\varepsilon = \left\{z + w(\varepsilon,z) \colon (\varepsilon,z) \in (-\varepsilon_0,\varepsilon_0) \times Z^R\right\}.$$

We have that Z_ε is locally diffeomorphic to Z. Moreover we have the following result.

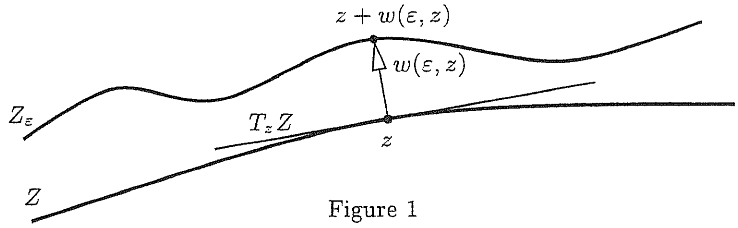

Figure 1

Lemma 2.2. *Z_ε is a natural constraint for f'_ε, namely: if $u_\varepsilon = z + w(\varepsilon, z) \in Z_\varepsilon$ and $f'_{\varepsilon|Z_\varepsilon}(u_\varepsilon) = 0$, then $f'_\varepsilon(u_\varepsilon) = 0$.*

Proof Suppose that $f'_{\varepsilon|Z_\varepsilon}(u_\varepsilon) = 0$, then $f'_\varepsilon(u_\varepsilon)$ is orthogonal to $T_{u_\varepsilon}Z_\varepsilon$. On the other side, $f'_\varepsilon(u_\varepsilon) \in T_zZ$ and $T_{u_\varepsilon}Z_\varepsilon$ is near to T_zZ for ε small. Therefore $f'_\varepsilon(u_\varepsilon) = 0$. □

In [4] (see also [3, 6]) the following theorem is proved.

Theorem 2.1. *Suppose* $(\mathbf{F_0} - \mathbf{F_3})$ *and* $(\mathbf{G_0} - \mathbf{G_3})$ *hold and assume there exist $\delta > 0$ and $z^* \in Z$ such that*

$$\textit{either} \quad \min_{\|z-z^*\|=\delta} \Gamma(z) > \Gamma(z^*), \quad \textit{or} \quad \max_{\|z-z^*\|=\delta} \Gamma(z) < \Gamma(z^*). \tag{2.1}$$

Then, for ε small, f_ε has a critical point u_ε.

Proof We give only a sketch of the proof, divided in two steps.

Step 1. Using the Taylor expansion we obtain, for $u = z + w(\varepsilon, z) \in Z_\varepsilon$,

$$f_\varepsilon(u) = c + \varepsilon^\alpha \Gamma(z) + o(\varepsilon^\alpha),$$

where c is a constant.

Step 2. It readily follows that, for small ε, f_ε has a local constrained minimum (or maximum) on Z_ε at some $u_\varepsilon = z_\varepsilon + w(\varepsilon, z_\varepsilon) \in Z_\varepsilon$, with $\|z_\varepsilon - z^*\| < \delta$. According to Lemma 2.1, such u_ε is a critical point of f_ε. □

Remark 2.1. Recently, the assumption $(\mathbf{F_1})$ has been relaxed to cover situations where the points of the manifold are near to be critical (see [9, 10, 11]).

In the next chapter (see also a joint paper with Marino Badiale [14]) we use this technique to study

$$\begin{cases} -\Delta\psi - \lambda\psi = a(x)|\psi|^{p-1}\psi + b(x)|\psi|^{q-1}\psi & x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} \psi(x) = 0, \end{cases} \tag{2.2}$$

where $N \geq 1$, λ is a negative parameter, $1 < p < q \leq \frac{N+2}{N-2}$ if $N \geq 3$ (and $q < +\infty$ if $N = 1, 2$), $p < 1 + 4/N$ and with $a, b : \mathbb{R}^N \to \mathbb{R}$ satisfying suitable hypotheses. Indeed if, roughly speaking, $A > 0$ is the limit of a at infinity and if we perform the change of variables $u(x) = \varepsilon^{2/(1-p)}\psi(x/\varepsilon)$ and $\lambda = -\varepsilon^2$, then equation (2.2) becomes

$$-\Delta u + u = A|u|^{p-1}u + (a(x/\varepsilon) - A)|u|^{p-1}u + \varepsilon^{2\frac{q-p}{p-1}} b(x/\varepsilon)|u|^{q-1}u, \quad (2.3)$$

and therefore can be considered as a perturbation problem. We find solutions bifurcating from the bottom of the essential spectrum of $-\Delta$. We will show that in some cases, this branch of solutions forms a smooth curve.

3 Bifurcation results for semilinear elliptic problems in $\mathbb{R}^N$

An interesting problem in bifurcation phenomena is to look for solutions bifurcating not from an eigenvalue but from a point of the continuous spectrum of the linearized operator of the involved equation. Typical examples of differential operators with continuous spectrum are the Laplace or the Schrödinger operators in all $\mathbb{R}^N$, and there are now many results on bifurcation of solutions for semilinear elliptic equations in $\mathbb{R}^N$, for example see [85, 86, 87, 84, 57]. See also [88], and the references therein, for the study of bifurcation into spectral gaps. Ambrosetti and Badiale have studied such kind of problems in [4] and [5], obtaining several results on bifurcation of solutions for a one-dimensional differential equation. In this chapter we pursue such a study, generalizing some of the results of [4] to higher dimensions, and considering also the case of a critical nonlinearity. In Section 3.3 we also fill a gap in the proof of Theorem 3.2 in [4] which was pointed out by S. Krömer. All the results of this chapter are contained in a joint paper with Marino Badiale [14].

We consider the equation

$$\begin{cases} -\Delta\psi - \lambda\psi = a(x)|\psi|^{p-1}\psi + b(x)|\psi|^{q-1}\psi & x \in \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} \psi(x) = 0, \end{cases} \tag{3.1}$$

where $N \geq 1$, λ is a negative parameter, $1 < p < q \leq \frac{N+2}{N-2}$ if $N \geq 3$ (and $q < +\infty$ if $N = 1, 2$), $p < 1 + 4/N$ and $a, b : \mathbb{R}^N \to \mathbb{R}$ satisfy suitable hypotheses (see below). Equation (3.1) is an homogeneous equation, so $\psi = 0$ is a solution for all λ, the line $\{(\lambda, \psi = 0) \mid \lambda \in \mathbb{R}\}$ is a line of trivial solutions and, as $q > p > 1$, the linearized operator at $\psi = 0$ is given by $\psi \to -\Delta\psi - \lambda\psi$. It is well known that $[0, +\infty)$ is the spectrum of $-\Delta$ on $\mathbb{R}^N$, and that it contains no eigenvalue. We will find solutions bifurcating from the bottom of the essential spectrum of $-\Delta$. To be precise, by "solution" we mean a couple (λ, ψ_λ) such that $\psi_\lambda \in H^1(\mathbb{R}^N)$ and ψ_λ is a solution of (3.1) in the weak sense of $H^1(\mathbb{R}^N)$. We look for solutions bifurcating from the origin in $H^1(\mathbb{R}^N)$, that is families (λ, ψ_λ) of solutions of (3.1) such that $\lambda \in (\lambda_0, 0)$ for some $\lambda_0 < 0$ and $\psi_\lambda \to 0$ in $H^1(\mathbb{R}^N)$ as $\lambda \to 0$.

Now let us state the hypotheses on the functions a, b. On a we assume that there is $A > 0$ such that either $a - A \in L^1(\mathbb{R}^N)$ or $a - A$ is asymptotic, at infinity, to $1/|x|^\gamma$, for suitable γ. To be precise, in the first case we assume the following set of hypotheses:

($\mathbf{a_1}$) $a - A$ is continuous, bounded and $a(x) - A \in L^1(\mathbb{R}^N)$;

($\mathbf{a_2}$) $\displaystyle\int_{\mathbb{R}^N} (a(x) - A)dx \neq 0$.

In the second case we assume the following hypothesis:

($\mathbf{a_3}$) $a - A$ is continuous and there exist $L \neq 0$ and $\gamma \in (0, N)$ such that $|x|^\gamma(a(x) - A) \to L$ as $|x| \to +\infty$.

Notice that ($\mathbf{a_1}$) of course implies that $a - A \in L^p(\mathbb{R}^N)$ for all $p \in [1, +\infty]$, while ($\mathbf{a_3}$) implies that $a - A$ is bounded. For b we use some of the following assumptions:

($\mathbf{b_1}$) b is continuous and bounded;

($\mathbf{b_2}$) $b \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ and, if $N \geq 2\frac{q-p}{p-1}$, we also assume that there exists $\beta \in [1, \beta^*)$ such that $b \in L^\beta(\mathbb{R}^N)$, where

$$\beta^* = \begin{cases} \dfrac{N(p-1)}{N(p-1) - 2(q-p)} & \text{if } N > 2\dfrac{q-p}{p-1}, \\ \\ +\infty & \text{if } N = 2\dfrac{q-p}{p-1}; \end{cases}$$

($\mathbf{b_3}$) $b \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ and, if $\gamma \geq 2\frac{q-p}{p-1}$, we also assume that there exists $\beta \in [1, \beta^*)$ such that $b \in L^\beta(\mathbb{R}^N)$, where

$$\beta^* = \begin{cases} \dfrac{N(p-1)}{\gamma(p-1) - 2(q-p)} & \text{if } \gamma > 2\dfrac{q-p}{p-1}, \\ \\ +\infty & \text{if } \gamma = 2\dfrac{q-p}{p-1}. \end{cases}$$

The value γ in ($\mathbf{b_3}$) is that given in ($\mathbf{a_3}$). We will assume either ($\mathbf{b_1}$) and ($\mathbf{b_2}$), or ($\mathbf{b_1}$) and ($\mathbf{b_3}$). Notice that, assuming ($\mathbf{b_1}$), hypotheses ($\mathbf{b_2}$) and ($\mathbf{b_3}$) are obviously satisfied when $b \in L^1(\mathbb{R}^N)$.

We can now state our main results.

Theorem 3.1. *Assume $1 < p < q \leq \frac{N+2}{N-2}$ if $N \geq 3$, and $q < +\infty$ if $N = 1, 2$. Suppose that ($\mathbf{a_1}$), ($\mathbf{a_2}$), ($\mathbf{b_1}$), ($\mathbf{b_2}$) hold. Then (3.1) has a family of solutions bifurcating from the origin in $L^\infty(\mathbb{R}^N)$. If, besides, $p < 1 + \frac{4}{N}$, this family of solutions bifurcates from the origin also in $H^1(\mathbb{R}^N)$.*

Theorem 3.2. *Assume $1 < p < q \leq \frac{N+2}{N-2}$ if $N \geq 3$, and $q < +\infty$ if $N = 1, 2$. Suppose that ($\mathbf{a_3}$), ($\mathbf{b_1}$) and ($\mathbf{b_3}$) hold. Then (3.1) has a family of solutions bifurcating from the origin in $L^\infty(\mathbb{R}^N)$. If, besides, $p < 1 + \frac{4}{N}$, this family of solutions bifurcates from the origin also in $H^1(\mathbb{R}^N)$.*

Remark 3.1. When $p \geq 1 + 4/N$, in $H^1(\mathbb{R}^N)$ the solutions can bifurcate from infinity or can be bounded away both from zero and infinity.

Remark 3.2. An interesting question is to know if the solutions that we find form a curve. We give some results in this direction in Section 3.3.

In the proof of Theorems 3.1 and 3.2 we follow the framework of [4], concerning the existence of critical points of perturbed functionals, presented also in Chapter 2. We start by a change of variables. Let us set $u(x) = \varepsilon^{2/(1-p)}\psi(x/\varepsilon)$, $\lambda = -\varepsilon^2$, so that equation (3.1) becomes

$$-\Delta u + u = A|u|^{p-1}u + (a(x/\varepsilon) - A)|u|^{p-1}u + \varepsilon^{2\frac{q-p}{p-1}}b(x/\varepsilon)|u|^{q-1}u. \quad (3.2)$$

It is obvious that to any family $u_\varepsilon \in H^1(\mathbb{R}^N)$ of solutions of (3.2), bounded as $\varepsilon \to 0$, there corresponds a family $\psi_\varepsilon(x) = \varepsilon^{2/(p-1)}u_\varepsilon(\varepsilon x)$ of solutions of (3.1). When $p < 1 + 4/N$ it is easy to check that $\psi_\varepsilon(x) \to 0$ in $H^1(\mathbb{R}^N)$, as $\varepsilon \to 0$. When $p \geq 1+4/N$ we still get solutions, and it is easy to see that they vanish, as $\varepsilon \to 0$, in $L^\infty(\mathbb{R}^N)$, but they do not vanish in $L^2(\mathbb{R}^N)$. Throughout this chapter we will look for bounded families of H^1-solutions of (3.2).

Finally we mention the paper of Badiale [12], where he goes on in the study of (3.1)

In Section 3.1 we prove Theorem 3.1 and in Section 3.2 we prove Theorem 3.2. In Section 3.3 we give some results on the existence of curves of solutions bifurcating from $(0, 0)$, and we fill a gap in the proof of Theorem 3.2 in [4].

Notation

- If E is a Banach space, $F : E \to E$, and $u \in E$, then $DF(u) : E \to E$, $D^2F(u) : E \times E \to E$ and $D^3F(u) : E \times E \times E \to E$ are the first, second and third differential of F at u, which are respectively linear, bilinear and three-times linear.
- $L(E, E)$ is the space of linear continuous operators from E to E.
- $2^* = \frac{2N}{N-2}$ is the critical exponent for the Sobolev embedding, when $N \geq 3$.
- We will use C to denote any positive constant, that can change from line to line.

3.1 First bifurcation result

In this section we prove Theorem 3.1. We want to apply the abstract tools of Chapter 2 and we refer to it for the notations. Let us set

$$E = H^1(\mathbb{R}^N), \quad \|u\|^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) dx, \quad F(u) = \frac{A}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx,$$

and $G = G_1 + G_2$ where

$$G_1(\varepsilon,u)=\begin{cases}-\dfrac{1}{p+1}\displaystyle\int_{\mathbb{R}^N}(a(x/\varepsilon)-A)|u|^{p+1}dx & \text{if } \varepsilon\neq 0,\\ 0 & \text{if } \varepsilon=0,\end{cases}$$

and

$$G_2(\varepsilon,u)=\begin{cases}-\dfrac{1}{q+1}\varepsilon^{2\frac{q-p}{p-1}}\displaystyle\int_{\mathbb{R}^N}b(x/\varepsilon)|u|^{q+1}dx & \text{if } \varepsilon\neq 0,\\ 0 & \text{if } \varepsilon=0.\end{cases}$$

Throughout this section we assume $N\geq 3$ and, of course, $1<p<q\leq \frac{N+2}{N-2}$. The cases $N=1,2$ can be handled in the same way, and in fact are easier. Let us recall the hypotheses on F and G introduced in Chapter 2:

($\mathbf{F_0}$) $F\in C^2$;
($\mathbf{F_1}$) there exists a d-dimensional C^2 manifold Z, $d\geq 1$, consisting of critical points of f_0, namely such that

$$z-F'(z)=0,\quad \forall\, z\in Z;$$

($\mathbf{F_2}$) $F''(z)$ is compact $\forall\, z\in Z$;
($\mathbf{F_3}$) $T_zZ=\ker[I_E-F''(z)]$, $\forall\, z\in Z$;
($\mathbf{G_0}$) G is continuous in $(\varepsilon,u)\in\mathbb{R}\times E$ and $G(0,u)=0$ for all $u\in E$;
($\mathbf{G_1}$) G is of class C^2 with respect to $u\in E$;
($\mathbf{G_2}$) the maps $(\varepsilon,u)\mapsto G'(\varepsilon,u)$, $(\varepsilon,u)\mapsto G''(\varepsilon,u)$ are continuous (as maps from $\mathbb{R}\times E$ to E, resp. to $L(E,E)$);
($\mathbf{G_3}$) there exist $\alpha>0$ and a continuous function $\Gamma: Z\to\mathbb{R}$ such that, for all $z\in Z$,

$$\Gamma(z)=\lim_{\varepsilon\to 0}\frac{G(\varepsilon,z)}{\varepsilon^\alpha}$$

and

$$G'(\varepsilon,z)=o(\varepsilon^{\alpha/2}).$$

We have now to verify that the hypotheses ($\mathbf{F_0}-\mathbf{F_3}$) and ($\mathbf{G_0}-\mathbf{G_3}$) are satisfied. The fact that $q>p>1$ gives of course ($\mathbf{F_0}$) and ($\mathbf{G_1}$). It is also well known (see [16, 17, 49]) that there exists a unique positive radial solution z_0 of

$$-\Delta u+u=A|u|^{p-1}u,\qquad x\in\mathbb{R}^N,$$

that z_0 is strictly radial decreasing, has an exponential decay at infinity together with its derivatives, and that f_0 possesses a $N-$dimensional manifold of critical points

$$Z=\left\{z_\theta(x)=z_0(x+\theta)\colon \theta\in\mathbb{R}^N\right\}.$$

Furthermore, we know (see [6, 70] and the references therein) that $T_{z_\theta}Z=\ker(I_E-F''(z_\theta))$ for all $z_\theta\in Z$. It is also easy to check that $F''(z_\theta)$ is compact, for all $z_\theta \in Z$. In this way all the hypotheses on F are satisfied, and the rest of this section is devoted to prove those on G. As to $(\mathbf{G_0})$, which is the assumption that G is continuous, it obviously derives from the continuity of G' and the homogeneity relations

$$G_1(\varepsilon,u) = \frac{1}{p+1}(G_1'(\varepsilon,u) \mid u), \qquad G_2(\varepsilon,u) = \frac{1}{q+1}(G_2'(\varepsilon,u) \mid u).$$

So we have only to prove $(\mathbf{G_2})$. This is done in the following lemma.

Lemma 3.1. *Assume* $(\mathbf{a_1})$ *and* $(\mathbf{b_1})$. *Then* G' *and* G'' *are continuous.*

Proof Let us consider G_1', and assume $(\varepsilon,u) \to (\varepsilon_0,u_0)$ with $\varepsilon_0 \neq 0$. We obtain

$$\|G_1'(\varepsilon,u) - G_1'(\varepsilon_0,u_0)\| = \sup_{\|v\|\le 1} \left\{ \left| (G_1'(\varepsilon,u) - G_1'(\varepsilon_0,u_0)) \mid v) \right| \right\}$$

$$= \sup_{\|v\|\le 1}\left\{ \left| \int_{\mathbb{R}^N} \left(a\left(\frac{x}{\varepsilon}\right) - A\right) |u|^{p-1}uv dx - \int_{\mathbb{R}^N}\left(a\left(\frac{x}{\varepsilon_0}\right) - A\right) |u_0|^{p-1}u_0 v dx \right| \right\}$$

$$\le \sup_{\|v\|\le 1}\left\{ \left| \int_{\mathbb{R}^N} \left(a\left(\frac{x}{\varepsilon}\right) - A\right) |u|^{p-1}uv dx - \int_{\mathbb{R}^N}\left(a\left(\frac{x}{\varepsilon_0}\right) - A\right) |u|^{p-1}uv dx \right| \right\}$$

$$+ \sup_{\|v\|\le 1}\left\{ \left| \int_{\mathbb{R}^N} \left(a\left(\frac{x}{\varepsilon_0}\right) - A\right) |u|^{p-1}uv dx - \int_{\mathbb{R}^N}\left(a\left(\frac{x}{\varepsilon_0}\right) - A\right) |u_0|^{p-1}u_0 v dx \right| \right\}.$$

For the first term we can write

$$\left| \int_{\mathbb{R}^N} \left(a\left(\frac{x}{\varepsilon}\right) - A\right) |u|^{p-1}u\, v\, dx - \int_{\mathbb{R}^N}\left(a\left(\frac{x}{\varepsilon_0}\right) - A\right) |u|^{p-1}u\, v\, dx \right|$$
$$\le C\left(\int_{\mathbb{R}^N} \left| a\left(\frac{x}{\varepsilon}\right) - a\left(\frac{x}{\varepsilon_0}\right)\right|^{\frac{p+1}{p}} |u|^{p+1}\right)^{\frac{p}{p+1}},$$

where C is independent of v, ($\|v\| \le 1$). As $\varepsilon \to \varepsilon_0 \neq 0$, $u \to u_0$ and a is continuous, by dominated convergence this term tends to zero. For the second term we have

$$\left| \int_{\mathbb{R}^N} \left(a\left(\frac{x}{\varepsilon_0}\right) - A\right) |u|^{p-1}u\, v\, dx - \int_{\mathbb{R}^N}\left(a\left(\frac{x}{\varepsilon_0}\right) - A\right) |u_0|^{p-1}u_0\, v\, dx \right|$$
$$\le C\left(\int_{\mathbb{R}^N} \left| |u|^{p-1}u - |u_0|^{p-1}u_0 \right|^{\frac{p+1}{p}} dx\right)^{\frac{p}{p+1}},$$

and this term vanishes as $u \to u_0$. Hence $\|G_1'(\varepsilon,u) - G_1'(\varepsilon_0,u_0)\| \to 0$ as $(\varepsilon,u) \to (\varepsilon_0,u_0)$, $\varepsilon_0 \neq 0$. Let us now assume $(\varepsilon,u) \to (0,u_0)$. By definition, $G_1'(0,u) = 0$ and, applying Hölder inequality and the change of variables $y = x/\varepsilon$, we obtain

$$\|G_1'(\varepsilon,u)\| = \sup_{\|v\|\le 1}\left\{\left|\int_{\mathbb{R}^N}\left(a\left(\frac{x}{\varepsilon}\right)-A\right)|u|^{p-1}u\,v\,dx\right|\right\}$$

$$\le \sup_{\|v\|\le 1}\left\{\left(\int_{\mathbb{R}^N}\left(\left|a\left(\frac{x}{\varepsilon}\right)-A\right|^{\frac{2^*}{2^*-p-1}}dx\right)\right)^{\frac{2^*-p-1}{2^*}}\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{\frac{p}{2^*}}\left(\int_{\mathbb{R}^N}|v|^{2^*}\right)^{\frac{1}{2^*}}\right\}$$

$$\le C\,\varepsilon^{N\frac{2^*-p-1}{2^*}}\left(\int_{\mathbb{R}^N}|a(y)-A|^{\frac{2^*}{2^*-p-1}}dy\right)^{\frac{2^*-p-1}{2^*}}\left(\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{\frac{p}{2^*}}$$

and this term vanishes as $\varepsilon\to 0$. In this way we have proved that G_1' is continuous.

Similar arguments work for G_2'. Indeed, if $(\varepsilon,u)\to(\varepsilon_0,u_0)$ with $\varepsilon_0\ne 0$, we obtain

$$\|G_2'(\varepsilon,u)-G_2'(\varepsilon_0,u_0)\| = \sup_{\|v\|\le 1}\left|(G_2'(\varepsilon,u)-G_2'(\varepsilon_0,u_0))\mid v)\right|$$

$$= \sup_{\|v\|\le 1}\left\{\left|\varepsilon^{2\frac{q-p}{p-1}}\int_{\mathbb{R}^N}b\left(\frac{x}{\varepsilon}\right)|u|^{p-1}uvdx-\varepsilon_0^{2\frac{q-p}{p-1}}\int_{\mathbb{R}^N}b\left(\frac{x}{\varepsilon_0}\right)|u_0|^{p-1}u_0vdx\right|\right\}$$

$$\le \sup_{\|v\|\le 1}\left\{\varepsilon^{2\frac{q-p}{p-1}}\left|\int_{\mathbb{R}^N}b\left(\frac{x}{\varepsilon}\right)|u|^{p-1}uvdx-\int_{\mathbb{R}^N}b\left(\frac{x}{\varepsilon_0}\right)|u_0|^{p-1}u_0vdx\right|\right\}$$

$$+\sup_{\|v\|\le 1}\left\{\left|\varepsilon^{2\frac{q-p}{p-1}}\int_{\mathbb{R}^N}b\left(\frac{x}{\varepsilon_0}\right)|u_0|^{p-1}u_0vdx-\varepsilon_0^{2\frac{q-p}{p-1}}\int_{\mathbb{R}^N}b\left(\frac{x}{\varepsilon_0}\right)|u_0|^{p-1}u_0vdx\right|\right\}$$

$$\le \varepsilon^{2\frac{q-p}{p-1}}\sup_{\|v\|\le 1}\left\{\left|\int_{\mathbb{R}^N}b\left(\frac{x}{\varepsilon}\right)|u|^{p-1}u\,v\,dx-\int_{\mathbb{R}^N}b\left(\frac{x}{\varepsilon_0}\right)|u_0|^{p-1}u_0\,v\,dx\right|\right\}$$

$$+\left|\varepsilon^{2\frac{q-p}{p-1}}-\varepsilon_0^{2\frac{q-p}{p-1}}\right|\sup_{\|v\|\le 1}\int_{\mathbb{R}^N}\left|b\left(\frac{x}{\varepsilon_0}\right)|u_0|^{p-1}u_0\,v\right|dx.$$

The first term can be treated exactly as before, the second term obviously vanishes as $\varepsilon\to\varepsilon_0$. Let us now assume $(\varepsilon,u)\to(0,u_0)$. We obtain

$$\begin{aligned}\|G_2'(\varepsilon,u)\| &= \sup_{\|v\|\le 1}\left|(G_2'(\varepsilon,u)\mid v)\right|\\ &\le \sup_{\|v\|\le 1}\varepsilon^{2\frac{q-p}{p-1}}\int_{\mathbb{R}^N}\left|b\left(\frac{x}{\varepsilon}\right)\right||u|^p|v|dx\\ &\le C\,\varepsilon^{2\frac{q-p}{p-1}}.\end{aligned}$$

Now we have proved that G' is continuous. The argument to prove the continuity of G'' is almost the same and we leave it to the reader. □

Let us now verify that $(\mathbf{G_3})$ is satisfied.

Lemma 3.2. *Let us assume* $(\mathbf{a_1})$, $(\mathbf{b_1})$ *and* $(\mathbf{b_2})$. *Let us define, for* $\theta \in \mathbb{R}^N$,

$$\Gamma(\theta) = -\frac{1}{p+1} z_0^{p+1}(\theta) \int_{\mathbb{R}^N} (a(y) - A)dy. \tag{3.3}$$

Then

$$\lim_{\varepsilon \to 0} \frac{G(\varepsilon, z_\theta)}{\varepsilon^N} = \Gamma(\theta) \tag{3.4}$$

and

$$G'(\varepsilon, z_\theta) = O(\varepsilon^{\frac{N}{2}+1}). \tag{3.5}$$

Proof As above we will study separately G_1 and G_2. By the change of variables $y = x/\varepsilon$ we have

$$\begin{aligned} G_1(\varepsilon, z_\theta) &= -\frac{1}{p+1} \int_{\mathbb{R}^N} \left(a\left(\frac{x}{\varepsilon}\right) - A \right) z_0^{p+1}(x+\theta)dx \\ &= -\frac{\varepsilon^N}{p+1} \int_{\mathbb{R}^N} (a(y) - A)\, z_0^{p+1}(\varepsilon y + \theta)\, dy. \end{aligned}$$

Since $a - A \in L^1(\mathbb{R}^N)$ and z_0 is bounded and continuous, by dominated convergence we get

$$\lim_{\varepsilon \to 0} \frac{G_1(\varepsilon, z_\theta)}{\varepsilon^N} = \Gamma(\theta). \tag{3.6}$$

Hence, to prove (3.4) we have to show that

$$\lim_{\varepsilon \to 0} \frac{G_2(\varepsilon, z_\theta)}{\varepsilon^N} = 0. \tag{3.7}$$

We distinguish two cases. Assume first $N < 2\frac{q-p}{p-1}$. In this case

$$\varepsilon^{-N} G_2(\varepsilon, z_\theta) = -\frac{1}{q+1} \varepsilon^{2\frac{q-p}{p-1} - N} \int_{\mathbb{R}^N} b\left(\frac{x}{\varepsilon}\right) z_0^{q+1}(x+\theta)dx,$$

and this expression of course vanishes as $\varepsilon \to 0$, because the integral is bounded. Hence, let us assume $N \geq 2\frac{q-p}{p-1}$. We obtain

$$\begin{aligned} &\varepsilon^{-N} G_2(\varepsilon, z_\theta) \\ &\quad = -\frac{1}{q+1} \varepsilon^{2\frac{q-p}{p-1} - N} \int_{\mathbb{R}^N} b\left(\frac{x}{\varepsilon}\right) z_0^{q+1}(x+\theta)dx \\ &\quad \leq C\varepsilon^{2\frac{q-p}{p-1} - N} \left(\int_{\mathbb{R}^N} \left| b\left(\frac{x}{\varepsilon}\right) \right|^\beta dx \right)^{\frac{1}{\beta}} \left(\int_{\mathbb{R}^N} z_0^{\frac{\beta(q+1)}{\beta-1}} (x+\theta)\, dx \right)^{\frac{\beta-1}{\beta}} \\ &\quad \leq C\, \varepsilon^{2\frac{q-p}{p-1} - N + \frac{N}{\beta}} \left(\int_{\mathbb{R}^N} |b(y)|^\beta\, dy \right)^{\frac{1}{\beta}}, \end{aligned}$$

where β is given by $(\mathbf{b_2})$. This term goes to zero since $2\frac{q-p}{p-1} - N + \frac{N}{\beta} > 0$. We have now proved (3.7), hence, by (3.6), (3.4) is also proved.

Let us go to the proof of (3.5). Again we will study separately G_1' and G_2'. We have

$$\|G_1'(\varepsilon, z_\theta)\| = \sup_{\|v\|\le 1} \left|(G_1'(\varepsilon, z_\theta)|v)\right| = \sup_{\|v\|\le 1} \left|\int_{\mathbb{R}^N} \left(a\left(\frac{x}{\varepsilon}\right) - A\right) z_\theta^p v dx\right|$$

$$\le \sup_{\|v\|\le 1} \left\{ \left(\int_{\mathbb{R}^N} \left|a\left(\frac{x}{\varepsilon}\right) - A\right|^{\frac{2N}{N+2}} z_\theta^{p\frac{2N}{N+2}} dx\right)^{\frac{N+2}{2N}} \left(\int_{\mathbb{R}^N} |v|^{2^*} dx\right)^{\frac{1}{2^*}} \right\}$$

$$\le C\varepsilon^{\frac{N}{2}+1} \left(\int_{\mathbb{R}^N} |a(y) - A|^{\frac{2N}{N+2}} dy\right)^{\frac{N+2}{2N}},$$

hence

$$G_1'(\varepsilon, z_\theta) = O(\varepsilon^{\frac{N}{2}+1}). \tag{3.8}$$

As to $G_2'(\varepsilon, z_\theta)$ we obtain, arguing ad before,

$$\|G_2'(\varepsilon, z_\theta)\| \le C\varepsilon^{2\frac{q-p}{p-1}} \left(\int_{\mathbb{R}^N} \left|b\left(\frac{x}{\varepsilon}\right)\right|^{\frac{2N}{N+2}} z_\theta^{q\frac{2N}{N+2}} dx\right)^{\frac{N+2}{2N}}.$$

By the usual change of variables $y = x/\varepsilon$ and using $(\mathbf{b_2})$, we obtain

$$G_2'(\varepsilon, z_\theta) = o(\varepsilon^{\frac{N}{2}+1}).$$

From this and (3.8) we readily get (3.5). □

Remark 3.3. Notice that in the abstract results of Chapter 2 the function Γ is defined on the manifold Z of critical point of the unperturbed functional f_0. In the present case this manifold is diffeomorphic to $\mathbb{R}^N$, so we consider Γ as a function defined on $\mathbb{R}^N$.

We can now conclude the proof of Theorem 3.1. We know that z_0 has a strict (global) maximum in $x = 0$, so that Γ has a (strict) global maximum or minimum (depending on the sign of $\int(a(y) - A)dy$) at $\theta = 0$. We can then apply Theorem 2.1, setting $z^* = 0$ and, for example, $\delta = 1$. We obtain a family $\{(\varepsilon, u_\varepsilon)\} \subset \mathbb{R} \times H^1(\mathbb{R}^N)$ such that u_ε is a critical point of f_ε, hence a solution of (2), and $\{u_\varepsilon\}$ is a bounded set in $H^1(\mathbb{R}^N)$. To be precise, we have

$$u_\varepsilon(x) = z_0(x + \theta_\varepsilon) + w(\varepsilon, \theta_\varepsilon)(x)$$

where $|\theta_\varepsilon| \le 1$ and $w(\varepsilon, \theta_\varepsilon) \to 0$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \to 0$. Of course $z_0(x + \theta_\varepsilon)$ is bounded in $L^\infty(\mathbb{R}^N)$, as $\varepsilon \to 0$. Writing down the equation satisfied by $w(\varepsilon, \theta_\varepsilon)$, and using standard elliptic estimates, it is easy to prove that also $w(\varepsilon, \theta_\varepsilon)$ is bounded in $L^\infty(\mathbb{R}^N)$. Hence we conclude that $\psi_\varepsilon(x) = \varepsilon^{2/p-1} u_\varepsilon(\varepsilon x) \to 0$ in $L^\infty(\mathbb{R}^N)$, as $\varepsilon \to 0$. When $p < 1 + 4/N$, recalling that $\lambda = -\varepsilon^2$, we obtain a family (λ, ψ_λ) of solutions of (1) such that $\psi_\lambda \to 0$ in $H^1(\mathbb{R}^N)$ as $\lambda \to 0$.

Remark 3.4. Let us observe that the hypothesis $(\mathbf{a_2})$ is not used to prove the properties $(\mathbf{G_0} - \mathbf{G_3})$. It is used to apply Theorem 2.1, and in particular to say that there are z^*, δ such that (2.1) holds. If $\int (a(y) - A)dy = 0$ then Γ, as defined in (3.3), is identically zero. It has critical points, but of course they are not stable under perturbations, so we can not conclude that they give rise to critical points of f_ε.

3.2 Second bifurcation result

In this section we prove Theorem 3.2. As before we have to prove that G, G' and G'' are continuous functions. Notice that in the proof of $(\mathbf{G_0})$ and $(\mathbf{G_2})$ we will consider just the function G_1, because the arguments of Lemma 3.1 for the function G_2 use only hypothesis $(\mathbf{b_1})$ which is unchanged. On the contrary in the proof of $(\mathbf{G_3})$ we will study both G_1 and G_2. As above we assume $N \geq 3$ and $1 < p < q \leq \frac{N+2}{N-2}$. The continuity of G is a consequence, as in the previous section, of the continuity of G' and the homogeneity relations between G_i and G'_i $(i = 1, 2)$. So let us prove $(\mathbf{G_2})$.

Lemma 3.3. *Assume* $(\mathbf{a_3})$. *Then* G'_1 *and* G''_1 *are continuous.*

Proof We will show the continuity of G'_1, the argument for G''_1 is similar. Assume $(\varepsilon, u) \to (\varepsilon_0, u_0)$ with $\varepsilon_0 \neq 0$. In this case one can argue exactly as in Lemma 3.1 to obtain $\|G'_1(\varepsilon, u) - G'_1(\varepsilon_0, u_0)\| \to 0$. Hence, let us now assume $(\varepsilon, u) \to (0, u_0)$. For each $\eta > 0$ let us fix $M_\eta > 0$ such that $|a(y) - A| \leq \eta$ if $|y| \geq M_\eta$. We obtain

$$
\begin{aligned}
\|G'_1(\varepsilon,u)\| &= \sup_{\|v\|\leq 1} \left| \left(a\left(\frac{x}{\varepsilon}\right) - A\right) |u|^{p-1}uv\,dx \right| \\
&\leq \sup_{\|v\|\leq 1} \left\{ \int_{|x/\varepsilon|\leq M_\eta} \left|a\left(\frac{x}{\varepsilon}\right) - A\right| |u|^p |v|\,dx \right\} \\
&\quad + \sup_{\|v\|\leq 1} \left\{ \int_{|x/\varepsilon|> M_\eta} \left|a\left(\frac{x}{\varepsilon}\right) - A\right| |u|^p |v|\,dx \right\} \\
&\leq C \sup_{\|v\|\leq 1} \left\{ \int_{|x|\leq \varepsilon M_\eta} |u|^p|v|\,dx \right\} + \eta \sup_{\|v\|\leq 1} \left\{ \int_{\mathbb{R}^N} |u|^p|v|\,dx \right\} \\
&\leq C \sup_{\|v\|\leq 1} \left\{ \left(\int_{|x|\leq \varepsilon M_\eta} |u|^{p+1}dx \right)^{p/(p+1)} \left(\int_{|x|\leq \varepsilon M_\eta} |v|^{p+1}dx \right)^{1/(p+1)} \right\} \\
&\quad + \eta \sup_{\|v\|\leq 1} \left\{ \left(\int_{\mathbb{R}^N} |u|^{p+1}dx \right)^{p/(p+1)} \left(\int_{\mathbb{R}^N} |v|^{p+1}dx \right)^{1/(p+1)} \right\} \\
&\leq C \left(\int_{|x|\leq \varepsilon M_\eta} |u|^{p+1}dx \right)^{p/(p+1)} + C\eta \left(\int_{\mathbb{R}^N} |u|^{p+1}dx \right)^{p/(p+1)} .
\end{aligned}
$$

We then obtain

$$\limsup_{(\varepsilon,u)\to(0,u_0)} \|G_1'(\varepsilon,u)\| \le C\eta$$

for all $\eta > 0$, hence $\lim_{(\varepsilon,u)\to(0,u_0)} \|G_1'(\varepsilon,u)\| = 0$ and the lemma is proved. □

In the next lemma we prove that $(\mathbf{G_3})$ is satisfied.

Lemma 3.4. *Assume* $(\mathbf{a_3})$, $(\mathbf{b_1})$ *and* $(\mathbf{b_3})$. *Define*

$$\Gamma(\theta) = -\frac{L}{p+1}\int_{\mathbb{R}^N} |x|^{-\gamma}\, z_0^{p+1}(x+\theta)dx.$$

Then, for all $\theta \in \mathbb{R}^N$, *we have*

$$\lim_{\varepsilon\to 0} \frac{G(\varepsilon, z_\theta)}{\varepsilon^\gamma} = \Gamma(\theta) \tag{3.9}$$

and

$$G'(\varepsilon, z_\theta) = o(\varepsilon^{\gamma/2}). \tag{3.10}$$

Proof As usual we will study separately G_1 and G_2. We have

$$\begin{aligned} G_1(\varepsilon, z_\theta) &= -\frac{1}{p+1}\int_{\mathbb{R}^N}\left(a\left(\frac{x}{\varepsilon}\right) - A\right) z_0^{p+1}(x+\theta)dx \\ &= -\frac{\varepsilon^\gamma}{p+1}\int_{\mathbb{R}^N}\left(a\left(\frac{x}{\varepsilon}\right) - A\right)\frac{|x|^\gamma}{\varepsilon^\gamma}\frac{z_0^{p+1}(x+\theta)}{|x|^\gamma}dx. \end{aligned}$$

We know that, for all $x \neq 0$,

$$\left(a\left(\frac{x}{\varepsilon}\right) - A\right)\frac{|x|^\gamma}{\varepsilon^\gamma} \to L \qquad \text{as } \varepsilon \to 0,$$

while, since $\gamma < N$ and z_0 has an exponential decay at infinity, $|x|^{-\gamma}\, z_0(x) \in L^1(\mathbb{R}^N)$. Hence by dominated convergence we get

$$\lim_{\varepsilon\to 0}\frac{G_1(\varepsilon, z_\theta)}{\varepsilon^\gamma} = \Gamma(\theta). \tag{3.11}$$

To study $\varepsilon^{-\gamma}G_2(\varepsilon, z_\theta)$ we can repeat the argument used in Lemma 3.2, distinguishing the cases $\gamma < 2\frac{q-p}{p-1}$ and $\gamma \ge 2\frac{q-p}{p-1}$, and using $(\mathbf{b_3})$ instead of $(\mathbf{b_2})$. We obtain

$$\lim_{\varepsilon\to 0}\frac{G_2(\varepsilon, z_\theta)}{\varepsilon^\gamma} = 0, \tag{3.12}$$

and (3.9) follows from (3.11) and (3.12).

Let us now prove (3.10). We study first G_1' then G_2'. With the same arguments of Lemma 3.2 we get

$$||G_1'(\varepsilon, z_\theta)|| \le C\left(\int_{\mathbb{R}^N}\left|a\left(\frac{x}{\varepsilon}\right)-A\right|^{\frac{2N}{N+2}} z_\theta^{p\frac{2N}{N+2}}(x)\,dx\right)^{\frac{N+2}{2N}}.$$

We have to distinguish three cases.

First case: $\gamma < \frac{N+2}{2}$.
We obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^N}\left|a\left(\frac{x}{\varepsilon}\right)-A\right|^{\frac{2N}{N+2}} z_\theta^{p\frac{2N}{N+2}}(x)\,dx\right)^{\frac{N+2}{2N}} \\ = \varepsilon^\gamma\left(\int_{\mathbb{R}^N}\left|\left(a\left(\frac{x}{\varepsilon}\right)-A\right)\frac{|x|^\gamma}{\varepsilon^\gamma}\right|^{\frac{2N}{N+2}}\frac{z_\theta^{p\frac{2N}{N+2}}(x)}{|x|^{\gamma\frac{2N}{N+2}}}\,dx\right)^{\frac{N+2}{2N}} \\ \le C\varepsilon^\gamma\left(\int_{\mathbb{R}^N}\frac{z_\theta^{p\frac{2N}{N+2}}(x)}{|x|^{\gamma\frac{2N}{N+2}}}\,dx\right)^{\frac{N+2}{2N}} \le C\varepsilon^\gamma,\end{aligned}$$

because $\gamma\frac{2N}{N+2} < N$, hence $z_\theta^{p\frac{2N}{N+2}}(x)\,|x|^{-\gamma\frac{2N}{N+2}} \in L^1(\mathbb{R}^N)$. Therefore in this case

$$||G_1'(\varepsilon, z_\theta)|| = O(\varepsilon^\gamma).$$

Second case: $\gamma > \frac{N+2}{2}$.
In this case the function $|a(x)-A|^{\frac{2N}{N+2}}$ is in $L^1(\mathbb{R}^N)$, because it is bounded and at infinity it is asymptotic to $|x|^{-\gamma\frac{2N}{N+2}}$, and $\gamma\frac{2N}{N+2} > N$. Therefore, by the usual change of variables $y = x/\varepsilon$ we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^N}\left|a\left(\frac{x}{\varepsilon}\right)-A\right|^{\frac{2N}{N+2}} z_\theta^{p\frac{2N}{N+2}}(x)\,dx\right)^{\frac{N+2}{2N}} \\ \le C\varepsilon^{\frac{N+2}{2}}\left(\int_{\mathbb{R}^N}|a(y)-A|^{\frac{2N}{N+2}}dy\right)^{\frac{N+2}{2N}} \le C\varepsilon^{\frac{N+2}{2}}.\end{aligned}$$

Hence, recalling that $\gamma < N$, we obtain

$$||G_1'(\varepsilon, z_\theta)|| = O(\varepsilon^{\frac{N}{2}+1}) = o(\varepsilon^{\gamma/2}).$$

Third case: $\gamma = \frac{N+2}{2}$.
In this case we apply Hölder inequality using as conjugate exponents, instead of $\frac{2N}{N+2}$ and $\frac{2N}{N-2}$, any s, s' such that s is smaller than $\frac{2N}{N-2}$ but near to it, so that s' is bigger than $\frac{2N}{N+2}$ but near to it. In this way, we obtain

$$\begin{aligned}
\|G_1'(\varepsilon, z_\theta)\| &= \sup_{\|v\|\le 1} |(G_1'(\varepsilon, z_\theta) \mid v)| \\
&= \sup_{\|v\|\le 1} \left| \int_{\mathbb{R}^N} \left(a\left(\frac{x}{\varepsilon}\right) - A \right) z_\theta^p v dx \right| \\
&\le \sup_{\|v\|\le 1} \left(\int_{\mathbb{R}^N} \left| a\left(\frac{x}{\varepsilon}\right) - A \right|^{s'} z_\theta^{ps'} dx \right)^{1/s'} \left(\int_{\mathbb{R}^N} |v|^s dx \right)^{1/s} \\
&\le C \left(\int_{\mathbb{R}^N} \left| a\left(\frac{x}{\varepsilon}\right) - A \right|^{s'} dx \right)^{1/s'} .
\end{aligned}$$

It is $s'\gamma > N$, so, as before, $|a - A|^{s'} \in L^1(\mathbb{R}^N)$. We can then apply the usual change of variables to obtain

$$\|G_1'(\varepsilon, z_\theta)\| \le C\varepsilon^{N/s'}.$$

We have that s' is near $\frac{2N}{N+2} = \frac{N}{\gamma}$, so N/s' is near γ. Hence we obtain

$$\|G_1'(\varepsilon, z_\theta)\| = O(\varepsilon^{N/s'}) = o(\varepsilon^{\frac{\gamma}{2}}).$$

We have concluded the study of $G_1'(\varepsilon, z_\theta)$. As to $G_2'(\varepsilon, z_\theta)$, the same argument of Lemma 3.2 gives

$$\|G_2'(\varepsilon, z_\theta)\| = o(\varepsilon^{\frac{N}{2}+1}) = o(\varepsilon^{\frac{\gamma}{2}}).$$

In this way the lemma is completely proved. □

We want now to complete the proof of Theorem 3.2. As in the previous section, we have only to prove that the function Γ satisfies the hypotheses of Theorem 2.1. Let us prove that

$$\exists R > 0 \text{ such that either } \min_{|\theta|=R} \Gamma(\theta) > \Gamma(0) \text{ or } \max_{|\theta|=R} \Gamma(\theta) < \Gamma(0). \quad (3.13)$$

To prove (3.13) we first notice that Γ is continuous and $\Gamma(\theta)$ is either positive on all $\mathbb{R}^N$ or negative on all $\mathbb{R}^N$. Then we claim that

$$\lim_{|\theta|\to+\infty} \Gamma(\theta) = 0. \quad (3.14)$$

To prove (3.14), let us write

$$\begin{aligned}
|\Gamma(\theta)| &= C \int_{\mathbb{R}^N} |x|^{-\gamma} z_0^{p+1}(x+\theta) dx \\
&= C \int_{|x|\le 1} |x|^{-\gamma} z_0^{p+1}(x+\theta) dx + C \int_{|x|>1} |x|^{-\gamma} z_0^{p+1}(x+\theta) dx.
\end{aligned}$$

It is $|x|^{-\gamma} \in L^1(B_1)$, where $B_1 = \{x \in \mathbb{R}^N \mid |x| < 1\}$, while $z_0^{p+1}(x+\theta) \to 0$ as $|\theta| \to +\infty$, for all x, so by dominated convergence the first integral vanishes as $|\theta| \to +\infty$. For the second integral we write

$$\int_{|x|>1} |x|^{-\gamma} z_0^{p+1}(x+\theta)dx = \int_{|y-\theta|>1} |y-\theta|^{-\gamma} z_0^{p+1}(y)dy$$
$$= \int_{\mathbb{R}^N} \chi_\theta(y)|y-\theta|^{-\gamma} z_0^{p+1}(y)dy,$$

where χ_θ is the characteristic function of the set $\{y \in \mathbb{R}^N \mid |y-\theta| > 1\}$. It is trivial to see that

$$\chi_\theta(y)|y-\theta|^{-\gamma} \leq 1$$

for all $y, \theta \in \mathbb{R}^N$ $(y \neq \theta)$, and that $\chi_\theta(y)|y-\theta|^{-\gamma} \to 0$ as $|\theta| \to +\infty$. Again by dominated convergence we obtain that also the second integral vanishes when $|\theta| \to +\infty$. So (3.14) is proved, hence also (3.13). We can apply Theorem 2.1 and argue as in the previous section.

3.3 Continuous branches of solutions

In this section we prove that in some cases the families of solutions bifurcating from $(0,0)$, that we have found in the previous sections, form a curve. We first will prove some abstract results (following the frame of Chapter 2), then we will apply these results to problems (3.1) and (3.2). So let us come back to the abstract frame of Chapter 2. To make easier the passage from the abstract frame to the applications, we will denote by z_θ, $\theta \in \mathbb{R}^d$, the elements of Z. Notice that our arguments are local in nature and we will work in the neighborhood of a fixed point, so we can assume, without loss of generality, that the manifold Z is given by a unique map $\theta \to z_\theta$. We will indicate with $\partial_i z$, $\partial_{ij} z$ the derivatives of z_θ with respect to the parameter θ, that is

$$\partial_i z = \frac{\partial z}{\partial \theta_i}(\theta), \qquad \partial_{ij} z = \frac{\partial^2 z}{\partial \theta_i \partial \theta_j}(\theta).$$

About the manifold Z of critical points we will also assume the following hypothesis, which is satisfied in our applications:

(H) $(\partial_i z \mid \partial_j z) = 0$ if $i \neq j$, $\|\partial_i z\| = c$ (with c independent of i and θ) and $(\partial_{ij} z \mid \partial_l z) = 0$ for all $i, j, l = 1, .., d$.

About the functionals F, G we will assume two different types of hypotheses. Recall that α, Γ are those given in hypothesis $(\mathbf{G_3})$.

$(\mathbf{F_4})$ F is of class C^4;

$(\mathbf{G_4})$ G is of class C^4 with respect to u and the map $(\varepsilon, u) \to G'''(\varepsilon, u)$ is continuous;

$(\mathbf{G_5})$ Γ is C^2 and, if θ_ε is a family such that $\theta_\varepsilon \to \theta$ as $\varepsilon \to 0$, then

$$\lim_{\varepsilon \to 0} \left[\varepsilon^{-\alpha}(G'(\varepsilon, z_{\theta_\varepsilon}) \mid \partial_{ij} z_{\theta_\varepsilon}) + \varepsilon^{-\alpha}(G''(\varepsilon, z_{\theta_\varepsilon})\partial_i z_{\theta_\varepsilon} \mid \partial_j z_{\theta_\varepsilon})\right] = \partial_{ij}\Gamma(\theta),$$
$$\lim_{\varepsilon \to 0} \varepsilon^{-\alpha/2} G''(\varepsilon, z_{\theta_\varepsilon})\partial_i z_{\theta_\varepsilon} = 0, \qquad \lim_{\varepsilon \to 0} \varepsilon^{-\alpha/2} G''(\varepsilon, z_{\theta_\varepsilon})\partial_{ij} z_{\theta_\varepsilon} = 0,$$
$$\lim_{\varepsilon \to 0} \varepsilon^{-\alpha/2} G'''(\varepsilon, z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] = 0.$$

$(\mathbf{F_4})'$ F is of class C^3;
$(\mathbf{G_4})'$ G is of class C^3 with respect to u and the map $(\varepsilon, u) \to G'''(\varepsilon, u)$ is continuous;
$(\mathbf{G_5})'$ $G'(\varepsilon, u) = O(\varepsilon^\alpha)$ for all $u \in E$, Γ is C^2 and, if θ_ε is a family such that $\theta_\varepsilon \to \theta$ as $\varepsilon \to 0$, then

$$\lim_{\varepsilon\to 0} \left[\varepsilon^{-\alpha}(G'(\varepsilon, z_{\theta_\varepsilon}) \mid \partial_{ij} z_{\theta_\varepsilon}) + \varepsilon^{-\alpha}(G''(\varepsilon, z_{\theta_\varepsilon})\partial_i z_{\theta_\varepsilon} \mid \partial_j z_{\theta_\varepsilon})\right] = \partial_{ij}\Gamma(\theta),$$

$$\lim_{\varepsilon\to 0} \varepsilon^{-\alpha/2} G''(\varepsilon, z_{\theta_\varepsilon})\partial_i z_{\theta_\varepsilon} = 0.$$

We can now prove two abstract theorems.

Theorem 3.3. *Assume* $(\mathbf{H}), (\mathbf{F_0} - \mathbf{F_4})$ *and* $(\mathbf{G_0} - \mathbf{G_5})$. *For a given* $\theta \in \mathbb{R}^d$, *and for any small* ε, *let us suppose that there is a critical point* $u_\varepsilon \in Z_\varepsilon$ *of* f_ε, *such that* $u_\varepsilon = z_{\theta_\varepsilon} + w(\varepsilon, \theta_\varepsilon)$ *and* $\theta_\varepsilon \to \theta$ *as* $\varepsilon \to 0$. *Assume that* $z_\theta = \lim_\varepsilon z_{\theta_\varepsilon}$ *is nondegenerate for the restriction of* f_0 *to* $(T_{z_\theta} Z)^\perp$, *with Morse index equal to* m_0, *and that the Hessian matrix* $D^2\Gamma(\theta)$ *is positive or negative definite.*

Then u_ε, *for small* ε, *is a nondegenerate critical point for* f_ε *with Morse index equal to* m_0 *if* $D^2\Gamma(\theta)$ *is positive, to* $m_0 + d$ *if* $D^2\Gamma(\theta)$ *is negative. As a consequence, the critical points of* f_ε *form a continuous curve.*

Proof Let us write

$$E = E^+ \oplus E^0 \oplus E^- \tag{3.15}$$

where $E^0 = T_{z_\theta} Z$, $\dim(E^-) = m_0$ and there exists $\delta > 0$ such that

$$\begin{cases} D^2 f_0(z_\theta)[v, v] > \delta \|v\|^2 & \forall\, v \in E^+, \\ D^2 f_0(z_\theta)[v, v] < -\delta \|v\|^2 & \forall\, v \in E^-. \end{cases}$$

From the hypothesis $f_0'(z_\eta) = 0$ for all $\eta \in \mathbb{R}^N$ it is easy to deduce

$$D^2 f_0(z_\theta)[\partial_i z_\theta, \partial_j z_\theta] = 0.$$

Let us define

$$\varphi_i^0 = \frac{1}{\|\partial_i z_\theta\|}\,\partial_i z_\theta.$$

The set $\{\varphi_i^0\}_{i=1,..,d}$ is an orthonormal base for E^0. Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of the symmetric matrix $D^2 f_0(z_\theta)$ on E^-. Of course $\lambda_i < 0$ for all i, and let $\lambda_0 = \max_i \lambda_i < 0$. Let $\{t_i^0\}_{i=1,..,m_0}$ be an orthonormal base for E^- such that $D^2 f_0[t_i^0, t_j^0] = 0$ if $i \neq j$, $D^2 f_0[t_i^0, t_i^0] = \lambda_i$. By orthogonality of the decomposition (3.15), we have $(\varphi_i^0 \mid t_j^0) = 0$ for all i, j. Define

$$\varphi_i^\varepsilon = \frac{1}{\|\partial_i z_{\theta_\varepsilon}\|}\,\partial_i z_{\theta_\varepsilon}.$$

The set $\{\varphi_i^\varepsilon\}_{i=1,..,d}$ is an orthonormal base for the tangent space $T_{z_{\theta_\varepsilon}} Z$, space that we denote E_ε^0. Notice that $\varphi_i^\varepsilon \to \varphi_i^0$ as $\varepsilon \to 0$.

For $i = 1, .., m_0$ we want to find τ_i^ε such that, setting $t_i^\varepsilon = t_i^0 + \tau_i^\varepsilon$, we obtain, for all i, j,

$$(t_i^\varepsilon \mid \varphi_j^\varepsilon) = 0. \tag{3.16}$$

That is, we want

$$0 = (t_i^\varepsilon \mid \varphi_j^\varepsilon) = (t_i^0 + \tau_i^\varepsilon \mid \varphi_j^0 + (\varphi_j^\varepsilon - \varphi_j^0)) = (t_i^0 \mid \varphi_j^\varepsilon - \varphi_j^0) + (\tau_i^\varepsilon \mid \varphi_j^\varepsilon),$$

hence

$$(\tau_i^\varepsilon \mid \varphi_j^\varepsilon) = -(t_i^0 \mid \varphi_j^\varepsilon - \varphi_j^0).$$

So, we define

$$\tau_i^\varepsilon = \sum_{j=1}^{d} -(t_i^0 \mid \varphi_j^\varepsilon - \varphi_j^0)\varphi_j^\varepsilon,$$

and (3.16) holds. Notice that $\tau_i^\varepsilon \to 0$ as $\varepsilon \to 0$, so that

$$t_i^\varepsilon \to t_i^0$$

as $\varepsilon \to 0$. As $\{t_i^0\}_i$ is an orthonormal base, the vectors $\{t_i^\varepsilon\}$ are linearly independent, for small ε. Let us define E_ε^- the m_0-dimensional space spanned by $\{t_i^\varepsilon\}$. For $v \in E_\varepsilon^-$, $\|v\| = 1$, we have $v = \sum_{k=1}^{m_0} \beta_k t_k^\varepsilon$ and we can write

$$\begin{aligned} D^2 f_\varepsilon(u_\varepsilon)[v, v] &= \sum_{l,k=1}^{m_0} \beta_l \beta_k D^2 f_\varepsilon(u_\varepsilon)[t_l^0 + \tau_l^\varepsilon, t_k^0 + \tau_k^\varepsilon] \\ &= \sum_{l,k=1}^{m_0} \beta_l \beta_k D^2 f_\varepsilon(u_\varepsilon)[t_l^0, t_k^0] + o(1), \end{aligned}$$

where $o(1)$ vanishes as $\varepsilon \to 0$, uniformly in v. By hypotheses $(\mathbf{F_0})$, $(\mathbf{G_1})$, $(\mathbf{G_2})$, we obtain

$$D^2 f_\varepsilon(u_\varepsilon)[t_l^0, t_k^0] \to D^2 f_0(z_\theta)[t_l^0, t_k^0].$$

As $t_k^\varepsilon \to t_k^0$, for $\varepsilon \to 0$, and $\{t_k^0\}$ is orthonormal, it is easy to see that, for small ε, $\|v\| = 1$ implies $\sum_{k=1}^{m_0} \beta_k^2 \geq \frac{1}{2}$. Hence

$$D^2 f_\varepsilon(u_\varepsilon)[v, v] = \sum_{k=1}^{m_0} \lambda_k \beta_k^2 + o(1) \leq \frac{\lambda_0}{2} + o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$, uniformly in v if $\|v\| = 1$. Hence, for small ε, $D^2 f_\varepsilon(u_\varepsilon)$ is negative definite in E_ε^-.

We now define

$$E_\varepsilon^+ = (E_\varepsilon^0 \oplus E_\varepsilon^-)^\perp,$$

so that

$$E = E_\varepsilon^+ \oplus E_\varepsilon^0 \oplus E_\varepsilon^-.$$

We want now to prove that $D^2 f_\varepsilon(u_\varepsilon)$ is positive definite on E^+_ε, for small ε. Let P^+ be the orthogonal projection of E to E^+.

We claim that there are $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$ and all $v \in E^+_\varepsilon$, $\|v\| = 1$, it holds

$$D^2 f_\varepsilon(u_\varepsilon)[v, v] > \delta_0.$$

We argue by contradiction. If the claim is not true, then there are sequences $\{\varepsilon_k\}$, $\{v_k\} \subset E^+_{\varepsilon_k}$, with $\|v_k\| = 1$ and $\varepsilon_k \to 0$ as $k \to \infty$, such that

$$\begin{aligned} \frac{1}{k} &> D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[v_k, v_k] \\ &= D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[P^+ v_k, P^+ v_k] \\ &\quad + 2D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[P^+ v_k, v_k - P^+ v_k] \\ &\quad + D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[v_k - P^+ v_k, v_k - P^+ v_k]. \end{aligned} \tag{3.17}$$

We recall that

$$v_k - P^+ v_k = \sum_{i=1}^{m_0} (v_k \mid t_i^0) t_i^0 + \sum_{i=1}^{d} (v_k \mid \varphi_i^0) \varphi_i^0,$$

and that, since $v_k \in E^+_{\varepsilon_k}$, $(v_k \mid t_i^{\varepsilon_k}) = 0$ and $(v_k \mid \varphi_i^{\varepsilon_k}) = 0$. Hence we have

$$(v_k \mid t_i^0) = (v_k \mid t_i^0 - t_i^{\varepsilon_k}) \to 0$$

as $k \to \infty$, because $t_i^{\varepsilon_k} \to t_i^0$ and $\{v_k\}$ is bounded. In the same way we get

$$(v_k \mid \varphi_i^0) \to 0,$$

hence

$$v_k - P^+ v_k \to 0. \tag{3.18}$$

Now (3.17) becomes

$$\begin{aligned} \frac{1}{k} &> D^2 f_{\varepsilon_k}(u_{\varepsilon_k})[P^+ v_k, P^+ v_k] + o(1) \\ &= D^2 f_0(z_\theta)[P^+ v_k, P^+ v_k] \\ &\quad + (D^2 f_{\varepsilon_k}(u_{\varepsilon_k}) - D^2 f_0(z_\theta))[P^+ v_k, P^+ v_k] + o(1). \end{aligned} \tag{3.19}$$

Thanks to the continuity hypotheses we have

$$(D^2 f_{\varepsilon_k}(u_{\varepsilon_k}) - D^2 f_0(z_\theta))[P^+ v_k, P^+ v_k] = o(1),$$

while

$$D^2 f_0(z_\theta)[P^+ v_k, P^+ v_k] > \delta \|P^+ v_k\|^2,$$

because $P^+ v_k \in E^+$. By (3.18) we also obtain

$$\|P^+ v_k\| \to 1.$$

Hence (3.19) gives

$$\frac{1}{k} > \delta + o(1),$$

a contradiction. So the claim is proved.

Up to now we have shown that, for small ε, $D^2 f_\varepsilon(u_\varepsilon)$ is negative definite on E_ε^- and positive definite on E_ε^+. We want now to study the behavior of $D^2 f_\varepsilon(u_\varepsilon)$ on E_ε^0. We will prove that $D^2 f_\varepsilon(u_\varepsilon)$ is positive or negative definite accordingly with $D^2\Gamma(\theta)$.

As first thing we recall that we have

$$\begin{aligned} &D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \\ &\qquad = (\partial_i z_{\theta_\varepsilon} \mid \partial_j z_{\theta_\varepsilon}) - (F''(u_\varepsilon)\partial_i z_{\theta_\varepsilon} \mid \partial_j z_{\theta_\varepsilon}) + (G''(\varepsilon, u_\varepsilon)\partial_i z_{\theta_\varepsilon} \mid \partial_j z_{\theta_\varepsilon}). \end{aligned}$$

As $\partial_i z_{\theta_\varepsilon} \in \ker[I_E - F''(z_{\theta_\varepsilon})]$ and $w(0, z_{\theta_\varepsilon}) = 0$, developing $F''(u_\varepsilon)$ and $G''(\varepsilon, u_\varepsilon)$ and setting $w_\varepsilon = w(\varepsilon, \theta_\varepsilon)$, we obtain

$$\begin{aligned} &D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \\ &\quad = (\partial_i z_{\theta_\varepsilon} \mid \partial_j z_{\theta_\varepsilon}) - (F''(z_{\theta_\varepsilon})\partial_i z_{\theta_\varepsilon} \mid \partial_j z_{\theta_\varepsilon}) - (F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \mid w_\varepsilon) \\ &\qquad + (G''(\varepsilon, z_{\theta_\varepsilon})\partial_i z_{\theta_\varepsilon} \mid \partial_j z_{\theta_\varepsilon}) + (G'''(\varepsilon, z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \mid w_\varepsilon) + O(\|w_\varepsilon\|^2) \\ &\quad = -(F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \mid w_\varepsilon) + (G''(\varepsilon, z_{\theta_\varepsilon})\partial_i z_{\theta_\varepsilon} \mid \partial_j z_{\theta_\varepsilon}) \\ &\qquad + (G'''(\varepsilon, z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \mid w_\varepsilon) + O(\|w_\varepsilon\|^2). \end{aligned}$$

We have $w_\varepsilon = o(\varepsilon^{\alpha/2})$ (see Lemma 2.1), hence from $(\mathbf{G_5})$ we deduce

$$(G'''(\varepsilon, z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \mid w_\varepsilon) = o(\varepsilon^\alpha)$$

so that

$$\begin{aligned} &D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] = \\ &\quad -(F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \mid w_\varepsilon) + (G''(\varepsilon, u_\varepsilon)\partial_i z_{\theta_\varepsilon} \mid \partial_j z_{\theta_\varepsilon}) + o(\varepsilon^\alpha). \end{aligned} \tag{3.20}$$

By Lemma 2.1 we have

$$z_{\theta_\varepsilon} + w_\varepsilon - F'(z_{\theta_\varepsilon} + w_\varepsilon) + G'(\varepsilon, z_{\theta_\varepsilon} + w_\varepsilon) = \sum_l a_l \partial_l z_{\theta_\varepsilon}.$$

Developing F' and G' we obtain

$$\begin{aligned} z_{\theta_\varepsilon} + w_\varepsilon - F'(z_{\theta_\varepsilon}) - F''(z_{\theta_\varepsilon})w_\varepsilon + G'(\varepsilon, z_{\theta_\varepsilon}) + G''(\varepsilon, z_{\theta_\varepsilon})w_\varepsilon + O(\|w_\varepsilon\|^2) \\ = \sum_l a_l \partial_l z_{\theta_\varepsilon}. \end{aligned}$$

By a scalar product with $\partial_{ij} z_{\theta_\varepsilon}$, recalling $(\mathbf{H})$ and the fact that $z_{\theta_\varepsilon} = F'(z_{\theta_\varepsilon})$, we get

$$(w_\varepsilon \mid \partial_{ij} z_{\theta_\varepsilon}) - (F''(z_{\theta_\varepsilon}) w_\varepsilon \mid \partial_{ij} z_{\theta_\varepsilon}) + (G'(\varepsilon, z_{\theta_\varepsilon}) \mid \partial_{ij} z_{\theta_\varepsilon}) \\ + (G''(\varepsilon, z_{\theta_\varepsilon}) w_\varepsilon \mid \partial_{ij} z_{\theta_\varepsilon}) + O(\|w_\varepsilon\|^2) = 0.$$

By $(\mathbf{G_5})$ it is

$$(G''(\varepsilon, z_{\theta_\varepsilon}) w_\varepsilon \mid \partial_{ij} z_{\theta_\varepsilon}) = (G''(\varepsilon, z_{\theta_\varepsilon}) \partial_{ij} z_{\theta_\varepsilon} \mid w_\varepsilon) = o(\varepsilon^\alpha),$$

so we obtain

$$(w_\varepsilon \mid \partial_{ij} z_{\theta_\varepsilon}) - (F''(z_{\theta_\varepsilon}) w_\varepsilon \mid \partial_{ij} z_{\theta_\varepsilon}) + (G'(\varepsilon, z_{\theta_\varepsilon}) \mid \partial_{ij} z_{\theta_\varepsilon}) + o(\varepsilon^\alpha) = 0. \quad (3.21)$$

Deriving twice the equation $z_\eta = F'(z_\eta)$ with respect to $\eta \in \mathbb{R}^N$ and computing the result at $\eta = \theta_\varepsilon$ we obtain

$$\partial_{ij} z_{\theta_\varepsilon} = F''(z_{\theta_\varepsilon}) \partial_{ij} z_{\theta_\varepsilon} + F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}].$$

By a scalar product with w_ε we get

$$(\partial_{ij} z_{\theta_\varepsilon} \mid w_\varepsilon) = (F''(z_{\theta_\varepsilon}) \partial_{ij} z_{\theta_\varepsilon}) \mid w_\varepsilon) + (F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \mid w_\varepsilon).$$

Substituting this last identity in (3.21) we obtain

$$-(F'''(z_{\theta_\varepsilon})[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \mid w_\varepsilon) = (G'(\varepsilon, z_{\theta_\varepsilon}) \mid \partial_{ij} z_{\theta_\varepsilon}) + o(\varepsilon^\alpha). \quad (3.22)$$

From this and (3.20) we have

$$D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] = (G'(\varepsilon, z_{\theta_\varepsilon}) \mid \partial_{ij} z_{\theta_\varepsilon}) + (G''(\varepsilon, u_\varepsilon) \partial_i z_{\theta_\varepsilon} \mid \partial_j z_{\theta_\varepsilon}) + o(\varepsilon^\alpha).$$

Hence, dividing by ε^α, passing to the limit and using hypothesis $(\mathbf{G_5})$ we obtain

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\alpha} D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}] \\ = \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon^\alpha} (G'(\varepsilon, z_{\theta_\varepsilon}) \mid \partial_{ij} z_{\theta_\varepsilon}) + \frac{1}{\varepsilon^\alpha} (G''(\varepsilon, u_\varepsilon) \partial_i z_{\theta_\varepsilon} \mid \partial_j z_{\theta_\varepsilon}) \right] = \partial_{ij} \Gamma(\theta).$$

As $D^2 \Gamma(\theta)$ is definite, so is $D^2 f_\varepsilon(u_\varepsilon)[\partial_i z_{\theta_\varepsilon}, \partial_j z_{\theta_\varepsilon}]$, for small ε.

Let us recall what we have proved up to now. Let us assume that $D^2 \Gamma(\theta)$ is definite positive (the other case is analogous). We have proved that there is a constant $\delta > 0$ such that

$$D^2 f_\varepsilon(u_\varepsilon)[v^-, v^-] \le -\delta \|v^-\|^2 \quad \text{for all } v^- \in E_\varepsilon^-,$$

$$D^2 f_\varepsilon(u_\varepsilon)[v^0, v^0] \ge \delta \varepsilon^\alpha \|v^0\|^2 \quad \text{for all } v^0 \in E_\varepsilon^0,$$

$$D^2 f_\varepsilon(u_\varepsilon)[v^+, v^+] \ge \delta \|v^+\|^2 \quad \text{for all } v^+ \in E_\varepsilon^+.$$

To conclude the proof we have to show that $D^2 f_\varepsilon(u_\varepsilon)$ is positive definite in $E_\varepsilon^+ + E_\varepsilon^o$. This does not derive directly from the previous statements because it is not true, in general, that $D^2 f_\varepsilon(u_\varepsilon)[v^+, v^0] = 0$. However, thanks to $(\mathbf{G_5})$, we have, for any $v^+ \in E_\varepsilon^+$, $v^0 \in E_\varepsilon^0$,

$$\left|D^2 f_\varepsilon(u_\varepsilon)[v^+, v^0]\right| \le o(\varepsilon^{\alpha/2}) \|v^+\| \, \|v^0\|.$$

Hence, for small ε, we obtain

$$\begin{aligned} & D^2 f_\varepsilon(u_\varepsilon)[v^+ + v^0, v^+ + v^0] \\ & \qquad \ge \delta \|v^+\|^2 + \delta\varepsilon^\alpha \|v^0\|^2 - o(\varepsilon^{\alpha/2}) \|v^+\| \|v^0\| \ge \varepsilon^\alpha \|v^+ + v^0\|^2. \end{aligned}$$

The proof is now complete. □

With small changes in the previous arguments one can prove the following theorem.

Theorem 3.4. *Assume* $(\mathbf{H}), (\mathbf{F_0} - \mathbf{F_3})$, $(\mathbf{G_0} - \mathbf{G_3})$, $(\mathbf{F_4})'$, $(\mathbf{G_4})'$, $(\mathbf{G_5})'$. *For a given* $\theta \in \mathbb{R}^d$, *and for any small* ε, *let us suppose that there is a critical point* $u_\varepsilon \in Z_\varepsilon$ *of* f_ε, *such that* $u_\varepsilon = z_{\theta_\varepsilon} + w(\varepsilon, \theta_\varepsilon)$ *and* $\theta_\varepsilon \to \theta$ *as* $\varepsilon \to 0$. *Assume that* $z_\theta = \lim_\varepsilon z_{\theta_\varepsilon}$ *is nondegenerate for the restriction of* f_0 *to* $(T_{z_\theta} Z)^\perp$, *with Morse index equal to* m_0, *and that the Hessian matrix* $D^2\Gamma(\theta)$ *is positive or negative definite.*

Then u_ε *is a nondegenerate critical point for* f_ε *with Morse index equal to* m_0 *if* $D^2\Gamma(\theta)$ *is positive, to* $m_0 + d$ *if* $D^2\Gamma(\theta)$ *is negative. As a consequence, the critical points of* f_ε *form a continuous curve.*

Proof To study the behavior of $D^2 f_\varepsilon(u_\varepsilon)$ on E_ε^+ and E_ε^- we repeat the arguments of the previous theorem. As to E_ε^0, we recall that the hypotheses imply $w_\varepsilon = O(\varepsilon^\alpha)$ (see [4, Lemma 2.2]), so (3.20) and (3.22) still hold and the proof goes on as in the previous theorem. □

We want now to apply these abstract results to our equation (3.1). In the following theorem we apply Theorem 3.3. To fit hypotheses $(\mathbf{F_4})$, $(\mathbf{G_4})$, we have to assume $p \ge 3$. Together with the hypothesis $p < \frac{N+2}{N-2}$, this of course implies $N \le 3$. Notice that in the following theorems we will treat curves of solutions bifurcating from 0, or ∞, or bounded away both from 0 and ∞. Recall that we refer in our claims to the H^1-norm, and that in any case the L^∞-norm is vanishing.

Theorem 3.5. *Let us suppose* $N = 1, 2, 3$ *and* $3 \le p < q < +\infty$ *if* $N = 1, 2$ *while* $3 \le p < q \le 5$ *if* $N = 3$. *Assume* $(\mathbf{a_1})$, $(\mathbf{a_2})$, $(\mathbf{b_1})$ *and* $(\mathbf{b_2})$. *Then we obtain a curve* (λ, ψ_λ) *of solutions of (3.1), where* $\lambda \in (\lambda_0, 0)$, *for a suitable* $\lambda_0 < 0$. *We have the following behavior of* ψ_λ *as* $\lambda \to 0$*:*

1. *if* $N = 1$ *and* $3 \le p < 5$, *then* $\|\psi_\lambda\| \to 0$, *so we have a curve of solutions bifurcating from the origin in* $H^1(\mathbb{R}^N)$*;*

2. *if $N = 1$ and $p = 5$, or if $N = 2$ and $p = 3$, then $\|\psi_\lambda\| \to c \neq 0$, so we have, in $H^1(\mathbb{R}^N)$, a curve of solutions bounded away from 0 and ∞;*
3. *if $N = 1$ and $p > 5$, or if $N = 2$ and $p > 3$, or if $N = 3$ and $3 \leq p < 5$, then $\|\psi_\lambda\| \to +\infty$, so we have, in $H^1(\mathbb{R}^N)$, a curve of solutions bifurcating from infinity.*

Proof By Theorem 3.1 we get for equation (3.2) a family of solutions $u_\varepsilon = z_{\theta_\varepsilon} + w(\varepsilon, \theta_\varepsilon)$. Using the general devices of Chapter 2, it is easy to see that θ_ε must converge, as $\varepsilon \to 0$, to a maximum or a minimum point of Γ. But Γ, as defined in Lemma 3.2, has the unique critical point $\theta = 0$, hence $\theta_\varepsilon \to 0$ as $\varepsilon \to 0$. To apply Theorem 3.3 we have to show that the hypotheses $(\mathbf{F_4})$, $(\mathbf{G_4})$, $(\mathbf{G_5})$ are satisfied. The assumption $p \geq 3$ gives that F, G are C^4. As to the asymptotic assumptions, they are easily checked with arguments similar to those of Section 3.1 and we leave this to the reader (notice that here $\alpha = N$). Recalling that z_0 is a nondegenerate critical point for f_0 we apply Theorem 3.3 and we find a curve $(\varepsilon, u_\varepsilon)$ of solutions of (3.2), such that $u_\varepsilon(x) = z_0(x + \theta_\varepsilon) + w(\varepsilon, \theta_\varepsilon)(x)$ with $\theta_\varepsilon \to 0$ and $w(\varepsilon, \theta_\varepsilon) \to 0$ as $\varepsilon \to 0$. By the change of variables $\lambda = -\varepsilon^2$ and $\psi_\lambda(x) = \varepsilon^{2/p-1} u_\varepsilon(\varepsilon x)$ we get a family of solutions of (3.1), which is still a curve. Noticing that $u_\varepsilon \to z_0$ in H^1, it is easy to verify the statements on $\lim_{\lambda \to 0} \|\psi_\lambda\|$, it is just a computation involving a change of variables, and we leave it to the reader. □

In the case $N = 1$ it is possible to relax some hypotheses. In the following theorem we assume $p \geq 2$ and we do not suppose a continuous and bounded. We apply Theorem 3.4.

Theorem 3.6. *Let us assume $N = 1$ and $2 \leq p < q < +\infty$. Assume $a - A \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} (a(x) - A)dx \neq 0$, and suppose also $(\mathbf{b_1})$ and $(\mathbf{b_2})$. Then we obtain a curve (λ, ψ_λ) of solutions of (3.1), where $\lambda \in (\lambda_0, 0)$, for a suitable $\lambda_0 < 0$. We have the following behavior of ψ_λ as $\lambda \to 0$:*

1. *if $2 \leq p < 5$, then $\|\psi_\lambda\| \to 0$, so we have a curve of solutions bifurcating from the origin in $H^1(\mathbb{R}^N)$;*
2. *if $p = 5$ then $\|\psi_\lambda\| \to c \neq 0$, so we have, in $H^1(\mathbb{R}^N)$, a curve of solutions bounded away from 0 and ∞;*
3. *if $p > 5$ then $\|\psi_\lambda\| \to +\infty$, so we have, in $H^1(\mathbb{R}^N)$, a curve of solutions bifurcating from infinity.*

Proof We want to apply Theorem 3.4. As $p \geq 2$, F, G are C^3. As to the asymptotic properties of G, in particular $(\mathbf{G_5})'$, notice first that here $\alpha = 1$. We use the arguments of [4] (in particular the proof of Lemma 4.1, p. 1142-1143) to study G_1' and G_1'', while the study of G_2' and G_2'' is the same as in the previous sections. Hence we obtain a curve $(\varepsilon, u_\varepsilon)$ of solutions of (3.2). Arguing as in the previous theorem, we obtain a curve (λ, ψ_λ) of solutions of (3.1), and we get its asymptotic properties as $\lambda \to 0$. □

Remark 3.5. Theorems 3.3 and 3.4 fill a gap in the proof of Theorem 3.2 in [4]. In that paper Theorem 3.2 was used only in Theorem 1.5, to prove that a family of solutions was a curve. Now this result is a particular case of Theorem 3.6. A similar correction to Theorem 3.2 of [4] was obtained by S. Krömer in his Diplomarbeit [48]. He also obtained there a bifurcation result analogous to Theorem 3.1.

Part II

The nonlinear Schrödinger equation

4 The nonlinear Schrödinger equation

The nonlinear Schrödinger equation with potential V is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - |\psi|^{p-1}\psi, \quad x \in \mathbb{R}^N, \tag{4.1}$$

where $\psi\colon (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N \mapsto \psi(t,x) \in \mathbb{C}$, i is the unit imaginary, Δ is the Laplace operator and $\hbar$ denotes the Planck constant

$$\hbar = 1.054 \times 10^{-27} erg \cdot sec.$$

Equations of this type arise naturally in several domain of Mathematical Physics, for example in self-focusing phenomena in nonlinear optics.

Let us consider first the "subcritical case", that is when $1 < p < 2^* - 1$, where

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N < 3. \end{cases}$$

We recall that 2^* is the critical exponent for the Sobolev embeddings.

In the last twenty years many mathematicians fixed their attentions to the searching of *solitary waves* of (4.1), namely the solutions of the form

$$\psi(t,x) = e^{i\frac{Et}{\hbar}} u(x),$$

with $E \in \mathbb{R}$ and $u\colon \mathbb{R}^N \to \mathbb{R}^+$. Such a ψ solves (4.1) provided the *standing waves* u satisfies

$$\begin{cases} -\dfrac{\hbar^2}{2m}\Delta u + (V(x) - E)u = u^p, & x \in \mathbb{R}^N, \\ u > 0. \end{cases} \tag{4.2}$$

The aim is to study the *semiclassical states* of the nonlinear Schrödinger equation, that is to study the behavior of the solutions when $\hbar \to 0^+$.

With an abuse of notations, let us write V instead of $V - E$ and ε^2 instead of $\hbar^2/(2m)$, then (4.2) becomes

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = u^p, & x \in \mathbb{R}^N, \\ u > 0, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{NLS}$$

Let us consider for the moment the case in which V is strictly positive, that is $V \geq V_0 > 0$.

This problem can be seen as a perturbation problem, indeed, performing the change of variables $x \mapsto \varepsilon x$, (NLS) becomes now

$$\begin{cases} -\Delta u + V(\varepsilon x)u = u^p, & x \in \mathbb{R}^N, \\ u > 0, \end{cases} \tag{4.3}$$

which can be considered as a perturbation of

$$\begin{cases} -\Delta u + V(0)u = u^p, & x \in \mathbb{R}^N, \\ u > 0. \end{cases} \tag{4.4}$$

Let us observe that, of course, if u is a solution of (4.3), then $u(\cdot/\varepsilon)$ is a solution of (NLS).

It is well known (see [16, 17, 49]) that (4.4) has a unique positive radial solution which as an exponential decay to 0 as $|x| \to \infty$.

In [36], Floer and Weinstein considered the case $N = 1$, $p = 3$ and they proved that, for ε sufficiently small, there exists a family of standing waves solutions which concentrates at each given nondegenerate critical points of the potential V, see also Oh [68, 69]. The arguments of all these papers are based on a Liapunov-Schmidt reduction and the solutions are found "near" the solutions of (4.4). Precisely, in Oh's papers, it was assumed that, for some $a > 0$, V belongs to the class $(V)_a$ introduced by Kato [46]: $V \in (V)_a$ if either $V(x) = a$ or $V(x) > a$ for all $x \in \mathbb{R}^N$ and $(V(x) - a)^{-1/2} \in \mathrm{Lip}(\mathbb{R}^N)$. Therefore classes of potentials, including "slowly" oscillating ones like $\sin(x)$, are permitted but "rapidly" oscillating ones like $\sin(x^2)$ are not. These papers deal with the one-dimensional case and do not use the variational nature of (NLS).

Rabinowitz, in [78], using variational methods, gives existence results for (NLS) for ε fixed, when

$$\lim_{|x|\to\infty} V(x) = +\infty.$$

Moreover, using arguments like concentration-compactness methods, of P.L. Lions [54], it is also proved in [78] the existence of a solution in the case

$$0 < V_0 \leq \inf_{\mathbb{R}^N} V < \liminf_{|x|\to\infty} V(x).$$

This solution concentrates around a global minimum of V as $\varepsilon \to 0$, as shown later by X.F. Wang in [90].

Later on, Ambrosetti, Badiale and Cingolani, [6], using perturbation methods (see also Chapter 2) and taking advantage of the variational structure of (NLS), obtained existence of standing wave solutions by assuming that the potential V is bounded above and below by positive constants and it has a local minimum or maximum with nondegenerate mth derivative, for some integer m.

Another result concerning degenerate critical points of V is given by Y.Y. Li, in [50], where existence results are proved by assuming that the critical points of V are stable with respect to a small C^1-perturbation of V.

Del Pino and Felmer, see [31], using a suitable penalization technique, can treat a more general nonlinearity $f(t)$ and a more general domain, and they find a solutions for minima of V. Subsequently, ins [32, 34], this result has been extended to deal also with local maxima and saddle points of V.

Roughly speaking, in all these paper it is been proved that if x_0 is a "topologically nontrivial" critical point of V, then, for ε sufficiently small, (NLS) admits solutions which concentrate at x_0 as $\varepsilon \to 0$. But also the converse is true, indeed, under suitable assumptions, some necessary conditions for the existence of concentrating solutions are also know (see for example [6, 90]).

It is also been studied the existence of multi-peak solutions of (NLS), that is solutions which concentrate in more than one points, as $\varepsilon \to 0$, see, for example, [25, 33, 70].

Some multiplicity results for (NLS) are due to Cingolani and Lazzo [23] and to Ambrosetti, Malchiodi and Secchi [11]. In the first paper, the authors prove that if $\liminf_{|x|\to\infty} V(x) > \inf_{\mathbb{R}^N} V > 0$, then, for ε sufficiently small, (NLS) has at least $\mathrm{cat}(M, M_\delta)$ solutions, where $M = \{x \in \mathbb{R}^N \mid V(x) = \inf_{\mathbb{R}^N} V\}$, M_δ is a δ-neighborhood of M and cat denotes the Lusternik-Schnirelman category. In the second paper, instead, the authors can treat not only the case of global minima of V, but also local minima or maxima of V and, moreover, the case of a nondegenerate (in a suitable sense) manifold of critical points of V.

We mention also a paper of Grossi [39], where the author gives the precise number of solutions concentrating at a critical point of V, under suitable assumptions on the potential.

In a recent paper, Squassina in [82] considers a more general problem then (NLS). More precisely he studies a general class of singularly perturbed quasilinear equation on $\mathbb{R}^N$,

$$-\varepsilon^2 \operatorname{div}\left(J(x,u)\nabla u\right) + \frac{\varepsilon^2}{2}\langle D_s J(x,u)\nabla u \mid \nabla u\rangle + V(x)u = u^p, \qquad (4.5)$$

by means of non-smooth critical points theory. Here, for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$, $J(x,s)$ is a uniformly elliptic, symmetric matrix $N \times N$. He proves that if x_0 is a common "minimum point" of V and J, then (4.5) admits a concentrating solution at x_0.

In a joint paper with Simone Secchi [74] (see also Chapter 5), we extend the results of Squassina, at least in the case of J depending only on x. Indeed we show that there exist solutions of

$$-\varepsilon^2 \operatorname{div}\big(J(x)\nabla u\big) + V(x)u = u^p, \tag{4.6}$$

which concentrates at a nondegenerate critical point of an auxiliary function $\Gamma\colon \mathbb{R}^N \to \mathbb{R}$ defined as follows:

$$\Gamma(z) = V(z)^{\frac{p+1}{p-1}-\frac{N}{2}}\,(\det J(z))^{\frac{1}{2}}\,.$$

We treat also the case of isolated local minima or maxima of Γ and we give also necessary conditions for the existence of concentrating solutions of (4.6). If x_0 is a common minimum point of J and V, then it is easy to see that x_0 is also a minimum point of Γ. But the converse is not true, indeed x_0 can be a minimum point of Γ, without being a minimum point of J and V. Let us observe, finally, that if J is constantly equal to the identity matrix $N \times N$, then (4.6) becomes (NLS).

Many of these results have been extended to the case of a nonlinear Schrödinger equation with more than one potential, namely

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = K(x)u^p + Q(x)u^\sigma, & \text{in } \mathbb{R}^N,\\ u > 0, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

with $1 < p < \sigma < 2^* - 1$, (see [24, 91]). Under certain conditions at infinity on the potentials, it is been proved existence of solutions which concentrates on the minima of a suitable auxiliary functional depending only on V, K and Q.

The critical case, that is when $\sigma = 2^* - 1$, has been studied by Alves, João Marcos do Ó and Souto in [1], proving the existence of solutions of

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = f(u) + u^\sigma & \text{in } \mathbb{R}^N,\\ u > 0, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{4.7}$$

concentrating on minima of V. In (4.7), $f(u)$ is a nonlinearity with subcritical growth.

On the other hand, when $K \equiv 0$ and $Q \equiv 1$, nonexistence results of single blow-up solutions have been proved in a recent work by Cingolani and Pistoia [26]. For a generalization of this result, see [73].

In [72] (see also Chapter 6), instead, we study

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = K(x)u^p + Q(x)u^\sigma & \text{in } \mathbb{R}^N,\\ u > 0, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

with V and K bounded below by a positive constant and with Q vanishing in zero. We show existence of concentrating solutions at minima and maxima and nondegenerate critical points of an auxiliary functional $\Gamma\colon \mathbb{R}^N \to \mathbb{R}$, so defined:

$$\Gamma(\xi) = C_1 \Gamma_1(\xi) - C_2 \Gamma_2(\xi),$$

where

$$\Gamma_1(\xi) \equiv V(\xi)^{\frac{p+1}{p-1} - \frac{N}{2}} K(\xi)^{-\frac{2}{p-1}},$$
$$\Gamma_2(\xi) \equiv Q(\xi)\, V(\xi)^{\frac{\sigma+1}{p-1} - \frac{N}{2}} K(\xi)^{-\frac{\sigma+1}{p-1}},$$

and with C_1 and C_2 suitable constants.

Recently, solutions concentrating on spheres have also been found for radial nonlinear Schrödinger equation. This problem is more complicated and is out of the content of this thesis. We limit to cite [9], where it is shown that concentration on spheres is governed by the auxiliary potential $M(r) = r^{N-1} V(r)^{\frac{p+1}{p-1} - \frac{1}{2}}$, see also [13] for a preliminary result. We also recall [29, 79] where concentration on spheres is proved for nonlinear Schrödinger equation coupled with Maxwell equations.

Finally let us remark that there has been obtained some existence results for (NLS) also in the case $V = 0$ in some point but $0 < \liminf_{|x| \to \infty} V(x) < +\infty$, see [18, 19], as well as in the case in which $\lim_{|x| \to \infty} V(x) = 0$, see [8]. For solutions in weighted Sobolev space see also [67, 80].

5 Singularly perturbed elliptic equations in divergence form

The aim of this chapter is to study the existence and the concentrating behavior of solutions to the following problem:

$$\begin{cases} -\varepsilon^2 \operatorname{div}\big(J(x)\nabla u\big) + V(x)u = u^p & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases} \tag{5.1}$$

where $N \geq 3$, $p \in \left(1, \frac{N+2}{N-2}\right)$, $V\colon \mathbb{R}^N \to \mathbb{R}$ and $J\colon \mathbb{R}^N \to \mathbb{R}^{N\times N}$ are C^1 functions. Here the symbol $\mathbb{R}^{N\times N}$ stands for the set of $(N \times N)$ real matrices.

As already observed on the previous chapter, such a problem, at least in the case $J \equiv I$, where I is the identity matrix in $\mathbb{R}^{N\times N}$, arises naturally when seeking *standing waves* of the nonlinear Schrödinger equation with bounded potential V, that are solutions of the form

$$\psi(t,x) = \mathrm{e}^{i\hbar^{-1}t}u(x)$$

of the following problem

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\Delta\psi + V(x)\psi - |\psi|^{p-1}\psi, \quad x \in \mathbb{R}^N,$$

where $\hbar$ denotes the Planck constant, i is the unit imaginary. The usual strategy is to put $\varepsilon = \hbar$ and then to study what happens when $\varepsilon \to 0$.

In the present chapter, we study (5.1) in the case $J \not\equiv I$. Our research is motivated by [82], where a general class of singularly perturbed quasilinear equation on $\mathbb{R}^N$,

$$-\varepsilon^2 \operatorname{div}\big(J(x,u)\nabla u\big) + \frac{\varepsilon^2}{2}\langle D_s J(x,u)\nabla u \mid \nabla u\rangle + V(x)u = f(u), \tag{5.2}$$

is studied by means of non-smooth critical points theory. If J depends only on x and $f(u) = u^p$, then (5.2) becomes (5.1).

We observe that it is in general impossible to reduce the second-order operator in equation (5.1) to the standard Laplace operator in the whole $\mathbb{R}^N$ by means of a single change of coordinates. This phenomenon especially appears in high dimension $N > 3$, as already remarked in [27, Chapter III].

On V and J we will make the following assumptions:

(V) $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V = \alpha > 0$;
(J) $J \in C^1(\mathbb{R}^N, \mathbb{R}^{N\times N})$, J is bounded; moreover, $J(x)$ is, for each $x \in \mathbb{R}^N$, a symmetric matrix and

$$(\exists \nu > 0)(\forall x \in \mathbb{R}^N)(\forall \xi \in \mathbb{R}^N \setminus \{0\}): \ \langle J(x)\xi, \xi\rangle \geq \nu|\xi|^2. \tag{5.3}$$

Let us introduce an auxiliary function which will play a crucial rôle in the study of (5.1). Let $\Gamma\colon \mathbb{R}^N \to \mathbb{R}$ be a function so defined:

$$\Gamma(z) = V(z)^{\frac{p+1}{p-1}-\frac{N}{2}} \left(\det J(z)\right)^{\frac{1}{2}}. \tag{5.4}$$

Let us observe that by **(J)**, Γ is well defined.

We now state the main results of this chapter. We will see that Γ gives, roughly speaking, a sufficient condition and a necessary one to have concentrating solutions around a point.

Theorem 5.1. *Suppose* **(V)** *and* **(J)** *hold. Suppose that there exists a compact domain* $\Lambda \subset \mathbb{R}^N$ *such that*

$$\min_{\Lambda} \Gamma < \min_{\partial\Lambda} \Gamma.$$

Then, for all $\varepsilon > 0$ *sufficiently small, there exists a solution* $u_\varepsilon \in H(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ *of* (5.1) *with* $\int_{\mathbb{R}^N} V(x)u^2 dx < +\infty$. *This solution has only one global maximum point* $x_\varepsilon \in \mathbb{R}^N$ *and we have that* $\Gamma(x_\varepsilon) \to \min_\Lambda \Gamma$ *as* $\varepsilon \to 0$ *and*

$$\lim_{\varepsilon\to 0} u_\varepsilon(x) = 0 \quad \text{for all } x \neq x_\varepsilon.$$

Theorem 5.2. *Assume, in addition to assumptions* **(V)** *and* **(J)**, *that* V *is bounded from above and there exist two positive constants* C, γ *such that*

$$|\nabla J(x)|,\ |\nabla V(x)| \leq Ce^{\gamma|x|}, \qquad \text{for all } x \in \mathbb{R}^N.$$

Let $\{u_{\varepsilon_j}\}$ *be a sequence of solutions of* (5.1) *such that for all* $\varepsilon > 0$ *there exist* $\rho > 0$ *and* $j_0 > 0$ *such that for all* $j \geq j_0$ *and for all points* x *with* $|x - z_0| \geq \varepsilon_j \rho$, *there results*

$$u_{\varepsilon_j}(x) \leq \varepsilon.$$

Then z_0 *is a critical point of* Γ.

Theorem 5.1 and 5.2 are simplified version of, respectively, Theorem 5.4 and 5.5, which hold under weaker assumptions. Indeed we can treat (5.1) in a more general domain, instead of $\mathbb{R}^N$, and with a more general nonlinearity. More precisely, let us consider:

$$\begin{cases} -\varepsilon^2 \operatorname{div}\left(J(x)\nabla u\right) + V(x)u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

where Ω is an open domain of $\mathbb{R}^N$, possibly unbounded, and $f\colon \mathbb{R}^+ \to \mathbb{R}$ is a C^1 such that:

(f1) $f(u) = o(u)$ as $u \to 0^+$;
(f2) for some $p \in \left(1, \frac{N+2}{N-2}\right)$ there holds

$$\lim_{u \to +\infty} \frac{f(u)}{u^p} = 0;$$

(f3) for some $\theta \in (2, p+1)$ we have

$$0 < \theta F(u) \leq f(u)u \quad \text{for all } u > 0,$$

where $F(u) = \int_0^u f(t)dt$;
(f4) the function

$$u \in (0, +\infty) \mapsto \frac{f(u)}{u}$$

is increasing.

The proof of Theorem 5.1 is based on the penalization technique used in [31], adapted to our case. See Section 5.1 and 5.2.

If z_0 is a common minimum point of V and J, J depends only by x and $f(u) = u^p$, then our result becomes a particular case of Theorem 1.1 of [82]. On the other side, [82] considers the case when V and J have a common minimum point, only.

In Section 5.3 we prove Theorem 5.2 using a recent version of Pucci-Serrin variational identity (see [30]).

In Section 5.4 we consider (5.1) assuming that V and J satisfy, in addition to hypotheses **(V)** and **(J)**:

(V1) $V \in C^2(\mathbb{R}^N, \mathbb{R})$, V and D^2V are bounded;
(J1) $J \in C^2(\mathbb{R}^N, \mathbb{R}^{N \times N})$, J and D^2J are bounded.

Then the following theorem holds.

Theorem 5.3. *Let* **(V-V1)** *and* **(J-J1)** *hold. Then for $\varepsilon > 0$ small, (5.1) has a solution concentrating in z_0, provided that one of the two following conditions holds:*

(a) z_0 an isolated local strict minimum or maximum of Γ;
(b) z_0 is a non-degenerate critical point of Γ.

The proof of Theorem (5.3) relies on a finite dimensional reduction, precisely on the perturbation technique developed in [4, 6, 11].

The last section of this chapter is devoted to the proof of some multiplicity results, see Theorem 5.7, 5.10 and 5.11. As before we distinguish between two cases, according to the use of penalization method of [31] or of the perturbation method of [11].

All the results contained in Section 5.4 and 5.5 seem to be new and do not appear in [82].

All the results of this chapter are contained in a joint paper with Simone Secchi [74].

Notation

- If F is a C^1 map on a Hilbert space H, we denote by $DF(u)$ its Fréchet derivative at $u \in H$.
- For $x, y \in \mathbb{R}^N$, we denote by $\langle x \mid y \rangle$ the ordinary inner product of x and y.
- C denotes a generic positive constant, which may also vary from line to line.
- $o_h(1)$ denotes a function that tends to 0 as $h \to 0$.

5.1 The ground-energy function Σ

In this section we present a more general version of Theorem 5.1 and we introduce the ground-energy function Σ that has a crucial rôle in the sequel and that, at least when $\Omega = \mathbb{R}^N$ and $f(u) = u^p$, is equal to Γ up to a constant factor.

We work in the weighted space

$$H_V(\Omega) = \left\{ u \in H_0^1(\Omega) \colon \int_\Omega V(x) u^2 dx < +\infty \right\},$$

endowed with the norm

$$\|u\|^2 = \int_\Omega \left(|\nabla u|^2 + V(x) u^2 \right) dx.$$

Our assumptions imply that the functional $I^\varepsilon \colon H_V(\Omega) \to \mathbb{R}$ defined by

$$I^\varepsilon(u) = \frac{\varepsilon^2}{2} \int_\Omega \langle J(x) \nabla u \mid \nabla u \rangle + \frac{1}{2} \int_\Omega V(x) |u|^2 - \int_\Omega F(u) \qquad (5.6)$$

is of class $C^2(\Omega)$. Moreover, (5.5) is the Euler-Lagrange equation associated to I^ε, so that we will find solutions of (5.5) as critical points of I^ε.

We define the *ground-energy function* $\Sigma(z)$ as the ground energy associated with

$$-\operatorname{div}\left(J(z) \nabla u \right) + V(z) u = f(u) \quad \text{in } \mathbb{R}^N, \qquad (5.7)$$

where $z \in \mathbb{R}^N$ is seen as a (fixed) parameter. More precisely, (5.7) is associated to the functional defined on $H^1(\mathbb{R}^N)$

$$I_z(u) = \frac{1}{2} \int_{\mathbb{R}^N} \langle J(z) \nabla u \mid \nabla u \rangle \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(z) |u|^2 dx - \int_{\mathbb{R}^N} F(u) dx. \quad (5.8)$$

If $\mathcal{N}_z$ is the *Nehari manifold* of (5.8), that is

$$\mathcal{N}_z = \left\{ u \in H^1(\mathbb{R}^N) \mid u \neq 0 \text{ and } DI_z(u)[u] = 0 \right\},$$

we have by definition

$$\Sigma(z) = \inf_{u \in \mathcal{N}_z} I_z(u). \tag{5.9}$$

Remark 5.1. As already said, when $\Omega = \mathbb{R}^N$ and $f(u) = u^p$, we have that there exists a positive constant $C > 0$ such that

$$\Sigma(z) = C\Gamma(z).$$

Indeed, if U is the unique radial solution in $H^1(\mathbb{R}^N)$ of

$$\begin{cases} -\Delta U + U = U^p & \text{in } \mathbb{R}^N \\ U > 0, \end{cases}$$

then it is easy to see that

$$\Sigma(z) = \left(\frac{1}{2} - \frac{1}{p+1} \right) V(z)^{\frac{p+1}{p-1} - \frac{N}{2}} (\det J(z))^{\frac{1}{2}} \int_{\mathbb{R}^N} U^{p+1}.$$

It is easy to see that $\mathcal{N}_z \neq \emptyset$ and moreover the following lemma holds.

Lemma 5.1. *For all $u \in H^1(\mathbb{R}^N)$ such that u is positive on a set of positive measure, there exists a unique maximum $t(u) > 0$ of*

$$\phi \colon t \in (0, +\infty) \mapsto I_z(tu).$$

In particular, $t(u)u \in \mathcal{N}_z$.

Proof Let us observe that if $\phi'(t) = 0$, then

$$\int_{\mathbb{R}^N} \langle J(z)\nabla u \mid \nabla u \rangle + V(z)u^2 = \int_{\mathbb{R}^N} \frac{f(tu)u}{t}\, dx$$

and so, by **(f4)**, ϕ has at most one critical value. By **(f1-2)**, $I_z(0) = 0$, $DI_z(0) = 0$ and $D^2 I_z(0)$ is strictly positive-definite in a neighborhood of 0 and so $\phi(t) > 0$ for t small. Moreover, since

$$\int_{\mathbb{R}^N} F(tu) = \int_{\{x \in \mathbb{R}^N \colon u(x) > 0\}} F(tu),$$

by **(f3)** there results $\phi(t) < 0$ for big t's. □

The following proposition gives us some useful properties of Σ (see also [91]).

Proposition 5.1. *Let the assumptions* **(V)**, **(J)**, **(f1-4)** *hold. Then:*

1. *the map Σ is well-defined and locally lipschitz;*
2. *the partial derivatives, from the left and the right, of Σ exist at every point, and moreover*

$$\left(\frac{\partial \Sigma}{\partial s_i}\right)^l = \sup_{v_s \in S^s} \left[\frac{1}{2}\int_{\mathbb{R}^N} \langle \frac{\partial J}{\partial s_i} \nabla v_s \mid \nabla v_s\rangle + \frac{1}{2}\frac{\partial V}{\partial s_i}\int_{\mathbb{R}^N} |v_s|^2\right],$$

$$\left(\frac{\partial \Sigma}{\partial s_i}\right)^r = \inf_{v_s \in S^s} \left[\frac{1}{2}\int_{\mathbb{R}^N} \langle \frac{\partial J}{\partial s_i} \nabla v_s \mid \nabla v_s\rangle + \frac{1}{2}\frac{\partial V}{\partial s_i}\int_{\mathbb{R}^N} |v_s|^2\right],$$

where S^s is the set of ground states corresponding to the energy level $\Sigma(s)$.

Proof First of all, the set S^s is non-empty. Indeed, since s is fixed, we can find a matrix $T = T(s) \in \mathrm{GL}(N)$ such that

$$T^t J(s) T = I \quad \text{(the identity matrix of order } N).$$

By the change of variables $x \mapsto Tx$, the equation

$$-\operatorname{div}(J(s)\nabla v) + V(s)v = f(v) \quad \text{in } \mathbb{R}^N$$

can be reduced to

$$-\Delta U + V(s)U = f(U) \quad \text{in } \mathbb{R}^N. \tag{5.10}$$

This change of variables rescales the functional I_z by a constant. Since it is well known that equation (5.10) has a ground state solution, for each $s \in \mathbb{R}^N$, it immediately follows that $S^s \neq \emptyset$.

Let us observe that if $v_t \in \mathcal{N}_t$, since it satisfies

$$\int_{\mathbb{R}^N} \langle J(t)\nabla v_t \mid \nabla v_t\rangle + V(t)|v_t|^2 = \int_{\mathbb{R}^N} f(v_t)v_t\, dx,$$

$v_t > 0$ on a set of positive measure and so we can apply the Lemma 5.1. Therefore, given s, $t \in \mathbb{R}^N$, there exists precisely one positive number $\theta(s,t)$ such that $\theta(s,t)v_t \in \mathcal{N}_s$. By definition, this means that

$$\int_{\mathbb{R}^N} \langle J(s)\nabla v_t \mid \nabla v_t\rangle + \int_{\mathbb{R}^N} V(s)|v_t|^2 = \int_{\mathbb{R}^N} \frac{f(\theta(s,t)v_t)v_t}{\theta(s,t)}.$$

Moreover, $\theta(t,t) = 1$. Collecting these facts, we see that, by the implicit function theorem, θ is differentiable with respect to the first variable. From its very definition, $\theta(s,t)$ is bounded for s and t bounded in $\mathbb{R}^N$. Let us now observe that

$$I_s(\theta(s,t)v_t) = \frac{\theta(s,t)^2}{2}\int_{\mathbb{R}^N} \langle J(s)\nabla v_t \mid \nabla v_t\rangle + \frac{\theta(s,t)^2}{2}\int_{\mathbb{R}^N} V(s)|v_t|^2 - \int_{\mathbb{R}^N} F(\theta(s,t)v_t).$$

The gradient of the function $s \mapsto I_s(\theta(s,t)v_t)$ is thus

$$\begin{aligned}\frac{\partial}{\partial s} I_s(\theta(s,t)v_t) &= \frac{\theta(s,t)^2}{2}\int_{\mathbb{R}^N} \langle \nabla J(s)\nabla v_t \mid \nabla v_t\rangle + \frac{\theta(s,t)^2}{2}\nabla V(s)\int_{\mathbb{R}^N}|v_t|^2 \\ &\quad + \theta(s,t)\frac{\partial\theta}{\partial s}\left(\int_{\mathbb{R}^N}\langle J(s)\nabla v_t \mid \nabla v_t\rangle + V(s)\int_{\mathbb{R}^N}|v_t|^2\right) \\ &\quad - \int_{\mathbb{R}^N} f(\theta(s,t)v_t)v_t\frac{\partial\theta}{\partial s} \\ &= \frac{\theta(s,t)^2}{2}\int_{\mathbb{R}^N}\langle \nabla J(s)\nabla v_t \mid \nabla v_t\rangle + \frac{\theta(s,t)^2}{2}\nabla V(s)\int_{\mathbb{R}^N}|v_t|^2\end{aligned}$$

because $\theta(s,t)v_t \in \mathcal{N}_s$. From this representation, the mean value theorem and the local boundedness of θ, it follows that for all $R > 0$ there exists $M > 0$ such that for all s_1 and s_2 with $|s_1| < R$, $|s_2| < R$:

$$|\Sigma(s_1) - \Sigma(s_2)| \leq M|s_1 - s_2|.$$

This proves the first statement. The proof of the second statement can be modeled on the similar results of [91] together with [77]. We omit the details. □

Remark 5.2. Some uniqueness conditions for the limiting equation appear in [81].

The next step is the proof of a fundamental equality between the ground state level and the mountain-pass one. This kind of result is well known at least in the case J equal to the identity matrix (see, for example, [78]).

Proposition 5.2.

$$\inf_{\gamma\in\mathcal{P}_z}\max_{0\leq t\leq 1} I_z(\gamma(t)) = \inf_{u\in\mathcal{N}_z} I_z(u) = \Sigma(z),$$

where

$$\mathcal{P}_z = \left\{\gamma \in C\left([0,1], H^1(\mathbb{R}^N)\right) : \gamma(0) = 0 \text{ and } I_z(\gamma(1)) < 0\right\}.$$

Proof We now show that for all path $\gamma \in \mathcal{P}_z$, there exists $t_0 > 0$ such that $\gamma(t_0) \in \mathcal{N}_z$.

First of all, let us observe that by our assumptions of f, together with the ellipticity of J and the definition of $\alpha = \inf V > 0$ we get

$$\begin{aligned}DI_z(u)[u] &= \int_{\mathbb{R}^N}\langle J(z)\nabla u \mid \nabla u\rangle + V(z)u^2 - \int_{\mathbb{R}^N} f(u)u\,dx \\ &\geq \nu\int_{\mathbb{R}^N}|\nabla u|^2 + \alpha\int_{\mathbb{R}^N}|u|^2 - c\int_{\mathbb{R}^N}|u|^{\theta+1} \\ &\geq \min\{\nu,\alpha\}\|u\|^2_{H^1_0(\Omega)} - c\|u\|_{L^{\theta+1}(\Omega)}.\end{aligned}$$

Therefore, fix a path $\gamma \in \mathcal{P}_z$ joining 0 to some $v \neq 0$ such that $I_z(v) < 0$. Hence

$$DI_z(\gamma(t))[\gamma(t)] > 0$$

for $t > 0$ small enough. On the other hand, since $v \neq 0$, we have

$$DI_z(v)[v] < 2I_z(v) \leq 0.$$

By the intermediate value theorem, there exists $t_0 \in [0,1]$ such that

$$\gamma(t_0) \in \mathcal{N}_z.$$

This shows also that

$$\inf_{\gamma \in \mathcal{P}_z} \max_{0 \leq t \leq 1} I_z(\gamma(t)) \geq \inf_{u \in \mathcal{N}_z} I_z(u) = \Sigma(z). \tag{5.11}$$

But equality actually holds in (5.11). Indeed, by Lemma 5.1,

$$u \in \mathcal{N}_z \quad \text{if and only if} \quad I_z(u) = \max_{\tau \geq 0} I_z(\tau u). \tag{5.12}$$

It then follows that the mountain-pass level corresponds to the least energy among the energies of all solutions, that is

$$\inf_{\gamma \in \mathcal{P}_z} \max_{0 \leq t \leq 1} I_z(\gamma(t)) = \inf_{u \in \mathcal{N}_z} I_z(u) = \Sigma(z).$$

□

Remark 5.3. It is shown in [45] that the level of mountain–pass of I_z coincides with the infimum of the energies among all critical points of I_z under less stringent assumptions on f. Anyway, to prove the regularity of Σ we are not able to weaken our set of hypotheses **(f1–4)**.

Our main result about existence for (5.5) is the following theorem.

Theorem 5.4. *Suppose* **(V)**, **(J)** *and* **(f1-4)** *hold. Suppose that there exists a compact domain* $\Lambda \subset \Omega$ *such that*

$$\min_{\Lambda} \Sigma < \min_{\partial \Lambda} \Sigma. \tag{5.13}$$

Then, for all $\varepsilon > 0$ *sufficiently small, there exists a solution* $u_\varepsilon \in H_V(\Omega) \cap C(\Omega)$ *of* (5.5). *Moreover, this solution has only one global maximum point* $x_\varepsilon \in \Lambda$ *and we have that* $\Sigma(x_\varepsilon) \to \min_\Lambda \Sigma$ *as* $\varepsilon \to 0$ *and*

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = 0 \quad \textit{for all } x \neq x_\varepsilon.$$

Remark 5.4. As noticed in [82], we have that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\| = 0.$$

The next corollary shows that results of [82], at least in the case of our differential operator, are a particular case of Theorem 5.4. More precisely, as said in the introduction, we will prove that if J and V have a local strict minimum in z_0, then z_0 is also a local strict minimum for Σ and so we can apply Theorem 5.4.

Corollary 5.1. *Suppose* **(V)**, **(J)** *and* **(f1-4)** *hold. Suppose that there exist a compact domain $\Lambda \subset \Omega$ and a $z_0 \in \Lambda$ which is a minimum point for V and J in Λ and a strict minimum point for V (resp. J), in the sense that*

$$V(z_0) < \min_{\partial\Lambda} V \qquad \left(\textit{resp.}\, V(z_0) \le \min_{\partial\Lambda} V \right) \tag{5.14}$$

and, for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and for all $z \in \partial\Lambda$,

$$\langle J(z_0)\xi \mid \xi\rangle \le \langle J(z)\xi \mid \xi\rangle \quad (\textit{resp.}\, \langle J(z_0)\xi \mid \xi\rangle < \langle J(z)\xi \mid \xi\rangle)\,. \tag{5.15}$$

Then (5.13) *holds and hence the conclusion of Theorem 5.4 continues to be true.*

We will prove Theorem 5.4 and Corollary 5.1 in the next section.

5.2 The penalization scheme

Since the domain Ω is in general unbounded, a direct application of critical point theory does not, as a rule, provide a solution to (5.5). Although the functional I^ε has a good geometric structure, it does not satisfy the Palais-Smale condition. Thus, as a first step, we replace I^ε with a different functional that satisfies $(\mathrm{PS})_c$ at all levels $c \in \mathbb{R}$, and finally prove that, as ε gets smaller, the critical points of this new functional are actually solutions of (5.5). This technique was introduced by del Pino and Felmer in [31], and then used by several authors. The main advantage of this scheme is that, unlike the direct application of some *Concentration-Compactness argument* as in [91], we do not have to impose any comparison assumption between the values of Σ at zero (say) and at infinity.

Following the scheme of [31] (see also [82]), we will define a penalization of the functional I^ε, which satisfies the Palais-Smale condition. Let θ be the number given in **(f3)**. Let $\ell > 0$ be the unique value such that $f(\ell)/\ell = \alpha/k$, where α is defined in **(V)** and $k > \theta/(\theta - 2)$.

We penalize the nonlinearity f in the following way. Define $\tilde{f} \colon \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(u) = \begin{cases} (\alpha/k)u & \text{if } u > \ell, \\ f(u) & \text{if } 0 \le u \le \ell, \\ 0 & \text{if } u < 0. \end{cases}$$

We now define $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ as

$$g(x,u) = \begin{cases} \chi_\Lambda(x) f(u) + (1 - \chi_\Lambda(x)) \tilde{f}(u) & \text{if } u \ge 0, \\ 0 & \text{if } u < 0, \end{cases}$$

where χ_Λ is the characteristic function of the set Λ, and let G be the primitive of g, that is

$$G(x,u) = \int_0^u g(x,\tau) d\tau.$$

By straightforward calculations, assumptions **(f1-4)** imply:

(g1) $g(x,u) = o(u)$ as $u \to 0^+$, uniformly in $x \in \Omega$;
(g2) $\lim_{u \to +\infty} g(x,u)/u^p = 0$ for some $p \in \left(1, \frac{N+2}{N-2}\right)$;
(g3-i) for some $\theta \in (2, p+1)$ we have

$$0 < \theta G(x,u) \le g(x,u)u \quad \text{for all } x \in \Lambda,\ u > 0;$$

(g3-ii) for some $k > \frac{\theta}{\theta - 2}$ there holds

$$0 \le 2G(x,u) \le g(x,u)u \le \frac{1}{k} V(x) u^2 \quad \text{for all } x \notin \Lambda,\ u > 0;$$

(g4) the function $u \mapsto \frac{g(x,u)}{u}$ is increasing for all $x \in \Lambda$.

The penalized functional will be $E^\varepsilon\colon H_V(\Omega) \to \mathbb{R}$, where

$$E^\varepsilon(u) = \frac{\varepsilon^2}{2} \int_\Omega \langle J(x) \nabla u \mid \nabla u \rangle + \frac{1}{2} \int_\Omega V(x)|u|^2 - \int_\Omega G(x,u). \tag{5.16}$$

Under our assumptions E^ε satisfies the (PS) condition, as we prove in the next lemma.

Lemma 5.2. *Let $\{u_h\}$ be a sequence in $H_V(\Omega)$ such that $E^\varepsilon(u_h) \to c \in \mathbb{R}$ and $DE^\varepsilon(u_h) \to 0$. Then $\{u_h\}$ has a strongly convergent subsequence.*

Proof As first step, we show that $\{u_h\}$ is bounded. Since $E^\varepsilon(u_h) \to c$ we have

$$\frac{\theta}{2} \varepsilon^2 \int_\Omega \langle J(x) \nabla u_h \mid \nabla u_h \rangle + \frac{\theta}{2} \int_\Omega V(x)|u_h|^2 \\ \le \int_\Lambda g(x,u_h)u_h + \frac{\theta}{2k} \int_{\Omega \setminus \Lambda} V(x)|u_h|^2 + \theta c + o(1).$$

Moreover, since $DE^\varepsilon(u_h)[u_h] = o(\|u_h\|)$,

$$\varepsilon^2 \int_\Omega \langle J(x)\nabla u_h \mid \nabla u_h\rangle + \int_\Omega V(x)|u_h|^2 \geq \int_\Lambda g(x,u_h)u_h + o(\|u_h\|).$$

Therefore

$$\min\left\{\left(\frac{\theta}{2}-1\right)\varepsilon^2, \frac{\theta}{2}-\frac{\theta}{2k}-1\right\}\int_\Omega \langle J(x)\nabla u_h \mid \nabla u_h\rangle + V(x)|u_h|^2 \\ \leq \theta c + o(1) + o(\|u_h\|)$$

and so the boundedness of $\{u_h\}$ follows from (5.3).

Up to subsequence, we have that $u_h \to u$ weakly and point-wise almost everywhere in Ω. To show that this convergence is actually strong, it suffices to prove that, for all $\delta > 0$, there exists $R > 0$ such that

$$\limsup_{k\to\infty} \int_{\Omega\setminus B_R} |\nabla u_h|^2 + V(x)|u_h|^2 < \delta.$$

We take $R > 0$ so large that $\Lambda \subset B_{R/2}$. Let $\eta_R \in C^2(\Omega)$ be a function such that, $\eta_R = 0$ in $B_{R/2}$, $\eta_R = 1$ in $\Omega \setminus B_R$, $0 \leq \eta_R \leq 1$ in Ω and $|\nabla\eta_R| \leq C/R$ in Ω. Since $\{u_h\}$ is bounded,

$$\lim_{k\to\infty} DE^\varepsilon(u_h)[\eta_R u_h] = 0.$$

Therefore

$$\int_\Omega \left(\langle J(x)\nabla u_h \mid \nabla u_h\rangle + V(x)|u_h|^2\right)\eta_R + \int_\Omega \langle J(x)\nabla u_h \mid \nabla\eta_R\rangle u_h \\ = \int_\Omega g(x,u_h)u_h\eta_R + o(1) \leq \frac{1}{k}\int_\Omega V(x)u_h^2\eta_R + o(1),$$

and so

$$\int_{\Omega\setminus B_R} |\nabla u_h|^2 + V(x)u_h^2 \leq \frac{C}{R}\|u_h\|_{L^2}\|\nabla u_h\|_{L^2} + o(1).$$

We conclude the proof by letting $R \to +\infty$. □

Since by Proposition 5.1 we know that Σ is a continuous function, we can assume without loss of generality that there exists $z_0 \in \Lambda$ such that

$$\Sigma(z_0) = \min_\Lambda \Sigma.$$

Hence the main assumption of the Theorem 5.4 can be stated as

$$\Sigma(z_0) < \min_{\partial\Lambda} \Sigma.$$

To save notation, we will write I_0 instead of I_{z_0}.

Let us set

$$\bar{c} = \inf_{\gamma \in \mathcal{P}_0} \max_{t \in [0,1]} I_0(\gamma(t)) = \Sigma(z_0),$$

where

$$\mathcal{P}_0 = \left\{\gamma \in C\left([0,1], H^1(\mathbb{R}^N)\right) : \gamma(0) = 0 \text{ and } I_0(\gamma(1)) < 0\right\}.$$

Lemma 5.3. *For all ε sufficiently small, there exists a critical point $u_\varepsilon \in H_V(\Omega)$ of E^ε such that*

$$E^\varepsilon(u_\varepsilon) \leq \varepsilon^N \bar{c} + o(\varepsilon^N).$$

Proof We already know that E^ε satisfies the (PS) condition at any level. By a standard minimax argument over the set of paths

$$\mathcal{P}_\varepsilon = \{\gamma \in C([0,1], H_V(\Omega)) : \gamma(0) = 0 \text{ and } E^\varepsilon(\gamma(1)) < 0\},$$

we can find a critical point u_ε such that

$$E^\varepsilon(u_\varepsilon) = \inf_{\gamma \in \mathcal{P}_\varepsilon} \max_{t \in [0,1]} E^\varepsilon(\gamma(t)).$$

Since $\bar{c}$ is a *mountain-pass level* of I_0, for all $\delta > 0$ there exists a path $\gamma\colon [0,1] \to H^1(\mathbb{R}^N)$ such that

$$\bar{c} \leq \max_{0 \leq t \leq 1} I_0(\gamma(t)) \leq \bar{c} + \delta, \quad \gamma(0) = 0, \quad I_0(\gamma(1)) < 0.$$

Let $\zeta \in C^2(\mathbb{R}^N)$ be a cut-off function such that $\zeta = 1$ in a neighborhood of z_0. Define a path in $H_V(\Omega)$ by

$$\Gamma_\varepsilon(\tau)\colon x \mapsto \zeta(x)\gamma(\tau)\left(\frac{x - z_0}{\varepsilon}\right).$$

By direct computation,

$$E^\varepsilon(\Gamma_\varepsilon(\tau)) = \varepsilon^N \left\{\frac{1}{2}\int_{\mathbb{R}^N} \langle J(z_0)\nabla\gamma(\tau) \mid \nabla\gamma(\tau)\rangle + \frac{1}{2}\int_{\mathbb{R}^N} V(z_0)|\gamma(\tau)|^2 - \int_{\mathbb{R}^N} F(\gamma(\tau)(\cdot))\right\} + o(\varepsilon^N),$$

that is

$$E^\varepsilon(\Gamma_\varepsilon(\tau)) = \varepsilon^N I_0(\gamma(\tau)) + o(\varepsilon^N)$$

as $\varepsilon \to 0$. But $\Gamma_\varepsilon \in \mathcal{P}_\varepsilon$, so that

$$\begin{aligned} E^\varepsilon(u_\varepsilon) &= \inf_{\gamma \in \mathcal{P}_\varepsilon} \max_{t \in [0,1]} E^\varepsilon(\gamma(t)) \\ &\leq \max_{t \in [0,1]} E^\varepsilon(\Gamma_\varepsilon(t)) = \varepsilon^N \max_{t \in [0,1]} I_0(\gamma(t)) + o(\varepsilon^N) \\ &\leq \varepsilon^N \bar{c} + \delta\varepsilon^N + o(\varepsilon^N). \end{aligned}$$

Since $\delta > 0$ was arbitrary, the proof is complete. □

Remark 5.5. By the uniform ellipticity of J and standard regularity theorems (see [38]), the element u_ε actually belongs to $C(\overline{\Lambda})$.

The next proposition is somehow the key ingredient.

Proposition 5.3. *Let $u_\varepsilon \in H_V(\Omega)$ be the critical point of E^ε found in the previous lemma. Then*

$$\lim_{\varepsilon \to 0} \max_{\partial \Lambda} u_\varepsilon = 0.$$

Moreover, u_ε has only one global maximum point $x_\varepsilon \in \Lambda$ and we have that $\Sigma(x_\varepsilon) \to \min_\Lambda \Sigma$ as $\varepsilon \to 0$ and

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = 0 \quad \text{for all } x \neq x_\varepsilon.$$

Proof The proof of this Proposition will be performed in several steps. First of all, the following claim implies the Proposition except for the uniqueness of the global maximum.

CLAIM 1: *If $\{\varepsilon_h\}$ is a sequence of positive real numbers converging to zero, and $\{x_h\} \subset \Lambda$ is a sequence of points in Λ such that*

$$u_{\varepsilon_h}(x_h) \geq c > 0,$$

then

$$\lim_{h \to \infty} \Sigma(x_h) = \min_\Lambda \Sigma.$$

Indeed, suppose that the statement of the Proposition is false. Then, up to a subsequence, we may suppose that there exists a sequence $\{x_n\} \subset \partial \Lambda$ such that $x_h \to \bar{x} \in \partial \Lambda$ as $\varepsilon_h \to 0$ and $u_{\varepsilon_h}(x_h) \geq c > 0$. Therefore

$$\min_{\partial \Lambda} \Sigma \leq \Sigma(\bar{x}) = \lim_{h \to \infty} \Sigma(x_h) = \min_\Lambda \Sigma,$$

which contradicts our assumptions on Λ. Here we have used the continuity of Σ, already proved. Hence we should just prove the Claim 1. By compactness, we assume without loss of generality that $x_h \to \hat{x} \in \Lambda$. We proceed by contradiction, assuming therefore that

$$\Sigma(\hat{x}) > \min_\Lambda \Sigma.$$

Let $v_h(x) = u_{\varepsilon_h}(x_h + \varepsilon_h x)$. By elliptic regularity, $\{v_h\}$ converges strongly in $H^1(K)$ towards some v, for each compact set $K \subset \mathbb{R}^N$. Let χ be the weak* limit (in L^∞) of the sequence $\{\chi_\Lambda(x_h + \varepsilon_h \cdot)\}_{h \in \mathbb{N}}$. Clearly, $0 \leq \chi \leq 1$. Therefore the function v is a weak solution of the equation

$$-\operatorname{div}(J(\hat{x})\nabla v) + V(\hat{x})v = g_0(\cdot, v) \quad \text{in } \mathbb{R}^N,$$

where

$$g_0(x,s) = \chi(x)f(s) + (1-\chi(x))\,\tilde{f}(s).$$

Let now $E^h : H_V(\Omega_h) \to \mathbb{R}$ be the functional

$$E^h(v) = \frac{1}{2}\int_{\Omega_h} \langle J(x_h + \varepsilon_h x)\nabla v|\nabla v\rangle + \frac{1}{2}\int_{\Omega_h} V(x_h+\varepsilon_h x)v^2 - \int_{\Omega_h} G(x_h+\varepsilon_h x, v),$$

where $\Omega_h = \varepsilon_h^{-1}(\Omega - x_h)$. Let us observe explicitly that

$$E^h(v_h) = \varepsilon_h^{-N} E^{\varepsilon_h}(u_{\varepsilon_h}). \tag{5.17}$$

It is clear that v_h is a critical point of E^h in $H_V(\Omega_h)$.

As remarked in [33] (see also [82]), v actually satisfies

$$-\operatorname{div}(J(\hat{x})\nabla v) + V(\hat{x})v = f(v) \quad \text{in } \mathbb{R}^N. \tag{5.18}$$

Indeed, without loss of generality, we may suppose that $\chi(x) = \chi_{\{x_1<0\}}(x)$ for all x. Consider the equation satisfied by v:

$$-\operatorname{div}(J(\hat{x})\nabla v) + V(\hat{x})v = \chi_{\{x_1<0\}}(x)f(v) + \chi_{\{x_1>0\}}(x)\tilde{f}(v) \quad \text{in } \mathbb{R}^N,$$

and multiply it by $T_k\partial_{x_1}v$, where T_k is a sequence of smooth functions such that $T_k(x) = 1$ if $|x| \le k$, $T_k(x) = 0$ if $|x| \ge 2k$, and $|\nabla T_k(x)| = O(1/k)$. Hence we get

$$\int_{\mathbb{R}^N} \langle J(\hat{x})\nabla v|\nabla(T_k\partial_{x_1}v)\rangle + \int_{\mathbb{R}^N} V(\hat{x})vT_k\partial_{x_1}v = \int_{\mathbb{R}^N} \varphi T_k\partial_{x_1}v,$$

where we have set $\varphi(x,v) = \chi_{\{x_1<0\}}(x)f(v(x)) + \chi_{\{x_1>0\}}(x)\tilde{f}(v(x))$. We observe that $v\partial_{x_1}v = \frac{1}{2}\partial_{x_1}v^2$, so that after an integration by parts we have

$$\int_{\mathbb{R}^N} \langle J(\hat{x})\nabla v|\partial_{x_1}v\nabla T_k\rangle + \int_{\mathbb{R}^N} \langle J(\hat{x})\nabla v|T_k\partial_{x_1}\nabla v\rangle - \int_{\mathbb{R}^N} V(\hat{x})\frac{1}{2}v^2\partial_{x_1}T_k = \int_{\mathbb{R}^N} \varphi T_k\partial_{x_1}v.$$

By a second integration by parts on the term $\int_{\mathbb{R}^N} \langle J(\hat{x})\nabla v|T_k\partial_{x_1}\nabla v\rangle$, and by the Dominated Convergence Theorem, we pass to the limit as $k \to +\infty$ to get

$$\int_{\mathbb{R}^N} \varphi(x,v)\partial_{x_1}v = 0.$$

This implies easily that $v(0,x_2,\dots,x_N) \le \ell$ for all $(x_2,\dots,x_N) \in \mathbb{R}^{N-1}$. It is now easy to check that we can choose

$$\max\{v(\cdot) - \ell, 0\}$$

as a barrier for the equation satisfied by v, and thus prove that $v \leq \ell$ in $\mathbb{R}^N$. In particular, $\varphi(x, v(x)) = f(v(x))$, so that v is a solution of (5.18).

CLAIM 2: *there results*

$$I_{\hat{x}}(v) \leq \liminf_{h\to\infty} E^h(v_h). \tag{5.19}$$

Indeed, consider now the function

$$\xi_h(x) = \frac{1}{2}\langle J(x_h + \varepsilon_h x)\nabla v_h | \nabla v_h\rangle + \frac{1}{2}V(x_h + \varepsilon_h x)v_h^2 = G(x_h + \varepsilon_h x, v_h).$$

We already know that $v_h \to v$ strongly on compact sets. Therefore

$$\lim_{h\to\infty} \int_{B_R} \xi_h = \frac{1}{2}\int_{B_R} \left(\langle J(\hat{x})\nabla v|\nabla v\rangle + V(\hat{x})v^2\right) - \int_{B_R} F(v)$$

for all $R > 0$. But $v \in H^1(\mathbb{R}^N)$, so that

$$I_{\hat{x}}(v) - \frac{1}{2}\int_{B_R} \left(\langle J(\hat{x})\nabla v|\nabla v\rangle + V(\hat{x})v^2\right) + \int_{B_R} F(v) = o(1)$$

as $R \to +\infty$. To show that $I_{\hat{x}}(v) \leq \liminf_{h\to\infty} E^h(v_h)$ it is enough to prove that for all $\delta > 0$ there exists $R > 0$ such that

$$\liminf_{h\to\infty} \int_{\Omega_h \setminus B_R} \xi_h \geq -\delta.$$

We introduce again a cut-off function $\eta_R \in C^\infty(\mathbb{R}^N)$ such that

$$\begin{aligned} \eta_R &= 0 && \text{in } B_{R-1}, \\ \eta_R &= 1 && \text{in } \mathbb{R}^N \setminus B_R, \\ |\nabla \eta_R| &\leq C && \text{in } \mathbb{R}^N. \end{aligned}$$

We test the equation satisfied by v_h against $\eta_R v_h$. After some computations we get

$$\liminf_{h\to\infty} \int_{\Omega_h \setminus B_R} \xi_h \geq -\frac{1}{2}\limsup_{h\to\infty} \left[\int_{B_R \setminus B_{R-1}} \langle J(x_h + \varepsilon_h x)\nabla v_h \mid \nabla(\eta_R v_h)\rangle \right.$$
$$\left. + \int_{B_R \setminus B_{R-1}} V(x_h + \varepsilon_h x)v_h^2 \eta_R - \int_{B_R \setminus B_{R-1}} g(x_h + \varepsilon_h x, v_h)\eta_R v_h \right] = o(1)$$

as $R \to +\infty$. This finally proves that $I_{\hat{x}}(v) \leq \liminf_{h\to\infty} E^h(v_h)$ and so Claim 2 holds.

We now complete the first part of the proof. First of all, from Lemma 5.3, (5.17) and from (5.19), it follows that

$$I_{\hat{x}}(v) \leq \bar{c} = \inf_{\gamma \in \mathcal{P}_0} \sup_{0 \leq t \leq 1} I_0(\gamma(t)).$$

On the other hand, since v is a critical point of $I_{\hat{x}}$, by Proposition 5.2, we have

$$\begin{aligned} I_{\hat{x}}(v) &\geq \inf_{\gamma \in \mathcal{P}_{\hat{x}}} \sup_{0 \leq t \leq 1} I_{\hat{x}}(\gamma(t)) \\ &= \inf_{u \in \mathcal{N}_{\hat{x}}} I_{\hat{x}}(u) = \Sigma(\hat{x}) \\ &> \inf_{u \in \mathcal{N}_{z_0}} I_0(u) = \Sigma(z_0) \\ &= \inf_{\gamma \in \mathcal{P}_0} \sup_{0 \leq t \leq 1} I_0(\gamma(t)) = \bar{c}. \end{aligned}$$

This contradiction proves Claim 1 and so also the first part of the proof.

As regards the last statements of the proposition, these follow easily from the corresponding properties of solutions in [31]. We just sketch the ideas. Let $\bar{z} = \lim_{\varepsilon \to 0} x_\varepsilon$ and take any critical point u of $I_{\bar{z}}$. By the change of variables $x \mapsto Tx$ introduced in the proof of Proposition 5.1, if v is a solution of equation (5.10), namely

$$-\Delta v + V(\bar{z})v = f(v) \quad \text{in } \mathbb{R}^N,$$

then $u(x) = v(Tx)$. It is well known by [37] that solutions of (5.10) are radially symmetric and decreasing. In particular, $x = 0$ is a nondegenerate maximum point of u. We are in a position to apply a reasoning similar to that of [31, page 133]. □

We can now prove Theorem 5.4.

Proof of Theorem 5.4 By Proposition 5.3, we know that if ε is small enough, then

$$u_\varepsilon(x) < \ell \quad \text{for all } x \in \partial \Lambda.$$

The function $(u_\varepsilon(\cdot) - \ell)^+ = \max\{u_\varepsilon(\cdot) - \ell, 0\}$ belongs to $H_0^1(\Omega)$, so that we can test the equation

$$-\varepsilon^2 \operatorname{div}(J \nabla u_\varepsilon) + V u_\varepsilon = g(\cdot, u_\varepsilon) \quad \text{in } \Omega$$

against it. By the divergence theorem,

$$\varepsilon^2 \int_{\Omega \setminus \Lambda} \langle J \nabla (u_\varepsilon - \ell)^+ \mid \nabla (u_\varepsilon - \ell)^+ \rangle + \int_{\Omega \setminus \Lambda} \ell \Phi_\varepsilon (u_\varepsilon - \ell)^+ + \int_{\Omega \setminus \Lambda} \Phi_\varepsilon \left((u_\varepsilon - \ell)^+ \right)^2 = 0, \quad (5.20)$$

where we have set

$$\Phi_\varepsilon(x) = V(x) - \frac{g(x, u_\varepsilon(x))}{u_\varepsilon(x)}.$$

The properties of g imply that $\Phi_\varepsilon > 0$ in $\Omega \setminus \Lambda$. Therefore all the terms in (5.20) must vanish, and in particular

$$u_\varepsilon \leq \ell \quad \text{in } \Omega \setminus \Lambda.$$

We conclude that u_ε, for ε small enough, is actually a critical point of I^ε, and hence a solution of (5.5). The regularity of u_ε follows again from [38]. The last statement of the theorem follows immediately by Proposition 5.3. □

Proof of Corollary 5.1 First of all, we remark that, for any $u \in H^1(\mathbb{R}^N)\setminus\{0\}$ and any fixed $z \in \partial\Lambda$, there results

$$I_z(u) > I_{z_0}(u),$$

because of (5.14) and (5.15). We claim that there exists a ground state v_z such that

$$I_z(v_z) = \inf_{u \in \mathcal{N}_z} I_z(u). \tag{5.21}$$

Indeed, the usual change of variables $x \mapsto Tx$ rescales the functional I_z by a constant factor $|\det T| > 0$. This reduces the search of a ground state for I_z to the search of a ground state for the equation

$$-\Delta v + V(z)v = f(v) \quad \text{in } \mathbb{R}^N,$$

whose existence follows easily from the results contained in [16] and [22]. This proves the claim.

Let $z \in \partial\Lambda$ and v_z as in (5.21). By Lemma 5.1 we know that there exists a positive constant τ such that $\tau v_z \in \mathcal{N}_{z_0}$. By our assumptions and (5.12), we easily get

$$\Sigma(z) = I_z(v_z) \geq I_z(\tau v_z) > I_{z_0}(\tau v_z) = \max_{t>0} I_{z_0}\left(t(\tau v_z)\right) \geq \Sigma(z_0).$$

Therefore $\Sigma(z_0) < \Sigma(z)$ and so (5.13) holds. □

5.3 Necessary condition for concentration

In this section we want to show that the function Σ also plays a *necessary* rôle for the existence of concentrating solutions of (5.5). We will give also a more general version of Theorem 5.2.

We suppose that $\Omega = \mathbb{R}^N$. Indeed, if Ω has a boundary, we do not expect that solutions must concentrate at critical points of Σ, but rather on critical point of some function connected to the geometry of $\partial\Omega$, see for example [66].

Theorem 5.5. *Assume, in addition to assumptions* (**V**), (**J**), (**f1-4**), *that* V *is bounded from above, there exist two positive constants* C, γ *such that*

$$|\nabla J(x)|,\ |\nabla V(x)| \leq Ce^{\gamma|x|}, \qquad \text{for all } x \in \mathbb{R}^N,$$

and $\Omega = \mathbb{R}^N$. *Let* $\{u_{\varepsilon_j}\}$ *be a sequence of solutions of* (5.5) *such that for all* $\varepsilon > 0$ *there exist* $\rho > 0$ *and* $j_0 > 0$ *such that for all* $j \geq j_0$ *and for all points* x *with* $|x - z_0| \geq \varepsilon_j \rho$, *there results*

$$u_{\varepsilon_j}(x) \leq \varepsilon.$$

If, for all $z \in \mathbb{R}^N$, *the functional* I_z *has only one positive ground-state (up to translations), then* z_0 *is a critical point of* Σ.

Before proving the theorem, we recall a recent version of Pucci–Serrin variational identity for lipschitz continuous solutions of a general class of Euler equations (see [30]).

Theorem 5.6. *Let* $\mathcal{L}\colon \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *be a* C^1 *function such that the function* $\xi \mapsto \mathcal{L}(x, s, \xi)$ *is strictly convex for every* $(x, s) \in \mathbb{R}^N \times \mathbb{R}$. *Let* $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$. *Let* $u\colon \mathbb{R}^N \to \mathbb{R}$ *be a locally Lipschitz weak solution of*

$$-\operatorname{div}\left(\partial_\xi \mathcal{L}(x, u, \nabla u)\right) + \partial_s \mathcal{L}(x, u, \nabla u) = \varphi \quad \text{in } \mathbb{R}^N.$$

Then

$$\sum_{i,\,k=1}^{N} \int_{\mathbb{R}^N} \partial_i h^k \partial_{\xi_i} \mathcal{L}(x, u, \nabla u) \partial_k u \\ - \int_{\mathbb{R}^N} \left[(\operatorname{div} h)\mathcal{L}(x, u, \nabla u) + h \cdot \partial_x \mathcal{L}(x, u, \nabla u)\right] = \int_{\mathbb{R}^N} (h \cdot \nabla u)\varphi,$$

for all $h \in C^1_{\mathrm{c}}\left(\mathbb{R}^N, \mathbb{R}^N\right)$.

Proof of Theorem 5.5 To save notation, we write u_j instead of u_{ε_j}. Define $w_j(x) = u_j(z_0 + \varepsilon_j x)$. Therefore

$$-\operatorname{div}\left(J(z_0 + \varepsilon_j x)\nabla w_j\right) + V(z_0 + \varepsilon_j x)w_j - f(w_j) = 0. \tag{5.22}$$

By assumption, w_j decays to zero uniformly with respect to $j \in \mathbb{N}$. It is not difficult to build an exponential barrier for w_j, proving in this way that w_j decays to zero exponentially fast at infinity. By elliptic regularity (see [38]), the sequence $\{w_j\}$ converges in C^2_{loc} to a solution w_0 of the equation

$$-\operatorname{div}\left(J(z_0)\nabla w_0\right) + V(z_0)w_0 - f(w_0) = 0.$$

Let us apply Theorem 5.6 to (5.22), with

$$\begin{aligned}\mathcal{L}(x,s,\xi) &= \frac{1}{2}\langle J(z_0+\varepsilon_j x)\xi \mid \xi\rangle + \frac{1}{2}V(z_0+\varepsilon_j x)s^2 - F(s),\\ h(x) &= (T(\varepsilon x), 0, \dots, 0),\\ \varphi(x) &= 0,\end{aligned}$$

where $T \in C^1_c(\mathbb{R}^N)$ such that $T(x) = 1$ if $|x| \le 1$ and $T(x) = 0$ if $|x| \ge 2$.

By Theorem 5.6, we have:[1]

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\mathbb{R}^N} \varepsilon\partial_i T(\varepsilon x)\langle J_i(z_0+\varepsilon_j x) \mid \nabla w_j\rangle \partial_1 w_j\\ &\quad - \int_{\mathbb{R}^N} \varepsilon\partial_1 T(\varepsilon x)\left[\frac{1}{2}\langle J(z_0+\varepsilon_j x)\nabla w_j \mid \nabla w_j\rangle + \frac{1}{2}V(z_0+\varepsilon_j x)w_j^2 - F(w_j)\right]\\ &\quad - \int_{\mathbb{R}^N} \varepsilon T(\varepsilon x)\left[\frac{1}{2}\langle \partial_1 J(z_0+\varepsilon_j x)\nabla w_j \mid \nabla w_j\rangle + \frac{1}{2}\partial_1 V(z_0+\varepsilon_j x)w_j^2\right] = 0.\end{aligned}$$

Passing to the limit in the previous relations, as $\varepsilon \to 0$, we get:

$$\frac{1}{2}\int_{\mathbb{R}^N} \left(\langle \partial_1 J(z_0)\nabla w_0 \mid \nabla w_0\rangle + \partial_1 V(z_0)|w_0|^2\right) = 0.$$

The proof is complete once we recall that if S^{z_0} consists of just one element, then, by Proposition 5.1, $\partial_1 \Sigma(z_0) = 0$. The proof for the other partial derivatives is identical. □

Remark 5.6. Let us observe that Theorem 5.2 is an immediate consequence of Theorem 5.5. Indeed by [22] we know that the problem (5.1) with frozen coefficients has only one positive ground-state (up to translations).

5.4 Existence via perturbation method

We have seen in the previous sections that the penalization technique of del Pino and Felmer provides at least a solution of (5.5) if the auxiliary map Σ possesses a minimum. Moreover, we could also treat the case of maximum point of Σ under some more restrictive, global, assumptions on the potentials J and V. In the present section we show that for (5.1) it is possible to find at least a solution just by differential methods if there exists a local maximum or minimum of Σ. More precisely, we will apply the perturbation technique in critical point theory as developed in [11]. Since this approach deals with the local behavior of the potentials J and V, we need a better knowledge about the derivatives of the potentials.

In addition to hypotheses **(V)** and **(J)**, in this section we will always assume:

[1] We denote by $J_i(x)$ the i–th row of $J(x)$.

(V1) $V \in C^2(\mathbb{R}^N, \mathbb{R})$, V and D^2V are bounded;
(J1) $J \in C^2(\mathbb{R}^N, \mathbb{R}^{N\times N})$, J and D^2J are bounded.

Since we follow closely [11], we will skip some proofs and we will give only the sketch of some others.

Without loss of generality we can assume that $V(0) = 1$. Moreover, using the change of variables introduced in the proof of Proposition 5.1, we can assume also $J(0) = I$, where I is the identity matrix of order $N \times N$.

Performing the change of variable $x \mapsto \varepsilon x$, equation

$$-\varepsilon^2 \operatorname{div}\big(J(x)\nabla u\big) + V(x)u = u^p \quad \text{in } \mathbb{R}^N$$

becomes

$$-\operatorname{div}\left(J(\varepsilon x)\nabla u\right) + V(\varepsilon x)u = u^p \quad \text{in } \mathbb{R}^N. \tag{5.23}$$

Solutions of (5.23) are the critical points $u \in H^1(\mathbb{R}^N)$ of

$$f_\varepsilon(u) = f_0(u) + \frac{1}{2}\int_{\mathbb{R}^N} \langle (J(\varepsilon x) - I))\nabla u \mid \nabla u\rangle dx + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x) - 1)\, u^2 dx,$$

where

$$f_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1}dx$$

and $\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + u^2$. The solutions of (5.23) will be found near a solution of

$$-\operatorname{div}\left(J(\varepsilon\xi)\nabla u\right) + V(\varepsilon\xi)u = u^p, \tag{5.24}$$

for an appropriate choice of $\xi \in \mathbb{R}^N$.

The solutions of (5.24) are critical points of the functional

$$F^{\varepsilon\xi}(u) = f_0(u) + \frac{1}{2}\int_{\mathbb{R}^N} \langle (J(\varepsilon\xi) - I)\nabla u \mid \nabla u\rangle + \frac{1}{2}\left(V(\varepsilon\xi) - 1\right)\int_{\mathbb{R}^N} u^2 dx \tag{5.25}$$

and can be found explicitly. First of all, by the usual change of variables, $x \mapsto T(\varepsilon\xi)$, equation (5.23) becomes

$$-\Delta v + V(\varepsilon\xi)v = v^p.$$

Let U denote the unique, positive, radial solution of

$$-\Delta u + u = u^p, \qquad u \in H^1(\mathbb{R}^N). \tag{5.26}$$

Then a straight calculation shows that $\alpha U(\beta T x)$ solves (5.23) whenever

$$\alpha = \alpha(\varepsilon\xi) = [V(\varepsilon\xi)]^{1/(p-1)}, \quad \beta = \beta(\varepsilon\xi) = [V(\varepsilon\xi)]^{1/2} \quad \text{and} \quad T = T(\varepsilon\xi).$$

We set

$$z^{\varepsilon\xi}(x) = \alpha(\varepsilon\xi)U\big(\beta(\varepsilon\xi)T(\varepsilon\xi)x\big) \tag{5.27}$$

and

$$Z^\varepsilon = \left\{ z^{\varepsilon\xi}(x-\xi) : \xi \in \mathbb{R}^N \right\}.$$

When there is no possible misunderstanding, we will write z, resp. Z, instead of $z^{\varepsilon\xi}$, resp Z^ε. We will also use the notation z_ξ to denote the function $z_\xi(x) := z^{\varepsilon\xi}(x-\xi)$. Obviously all the functions in $z_\xi \in Z$ are solutions of (5.24) or, equivalently, critical points of $F^{\varepsilon\xi}$.

Remark 5.7. Before we proceed, a remark is in order concerning the definition of the manifold Z^ε. Indeed, there is a lot of freedom in the choice of the diagonalizing matrix T. Moreover, Z^ε should be a regular manifold. We claim that, thanks to the uniform ellipticity assumption on J, see **(J)**, it is possible to choose $T(\varepsilon\xi)$ with the same regularity as J itself. We will not supply a complete proof of this fact. However, the best way to convince oneselves of this is to remember the celebrated Householder algorithm that diagonalizes a symmetric matrix J by means of arithmetic operations on the rows and the columns of J. We refer to [83] for an explanation of the method. Each of these operations corresponds to an orthogonal change of variables which preserves the uniform ellipticity of J, and at each step the only possible lack of regularity can be due to the division by an entry on the main diagonal of J. By **(J)**, each such entry is a function of $\varepsilon\xi$ strictly bounded away from zero, so that it cannot introduce any singularity in the algorithm. A repeated application of this argument can now applied to prove the regularity of T.

For future references, let us point out some estimates. First of all, by straightforward calculations, we get:

$$\partial_\xi z^{\varepsilon\xi}(x-\xi) = -\partial_x z^{\varepsilon\xi}(x-\xi) + O(\varepsilon). \tag{5.28}$$

Moreover, using **(J1)** and **(V1)**, we can infer that $\nabla f_\varepsilon(z_\xi)$ is close to zero when ε is small. Indeed we have:

$$||\nabla f_\varepsilon(z_\xi)|| \le C\left(\varepsilon|DJ(\varepsilon\xi)| + \varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon^2\right), \quad C > 0. \tag{5.29}$$

In the next lemma we will show that $D^2 f_\varepsilon$ is invertible on $\left(T_{z_\xi}Z^\varepsilon\right)^\perp$, where $T_{z_\xi}Z^\varepsilon$ denotes the tangent space to Z^ε at z_ξ.

Let $L_{\varepsilon,\xi} : (T_{z_\xi}Z^\varepsilon)^\perp \to (T_{z_\xi}Z^\varepsilon)^\perp$ denote the operator defined by setting $(L_{\varepsilon,\xi}v|w) = D^2 f_\varepsilon(z_\xi)[v,w]$.

Lemma 5.4. *Given $\overline{\xi} > 0$ there exists $C > 0$ such that for ε small enough one has that*

$$||L_{\varepsilon,\xi}v|| \ge C||v||, \qquad \forall\, |\xi| \le \overline{\xi},\ \forall\, v \in (T_{z_\xi}Z^\varepsilon)^\perp. \tag{5.30}$$

Proof We recall that $T_{z_\xi}Z^\varepsilon = \text{span}\{\partial_{\xi_1}z_\xi, \dots, \partial_{\xi_N}z_\xi\}$. Let $\mathcal{V} = \text{span}\{z_\xi, \partial_{x_1}z_\xi, \dots, \partial_{x_N}z_\xi\}$, by (5.28) it suffices to prove (5.30) for all $v \in \text{span}\{z_\xi, \phi\}$, where ϕ is orthogonal to $\mathcal{V}$. Precisely we shall prove that there exist $C_1, C_2 > 0$ such that for all $\varepsilon > 0$ small and all $|\xi| \le \overline{\xi}$ one has:

$$(L_{\varepsilon,\xi} z_\xi | z_\xi) \leq -C_1 < 0, \tag{5.31}$$
$$(L_{\varepsilon,\xi} \phi | \phi) \geq C_2 \|\phi\|^2. \tag{5.32}$$

The proof of (5.31) follows easily from the fact that z_ξ is a Mountain Pass critical point of $F^{\varepsilon\xi}$ and so from the fact that, given $\overline{\xi}$, there exists $c_0 > 0$ such that for all $\varepsilon > 0$ small and all $|\xi| \leq \overline{\xi}$ one finds:

$$D^2 F^{\varepsilon\xi}(z_\xi)[z_\xi, z_\xi] < -c_0 < 0.$$

Let us prove (5.32). As before, the fact that z_ξ is a Mountain Pass critical point of $F^{\varepsilon\xi}$ implies that

$$D^2 F^{\varepsilon\xi}(z_\xi)[\phi, \phi] > c_1 \|\phi\|^2 \quad \forall \phi \perp \mathcal{V}. \tag{5.33}$$

Let $R \gg 1$ and consider a radial smooth function $\chi_1 : \mathbb{R}^N \mapsto \mathbb{R}$ such that

$$\chi_1(x) = 1, \quad \text{for } |x| \leq R; \qquad \chi_1(x) = 0, \quad \text{for } |x| \geq 2R;$$

$$|\nabla \chi_1(x)| \leq \frac{2}{R}, \quad \text{for } R \leq |x| \leq 2R.$$

We also set $\chi_2(x) = 1 - \chi_1(x)$. Given ϕ let us consider the functions

$$\phi_i(x) = \chi_i(x - \xi)\phi(x), \quad i = 1, 2.$$

Therefore we need to evaluate the three terms in the equation below:

$$(L_{\varepsilon,\xi}\phi|\phi) = (L_{\varepsilon,\xi}\phi_1|\phi_1) + (L_{\varepsilon,\xi}\phi_2|\phi_2) + 2(L_{\varepsilon,\xi}\phi_1|\phi_2).$$

Using (5.33) and the definition of χ_i, we easily get

$$(L_{\varepsilon,\xi}\phi_1|\phi_1) \geq \varepsilon c_1 \|\phi_1\|^2 - \varepsilon c_2 \|\phi\|^2 + o_R(1)\|\phi\|^2,$$
$$(L_{\varepsilon,\xi}\phi_2|\phi_2) \geq c_3 \|\phi_2\|^2 + o_R(1)\|\phi\|^2,$$
$$(L_{\varepsilon,\xi}\phi_1|\phi_2) \geq o_R(1)\|\phi\|^2.$$

Therefore, since

$$\|\phi\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + 2\int_{\mathbb{R}^N} \chi_1\chi_2(\phi^2 + |\nabla\phi|^2) + o_R(1)\|\phi\|^2,$$

we get

$$(L_{\varepsilon,\xi}\phi|\phi) \geq c_4\|\phi\|^2 - c_5 R\varepsilon\|\phi\|^2 + o_R(1)\|\phi\|^2.$$

Taking $R = \varepsilon^{-1/2}$, and choosing ε small, (5.32) follows.

This completes the proof of the lemma. □

We will show that the existence of critical points of f_ε can be reduced to the search of critical points of an auxiliary finite dimensional functional. First of all we will make a Liapunov-Schmidt reduction, and successively we will study the behavior of an auxiliary finite dimensional functional.

Lemma 5.5. *For $\varepsilon > 0$ small and $|\xi| \leq \overline{\xi}$ there exists a unique $w = w(\varepsilon, \xi) \in (T_{z_\xi} Z)^\perp$ such that $\nabla f_\varepsilon(z_\xi + w) \in T_{z_\xi} Z$. Such a $w(\varepsilon, \xi)$ is of class C^2, resp. $C^{1,p-1}$, with respect to ξ, provided that $p \geq 2$, resp. $1 < p < 2$. Moreover, the functional $\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon, \xi))$ has the same regularity of w and satisfies:*

$$\nabla \Phi_\varepsilon(\xi_0) = 0 \quad \Longleftrightarrow \quad \nabla f_\varepsilon\left(z_{\xi_0} + w(\varepsilon, \xi_0)\right) = 0.$$

Proof Let $P = P_{\varepsilon\xi}$ denote the projection onto $(T_{z_\xi} Z)^\perp$. We want to find a solution $w \in (T_{z_\xi} Z)^\perp$ of the equation $P\nabla f_\varepsilon(z_\xi + w) = 0$. One has that $\nabla f_\varepsilon(z + w) = \nabla f_\varepsilon(z) + D^2 f_\varepsilon(z)[w] + R(z, w)$ with $\|R(z, w)\| = o(\|w\|)$, uniformly with respect to $z = z_\xi$, for $|\xi| \leq \overline{\xi}$. Therefore, our equation is:

$$L_{\varepsilon,\xi} w + P\nabla f_\varepsilon(z) + PR(z, w) = 0.$$

According to Lemma 5.4, this is equivalent to

$$w = N_{\varepsilon,\xi}(w), \quad \text{where} \quad N_{\varepsilon,\xi}(w) = -\left(L_{\varepsilon,\xi}\right)^{-1}\left(P\nabla f_\varepsilon(z) + PR(z, w)\right).$$

By (5.29) it follows that

$$\|N_{\varepsilon,\xi}(w)\| \leq c_1\left(\varepsilon|DJ(\varepsilon\xi)| + \varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon^2\right) + o(\|w\|). \tag{5.34}$$

Then one readily checks that $N_{\varepsilon,\xi}$ is a contraction on some ball in $(T_{z_\xi} Z)^\perp$ provided that $\varepsilon > 0$ is small enough and $|\xi| \leq \overline{\xi}$. Then there exists a unique w such that $w = N_{\varepsilon,\xi}(w)$. Let us point out that we cannot use the Implicit Function Theorem to find $w(\varepsilon, \xi)$, because the map $(\varepsilon, u) \mapsto P\nabla f_\varepsilon(u)$ fails to be C^2. However, fixed $\varepsilon > 0$ small, we can apply the Implicit Function Theorem to the map $(\xi, w) \mapsto P\nabla f_\varepsilon(z_\xi + w)$. Then, in particular, the function $w(\varepsilon, \xi)$ turns out to be of class C^1 with respect to ξ. Finally, it is a standard argument, see [4, 6], to check that the critical points of $\Phi_\varepsilon(\xi) = f_\varepsilon(z + w)$ give rise to critical points of f_ε. □

Remark 5.8. From (5.34) it immediately follows that:

$$\|w\| \leq C\left(\varepsilon|DJ(\varepsilon\xi)| + \varepsilon|\nabla V(\varepsilon\xi)| + \varepsilon^2\right), \tag{5.35}$$

where $C > 0$.

With easy calculations (see [11, Lemma 4]), we can give an estimate of the derivative $\partial_\xi w$.

Lemma 5.6. *One has that:*

$$\|\partial_\xi w\| \leq c\left(\varepsilon|DJ(\varepsilon\xi)| + \varepsilon|\nabla V(\varepsilon\xi)| + O(\varepsilon^2)\right)^\gamma, \tag{5.36}$$

with $c > 0$ and $\gamma = \min\{1, p-1\}$.

Now we will use the estimates on w and $\partial_\xi w$ established above to find an expansion of $\nabla \Phi_\varepsilon(\xi)$, where $\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon, \xi))$. In the sequel, to be short, we will often write z instead of z_ξ and w instead of $w(\varepsilon, \xi)$. It is always understood that ε is taken in such a way that all the results discussed previously hold.

We have:

$$\Phi_\varepsilon(\xi) = \frac{1}{2}\|z+w\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} \langle (J(\varepsilon x) - I)\, \nabla(z+w) \mid \nabla(z+w)\rangle + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x) - 1)\,(z+w)^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} (z+w)^{p+1}.$$

Since

$$-\operatorname{div}(J(\varepsilon\xi)\nabla z) + V(\varepsilon\xi) z = z^p,$$

we infer that:

$$\|z\|^2 = -\int_{\mathbb{R}^N} \langle (J(\varepsilon\xi) - I)\,\nabla z \mid \nabla z\rangle - (V(\varepsilon\xi) - 1)\int_{\mathbb{R}^N} z^2 + \int_{\mathbb{R}^N} z^{p+1},$$

$$(z|w) = -\int_{\mathbb{R}^N} \langle (J(\varepsilon\xi) - I)\,\nabla z \mid \nabla w\rangle - (V(\varepsilon\xi) - 1)\int_{\mathbb{R}^N} zw + \int_{\mathbb{R}^N} z^p w.$$

Then we find:

$$\begin{aligned}\Phi_\varepsilon(\xi) = &\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N} z^{p+1} \\ &+\frac{1}{2}\int_{\mathbb{R}^N} \langle (J(\varepsilon x) - J(\varepsilon\xi))\,\nabla z \mid \nabla z\rangle + \frac{1}{2}\int_{\mathbb{R}^N} [V(\varepsilon x) - V(\varepsilon\xi)]\, z^2 \\ &+\int_{\mathbb{R}^N} \langle (J(\varepsilon x) - J(\varepsilon\xi))\,\nabla z \mid \nabla w\rangle + \int_{\mathbb{R}^N} [V(\varepsilon x) - V(\varepsilon\xi)]\, zw \\ &+\frac{1}{2}\int_{\mathbb{R}^N} \langle J(\varepsilon x)\nabla w \mid \nabla w\rangle + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x) w^2 \\ &+\frac{1}{2}\|w\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} \left[(z+w)^{p+1} - z^{p+1} - (p+1)z^p w\right].\end{aligned}$$

Since $z(x) = \alpha(\varepsilon\xi) U\big(\beta(\varepsilon\xi) T(\varepsilon\xi) x\big)$, see (5.27), it follows

$$\int_{\mathbb{R}^N} z^{p+1} dx = C_0 V(\varepsilon\xi)^{\frac{p+1}{p-1} - \frac{N}{2}} \left(\det J(\varepsilon\xi)\right)^{\frac{1}{2}} = C_0 \Gamma(\varepsilon\xi),$$

where $C_0 = \int_{\mathbb{R}^N} U^{p+1}$ and Γ is the auxiliary function introduced in (5.4). Letting $C_1 = C_0[1/2 - 1/(p+1)]$ and recalling the estimates (5.35) and (5.36) on w and $\nabla_\xi w$, respectively, we readily find:

$$\Phi_\varepsilon(\xi) = C_1 \Gamma(\varepsilon\xi) + \rho_\varepsilon(\xi), \tag{5.37}$$

where $|\rho_\varepsilon(\xi)| \le \text{const}\,\left(\varepsilon |DJ(\varepsilon\xi)| + \varepsilon |\nabla V(\varepsilon\xi)| + \varepsilon^2\right)$ and

$$\nabla \Phi_\varepsilon(\xi) = C_1 \varepsilon \nabla \Gamma(\varepsilon\xi) + \varepsilon^{1+\gamma} R_\varepsilon(\xi), \tag{5.38}$$

where $|R_\varepsilon(\xi)| \leq \text{const}$ and $\gamma = \min\{1, p-1\}$.

Remark 5.9. We highlight that, as observed in Remark 5.1, $C_1\Gamma = \Sigma$, where Σ is the ground-state function.

Now we can prove Theorem 5.3 at least in the case (a). The other is an easy consequence of Theorem 5.10.

Proof of Theorem 5.3 Let z_0 be a minimum point of Γ (the other case is similar) and let $\Lambda \subset \mathbb{R}^N$ be a compact neighborhood of z_0 such that

$$\min_\Lambda \Gamma < \min_{\partial\Lambda} \Gamma.$$

By (5.37), it is easy to see that for ε sufficiently small, there results:

$$\min_\Lambda \Phi(\cdot/\varepsilon) < \min_{\partial\Lambda} \Phi(\cdot/\varepsilon).$$

Hence, $\Phi(\cdot/\varepsilon)$ possesses a critical point ξ in Λ. By Lemma 5.5 we have that $u_{\varepsilon,\xi} = z^\xi(\cdot - \xi/\varepsilon) + w(\varepsilon,\xi)$ is a critical point of f_ε and so a solution of problem (5.23). Therefore

$$u_{\varepsilon,\xi}(x/\varepsilon) \simeq z^\xi\left(\frac{x-\xi}{\varepsilon}\right)$$

is a solution of (5.1). This ξ converges to some $\bar\xi$ as $\varepsilon \to 0$, but by (5.38) it follows that $\bar\xi = z_0$. □

5.5 Existence of multiple solutions

In this section we will study the problem of the multiplicity of solutions. In the first subsection we will prove that under a more stringent assumption on the function Σ, our problem (5.5) possesses generically more than one solution. In the second subsection, instead, we will deal with the problem (5.1) and, as done in Section 5.4, we will treat also the case of maximum points for Σ.

5.5.1 Using penalization method

Since our arguments are inspired to those of [24] and [32], we will skip some easy details.

Let $c_0 = \min_{\mathbb{R}^N} \Sigma(z)$. Let $M \subset \Sigma^{-1}(c_0) \cap \Omega$.

We state our main result for multiple solutions.

Theorem 5.7. *Suppose* **(V)**, **(J)**, **(f1-4)**. *Suppose that* M *is compact and let* $\Lambda \subset \Omega$ *be the closure of a bounded neighborhood of* M *such that* $c_0 < \inf_{\partial\Lambda} \Sigma$.

Suppose, in addition, that there exists a point $z_0 \in M$ *such that:*

(V2) $V(z_0) = \min_\Lambda V$;
(J2) *the matrix* $J(z) - J(z_0)$ *is positive-definite for all* $z \in \mathbb{R}^N$.

Then there exists $\varepsilon(\Lambda) > 0$ *such that, for any* $\varepsilon < \varepsilon(\Lambda)$, *problem* (5.5) *has at least* $\operatorname{cat}(M, \Lambda)$ *solutions concentrating at some points of* M. *Here* $\operatorname{cat}(M, \Lambda)$ *denotes the Lusternik-Schnirelman category of* M *with respect to* Λ.

The proof of theorem 5.7 requires some preliminary lemmas. The main ingredient is the following result in abstract critical point theory (see for example [95]).

Theorem 5.8. *Let* X *be a complete Riemannian manifold of class* $C^{1,1}$, *and assume that* $\phi \in C^1(X)$ *is bounded from below. Let*

$$-\infty < \inf_X \phi < a < b < +\infty.$$

Suppose that ϕ *satisfies the Palais–Smale condition on the sublevel* $\{u \in X \mid \phi(u) \leq b\}$ *and that* a *is not a critical level for* ϕ. *Then the number of critical points of* ϕ *in* $\phi^a = \{u \in X \mid \phi(u) \leq a\}$ *is at least* $\operatorname{cat}(\phi^a, \phi^a)$.

We shall apply this theorem to the penalized functional E^ε, introduced in (5.16), constrained to its Nehari manifold $\mathcal{N}^\varepsilon$, so that it satisfies (PS) and it is bounded from below. The crucial step is therefore to link the topological richness of the sublevels of E^ε with that of M. For this purpose we make use of the following elementary result. For the proof we refer to [15].

Lemma 5.7. *Let* H, Ω^+, Ω^- *be closed sets with* $\Omega^- \subset \Omega^+$; *let* $\beta \colon H \to \Omega^+$, $\psi \colon \Omega^- \to H$ *be two continuous maps such that* $\beta \circ \psi$ *is homotopically equivalent to the embedding* $j \colon \Omega^- \to \Omega^+$. *Then* $\operatorname{cat}(H, H) \geq \operatorname{cat}(\Omega^-, \Omega^+)$.

Let $\eta > 0$ be a smooth, non-increasing cut–off function, defined in $[0, +\infty)$, such that $\eta(|x|) = 1$ if $x \in \Lambda$, and $|\eta'| \leq C$ for some $C > 0$. For any $\xi \in M$ let

$$\psi_{\varepsilon,\xi} \colon x \mapsto \eta(|x - \xi|)\, \omega\left(\frac{x - \xi}{\varepsilon}\right),$$

where ω is a positive ground state of the functional I_ξ. Now define $\Phi_\varepsilon \colon M \to \mathcal{N}^\varepsilon$ by

$$\Phi_\varepsilon(\xi) = \theta_\varepsilon \psi_{\varepsilon,\xi},$$

where $\theta_\varepsilon \in \mathbb{R}$ is such that $\theta_\varepsilon \psi_{\varepsilon,\xi} \in \mathcal{N}^\varepsilon$. By Lemma 5.1 with minor changes, we infer that there exists such a θ_ε.

Lemma 5.8. *Uniformly in* $\xi \in M$ *we have*

$$\lim_{\varepsilon \to 0} \varepsilon^{-N} E^\varepsilon(\Phi_\varepsilon(\xi)) = c_0.$$

Proof The proof is similar to the one of Lemma 4.1 in [24], taking into account the monotonicity property **(f4)** of f and the fact that $\xi \in M \subset \Lambda$. □

We now construct a second auxiliary map which proves to be useful for the comparison of the topologies of M and of the sublevels of E^ε.

Let $R > 0$ be such that $\Lambda \subset \{x \in \mathbb{R}^N : |x| \leq R\}$. Let $\chi \colon \mathbb{R}^N \to \mathbb{R}^N$ be defined by

$$\chi(x) = \begin{cases} x & \text{for } |x| \leq R, \\ \dfrac{Rx}{|x|} & \text{for } |x| > R. \end{cases}$$

Finally, define $\beta \colon \mathcal{N}^\varepsilon \to \mathbb{R}^N$ by

$$\beta(u) = \frac{\int_{\mathbb{R}^N} \chi \cdot |u|^2}{\int_{\mathbb{R}^N} |u|^2}.$$

As in [24], it is easy to show that $\beta(\Phi_\varepsilon(\xi)) = \xi + o(1)$ as $\varepsilon \to 0$, uniformly with respect to $\xi \in M$.

Let us define a suitable sublevel of E^ε:

$$\tilde{\mathcal{N}}^\varepsilon = \left\{ u \in \mathcal{N}^\varepsilon : E^\varepsilon(u) \leq \varepsilon^N (c_0 + o(1)) \right\}.$$

As already stated, we know that E^ε verifies the (PS) condition at all levels.

Lemma 5.9. *Let $\tilde{\Lambda}$ a sufficiently small homotopically equivalent neighborhood of Λ. For all ε sufficiently small, we get*

$$\beta(\tilde{\mathcal{N}}^\varepsilon) \subset \tilde{\Lambda}.$$

Proof The proof proceeds by contradiction. If the claim does not hold, then we may find sequences $\{\varepsilon_n\}$, $\{u_n\}$ such that $\varepsilon_n \to 0$, $u_n \in \tilde{\mathcal{N}}^{\varepsilon_n}$ but $\beta(u_n) \notin \tilde{\Lambda}$. We claim that

$$\lim_{n \to \infty} \varepsilon_n^{-N} \int\limits_{\Omega \setminus \tilde{\Lambda}} |u_n|^2 = 0. \tag{5.39}$$

Indeed, since $u_n \in \mathcal{N}^{\varepsilon_n}$, we have that

$$E^{\varepsilon_n}(u_n) \geq E^{\varepsilon_n}(t u_n)$$

for any $t > 0$. Let us set

$$\tilde{E}_n(v) = \frac{1}{2} \int_{\tilde{\Lambda}} \varepsilon_n^2 \langle J(x) \nabla v \mid \nabla v \rangle + V(x)|v|^2 - \int_{\tilde{\Lambda}} G(x, v)\, dx.$$

Choose $t_n > 0$ such that

$$\tilde{E}_n(t_n u_n) = \max_{t>0} \tilde{E}_n(t u_n).$$

Since $u_n \in \tilde{\mathcal{N}}^{\varepsilon_n}$ and that fact that

$$\frac{V(x)}{2}u^2 - G(x,u) \geq Cu^2$$

for all $x \in \Omega \setminus \tilde{\Lambda}$ and all $u > 0$, we obtain

$$\tilde{E}_n(t_n u_n) + Ct_n^2 \int\limits_{\Omega \setminus \tilde{\Lambda}} |u_n|^2 \leq \varepsilon_n^N (c_0 + o(1)). \tag{5.40}$$

From our assumptions on V, J and f, since $E^{\varepsilon_n}(u_n) \leq C\varepsilon_n^N$ and $u_n \in \mathcal{N}^{\varepsilon_n}$, we see that

$$\int_\Omega \varepsilon_n^2 |\nabla u_n|^2 + |u_n|^2 \leq C\varepsilon_n^N. \tag{5.41}$$

Set $v_n \colon x \mapsto t_n u_n(\varepsilon_n x)$. From the definition of t_n it follows

$$\begin{aligned} \int\limits_{\varepsilon_n^{-1}\tilde{\Lambda}} \langle J(\varepsilon_n x)\nabla v_n \mid \nabla v_n \rangle + V(\varepsilon_n x)|v_n|^2 &= \int\limits_{\varepsilon_n^{-1}\tilde{\Lambda}} g(\varepsilon_n x, v_n)v_n \\ &\leq \int\limits_{\varepsilon_n^{-1}\tilde{\Lambda}} C|v_n|^{p+1} + \rho|v_n|^2, \end{aligned}$$

where $\rho > 0$ can be taken arbitrarily small. Now, Sobolev's theorem yields that

$$\int\limits_{\varepsilon_n^{-1}\tilde{\Lambda}} |v_n|^{p+1} \leq C\Big(\int\limits_{\varepsilon_n^{-1}\tilde{\Lambda}} |\nabla v_n|^2 + |v_n|^2 \Big)^{\frac{p+1}{2}}$$

and the constant C can be taken the same for all n, since it generally depends only on the geometry of the domain of integration but not on its volume. Combining the two last inequalities, since J and V are bounded below, we find that there exists $\sigma > 0$ such that for all n,

$$\int\limits_{\varepsilon_n^{-1}\tilde{\Lambda}} |v_n|^{p+1} \geq \sigma > 0.$$

Hence

$$t_n^2 \int\limits_{\tilde{\Lambda}} \varepsilon_n^2 |\nabla u_n|^2 + |u_n|^2 \geq \varepsilon_n^N \sigma',$$

with $\sigma' > 0$. Combining this with (5.41) we see that

$$t_n \geq \sigma'' > 0 \quad \text{for all } n \in \mathbb{N}, \tag{5.42}$$

with $\sigma'' > 0$. Now, by the definition of t_n, we have that

$$\tilde{E}_n(t_n u_n) \geq \inf_{u \in H^1(\bar{\Lambda})} \sup_{t>0} \tilde{E}_n(tu) =: b_n. \tag{5.43}$$

But it follows from [33, Lemma 1.3], with obvious modifications, that

$$\lim_{n\to\infty} \varepsilon_n^{-N} b_n = c_0$$

and this, together with (5.40), (5.42) and (5.43), easily implies the validity of Claim (5.39).

We now proceed to prove Lemma 5.9. Set $v_n \colon x \mapsto u_n(\varepsilon_n x)$. We claim that

$$\sup_{t>0} I_{z_0}(tv_n) \leq c_0 + o(1). \tag{5.44}$$

To see this, we recall that $\{v_n\}$ is bounded in the H^1 norm. Since

$$\int_{\varepsilon_n^{-1}\Omega} \langle J(\varepsilon_n x)\nabla v_n \mid \nabla v_n\rangle + V(\varepsilon_n x)|v_n|^2 = \int_{\varepsilon_n^{-1}\Omega} g(\varepsilon_n x, v_n)v_n \leq \int_{\varepsilon_n^{-1}\Omega} f(v_n)v_n,$$

similar arguments as those above show that

$$\int_{\varepsilon_n^{-1}\Omega} |v_n|^{p+1} \geq \sigma > 0.$$

Hence, by the first lemma of Concentration–Compactness (see [54, Lemma I.1]), there is a sequence B_n of balls of radius one such that

$$\int_{B_n} |v_n|^2 \geq \sigma > 0. \tag{5.45}$$

We now select $t_n > 0$ such that $I_{z_0}(t_n v_n) = \sup_{t>0} I_{z_0}(tv_n)$. Since $\{v_n\}$ is bounded in H^1 norm we get

$$Ct_n^2 - \int_{\varepsilon_n^{-1}\Omega} F(t_n v_n) \geq I_{z_0}(t_n v_n) \geq c_0.$$

But from assumption **(f3)** we see that $F(u) \geq Cu^\theta$, so that

$$t_n^{\theta-2} \int_{\varepsilon_n^{-1}\Omega} |v_n|^\theta \leq C.$$

This and (5.45) imply that $\{t_n\}$ is bounded. Therefore from (5.39) we deduce

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N \setminus \varepsilon_n^{-1}\tilde{\Lambda}} |t_n v_n|^2 = 0. \tag{5.46}$$

From the properties of z_0 we easily get (here we use **(J1)** to get rid of the contribution of J both inside and outside $\tilde{\Lambda}$)

$$c_0 + o(1) \geq \varepsilon_n^{-N} E^{\varepsilon_n}(t_n u_n) \geq I_{z_0}(t_n v_n) - \frac{t_n^2}{2} \int\limits_{\mathbb{R}^N \setminus \varepsilon_n^{-1}\tilde{\Lambda}} V(z_0)|v_n|^2.$$

and so we get (5.44).

If we set $w_n = t_n v_n$, we see that $\{w_n\}$ is a minimizing sequence for I_{z_0} constrained to its Nehary manifold $\mathcal{N}_{z_0}$. By a straightforward application of the Ekeland variational principle, we can build a Palais–Smale sequence $\{\tilde{w_n}\}$ of I_{z_0} such that $\tilde{w_n} - w_n \to 0$ strongly in H^1. Thus there exists a sequence of points $\{z_n\}$ such that $\{w_n(\cdot + z_n)\}$ converges strongly to a positive critical point w_∞ of I_{z_0}. Let $\bar{y}_n = \varepsilon_n z_n$. If $\liminf_{n\to\infty} \operatorname{dist}(\bar{y}_n, \Lambda) > 0$ then, since we can take $\tilde{\Lambda}$ sufficiently small, we have also $\liminf_{n\to\infty} \operatorname{dist}(\bar{y}_n, \tilde{\Lambda}) > 0$ and so from (5.46) we get

$$\begin{aligned} o(1) = \int\limits_{\mathbb{R}^N \setminus \varepsilon_n^{-1}\tilde{\Lambda}} |t_n v_n|^2 &= \int\limits_{\mathbb{R}^N \setminus \varepsilon_n^{-1}\tilde{\Lambda}} |w_n|^2 = \\ &= \int\limits_{\mathbb{R}^N \setminus \varepsilon_n^{-1}(\tilde{\Lambda} - \bar{y}_n)} |w_n(\cdot + z_n)|^2 = \int_{\mathbb{R}^N} |w_\infty|^2 + o(1), \end{aligned}$$

which contradicts the positivity of w_∞. Hence we may assume that $\bar{y}_n \to \bar{y} \in \Lambda$. But then

$$\beta(u_n) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n x + \bar{y}_n)|w_n(x + z_n)|^2\, dx}{\int_{\mathbb{R}^N} |w_n(x + z_n)|^2\, dx} \to \bar{y} \in \Lambda,$$

against our (absurd) assumptions $\beta(u_n) \notin \tilde{\Lambda}$. □

Proof of Theorem 5.7 By Lemma 5.8, the map $\xi \mapsto \Phi_\varepsilon(\xi)$ sends M into $\tilde{\mathcal{N}}^\varepsilon$. Moreover, by Lemma 5.9 we know that $\beta(\tilde{\mathcal{N}}^\varepsilon) \subset \tilde{\Lambda}$. Then the map $\xi \mapsto \beta \circ \Phi_\varepsilon(\xi)$ is homotopic to the inclusion $j\colon M \to \tilde{\Lambda}$, for any ε sufficiently small. We now combine Theorem 5.8 with Lemma 5.7 to get that E^ε has at least $\operatorname{cat}(M, \tilde{\Lambda}) = \operatorname{cat}(M, \Lambda)$ critical points on the manifold $\mathcal{N}^\varepsilon$. The verification that each one of these critical points is actually a solutions of (5.5) follows again from Section 5.2, once we recall that the main formula in Lemma 5.3 holds true for each one of the critical points just found by definition of $\tilde{\mathcal{N}}^\varepsilon$. This completes the proof. □

5.5.2 Using perturbation method

Let us introduce a topological invariant related to Conley theory.

Definition 5.1. *Let M be a subset of $\mathbb{R}^N$, $M \neq \emptyset$. The cup long $l(M)$ of M is defined by*

$$l(M) = 1 + \sup\{k \in \mathbb{N} : \exists\, \alpha_1, \ldots, \alpha_k \in \check{H}^*(M) \setminus 1,\ \alpha_1 \cup \ldots \cup \alpha_k \neq 0\}.$$

If no such class exists, we set $l(M) = 1$. Here $\check{H}^*(M)$ is the Alexander cohomology of M with real coefficients and $\cup$ denotes the cup product.

Let us recall Theorem 6.4 in Chapter II of [21].

Theorem 5.9. *Let N a Hilbert-Riemannian manifold. Let $g \in C^2(N)$ and let $M \subset N$ be a smooth compact nondegenerate manifold of critical points of g. Let U be a neighborhood of M and let $h \in C^1(N)$. Then, if $\|g - h\|_{C^1(\bar{U})}$ is sufficiently small, the function g possesses at least $l(M)$ critical points in U.*

Let us suppose that Γ has a smooth manifold of critical points M. We say that M is nondegenerate (for Γ) if every $x \in M$ is a nondegenerate critical point of $\Gamma_{|M^\perp}$. The Morse index of M is, by definition, the Morse index of any $x \in M$, as critical point of $\Gamma_{|M^\perp}$.

We now can state our multiplicity result.

Theorem 5.10. *Let* **(V-V1)** *and* **(J-J1)** *hold and suppose Γ has a nondegenerate smooth manifold of critical points M. Then for $\varepsilon > 0$ small, (5.1) has at least $l(M)$ solutions that concentrate near points of M.*

Proof First of all, we fix $\bar{\xi}$ in such a way that $|x| < \bar{\xi}$ for all $x \in M$. We will apply the finite dimensional procedure with such $\bar{\xi}$ fixed.

In order to use Theorem 5.9, we set $g(\xi) = C_1\Gamma(\xi)$ and $h(\xi) = \Phi_\varepsilon(\xi/\varepsilon)$. Fix a δ-neighborhood M_δ of M such that $M_\delta \subset \{|x| < \bar{\xi}\}$ and the only critical points of Γ in M_δ are those in M. We will take $U = M_\delta$.

By (5.37) and (5.38), $\Phi_\varepsilon(\cdot/\varepsilon)$ converges to $C_1\Gamma(\cdot)$ in $C^1(\bar{U})$ and so, by Theorem 5.9 we have at least $l(M)$ critical points of g provided ε sufficiently small. The concentration statement follows as in [11]. □

Clearly, this theorem shows that there is no essential difficulty in dealing with local maxima of Γ instead of minima. Moreover, when we deal with local minima (resp. maxima) of Γ, the preceding results can be improved because the number of positive solutions of (5.1) can be estimated by means of the category and M does not need to be a manifold.

Theorem 5.11. *Let* **(V-V1)** *and* **(J-J1)** *hold and suppose Γ has a compact set X where Γ achieves a strict local minimum (resp. maximum), in the sense that there exists $\delta > 0$ and a δ-neighborhood X_δ of X such that*

$$b := \inf\{\Gamma(x) : x \in \partial X_\delta\} > a := \Gamma_{|X}, \quad (\text{resp. } \sup\{\Gamma(x) : x \in \partial X_\delta\} < a).$$

Then there exists $\varepsilon_\delta > 0$ such that (5.1) *has at least* $\text{cat}(X, X_\delta)$ *solutions that concentrate near points of X_δ, provided $\varepsilon \in (0, \varepsilon_\delta)$.*

Proof We will treat only the case of minima, being the other one similar. Fix again $\bar{\xi}$ in such a way that X_δ is contained in $\{x \in \mathbb{R}^N : |x| < \bar{\xi}\}$. We set $X^\varepsilon = \{\xi : \varepsilon\xi \in X\}$, $X_\delta^\varepsilon = \{\xi : \varepsilon\xi \in X_\delta\}$ and $Y^\varepsilon = \{\xi \in X_\delta^\varepsilon : \Phi_\varepsilon(\xi) \le C_1(a+b)/2\}$. By (5.37) it follows that there exists $\varepsilon_\delta > 0$ such that

$$X^\varepsilon \subset Y^\varepsilon \subset X_\delta^\varepsilon, \tag{5.47}$$

provided $\varepsilon \in (0, \varepsilon_\delta)$. Moreover, if $\xi \in \partial X_\delta^\varepsilon$ then $\Gamma(\varepsilon\xi) \ge b$ and hence

$$\Phi_\varepsilon(\xi) \ge C_1 \Gamma(\varepsilon\xi) + o_\varepsilon(1) \ge C_1 b + o_\varepsilon(1).$$

On the other side, if $\xi \in Y^\varepsilon$ then $\Phi_\varepsilon(\xi) \le C_1(a+b)/2$. Hence, for ε small, Y^ε cannot meet $\partial X_\delta^\varepsilon$ and this readily implies that Y^ε is compact. Then Φ_ε possesses at least $\text{cat}(Y^\varepsilon, X_\delta^\varepsilon)$ critical points in X_δ. Using (5.47) and the properties of the category one gets

$$\text{cat}(Y^\varepsilon, Y^\varepsilon) \ge \text{cat}(X^\varepsilon, X_\delta^\varepsilon) = \text{cat}(X, X_\delta),$$

and the result follows. □

Remark 5.10. Let us observe that Theorem 5.3 is a particular case of Theorems 5.10 and 5.11.

6 Schödinger equation with critical Sobolev exponent

In this chapter we study the existence of solutions and their concentration phenomena of a singularly perturbed semilinear Schrödinger equation with the presence of the critical Sobolev exponent, that is:

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = K(x)u^p + Q(x)u^\sigma & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{6.1}$$

where $N \geq 3$, $1 < p < \sigma = \frac{N+2}{N-2}$, V, K and Q are C^2 function from $\mathbb{R}^N$ to $\mathbb{R}$. We will show that there exist solutions of (6.1) concentrating near the maximum and minimum points of an auxiliary functional which depends only on V, K and Q.

On the potentials, we will make the following assumptions:

(V) $V \in C^2(\mathbb{R}^N, \mathbb{R})$, V and D^2V are bounded; moreover,

$$V(x) \geq C > 0 \quad \text{for all } x \in \mathbb{R}^N;$$

(K) $K \in C^2(\mathbb{R}^N, \mathbb{R})$, K and D^2K are bounded; moreover,

$$K(x) \geq C > 0 \quad \text{for all } x \in \mathbb{R}^N;$$

(Q) $Q \in C^2(\mathbb{R}^N, \mathbb{R})$, Q and D^2Q are bounded; moreover, $Q(0) = 0$.

We point out that while V and K must be strictly positive, Q can change sign and must vanish in 0.

Let us introduce an auxiliary function which will play a crucial rôle in the study of (6.1). Let $\Gamma\colon \mathbb{R}^N \to \mathbb{R}$ be a function so defined:

$$\Gamma(\xi) = \bar{C}_1 \Gamma_1(\xi) - \bar{C}_2 \Gamma_2(\xi), \tag{6.2}$$

where

$$\begin{aligned} \Gamma_1(\xi) &\equiv V(\xi)^{\frac{p+1}{p-1} - \frac{N}{2}} K(\xi)^{-\frac{2}{p-1}}, \\ \Gamma_2(\xi) &\equiv Q(\xi)\, V(\xi)^{\frac{\sigma+1}{p-1} - \frac{N}{2}} K(\xi)^{-\frac{\sigma+1}{p-1}}, \\ \bar{C}_1 &\equiv \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} U^{p+1}, \\ \bar{C}_2 &\equiv \frac{1}{\sigma+1} \int_{\mathbb{R}^N} U^{\sigma+1}, \end{aligned}$$

and U is the unique solution of

$$\begin{cases} -\Delta U + U = U^p & \text{in } \mathbb{R}^N, \\ U > 0 & \text{in } \mathbb{R}^N, \\ U(0) = \max_{\mathbb{R}^N} U. \end{cases} \tag{6.3}$$

Let us observe that by **(V)** and **(K)**, Γ is well defined.

Our main result is:

Theorem 6.1. *Let $\xi_0 \in \mathbb{R}^N$. Suppose* **(V)**, **(K)** *and* **(Q)**. *There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then* (6.1) *possesses a solution u_ε which concentrates on ξ_ε with $\xi_\varepsilon \to \xi_0$, as $\varepsilon \to 0$, provided that one of the two following conditions holds:*

(a) ξ_0 is a non-degenerate critical point of Γ;
(b) ξ_0 is an isolated local strict minimum or maximum of Γ.

In the case $V \equiv K \equiv 1$, by Theorem 6.1 and by the expression of Γ, see (6.2), we easily get:

Corollary 6.1. *Let $\xi_0 \in \mathbb{R}^N$. Let $V \equiv K \equiv 1$ and suppose* **(Q)**. *There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then* (6.1) *possesses a solution u_ε which concentrates on ξ_ε with $\xi_\varepsilon \to \xi_0$, as $\varepsilon \to 0$, provided that one of the two following conditions holds:*

(a) ξ_0 is a non-degenerate critical point of Q;
(b) ξ_0 is an isolated local strict minimum or maximum of Q.

The new feature of the present chapter is that the coefficient Q of $u^{\frac{N+2}{N-2}}$ vanishes at $x = 0$. After the rescaling $x \mapsto \varepsilon x$, equation (6.1) becomes

$$-\Delta u + V(\varepsilon x)u = K(\varepsilon x)u^p + Q(\varepsilon x)u^\sigma.$$

Then, assumption $Q(0) = 0$ implies that, roughly, the unperturbed problem, with $\varepsilon = 0$ is unaffected by the critical nonlinearity.

Theorem 6.1 will be proved as a particular case of two multiplicity results in Section 6.4. The proof of the theorem relies on a finite dimensional reduction, precisely on the perturbation technique developed in [11], where (6.1) with $Q \equiv 0$ is studied. For the sake of brevity, we will refer to [11] for some details. In Section 6.1 we present the variational framework. In Section 6.2 we perform the Liapunov-Schmidt reduction and in Section 6.3 we make the asymptotic expansion of the finite dimensional functional.

All the results of this chapter are contained in a recent paper [72].

Notation

- With $o_\varepsilon(1)$ we denote a function which tends to 0 as $\varepsilon \to 0$.
- We set $2^* = \frac{2N}{N-2}$, the critical Sobolev exponent.

6.1 The variational framework

Performing the change of variable $x \mapsto \varepsilon x$, equation (6.1) becomes

$$\begin{cases} -\Delta u + V(\varepsilon x)u = K(\varepsilon x)u^p + Q(\varepsilon x)u^\sigma & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N. \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{6.4}$$

Of course, if u is a solution of (6.4), then $u(\cdot/\varepsilon)$ is solution of (6.1).

Solutions of (6.4) are critical points $u \in H^1(\mathbb{R}^N)$ of

$$f_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon x)u^{p+1}dx - \frac{1}{\sigma+1}\int_{\mathbb{R}^N} Q(\varepsilon x)u^{\sigma+1}dx.$$

The solutions of (6.4) will be found near the solutions of

$$-\Delta u + V(\varepsilon\xi)u = K(\varepsilon\xi)u^p \qquad \text{in } \mathbb{R}^N, \tag{6.5}$$

for an appropriate choice of $\xi \in \mathbb{R}^N$.

The solutions of (6.5) are critical points of the functional

$$F^{\varepsilon\xi}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon\xi)u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon\xi)u^{p+1}dx \tag{6.6}$$

and can be found explicitly.

Let U denote the unique, positive, radial solution of (6.3), then a straight calculation shows that $\alpha U(\beta x)$ solves (6.5) whenever

$$\alpha = \alpha(\varepsilon\xi) = \left[\frac{V(\varepsilon\xi)}{K(\varepsilon\xi)}\right]^{1/(p-1)} \quad \text{and} \quad \beta = \beta(\varepsilon\xi) = [V(\varepsilon\xi)]^{1/2}.$$

We set

$$z^{\varepsilon\xi}(x) = \alpha(\varepsilon\xi)U\big(\beta(\varepsilon\xi)x\big) \tag{6.7}$$

and

$$Z^\varepsilon = \left\{z^{\varepsilon\xi}(x-\xi) : \xi \in \mathbb{R}^N\right\}.$$

When there is no possible misunderstanding, we will write z, resp. Z, instead of $z^{\varepsilon\xi}$, resp Z^ε. We will also use the notation z_ξ to denote the function $z_\xi(x) \equiv z^{\varepsilon\xi}(x-\xi)$. Obviously all the functions in $z_\xi \in Z$ are solutions of (6.5) or, equivalently, critical points of $F^{\varepsilon\xi}$.

The next lemma shows that z_ξ is an "almost solution" of (6.4).

Lemma 6.1. *Given $\overline{\xi}$, for all $|\xi| \le \overline{\xi}$ and for all ε sufficiently small, we have*

$$\|\nabla f_\varepsilon(z_\xi)\| = O(\varepsilon). \tag{6.8}$$

Proof Let $v \in H^1(\mathbb{R}^N)$, recalling that z_ξ is solution of (6.5), we have:

$$
\begin{aligned}
(\nabla f_\varepsilon(z_\xi) \mid v) &= \int_{\mathbb{R}^N} \nabla z_\xi \cdot \nabla v\, dx + \int_{\mathbb{R}^N} V(\varepsilon x) z_\xi v\, dx - \int_{\mathbb{R}^N} K(\varepsilon x) z_\xi^p v\, dx \\
&\quad - \int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^\sigma v\, dx \\
&= \int_{\mathbb{R}^N} \left[\nabla z_\xi \cdot \nabla v\, dx + V(\varepsilon\xi) z_\xi v\, dx - K(\varepsilon\xi) z_\xi^p v\, dx\right] \\
&\quad + \int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon\xi)) z_\xi v\, dx - \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon\xi)) z_\xi^p v\, dx \\
&\quad - \int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^\sigma v\, dx \\
&= \int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon\xi)) z_\xi v\, dx - \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon\xi)) z_\xi^p v\, dx \\
&\quad - \int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^\sigma v\, dx. \qquad (6.9)
\end{aligned}
$$

Following [11], we infer that

$$\int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon\xi)) z_\xi v\, dx - \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon\xi)) z_\xi^p v\, dx = O(\varepsilon)\|v\|.$$

Let us study the last term in (6.9). We get

$$\left|\int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^\sigma v\, dx\right| \le \left(\int_{\mathbb{R}^N} Q(\varepsilon x)^{\frac{2^*}{\sigma}} z_\xi^{2^*}\, dx\right)^{\frac{\sigma}{2^*}} \|v\|.$$

By assumption **(Q)**, we know that

$$|Q(\varepsilon x)| \le \varepsilon |\nabla Q(0)|\, |x| + C\varepsilon^2 |x|^2,$$

therefore

$$
\begin{aligned}
&\int_{\mathbb{R}^N} Q(\varepsilon x)^{\frac{2^*}{\sigma}} z_\xi^{2^*}\, dx \\
&\qquad \le C_1\, \varepsilon^{\frac{2^*}{\sigma}} \int_{\mathbb{R}^N} |x|^{\frac{2^*}{\sigma}} z^{2^*}(x-\xi) dx + C_2\, \varepsilon^{2\frac{2^*}{\sigma}} \int_{\mathbb{R}^N} |x|^{2\frac{2^*}{\sigma}} z^{2^*}(x-\xi) dx.
\end{aligned}
$$

By the exponential decay of z, it is easy to see that, if $|\xi| \le \overline{\xi}$, then

$$\left(\int_{\mathbb{R}^N} Q(\varepsilon x)^{\frac{2^*}{\sigma}} z_\xi^{2^*}\, dx\right)^{\frac{\sigma}{2^*}} \|v\| = O(\varepsilon)\|v\|$$

and so the lemma is proved. □

6.2 The finite dimensional reduction

In the next lemma we will show that $D^2 f_\varepsilon$ is invertible on $\left(T_{z_\xi} Z^\varepsilon\right)^\perp$, where $T_{z_\xi} Z^\varepsilon$ denotes the tangent space to Z^ε at z_ξ.

Let $L_{\varepsilon,\xi} : (T_{z_\xi} Z^\varepsilon)^\perp \to (T_{z_\xi} Z^\varepsilon)^\perp$ denote the operator defined by setting $(L_{\varepsilon,\xi} v \mid w) = D^2 f_\varepsilon(z_\xi)[v, w]$.

Lemma 6.2. *Given $\overline{\xi} > 0$, there exists $C > 0$ such that for ε small enough one has that*

$$\|L_{\varepsilon,\xi} v\| \geq C\|v\|, \qquad \forall\, |\xi| \leq \overline{\xi},\ \forall\, v \in (T_{z_\xi} Z^\varepsilon)^\perp. \tag{6.10}$$

Proof We recall that $T_{z_\xi} Z^\varepsilon = \text{span}\{\partial_{\xi_1} z_\xi, \ldots, \partial_{\xi_N} z_\xi\}$ and, moreover, by straightforward calculations, (see [11]), we get:

$$\partial_{\xi_i} z^{\varepsilon\xi}(x-\xi) = -\partial_{x_i} z^{\varepsilon\xi}(x-\xi) + O(\varepsilon). \tag{6.11}$$

Therefore, let $\mathcal{V} = \text{span}\{z_\xi,\ \partial_{x_1} z_\xi, \ldots, \partial_{x_N} z_\xi\}$, by (6.11) it suffices to prove (6.10) for all $v \in \text{span}\{z_\xi, \phi\}$, where ϕ is orthogonal to $\mathcal{V}$. Precisely we shall prove that there exist $C_1, C_2 > 0$ such that, for all $\varepsilon > 0$ small and all $|\xi| \leq \overline{\xi}$, one has:

$$(L_{\varepsilon,\xi} z_\xi \mid z_\xi) \leq -C_1 < 0, \tag{6.12}$$

$$(L_{\varepsilon,\xi} \phi \mid \phi) \geq C_2 \|\phi\|^2, \quad \text{for all } \phi \perp \mathcal{V}. \tag{6.13}$$

The proof of (6.12) follows easily from the fact that z_ξ is a Mountain Pass critical point of $F^{\varepsilon\xi}$ and so from the fact that, given $\overline{\xi}$, there exists $c_0 > 0$ such that for all $\varepsilon > 0$ small and all $|\xi| \leq \overline{\xi}$ one finds:

$$D^2 F^{\varepsilon\xi}(z_\xi)[z_\xi, z_\xi] < -c_0 < 0.$$

Indeed, arguing as in the proof of Lemma 6.1, we have

$$\begin{aligned}(L_{\varepsilon,\xi} z_\xi \mid z_\xi) = {} & D^2 F^{\varepsilon\xi}(z_\xi)[z_\xi, z_\xi] + \int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon\xi)) z_\xi^2 \, dx \\ & - p \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon\xi)) z_\xi^{p+1} \, dx - \sigma \int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^{\sigma+1} \, dx \\ < {} & -c_0 + O(\varepsilon) < -C_1.\end{aligned}$$

Let us prove (6.13). As before, the fact that z_ξ is a Mountain Pass critical point of $F^{\varepsilon\xi}$ implies that

$$D^2 F^{\varepsilon\xi}(z_\xi)[\phi, \phi] > c_1 \|\phi\|^2 \qquad \text{for all } \phi \perp \mathcal{V}. \tag{6.14}$$

Consider a radial smooth function $\chi_1 : \mathbb{R}^N \to \mathbb{R}$ such that

$$\chi_1(x) = 1, \quad \text{for } |x| \leq \varepsilon^{-1/2}; \qquad \chi_1(x) = 0, \quad \text{for } |x| \geq 2\varepsilon^{-1/2};$$

$$|\nabla\chi_1(x)| \le 2\varepsilon^{1/2}, \quad \text{for } \varepsilon^{-1/2} \le |x| \le 2\varepsilon^{-1/2}.$$

We also set $\chi_2(x) = 1 - \chi_1(x)$. Given ϕ let us consider the functions

$$\phi_i(x) = \chi_i(x-\xi)\phi(x), \quad i = 1, 2.$$

As observed in [11], we have

$$||\phi||^2 = ||\phi_1||^2 + ||\phi_2||^2 + 2\underbrace{\int_{\mathbb{R}^N} \chi_1\chi_2(\phi^2 + |\nabla\phi|^2)}_{I_\phi} + o_\varepsilon(1)||\phi||^2.$$

We need to evaluate the three terms in the equation below:

$$(L_{\varepsilon,\xi}\phi \mid \phi) = (L_{\varepsilon,\xi}\phi_1 \mid \phi_1) + (L_{\varepsilon,\xi}\phi_2 \mid \phi_2) + 2(L_{\varepsilon,\xi}\phi_1 \mid \phi_2).$$

We have:

$$(L_{\varepsilon,\xi}\phi_1 \mid \phi_1) = D^2F^{\varepsilon\xi}(z_\xi)[\phi_1,\phi_1] + \int_{\mathbb{R}^N}(V(\varepsilon x) - V(\varepsilon\xi))\phi_1^2\,dx$$
$$- p\int_{\mathbb{R}^N}(K(\varepsilon x) - K(\varepsilon\xi))z_\xi^{p-1}\phi_1^2\,dx - \sigma\int_{\mathbb{R}^N} Q(\varepsilon x) z_\xi^{\sigma-1}\phi_1^2\,dx.$$

Following [11], using (6.14) and the definition of χ_i, it is easy to see that

$$D^2F^{\varepsilon\xi}(z_\xi)[\phi_1,\phi_1] \ge c_1||\phi_1||^2 + o_\varepsilon(1)||\phi||^2$$

and

$$\Bigg|\int_{\mathbb{R}^N}(V(\varepsilon x) - V(\varepsilon\xi))\phi_1^2\,dx - p\int_{\mathbb{R}^N}(K(\varepsilon x) - K(\varepsilon\xi))z_\xi^{p-1}\phi_1^2\,dx$$
$$-\sigma\int_{\mathbb{R}^N} Q(\varepsilon x)z_\xi^{\sigma-1}\phi_1^2\,dx\Bigg| \le \varepsilon^{1/2}c_2||\phi||^2,$$

hence

$$(L_{\varepsilon,\xi}\phi_1 \mid \phi_1) \ge c_1||\phi_1||^2 - \varepsilon^{1/2}c_2||\phi||^2 + o_\varepsilon(1)||\phi||^2.$$

Analogously

$$(L_{\varepsilon,\xi}\phi_2 \mid \phi_2) \ge c_3||\phi_2||^2 + o_\varepsilon(1)||\phi||^2,$$
$$(L_{\varepsilon,\xi}\phi_1 \mid \phi_2) \ge c_4 I_\phi + o_\varepsilon(1)||\phi||^2.$$

Therefore, we get

$$(L_{\varepsilon,\xi}\phi \mid \phi) \ge c_5||\phi||^2 - c_6\varepsilon^{1/2}||\phi||^2 + o(\varepsilon)||\phi||^2.$$

This proves (6.13) and completes the proof of the lemma. □

We will show that the existence of critical points of f_ε can be reduced to the search of critical points of an auxiliary finite dimensional functional. First of all we will make a Liapunov-Schmidt reduction, and successively we will study the behavior of an auxiliary finite dimensional functional.

Lemma 6.3. *For* $\varepsilon > 0$ *small and* $|\xi| \le \overline{\xi}$ *there exists a unique* $w = w(\varepsilon, \xi) \in (T_{z_\xi} Z)^\perp$ *such that* $\nabla f_\varepsilon(z_\xi + w) \in T_{z_\xi} Z$. *Such a* $w(\varepsilon, \xi)$ *is of class* C^2, *resp.* $C^{1,p-1}$, *with respect to* ξ, *provided that* $p \ge 2$, *resp.* $1 < p < 2$. *Moreover, the functional* $\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon, \xi))$ *has the same regularity of* w *and satisfies:*

$$\nabla \Phi_\varepsilon(\xi_0) = 0 \quad \Longleftrightarrow \quad \nabla f_\varepsilon\left(z_{\xi_0} + w(\varepsilon, \xi_0)\right) = 0.$$

Proof Let $P \equiv P_{\varepsilon,\xi}$ denote the projection onto $(T_{z_\xi} Z)^\perp$. We want to find a solution $w \in (T_{z_\xi} Z)^\perp$ of the equation $P \nabla f_\varepsilon(z_\xi + w) = 0$. One has that $\nabla f_\varepsilon(z_\xi + w) = \nabla f_\varepsilon(z_\xi) + D^2 f_\varepsilon(z_\xi)[w] + R(z_\xi, w)$ with $||R(z, w)|| = o(||w||)$, uniformly with respect to z_ξ, for $|\xi| \le \overline{\xi}$. Therefore, our equation is:

$$L_{\varepsilon,\xi} w + P \nabla f_\varepsilon(z_\xi) + P R(z_\xi, w) = 0. \tag{6.15}$$

According to Lemma 6.2, this is equivalent to

$$w = N_{\varepsilon,\xi}(w), \quad \text{where} \quad N_{\varepsilon,\xi}(w) = -\left(L_{\varepsilon,\xi}\right)^{-1}\left(P \nabla f_\varepsilon(z_\xi) + P R(z_\xi, w)\right).$$

By (6.8) it follows that

$$||N_{\varepsilon,\xi}(w)|| = O(\varepsilon) + o(||w||). \tag{6.16}$$

Now the proof goes on as in Lemma 5.5. □

Now we will give two estimates on w and $\partial_{\xi_i} w$ which will be useful to study the finite dimensional functional $\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon, \xi))$.

Remark 6.1. From (6.16) it immediately follows that:

$$||w|| = O(\varepsilon). \tag{6.17}$$

Moreover, repeating the arguments of [11], if $\gamma = \min\{1, p-1\}$ and $i = 1, \dots, N$, we infer that

$$||\partial_{\xi_i} w|| = O(\varepsilon^\gamma). \tag{6.18}$$

6.3 The finite dimensional functional

Now we will use the estimates on w and $\partial_{\xi_i} w$ established in the previous section to find the expansion of $\nabla \Phi_\varepsilon(\xi)$, where $\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon, \xi))$.

Lemma 6.4. *Let* $|\xi| \le \overline{\xi}$. *Suppose* **(V)**, **(K)** *and* **(Q)**. *Then, for* ε *sufficiently small, we get:*

$$\Phi_\varepsilon(\xi) = f_\varepsilon(z_\xi + w(\varepsilon, \xi)) = \Gamma(\varepsilon\xi) + O(\varepsilon), \tag{6.19}$$

where Γ *is the auxiliary function introduced in* (6.2).

Moreover, for all $i = 1, \dots, N$, *we get:*

$$\partial_{\xi_i} \Phi_\varepsilon(\xi) = \varepsilon \partial_{\xi_i} \Gamma(\varepsilon\xi) + o(\varepsilon). \tag{6.20}$$

Proof In the sequel, to be short, we will often write w instead of $w(\varepsilon, \xi)$. It is always understood that ε is taken in such a way that all the results discussed previously hold.

Since z_ξ is a solution of (6.5), we have:

$$\begin{aligned}\Phi_\varepsilon(\xi) &= f_\varepsilon(z_\xi + w(\varepsilon, Q)) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(z_\xi + w)|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)(z_\xi + w)^2\, dx \\ &\quad -\frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon x)(z_\xi + w)^{p+1}\, dx - \frac{1}{\sigma+1}\int_{\mathbb{R}^N} Q(\varepsilon x)(z_\xi + w)^{\sigma+1}\, dx \\ &= \Sigma_\varepsilon(\xi) + \Lambda_\varepsilon(\xi) + \Theta_\varepsilon(\xi), \end{aligned} \tag{6.21}$$

where

$$\Sigma_\varepsilon(\xi) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla z_\xi|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon\xi) z_\xi^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon\xi) z_\xi^{p+1} dx, \tag{6.22}$$

$$\Theta_\varepsilon(\xi) = -\frac{1}{\sigma+1}\int_{\mathbb{R}^N} Q(\varepsilon\xi) z_\xi^{\sigma+1}\, dx \tag{6.23}$$

and

$$\begin{aligned}\Lambda_\varepsilon(\xi) &= \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon\xi)) z_\xi^2\, dx + \int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon\xi)) z_\xi w\, dx \\ &\quad -\frac{1}{p+1}\int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon\xi)) z_\xi^{p+1}\, dx - \int_{\mathbb{R}^N} (K(\varepsilon x) - K(\varepsilon\xi)) z_\xi^p w\, dx \\ &\quad -\frac{1}{\sigma+1}\int_{\mathbb{R}^N} (Q(\varepsilon x) - Q(\varepsilon\xi)) z_\xi^{\sigma+1}\, dx - \int_{\mathbb{R}^N} (Q(\varepsilon x) - Q(\varepsilon\xi)) z_\xi^\sigma w\, dx \\ &\quad +\frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x) w^2\, dx \\ &\quad -\frac{1}{p+1}\int_{\mathbb{R}^N} K(\varepsilon x)\left[(z_\xi + w)^{p+1} - z_\xi^{p+1} - (p+1) z_\xi^p w\, dx\right] \\ &\quad -\frac{1}{\sigma+1}\int_{\mathbb{R}^N} Q(\varepsilon x)\left[(z_\xi + w)^{\sigma+1} - z_\xi^{\sigma+1} - (\sigma+1) z_\xi^\sigma w\, dx\right] \\ &\quad -\int_{\mathbb{R}^N} Q(\varepsilon\xi) z_\xi^\sigma w\, dx. \end{aligned}$$

Let us observe that, since, $Q(0) = 0$, arguing as in the proof of Lemma 6.1 and recalling (6.17), we get

$$\left|\int_{\mathbb{R}^N} Q(\varepsilon\xi) z_\xi^\sigma w\, dx\right| \le \left(\int_{\mathbb{R}^N} Q(\varepsilon\xi)^{\frac{2^*}{\sigma}} z_\xi^{2^*}\, dx\right)^{\frac{\sigma}{2^*}} \|w\| = o(\varepsilon).$$

By this and with easy calculations, see also [11], we infer

$$\Lambda_\varepsilon(\xi) = O(\varepsilon). \tag{6.24}$$

Moreover, since z_ξ is solution of (6.5), we get

$$\Sigma_\varepsilon(\xi) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} K(\varepsilon\xi) z_\xi^{p+1}\, dx.$$

By (6.7), we have

$$\int_{\mathbb{R}^N} K(\varepsilon\xi) z_\xi^{p+1}\, dx = V(\varepsilon\xi)^{\frac{p+1}{p-1}-\frac{N}{2}} K(\varepsilon\xi)^{-\frac{2}{p-1}} \int_{\mathbb{R}^N} U^{p+1}.$$
$$\int_{\mathbb{R}^N} Q(\varepsilon\xi) z_\xi^{\sigma+1}\, dx = Q(\varepsilon\xi) V(\varepsilon\xi)^{\frac{\sigma+1}{p-1}-\frac{N}{2}} K(\varepsilon\xi)^{-\frac{\sigma+1}{p-1}} \int_{\mathbb{R}^N} U^{\sigma+1}.$$

By these two equations and by (6.21), (6.22), (6.23) and (6.24) we prove the first part of the lemma.

Let us prove now the estimate on the derivatives of Φ_ε.

It is easy to see that

$$\nabla \Theta_\varepsilon(\xi) = o(\varepsilon).$$

With calculations similar to those of [11], we infer that

$$\nabla \Lambda_\varepsilon(\xi) = o(\varepsilon),$$

and so (6.20) follows immediately. □

6.4 Proof of Theorem 6.1

In this section we will give two multiplicity results. Theorem 6.1 will follow from those as a particular case.

Theorem 6.2. *Let* **(V)**, **(K)** *and* **(Q)** *hold. Suppose Γ has a nondegenerate smooth manifold of critical points M. Then for $\varepsilon > 0$ small,* (6.1) *has at least $l(M)$ solutions that concentrate near points of M. Here $l(M)$ denotes the cup long of M (see Definition 5.1).*

Proof First of all, we fix $\overline{\xi}$ in such a way that $|x| < \overline{\xi}$ for all $x \in M$. We will apply the finite dimensional procedure with such $\overline{\xi}$ fixed.

Fix a δ-neighborhood M_δ of M such that $M_\delta \subset \{|x| < \overline{\xi}\}$ and the only critical points of Γ in M_δ are those in M. We will take $U = M_\delta$.

By (6.19) and (6.20), $\Phi_\varepsilon(\cdot/\varepsilon)$ converges to $\Gamma(\cdot)$ in $C^1(\bar{U})$ and so, by Theorem 5.9, we have at least $l(M)$ critical points of l provided ε sufficiently small.

Let ξ be one of these critical points of Ψ_ε, then $u_\varepsilon^\xi = z_\xi + w(\varepsilon, \xi)$ is a solution of (6.4) and so

$$u_\varepsilon^\xi(x/\varepsilon) \simeq z_\xi(x/\varepsilon) = z^{\varepsilon\xi}\left(\frac{x-\xi}{\varepsilon}\right)$$

is a solution of (6.1) and concentrates on ξ. □

Moreover, when we deal with local minima (resp. maxima) of Γ, the preceding results can be improved because the number of positive solutions of (6.1) can be estimated by means of the category and M does not need to be a manifold.

Theorem 6.3. *Let* **(V)**, **(K)** *and* **(Q)** *hold and suppose Γ has a compact set X where Γ achieves a strict local minimum (resp. maximum), in the sense that there exists $\delta > 0$ and a δ-neighborhood X_δ of X such that*

$$b \equiv \inf\{\Gamma(x) : x \in \partial X_\delta\} > a \equiv \Gamma_{|X}, \quad (\text{resp. } \sup\{\Gamma(x) : x \in \partial X_\delta\} < a)\,.$$

Then there exists $\varepsilon_\delta > 0$ such that (6.1) *has at least* $\operatorname{cat}(X, X_\delta)$ *solutions that concentrate near points of X_δ, provided $\varepsilon \in (0, \varepsilon_\delta)$.*

Proof We will treat only the case of minima, being the other one similar. Fix again $\bar\xi$ in such a way that X_δ is contained in $\{x \in \mathbb{R}^N : |x| < \bar\xi\}$. We set $X^\varepsilon = \{\xi : \varepsilon\xi \in X\}$, $X_\delta^\varepsilon = \{\xi : \varepsilon\xi \in X_\delta\}$ and $Y^\varepsilon = \{\xi \in X_\delta^\varepsilon : \Phi_\varepsilon(\xi) \le (a+b)/2\}$. By (6.19) it follows that there exists $\varepsilon_\delta > 0$ such that

$$X^\varepsilon \subset Y^\varepsilon \subset X_\delta^\varepsilon, \tag{6.25}$$

provided $\varepsilon \in (0, \varepsilon_\delta)$. Moreover, if $\xi \in \partial X_\delta^\varepsilon$ then $\Gamma(\varepsilon\xi) \ge b$ and hence

$$\Phi_\varepsilon(\xi) \ge \Gamma(\varepsilon\xi) + O(\varepsilon) \ge b + o_\varepsilon(1).$$

On the other side, if $\xi \in Y^\varepsilon$ then $\Phi_\varepsilon(\xi) \le (a+b)/2$. Hence, for ε small, Y^ε cannot meet $\partial X_\delta^\varepsilon$ and this readily implies that Y^ε is compact. Then Φ_ε possesses at least $\operatorname{cat}(Y^\varepsilon, X_\delta^\varepsilon)$ critical points in X_δ. Using (6.25) and the properties of the category one gets

$$\operatorname{cat}(Y^\varepsilon, Y^\varepsilon) \ge \operatorname{cat}(X^\varepsilon, X_\delta^\varepsilon) = \operatorname{cat}(X, X_\delta).$$

The concentration statement follows as before. □

Remark 6.2. Let us observe that the (a) of Theorem 6.1 is a particular case of Theorem 6.2, while the (b) is a particular case of Theorem 6.3.

Part III

Singularly perturbed Neumann problems

7 Singularly perturbed Neumann problems

For singularly perturbed Neumann problem, we mean

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_ε}$$

with Ω is a smooth domain of $\mathbb{R}^N$ and $1 < p < 2^* - 1$, where we recall that 2^* is the critical exponent for the Sobolev embeddings:

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N < 3. \end{cases}$$

This equation is known as the stationary equation of the Keller-Segel system in chemotaxis, that is the oriented movement of cells in response to chemicals in their environment. Cellular slime molds (amoebae) release a certain chemical, move toward places of its higher concentration and eventually form aggregates. Keller and Segel, in [47], proposed a model to describe the chemotactic aggregation stage of cellular slime molds. Let $u(x,t)$ be the population of amoebae at place x and at time t and $v(x,t)$ the concentration of the chemical. Then the simplified Keller-Segel system is written as

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u - \chi \nabla \cdot (u \nabla \phi(v)) & \text{in } \Omega \times \mathbb{R}_+, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + k(u,v) & \text{in } \Omega \times \mathbb{R}_+, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{in } \partial\Omega \times \mathbb{R}_+, \\ u(x,0) > 0 & \text{in } \Omega, \\ v(x,0) > 0 & \text{in } \Omega, \end{cases} \tag{7.1}$$

where d_1, d_2 and χ are positive constants, ϕ is a smooth function such that $\phi'(r) > 0$ for $r > 0$, k is a smooth function with $\frac{\partial k}{\partial u} \geq 0$ and $\frac{\partial k}{\partial v} \leq 0$ and ν

denotes the outer unit normal to $\partial\Omega$. In [52], Li, Ni and Takagi have studied the existence of stationary solutions of (7.1) in the case $\phi(v) = \log v$ and $k(u,v) = -av + bu$, where a and b are positive constants. Since $\int_\Omega u(x,t)dx = \int_\Omega u(x,0)dx$, for all $t > 0$, by virtue of the first equation of (7.1) and of the Dirichlet boundary condition, then the problem becomes:

$$\begin{cases} d_1\Delta u - \chi\nabla\cdot(u\nabla\log v) = 0 & \text{in } \Omega, \\ d_2\Delta v - av + bu = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0 & \text{in } \partial\Omega, \\ u > 0 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ |\Omega|^{-1}\displaystyle\int_\Omega u(x)dx = \bar{u}. \end{cases} \tag{7.2}$$

Solving (7.2) is reduced to finding solutions to a single equation. Indeed writing the first equation of (7.2) as $\nabla\cdot\{d_1 u\nabla[\log u - \chi d_1^{-1}\log v]\} = 0$ and using the Dirichlet boundary condition, we have that $u = \lambda v^{\chi/d_1}$, for some positive constant λ. Therefore (7.2) is equivalent to the following system for (v,λ):

$$\begin{cases} d_2\Delta v - av + b\lambda v^{\chi/d_1} = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial\nu} = 0 & \text{in } \partial\Omega, \\ v > 0 & \text{in } \Omega, \\ |\Omega|^{-1}\displaystyle\int_\Omega v(x)dx = \bar{v}. \end{cases} \tag{7.3}$$

Finally, putting $p = \chi/d_1$, $\varepsilon^2 = d_2/a$, $\mu = (a^{-1}b\lambda)^{1/(p-1)}$ and $w(x) = \mu v(x)$, we need to find the positive solution of

$$\begin{cases} \varepsilon^2\Delta w - w + w^p = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial\nu} = 0 & \text{in } \partial\Omega. \end{cases}$$

Moreover equation (P_ε) can also be seen as a simplified model for the description of steady state solutions of the Gierer-Meinhardt system in biological pattern formation, namely

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u - u + \dfrac{u^p}{v^q} & \text{in } \Omega \times \mathbb{R}_+ \\ \tau \dfrac{\partial v}{\partial t} = d_2 \Delta v - v + \dfrac{u^r}{v^s} & \text{in } \Omega \times \mathbb{R}_+ \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{in } \partial\Omega \times \mathbb{R}_+, \end{cases} \tag{7.4}$$

where d_1, d_2, p, q, r and τ are positive constants and s is nonnegative. The function u represents the density of an activator that is slowly diffusing (d_1 is small), while v represents the density of an inhibitor, which is rapidly diffusing (d_2 is large). See [63], for more details. For biological aspects, see Meinhardt [62] or references in [89].

Equation (P_ε) was intensively studied in several works. For example, Ni & Takagi, in [64, 65], show that, for ε sufficiently small, there exists a solution u_ε of (P_ε) which concentrates in a point $Q_\varepsilon \in \partial\Omega$ and $H(Q_\varepsilon) \to \max_{\partial\Omega} H$, here H denotes the mean curvature of $\partial\Omega$. Moreover in [51], using the Liapunov-Schmidt reduction, Li constructs solutions with single peak and multi-peaks on $\partial\Omega$ located near any stable critical points of H. Since the publication of [64, 65], there have been many works on spike-layer solutions of (P_ε), see for example [28, 35, 40, 41, 92] and references therein.

Many authors studied also the existence of solutions of (P_ε) which concentrate not on the boundary $\partial\Omega$ but inside Ω. For example interior spikes have been found by Wei (see [94]) showing that concentration occurs at local maxima of the distance function $\text{dist}(\cdot, \partial\Omega)$. See also [20, 42, 93].

We mention the paper of Gui and Wei [43], where they prove that for any given pair of nonnegative integers k and l, problem (P_ε) has a solution concentrating on k interior points and l boundary points.

In the same spirit of Chapter 5, in Chapter 8 and in Chapter 9 (see also [71] and a joint work with Simone Secchi [75]) we study the following equation:

$$\begin{cases} -\varepsilon^2 \operatorname{div}(J(x)\nabla u) + V(x)u = u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.5}$$

where $J : \mathbb{R}^N \to \mathbb{R}$ and $V : \mathbb{R}^N \to \mathbb{R}$ are C^2 strictly positive functions. Let us observe that if $J \equiv 1$ and $V \equiv 1$, then (7.5) becomes (P_ε).

In Chapter 8 we prove that for the existence of concentrating solutions, one has to check if at least one between J and V is not constant on $\partial\Omega$. In this case the concentration point is determined by J and V through the function $\Gamma_1 : \partial\Omega \to \mathbb{R}$ defined as

$$\Gamma_1(Q) \equiv V(Q)^{\frac{p+1}{p-1} - \frac{N}{2}} J(Q)^{\frac{N}{2}}.$$

If x_0 is a non-degenerate critical point of Γ_1 or an isolated local strict minimum or maximum of Γ_1, then there exists a solution u_ε concentrating near x_0.

In the other case the concentration point is determined by an interplay among the derivatives of J and V calculated on $\partial\Omega$ and the mean curvature H. More precisely, if J and V (and so also Γ_1) are constant on the boundary, then the concentration phenomena are due by another auxiliary function which depends on the derivatives of J and V on the boundary and by the mean curvature H, that is $\bar{\Sigma}\colon \partial\Omega \to \mathbb{R}$ the function so defined:

$$\bar{\Sigma}(Q) \equiv k_1 \int_{\mathbb{R}^-_{\nu(Q)}} J'(Q)[x] \left|(\nabla \bar{U})(k_2 x)\right|^2 dx \\ + k_3 \int_{\mathbb{R}^-_{\nu(Q)}} V'(Q)[x] \left[\bar{U}(k_2 x)\right]^2 dx - k_4 H(Q),$$

where $\bar{U}$ is the unique solution of

$$\begin{cases} -\Delta \bar{U} + \bar{U} = \bar{U}^p & \text{in } \mathbb{R}^N, \\ \bar{U} > 0 & \text{in } \mathbb{R}^N, \\ \bar{U}(0) = \max_{\mathbb{R}^N} \bar{U}, \end{cases}$$

$\nu(Q)$ is the outer normal in Q at Ω,

$$\mathbb{R}^-_{\nu(Q)} \equiv \left\{x \in \mathbb{R}^N : x \cdot \nu(Q) \leq 0\right\},$$

and, for $i = 1, \ldots, 4$, k_i are constants which depend only on J and V and not on Q.

In Chapter 9, instead, we study the existence of interior spikes of (7.5) and we prove that there exist solutions concentrating near non-degenerate critical points and near minima and maxima of an auxiliary functional $\Gamma_2\colon \Omega \to \mathbb{R}$ be a function so defined:

$$\Gamma_2(Q) \equiv V(Q)^{\frac{p+1}{p-1} - \frac{N}{2}} J(Q)^{\frac{N}{2}}.$$

Let us observe that Γ_2 is similar to Γ_1 but it is defined in the whole Ω and not on the boundary $\partial\Omega$.

In all these results we have solutions of (P_ε) concentrating at a point and so the concentration set is zero-dimensional. In the last years it has been proved the existence of solutions which concentrate on a k-dimensional subset of $\bar{\Omega}$, with $1 \leq k \leq N-1$, see [58, 59, 60] and [10, 61] which deal with the radial case.

8 Neumann problems with potentials: boundary peak solutions

In this chapter we study the following problem:

$$\begin{cases} -\varepsilon^2 \operatorname{div}(J(x)\nabla u) + V(x)u = u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.1}$$

where Ω is a smooth bounded domain with external normal ν, $N \geq 3$, $1 < p < (N+2)/(N-2)$, $J\colon \mathbb{R}^N \to \mathbb{R}$ and $V\colon \mathbb{R}^N \to \mathbb{R}$ are C^2 functions.

When $J \equiv 1$ and $V \equiv 1$, then (8.1) becomes

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.2}$$

and, as observed in the previous chapter, such a problem was intensively studied in several works showing the existence of solutions concentrating in minima and maxima of the mean curvature H of $\partial\Omega$.

What happens in presence of potentials J and V?

In this chapter we try to give an answer to this question and we will show that, for the existence of concentrating solutions, one has to check if at least one between J and V is not constant on $\partial\Omega$. In this case the concentration point is determined by J and V only. In the other case the concentration point is determined by an interplay among the derivatives of J and V calculated on $\partial\Omega$ and the mean curvature H.

On J and V we will do the following assumptions:

(J) $J \in C^2(\Omega, \mathbb{R})$, J and $D^2 J$ are bounded; moreover,

$$J(x) \geq C > 0 \quad \text{for all } x \in \Omega;$$

(V) $V \in C^2(\Omega, \mathbb{R})$, V and $D^2 V$ are bounded; moreover,

$$V(x) \geq C > 0 \quad \text{for all } x \in \Omega.$$

Let us introduce an auxiliary function which will play a crucial rôle in the study of (8.1). Let $\Gamma\colon \partial\Omega \to \mathbb{R}$ be a function so defined:

$$\Gamma(Q) = V(Q)^{\frac{p+1}{p-1}-\frac{N}{2}} J(Q)^{\frac{N}{2}}. \tag{8.3}$$

Let us observe that by **(J)** and **(V)**, Γ is well defined.

Our first result is:

Theorem 8.1. *Let $Q_0 \in \partial\Omega$. Suppose* **(J)** *and* **(V)**. *There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then* (8.1) *possesses a solution u_ε which concentrates in Q_ε with $Q_\varepsilon \to Q_0$, as $\varepsilon \to 0$, provided that one of the two following conditions holds:*

(a) Q_0 is a non-degenerate critical point of Γ;
(b) Q_0 is an isolated local strict minimum or maximum of Γ.

Hence, if J and V are not constant on the boundary $\partial\Omega$, the concentration phenomena depend only by J and V and not by the mean curvature H. Our second result deals with the other case and, more precisely, we will show that, if J and V (and so also Γ) are constant on the boundary, then the concentration phenomena are due by another auxiliary function which depends on the derivatives of J and V on the boundary and by the mean curvature H. Let $\bar{\Sigma}\colon \partial\Omega \to \mathbb{R}$ be the function so defined:

$$\bar{\Sigma}(Q) \equiv k_1 \int_{\mathbb{R}^-_{\nu(Q)}} J'(Q)[x] \left|(\nabla\bar{U})(k_2 x)\right|^2 dx + k_3 \int_{\mathbb{R}^-_{\nu(Q)}} V'(Q)[x] \left[\bar{U}(k_2 x)\right]^2 dx - k_4 H(Q), \tag{8.4}$$

where $\bar{U}$ is the unique solution of

$$\begin{cases} -\Delta\bar{U} + \bar{U} = \bar{U}^p & \text{in } \mathbb{R}^N, \\ \bar{U} > 0 & \text{in } \mathbb{R}^N, \\ \bar{U}(0) = \max_{\mathbb{R}^N} \bar{U}, \end{cases}$$

$\nu(Q)$ is the outer normal in Q at Ω,

$$\mathbb{R}^-_{\nu(Q)} \equiv \left\{x \in \mathbb{R}^N : x \cdot \nu(Q) \le 0\right\},$$

and, for $i = 1, \ldots, 4$, k_i are constants which depend only on J and V and not on Q (see Remark 8.3 for an explicit formula).

Our second result is:

Theorem 8.2. *Suppose* **(J)** *and* **(V)** *with J and V constant on the boundary $\partial\Omega$. Let $Q_0 \in \partial\Omega$ be an isolated local strict minimum or maximum of $\bar{\Sigma}$. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then* (8.1) *possesses a solution u_ε which concentrates in Q_ε with $Q_\varepsilon \to Q_0$, as $\varepsilon \to 0$.*

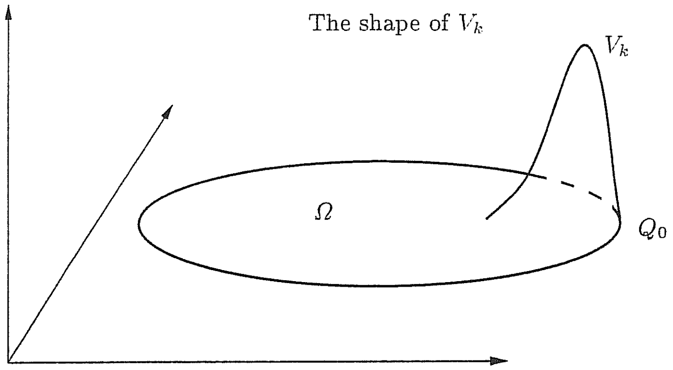

Figure 2

Example 8.1. Suppose that $J \equiv 1$ and fix any $Q_0 \in \partial\Omega$. For $k \in \mathbb{N}$, let V_k be a bounded smooth function constantly equal to 1 on the $\partial\Omega$ and in the whole Ω, except a little ball tangent at $\partial\Omega$ in Q_0, with $\nabla V_k(Q_0) = -k\nu(Q_0)$ (see Figure 2).
It is easy to see that, outside a little neighborhood of Q_0 in $\partial\Omega$, we have

$$\bar{\Sigma}(Q) = -C_1 H(Q),$$

while

$$\bar{\Sigma}(Q_0) = -C_1 H(Q_0) + kC_2,$$

where

$$C_1 = \frac{1}{2}\bar{B} + \left(\frac{1}{2} - \frac{1}{p+1}\right)\bar{A},$$
$$C_2 = -\frac{1}{2}\int_{\{\nu(Q_0)\cdot x \leq 0\}} \nu(Q_0)\cdot x\, \bar{U}^2 dx.$$

Since $C_2 > 0$, we can choose $k \gg 1$ such that Q_0 is the absolute maximum point for $\bar{\Sigma}$ and hence there exists a solution concentrating at Q_0.

Theorem 8.1 will be proved as a particular case of two multiplicity results in Section 8.5, where we will prove also Theorem 8.2. The proof of the theorems relies on a finite dimensional reduction, precisely on the perturbation technique developed in [4, 6, 11]. In Section 8.1 we give some preliminary lemmas and some estimates which will be useful in Section 8.2 and Section 8.3, where we perform the Liapunov-Schmidt reduction, and in Section 8.4, where we make the asymptotic expansion of the finite dimensional functional.

Finally we recall that problem (8.1), but with the Dirichlet boundary conditions, is studied in Chapter 5 (see also a joint work with Simone Secchi [74]), where we show that there are solutions which concentrate in minima of an auxiliary function, which depends only on J and V.

All the results in this chapter are contained in [71].

Notation

- $\mathbb{R}^N_+ \equiv \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\}$.
- If $\mu \in \mathbb{R}^N$, then $\mathbb{R}^-_\mu \equiv \{x \in \mathbb{R}^N : x \cdot \mu \le 0\}$, where with $x \cdot \mu$ we denote the scalar product in $\mathbb{R}^N$ between x and μ.
- If $r > 0$ and $x_0 \in \mathbb{R}^N$, $B_r(x_0) \equiv \{x \in \mathbb{R}^N : |x - x_0| < r\}$. We denote with B_r the ball of radius r centered in the origin.
- If $u\colon \mathbb{R}^N \to \mathbb{R}$ and $P \in \mathbb{R}^N$, we set $u_P \equiv u(\cdot - P)$.
- If U^Q is the function defined in (8.6), when there is no misunderstanding, we will often write U instead of U^Q. Moreover if $P = Q/\varepsilon$, then $U_P \equiv U^Q(\cdot - P)$.
- If $Q \in \partial\Omega$, we denote with $\nu(Q)$ the outer normal in Q at Ω and with $H(Q)$ the mean curvature of $\partial\Omega$ in Q.
- If $\varepsilon > 0$, we set $\Omega_\varepsilon \equiv \Omega/\varepsilon \equiv \{x \in \mathbb{R}^N : \varepsilon x \in \Omega\}$.
- We denote with $\|\cdot\|$ and with $(\cdot \mid \cdot)$ respectively the norm and the scalar product of $H^1(\Omega_\varepsilon)$. While we denote with $\|\cdot\|_+$ and with $(\cdot \mid \cdot)_+$ respectively the norm and the scalar product of $H^1(\mathbb{R}^N_+)$.
- If $P \in \partial\Omega_\varepsilon$, we set $\partial_{P_i} \equiv \frac{\partial}{\partial e_i}$, where $\{e_1, \ldots, e_{N-1}\}$ is an orthonormal basis of $T_P(\partial\Omega_\varepsilon)$. Analogously, if $Q \in \partial\Omega$, we set $\partial_{Q_i} \equiv \frac{\partial}{\partial \tilde{e}_i}$, where $\{\tilde{e}_1, \ldots, \tilde{e}_{N-1}\}$ is an orthonormal basis of $T_Q(\partial\Omega)$.

8.1 Preliminary lemmas and some estimates

First of all we perform the change of variable $x \mapsto \varepsilon x$ and so problem (8.1) becomes

$$\begin{cases} -\operatorname{div}(J(\varepsilon x)\nabla u) + V(\varepsilon x)u = u^p & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{8.5}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. Of course if u is a solution of (8.5), then $u(\cdot/\varepsilon)$ is a solution of (8.1).

Solutions of (8.5) are critical points $u \in H^1(\Omega_\varepsilon)$ of

$$f_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} J(\varepsilon x)|\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_\varepsilon} V(\varepsilon x)u^2 dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} |u|^{p+1}.$$

The solutions of (8.5) will be found near a U^Q, the unique solution of

$$\begin{cases} -J(Q)\Delta u + V(Q)u = u^p & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(0) = \max_{\mathbb{R}^N} u, \end{cases}$$

for an appropriate choice of $Q \in \partial\Omega$. It is easy to see that

$$U^Q(x) = V(Q)^{\frac{1}{p-1}}\,\bar{U}\left(x\sqrt{V(Q)/J(Q)}\right), \tag{8.6}$$

where $\bar{U}$ is the unique solution of

$$\begin{cases} -\Delta\bar{U} + \bar{U} = \bar{U}^p & \text{in } \mathbb{R}^N, \\ \bar{U} > 0 & \text{in } \mathbb{R}^N, \\ \bar{U}(0) = \max_{\mathbb{R}^N}\bar{U}, \end{cases}$$

which is radially symmetric and decays exponentially at infinity with its derivatives.

We remark that U^Q is a solution also of the "problem to infinity":

$$\begin{cases} -J(Q)\Delta u + V(Q)u = u^p & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{8.7}$$

The solutions of (8.7) are critical points of the functional defined on $H^1(\mathbb{R}^N_+)$

$$F^Q(u) = \frac{1}{2}J(Q)\int_{\mathbb{R}^N_+}|\nabla u|^2 + \frac{1}{2}V(Q)\int_{\mathbb{R}^N_+}u^2 - \frac{1}{p+1}\int_{\mathbb{R}^N_+}|u|^{p+1}. \tag{8.8}$$

We recall that we will often write U instead of U^Q. If $P = \varepsilon^{-1}Q \in \partial\Omega_\varepsilon$, we set $U_P \equiv U^Q(\cdot - P)$ and

$$Z^\varepsilon \equiv \{U_P : P \in \partial\Omega_\varepsilon\}.$$

Lemma 8.1. *For all $Q \in \partial\Omega$ and for all ε sufficiently small, if $P = Q/\varepsilon \in \partial\Omega_\varepsilon$, then*

$$\|\nabla f_\varepsilon(U_P)\| = O(\varepsilon). \tag{8.9}$$

Proof Let $v \in H^1(\Omega_\varepsilon)$. We have:

$$\begin{aligned}
(\nabla f_\varepsilon(U_P) \mid v) &= \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla U_P \cdot \nabla v + \int_{\Omega_\varepsilon} V(\varepsilon x)U_P v - \int_{\Omega_\varepsilon} U_P^p v \\
&= \int_{\frac{\Omega-Q}{\varepsilon}} J(\varepsilon x + Q)\nabla U \cdot \nabla v_{-P} + \int_{\frac{\Omega-Q}{\varepsilon}} V(\varepsilon x + Q)U v_{-P} \\
&\quad - \int_{\frac{\Omega-Q}{\varepsilon}} U^p v_{-P} \\
&= \int_{\frac{\Omega-Q}{\varepsilon}} J(Q)\nabla U \cdot \nabla v_{-P} + \int_{\frac{\Omega-Q}{\varepsilon}} V(Q)U v_{-P} - \int_{\frac{\Omega-Q}{\varepsilon}} U^p v_{-P} \\
&\quad + \int_{\frac{\Omega-Q}{\varepsilon}} (J(\varepsilon x + Q) - J(Q))\nabla U \cdot \nabla v_{-P} \\
&\quad + \int_{\frac{\Omega-Q}{\varepsilon}} (V(\varepsilon x + Q) - V(Q))U v_{-P}
\end{aligned}$$

$$= \int_{\frac{\Omega-Q}{\varepsilon}} [-J(Q)\Delta U + V(Q)U - U^p]\, v_{-P} + J(Q)\int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial\nu} v$$
$$+ \int_{\frac{\Omega-Q}{\varepsilon}} (J(\varepsilon x+Q) - J(Q))\nabla U\cdot\nabla v_{-P}$$
$$+ \int_{\frac{\Omega-Q}{\varepsilon}} (V(\varepsilon x+Q) - V(Q))U v_{-P}.$$

Hence, since $U\equiv U^Q$ is solution of (8.7), we get

$$(\nabla f_\varepsilon(U_P)\mid v) = J(Q)\int_{\partial\Omega_\varepsilon}\frac{\partial U_P}{\partial\nu}v + \int_{\frac{\Omega-Q}{\varepsilon}}(J(\varepsilon x+Q)-J(Q))\nabla U\cdot\nabla v_{-P}$$
$$+ \int_{\frac{\Omega-Q}{\varepsilon}}(V(\varepsilon x+Q)-V(Q))Uv_{-P}. \quad (8.10)$$

Let us estimate the first of these three terms:

$$\left|J(Q)\int_{\partial\Omega_\varepsilon}\frac{\partial U_P}{\partial\nu}v\right| \le C\|v\|_{L^2(\partial\Omega_\varepsilon)}\left(\int_{\partial\Omega_\varepsilon}\left|\frac{\partial U_P}{\partial\nu}\right|^2\right)^{1/2}.$$

First of all, we observe that there exist $\varepsilon_0>0$ and $C>0$ such that, for all $\varepsilon\in(0,\varepsilon_0)$ and for all $v\in H^1(\Omega_\varepsilon)$, we have

$$\|v\|_{L^2(\partial\Omega_\varepsilon)} \le C\|v\|_{H^1(\Omega_\varepsilon)}.$$

Moreover, after making a translation and rotation, we can assume that Q coincides with the origin $\mathcal{O}$ and that part of $\partial\Omega$ is given by $x_N=\psi(x')=\frac{1}{2}\sum_{i=1}^{N-1}\lambda_i x_i^2 + O(|x'|^3)$ for $|x'|<\mu$, where μ is some constant depending only on Ω. Then for $|y'|<\mu/\varepsilon$, the corresponding part of $\partial\Omega_\varepsilon$ is given by $y_N=\Psi(y')=\varepsilon^{-1}\psi(\varepsilon y')=\frac{\varepsilon}{2}\sum_{i=1}^{N-1}\lambda_i y_i^2+O(\varepsilon^2|y'|^3)$. Then it is easy to see that

$$\frac{\partial U}{\partial\nu}(y',\Psi(y')) = \varepsilon\left[\sum_{i=1}^{N-1}\lambda_i y_i\frac{\partial U}{\partial y_i}(y',0) - \frac{1}{2}\frac{\partial^2 U}{\partial y_N^2}(y',0)\sum_{i=1}^{N-1}\lambda_i y_i^2\right] + O(\varepsilon^2).$$

Let us observe that by the exponential decay of U and of its derivatives, we get:

$$\int_{\partial\tilde\Omega_\varepsilon}\left|\frac{\partial U}{\partial\nu}\right|^2 = \varepsilon^2\int_{\partial\tilde\Omega_\varepsilon}\left[\sum_{i=1}^{N-1}\lambda_i y_i\frac{\partial U}{\partial y_i}(y',0) - \frac{1}{2}\frac{\partial^2 U}{\partial y_N^2}(y',0)\sum_{i=1}^{N-1}\lambda_i y_i^2\right]^2 + o(\varepsilon^2)$$
$$= O(\varepsilon^2),$$

where $\partial\tilde\Omega_\varepsilon\equiv\partial\Omega_\varepsilon\cap B_{\varepsilon^{-1/2}}$. Therefore

$$\left(\int_{\partial\Omega_\varepsilon}\left|\frac{\partial U}{\partial\nu}\right|^2\right)^{1/2}=\left(\int_{\partial\Omega_\varepsilon\cap B_{\varepsilon^{-1/2}}}\left|\frac{\partial U}{\partial\nu}\right|^2\right)^{1/2}+o(\varepsilon)=O(\varepsilon). \tag{8.11}$$

Let us calculate the second term of (8.10). We start observing that, from the assumption D^2J bounded, we infer that

$$|J(\varepsilon x+Q)-J(Q)|\le\varepsilon|J'(Q)||x|+c_1\varepsilon^2|x|^2,$$

and so, using again the exponential decay of U and of its derivatives,

$$\begin{aligned}\int_{\frac{\Omega-Q}{\varepsilon}}&(J(\varepsilon x+Q)-J(Q))\nabla U\cdot\nabla v_{-P}\\&\le\|v\|\left(\int_{\frac{\Omega-Q}{\varepsilon}}|J(\varepsilon x+Q)-J(Q)|^2|\nabla U|^2\right)^{1/2}\\&\le c_2\|v\|\left[\int_{\mathbb{R}^N_+}\varepsilon^2|J'(Q)|^2|x|^2|\nabla U|^4+\int_{\mathbb{R}^N_+}\varepsilon^4|x|^4|\nabla U|^4\right]^{1/2}\\&=O(\varepsilon)\|v\|. \end{aligned}\tag{8.12}$$

Analogously, we can say that:

$$\int_{\frac{\Omega-Q}{\varepsilon}}(V(\varepsilon x+Q)-V(Q))Uv_{-P}=O(\varepsilon)\|v\|. \tag{8.13}$$

Now the conclusion follows immediately by (8.10), (8.11), (8.12) and (8.13). □

We here present some useful estimates that will be used in the sequel.

Proposition 8.1. *Let $P=Q/\varepsilon\in\partial\Omega_\varepsilon$. Then we have:*

$$\int_{\Omega_\varepsilon}U_P^{p+1}=\int_{\mathbb{R}^N_+}(U^Q)^{p+1}-\varepsilon\frac{H(Q)}{2}\int_{\mathbb{R}^{N-1}}\left[U^Q(y',0)\right]^{p+1}|y'|^2dy'+o(\varepsilon), \tag{8.14}$$

$$\int_{\partial\Omega_\varepsilon}\frac{\partial U_P}{\partial\nu}U_P=-\varepsilon\frac{(N-1)H(Q)}{4}\int_{\mathbb{R}^{N-1}}\left[U^Q(y',0)\right]^2dy'+o(\varepsilon), \tag{8.15}$$

$$\begin{aligned}J(Q)\int_{\Omega_\varepsilon}|\nabla U_P|^2+V(Q)\int_{\Omega_\varepsilon}U_P^2\\=\int_{\mathbb{R}^N_+}(U^Q)^{p+1}-\varepsilon\frac{H(Q)}{2}\int_{\mathbb{R}^{N-1}}\left[U^Q(y',0)\right]^{p+1}|y'|^2dy'\\-\varepsilon J(Q)\frac{(N-1)H(Q)}{4}\int_{\mathbb{R}^{N-1}}\left[U^Q(y',0)\right]^2dy'+o(\varepsilon),\end{aligned}\tag{8.16}$$

$$\int_{\Omega_\varepsilon} J(\varepsilon x)|\nabla U_P|^2 = J(Q)\int_{\Omega_\varepsilon}|\nabla U_P|^2 + \varepsilon\int_{\mathbb{R}^-_{\nu(Q)}} J'(Q)[x]|\nabla U^Q|^2 + o(\varepsilon), \tag{8.17}$$

$$\int_{\Omega_\varepsilon} V(\varepsilon x)U_P^2 = V(Q)\int_{\Omega_\varepsilon} U_P^2 + \varepsilon\int_{\mathbb{R}^-_{\nu(Q)}} V'(Q)[x]\left(U^Q\right)^2 + o(\varepsilon). \tag{8.18}$$

Moreover, we have

$$\int_{\Omega_\varepsilon} U_P^p\, \partial_{P_i} U_P = \varepsilon\frac{1}{p+1}\bar{C}\partial_{Q_i}\Gamma(Q) + o(\varepsilon), \tag{8.19}$$

$$\partial_{P_i}\left[J(Q)\int_{\Omega_\varepsilon}|\nabla U_P|^2 + V(Q)\int_{\Omega_\varepsilon} U_P^2\right] = \varepsilon\bar{C}\partial_{Q_i}\Gamma(Q) + o(\varepsilon). \tag{8.20}$$

where $\bar{C} = \int_{\mathbb{R}^N_+}\bar{U}^{p+1}$ *and* Γ *is defined in* (8.3).

Proof The first two formulas can be proved repeating the arguments of [51, Lemma 1.2]. Equation (8.16) follows easily by (8.14) and (8.15) observing that

$$J(Q)\int_{\Omega_\varepsilon}|\nabla U_P|^2 + V(Q)\int_{\Omega_\varepsilon} U_P^2 = \int_{\Omega_\varepsilon} U_P^{p+1} + J(Q)\int_{\partial\Omega_\varepsilon}\frac{\partial U_P}{\partial\nu}U_P.$$

Let us prove (8.17). Arguing as in the proof of (8.12), we infer:

$$\begin{aligned}\int_{\Omega_\varepsilon} J(\varepsilon x)|\nabla U_P|^2 &= \int_{\frac{\Omega-Q}{\varepsilon}} J(\varepsilon x+Q)|\nabla U^Q|^2\\ &= J(Q)\int_{\frac{\Omega-Q}{\varepsilon}}|\nabla U^Q|^2 + \varepsilon\int_{\frac{\Omega-Q}{\varepsilon}} J'(Q)[x]|\nabla U^Q|^2 + o(\varepsilon)\\ &= J(Q)\int_{\Omega_\varepsilon}|\nabla U_P|^2 + \varepsilon\int_{\mathbb{R}^-_{\nu(Q)}} J'(Q)[x]|\nabla U^Q|^2 + o(\varepsilon).\end{aligned}$$

We can prove equation (8.18) repeating the arguments of (8.17).

Since

$$\int_{\Omega_\varepsilon} U_P^p\, \partial_{P_i} U_P = \frac{1}{p+1}\partial_{P_i}\int_{\Omega_\varepsilon} U_P^{p+1},$$

equations (8.19) and (8.20) follow easily because, as observed by [51], the error terms $O(\varepsilon)$ in (8.14) and (8.16) become of order $o(\varepsilon)$ after applying ∂_{P_i} to them. □

8.2 Invertibility of $D^2 f_\varepsilon$ on $(T_{U_P} Z^\varepsilon)^\perp$

In this section we will show that $D^2 f_\varepsilon$ is invertible on $(T_{U_P} Z^\varepsilon)^\perp$, where $T_{U_P} Z^\varepsilon$ denotes the tangent space to Z^ε at U_P.

Let $L_{\varepsilon,Q} : (T_{U_P} Z^\varepsilon)^\perp \to (T_{U_P} Z^\varepsilon)^\perp$ denote the operator defined by setting $(L_{\varepsilon,Q} v \mid w) = D^2 f_\varepsilon(U_P)[v, w]$.

Lemma 8.2. *There exists $C > 0$ such that for ε small enough one has that*

$$\|L_{\varepsilon,Q} v\| \geq C\|v\|, \qquad \forall\, v \in (T_{U_P} Z^\varepsilon)^\perp. \tag{8.21}$$

Proof By (8.6), if we set $\alpha(Q) = V(Q)^{\frac{1}{p-1}}$ and $\beta(Q) = \sqrt{V(Q)/J(Q)}$, we have that $U^Q(x) = \alpha(Q)\bar{U}(\beta(Q)x)$. Therefore, we have:

$$\begin{aligned}
\partial_{P_i} U^Q(x-P) &= \partial_{P_i}\left[\alpha(\varepsilon P)\bar{U}(\beta(\varepsilon P)(x-P))\right] \\
&= \varepsilon \partial_{P_i}\alpha(\varepsilon P) U^Q(\beta(\varepsilon P)(x-P)) \\
&\quad + \varepsilon \alpha(\varepsilon P)\partial_{P_i}\beta(\varepsilon P)\nabla U^Q(\beta(\varepsilon P)(x-P)) \cdot (x-P) \\
&\quad - \alpha(\varepsilon P)\beta(\varepsilon P)(\partial_{x_i} U^Q)(\beta(\varepsilon P)(x-P)).
\end{aligned}$$

Hence

$$\partial_{P_i} U^Q(x-P) = -\partial_{x_i} U^Q(x-P) + O(\varepsilon). \tag{8.22}$$

For simplicity, we can assume that $Q = \varepsilon P$ is the origin $\mathcal{O}$.

Following [51], without loss of generality, we assume that $Q = \varepsilon P$ is the origin $\mathcal{O}$, x_N is the tangent plane of $\partial\Omega$ at Q and $\nu(Q) = (0, \ldots, 0, -1)$. We also assume that part of $\partial\Omega$ is given by $x_N = \psi(x') = \frac{1}{2}\sum_{i=1}^{N-1} \lambda_i x_i^2 + O(|x'|^3)$ for $|x'| < \mu$, where μ is some constant depending only on Ω. Then for $|y'| < \mu/\varepsilon$, the corresponding part of $\partial\Omega_\varepsilon$ is given by $y_N = \Psi(y') = \varepsilon^{-1}\psi(\varepsilon y') = \frac{\varepsilon}{2}\sum_{i=1}^{N-1} \lambda_i y_i^2 + O(\varepsilon^2|y'|^3)$.

We recall that $T_{U^\mathcal{O}} Z^\varepsilon = \mathrm{span}_{H^1(\Omega_\varepsilon)}\{\partial_{P_1} U^\mathcal{O}, \ldots, \partial_{P_{N-1}} U^\mathcal{O}\}$. We set

$$\begin{aligned}
\mathcal{V}_\varepsilon &= \mathrm{span}_{H^1(\Omega_\varepsilon)}\{U^\mathcal{O}, \partial_{x_1} U^\mathcal{O}, \ldots, \partial_{x_{N-1}} U^\mathcal{O}\}, \\
\mathcal{V}_+ &= \mathrm{span}_{H^1(\mathbb{R}^N_+)}\{U^\mathcal{O}, \partial_{x_1} U^\mathcal{O}, \ldots, \partial_{x_{N-1}} U^\mathcal{O}\}.
\end{aligned}$$

By (8.22) it suffices to prove (8.21) for all $v \in \mathrm{span}\{U^\mathcal{O}, \phi\}$, where ϕ is orthogonal to $\mathcal{V}_\varepsilon$. Precisely we shall prove that there exist $C_1, C_2 > 0$ such that, for all $\varepsilon > 0$ small enough, one has:

$$(L_{\varepsilon,\mathcal{O}} U^\mathcal{O} \mid U^\mathcal{O}) \leq -C_1 < 0, \tag{8.23}$$

$$(L_{\varepsilon,\mathcal{O}} \phi \mid \phi) \geq C_2\|\phi\|^2. \tag{8.24}$$

The proof of (8.23) follows easily from the fact that $U^\mathcal{O}$ is a Mountain Pass critical point of $F^\mathcal{O}$ and so from the fact that there exists $c_0 > 0$ such that, for all $\varepsilon > 0$ small enough, one finds:

$$D^2F^{\mathcal{O}}(U^{\mathcal{O}})[U^{\mathcal{O}},U^{\mathcal{O}}] < -c_0 < 0.$$

Indeed, arguing as in the proof of Lemma 8.1 (see (8.12) and (8.13)) and by (8.14) and (8.16), we have:

$$\begin{aligned}(L_{\varepsilon,\mathcal{O}}U^{\mathcal{O}} \mid U^{\mathcal{O}}) &= \int_{\Omega_\varepsilon} J(\varepsilon x)|\nabla U^{\mathcal{O}}|^2 + \int_{\Omega_\varepsilon} V(\varepsilon x)(U^{\mathcal{O}})^2 - p\int_{\Omega_\varepsilon}(U^{\mathcal{O}})^{p+1} \\ &= J(\mathcal{O})\int_{\Omega_\varepsilon}|\nabla U^{\mathcal{O}}|^2 + V(\mathcal{O})\int_{\Omega_\varepsilon}(U^{\mathcal{O}})^2 - p\int_{\Omega_\varepsilon}(U^{\mathcal{O}})^{p+1} + O(\varepsilon) \\ &= D^2F^{\mathcal{O}}(U^{\mathcal{O}})[U^{\mathcal{O}},U^{\mathcal{O}}] + O(\varepsilon) < -c_0 + O(\varepsilon) < -C_1.\end{aligned}$$

Let us prove (8.24).

As before, the fact that $U^{\mathcal{O}}$ is a Mountain Pass critical point of $F^{\mathcal{O}}$ implies that

$$D^2F^{\mathcal{O}}(U^{\mathcal{O}})[\tilde\phi,\tilde\phi] > c_1\|\tilde\phi\|_+^2 \quad \forall\tilde\phi \perp \mathcal{V}_+. \tag{8.25}$$

Let us consider a smooth function $\chi_1 : \mathbb{R}^N \to \mathbb{R}$ such that

$$\chi_1(x) = 1, \quad \text{for } |x| \le \varepsilon^{-1/8}; \qquad \chi_1(x) = 0, \quad \text{for } |x| \ge 2\varepsilon^{-1/8};$$
$$|\nabla\chi_1(x)| \le 2\varepsilon^{1/8}, \quad \text{for } \varepsilon^{-1/8} \le |x| \le 2\varepsilon^{-1/8}.$$

We also set $\chi_2(x) = 1 - \chi_1(x)$. Given $\phi \perp \mathcal{V}_\varepsilon$, let us consider the functions

$$\phi_i(x) = \chi_i(x)\phi(x), \quad i = 1, 2.$$

If $Q \ne \mathcal{O}$, then we would take

$$\phi_i(x) = \chi_i(x - P)\phi(x), \quad i = 1, 2.$$

With calculations similar to those of [11], we have

$$\|\phi\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + \underbrace{2\int_{\mathbb{R}^N}\chi_1\chi_2(\phi^2 + |\nabla\phi|^2)}_{I_\phi} + O(\varepsilon^{1/8})\|\phi\|^2. \tag{8.26}$$

We need to evaluate the three terms in the equation below:

$$(L_{\varepsilon,\mathcal{O}}\phi \mid \phi) = (L_{\varepsilon,\mathcal{O}}\phi_1 \mid \phi_1) + (L_{\varepsilon,\mathcal{O}}\phi_2 \mid \phi_2) + 2(L_{\varepsilon,\mathcal{O}}\phi_1 \mid \phi_2). \tag{8.27}$$

Let us start with $(L_{\varepsilon,\mathcal{O}}\phi_1 \mid \phi_1)$.

Let $\eta = \eta_\varepsilon$ a smooth cutoff function satisfying

$$\eta(y) = 1, \quad \text{for } |y| \le \varepsilon^{-1/4}; \qquad \eta(y) = 0, \quad \text{for } |y| \ge 2\varepsilon^{-1/4};$$
$$|\nabla\eta(y)| \le 2\varepsilon^{1/4}, \quad \text{for } \varepsilon^{-1/4} \le |y| \le 2\varepsilon^{-1/4}.$$

Now we will straighten $\partial\Omega_\varepsilon$ in the following way: let Φ: $\mathbb{R}^N_+ \cap B_{\varepsilon^{-1/2}} \to \Omega_\varepsilon$ be a function so defined:

$$\Phi(y', y_N) = (y', y_N + \Psi(y')).$$

We observe that:

$$D\Phi(y) = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ \hline & \nabla_{y'}\Psi(y') & & 1 \end{pmatrix}.$$

Let us defined $\tilde{\phi_1} \in H^1(\mathbb{R}^N_+)$ as:

$$\tilde{\phi_1}(y) = \begin{cases} \phi_1(\Phi(y))\,\eta(y) & \text{if } |y| \le \varepsilon^{-1/2}, \\ 0 & \text{if } |y| > \varepsilon^{-1/2}. \end{cases}$$

We get:

$$\int_{\mathbb{R}^N_+} |\nabla\tilde{\phi_1}|^2 = \int_{\mathbb{R}^N_+ \cap B_{2\varepsilon^{-1/4}}} |\nabla\left[\phi_1(\Phi(y))\right]|^2\, dy$$

$$= \int_{\mathbb{R}^N_+ \cap B_{2\varepsilon^{-1/4}}} \sum_{i=1}^{N-1} \left|\frac{\partial\phi_1}{\partial x_i}(\Phi) + \varepsilon\lambda_i y_i \frac{\partial\phi_1}{\partial x_N}(\Phi)\right|^2 + \left|\frac{\partial\phi_1}{\partial x_N}(\Phi)\right|^2 + o(\varepsilon)\|\phi\|^2$$

$$= \int_{\mathbb{R}^N_+ \cap B_{2\varepsilon^{-1/4}}} |(\nabla\phi_1)(\Phi)|^2 + O(\varepsilon^{7/8})\|\phi\|^2 = \int_{\Omega_\varepsilon} |\nabla\phi_1|^2 + O(\varepsilon^{7/8})\|\phi\|^2.$$

Analogously, we have:

$$\int_{\mathbb{R}^N_+} |\tilde{\phi_1}|^2 = \int_{\Omega_\varepsilon} |\phi_1|^2,$$

and so

$$\|\tilde{\phi_1}\|_+^2 = \|\phi_1\|^2 + O(\varepsilon^{7/8})\|\phi\|^2.$$

Let us now evaluate $(L_{\varepsilon,\mathcal{O}}\phi_1 | \phi_1)$:

$$\begin{aligned}(L_{\varepsilon,\mathcal{O}}\phi_1 \mid \phi_1) &= \int_{\Omega_\varepsilon} J(\varepsilon x)|\nabla\phi_1|^2 + \int_{\Omega_\varepsilon} V(\varepsilon x)\phi_1^2 - p\int_{\Omega_\varepsilon} (U^{\mathcal{O}})^{p-1}\phi_1^2 \\ &= J(\mathcal{O})\int_{\Omega_\varepsilon} |\nabla\phi_1|^2 + V(\mathcal{O})\int_{\Omega_\varepsilon} \phi_1^2 - p\int_{\Omega_\varepsilon} (U^{\mathcal{O}})^{p-1}\phi_1^2 \\ &\quad + \varepsilon\int_{\Omega_\varepsilon} J'(\mathcal{O})[x]|\nabla\phi_1|^2 + \varepsilon\int_{\Omega_\varepsilon} V'(\mathcal{O})[x]\phi_1^2 + o(\varepsilon)\|\phi\|^2 \\ &= J(\mathcal{O})\int_{\Omega_\varepsilon} |\nabla\phi_1|^2 + V(\mathcal{O})\int_{\Omega_\varepsilon} \phi_1^2 - p\int_{\Omega_\varepsilon} (U^{\mathcal{O}})^{p-1}\phi_1^2 \\ &\quad + O(\varepsilon^{7/8})\|\phi\|^2\end{aligned}$$

$$= J(\mathcal{O}) \int_{\mathbb{R}^N_+} |\nabla \tilde{\phi_1}|^2 + V(\mathcal{O}) \int_{\mathbb{R}^N_+} \tilde{\phi_1}^2 - p \int_{\mathbb{R}^N_+} [U^{\mathcal{O}}(\Phi)]^{p-1} \tilde{\phi_1}^2$$
$$+ O(\varepsilon^{7/8}) ||\phi||^2$$
$$= D^2 F^{\mathcal{O}}(U^{\mathcal{O}})[\tilde{\phi_1}, \tilde{\phi_1}] - p \int_{\mathbb{R}^N_+} \left([U^{\mathcal{O}}(\Phi)]^{p-1} - (U^{\mathcal{O}})^{p-1}\right) \tilde{\phi_1}^2$$
$$+ O(\varepsilon^{7/8}) ||\phi||^2.$$

We have:

$$\left| \int_{\mathbb{R}^N_+} \left([U^{\mathcal{O}}(\Phi)]^{p-1} - (U^{\mathcal{O}})^{p-1}\right) \tilde{\phi_1}^2 \right| \leq C \int_{\mathbb{R}^N_+} |\Psi(y')| \tilde{\phi_1}^2$$
$$= O(\varepsilon^{3/4}) ||\tilde{\phi_1}||^2 = O(\varepsilon^{3/4}) ||\phi||^2.$$

Therefore, we have that

$$(L_{\varepsilon, \mathcal{O}} \phi_1 \mid \phi_1) = D^2 F^{\mathcal{O}}(U^{\mathcal{O}})[\tilde{\phi_1}, \tilde{\phi_1}] + O(\varepsilon^{3/4}) ||\phi||^2. \tag{8.28}$$

We can write $\tilde{\phi_1} = \xi + \zeta$, where $\xi \in \mathcal{V}_+$ and $\zeta \perp \mathcal{V}_+$. More precisely

$$\xi = (\tilde{\phi_1} \mid U^{\mathcal{O}})_+ U^{\mathcal{O}} ||U^{\mathcal{O}}||_+^{-2} + \sum_{i=1}^{N-1} (\tilde{\phi_1} \mid \partial_{P_i} U^{\mathcal{O}})_+ \partial_{P_i} U^{\mathcal{O}} ||\partial_{P_i} U^{\mathcal{O}}||_+^{-2}.$$

Let us calculate $(\tilde{\phi_1} | U^{\mathcal{O}})_+$.

$$(\tilde{\phi_1} \mid U^{\mathcal{O}})_+ = \int_{\mathbb{R}^N_+} \nabla \tilde{\phi_1} \cdot \nabla U^{\mathcal{O}} + \int_{\mathbb{R}^N_+} \tilde{\phi_1} U^{\mathcal{O}}$$
$$= \int_{\mathbb{R}^N_+ \cap B_{2\varepsilon^{-1/4}}} \nabla [\phi_1(\Phi(y))] \cdot \nabla U^{\mathcal{O}} + \int_{\mathbb{R}^N_+ \cap B_{2\varepsilon^{-1/4}}} \phi_1(\Phi(y)) \, U^{\mathcal{O}}$$
$$= \int_{\mathbb{R}^N_+ \cap B_{2\varepsilon^{-1/4}}} \left[(\nabla \phi_1)(\Phi) \cdot \nabla U^{\mathcal{O}} + \phi_1(\Phi) \, U^{\mathcal{O}} \right]$$
$$+ \varepsilon \sum_{i=1}^{N-1} \int_{\mathbb{R}^N_+ \cap B_{2\varepsilon^{-1/4}}} \lambda_i y_i \frac{\partial \phi_1}{\partial x_N}(\Phi) \frac{\partial U^{\mathcal{O}}}{\partial x_i}$$
$$= \int_{\Omega_\varepsilon} \nabla \phi_1 \cdot \nabla U^{\mathcal{O}}(\Phi^{-1}) + \int_{\Omega_\varepsilon} \phi_1 U^{\mathcal{O}}(\Phi^{-1}) + O(\varepsilon^{7/8}) ||\phi||^2$$
$$= \int_{\Omega_\varepsilon} \nabla \phi_1 \cdot \nabla U^{\mathcal{O}} + \int_{\Omega_\varepsilon} \phi_1 U^{\mathcal{O}} + O(\varepsilon^{3/4}) ||\phi|| = O(\varepsilon^{3/4}) ||\phi||.$$

In an analogous way, we can prove also that $(\tilde{\phi_1} \mid \partial_{P_i} U^{\mathcal{O}})_+ = O(\varepsilon^{3/4}) ||\phi||$, and so

$$||\xi||_+ = O(\varepsilon^{3/4}) ||\phi||, \tag{8.29}$$
$$||\zeta||_+ = ||\phi_1|| + O(\varepsilon^{3/4}) ||\phi||. \tag{8.30}$$

Let us estimate $D^2F^{\mathcal{O}}(U^{\mathcal{O}})[\tilde{\phi_1}, \tilde{\phi_1}]$. We get:

$$D^2F^{\mathcal{O}}(U^{\mathcal{O}})[\tilde{\phi_1}, \tilde{\phi_1}] = D^2F^{\mathcal{O}}(U^{\mathcal{O}})[\zeta,\zeta]+2D^2F^{\mathcal{O}}(U^{\mathcal{O}})[\zeta,\xi]+D^2F^{\mathcal{O}}(U^{\mathcal{O}})[\xi,\xi]. \tag{8.31}$$

By (8.25) and (8.30), we know that

$$D^2F^{\mathcal{O}}(U^{\mathcal{O}})[\zeta,\zeta] > c_1\|\zeta\|_+^2 = c_1\|\phi_1\|^2 + O(\varepsilon^{3/4})\|\phi\|^2,$$

while, by (8.29) and straightforward calculations, we have

$$D^2F^{\mathcal{O}}(U^{\mathcal{O}})[\zeta,\xi] = O(\varepsilon^{3/4})\|\phi\|^2,$$
$$D^2F^{\mathcal{O}}(U^{\mathcal{O}})[\xi,\xi] = O(\varepsilon^{3/2})\|\phi\|^2.$$

By these estimates, (8.31) and (8.28), we can say that

$$(L_{\varepsilon,\mathcal{O}}\phi_1 \mid \phi_1) > c_1\|\phi_1\|^2 + O(\varepsilon^{3/4})\|\phi\|^2. \tag{8.32}$$

Using the definition of χ_i and the exponential decay of $U^{\mathcal{O}}$, we easily get

$$(L_{\varepsilon,\mathcal{O}}\phi_2 \mid \phi_2) \geq c_2\|\phi_2\|^2 + o(\varepsilon)\|\phi\|^2, \tag{8.33}$$
$$(L_{\varepsilon,\mathcal{O}}\phi_1 \mid \phi_2) \geq c_3 I_\phi + O(\varepsilon^{1/8})\|\phi\|^2, \tag{8.34}$$

where I_ϕ is defined in (8.26). Therefore by (8.27), (8.32), (8.33), (8.34) and recalling (8.26) we get

$$(L_{\varepsilon,\mathcal{O}}\phi \mid \phi) \geq c_4\|\phi\|^2 + O(\varepsilon^{1/8})\|\phi\|^2.$$

This completes the proof of the lemma. □

8.3 The finite dimensional reduction

Lemma 8.3. *For $\varepsilon > 0$ small enough, there exists a unique $w = w(\varepsilon, Q) \in (T_{U_P}Z^\varepsilon)^\perp$ such that $\nabla f_\varepsilon(U_P+w) \in T_{U_P}Z$. Such a $w(\varepsilon, Q)$ is of class C^2, resp. $C^{1,p-1}$, with respect to Q, provided that $p \geq 2$, resp. $1 < p < 2$. Moreover, the functional $\mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_{Q/\varepsilon} + w(\varepsilon, Q))$ has the same regularity of w and satisfies:*

$$\nabla\mathcal{A}_\varepsilon(Q_0) = 0 \iff \nabla f_\varepsilon\left(U_{Q_0/\varepsilon} + w(\varepsilon, Q_0)\right) = 0.$$

Proof Let $\mathcal{P} = \mathcal{P}_{\varepsilon,Q}$ denote the projection onto $(T_{U_P}Z^\varepsilon)^\perp$. We want to find a solution $w \in (T_{U_P}Z^\varepsilon)^\perp$ of the equation $\mathcal{P}\nabla f_\varepsilon(U_P + w) = 0$. One has that $\nabla f_\varepsilon(U_P+w) = \nabla f_\varepsilon(U_P)+D^2 f_\varepsilon(U_P)[w]+R(U_P,w)$ with $\|R(U_P,w)\| = o(\|w\|)$, uniformly with respect to U_P. Therefore, our equation is:

$$L_{\varepsilon,Q}w + \mathcal{P}\nabla f_\varepsilon(U_P) + \mathcal{P}R(U_P,w) = 0. \tag{8.35}$$

According to Lemma 8.2, this is equivalent to

$$w = N_{\varepsilon,Q}(w), \quad \text{where} \quad N_{\varepsilon,Q}(w) = -\left(L_{\varepsilon,Q}\right)^{-1}\left(\mathcal{P}\nabla f_\varepsilon(U_P) + \mathcal{P}R(U_P,w)\right).$$

By (8.9) it follows that

$$\|N_{\varepsilon,Q}(w)\| = O(\varepsilon) + o(\|w\|). \tag{8.36}$$

Now the proof is similar to that of Lemma 5.5. □

Remark 8.1. From (8.36) it immediately follows that:

$$\|w\| = O(\varepsilon). \tag{8.37}$$

For future references, it is convenient to estimate the derivative $\partial_{P_i} w$.

Lemma 8.4. *If* $\gamma = \min\{1, p-1\}$*, then, for* $i = 1, \dots, N-1$*, one has that:*

$$\|\partial_{P_i} w\| = O(\varepsilon^\gamma). \tag{8.38}$$

Proof We will set $h(U_P, w) = (U_P + w)^p - U_P^p - pU_P^{p-1}w$. With these notations, and recalling that $L_{\varepsilon,Q}w = -\operatorname{div}(J(\varepsilon x)\nabla w) + V(\varepsilon x)w - pU_P^{p-1}w$, it follows that, for all $v \in (T_{U_P}Z^\varepsilon)^\perp$, since w satisfies (8.35), then:

$$\begin{aligned}
&\int_{\Omega_\varepsilon} J(\varepsilon x)\nabla U_P \cdot \nabla v + \int_{\Omega_\varepsilon} V(\varepsilon x)U_P v - \int_{\Omega_\varepsilon} U_P^p v \\
&+ \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla w \cdot \nabla v + \int_{\Omega_\varepsilon} V(\varepsilon x) w v - p\int_{\Omega_\varepsilon} U_P^{p-1} w v - \int_{\Omega_\varepsilon} h(U_P, w) v = 0.
\end{aligned}$$

Hence $\partial_{P_i} w$ verifies:

$$\begin{aligned}
&\int_{\Omega_\varepsilon} J(\varepsilon x)\nabla(\partial_{P_i}U_P) \cdot \nabla v + \int_{\Omega_\varepsilon} V(\varepsilon x)(\partial_{P_i}U_P)v - p\int_{\Omega_\varepsilon} U_P^{p-1}(\partial_{P_i}U_P)v \\
&+ \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla(\partial_{P_i}w) \cdot \nabla v + \int_{\Omega_\varepsilon} V(\varepsilon x)(\partial_{P_i}w)v - p\int_{\Omega_\varepsilon} U_P^{p-1}(\partial_{P_i}w)v \\
&-p(p-1)\int_{\Omega_\varepsilon} U_P^{p-2}(\partial_{P_i}U_P)wv - \int_{\Omega_\varepsilon}\left[h_{U_P}(\partial_{P_i}U_P) + h_w(\partial_{P_i}w)\right]v = 0.
\end{aligned} \tag{8.39}$$

Let us set $L' = L_{\varepsilon,Q} - h_w$. Then (8.39) can be written as

$$\begin{aligned}
(L'(\partial_{P_i}w) \mid v) = {} & p(p-1)\int_{\Omega_\varepsilon} U_P^{p-2}(\partial_{P_i}U_P)wv + \int_{\Omega_\varepsilon} h_{U_P}(\partial_{P_i}U_P)v \\
&- \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla(\partial_{P_i}U_P) \cdot \nabla v - \int_{\Omega_\varepsilon} V(\varepsilon x)(\partial_{P_i}U_P)v \\
&+ p\int_{\Omega_\varepsilon} U_P^{p-1}(\partial_{P_i}U_P)v.
\end{aligned} \tag{8.40}$$

It is easy to see that

$$\left| p(p-1) \int_{\Omega_\varepsilon} U_P^{p-2} (\partial_{P_i} U_P) w v \right| \le c_1 \|w\| \|v\| \tag{8.41}$$

and, if $\gamma = \min\{1, p-1\}$,

$$\left| \int_{\Omega_\varepsilon} h_{U_P} (\partial_{P_i} U_P) v \right| \le c_2 \|w\|^\gamma \|v\|. \tag{8.42}$$

Let us study the second line of (8.40). We recall that often we will write U instead of U^Q. Reasoning as in the proof of Lemma 8.1 (see (8.12) and (8.13)), we infer:

$$\begin{aligned}
I &\equiv \int_{\Omega_\varepsilon} J(\varepsilon x) \nabla(\partial_{P_i} U_P) \cdot \nabla v + \int_{\Omega_\varepsilon} V(\varepsilon x)(\partial_{P_i} U_P) v - p \int_{\Omega_\varepsilon} U_P^{p-1} (\partial_{P_i} U_P) v \\
&= \int_{\frac{\Omega - Q}{\varepsilon}} J(Q) \nabla(\partial_{P_i} U) \cdot \nabla v_{-P} + \int_{\frac{\Omega - Q}{\varepsilon}} V(Q)(\partial_{P_i} U) v_{-P} \\
&\quad + \varepsilon \int_{\Omega_\varepsilon} J'(Q)[x-P] \nabla(\partial_{P_i} U_P) \cdot \nabla v + \varepsilon \int_{\Omega_\varepsilon} V'(Q)[x-P](\partial_{P_i} U_P) v \\
&\quad - p \int_{\Omega_\varepsilon} U_P^{p-1} (\partial_{P_i} U_P) v + O(\varepsilon)\|v\|.
\end{aligned}$$

Suppose, for simplicity, Q coincides with the origin $\mathcal{O}$ and that part of $\partial\Omega$ is given by $x_N = \psi(x') = \frac{1}{2}\sum_{i=1}^{N-1} \lambda_i x_i^2 + O(|x'|^3)$ for $|x'| < \mu$, where μ is some constant depending only on Ω. Then for $|y'| < \mu/\varepsilon$, the corresponding part of $\partial\Omega_\varepsilon$ is given by $y_N = \Psi(y') = \varepsilon^{-1}\psi(\varepsilon y') = \frac{\varepsilon}{2}\sum_{i=1}^{N-1} \lambda_i y_i^2 + O(\varepsilon^2 |y'|^3)$.

Since by (8.22) $\partial_{P_i} U_P = -\partial_{x_i} U_P + O(\varepsilon)$, by integration by parts, we get:

$$\begin{aligned}
\varepsilon \int_{\Omega_\varepsilon} J'(Q)[x-P] \nabla(\partial_{P_i} U_P) \cdot \nabla v &= \varepsilon \int_{\Omega_\varepsilon} \partial_{Q_i} J(Q) \nabla U_P \cdot \nabla v + O(\varepsilon)\|v\|, \\
\varepsilon \int_{\Omega_\varepsilon} V'(Q)[x-P](\partial_{P_i} U_P) v &= \varepsilon \int_{\Omega_\varepsilon} \partial_{Q_i} V(Q) U_P v + O(\varepsilon)\|v\|.
\end{aligned}$$

Hence

$$\begin{aligned}
I = &\int_{\Omega_\varepsilon} J(Q) \nabla(\partial_{P_i} U_P) \cdot \nabla v + \varepsilon \int_{\Omega_\varepsilon} \partial_{Q_i} J(Q) \nabla U_P \cdot \nabla v \\
&+ \int_{\Omega_\varepsilon} V(Q)(\partial_{P_i} U_P) v + \varepsilon \int_{\Omega_\varepsilon} \partial_{Q_i} V(Q) U_P v \\
&- p \int_{\Omega_\varepsilon} U_P^{p-1} (\partial_{P_i} U_P) v + O(\varepsilon)\|v\|.
\end{aligned}$$

Being $U = U^Q$ solution of (8.7), we have that

$$- J(Q)\Delta(\partial_{P_i}U) - \varepsilon\partial_{Q_i}J(Q)\Delta U + V(Q)(\partial_{P_i}U) + \varepsilon\partial_{Q_i}V(Q)U - pU^{p-1}(\partial_{P_i}U) = 0$$

and so

$$I = J(Q)\int_{\partial\Omega_\varepsilon}\frac{\partial}{\partial\nu}(\partial_{P_i}U_P)v + \varepsilon\partial_{Q_i}J(Q)\int_{\partial\Omega_\varepsilon}\frac{\partial U_P}{\partial\nu}v + O(\varepsilon)||v||.$$

Arguing again as in the proof of Lemma 8.1 (see (8.11)), we can prove that

$$\left|J(Q)\int_{\partial\Omega_\varepsilon}\frac{\partial}{\partial\nu}(\partial_{P_i}U_P)v + \varepsilon\partial_{Q_i}J(Q)\int_{\partial\Omega_\varepsilon}\frac{\partial U_P}{\partial\nu}v\right| = O(\varepsilon)||v||.$$

Hence

$$I = O(\varepsilon^{3/4})||v||. \tag{8.43}$$

Putting together (8.40), (8.41), (8.42) and (8.43), we find

$$|(L'(\partial w_i) \mid v)| = (c_3||w||^\gamma + O(\varepsilon))\,||v||.$$

Since $h_w \to 0$ as $w \to 0$, the operator L', likewise L, is invertible for $\varepsilon > 0$ small and therefore one finds

$$||\partial_{P_i}w|| \le c_4||w||^\gamma + O(\varepsilon).$$

Finally, by Remark 8.1, the lemma follows. □

8.4 The finite dimensional functional

Theorem 8.3. *Let $Q \in \partial\Omega$ and $P = Q/\varepsilon \in \partial\Omega_\varepsilon$. Suppose* **(J)** *and* **(V)**. *Then, for ε sufficiently small, we get:*

$$\mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_P + w(\varepsilon, Q)) = c_0\Gamma(Q) + \varepsilon\Sigma(Q) + o(\varepsilon), \tag{8.44}$$

where Γ is the auxiliary functions introduced in (8.3),

$$c_0 \equiv \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N_+}\bar{U}^{p+1},$$

and $\Sigma \colon \partial\Omega \to \mathbb{R}$ is so defined:

$$\Sigma(Q) \equiv \frac{1}{2}\int_{\mathbb{R}^-_{\nu(Q)}} J'(Q)[x]|\nabla U^Q|^2 dx + \frac{1}{2}\int_{\mathbb{R}^-_{\nu(Q)}} V'(Q)[x]\left(U^Q\right)^2 dx - \frac{1}{2}\bar{B}^Q J(Q)H(Q) - \left(\frac{1}{2} - \frac{1}{p+1}\right)\bar{A}^Q H(Q), \tag{8.45}$$

with

$$\bar{A}^Q \equiv \frac{1}{2}\int_{\mathbb{R}^{N-1}} \left[U^Q(x',0)\right]^{p+1} |x'|^2 dx',$$
$$\bar{B}^Q \equiv \frac{(N-1)}{4}\int_{\mathbb{R}^{N-1}} \left[U^Q(x',0)\right]^2 dx.$$

Moreover, for all $i = 1, \ldots, N-1$, *we get:*

$$\partial_{P_i}\mathcal{A}_\varepsilon(Q) = \varepsilon c_0 \partial_{Q_i}\Gamma(Q) + o(\varepsilon). \tag{8.46}$$

Proof In the sequel, to be short, we will often write w instead of $w(\varepsilon, Q)$. It is always understood that ε is taken in such a way that all the results discussed previously hold.

First of all, reasoning as in the proofs of (8.17) and (8.18) and by (8.37), we can observe that

$$\int_{\Omega_\varepsilon} J(\varepsilon x)\nabla U_P \cdot \nabla w = J(Q)\int_{\Omega_\varepsilon} \nabla U_P \cdot \nabla w + o(\varepsilon), \tag{8.47}$$
$$\int_{\Omega_\varepsilon} V(\varepsilon x) U_P\, w = V(Q)\int_{\Omega_\varepsilon} U_P\, w + o(\varepsilon). \tag{8.48}$$

We have:

$$\begin{aligned}\mathcal{A}_\varepsilon(Q) &= f_\varepsilon(U_P + w(\varepsilon, Q)) \\ &= \frac{1}{2}\int_{\Omega_\varepsilon} J(\varepsilon x)|\nabla(U_P + w)|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} V(\varepsilon x)(U_P + w)^2 \\ &\quad - \frac{1}{p+1}\int_{\Omega_\varepsilon}(U_P + w)^{p+1}\end{aligned}$$

[by (8.37)]

$$\begin{aligned}&= \frac{1}{2}\int_{\Omega_\varepsilon} J(\varepsilon x)|\nabla U_P|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} V(\varepsilon x) U_P^2 - \frac{1}{2}\int_{\Omega_\varepsilon} U_P^{p+1} \\ &\quad + \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla U_P \cdot \nabla w + \int_{\Omega_\varepsilon} V(\varepsilon x) U_P\, w - \int_{\Omega_\varepsilon} U_P^p\, w + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon} U_P^{p+1} \\ &\quad - \frac{1}{p+1}\int_{\Omega_\varepsilon}\left[(U_P + w)^{p+1} - U_P^{p+1} - (p+1)U_P^p\, w\right] + o(\varepsilon) =\end{aligned}$$

[by (8.16), (8.17), (8.18), (8.47) and (8.48) and with our notations]

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^N_+} U^{p+1} - \frac{\varepsilon}{2}\bar{A}^Q H(Q) - \frac{\varepsilon}{2}\bar{B}^Q J(Q)H(Q) + \frac{\varepsilon}{2}\int_{\mathbb{R}^-_{\nu(Q)}} J'(Q)[x]|\nabla U|^2 \\
&+\frac{\varepsilon}{2}\int_{\mathbb{R}^-_{\nu(Q)}} V'(Q)[x]U^2 - \frac{1}{2}\int_{\mathbb{R}^N_+} U^{p+1} + \frac{\varepsilon}{2}\bar{A}^Q H(Q) \\
&+J(Q)\int_{\Omega_\varepsilon} \nabla U_P \cdot \nabla w + V(Q)\int_{\Omega_\varepsilon} U_P\, w - \int_{\Omega_\varepsilon} U_P^p\, w \\
&+\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N_+} U^{p+1} - \varepsilon\left(\frac{1}{2} - \frac{1}{p+1}\right)\bar{A}^Q H(Q) + o(\varepsilon).
\end{aligned}
$$

From the fact that U is solution of (8.7), we infer

$$
\begin{aligned}
J(Q)&\int_{\Omega_\varepsilon} \nabla U_P \cdot \nabla w + V(Q)\int_{\Omega_\varepsilon} U_P\, w - \int_{\Omega_\varepsilon} U_P^p\, w \\
&= \int_{\Omega_\varepsilon} [-J(Q)\Delta U_P + V(Q)U_P - U_P^p]\, w + J(Q)\int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} w \\
&= J(Q)\int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} w = o(\varepsilon).
\end{aligned}
$$

By these considerations we can say that

$$
\begin{aligned}
\mathcal{A}_\varepsilon(Q) = &\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N_+} U^{p+1} \\
&+\varepsilon\Bigg[\frac{1}{2}\int_{\mathbb{R}^-_{\nu(Q)}} J'(Q)[x]|\nabla U|^2 + \frac{1}{2}\int_{\mathbb{R}^-_{\nu(Q)}} V'(Q)[x]U^2 \\
&-\frac{1}{2}\bar{B}^Q J(Q)H(Q) - \left(\frac{1}{2} - \frac{1}{p+1}\right)\bar{A}^Q H(Q)\Bigg] + o(\varepsilon).
\end{aligned}
$$

Now the conclusion of the first part of the theorem follows observing that, since by (8.6)

$$
U^Q(x) = V(Q)^{\frac{1}{p-1}}\, \bar{U}\left(x\sqrt{V(Q)/J(Q)}\right),
$$

then

$$
\int_{\mathbb{R}^N_+} U^{p+1} = V(Q)^{\frac{p+1}{p-1} - \frac{N}{2}} J(Q)^{\frac{N}{2}} \int_{\mathbb{R}^N_+} \bar{U}^{p+1}.
$$

Let us prove now the estimate on the derivatives of $\mathcal{A}_\varepsilon$. First of all, we observe that by (8.9) and by (8.38), we infer that

$$
|\nabla f_\varepsilon(U_P)[\partial_{P_i} w]| = O(\varepsilon^{1+\gamma}),
$$

and so, by (8.37) and (8.38), we have:

$$\begin{aligned}\partial_{P_i}\mathcal{A}_\varepsilon(Q) &= \nabla f_\varepsilon(U_P+w)[\partial_{P_i}U_P+\partial_{P_i}w]\\ &= \nabla f_\varepsilon(U_P+w)[\partial_{P_i}U_P]+O(\varepsilon^{1+\gamma})\\ &= \nabla f_\varepsilon(U_P)[\partial_{P_i}U_P]+D^2 f_\varepsilon(U_P)[w,\partial_{P_i}U_P]\\ &\quad+\left(\nabla f_\varepsilon(U_P+w)-\nabla f_\varepsilon(U_P)-D^2 f_\varepsilon(U_P)[w]\right)[\partial_{P_i}U_P]\\ &\quad+O(\varepsilon^{1+\gamma}).\end{aligned}$$

But

$$\|\nabla f_\varepsilon(U_P+w)-\nabla f_\varepsilon(U_P)-D^2 f_\varepsilon(U_P)[w]\| = o(\|w\|) = o(\varepsilon)$$

and, moreover, by (8.35) also $D^2 f_\varepsilon(U_P)[w,\partial_{P_i}U_P] = O(\varepsilon^{1+\gamma})$, therefore

$$\partial_{P_i}\mathcal{A}_\varepsilon(Q) = \nabla f_\varepsilon(U_P)[\partial_{P_i}U_P]+O(\varepsilon^{1+\gamma}). \tag{8.49}$$

Let us calculate $\nabla f_\varepsilon(U_P)[\partial_{P_i}U_P]$.

$$\begin{aligned}\nabla f_\varepsilon(U_P)[\partial_{P_i}U_P] &= \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla U_P\cdot\nabla(\partial_{P_i}U_P)+\int_{\Omega_\varepsilon} V(\varepsilon x)U_P(\partial_{P_i}U_P)\\ &\quad-\int_{\Omega_\varepsilon} U_P^p(\partial_{P_i}U_P)\\ &= J(Q)\int_{\Omega_\varepsilon}\nabla U_P\cdot\nabla(\partial_{P_i}U_P)+V(Q)\int_{\Omega_\varepsilon}U_P(\partial_{P_i}U_P)\\ &\quad+\varepsilon\int_{\mathbb{R}^-_{\nu(Q)}} J'(Q)[x]\nabla U\cdot\nabla(\partial_{P_i}U)+\varepsilon\int_{\mathbb{R}^-_{\nu(Q)}} V'(Q)[x]U(\partial_{P_i}U)\\ &\quad-\int_{\Omega_\varepsilon} U_P^p(\partial_{P_i}U_P)+o(\varepsilon).\end{aligned}$$

Suppose, for simplicity, Q coincides with the origin $\mathcal{O}$ and that part of $\partial\Omega$ is given by $x_N=\psi(x')=\frac{1}{2}\sum_{i=1}^{N-1}\lambda_i x_i^2+O(|x'|^3)$ for $|x'|<\mu$, where μ is some constant depending only on Ω. Then for $|y'|<\mu/\varepsilon$, the corresponding part of $\partial\Omega_\varepsilon$ is given by $y_N=\Psi(y')=\varepsilon^{-1}\psi(\varepsilon y')=\frac{\varepsilon}{2}\sum_{i=1}^{N-1}\lambda_i y_i^2+O(\varepsilon^2|y'|^3)$.

Since by (8.22) $\partial_{P_i}U_P=-\partial_{x_i}U_P+O(\varepsilon)$, by integration by parts, we get:

$$\int_{\mathbb{R}^-_{\nu(Q)}} J'(Q)[x]\nabla U\cdot\nabla(\partial_{P_i}U) = \frac{1}{2}\int_{\mathbb{R}^-_{\nu(Q)}}\partial_{Q_i}J(Q)|\nabla U|^2,$$

$$\int_{\mathbb{R}^-_{\nu(Q)}} V'(Q)[x]U(\partial_{P_i}U) = \frac{1}{2}\int_{\mathbb{R}^-_{\nu(Q)}}\partial_{Q_i}V(Q)U^2.$$

Therefore we infer

$$\begin{aligned}&\nabla f_\varepsilon(U_P)[\partial_{P_i}U_P]\\ &= \frac{1}{2}\partial_{P_i}\left[J(Q)\int_{\Omega_\varepsilon}|\nabla U_P|^2+V(Q)\int_{\Omega_\varepsilon}U_P^2\right]-\int_{\Omega_\varepsilon}U_P^p\,(\partial_{P_i}U_P)+o(\varepsilon),\end{aligned}$$

and so, by (8.19) and (8.20),

$$\nabla f_\varepsilon(U_P)[\partial_{P_i} U_P] = \varepsilon \left[\left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N_+} \bar{U}^{p+1} \right] \partial_{Q_i} \Gamma(Q)$$
$$= \varepsilon c_0 \partial_{Q_i} \Gamma(Q) + o(\varepsilon).$$

By this equation and by (8.49), (8.46) follows immediately. □

Remark 8.2. Let us observe that by (8.44) and (8.46), for ε sufficiently small, we have

$$\|\mathcal{A}_\varepsilon - c_0 \Gamma\|_{C^1(\partial\Omega)} = O(\varepsilon). \tag{8.50}$$

Remark 8.3. By (8.6), it is easy to see that, if J and V are constant on the boundary $\partial\Omega$, then $\bar{\Sigma}$, defined in (8.4), coincides with Σ, defined in (8.45) with the following definitions:

$$\begin{array}{ll} C_J \equiv J_{|\partial\Omega}, & C_V \equiv V_{|\partial\Omega}, \\ k_1 \equiv \frac{(C_V)^{\frac{p+1}{p-1}}}{2C_J}, & k_2 \equiv \sqrt{C_V/C_J}, \\ k_3 \equiv \frac{(C_V)^{\frac{2}{p-1}}}{2}, & k_4 \equiv -\frac{1}{2}\bar{B}C_J - \left(\frac{1}{2} - \frac{1}{p+1}\right)\bar{A}, \end{array}$$

where

$$\bar{A} \equiv \frac{(C_V)^{\frac{p+1}{p-1}}}{2} \int_{\mathbb{R}^{N-1}} \left[\bar{U}\left(x'\sqrt{C_V/C_J}, 0 \right) \right]^{p+1} |x'|^2 dx',$$
$$\bar{B} \equiv \frac{(N-1)(C_V)^{\frac{2}{p-1}}}{4} \int_{\mathbb{R}^{N-1}} \left[\bar{U}\left(x'\sqrt{C_V/C_J}, 0 \right) \right]^2 dx'.$$

8.5 Proofs of Theorem 8.1 and Theorem 8.2

In this section we will state and prove two multiplicity results for (8.1) whose Theorem 8.1 is a particular case. Finally we will prove also Theorem 8.2.

Let us suppose that Γ has a smooth manifold of critical points M. We say that M is nondegenerate (for Γ) if every $x \in M$ is a nondegenerate critical point of $\Gamma_{|M^\perp}$. The Morse index of M is, by definition, the Morse index of any $x \in M$, as critical point of $\Gamma_{|M^\perp}$.

We now can state our first multiplicity result.

Theorem 8.4. *Let* **(J)** *and* **(V)** *hold and suppose Γ has a nondegenerate smooth manifold of critical points $M \subset \partial\Omega$. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then* (8.1) *has at least $l(M)$ solutions that concentrate near points of M. Here $l(M)$ is denotes the cup long of M (see Definition 5.1).*

Proof Fix a δ-neighborhood M_δ of M such that the only critical points of Γ in M_δ are those in M. We will take $U = M_\delta$.

For ε sufficiently small, by (8.50) and Theorem 5.9, $\mathcal{A}_\varepsilon$ possesses at least $l(M)$ critical points, which are solutions of (8.5) by Lemma 8.3. Let $Q_\varepsilon \in M$ be one of these critical points, then $u_\varepsilon^{Q_\varepsilon} = U_{Q_\varepsilon/\varepsilon} + w(\varepsilon, Q_\varepsilon)$ is a solution of (8.5). Therefore

$$u_\varepsilon^{Q_\varepsilon}(x/\varepsilon) \simeq U_{Q_\varepsilon/\varepsilon}(x/\varepsilon) = U^{Q_\varepsilon}\left(\frac{x - Q_\varepsilon}{\varepsilon}\right)$$

is a solution of (8.1). □

Moreover, when we deal with local minima (resp. maxima) of Γ, the preceding results can be improved because the number of positive solutions of (8.1) can be estimated by means of the category and M does not need to be a manifold.

Theorem 8.5. *Let* **(J)** *and* **(V)** *hold and suppose Γ has a compact set $X \subset \partial\Omega$ where Γ achieves a strict local minimum (resp. maximum), in the sense that there exist $\delta > 0$ and a δ-neighborhood $X_\delta \subset \partial\Omega$ of X such that*

$$b \equiv \inf\{\Gamma(Q) : Q \in \partial X_\delta\} > a \equiv \Gamma_{|X}, \quad \left(\text{resp. } \sup\{\Gamma(Q) : Q \in \partial X_\delta\} < \Gamma_{|X}\right).$$

Then there exists $\varepsilon_0 > 0$ such that (8.1) *has at least* $\mathrm{cat}(X, X_\delta)$ *solutions that concentrate near points of X_δ, provided $\varepsilon \in (0, \varepsilon_0)$. Here* $\mathrm{cat}(X, X_\delta)$ *denotes the Lusternik-Schnirelman category of X with respect to X_δ.*

Proof We will treat only the case of minima, being the other one similar. We set $Y = \{Q \in X_\delta : \mathcal{A}_\varepsilon(Q) \le c_0(a+b)/2\}$. By (8.44) it follows that there exists $\varepsilon_0 > 0$ such that

$$X \subset Y \subset X_\delta, \tag{8.51}$$

provided $\varepsilon \in (0, \varepsilon_0)$. Moreover, if $Q \in \partial X_\delta$ then $\Gamma(Q) \ge b$ and hence

$$\mathcal{A}_\varepsilon(Q) \ge c_0\Gamma(Q) + O(\varepsilon) \ge c_0 b + O(\varepsilon).$$

On the other side, if $Q \in Y$ then $\mathcal{A}_\varepsilon(Q) \le c_0(a+b)/2$. Hence, for ε small, Y cannot meet ∂X_δ and this readily implies that Y is compact. Then $\mathcal{A}_\varepsilon$ possesses at least $\mathrm{cat}(Y, X_\delta)$ critical points in X_δ. Using (8.51) and the properties of the category one gets

$$\mathrm{cat}(Y, Y) \ge \mathrm{cat}(X, X_\delta),$$

and the result follows. □

Remark 8.4. Let us observe that the (a) of Theorem 8.1 is a particular case of Theorem 8.4 while the (b) of Theorem 8.1 is a particular case of Theorem 8.5.

Let us now prove Theorem 8.2.

Proof of Theorem 8.2 Let Q be a minimum point of $\bar{\Sigma}$ (the other case is similar) and let $\Lambda \subset \partial\Omega$ be a compact neighborhood of Q such that

$$\min_{\Lambda} \bar{\Sigma} < \min_{\partial\Lambda} \bar{\Sigma}.$$

By (8.44) and Remark 8.3, it is easy to see that for ε sufficiently small, there results:

$$\min_{\Lambda} \mathcal{A}_\varepsilon < \min_{\partial\Lambda} \mathcal{A}_\varepsilon.$$

Hence, $\mathcal{A}_\varepsilon$ possesses a critical point Q_ε in Λ. By Lemma 8.3 we have that $u_{\varepsilon,Q_\varepsilon} = U_{Q_\varepsilon/\varepsilon} + w(\varepsilon, Q_\varepsilon)$ is a critical point of f_ε and so a solution of problem (8.5). Therefore

$$u_{\varepsilon,Q_\varepsilon}(x/\varepsilon) \simeq U_{Q_\varepsilon/\varepsilon}(x/\varepsilon) = U^{Q_\varepsilon}\left(\frac{x - Q_\varepsilon}{\varepsilon}\right)$$

is a solution of (8.1). □

9 Neumann problems with potentials: interior peak solutions

In this chapter we go on with the study done in the previous chapter about the singularly perturbed Neumann problem with potentials:

$$\begin{cases} -\varepsilon^2 \operatorname{div}(J(x)\nabla u) + V(x)u = u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ with external normal ν, $N \geq 3$, $1 < p < (N+2)/(N-2)$, $J\colon \mathbb{R}^N \to \mathbb{R}$ and $V\colon \mathbb{R}^N \to \mathbb{R}$ are C^2 functions.

In the previous chapter, extending the classical results by Ni and Takagi, in [64, 65], we proved that there exist solutions of (9.1) that concentrate at maximum and minimum points of a suitable auxiliary function defined on the boundary $\partial\Omega$ and depending only on J and V. Here we study the existence of solutions which concentrate in the interior of Ω and we will show that the concentration occurs at maximum and minimum points of the same auxiliary function introduced in Chapter 8, but now defined in Ω. We will often refer to the previous chapter.

When $J \equiv 1$ and $V \equiv 1$, interior spikes have been found by Wei (see [94]) showing that concentration occurs at local maxima of the distance function $\operatorname{dist}(\cdot, \partial\Omega)$. See also [20, 42, 93].

On J and V we will do the following assumptions:

(J) $J \in C^2(\Omega, \mathbb{R})$, J and D^2J are bounded; moreover, $J(x) \geq C > 0$ for all $x \in \Omega$;

(V) $V \in C^2(\Omega, \mathbb{R})$, V and D^2V are bounded; moreover, $V(x) \geq C > 0$ for all $x \in \Omega$.

Let us introduce an auxiliary function which will play a crucial rôle in the study of (9.1). Let $\Gamma\colon \Omega \to \mathbb{R}$ be the function defined by:

$$\Gamma(Q) = V(Q)^{\frac{p+1}{p-1} - \frac{N}{2}} J(Q)^{\frac{N}{2}}. \tag{9.2}$$

Let us observe that by **(J)** and **(V)**, Γ is well defined. Moreover this functional is the same one introduced in (8.3), but now defined in the whole Ω and not only on the boundary $\partial\Omega$.

Our main result is:

Theorem 9.1. *Let $Q_0 \in \Omega$. Suppose* **(J)** *and* **(V)** *hold. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then* (9.1) *possesses a solution u_ε concentrating at Q_ε with $Q_\varepsilon \to Q_0$, as $\varepsilon \to 0$, provided that one of the two following conditions holds:*

(a) Q_0 is a non-degenerate critical point of Γ;
(b) Q_0 is an isolated local strict minimum or maximum of Γ.

All the results in this chapter are contained in a joint paper with Simone Secchi [75].

Notation

- If $u\colon \mathbb{R}^N \to \mathbb{R}$ and $P \in \mathbb{R}^N$, we set $u_P \equiv u(\cdot - P)$.
- If U^Q is the function defined in (9.5), when there is no misunderstanding, we will often write U instead of U^Q. Moreover if $P = Q/\varepsilon$, then $U_P \equiv U^Q(\cdot - P)$.
- If $\varepsilon > 0$, we set $\Omega_\varepsilon \equiv \Omega/\varepsilon \equiv \{x \in \mathbb{R}^N : \varepsilon x \in \Omega\}$.
- With $o_\varepsilon(1)$ we denote a quantity which tends to zero as $\varepsilon \to 0$.

9.1 Preliminary lemmas and some estimates

First of all we perform the change of variables $x \mapsto \varepsilon x$ and so problem (9.1) becomes

$$\begin{cases} -\operatorname{div}(J(\varepsilon x)\nabla u) + V(\varepsilon x)u = u^p & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{9.3}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. Of course if u is a solution of (9.3), then $u(\cdot/\varepsilon)$ is a solution of (9.1).

Solutions of (9.3) are critical points $u \in H^1(\Omega_\varepsilon)$ of

$$f_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} J(\varepsilon x)|\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_\varepsilon} V(\varepsilon x)u^2 dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} |u|^{p+1}.$$

We look for solutions of (9.3) near a U^Q, the unique solution of the *limiting problem*

$$\begin{cases} -J(Q)\Delta u + V(Q)u = u^p & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(0) = \max_{\mathbb{R}^N} u, \end{cases} \tag{9.4}$$

for an appropriate choice of $Q \in \Omega$. It is easy to see that

$$U^Q(x) = V(Q)^{\frac{1}{p-1}}\, \bar{U}\left(x\sqrt{V(Q)/J(Q)}\right), \tag{9.5}$$

where $\bar{U}$ is the unique solution of

$$\begin{cases} -\Delta \bar{U} + \bar{U} = \bar{U}^p & \text{in } \mathbb{R}^N, \\ \bar{U} > 0 & \text{in } \mathbb{R}^N, \\ \bar{U}(0) = \max_{\mathbb{R}^N} \bar{U}, \end{cases}$$

which is radially symmetric and decays exponentially at infinity together with its first derivatives.

For the sake of brevity, we will often write U instead of U^Q. If $P = \varepsilon^{-1} Q \in \Omega_\varepsilon$, we set $U_P \equiv U^Q(\cdot - P)$ and

$$Z^\varepsilon \equiv \{U_P : P \in \Omega_\varepsilon\}.$$

Lemma 9.1. *For all $Q \in \Omega$ and for all ε sufficiently small, if $P = Q/\varepsilon \in \Omega_\varepsilon$, then*

$$\|\nabla f_\varepsilon(U_P)\| = O(\varepsilon). \tag{9.6}$$

Proof Repeating the calculations of the proof of Lemma 8.1, for all $v \in H^1(\Omega_\varepsilon)$, we get:

$$\begin{aligned} (\nabla f_\varepsilon(U_P) \mid v) = & \int_{\frac{\Omega - Q}{\varepsilon}} [-J(Q)\Delta U + V(Q)U - U^p] v_{-P} + J(Q) \int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} v \\ & + \int_{\frac{\Omega - Q}{\varepsilon}} (J(\varepsilon x + Q) - J(Q)) \nabla U \cdot \nabla v_{-P} \\ & + \int_{\frac{\Omega - Q}{\varepsilon}} (V(\varepsilon x + Q) - V(Q)) U v_{-P}. \end{aligned}$$

Hence, since $U \equiv U^Q$ is solution of (9.4), we get

$$\begin{aligned} (\nabla f_\varepsilon(U_P) \mid v) = J(Q) & \int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} v + \int_{\frac{\Omega - Q}{\varepsilon}} (J(\varepsilon x + Q) - J(Q)) \nabla U \cdot \nabla v_{-P} \\ & + \int_{\frac{\Omega - Q}{\varepsilon}} (V(\varepsilon x + Q) - V(Q)) U v_{-P}. \end{aligned} \tag{9.7}$$

Let us estimate the first of these three terms:

$$\left| J(Q) \int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} v \right| \le C \|v\|_{L^2(\partial\Omega_\varepsilon)} \left(\int_{\partial\Omega_\varepsilon} \left| \frac{\partial U_P}{\partial \nu} \right|^2 \right)^{1/2}.$$

First of all, we observe that there exist $\varepsilon_0 > 0$ and $C > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $v \in H^1(\Omega_\varepsilon)$, we have

$$\|v\|_{L^2(\partial\Omega_\varepsilon)} \le C \|v\|_{H^1(\Omega_\varepsilon)}.$$

Using the exponential decay of U and its derivatives, it is easy to see that

$$\int_{\partial\Omega_\varepsilon}\left|\frac{\partial U_P}{\partial\nu}\right|^2=\int_{\partial(\frac{\Omega-Q}{\varepsilon})}\left|\frac{\partial U}{\partial\nu}\right|^2=o(\varepsilon). \tag{9.8}$$

Arguing as in the previous chapter, one can prove that:

$$\int_{\frac{\Omega-Q}{\varepsilon}}(J(\varepsilon x+Q)-J(Q))\nabla U\cdot\nabla v_{-P}+\int_{\frac{\Omega-Q}{\varepsilon}}(V(\varepsilon x+Q)-V(Q))Uv_{-P}=O(\varepsilon)\|v\|. \tag{9.9}$$

Now the conclusion follows immediately from (9.7), (9.8) and (9.9). □

We here present some useful estimates that will be used in the sequel.

Proposition 9.1. *Let $P=Q/\varepsilon\in\Omega_\varepsilon$. Then we have:*

$$\int_{\Omega_\varepsilon}U_P^{p+1}=\int_{\mathbb{R}^N}\left(U^Q\right)^{p+1}+o(\varepsilon)$$

$$\int_{\partial\Omega_\varepsilon}\frac{\partial U_P}{\partial\nu}U_P=o(\varepsilon),$$

$$J(Q)\int_{\Omega_\varepsilon}|\nabla U_P|^2+V(Q)\int_{\Omega_\varepsilon}U_P^2=\int_{\mathbb{R}^N}\left(U^Q\right)^{p+1}+o(\varepsilon),$$

$$\int_{\Omega_\varepsilon}J(\varepsilon x)|\nabla U_P|^2=J(Q)\int_{\Omega_\varepsilon}|\nabla U_P|^2+\varepsilon\int_{\mathbb{R}^N}J'(Q)[x]|\nabla U^Q|^2+o(\varepsilon), \tag{9.10}$$

$$\int_{\Omega_\varepsilon}V(\varepsilon x)U_P^2=V(Q)\int_{\Omega_\varepsilon}U_P^2+\varepsilon\int_{\mathbb{R}^N}V'(Q)[x]\left(U^Q\right)^2+o(\varepsilon). \tag{9.11}$$

Proof Let us prove the first formula. We have:

$$\int_{\Omega_\varepsilon}U_P^{p+1}=\int_{\frac{\Omega-Q}{\varepsilon}}\left(U^Q\right)^{p+1}=\int_{\mathbb{R}^N}\left(U^Q\right)^{p+1}-\int_{\mathbb{R}^N\setminus\frac{\Omega-Q}{\varepsilon}}\left(U^Q\right)^{p+1}.$$

Using the exponential decay of U^Q, it is easy to see that

$$\int\limits_{\mathbb{R}^N\setminus\frac{\Omega-Q}{\varepsilon}}\left(U^Q\right)^{p+1}\le\int\limits_{\mathbb{R}^N\setminus B_{1/\sqrt{\varepsilon}}}\left(U^Q\right)^{p+1}=C\int_{\frac{1}{\sqrt{\varepsilon}}}^{\infty}r^{N-1}(U^Q(r))^{p+1}\,dr=o(\varepsilon).$$

Using the exponential decay of U^Q, also the second formula can be proved in a similar way. The other equations can be proved as in the proof of Proposition 8.1. □

9.2 The finite dimensional reduction

In this section we perform a finite dimensional reduction, following some ideas introduced in [11]. The symbol $T_{U_P}Z^\varepsilon$ denotes the tangent space to Z^ε at U_P. Let $L_{\varepsilon,Q} : (T_{U_P}Z^\varepsilon)^\perp \to (T_{U_P}Z^\varepsilon)^\perp$ denote the operator defined by setting $(L_{\varepsilon,Q}v \mid w) = D^2 f_\varepsilon(U_P)[v,w]$.

Lemma 9.2. *There exists $C > 0$ such that for ε small enough one has that*

$$||L_{\varepsilon,Q}v|| \geq C||v||, \qquad \forall\, v \in (T_{U_P}Z^\varepsilon)^\perp. \tag{9.12}$$

Proof The proof of (9.12) is completely analogous to that of equation (8.21) in Lemma 8.2, so we omit the details. □

Lemma 9.3. *For $\varepsilon > 0$ small enough, there exists a unique $w = w(\varepsilon, Q) \in (T_{U_P}Z^\varepsilon)^\perp$ such that $\nabla f_\varepsilon(U_P+w) \in T_{U_P}Z$. Such a $w(\varepsilon,Q)$ is of class C^2, resp. $C^{1,p-1}$, with respect to Q, provided that $p \geq 2$, resp. $1 < p < 2$. Moreover, the functional $\mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_{Q/\varepsilon} + w(\varepsilon,Q))$ has the same regularity of w and satisfies:*

$$\nabla \mathcal{A}_\varepsilon(Q_0) = 0 \quad \Longleftrightarrow \quad \nabla f_\varepsilon\left(U_{Q_0/\varepsilon} + w(\varepsilon, Q_0)\right) = 0.$$

Proof Let $\mathcal{P} = \mathcal{P}_{\varepsilon,Q}$ denote the projection onto $(T_{U_P}Z^\varepsilon)^\perp$. We want to find a solution $w \in (T_{U_P}Z^\varepsilon)^\perp$ of the equation $\mathcal{P}\nabla f_\varepsilon(U_P + w) = 0$. One has that $\nabla f_\varepsilon(U_P+w) = \nabla f_\varepsilon(U_P) + D^2 f_\varepsilon(U_P)[w] + R(U_P,w)$ with $||R(U_P,w)|| = o(||w||)$, uniformly with respect to U_P. Therefore, our equation is:

$$L_{\varepsilon,Q}w + \mathcal{P}\nabla f_\varepsilon(U_P) + \mathcal{P}R(U_P,w) = 0.$$

According to Lemma 9.2, this is equivalent to

$$w = N_{\varepsilon,Q}(w), \quad \text{where} \quad N_{\varepsilon,Q}(w) = -\left(L_{\varepsilon,Q}\right)^{-1}\left(\mathcal{P}\nabla f_\varepsilon(U_P) + \mathcal{P}R(U_P,w)\right).$$

By (9.6) it follows that

$$||N_{\varepsilon,Q}(w)|| = O(\varepsilon) + o(||w||). \tag{9.13}$$

Now the proof goes on as in Lemma 5.5. □

Remark 9.1. From (9.13) it follows immediately that:

$$||w|| = O(\varepsilon). \tag{9.14}$$

Moreover repeating the arguments of the previous chapter, if $\gamma = \min\{1, p-1\}$, then, for $i = 1, \ldots, N$, we infer that

$$||\partial_{P_i} w|| = O(\varepsilon^\gamma).$$

9.3 The finite dimensional functional

Theorem 9.2. *Let $Q \in \Omega$ and $P = Q/\varepsilon \in \Omega_\varepsilon$. Suppose* **(J)** *and* **(V)**. *Then, for ε sufficiently small, we get:*

$$\begin{aligned}\mathcal{A}_\varepsilon(Q) &= f_\varepsilon(U_P + w(\varepsilon, Q)) \\ &= c_0\Gamma(Q) + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} J'(Q)[x]|\nabla U|^2 + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} V'(Q)[x]U^2 + o(\varepsilon),\end{aligned}$$

where Γ is the auxiliary function introduced in (9.2) *and*

$$c_0 \equiv \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N} \bar{U}^{p+1}.$$

Moreover, for all $i = 1, \dots, N$, we get:

$$\partial_{P_i}\mathcal{A}_\varepsilon(Q) = \varepsilon c_0 \partial_{Q_i}\Gamma(Q) + o(\varepsilon), \tag{9.15}$$

and hence

$$\|\mathcal{A}_\varepsilon - c_0\Gamma\|_{C^1(\Omega)} = O(\varepsilon). \tag{9.16}$$

Proof In the sequel, to be short, we will often write w instead of $w(\varepsilon, Q)$. It is always understood that ε is taken in such a way that all the results discussed previously hold.

First of all, reasoning as in the proofs of (9.10) and (9.11) and by (9.14), we can observe that

$$\int_{\Omega_\varepsilon} J(\varepsilon x)\nabla U_P \cdot \nabla w = J(Q)\int_{\Omega_\varepsilon} \nabla U_P \cdot \nabla w + o(\varepsilon), \tag{9.17}$$

$$\int_{\Omega_\varepsilon} V(\varepsilon x)U_P\, w = V(Q)\int_{\Omega_\varepsilon} U_P\, w + o(\varepsilon). \tag{9.18}$$

Recalling (9.14), we have:

$$\begin{aligned}\mathcal{A}_\varepsilon(Q) &= \frac{1}{2}\int_{\Omega_\varepsilon} J(\varepsilon x)|\nabla(U_P + w)|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} V(\varepsilon x)(U_P + w)^2 \\ &\quad - \frac{1}{p+1}\int_{\Omega_\varepsilon}(U_P + w)^{p+1} \\ &= \frac{1}{2}\int_{\Omega_\varepsilon} J(\varepsilon x)|\nabla U_P|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} V(\varepsilon x)U_P^2 - \frac{1}{2}\int_{\Omega_\varepsilon} U_P^{p+1} \\ &\quad + \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla U_P \cdot \nabla w + \int_{\Omega_\varepsilon} V(\varepsilon x)U_P\, w \\ &\quad - \int_{\Omega_\varepsilon} U_P^p\, w + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon} U_P^{p+1} \\ &\quad - \frac{1}{p+1}\int_{\Omega_\varepsilon}\left[(U_P + w)^{p+1} - U_P^{p+1} - (p+1)U_P^p\, w\right] + o(\varepsilon) =\end{aligned}$$

[by Proposition 9.1, (9.17) and (9.18) and with our notations]

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^N} U^{p+1} + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} J'(Q)[x]|\nabla U|^2 + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} V'(Q)[x]U^2 - \frac{1}{2}\int_{\mathbb{R}^N} U^{p+1} \\
&\quad + J(Q)\int_{\Omega_\varepsilon} \nabla U_P \cdot \nabla w + V(Q)\int_{\Omega_\varepsilon} U_P\, w - \int_{\Omega_\varepsilon} U_P^p\, w \\
&\quad + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N} U^{p+1} + o(\varepsilon).
\end{aligned}
$$

Since that U is solution of (9.4), we infer

$$
\begin{aligned}
J(Q)&\int_{\Omega_\varepsilon} \nabla U_P \cdot \nabla w + V(Q)\int_{\Omega_\varepsilon} U_P\, w - \int_{\Omega_\varepsilon} U_P^p\, w \\
&= \int_{\Omega_\varepsilon} [-J(Q)\Delta U_P + V(Q)U_P - U_P^p]\, w + J(Q)\int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} w \\
&= J(Q)\int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} w = o(\varepsilon).
\end{aligned}
$$

By these considerations we can say that

$$
\mathcal{A}_\varepsilon(Q) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N} U^{p+1} + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} J'(Q)[x]|\nabla U|^2 + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} V'(Q)[x]U^2 + o(\varepsilon).
$$

Now the conclusion of the first part of the theorem follows observing that, since by (9.5)

$$
U^Q(x) = V(Q)^{\frac{1}{p-1}}\, \bar{U}\left(x\sqrt{\tfrac{V(Q)}{J(Q)}}\right),
$$

then

$$
\int_{\mathbb{R}^N} U^{p+1} = V(Q)^{\frac{p+1}{p-1} - \frac{N}{2}} J(Q)^{\frac{N}{2}} \int_{\mathbb{R}^N} \bar{U}^{p+1}.
$$

The estimate (9.15) on the derivatives of $\mathcal{A}_\varepsilon$ follows easily by repeating the arguments of the proof of Theorem 8.3. □

9.4 Proof of Theorem 9.1

In this section we will state and prove two multiplicity results for (9.1). Theorem 9.1 will follow as a particular case.

Theorem 9.3. *Let* **(J)** *and* **(V)** *hold and suppose Γ has a nondegenerate smooth manifold of critical points $M \subset \Omega$. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then* (9.1) *has at least $l(M)$ solutions that concentrate near points of M. Here $l(M)$ denotes the cup long of M (see Definition 5.1).*

Proof Fix a δ-neighborhood M_δ of M such that the only critical points of Γ in M_δ are those in M. We will take $U = M_\delta$. For ε sufficiently small, by (9.16) and Theorem 5.9, $\mathcal{A}_\varepsilon$ possesses at least $l(M)$ critical points, which are solutions of (9.3) by Lemma 9.3. Let $Q_\varepsilon \in M$ be one of these critical points, then $u_\varepsilon^{Q_\varepsilon} = U_{Q_\varepsilon/\varepsilon} + w(\varepsilon, Q_\varepsilon)$ is a solution of (9.3). Therefore

$$u_\varepsilon^{Q_\varepsilon}(x/\varepsilon) \simeq U_{Q_\varepsilon/\varepsilon}(x/\varepsilon) = U^{Q_\varepsilon}\left(\frac{x - Q_\varepsilon}{\varepsilon}\right)$$

is a solution of (9.1). □

Moreover, when we deal with local minima (resp. maxima) of Γ, the preceding results can be improved because the number of positive solutions of (9.1) can be estimated by means of the category and M does not need to be a manifold. We will give only the statement of the theorem; for the proof, see the previous chapter.

Theorem 9.4. *Let* **(J)** *and* **(V)** *hold and suppose Γ has a compact set $X \subset \Omega$ where Γ achieves a strict local minimum (resp. maximum), in the sense that there exist $\delta > 0$ and a δ-neighborhood $X_\delta \subset \Omega$ of X such that*

$$b \equiv \inf\{\Gamma(Q) : Q \in \partial X_\delta\} > a \equiv \Gamma_{|X}, \quad \left(\text{resp. } \sup\{\Gamma(Q) : Q \in \partial X_\delta\} < \Gamma_{|X}\right).$$

Then there exists $\varepsilon_0 > 0$ such that (9.1) *has at least* $\mathrm{cat}(X, X_\delta)$ *solutions that concentrate near points of X_δ, provided $\varepsilon \in (0, \varepsilon_0)$. Here* $\mathrm{cat}(X, X_\delta)$ *denotes the Lusternik-Schnirelman category of X with respect to X_δ.*

Remark 9.2. Let us observe that part (a) of Theorem 9.1 is a particular case of Theorem 9.3 while part (b) of Theorem 9.1 is a particular case of Theorem 9.4.

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