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Scuola Internazionale Superiore di Studi Avanzati - Trieste

Stability of Neumann problems and applications to the variational theory of crack propagation

CANDIDATE
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SUPERVISOR
Prof. Gianni Dal Maso

Thesis submitted for the degree of *Doctor Philosophiae*
Academic Year 2003-2004

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Contents

Introduction

In this Ph.D. thesis we deal with mathematical problems concerning the variational theory for quasi static crack propagation in elastic bodies proposed by Francfort and Marigo in [48]. The main items are stability properties for elliptic problems under boundary variations, study of functionals which present an interplay between volume and surface energies, semicontinuity and relaxation for these kinds of functionals.

The model for quasi static crack propagation proposed in [48] is inspired by the classical *Griffith's criterion*, and its main characteristic is that the path of the crack is not preassigned, whereas it is determined through a competition between the *surface energy* spent to increase the crack and the corresponding release of *bulk energy*. The authors considered an hyperelastic body, which in its reference configuration is represented by a bounded open subset Ω of $\mathbb{R}^3$. The body is subject to a time dependent load $\psi(t)$ on a part of its boundary. The quasi static crack evolution during the load process can be described by a pair $(u(t), K(t))$ where the crack $K(t)$ growths in time, and the deformation $u(t)$ provides at each time the static equilibrium of the cracked body subject to the load $\psi(t)$. The total energy associated to the pair $(u(t), K(t))$ is denoted by $E(u, K)$ and is given by

$$E(u,K) := \int_{\Omega\setminus K} f(x, \nabla u(x))\, dx + \int_K g(x,\nu)\, d\mathcal{H}^{N-1}(x). \tag{1}$$

The first term is referred to as *bulk energy* of the body and it is the internal energy of the hyperelastic body, while the second term is referred to as *surface energy* of the crack. The surface energy has to be considered as the energy dissipated during the cracking process, and in the case where g is constant it is proportional to the cracked surface. The presence of x in f and g takes into account possible inhomogeneities, while the presence of the normal ν in g takes into account a possible anisotropy of the body. In order to be a quasi static crack growth the pair $(u(t), K(t))$ has to satisfy the following conditions.

(a) irreversibility: $K(s)$ is contained in $K(t)$ for $0 \le s < t \le T$;

(b) static equilibrium: $(u(t), K(t))$ at each time minimize the *total energy* between all configurations with enlarged crack, i.e.

$$\int_{\Omega\setminus K(t)} f(x, \nabla u(t))\, dx + \int_{K(t)} g(x,\nu)\, ds \le \int_{\Omega\setminus H} f(x, \nabla v)\, dx + \int_H g(x,\nu)\, ds, \tag{2}$$

for every crack H containing $K(t)$, and for every admissible function v, i.e. with v satisfying the boundary conditions and with discontinuities contained in H.

(c) energy balance: the increment in stored energy plus the energy spent in crack increase equals the work of external forces.

We say that the pair $(u(t), K(t))$ is an *unilateral minimizer*, because it is a minimum only among pairs (v, H) with H larger than K. In view of these three properties, this model of quasi static crack propagation can be interpreted in the general framework of the theory of rate independent processes proposed by A. Mielke and coauthors: in this direction, we refer to [58] and references therein.

In [48] Francfort and Marigo suggest that the quasistatic evolution $(u(t), K(t))$ during the loading process can be obtained as a limit of a discretized in time evolution $(u_n(t), K_n(t))$ which by construction satisfies at each time the unilateral minimality property (2). We are thus led to a problem of *stability* for unilateral minimizers, i.e. whether the minimality property (2) is conserved in the passage from $(u_n(t), K_n(t))$ to $(u(t), K(t))$. These kind of stability results represent typically the main difficulty in order to prove existence results for quasi static crack growth, and their proof is often the key step. This analysis is the main topic of this thesis.

The first mathematical existence result has been obtained by Dal Maso and Toader in [36] in a two dimensional setting, in the case of linearized elasticity for anti-plane shear; more precisely with $f(x,\xi) := |\xi|^2$ and $g \equiv 1$, and then extended by Chambolle in [24] for energies involving the symmetrized gradients of planar elasticity. In order to prove the stability property for unilateral minimizers the authors used the technical assumption that the crack has an a priori bounded number of connected components, and the stability is proved with respect to the Hausdorff convergence of the cracks, while the displacement u is in the Deny-Lions spaces, usually involved in minimization problems in non smooth domains where Poincaré inequality does not hold in general. We refer to stability results (and therefore to existence result for quasi static crack growth) obtained in this setting as the *strong formulation of the problem*; this approach will be the subject of the Part I of this thesis.

In [36] the stability of unilateral minimizers for problem (2) is obtained as a consequence of the stability result for the Laplace operator with Neumann type condition on the crack, under boundary variations. The authors give a new independent proof of this stability property, in order to treat the mixed type boundary conditions on $\partial\Omega$ involved in their existence result of a quasi static growth of crack.

For an elliptic operator with prescribed boundary conditions, the notion of stability under boundary variations is defined as the continuity of the mapping $\Omega \mapsto u_\Omega$ which associates to every Ω the corresponding solution u_Ω of the elliptic problem, along a sequence of domains Ω_h. In the strong formulation the stability of the mapping is studied only along sequences Ω_h such that the complementaries K_h (which in the applications to fracture mechanics represent the cracks of a fixed domain) are in $\mathcal{K}_m(\overline{\Omega})$ for some positive integer m, where $\mathcal{K}_m(\overline{\Omega})$ is the class of all closed subsets of $\overline{\Omega}$ having at most m connected components. The convergence considered on the domains is the Hausdorff complementary topology, i.e. the Hausdorff convergence of the complementaries K_h.

The problem of stability for the Laplace operator with Neumann conditions was first studied by Chenais [28] under the assumption that the domains Ω_h satisfy a uniform cone condition, which allows to use extension operators with uniformly bounded norms. This condition excludes domains with cracks which we are interested in. The stability property in nonsmooth domains is studied by Bucur-Zolesio in [19], [20], [21] under various assumptions on Ω_h. In Chambolle-Doveri [25] the problem is studied assuming a uniform bound for the lengths of the boundaries $\mathcal{H}^1(\partial\Omega_h)$. This result has been improved by Bucur-Varchon in [16], [17], where the bound on $\mathcal{H}^1(\partial\Omega_h)$ is replaced by the weaker assumption of convergence of the two-dimensional measures of the domains, i.e., $|\Omega_h| \to |\Omega|$, which is also necessary for the stability result. The celebrated example of Neumann sieve (see [37], [60], [38], [29], [27]) shows that also the assumption of the uniform bound on the connected components of K_h is essential.

In Chapter 1 we present a generalization of these stability results to the nonlinear case obtained by Dal Maso-Ebobisse-Ponsiglione in [34]. The main difference with respect to the linear case studied in [16] and [17] is that for the nonlinear problems we can not use the method of conformal mappings. We consider nonlinear elliptic equations with Neumann boundary conditions of the form

$$\begin{cases} -\operatorname{div} a(x,\nabla u) + b(x,u) = 0 & \text{in} \quad \Omega, \\ a(x,\nabla u)\cdot\nu = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded open subset of $\mathbb{R}^2$ and $a\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$ and $b\colon \mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ are two Carathéodory functions which satisfy suitable monotonicity, coerciveness, and growth conditions. We study the stability for the nonlinear problems (3) in the class $\mathcal{K}_m(\overline{\Omega})$ with respect to the Hausdorff convergence of the cracks K_h, assuming as in [16] that $|\Omega_h| \to |\Omega|$. The convergence considered on the solutions u_Ω is the convergence in L^p spaces of the functions and of their gradients, where the exponent p is related in the usual way to the growth conditions of the functions a and b. We reduce the problem to the convergence in the sense of Mosco of the Sobolev spaces $W^{1,p}(\Omega_h)$ to the Sobolev space $W^{1,p}(\Omega)$, and we establish that stability holds for every $1 < p \leq 2$, while in the case $p > 2$ the stability result for (3) is not true under our hypotheses, as shown in Remarks 1.2.7 and 1.3.6. In the last section we present more explicit examples given by Ebobisse-Ponsiglione in [42] of non stability and we study the limit problem using the tool of Γ-convergence. The natural ask now is: For $p > 2$ are unilateral minimizer for problem (2) stable even if the elliptic problem is not stable?

In Chapter 2, following Ebobisse-Ponsiglione in [41], we give a positive answer to this question. We consider for simplicity the case $g \equiv 1$ in (2), and we prove the following stability result. Let (u_h, K_h) be a sequence of unilateral minimizers for (2), where K_h have a uniformly bounded number of connected components and are uniformly bounded in length. If ∇u_h weakly converges to some ∇u in $L^p(\Omega)$ and K_h converges to some K in the Hausdorff metric, then the pair (u, K) is a unilateral minimizer for problem (2). Moreover $\nabla u_h \to \nabla u$ strongly in $L^p(\Omega)$. The obstruction to the stability of elliptic problems when $p > 2$ is due to the fact that two connected components of the approximating sequence (K_h) can approach and touch each other in the limit fracture K, leading then to the appearance of a transmission term in the limit problem. Actually we prove that such phenomena can not appear if (u_h, K_h) are unilateral minimizers.

In Chapter 3, following Acanfora-Ponsiglione in [1], we consider the problem of quasi static crack growth for a homogeneous isotropic flexural plate subject to a time dependent vertical displacement on a part of its boundary. We formulate the model in the same spirit of [48] and we adopt the strong formulation where the cracks are supposed to have a finite number of connected components. According to Griffith's criterion, our model takes into account the competition between bulk and surface energies in the process of cracking, while it does not allow the appearing of kinks. We stress that (as far as we know) it is not completely decided in the literature whether the investigated model adequately portrays brittle fracture evolution in a plate submitted to bending.

In our model the middle surface of the plate in its reference configuration is given by an open bounded subset Ω of $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$. Let $m > 0$ be a fixed integer. The set of admissible cracks is the set $\mathcal{K}_m(\overline{\Omega})$ of all closed subsets K of $\overline{\Omega}$ whose elements have at most m connected components, while the admissible displacements are in the Deny-Lions Space $L^{2,2}(\Omega)$. The bulk energy associated to a displacement v is given by

$$B(v,v) := \frac{2E}{3(1-k^2)} \int_\Omega |v_{xx}|^2 + |v_{yy}|^2 + (2-2k)|v_{xy}|^2 \, dxdy, \tag{4}$$

(see [40]), where the Poisson's coefficient $0 < k \leq 1/2$ and the Young's modulus E measure the rigidity of the constituting material. Finally, for every admissible crack $K \in \mathcal{K}_m(\overline{\Omega})$ and for every boundary datum ψ, the associated *total energy* is given by

$$E(\psi, K) := B(u,u) + \mathcal{H}^1(K), \tag{5}$$

where u is the displacement associated to K and ψ, and $\mathcal{H}^1(K)$ is the one dimensional Hausdorff measure of K.

We now describe our model of irreversible quasi static crack growth under the action of a time dependent boundary datum. Let $\psi(t) \in AC([0,1]; W^{2,2}(\Omega))$ (i.e. the function $t \to \psi(t)$ is absolutely continuous) and let $K_0 \in \mathcal{K}_m(\overline{\Omega})$ be a preexisting crack. In our model, the *irreversible quasi static crack growth* relative to the boundary datum ψ and to the preexisting crack K_0, is a function $\Gamma : (0,1) \to \mathcal{K}_m(\overline{\Omega})$ which satisfies the following three properties:

(1) *Irreversibility of the process:*

$K_0 \subseteq \Gamma(0) \subseteq \Gamma(t_1) \subseteq \Gamma(t_2) \quad \forall 0 \leq t_1 \leq t_2 \leq 1;$

(2) *Static equilibrium:*

$E(\psi(0), \Gamma(0)) \leq E(\psi(0), H) \quad \forall H \in \mathcal{K}_m(\overline{\Omega}) : K_0 \subseteq H$ and

$E(\psi(t), \Gamma(t)) \leq E(\psi(t), H) \quad \forall t \in (0,1], \forall H \in \mathcal{K}_m(\overline{\Omega}) : \cup_{s<t}\Gamma(s) \subseteq H;$

(3) *Nondissipativity:*

the function $t \to E(g(t), \Gamma(t))$ is absolutely continuous and

$\frac{d}{dt}E(\psi(t), \Gamma(t)) = 2B(u(t), \dot{\psi}(t)),$

where $u(t)$ is the displacement relative to $\Gamma(t)$ and to $\psi(t)$.

The main result is Theorem 3.5.1, which establishes the existence of a quasi static evolution that verifies properties (1), (2) and (3) above. The static equilibrium property says that $(u(t), \Gamma(t))$ is a unilateral minimizer for the following problem

$$B\big(u(t), u(t)\big) + \mathcal{H}^1\big(\Gamma(t)\big) \leq B\big(v, v\big) + \mathcal{H}^1\big(H\big) \qquad \text{for every } H \in \mathcal{K}_m(\overline{\Omega}) : \Gamma(t) \subseteq H, \tag{6}$$

and for every admissible function v. Therefore, in order to approximate the quasi static growth with discretized in time evolutions $(u_n(t), \Gamma_n(t))$ as suggested in [48], we need a stability result for unilateral minimizers for problems involving higher order derivatives. This is done in the following theorem.

Theorem. *Let (ψ_h) be a sequence in $W^{2,2}(\Omega)$ which converges to some ψ strongly in $W^{2,2}(\Omega)$. Let $(K_h) \subset \mathcal{K}_m(\overline{\Omega})$ with $\mathcal{H}^1(K_h) \leq C$, and let $u_h \in L^{2,2}(\Omega \setminus K_h)$ be such that the pair (u_h, K_h) is a unilateral minimum relative to the boundary condition ψ_h for problem (6). Assume that*

$$D^2 u_h \rightharpoonup D^2 u \quad \text{weakly in } L^2(\Omega, \mathcal{M}^{2\times 2}), \qquad K_h \to K \text{ in the Hausdorff metric.}$$

Then the pair (u, K) is a unilateral minimum relative to the boundary condition ψ for problem (6). Moreover $D^2 u_h$ converges to $D^2 u$ strongly in $L^2(\Omega, \mathcal{M}^{2\times 2})$.

As remarked in the last section of the chapter, this stability result can be easily generalized to more general energies $E : L^{k,p}(\Omega) \to \mathbb{R}$ depending on the k-order derivatives of u and with standard p-growth hypothesis. It is also possible to treat energies depending on the point x of the reference configuration Ω, as in the case of shells. Actually it is a generalization of the stability result for energies depending on the gradient described in Chapter 2; however its proof is more complex, using some Vitali-Besicovitch covering argument in the same spirit of [47].

Finally we check that, given a rectilinearly growing crack, the classic Griffith's propagation criteria involving the stress intensity factor are recovered. The obtained expression (7) below for the energy release rate is to our knowledge new. More precisely we consider the particular case where Γ is rectilinear. In this case in [40] is given a formula for the derivative of bulk energy with respect to the growth of the crack through a $3D - 2D$ dimension reduction, under very strong regularity assumptions. Moreover in [64] is proved that this asymptotic quantity coincides with the derivative of the bulk energy $B(u_K, u_K)$ with respect to the growth of the crack (here u_K is the displacement relative to the crack K). We prove that this quantity depends only on the singular part of the displacement u, and its explicit computation leads to

$$9\pi(1+k)^2\left(\frac{b_1^2}{(7+k)^2}+\frac{b_2^2}{(5+3k)^2}\right), \tag{7}$$

where b_1 and b_2 are coefficients which appear in the singular part of u around the tip (see [40]), and play a role analogous of the so called *mode III stress intensity factor* in elasticity. Moreover, we prove that during the load process

$$9\pi(1+k)^2\left(\frac{b_1(t)^2}{(7+k)^2}+\frac{b_2(t)^2}{(5+3k)^2}\right)\leq 1, \tag{8}$$

and that the tip moves if and only if (8) is satisfied with the equality. This is the Griffith's criterion for crack propagation in our model.

The first existence result for quasi static evolutions without any technical connectedness assumption on the cracks was obtained by Francfort and Larsen in [47] in the framework of *generalized antiplanar shear* (i.e. $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$). They employed the framework of SBV functions (special functions with bounded variation), where the crack K is related with the set $S(u)$ of discontinuities of the displacement $u \in SBV(\Omega)$. We will refer to stability results (and to the relative existence results for quasi static crack growth) as the weak formulation of the problem; this will be the subject of the Part II of this thesis. In this formulation the cracks are rectifiable sets, and not connectedness assumptions are required; therefore, the celebrated examples of Neumann sieve suggest that stability properties for the solutions have to be approached through unilateral minimality assumptions. The unilateral minimality property involved in [47] is the following.

$$\int_\Omega |\nabla u|^2\,dx \leq \int_\Omega |\nabla v|^2\,dx + \mathcal{H}^{N-1}(S(v)\setminus S(u)) \qquad \text{for all } v\in SBV(\Omega), \tag{9}$$

(which corresponds to (2) with $H = S(u_n)\cup S(v)$). The authors proved the corresponding stability result through a geometrical construction which they called Transfer of Jump Sets [47, Theorem 2.1]. This was actually the new tool which permitted to solve the problem in the most general situation.

The case in which $S(u_n)$ is replaced by a rectifiable set K_n has been treated by Dal Maso, Francfort and Toader in [35], where they considered also a Carathéodory bulk energy $f(x,\xi)$ quasiconvex and with p growth assumptions in ξ, and a Borel surface energy $g(x,\nu)$ bounded and bounded away from zero. They employed a variational notion of convergence for rectifiable sets which they called σ^p-convergence to recover a crack K in the limit, and they proved a Transfer of Jump Sets theorem for $(K_n)_{n\in\mathbb{N}}$ satisfying $\mathcal{H}^{N-1}(K_n)\leq C$ [35, Theorem 5.1].

In Chapter 4 we present a different approach to the problem of stability of unilateral minimizer done by Giacomini-Ponsiglione in [52], based on Γ-convergence, and which will permit also to treat the case of varying bulk and surface energies f_n and g_n. We restrict our analysis to the scalar case. Our approach is based on the observation that the problem has a variational character. In fact, considering for a while the case of fixed energies f and g with f convex in ξ, we have that if (u_n, K_n) is a unilateral minimizer for the energy (1), then u_n is a minimum for the functional

$$\mathcal{E}_n(v) := \int_\Omega f(x,\nabla v(x))\,dx + \int_{S(v)\setminus K_n} g(x,\nu)\,d\mathcal{H}^{N-1}(x). \tag{10}$$

Then the problem of stability of unilateral minimizers can be treated in the framework of Γ-convergence which ensures the convergence of minimizers. In Section 4.3, using an abstract representation result by

Bouchitté, Fonseca, Leoni and Mascarenhas [11], we prove that the Γ-limit (up to a subsequence) of the functional $\mathcal{E}_n$ can be represented as

$$\mathcal{E}(v) := \int_\Omega f(x, \nabla v(x))\, dx + \int_{S(v)} g^-(x,\nu)\, d\mathcal{H}^{N-1}(x), \tag{11}$$

where g^- is a suitable function defined on $\Omega \times S^{N-1}$ determined only by g and $(K_n)_{n\in\mathbb{N}}$, and such that $g^- \leq g$. If we assume that $u_n \rightharpoonup u$ weakly in $SBV(\Omega)$, then by Γ-convergence we get that u is a minimizer for $\mathcal{E}$. Suppose now that K is a rectifiable set in Ω such that $S(u) \subseteq K$ and

$$g^-(x, \nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K. \tag{12}$$

Then we have immediately that the pair (u, K) is a unilateral minimizer for f and g because for all pairs (v, H) with $S(v) \subseteq H$ and $K \subseteq H$ we have

$$\begin{aligned}\int_\Omega f(x, \nabla u(x))\, dx = \mathcal{E}(u) \leq \mathcal{E}(v) &= \int_\Omega f(x, \nabla v(x))\, dx + \int_{S(v)} g^-(x,\nu)\, d\mathcal{H}^{N-1}\\ &= \int_\Omega f(x, \nabla v(x))\, dx + \int_{S(v)\setminus K} g^-(x,\nu) \leq \int_\Omega f(x, \nabla v(x))\, dx + \int_{H\setminus K} g(x,\nu).\end{aligned}$$

The rectifiable set K satisfying (12) is provided in Section 4.4, where we define a new variational notion of convergence for rectifiable sets which we call *σ-convergence*, and which departs from the notion of σ^p-convergence given in [35]. The σ-limit K of a sequence of rectifiable sets $(K_n)_{n\in\mathbb{N}}$ is constructed looking for the Γ-limit $\mathcal{H}^-$ in the strong topology of $L^1(\Omega)$ of the functionals

$$\mathcal{H}^-_n(u) := \begin{cases}\mathcal{H}^{N-1}(S(u) \setminus K_n) & u \in P(\Omega),\\ +\infty & \text{otherwise},\end{cases} \tag{13}$$

where $P(\Omega)$ is the space of piecewise constant function in Ω (see (4.19)). Roughly, the σ-limit K is the maximal rectifiable set on which the density h^- representing $\mathcal{H}^-$ vanishes. By the growth estimate on g it turns out that K is also the maximal rectifiable set on which the density g^- vanishes, so that K is the natural limit candidate for K_n in order to preserve the unilateral minimality property. The definition of σ-convergence involves only the surface energies $\mathcal{H}^-_n$, and as a consequence it does not depend on the exponent p and it is stable with respect to infinitesimal perturbations in length (see Remark 4.4.8). Moreover it turns out that the σ-limit K contains the σ^p-limit points of $(K_n)_{n\in\mathbb{N}}$, so that our Γ-convergence approach improves also the minimality property given by the previous approaches.

Our method naturally extends to the case of varying bulk and surface energies f_n and g_n, and this is indeed the main motivation for which we developed our Γ-convergence approach. The key point to recover effective energies f and g for the minimality property in the limit is a Γ-convergence result for functionals of the form

$$\int_\Omega f_n(x, \nabla u_n(x))\, dx + \int_{S(u_n)} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x). \tag{14}$$

In Section 4.3, we prove that the Γ-limit has the form

$$\int_\Omega f(x, \nabla u(x))\, dx + \int_{S(u)} g(x,\nu)\, d\mathcal{H}^{N-1}(x),$$

where f is determined only by $(f_n)_{n\in\mathbb{N}}$, and g is determined only by $(g_n)_{n\in\mathbb{N}}$, that is no interaction occurs between the bulk and the surface part of the functionals in the Γ-convergence process. A result of this type has been proved in the case of periodic homogenization (in the vectorial case, and with dependence on the trace of u in the surface part of the energy) by Braides, Defranceschi and Vitali [13].

We notice that an approach to stability in the line of Dal Maso, Francfort and Toader in the case of varying energies would have required a Transfer of Jump Sets for f_n, g_n and f, g, which seems difficult to be derived directly. Our Γ-convergence approach also provides this result (Proposition 4.5.4).

In section 4.7 we deal with the study of quasistatic crack evolution in composite materials. More precisely we study the asymptotic behavior of a quasistatic evolution $t \to (u_n(t), K_n(t))$ relative to the bulk energy f_n and the surface energy g_n. Using our stability result we prove (Theorem 4.7.1) that $t \to (u_n(t), K_n(t))$ converges to a quasistatic evolution $t \to (u(t), K(t))$ relative to the effective bulk and surface energies f and g. Moreover convergence for bulk and surface energies for all times holds. This analysis applies to the case of composite materials, i.e. materials obtained through a fine mixture of different phases. The model case is that of periodic homogenization, i.e. materials with total energy given by

$$\mathcal{E}_\varepsilon(u, K) := \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u(x)\right)\, dx + \int_K g\left(\frac{x}{\varepsilon}, \nu\right)\, d\mathcal{H}^{N-1}(x),$$

where ε is a small parameter giving the size of the mixture, and f, g are periodic in x. Our result implies that a quasisistatic crack evolution $t \to (u_\varepsilon(t), K_\varepsilon(t))$ for ε small is very near to a quasistatic evolution for the homogeneous material having bulk and surface energies f_{hom} and g_{hom}, which are obtained from f and g through periodic homogenization formulas available in the literature (see for example [13]).

In the last two chapters of the thesis we provide a discontinuous finite element approximation for the quasi static crack growth proposed in Francfort-Larsen [47] and in Dal Maso-Francfort-Toader [35] respectively. We restrict our analysis to a two dimensional setting considering only a polygonal reference configuration $\Omega \subseteq \mathbb{R}^2$, and the discretization of the domain Ω is carried out employing *adaptive triangulations* introduced by M. Negri in [62] (see also [63]). More precisely we fix two parameters $\varepsilon > 0$ and $a \in]0, \frac{1}{2}[$ and we consider a regular triangulation $\mathbf{R}_\varepsilon$ of size ε of Ω, i.e. we assume that there exist two constants c_1 and c_2 so that every triangle $T \in \mathbf{R}_\varepsilon$ contains a ball of radius $c_1\varepsilon$ and is contained in a ball of radius $c_2\varepsilon$. In order to treat the boundary data, we assume also that $\partial_D\Omega$ is composed of edges of $\mathbf{R}_\varepsilon$. On each edge $[x, y]$ of $\mathbf{R}_\varepsilon$ we consider a point z such that $z = tx + (1-t)y$ with $t \in [a, 1-a]$; these points are called *adaptive vertices*. Connecting together the adaptive vertices, we divide every $T \in \mathbf{R}_\varepsilon$ into four triangles. We take the new triangulation $\mathbf{T}$ obtained after this division as the discretization of Ω. The family of all such triangulations is denoted by $\mathcal{T}_{\varepsilon,a}(\Omega)$.

The discretization of the energy functional is obtained restricting the total energy to the family of functions u which are affine on the triangles of some triangulation $\mathbf{T}(u) \in \mathcal{T}_{\varepsilon,a}(\Omega)$ and are allowed to jump across the edges of $\mathbf{T}(u)$. We indicate this space by $\mathcal{A}_{\varepsilon,a}(\Omega)$. The main goal of our approximation results is that the weak formulation in the framework of SBV spaces adopted in [47] and [35] can be obtained as "relaxation" of a fairly simple setting involving piecewise affine deformations and piecewise linear fractures, providing also an approximation of the relevant physical quantities (total, bulk and surface energies). Our results provides a theoretical justification for numerical implementations with which however we were not concerned.

In Chapter 5 we present the approximation result obtained in Giacomini-Ponsiglione [50] for the model proposed in [47]. The boundary datum is assumed to belong to the space $\mathcal{AF}_\varepsilon(\Omega)$ of continuous functions which are affine on every triangle $T \in \mathbf{R}_\varepsilon$.

Given the boundary data $\psi \in W^{1,1}([0,1], H^1(\Omega))$ with $\psi(t) \in \mathcal{AF}_\varepsilon(\Omega)$ for all $t \in [0,1]$, we divide $[0,1]$ into subintervals $[t_i^\delta, t_{i+1}^\delta]$ of size $\delta > 0$ for $i = 0, \dots, N_\delta$, we set $\psi_i^\delta = \psi(t_i^\delta)$, and for all $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$ we indicate by $S_D^{\psi_i^\delta}(u)$ the edges of the triangulation $\mathbf{T}(u)$ contained in $\partial_D\Omega$ on which $u \neq \psi_i^\delta$. Using a variational argument we construct a *discrete evolution* $\{u^{\delta,i}_{\varepsilon,a} : i = 0, \dots, N_\delta\}$ such that $u^{\delta,i}_{\varepsilon,a} \in \mathcal{A}_{\varepsilon,a}(\Omega)$ for all $i = 0, \dots, N_\delta$, and such that considering the *discrete fracture*

$$\Gamma^{\delta,i}_{\varepsilon,a} := \bigcup_{r=0}^{i} \left[S(u^{\delta,r}_{\varepsilon,a}) \cup S^{\psi^\delta_r}_D(u^{\delta,r}_{\varepsilon,a}) \right],$$

the following *unilateral minimality property* holds:

$$\int_\Omega |\nabla u^{\delta,i}_{\varepsilon,a}|^2 \, dx \le \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^1 \left(\left(S(v) \cup S^{\psi^\delta_i}_D(v) \right) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a} \right). \tag{15}$$

Moreover we get suitable estimates for the discrete total energy

$$\mathcal{E}^{\delta,i}_{\varepsilon,a} := \|\nabla u^{\delta,i}_{\varepsilon,a}\|^2_{L^2(\Omega;\mathbb{R}^2)} + \mathcal{H}^1 \left(\Gamma^{\delta,i}_{\varepsilon,a} \right).$$

The main result of the chapter is the following theorem (Theorem 5.1.1).

Theorem 0.0.1. *Let $\psi \in W^{1,1}([0,1], H^1(\Omega))$ be such that $\|\psi(t)\|_\infty \le C$ for all $t \in [0,1]$ and let $\psi_\varepsilon \in W^{1,1}([0,1], H^1(\Omega))$ be such that $\|\psi_\varepsilon(t)\|_\infty \le C$, $\psi_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega)$ for all $t \in [0,1]$ and*

$$\psi_\varepsilon \to \psi \quad \text{strongly in } W^{1,1}([0,1], H^1(\Omega)). \tag{16}$$

Given the discrete evolution $\{t \to u^\delta_{\varepsilon,a}(t)\}$ relative to the boundary data ψ_ε, let $\Gamma^\delta_{\varepsilon,a}$ and $\mathcal{E}^\delta_{\varepsilon,a}$ be the associated fracture and total energy.

Then there exist $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, and a quasi-static evolution in the sense of [47] $\{t \to (u(t), \Gamma(t)),\ t \in [0,1]\}$ relative to the boundary data ψ, such that setting $u_n := u^{\delta_n}_{\varepsilon_n,a_n}$, $\Gamma_n := \Gamma^{\delta_n}_{\varepsilon_n,a_n}$, $\mathcal{E}_n := \mathcal{E}^{\delta_n}_{\varepsilon_n,a_n}$, the following hold:

(a) if $\mathcal{N}$ is the set of discontinuities of $\mathcal{H}^1(\Gamma(\cdot))$, for all $t \in [0,1] \setminus \mathcal{N}$ we have

$$\nabla u_n(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega; \mathbb{R}^2) \tag{17}$$

and

$$\lim_n \mathcal{H}^1(\Gamma_n(t)) = \mathcal{H}^1(\Gamma(t)); \tag{18}$$

(b) for all $t \in [0,1]$ we have

$$\lim_n \mathcal{E}_n(t) = \mathcal{E}(t). \tag{19}$$

We conclude that we have the convergence of the total energy at each time $t \in [0,1]$, and the separate convergence of bulk and surface energy for all $t \in [0,1]$ except a countable set.

In order to prove Theorem 5.1.1 we proceed in two steps. Firstly we fix a and let $\delta \to 0$ and $\varepsilon \to 0$. We obtain a crack evolution which satisfies static equilibrium and nondissipativity properties in an approximate sense, i.e. with an error depending on a which takes into account possible anisotropies that could be generated as δ and $\varepsilon \to 0$: in fact, since a is fixed, we have that the angles of the triangles in $\mathcal{T}_{\varepsilon,a}(\Omega)$ are between fixed values (determined by a), and so fractures with certain directions cannot be approximated in length. In the second step, we let $a \to 0$ and determine from $\{t \to u_a(t) : t \in [0,1]\}$ a quasi-static evolution $\{t \to u(t) : t \in [0,1]\}$ in the sense of Francfort and Larsen. Then, using a diagonal argument, we find sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, and $a_n \to 0$ satisfying Theorem 0.0.1.

The main difficulties arise in the first part of our analysis, namely when $\delta, \varepsilon \to 0$. The convergence $u^\delta_{\varepsilon,a}(t) \to u_a(t)$ in $SBV(\Omega)$ for $t \in D \subseteq [0,1]$ countable and dense is easily obtained by means of Ambrosio's Compactness Theorem. The minimality property derives from its discrete version using a variant of the transfer of jump Theorem of [47]. In all the geometric operations involved in the transfer of jump Theorem, we need to avoid degeneration of the triangles of $\mathbf{T}(u^\delta_{\varepsilon,a}(t))$ which is guaranteed from the fact that a is constant: this is the principal reason to keep a fixed in the first step. A second difficulty arises when $u_a(\cdot)$ is extended from D to the entire interval $[0,1]$: indeed it is no longer clear whether $\nabla u^\delta_{\varepsilon,a}(t) \to \nabla u_a(t)$ for $t \notin D$. Since the space $\mathcal{A}_{\varepsilon,a}(\Omega)$ is not a vector space, we cannot provide an estimate on $\|\nabla u^\delta_{\varepsilon,a}(t) - \nabla u^\delta_{\varepsilon,a}(s)\|$ with $s \in D$ and $s < t$: we thus cannot expect to recover the convergence at time t from the convergence at time s. We overcome this difficulty observing that $\nabla u^\delta_{\varepsilon,a}(t) \to \nabla \tilde{u}_a$ with $\tilde{u}_a$ satisfying a minimality property similar to (5.6) and then proving $\nabla \tilde{u}_a = \nabla u_a(t)$ by a uniqueness argument for the gradients of the solutions.

In Chapter 6 we provide a discontinuous finite element approximation for the model proposed in [35] by Dal Maso, Francfort, and Toader, and which takes into account possible volume and traction forces applied to the elastic body. The bulk energy considered in [35] is of the form

$$\int_\Omega W(x, \nabla u(x))\, dx,$$

where $W(x,\xi)$ is quasiconvex in ξ, and satisfies suitable regularity and growth assumptions (see (3.66) and (3.67)). Moreover the time dependent body and traction forces are supposed to be conservative with work given by

$$-\int_{\Omega\setminus\Gamma} F(t,x,u(x))\, dx - \int_{\partial_S\Omega} G(t,x,u(x))\, d\mathcal{H}^{N-1}(x),$$

where F and G satisfy suitable regularity and growth conditions. Finally the work made to produce the crack Γ is given by

$$\mathcal{E}^s(\Gamma) := \int_\Gamma k(x,\nu(x))\, d\mathcal{H}^{N-1}(x),$$

where $\nu(x)$ is the normal to Γ at x, and $k(x,\nu)$ satisfies standard hypotheses which guarantee lower semicontinuity. Clearly, W, F, G and k depend on the material. Let us set

$$\mathcal{E}^b(t)(u) := \int_\Omega W(x,\nabla u(x))\, dx - \int_{\Omega\setminus\Gamma} F(t,x,u(x))\, dx - \int_{\partial_S\Omega} G(t,x,u(x))\, d\mathcal{H}^{N-1}(x),$$

and

$$\mathcal{E}(t)(u,\Gamma) := \mathcal{E}^b(t)(u) + \mathcal{E}^s(\Gamma). \tag{20}$$

As in Chapter 5 we consider the finite element space associated to the parameters ε and a, we divide $[0,1]$ into subintervals $[t^\delta_i, t^\delta_{i+1}]$ of size $\delta > 0$ for $i = 0, \dots, N_\delta$, and we construct the step functions $u^\delta_{\varepsilon,a}(t)$ and $\Gamma^\delta_{\varepsilon,a}(t)$. The main result of the chapter (Theorem 5.1.1) states that there exist a quasistatic evolution $\{t \to (u(t),\Gamma(t)) : t \in [0,T]\}$ in the sense of [35] relative to the boundary deformation g and the preexisting crack Γ^0 and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, such that setting

$$u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t), \qquad \Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t),$$

for all $t \in [0,T]$ the following facts hold:

(a) $(u_n(t))_{n\in\mathbb{N}}$ is weakly precompact in $GSBV_q^p(\Omega;\mathbb{R}^2)$, and every accumulation point $\tilde{u}(t)$ is such that $\tilde{u}(t) \in AD(g(t),\Gamma(t))$, and $(\tilde{u}(t),\Gamma(t))$ satisfy the static equilibrium; moreover there exists a subsequence $(\delta_{n_k},\varepsilon_{n_k},a_{n_k})_{k\in\mathbb{N}}$ of $(\delta_n,\varepsilon_n,a_n)_{n\in\mathbb{N}}$ (depending on t) such that

$$u_{n_k}(t) \rightharpoonup u(t) \quad \text{weakly in } GSBV_q^p(\Omega;\mathbb{R}^2)$$

(see Section 3.1 for a precise definition of $GSBV_q^p(\Omega;\mathbb{R}^2)$, and of weak convergence in this space);

(b) convergence of the total energy holds, and more precisely elastic and surface energies converge separately, that is

$$\mathcal{E}^b(t)(u_n(t)) \to \mathcal{E}^b(t)(u(t)), \qquad \mathcal{E}^s(\Gamma_n(t)) \to \mathcal{E}^s(\Gamma(t)).$$

By point (a), the approximation of the deformation $u(t)$ is available only up to a subsequence depending on t: this is due to the possible non uniqueness of the minimum energy deformation associated to $\Gamma(t)$. In the case $\mathcal{E}^b(t)(u)$ is strictly convex, it turns out that the deformation $u(t)$ is uniquely determined, and we prove that (Theorem 6.7.1)

$$\nabla u_n(t) \to \nabla u(t) \qquad \text{strongly in } L^p(\Omega;\mathcal{M}^{2\times 2}),$$

and

$$u_n(t) \to u(t) \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2).$$

In order to find the fracture $\Gamma(t)$ in the limit, in Lemma 6.6.2 and Lemma 6.6.4 we adapt to the context of finite elements the notion of σ^p- convergence of sets formulated in [35]. This is the key tool to obtain the convergence of elastic and surface energies at all times $t \in [0,T]$ (while in [50] this was available only at the continuity points of $\mathcal{H}^1(\Gamma(t))$). In order to infer the static equilibrium of $\Gamma(t)$ from that of $\Gamma_n(t)$, we employ a generalization of the piecewise affine transfer of jumps [50, Proposition 5.1] (see Proposition 6.3.2).

Part I

The strong formulation

Preliminaries to Part I

In this section we state the notation and recall the preliminary results employed in the Part I of this thesis.

Deny-Lions spaces. Given an open subset U of $\mathbb{R}^2$, for every positive integers k and p the Deny-Lions space $L^{k,p}(U)$ is the space of functions in $L^p_{loc}(U)$ with derivatives of order k in $L^p(U)$.

It is well-known that Deny-Lions spaces coincide with the Sobolev spaces whenever U is bounded and has a Lipschitz boundary. It is also known that the set $\{D^k u : u \in L^{k,p}(U)\}$ is closed in the strong topology of L^p. For further properties of Deny-Lions spaces the reader is referred to [39] and [57].

In many problems it is useful to consider the following equivalence relation in $L^{1,p}(U)$:

$$v_1 \sim v_2 \qquad \text{if and only if} \qquad \nabla v_1 = \nabla v_2 \ \text{ a.e. in } U. \tag{21}$$

The corresponding quotient space is denoted by $L^{1,p}(U)/_\sim$.

Capacity. Let $1 < r < \infty$. For every subset E of $\mathbb{R}^2$, the $(1,r)$-*capacity* of E in $\mathbb{R}^2$, denoted by $C_r(E)$, is defined as the infimum of $\int_{\mathbb{R}^2}(|\nabla u|^r + |u|^r)\,dx$ over the set of all functions $u \in W^{1,r}(\mathbb{R}^2)$ such that $u \geq 1$ a.e. in a neighborhood of E. If $r > 2$, then $C_r(E) > 0$ for every nonempty set E. On the contrary, if $r = 2$ there are nonempty sets E with $C_r(E) = 0$ (for instance, $C_r(\{x\}) = 0$ for every $x \in \mathbb{R}^2$).

We say that a property $\mathcal{P}(x)$ holds C_r-*quasi everywhere* (abbreviated as C_r-*q.e.*) in a set E if it holds for all $x \in E$ except a subset N of E with $C_r(N) = 0$. We recall that the expression *almost everywhere* (abbreviated as *a.e.*) refers, as usual, to the Lebesgue measure.

A function $u\colon E \to \overline{\mathbb{R}}$ is said to be *quasi-continuous* if for every ε there exists $A_\varepsilon \subset \mathbb{R}^2$, with $C_r(A_\varepsilon) < \varepsilon$, such that the restriction of u to $E \setminus A_\varepsilon$ is continuous. If $r > 2$ every quasi-continuous function is continuous, while for $r = 2$ there are quasi-continuous functions that are not continuous.

It is well known that, for every open bounded set U, any function $u \in L^{1,r}(U)$ has a *quasi-continuous representative* $\overline{u} : U \cup \partial_L U \to \mathbb{R}$ which satisfies

$$\lim_{\rho \to 0^+} \fint_{B_\rho(x)\cap U} |u(y) - \overline{u}(x)|\,dy = 0 \quad \text{for } C_r\text{-q.e. } x \in U \cup \partial_D U,$$

where $\partial_L U$ denotes the Lipschitz part of the boundary ∂U of U and $B(x,\rho)$ is the open ball with centre x and radius ρ. We recall that if u_h converges strongly to u in $W^{1,r}(U)$, then a subsequence of $\overline{u}_h$ converges to $\overline{u}$ pointwise C_r-q.e. on U. To simplify the notation, we always identify each function $u \in W^{1,r}(U)$ with its quasi-continuous representative $\overline{u}$. For these and other properties on quasi-continuous representatives the reader is referred to [43], [54], [57], [66].

Hausdorff metric. We recall here the notion of convergence in the sense of *Kuratowski*. We say that a sequence (C_h) of closed subsets of $\mathbb{R}^2$ converges to a closed set C in the sense of Kuratowski if the following two properties hold:

(K_1) for every $x \in C$, there exists a sequence $x_h \in C_h$ such that $x_h \to x$;

(K_2) if (h_k) is a sequence of indices converging to ∞, (x_k) is a sequence such that $x_k \in C_{h_k}$ for every k, and x_k converges to some $x \in \mathbb{R}^2$, then $x \in C$.

Let us recall also that the *Hausdorff distance* between two nonempty closed subsets C_1 and C_2 of $\mathbb{R}^2$ is defined by

$$d_H(C_1, C_2) := \max\left\{ \sup_{x\in C_1} \operatorname{dist}(x, C_2)\,,\ \sup_{x\in C_2} \operatorname{dist}(x, C_1) \right\},$$

with the conventions $\operatorname{dist}(x, \emptyset) = \operatorname{diam}(\Omega)$ and $\sup \emptyset = 0$, so that

$$d_H(\emptyset\,, K) = \begin{cases} 0 & \text{if } K = \emptyset, \\ \operatorname{diam}(\Omega) & \text{if } K \neq \emptyset. \end{cases}$$

We say that a sequence (C_h) of nonempty closed subsets of $\mathbb{R}^2$ converges to a nonempty closed subset C in the *Hausdorff metric* if $d_H(C_h\,, C)$ converges to 0.

A sequence of subsets of $\mathbb{R}^2$ is said to be *uniformly bounded* if there exists a bounded subset of $\mathbb{R}^2$ which contains all sets of the sequence.

The convergence in the Hausdorff metric implies the convergence in the sense of Kuratowski, while in general the converse is false. However, if (C_h) is a uniformly bounded sequence of nonempty closed sets in $\mathbb{R}^2$, then (C_h) converges to a closed set C in the Hausdorff metric if and only if (C_h) converges to C in the sense of Kuratowski.

Γ-convergence. Let us recall the definition of De Giorgi's Γ-*convergence* in metric spaces: we refer the reader to [33] for an exhaustive treatment of this subject. Let (X, d) be a metric space. We say that a sequence $F_h : X \to [-\infty, +\infty]$ Γ-converges to $F : X \to [-\infty, +\infty]$ (as $h \to +\infty$) if for all $u \in X$ we have

(i) (Γ-*liminf inequality*) for every sequence $(u_h)_{h\in\mathbb{N}}$ converging to u in X,

$$\liminf_{h\to+\infty} F_h(u_h) \geq F(u);$$

(ii) (Γ-*limsup inequality*) there exists a sequence $(u_h)_{h\in\mathbb{N}}$ converging to u in X, such that

$$\limsup_{h\to+\infty} F_h(u_h) \leq F(u).$$

The function F is called the Γ-limit of (F_h) (with respect to d), and we write $F = \Gamma\text{-}\lim_h F_h$.

We say that a family of functionals $\{F_\varepsilon\}$ Γ-converges to F as $\varepsilon \to 0$ if for every sequence $\varepsilon_h \to 0$ as $h \to +\infty$ we have $\Gamma\text{-}\lim_h F_{\varepsilon_h} = F$.

The peculiarity of this type of convergence is its variational character explained in the following proposition.

Proposition 0.0.2. *Assume that the sequence $(F_h)_{h\in\mathbb{N}}$ Γ-converges to F and that there exists a compact set $K \subseteq X$ such that for all $h \in \mathbb{N}$*

$$\inf_{u\in K} F_h(u) = \inf_{u\in X} F_h(u).$$

Then F admits a minimum on X, $\inf_X F_h \to \min_X F$, and any limit point of any sequence $(u_h)_{h\in\mathbb{N}}$ such that

$$\lim_{h\to+\infty} \Big(F_h(u_h) - \inf_{u\in X} F_h(u) \Big) = 0$$

is a minimizer of F.

Chapter 1

Stability of nonlinear Neumann problems

Introduction

In this chapter [1] we study, in dimension two, the stability of the solutions of some nonlinear elliptic equations with Neumann boundary conditions, under perturbations of the domains in the Hausdorff complementary topology. More precisely, for every bounded open subset Ω of $\mathbb{R}^2$, we consider the problem

$$\begin{cases} -\operatorname{div} a(x, \nabla u_\Omega) + b(x, u_\Omega) = 0 & \text{in} \quad \Omega, \\ a(x, \nabla u_\Omega) \cdot \nu = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $a\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $b\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions which satisfy the standard monotonicity and growth conditions of order p, with $1 < p \leq 2$. Let Ω_n be a uniformly bounded sequence of open sets in $\mathbb{R}^2$, whose complements Ω_n^c have a uniformly bounded number of connected components. We prove that, if $\Omega_n^c \to \Omega^c$ in the Hausdorff metric and $|\Omega_n| \to |\Omega|$, then $u_{\Omega_n} \to u_\Omega$ and $\nabla u_{\Omega_n} \to \nabla u_\Omega$ strongly in L^p. The proof is obtained by showing the Mosco convergence of the Sobolev spaces $W^{1,p}(\Omega_n)$ to the Sobolev space $W^{1,p}(\Omega)$. The proof of stability for $1 < p \leq 2$ is obtained in two steps. First, under the same assumptions on Ω_h and Ω, we prove the continuity of the map $\Omega \mapsto \nabla u_\Omega$ for the solutions u_Ω of following nonlinear Neumann problems

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = 0 & \text{in} \quad \Omega, \\ a(x, \nabla u) \cdot \nu = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

This result is obtained by using the fact that the rotation by $\pi/2$ of the vectorfield $a(x, \nabla u)$ (extended to 0 on the complement Ω^c) is the gradient of a function v_Ω which is constant on each connected component of Ω^c (Proposition 1.2.6). This function plays the role of the conjugate of u_Ω used in [17] and [36] in the linear case. Another important ingredient in the proof is a result on the stability of nonlinear Dirichlet problems proved in [18], which allows to show that, if each function v_{Ω_h} is constant on each connected component of Ω_h^c, then their weak limit is constant on each connected component of Ω^c (Lemmas 1.2.3

[1] The results of this chapter are contained in Dal maso-Ebobisse-Ponsiglione [34]. We presente also some examples contained in Ebobisse-Ponsiglione [42]

and 1.2.5). The second step in the proof of the convergence of the Sobolev spaces $W^{1,p}(\Omega_h)$ is the approximation of locally constant functions in Ω by functions belonging to $W^{1,p}(\Omega_h)$, which relies on a result obtained in [16]. The hypothesis $p \leq 2$ is used both in the first and in the second step. Then we consider the case of unbounded open sets and the case of mixed boundary value problems, with a Dirichlet condition on a fixed part of the boundary.

In the case $p > 2$ the stability result for (3) and (1.1) is not true under our hypotheses, as shown in Remarks 1.2.7 and 1.3.6. In the last section we present more explicit examples given by Ebobisse-Ponsiglione in [42] of non stability and we study the limit problem using the tool of Γ-convergence.

1.1 Notation and preliminaries

Throughout the chapter p and q are real numbers, with $1 < p \leq 2 \leq q < +\infty$ and $p^{-1} + q^{-1} = 1$. The scalar product of two vectors ξ, $\zeta \in \mathbb{R}^2$ is denoted by $\xi \cdot \zeta$, and the norm of ξ by $|\xi|$. For any E, $F \subset \mathbb{R}^2$, $E \triangle F := (E \setminus F) \cup (F \setminus E)$ is the symmetric difference of E and F, and $|E|$ is the Lebesgue (outer) measure of E.

We say that a sequence (Ω_h) of open subsets of $\mathbb{R}^2$ converges to an open set Ω in the *Hausdorff complementary topology*, if $d_H(\Omega_h^c, \Omega^c)$ converges to 0, where Ω_h^c and Ω^c are the complements of Ω_h and Ω in $\mathbb{R}^2$, It is well-known (see, e.g., [44, Blaschke's Selection Theorem]) that every uniformly bounded sequence of nonempty closed sets is compact with respect to the Hausdorff convergence. This implies that every uniformly bounded sequence of open sets is compact with respect to the Hausdorff complementary topology. Moreover, a uniformly bounded sequence of open sets (Ω_h) converges to an open set Ω in the Hausdorff complementary topology, if and only if the sequence (Ω_h^c) converges to Ω^c in the sense of Kuratowski.

1.1.1 The Neumann problems

Let $a\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $b\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be two Carathéodory functions that satisfy the following assumptions: there exist $0 < c_1 \leq c_2$, $\alpha \in L^q(\mathbb{R}^2)$, and $\beta \in L^1(\mathbb{R}^2)$ such that, for almost every $x \in \mathbb{R}^2$ and for every ξ, $\xi_1, \xi_2 \in \mathbb{R}^2$ with $\xi_1 \neq \xi_2$

$$(a(x,\xi_1) - a(x,\xi_2)) \cdot (\xi_1 - \xi_2) > 0; \tag{1.2}$$

$$|a(x,\xi)| \leq \alpha(x) + c_2|\xi|^{p-1}; \tag{1.3}$$

$$a(x,\xi) \cdot \xi \geq -\beta(x) + c_1|\xi|^p. \tag{1.4}$$

We assume that b satisfies the same inequalities for every ξ, $\xi_1, \xi_2 \in \mathbb{R}$.

For every open set $\Omega \subset \mathbb{R}^2$, we consider the following nonlinear Neumann boundary value problems, where ν denotes the outward unit normal to $\partial\Omega$:

$$\begin{cases} -\operatorname{div} a(x,\nabla u) + b(x,u) = 0 & \text{in} \quad \Omega, \\ a(x,\nabla u) \cdot \nu = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

and

$$\begin{cases} -\operatorname{div} a(x,\nabla v) = 0 & \text{in} \quad \Omega\,, \\ a(x,\nabla v) \cdot \nu = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{1.6}$$

A function u is a solution of (1.5) if

$$\begin{cases} u \in W^{1,p}(\Omega), \\ \displaystyle\int_\Omega \big[a(x,\nabla u)\cdot\nabla z + b(x,u)z\big]dx = 0 \quad \forall z \in W^{1,p}(\Omega). \end{cases} \tag{1.7}$$

while v is a solution of (1.6) if

$$\begin{cases} v \in L^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(x,\nabla v)\cdot\nabla z \; dx = 0 \quad \forall z \in L^{1,p}(\Omega). \end{cases} \tag{1.8}$$

By well-known existence results for nonlinear elliptic equations with strictly monotone operators (see, e.g., Lions [56]), one can easily see that (1.7) has a unique solution in $W^{1,p}(\Omega)$. Similarly one can prove that (1.8) has a solution, and that if v_1 and v_2 are solutions of (1.8), then $\nabla v_1 = \nabla v_2$ a.e. in Ω. Note that problem (1.8) can be formulated in the quotient space $L^{1,p}(\Omega)/_\sim$ defined in (21), where a uniqueness result holds.

1.1.2 Stability of Neumann problems

In order to study the stability of (1.5) and (1.6) with respect to the variations of the open set Ω, we should be able to compare two solutions defined in two different domains. For any subset E of $\mathbb{R}^2$, the characteristic function 1_E of E is defined by $1_E(x) := 1$ for $x \in E$ and $1_E(x) := 0$ for $x \in E^c$. For every $u \in L^{1,p}(\Omega)$, the functions $u1_\Omega$ and $\nabla u 1_\Omega$ are the extensions of the functions u and ∇u which vanish in Ω^c. By means of these extensions, $W^{1,p}(\Omega)$ will be identified with the closed linear subspace X_Ω of $L^p(\mathbb{R}^2)\times L^p(\mathbb{R}^2,\mathbb{R}^2)$ defined by

$$X_\Omega := \{(u1_\Omega, \nabla u1_\Omega) : u \in W^{1,p}(\Omega)\}, \tag{1.9}$$

while the quotient space $L^{1,p}(\Omega)/_\sim$ will be identified with the closed linear subspace Y_Ω of $L^p(\mathbb{R}^2,\mathbb{R}^2)$ defined by

$$Y_\Omega := \{\nabla u1_\Omega : u \in L^{1,p}(\Omega)\}. \tag{1.10}$$

Let Ω be an open subset of $\mathbb{R}^2$ and let (Ω_h) be a sequence of open subsets of $\mathbb{R}^2$. Given a pair of Carathéodory functions $a\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$ and $b\colon \mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ satisfying (1.2)–(1.4), let u be the solution to problem (1.5) in Ω and, for every h, let u_h be the solution to problem (1.5) in Ω_h.

Definition 1.1.1. *We say that Ω is stable for the Neumann problems (1.5) along the sequence (Ω_h) if for every pair of functions a, b satisfying (1.2)–(1.4) the sequence $(u_h1_{\Omega_h})$ converges to $u1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and the sequence $(\nabla u_h 1_{\Omega_h})$ converges to $\nabla u1_\Omega$ strongly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$.*

Similarly, let v be a solution to problem (1.6) in Ω and, for every h, let v_h be a solution to problem (1.6) in Ω_h.

Definition 1.1.2. *We say that Ω is stable for the Neumann problems (1.6) along the sequence (Ω_h) if for every function a satisfying (1.2)–(1.4) the sequence $(\nabla v_h 1_{\Omega_h})$ converges to $\nabla v1_\Omega$ strongly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$.*

1.1.3 Mosco convergence

We shall prove that the notion of stability introduced in the previous definitions is equivalent to a notion of convergence for subspaces of a Banach space introduced by Mosco in [59].

Let Ω_h and Ω be open subsets of $\mathbb{R}^2$, and let X_{Ω_h} and X_Ω be the corresponding subspaces defined by (1.9). We recall that X_{Ω_h} converges to X_Ω in the sense of Mosco (see [59, Definition 1.1]) if the following two properties hold:

(M_1) for every $u \in W^{1,p}(\Omega)$, there exists a sequence $u_h \in W^{1,p}(\Omega_h)$ such that $u_h 1_{\Omega_h}$ converges to $u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla u_h 1_{\Omega_h}$ converges to $\nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$;

(M_2) if (h_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, and $u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ, then there exists $u \in W^{1,p}(\Omega)$ such that $\phi = u 1_\Omega$ and $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^2$.

Analogously, the convergence in the sense of Mosco of the spaces Y_{Ω_h} to Y_Ω defined by (1.10) is obtained by using only the convergence of the extensions of gradients, that is:

(M_1') for every $u \in L^{1,p}(\Omega)$, there exists a sequence $u_h \in L^{1,p}(\Omega_h)$ such that $\nabla u_h 1_{\Omega_h}$ converges strongly to $\nabla u 1_\Omega$ in $L^p(\mathbb{R}^2, \mathbb{R}^2)$;

(M_2') if (h_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in L^{1,p}(\Omega_{h_k})$ for every k, and $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ, then there exists $u \in L^{1,p}(\Omega)$ such that $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^2$.

Theorem 1.1.3. *Let Ω_h and Ω be open subsets of $\mathbb{R}^2$, and let X_{Ω_h} and X_Ω be the corresponding subspaces defined by (1.9). Then Ω is stable for the Neumann problems (1.5) along the sequence (Ω_h) if and only if X_{Ω_h} converges to X_Ω in the sense of Mosco.*

Proof. Assume that Ω is stable for the Neumann problems (1.5) along the sequence (Ω_h). We want to prove that X_{Ω_h} converges to X_Ω in the sense of Mosco by using only the stability of the solutions corresponding to functions a and b of the special form

$$a(x,\xi) := a_0(x) + a_1(\xi), \qquad b(x,t) := b_0(x) + b_1(t), \tag{1.11}$$

with

$$a_1(\xi) := |\xi|^{p-2}\xi, \qquad b_1(t) := |t|^{p-2}t, \qquad a_0 \in L^q(\mathbb{R}^2, \mathbb{R}^2), \qquad b_0 \in L^q(\mathbb{R}^2). \tag{1.12}$$

Let us prove (M_1). Given $u \in W^{1,p}(\Omega)$, let $a_0 := -|\nabla u|^{p-2}\nabla u 1_\Omega$ and $b_0 := -|u|^{p-2}u 1_\Omega$. Then u is the solution of (1.7) in Ω with a and b given by (1.11). Let $u_h \in W^{1,p}(\Omega_h)$ be the solution of (1.7) in Ω_h with the same a and b. By Definition 1.1.1 the sequence $(u_h 1_{\Omega_h})$ converges to $u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $(\nabla u_h 1_{\Omega_h})$ converges to $\nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. This proves (M_1).

Let us prove (M_2). Let (h_k) be a sequence of indices converging to ∞ and let (u_k) be a sequence, with $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, such that $(u_k 1_{\Omega_{h_k}})$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $(\nabla u_k 1_{\Omega_{h_k}})$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ. Let $a_0 := -a_1(\psi) = -|\psi|^{p-2}\psi$ and $b_0 := -b_1(\phi) = -|\phi|^{p-2}\phi$, let a and b be defined by (1.11), and let u^* and $u^*_{h_k}$ be the solutions of problems (1.7) in Ω and Ω_{h_k} respectively. By the stability assumption the sequence $(u^*_{h_k} 1_{\Omega_{h_k}})$ converges to $u^* 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $(\nabla u^*_{h_k} 1_{\Omega_{h_k}})$ converges to $\nabla u^* 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. This implies that $a(x, \nabla u^*_{h_k} 1_{\Omega_{h_k}})$ converges to $a(x, \nabla u^* 1_\Omega)$ strongly in $L^q(\mathbb{R}^2, \mathbb{R}^2)$ and $b(x, u^*_{h_k} 1_{\Omega_{h_k}})$ converges to $b(x, u^* 1_\Omega)$ strongly in $L^q(\mathbb{R}^2)$. Therefore

$$\lim_{k\to\infty} \int_{\Omega_{h_k}} \left[a(x, \nabla u^*_{h_k}) \cdot (\nabla u_{h_k} - \nabla u^*_{h_k}) + b(x, u^*_{h_k})(u_{h_k} - u^*_{h_k})\right] dx =$$
$$= \int_{\mathbb{R}^2} \left[a(x, \nabla u^* 1_\Omega) \cdot (\psi - \nabla u^* 1_\Omega) + b(x, u^* 1_\Omega)(\phi - u^* 1_\Omega)\right] dx. \tag{1.13}$$

By (1.7) the left hand side of (1.13) is zero. Therefore, using (1.11) and (1.12) we obtain

$$\int_{\mathbb{R}^2} \left[(a_1(\nabla u^* 1_\Omega) - a_1(\psi)) \cdot (\psi - \nabla u^* 1_\Omega) + (b_1(u^* 1_\Omega) - b_1(\phi))(\phi - u^* 1_\Omega)\right] dx = 0.$$

Using the strict monotonicity of a_1 and b_1 we obtain that $\psi = \nabla u^* 1_\Omega$ and $\phi = u^* 1_\Omega$ a.e. in $\mathbb{R}^2$.

Conversely, assume now that X_{Ω_h} converges to X_Ω in the sense of Mosco and let us prove the stability. Let a and b be two Carathéodory functions satisfying (1.2)–(1.4) and let u_h and u be the solutions to problems (1.5) in Ω_h and Ω. The weak convergence in $L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2, \mathbb{R}^2)$ of $(u_h 1_{\Omega_h}, \nabla u_h 1_{\Omega_h})$ to $(u 1_\Omega, \nabla u 1_\Omega)$ is a particular case of [59, Theorem A]. For the reader's convenience, we give here the simple proof.

Using $z := u_h$ as test function in (1.7) for Ω_h, from (1.4) we obtain that $\|u_h\|_{W^{1,p}(\Omega_h)}$ is bounded. Passing to a subsequence, we have that $u_h 1_{\Omega_h}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $\nabla u_h 1_{\Omega_h}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ. By (M_2) there exists $u^* \in W^{1,p}(\Omega)$ such that $\phi = u^* 1_\Omega$ and $\psi = \nabla u^* 1_\Omega$ a.e. in $\mathbb{R}^2$. By monotonicity for every $v \in W^{1,p}(\Omega)$ we have

$$\int_{\mathbb{R}^2} \left[a(x, \nabla v 1_\Omega) \cdot (\nabla v 1_\Omega - \nabla u_h 1_{\Omega_h}) + b(x, v 1_\Omega)(v 1_\Omega - u_h 1_{\Omega_h})\right] dx \geq \tag{1.14}$$
$$\geq \int_{\mathbb{R}^2} \left[a(x, \nabla u_h 1_{\Omega_h}) \cdot (\nabla v 1_\Omega - \nabla u_h 1_{\Omega_h}) + b(x, u_h 1_{\Omega_h})(v 1_\Omega - u_h 1_{\Omega_h})\right] dx.$$

By (M_1) there exists a sequence $v_h \in W^{1,p}(\Omega_h)$ such that $v_h 1_{\Omega_h}$ converges to $v 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla v_h 1_{\Omega_h}$ converges to $\nabla v 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. As $v_h - u_h \in W^{1,p}(\Omega_h)$, by (1.7) we have

$$\int_{\mathbb{R}^2} \left[a(x, \nabla u_h 1_{\Omega_h}) \cdot (\nabla v 1_\Omega - \nabla u_h 1_{\Omega_h}) + b(x, u_h 1_{\Omega_h})(v 1_\Omega - u_h 1_{\Omega_h})\right] dx = \tag{1.15}$$
$$= \int_{\mathbb{R}^2} \left[a(x, \nabla u_h 1_{\Omega_h}) \cdot (\nabla v 1_\Omega - \nabla v_h 1_{\Omega_h}) + b(x, u_h 1_{\Omega_h})(v 1_\Omega - v_h 1_{\Omega_h})\right] dx.$$

Since $a(x, \nabla u_h 1_{\Omega_h})$ is bounded in $L^q(\mathbb{R}^2, \mathbb{R}^2)$ and $b(x, u_h 1_{\Omega_h})$ is bounded in $L^q(\mathbb{R}^2)$, passing to the limit in (1.14) and (1.15) we obtain

$$\int_\Omega \left[a(x, \nabla v) \cdot (\nabla v - \nabla u^*) + b(x, v)(v - u^*)\right] dx \geq \tag{1.16}$$
$$\geq \lim_{h\to\infty} \int_{\mathbb{R}^2} \left[a(x, \nabla u_h 1_{\Omega_h}) \cdot (\nabla v 1_\Omega - \nabla v_h 1_{\Omega_h}) + b(x, u_h 1_{\Omega_h})(v 1_\Omega - v_h 1_{\Omega_h})\right] dx = 0.$$

Then we take $v = u^* \pm \varepsilon z$ in (1.16), with $z \in W^{1,p}(\Omega)$ and $\varepsilon > 0$. Dividing by ε, and passing to the limit as ε tends to 0, we obtain that u^* satisfies (1.7) in Ω. This proves that $u^* = u$. Therefore $u_h 1_{\Omega_h}$ converges to $u 1_\Omega$ weakly in $L^p(\mathbb{R}^2)$ and $\nabla u_h 1_{\Omega_h}$ converges to $\nabla u 1_\Omega$ weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$.

Taking $v := u$ in (1.16) we obtain that

$$\int_{\mathbb{R}^2} (a(x, \nabla u_h 1_{\Omega_h}) - a(x, \nabla u 1_\Omega)) \cdot (\nabla u_h 1_{\Omega_h} - \nabla u 1_\Omega) dx + \\ + \int_{\mathbb{R}^2} (b(x, u_h 1_{\Omega_h}) - b(x, u 1_\Omega))(u_h 1_{\Omega_h} - u 1_\Omega) dx$$

tends to 0 as $h \to \infty$. Using the monotonicity of a and b we conclude that each integral tends to 0. The strong convergence of $(u_h 1_{\Omega_h}, \nabla u_h 1_{\Omega_h})$ is now a consequence of the following lemma. □

Lemma 1.1.4. *Let (ψ_h) be a sequence in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ converging weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ. Assume that*

$$\lim_{h \to \infty} \int_{\mathbb{R}^2} (a(x, \psi_h) - a(x, \psi)) \cdot (\psi_h - \psi) dx = 0. \tag{1.17}$$

Then ψ_h converges to ψ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$.

Proof. Various forms of this lemma have been used in the study of Leray-Lions operators (see, e.g., [8, Lemma 5]). For the sake of completeness, we give here the short proof of the present version.

Let $g_h := (a(x, \psi_h) - a(x, \psi)) \cdot (\psi_h - \psi)$. By monotonicity we have $g_h \geq 0$ a.e. in $\mathbb{R}^2$, thus (1.17) implies that g_h converges to 0 strongly in $L^1(\mathbb{R}^2)$. Passing to a subsequence, we may assume that g_h converges to 0 a.e. in $\mathbb{R}^2$. Using the Cauchy inequality, from (1.3) and (1.4) we obtain for every $\varepsilon > 0$

$$c_1 |\psi_h|^p \leq \beta + a(x, \psi_h) \cdot \psi_h = \beta + g_h + a(x, \psi_h) \cdot \psi + a(x, \psi) \cdot (\psi_h - \psi) \leq \\ \leq g_h + \beta + \alpha |\psi| + |a(x, \psi)||\psi| + c_2 (\frac{\varepsilon^q}{q} + \frac{\varepsilon^p}{p}) |\psi_h|^p + c_2 (\frac{|\psi|^p}{p \varepsilon^p} + \frac{|a(x, \psi)|^q}{q \varepsilon^q}).$$

Choosing ε small enough, we obtain that there exist a constant $c_3 > 0$ and a function $\gamma \in L^1(\mathbb{R}^2)$ such that

$$c_3 |\psi_h|^p \leq g_h + \gamma. \tag{1.18}$$

Let us fix a point $x \in \mathbb{R}^2$ where $\gamma(x) < +\infty$ and where $g_h(x)$ tends to 0. By (1.18) the sequence $\psi_h(x)$ is bounded in $\mathbb{R}^2$, thus a subsequence (depending on x) converges to a vector $\xi \in \mathbb{R}^2$. By the definition of $g_h(x)$ and by the continuity of $a(x, \cdot)$ we get $(a(x, \xi) - a(x, \psi(x))) \cdot (\xi - \psi(x)) = 0$, which implies $\xi = \psi(x)$ by (1.2). Therefore the whole sequence $\psi_h(x)$ converges to $\psi(x)$. Since this is true for a.e. $x \in \mathbb{R}^2$, the strong convergence in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ follows from (1.18) by the dominated convergence theorem. □

Remark 1.1.5. Let us observe that, if (Ω_h) is a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ such that X_{Ω_h} converges to X_Ω in the sense of Mosco, then $|\Omega_h \triangle \Omega|$ converges to 0.

Let $\Sigma \subset \mathbb{R}^2$ be a bounded closed set such that $\Omega_h \subset \Sigma$ for every h. From property (M_1) it follows that $\Omega \subset \Sigma$. Indeed, if $\Omega \setminus \Sigma \neq \emptyset$, let $B \subset \Omega \setminus \Sigma$ be an open ball and let $\varphi \in C_c^\infty(B)$ with $\varphi \neq 0$. By property (M_1), there exists $u_h \in W^{1,p}(\Omega_h)$ such that $u_h 1_{\Omega_h} \to \varphi$ strongly in $L^p(\mathbb{R}^2)$. So,

$$0 < \int_B |\varphi|^p \, dx = \lim_{h \to \infty} \int_B |u_h 1_{\Omega_h}|^p \, dx = 0,$$

which is absurd.

Now, let $u := 1_\Omega$. Since $u \in W^{1,p}(\Omega)$, by property (M_1) there exists $u_h \in W^{1,p}(\Omega_h)$ such that $u_h 1_{\Omega_h} \to 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$. As $|u_h 1_{\Omega_h} - 1_\Omega|^p = 1$ a.e. on $\Omega \setminus \Omega_h$, we have

$$\lim_{h \to \infty} |\Omega \setminus \Omega_h| \leq \lim_{h \to \infty} \int_{\mathbb{R}^2} |u_h 1_{\Omega_h} - 1_\Omega|^p \, dx = 0. \tag{1.19}$$

On the other hand, up to a subsequence, 1_{Ω_h} converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to some ϕ. Hence from property (M_2), there exists $v \in W^{1,p}(\Omega)$ such that $\phi = v1_\Omega$ a.e. in $\mathbb{R}^2$. So we have that

$$\lim_{h\to\infty} |\Omega_h \setminus \Omega| = \lim_{h\to\infty} \int_{\mathbb{R}^2} 1_{\Omega_h} 1_{\Sigma\setminus\Omega}\, dx = \int_{\mathbb{R}^2} v 1_\Omega 1_{\Sigma\setminus\Omega}\, dx = 0,$$

which together with (1.19) gives $|\Omega_h \triangle \Omega| \to 0$.

Note that, if the open sets Ω_h are not uniformly bounded, it is possible that X_{Ω_h} converges to X_Ω in the sense of Mosco while $|\Omega_h \triangle \Omega|$ does not converge to 0. Consider, for example, the sequence of open sets $\Omega_h := B(0,1) \cup ((B(0, h+h^{-1}) \setminus \overline{B(0,h)})$ and $\Omega := B(0,1)$. We have that $|\Omega_h \triangle \Omega| = 2\pi + h^{-2}\pi \to 2\pi$.

Let us verify that X_{Ω_h} converges to X_Ω in the sense of Mosco. For every $u \in W^{1,p}(\Omega)$, property (M_1) is satisfied by the sequence $u_h := u1_\Omega$. For property (M_2), let (h_k) be a sequence of indices converging to ∞; let (u_k) be a sequence, with $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, such that $u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ. We set $u := \phi|_\Omega$. As $u_k \rightharpoonup u$ weakly in $L^p(\Omega)$ and $\nabla u_k \rightharpoonup \psi|_\Omega$ weakly in $L^p(\Omega, \mathbb{R}^2)$, we have $u \in W^{1,p}(\Omega)$ and $\nabla u = \psi|_\Omega$ a.e. on Ω. Now, for every ball $D \subset \Omega^c$ we have $D \subset \Omega_h^c$ for h large enough, hence

$$\int_D \phi\, dx = \lim_{k\to\infty} \int_D u_k 1_{\Omega_{h_k}}\, dx = 0.$$

So, $\phi = 0$ a.e. in Ω^c and similarly also $\psi = 0$ a.e. in Ω^c. Hence, $\phi = u1_\Omega$ and $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^2$.

Note that, in this case, Ω_h converges to Ω in the Hausdorff complementary topology, since $d_H(\Omega_h^c, \Omega^c) = h^{-1} \to 0$. By adding a small strip whose width tends to zero one can obtain an example with connected sets.

The following theorem can be proved as Theorem 1.1.3.

Theorem 1.1.6. *Let Ω_h and Ω be open subsets of $\mathbb{R}^2$, and let Y_{Ω_h} and Y_Ω be the corresponding subspaces defined by (1.10). Then Ω is stable for the Neumann problems (1.6) along the sequence (Ω_h) if and only if Y_{Ω_h} converges to Y_Ω in the sense of Mosco.*

Remark 1.1.7. If (Ω_h) is a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ such that Y_{Ω_h} converges to Y_Ω in the sense of Mosco, then $|\Omega_h \triangle \Omega|$ converges to 0. Let $\Sigma \subset \mathbb{R}^2$ be a bounded set such that $\Omega_h \subset \Sigma$ for every h. Arguing as in Remark 1.1.5, we get that $\Omega \subset \Sigma$.

Now, let $u(x) := \xi \cdot x$ with $\xi \in \mathbb{R}^2$ and $|\xi| = 1$. Since $u \in L^{1,p}(\Omega)$, by property (M_1') there exists $u_h \in L^{1,p}(\Omega_h)$ such that $\nabla u_h 1_{\Omega_h} \to \nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. As $|\nabla u_h 1_{\Omega_h} - \nabla u 1_\Omega|^p = 1$ a.e. on $\Omega \setminus \Omega_h$, we have

$$\lim_{h\to\infty} |\Omega \setminus \Omega_h| \le \lim_{h\to\infty} \int_{\mathbb{R}^2} |\nabla u_h 1_{\Omega_h} - \nabla u 1_\Omega|^p\, dx = 0. \tag{1.20}$$

On the other hand, we consider the functions $v_h \in L^{1,p}(\Omega_h)$ defined by $v_h(x) := \xi \cdot x$. Up to a subsequence, $\nabla v_h 1_{\Omega_h} = \xi 1_{\Omega_h}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to some function ψ. By property (M_2'), there exists a function $v \in L^{1,p}(\Omega)$ such that $\psi = \nabla v 1_\Omega$ a.e. in $\mathbb{R}^2$. So, it follows that

$$\xi \lim_{h\to\infty} |\Omega_h \setminus \Omega| = \lim_{h\to\infty} \int_{\mathbb{R}^2} \nabla v_h 1_{\Omega_h} 1_{\Sigma\setminus\Omega}\, dx = \int_{\mathbb{R}^2} \nabla v 1_\Omega 1_{\Sigma\setminus\Omega}\, dx = 0,$$

which together with (1.20) gives $|\Omega \triangle \Omega_h| \to 0$.

Remark 1.1.8. Theorems 1.1.3 and 1.1.6 can be applied in the following easy case. If (Ω_h) is increasing and Ω is the union of the sequence, then it is easy to see that X_{Ω_h} converges to X_Ω and Y_{Ω_h} converges to Y_Ω in the sense of Mosco. Therefore every open set is stable for the Neumann problems (1.5) and (1.6) along any increasing sequence converging to it.

1.2 Mosco convergence of Deny-Lions spaces

In this section we study the Mosco convergence of the subspaces Y_Ω introduced in (1.10) and corresponding to the Deny-Lions spaces $L^{1,p}(\Omega)$. By Theorem 1.1.6, this is equivalent to the stability for the Neumann problems (1.6).

Theorem 1.2.1. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ that converges to an open set Ω in the Hausdorff complementary topology. Assume that $|\Omega_h|$ converges to $|\Omega|$ and that Ω_h^c has a uniformly bounded number of connected components. Then Ω is stable for the Neumann problems (1.6) along the sequence (Ω_h).*

To prove Theorem 1.2.1 we use the following lemmas.

Lemma 1.2.2. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ which converges to an open set Ω in the Hausdorff complementary topology. Assume that $|\Omega_h| \to |\Omega|$. Then $1_{\Omega_h} \to 1_\Omega$ in measure, i.e., $|\Omega_h \Delta \Omega| \to 0$. Moreover, if $\varphi_h \rightharpoonup \varphi$ weakly in $L^r(\mathbb{R}^2)$ for some $1 < r < +\infty$, and $\varphi_h = 0$ a.e. in Ω_h^c, then $\varphi = 0$ a.e. in Ω^c.*

Proof. From the convergence of Ω_h to Ω in the Hausdorff complementary topology we have that $1_{\Omega \setminus \Omega_h} \to 0$ pointwise, hence $|\Omega \setminus \Omega_h| \to 0$ by the dominated convergence theorem. Since $|\Omega_h| - |\Omega| = |\Omega_h \setminus \Omega| - |\Omega \setminus \Omega_h|$, and the left hand side tends to 0 by hypothesis, we conclude that $|\Omega_h \setminus \Omega| \to 0$ too.

Now, let $\psi \in L^\infty(\mathbb{R}^2)$. As $1_{\Omega_h} \to 1_\Omega$ strongly in $L^s(\mathbb{R}^2)$ for every $1 \le s < +\infty$, we have

$$\int_{\Omega^c} \varphi\psi\, dx = \lim_{h\to\infty} \int_{\Omega^c} \varphi_h \psi\, dx = \lim_{h\to\infty} \int_{\Omega^c} 1_{\Omega_h}\varphi_h\psi\, dx = \int_{\Omega^c} 1_\Omega \varphi\psi\, dx = 0,$$

which implies $\varphi = 0$ a.e. in Ω^c. □

Lemma 1.2.3. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$, converging to an open set Ω in the Hausdorff complementary topology. If $q = 2$, assume also that the sets Ω_h^c have a uniformly bounded number of connected components. Let (v_h) be a sequence in $W^{1,q}(\mathbb{R}^2)$ converging weakly in $W^{1,q}(\mathbb{R}^2)$ to a function v and with $v_h = 0$ C_q-q.e. on Ω_h^c. Then $v = 0$ C_q-q.e. on Ω^c.*

Proof. If $q > 2$ we have to prove that (the continuous representative of) v vanishes everywhere on Ω^c. This follows easily from our hypotheses, since (v_h) converges to v uniformly on $\mathbb{R}^2$.

The case $q = 2$ is considered in [36, Lemma 5.2]. For the reader's convenience, we show here that the conclusion of the lemma follows directly from Šverák's result [65, Theorem 4.1] on the convergence of solutions to Dirichlet problems. Let w_h and w be the solutions of the problems

$$\begin{aligned} &w_h \in W^{1,2}(\mathbb{R}^2), \qquad \Delta w_h = \Delta v \quad \text{in} \quad \Omega_h, \qquad w_h = 0 \quad C_2\text{-q.e. in } \Omega_h^c, \\ &w \in W^{1,2}(\mathbb{R}^2), \qquad \Delta w = \Delta v \quad \text{in} \quad \Omega, \qquad w = 0 \quad C_2\text{-q.e. in } \Omega^c. \end{aligned} \tag{1.21}$$

Then w_h converges to w strongly in $W^{1,2}(\mathbb{R}^2)$ by [65, Theorem 4.1]. Taking $v_h - w_h$ as test function in (1.21), we obtain $\langle \Delta w_h, v_h - w_h \rangle = \langle \Delta v, v_h - w_h \rangle$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between $W^{-1,2}(\mathbb{R}^2)$ and $W^{1,2}(\mathbb{R}^2)$. Passing to the limit we obtain $\langle \Delta w, v - w \rangle = \langle \Delta v, v - w \rangle$, which implies $v = w$. Since, by definition, $w = 0$ C_2-q.e. in Ω^c, we conclude that $v = 0$ C_2-q.e. in Ω^c. □

Lemma 1.2.4. *Let $v \in W^{1,q}(\mathbb{R}^2)$ and let C_1 and C_2 be two connected closed subsets of $\mathbb{R}^2$ with $C_1 \cap C_2 \neq \emptyset$. If v is constant C_q-q.e. in C_1 and in C_2, then v is constant C_q-q.e. in $C_1 \cup C_2$.*

Proof. For $q = 2$ we the reader is referred to Proposition 2.5 in [36], while for $q > 2$ the result follows from the Sobolev embedding theorem, which yields the continuity of v. □

Lemma 1.2.5. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ which converges to an open set Ω in the Hausdorff complementary topology, and let (v_h) be a sequence in $W^{1,q}(\mathbb{R}^2)$, which converges to a function v weakly in $W^{1,q}(\mathbb{R}^2)$. Assume that Ω_h^c has a uniformly bounded number of connected components and that every function v_h is constant C_q-q.e. in each connected component of Ω_h^c. Then v is constant C_q-q.e. in each connected component of Ω^c.*

Proof. Let $C_h^1, \ldots, C_h^{n_h}$ be the connected components of Ω_h^c. Passing to a subsequence, we can assume that n_h does not depend on h, and that the sets C_h^i converge in the Hausdorff metric to some connected sets C^i as $h \to \infty$. Let us prove that v is constant q.e. in each C^i.

This is trivial if $q > 2$, since in this case (v_h) converges to v uniformly. Let us assume now $q = 2$. If C^i contains only a single point, there is nothing to prove. If C^i has more than one point, there exists $r > 0$ such that $\operatorname{diam}(C_h^i) > 2r$ for h large enough. Let us prove that the constant values c_h^i taken by v_h on C_h^i are bounded uniformly with respect to h. To this aim let us consider a point $x_h \in C_h^i$. Since $\operatorname{diam}(C_h^i) > 2r$, we have $C_h^i \setminus B(x_h, r) \neq \emptyset$, and by connectedness

$$C_h^i \cap \partial B(x_h, \rho) \neq \emptyset \qquad \text{for every } 0 < \rho < r. \tag{1.22}$$

As $v_h = c_h^i$ C_2-q.e. on C_h^i, by using polar coordinates we deduce from (1.22) the Poincaré inequality

$$\int_{B(x_h,r)} |v_h - c_h^i|^2 \, dx \leq M r^2 \int_{B(x_h,r)} |\nabla v_h|^2 \, dx,$$

where the constant M is independent of h, i, and r. Since v_h is bounded in $W^{1,2}(\mathbb{R}^2)$, it follows that c_h^i is bounded, and so it converges (up to a subsequence) to some constant c^i.

To prove that $v = c^i$ C_2-q.e. on C^i, we fix two open balls B_1 and B_2 with $B_1 \subset\subset B_2$, and a cut-off function $\varphi \in C_c^\infty(B_2)$ with $\varphi = 1$ in B_1. Then we have that $\varphi(v_h - c_h^i) = 0$ C_2-q.e. on $(B_2 \setminus C_h^i)^c$. By Lemma 1.2.3 we get $\varphi(v - c^i) = 0$ C_2-q.e. in $(B_2 \setminus C^i)^c$, hence $v = c^i$ C_2-q.e on $B_1 \cap C^i$. As B_1 is arbitrary, we obtain $v = c^i$ C_2-q.e. on C^i. If $C^i \cap C^j \neq \emptyset$, by Lemma 1.2.4 we have that v is constant C_2-q.e. on $C^i \cup C^j$. As Ω^c is the union of the sets C^i, we conclude that v is constant C_2-q.e. on each connected component of Ω^c. □

Lemma 1.2.6. *Let Ω be a bounded open subset of $\mathbb{R}^2$ and let u be a solution of problem (1.6). Let R be the rotation on $\mathbb{R}^2$ defined by $R(y_1, y_2) := (-y_2, y_1)$. Then there exists a unique function $v \in W^{1,q}(\mathbb{R}^2)$ such that $\nabla v = Ra(x, \nabla u)1_\Omega$ a.e. in $\mathbb{R}^2$. Moreover v is constant C_q-q.e. on each connected component of Ω^c.*

Proof. We consider the vector field $\Phi \in L^q(\mathbb{R}^2, \mathbb{R}^2)$ defined by $\Phi := a(x, \nabla u)1_\Omega$. By (1.8) we have $\operatorname{div} \Phi = 0$ in $\mathcal{D}'(\mathbb{R}^2)$, hence $\operatorname{rot}(R\Phi) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Since Ω is bounded, there exists a potential $v \in W^{1,q}(\mathbb{R}^2)$ such that $\nabla v = R\Phi$ a.e. in $\mathbb{R}^2$ and $v = 0$ a.e. in the interior of the unbounded connected component of Ω^c.

Given a connected component C of Ω^c, it remains to prove that v is constant C_q-q.e. on C. For every $\varepsilon > 0$ let $C_\varepsilon := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, C) < \varepsilon\}$, and let u_ε be a solution of problem (1.6) in $\Omega_\varepsilon := \Omega \setminus \overline{C}_\varepsilon$. Let v_ε be the unique function in $W^{1,q}(\mathbb{R}^2)$ such that $\nabla v_\varepsilon = Ra(x, \nabla u_\varepsilon)1_{\Omega_\varepsilon}$ a.e. in $\mathbb{R}^2$. By Remark 1.1.8, ∇u_ε converges to ∇u strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ and so v_ε converges to v strongly in $W^{1,q}(\mathbb{R}^2)$. By construction $\nabla v_\varepsilon = 0$ in C_ε. As C_ε is a connected open set containing C, we have that v_ε is constant C_q-q.e. on C. Since a subsequence of v_ε converges to v C_q-q.e. on $\mathbb{R}^2$, we conclude that v is constant C_q-q.e. on C. □

Proof of Theorem 1.2.1. Let $a \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be a Carathéodory function satisfying (1.2)–(1.4) and let u_h and u be solutions to problems (1.6) in Ω_h and Ω. Taking u_h as test function in (1.8) in Ω_h and using (1.4) we obtain that $\nabla u_h 1_{\Omega_h}$ is bounded in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. By (1.3) we obtain also that $a(x, \nabla u_h)1_{\Omega_h}$ is bounded in $L^q(\mathbb{R}^2)$. Passing to a subsequence, we may assume that $\nabla u_h 1_{\Omega_h} \rightharpoonup \Psi$ weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ and $a(x, \nabla u_h)1_{\Omega_h} \rightharpoonup \Phi$ weakly in $L^q(\mathbb{R}^2, \mathbb{R}^2)$. By (1.8) we have $\operatorname{div}(a(x, \nabla u_h)1_{\Omega_h}) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$, hence $\operatorname{div} \Phi = 0$ in $\mathcal{D}'(\mathbb{R}^2)$.

If $\Omega' \subset\subset \Omega$, by the Hausdorff complementary convergence we have $\Omega' \subset\subset \Omega_h$ for h large enough. Since the set of gradients of functions of $L^{1,p}(\Omega')$ is closed in $L^p(\Omega', \mathbb{R}^2)$, the vectorfield Ψ is the gradient of a function of $L^{1,p}(\Omega')$. As Ω' is arbitrary, we can construct $u^* \in L^{1,p}(\Omega)$ such that $\Psi = \nabla u^*$ a.e. in Ω. On the other hand, by Lemma 1.2.2 we have $\Psi = 0$ a.e. in Ω^c, hence $\Psi = \nabla u^* 1_\Omega$ a.e. in $\mathbb{R}^2$.

Let us prove that $\Phi = a(x, \nabla u^*)1_\Omega$ a.e. in $\mathbb{R}^2$. By Lemma 1.2.2 it is enough to prove the equality in every open ball $B \subset\subset \Omega$. Note that by the Hausdorff complementary convergence we have $B \subset\subset \Omega_h$ for h large enough. By adding suitable constants, we may assume that the mean values of u_h and u^* on B are zero. Thus the Poincaré inequality and the Rellich theorem imply that u_h converges to u^* strongly in $L^p(B)$.

Let $z \in W^{1,p}(B)$ and let $\varphi \in C_c^\infty(B)$ with $\varphi \geq 0$. For h large enough we have $B \subset\subset \Omega_h$, thus by monotonicity we have

$$\int_B (a(x, \nabla z) - a(x, \nabla u_h)) \cdot (\nabla z - \nabla u_h)\varphi dx \geq 0. \tag{1.23}$$

By (1.8) we have also

$$\int_B a(x, \nabla u_h) \cdot \nabla((z - u_h)\,\varphi) dx = 0,$$

which, together with (1.23), gives

$$\int_B a(x, \nabla z) \cdot \nabla((z - u_h)\varphi)dx - \int_B (a(x, \nabla z) - a(x, \nabla u_h)) \cdot \nabla\varphi\,(z - u_h) dx \geq 0. \tag{1.24}$$

We can pass to the limit in each term of (1.24) and we get

$$\int_B a(x, \nabla z) \cdot \nabla((z - u^*)\varphi)dx - \int_B (a(x, \nabla z) - \Phi) \cdot \nabla\varphi\,(z - u^*) dx \geq 0. \tag{1.25}$$

As $\operatorname{div} \Phi = 0$ in $\mathcal{D}'(B)$, we have

$$\int_B \Phi \cdot \nabla((z - u^*)\varphi)dx = 0. \tag{1.26}$$

From (1.25) and (1.26) we obtain

$$\int_B (a(x, \nabla z) - \Phi) \cdot (\nabla z - \nabla u^*)\,\varphi dx \geq 0.$$

As φ is arbitrary, we get $(a(x, \nabla z) - \Phi) \cdot (\nabla z - \nabla u^*) \geq 0$ a.e. in B. In particular, taking $z(x) := u^*(x) \pm \varepsilon\xi \cdot x$, with $\xi \in \mathbb{R}^2$ and $\varepsilon > 0$, we obtain $\pm(a(x, \nabla u^* \pm \varepsilon\xi) - \Phi) \cdot \xi \geq 0$ a.e. in B. As ε tends to zero we get $(a(x, \nabla u^*) - \Phi) \cdot \xi = 0$ a.e. in B, which implies that $a(x, \nabla u^*) = \Phi$ a.e. in B by the arbitrariness of ξ.

Let us prove now that

$$\int_\Omega a(x, \nabla u^*) \cdot \nabla z\, dx = 0 \qquad \forall z \in L^{1,p}(\Omega). \tag{1.27}$$

By Lemma 1.2.6 for every h there exists $v_h \in W^{1,q}(\mathbb{R}^2)$ such that $\nabla v_h = Ra(x, \nabla u_h)1_{\Omega_h}$ a.e. in $\mathbb{R}^2$. Moreover v_h is constant C_q-q.e. on each connected component of Ω_h^c. As $a(x, \nabla u_h)1_{\Omega_h}$ converges to $a(x, \nabla u^*)1_\Omega$ weakly in $L^q(\mathbb{R}^2, \mathbb{R}^2)$, there exists a function $v \in W^{1,q}(\mathbb{R}^2)$ such that $v_h \rightharpoonup v$ weakly in $W^{1,q}(\mathbb{R}^2)$ and $\nabla v = Ra(x, \nabla u^*)1_\Omega$ a.e. in $\mathbb{R}^2$. So, we have to prove that

$$\int_\Omega R\nabla v \cdot \nabla z\, dx = 0 \qquad \forall z \in L^{1,p}(\Omega).$$

From the Lemma 1.2.5 it follows that v is constant C_q-q.e. on the connected components of Ω^c. By [54, Theorem 4.5] we can approximate v strongly in $W^{1,q}(\mathbb{R}^2)$ by a sequence of functions $v_h \in C_c^\infty(\mathbb{R}^2)$ that are constant in suitable neighborhoods U_h^i of each connected component C^i of Ω^c. Let $z \in L^{1,p}(\Omega)$ and $z_h \in W_0^{1,p}(\Omega)$ such that $z_h = z$ in $\Omega \setminus \bigcup_i U_h^i$. Then, we have

$$\int_\Omega R\nabla v_h \cdot \nabla z \, dx = \int_\Omega R\nabla v_h \cdot \nabla z_h \, dx = 0, \tag{1.28}$$

where the last equality follows from the fact that the vector field $R\nabla v_h$ is divergence free. Then, passing to the limit in (1.28) for $h \to \infty$, we get

$$\int_\Omega a(x, \nabla u^*) \cdot \nabla z \, dx = -\int_\Omega R\nabla v \cdot \nabla z dx = 0.$$

So u^* is a solution of (1.8) in Ω, hence $\nabla u^* = \nabla u$ a.e. in Ω by uniqueness of the gradients. This implies that $\nabla u_h 1_{\Omega_h}$ converges to $\nabla u 1_\Omega$ weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ and $a(x, \nabla u_h)1_{\Omega_h}$ converges to $a(x, \nabla u)1_\Omega$ weakly in $L^q(\mathbb{R}^2, \mathbb{R}^2)$. Since $|\Omega_h \triangle \Omega|$ tends to 0 by Lemma 1.2.2, from the identity $a(x, \nabla u_h 1_{\Omega_h}) = a(x, \nabla u_h)1_{\Omega_h} + a(x, 0)1_{\Omega_h^c}$ we conclude also that $a(x, \nabla u_h 1_{\Omega_h})$ converges to $a(x, \nabla u 1_\Omega)$ weakly in $L^q(\mathbb{R}^2, \mathbb{R}^2)$.

To prove the strong convergence, we consider the integral

$$I_h := \int_{\mathbb{R}^2} (a(x, \nabla u_h 1_{\Omega_h}) - a(x, \nabla u 1_\Omega)) \cdot (\nabla u_h 1_{\Omega_h} - \nabla u 1_\Omega) dx.$$

Since by (1.8)

$$\int_{\Omega_h} a(x, \nabla u_h) \cdot \nabla u_h \, dx = 0 \qquad \text{and} \qquad \int_\Omega a(x, \nabla u) \cdot \nabla u \, dx = 0,$$

we have

$$I_h = -\int_\Omega a(x, \nabla u_h 1_{\Omega_h}) \cdot \nabla u \, dx - \int_{\Omega_h} a(x, \nabla u 1_\Omega) \cdot \nabla u_h 1_{\Omega_h} \, dx.$$

Therefore

$$\lim_{h \to \infty} I_h = -2 \int_\Omega a(x, \nabla u) \cdot \nabla u \, dx = 0, \tag{1.29}$$

where the last equality can be deduced from (1.8). The strong convergence in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ of $\nabla u_h 1_{\Omega_h}$ to $\nabla u 1_\Omega$ follows now from (1.29) and from Lemma 1.1.4. □

Remark 1.2.7. In the case $p > 2$ the stability result for problem (1.6) is not true under our hypotheses. Indeed, let us consider

$$S := [1,3] \times \{0\}, \quad S_h := ([1, 2 - 1/h] \cup [2 + 1/h, 3]) \times \{0\},$$

$$\Omega := B(0,3) \setminus (\overline{B(0,1)} \cup S) \quad \text{and} \quad \Omega_h := B(0,3) \setminus (\overline{B(0,1)} \cup S_h).$$

Let $\varphi \in C_c^\infty(0, \infty)$ be such that $\varphi(\rho) = \rho^{-p}$ for $1 \le \rho \le 3$. We set

$$a(x, \xi) := |\xi|^{p-2}\xi - \varphi(|x|)Rx,$$

where R is the rotation by $\pi/2$ defined by $Rx := (-x_2, x_1)$. Let u_h and u be solutions of problems (1.6) in Ω and Ω_h, with $\int_{\Omega_h} u_h \, dx = \int_\Omega u \, dx = 0$. For every $x \in \Omega$, let $0 < \theta(x) < 2\pi$ be the angle between x and the positive x_1-axis. As $\nabla\theta(x) = Rx/|x|^2$, we have that $u = \theta - \pi$ in Ω.

If the open set Ω were stable for problem (1.6) along the sequence (Ω_h), then ∇u_h would converge strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$. By the Poincaré inequality we would have that u_h converges strongly to u in $W^{1,p}(\Omega)$.

For every $v \in W^{1,p}(\Omega)$, let v^+ and v^- be the upper and lower traces of v on S, defined by

$$v^+(x) := \lim_{\substack{y \to x \\ y_2 > 0}} v(y) \quad \text{and} \quad v^-(x) := \lim_{\substack{y \to x \\ y_2 < 0}} v(y). \tag{1.30}$$

From the convergence of u_h to u in $W^{1,p}(\Omega)$, we obtain that $u_h^+ \to u^+$ and $u_h^- \to u^-$ uniformly on S (recall that $p > 2$ here). Since $u_h^+(2,0) = u_h^-(2,0)$ by the continuity of u_h, we obtain $u^+(2,0) = u^-(2,0)$, which contradicts the fact that $u^+(2,0) = -\pi$ and $u^-(2,0) = \pi$, being $u = \theta - \pi$.

1.3 Mosco convergence of Sobolev spaces

In this section we study the convergence in the sense of Mosco of the subspaces X_Ω introduced in (1.9) and corresponding to the Sobolev spaces $W^{1,p}(\Omega)$. The convergence of X_{Ω_h} to X_Ω will be obtained from the convergence of Y_{Ω_h} to Y_Ω and from the following approximation theorem for functions which are locally constant on the limit open set Ω.

Theorem 1.3.1. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ which converges to an open set Ω in the Hausdorff complementary topology. Assume that $|\Omega_h|$ converges to $|\Omega|$ and that Ω_h^c is connected for every h. Then for every $u \in W^{1,p}(\Omega)$ with $\nabla u = 0$ a.e. in Ω, there exists a sequence $u_h \in W^{1,p}(\Omega_h)$ such that $u_h 1_{\Omega_h}$ converges to $u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla u_h 1_{\Omega_h}$ converges to 0 strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$.*

The proof of this theorem is postponed. We are now in a position to state the main result of the chapter.

Theorem 1.3.2. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$ which converges to an open set Ω in the Hausdorff complementary topology, with $|\Omega_h|$ converging to $|\Omega|$. Assume that Ω_h^c has a uniformly bounded number of connected components. Then X_{Ω_h} converges to X_Ω in the sense of Mosco.*

To prove Theorem 1.3.2 we need the following localization lemma.

Lemma 1.3.3. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$, and let Ω be a bounded open subset of $\mathbb{R}^2$. Assume that for every $x \in \mathbb{R}^2$ there exists $\varepsilon > 0$ such that the sequence $X_{B(x,\varepsilon)\cap\Omega_h}$ converges to $X_{B_\varepsilon(x)\cap\Omega}$ in the sense of Mosco. Then X_{Ω_h} converges to X_Ω in the sense of Mosco.*

Proof. Condition (M_2) is easy, and condition (M_1) can be obtained by using a partition of unity. □

Proof of Theorem 1.3.2. Step 1. Assume that Ω_h^c is connected for every h. Let us prove (M_2). Let (h_k) be a sequence of indices converging to ∞, (u_k) be a sequence, with $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, such that $u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ. From Lemma 1.2.2 it follows that ϕ and ψ vanish a.e. in Ω^c.

Let $\Omega' \subset\subset \Omega$ be an open set. By the Hausdorff complementary convergence we have $\Omega' \subset\subset \Omega_h$ for h large enough. So, $u_k|_{\Omega'}$ converges weakly to $\phi|_{\Omega'}$ in $L^p(\Omega')$ and $\nabla u_k|_{\Omega'}$ converges weakly to $\psi|_{\Omega'}$ in $L^p(\Omega', \mathbb{R}^2)$. Hence $\phi|_{\Omega'} \in W^{1,p}(\Omega')$ and $\psi|_{\Omega'} = \nabla\phi|_{\Omega'}$ in Ω'. From the arbitrariness of Ω', it follows that the function $u := \phi|_\Omega$ belongs to $W^{1,p}(\Omega)$, $\phi = u 1_\Omega$ and $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^2$.

Now let us prove (M_1). Let $u \in W^{1,p}(\Omega)$. We write $\Omega := \bigcup_{i=1}^m \Omega_i$, where $1 \leq m \leq \infty$ and (Ω_i) is the family of connected components of Ω. Since the set of functions u satisfying (M_1) is a closed linear subspace of $W^{1,p}(\Omega)$, by a density argument it is enough to prove (M_1) when u belongs to $L^\infty(\Omega)$ and vanishes on all connected components of Ω except one. By renumbering the sequence (Ω_i), we may assume that u vanishes on Ω_i for every $i \geq 2$.

From Theorem 1.2.1 on the convergence of Y_{Ω_h} to Y_Ω in the sense of Mosco, there exists a sequence $z_h \in L^{1,p}(\Omega_h)$ such that $\nabla z_h 1_{\Omega_h}$ converges strongly to $\nabla u 1_\Omega$ in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. Let us fix a nonempty open set $A_0 \subset\subset \Omega_1$. We can assume that $\int_{A_0} z_h dx = \int_{A_0} u dx$. For every smooth connected open set A, with $A_0 \subset\subset A \subset\subset \Omega_1$, by the Poincaré inequality we have that $z_h|_A \to u|_A$ strongly in $W^{1,p}(A)$.

We consider now $w_h := (-\|u\|_\infty) \vee z_h \wedge \|u\|_\infty$. We have that $w_h|_A \to u|_A$ strongly in $W^{1,p}(A)$ for every open set $A \subset\subset \Omega_1$. Moreover, for every open set $E \subset\subset \Omega$ the function $w_h|_E$ belongs to $W^{1,p}(E)$ for h large enough. As $\|w_h\|_\infty \leq \|u\|_\infty$ and $|\nabla w_h|_E| \leq |\nabla z_h|_E|$, the sequence $(w_h|_E)$ is bounded in $W^{1,p}(E)$. By the Rellich theorem, there exists $w \in W^{1,p}(\Omega)$ such that $w_h|_E$ converges to $w|_E$ strongly in $L^p(E)$ for every open set $E \subset\subset \Omega$ with smooth boundary. As $w_h|_A \to u|_A$ strongly in $W^{1,p}(A)$ for every open set $A \subset\subset \Omega_1$, we have that $w = u$ a.e. in Ω_1.

For every open set $E \subset\subset \Omega \setminus \Omega_1$, since $|\nabla w_h| \leq |\nabla z_h|$ and $\nabla z_h|_E \to \nabla u|_E = 0$ strongly in $L^p(E, \mathbb{R}^2)$, we have $\nabla w_h|_E \to 0$ strongly in $L^p(E, \mathbb{R}^2)$. Therefore we get $\nabla w = 0 = \nabla u$ a.e. in $\Omega \setminus \Omega_1$, which together with the result obtained in Ω_1 implies that $\nabla w_h|_E$ converges to $\nabla u|_E$ strongly in $L^p(E, \mathbb{R}^2)$ for every $E \subset\subset \Omega$. In particular, we obtain that the function $u - w$ is locally constant in Ω.

We claim that $\nabla w_h 1_{\Omega_h}$ converges to $\nabla w 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. Indeed for every $E \subset\subset \Omega$ we have $\|\nabla w_h 1_{\Omega_h} - \nabla u 1_\Omega\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)} \leq \|\nabla w_h 1_E - \nabla u 1_E\|_{L^p(E,\mathbb{R}^2)} + \|\nabla z_h 1_{\Omega_h} 1_{E^c}\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)} + \|\nabla u 1_{\Omega\setminus E}\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)}$. Hence

$$\limsup_{h\to\infty} \|\nabla w_h 1_{\Omega_h} - \nabla u 1_\Omega\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)} \leq 2\|\nabla u 1_{\Omega\setminus E}\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)},$$

and by letting $E \nearrow \Omega$ we prove the claim. In a similar way, we obtain also that $w_h 1_{\Omega_h}$ converges to $w 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$.

Since $u - w$ is locally constant in Ω, from Theorem 1.3.1 there exists $v_h \in W^{1,p}(\Omega_h)$ such that $v_h 1_{\Omega_h} \to (u - w)1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla v_h 1_{\Omega_h} \to 0$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. Therefore $w_h + v_h \in W^{1,p}(\Omega_h)$, $(w_h + v_h)1_{\Omega_h} \to u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$, and $\nabla(w_h + v_h)1_{\Omega_h} \to \nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$, which give property (M_1).

Step 2. We now remove the hypothesis that Ω_h^c is connected. Let $C_h^1, \ldots, C_h^{n_h}$ be the connected components of Ω_h^c. Passing to a subsequence we can assume that n_h does not depend on h and that the sets C_h^i converge in the Hausdorff metric to some connected sets C^i as $h \to \infty$. Let $C^{i_1}, \ldots, C^{i_d}$ be those C^i having at least two points. We set

$$\Omega^* := \Big(\bigcup_{j=1}^d C^{i_j}\Big)^c \quad \text{and} \quad \Omega_h^* := \Big(\bigcup_{j=1}^d C_h^{i_j}\Big)^c.$$

We have that $\Omega \subset \Omega^*$, $\Omega_h \subset \Omega_h^*$, and, by construction, Ω_h^* converges in the Hausdorff complementary topology to Ω^* and $|\Omega_h^*| \to |\Omega^*|$ (because $|C_h^i| \to 0$ if $i \neq i_1, \ldots, i_d$). There exists some $\eta > 0$ such that $\operatorname{diam} C^{i_j} > \eta$ for every j, hence $\operatorname{diam} C_h^{i_j} > \eta$ for h large enough. Let us observe that, for every $x \in \mathbb{R}^2$, the sequence $B(x, \eta/2) \cap \Omega_h^*$ converges to $B(x, \eta/2) \cap \Omega^*$ in the Hausdorff complementary topology and also $|B(x, \eta/2) \cap \Omega_h^*| \to |B(x, \eta/2) \cap \Omega^*|$ (by Lemma 1.2.2). As $\operatorname{diam} C_h^{i_j} > \eta$, it is easy to see that $(B(x, \eta/2) \cap \Omega_h^*)^c$ is connected for h large enough. So, from Step 1 we obtain the Mosco convergence of $X_{B(x,\eta/2)\cap\Omega_h^*}$ to $X_{B(x,\eta/2)\cap\Omega^*}$. Now, using Lemma 1.3.3 we get the Mosco convergence of $X_{\Omega_h^*}$ to X_{Ω^*}.

As $\Omega_h^* \setminus \Omega_h$ is the union of a uniformly bounded number of sets with diameter tending to 0, using the fact that $1 < p \leq 2$, we deduce that $C_p(\Omega_h^* \setminus \Omega_h) \to 0$. Let us show that this implies that X_{Ω_h} converges to X_Ω in the sense of Mosco.

For property (M_1), let $u \in W^{1,p}(\Omega)$. Since the set $\Omega^* \setminus \Omega$ is finite, we have that $C_p(\Omega^* \setminus \Omega) = 0$. Hence, $u \in W^{1,p}(\Omega^*)$. So, there exists $u_h^* \in W^{1,p}(\Omega_h^*)$ such that $u_h^* 1_{\Omega_h^*}$ converges to $u 1_{\Omega^*}$ strongly in

$L^p(\mathbb{R}^2)$ and $\nabla u_h^* 1_{\Omega_h^*}$ converges to $\nabla u 1_{\Omega^*}$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. Setting $u_h = u_h^*|_{\Omega_h}$, we obtain that $u_h 1_{\Omega_h}$ converges to $u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla u_h 1_{\Omega_h}$ converges to $\nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$, and so property (M_1) holds.

Let us prove property (M_2). Let (h_k) be a sequence of indices converging to ∞, (u_k) be a sequence, with $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, such that $u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ. From Lemma 1.2.2 it follows that ϕ and ψ vanish a.e. in Ω^c.

As $C_p(\Omega_{h_k}^* \setminus \Omega_{h_k}) \to 0$, there exists a sequence $\varphi_k \in W^{1,p}(\mathbb{R}^2)$ converging strongly to 0 in $W^{1,p}(\mathbb{R}^2)$ such that $\varphi_k = 1$ a.e. in a neighborhood of $\Omega_{h_k}^* \setminus \Omega_{h_k}$.

We set

$$u_k^* := u_k(1 - \varphi_k).$$

Note that $u_k^* \in W^{1,p}(\Omega_{h_k}^*)$ and that $u_k^* 1_{\Omega_{h_k}^*}$ converges weakly in $L^p(\mathbb{R}^2)$ to ϕ, while $\nabla u_k^* 1_{\Omega_{h_k}^*}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to ψ. So, from the Mosco convergence of $X_{\Omega_h^*}$ to X_{Ω^*}, it follows that there exists $u^* \in W^{1,p}(\Omega^*)$ such that $\phi = u^* 1_{\Omega^*}$ and $\psi = \nabla u^* 1_{\Omega^*}$ a.e. in $\mathbb{R}^2$. By setting $u = u^*|_\Omega$, we get that $\phi = u 1_\Omega$ and $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^2$ and the proof of (M_2) is complete. □

The rest of this section is devoted to the proof of Theorem 1.3.1. To this aim we will need some preliminary results.

Lemma 1.3.4. *Let (Ω_h) be a uniformly bounded sequence of open subsets of $\mathbb{R}^2$, which converges to an open set Ω in the Hausdorff complementary topology, with $|\Omega_h|$ converging to $|\Omega|$. Assume that $\Omega_h = \Omega_h^1 \cup \Omega_h^2$ with Ω_h^i open and $\Omega_h^1 \cap \Omega_h^2 = \emptyset$. Assume also that (Ω_h^i) converges to an open set Ω^i, $i = 1, 2$, in the Hausdorff complementary topology. Then*

(i) $\Omega^1 \cap \Omega^2 = \emptyset$,

(ii) $\Omega^1 \cup \Omega^2 = \Omega$,

(iii) $|\Omega^i| = \lim_h |\Omega_h^i|$, $i = 1, 2$.

In particular, if Ω_h^0 is union of connected components of Ω_h and converges to an open set Ω^0 in the Hausdorff complementary topology, then Ω^0 is union of connected components of Ω and $|\Omega_h^0|$ converges to $|\Omega^0|$.

Proof. (i) and (ii) are easy consequences of the convergence in the Hausdorff complementary topology, while (iii) follows from Lemma 1.2.2. As Ω_h^0 is union of connected components of Ω_h, then the set $\Omega_h' := \Omega_h \setminus \Omega_h^0$ is open in the relative topology of Ω_h. Up to a subsequence, Ω_h' converges to an open set Ω' in the Hausdorff complementary topology. From (i) and (ii) we have that $\Omega^0 \cap \Omega' = \emptyset$, and $\Omega^0 \cup \Omega' = \Omega$; hence Ω^0 is union of connected components of Ω. The last assertion follows from (iii). □

The following lemma, proved by Bucur and Varchon in [16], will also be used in the proof of Theorem 1.3.1.

Lemma 1.3.5. *Let Ω be a bounded open set in $\mathbb{R}^2$ and let a and b be two points in two different connected components Ω_a and Ω_b of Ω, whose distance from Ω^c is greater than 10δ for some $\delta > 0$. Let U be an open subset of $\mathbb{R}^2$ such that U^c is connected and $d_H(U^c, \Omega^c) < \delta$. Then there exists $x \in \Omega^c$ such that the closed square $Q(x, 9\delta)$, with centre x and side length 9δ, intersects any curve contained in U and joining the points a and b.*

Proof of Theorem 1.3.1. By a density argument, it is sufficient to prove that for every connected component Ω^0 of Ω there exists a sequence $u_h \in W^{1,p}(\Omega_h)$ such that $u_h 1_{\Omega_h}$ converges strongly to 1_{Ω^0} in $L^p(\mathbb{R}^2)$ and $\nabla u_h 1_{\Omega_h}$ converges strongly to 0 in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. Let $a^0 \in \Omega^0$ and let Ω_h^0 be the connected component of Ω_h which contains a^0 (which is defined for h large enough). Up to a subsequence, Ω_h^0 converges in the Hausdorff complementary topology to some open set E. From Lemma 1.3.4 it follows that

$$E = \bigcup_{i=0}^{m} \Omega^i,$$

where $0 \leq m \leq \infty$ and (Ω^i) is a family of connected components of Ω (including Ω^0), and

$$|\Omega_h^0| \to |E|. \tag{1.31}$$

Let $0 < \varepsilon < |\Omega^0|$ be fixed. There exist a finite integer $n_\varepsilon \geq 1$ and an open set Ω_ε such that

$$E = \bigcup_{i=0}^{n_\varepsilon} \Omega^i \cup \Omega_\varepsilon, \tag{1.32}$$

where $|\Omega_\varepsilon| \leq \varepsilon$ and $\Omega^i \cap \Omega_\varepsilon = \emptyset$ for every $i \leq n_\varepsilon$.

We fix now a point a^i in each set Ω^i. Let $\delta > 0$ be such that $\mathrm{dist}(a^i, \Omega^c) > 10\delta$ for every $i \leq n_\varepsilon$. From Lemma 1.3.5, for h big enough there exist some points $(x_h^{\delta,i})_{i=1}^{n_\varepsilon}$, uniformly bounded in Ω^c, such that for every $i \leq n_\varepsilon$ the square $Q(x_h^{\delta,i}, 9\delta)$ intersects any curve contained in Ω_h^0 and joining the points a^0 and a^i. Up to a subsequence, we have that $x_h^{\delta,i} \to x^{\delta,i}$ as $h \to \infty$, for some $x^{\delta,i} \in \Omega^c$. Once again up to a subsequence, we have that $x^{\delta,i} \to x^i$ as $\delta \to 0$, for some $x^i \in \Omega^c$. Let

$$K^{\delta,\varepsilon} := \bigcup_{i=1}^{n_\varepsilon} \overline{B(x^i, 10\delta)}$$

and, for $i = 0, \ldots, n_\varepsilon$, let $\Omega_h^{\delta,\varepsilon,i}$ be the connected component of $\Omega_h^0 \setminus K^{\delta,\varepsilon}$ containing a^i. As $K^{\delta,\varepsilon} \supset Q(x_h^{\delta,i}, 9\delta)$ for δ small enough and h large enough, we have

$$\Omega_h^{\delta,\varepsilon,0} \neq \Omega_h^{\delta,\varepsilon,i} \quad \text{for } i \neq 0. \tag{1.33}$$

Let $\varphi^{\delta,\varepsilon}$ be the C_p-capacitary potential of $K^{\delta,\varepsilon}$, i.e., the solution of the minimum problem

$$\min\left\{\int_{\mathbb{R}^2} [|\nabla\varphi|^p + |\varphi|^p]\,dx : \varphi \in W^{1,p}(\mathbb{R}^2),\ \varphi = 1\ C_p\text{-a.e. on } K^{\delta,\varepsilon}\right\}.$$

We set

$$u_h^{\varepsilon,\delta} := \begin{cases} 1 & \text{in } \Omega_h^{\delta,\varepsilon,0}, \\ \varphi^{\delta,\varepsilon} & \text{in } \Omega_h \setminus \Omega_h^{\delta,\varepsilon,0}. \end{cases} \tag{1.34}$$

As $\Omega_h \cap \partial\Omega_h^{\delta,\varepsilon,0} \subset K^{\delta,\varepsilon}$, we have that $u_h^{\varepsilon,\delta} \in W^{1,p}(\Omega_h)$. We observe that

$$\|u_h^{\delta,\varepsilon} 1_{\Omega_h} - 1_{\Omega_h^{\delta,\varepsilon,0}}\|_{L^p(\mathbb{R}^2)} + \|\nabla u_h^{\delta,\varepsilon} 1_{\Omega_h}\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)} \leq 2C_p(K^{\delta,\varepsilon})^{\frac{1}{p}}.$$

As $C_p(K^{\delta,\varepsilon}) \leq n_\varepsilon C_p(B(0, 10\delta))$ and $p \leq 2$, we conclude that

$$\limsup_{\delta\to 0}\ \limsup_{h\to\infty} \big[\|u_h^{\delta,\varepsilon,0} 1_{\Omega_h} - 1_{\Omega_h^{\delta,\varepsilon}}\|_{L^p(\mathbb{R}^2)} + \|\nabla u_h^{\delta,\varepsilon} 1_{\Omega_h}\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)}\big] = 0. \tag{1.35}$$

We claim that

$$\limsup_{\delta\to 0}\limsup_{h\to\infty}\|u_h^{\varepsilon,\delta,0}1_{\Omega_h}-1_{\Omega^0}\|^p_{L^p(\mathbb{R}^2)}\le\varepsilon,\tag{1.36}$$

and

$$\limsup_{\delta\to 0}\limsup_{h\to\infty}\|\nabla u_h^{\varepsilon,\delta}1_{\Omega_h}\|_{L^p(\mathbb{R}^2,\mathbb{R}^2)}=0,\tag{1.37}$$

from which the proof of the theorem is achieved by the arbitrariness of ε.

It is easy to see that (1.37) follows from (1.35), while (1.36) is a consequence of (1.35) and of the following inequality

$$\limsup_{\delta\to 0}\limsup_{h\to\infty}|\Omega_h^{\delta,\varepsilon,0}\triangle\Omega^0|\le\varepsilon.\tag{1.38}$$

So, let us prove (1.38). For every $i=0,\dots n_\varepsilon$, up to a subsequence, $\Omega_h^{\delta,\varepsilon,i}$ converges in the Hausdorff complementary topology, when $h\to\infty$, to some open set $\Omega^{\delta,\varepsilon,i}\subset E$. We observe that $\Omega_h^0\setminus K^{\delta,\varepsilon}$ converges to $E\setminus K^{\delta,\varepsilon}$ in the Hausdorff complementary topology when $h\to\infty$. Let $E^{\delta,\varepsilon,i}$ be the connected component of $E\setminus K^{\delta,\varepsilon}$ which contains a^i. It is easy to see that

$$E^{\delta,\varepsilon,i}\subset\Omega^{\delta,\varepsilon,i}.\tag{1.39}$$

Note that, as $\delta\searrow 0$, $K^{\delta,\varepsilon}$ converges decreasingly to the set $\{x_1,\dots,x_{n_\varepsilon}\}$, $E^{\delta,\varepsilon,i}$ converges increasingly to Ω^i, and $\Omega^{\delta,\varepsilon,i}$ converges increasingly to some open set $\Omega^{\varepsilon,i}\subset E$. From (1.39), it follows that $\Omega^i\subset\Omega^{\varepsilon,i}$. From (1.31) and from Lemma 1.2.2 it follows that

$$|\Omega_h^0\setminus K^{\delta,\varepsilon}|\to|E\setminus K^{\delta,\varepsilon}|.\tag{1.40}$$

By Lemma 1.3.4 applied to $\Omega_h^1:=\Omega_h^{\delta,\varepsilon,0}$ and $\Omega_h^2:=(\Omega_h^0\setminus K^{\delta,\varepsilon})\setminus\Omega_h^{\delta,\varepsilon,0}$, we have that $\Omega^{\delta,\varepsilon,0}\cap\Omega^{\delta,\varepsilon,i}=\emptyset$ for every $i\neq 0$, from which it follows that $\Omega^{\varepsilon,0}\cap\Omega^{\varepsilon,i}=\emptyset$ and hence $\Omega^{\varepsilon,0}\cap\Omega^i=\emptyset$ for every $1\le i\le n_\varepsilon$. Therefore, there exists an open set Ω'_ε, contained in the set Ω_ε introduced in (1.32), such that

$$\Omega^{\varepsilon,0}=\Omega^0\cup\Omega'_\varepsilon.$$

From (1.40) and from Lemmas 1.3.4 and 1.2.2, it follows that

$$|\Omega_h^{\delta,\varepsilon,0}\triangle\Omega^{\delta,\varepsilon,0}|\to 0.\tag{1.41}$$

As $\Omega^{\delta,\varepsilon,0}\subset\Omega^{\varepsilon,0}=\Omega^0\cup\Omega'_\varepsilon$, it follows that

$$|\Omega_h^{\delta,\varepsilon,0}\setminus\Omega^0|\le|\Omega_h^{\delta,\varepsilon,0}\setminus\Omega^{\delta,\varepsilon,0}|+|\Omega^{\delta,\varepsilon,0}\setminus\Omega^0|\le|\Omega_h^{\delta,\varepsilon,0}\setminus\Omega^{\delta,\varepsilon,0}|+|\Omega'_\varepsilon|,\tag{1.42}$$

and

$$|\Omega^0\setminus\Omega_h^{\delta,\varepsilon,0}|\le|\Omega^0\setminus\Omega^{\delta,\varepsilon,0}|+|\Omega^{\delta,\varepsilon,0}\setminus\Omega_h^{\delta,\varepsilon,0}|\le|\Omega^{\varepsilon,0}\setminus\Omega^{\delta,\varepsilon,0}|+|\Omega^{\delta,\varepsilon,0}\setminus\Omega_h^{\delta,\varepsilon,0}|.\tag{1.43}$$

As $\Omega^{\delta,\varepsilon,0}$ converges increasingly to $\Omega^{\varepsilon,0}$ as $\delta\to 0^+$, we have $|\Omega^{\varepsilon,0}\setminus\Omega^{\delta,\varepsilon,0}|\to 0$ as $\delta\to 0$. By (1.41), passing to the limit in (1.42) and (1.43) first as $h\to\infty$ and then as $\delta\to 0^+$ we obtain

$$\limsup_{\delta\to 0}\limsup_{h\to\infty}|\Omega_h^{\delta,\varepsilon,0}\setminus\Omega^0|\le|\Omega_\varepsilon|+\limsup_{h\to\infty}|\Omega_h^{\delta,\varepsilon,0}\setminus\Omega^{\delta,\varepsilon,0}|\le\varepsilon$$

and

$$\limsup_{\delta\to 0}\limsup_{h\to\infty}|\Omega^0\setminus\Omega_h^{\delta,\varepsilon,0}|=0,$$

which give inequality (1.38). □

Remark 1.3.6. In the case $p > 2$ the stability result for problem (3) is not true under our hypotheses. Indeed, let us consider

$$S := [-1,1] \times \{0\}, \quad S_h := ([-1,-1/h] \cup [1/h,1]) \times \{0\},$$

$$\Omega := (-1,1)^2 \setminus S, \quad \text{and} \quad \Omega_h := (-1,1)^2 \setminus S_h.$$

We set

$$a(x,\xi) := |\xi|^{p-2}\xi \quad \text{and} \quad b(x,\eta) := |\eta|^{p-2}\eta - x_2,$$

where $x = (x_1, x_2)$. Let u_h and u be solutions of problems (3) in Ω_h and Ω respectively. By the symmetry of Ω, the solution u will depend only on x_2. Therefore, for every $x = (x_1, x_2) \in \Omega$

$$u(x) = \begin{cases} w(x_2) & \text{if } x_2 \in (0,1), \\ -w(-x_2) & \text{if } x_2 \in (-1,0) \end{cases}$$

where w is the solution of the one-dimensional problem

$$\begin{cases} -(|w'|^{p-2}w')' + |w|^{p-2}w = t & \text{in } (0,1), \\ w'(0) = w'(1) = 0, \end{cases} \tag{1.44}$$

which turns out to be of class $C^1([0,1])$. For every $v \in W^{1,p}(\Omega)$, let v^+ and v^- be the upper and lower traces of v on S, defined as in (1.30).

If the open set Ω were stable for problem (3) along the sequence (Ω_h), then u_h would converge strongly to u in $W^{1,p}(\Omega)$. Hence we would have that $u_h^+ \to u^+$ and $u_h^- \to u^-$ uniformly on S (recall that $p > 2$ here). Since $u_h^+(0,0) = u_h^-(0,0)$ by the continuity of u_h, we would obtain $u^+(0,0) = u^-(0,0)$, which implies that $w(0) = 0$. Let us prove that this is false. Indeed, by the maximum principle we have that $w(t) \geq 0$ for every $t \in [0,1]$. Since $w'(0) = 0$ and $p > 2$, we have that $w^{p-1}(t) - t < 0$ in a small neighborhood I of 0 in $[0,1]$. So, from equation (1.44) the function $|w'|^{p-2}w'$ is decreasing in I and hence $w'(t) < 0$ for every $t \in I$. If $w(0)$ were equal to 0, we would obtain $w(t) < 0$ for every $t \in I$, which contradicts the fact that $w(t) \geq 0$ for every $t \in [0,1]$. This proves that $w(0) > 0$, and hence Ω is not stable for problem (3) along the sequence (Ω_h) for $p > 2$.

1.4 The case of unbounded domains

We now extend the results of the previous sections to the case of unbounded domains.

Theorem 1.4.1. *Let (Ω_h) be a sequence of open subsets of $\mathbb{R}^2$ such that (Ω_h^c) converges to Ω^c in the sense of Kuratowski for some open subset Ω. Assume that, for every $R > 0$, $|\Omega_h \cap B(0,R)|$ converges to $|\Omega \cap B(0,R)|$ and that the number of connected components of $(\Omega_h \cap B(0,R))^c$ is uniformly bounded with respect to h. Then the sequence of subspaces X_{Ω_h} (resp. Y_{Ω_h}) converges to X_Ω (resp. Y_Ω).*

Proof. We prove only the Mosco convergence of X_{Ω_h} to X_Ω, since the convergence of Y_{Ω_h} to Y_Ω can be proved in the same way. First of all note that, from the convergence of Ω_h^c to Ω^c in the sense of Kuratowski, it follows that the sequence $\Omega_h \cap B(0,R)$ converges to $\Omega \cap B(0,R)$ in the Hausdorff complementary topology. Moreover, by the assumptions of the theorem, we can apply Theorem 1.3.2 to the sequence $\Omega_h \cap B(0,R)$. So, we get that $X_{\Omega_h \cap B(0,R)}$ converges to $X_{\Omega \cap B(0,R)}$ in the sense of Mosco.

Now let us prove (M_2) for X_{Ω_h} and X_Ω. Let (h_k) be a sequence of indices converging to ∞, (u_k) be a sequence, with $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, such that $u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2)$ to a function ϕ, while $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ to a function ψ. It follows that $u_k 1_{\Omega_{h_k}\cap B(0,R)}$ converges to $\phi 1_{B(0,R)}$ weakly in $L^p(\mathbb{R}^2)$, while $\nabla u_k 1_{\Omega_{h_k}\cap B(0,R)}$ converges to $\psi 1_{B(0,R)}$ weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. So, by property (M_2) relative to the Mosco convergence of $X_{\Omega_h \cap B(0,R)}$ to $X_{\Omega\cap B(0,R)}$, there exists a function $u_R \in W^{1,p}(\Omega \cap B(0,R))$ such that $\phi 1_{B(0,R)} = u_R 1_{\Omega\cap B(0,R)}$ and $\psi 1_{B(0,R)} = \nabla u_R 1_{\Omega\cap B(0,R)}$ a.e. in $\mathbb{R}^2$. Since R is arbitrary, it is easy to construct $u \in W^{1,p}(\Omega)$ such that $\phi = u 1_\Omega$ and $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^2$.

Let us prove property (M_1). Let $u \in W^{1,p}(\Omega)$ and let $\varepsilon > 0$. There exists $R_\varepsilon > 0$ such that

$$\int_{\Omega\setminus B(0,R_\varepsilon)} [|u|^p + |\nabla u|^p]\, dx \le \varepsilon.$$

By property (M_1) relative to the Mosco convergence of $X_{\Omega_h\cap B(0,R_\varepsilon+1)}$ to $X_{\Omega\cap B(0,R_\varepsilon+1)}$, there exists a sequence $w_h^\varepsilon \in W^{1,p}(\Omega_h \cap B(0,R_\varepsilon+1))$ such that $w_h^\varepsilon 1_{\Omega_h\cap B(0,R_\varepsilon+1)}$ converges strongly to $u 1_{\Omega\cap B(0,R_\varepsilon+1)}$ in $L^p(\mathbb{R}^2)$ and $\nabla w_h^\varepsilon 1_{\Omega_h\cap B(0,R_\varepsilon+1)}$ converges strongly to $\nabla u 1_{\Omega\cap B(0,R_\varepsilon+1)}$ in $L^p(\mathbb{R}^2,\mathbb{R}^2)$. Let $\varphi_\varepsilon \in C_c^1(B(0,R_\varepsilon+1))$ such that $0 \le \varphi_\varepsilon \le 1$, $\varphi_\varepsilon = 1$ in $B(0,R_\varepsilon)$, and $\|\nabla\varphi_\varepsilon\|_\infty \le C$. Now we set $u_h^\varepsilon := \varphi_\varepsilon w_h^\varepsilon$. By construction $u_h^\varepsilon \in W^{1,p}(\Omega_h)$, $u_h^\varepsilon 1_{\Omega_h} \to \varphi_\varepsilon u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$, and $\nabla u_h^\varepsilon 1_{\Omega_h} \to \varphi_\varepsilon \nabla u 1_\Omega + u\nabla\varphi_\varepsilon 1_\Omega$ strongly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$. On the other hand

$$\limsup_{h\to\infty} \int_{\mathbb{R}^2} |u_h^\varepsilon 1_{\Omega_h} - u1_\Omega|^p + |\nabla u_h^\varepsilon 1_{\Omega_h} - \nabla u 1_\Omega|^p\, dx \;\le\; 2^{p-1}(C^p+1)\varepsilon,$$

$\varphi_\varepsilon u 1_\Omega \to u 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$, and $\varphi_\varepsilon \nabla u 1_\Omega + u\nabla\varphi_\varepsilon 1_\Omega \to \nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^2,\mathbb{R}^2)$. Therefore, we can construct a sequence $u_h \in W^{1,p}(\Omega_h)$ which satisfies (M_1) by a standar argument on double sequences. □

1.5 Problems with Dirichlet boundary conditions

In this section we study the Mosco convergence of Sobolev and Deny-Lions spaces with prescribed Dirichlet conditions on part of the boundary.

Let $A \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary ∂A, and let $\partial_D A$ be a relatively open subset of ∂A with a finite number of connected components. For every compact set $K \subset \overline{A}$, for every $g \in W^{1,p}(A)$, and for every pair of function a and b satisfying the properties (1.2)–(1.4), we consider the solutions u and v of the mixed problems

$$\begin{cases} -\operatorname{div} a(x,\nabla u) + b(x,u) = 0 & \text{in} \quad A\setminus K, \\ u = g & \text{on} \quad \partial_D A \setminus K, \\ a(x,\nabla u)\cdot\nu = 0 & \text{on} \quad \partial(A\setminus K)\setminus(\partial_D A\setminus K), \end{cases} \tag{1.45}$$

and

$$\begin{cases} -\operatorname{div} a(x,\nabla v) = 0 & \text{in} \quad A\setminus K, \\ v = g & \text{on} \quad \partial_D A \setminus K, \\ a(x,\nabla v)\cdot\nu = 0 & \text{on} \quad \partial(A\setminus K)\setminus(\partial_D A\setminus K). \end{cases} \tag{1.46}$$

Let (K_h) be a sequence of compact subsets of $\overline{A}$, let (g_h) be a sequence in $W^{1,p}(A)$, and let (u_h) be the sequence of the solutions of problems (1.45) corresponding to K_h and g_h.

Definition 1.5.1. *We say that the pair (K, g) is stable for the mixed problems (1.45) along the sequence (K_h, g_h) if for every pair of functions a, b satisfying (1.2)–(1.4) the sequence $(u_h 1_{K_h^c})$ converges to $u 1_{K^c}$ strongly in $L^p(A)$ and the sequence $(\nabla u_h 1_{K_h^c})$ converges to $\nabla u 1_{K^c}$ strongly in $L^p(A, \mathbb{R}^2)$.*

The stability for problems (1.46) is defined in a similar way by using only the convergence of the gradients (as in Definition 1.1.2).

The stabilty for problems (1.46) has been recently studied in [36] in the case $a(x, \xi) = \xi$. In this section we will study the stability in the general case by using again the notion of Mosco convergence.

We set

$$W_g^{1,p}(A \setminus K, \partial_D A \setminus K) := \{u \in W^{1,p}(A \setminus K) : u = g \text{ on } \partial_D A \setminus K\},$$

and

$$L_g^{1,p}(A \setminus K, \partial_D A \setminus K) := \{u \in L^{1,p}(A \setminus K) : u = g \text{ on } \partial_D A \setminus K\},$$

where the equality $u = g$ on $\partial_D A \setminus K$ is intended in the usual sense of traces.

As in Section 1.1.2 the space $W_g^{1,p}(A \setminus K, \partial_D A \setminus K)$ will be identified with the closed linear subspace $X_K^g(A)$ of $L^p(A) \times L^p(A, \mathbb{R}^2)$ defined by

$$X_K^g(A) := \{(u 1_{K^c}, \nabla u 1_{K^c}) : u \in W_g^{1,p}(A \setminus K, \partial_D A \setminus K)\}. \tag{1.47}$$

For problem (1.46), we consider in $L_g^{1,p}(A \setminus K, \partial_D A \setminus K)$ the equivalence relation $\sim$ defined in (21). Note that in this case $v_1 \sim v_2$ if and only if $v_1 = v_2$ a.e. in those connected components of $A \setminus K$ whose boundary intersects $\partial_D A \setminus K$ and $\nabla v_1 = \nabla v_2$ a.e. in the other connected components of $A \setminus K$. The corresponding quotient space, denoted by $L_g^{1,p}(A \setminus K, \partial_D A \setminus K)/_\sim$, will be identified with the closed linear subspace $Y_K^g(A)$ of $L^p(A, \mathbb{R}^2)$ defined by

$$Y_K^g(A) := \{\nabla u 1_{K^c} : u \in L_g^{1,p}(A \setminus K, \partial_D A \setminus K)\}. \tag{1.48}$$

Let K_h, K be compact subsets of $\overline{A}$ and let g_h, $g \in W^{1,p}(A)$. Let $X_{K_h}^{g_h}(A)$ and $X_K^g(A)$ be the corresponding subspaces defined by (1.47). We recall that $X_{K_h}^{g_h}(A)$ converges to $X_K^g(A)$ in the sense of Mosco if the following two properties hold:

(M_1'') for every $u \in W_g^{1,p}(A \setminus K, \partial_D A \setminus K)$, there exists a sequence $u_h \in W_{g_h}^{1,p}(A \setminus K_h, \partial_D A \setminus K_h)$ such that $u_h 1_{K_h^c}$ converges strongly to $u 1_{K^c}$ in $L^p(A)$ and $\nabla u_h 1_{K_h^c}$ converges strongly to $\nabla u 1_{K^c}$ in $L^p(A, \mathbb{R}^2)$;

(M_2'') if (h_k) is a sequence of indices converging to ∞, (u_k) is a sequence, with $u_k \in W_{g_{h_k}}^{1,p}(A \setminus K_{h_k}, \partial_D A \setminus K_{h_k})$ for every k, such that $u_k 1_{K_{h_k}^c}$ converges weakly in $L^p(A)$ to a function ϕ, while $\nabla u_k 1_{K_{h_k}^c}$ converges weakly in $L^p(A, \mathbb{R}^2)$ to a function ψ, then there exists $u \in W_g^{1,p}(A \setminus K, \partial_D A \setminus K)$ such that $\phi = u 1_{K^c}$ and $\psi = \nabla u 1_{K^c}$ a.e. in A.

Analogously, the convergence of $Y_{K_h}^{g_h}(A)$ to $Y_K^g(A)$ in the sense of Mosco can be characterized by using only the convergence of the extensions of the gradients.

Remark 1.5.2. As in Section 1.1.3 we can prove that the Mosco convergence of $X_{K_h}^{g_h}(A)$ to $X_K^g(A)$ (resp. of $Y_{K_h}^{g_h}(A)$ to $Y_K^g(A)$) is equivalent to the stability of (K, g) for the mixed problems (1.45) (resp. (1.46) along the sequence (K_h, g_h)).

The following theorem is the main result of this section.

Theorem 1.5.3. *Let A be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary ∂A and let $\partial_D A$ be a relatively open subset of ∂A with a finite number of connected components. Let (g_h) be a sequence in $W^{1,p}(A)$ converging strongly to a function g in $W^{1,p}(A)$, and let (K_h) be a sequence of compact subsets of $\overline{A}$ converging to a set K in the Hausdorff metric. Assume that $|K_h|$ converges to $|K|$ and that the sets K_h have a uniformly bounded number of connected components. Then $X_{K_h}^{g_h}(A)$ converges to $X_K^g(A)$ (resp. $Y_{K_h}^{g_h}(A)$ converges to $Y_K^g(A)$) in the sense of Mosco.*

Proof. The main idea of this proof is due to Chambolle. Let us first prove the Mosco convergence of $X^{g_h}_{K_h}(A)$ to $X^g_K(A)$. Let Σ be an open ball in $\mathbb{R}^2$ such that $\overline{A} \subset \Sigma$. Let $\tilde{g}_h$, $\tilde{g} \in W^{1,p}(\Sigma)$ be extensions of g_h and g to Σ such that $\tilde{g}_h$ converges to $\tilde{g}$ strongly in $W^{1,p}(\Sigma)$. We set

$$\Omega_h := \Sigma \setminus (K_h \cup (\partial A \setminus \partial_D A)) \quad \text{and} \quad \Omega := \Sigma \setminus (K \cup (\partial A \setminus \partial_D A)).$$

Note that Ω_h and Ω satisfy the assumptions of Theorem 1.3.2. Let us prove property $(M2'')$. Let (h_k) be a sequence of indices that tends to ∞, and $u_k \in W^{1,p}_{g_{h_k}}(A \setminus K_{h_k}, \partial_D A \setminus K_{h_k})$ such that $u_k 1_{K^c_{h_k}}$ converges weakly to ϕ in $L^p(A)$ and $\nabla u_k 1_{K^c_{h_k}}$ converges weakly to ψ in $L^p(A, \mathbb{R}^2)$. Let $\tilde{u}_k$ be the extension of u_k defined by

$$\tilde{u}_k := \begin{cases} u_k 1_{K^c_{h_k}} & \text{in } A, \\ \tilde{g}_{h_k} & \text{in } \Sigma \setminus A, \end{cases}$$

and let $\tilde{\phi}$ and $\tilde{\psi}$ be defined by

$$\tilde{\phi} := \begin{cases} \phi & \text{in } A, \\ \tilde{g} & \text{in } \Sigma \setminus A, \end{cases} \qquad \text{and} \qquad \tilde{\psi} := \begin{cases} \psi & \text{in } A, \\ \nabla \tilde{g} & \text{in } \Sigma \setminus A. \end{cases}$$

As $u_k = g_{h_k}$ on $\partial_D A \setminus K_{h_k}$, we have $\tilde{u}_k \in W^{1,p}(\Omega_{h_k})$. Since $\tilde{u}_k 1_{\Omega_{h_k}}$ converges to $\tilde{\phi} 1_\Sigma$ weakly in $L^p(\mathbb{R}^2)$ and $\nabla \tilde{u}_k 1_{\Omega_{h_k}}$ converges to $\tilde{\psi} 1_\Sigma$ weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$, by Theorem 1.3.2 we conclude that there exists $\tilde{u} \in W^{1,p}(\Omega)$ such that $\tilde{\phi} 1_\Sigma = \tilde{u} 1_\Omega$ and $\tilde{\psi} 1_\Sigma = \nabla \tilde{u} 1_\Omega$. Let u be the restriction of $\tilde{u}$ to $A \setminus K$. Then $u \in W^{1,p}(A \setminus K)$ and we have that $\phi = u 1_{K^c}$ and $\psi = \nabla u 1_{K^c}$ a.e. in A. As $\tilde{u} \in W^{1,p}(\Omega)$, the traces of $\tilde{u}$ on both sides of $\partial_D A \setminus K$ coincide. Since $\tilde{u} = \tilde{g}$ a.e. in $\Sigma \setminus A$, we conclude that $u = \tilde{g} = g$ in the sense of traces on $\partial_D A \setminus K$. Therefore $u \in W^{1,p}_g(A \setminus K, \partial_D A \setminus K)$.

Now we prove property (M_1''). Let $u \in W^{1,p}_g(A \setminus K, \partial_D A \setminus K)$. Let $\tilde{u}$ be the extension of u defined by

$$\tilde{u} = \begin{cases} u 1_{K^c} & \text{a.e. in } A, \\ \tilde{g} & \text{a.e. in } \Sigma \setminus A. \end{cases}$$

As $u = g$ on $\partial_D A \setminus K$, we have that $\tilde{u} \in W^{1,p}(\Omega)$. By Theorem 1.3.2 there exists a sequence $\tilde{u}_h \in W^{1,p}(\Omega_h)$ such that $\tilde{u}_h 1_{\Omega_h}$ converges to $\tilde{u} 1_\Omega$ strongly in $L^p(\mathbb{R}^2)$ and $\nabla \tilde{u}_h 1_{\Omega_h}$ converges to $\nabla \tilde{u} 1_\Omega$ strongly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. We consider the function

$$\phi_h := (\tilde{u}_h - \tilde{g}_h)|_{\Sigma \setminus \overline{A}}.$$

By construction, $\phi_h \to 0$ strongly in $W^{1,p}(\Sigma \setminus \overline{A})$. Therefore there exists a sequence $v_h \in W^{1,p}(\Sigma)$, converging to 0 strongly in $W^{1,p}(\Sigma)$, such that $v_h|_{\Sigma \setminus \overline{A}} = \phi_h$ a.e. in $\Sigma \setminus \overline{A}$. We set

$$u_h := (\tilde{u}_h - v_h)|_{A \setminus K_h}.$$

By construction, we have that $u_h \in W^{1,p}(A \setminus K_h)$ and $u_h = g_h$ in the sense of traces on $\partial_D A \setminus K_h$. Moreover, we have that $u_h 1_{K^c_h}$ converges to $u 1_{K^c}$ strongly in $L^p(A)$ and $\nabla u_h 1_{K^c_h}$ converges to $\nabla u 1_{K^c}$ strongly in $L^p(A, \mathbb{R}^2)$.

Now let us prove that $Y^{g_h}_{K_h}(A)$ converges to $Y^g_K(A)$ in the sense of Mosco. Property (M_2'') is obtained arguing as in [36, Lemma 4.1]. So, let us prove (M_1''). Let $u \in L^{1,p}_g(A \setminus K, \partial_D A \setminus K)$. We set for every $k \in \mathbb{N}$

$$u^k := (g - k) \vee u \wedge (g + k).$$

Then $u^k \in W^{1,p}_g(A \setminus K, \partial_D A \setminus K)$ and $\nabla u^k 1_{K^c} \to \nabla u 1_{K^c}$ strongly in $L^p(A, \mathbb{R}^2)$. From property (M_1'') proved above for the Mosco convergence of $X^{g_h}_{K_h}(A)$ to $X^g_K(A)$, for every k there exists $u^k_h \in W^{1,p}_{g_h}(A \setminus K_h, \partial_D A \setminus K_h)$ such that $\nabla u^k_h 1_{K^c_h} \to \nabla u^k 1_{K^c}$ in $L^p(A, \mathbb{R}^2)$. Hence, by a standard argument on double sequences, we obtain a sequence of indices k_h converging to ∞ such that, setting $u_h := u^{k_h}_h$, we get $\nabla u_h 1_{K^c_h} \to \nabla u 1_{K^c}$ in $L^p(A, \mathbb{R}^2)$. □

1.6 Some examples of non stability

Remarks 1.2.7 and 1.3.6 show that in general the stability results stated in the previous sections does not hold in the case $p > 2$. Here we will give more explicit examples using the tool of Γ-convergence. We will refer to problem (1.46) with $a(x, \xi) := \xi^{p-2}\xi$, where $p > 2$, and $g_h \equiv g$. In order to use the tool of Γ-convergence, let us observe that problem (1.46) with this choice of a can be written in the following variational form:

$$(P) \qquad \min_{v=g \text{ on } \partial_D\Omega \setminus K} \int_{\Omega \setminus K} \frac{1}{p} |\nabla v(x)|^p \, dx.$$

In the first example the stability result for problem (P) holds.

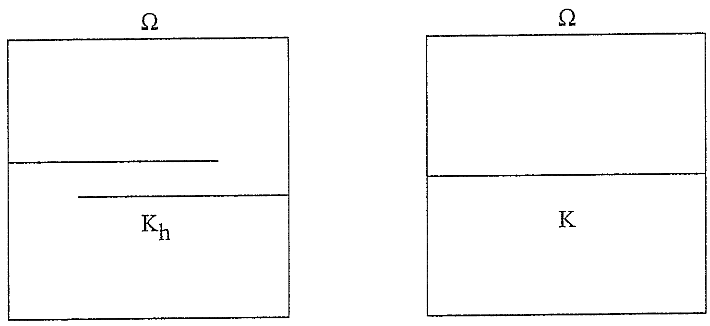

Fig. 1

Example 1.6.1. Let $\Omega := (-1,1) \times (-1,1)$, $\partial_D \Omega := (-1,1) \times \{-1, 1\}$, $K = [-1,1] \times \{0\}$ and let

$$K_h := \left[-1, \frac{1}{2}\right] \times \left\{\frac{1}{h}\right\} \cup \left[-\frac{1}{2}, 1\right] \times \left\{-\frac{1}{h}\right\},$$

(see Fig. 1). We consider the sequence of functionals F_h defined in $L^p(\Omega)$ by:

$$F_h(u) := \begin{cases} \dfrac{1}{p} \displaystyle\int_{\Omega \setminus K_h} |\nabla u|^p dx dy & \text{if } u \in W^{1,p}(\Omega \setminus K_h) \text{ and } u = g \text{ on } \partial_D \Omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.49}$$

Then, F_h Γ-converges to F_∞ in the strong topology of $L^p(\Omega)$, where

$$F_\infty(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_{\Omega\setminus K} |\nabla u|^p \, dxdy & \text{if } u \in W^{1,p}(\Omega\setminus K) \text{ and } u = g \text{ on } \partial_D\Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Hence in this case the conclusion of Theorem 1.5.3 follows from the general result on convergence of minima in the theory of Γ-convergence.

Proof. (i) **Γ-liminf**: Let $u_h \to u$ strongly in $L^p(\Omega)$, we want to prove that $\liminf_{h\to\infty} F_h(u_h) \geq F(u)$. We can assume that $\liminf_{h\to\infty} F_h(u_h) = \lim_{h\to\infty} F_h(u_h) < \infty$. So, for any $\Omega' \subset\subset \Omega\setminus K$ with $\partial_D\Omega \subset \partial\Omega'$, we have that $u_h \in W^{1,p}(\Omega')$ for h big enough, $u_h \rightharpoonup u$ in $W^{1,p}(\Omega')$ and $u = g$ on $\partial_D\Omega$. Now from the lower semicontinuity of the L^p−norm of the gradients and from the arbitrariness of Ω' we get that $u \in W^{1,p}(\Omega\setminus K)$ and the Γ-liminf inequality holds.
(ii) **Γ-limsup**: Let $u \in L^p(\Omega)$. We want to construct a sequence $(u_h) \subset L^p(\Omega)$ converging strongly to u in $L^p(\Omega)$ such that $\lim_h F_h(u_h) \leq F(u)$. We can assume that $u \in W^{1,p}(\Omega\setminus K)$ and $u = g$ on $\partial_D\Omega$. We set $u_h := u$ in $\Omega \setminus R_h$ where $R_h := (-1,1)\times\left[-\frac{1}{h},\frac{1}{h}\right]$. Now let us define the function u_h in R_h. To this aim, we consider the function v_h defined in R_h by

$$v_h(x,y) := u\Big(x, \frac{2}{h} sgn(x) - y\Big)$$

where $sgn(x)$ denotes the sign of x. In other words, the function v_h is obtained from u by symmetry with respect to the segment $[0,1]\times\{\frac{1}{h}\}$ for x positive and by symmetry with respect to the segment $[-1,0]\times\{-\frac{1}{h}\}$ for x negative.

Such a function v_h may jump on the segment $\{0\}\times\left[-\frac{1}{h},\frac{1}{h}\right]$. So, we consider the function $\varphi \in C^0(R_h)$ defined by

$$\varphi(x,y) := \begin{cases} 1 & \text{if } |x| > \frac{1}{2}, \\ -2x & \text{if } -\frac{1}{2} \leq x \leq 0, \\ 2x & \text{if } 0 \leq x \leq \frac{1}{2}. \end{cases}$$

Now we set $u_h := \varphi v_h$ on R_h. For this choice of u_h, it is easy to see that $u_h \in W^{1,p}(\Omega\setminus K_h)$ with $u_h = g$ on $\partial_D\Omega$ and that the Γ-limsup inequality holds. □

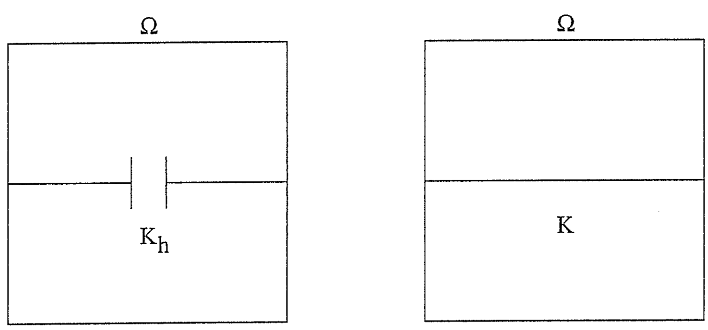

Fig.2

In the following example, we consider a sequence of compact sets K_h along which the problem (P) is not stable. More precisely in the limit problem, that is the problem solved by the limit function u, there is an additional term involving the jump of u on a point of K.

Example 1.6.2. Let Ω, $\partial_D\Omega$ and K be as in the previous example and let

$$K_h := \left[-1, -\frac{a_h}{2}\right] \times \left\{0\right\} \cup \left[\frac{a_h}{2}, 1\right] \times \left\{0\right\} \cup \left\{-\frac{a_h}{2}\right\} \times \left[-\frac{b_h}{2}, \frac{b_h}{2}\right] \cup \left\{\frac{a_h}{2}\right\} \times \left[-\frac{b_h}{2}, \frac{b_h}{2}\right]$$

be as in Fig. 2 with (a_h) and (b_h) being two sequences of positive numbers converging to 0. In this way (K_h) converges to K in the Hausdorff metric. Let F_h be defined as in (1.49).

Assume that the sequence $(\frac{1}{p}\, a_h\, b_h^{1-p})$ converges to some $c \in [0, +\infty]$. Then F_h Γ-converges in the strong topology of $L^p(\Omega)$ to F_∞ defined in $L^p(\Omega)$ in the following way (with the convention that $0{\cdot}\infty = 0$).

$$F_\infty(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_{\Omega\setminus K} |\nabla u|^p dxdy + c\left|u^+(0,0) - u^-(0,0)\right|^p & \text{if } \left\{\begin{array}{l} u \in W^{1,p}(\Omega\setminus K) \text{ and} \\ u = g \text{ on } \partial_D\Omega, \end{array}\right. \\ +\infty & \text{otherwise,} \end{cases} \tag{1.50}$$

where $u^+(0,0)$ and $u^-(0,0)$ are respectively the values in $(0,0)$ of the traces of $u|_{\Omega^+}$ and $u|_{\Omega^-}$ on K, Ω^+ and Ω^- being respectively the upper and the lower connected components of $\Omega \setminus K$.

Proof. **(i) Γ-liminf**: Let $u_h \to u$ in $L^p(\Omega)$, we want to prove that $\liminf_{h\to\infty} F_h(u_h) \geq F(u)$. We can assume that $\liminf_{h\to\infty} F_h(u_h) = \lim_{h\to\infty} F_h(u_h) < \infty$. So, for any $\Omega' \subset\subset \Omega \setminus K$ with $\partial_D\Omega \subset \partial\Omega'$, we have that $u_h \in W^{1,p}(\Omega')$ for h big enough, $u_h \rightharpoonup u$ in $W^{1,p}(\Omega')$ and $u = g$ on $\partial_D\Omega$. Now from the lower semicontinuity of the L^p-norm of the gradients and from the arbitrariness of Ω' we get that $u \in W^{1,p}(\Omega \setminus K)$.

We set $R_h := \left(-\frac{a_h}{2}, \frac{a_h}{2}\right) \times \left(-\frac{b_h}{2}, \frac{b_h}{2}\right)$. We have

$$\begin{aligned} F_h(u_h) &= \frac{1}{p}\int_{\Omega\setminus K_h} |\nabla u_h|^p\, dxdy = \frac{1}{p}\int_{\Omega\setminus R_h} |\nabla u_h|^p\, dxdy + \frac{1}{p}\int_{R_h} |\nabla u_h|^p\, dxdy \\ &\geq \frac{1}{p}\int_{\Omega\setminus R_h} |\nabla u_h|^p\, dxdy + \frac{1}{p} a_h b_h^{1-p} \fint_{-\frac{a_h}{2}}^{\frac{a_h}{2}} \left|u_h\left(x, -\frac{b_h}{2}\right) - u_h\left(x, \frac{b_h}{2}\right)\right|^p dx. \end{aligned} \tag{1.51}$$

Now let us fix $\Omega' \subset\subset \Omega \setminus K$. We have that $\Omega' \subset\subset \Omega \setminus R_h$ definitively, and

$$\liminf_{h\to\infty} \int_{\Omega\setminus R_h} |\nabla u_h|^p\, dxdy \geq \liminf_{h\to\infty} \int_{\Omega'} |\nabla u_h|^p\, dxdy \geq \int_{\Omega'} |\nabla u|^p\, dxdy.$$

By the arbitrariness of Ω', we get

$$\liminf_{h\to\infty} \int_{\Omega\setminus R_h} |\nabla u_h|^p\, dxdy \geq \int_{\Omega\setminus K} |\nabla u|^p\, dxdy. \tag{1.52}$$

Let us consider the functions $\tilde{u}_h^1$ defined in $(-1,1) \times (0, 1 - \frac{b_h}{2})$ by

$$\tilde{u}_h^1(x,y) := u_h|_{(-1,1)\times(\frac{b_h}{2},1)}(x, y + \frac{b_h}{2})$$

and $\tilde{u}_h^2$ defined in $(-1,1)\times(-1+\frac{b_h}{2},0)$ by

$$\tilde{u}_h^2(x,y) := u_h|_{(-1,1)\times(-1,-\frac{b_h}{2})}(x, y-\frac{b_h}{2}).$$

We extend $\tilde{u}_h^1$ and $\tilde{u}_h^2$ respectively in Ω^+ and Ω^- in such a way those extensions converge weakly to u respectively in $W^{1,p}(\Omega^+)$ and in $W^{1,p}(\Omega^-)$. Recalling that $p>2$, We have the uniform convergence of their traces on K. So,

$$\begin{aligned}\frac{1}{p}a_h b_h^{1-p} \fint_{-\frac{a_h}{2}}^{\frac{a_h}{2}} \left|u_h\Big(x,-\frac{b_h}{2}\Big) - u_h\Big(x,\frac{b_h}{2}\Big)\right|^p dx &= \frac{1}{p}a_h b_h^{1-p} \fint_{-\frac{a_h}{2}}^{\frac{a_h}{2}} \left|\tilde{u}_h^1(x,0) - \tilde{u}_h^2(x,0)\right|^p dx\\ &= \frac{1}{p}a_h b_h^{1-p} \fint_{-\frac{a_h}{2}}^{\frac{a_h}{2}} \left|u^+(x,0) - u^-(x,0) + w_h(x)\right|^p dx,\end{aligned}$$

with (w_h) converging uniformly to 0 on K. From this, it follows that

$$\lim_{h\to\infty}\frac{1}{p}a_h b_h^{1-p} \fint_{-\frac{a_h}{2}}^{\frac{a_h}{2}} \left|u_h\Big(x,-\frac{b_h}{2}\Big) - u_h\Big(x,\frac{b_h}{2}\Big)\right|^p dx = c\left|u^+(0,0)-u^-(0,0)\right|^p. \tag{1.53}$$

Therefore, the Γ-liminf inequality follows from (1.51), (1.52) and (1.53).

(ii) Γ-limsup: Let $u \in L^p(\Omega)$. We want to construct a sequence $(u_h)\subset L^p(\Omega)$ which converges to u such that $\lim_h F_h(u_h) \le F(u)$. We can assume that $u\in W^{1,p}(\Omega\setminus K)$ and $u=g$ on $\partial_D\Omega$.

We set $u_h = u$ in $(\Omega\setminus K)\setminus R_h$ and we modify suitably u in $R_h\setminus K$ in order to get a new function which does not jump on $K\cap R_h$. To this aim let $R_h^1 := R_h\cap\{y>0\}$ and $R_h^2 := R_h\cap\{y<0\}$. Let us define u_h in R_h^1. We set

$$v_h := u|_{\left(-\frac{a_h}{2},\frac{a_h}{2}\right)\times\left(\frac{b_h}{2},b_h\right)}$$

and

$$\tilde{u}_h(x,y) := v_h(x, b_h - y) \quad \text{for any } (x,y)\in R_h^1.$$

In other words, $\tilde{u}_h$ is defined by taking the reflection of the restriction of u on the rectangle symmetric to R_h^1 with respect to the horizontal line $y=\frac{b_h}{2}$. Now we consider the linear function $\varphi_1(x,y) := \frac{2}{b_h}y$. For any $(x,y)\in R_h^1$ we set

$$u_h(x,y) := \varphi_1(x,y)\Big(\tilde{u}_h(x,y) - \frac{u^+(0,0)+u^-(0,0)}{2}\Big) + \frac{u^+(0,0)+u^-(0,0)}{2}.$$

In the similar way, we define u_h in R_h^2 using

$$u|_{\left(-\frac{a_h}{2},\frac{a_h}{2}\right)\times\left(-b_h,-\frac{b_h}{2}\right)} \quad\text{and}\quad \varphi_2(x,y) := -\frac{2}{b_h}y.$$

It is easy to check that $u_h\in W^{1,p}(\Omega\setminus K_h)$, $u_h = g$ on $\partial_D\Omega$ and by construction

$$\lim_{h\to\infty}\frac{1}{p}\int_{R_h^1}|\nabla u_h|^p\,dxdy = \lim_{h\to\infty}\frac{1}{p}\int_{R_h^1}|\nabla u_h|^p\,dxdy = \frac{c}{2}\left|u^+(0,0)-u^-(0,0)\right|^p.$$

Therefore,

$$\begin{aligned}\lim_h F_h(u_h) &= \lim_h \frac{1}{p}\int_{\Omega\setminus K_h} |\nabla u_h|^p\,dxdy \\ &= \lim_h \frac{1}{p}\int_{(\Omega\setminus K)\setminus R_h} |\nabla u_h|^p\,dxdy + \lim_h \frac{1}{p}\int_{R_h^1} |\nabla u_h|^p\,dxdy + \lim_h \frac{1}{p}\int_{R_h^2} |\nabla u_h|^p\,dxdy \\ &= \frac{1}{p}\int_{\Omega\setminus K} |\nabla u|^p\,dxdy + c\left|u^+(0,0)-u^-(0,0)\right|^p = F_\infty(u).\end{aligned}$$

□

Remark 1.6.3. Starting from Example 1.6.2, one can construct examples in which the Γ-limit involves traces at the origin from more than two subdomains, as shown in fig. 3.

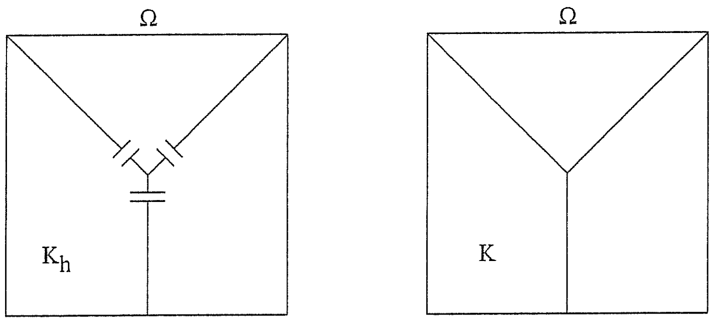

Fig. 3

In this case, we can obtain a Γ-limit of the form

$$F_\infty(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_{\Omega\setminus K} |\nabla u|^p dxdy + \sum_{1\le i<j\le 3} c_{i,j}\left|u^i(0,0)-u^j(0,0)\right|^p & \text{if } \begin{cases} u\in W^{1,p}(\Omega\setminus K) \\ u = g \text{ on } \partial\Omega \end{cases} \\ +\infty & \text{otherwise ,} \end{cases}$$

where $u^i(0,0)$ is the value at $(0,0)$ of the trace of $u|_{\Omega_i}$, Ω_i being the connected components of $\Omega\setminus K$.

Note that in the first example, where stability holds, the intersection of the limits of the two different connected components of $K_h \cup \partial_N\Omega$ has positive length, and hence positive $(1,q)$-capacity. In Ebobisse-Ponsiglione [42] authors give a sufficient condition to the stability of some variational problems with growth assumption $p > 2$ in simply connected domains: actually they prove that the stability result holds if the intersection of the limits of two different connected components of $K_h \cup \partial_N\Omega$ is either empty or has positive $(1,q)$-capacity. More precisely let Ω be a simply connected open set with Lipschitz continuous boundary. Let moreover $f : \Omega\times\mathbb{R}^2 \to \mathbb{R}$ be a Borel function which satisfies the following assumptions: there exist positive constants α, β, γ such that, for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^2$

$$\alpha|\xi|^p \leq f(x,\xi) \leq \beta|\xi|^p + \gamma;$$

$$f(x,\cdot) \text{ is strictly convex.}$$

Given $K \in \mathcal{K}(\overline{\Omega})$ and a function $g \in W^{1,p}(\Omega)$, we consider the following minimization problem

$$\min_w \left\{ \int_{\Omega\setminus K} f(x,\nabla w)\,dx \colon\ w \in L^{1,p}(\Omega\setminus K)\,, \quad w = g \text{ on } \partial_D\Omega \setminus K \right\} \tag{1.54}$$

whose solution exits from direct methods of the calculus of variations and is unique in the sense of gradients. The following theorem holds (the proof is omitted).

Theorem 1.6.4. *[42, Theorem 4.2] Let $(K_h) \subset \mathcal{K}_m(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric and such that $|K_h|$ converges to $|K|$. Let $g \in W^{1,p}(\Omega)$. Let u_h and u be solutions of (1.54) in $\Omega \setminus K_h$ and in $\Omega \setminus K$ respectively. Assume that the intersection of the limits of two different connected components of $K_h \cup \partial_N\Omega$ is either empty or has positive $(1,q)$-capacity. Then ∇u_h converges strongly to ∇u in $L^p(\Omega,\mathbb{R}^2)$.*

Chapter 2

Stability of unilateral free-discontinuity problems

Introduction

The purpose of this chapter [1] is to study the stability of some unilateral free-discontinuity problems in two-dimensional domains, with the density of the volume part having p-growth, with $1 < p < \infty$, under perturbations of the discontinuity sets in the Hausdorff metric. We adopt the strong formulation assuming that the discontinuity sets have a uniformly bounded number of connected components.

Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set with Lipschitz continuous boundary $\partial\Omega$ and let $\partial_D\Omega \subset \partial\Omega$ be a (non-empty) relatively open subset of $\partial\Omega$ composed of a finite number of connected components and let $\mathcal{K}_m(\overline{\Omega})$ be the class of all closed subset of $\overline{\Omega}$ whose elements have at most m connected components. Let $g \in W^{1,p}(\Omega)$ and let $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a Borel function which satisfies the assumptions (2.3)-(2.4) below. We consider pairs (u, K) with $K \in \mathcal{K}_m(\overline{\Omega})$ and $u \in L^{1,p}(\Omega \setminus K) := \{v \in L^p_{loc}(\Omega \setminus K), \nabla v \in L^p(\Omega \setminus K, \mathbb{R}^2)\}$ with $u = g$ on $\partial_D\Omega \setminus K$, which satisfy the following unilateral minimality condition:

$$\int_{\Omega\setminus K} f(x, \nabla u)\, dx + \mathcal{H}^1(K) \leq \int_{\Omega\setminus H} f(x, \nabla w)\, dx + \mathcal{H}^1(H), \tag{2.1}$$

among all $H \in \mathcal{K}_m(\overline{\Omega})$ and all functions $w \in L^{1,p}(\Omega \setminus H)$ with $w = g$ on $\partial_D\Omega \setminus H$.

Our main result is the following. Let (K_h) be a sequence in $\mathcal{K}_m(\overline{\Omega})$ which converges to a compact set K in the Hausdorff metric, and let $(g_h) \subset W^{1,p}(\Omega)$ be a sequence which converges strongly to a function g in $W^{1,p}(\Omega)$. Let u_h be such that the pair (u_h, K_h) is a unilateral minimizer, i.e. satisfies property (2.1) relative to the boundary datum g_h. Then ∇u_h converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$ for some function u such that the pair (u, K) is a unilateral minimizer relative to the boundary datum g.

One of the key points in the existence result of a quasi static crack growth of cracks in Dal Maso-Toader [36] is the stability of (2.1) for $f(x, \xi) = |\xi|^2$, which follows from the stability for $p \leq 2$ of the following minimization problem:

$$\min_v \left\{ \int_{\Omega\setminus K} f(x, \nabla v)\, dx \colon\ v \in L^{1,p}(\Omega \setminus K), \quad v = g \text{ on } \partial_D\Omega \setminus K \right\}. \tag{2.2}$$

[1] The results presented in this chapter are contained in Ebobisse-Ponsiglione [41].

As seen in the previous chapter, the stability of (2.2) holds for every $p \leq 2$, while in the case $p > 2$ it does not hold. The strategy to get the stability of problem (2.1) for every $1 < p < \infty$ is to obtain the stability of (2.2) using the unilateral minimality condition.

The obstruction to the stability of (2.2) when $p > 2$ is due to the fact that two connected components of the approximating sequence (K_h) can approach and touch each other in the limit fracture K, leading then to the appearance of a transmission term in the limit problem. To avoid such phenomena we joint these two connected components by curves of infinitesimal length, obtaining then a new sequence of cracks (H_h) having the properties that $K_h \subset H_h$, H_h converges to K, $\mathcal{H}^1(H_h \setminus K_h) \to 0$ and any connected component of H_h converges to a connected component of the limit fracture K. Then the stability of (2.2) along this new sequence of cracks (H_h) will follow from Proposition 2.3.1 (see also Theorem 1.6.4). Now, using the unilateral constraint, we obtain the stability of (2.2) also along the original sequence of cracks (K_h). We prove our main results (see Theorems 2.3.2 and Theorem 2.3.3) following the duality approach, i.e., through the conjugates (see Section 3), performed in [17], [36] for linear problems, and extended in [34] to nonlinear problems.

2.1 Notation and preliminaries

Let Ω be a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$. Let $\partial_D\Omega \subset \partial\Omega$ be a (non-empty) relatively open subset of $\partial\Omega$ composed of a finite number of connected components and $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$.

Let $\mathcal{K}(\overline{\Omega})$ be the class of compact subsets of $\overline{\Omega}$ and $\mathcal{K}_m(\overline{\Omega})$ be the subset of $\mathcal{K}(\overline{\Omega})$ whose elements have at most m connected components. We denote $\mathcal{K}_m^f(\overline{\Omega})$ the subclass of $\mathcal{K}_m(\overline{\Omega})$ whose elements have finite one-dimensional Hausdorff measure $\mathcal{H}^1$. For every $\lambda > 0$, $\mathcal{K}_m^\lambda(\overline{\Omega})$ denotes the class of sets K in $\mathcal{K}_m(\overline{\Omega})$ such that $\mathcal{H}^1(K) \leq \lambda$.

For any $x \in \Omega$ and $\rho > 0$, $B(x,\rho)$ denotes the open ball of $\mathbb{R}^2$ centered at x with radius ρ. For any subset E of $\mathbb{R}^2$, 1_E is the characteristic function of E, E^c is the complement of E, and $|E|$ is the Lebesgue measure of E. Throughout the chapter p and q are real numbers, with $1 < p,\ q < +\infty$ and $p^{-1} + q^{-1} = 1$.

2.1.1 The minimization problem

Let $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a Borel function which satisfies the following assumptions: there exist positive constants α, β, γ such that, for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^2$

$$\alpha|\xi|^p \leq f(x,\xi) \leq \beta|\xi|^p + \gamma; \tag{2.3}$$

$$f(x,\cdot) \text{ is strictly convex and is of class } C^1. \tag{2.4}$$

Given $K \in \mathcal{K}(\overline{\Omega})$ and a function $g \in W^{1,p}(\Omega)$, we consider the following minimization problem

$$\min_v \left\{ \int_{\Omega\setminus K} f(x,\nabla v)\,dx :\ v \in L^{1,p}(\Omega \setminus K), \quad v = g \text{ on } \partial_D\Omega \setminus K \right\}, \tag{2.5}$$

whose weak Euler-Lagrange equation is given by

$$\begin{cases} u \in L^{1,p}(\Omega \setminus K), & u = g \quad \text{on} \quad \partial_D\Omega \setminus K, \\ \displaystyle\int_{\Omega\setminus K} f_\xi(x,\nabla u)\cdot\nabla\varphi\,dx = 0 & \forall \varphi \in L^{1,p}(\Omega\setminus K), \quad \varphi = 0 \quad \text{on} \quad \partial_D\Omega \setminus K. \end{cases} \tag{2.6}$$

By well-known existence results for nonlinear elliptic equations involving strictly monotone operators (see e.g. Lions [56]), one can easily see that (2.6) has a unique solution in the sense that the gradient is always unique.

From now on, given $K \in \mathcal{K}_m(\overline{\Omega})$ and $u \in L^{1,p}(\Omega \setminus K)$, we set

$$E(u, K) := \int_{\Omega\setminus K} f(x, \nabla u)\, dx + \mathcal{H}^1(K). \tag{2.7}$$

Definition 2.1.1. *Let $g \in W^{1,p}(\Omega)$ and let m be a positive integer. We say that a pair (u, K), with $K \in \mathcal{K}_m(\overline{\Omega})$, $u \in L^{1,p}(\Omega \setminus K)$ and $u = g$ on $\partial_D\Omega \setminus K$ is a unilateral minimum of (2.7) if*

$$E(u, K) \leq E(v, H) \tag{2.8}$$

among all $H \in \mathcal{K}_m(\overline{\Omega})$, $H \supset K$ and $v \in L^{1,p}(\Omega \setminus H)$ with $v = g$ on $\partial_D\Omega \setminus H$.

In the following we will need some Lemma on sequences of closed sets converging in the Hausdorff metric. The following Lemma is proved in [36].

Lemma 2.1.2. *Let U be a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. Let K be a closed connected subset of $\overline{U}$. Let $\lambda > 0$ and let $(K_h) \subset \mathcal{K}_m^\lambda(\overline{U})$ be a sequence which converges to K in the Hausdorff metric. Then there exists a sequence (H_h) of closed connected subsets of $\overline{U}$ which converges to K in the Hausdorff metric, with $K_h \subset H_h$ for every h and $\mathcal{H}^1(H_h \setminus K_h) \to 0$.*

Lemma 2.1.3. *Let U be a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary and let $(K_h) \subset \mathcal{K}_m^f(\overline{U})$ be a sequence which converges to a compact set K in the Hausdorff metric. Let Γ be a compact subset of $\overline{U}$ with a finite number of connected components. Then there exists a sequence $(H_h) \subset \mathcal{K}_m^f(\overline{U})$ which converges to K in the Hausdorff metric, with $K_h \subset H_h$ for every h, $\mathcal{H}^1(H_h \setminus K_h) \to 0$ and such that any connected component of $H_h \cup \Gamma$ converges to a connected component of $K \cup \Gamma$ in the Hausdorff metric.*

The proof of this lemma follows the lines of [36, Lemma 3.6]. Precisely, we apply Lemma 2.1.2 to every connected component C of $K \cup \Gamma$ and the union of those connected components of $K_h \cup \Gamma$ whose limits in the Hausdorff metric are contained in C.

The following Lemma proved in [36] will also be useful in the proof of our main results.

Lemma 2.1.4. *Let p and m be two positive integers. Let (K_h) be a sequence in $\mathcal{K}_p^f(\overline{\Omega})$ which converges in the Hausdorff metric to $K \in \mathcal{K}_p^f(\overline{\Omega})$, and let $H \in \mathcal{K}_m^f(\overline{\Omega})$ with $H \supset K$. Then there exists a sequence $(H_h) \subset \mathcal{K}_m^f(\overline{\Omega})$ such that $H_h \to H$ in the Hausdorff metric, $K_h \subset H_h$, and $\mathcal{H}^1(H_h \setminus K_h) \to \mathcal{H}^1(H \setminus K)$.*

In order to study the continuity of the solution u of (2.5) with respect to the variations of the compact set K, we should be able to compare two solutions defined in two different domains. This is why, throughout this chapter, given a function $u \in L^{1,p}(\Omega \setminus K)$, we extend ∇u in Ω by setting $\nabla u = 0$ in $\Omega \cap K$.

The following lemma is proved in [36, Lemma 4.1] for $p = 2$. The case $p \neq 2$ can be proved in the same way.

Lemma 2.1.5. *Let (K_h) be a sequence in $\mathcal{K}(\overline{\Omega})$ which converges to a compact set K in the Hausdorff metric. Let $u_h \in L^{1,p}(\Omega \setminus K_h)$ be a sequence such that $u_h = 0$ C_p-q.e. on $\partial_D\Omega \setminus K_h$ and (∇u_h) is bounded in $L^p(\Omega, \mathbb{R}^2)$. Then, there exists a function $u \in L^{1,p}(\Omega \setminus K)$ with $u = 0$ C_p-q.e. on $\partial_D\Omega \setminus K$ such that, up to a subsequence, ∇u_h converges weakly to ∇u in $L^p(A, \mathbb{R}^2)$ for every $A \subset\subset \Omega \setminus K$. If, in addition, $|K_h|$ converges to $|K|$, then ∇u_h converges weakly to ∇u in $L^p(\Omega, \mathbb{R}^2)$.*

The following three lemmas will be crucial in the proof of our main result.

Lemma 2.1.6. *Let $(K_h) \subset \mathcal{K}_1(\overline{\Omega})$ converging to a compact set K in the Hausdorff metric. Let (v_h) be a sequence in $W^{1,q}(\Omega)$ converging weakly in $W^{1,q}(\Omega)$ to a function v, with $v_h = 0$ C_q-q.e. in K_h. Then $v = 0$ C_q-q.e. in K.*

Proof. We consider an open ball B containing $\overline{\Omega}$ and we extend both functions v_h and v to functions still denoted respectively by v_h and v such that the two extensions belong two $W_0^{1,q}(B)$ and $v_h \rightharpoonup v$ in $W^{1,q}(B)$. Let w_h and w be the solutions of the problems

$$\begin{aligned} w_h \in W_0^{1,q}(B \setminus K_h), \qquad & \Delta_q w_h = \Delta_q v \quad \text{in} \quad B \setminus K_h, \\ w \in W_0^{1,q}(B \setminus K), \qquad & \Delta_q w = \Delta_q v \quad \text{in} \quad B \setminus K. \end{aligned} \tag{2.9}$$

Using a result on the stability of Dirichlet problems by Bucur and Trebeschi [18] (see also Šverák [65] for the case $q = 2$), we obtain that w_h converges to w strongly in $W_0^{1,q}(B)$. Taking $v_h - w_h$ as test function in (2.9), which is possible since $v_h - w_h \in W_0^{1,q}(B \setminus K_h)$ (see, e.g., [54, Theorem 4.5]), we obtain

$$\langle \Delta_q w_h, v_h - w_h \rangle = \langle \Delta_q v, v_h - w_h \rangle, \tag{2.10}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $W^{-1,p}(B)$ and $W_0^{1,q}(B)$. Passing to the limit in (2.10) we obtain

$$\langle \Delta_q w, v - w \rangle = \langle \Delta_q v, v - w \rangle,$$

which implies $v = w$ by the strict monotonicity of $-\Delta_q$. Since, by definition, $w = 0$ C_q-q.e. in K, we conclude that $v = 0$ C_q-q.e. in K. □

Lemma 2.1.7. *Let $(K_h) \subset \mathcal{K}_1(\overline{\Omega})$ converging to a compact set K in the Hausdorff metric. Let (v_h) be a sequence in $W^{1,q}(\Omega)$, converging weakly in $W^{1,q}(\Omega)$ to a function v. Assume that every function v_h is constant C_q-q.e. in K_h. Then v is constant C_q-q.e. in K.*

Proof. This is trivial if K contains only a single point. If K has more than one point, there exists $r > 0$ such that $\operatorname{diam}(K_h) > 2r$ for h large enough. Let us prove that the constant values c_h taken by v_h on K_h are bounded uniformly with respect to h. To this aim let us consider a point $x_h \in K_h$. Since $\operatorname{diam}(K_h) > 2r$, we have $K_h \setminus B(x_h, r) \neq \emptyset$, and by connectedness

$$K_h \cap \partial B(x_h, \rho) \neq \emptyset \qquad \text{for every } 0 < \rho < r. \tag{2.11}$$

As $v_h = c_h$ C_q-q.e. on K_h, by using polar coordinates we deduce from (2.11) the Poincaré inequality

$$\int_{B(x_h,r)} |v_h - c_h|^q \, dx \leq M r^q \int_{B(x_h,r)} |\nabla v_h|^q \, dx,$$

where the constant M is independent of h and r. Since the sequence v_h is bounded in $W^{1,q}(\Omega)$, it follows that c_h is bounded, and so it converges (up to a subsequence) to some constant c. So, the sequence $v_h - c_h$ converges weakly to $v - c$ in $W^{1,q}(\Omega)$, and by Lemma 2.1.6 we get that $v = c$ C_q-q.e. on K. □

2.2 Conjugates and their properties

Let R be the rotation on $\mathbb{R}^2$ defined by $R(y_1, y_2) := (-y_2, y_1)$. The following proposition on the global construction of conjugates will be crucial in the proof of Theorem 2.3.2.

Proposition 2.2.1. *Let $K \in \mathcal{K}(\overline{\Omega})$ and let u be a solution of the problem (2.5). Assume that Ω is simply connected. Then there exists a function $v \in W^{1,q}(\Omega)$ such that $\nabla v = Rf_\xi(x, \nabla u)1_{K^c}$ a.e. in Ω. Moreover, v is constant C_q-q.e. on each connected component of $K \cup \partial_N \Omega$.*

Proof. Let u be a solution of (2.5). We consider the vector field $\Phi \in L^q(\Omega, \mathbb{R}^2)$ defined by

$$\Phi := f_\xi(x, \nabla u)1_{K^c}.$$

We have that $\operatorname{div}(\Phi) = 0$ in $\mathcal{D}'(\Omega)$; hence $\operatorname{rot}(R\Phi) = 0$ in $\mathcal{D}'(\Omega)$. As Ω is simply connected and has a Lipschitz boundary, there exists $v \in W^{1,q}(\Omega)$ such that $\nabla v = R\Phi$ a.e. on Ω.

Let us now prove that v is constant C_q-q.e. on each connected component of $K \cup \partial_N \Omega$ we proceed as follows. Let C be a connected component of $K \cup \partial_N \Omega$ with $C_{1,q}(C) > 0$ and let $\varepsilon > 0$. We set

$$C_\varepsilon := \{x \in \overline{\Omega} \colon \operatorname{dist}(x\,,\, C) < \varepsilon\} \quad \text{and} \quad K_\varepsilon := (K \cup \partial_N \Omega) \cup \overline{C}_\varepsilon.$$

Let u_ε be the solution of the problem (2.5) in $\Omega \setminus K_\varepsilon$. From Lemma 2.1.5 applied to $u_\varepsilon - g$ and by the monotonicity of K_ε, we have that ∇u_ε converges (up to a subsequence) to ∇u^* weakly in $L^p(\Omega \setminus K, \mathbb{R}^2)$ for some $u^* \in L^{1,p}(\Omega \setminus K)$ with $u^* = g$ on $\partial_D \Omega \setminus K$.

We claim that $\nabla u^* = \nabla u$ a.e. in Ω. Indeed, by reformulating the problem (2.6) as a variational inequality in $\Omega \setminus K_\varepsilon$ and using Minty's lemma, we get

$$\int_{\Omega \setminus K_\varepsilon} f_\xi(x, \nabla z) \cdot (\nabla z - \nabla u_\varepsilon)\, dx \geq 0 \qquad \forall z \in L^{1,p}(\Omega \setminus K_\varepsilon),\ z = g \text{ on } \partial_D \Omega \setminus K_\varepsilon.$$

Now, let $z \in L^{1,p}(\Omega \setminus K)$ with $z = g$ on $\partial_D \Omega \setminus K$. By the monotonicity of K_ε, we have that $z \in L^{1,p}(\Omega \setminus K_\varepsilon)$ and $z = g$ on $\partial_D \Omega \setminus K_\varepsilon$. So,

$$\int_{\Omega \setminus K_\varepsilon} f_\xi(x, \nabla z) \cdot (\nabla z - \nabla u_\varepsilon)\, dx \geq 0.$$

Using the convention that $\nabla u_\varepsilon = 0$ in $\Omega \cap K_\varepsilon$ we obtain

$$\int_{\Omega \setminus K} f_\xi(x, \nabla z) \cdot (\nabla z - \nabla u_\varepsilon)\, dx \geq - \int_{K_\varepsilon \setminus K} f_\xi(x, \nabla z) \cdot \nabla z\, dx. \tag{2.12}$$

Now, letting $\varepsilon \to 0$ in (2.12) we obtain

$$\int_{\Omega \setminus K} f_\xi(x, \nabla z) \cdot (\nabla z - \nabla u^*)\, dx \geq 0 \qquad \forall z \in L^{1,p}(\Omega \setminus K),\ z = g \text{ on } \partial_D \Omega \setminus K.$$

which, using again Minty's lemma is equivalent to

$$\int_{\Omega \setminus K} f_\xi(x, \nabla u^*) \cdot \nabla \varphi\, dx = 0 \qquad \forall \varphi \in L^{1,p}(\Omega \setminus K),\ \varphi = 0 \text{ on } \partial_D \Omega \setminus K.$$

By the uniqueness of solution of (2.6) in $\Omega \setminus K$, we get that $\nabla u^* = \nabla u$. So, we have proved that all the sequence (∇u_ε) converges to ∇u weakly in $L^p(\Omega, \mathbb{R}^2)$. On the other hand, one can see that

$$\lim_{\varepsilon \to 0} \int_\Omega [f_\xi(x, \nabla u_\varepsilon) - f_\xi(x, \nabla u)] \cdot (\nabla u - \nabla u_\varepsilon)\, dx = 0. \tag{2.13}$$

Hence arguing as in [34, Lemma 2.4] (recall that $f_\xi(x, \cdot)$ is strictly monotone), it follows that ∇u_ε converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$.

Now, from the first part of the proof, we consider a function $v_\varepsilon \in W^{1,q}(\Omega)$ such that $\nabla v_\varepsilon = Rf_\xi(x, \nabla u_\varepsilon)1_{K_\varepsilon^c}$ a.e. in Ω. We can assume that $\int_\Omega v_\varepsilon \, dx = \int_\Omega v \, dx = 0$. So, by Poincaré inequality we obtain that v_ε converges strongly to v in $W^{1,q}(\Omega)$. By construction $\nabla v_\varepsilon = 0$ in C_ε from which it follows that v_ε is constant C_q-q.e. on $C_\varepsilon \cup \partial_L C_\varepsilon$. Hence v_ε is constant C_q-q.e. on C. Since a subsequence of v_ε converges to v C_q-q.e. on $\overline{\Omega}$, we conclude that v is constant C_q-q.e. on C and this completes the proof. □

Definition 2.2.2. *The function v in Proposition 2.2.1 is called a conjugate of the function u.*

The following lemma is proved like in [36, Theorem 4.3] for $f(x, \xi) = |\xi|^2$. For the reader's convenience we will give here the proof of the present version.

Lemma 2.2.3. *Let $K \in \mathcal{K}_m(\overline{\Omega})$ and $u \in L^{1,p}(\Omega \setminus K)$ with $u = g$ on $\partial_D\Omega \setminus K$. Assume that there exists $v \in W^{1,q}(\Omega)$ such that $\nabla v = Rf_\xi(x, \nabla u)1_{K^c}$ a.e. in Ω and that v is constant C_q-q.e. on every connected component of $K \cup \partial_N\Omega$. Then u is solution of (2.6).*

Proof. Let $C^1, \dots, C^l$ be the connected components of $K \cup \partial_N\Omega$. Since $v = c^i$ C_q-q.e on C^i, by [54, Theorem 4.5], we can approximate v strongly in $W^{1,q}(\Omega)$ by a sequence of functions $v_n \in C_c^\infty(\mathbb{R}^2)$ that are constant in a suitable neighborhood V_n^i of C^i. Let $\varphi \in L^{1,p}(\Omega \setminus K)$ with $\varphi = 0$ on $\partial_D\Omega \setminus K$ and let $\varphi_n \in W_0^{1,p}(\Omega \setminus K)$ such that $\varphi_n = \varphi$ in $\Omega \setminus \bigcup_i V_n^i$. Then we have that

$$\int_\Omega R\nabla v_n \nabla\varphi \, dx = \int_{\Omega\setminus K} R\nabla v_n \nabla\varphi_n dx = 0, \tag{2.14}$$

where the last equality follows from the fact that the vector field $R\nabla v_n$ is divergence free. Then passing to the limit in (2.14) for $n \to \infty$, we get

$$\int_{\Omega\setminus K} f_\xi(x, \nabla u)\nabla\varphi \, dx = \int_\Omega R\nabla v \nabla\varphi \, dx = 0 \quad \forall\varphi \in L^{1,p}(\Omega \setminus K) \text{ with } \varphi = 0 \text{ on } \partial_D\Omega \setminus K.$$

So u is a solution of (2.6). □

The following Lemma on the local construction of conjugates will be used in the proof of Theorem 2.3.3.

Lemma 2.2.4. *Let $K \in \mathcal{K}(\overline{\Omega})$ and let u be a solution of (2.5) in $\Omega \setminus K$. Let U be an open rectangle such that $U \cap \Omega$ is a non empty simply connected set. Then there exists a function $v \in W^{1,q}(U \cap \Omega)$ such that $\nabla v = Rf_\xi(x, \nabla u)1_{K^c}$ a.e. in $U \cap \Omega$. Moreover, v is constant C_q-q.e. on each connected component of $\overline{U} \cap (K \cup \partial_N\Omega)$.*

Proof. We note that u is solution of the following problem

$$\min\left\{\int_{(U\cap\Omega)\setminus K} f(x, \nabla w) \, dx \colon \ w \in L^{1,p}(U \cap \Omega) \setminus K) \text{ and } w = u \text{ on } \partial(U \cap \Omega) \setminus K\right\}.$$

Since $U \cap \Omega$ is simply connected, we can apply Proposition 2.2.1 with Ω replaced by $U \cap \Omega$. So, there exists a function $v \in W^{1,q}(U \cap \Omega)$ such that $\nabla v = Rf_\xi(x, \nabla u)1_{K^c}$ a.e. in $U \cap \Omega$, with v constant C_q-q.e. on each connected component of $\overline{U} \cap (K \cup \partial_N\Omega)$. □

2.3 The stability results relative to problem (2.8)

In this section we give the stability results relative to problem (2.8). First of all, we prove in the following proposition, the stability of problem (2.5) under the condition that any connected component of $K_h \cup \partial_N \Omega$ converges to a connected component of $K \cup \partial_N \Omega$ in the Hausdorff metric.

Proposition 2.3.1. *Let Ω be a simply connected and bounded open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. Assume that $\partial_N \Omega$ has M connected components. Let $\lambda > 0$ and let $(K_h) \subset \mathcal{K}_m^{\lambda}(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric. Let (g_h) be a sequence in $W^{1,p}(\Omega)$ which converges to g strongly in $W^{1,p}(\Omega)$. Let $u_h \in L^{1,p}(\Omega \setminus K_h)$ and $u \in L^{1,p}(\Omega \setminus K)$ be the solutions of the minimization problem (2.5) with boundary data g_h and g respectively. Assume that any connected component of $K_h \cup \partial_N \Omega$ converges to a connected component of $K \cup \partial_N \Omega$ in the Hausdorff metric. Then ∇u_h converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$.*

Proof. By the growth assumptions (2.3) on the function f, we have that ∇u_h and $f_\xi(x, \nabla u_h)$ are bounded respectively in $L^p(\Omega, \mathbb{R}^2)$ and in $L^q(\Omega, \mathbb{R}^2)$. So, applying Lemma 2.1.5 to $u_h - g_h$, we obtain that ∇u_h converges (up to a subsequence) to ∇u^* weakly in $L^p(\Omega, \mathbb{R}^2)$ for some function $u^* \in L^{1,p}(\Omega \setminus K)$ and $u^* = g$ on $\partial_D \Omega \setminus K$.

On the other hand, there exists a vector field $\Psi \in L^q(\Omega, \mathbb{R}^2)$ such that $f_\xi(x, \nabla u_h) \rightharpoonup \Psi$ weakly in $L^q(\Omega, \mathbb{R}^2)$. Let us prove that $\Psi = f_\xi(x, \nabla u^*)$ a.e. in Ω. Since $|K_h| = |K| = 0$ it is sufficient to prove that for every open ball $B \subset\subset \Omega \setminus K$, $\Psi = f_\xi(x, \nabla u^*)$ a.e. in B. Note that by the Hausdorff complementary convergence we have $B \subset\subset \Omega \setminus K_h$ for h large enough.

We may assume that the mean values of u_h and u^* on B are zero. Thus the Poincaré inequality and the Rellich theorem imply that $u_h \to u^*$ strongly in $L^p(B)$. Let $z \in W^{1,p}(B)$ and $\varphi \in C_c^\infty(B)$ with $\varphi \geq 0$. For h large enough we have $B \subset\subset \Omega \setminus K_h$, thus by the monotonicity of $f_\xi(x, \cdot)$ we have

$$\int_B (f_\xi(x, \nabla z) - f_\xi(x, \nabla u_h)) \cdot (\nabla z - \nabla u_h)\varphi \, dx \geq 0. \tag{2.15}$$

We have also

$$\int_B f_\xi(x, \nabla u_h) \cdot \nabla((z - u_h)\, \varphi) \, dx = 0,$$

which, together with (2.15), gives

$$\int_B f_\xi(x, \nabla z) \cdot \nabla((z - u_h)\varphi) \, dx - \int_B (f_\xi(x, \nabla z) - f_\xi(x, \nabla u_h)) \cdot \nabla\varphi \, (z - u_h) \, dx \geq 0. \tag{2.16}$$

We can pass to the limit in each term of (2.16) and we get

$$\int_B f_\xi(x, \nabla z) \cdot \nabla((z - u^*)\varphi) \, dx - \int_B (f_\xi(x, \nabla z) - \Psi) \cdot \nabla\varphi \, (z - u^*) \, dx \geq 0. \tag{2.17}$$

As $\operatorname{div} \Psi = 0$ in $\mathcal{D}'(B)$, we have

$$\int_B \Psi \cdot \nabla((z - u^*)\varphi) \, dx = 0. \tag{2.18}$$

From (2.17) and (2.18) we obtain

$$\int_B (f_\xi(x, \nabla z) - \Psi) \cdot (\nabla z - \nabla u^*) \, \varphi \, dx \geq 0.$$

As φ is arbitrary, we get $(f_\xi(x, \nabla z) - \Psi) \cdot (\nabla z - \nabla u^*) \geq 0$ a.e. in B. In particular, taking $z(x) := u^*(x) \pm \varepsilon\eta \cdot x$, with $\eta \in \mathbb{R}^2$ and $\varepsilon > 0$, we obtain $\pm(f_\xi(x, \nabla u^* \pm \varepsilon\eta) - \Psi) \cdot \eta \geq 0$ a.e. in B. As ε tends

to zero we get $(f_\xi(x,\nabla u^*) - \Psi)\cdot\eta = 0$ a.e. in B, which implies that $f_\xi(x,\nabla u^*) = \Psi$ a.e. in B by the arbitrariness of η.
So we have proved that $f_\xi(x,\nabla u_h) \rightharpoonup f_\xi(x,\nabla u^*)$ weakly in $L^q(\Omega,\mathbb{R}^2)$. Now let us prove that u^* is a solution of (2.5) in $\Omega\setminus K$.

Now we use the assumption that K_h^i converges to K^i for every i. By Proposition 2.2.1 there exists $v_h \in W^{1,q}(\Omega)$ such that $\nabla v_h = Rf_\xi(x,\nabla u_h)$ a.e. in Ω with v_h constant C_q-q.e. on each connected component of $K_h \cup \partial_N\Omega$. Since $f_\xi(x,\nabla u_h)$ converges to $f_\xi(x,\nabla u^*)$ weakly in $L^q(\Omega,\mathbb{R}^2)$, there exists a function $v \in W^{1,q}(\Omega)$ such that $v_h \rightharpoonup v$ weakly in $W^{1,q}(\Omega)$ and $\nabla v = Rf_\xi(x,\nabla u^*)$ a.e. in Ω. Moreover, by Lemma refmoscostante1bis we get that v is constant C_q-q.e. on K^i for every i. So from Lemma 2.2.3 it follows that u^* is a solution of (2.5) in $\Omega\setminus K$ and hence Thus, $\nabla u^* = \nabla u$ a.e. in Ω. Therefore, all the sequence ∇u_h converges to ∇u weakly in $L^p(\Omega,\mathbb{R}^2)$.

Now let us prove that ∇u_h converges to ∇u strongly in $L^p(\Omega,\mathbb{R}^2)$. First of all, by lower semicontinuity we have that

$$\int_\Omega f(x,\nabla u)\,dx \le \liminf_{h\to\infty}\int_\Omega f(x,\nabla u_h)\,dx. \tag{2.19}$$

By the convexity of $f(x,\cdot)$ we have also that

$$\int_\Omega f(x,\nabla u)\,dx \ge \int_\Omega f(x,\nabla u_h)\,dx + \int_\Omega f_\xi(x,\nabla u_h)\cdot(\nabla u - \nabla u_h)\,dx. \tag{2.20}$$

Since

$$\int_\Omega f_\xi(x,\nabla u_h)\cdot(\nabla u_h - \nabla g_h)\,dx = 0 \quad\text{and}\quad \int_\Omega f_\xi(x,\nabla u)\cdot(\nabla u - \nabla g)\,dx = 0,$$

it follows that

$$\begin{aligned}\lim_{h\to\infty}\int_\Omega f_\xi(x,\nabla u_h)\cdot(\nabla u - \nabla u_h)\,dx &= \lim_{h\to\infty}\int_\Omega f_\xi(x,\nabla u_h)\cdot(\nabla u - \nabla g_h)\,dx\\ &= \int_\Omega f_\xi(x,\nabla u)\cdot(\nabla u - \nabla g)\,dx = 0.\end{aligned}$$

Hence passing to the limit in (2.20) we get

$$\int_\Omega f(x,\nabla u)\,dx \ge \limsup_{h\to\infty}\int_\Omega f(x,\nabla u_h)\,dx,$$

which together with (2.19) implies

$$\lim_{h\to\infty}\int_\Omega f(x,\nabla u_h)\,dx = \int_\Omega f(x,\nabla u)\,dx. \tag{2.21}$$

Since $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega,\mathbb{R}^2)$, using the strict convexity of $f(x,\cdot)$, it follows from (2.21) that ∇u_h converges to ∇u strongly in $L^p(\Omega,\mathbb{R}^2)$. □

We are now in a position to prove the main results of the chapter.

2.3.1 The case Ω simply connected

Theorem 2.3.2. *Let Ω be a simply connected and bounded open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. Assume that $\partial_N\Omega$ has M connected components. Let $\lambda > 0$ and $(K_h) \subset \mathcal{K}_m^\lambda(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric. Let (g_h) be a sequence in $W^{1,p}(\Omega)$ which* *converges to g strongly in $W^{1,p}(\Omega)$. Let u_h be such that (u_h, K_h) is a unilateral minimum relative to g_h of the functional E defined in (2.7) and let $u \in L^{1,p}(\Omega\backslash K)$ be the solution of the minimization problem (2.5). Then ∇u_h converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$. Moreover, the pair (u, K) is a unilateral minimum of the functional E relative to g.*

The proof of Theorem 2.3.2. *Step 1.* Let us prove that ∇u_h converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$. Let $K_h^1, \dots, K_h^{n_h}$ be the connected components of $K_h \cup \partial_N \Omega$. As by assumption $n_h \leq m + M$, passing to a subsequence we can assume that $n_h = n$ for every h and that, for every $i \in \{1, \dots, n\}$, K_h^i converges to some compact connected set K^i in the Hausdorff metric.

If $K^i \cap K^j = \emptyset$ for every $i \neq j$, then $K^1, \dots, K^n$ are exactly the connected components of $K \cup \partial_N \Omega$. So, by Proposition 2.3.1 it follows that ∇u_h converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$.

Now we remove the assumption that $K^i \cap K^j = \emptyset$ for every $i \neq j$.

Applying Lemma 2.1.3 for $U = \Omega$ and $\Gamma = \partial_N \Omega$, we obtain a sequence $(H_h) \subset \mathcal{K}_m^f(\overline{\Omega})$ which converges to K in the Hausdorff metric, with $K_h \subset H_h$ for every h, $\mathcal{H}^1(H_h \setminus K_h) \to 0$ and such that any connected component of $H_h \cup \partial_N \Omega$ converges to a connected component of $K \cup \partial_N \Omega$ in the Hausdorff metric.

We consider now the following minimization problem

$$\min_w \left\{ \int_{\Omega\setminus H_h} f(x, \nabla w)\, dx : \; w \in L^{1,p}(\Omega \setminus H_h), \quad w = g_h \text{ on } \partial_D \Omega \setminus H_h \right\}. \tag{2.22}$$

Let $w_h \in L^{1,p}(\Omega \setminus H_h)$ be the solution of (2.22). From Proposition 2.3.1, it follows that ∇w_h converges to ∇u strongly in $L^p(\Omega, \mathbb{R}^2)$. Now using the fact that the pair (u_h, K_h) is a unilateral minimum of the functional (2.7), we get that

$$\begin{aligned} \limsup_{h\to\infty} \int_{\Omega\setminus K_h} f(x, \nabla u_h)\, dx &\leq \lim_{h\to\infty} \int_{\Omega\setminus H_h} f(x, \nabla w_h)\, dx + \lim_{h\to\infty} \mathcal{H}^1(H_h \setminus K_h) \qquad (2.23) \\ &= \int_{\Omega\setminus K} f(x, \nabla u)\, dx. \end{aligned}$$

Hence, recalling that ∇u_h converges to ∇u^* weakly in $L^p(\Omega, \mathbb{R}^2)$, we obtain

$$\begin{aligned} \int_{\Omega\setminus K} f(x, \nabla u^*)\, dx &\leq \liminf_{h\to\infty} \int_{\Omega\setminus K_h} f(x, \nabla u_h)\, dx \leq \\ \leq \limsup_{h\to\infty} \int_{\Omega\setminus K_h} f(x, \nabla u_h)\, dx &\leq \int_{\Omega\setminus K} f(x, \nabla u)\, dx, \end{aligned}$$

which implies (by the uniqueness of solution of (2.5) in $\Omega \setminus K$) that $\nabla u^* = \nabla u$ a.e. in Ω. So, all the sequence ∇u_h converges to ∇u weakly in $L^p(\Omega, \mathbb{R}^2)$ and

$$\lim_{h\to\infty} \int_\Omega f(x, \nabla u_h)\, dx = \int_\Omega f(x, \nabla u)\, dx.$$

Since $\xi \to f(x, \xi)$ is strictly convex, it follows that ∇u_h converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$ and this achieves the proof of Step 1.

Step 2. Let us prove that the pair (u, K) is a unilateral minimum of the functional E relative to g. Let $H \in \mathcal{K}_m(\overline{\Omega})$ with $K \subset H$ and let $w \in L^{1,p}(\Omega \setminus H)$ with $w = g$ on $\partial_D \Omega \setminus H$. By Lemma 2.1.4, there exists a sequence $(H_h) \subset \mathcal{K}_m(\overline{\Omega})$ such that $H_h \to H$ in the Hausdorff metric, $K_h \subset H_h$, and $\mathcal{H}^1(H_h \setminus K_h) \to \mathcal{H}^1(H \setminus K)$. From Lemma 2.1.3, we have a sequence $(\tilde{H}_h) \subset \mathcal{K}_m(\overline{\Omega})$ which converges to

H in the Hausdorff metric and such that $H_h \subset \tilde{H}_h$, $\mathcal{H}^1(\tilde{H}_h \setminus H_h) \to 0$ and, every connected component of $H_h \cup \partial_N \Omega$ converges in the Hausdorff metric to a connected component of $H \cup \partial_N \Omega$. Let $z_h \in L^{1,p}(\Omega \setminus \tilde{H}_h)$ and $z \in L^{1,p}(\Omega \setminus H)$ be the solutions of (2.5) with boundary data g_h and g respectively. From Proposition 2.3.1 it follows that $\nabla z_h \to \nabla z$ strongly in $L^p(\Omega, \mathbb{R}^2)$.

Now using the fact that the pair (u_h, K_h) is a unilateral minimum of the functional (2.7), we get that

$$\int_{\Omega \setminus K_h} f(x, \nabla u_h)\, dx \leq \int_{\Omega \setminus \tilde{H}_h} f(x, \nabla z_h)\, dx + \mathcal{H}^1(\tilde{H}_h \setminus K_h). \tag{2.24}$$

So, passing to the limit in (2.24) and using the fact that $\nabla u_h \to \nabla u$ strongly in $L^p(\Omega, \mathbb{R}^2)$ and $\nabla z_h \to \nabla z$ strongly in $L^p(\Omega, \mathbb{R}^2)$, we obtain

$$\begin{aligned}\int_{\Omega \setminus K} f(x, \nabla u)\, dx = \lim_{h \to \infty} \int_{\Omega \setminus K_h} f(x, \nabla u_h)\, dx \leq \lim_{h \to \infty} \int_{\Omega \setminus \tilde{H}_h} f(x, \nabla z_h)\, dx + \lim_{h \to \infty} \mathcal{H}^1(\tilde{H}_h \setminus K_h) \leq \\ \int_{\Omega \setminus H} f(x, \nabla z)\, dx + \mathcal{H}^1(H \setminus K) \leq \int_{\Omega \setminus H} f(x, \nabla w)\, dx + \mathcal{H}^1(H \setminus K),\end{aligned}$$

which gives Step 2 and achieves the proof of the theorem. □

2.3.2 The general case

Here we remove the assumption that Ω is simply connected and we prove the stability theorem below using the local conjugates in Lemma 2.2.4.

Theorem 2.3.3. *Let Ω be a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. Assume that $\partial_N \Omega$ has M connected components. Let $\lambda > 0$ and $(K_h) \subset \mathcal{K}_m^\lambda(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric. Let (g_h) be a sequence in $W^{1,p}(\Omega)$ which converges to g strongly in $W^{1,p}(\Omega)$. Let u_h be such that (u_h, K_h) is a unilateral minimum relative to g_h of the functional E defined in (2.7) and let $u \in L^{1,p}(\Omega \setminus K)$ be the solution of the minimization problem (2.5). Then ∇u_h converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$. Moreover, the pair (u, K) is a unilateral minimum of the functional E relative to g.*

Proof of Theorem 2.3.3.
First of all let us prove that ∇u_h converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$. By the growth assumptions (2.3) on f, we have that ∇u_h is bounded in $L^p(\mathbb{R}^2, \mathbb{R}^2)$. By Lemma 2.1.5 applied to $u_h - g_h$, we have that ∇u_h converges (up to a subsequence) to ∇u^* weakly in $L^p(\mathbb{R}^2, \mathbb{R}^2)$ for some $u^* \in L^{1,p}(\Omega \setminus K)$ with $u^* = g$ on $\partial_D \Omega \setminus K$. We claim that $\nabla u^* = \nabla u$ a.e. in Ω.

To this aim, we fix $r > 0$ such that the minimum of the diameters of the connected components of Ω^c is equal to $3r$. Using the fact that Ω has a Lipschitz continuous boundary, we may find two families of open rectangles $(Q_i)_{i=1}^n$ and $(U_i)_{i=1}^n$ such that, for every $i \in \{1, \ldots, n\}$, $Q_i \subset\subset U_i$, $Q_i \cap \Omega \neq \emptyset$ and $U_i \cap \Omega$ is a Lipschitz domain and,

$$\overline{\Omega} \subset \bigcup_{i=1}^n Q_i \quad \text{and} \quad \max_{1 \leq i \leq n} \operatorname{diam}(U_i) = 2r.$$

We set

$$\eta := \min_{1 \leq i \leq n} d(Q_i, \partial U_i).$$

For every $i \in \{1, \ldots, n\}$, the number of connected components C of $\overline{U}_i \cap K_h$ which intersect Q_i is less or equal to $m + \lambda/\eta$. Indeed, if C intersects ∂U_i, then $\mathcal{H}^1(C) \geq \eta$ and hence, their number is at most λ/η.

If $C \cap \partial U_i = \emptyset$, then C is a connected component of K_h, and their number is less or equal to m. Similarly the number of connected components of $\overline{U}_i \cap \partial_N \Omega$ which intersect Q_i is less or equal to $M + \mathcal{H}^1(\partial_N \Omega)/\eta$. Let $K_h^{i,1}, \ldots, K_h^{i,k_h}$ be all the connected components of $\overline{U}_i \cap K_h$ which intersect Q_i. Since $k_h \leq m + \lambda/\eta$, passing to a subsequence, we can assume that $k_h = k$ for every h. We set

$$K_h^i := \bigcup_{j=1}^{k} K_h^{i,j}.$$

Up to a subsequence, we have that K_h^i converges in the Hausdorff metric to some compact set $K^i \in \mathcal{K}_k^\lambda(\overline{U_i \cap \Omega})$. Let Γ^i be the union of those connected components of $\overline{U}_i \cap \partial_N \Omega$ which intersect Q_i. By Lemma 2.1.3 applied to $U = U_i \cap \Omega$ and $\Gamma = \Gamma_i$, we get a sequence $(H_h^i) \subset \mathcal{K}_k^f(\overline{U_i \cap \Omega})$ which converges to K^i in the Hausdorff metric, with $K_h^i \subset H_h^i$ for every h, $\mathcal{H}^1(H_h^i \setminus K_h^i) \to 0$ and such that any connected component of $H_h^i \cup \Gamma^i$ converges to a connected component of $K^i \cup \Gamma^i$ in the Hausdorff metric. We set

$$H_h := \bigcup_{i=1}^{n} H_h^i.$$

Note that

$$(H_h) \in \mathcal{K}_m(\overline{\Omega}), \quad H_h \supset K_h, \quad \mathcal{H}^1(H_h \setminus K_h) \to 0$$

and H_h converges in the Hausdorff metric to the compact set $\tilde{K} := \bigcup_{i=1}^n K^i$. Moreover it is easy to see that $\tilde{K} = K$.

We consider now the minimization problem

$$\min_w \left\{ \int_{\Omega \setminus H_h} f(x, \nabla w)\, dx \colon\ w \in L^{1,p}(\Omega \setminus H_h), \quad w = g_h \text{ on } \partial_D \Omega \setminus H_h \right\}. \tag{2.25}$$

Let $\tilde{u}_h \in L^{1,p}(\Omega \setminus H_h)$ be the solution of problem (2.25). Applying Lemma 2.1.5 to $\tilde{u}_h - g_h$, we get that $\nabla \tilde{u}_h$ converges (up to a subsequence) to $\nabla \tilde{u}^*$ weakly in $L^p(\Omega, \mathbb{R}^2)$, for some function $\tilde{u}^*$ in $L^{1,p}(\Omega \setminus K)$ with $\tilde{u}^* = g$ on $\partial_D \Omega \setminus K$. As in the proof of Proposition 2.3.1, we have also that $f_\xi(x, \nabla \tilde{u}_h)$ converges to $f_\xi(x, \nabla \tilde{u}^*)$ weakly in $L^q(\Omega, \mathbb{R}^2)$.

Let us prove that $\nabla \tilde{u}^* = \nabla u$ a.e. in Ω. By a localization argument, it is sufficient to prove that the function $\tilde{u}^*$ satisfies:

$$\begin{cases} \displaystyle\int_{(Q_i \cap \Omega) \setminus K} f_\xi(x, \nabla \tilde{u}^*) \cdot \nabla \varphi\, dx = 0, \\ \forall \varphi \in C_c^\infty(Q_i) \text{ with } \varphi = 0 \text{ on } (Q_i \cap \partial_D \Omega) \setminus K. \end{cases} \tag{2.26}$$

Let $\tilde{H}_h^i := H_h \cap \overline{U_i \cap \Omega}$ and $\tilde{\Gamma}^i := \partial_N \Omega \cap \overline{U_i}$. Since the diameter of U_i is strictly less than the minimum of the diameters of the connected components of Ω^c, we have that the open set $U_i \cap \Omega$ is simply connected. So, by Lemma 2.2.4, there exists a function $v_h^i \in W^{1,q}(U_i \cap \Omega)$ such that $\nabla v_h^i = R f_\xi(x, \nabla \tilde{u}_h)$ a.e. in $U_i \cap \Omega$ and v_h^i is constant C_q-q.e. on the connected components of $\tilde{H}_h^i \cup \tilde{\Gamma}^i$. Since $H_h^i \cup \Gamma^i \subset \tilde{H}_h^i \cup \tilde{\Gamma}^i$, we have that any connected component of $H_h^i \cup \Gamma^i$ is contained in a connected component of $\tilde{H}_h^i \cup \tilde{\Gamma}^i$. So we have also that v_h^i is constant C_q-q.e. on the connected components of $H_h^i \cup \Gamma^i$. From the fact that $f_\xi(x, \nabla \tilde{u}_h)$ converges $f_\xi(x, \nabla \tilde{u}^*)$ weakly in $L^q(\Omega, \mathbb{R}^2)$, it follows that v_h^i converges weakly to some function v^i in $W^{1,q}(U_i \cap \Omega)$ such that $\nabla v^i = R f_\xi(x, \nabla \tilde{u}^*)$ a.e. in $U_i \cap \Omega$. Since any connected component of $H_h^i \cup \Gamma^i$ converges to a connected component of $K^i \cup \Gamma^i$ in the Hausdorff metric and v_h^i is constant C_q-q.e. on the connected components of $H_h^i \cup \Gamma^i$, we get from Lemma refmoscostante1bis that v^i is constant C_q-q.e. on every connected component of $K^i \cup \Gamma^i$. Now applying Lemma 2.2.3 with Ω replaced by $Q_i \cap \Omega$, we get that $\tilde{u}^*$ satisfies (2.26). Therefore $\nabla\tilde{u}^* = \nabla u$ a.e. in Ω. So, $\nabla\tilde{u}_h$ converges weakly to ∇u in $L^p(\Omega,\mathbb{R}^2)$ and $f_\xi(x,\nabla\tilde{u}_h)$ converges to $f_\xi(x,\nabla u)$ weakly in $L^q(\Omega,\mathbb{R}^2)$. Thus, arguing as in the proof of Proposition 2.3.1, we get that $\nabla\tilde{u}_h$ converges to ∇u strongly in $L^p(\Omega,\mathbb{R}^2)$.

Now, from the minimality of the pair (u_h, K_h), we have that

$$\int_{\Omega\setminus K_h} f(x,\nabla u_h)\,dx \le \int_{\Omega\setminus H_h} f(x,\nabla\tilde{u}_h)\,dx + \mathcal{H}^1(H_h\setminus K_h). \tag{2.27}$$

So, passing to the limit in (2.27) and using the fact that $\nabla u_h \rightharpoonup \nabla u^*$ weakly in $L^p(\Omega,\mathbb{R}^2)$, we obtain

$$\begin{aligned} &\int_{\Omega\setminus K} f(x,\nabla u^*)\,dx \le \liminf_{h\to\infty}\int_{\Omega\setminus K_h} f(x,\nabla u_h)\,dx \le \\ \le\ &\lim_{h\to\infty}\int_{\Omega\setminus H_h} f(x,\nabla\tilde{u}_h)\,dx\,dx + \lim_{h\to\infty}\mathcal{H}^1(H_h\setminus K_h) = \int_{\Omega\setminus K} f(x,\nabla u)\,dx, \end{aligned}$$

which implies (by the uniqueness of solution of (2.5) in $\Omega\setminus K$) that $\nabla u^* = \nabla u$ a.e. in Ω and

$$\lim_{h\to\infty}\int_\Omega f(x,\nabla u_h)\,dx = \int_\Omega f(x,\nabla u)\,dx. \tag{2.28}$$

Since $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega,\mathbb{R}^2)$, using the strict convexity of $f(x,\cdot)$, it follows from (2.28) that ∇u_h converges to ∇u strongly in $L^p(\Omega,\mathbb{R}^2)$. This achieves the proof of the first part of the theorem.

Now let us prove that the pair (u,K) is a unilateral minimum of the functional E relative to g. Let $H\in\mathcal{K}_m(\overline{\Omega})$ with $K\subset H$ and let $w\in L^{1,p}(\Omega\setminus H)$ with $w=g$ on $\partial_D\Omega\setminus H$. It is not restrictive to assume that $H\in\mathcal{K}_m^f(\overline{\Omega})$. By Lemma 2.1.4, there exists a sequence $(H_h)\subset\mathcal{K}_m(\overline{\Omega})$ such that $H_h\to H$ in the Hausdorff metric, $K_h\subset H_h$, and

$$\mathcal{H}^1(H_h\setminus K_h)\to\mathcal{H}^1(H\setminus K).$$

Since $(K_h)\subset\mathcal{K}_m^\lambda(\overline{\Omega})$ and $H\in\mathcal{K}_m^f(\overline{\Omega})$, we have that $(H_h)\subset\mathcal{K}_m^{\lambda+\varepsilon}(\overline{\Omega})$ for some $\varepsilon>0$. Arguing as in the first part of the proof, we can construct a sequence $(\tilde{H}_h)\subset\mathcal{K}_m(\overline{\Omega})$ such that $H_h\subset\tilde{H}_h$, $\mathcal{H}^1(\tilde{H}_h\setminus H_h)\to 0$, and denoting $z_h\in L^{1,p}(\Omega\setminus\tilde{H}_h)$ and $z\in L^{1,p}(\Omega\setminus H)$ the solutions of (2.5) with boundary data g_h and g respectively, we get that $\nabla z_h\to\nabla z$ strongly in $L^p(\Omega,\mathbb{R}^2)$. Now we can achieve the proof as in Step 2 of Theorem 2.3.2. □

Chapter 3

Quasi static growth of brittle cracks in a plate

Introduction

In this chapter [1] we propose a variational model for the irreversible quasi static growth in brittle fractures for a linearly elastic homogeneous isotropic plate, subject to a time dependent vertical displacement on a part of its lateral surface. The model is based on the Griffith's criterion for crack growth and is inspired by the model proposed in [48] by G.A. Francfort and J.-J. Marigo in the case of 3-D elasticity. We give a precise mathematical formulation of the model (we adopt the setting of the strong formulation, assuming that the cracks have a uniformly bounded number of connected components) and in this framework we prove an existence result.

The reference configuration is a bounded open set Ω of $\mathbb{R}^2$, which represents the middle surface of the plate, with Lipschitz continuous boundary $\partial\Omega$. Let $m > 0$ be a fixed integer. The set of admissible cracks is the set $\mathcal{K}_m(\overline{\Omega})$ of all closed subsets K of $\overline{\Omega}$ whose elements have at most m connected components. Let $\partial_D\Omega$ be open and with a finite number of connected components. Given a crack $K \in \mathcal{K}_m(\overline{\Omega})$, the boundary datum is prescribed in the set $\partial_D\Omega\backslash K$, and is given by (the trace of) a function $g \in W^{2,2}(\Omega)$; we can not prescribe a boundary condition on $\partial_D\Omega \cap K$ because it is not transmitted through the crack. The remaining part of the boundary $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$ and the cracks K are *traction free.* The displacement u relative to the crack K and subject to the boundary condition g is a function which may jump across K (we will introduce rigorously the space of admissible displacements in Section 3), which verifies the boundary condition and minimize the quadratic form

$$B(v,v) := \frac{2E}{3(1-k^2)} \int_\Omega |v_{xx}|^2 + |v_{yy}|^2 + (2-2k)|v_{xy}|^2 \, dxdy, \tag{3.1}$$

(see [40]), where the Poisson's coefficient $0 < k \leq 1/2$ and the Young's modulus E measure the rigidity of the constituting material. We will consider for simplicity of notations $E = \frac{3(1-k^2)}{2}$, so that the leading coefficient in (4) is equal to 1. Finally, for every admissible crack $K \in \mathcal{K}_m(\overline{\Omega})$ and for every boundary datum g, let us introduce the *bulk energy* $\mathcal{E}^b$ and the *total energy* E defined by

$$\mathcal{E}^b(g,K) := B(u,u), \qquad E(g,K) := B(u,u) + \mathcal{H}^1(K), \tag{3.2}$$

[1] The results presented in this chapter are contained in Acanfora-Ponsiglione [1].

where u is the displacement relative to K and g, and $\mathcal{H}^1(K)$ is the one dimensional Hausdorff measure of K.

We now describe our model of irreversible quasi static crack growth under the action of a time dependent boundary datum. Let $g(t) \in AC([0,1]; W^{2,2}(\Omega))$ (i.e. the function $t \to g(t)$ is absolutely continuous) and let $K_0 \in \mathcal{K}_m(\overline{\Omega})$ be a preexisting crack. In our model, the *irreversible quasi static crack growth* relative to the boundary datum g and to the preexisting crack K_0, is a function $\Gamma : (0,1) \to \mathcal{K}_m(\overline{\Omega})$ which verifies the following three properties:

(1) *Irreversibility of the process:*

$K_0 \subseteq \Gamma(0) \subseteq \Gamma(t_1) \subseteq \Gamma(t_2) \quad \forall 0 \leq t_1 \leq t_2 \leq 1;$

(2) *Static equilibrium:*

$E(g(0), \Gamma(0)) \leq E(g(0), H) \quad \forall H \in \mathcal{K}_m(\overline{\Omega}) : K_0 \subseteq H$ and

$E(g(t), \Gamma(t)) \leq E(g(t), H) \quad \forall t \in (0,1], \forall H \in \mathcal{K}_m(\overline{\Omega}) : \cup_{s<t}\Gamma(s) \subseteq H;$

(3) *Nondissipativity:*

the function $t \to E(g(t), \Gamma(t))$ is absolutely continuous and

$\frac{d}{dt}E(g(t), \Gamma(t)) = 2B(u(t), \dot{g}(t)),$

where $u(t)$ is the displacement relative to $\Gamma(t)$ and to $g(t)$.

The main result of this chapter is Theorem 3.5.1, which establishes the existence of a quasi static evolution that verifies properties (1), (2) and (3) above. This quasi static growth is obtained as limit of a discrete in time growth $\Gamma_\delta(t)$. The construction of the step function $\Gamma_\delta(t)$ is inspired by the Griffith's criterion; more precisely, supposing to have constructed Γ_δ in the interval $[(i-1)\delta, i\delta)$, we define Γ_δ in $[i\delta, (i+1)\delta)$ as a solution of the minimum problem

$$\min_K \{E(g(i\delta), K),\ K \in \mathcal{K}_m(\overline{\Omega}) : \Gamma^\delta_{i-1} \subseteq K\}. \tag{3.3}$$

The main tool used is the stability of these kind of *unilateral free-discontinuity problems* as $\delta \to 0$, that leads to the static equilibrium property of Γ; this is the subject of Section 3.4. The stability result is obtained through Theorem 3.4.1, that we will prove in Section 3.7. It is a new version in the framework of Sobolev spaces of the transfer of jumps Theorem given in [47], which enables to treat energies with derivatives of order greater than one. In fact the proof of the transfer of jumps Theorem given in [47] is based on a geometrical construction which uses the coarea formula, and therefore it needs an a priori bound on $\|\nabla u\|_{L^p(\Omega)}$, given by the fact that u minimizes the bulk energy. In our case the bulk energy involves only second derivatives and the domain (by the presence of cracks) is not regular, and hence Poincaré type inequalities does not hold in general. Therefore it is not clear how to provide a weak formulation in SBV spaces which guarantees such a priori bound on the gradient of u in order to perform the same construction of [47]. These considerations motivated us to choose a strong formulation in the setting of Deny-Lions space

$$L^{2,2}(U) := \{u \in L^2_{\mathrm{loc}}(U) : D^2u \in L^2(u; \mathcal{M}^{2\times 2})\},$$

and with the assumptions on the cracks to be in $\mathcal{K}_m(\overline{\Omega})$. We can thus perform a geometrical construction which does not use coarea formula. Actually our stability result is a generalization of the stability result for energies depending on the gradient described in Chapter 2. However its proof is more complicate, using the tool of the transfer of jumps theorem introduced in [47].

In Section 3.6 we consider the particular case where Γ is rectilinear. In this case in [40] is given a formula for the derivative of bulk energy with respect to the growth of the crack through a $3D-2D$ dimension reduction, under very strong regularity assumptions. Moreover in [64] is proved that this asymptotic quantity coincides with the derivative of the bulk energy $B(u_K, u_K)$ with respect to the growth of the crack (here u_K is the displacement relative to the crack K).

We prove that this quantity depends only on the singular part of the displacement u, and its explicit computation leads to

$$9\pi(1+k)^2\left(\frac{b_1^2}{(7+k)^2}+\frac{b_2^2}{(5+3k)^2}\right),$$

where b_1 and b_2 are coefficients which appear in the singular part of u around the tip (see [40]), and play a role analogous of the so called *mode III stress intensity factor* in elasticity. Moreover, we prove that during the load process

$$9\pi(1+k)^2\left(\frac{b_1(t)^2}{(7+k)^2}+\frac{b_2(t)^2}{(5+3k)^2}\right)\leq 1, \tag{3.4}$$

and that the tip moves if and only if (3.4) is satisfied with the equality. This is the Griffith's criterion for crack propagation in our model.

3.1 Notation and Preliminaries

In this section we introduce the main notations and the preliminary results employed in the rest of the chapter.

From now on Ω is an open bounded subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. For every $x \in \Omega$, we denote the open ball of radius r an centered at x by $B_r(x)$. Let $\mathcal{K}_m(\overline{\Omega})$ be the class of all closed subsets K of $\overline{\Omega}$ whose elements have at most m connected components.

The following semicontinuity result holds (for the proof see [36]).

Lemma 3.1.1. *Let $(K_n) \subset \mathcal{K}_m(\overline{\Omega})$ be such that $\mathcal{H}^1(K_n)$ is uniformly bounded. Then there exists $K \subset \mathcal{K}_m(\overline{\Omega})$ such that K_n converges to K in the Hausdorff metric, and*

$$\mathcal{H}^1(K) \leq \liminf_n \mathcal{H}^1(K_n).$$

Let S be a subset of $\mathbb{R}^2$ and let $x \in S$. For every positive $\lambda \in \mathbb{R}^+$ we set

$$D_\lambda(x)(S) := \Big\{x+\lambda(\xi - x), \quad \xi \in S\Big\}.$$

It is known that if K is connected and $\mathcal{H}^1(K)$ is finite, then $\mathcal{H}^1$-a.e. $x \in K$ admits an approximate normal vector in the sense of measure (see for instance [44]). Moreover the following lemma, proved in [49], holds.

Lemma 3.1.2. *Let $K \in \mathcal{K}_m(\overline{\Omega})$ with $\mathcal{H}^1(K) < \infty$. Then for $\mathcal{H}^1$-a.e. $x \in K$ there exists a vector $\nu(x)$ with $|\nu(x)| = 1$ such that*

$$D_\lambda(x)(K \cap \overline{B_{1/\lambda}(x)}) \to \{x + \nu(x)^\perp\} \cap \overline{B_1(x)} \tag{3.5}$$

in the Hausdorff metric as $\lambda \to \infty$, where $\nu(x)^\perp$ is the space spanned by a vector orthogonal to $\nu(x)$.

The vector $\nu(x)$ is the so called *approximate normal to K at x*. We will need the following Lemma which easily follows by Lemma 3.1.2.

Lemma 3.1.3. *Let K, $H \in \mathcal{K}_m(\overline{\Omega})$ be such that $K \subset H$ and $\mathcal{H}^1(H) \leq \infty$. Then, for $\mathcal{H}^1$-a.e. $x \in K$ there exists a vector $\nu(x)$ with $|\nu(x)| = 1$ such that*

$$D_\lambda(x)(K \cap \overline{B_{1/\lambda}(x)}) \to \{x + \nu(x)^\perp\} \cap \overline{B_1(x)}, \tag{3.6}$$

and

$$D_\lambda(x)(H \cap \overline{B_{1/\lambda(x)}}) \to \{x + \nu(x)^\perp\} \cap \overline{B_1(x)}, \tag{3.7}$$

in the Hausdorff metric as $\lambda \to \infty$.

We recall that given an open subset U of $\mathbb{R}^2$ the Deny-Lions space $L^{2,2}(U)$ is defined by

$$L^{2,2}(U) := \{u \in L^2_{\text{loc}}(U) : \ D^2u \in L^2(u; \mathcal{M}^{2\times 2})\},$$

where $\mathcal{M}^{2\times 2}$ are the 2×2 real matrices. The spaces $L^{2,2}(u)$ are endowed with the seminorm

$$\|u\|_{L^{2,2}(U)} := \|D^2u\|_{L^2(U;\mathcal{M}^{2\times 2})} \qquad \forall u \in L^{2,2}(U).$$

It is well known that $L^{2,2}(U)$ coincides with the Sobolev space $W^{2,2}(U)$ whenever U is bounded and has a Lipschitz continuous boundary, and that the set $\{D^2u \,:\, u \in L^{2,2}(U)\}$ is a closed subspace of $L^2(U; \mathcal{M}^{2\times 2})$.

It is also known (see [57]) that if A is an open subset of $\mathbb{R}^2$ with Lipschitz boundary, there exists a continuous extension operator $E : L^{2,2}(A) \to L^{2,2}(\mathbb{R}^2)$. For every open set $A \subset \mathbb{R}^2$ and every $\varepsilon > 0$, let us now set

$$A_\varepsilon := \{\varepsilon\xi, \, \xi \in A\}.$$

From the existence of a continuous extension operator for a fixed domain, we deduce the following Lemma.

Lemma 3.1.4. *Let A be an open bounded subset of $\mathbb{R}^2$ with Lipschitz boundary. Then for every $\varepsilon > 0$ there exists a continuous extension operator $E_\varepsilon : L^{2,2}(A_\varepsilon) \to L^{2,2}(\mathbb{R}^2)$, such that*

$$\|E_\varepsilon(u)\|^2_{L^{2,2}(\mathbb{R}^2)} \leq C \, \|u\|^2_{L^{2,2}(A_\varepsilon)} \tag{3.8}$$

where C is a constant independent on ε.

Proof. Let E_1 be the extension operator relative to the set A. For every function $u \in L^{2,2}(A_\varepsilon)$, we define the function $\tilde{u} \in L^{2,2}(A)$ as follows: $\tilde{u}(x) := u(\varepsilon x)$. We consider the extension operator E_ε given by

$$E_\varepsilon(u)(x) := E_1(\tilde{u})(\frac{x}{\varepsilon}).$$

Then, by change of variable we get

$$\begin{aligned}\int_{\mathbb{R}^2} |D^2 E_\varepsilon(u)(x)|^2 \, dx &= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} |D^2 E_1(\tilde{u})(y)|^2 \, dy \leq \\ &\leq \frac{1}{\varepsilon^2} C \int_A |D^2 \tilde{u}(y)|^2 \, dy = \frac{1}{\varepsilon^2} \, \varepsilon^2 C \int_{A_\varepsilon} |D^2 u(x)|^2 \, dx,\end{aligned}$$

where C is the constant in (3.8) relative to the extension operator E_1, and this concludes the proof. □
For further properties of the spaces $L^{2,2}$ we refer the reader to [57].

3.2 Formulation of the problem

Static equilibrium for a clamped plate with cracks. We recall here the variational formulation for the static equilibrium of a homogeneous isotropic plate with crack $K \in \mathcal{K}_m(\overline{\Omega})$, subject to vertical displacement on a part of its boundary.

Let Ω be a bounded open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$. We fix a subset $\partial_D\Omega$ of $\partial\Omega$ on which we prescribe a boundary condition; we assume that $\partial_D\Omega$ is non-empty, relatively open in $\partial\Omega$ and composed of a finite number of connected components, and we set $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. For every function $g \in W^{2,2}(\Omega)$ and for every crack $K \in \mathcal{K}_m(\overline{\Omega})$, we set

$$L^{2,2}_{g,\partial_D\Omega}(\Omega \setminus K) := \{u \in L^{2,2}(\Omega \setminus K) : u - g = 0,\ \frac{\partial}{\partial\nu}(u - g) = 0 \text{ a.e. on } \partial_D\Omega \setminus K\}.$$

Here the equality $u - g = 0$ and $\frac{\partial}{\partial\nu}(u - g) = 0$ a.e. on $\partial_D\Omega \setminus K$ are intended in the sense of traces as in [36].

Let us fix the so called Poisson coefficient $0 < k \leq 1/2$. However for most of materials (see [40]) k is strictly less than $1/2$, the case $k = 1/2$ corresponding to incompressible materials. Let us consider the bilinear form $B : L^{2,2}(\Omega \setminus K) \times L^{2,2}(\Omega \setminus K) \to \mathbb{R}$ defined by

$$B(u,v) := \int_{\Omega\setminus K} u_{xx}v_{xx} + u_{yy}v_{yy} + (2-2k)u_{xy}v_{xy}\,dxdy \quad \text{for every } u,\, v \in L^{2,2}(\Omega \setminus K).$$

Note that by definition

$$\|D^2v\|^2 \leq B(v,v) \leq 2\|D^2v\|^2, \tag{3.9}$$

where $\|\cdot\|$ denotes the L^2 norm. The displacement u corresponding to the boundary condition g is a solution of the following minimization problem:

$$\min_{v \in L^{2,2}_{g,\partial_D\Omega}(\Omega\setminus K)} B(v,v). \tag{3.10}$$

Using that the set $\{D^2u : u \in L^{2,2}_{g,\partial_D\Omega}(\Omega)\}$ is closed (see [61]) and (3.9), from the direct method of calculus of variations it follows that the minimum problem (3.10) admits a solution $u \in L^{2,2}_{g,\partial_N\Omega}(\Omega \setminus K)$; moreover the functional $D^2u \to B(u,u)$ is strictly convex, so that D^2u is uniquely determined. We remark also that u and ∇u are uniquely determined in every connected component A of Ω such that $\mathcal{H}^1(\partial A \cap \partial_D\Omega) > 0$, and that problem (3.10) is equivalent to finding $u \in L^{2,2}_{g,\partial_D\Omega}(\Omega)$ such that

$$B(u,v) = 0 \qquad \forall v \in L^{2,2}_{0,\partial_D\Omega}(\Omega). \tag{3.11}$$

For more details on the subject see for instance [40], [45],[64].

Finally let us introduce the *bulk energy* $\mathcal{E}^b : W^{2,2}(\Omega) \times \mathcal{K}_m(\overline{\Omega}) \to \mathbb{R}$ and the *total energy* $E : W^{2,2}(\Omega) \times \mathcal{K}_m(\overline{\Omega}) \to \mathbb{R}$ defined by

$$\mathcal{E}^b(g,K) := B(u,u), \qquad E(g,K) := B(u,u) + \mathcal{H}^1(K), \tag{3.12}$$

where u is a solution of problem (3.10).

Irreversible quasi static growth. We consider now the case of time-dependent boundary conditions and we introduce the notion of *irreversible quasi static growth*. Let $g \in AC([0,1]; W^{2,2}(\Omega))$, where $AC([0,1]; W^{2,2}(\Omega))$ is the space of all absolutely continuous functions defined in $[0,1]$ with values in $W^{2,2}(\Omega)$ (for details on the spaces of absolutely continuous functions see [14]). It is well-known that for a.e. $x \in [0,1]$ there exists the time derivative of g, denoted by $\dot{g}$, and that $\dot{g}$ is a Bochner integrable function with values in $W^{2,2}(\Omega)$.

Let us fix a positive integer $m > 0$. Given a pre-existing crack $K_0 \in \mathcal{K}_m(\overline{\Omega})$ with finite length, an *irreversible quasi static growth* relative to the initial crack K_0 and to the boundary datum $g(t)$, is a function

$$\Gamma : [0,1] \to \mathcal{K}_m(\overline{\Omega})$$

such that the following three properties hold.

(1) *Irreversibility of the process:*

$$K_0 \subseteq \Gamma(0) \subseteq \Gamma(t_1) \subseteq \Gamma(t_2) \quad \forall\, 0 \le t_1 \le t_2 \le 1;$$

(2) *Static equilibrium:*

$$E(g(0), \Gamma(0)) \le E(g(0), H) \quad \forall H \in \mathcal{K}_m(\overline{\Omega}) : K_0 \subseteq H \text{ and}$$

$$E(g(t), \Gamma(t)) \le E(g(t), H) \quad \forall t \in (0,1], \forall H \in \mathcal{K}_m(\overline{\Omega}) : \cup_{s<t}\Gamma(s) \subseteq H;$$

(3) *Nondissipativity:*

the function $t \to E(g(t), \Gamma(t))$ is absolutely continuous and

$$\tfrac{d}{dt}E(g(t), \Gamma(t)) = 2B\big(u(t), \dot{g}(t)\big),$$

where $u(t)$ is the solution of the minimum problem in (3.10) with K replaced by $\Gamma(t)$ and g replaced by $g(t)$.

3.3 Discrete growth of the cracks

In this section we construct a discrete in time approximation of the quasi static growth described previously.

Let Ω, $\partial_D\Omega$ and $\partial_N\Omega$ be as defined in the previous section. Let m be a fixed positive integer, let $K_0 \in \mathcal{K}_m(\overline{\Omega})$ with $\mathcal{H}^1(K_0) \le \infty$, and let $g \in AC([0,1]; W^{2,2}(\Omega))$. For any $\delta > 0$, let N_δ be the largest integer such that $\delta(N_\delta - 1) \le 1$; for $0 \le i \le N_\delta - 1$ we set $t_i^\delta := i\delta$, and $t_{N_\delta}^\delta = 1$. We discretize the boundary data setting $g_i^\delta := g(t_i^\delta)$, and we construct the discrete growth as follows: we set $\Gamma_{-1}^\delta = K_0$ and, supposing to have constructed Γ_{i-1}^δ, we proceed recursively setting $\Gamma_i^\delta \in \mathcal{K}_m(\overline{\Omega})$ as a solution of

$$\min_K \left\{ E(g_i^\delta, K),\ K \in \mathcal{K}_m(\overline{\Omega}) : \Gamma_{i-1}^\delta \subseteq K \right\}, \tag{3.13}$$

and setting u_i^δ as a solution of the minimum problem 3.10 in $\Omega \setminus \Gamma_i^\delta$ with boundary datum g_i^δ.

Lemma 3.3.1. *Problem* (3.13) *admits a solution.*

Proof. Let (K_n) be a minimizing sequence for problem (3.13) and let u_n be a solution of problem (3.10) in $\Omega \setminus K_n$. By the fact that g_i^δ is an admissible function in (3.10) and by (3.9), we can assume that there exists a positive constant C such that

$$\int_{\Omega \setminus K_n} |D^2 u_n|^2 \le C, \qquad \mathcal{H}^1(K_n) \le C. \tag{3.14}$$

By Lemma 3.1.1 there exists $K \in \mathcal{K}_m(\overline{\Omega})$ such that, up to a subsequence, $K_n \to K$ in the Hausdorff metric and

$$\mathcal{H}^1(K) \leq \liminf \mathcal{H}^1(K_n). \tag{3.15}$$

Moreover, by the fact that $\Gamma_{i-1}^\delta \subset K_n$ for every n and $K_n \to K$, we have that $\Gamma_{i-1}^\delta \subset K$. Now, let $A \subset \overline{A} \subset \Omega \setminus K$ be open; by the Hausdorff convergence of K_n to K and since $K \cap A = \emptyset$, it follows that for n big enough $K_n \cap A = \emptyset$. By (3.14) we have that

$$\int_A |D^2 u_n|^2 \, dx \leq C,$$

so that there exists $u \in L^{2,2}(A)$ such that, up to a subsequence, $D^2 u_n$ converges to $D^2 u$ weakly in $L^{2,2}(A, \mathcal{M}^{2\times 2})$. Since A is arbitrary, u can actually be defined in $L^{2,2}(\Omega \setminus K)$. Moreover it is easy to see that $u \in L^{2,2}_{g_i^\delta, \partial_D \Omega}(\Omega \setminus K)$. By lower semicontinuity

$$B(u,u) \leq \liminf B(u_n, u_n). \tag{3.16}$$

From (3.15) and (3.16) it follows that the pair (u, K) minimizes $B(v,v) + \mathcal{H}^1(H)$ among all $H \in \mathcal{K}_m(\overline{\Omega})$ with $\Gamma_{i-1}^\delta \subset H$, and all $v \in L^{2,2}_{g_i^\delta, \partial_D \Omega}(\Omega \setminus H)$, so that the proof is concluded. □

Note that by construction we have that

$$K_0 \subseteq \Gamma_i^\delta \subseteq \Gamma_j^\delta \qquad \forall\, 0 \leq i \leq j \leq N_\delta.$$

Moreover, the minimality property (3.13) is equivalent to

$$B(u_i^\delta, u_i^\delta) + \mathcal{H}^1(\Gamma_i^\delta) \leq B(v,v) + \mathcal{H}^1(H), \tag{3.17}$$

for every $H \in \mathcal{K}_m(\overline{\Omega})$ which contains Γ_i^δ and for every $v \in L^{2,2}_{g_i^\delta, \partial_D \Omega}(\Omega \setminus H)$. From (3.17), comparing u_i^δ with g_i^δ and by (3.9), we have that for every i

$$\int_\Omega |D^2 u_i^\delta|^2 \, dx \leq C \qquad \forall\, 0 \leq i \leq N_\delta, \tag{3.18}$$

where C is a constant independent on i, δ.

Now we define the step functions

$$g^\delta := g_i^\delta, \quad u^\delta := u_i^\delta, \quad \Gamma^\delta := \Gamma_i^\delta, \tag{3.19}$$

for $t_i^\delta \leq t < t_{i+1}^\delta$. By construction and by (3.18), we have that

$$K_0 \subseteq \Gamma^\delta(t_1) \subseteq \Gamma^\delta(t_2) \qquad \forall 0 \leq t_1 \leq t_2 \leq 1,$$

and

$$\int_\Omega |D^2 u^\delta(t)|^2 \, dx \leq C \qquad \forall\, 0 \leq t \leq 1.$$

Next lemma gives an estimate from above for the discrete energy $E(g_i^\delta, \Gamma_i^\delta)$.

Lemma 3.3.2. *For every $1 \leq i \leq N_\delta$ we have*

$$E(g_i^\delta, \Gamma_i^\delta) \leq E(g_0^\delta, \Gamma_0^\delta) + 2\int_0^{t_i^\delta} B(u^\delta(t), \dot{g}(t)) \, dt + o(\delta), \tag{3.20}$$

where $o(\delta) \to 0$ as $\delta \to 0$.

Proof. By (3.17), comparing u^δ_{j+1} with $u^\delta_j - g^\delta_j + g^\delta_{j+1}$, we have that for every $0 \le j \le N_\delta - 1$

$$B\Big(u^\delta_{j+1}, u^\delta_{j+1}\Big) + \mathcal{H}^1\Big(\Gamma^\delta_{j+1}\Big) \le B\Big(u^\delta_j + g^\delta_{j+1} - g^\delta_j, u^\delta_j + g^\delta_{j+1} - g^\delta_j\Big) + \mathcal{H}^1\Big(\Gamma^\delta_i\Big) \le \tag{3.21}$$
$$\le B\Big(u^\delta_j, u^\delta_j\Big) + \mathcal{H}^1\Big(\Gamma^\delta_j\Big) + 2B\Big(u^\delta_j, \int_{t^\delta_j}^{t^\delta_{j+1}} \dot g(t)\, dt\Big) + 2\|D^2 g^\delta_{j+1} - D^2 g^\delta_j\|^2 \le$$
$$\le B\Big(u^\delta_j, u^\delta_j\Big) + \mathcal{H}^1\Big(\Gamma^\delta_j\Big) + 2\int_{t^\delta_j}^{t^\delta_{j+1}} B\Big(u^\delta_j, \dot g(t)\Big)\, dt + S(\delta)\int_{t^\delta_j}^{t^\delta_{j+1}} \|D^2 \dot g(t)\|\, dt,$$

where

$$S(\delta) := 2 \max_{0 \le l \le N_\delta - 1} \int_{t^\delta_l}^{t^\delta_{l+1}} \|D^2 \dot g(t)\|\, dt.$$

Considering the sum for $j = 0$ to $i - 1$ in (3.21), we obtain

$$B\Big(u^\delta_i, u^\delta_i\Big) + \mathcal{H}^1(\Gamma^\delta_i) \le B\Big(u^\delta_0, u^\delta_0\Big) + \mathcal{H}^1(\Gamma^\delta_0) + 2\int_0^{t^\delta_i} B\Big(u^\delta(t), \dot g(t)\Big)\, dt + S(\delta)\int_0^{t^\delta_i} \|D^2 \dot g\|\, dt,$$

that implies (3.20) by choosing $o(\delta) := S(\delta)\int_0^1 \|D^2 \dot g(t)\|\, dt$. □

3.4 Stability of the unilateral free-discontinuity problem

In the minimum problem (3.13), the unknown set Γ^δ_i minimizes the energy $E(g, H)$ among all $H \in \mathcal{K}_m(\overline{\Omega})$ such that $\Gamma^\delta_{i-1} \subseteq H$. In particular Γ^δ_i minimizes the energy among all H larger then Γ^δ_i. This is a so called *unilateral free-discontinuity problem.*

More precisely let $g \in W^{2,2}(\Omega)$, let $K \in \mathcal{K}_m(\overline{\Omega})$ with $\mathcal{H}^1(K) \le \infty$ and let u be a solution of the minimum problem (3.10). We say that the pair (u, K) is a unilateral minimum relative to the boundary condition g if

$$E(g, K) \le E(g, H) \text{ for all } H \in \mathcal{K}_m(\overline{\Omega}) \text{ such that } K \subset H. \tag{3.22}$$

The aim of this section is to study the stability of the unilateral minimality property (3.22) among a sequence of closed sets (K_h) (Theorem 3.4.2), and this result will be a key point for the proof of the equilibrium condition for the crack $\Gamma(t)$. We need the following version of the transfer of jumps Theorem, proved in the setting of BV functions in [47]. The proof is postponed to Section 8.

Theorem 3.4.1 (transfer of jumps Theorem). *Let $(K_h) \subset \mathcal{K}_m(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric and such that $\mathcal{H}^1(K_h) \le C$ for some fixed positive constant C. Let (g_h) be a sequence in $W^{2,2}(\Omega)$ which converges to g strongly in $W^{2,2}(\Omega)$ and let H in $\mathcal{K}_m(\overline{\Omega})$ with $K \subseteq H$ and $\mathcal{H}^1(H) \le C$. Then there exists a sequence $(H_h) \subseteq \mathcal{K}_m(\overline{\Omega})$ converging to H in the Hausdorff metric, with $K_h \subseteq H_h$ for every h, and such that the following properties hold.*

i) $\mathcal{H}^1(H_h \setminus K_h) \to \mathcal{H}^1(H \setminus K)$.

ii) For every $v \in L^{2,2}_{g,\partial_D\Omega}(\Omega \setminus H)$ there exists $v_h \in L^{2,2}_{g_h,\partial_D\Omega}(\Omega \setminus H_h)$ such that

$$D^2 v_h \to D^2 v \qquad \text{strongly in } L^2(\Omega, \mathcal{M}^{2\times 2}).$$

We are now in position to prove the main result of this section.

Theorem 3.4.2. *Let (g_h) be a sequence in $W^{2,2}(\Omega)$ which converges to some g strongly in $W^{2,2}(\Omega)$. Let $(K_h) \subset \mathcal{K}_m(\overline{\Omega})$ with $\mathcal{H}^1(K_h) \leq C$, and let $u_h \in L^{2,2}(\Omega \setminus K_h)$ be such that the pair (u_h, K_h) is a unilateral minimum relative to the boundary condition g_h. Finally let us assume that*

$$D^2 u_h \rightharpoonup D^2 u \quad \text{weakly in } L^2(\Omega, \mathcal{M}^{2\times 2}), \qquad K_h \to K \text{ in the Hausdorff metric.}$$

Then the pair (u, K) is a unilateral minimum relative to the boundary condition g. Moreover $D^2 u_h$ converges to $D^2 u$ strongly in $L^2(\Omega, \mathcal{M}^{2\times 2})$.

Proof. Let us prove that the pair (u, K) is a unilateral minimum relative to the boundary condition g. To this aim, let $H \in \mathcal{K}_m(\overline{\Omega})$ with $K \subset H$ and let $v \in L^{2,2}(\Omega \setminus K)$. Let us consider the sequences (H_h) and (v_h) given by Theorem 3.4.1. By the fact that $K_h \subset H_h$, we have that $\mathcal{H}^1(H_h \setminus K_h) = \mathcal{H}^1(H_h) - \mathcal{H}^1(K_h)$. Hence by the unilateral minimality of the pair (u_h, K_h), we get

$$B(u_h, u_h) \leq B(v_h, v_h) + \mathcal{H}^1(H_h \setminus K_h). \tag{3.23}$$

Passing to the limit for $h \to \infty$ and using Theorem 3.4.1, we get

$$B(u,u) \leq \liminf_h B(u_h, u_h) \leq \limsup_h B(v_h, v_h) + \limsup \mathcal{H}^1(H_h \setminus K_h) = B(v,v) + \mathcal{H}^1(H \setminus K), \tag{3.24}$$

which, by the fact that $K \subset H$, is equivalent to the unilateral minimality condition. Choosing now $H = K$ and $v = u$ in (3.24), we obtain

$$B(u,u) \leq \liminf_h B(u_h, u_h) \leq \limsup B(u_h, u_h) \leq$$
$$\limsup_h B(v_h, v_h) + \limsup \mathcal{H}^1(H_h \setminus K_h) = B(v,v) + \mathcal{H}^1(H \setminus K) = B(u,u).$$

We deduce that $B(u_h, u_h) \to B(u,u)$, which (together with $D^2 u_h \rightharpoonup D^2 u$) implies that $D^2 u_h$ converges to $D^2 u$ strongly in $L^2(\Omega, \mathcal{M}^{2\times 2})$, and this concludes the proof. □

Remark. In Theorem 3.4.2 the assumption that the minima u_h are unilateral can not be removed in order to get the stability. In fact let us consider $\Omega := (-1,1)^2$, and

$$K_h := [-1, -1/h] \cup [1/h, 1] \times 0,$$

which converges to $K := [-1,1] \times 0$ in the Hausdorff metric. Let moreover

$$\partial_D \Omega := [-1,1] \times \{-1\} \cup [-1,1] \times \{1\},$$

and let $g_h \equiv g$ be a fixed function with normal derivative equal to 0 on $\partial_D \Omega$ and with

$$g = -1 \text{ on } [-1,1] \times \{-1\} \quad \text{and} \quad g = 1 \text{ on } [-1,1] \times \{1\}.$$

Let $Q^- := \Omega \cap \{x_2 < 0\}$ and $Q^+ := \Omega \cap \{x_2 > 0\}$. The solution u of (3.10) in $\Omega \setminus K$ is clearly the function with zero energy defined by

$$u(x) := \begin{cases} 1 & \text{if } x \in Q^+; \\ -1 & \text{if } x \in Q^-. \end{cases}$$

If u_h are the solutions of (3.10) in $\Omega \setminus K_h$, it is easy to see that $D^2 u_h$ does not converge to $D^2 u$ weakly in $L^2(\Omega, \mathcal{M}^{2\times 2})$; otherwise we will have that u_h converges to -1 uniformly in Q^- and to $+1$ uniformly in Q^+. On the other hand by symmetry we have that $u_h(0,0) \equiv 0$, and this gives a contradiction.

3.5 Irreversible quasi static growth of the cracks

In this Section we prove the main result of the chapter, that is the existence of an irreversible quasi static growth of brittle cracks as formulated in Section 2.

Theorem 3.5.1. *Let m be a fixed positive integer, let $K_0 \in \mathcal{K}_m(\overline{\Omega})$ with finite length and let $g \in AC([0,1]; W^{2,2}(\Omega))$. Then there exists an irreversible quasi static growth $\Gamma : [0,1] \to \mathcal{K}_m(\overline{\Omega})$ relative to the initial crack K_0 and to the boundary datum g.*

Proof. Let Γ_δ be the step function defined in (3.19). As proved in [36, Theorem 6.3.], there exists a sequence $\delta_n \to 0$ and an increasing function $\Gamma : [0,1] \to \mathcal{K}_m(\overline{\Omega})$, such that for every $t \in [0,1]$

$$\Gamma_{\delta_n}(t) \to \Gamma(t) \quad \text{in the Hausdorff metric.} \tag{3.25}$$

We claim that Γ is a quasi static growth. For every $t \in [0,1]$, we set $u(t)$ as a solution of (3.10) in $\Omega \setminus \Gamma(t)$ relative to the boundary condition $g(t)$. We have that

$$E\Big(g(0), \Gamma(0)\Big) \leq E\Big(g(t), H\Big) \quad \forall H \in \mathcal{K}_m(\overline{\Omega}) : K_0 \subseteq H. \tag{3.26}$$

In fact $\Gamma_\delta(0)$ does not depend on δ, that is $\Gamma_\delta(0) \equiv \Gamma(0)$ for every δ. Then (3.26) follows directly by (3.17) with $i = 0$. Now we prove that

$$E(g(t), \Gamma(t)) \leq E(g(t), H) \quad \forall t \in (0,1], \quad \forall H \in \mathcal{K}_m(\overline{\Omega}) : \cup_{s \leq t} \Gamma(s) \subseteq H. \tag{3.27}$$

To this aim, note that for every fixed $t \in [0,1]$ the pair $(u_{\delta_n}(t), \Gamma_{\delta_n}(t))$ is a unilateral minimum relative to the boundary condition $g_{\delta_n}(t)$. For every t we have by construction that $\Gamma^{\delta_n}(t) \to \Gamma(t)$ in the Hausdorff metric. Moreover, up to a subsequence $D^2 u^{\delta_n}(t) \rightharpoonup D^2 \tilde{u}$ for some $\tilde{u} \in L^{2,2}(\Omega \setminus \Gamma(t))$. Recalling that $g_{\delta_n}(t)$ converges to $g(t)$ strongly in $W^{2,2}(\Omega)$, we are in position to apply Theorem 3.4.2, so that the pair $(\tilde{u}, \Gamma(t))$ is a unilateral minimum relative to the boundary condition $g(t)$. By the fact that both $\tilde{u}$ and u are minimizers of (3.10), we deduce that $D^2\tilde{u} = D^2 u(t)$, and hence for every t the pair $(u(t), \Gamma(t))$ is a unilateral minimum relative to the boundary condition $g(t)$, that is (3.27) holds. Moreover, as a consequence of Theorem 3.4.2 we also get

$$D^2 u^{\delta_n}(t) \to D^2 u(t) \quad \text{strongly in } L^2(\Omega, \mathcal{M}^{2\times 2}) \text{ for every } t \in [0,1]. \tag{3.28}$$

Now we prove that all properties defining the quasi static growth are satisfied.

1) Irreversibility of the process. This property holds by construction.

3) Nondissipativity. Using (3.28) and (3.20), we easily get

$$\begin{aligned} B(u(t), u(t)) + \mathcal{H}^1(\Gamma(t)) &\leq \liminf \left(B(u^{\delta_n}(t), u^{\delta_n}(t)) + \mathcal{H}^1(\Gamma^{\delta_n}(t)\right) \leq \\ &\leq E(g_0^{\delta_n}, \Gamma_0^{\delta_n}) + \liminf 2 \int_0^t B(u^{\delta_n}(\tau), \dot{g}(\tau))\, d\tau = E(g(0), \Gamma(0)) + 2 \int_0^t B(u(\tau), \dot{g}(\tau)) d\tau. \end{aligned}$$

To prove the inverse inequality, given $t \in [0,1]$ and given a positive integer k, let us set $s_i^k := \frac{i}{k} t$ for all $i = 0, \ldots, k$. By (3.27), comparing $u(s_i^k)$ with $u(s_{i+1}^k) - g(s_{i+1}^k) + g(s_i^k)$, we get

$$\begin{aligned} &B(u(s_i^k), u(s_i^k)) + \mathcal{H}^1(\Gamma((s_i^k)) \leq \\ B\left((u(s_{i+1}^k) - g(s_{i+1}^k) + g(s_i^k)), \right.&\left.(u(s_{i+1}^k) - g(s_{i+1}^k) + g(s_i^k))\right) + \mathcal{H}^1(\Gamma(s_{i+1}^k)) = \\ B(u(s_{i+1}^k), u(s_{i+1}^k)) + \mathcal{H}^1(\Gamma(s_{i+1}^k)) &+ B\left((g(s_{i+1}^k) - g(s_i^k)), (g(s_{i+1}^k) - g(s_i^k))\right) - \\ &2 \int_{s_i^k}^{s_{i+1}^k} B\left(u(s_{i+1}^k), \dot{g}(\tau)\right) d\tau. \end{aligned} \tag{3.29}$$

Summing for $i = 0$ to k in (3.29), and setting $u^k(t) = u(s^k_{i+1})$ for $s^k_i \leq t < s^k_{i+1}$, we get

$$B(u(0), u(0)) + \mathcal{H}^1(\Gamma(0)) + 2\int_0^t B(u^k(\tau), \dot{g}(\tau))\, d\tau \leq B(u(t), u(t)) + \mathcal{H}^1(\Gamma(t)) + o_k, \tag{3.30}$$

where $o_k \to 0$ as $k \to \infty$. Let us set now $\Gamma^k(t) = \Gamma(s^k_{i+1})$ for $s^k_i \leq t < s^k_{i+1}$. By construction we have that $(u^k(t), \Gamma^k(t))$ is a unilateral minimum for every t. If t is a continuity point for the function $l \to \mathcal{H}^1(\Gamma(l))$ it is easy to check that $\Gamma^k(t)$ converges to $\Gamma(t)$ in the Hausdorff metric. Arguing as in the proof of (3.28) we have that $D^2u^k(t)$ converges to $D^2u(t)$ strongly in $L^2(\Omega, \mathcal{M}^{2\times 2})$ so that $B(u^k(\tau, \dot{g}(\tau))$ converges to $B(u^k(\tau, \dot{g}(\tau))$ for a.e. τ. Therefore passing to the limit for $k \to \infty$ in (3.30) we obtain

$$E(g(t), \Gamma(t)) \geq E(g(0), \Gamma(0)) + 2\int_0^t B(u(\tau), \dot{g}(\tau)) d\tau. \tag{3.31}$$

2) Static equilibrium. Let us fix $t \in (0, 1)$ and let (s_h) be an increasing sequence converging to t. By (3.27) we get

$$B(u(s_h), u(s_h)) + \mathcal{H}^1(\Gamma(s_h)) \leq B(v - g(t) + g(s_h), v - g(t) + g(s_h)) + \mathcal{H}^1(H)$$

for every $H \in \mathcal{K}_m(\overline{\Omega}) : \cup_{s<t}\Gamma(s) \subseteq H$ and for every $v \in L^{2,2}_{g(t),\partial_D\Omega}(\Omega \setminus H)$. Letting $h \to \infty$ and using that the function $t \to E(g(t), \Gamma(t))$ is continuous by the nondissipativity condition, we deduce

$$B(u(t), u(t)) + \mathcal{H}^1(\Gamma(t)) \leq B(v, v) + \mathcal{H}^1(H),$$

for every $H \in \mathcal{K}_m(\overline{\Omega}) : \cup_{s<t}\Gamma(s) \subseteq H$ and for every $v \in L^{2,2}_{g(t),\partial_D\Omega}(\Omega \setminus H)$, so that also static equilibrium property holds. □

3.6 Griffith's criterion for crack growth

In this section we shall see that, in the model case where the crack $\Gamma(t)$ is rectilinear, it satisfies Griffith's criterion for crack growth. More precisely let Ω be open and connected, and let $\partial_D\Omega$ be a (non empty) open subset of $\partial\Omega$ composed of a finite number of connected components. We consider a quasi static growth $\Gamma(t)$, relative to a boundary datum $g \in AC([0, 1]; W^{2,2}(\Omega))$, of the following type (see Fig. 1):

$$\Gamma(t) := [0, x_1(t)] \times \{x_2\}, \tag{3.32}$$

where $x_1 : [0, 1] \to [l_1, l_2]$ is an increasing function and $[0, l_1] \times \{x_2\}$ is a preexisting crack which touches the boundary of Ω at the point $(0, x_2)$. For every $x_1 \in [l_1, l_2]$ we set

$$K(x_1) := [0, x_1] \times x_2.$$

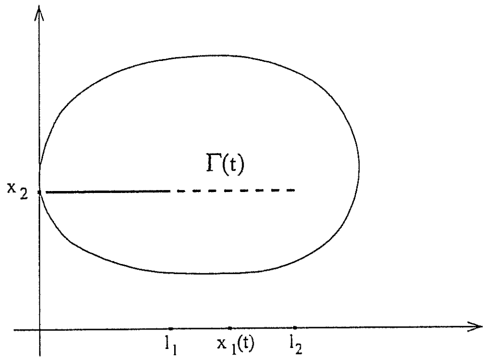

Fig. 1

We want to compute the derivative of the bulk energy $\mathcal{E}^b(g(t), K(x_1))$ defined in (3.12) with respect to the growth of the crack (that is with respect to x_1) at the point $x_1(t)$. For every function $v \in L^{2,2}(\Omega \setminus \Gamma(t))$, we set

$$
\begin{aligned}
M_{11}[v] &:= v_{x_1x_1} + k\, v_{x_2x_2};\\
M_{22}[v] &:= v_{x_2x_2} + k\, v_{x_1x_1};\\
M_{12}[v] &= M_{21}[v] := (1-k)\, v_{x_1x_2}.
\end{aligned}
$$

Let now C be a smooth closed path around the point $(x_1(t), x_2)$ and let $u(t)$ be a solution of (3.10) in $\Omega \setminus \Gamma(t)$. In [64], [40], is proved that the functional $x_1 \to \mathcal{E}^b(g(t)), K(x_1))$ is C^1, and that the following formula holds.

$$
\begin{aligned}
\frac{d}{dx_1}\mathcal{E}^b(g(t)), K(x_1))\,|_{x_1=x_1(t)} &= \sum_{i,j\in\{1,2\}} \Big(-\frac{1}{2}\int_C M_{ij}[u(t)]\, u_{x_ix_j}(t)\nu_1 \\
&+ \int_C M_{ij}[u(t)]\, u_{x_1x_j}(t)\nu_i - \int_C \frac{\partial}{\partial x_j} M_{ij}[u(t)] \frac{\partial}{\partial x_1} u(t)\nu_i(t)\Big) \\
&+ (M_{12}[u(t)]\frac{\partial}{\partial x_1} u(x^+)(t)) - (M_{12}[u(t)]\frac{\partial}{\partial x_1} u(x^-)(t)),
\end{aligned} \tag{3.33}
$$

where $\nu = (\nu_1, \nu_2)$ is the inner normal to C and x^+ and x^- are as in Fig. 2.

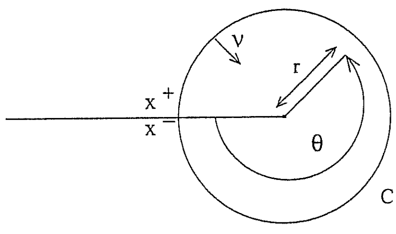

Fig. 2

It is well known that the solution $u(t)$ has the following behavior in a neighborhood of the tip $(x_1(t), x_2)$ (see [40], [53]).

$$u(t)(r,\theta) = r^{3/2}\Big(b_1(t)\big(\sin(\frac{3\theta}{2}) + \frac{3(1-k)}{7+k}\sin(\frac{\theta}{2})\big) + b_2(t)\big(\cos(\frac{3\theta}{2}) + \frac{3(1-k)}{5+3k}\cos(\frac{\theta}{2})\big)\Big) + u^R(r,\theta), \tag{3.34}$$

where (r,θ) are the polar coordinates as in Figure 2, and $u^R \in W^{3,2}(\Omega)$.

For every b_1, $b_2 \in \mathbb{R}$ we set

$$u^S(b_1,b_2)(r,\theta) := r^{3/2}\Big(b_1\big(\sin(\frac{3\theta}{2}) + \frac{3(1-k)}{7+k}\sin(\frac{\theta}{2})\big) + b_2\big(\cos(\frac{3\theta}{2}) + \frac{3(1-k)}{5+3k}\cos(\frac{\theta}{2})\big)\Big), \tag{3.35}$$

so that $u(t) = u^S(b_1(t), b_2(t)) + u^R(t)$. Now let us fix a radius $\varepsilon > 0$. For every v, $w \in L^{2,2}(\Omega \setminus \Gamma(t))$, we consider the bilinear form $b^\varepsilon : L^{2,2}(\Omega \setminus \Gamma(t)) \times L^{2,2}(\Omega \setminus \Gamma(t)) \to \mathbb{R}$ defined by

$$b^\varepsilon(v,w) := \sum_{i,j\in\{1,2\}} \Big(-\frac{1}{2}\int_{B_\varepsilon((x_1(t),x_2))} M_{ij}[v] w_{x_i x_j}\nu_1 + \int_{B_\varepsilon((x_1(t),x_2))} M_{ij}[v] w_{x_1 x_j}\nu_i \tag{3.36}$$
$$- \int_{B_\varepsilon((x_1(t),x_2))} \frac{\partial}{\partial x_j} M_{ij}[v] \frac{\partial}{\partial x_1} w\nu_i(t)\Big) + \big(M_{12}[u(t)]\frac{\partial}{\partial x_1} w(x^+)(t)\big) - \big(M_{12}[u(t)]\frac{\partial}{\partial x_1} w(x^-)(t)\big),$$

Finally, for every b_1, $b_2 \in \mathbb{R}$ we define the quadratic form $q : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$q(b_1,b_2) := -b^\varepsilon(u^S(b_1,b_2), u^S(b_1,b_2)). \tag{3.37}$$

From (3.35) and (3.36) it easily follows that q does not depend on ε. The explicit computation of the right hand-side of (3.37), leads to the following expression

$$q(b_1,b_2) = 9\pi(1+k)^2\left(\frac{b_1^2}{(7+k)^2} + \frac{b_2^2}{(5+3k)^2}\right).$$

In order to prove that $q(b_1(t), b_2(t))$ is the only contribution that does not vanish in (3.33) as ε tends to zero, we will need the following Lemma.

Lemma 3.6.1. *Let $B_h(z)$ be an open ball in $\mathbb{R}^2$ and let $f \in L^p(B_h(z))$. Then there exists a subset $I \subset [0,h]$ such that*

$$\lim_{l\to 0} \frac{|I \cap [0,l]|}{l} = 1,$$

and such that for every sequence $\{\varepsilon_n\} \subset I$ with $\{\varepsilon_n\} \to 0$, we have that f is defined for $\mathcal{H}^1$-a.e. $x \in \partial B_{\varepsilon_n}(z)$, and

$$\lim_{n\to\infty} \quad \varepsilon_n^{\frac{2-p}{p}} \int_{\partial B_{\varepsilon_n}(z)} |f(x)|\, dx \ \to 0.$$

Proof. We have

$$\int_{B_h(z)} |f|^p = \int_0^h \int_{\partial B_r(z)} |f|^p = \int_0^h 2\pi r\Big(\frac{1}{2\pi r}\int_{\partial B_r(z)} |f|^p\Big).$$

By the fact that $f \in L^p(B_h(z))$, using Jensen inequality, there exists a positive constant C such that

$$\int_0^h r^{1-p}\left(\int_{\partial B_r(z)} |f|\right)^p \le C.$$

Now suppose by contradiction that there exist $\delta_1, \delta_2 > 0$ such that, setting

$$U := \{r : r^{(2-p)/p} \int_{\partial B_r(z)} |f| \geq \delta_1\},$$

there exist arbitrary small intervals $[0, J]$ with

$$\frac{|U \cap [0, J]|}{J} \geq \delta_2.$$

We deduce that

$$\int_{[0,J]} r^{1-p} \left(\int_{\partial B_r(z)} |f| \right)^p = \int_{[0,J]} \frac{1}{r} \left(r^{(2-p)/p} \int_{\partial B_r(z)} |f| \right)^p \geq \int_{[0,J] \cap U} \frac{1}{r} \delta_1^p \geq \frac{1}{J} \delta_1^p \, |[0,J] \cap U| \geq \delta_1^p \delta_2,$$

and this, by the arbitrariness of J, is in contradiction with the equi-integrability of the L^1 function $r^{1-p} \big(\int_{\partial B_r(z)} |f| \big)^p$. □

Next theorem gives a more explicit formula then (3.33) for the derivative of the bulk energy with respect to the growth of the crack. We will see that this derivative actually depends only on the coefficients in (3.34) of the singular part of u.

Theorem 3.6.2. *Let $\Gamma(\cdot)$ be a quasi static growth of the type* (3.32)*, and let $b_1(\cdot)$, $b_2(\cdot)$ be the coefficients in* (3.34)*. Then for every $t \in (0, 1)$*

$$\frac{d}{dx_1} \mathcal{E}^b(g(t)), K(x_1)) \,|_{x_1 = x_1(t)} = -q(b_1(t), b_2(t)).$$

Proof. By (3.33), we have that there exists $h > 0$ such that for every $\varepsilon \leq h$

$$\begin{aligned} &\frac{d}{dx_1} \mathcal{E}^b(g(t)), K(x_1)) \,|_{x_1 = x_1(t)} = b^\varepsilon \big(u^S(b_1(t), b_2(t)) + u^R(t), u^S(b_1(t), b_2(t)) + u^R(t) \big) = \\ &-q\big(b_1(t), b_2(t)\big) + b^\varepsilon \big(u^S(b_1(t), b_2(t)), u^R(t) \big) + b^\varepsilon \big(u^R(t), u^S(b_1(t), b_2(t)) \big) + b^\varepsilon \big(u^R(t), u^R(t) \big). \end{aligned} \tag{3.38}$$

We claim that there exists subsets $I, L, M \subset (0, h)$ with the property

$$\lim_{l \to 0} \frac{I \cap [0, l]}{l} = 1, \qquad \lim_{l \to 0} \frac{L \cap [0, l]}{l} = 1, \qquad \lim_{l \to 0} \frac{M \cap [0, l]}{l} = 1, \tag{3.39}$$

such that the following hold:

1) $\lim_{n \to \infty} b^{\varepsilon_n} (u^S(b_1(t), b_2(t)), u^R(t)) = 0 \qquad \forall \{\varepsilon_n\} \subset I, \ \varepsilon_n \to 0;$

2) $\lim_{n \to \infty} b^{\varepsilon_n} (u^R(t), u^S(b_1(t), b_2(t))) = 0 \qquad \forall \{\varepsilon_n\} \subset L, \ \varepsilon_n \to 0;$

3) $\lim_{n \to \infty} b^{\varepsilon_n} (u^R(t), u^R(t)) = 0 \qquad \forall \{\varepsilon_n\} \subset M, \ \varepsilon_n \to 0.$

Therefore, it is sufficient to consider a sequence (ε_n) in $I \cap L \cap M$ (which exists in view of (3.39)): along this sequence, the right hand side of (3.38) tends to $-q(b_1(t), b_2(t))$, and this concludes the proof. So let us prove 1), the proof of 2) and 3) being similar.

To this aim, note that we can always assume that $\nabla u^R(t)(x_1(t), x_2) = 0$. In fact, for every fixed $\xi \in \mathbb{R}^n$ and for every $s \in (0, 1)$, we have that the function

$$u^S(b_1(s), b_2(s)) + (u^R(s) + \xi \cdot x)$$

is a solution of (3.10) relative to the boundary condition $g(s)+\xi \cdot x$, and

$$\mathcal{E}^b((g(s)), K(x_1(s))) = \mathcal{E}^b((g(s)+\xi\cdot x(s)), K(x_1(s))).$$

We have

$$b^\varepsilon(u^S(b_1(t),b_2(t)), u^R(t)) \leq C\int_{\partial B_\varepsilon((x_1(t),x_2))} \varepsilon^{-1/2}|f(x)| + \sum_{i,j\in\{1,2\}}\int_{\partial B_\varepsilon((x_1(t),x_2))}\varepsilon^{-3/2}|\frac{\partial}{\partial x_1}u^R(t)| +$$
$$(M_{12}[u^S(b_1(t),b_2(t))]\frac{\partial}{\partial x_1}u^R(x^+)(t)) - (M_{12}[u^S(b_1(t),b_2(t))]\frac{\partial}{\partial x_1}u^R(x^-)(t)),$$

where C is a positive constant and $f \in H^1(\Omega)$. By Lemma 3.6.1, noting that by the Sobolev embedding Theorem $f \in L^p(B_\varepsilon(x_1(t)))$ for every $p \geq 1$, there exists a subset $I \subset (0,h)$ that verifies (3.39) and such that

$$\lim_{n\to\infty} C\int_{\partial B_{\varepsilon_n}(x_1(t)}\varepsilon_n^{-1/2}f(x) = 0 \qquad \forall\{\varepsilon_n\}\subset I,\ \varepsilon_n \to 0.$$

Concerning the second term, note that the function $\frac{\partial}{\partial x_1}u^R(t)$ is in $H^2(\Omega)$, so that it is holder continuous with coefficient greater than $1/2$ (it is for instance in $C^{0,2/3}(\Omega)$), and hence, recalling that $\nabla u^R(t)(x_1(t),x_2) = 0$, we have that for every i,j and for ε small enough

$$\int_{\partial B_\varepsilon(x_1(t))}\varepsilon^{-3/2}|\frac{\partial}{\partial x_1}u^R(t)| \leq C\int_{\partial B_\varepsilon(x_1(t))}\varepsilon^{-3/2}\varepsilon^{2/3} = C\int_{\partial B_\varepsilon(x_1(t))}\varepsilon^{-5/6}$$

which tends to zero as $\varepsilon \to 0$. Finally last term is equal to zero for every ε because

$$\frac{\partial}{\partial x_1}u^R(x^+)(t) = \frac{\partial}{\partial x_1}u^R(x^-)(t), \qquad M_{12}[u^S(b_1(t),b_2(t))](x^+) = M_{12}[u^S(b_1(t),b_2(t))](x^-).$$

□

By Theorem 3.6.2, we deduce the following formula for the derivative of the total energy with respect to the growth of the crack.

$$\frac{d}{dx_1}E(g(t)), K(x_1))\,|_{x_1=x_1(t)} = 1 - q(b_1(t),b_2(t)). \tag{3.40}$$

Moreover, arguing as in [36], it is possible to prove that for every $t \in (0,1)$

$$\frac{d}{ds}E(g(t),\Gamma(s))\,|_{s=t} = 0.$$

We are now in position to state the main result of this section.

Theorem 3.6.3. *Let $\Gamma(t)$ be a quasi static growth of the type* (3.32). *Then*

$$\dot{x}_1(t) \geq 0 \qquad \textit{for a.e. } t\in(0,1), \tag{3.41}$$
$$1 - q(b_1(t),b_2(t)) \geq 0 \qquad \textit{for every } t\in(0,1), \tag{3.42}$$
$$(1 - q(b_1(t),b_2(t)))\dot{x_1}(t) = 0 \qquad \textit{for a.e. } t\in(0,1). \tag{3.43}$$

Proof. The first condition comes directly from the irreversibility of the process. The second condition comes directly from (3.40) and from the static equilibrium condition.

So, let us pass to the prove of the third condition, that actually is the Griffith's criterion for crack growth in our model. let $t \in [0,1]$ be a point of differentiability for $x_1(t)$. We have

$$0 = \frac{d}{ds}E(g(t),\Gamma(s))\,|_{s=t} = \frac{d}{dx_1}E(g(t),K(x_1))\,|_{x_1=x_1(t)}\,\dot{x}_1(t) = (1-q(b_1(t),b_2(t)))\,\dot{x}_1(t).$$

□

and this concludes the proof.

3.7 Proof of the transfer of jumps Theorem

In this section we prove the transfer of jumps Theorem (Theorem 3.4.1). In the proof we will need the following lemma, which is a particular case of [36][Lemma 3.6].

Lemma 3.7.1. *Let U be an open bounded subset of $\mathbb{R}^2$ with Lipschitz continuous boundary, let $p \geq m \geq 0$ and let (K_h) be a sequence in $\mathcal{K}_p(\overline{U})$ converging to some $K \in \mathcal{K}_m(\overline{U})$ and uniformly bounded in length. Then there exists $(J_h) \subset \mathcal{K}_m(\overline{U})$ converging to K, with $K_h \subset J_h$, and such that*

$$\lim_h \mathcal{H}^1(J_h \setminus K_h) \to 0.$$

We are now in position to prove the transfer of jumps Theorem.

Proof of Theorem 3.4.1. For every $x \in K$ which satisfies (3.6), for every $0 < \delta < 1$ and for every $r > 0$ let us set

$$R_r(x) := B_r(x) \cap \{z \in \mathbb{R}^2 : |(z-x) \cdot \nu(x)| < (\delta/2)r\};$$

$$B_r^+(x) := B_r(x) \cap \{z \in \mathbb{R}^2 : (z-x) \cdot \nu(x) > \delta r\};$$

$$B_r^-(x) := B_r(x) \cap \{z \in \mathbb{R}^2 : (z-x) \cdot \nu(x) < -\delta r\};$$

$$L_r(x) := \partial B_r(x) \cap \partial R_r(x).$$

The idea of the proof is the following. We would like to recover K with small balls $B_r(x)$ such that (up to small errors in length) K cuts every $B_r(x)$ into two connected components. Then, in order to have the same geometrical configuration for the sequence K_h, we have to enlarge a bit them, obtaining a new sequence of closed sets which still cut every $B_r(x)$ into two connected components which we denote now by $D_r^+(x)$ and $\underline{D_r^-(x)}$, so that $B_r^+(x) \subset D_r^+(x)$ and $B_r^-(x) \subset D_r^-(x)$. We add to this sequence of enlarged K_h the set $\overline{H \setminus K}$, obtaining a sequence which looks like the H_h of Theorem 3.4.1. Now we have to approximate v with functions $v_h \in L^{2,2}_{g_h,\partial_D\Omega}(\Omega \setminus H_h)$. This procedure is called *transfer of the jump's set*. We set $v_h = v$ far from K, while around K we consider the restriction of v on every $B_r^+(x)$ (respectively on $B_r^-(x)$) and we extend it on $D_r^+(x)$ (respectively on $D_r^-(x)$), obtaining in this way a function whose jumps are contained in H_h. With a further modification we also obtain the right boundary datum. However the rigorous proof presents some additional difficulties; for instance it will need some technical effort in order to ensure H_h to be in $\mathcal{K}_m(\overline{\Omega})$. In order to keep rigorous this rough idea, let us claim as follows.

Claim. For every $0 < \delta < 1$ and for every $\varepsilon > 0$ there exists a finite family of disjoint balls $\{B_{r_1}(x_1), \dots B_{r_N}(x_N)\}$ (where N depends on ε), and there exists a sequence $(H_h^{\delta,\varepsilon}) \subset \mathcal{K}_m(\overline{\Omega})$ of closed sets, such that for every i the following properties hold.

a) $H \cap B_{r_i}(x_i) \subset R_{r_i}(x_i)$;

b) Either $B_{r_i}^+(x_i) \subset \Omega$ or $B_{r_i}^+(x_i) \subset \mathbb{R}^2 \setminus \overline{\Omega}$;

c) Either $B_{r_i}^-(x_i) \subset \Omega$ or $B_{r_i}^-(x_i) \subset \mathbb{R}^2 \setminus \overline{\Omega}$;

d) For h large enough $H_h^{\delta,\varepsilon} \cap (B_{r_i}^+(x_i) \cup B_{r_i}^-(x_i)) = \emptyset$. Moreover $B_{r_i}^+(x_i)$ and $B_{r_i}^-(x_i)$ are in two different connected components of $B_{r_i}(x_i) \setminus H_h^{\delta,\varepsilon}$;

e) $\mathcal{H}^1(K \setminus \cup_{i=1}^N B_{r_i}(x_i)) \leq \varepsilon$;

f) $r_i \le \varepsilon$;

g) $K_h \cup \overline{H \setminus K} \subseteq H_h^{\delta,\varepsilon}$. Moreover

$$\lim_h \mathcal{H}^1(H_h^{\delta,\varepsilon} \setminus K_h) = \mathcal{H}^1(H \setminus K) + o(\delta), \qquad \text{where } o(\delta) \to 0 \text{ as } \delta \to 0.$$

Using the claim, we construct a sequence $v_h^{\delta,\varepsilon} \in L^{2,2}(\Omega \setminus H_h^{\delta,\varepsilon})$ as follows. For every $1 \le i \le N$, by property d) we can define (for h large enough) $D_{i,h}^+$ as the connected component of $B_{r_i}(x_i) \setminus H_h^{\delta,\varepsilon}$ containing $B_{r_i}^+(x_i)$, and similarly $D_{i,h}^-$ as the connected component of $B_{r_i}(x_i) \setminus H_h^{\delta,\varepsilon}$ containing $B_{r_i}^-(x_i)$. Let us define the function $v_h^{\delta,\varepsilon}$ on every $D_{i,h}^+ \cap \Omega$ (the case $D_{i,h}^- \cap \Omega$ being similar). If $B_{i,h}^+$ is contained in $\mathbb{R}^2 \setminus \overline{\Omega}$, we define $v_h^{\delta,\varepsilon} = g$ on $D_{i,h}^+ \cap \Omega$. Otherwise, by property b) we have that $B_{i,h}^+ \subset \Omega$. Let v_i^+ be the restriction of v on $B_{r_i}^+(x_i)$. By property a) we have that $v_i^+ \in L^{2,2}(B_{r_i}^+(x_i))$, so that we can consider its extension $E(v_i^+)$ on $\mathbb{R}^2$ given by Lemma 3.1.4. We define $v_h^{\delta,\varepsilon} = E(v_i^+)$ on $D_{i,h}^+$. Finally we define $v_h^{\delta,\varepsilon} = v$ on

$$\Omega \setminus \bigcup_i \left(D_{i,h}^+ \cup D_{i,h}^- \right).$$

Note that by construction $v_h^{\delta,\varepsilon} \in L^{2,2}_{g,\partial_D\Omega}(\Omega \setminus H_h^{\delta,\varepsilon})$. Moreover by Lemma 3.1.4 there exists a positive constant C_δ (independent on ε) such that

$$\int_\Omega |D^2 v - D^2 v_h^{\delta,\varepsilon}|^2 \, dx \le C_\delta \sum_i \int_{(D_i^+ \cup D_i^-)\cap\Omega} \|D^2 v\|^2 \, dx. \tag{3.44}$$

Let us fix two sequences $(\delta_h) \to 0$ and $(\varepsilon_h) \to 0$, and let us repeat the construction of the sets $H_h^{\delta_h,\varepsilon_h}$ as described above. Using property f) and the equi-integrability of $\|D^2 v\|^2$, we can assume without loss of generality that (δ_h) and (ε_h) are chosen such that right hand side of (3.44) tends to zero as h tends to infinity. Moreover by a diagonal argument (i.e. by freezing δ_h and ε_h) we can also assume that property d) holds for every h with $H_h^{\delta_h,\varepsilon_h}$ in place of $H_h^{\delta,\varepsilon}$, so that for every h we can construct the functions $v_h^{\delta_h,\varepsilon_h}$ as described previously. We set

$$H_h := H_h^{\delta_h,\varepsilon_h} \qquad v_h := v_h^{\delta_h,\varepsilon_h} - g + g_h. \tag{3.45}$$

By property g) we have $K_h \subset H_h$, $H_h \to H$ in the Hausdorff metric, and

$$\lim_h \mathcal{H}^1(H_h \setminus K_h) = \mathcal{H}^1(H \setminus K).$$

On the other hand $v_h \in L^{2,2}_{g_h,\partial_D\Omega}(\Omega \setminus H_h)$, and by (3.44) and the choice of δ_h, ε_h we have that $D^2 v_h \to D^2 v$ strongly in $L^2(\Omega; \mathcal{M}^{2\times 2})$, so that using the claim the proof of the Theorem is completed.

Let us pass to the proof of the claim. From now on $0 < \delta < 1$ and $\varepsilon > 0$ are keep fixed. For almost every $x \in K$, if r is small enough (depending on x) the following properties hold.

i) $H \cap B_r(x) \subset R_r(x)$;

ii) Either $B_r^+(x) \subset \Omega$ or $B_r^+(x) \subset \mathbb{R}^2 \setminus \overline{\Omega}$;

iii) Either $B_r^-(x) \subset \Omega$ or $B_r^-(x) \subset \mathbb{R}^2 \setminus \overline{\Omega}$;

iv) There exists a closed segment $S_r(x) \subset R_r(x)$ with $\mathcal{H}^1(S_r(x)) \le \delta r$ and such that $(K \cap B_r(x)) \cup L_r(x) \cup S_r(x)$ is connected.

In fact, by Lemma 3.1.2 we can assume that $x \in K$ is a point satisfying (3.6) and (3.7). Property i) follows by (3.7) for r small enough. Properties ii) and iii) are trivial if $x \in \Omega$ and $r < d(x, \partial\Omega)$, while if $x \in \partial\Omega$, it holds at every x which admits the approximate normal to $\partial\Omega$ with r small enough; the fact that $\partial\Omega$ is Lipschitz ensure that such x have full measure in $K \cap \partial\Omega$. Let us pass to the proof of iv). Let m be the minimum of the diameter of the connected components of K which are not single points (so that $m > 0$). We can always assume that there are not isolated points in $K \cap B_r(x)$ and that $2r < m$. We deduce that every connected component of $K \cap B_r(x)$ intersect $\partial B_r(x)$, otherwise there will be a connected component of K with diameter smaller than m. On the other hand by (3.6) for r small enough $K \cap B_r(x) \subset R_r(x)$, and hence every connected component of $K \cap B_r(x)$ intersects $L_r(x)$. Let us denote by $L_r^L(x)$ and $L_r^R(x)$ the two connected components of $L_r(x)$ and let $K_r^L(x)$ (respectively $K_r^R(x)$) be the union of all connected components of $K \cap B_r(x)$ which intersect $L_r^L(x)$ (respectively $L_r^R(x)$). By (3.6) we have that

$$\frac{d_H(K_r^L(x) \cup L_r^L(x), K_r^R(x) \cup L_r^R(x))}{r} \to 0 \quad \text{as } r \to 0.$$

We deduce that there are two points $a_r \in K_r^L(x) \cup L_r^L(x)$ and $b_r \in K_r^R(x) \cup L_r^R(x)$ with

$$\frac{|a_r - b_r|}{r} \to 0 \quad \text{as } r \to 0. \tag{3.46}$$

We set S_r as the closed segment with end points a_r and b_r. By construction we have that $(K \cap B_r(x)) \cup L_r(x) \cup S_r$ is connected, and by (3.46) for r small enough $\mathcal{H}^1(S_r) \le \delta r$, and this concludes the proof of property iv).

By properties i)-iv) above, applying Vitali-Besicovitch covering Theorem (see for instance [7]), we can consider a finite family of disjoint balls, $\{B_{r_1}(x_1), \dots B_{r_N}(x_N)\}$ (where N depends on ε), such that for every $1 \le i \le N$ the following properties hold:

1) $H \cap B_{r_i}(x_i) \subset R_{r_i}(x_i)$;
2) Either $B_{r_i}^+(x_i) \subset \Omega$ or $B_{r_i}^+(x_i) \subset \mathbb{R}^2 \setminus \overline{\Omega}$;
3) Either $B_{r_i}^-(x_i) \subset \Omega$ or $B_{r_i}^-(x_i) \subset \mathbb{R}^2 \setminus \overline{\Omega}$;
4) There exists a closed segment $S_{r_i}(x_i) \subset R_{r_i}(x_i)$ with $\mathcal{H}^1(S_{r_i}(x_i)) \le \delta r_i$ and such that $(K \cap B_{r_i}(x_i)) \cup L_{r_i}(x_i) \cup S_{r_i}(x_i)$ is connected;
5) $\mathcal{H}^1(K \setminus \cup_{i=1}^N B_{r_i}(x_i)) \le \varepsilon$;
6) $r_i \le \varepsilon$.

Properties 1), 2), 3), 5) and 6) are exactly properties a), b), c) e) and f) of the Claim. In order to prove properties d) and g) let us fix $1 \le i \le N$ and let us set

$$\tilde{K} := K \cup S_{r_i}(x_i) \cup L_{r_i}(x_i), \qquad \tilde{K}_h := K_h \cup S_{r_i}(x_i) \cup L_{r_i}(x_i).$$

Note that $\tilde{K}$ and $\tilde{K}_h$ have at most $m+3$ connected components. By property 4) $\tilde{K} \cap \overline{B}_{r_i}(x_i)$ is connected, and hence there exists a connected component $\tilde{K}^i$ of $\tilde{K}$ which contains $\tilde{K} \cap \overline{B}_{r_i}(x_i)$. Let $C_h^1, \dots, C_h^l$ be the connected components of $\tilde{K}_h$ converging to the sets $C^1, \dots, C^l$ composing $\tilde{K}^i$, i.e. such that $\cup_{j=1}^l C^j = \tilde{K}^i$. We have $l \le m+3$ and we can thus apply Lemma 3.7.1 to the sequence $\cup_{j=1}^l C_h^j$, obtaining that there exists a sequence of connected sets J_h^i in $\overline{\Omega}$ which still converges to $\tilde{K}^i$ in the Hausdorff metric and such that $\lim_h \mathcal{H}^1(J_h^i \setminus \tilde{K}_h) = 0$. Therefore we have

$$\limsup_h \mathcal{H}^1(J_h^i \setminus K_h) \le \mathcal{H}^1(L_{r_i}(x_i) \cup S_{r_i}(x_i)). \tag{3.47}$$

Let us enlarge $L_{r_i}(x_i)$; more precisely let us set

$$\tilde{L}_{r_i}(x_i) := \{x \in \partial B_{r_i}(x_i) : d(x, L_{r_i}(x_i)) \leq a\},$$

where a is a positive constant chosen such that $\tilde{L}_{r_i}(x_i)$ does not intersect neither $B^+_{r_i}(x_i)$ nor $B^-_{r_i}(x_i)$. The sequence $J^i_h \cap \overline{B}_{r_i}(x_i)$ converges to $\tilde{K}^i \cap \overline{B}_{r_i}(x_i) = \tilde{K} \cap \overline{B}_{r_i}(x_i)$ in the Hausdorff metric, which is contained in $\overline{R}_{r_i}(x_i)$. We deduce that for h large enough every connected component of $J^i_h \cap \overline{B}_{r_i}(x_i)$ can intersect $B_{r_i}(x_i)$ only on $\tilde{L}_{r_i}(x_i)$. Therefore, recalling that J^i_h is connected, we have that $(J^i_h \cup \tilde{L}_{r_i}(x_i)) \cap \overline{B}_{r_i}(x_i)$ has at most three connected components and it converges to the connected set $(\tilde{K} \cup \tilde{L}_{r_i}(x_i)) \cap \overline{B}_{r_i}(x_i)$ in the Hausdorff metric. Applying again Lemma 3.7.1 to the sequence $(J^i_h \cup \tilde{L}_{r_i}(x_i)) \cap \overline{B}_{r_i}(x_i)$ we deduce that there exists a sequence of connected sets I^i_h in $\overline{B}_{r_i}(x_i)$ converging to $(\tilde{K} \cup \tilde{L}_{r_i}(x_i)) \cap \overline{B}_{r_i}(x_i)$ and such that

$$\lim_h \mathcal{H}^1\Big(I^i_h \setminus \big((J^i_h \cup \tilde{L}_{r_i}(x_i)) \cap \overline{B}_{r_i}(x_i)\big)\Big) = 0. \tag{3.48}$$

Note that by the fact that I^i_h is connected, contains $\tilde{L}_{r_i}(x_i)$, and for h large enough does not intersect neither $B^+_{r_i}(x_i)$ nor $B^-_{r_i}(x_i)$, it follows that

$$B^+_{r_i}(x_i) \text{ and } B^-_{r_i}(x_i) \text{ are in two different connected components of } B_{r_i}(x_i) \setminus I^i_h. \tag{3.49}$$

By (3.47) we have

$$\limsup_h \mathcal{H}^1\big((J^i_h \cup \tilde{L}_{r_i}(x_i)) \cap \overline{B}_{r_i}(x_i) \setminus K_h\big) \leq \mathcal{H}^1(\tilde{L}_{r_i}(x_i) \cup S_{r_i}(x_i)). \tag{3.50}$$

By (3.48) and (3.50) we obtain

$$\limsup_h \mathcal{H}^1(I^i_h \setminus K_h) \leq \mathcal{H}^1(\tilde{L}_{r_i}(x_i) \cup S_{r_i}(x_i)). \tag{3.51}$$

Let us repeat the construction above for every $1 \leq i \leq N$, and let us set

$$I^{\delta,\varepsilon}_h := \bigcup_{i=1}^N I^i_h \cup K_h \cup \overline{H \setminus K}. \tag{3.52}$$

For every i we have

$$\mathcal{H}^1(\tilde{L}_{r_i}(x_i) \cup S_{r_i}(x_i)) \leq C\delta r_i, \tag{3.53}$$

for a positive constant C (independent on δ and ε). Moreover by property 4) it follows that

$$\mathcal{H}^1(K \cap B_{r_i}(x_i)) \geq (1-\delta) r_i,$$

and hence

$$|\cup_{i=1}^N \left(\tilde{L}_{r_i}(x_i) \cup S_{r_i}(x_i)\right)| \leq C\frac{\delta}{1-\delta}\mathcal{H}^1(K). \tag{3.54}$$

By (3.51), (3.52) and (3.54) we obtain

$$\lim_h \mathcal{H}^1(I^{\delta,\varepsilon}_h \setminus K_h) = \mathcal{H}^1(H \setminus K) + o(\delta), \tag{3.55}$$

where $o(\delta) \to 0$ as $\delta \to 0$.

By the fact that $K \subset H$, $K \in \mathcal{K}_m(\overline{\Omega})$, $H \in \mathcal{K}_m(\overline{\Omega})$, and that every I^i_h is connected, we deduce that the number of connected components of $I^{\delta,\varepsilon}_h$ is uniformly bounded with respect to h. Moreover

$I_h^{\delta,\varepsilon}$ converges to $H \cup_{i=1}^N \left(\tilde{L}_{r_i}(x_i) \cup S_{r_i}(x_i)\right)$, which by construction and by property 4) has at most m connected components. By Lemma 3.7.1 there exists a sequence $\tilde{I}_h^{\delta,\varepsilon} \in \mathcal{K}_m(\overline{\Omega})$ which contain $I_h^{\delta,\varepsilon}$, which still converge to $H \cup_{i=1}^N \left(\tilde{L}_{r_i}(x_i) \cup S_{r_i}(x_i)\right)$ in the Hausdorff metric and with

$$\lim_h \mathcal{H}^1(\tilde{I}_h^{\delta,\varepsilon} \setminus K_h) = \mathcal{H}^1(H \setminus K) + o(\delta). \tag{3.56}$$

The construction of $\tilde{I}_h^{\delta,\varepsilon}$ does not ensure that $\tilde{I}_h^{\delta,\varepsilon}$ are contained in $\overline{\Omega}$. Therefore we have to project every $\tilde{I}_h^{\delta,\varepsilon} \cap (\mathbb{R}^2 \setminus \Omega)$ on $\partial\Omega$ as follows. For every connected component C of $\tilde{I}_h^{\delta,\varepsilon} \cap (\mathbb{R}^2 \setminus \Omega)$ we set $\partial_C\Omega$ as the connected subset of $\partial\Omega$ with minimal length which contains $C \cap \partial\Omega$. By the fact that $\partial\Omega$ is Lipschitz, we deduce that there exists a positive constant L such that

$$\mathcal{H}^1(\partial_C\Omega) \leq L\mathcal{H}^1(C). \tag{3.57}$$

Therefore, substituting every connected component C of $\tilde{I}_h^{\delta,\varepsilon} \cap (\mathbb{R}^2 \setminus \Omega)$ with the corresponding $\partial_C\Omega$ we obtain a sequence $H_h^{\delta,\varepsilon}$ which by construction is in $\mathcal{K}_m(\overline{\Omega})$, by (3.52) contains $K_h \cup \overline{H \setminus K}$, so that by (3.56) and (3.57) satisfies property g) of the Claim. Moreover by construction and by (3.49) we deduce that also property d) holds. This concludes the proof of the Claim and therefore of the Theorem. □

3.8 Conclusions and remarks

In Theorem 3.5.1 we proved the existence of an irreversible quasi static growth of cracks for a plate clamped on a part of its boundary. More general boundary conditions can be treated with these methods. We mention for instance the case of the so called *hinged plate*, where no conditions are imposed on the normal derivative of the displacement u on $\partial_D\Omega$. In this case, it is sufficient to set the minimum problem (3.10) in the space

$$\{u \in L^{2,2}(\Omega \setminus K) : u - g = 0 \text{ on } \partial_D\Omega \text{ in the sense of traces}\}.$$

The main tool used is Theorem 3.4.1, which leads to the stability of unilateral minimality problems like (3.13). Note that the proof of Theorem 3.4.1 is based on a geometrical construction, and can be extended in the framework of $L^{k,p}$ spaces (i.e. the space of functions in L^p_{loc} with derivatives of order k in L^p (see [57]). Therefore the stability of unilateral minimum problems like (3.13) holds for more general energies $E : L^{k,p}(\Omega) \to \mathbb{R}$ depending on the k-order derivatives of u and with standard p-growth hypothesis. It is also possible to treat energies depending on the point x of the reference configuration Ω, as in the case of shells.

Part II

The weak formulation

In this section we state the notation and recall the preliminary results employed in the Part II of this thesis. We also give the notion of quasi static crack growth proposed by Francfort-Larsen in [47] and by Dal Maso-Francfort-Toader in [35]. From now on we suppose that Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary.

SBV spaces. For the general theory of functions of bounded variation, we refer to [7]; here we recall some basic definitions and theorems we need in the sequel. We say that $u \in BV(\Omega)$ if $u \in L^1(\Omega)$, and its distributional derivative Du is a bounded vector-valued Radon measure on Ω. In this case it turns out that the set $S(u)$ of points $x \in \Omega$ which are not Lebesgue points of u is rectifiable, that is there exists a sequence of C^1 manifolds $(M_i)_{i\in\mathbb{N}}$ such that $S(u) \subseteq \cup_i M_i$ up to a set of $\mathcal{H}^{N-1}$-measure zero. As a consequence $S(u)$ admits a normal $\nu_u(x)$ at $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$. Moreover $\mathcal{H}^{N-1}$ a.e. $x \in S(u)$ is a point of approximate jump for u, i.e. there exist $u^+(x), u^-(x) \in \mathbb{R}$ such that

$$\lim_{r\to 0} \frac{1}{|B_r^\pm(x)|} \int_{B_r^\pm(x)} |u(y) - u^\pm(x)|\, dy = 0,$$

where $B_r^+(x) := \{y \in B_r(x) \,:\, (y-x)\cdot \nu_u(x) > 0\}$, $B_r^-(x)$ is defined similarly and $B_r(x)$ is the ball with center x and radius r. It turns out that Du can be represented as

$$Du(A) = \int_A \nabla u(x)\, dx + \int_{A\cap S(u)} (u^+(x) - u^-(x))\nu_u(x)\, d\mathcal{H}^{N-1}(x) + D^c u(A), \qquad A \in \mathcal{A}(\Omega)$$

where ∇u denotes the approximate gradient of u, $D^c u$ is the Cantor part of Du, and $\mathcal{A}(\Omega)$ is the family of open subsets of Ω. We say that $u \in SBV(\Omega)$ if $u \in BV(\Omega)$ and $D^c u = 0$. The space $SBV(\Omega)$ is called the space of *special functions of bounded variation.* Note that if $u \in SBV(\Omega)$, then the singular part of Du is concentrated on $S(u)$. The space SBV is very useful when dealing with variational problems involving volume and surface energies because of the following compactness and lower semicontinuity result due to L.Ambrosio (see [2], [3], [4], [7]).

Theorem 3.8.1. *Let A be an open and bounded subset of $\mathbb{R}^N$, and let (u_k) be a sequence in $SBV(A;\mathbb{R}^n)$. Assume that there exists $q > 1$ and $c \geq 0$ such that*

$$\int_A |\nabla u_k|^q\, dx + \mathcal{H}^{N-1}(S(u_k)) + ||u_k||_\infty \leq c$$

for every $k \in \mathbb{N}$. Then there exists a subsequence (u_{k_h}) and a function $u \in SBV(A;\mathbb{R}^n)$ such that

$$\begin{aligned} &u_{k_h} \to u \quad \text{strongly in } L^1(A;\mathbb{R}^n);\\ &\nabla u_{k_h} \rightharpoonup \nabla u \quad \text{weakly in } L^1(A;M^{N\times n});\\ &\mathcal{H}^{N-1}(S(u)) \leq \liminf_h \mathcal{H}^{N-1}(S(u_{k_h})). \end{aligned} \tag{3.58}$$

For all $1 < p < +\infty$ we set

$$SBV^p(\Omega) := \{u \in SBV(\Omega) \,:\, \nabla u \in L^p(A, \mathbb{R}^N), \mathcal{H}^{N-1}(S(u)) < +\infty\}.$$

We will consider weak convergence in $SBV^p(\Omega)$ defined in the following way: $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$ if

$$\begin{aligned} &u_n \to u \quad \text{strongly in } L^1(\Omega),\\ &\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^p(\Omega;\mathbb{R}^N),\\ &\mathcal{H}^{N-1}(S(u_n)) \leq C. \end{aligned}$$

GSBV **spaces.** We indicate with $SBV_{loc}(A,\mathbb{R}^m)$ the space of functions which belong to $SBV(A',\mathbb{R}^m)$ for every open set A' with compact closure in A.

The set $GSBV(A,\mathbb{R}^m)$ is defined as the set of functions $u : A \to \mathbb{R}^m$ such that $\varphi(u) \in SBV_{loc}(A)$ for every $\varphi \in C^1(\mathbb{R}^m)$ such that the support of $\nabla\varphi$ has compact closure in $\mathbb{R}^m$. If $p \in]1,+\infty[$, we set

$$GSBV^p(A,\mathbb{R}^m) := \{u \in GSBV(A,\mathbb{R}^m) \, : \, \nabla u \in L^p(A,\mathcal{M}^{m\times n}),\, \mathcal{H}^{n-1}(S(u)) < +\infty\}.$$

By [35, Proposition 2.2] the space $GSBV^p(A,\mathbb{R}^m)$ coincide with $(GSBV^p(A,\mathbb{R}))^m$, that is $u := (u_1,\dots,u_m) \in GSBV^p(A,\mathbb{R}^m)$ if and only if $u_i \in GSBV^p(A,\mathbb{R})$ for every $i = 1,\dots,m$.

The following compactness and lower semicontinuity result will be used in the following. For a proof, we refer to [3].

Theorem 3.8.2. *Let A be an open and bounded subset of $\mathbb{R}^n$. Let $g(x,u) : A \times \mathbb{R}^m \to [0,\infty]$ be a Borel function, lower semicontinuous in u and satisfying the condition*

$$\lim_{|u|\to\infty} g(x,u) = +\infty \textit{ for a.e. } x \in A.$$

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $GSBV^p(A;\mathbb{R}^m)$ such that

$$\sup_k \int_A |\nabla u_k(x)|^p \, dx + \mathcal{H}^{n-1}\left(S(u_k)\right) + \int_A g(x,u_k(x))\, dx < +\infty.$$

Then there exists a subsequence $(u_{k_h})_{h\in\mathbb{N}}$ and a function $u \in GSBV^p(A;\mathbb{R}^m)$ such that

$$\begin{aligned} u_{k_h} &\to u && \textit{in measure,} \\ \nabla u_{k_h} &\rightharpoonup \nabla u && \textit{weakly in } L^p(A;M^{m\times n}). \end{aligned} \tag{3.59}$$

Moreover we have that

$$\mathcal{H}^{n-1}\left(S(u)\right) \le \liminf_h \mathcal{H}^{n-1}\left(S(u_{k_h})\right).$$

Let $q \in]1,+\infty[$ and let us set

$$GSBV^p_q(A;\mathbb{R}^m) := GSBV^p(A;\mathbb{R}^m) \cap L^q(A;\mathbb{R}^m). \tag{3.60}$$

We say that $u_k \rightharpoonup u$ weakly in $GSBV^p_q(A;\mathbb{R}^m)$ if

$$\begin{aligned} u_k &\to u && \text{in measure} \\ \nabla u_k &\rightharpoonup \nabla u && \text{weakly in } L^p(A;M^{m\times n}) \\ u_k &\rightharpoonup u && \text{weakly in } L^q(A;\mathbb{R}^m). \end{aligned} \tag{3.61}$$

We will often use the following fact: if $u_k \rightharpoonup u$ weakly in $GSBV^p_q(A;\mathbb{R}^m)$ and $\Gamma \subseteq A$ is such that $\mathcal{H}^{N-1}(\Gamma) < +\infty$ and $S(u_k) \subseteq \Gamma$ up to a set of $\mathcal{H}^{N-1}$-measure zero for all k, then $S(u) \subseteq \Gamma$ up to a set of $\mathcal{H}^{N-1}$-measure zero.

Quasi-static evolution of brittle fracture by Francfort-Larsen [47]. Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, and let $\partial_D\Omega$ be a subset of $\partial\Omega$ open in the relative topology. Let $g : [0,1] \to H^1(\Omega)$ be absolutely continuous; we indicate the gradient of g at time t by $\nabla g(t)$, and the time derivative of g at time t by $\dot{g}(t)$. The main result of [47] is the following theorem.

Theorem 3.8.3. *There exists a crack $\Gamma(t) \subseteq \overline{\Omega}$ and a field $u(t) \in SBV(\Omega)$ such that:*

(a) $\Gamma(t)$ *increases with* t;

(b) $u(0)$ *minimizes*

$$\int_\Omega |\nabla v|^2\,dx + \mathcal{H}^{N-1}(S(v) \cup \{x \in \partial_D\Omega : v(x) \neq g(0)(x)\})$$

among all $v \in SBV(\Omega)$ *(inequalities on* $\partial_D\Omega$ *are intended for the traces of* v *and* g*);*

(c) for $t > 0$, $u(t)$ *minimizes*

$$\int_\Omega |\nabla v|^2\,dx + \mathcal{H}^{N-1}\left([S(v) \cup \{x \in \partial_D\Omega : v(x) \neq g(t)(x)\}] \setminus \Gamma(t)\right)$$

among all $v \in SBV(\Omega)$;

(d) $S(u(t)) \cup \{x \in \partial_D\Omega\, :\, u(t)(x) \neq g(t)(x)\} \subseteq \Gamma(t)$*, up to a set of* $\mathcal{H}^{N-1}$*-measure* 0.

Furthermore, the total energy

$$\mathcal{E}(t) := \int_\Omega |\nabla u(t)|^2\,dx + \mathcal{H}^{N-1}(\Gamma(t))$$

is absolutely continuous and satisfies

$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau$$

for every $t \in [0,1]$*. Finally, for any countable, dense set* $I \subseteq [0,1]$*, the crack* $\Gamma(t)$ *and the displacement* $u(t)$ *can be chosen so that*

$$\Gamma(t) = \bigcup_{\tau \in I, \tau \leq t} \left(S(u(\tau)) \cup \{x \in \partial_D\Omega\, :\, u(\tau)(x) \neq g(\tau)(x)\}\right).$$

The main tool in the proof of Theorem 3.8.3 is the following result [47, Theorem 2.1], which is useful also in our analysis.

Theorem 3.8.4. *Let* $\overline{\Omega} \subseteq \Omega'$*, with* $\partial\Omega$ *Lipschitz, and let for* $r = 1, \dots, i$ (u_n^r) *be a sequence in* $SBV(\Omega')$ *such that*

(1) $S(u_n^r) \subseteq \overline{\Omega}$;

(2) $|\nabla u_n^r|$ *weakly converges in* $L^1(\Omega')$*; and*

(3) $u_n^r \to u^r$ *strongly in* $L^1(\Omega')$,

where $u^r \in BV(\Omega')$ *with* $\mathcal{H}^{N-1}(S(u^r)) < \infty$. *Then for every* $\phi \in SBV(\Omega')$ *with* $\mathcal{H}^{N-1}(S(\phi)) < \infty$ *and* $\nabla\phi \in L^q(\Omega;\mathbb{R}^N)$ *for some* $q \in [1, +\infty[$, *there exists a sequence* (ϕ_n) *in* $SBV(\Omega')$ *with* $\phi_n \equiv \phi$ *on* $\Omega' \setminus \overline{\Omega}$ *such that*

(a) $\phi_n \to \phi$ *strongly in* $L^1(\Omega')$;

(b) $\nabla\phi_n \to \nabla\phi$ *strongly in* $L^q(\Omega')$; *and*

(c) $\mathcal{H}^{N-1}\left([S(\phi_n) \setminus \bigcup_{r=1}^i S(u_n^r)] \setminus [S(\phi) \setminus \bigcup_{r=1}^i S(u^r)]\right) \to 0$.

In particular

$$\limsup_n \mathcal{H}^{N-1}\left(S(\phi_n) \setminus \bigcup_{r=1}^i S(u_n^r)\right) \le \mathcal{H}^{N-1}\left(S(\phi) \setminus \bigcup_{r=1}^i S(u^r)\right). \tag{3.62}$$

The quasistatic crack growth of Dal Maso-Francfort-Toader [35].

We now describe the quasistatic evolution of brittle fractures proposed in [35]. They consider the case of n-dimensional nonlinear elasticity, for an arbitrary $n \ge 1$, with a quasiconvex bulk energy and with prescribed boundary deformations and applied loads, depending on time. Since we will approximate the case $n = 2$, we prefer to introduce the model in this particular case. For more details, we refer the reader to [35].

Let Ω be a bounded open set of $\mathbb{R}^2$ with Lipschitz boundary and let Ω_B be an open subset of Ω. Let $\partial_N\Omega \subseteq \partial\Omega$ be closed in the relative topology, and let $\partial_D\Omega := \partial\Omega \setminus \partial_N\Omega$. Let $\partial_S\Omega \subseteq \partial_N\Omega$ be closed in the relative topology and such that $\overline{\Omega}_B \cap \partial_S\Omega = \emptyset$. In the model proposed in [35], Ω_B represents the brittle part of Ω, and $\partial_D\Omega$ the part of the boundary on which the deformation is prescribed. Moreover the elastic body Ω is supposed to be subject to surface forces acting on $\partial_S\Omega$.

Admissible cracks and deformations. The set of admissible cracks is given by

$$\mathcal{R}(\overline{\Omega}_B; \partial_N\Omega) := \{\Gamma : \Gamma \text{ is rectifiable }, \Gamma \,\tilde{\subseteq}\, (\overline{\Omega}_B \setminus \partial_N\Omega),\ \mathcal{H}^1(\Gamma) < +\infty\}.$$

Here $A \,\tilde{\subseteq}\, B$ means that $A \subseteq B$ up to a set of $\mathcal{H}^1$-measure zero, and Γ rectifiable means that there exists a sequence (M_i) of C^1-manifolds such that $\Gamma \,\tilde{\subseteq}\, \bigcup_i M_i$. If Γ is rectifiable, we can define normal vector fields ν to Γ in the following way: if $\Gamma = \bigcup_i \Gamma_i$ with $\Gamma_i \,\tilde{\subseteq}\, M_i$ and $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \ne j$, given $x \in \Gamma_i$, we take $\nu(x) = \nu_{M_i}(x)$, where $\nu_{M_i}(x)$ is a normal vector to the C^1-manifold M_i at x. It turns out that two normal vector fields associated to different decompositions $\bigcup_i \Gamma_i$ of Γ coincide up to their sign $\mathcal{H}^1$ almost everywhere.

Given a crack Γ, an admissible deformation is given by any function $u \in GSBV(\Omega;\mathbb{R}^2)$ such that $S(u) \,\tilde{\subseteq}\, \Gamma$.

The surface energy. The surface energy of a crack Γ is given by

$$\mathcal{E}^s(\Gamma) := \int_\Gamma k(x, \nu(x))\, d\mathcal{H}^1(x), \tag{3.63}$$

where ν is a unit normal vector field on Γ. Here $k : \overline{\Omega}_B \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $k(x, \cdot)$ is a norm in $\mathbb{R}^2$ for all $x \in \overline{\Omega}_B$ and for all $x \in \overline{\Omega}_B$ and $\nu \in \mathbb{R}^2$

$$K_1|\nu| \le k(x,\nu) \le K_2|\nu|, \tag{3.64}$$

where $K_1, K_2 > 0$. Notice that since k is even in the second variable, we have that the integral (3.63) is independent of the orientation given to Γ, that is independent of the particular choice of the unit normal vector field ν.

The bulk energy. Let $p > 1$ be fixed. Given a deformation $u \in GSBV^p(\Omega;\mathbb{R}^2)$ the associated *bulk energy* is given by

$$\mathcal{W}(\nabla u) := \int_\Omega W(x, \nabla u(x))\, dx, \tag{3.65}$$

where $W : \Omega \times \mathcal{M}^{2\times 2} \to [0, +\infty)$ is a Carathéodory function satisfying

$$\text{for every } x \in \Omega: \ W(x,\cdot) \text{ is quasiconvex and } C^1 \text{ on } \mathcal{M}^{2\times 2}, \tag{3.66}$$

$$\text{for every } (x,\xi) \in \Omega \times \mathcal{M}^{2\times 2}: a_0^W|\xi|^p - b_0^W(x) \le W(x,\xi) \le a_1^W|\xi|^p + b_1^W(x). \tag{3.67}$$

Here $a_0^W, a_1^W > 0$, and $b_0^W, b_1^W \in L^1(\Omega)$ are nonnegative functions. Quasiconvexity of W means that for all $\xi \in \mathcal{M}^{2\times 2}$ and for all $\varphi \in C_c^\infty(\Omega;\mathbb{R}^2)$

$$W(\xi) \le \int_\Omega W(\xi + \nabla\varphi)\, dx.$$

If we denote by $\partial_\xi W : \Omega \times \mathcal{M}^{2\times 2} \to \mathcal{M}^{2\times 2}$ the partial derivative of W with respect to ξ, since $\xi \to W(x,\xi)$ is rank one convex on $\mathcal{M}^{2\times 2}$, under the growth assumption (3.67) it turns out that (see for example [32]) there exists a positive constant $a_2^W > 0$ and a nonnegative function $b_2^W \in L^{p'}(\Omega)$, with $p' := p/(p-1)$, such that for all $(x,\xi) \in \Omega \times \mathcal{M}^{2\times 2}$

$$|\partial_\xi W(x,\xi)| \le a_2^W|\xi|^{p-1} + b_2^W(x). \tag{3.68}$$

By (3.67) and (3.68) the functional $\mathcal{W}$, defined for all $\Phi \in L^p(\Omega;\mathcal{M}^{2\times 2})$ by

$$\mathcal{W}(\Phi) := \int_\Omega W(x, \Phi(x))\, dx,$$

is of class C^1 on $L^p(\Omega;\mathcal{M}^{2\times 2})$, and its differential $\partial\mathcal{W} : L^p(\Omega;\mathcal{M}^{2\times 2}) \to L^{p'}(\Omega;\mathcal{M}^{2\times 2})$ is given by

$$\langle \partial\mathcal{W}(\Phi), \Psi\rangle = \int_\Omega \partial_\xi W(x, \Phi(x))\Psi(x)\, dx, \qquad \Phi, \Psi \in L^p(\Omega;\mathcal{M}^{2\times 2}),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the spaces $L^{p'}(\Omega;\mathcal{M}^{2\times 2})$ and $L^p(\Omega;\mathcal{M}^{2\times 2})$. By (3.67) and (3.68), there exist six positive constants $\alpha_0^{\mathcal{W}} > 0$, $\alpha_1^{\mathcal{W}} > 0$, $\alpha_2^{\mathcal{W}} > 0$, $\beta_0^{\mathcal{W}} \ge 0$, $\beta_1^{\mathcal{W}} \ge 0$, $\beta_2^{\mathcal{W}} \ge 0$ such that for every Φ, $\Psi \in L^p(\Omega;\mathcal{M}^{2\times 2})$

$$\alpha_0^{\mathcal{W}}\|\Phi\|_p^p - \beta_0^{\mathcal{W}} \le \mathcal{W}(\Phi) \le \alpha_1^{\mathcal{W}}\|\Phi\|_p^p + \beta_1^{\mathcal{W}},$$

$$|\langle \partial\mathcal{W}(\Phi), \Psi\rangle| \le (\alpha_2^{\mathcal{W}}\|\Phi\|_p^{p-1} + \beta_2^{\mathcal{W}})\|\Psi\|_p. \tag{3.69}$$

The body forces. Let $q > 1$ be fixed. The density of applied body forces per unit volume in the reference configuration relative to the deformation u at time $t \in [0,T]$ is given by $\partial_z F(t,x,u(x))$. Here $F : [0,T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is such that:

$$\text{for every } z \in \mathbb{R}^2: \ (t,x) \to F(t,x,z) \text{ is } \mathcal{L}^1 \times \mathcal{L}^2 \text{ measurable on } [0,T] \times \Omega,$$

$$\text{for every } (t,x) \in [0,T] \times \Omega: \ z \to F(t,x,z) \text{ belongs to } C^1(\mathbb{R}^2),$$

and satisfies the following growth conditions

$$a_0^F|z|^q - b_0^F(t,x) \le -F(t,x,z) \le a_1^F|z|^q + b_1^F(t,x), \qquad (3.70)$$
$$|\partial_z F(t,x,z)| \le a_2^F|z|^{q-1} + b_2^F(t,x)$$

for every $(t,x,z) \in [0,T]\times\Omega\times\mathbb{R}^2$, with $a_0^F > 0$, $a_1^F > 0$ and $a_2^F > 0$, and where $b_0^F, b_1^F \in C^0([0,T]; L^1(\Omega))$, $b_2^F \in C^0([0,T]; L^{q'}(\Omega))$ are nonnegative functions, with $q' := q/(q-1)$.

In order to deal with time variations, we assume also that for every $(t,z) \in [0,T]\times\mathbb{R}^2$

$$F(t,x,z) = F(0,x,z) + \int_0^t \dot F(s,x,z)\,ds \qquad \text{for a.e. } x \in \Omega,$$
$$\partial_z F(t,x,z) = \partial_z F(0,x,z) + \int_0^t \partial_z \dot F(s,x,z)\,ds \qquad \text{for a.e. } x \in \Omega,$$

where $\dot F : [0,T]\times\Omega\times\mathbb{R}^2 \to \mathbb{R}$ is such that

for all $z \in \mathbb{R}^2$: $(t,x) \to \dot F(t,x,z)$ is $\mathcal{L}^1 \times \mathcal{L}^2$ measurable on $[0,T]\times\Omega$,
for all $(t,x) \in [0,T]\times\Omega$: $z \to \dot F(t,x,z)$ is of class C^1 on $\mathbb{R}^2$,

and satisfies the growth conditions

$$|\dot F(t,x,z)| \le a_3^F(t)|z|^{\dot q} + b_3^F(t,x),$$
$$|\partial_z \dot F(t,x,z)| \le a_4^F(t)|z|^{\dot q-1} + b_4^F(t,x)$$

for all $(t,x,z) \in [0,T]\times\Omega\times\mathbb{R}^2$. Here $1 \le \dot q < q$, and $a_3^F, a_4^F \in L^1([0,T])$, $b_3^F \in L^1([0,T]; L^1(\Omega))$, $b_4^F \in L^1([0,T]; L^{\dot q'}(\Omega))$ are nonnegative functions with $\dot q' := \frac{\dot q}{\dot q-1}$.

Under the previous assumptions, for every $t \in [0,T]$ the functionals

$$\mathcal{F}(t)(u) := \int_\Omega F(t,x,u(x))\,dx, \qquad \dot{\mathcal{F}}(t)(u) := \int_\Omega \dot F(t,x,u(x))\,dx \qquad (3.71)$$

are well defined on $L^q(\Omega;\mathbb{R}^2)$ and $L^{\dot q}(\Omega;\mathbb{R}^2)$ respectively. Moreover we have that $\mathcal{F}(t)$ is of class C^1 on $L^q(\Omega;\mathbb{R}^2)$, with differential $\partial\mathcal{F}(t) : L^q(\Omega;\mathbb{R}^2) \to L^{q'}(\Omega;\mathbb{R}^2)$ defined by

$$\langle \partial\mathcal{F}(t)(u), v\rangle = \int_\Omega \partial_z F(t,x,u(x))v(x)\,dx, \qquad u,v \in L^q(\Omega;\mathbb{R}^2),$$

where $\langle\cdot,\cdot\rangle$ denotes now the duality pairing between $L^{q'}(\Omega;\mathbb{R}^2)$ and $L^q(\Omega;\mathbb{R}^2)$. $\dot{\mathcal{F}}(t)$ is C^1 on $L^{\dot q}(\Omega;\mathbb{R}^2)$ with differential defined by

$$\langle \partial\dot{\mathcal{F}}(t)(u), v\rangle = \int_\Omega \partial_z \dot F(t,x,u(x))v(x)\,dx, \qquad u,v \in L^{\dot q}(\Omega;\mathbb{R}^2),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^{\dot q'}(\Omega;\mathbb{R}^2)$ and $L^{\dot q}(\Omega;\mathbb{R}^2)$. For all $u,v \in L^q(\Omega;\mathbb{R}^2)$ and for all $t \in [0,T]$ we have

$$\mathcal{F}(t)(u) = \mathcal{F}(0)(u) + \int_0^t \dot{\mathcal{F}}(s)(u)\,ds,$$
$$\langle \partial\mathcal{F}(t)(u), v\rangle = \langle \partial\mathcal{F}(0)(u), v\rangle + \int_0^t \langle \partial\dot{\mathcal{F}}(s)(u), v\rangle\,ds. \qquad (3.72)$$

Moreover we have that for every $t \in [0,T]$ and for every u, $v \in L^q(\Omega;\mathbb{R}^n)$

$$\begin{aligned}
&\alpha_0^{\mathcal{F}}\|u\|_q^q - \beta_0^{\mathcal{F}} \le -\mathcal{F}(t)(u) \le \alpha_1^{\mathcal{F}}\|u\|_q^q + \beta_1^{\mathcal{F}}, \\
&|\langle \partial\mathcal{F}(t)(u), v\rangle| \le (\alpha_2^{\mathcal{F}}\|u\|_q^{q-1} + \beta_2^{\mathcal{F}})\|v\|_q, && (3.73)\\
&|\dot{\mathcal{F}}(t)(u)| \le \alpha_3^{\mathcal{F}}(t)\|u\|_{\dot q}^{\dot q} + \beta_3^{\mathcal{F}}(t), && (3.74)\\
&|\langle \partial\dot{\mathcal{F}}(t)(u), v\rangle| \le (\alpha_4^{\mathcal{F}}(t)\|u\|_{\dot q}^{\dot q-1} + \beta_4^{\mathcal{F}}(t))\|v\|_{\dot q}, && (3.75)
\end{aligned}$$

where $\alpha_0^{\mathcal{F}} > 0$, $\alpha_1^{\mathcal{F}} > 0$, $\alpha_2^{\mathcal{F}} > 0$, $\beta_0^{\mathcal{F}} \ge 0$, $\beta_1^{\mathcal{F}} \ge 0$, $\beta_2^{\mathcal{F}} \ge 0$ are positive constants, and $\alpha_3^{\mathcal{F}}, \alpha_4^{\mathcal{F}}, \beta_3^{\mathcal{F}}, \beta_4^{\mathcal{F}} \in L^1([0,T])$ are nonnegative functions.

The surface forces. The density of the surface forces on $\partial_S\Omega$ at time t under the deformation u is given by $\partial_z G(t,x,u(x))$, where $G : [0,T] \times \partial_S\Omega \times \mathbb{R}^2 \to \mathbb{R}$ is such that

$$\begin{aligned}
&\text{for every } z \in \mathbb{R}^2 : \ (t,x) \to G(t,x,z) \text{ is } \mathcal{L}^1 \times \mathcal{H}^1\text{-measurable},\\
&\text{for every } (t,x) \in [0,T] \times \partial_S\Omega : \ z \to G(t,x,z) \text{ belongs to } C^1(\mathbb{R}^2),
\end{aligned}$$

and satisfies the growth conditions

$$\begin{aligned}
&- a_0^G(t,x)|z| - b_0^G(t,x) \le -G(t,x,z) \le a_1^G|z|^r + b_1^G(t,x),\\
&|\partial_z G(t,x,z)| \le a_2^G|z|^{r-1} + b_2^G(t,x),
\end{aligned}$$

for every $(t,x,z) \in [0,T] \times \partial_S\Omega \times \mathbb{R}^2$. Here r is an exponent related to the trace operators on Sobolev spaces: if $p < 2$, then we suppose that $p \le r \le \frac{p}{2-p}$, while if $p \ge 2$, we suppose $p \le r$. Moreover $a_1^G \ge 0$, $a_2^G \ge 0$ are two nonnegative constants, and $a_0^G \in L^\infty([0,T];L^{r'}(\partial_S\Omega))$, $b_0^G, b_1^G \in C^0([0,T];L^1(\partial_S\Omega))$, and $b_2^G \in C^0([0,T];L^{r'}(\partial_S\Omega))$ are nonnegative functions with $r' := r/(r-1)$

We assume that for every $(t,z) \in [0,T] \times \mathbb{R}^2$

$$\begin{aligned}
&G(t,x,z) = G(0,x,z) + \int_0^t \dot G(s,x,z)ds \qquad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial_S\Omega,\\
&\partial_z G(t,x,z) = \partial_z G(0,x,z) + \int_0^t \partial_z \dot G(s,x,z)ds \qquad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial_S\Omega,
\end{aligned}$$

where $\dot G : [0,T] \times \partial_S\Omega \times \mathbb{R}^2 \to \mathbb{R}$ is such that

$$\begin{aligned}
&\text{for all } z \in \mathbb{R}^2 : \ (t,x) \to \dot G(t,x,z) \text{ is } \mathcal{L}^1 \times \mathcal{H}^1\text{-measurable},\\
&\text{for all } (t,x) \in [0,T] \times \partial_S\Omega : \ z \to \dot G(t,x,z) \text{ belongs to } C^1(\mathbb{R}^2),
\end{aligned}$$

and satisfies the the growth conditions

$$\begin{aligned}
&|\dot G(t,x,z)| \le a_3^G(t)|z|^r + b_3^G(t,x),\\
&|\partial_z \dot G(t,x,z)| \le a_4^G(t)|z|^{r-1} + b_4^G(t,x)
\end{aligned}$$

for all $(t,x,z) \in [0,T] \times \partial_S\Omega \times \mathbb{R}^2$. Here a_3^G, $a_4^G \in L^1([0,T])$, $b_3^G \in L^1([0,T];L^1(\partial_S\Omega))$ and $b_4^G \in L^1([0,T];L^{r'}(\partial_S\Omega))$ are nonnegative functions.

By the previous assumptions, the following functionals on $L^r(\partial_S\Omega;\mathbb{R}^2)$

$$\mathcal{G}(t)(u) := \int_{\partial_S\Omega} G(t,x,u(x))\,d\mathcal{H}^1(x), \quad \dot{\mathcal{G}}(t)(u) := \int_{\partial_S\Omega} \dot G(t,x,u(x))\,d\mathcal{H}^1(x) \tag{3.76}$$

are well defined. For every $t \in [0,T]$ we have that $\mathcal{G}(t)$ is of class C^1 on $L^r(\partial_S\Omega;\mathbb{R}^2)$ and its differential is given by

$$\langle \partial\mathcal{G}(t)(u), v\rangle = \int_{\partial_S\Omega} \partial_z G(t,x,u(x))v(x)\, d\mathcal{H}^1(x), \qquad u,v \in L^r(\partial_S\Omega;\mathbb{R}^2),$$

where $\langle\cdot,\cdot\rangle$ denotes now the duality pairing between $L^{r'}(\partial_S\Omega;\mathbb{R}^2)$ and $L^r(\partial_S\Omega;\mathbb{R}^2)$. Moreover, $\dot{\mathcal{G}}(t)$ is of class C^1 on $L^r(\partial_S\Omega;\mathbb{R}^2)$, and its differential is given by

$$\langle \partial\dot{\mathcal{G}}(t)(u), v\rangle = \int_{\partial_S\Omega} \partial_z \dot{G}(t,x,u(x))v(x)\, d\mathcal{H}^1(x)$$

for all $u,v \in L^r(\partial_S\Omega;\mathbb{R}^2)$. Finally we have

$$\mathcal{G}(t)(u) = \mathcal{G}(0)(u) + \int_0^t \dot{\mathcal{G}}(s)(u)\, ds, \qquad \langle \partial\mathcal{G}(t)(u), v\rangle = \langle \partial\mathcal{G}(0)(u), v\rangle + \int_0^t \langle \partial\dot{\mathcal{G}}(s)(u), v\rangle\, ds,$$

for every $u,v \in L^r(\partial_S\Omega;\mathbb{R}^2)$.

Let $\Omega_S \subseteq \Omega \setminus \overline{\Omega}_B$ be open with Lipschitz boundary, and such that $\partial_S\Omega \subseteq \partial\Omega_S$; the trace operator from $W^{1,p}(\Omega_S;\mathbb{R}^2)$ into $L^r(\partial\Omega_S;\mathbb{R}^2)$ is then compact, and so there exists a constant $\gamma_S > 0$ such that

$$\|u\|_{r,\partial_S\Omega} \leq \gamma_S(\|\nabla u\|_{p,\Omega_S} + \|u\|_{p,\Omega_S}) \tag{3.77}$$

for every $u \in W^{1,p}(\Omega_S;\mathbb{R}^2)$. By the previous assumptions, we have that there exist six nonnegative constants $\alpha_0^{\mathcal{G}}$, $\alpha_1^{\mathcal{G}}$, $\alpha_2^{\mathcal{G}}$, $\beta_0^{\mathcal{G}}$, $\beta_1^{\mathcal{G}}$, $\beta_2^{\mathcal{G}}$ and four nonnegative functions $\alpha_3^{\mathcal{G}}$, $\alpha_4^{\mathcal{G}}$, $\beta_3^{\mathcal{G}}$, $\beta_4^{\mathcal{G}} \in L^1([0,T])$, such that

$$\begin{aligned}
&- \alpha_0^{G}\|u\|_{r,\partial_S\Omega} - \beta_0^{G} \leq -\mathcal{G}(t)(u) \leq \alpha_1^{\mathcal{G}}\|u\|^r_{r,\partial_S\Omega} + \beta_1^{G}, \\
&|\langle \partial\mathcal{G}(t)(u), v\rangle| \leq (\alpha_2^{\mathcal{G}}\|u\|^{r-1}_{r,\partial_S\Omega} + \beta_2^{\mathcal{G}})\|v\|_{r,\partial_S\Omega}, && (3.78)\\
&|\dot{\mathcal{G}}(t)(u)| \leq \alpha_3^{\mathcal{G}}(t)\|u\|^r_{r,\partial_S\Omega} + \beta_3^{\mathcal{G}}(t), && (3.79)\\
&|\langle \partial\dot{\mathcal{G}}(t)(u), v\rangle| \leq (\alpha_4^{\mathcal{G}}(t)\|u\|^{r-1}_{r,\partial_S\Omega} + \beta_4^{\mathcal{G}}(t))\|v\|_{r,\partial_S\Omega}
\end{aligned}$$

for every $t \in [0,T]$ and $u,v \in L^r(\partial_S\Omega;\mathbb{R}^2)$.

Configurations with finite energy. The deformations on the boundary $\partial_D\Omega$ are given by (the traces of) functions $g \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$, where p,q are the exponents in (3.67) and (3.70) respectively. Given a crack $\Gamma \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ and a boundary deformation g, the set of *admissible deformations with finite energy* relative to (g,Γ) is defined by

$$AD(g,\Gamma) := \{u \in GSBV^p_q(\Omega;\mathbb{R}^2) : S(u) \tilde{\subseteq} \Gamma, u = g \text{ on } \partial_D\Omega \setminus \Gamma\},$$

where we recall that

$$GSBV^p_q(\Omega;\mathbb{R}^2) := GSBV^p(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2),$$

and the equality $u = g$ on $\partial_D\Omega \setminus \Gamma$ is intended in the sense of traces (see [35, Section 2]).

Note that if $u \in GSBV^p_q(\Omega;\mathbb{R}^2)$, then $\mathcal{W}(u) < +\infty$ and $|\mathcal{F}(t)(u)| < +\infty$ for all $t \in [0,T]$. Moreover since $\Gamma \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ and $S(u) \tilde{\subseteq} \Gamma$, we have that $u \in W^{1,p}(\Omega_S;\mathbb{R}^2) \cap L^q(\Omega_S;\mathbb{R}^2)$ so that $\mathcal{G}(t)(u)$ is well defined and $|\mathcal{G}(t)(u)| < +\infty$ for all $t \in [0,T]$. Notice that there exists always a deformation without crack which satisfies the boundary condition, namely the function g itself.

The total energy. For every $t \in [0,T]$, the total energy relative to the configuration (u,Γ) with $u \in AD(g,\Gamma)$ is given by

$$\mathcal{E}(t)(u,\Gamma) := \mathcal{E}^b(t)(u) + \mathcal{E}^s(\Gamma), \tag{3.80}$$

where

$$\mathcal{E}^b(t)(u) := \mathcal{W}(u) - \mathcal{F}(t)(u) - \mathcal{G}(t)(u), \tag{3.81}$$

and $\mathcal{W}$, $\mathcal{F}(t)$, $\mathcal{G}(t)$ and $\mathcal{E}^s$ are defined in (3.65), (3.71), (3.76) and (3.63) respectively. It turns out that there exist four constants $\alpha_0^{\mathcal{E}} > 0$, $\alpha_1^{\mathcal{E}} > 0$, $\beta_0^{\mathcal{E}} \geq 0$, $\beta_1^{\mathcal{E}} \geq 0$ such that

$$\mathcal{E}^b(t)(u) \geq \alpha_0^{\mathcal{E}}(\|\nabla u\|_p^p + \|u\|_q^q) - \beta_0^{\mathcal{E}}, \tag{3.82}$$

$$\mathcal{E}^b(t)(u) \leq \alpha_1^{\mathcal{E}}(\|\nabla u\|_p^p + \|u\|_q^q + \|u\|_{r,\partial_S\Omega}^r) + \beta_1^{\mathcal{E}},$$

for every $t \in [0,T]$ and $u \in GSBV_q^p(\Omega;\mathbb{R}^2)$.

The time dependent boundary deformations. We will consider boundary deformations $g(t)$ such that

$$t \to g(t) \in AC([0,T]; W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)),$$

so that

$$t \to \dot{g}(t) \in L^1([0,T]; W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)),$$

and

$$t \to \nabla\dot{g}(t) \in L^1([0,T]; L^p(\Omega;\mathcal{M}^{2\times 2})).$$

Convergence of sets and dispacements. Dal Maso, Francfort and Toader defined a variational notion of convergence for sets in $\mathbb{R}^N$ which they called σ^p*-convergence.* With respect to this notion they proved the associated transfer of jump theorem, involved in the existence result for a quasi static crack growth.

Definition 3.8.5. *Let $(K_n)_{n\in\mathbb{N}}$ and K be subsets of Ω. We say that K_n σ^p-converges in Ω to K if the following hold*

(1) if $u_h \rightharpoonup u$ weakly in $SBV^p(\Omega)$ with $S(u_h) \subseteq K_{n_h}$, then $S(u) \subseteq K$;

(2) $K = S(u)$ and there exists $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$ with $S(u_n) \subseteq K_n$.

In the same paper the authors proved the following compactness property.

Theorem 3.8.6. *If $\mathcal{H}^{N-1}(K_n) \leq C$, then up to a subsequence $K_n \to K$ in the sense of σ^p -convergence.*

The next theorem gives a compactness and lower semicontinuity result with respect to weak convergence in $GSBV_q^p(\Omega,\mathbb{R}^2)$ of the displacements.

Theorem 3.8.7. *Let $t_k \in [0,T]$ with $t_k \to t$, and let $(u_k) \subset GSBV_q^p(\Omega;\mathbb{R}^2)$, $C \in]0;+\infty[$ such that $S(u_k) \tilde{\subseteq} \overline{\Omega}_B$ and*

$$\mathcal{E}^b(t_k)(u_k) + \mathcal{E}^s(S(u_k)) \leq C,$$

where $\mathcal{E}^b$ and $\mathcal{E}^s$ are defined as in (3.81) *and* (3.63). *Then there exists a subsequence $(u_{k_h})_{h\in\mathbb{N}}$ converging to some u weakly in $GSBV_q^p(\Omega;\mathbb{R}^2)$ such that $S(u) \tilde{\subseteq} \overline{\Omega}_B$,*

$$\mathcal{E}^b(t)(u) \leq \liminf_{h\to\infty} \mathcal{E}^b(t_{k_h})(u_{k_h}) \qquad \text{and} \qquad \mathcal{E}^s(S(u)) \leq \liminf_{h\to\infty} \mathcal{E}^s(S(u_{k_h})).$$

Proof. By (3.82) and (3.64), we have that there exists $C' \in]0,+\infty[$ such that

$$\|\nabla u_k\|_p^p + \|u_k\|_q^q + \mathcal{H}^1(S(u_k)) \leq C'.$$

Then we can apply Theorem 3.8.2 with $g(x,u_k) = |u_k|^q$, obtaining a subsequence $(u_{k_h})_{h\in\mathbb{N}}$ and $u \in GSBV^p(\Omega;\mathbb{R}^2)$ such that (3.59) holds: in particular we may assume that $u_{k_h} \to u$ pointwise a.e.. We have $u_{k_h} \to u$ strongly in $L^1(\Omega;\mathbb{R}^2)$, and by Fatou's Lemma we have that $u \in L^q(\Omega;\mathbb{R}^2)$ so that $u \in GSBV^p_q(\Omega;\mathbb{R}^2)$. We conclude $u_{k_h} \rightharpoonup u$ weakly in $GSBV^p_q(\Omega;\mathbb{R}^2)$. By [3, Theorem 3.7] we have that

$$\mathcal{E}^s(S(u)) \le \liminf_h \mathcal{E}^s(S(u_{k_h})),$$

and by [55] we have that

$$\int_\Omega W(x,\nabla u)\,dx \le \liminf_h \int_\Omega W(x,\nabla u_{k_h})\,dx.$$

Since by assumption the functions $z \to F(0,x,z)$ and $z \to \dot F(s,x,z)$ are continuous for all $s \in [0,T]$ and for a.e. $x \in \Omega$, and

$$F(t_{k_h},x,u_{k_h}(x)) = F(0,x,u_{k_h}(x)) + \int_0^{t_{k_h}} \dot F(s,x,u_{k_h}(x))\,ds,$$

we have that $F(t_{k_h},x,u_{k_h}(x)) \to F(t,x,u(x))$ for a.e. $x \in \Omega$. By Fatou's Lemma (in the limsup version) we deduce

$$\limsup_h \int_\Omega F(t_{k_h},x,u_{k_h}(x))\,dx \le \int_\Omega F(t,x,u(x))\,dx.$$

Since $(u_{k_h})_{|\Omega_S}$ is bounded in $W^{1,p}(\Omega_S;\mathbb{R}^2)\cap L^q(\Omega_S;\mathbb{R}^2)$, and the trace operator from $W^{1,p}(\Omega_S;\mathbb{R}^2)$ into $L^r(\Omega_S;\mathbb{R}^2)$ is compact, we get

$$\lim_h \mathcal{G}(t_{k_h})(u_{k_h}) = \mathcal{G}(t)(u),$$

and so the proof is thus concluded. □

The existence result. Let $\Gamma_0 \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ be a preexisting crack. The next Theorem proved in [35] establishes the existence of a quasistatic evolution with preexisting crack Γ_0.

Theorem 3.8.8. *Let $\Gamma_0 \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ be a preexisting crack. Then there exists a quasistatic evolution with preexisting crack Γ_0 and boundary deformation $g(t)$, i.e., there exists a function $t \to (u(t),\Gamma(t))$ from $[0,T]$ to $GSBV^p_q(\Omega;\mathbb{R}^2)\times\mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ with the following properties:*

(a) $(u(0),\Gamma(0))$ *is such that*

$$\mathcal{E}(0)(u(0),\Gamma(0)) = \min\{\mathcal{E}(0)(v,\Gamma) : v \in AD(g(0),\Gamma), \Gamma_0 \tilde\subseteq \Gamma\};$$

(b) $u(t) \in AD(g(t),\Gamma(t))$ *for all* $t \in [0,T]$;

(c) irreversibility: $\Gamma_0 \tilde\subseteq \Gamma(s) \tilde\subset \Gamma(t)$ *whenever* $0 \le s < t \le T$;

(d) static equilibrium: for all $t \in [0,T]$

$$\mathcal{E}(t)(u(t),\Gamma(t)) = \min\{\mathcal{E}(t)(v,\Gamma) : v \in AD(g(t),\Gamma), \Gamma(t) \tilde\subseteq \Gamma\};$$

(e) nondissipativity: the function $t \to E(t) := \mathcal{E}(t)(u(t),\Gamma(t))$ *is absolutely continuous on* $[0,T]$, *and for a.e.* $t \in [0,T]$

$$\dot E(t) = \langle \partial\mathcal{W}(\nabla u(t)),\nabla \dot g(t)\rangle - \langle \partial\mathcal{F}(t)(u(t)),\dot g(t)\rangle - \dot{\mathcal{F}}(t)(u(t)) - \langle \partial\mathcal{G}(t)(u(t)),\dot g(t)\rangle - \dot{\mathcal{G}}(t)(u(t)). \tag{3.83}$$

Chapter 4

A Γ-convergence approach to stability

Introduction

In this chapter [1] we provide a new approach to the problem of stability of *unilateral minimality properties* based on Γ-convergence, which permits to treat also the case of varying volume and surface energies. Finally we give an application to the study of crack propagation in composite materials. Our approach is based on the observation that the problem has a variational character. In fact, considering for a while the case of fixed energies f and g with f convex in ξ, we have that if (u_n, K_n) is a unilateral minimizer for the energy (1), then u_n is a minimum for the functional

$$\mathcal{E}_n(v) := \int_\Omega f(x, \nabla v(x))\, dx + \int_{S(v)\setminus K_n} g(x, \nu)\, d\mathcal{H}^{N-1}(x). \tag{4.1}$$

Then the problem of stability of unilateral minimizers can be treated in the framework of Γ-convergence which ensures the convergence of minimizers. In Section 4.3, using an abstract representation result by Bouchitté, Fonseca, Leoni and Mascarenhas [11], we prove that the Γ-limit (up to a subsequence) of the functional $\mathcal{E}_n$ can be represented as

$$\mathcal{E}(v) := \int_\Omega f(x, \nabla v(x))\, dx + \int_{S(v)} g^-(x, \nu)\, d\mathcal{H}^{N-1}(x), \tag{4.2}$$

where g^- is a suitable function defined on $\Omega \times S^{N-1}$ determined only by g and $(K_n)_{n\in\mathbb{N}}$, and such that $g^- \le g$. If we assume that $u_n \rightharpoonup u$ weakly in $SBV(\Omega)$, then by Γ-convergence we get that u is a minimizer for $\mathcal{E}$. Suppose now that K is a rectifiable set in Ω such that $S(u) \subseteq K$ and

$$g^-(x, \nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K. \tag{4.3}$$

[1] The results presented in this chapter are contained in Giacomini-Ponsiglione [52].

Then we have immediately that the pair (u, K) is a unilateral minimizer for f and g because for all pairs (v, H) with $S(v) \subseteq H$ and $K \subseteq H$ we have

$$\int_\Omega f(x, \nabla u(x))\, dx = \mathcal{E}(u) \leq \mathcal{E}(v) = \int_\Omega f(x, \nabla v(x))\, dx + \int_{S(v)} g^-(x, \nu)\, d\mathcal{H}^{N-1}$$
$$= \int_\Omega f(x, \nabla v(x))\, dx + \int_{S(v)\setminus K} g^-(x, \nu) \leq \int_\Omega f(x, \nabla v(x))\, dx + \int_{H\setminus K} g(x, \nu).$$

The rectifiable set K satisfying (4.3) is provided in Section 4.4, where we define a new variational notion of convergence for rectifiable sets which we call *σ-convergence*, and which departs from the notion of σ^p-convergence given in [35]. The σ-limit K of a sequence of rectifiable sets $(K_n)_{n\in\mathbb{N}}$ is constructed looking for the Γ-limit $\mathcal{H}^-$ in the strong topology of $L^1(\Omega)$ of the functionals

$$\mathcal{H}_n^-(u) := \begin{cases} \mathcal{H}^{N-1}(S(u) \setminus K_n) & u \in P(\Omega), \\ +\infty & \text{otherwise}, \end{cases} \tag{4.4}$$

where $P(\Omega)$ is the space of piecewise constant function in Ω (see (4.19)). Roughly, the σ-limit K is the maximal rectifiable set on which the density h^- representing $\mathcal{H}^-$ vanishes. By the growth estimate on g it turns out that K is also the maximal rectifiable set on which the density g^- vanishes, so that K is the natural limit candidate for K_n in order to preserve the unilateral minimality property. The definition of σ-convergence involves only the surface energies $\mathcal{H}_n^-$, and as a consequence it does not depend on the exponent p and it is stable with respect to infinitesimal perturbations in length (see Remark 4.4.8). Moreover it turns out that the σ-limit K contains the σ^p-limit points of $(K_n)_{n\in\mathbb{N}}$, so that our Γ-convergence approach improves also the minimality property given by the previous approaches.

Our method naturally extends to the case of varying bulk and surface energies f_n and g_n, and this is indeed the main motivation for which we developed our Γ-convergence approach. The key point to recover effective energies f and g for the minimality property in the limit is a Γ-convergence result for functionals of the form

$$\int_\Omega f_n(x, \nabla u_n(x))\, dx + \int_{S(u_n)} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x). \tag{4.5}$$

In Section 4.3, we prove that the Γ-limit has the form

$$\int_\Omega f(x, \nabla u(x))\, dx + \int_{S(u)} g(x, \nu)\, d\mathcal{H}^{N-1}(x),$$

where f is determined only by $(f_n)_{n\in\mathbb{N}}$, and g is determined only by $(g_n)_{n\in\mathbb{N}}$, that is no interaction occurs between the bulk and the surface part of the functionals in the Γ-convergence process. A result of this type has been proved in the case of periodic homogenization (in the vectorial case, and with dependence on the trace of u in the surface part of the energy) by Braides, Defranceschi and Vitali [13].

We notice that an approach to stability in the line of Dal Maso, Francfort and Toader in the case of varying energies would have required a Transfer of Jump Sets for f_n, g_n and f, g, which seems difficult to be derived directly. Our Γ-convergence approach also provides this result (Proposition 4.5.4).

In section 4.7 we deal with the study of quasistatic crack evolution in composite materials. More precisely we study the asymptotic behavior of a quasistatic evolution $t \to (u_n(t), K_n(t))$ relative to the bulk energy f_n and the surface energy g_n. Using our stability result we prove (Theorem 4.7.1) that $t \to (u_n(t), K_n(t))$ converges to a quasistatic evolution $t \to (u(t), K(t))$ relative to the effective bulk and surface energies f and g. Moreover convergence for bulk and surface energies for all times holds. This analysis applies to the case of composite materials, i.e. materials obtained through a fine mixture of different phases. The model case is that of periodic homogenization, i.e. materials with total energy given by

$$\mathcal{E}_\varepsilon(u,K) := \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx + \int_K g\left(\frac{x}{\varepsilon}, \nu\right) d\mathcal{H}^{N-1}(x),$$

where ε is a small parameter giving the size of the mixture, and f, g are periodic in x. Our result implies that a quasisistatic crack evolution $t \to (u_\varepsilon(t), K_\varepsilon(t))$ for ε small is very near to a quasistatic evolution for the homogeneous material having bulk and surface energies $f_{\hom}$ and $g_{\hom}$, which are obtained from f and g through periodic homogenization formulas available in the literature (see for example [13]).

The chapter is organized as follows. In Section 4.1 we prove a blow up result for Γ-limits which will be employed in the proof of the main results. In Section 4.2 we prove some representation results which we use in Section 4.3 where we deal with the Γ-convergence of free discontinuity problems like (4.5). The notion of σ-convergence for rectifiable sets is contained in Section 4.4, while the main result on stability for unilateral minimizer is contained in Section 4.5. In Section 4.6 we prove a stability result for unilateral minimality properties with boundary conditions which will be employed in Section 4.7 for the study of quasistatic crack evolution in composite materials.

4.1 Blow-up for Γ-limits

Let $1 < p < +\infty$ and let $f : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function such that

$$a_1(x) + \alpha|\xi|^p \leq f(x,\xi) \leq a_2(x) + \beta|\xi|^p, \tag{4.6}$$

where $a_1, a_2 \in L^1(\Omega)$ and $\alpha, \beta > 0$. Let us assume that

$$\xi \to f(x,\xi) \quad \text{is convex for a.e. } x \in \Omega.$$

Let B_1 be the unit ball in $\mathbb{R}^N$ with center 0 and radius 1. The following blow up result in the sense of Γ-convergence holds.

Lemma 4.1.1. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence converging to zero. Then for a.e. $x \in \Omega$ the functionals*

$$F_k(u) := \begin{cases} \int_{B_1} f(x + \rho_k y, \nabla u(y))\, dy & u \in W^{1,p}(B_1), \\ +\infty & \text{otherwise in } L^1(B_1) \end{cases} \tag{4.7}$$

Γ-converge in the strong topology of $L^1(B_1)$ to the functional

$$F(u) := \begin{cases} \int_{B_1} f(x, \nabla u(y))\, dy & u \in W^{1,p}(B_1), \\ +\infty & \text{otherwise in } L^1(B_1). \end{cases} \tag{4.8}$$

Proof. By Scorza-Dragoni Theorem there exists a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ such that $|\Omega \setminus K_n| \to 0$ and such that f restricted to $K_n \times \mathbb{R}^N$ is continuous. Let us define

$$\mathcal{N} := \Omega \setminus \{x \in \Omega : \text{ there exists } n \in \mathbb{N} \text{ such that } x \text{ is of density 1 for } K_n\}. \tag{4.9}$$

We clearly have $|\mathcal{N}| = 0$, and every $x \in \Omega \setminus \mathcal{N}$ is a Lebesgue point for $f(\cdot, \xi)$ for every $\xi \in \mathbb{R}^N$.

Let us fix $x \in \Omega \setminus \mathcal{N}$ with $x \in K_n$ for some $n \in \mathbb{N}$, and let us begin with the proof of the Γ-limsup inequality. We can prove it for a dense set in $W^{1,p}(B_1)$, for example for the piecewise affine functions.

So let u be piecewise affine, and let $\nabla u(y) \in \{\xi_1, \dots, \xi_m\}$ for all $y \in B_1$. Since x is of density 1 for K_n and f is continuous on $K_n \times \mathbb{R}^N$, we have that for all $\varepsilon > 0$

$$|\{y \in B_1 \,:\, |f(x+\rho_k y, \xi_i) - f(x, \xi_i)| > \varepsilon\}| \to 0.$$

Then considering as recovering sequence $u_k = u$, we get

$$\limsup_{k \to +\infty} \int_{B_1} f(x+\rho_k y, Du)\, dy \le \int_{B_1} f(x, Du)\, dy,$$

so that the inequality is proved.

Let us come to the Γ-liminf inequality. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $L^1(B_1)$ such that $u_k \to u$ strongly in $L^1(B_1)$. We can assume that $\sup_{k\in\mathbb{N}} F_k(u_k) < +\infty$, so that $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(B_1;\mathbb{R}^N)$. Let $M > 0$ be fixed, and let δ be such that $|\{|\nabla u| \ge M\}| \le \delta$. Let us consider

$$\Phi_k^M(y) := \begin{cases} \nabla u_k(y) & \text{if } |\nabla u_k(y)| \le M, \\ 0 & \text{otherwise,} \end{cases}$$

and let us denote by Φ^M its weak limit (up to a further subsequence) in $L^p(B_1;\mathbb{R}^N)$. Since by assumption on x we have that for all $\varepsilon > 0$

$$\lim_{k\to+\infty} |\{y \in B_1 \,:\, |f(x+\rho_k y, \Phi_k^M(y)) - f(x, \Phi_k^M(y))| > \varepsilon\}| = 0,$$

we obtain

$$\begin{aligned}\liminf_{k\to+\infty} \int_{B_1} f(x+\rho_k y, \nabla u_k(y))\, dy &\ge \liminf_{k\to+\infty} \int_{B_1} f(x+\rho_k y, \Phi_k^M(y))\, dy - e(\delta) \\ &\ge \liminf_{k\to+\infty} \int_{B_1} f(x, \Phi_k^M(y))\, dy - \tilde{e}(\delta) \ge \int_{B_1} f(x, \Phi^M(y))\, dy - \tilde{e}(\delta),\end{aligned}$$

where $\tilde{e}(\delta) \to 0$ as $\delta \to 0$. Letting $M \to +\infty$, we get $\delta \to 0$ and $\Phi^M \rightharpoonup \nabla u$ weakly in $L^p(B_1, \mathbb{R}^N)$. The result follows by lower semicontinuity since $f(x, \cdot)$ is convex. □

Let us consider now $f_n : \Omega \times \mathbb{R}^N \to [0, +\infty[$ Carathéodory function satisfying the growth estimate (4.6) uniformly in n. Let us assume that for all $A \in \mathcal{A}(\Omega)$ the localized functionals

$$\mathcal{F}_n(u, A) := \begin{cases} \int_A f_n(x, \nabla u(x))\, dx & u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.10}$$

Γ-converge with respect to the strong topology of $L^1(\Omega)$ to

$$\mathcal{F}(u, A) := \begin{cases} \int_A f(x, \nabla u(x))\, dx & u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise} \end{cases} \tag{4.11}$$

for some Carathéodory function f which satisfies estimate (4.6). Using a diagonal argument we may conclude that the following theorem holds.

Theorem 4.1.2. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence converging to zero. Then for a.e. $x \in \Omega$ there exists $(n_k)_{k\in\mathbb{N}}$ such that the functionals*

$$F_k(u) := \begin{cases} \int_{B_1} f_{n_k}(x+\rho_k y, \nabla u(y))\, dy & u \in W^{1,p}(B_1), \\ +\infty & \textit{otherwise in } L^1(B_1) \end{cases} \tag{4.12}$$

Γ-converge in the strong topology of $L^1(B_1)$ to the functional

$$F(u) := \begin{cases} \int_{B_1} f(x, \nabla u(y))\, dy & u \in W^{1,p}(B_1), \\ +\infty & \textit{otherwise in } L^1(B_1). \end{cases} \tag{4.13}$$

Remark 4.1.3. In the case of periodic homogenization, i.e. in the case in which $f_n(x,\xi) := f(nx,\xi)$ with f periodic in x, it is sufficient to choose n_k in such a way that $n_k\rho_k \to +\infty$. In fact for $x = 0$ we have

$$F_k(u) := \begin{cases} \int_{B_1} f((n_k\rho_k)y, \nabla u(y))\, dy & u \in W^{1,p}(B_1), \\ +\infty & \text{otherwise in } L^1(B_1) \end{cases}$$

which still Γ-converges to (see for instance [33])

$$F(u) := \begin{cases} \int_{B_1} f_{\text{hom}}(\nabla u(y))\, dy & u \in W^{1,p}(B_1), \\ +\infty & \text{otherwise in } L^1(B_1). \end{cases}$$

In the rest of this section we prove a regularity result for the density f defined in (4.11) under additional hypothesis on f_n which will be employed in Section 4.7. Let us assume that for a.e. $x \in \Omega$

(1) $f_n(x,\cdot)$ is convex;

(2) $f_n(x,\cdot)$ is of class C^1;

(3) for all $M \geq 0$ and for all ξ_n^1, ξ_n^2 such that $|\xi_n^1| \leq M$, $|\xi_n^1| \leq M$, $|\xi_n^1 - \xi_n^2| \to 0$ we have

$$|\nabla_\xi f_n(x,\xi_n^1) - \nabla_\xi f_n(x,\xi_n^2)| \to 0. \tag{4.14}$$

Notice that for instance $f_n(x,\xi) := a_n(x)|\xi|^p$ with $\alpha \leq a_n(x) \leq \beta$ satisfies the assumptions above. Notice moreover that by semicontinuity of Γ-limits $\xi \to f(x,\xi)$ is convex for a.e. $x \in \Omega$.

We need the following lemma which is a straightforward variant of [35, Lemma 4.9].

Lemma 4.1.4. *Let (X, A, μ) be a finite measure space, $p > 1$, $N \geq 1$, and let $H_n : X \times \mathbb{R}^N \to \mathbb{R}$ be a sequence of Carathéodory functions which satysfies the following properties: there exist a positive constant $a \geq 0$ and a nonnegative function $b \in L^{p'}(X)$, with $p' = p/(p-1)$ such that*

(1) $|H_n(x,\xi)| \leq a|\xi|^{p-1} + b(x)$ for every $x \in X$, $\xi \in \mathbb{R}^N$;

(2) for all $M \geq 0$ and for a.e. $x \in \Omega$, for all ξ_n^1, ξ_n^2 such that $|\xi_n^1| \leq M$, $|\xi_n^1| \leq M$, $|\xi_n^1 - \xi_n^2| \to 0$ we have

$$|H_n(x,\xi_n^1) - H_n(x,\xi_n^2)| \to 0.$$

Assume that $(\Phi_n)_{n\in\mathbb{N}}$ is bounded in $L^p(X,\mathbb{R}^N)$ and that $(\Psi_n)_{n\in\mathbb{N}}$ converges to 0 strongly in $L^p(X,\mathbb{R}^N)$. Then

$$\int_X [H_n(x, \Phi_n(x) + \Psi_n(x)) - H_n(x, \Phi_n(x))]\Phi(x)\, d\mu(x) \to 0, \tag{4.15}$$

for every $\Phi \in L^p(X,\mathbb{R}^N)$.

The following regularity result on f holds.

Proposition 4.1.5. *For a.e. $x \in \Omega$ the function $\xi \to f(x,\xi)$ is of class C^1.*

Proof. Let $x \in \Omega \setminus \mathcal{N}$, where $\mathcal{N}$ is defined in (4.9). Let $\rho_k \to 0$ and let $(n_k)_{k\in\mathbb{N}}$ be a sequence such that, according to Theorem 4.1.2, $(F_k)_{k\in\mathbb{N}}$ Γ-converges with respect to the strong topology of $L^1(B_1)$ to F.

Let $(\phi_k)_{k\in\mathbb{N}}$ be a recovering sequence for the affine function $y \to \xi \cdot y$ with $\xi \in \mathbb{R}^N$. Up to a further subsequence, we can always assume that there exists $\psi \in \mathbb{R}^N$ such that

$$\frac{1}{|B_1|}\int_{B_1} \nabla_\xi f_{n_k}(x+\rho_k y, \nabla\phi_k(y))\,dy \to \psi. \tag{4.16}$$

Let $t_j \searrow 0$ and let $\eta \in \mathbb{R}^N$. By the convexity of f_{n_k} in the second variable, we have

$$\int_{B_1} f_{n_k}(x+\rho_k y, \nabla\phi_k(y)+t_j\eta) - f_{n_k}(x+\rho_k y, \nabla\phi_k(y))\,dy \\ \leq t_j \int_{B_1} \nabla_\xi f_{n_k}(x+\rho_k y, \nabla\phi_k(y)+t_j\eta)\eta\,dy. \tag{4.17}$$

By Γ-convergence we can find k_j such that

$$\frac{f(x,\xi+t_j\eta)-f(x,\xi)}{t_j} - \frac{1}{j} \leq \frac{1}{|B_1|}\int_{B_1} \nabla_\xi f_{n_{k_j}}(x+\rho_{k_j} y, \nabla\phi_{k_j}(y)+t_j\eta)\eta\,dy,$$

so that we have

$$\limsup_{j\to+\infty} \frac{f(x,\xi+t_j\eta)-f(x,\xi)}{t_j} \leq \frac{1}{|B_1|}\limsup_{j\to+\infty}\int_{B_1} \nabla_\xi f_{n_{k_j}}(x+\rho_{k_j} y, \nabla\phi_{k_j}(y)+t_j\eta)\eta\,dy. \tag{4.18}$$

Notice that by Lemma 4.1.4 and by (4.16) we have that

$$\lim_{j\to+\infty}\int_{B_1} \nabla_\xi f_{n_{k_j}}(x+\rho_{k_j} y, \nabla\phi_{k_j}(y)+t_j\eta)\eta\,dy \\ = \lim_{j\to+\infty}\int_{B_1} \nabla_\xi f_{n_{k_j}}(x+\rho_{k_j} y, \nabla\phi_{k_j}(y))\eta\,dy = |B_1|\psi\eta,$$

and so for every subgradient ζ of $f(x,\cdot)$ at ξ by (4.18) we have

$$\zeta\eta \leq \limsup_{j\to+\infty} \frac{f(x,\xi+t_j\eta)-f(x,\xi)}{t_j} \leq \psi\eta.$$

We deduce that $\zeta = \psi$, so that $f(x,\cdot)$ is Gateaux differentiable at ξ with $\nabla_\xi f(x,\xi) = \psi$: since $f(x,\cdot)$ is convex, we get that $f(x,\cdot)$ is of class C^1. □

Remark 4.1.6. Notice that an hypothesis of *equiuniform continuity* for $(\nabla_\xi f_n(x,\xi))_{n\in\mathbb{N}}$ like (4.14) is needed in order to preserve C^1-regularity in the passage from f_n to f: in fact if $\xi \to f_n(\xi)$ are smooth convex functions uniformly converging to a nondifferentiable convex function $\xi \to f(\xi)$, the associated integral functionals Γ-converge, and this provides a counterexample.

4.2 Some representation lemmas

We indicate by $P(\Omega)$ the family of sets with finite perimeter in Ω, that is the class of sets $E \subseteq \Omega$ such that $1_E \in BV(\Omega)$. In view of the applications of Sections 4.2, 4.3 and 4.4, it will be useful to look at $P(\Omega)$ in term of functions, that is to use the following equivalent description:

$$P(\Omega) = \{u \in BV(\Omega) : u(x) \in \{0,1\} \text{ for a.e. } x \in \Omega\}. \tag{4.19}$$

In order to take into account a boundary datum, we will use the following notation: if $\partial_D\Omega \subseteq \partial\Omega$, then for all $u, g \in BV(\Omega)$ we set

$$S^g(u) := S(u) \cup \{x \in \partial_D\Omega \,:\, u(x) \neq g(x)\}, \tag{4.20}$$

where the inequality on $\partial_D\Omega$ is intended in the sense of traces. Moreover, we set for all $x \in S(u)$

$$[u](x) := u^+(x) - u^-(x),$$

and for all $x \in \partial_D\Omega$ we set $[u](x) := u(x) - g(x)$, where the traces of u and g on $\partial\Omega$ are used.

Let $a_1, a_2 \in L^1(\Omega)$, $1 < p < +\infty$, and let $\alpha, \beta > 0$. For all $n \in \mathbb{N}$ let $f_n : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}$

$$a_1(x) + \alpha|\xi|^p \leq f_n(x, \xi) \leq a_2(x) + \beta|\xi|^p, \tag{4.21}$$

and let $g_n : \Omega \times S^{N-1} \to [0, +\infty[$ be a Borel function such that for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ and for all $\nu \in S^{N-1} := \{\eta \in \mathbb{R}^N : |\eta| = 1\}$

$$\alpha \leq g_n(x, \nu) \leq \beta. \tag{4.22}$$

In Section 4.3 we will be interested the functionals on $L^1(\Omega) \times \mathcal{A}(\Omega)$

$$\mathcal{E}_n(u, A) := \begin{cases} \int_A f_n(x, \nabla u(x))\, dx + \int_{A \cap (S(u) \setminus K_n)} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x) & u \in SBV^p(A), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.23}$$

where $\mathcal{A}(\Omega)$ denotes the family of open subsets of Ω, and $(K_n)_{n\in\mathbb{N}}$ is a sequence of rectifiable sets in Ω such that

$$\mathcal{H}^{N-1}(K_n) \leq C. \tag{4.24}$$

In particular we will be interested in the Γ-limit in the strong topology of $L^1(\Omega)$ of $(\mathcal{E}_n(\cdot, A))_{n\in\mathbb{N}}$ for every $A \in \mathcal{A}(\Omega)$. To this extend we consider the functionals $\mathcal{F}_n : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$

$$\mathcal{F}_n(u, A) := \begin{cases} \int_A f_n(x, \nabla u(x))\, dx & u \in W^{1,p}(A), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.25}$$

and the functionals $\mathcal{G}_n^- : P(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty[$

$$\mathcal{G}_n^-(u, A) := \int_{A \cap (S(u) \setminus K_n)} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x) \tag{4.26}$$

defined on Sobolev and piecewise constant functions with values in $\{0, 1\}$ (see (4.19)) respectively, and we will reconstruct the Γ-limit of $(\mathcal{E}_n(\cdot, A))_{n\in\mathbb{N}}$ through the Γ-limits of $(\mathcal{F}_n(\cdot, A))_{n\in\mathbb{N}}$ and $(\mathcal{G}_n^-(\cdot, A))_{n\in\mathbb{N}}$.

For the results of Section 4.5, we will need also the functionals $\mathcal{G}_n^- : P(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty[$

$$\mathcal{G}_n(u, A) := \int_{A \cap S(u)} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x) \tag{4.27}$$

In this section we provide some integral representation results for the Γ-limits of the functionals $\mathcal{F}_n, \mathcal{G}_n^-, \mathcal{G}_n$ and $\mathcal{E}_n$. In the following, for every functional $\mathcal{H}$ defined on $L^1(\Omega) \times \mathcal{A}(\Omega)$ or on $P(\Omega) \times \mathcal{A}(\Omega)$ with values to $[0, +\infty]$, for every $A \in \mathcal{A}(\Omega)$ and $\psi \in L^1(A)$ we will use the notation

$$\mathbf{m}_{\mathcal{H}}(A, \psi) = \inf_{u \in L^1(\Omega)} \{\mathcal{H}(u, A) \,:\, u = \psi \text{ in a neighborhood of } \partial A\}. \tag{4.28}$$

Moreover for all $x \in \mathbb{R}^N$, $a, b \in \mathbb{R}$ and $\nu \in S^{N-1}$ we consider $u_{x,a,b,\nu} : B_1(x) \to \mathbb{R}$ defined by

$$u_{x,a,b,\nu}(y) := \begin{cases} b & \text{if } (y-x)\nu \geq 0, \\ a & \text{if } (y-x)\nu < 0, \end{cases} \tag{4.29}$$

where $B_1(x)$ is the ball of center x and radius 1.

As for the functionals $\mathcal{F}_n$ the following Γ-convergence and representation result holds.

Proposition 4.2.1. *There exists $\mathcal{F} : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that up to a subsequence the functionals $\mathcal{F}_n(\cdot, A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. Moreover for all $u \in W^{1,p}(\Omega)$ we have that*

$$\mathcal{F}(u, A) = \int_A f(x, \nabla u(x))\, dx \tag{4.30}$$

where

$$f(x, \xi) := \limsup_{\rho \to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(B_\rho(x), \xi(z-x))}{\omega_N \rho^N}, \tag{4.31}$$

$\mathbf{m}_{\mathcal{F}}$ is defined in (4.28), and ω_N is the volume of the unit ball in $\mathbb{R}^N$. Finally f is a Carathéodory function satisfying the growth conditions (4.21).

Proof. Let us consider the restriction $\tilde{\mathcal{F}}_n$ of $\mathcal{F}_n$ to $L^p(\Omega) \times \mathcal{A}(\Omega)$. Then in view of the growth estimate (4.21), by [33] we deduce that there exists $\tilde{\mathcal{F}} : L^p(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that up to a subsequence $\tilde{\mathcal{F}}_n(\cdot, A)$ Γ-converges in the strong topology of $L^p(\Omega)$ to $\tilde{\mathcal{F}}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$.

For every $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$ let us set

$$\mathcal{F}(u, A) := \limsup_{M \to +\infty} \tilde{\mathcal{F}}(T_M(u), A)$$

where $T_M(u) := (u \vee -M) \wedge M$. Let us prove that along the same subsequence $\mathcal{F}_n(\cdot, A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. As for the Γ-liminf inequality, let us consider a sequence $(u_n)_{n\in\mathbb{N}}$ in $L^1(\Omega)$ with $u_n \to u$ strongly in $L^1(\Omega)$. Then for every $M > 0$ we have that $T_M(u_n) \to T_M(u)$ strongly in $L^p(\Omega)$, so that for every $A \in \mathcal{A}(\Omega)$ we have

$$\tilde{\mathcal{F}}(T_M(u), A) \leq \liminf_{n\to+\infty} \tilde{\mathcal{F}}_n(T_M(u_n), A) \leq \liminf_{n\to+\infty} \mathcal{F}_n(u_n, A).$$

Taking the limsup for $M \to +\infty$, we get that the Γ-liminf inequality holds.

Let us come to Γ-limsup inequality. It is sufficient to consider $u \in L^\infty(\Omega)$, since $L^\infty(\Omega)$ is a subset of $L^1(\Omega)$ dense in energy with respect to $\mathcal{F}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. A recovering sequence for u with respect to $\tilde{\mathcal{F}}_n(\cdot, A)$ and the strong topology of $L^p(\Omega)$ is a good recovering sequence for $\mathcal{F}_n(\cdot, A)$ and the strong topology of $L^1(\Omega)$.

Let us consider the restriction of $\mathcal{F}$ to $W^{1,p}(\Omega) \times \mathcal{A}(\Omega)$. We have that

($\mathcal{F}1$) for all $u \in W^{1,p}(\Omega)$, $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;

($\mathcal{F}2$) $\mathcal{F}(u, A) = \mathcal{F}(v, A)$ if $u = v$ on A;

($\mathcal{F}3$) $\mathcal{F}(u + C, A) = \mathcal{F}(u, A)$ for every constant C;

($\mathcal{F}4$) $\mathcal{F}(\cdot, A)$ is lower semicontinuous with respect to L^1 convergence;

($\mathcal{F}5$) we have the growth estimate

$$\int_A a_1(x)\,dx + \alpha\|\nabla u\|_A^p \le \mathcal{F}(u,A) \le \int_A a_2(x)\,dx + \beta\|\nabla u\|_A^p.$$

We can thus apply the representation result by Buttazzo and Dal Maso [23] (see also [11, Theorem 2]) and deduce that the representation results (4.30) and (4.31) hold. □

Let us come to the functionals $\mathcal{G}_n$ defined in (4.27). The following proposition holds.

Proposition 4.2.2. *There exists $\mathcal{G} : P(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty[$ such that up to a subsequence $\mathcal{G}_n(\cdot,A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{G}(\cdot,A)$ for all $A \in \mathcal{A}(\Omega)$. Moreover for all $u \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we have that*

$$\mathcal{G}(u,A) = \int_{A\cap S(u)} g(x,\nu)\,dx \tag{4.32}$$

with

$$g(x,\nu) := \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{G}}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}}, \tag{4.33}$$

where $\mathbf{m}_{\mathcal{G}}$ is defined in (4.28) and $u_{x,0,1,\nu}$ is as in (4.29).

Proof. The existence of $\mathcal{G} : P(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty[$ such that up to a subsequence $\mathcal{G}_n(\cdot,A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{G}(\cdot,A)$ for all $A \in \mathcal{A}(\Omega)$ has been proved by Ambrosio and Braides [5, Theorem 3.2]. By the growth estimate on g_n we get that $\mathcal{G}$ satisfies the following properties:

($\mathcal{G}1$) for all $u \in P(\Omega)$, $\mathcal{G}(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;

($\mathcal{G}2$) $\mathcal{G}(u,A) = \mathcal{G}(v,A)$ if $u = v$ on A;

($\mathcal{G}3$) $\mathcal{G}(\cdot,A)$ is lower semicontinuous with respect to the strong topology of $L^1(\Omega)$;

($\mathcal{G}4$) we have the growth estimate

$$\alpha\mathcal{H}^{N-1}(S(u)\cap A) \le \mathcal{G}(u,A) \le \beta\mathcal{H}^{N-1}(S(u)\cap A).$$

Then the representation formulas (4.32) and (4.33) come from [11, Theorem 3]. □

Let us come to the functionals $\mathcal{G}_n^-$ defined in (4.26). The following proposition holds.

Proposition 4.2.3. *There exists $\mathcal{G}^- : P(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty[$ such that up to a subsequence $\mathcal{G}_n^-(\cdot,A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{G}^-(\cdot,A)$ for all $A \in \mathcal{A}(\Omega)$. Moreover for all $u \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we have that*

$$\mathcal{G}^-(u,A) = \int_{A\cap S(u)} g^-(x,\nu)\,d\mathcal{H}^{N-1}(x) \tag{4.34}$$

with

$$g^-(x,\nu) := \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{G}^-}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}}, \tag{4.35}$$

where $\mathbf{m}_{\mathcal{G}^-}$ is defined in (4.28) and $u_{x,0,1,\nu}$ is as in (4.29).

Proof. By the growth estimate (4.22) on g_n, by the result of Ambrosio and Braides [5] there exists $\mathcal{G}^- : P(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty]$ such that up to a subsequence $(\mathcal{G}_n^-(\cdot,A))_{n\in\mathbb{N}}$ Γ-converges in the strong topology of $L^1(\Omega)$ to $\mathcal{G}^-(\cdot,A)$ for every $A \in \mathcal{A}(\Omega)$, and such that the following properties hold:

$(\mathcal{G}^-1)$ for all $u \in P(\Omega)$, $\mathcal{G}^-(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;

$(\mathcal{G}^-2)$ $\mathcal{G}^-(u,A) = \mathcal{G}^-(v,A)$ if $u = v$ on A;

$(\mathcal{G}^-3)$ $\mathcal{G}^-(\cdot,A)$ is lower semicontinuous with respect to the strong convergence in $L^1(\Omega)$;

$(\mathcal{G}^-4)$ we have the growth estimate

$$0 \le \mathcal{G}^-(u,A) \le \beta\mathcal{H}^{N-1}(S(u)\cap A).$$

The integral representation formula (4.34) for $\mathcal{G}^-(\cdot,A)$ is given by the result of Ambrosio and Braides in view of properties $(\mathcal{G}^-1)$-$(\mathcal{G}^-4)$ (see also Bouchitté, Fonseca, Leoni and Mascarenhas [11]). For the sequel we need also the explicit formula (4.35) for the density g^- which is not given directly by the results of [5] and [11] because of a lack of coercivity from below. So in what follows we modify the concrete approximation $\mathcal{G}_n^-(\cdot,A)$ for $\mathcal{G}^-(\cdot,A)$ in order to get the coerciveness we need, and to obtain in the end formula (4.35).

Let us consider the functionals

$$\mathcal{G}_n^\varepsilon(u,A) := \int_{A\cap S(u)} g_n^\varepsilon(x,\nu)\,d\mathcal{H}^{N-1}(x) \tag{4.36}$$

where

$$g_n^\varepsilon(x,\nu) := \begin{cases} \varepsilon & \text{if } x \in K_n, \nu = \nu_{K_n}(x), \\ g_n(x,\nu) & \text{otherwise.} \end{cases} \tag{4.37}$$

Let us denote by $\mathcal{G}^\varepsilon(\cdot,A)$ the Γ-limit (up to a subsequence) of $\mathcal{G}_n^\varepsilon(\cdot,A)$ for all $A \in \mathcal{A}(\Omega)$. Since $\mathcal{G}^\varepsilon$ is such that for ε small

$$\varepsilon\mathcal{H}^{N-1}(S(u)\cap A) \le \mathcal{G}^\varepsilon(u,A) \le \beta\mathcal{H}^{N-1}(S(u)\cap A),$$

by the representation result of [12] we have that

$$\mathcal{G}^\varepsilon(u,A) = \int_{S(u)\cap A} g^\varepsilon(x,\nu)\,d\mathcal{H}^{N-1}(x),$$

where $g^\varepsilon : \Omega \times S^{N-1} \to [0,+\infty]$ is given by

$$g^\varepsilon(x,\nu) := \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{G}^\varepsilon}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}}. \tag{4.38}$$

We have for all $u \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$

$$\mathcal{G}_n^\varepsilon(u,A) \le \mathcal{G}_n^-(u,A) + \varepsilon\mu_n(A),$$

where $\mu_n := \mathcal{H}^{N-1} \llcorner K_n$, so that for $n \to +\infty$ by Γ-convergence we have

$$\mathcal{G}^\varepsilon(u,A) \le \mathcal{G}^-(u,A) + \varepsilon\mu(\bar{A}), \tag{4.39}$$

where μ is the weak* limit of $(\mu_n)_{n\in\mathbb{N}}$ (up to a subsequence) in the sense of measures. Notice that (see for instance [7, Theorem 2.56]) up to a set of $\mathcal{H}^{N-1}$-measure zero we have

$$H(x) := \limsup_{\rho\to 0^+} \frac{\mu(\bar{B}_\rho(x))}{\omega_{N-1}\rho^{N-1}} < +\infty. \tag{4.40}$$

Let us prove that for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ we have

$$g^-(x,\nu) = \lim_{\varepsilon\to 0} g^\varepsilon(x,\nu), \tag{4.41}$$

where $g^-(x,\nu)$ is defined in (4.35). In fact, notice that $\{g^\varepsilon\}_\varepsilon$ is monotone decreasing in ε and that $g^- \le g^\varepsilon$ for all $\varepsilon > 0$, so that for all x and ν

$$g^-(x,\nu) \le \lim_{\varepsilon\to 0} g^\varepsilon(x,\nu).$$

Let us set for every $\rho > 0$, $x \in \Omega$ and $\nu \in S^{N-1}$

$$m_\rho^\varepsilon(x,\nu) := \frac{\mathbf{m}_{\mathcal{G}^\varepsilon}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}} \quad \text{and} \quad m_\rho^-(x,\nu) := \frac{\mathbf{m}_{\mathcal{G}^-}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}}.$$

Then by (4.39) we have that

$$m_\rho^\varepsilon(x,\nu) \le m_\rho^-(x,\nu) + \varepsilon\frac{\mu(\bar{B}_\rho(x))}{\omega_{N-1}\rho^{N-1}}.$$

Taking the lim sup for $\rho \to 0^+$ we have

$$g^\varepsilon(x,\nu) \le g^-(x,\nu) + \varepsilon H(x),$$

and so letting $\varepsilon \to 0$ we obtain for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$

$$\lim_{\varepsilon\to 0} g^\varepsilon(x,\nu) \le g^-(x,\nu)$$

which gives (4.41). Since for all $u \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we have $\mathcal{G}^\varepsilon(u,A) \to \mathcal{G}^-(u,A)$ as $\varepsilon \to 0$, we conclude that

$$\mathcal{G}^-(u,A) = \lim_{\varepsilon\to 0}\mathcal{G}^\varepsilon(u,A) = \lim_{\varepsilon\to 0}\int_{S(u)\cap A} g^\varepsilon(x,\nu)\,d\mathcal{H}^{N-1}(x) = \int_{S(u)\cap A} g^-(x,\nu)\,d\mathcal{H}^{N-1}(x), \tag{4.42}$$

so that the representation formulas (4.34) and (4.35) hold. □

Remark 4.2.4. It is immediate to check that if we replace $P(\Omega)$ in Proposition 4.2.3 by the space $P_{a,b}(\Omega) := \{u \in BV(\Omega) : u(x) \in \{a,b\} \text{ for a.e. } x \in \Omega\}$, with $a, b \in \mathbb{R}$, then the Γ-limit in the strong topology of $L^1(\Omega)$ of $\mathcal{G}_n^-(\cdot,A)$ can still be represented by the density g^- defined in (4.35).

Let us finally come to the functionals $\mathcal{E}_n$ defined in (4.23). Using the growth estimates (4.21) and (4.22) on f_n and g_n (see [13]), there exists $\mathcal{E} : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty[$ such that up to a subsequence $\mathcal{E}_n(\cdot,A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{E}(\cdot,A)$ for all $A \in \mathcal{A}(\Omega)$. The restriction of the functional $\mathcal{E}$ to $SBV^p(\Omega) \times \mathcal{A}(\Omega)$ satisfies the following properties:

($\mathcal{E}1$) for all $u \in SBV^p(\Omega)$, $\mathcal{E}(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;

($\mathcal{E}2$) $\mathcal{E}(u,A) = \mathcal{E}(v,A)$ if $u = v$ on A;

($\mathcal{E}3$) $\mathcal{E}(\cdot,A)$ is lower semicontinuous with respect the strong topology of $L^1(\Omega)$;

($\mathcal{E}4$) we have the growth estimate

$$\alpha\int_A |\nabla u|^p\,dx \le \mathcal{E}(u,A) \le \beta\int_A |\nabla u|^p\,dx + \beta\mathcal{H}^{N-1}(S(u)).$$

For every $\varepsilon > 0$ let us set

$$\mathcal{E}_\varepsilon(u, A) := \mathcal{E}(u, A) + \varepsilon \int_{S(u)\cap A} 1 + |[u]| \, d\mathcal{H}^{N-1}.$$

We have that $\mathcal{E}_\varepsilon$ satisfies the following properties:

($\mathcal{E}_\varepsilon 1$) for all $u \in SBV^p(\Omega)$, $\mathcal{E}_\varepsilon(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;

($\mathcal{E}_\varepsilon 2$) $\mathcal{E}_\varepsilon(u, A) = \mathcal{E}_\varepsilon(v, A)$ if $u = v$ on A;

($\mathcal{E}_\varepsilon 3$) $\mathcal{E}_\varepsilon(\cdot, A)$ is lower semicontinuous with respect the strong topology of $L^1(\Omega)$;

($\mathcal{E}_\varepsilon 4$) we have the growth estimate

$$\int_A a_1 \, dx + \varepsilon \left(\int_A |\nabla u|^p \, dx + \int_{S(u)\cap A} 1 + |[u]| \, d\mathcal{H}^{N-1} \right) \leq \mathcal{E}_\varepsilon(u, A)$$
$$\leq \int_A a_2 \, dx + \beta \left(\int_A |\nabla u|^p \, dx + \int_{S(u)\cap A} 1 + |[u]| \, d\mathcal{H}^{N-1} \right).$$

Then by the result of Bouchitté, Fonseca, Leoni and Mascarenhas we get that

$$\mathcal{E}_\varepsilon(u, A) = \int_A f^\varepsilon_\infty(x, \nabla u(x)) \, dx + \int_{A\cap S(u)} g^\varepsilon_\infty(x, u^-(x), u^+(x), \nu) \, d\mathcal{H}^{N-1}(x)$$

with f^ε_∞ and g^ε_∞ satisfying the following formulas

$$f^\varepsilon_\infty(x, \xi) := \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{E}_\varepsilon}(B_\rho(x), \xi(z-x))}{\omega_N \rho^N}, \tag{4.43}$$

and

$$g^\varepsilon_\infty(x, a, b, \nu) := \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{E}_\varepsilon}(B_\rho(x), u_{x,a,b,\nu})}{\omega_{N-1} \rho^{N-1}}, \tag{4.44}$$

where $\mathbf{m}_{\mathcal{E}_\varepsilon}$ is defined in (4.28) and $u_{x,a,b,\nu}$ is as in (4.29).

Notice that f^ε_∞ and g^ε_∞ are monotone decreasing in ε, and that $\mathcal{E}_\varepsilon(\cdot, A)$ converges pointwise to $\mathcal{E}(\cdot, A)$ as $\varepsilon \to 0$ for all $A \in \mathcal{A}(\Omega)$. We conclude that the representation result for $\mathcal{E}_\varepsilon$ implies a representation result for the functional $\mathcal{E}$.

Summarizing we have that the following proposition holds.

Proposition 4.2.5. *There exists $\mathcal{E} : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that up to a subsequence $\mathcal{E}_n(\cdot, A)$ Γ-converges in the strong topology of $L^1(\Omega)$ to $\mathcal{E}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. Moreover, for every $u \in SBV^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we have that*

$$\mathcal{E}(u, A) = \int_A f_\infty(x, \nabla u(x)) \, dx + \int_{A\cap S(u)} g_\infty(x, u^-(x), u^+(x), \nu) \, d\mathcal{H}^{N-1}(x)$$

with

$$f_\infty(x, \xi) := \lim_{\varepsilon\to 0} f^\varepsilon_\infty(x, \xi) \qquad g_\infty(x, a, b, \nu) := \lim_{\varepsilon\to 0} g^\varepsilon_\infty(x, a, b, \nu), \tag{4.45}$$

where f^ε_∞ and g^ε_∞ are defined in (4.43) *and* (4.44) *respectively.*

Remark 4.2.6. In the rest of the chapter we will often make use the following property which is implied by the fact that $\mathcal{E}(u, \cdot)$ is a Radon measure for every $u \in SBV^p(\Omega)$. If $(u_n)_{n\in\mathbb{N}}$ is a recovering sequence for u with respect to $\mathcal{E}_n(\cdot, \Omega)$, then $(u_n)_{n\in\mathbb{N}}$ is optimal for u with respect to $\mathcal{E}_n(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$ such that the measure $\mathcal{E}(u, \cdot)$ vanishes on ∂A.

4.3 A Γ-convergence result for free discontinuity problems

The main result of this section is the following Γ-convergence theorem concerning the functionals $\mathcal{E}_n$ defined in (4.23).

Theorem 4.3.1. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that $\mathcal{H}^{N-1}(K_n) \le C$ for all $n \in \mathbb{N}$. Let us assume that for all $A \in \mathcal{A}(\Omega)$ the functionals $\mathcal{F}_n(\cdot, A)$ and $\mathcal{G}_n^-(\cdot, A)$ defined in (4.25) and (4.26) Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot, A)$ and $\mathcal{G}^-(\cdot, A)$ respectively. Then for all $A \in \mathcal{A}(\Omega)$ the functionals $\mathcal{E}_n(\cdot, A)$ defined in (4.23) Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{E}(\cdot, A)$ such that for all $u \in SBV^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$*

$$\mathcal{E}(u, A) = \int_A f(x, \nabla u(x))\, dx + \int_{A\cap S(u)} g^-(x, \nu)\, d\mathcal{H}^{N-1}(x) \tag{4.46}$$

where f and g^- are the densities of $\mathcal{F}$ and $\mathcal{G}^-$ according to Propositions 4.2.1 and 4.2.3.

Proof. We know that up to a subsequence the functionals $\mathcal{E}_n(\cdot, A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to a functional $\mathcal{E}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$, and that by Proposition 4.2.5 for all $u \in SBV^p(\Omega)$ and for all $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{E}(u, A) = \int_A f_\infty(x, \nabla u)\, dx + \int_{S(u)\cap A} g_\infty(x, u^-(x), u^+(x), \nu)\, d\mathcal{H}^{N-1}(x),$$

where f_∞ and g_∞ satisfy formula (4.45). The theorem will be proved if we show that for all $u \in SBV^p(\Omega)$ we have

(a) for a.e. $x \in \Omega$
$$f_\infty(x, \nabla u(x)) = f(x, \nabla u(x)); \tag{4.47}$$

(b) for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$
$$g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) = g^-(x, \nu_{S(u)}(x)), \tag{4.48}$$

where $\nu_{S(u)}(x)$ is the normal to $S(u)$ at x.

The proof will be divided into four steps.

Step 1: $\mathbf{f_\infty(x, \nabla u(x)) \le f(x, \nabla u(x))}$ for a.e. $\mathbf{x \in \Omega}$.

This inequalty can be derived using the explicite formulas for f_∞ and f. Let $x \in \Omega$, $\xi \in \mathbb{R}^N$, and let us fix $\varepsilon > 0$. For every $\rho > 0$ let $u_{\varepsilon,\rho} \in W^{1,p}(B_\rho(x))$ be such that $u_{\varepsilon,\rho}(z) = \xi(z-x)$ in a neighborhood of $\partial B_\rho(x)$ and

$$\mathcal{F}(u_{\varepsilon,\rho}, B_\rho(x)) \le \mathbf{m}_{\mathcal{F}}(B_\rho(x), \xi(z-x)) + \varepsilon\omega_N\rho^N.$$

Then we get

$$\begin{aligned} f_\infty^\varepsilon(x,\xi) = \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{E}_\varepsilon}(B_\rho(x), \xi(z-x))}{\omega_N\rho^N} &\le \limsup_{\rho\to 0^+} \frac{\mathcal{E}(u_{\varepsilon,\rho}, B_\rho(x))}{\omega_N\rho^N} \\ &\le \limsup_{\rho\to 0^+} \frac{\mathcal{F}(u_{\varepsilon,\rho}, B_\rho(x))}{\omega_N\rho^N} \le \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(B_\rho(x), \xi(z-x))}{\omega_N\rho^N} + \varepsilon = f(x,\xi) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, we obtain that $f_\infty(x,\xi) \le f(x,\xi)$, so that the step is concluded.

Step 2: $\mathbf{f_\infty(x, \nabla u(x)) \geq f(x, \nabla u(x))}$ for a.e. $\mathbf{x \in \Omega}$.

We can consider those $x \in \Omega$ such that u is approximatively differentiable at x, x is a Lebesgue point for $f(\cdot, \xi)$ for all $\xi \in \mathbb{R}^N$ and such that

$$f_\infty(x, \nabla u(x)) = \lim_{\rho \to 0^+} \frac{\mathcal{E}(u, B_\rho(x))}{\omega_N \rho^N} < +\infty. \tag{4.49}$$

Let moreover $(u_n)_{n\in\mathbb{N}}$ be a recovering sequence for $\mathcal{E}(u, \Omega)$: by (4.22) and since $\mathcal{H}^{N-1}(K_n) \leq C$, we have that $\mathcal{H}^{N-1}(S(u_n))$ is bounded and so up to a subsequence

$$\mu_n := \mathcal{H}^{N-1} \llcorner S(u_n) \xrightarrow{*} \mu \qquad \text{weakly* in the sense of measures}$$

for some Borel measure μ. We can assume that (see for instance [7, Theorem 2.56])

$$\limsup_{\rho \to 0^+} \frac{\mu(\bar{B}_\rho(x))}{\rho^{N-1}} = 0. \tag{4.50}$$

Let $\rho_i \searrow 0$ be such that $\mathcal{E}(u, \partial B_{\rho_i}(x)) = 0$. In view of Remark 4.2.6, for every i there exists n_i such that for $n \geq n_i$

$$\begin{aligned} \frac{\mathcal{E}(u, B_{\rho_i}(x))}{\omega_N \rho_i^N} &\geq \frac{\mathcal{E}_n(u_n, B_{\rho_i}(x))}{\omega_N \rho_i^N} - \frac{1}{i} \\ &\geq \frac{\int_{B_{\rho_i}(x)} f_n(x, \nabla u_n(x))\, dx}{\omega_N \rho_i^N} - \frac{1}{i} = \frac{1}{\omega_N} \int_{B_1} f_n(x + \rho_i y, \nabla v_n^i(y))\, dy - \frac{1}{i} \end{aligned} \tag{4.51}$$

where

$$v_n^i(y) := \frac{u_n(x + \rho_i y) - u(x)}{\rho_i}.$$

Taking into account the assumptions on x and (4.50), we can choose $(n_i)_{i\in\mathbb{N}}$ is such a way that

$$v_{n_i}^i \to \nabla u(x) \cdot y \qquad \text{strongly in } L^1(B_1) \text{ for } i \to +\infty, \tag{4.52}$$

$$(\nabla v_{n_i}^i)_{i\in\mathbb{N}} \text{ is bounded in } L^p(B_1, \mathbb{R}^N), \tag{4.53}$$

$$\lim_{i \to +\infty} \mathcal{H}^{N-1}(S(v_{n_i}^i)) = 0, \tag{4.54}$$

and

$$f_\infty(x, \nabla u(x)) = \lim_{i \to +\infty} \frac{\mathcal{E}(u, B_{\rho_i}(x))}{\omega_N \rho_i^N} \geq \liminf_{i \to +\infty} \frac{1}{\omega_N} \int_{B_1} f_{n_i}(x + \rho_i y, \nabla v_{n_i}^i(y))\, dy. \tag{4.55}$$

Moreover by a truncation argument we can assume that $(v_{n_i}^i)_{i\in\mathbb{N}}$ is uniformly bounded in $L^\infty(B_1)$, so that we get

$$\|\nabla v_{n_i}^i\|_{L^p(B_1, \mathbb{R}^N)}^p + \int_{S(v_{n_i}^i)} |[v_{n_i}^i]|\, d\mathcal{H}^{N-1} \leq C \quad \text{and} \quad \lim_{i \to +\infty} \mathcal{H}^{N-1}(S(v_{n_i}^i)) = 0.$$

Following Kristensen [55] we get that there exists $w_i \in W^{1,\infty}(B_1)$ such that $w_i \to \nabla u(x) \cdot y$ strongly in $L^1(B_1)$ as $i \to +\infty$ and such that

$$\liminf_{i \to +\infty} \int_{B_1} f_{n_i}(x + \rho_i y, \nabla v_{n_i}^i(y))\, dy = \liminf_{i \to +\infty} \int_{B_1} f_{n_i}(x + \rho_i y, \nabla w_i(y))\, dy. \tag{4.56}$$

If n_i is choosen such that the blow-up for Γ-limits given by Theorem 4.1.2 holds, we get that

$$\liminf_{i\to+\infty}\int_{B_1} f_{n_i}(x+\rho_i y,\nabla w_i(y))\,dy \ge \omega_N f(x,\nabla u(x)),$$

so that in view of (4.55) we obtain

$$f_\infty(x,\nabla u(x)) \ge f(x,\nabla u(x)).$$

Step 3: $\mathbf{g_\infty(x,u^-(x),u^+(x),\nu_{S(u)}(x)) \le g^-(x,\nu_{S(u)}(x))}$ for $\mathbf{\mathcal{H}^{N-1}}$-a.e. $\mathbf{x\in S(u)}$.
Up to a subsequence, we have that

$$\mu_n := \mathcal{H}^{N-1}\llcorner K_n \stackrel{*}{\rightharpoonup} \mu$$

weakly* in the sense of measures. Since $\mathcal{H}^{N-1}(K_n)\le C$ we have that for $\mathcal{H}^{N-1}$-a.e. $x\in\Omega$ (see for instance [7, Theorem 2.56])

$$H(x) := \limsup_{\rho\to0^+}\frac{\mu(\bar{B}_\rho(x))}{\omega_{N-1}\rho^{N-1}} < +\infty. \tag{4.57}$$

We claim that for all $A\in\mathcal{A}(\Omega)$ such that $\bar{A}\subseteq\Omega$

$$\alpha\mathcal{H}^{N-1}(S(u)\cap A) \le \mathcal{G}^-(u,A)+\mu(\bar{A}). \tag{4.58}$$

In fact we have that for all $n\in\mathbb{N}$

$$\alpha\mathcal{H}^{N-1}\left((S(u)\setminus K_n)\cap A\right) \le \mathcal{G}_n^-(u,A)$$

so that

$$\alpha\mathcal{H}^{N-1}\left(S(u)\cap A\right) \le \mathcal{G}_n^-(u,A)+\mu_n(A)$$

and so passing to the Γ-limit for $n\to+\infty$ we obtain that (4.58) holds.
Let us choose $x\in S(u)$ in such a way that (4.57) holds and such that

$$\limsup_{\rho\to0^+}\frac{\int_{B_\rho(x)} a_2\,dx}{\rho^{N-1}} = 0.$$

Let us indicate $u^-(x), u^+(x)$ and $\nu_{S(u)}(x)$ simply by u^-, u^+ and ν. Let us moreover set $[u] := u^+ - u^-$.
Following Remark 4.2.4, let us consider the functionals $\mathcal{G}_n^-$ defined in (4.26) acting on the space $P_{u^-,u^+}(\Omega) := \{u\in BV(\Omega) : u(y)\in\{u^-,u^+\}$ for a.e. $y\in\Omega\}$.
Let us fix $\varepsilon>0$. For every $\rho>0$, let $u_{\varepsilon,\rho}\in P_{u^-,u^+}(B_\rho(x))$ be such that $u_{\varepsilon,\rho} = u_{x,u^-,u^+,\nu}$ in a neighborhood of $B_\rho(x)$ and

$$\mathcal{G}^-(u_{\varepsilon,\rho},B_\rho(x)) \le \mathbf{m}_{\mathcal{G}^-}(B_\rho(x),u_{x,u^-,u^+,\nu}) + \varepsilon\omega_{N-1}\rho^{N-1}.$$

Then we get in view of (4.58)

$$
\begin{aligned}
g^{\varepsilon}_{\infty}(x,u^-,u^+,\nu) &= \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{E}_\varepsilon}(B_\rho(x),u_{x,u^-,u^+,\nu})}{\omega_{N-1}\rho^{N-1}} \\
&\le \limsup_{\rho\to 0^+} \frac{\mathcal{E}(u_{\varepsilon,\rho},B_\rho(x))+\varepsilon(1+|[u]|)\mathcal{H}^{N-1}(S(u_{\varepsilon,\rho})\cap B_\rho(x))}{\omega_{N-1}\rho^{N-1}} \\
&\le \limsup_{\rho\to 0^+} \frac{\int_{B_\rho(x)} a_2\,dx + \mathcal{G}^-(u_{\varepsilon,\rho},B_\rho(x))+\varepsilon(1+|[u]|)(\mathcal{G}^-(u_{\varepsilon,\rho},B_\rho(x))+\mu(\bar B_\rho(x)))}{\omega_{N-1}\rho^{N-1}} \\
&\le \limsup_{\rho\to 0^+} \frac{(1+\varepsilon+\varepsilon|[u]|)\mathbf{m}_{\mathcal{G}^-}(B_\rho(x),u_{x,u^-,u^+,\nu})+\varepsilon(1+|[u]|)\mu(\bar B_\rho(x)))}{\omega_{N-1}\rho^{N-1}} \\
&\le \varepsilon+(1+\varepsilon+\varepsilon|[u]|)g^-(x,\nu)+\varepsilon(1+|[u]|)H(x).
\end{aligned}
$$

Letting $\varepsilon\to 0$ we obtain $g_\infty(x,u^-,u^+,\nu)\le g^-(x,\nu)$, so that the step is concluded.

Step 4: $\mathbf{g_\infty(x,u^-(x),u^+(x),\nu_{S(u)}(x))\ge g^-(x,\nu_{S(u)}(x))}$ for $\mathcal{H}^{N-1}$-a.e. $\mathbf{x\in S(u)}$.
Let us choose $x\in S(u)$ which is an approximate jump point for u,

$$
g_\infty(x,u^-(x),u^+(x),\nu_{S(u)}(x)) = \lim_{\rho\to 0^+}\frac{\mathcal{E}(u,B_\rho(x))}{\omega_{N-1}\rho^{N-1}}<+\infty, \tag{4.59}
$$

and such that

$$
\lim_{\rho\to 0^+}\frac{\int_{B_\rho(x)}|a_1(y)|\,dy}{\rho^{N-1}}=0, \tag{4.60}
$$

where a_1 is defined in (4.21).
Since $\mathcal{H}^{N-1}(K_n)\le C$, up to a subsequence we have

$$
\mu_n := \mathcal{H}^{N-1}\llcorner K_n \stackrel{*}{\rightharpoonup} \mu \qquad \text{weakly}^* \text{ in the sense of measures}
$$

for some Borel measure μ. We can assume that (see for instance [7, Theorem 2.56])

$$
\limsup_{\rho\to 0^+}\frac{\mu(B_\rho(x))}{\rho^{N-1}}<+\infty. \tag{4.61}
$$

Let $(u_n)_{n\in\mathbb{N}}$ be a recovering sequence for $\mathcal{E}(u,\Omega)$, and let $\rho_i\searrow 0$ be such that $\mathcal{E}(u,\partial B_{\rho_i}(x))=0$. For every $i\in\mathbb{N}$ there exists $n_i\in\mathbb{N}$ such that for $n\ge n_i$ we have

$$
\begin{aligned}
\frac{\mathcal{E}(u,B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}} &\ge \frac{\mathcal{E}_n(u_n,B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}}-\frac{1}{i} \\
&\ge \frac{\int_{B_{\rho_i}(x)\cap[S(u_n)\setminus K_n]} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x)}{\omega_{N-1}\rho_i^{N-1}}+\frac{\int_{B_{\rho_i}(x)}a_1(y)\,dy}{\omega_{N-1}\rho_i^{N-1}}-\frac{1}{i} \\
&= \frac{1}{\omega_{N-1}}\int_{B_1\cap[S(v_n^i)\setminus K_n^i]} g_n(x+\rho_i y,\nu)\,d\mathcal{H}^{N-1}(y)+\frac{\int_{B_{\rho_i}(x)}a_1(y)\,dy}{\omega_{N-1}\rho_i^{N-1}}-\frac{1}{i}
\end{aligned} \tag{4.62}
$$

where

$$
v_n^i(y) := u_n(x+\rho_i y) \quad \text{and} \quad K_n^i := \frac{\{K_n\cap B_{\rho_i}(x)\}-x}{\rho_i}. \tag{4.63}
$$

We claim that we can find w_n^i piecewise constant in B_1 such that for $n \to +\infty$

$$w_n^i \to w^i \qquad \text{strongly in } L^1(B_1),$$

where w^i is piecewise constant and $w^i = u_{0,0,1,\nu_{S(u)}(x)}$ in a neighborhood of the boundary, and such that for n large

$$\int_{B_1 \cap [S(v_n^i) \setminus K_n^i]} g_n(x + \rho_i y, \nu)\, d\mathcal{H}^{N-1}(y) \geq \int_{B_1 \cap [S(w_n^i) \setminus K_n^i]} g_n(x + \rho_i y, \nu)\, d\mathcal{H}^{N-1}(y) - e_i \tag{4.64}$$

with $e_i \to 0$ for $i \to +\infty$.

Using the claim, by (4.62), (4.64) and (4.60) we have that for n large

$$g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) \geq \frac{\int_{B_{\rho_i} \cap [S(z_n^i) \setminus K_n]} g_n(\zeta, \nu)\, d\mathcal{H}^{N-1}(\zeta)}{\omega_{N-1}\rho_i^{N-1}} - \hat{e}_i = \frac{\mathcal{G}_n^-(z_n^i, B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}} - \hat{e}_i$$

where $\hat{e}_i \to 0$ and

$$z_n^i(\zeta) := w_n^i\left(\frac{\zeta - x}{\rho_i}\right) \to z^i(\zeta) := w^i\left(\frac{\zeta - x}{\rho_i}\right) \qquad \text{strongly in } L^1(B_{\rho_i}(x)).$$

By the Γ-convergence assumption on $\mathcal{G}_n^-$, using Γ-liminf inequality we have that

$$g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) \geq \frac{\mathcal{G}^-(z^i, B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}} - \hat{e}_i \geq \frac{\mathbf{m}_{\mathcal{G}^-}(B_{\rho_i}, u_{x,0,1,\nu_{S(u)}(x)})}{\omega_{N-1}\rho_i^{N-1}} - \hat{e}_i.$$

Letting $i \to +\infty$, and recalling the representation formula (4.35) for $g^-(x, \nu)$, we have that the result is proved.

In order to complete the proof of the step, we have to prove the claim. Since

$$\nabla v_n^i(y) = \rho_i \nabla u_n(x + \rho_i y),$$

we get by the coercivity assumption (4.21)

$$\int_{B_1} |\nabla v_n^i(y)|^p\, dy = \rho_i^p \int_{B_1} |\nabla u_n(x + \rho_i y)|^p\, dy = \rho_i^p \frac{\int_{B_{\rho_i}(x)} |\nabla u_n(z)|^p\, dz}{\rho_i^N}$$
$$\leq \frac{\rho_i^{p-1}}{\alpha}\left(\frac{\mathcal{E}_n(u_n, B_{\rho_i}(x))}{\rho_i^{N-1}} - \frac{\int_{B_{\rho_i}(x)} a_1(y)\, dy}{\rho_i^{N-1}}\right).$$

Since u_n is optimal for u we have that

$$\frac{\mathcal{E}_n(u_n, B_{\rho_i}(x))}{\rho_i^{N-1}} \xrightarrow{n \to +\infty} \frac{\mathcal{E}(u, B_{\rho_i}(x))}{\rho_i^{N-1}} \xrightarrow{i \to +\infty} \omega_{N-1} g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) < +\infty.$$

In view also of (4.60), we conclude that we can choose n_i so that for $n \geq n_i$.

$$\int_{B_1} |\nabla v_n^i(y)|^p\, dy \leq C\rho_i^{p-1} \tag{4.65}$$

for some constant $C \geq 0$. By Coarea formula for BV functions (see [7, Theorem 3.40]) we get

$$\int_{u^-(x)}^{u^+(x)} \mathcal{H}^{N-1}\left(\partial^* E_n^i(t) \setminus S(v_n^i)\right) dt \leq \int_{B_1} |\nabla v_n^i| \, dy \leq \tilde{C} \rho_i^{1-\frac{1}{p}},$$

for a suitable constant $\tilde{C}$, where

$$E_n^i(t) := \{x \in B_1 \, : \, x \text{ is a Lebesgue point for } v_n^i \text{ and } v_n^i(x) > t\} \tag{4.66}$$

and ∂^* denotes the reduced boundary. By the Mean Value Theorem there exists $t_n^i \in [u^-(x), u^+(x)]$ such that

$$\mathcal{H}^{N-1}\left(\partial^* E_n^i(t_n^i) \setminus S(v_n^i)\right) \leq \frac{\tilde{C}}{u^+(x) - u^-(x)} \rho_i^{1-\frac{1}{p}}. \tag{4.67}$$

We now employ a construction similar to that employed by Francfort and Larsen in their Transfer of Jump Sets Theorem [47, Theorem 2.3]. Since x is a jump point for u we have that for $i \to +\infty$

$$u(x + \rho_i y) \to u_{0,u^-(x),u^+(x),\nu_{S(u)}(x)} \qquad \text{strongly in } L^1(B_1).$$

Then we have that for n large

$$|B_1^+ \triangle E_n^i(t_n^i)| \leq e_i,$$

where $B_1^+ := \{y \in B_1 \, : \, y \cdot \nu_{S(u)}(x) \geq 0\}$, $A \triangle B := (A \setminus B) \cup (B \setminus A)$, and $e_i \to 0$ for $i \to +\infty$. By Fubini's Theorem we have

$$\int_0^{\sqrt{e_i}} \mathcal{H}^{N-1}\left((B_1^+ \setminus E_n^i(t_n^i)) \cap H^+(s)\right) ds \leq \int_{-\infty}^{+\infty} \mathcal{H}^{N-1}\left((B_1^+ \setminus E_n^i(t_n^i)) \cap H^+(s)\right) ds \leq e_i,$$

where $H^+(s) := \{y \in B_1 \, : \, y \cdot \nu_{S(u)}(x) = s\}$, and by the Mean Value Theorem we get that there exists $0 < s_n^{i,+} < \sqrt{e_i}$ such that setting $H_n^{i,+} := H^+(s_n^{i,+})$ we have

$$\mathcal{H}^{N-1}\left((B_1^+ \setminus E_n^i(t_n^i)) \cap H_n^{i,+}\right) \leq \sqrt{e_i}.$$

Similarly we obtain $-\sqrt{e_i} < s_n^{i,-} < 0$ such that setting $H_n^{i,-} := H^+(s_n^{i,-})$ we have

$$\mathcal{H}^{N-1}\left((E_n^i(t_n^i) \setminus B_1^+) \cap H_n^{i,-}\right) \leq \sqrt{e_i}.$$

Let us write $y = (y', y_N)$, where y_N is the coordinate along $\nu_{S(u)}(x)$ and y' the coordinates in the hyperplain orthogonal to $\nu_{S(u)}(x)$. Let l_i be such that for every $y \in B_1$

$$|y_N| \geq 2\sqrt{e_i} \Longrightarrow |y'| \leq 1 - l_i.$$

Let us set

$$D_n^i := (E_n^i(t_n^i) \cup \{y \in B_1 \, : \, y_N \geq s_n^{i,+}\}) \setminus \{y \in B_1 \, : \, y_N \leq s_n^{i,-}\}.$$

We set

$$w_n^i := \begin{cases} 1 & |y'| \geq 1 - l_i; y_N \geq 0, \\ 0 & |y'| \geq 1 - l_i, y_N < 0, \\ 1 & |y'| \leq 1 - l_i, y \in D_n^i, \\ 0 & \text{otherwise}. \end{cases} \tag{4.68}$$

Notice that w_i^n is piecewise constant, with $w_n^i = u_{0,0,1,\nu_{S(u)}(x)}$ in a neighborhood of the boundary, and such that

$$\int_{B_1\cap[S(v_n^i)\setminus K_n^i]} g_n(x+\rho_i y,\nu)\,d\mathcal{H}^{N-1}(y) \geq \int_{B_1\cap[S(w_n^i)\setminus K_n^i]} g_n(x+\rho_i y,\nu)\,d\mathcal{H}^{N-1}(y) - \tilde{e}_i \tag{4.69}$$

with $\tilde{e}_i \to 0$ for $i \to +\infty$.

In view of (4.67) and of the assumption (4.61) we have that $\mathcal{H}^{N-1}(S(w_n^i)) \leq C_i$ uniformly in n for some finite constant C_i. By Ambrosio's Compactness Theorem we get for $n \to +\infty$

$$w_n^i \to w^i \qquad \text{strongly in } L^1(B_1),$$

where w^i is piecewise constant and $w^i = u_{0,0,1,\nu_{S(u)}(x)}$ in a neighborhood of the boundary, so that the claim is proved.

□

Remark 4.3.2. Theorem 4.3.1 states that in the Γ-limit process there is no interaction between bulk and surface energies, since they are constructed looking at Γ-convergence problems in Sobolev space and in the space of piecewise constant functions respectively. As a consequence, considering bulk and surface energies of the form $c_1 f_n$ and $c_2 g_n$ with $c_1, c_2 > 0$, we get in the limit $c_1 f$ and $c_2 g$ as bulk and surface energy densities. We remark that a key assumption for non interaction is given by equi-boundness of $\mathcal{H}^{N-1}(K_n)$: dropping this assumption, interaction can occur even in the case of constant densities, for example $f(\xi) := |\xi|^p$ and $g(x,\nu) \equiv 1$ (if we consider in $]0,1[$ the set $K_n := \{\frac{i}{n} : i = 1,\dots,n-1\}$, we get as Γ-limit the zero functional). As mentioned in the Introduction, non interaction between bulk and surface energies was noticed in the case of periodic homogenization (with $K_n = \emptyset$) by Braides, Defranceschi and Vitali in [13].

In the rest of this section we employ Theorem 4.3.1 to obtain a lower semicontinuity result for SBV functions in the case of varying bulk and surface energies in the same spirit of Ambrosio's lower semicontinuity theorem [3].

From Theorem 4.3.1 we get that the following semicontinuity result holds.

Proposition 4.3.3. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that $\mathcal{H}^{N-1}(K_n) \leq C$ for all $n \in \mathbb{N}$. Let us assume that for all $A \in \mathcal{A}(\Omega)$ the functionals $\mathcal{F}_n(\cdot,A)$ and $\mathcal{G}_n^-(\cdot,A)$ defined in (4.25) and (4.26) Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot,A)$ and $\mathcal{G}^-(\cdot,A)$ respectively. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^p(\Omega)$ such that*

$$u_n \rightharpoonup u \qquad \text{weakly in } SBV^p(\Omega).$$

Then for all $A \in \mathcal{A}(\Omega)$ we have

$$\int_A f(x,\nabla u(x))\,dx \leq \liminf_{n\to+\infty} \int_A f_n(x,\nabla u_n(x))\,dx, \tag{4.70}$$

and

$$\int_{S(u)\cap A} g^-(x,\nu)\,d\mathcal{H}^{N-1} \leq \liminf_{n\to+\infty} \int_{(S(u_n)\setminus K_n)\cap A} g_n(x,\nu)\,d\mathcal{H}^{N-1}, \tag{4.71}$$

where f and g^- are the densities of $\mathcal{F}$ and $\mathcal{G}^-$ respectively.

Moreover let us assume $\mathcal{G}_n(\cdot,A)$ defined in (4.27) Γ-converges in the strong topology of $L^1(\Omega)$ to $\mathcal{G}(\cdot,A)$. Then

$$\int_{S(u)\cap A} g(x,\nu)\,d\mathcal{H}^{N-1} \leq \liminf_{n\to+\infty} \int_{S(u_n)\cap A} g_n(x,\nu)\,d\mathcal{H}^{N-1}, \tag{4.72}$$

where g is the density of $\mathcal{G}$.

Proof. By Theorem 4.3.1, we have that for all $h, k \in \mathbb{N}$ and for all $A \in \mathcal{A}(\Omega)$ the functionals

$$\mathcal{E}_n^{h,k}(u, A) := h \int_A f_n(x, \nabla u(x))\, dx + k \int_{(S(u)\setminus K_n)\cap A} g_n(x, \nu)\, d\mathcal{H}^{N-1}$$

Γ-converge in the strong topology of $L^1(\Omega)$ to

$$\mathcal{E}^{h,k}(u, A) := h \int_A f(x, \nabla u(x))\, dx + k \int_{S(u)\cap A} g^-(x, \nu)\, d\mathcal{H}^{N-1}.$$

In particular by Γ-liminf inequality we have

$$\mathcal{E}^{h,k}(u, A) \leq \liminf_{n\to+\infty} \mathcal{E}_n^{h,k}(u_n, A).$$

Then we get

$$\begin{aligned}\int_A f(x, \nabla u(x))\, dx &\leq \liminf_{n\to+\infty} \int_A f_n(x, \nabla u_n(x))\, dx + \frac{k}{h} \int_{(S(u_n)\setminus K_n)\cap A} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x) \\ &\leq \liminf_{n\to+\infty} \int_A f_n(x, \nabla u_n(x))\, dx + \frac{k}{h} C\end{aligned}$$

for some constant C independent of h and k. Since h, k are arbitrary we get that (4.70) holds. The proof of (4.71) is analogous. Finally (4.72) derives from (4.71) in the case $K_n \equiv \emptyset$. □

4.4 A new variational convergence for rectifiable sets

In this section we use the Γ-convergence results of Section 4.3 to introduce a variational notion of convergence for rectifiable sets which will be employed in the study of stability of unilateral minimality properties.

Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω, and let us assume following Ambrosio and Braides [5, Theorem 3.2] that the functionals $\mathcal{H}_n^- : P(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty)$ defined by

$$\mathcal{H}_n^-(u, A) := \mathcal{H}^{N-1}\left((S(u) \setminus K_n) \cap A\right) \tag{4.73}$$

Γ-converge with respect to the strong topology of $L^1(\Omega)$ for every $A \in \mathcal{A}(\Omega)$ to a functional $\mathcal{H}^-(\cdot, A)$, which by the representation result of Bouchitté, Fonseca, Leoni and Mascarenhas [11, Theorem 3] is of the form

$$\mathcal{H}^-(u, A) := \int_{S(u)\cap A} h^-(x, \nu)\, d\mathcal{H}^{N-1}(x) \tag{4.74}$$

for some function $h^- : \Omega \times S^{N-1} \to [0, +\infty)$.

Definition 4.4.1 (**σ-convergence of rectifiable sets**). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω. We say that K_n σ-converges in Ω to K if the functionals $(\mathcal{H}_n^-)_{n\in\mathbb{N}}$ defined in (4.73) Γ-converge in the strong topology of $L^1(\Omega)$ to the functional $\mathcal{H}^-$ defined in (4.74), and K is the unique rectifiable set in Ω such that*

$$h^-(x, \nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K, \tag{4.75}$$

and such that for every rectifiable set $H \subseteq \Omega$ we have

$$h^-(x, \nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in H \Longrightarrow H \tilde{\subseteq} K. \tag{4.76}$$

In order to prove the main properties of σ-convergence of rectifiable sets, we need the following covering argument.

Lemma 4.4.2. *Let $H \subseteq \Omega$ be a rectifiable set with $\mathcal{H}^{N-1}(H) < +\infty$. Then for all $\varepsilon > 0$ there exist an open set $U \in \mathcal{A}(\Omega)$ and $u \in P(\Omega)$ such that $\mathcal{H}^{N-1}(H \setminus U) < \varepsilon$ and*

$$\mathcal{H}^{N-1}\left((S(v) \triangle H) \cap U\right) < \varepsilon,$$

where $\triangle$ denotes the symmetric difference of sets.

Proof. Since H is rectifiable, we have that $H = H_0 \cup \bigcup_{i \in \mathbb{N}} K_i$, where $\mathcal{H}^{N-1}(H_0) = 0$, K_i is compact, and $K_i \subseteq M_i$ for a suitable C^1 hypersurface M_i of $\mathbb{R}^N$. For all $i \in \mathbb{N}$, let us denote by $\tilde{K}_i$ the set of point x such that x has $(N-1)$-dimensional density 1 with respect to K_i. We have that $\mathcal{H}^{N-1}(H \setminus \bigcup_{i \in \mathbb{N}} \tilde{K}_i) = 0$.

Let us fix $\varepsilon > 0$. Then for all $x \in \tilde{K}_i$, there exists $\rho(x) > 0$ such that for all $\rho < \rho(x)$ we have

$$\omega_{N-1}\rho^{N-1} < (1+\varepsilon)\mathcal{H}^{N-1}\left(\tilde{K}_i \cap B_\rho(x)\right) \tag{4.77}$$

and

$$\mathcal{H}^{N-1}\left((M_i \setminus \tilde{K}_i) \cap B_\rho(x)\right) < \varepsilon\omega_{N-1}\rho^{N-1}, \tag{4.78}$$

Since M_i is of class C^1, we can assume that $\rho(x)$ is so small that $B_\rho(x) \setminus M_i$ has exactly two connected components $B^+_\rho(x)$ and $B^-_\rho(x)$ for every $\rho < \rho(x)$.

We can apply now the Vitali-Besicovitch Covering Theorem (see [7, Theorem 2.19]), and deduce that there exists a disjoint family of balls $(B_{\rho_j}(x_j))_{j \in \mathbb{N}}$ such that

$$\mathcal{H}^{N-1}\Big(H \setminus \bigcup_{j \in \mathbb{N}} B_{\rho_j}(x_j)\Big) = 0.$$

Let us choose $n \in \mathbb{N}$ such that

$$\mathcal{H}^{N-1}\Big(H \setminus \bigcup_{j=0}^{n} B_{\rho_i}(x_i)\Big) \le \varepsilon,$$

and let us set $U := \bigcup_{j=0}^{n} B_{\rho_i}(x_i)$. Let us consider $u \in P(\Omega)$ defined setting $u = 1$ in $B^+_{\rho_j}(x_j)$, $j = 1, \dots, n$, and $u = 0$ otherwise. We have that

$$(H \triangle S(u)) \cap U \subseteq \bigcup_{j \in \mathbb{N}} (M_{i_j} \setminus \tilde{K}_{i_j}) \cap B_{\rho_j}(x_j),$$

where K_{i_j} is the compact set relative to x_j, and M_{i_j} is the associated $(N-1)$-dimensional hypersurface. In view of (4.77) and (4.78) we conclude that

$$\mathcal{H}^{N-1}((H \triangle S(u)) \cap U) < \varepsilon(1+\varepsilon)\mathcal{H}^{N-1}(H),$$

so that the theorem is proved. ◻

Let us now come to the main properties of σ-convergence for rectifiable sets. By compactness of Γ-convergence, we deduce the following compactness result for σ-convergence.

Proposition 4.4.3 (compactness). *Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of rectifiable sets in Ω with $\mathcal{H}^{N-1}(K_n) \le C$. Then there exists a subsequence $(n_h)_{h \in \mathbb{N}}$ and a rectifiable set K in Ω such that K_{n_h} σ-converges in Ω to K. Moreover*

$$\mathcal{H}^{N-1}(K) \le \liminf_{n \to +\infty} \mathcal{H}^{N-1}(K_n). \tag{4.79}$$

Proof. By Proposition 4.2.3, up to a subsequence we have that for all $A \in \mathcal{A}(\Omega)$ the functionals $\mathcal{H}_n^-(\cdot, A)$ defined in (4.73) Γ-converge in the strong topology of $L^1(\Omega)$ to a functional $\mathcal{H}^-(\cdot, A)$ which can be represented through a density h^- according to (4.74).

Let us consider the class

$$\mathcal{K} := \{H \subseteq \Omega : H \text{ is rectifiable and } h^-(x, \nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in H\}.$$

Notice that $\mathcal{K}$ contains at least the empty set. Moreover for all $H \in \mathcal{K}$ we have

$$\mathcal{H}^{N-1}(H) \le L := \liminf_{n \to +\infty} \mathcal{H}^{N-1}(K_n). \tag{4.80}$$

In fact let $H \in \mathcal{K}$. Since $H = \cup_i H_i$ with H_i compact and rectifiable with $\mathcal{H}^{N-1}(H_i) < +\infty$, it is not restrictive to consider $\mathcal{H}^{N-1}(H) < +\infty$. Given $\varepsilon > 0$, by Lemma 4.4.2 we can find an open set U and a piecewise constant function $v \in P(\Omega)$ such that

$$\mathcal{H}^{N-1}(H \setminus U) < \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}\left((S(v) \triangle H) \cap U\right) < \varepsilon,$$

where $\triangle$ denotes the symmetric difference of sets. Since $h^- \le 1$ we have

$$\mathcal{H}^-(v, U) = \int_{S(v)\cap U} h^-(x,\nu)\, d\mathcal{H}^{N-1}(x) = \int_{(S(v)\setminus H)\cap U} h^-(x,\nu)\, d\mathcal{H}^{N-1}(x) < \varepsilon.$$

Let $(v_n)_{n\in\mathbb{N}}$ be a recovering sequence for v with respect to $\mathcal{H}^-(\cdot, U)$. Then we have that

$$\limsup_{n\to+\infty} \mathcal{H}^{N-1}\left((S(v_n) \setminus K_n) \cap U\right) < \varepsilon.$$

By Ambrosio's Theorem we deduce that

$$\begin{aligned}\mathcal{H}^{N-1}(H) &\le \mathcal{H}^{N-1}(H \cap U) + \mathcal{H}^{N-1}(H \setminus U) \le \mathcal{H}^{N-1}(S(v) \cap U) + 2\varepsilon \\ &\le \liminf_{n\to+\infty} \mathcal{H}^{N-1}(S(v_n) \cap U) + 2\varepsilon \le \liminf_{n\to+\infty} \mathcal{H}^{N-1}(K_n) + 3\varepsilon = L + 3\varepsilon.\end{aligned}$$

Since ε is arbitrary we get that (4.80) holds.

Let us now consider

$$\tilde{L} := \sup\{\mathcal{H}^{N-1}(H) \,:\, H \in \mathcal{K}\} < +\infty,$$

and let $(H_k)_{k\in\mathbb{N}}$ be a maximizing sequence for $\tilde{L}$. We set

$$K := \bigcup_{k=1}^{\infty} H_k.$$

Clearly (4.79) and (4.75) hold. Moreover, since $\mathcal{H}^{N-1}(K) = \tilde{L}$ we have that (4.76) holds, and the proof is concluded. □

Remark 4.4.4. Let $\Omega := (-1,1) \times (-1,1)$ in $\mathbb{R}^2$, and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of closed sets with $K_n \to K := \{(-1,1)\} \times \{0\}$ in the Hausdorff metric and such that

$$\mathcal{H}^1 \llcorner K_n \stackrel{*}{\rightharpoonup} a\mathcal{H}^1 \llcorner K$$

weakly* in the sense of measures. If $a < 1$ by (4.79) we deduce that K_n σ-converges in Ω to the empty set. We stress that the condition $a \geq 1$ is not enough to guarantee that K is the σ-limit of $(K_n)_{n\in\mathbb{N}}$ because the σ-limit is also affected by the behavior of the normal vectors to K_n. In fact considering

$$K_n := \bigcup_{i=-n}^{n} \left\{\frac{i}{n}\right\} \times \left[-\frac{1}{n}, \frac{1}{n}\right]$$

we have

$$\mathcal{H}^1 \llcorner K_n \overset{*}{\rightharpoonup} 2\mathcal{H}^1 \llcorner K$$

weakly* in the sense of measures. However also in this case we have that K_n σ-converges in Ω to the empty set. In fact let us consider $u \in P(\Omega)$ such that $u = 1$ in $\Omega^+ := (-1,1) \times (0,1)$ and $u = 0$ in $\Omega^- := (-1,1) \times (-1,0)$, and let u_n be a sequence in $P(\Omega)$ such that $u_n \to u$ strongly in $L^1(\Omega)$ and with $\mathcal{H}^{N-1}(S(u_n)) \leq C$. Let (e_1, e_2) be the canonical base of $\mathbb{R}^2$. By Ambrosio's theorem we get that

$$\nu[u_n]\mathcal{H}^1 \llcorner S(u_n) \overset{*}{\rightharpoonup} e_2\mathcal{H}^1 \llcorner S(u)$$

weakly* in the sense of measures. Considering the vector field φe_2 with $\varphi \in C_c^\infty(\Omega)$ we get

$$\int_{S(u_n)\setminus K_n} \varphi e_2 \cdot \nu[u_n]\, d\mathcal{H}^1 = \int_{S(u_n)} \varphi e_2 \cdot \nu[u_n]\, d\mathcal{H}^1 \to \int_K \varphi\, d\mathcal{H}^1.$$

Since φ is arbitrary, we deduce that $\liminf_{n\to+\infty} \mathcal{H}_n^-(u_n) = \liminf_{n\to+\infty} \mathcal{H}^1(S(u_n) \setminus K_n) \geq 1$. By Γ-liminf we conclude that $\mathcal{H}^-(u) = 1$ that is $h^-(x, e_2) = 1$ for $\mathcal{H}^1$-a.e. $x \in K$. Since the σ-limit of $(K_n)_{n\in\mathbb{N}}$ can be only contained in K, we deduce that the σ-limit is the empty set.

The following proposition shows that the σ-limit is a natural limit candidate for a sequence of rectifiable sets in connection with unilateral minimality properties (see the Introduction).

Proposition 4.4.5. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω with K_n σ-converging in Ω to K. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of Borel functions satisfying the growth estimates (4.22), and let g^- be the energy density of the Γ-limit in the strong topology of $L^1(\Omega)$ of the functionals $(\mathcal{G}_n^-)_{n\in\mathbb{N}}$ defined in (4.26). Then we have*

$$g^-(x, \nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K, \tag{4.81}$$

and for every rectifiable set $H \subseteq \Omega$

$$g^-(x, \nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in H \Longrightarrow H \tilde{\subseteq} K. \tag{4.82}$$

Proof. By growth estimates on g_n we have for all $u \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$

$$\alpha\mathcal{H}^-(u, A) \leq \mathcal{G}^-(u, A) \leq \beta\mathcal{H}^-(u, A).$$

The proof follows if we prove that for all rectifiable set $H \subseteq \Omega$

$$h^-(x, \nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in H \tag{4.83}$$

is equivalent to

$$g^-(x, \nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in H. \tag{4.84}$$

Let us show that (4.83) implies (4.84), the reverse implication being similar. It is not restrictive to assume $\mathcal{H}^{N-1}(H) < +\infty$. Given $\varepsilon > 0$, by Lemma 4.4.2 we can find an open set U and a piecewise constant function $v \in P(\Omega)$ such that

$$\mathcal{H}^{N-1}(H \setminus U) < \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}\left((S(v) \triangle H) \cap U\right) < \varepsilon,$$

where $\triangle$ denotes the symmetric difference of sets. Then we get

$$\int_H g^-(x,\nu)\,d\mathcal{H}^{N-1}(x) = \int_{H\cap U} g^-(x,\nu)\,d\mathcal{H}^{N-1}(x) + \int_{H\setminus U} g^-(x,\nu)\,d\mathcal{H}^{N-1}(x)$$
$$\le \int_{S(v)\cap U} g^-(x,\nu)\,d\mathcal{H}^{N-1}(x) + 2\beta\varepsilon \le \beta\int_{S(v)\cap U} h^-(x,\nu)\,d\mathcal{H}^{N-1}(x) + 2\beta\varepsilon = 2\beta\varepsilon.$$

Since ε is arbitrary we get that

$$g^-(x,\nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x\in H$$

so that the proof is concluded. □

The following proposition is essential in the study of stability of unilateral minimality properties.

Proposition 4.4.6. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that K_n σ-converges in Ω to K. Let $1<p<+\infty$, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^p(\Omega)$ with $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$ and $\mathcal{H}^{N-1}(S(u_n)\setminus K_n)\to 0$. Then $S(u)\tilde{\subseteq} K$.*

Proof. By lower semicontinuity given by Proposition 4.3.3 we have

$$\int_{S(u)} h^-(x,\nu)\,d\mathcal{H}^{N-1}(x) \le \liminf_{n\to+\infty}\mathcal{H}^{N-1}\left(S(u_n)\setminus K_n\right) = 0.$$

We deduce that

$$h^-(x,\nu_{S(u)}(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x\in S(u),$$

so that by definition of σ-limit we deduce $S(u)\tilde{\subseteq} K$. □

In the next corollary, we prove that our σ-limit always contains the σ^p-limit introduced by Dal Maso, Francfort and Toader in [35] to study quasistatic crack growth in nonlinear elasticity (see the section of preliminaries for a definition).

Corollary 4.4.7. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that K_n σ-converges in Ω to K. Let $1<p<+\infty$, and let us assume that K_n σ^p-converges in Ω to some rectifiable set $\tilde{K}$. Then $\tilde{K}\tilde{\subseteq} K$.*

Proof. Recall that by definition of σ^p-convergence we have $K = S(z)$ for some $z\in SBV^p(\Omega)$, and there exists $(z_n)_{n\in\mathbb{N}}$ sequence in $SBV^p(\Omega)$ with $z_n \rightharpoonup z$ weakly in $SBV^p(\Omega)$ and $S(z_n)\tilde{\subseteq} K_n$. The result follows applying Proposition 4.4.6 to $(z_n)_{n\in\mathbb{N}}$. □

Remark 4.4.8. Notice that is general we can have that the σ^p-limit $\tilde{K}$ of $(K_n)_{n\in\mathbb{N}}$ is strictly contained in K. In fact we can consider $\Omega := (-1,1)\times(-1,1)$ in $\mathbb{R}^2$, and

$$K_n := \{(-1,1)\setminus L_n\}\times\{0\}$$

with $L_n\subseteq(-1,1)$ and $|L_n|\to 0$. In this case we get $K=(-1,1)\times\{0\}$, while if L_n is chosen in such a way that its *c_p-capacity* is big enough (see the celebrated example of the Neumann sieve, we refer to [60]) we get $\tilde{K}=\emptyset$.

This example is based on the fact that the σ^p-limit is influenced by infinitesimal perturbations of the K_n, while the set K is not. To be precise we have that if $\mathcal{H}^{N-1}(K_n\triangle K_n')\to 0$, and $K_n\to K$ in the sense of σ-convergence, then $\tilde{K}_n$ still σ-converges to K.

The following lower semicontinuity result for surface energies along sequences of rectifiable sets converging in the sense of σ-convergence will be employed in Section 4.7.

Proposition 4.4.9 (lower semicontinuity). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that K_n σ-converges in Ω to K. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of Borel functions satisfying the growth estimates* (4.22), *and let g be the associated function according to Proposition 4.2.2. Then we have*

$$\int_K g(x,\nu)\, d\mathcal{H}^{N-1}(x) \leq \liminf_{n\to+\infty} \int_{K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x).$$

Proof. Let $H \tilde{\subseteq} K$ with $\mathcal{H}^{N-1}(H) < +\infty$. Given $\varepsilon > 0$, by Lemma 4.4.2 we can find an open set U and a piecewise constant function $v \in P(\Omega)$ such that

$$\mathcal{H}^{N-1}(H \setminus U) < \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}\left((S(v) \,\triangle\, H) \cap U\right) < \varepsilon,$$

where $\triangle$ denotes the symmetric difference of sets. If $(v_n)_{n\in\mathbb{N}}$ is a recovering sequence for v with respect to $\mathcal{H}^{-}(\cdot, U)$ defined in (4.74) we have

$$\limsup_{n\to+\infty} \mathcal{H}^{N-1}\left((S(v_n) \setminus K_n) \cap U\right) < \varepsilon.$$

We deduce by the lower semicontinuity result of Proposition 4.3.3 that

$$\begin{aligned}
\int_H g(x,\nu)\, d\mathcal{H}^{N-1}(x) &= \int_{H\cap U} g(x,\nu)\, d\mathcal{H}^{N-1}(x) + \int_{H\setminus U} g(x,\nu)\, d\mathcal{H}^{N-1}(x) \\
&\leq \int_{S(v)\cap U} g(x,\nu)\, d\mathcal{H}^{N-1}(x) + 2\beta\varepsilon \leq \liminf_{n\to+\infty} \int_{S(v_n)\cap U} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) + 2\beta\varepsilon \\
&\leq \liminf_{n\to+\infty} \int_{K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) + 3\beta\varepsilon.
\end{aligned}$$

Since ε is arbitrary we deduce

$$\int_H g(x,\nu)\, d\mathcal{H}^{N-1}(x) \leq \liminf_{n\to+\infty} \int_{K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x),$$

and since H is arbitrary in K the proof is concluded. □

In Section 4.6 and Section 4.7, we will need a definition of σ-convergence in the closed set $\overline{\Omega}$.

Definition 4.4.10 (σ-convergence in $\overline{\Omega}$). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in $\overline{\Omega}$. We say that K_n σ-converges in $\overline{\Omega}$ to $K \tilde{\subseteq} \overline{\Omega}$ if K_n σ-converges in Ω' to K for every open bounded set Ω' such that $\overline{\Omega} \subseteq \Omega'$.*

Notice that to check the σ-convergence in $\overline{\Omega}$ of rectifiable sets, it is enough check σ-convergence in Ω' for just one Ω' with $\overline{\Omega} \subseteq \Omega'$.

4.5 Stability of unilateral minimality properties

In this section we apply the results of Section 4.3 and Section 4.4 to obtain the stability result of unilateral minimality properties under Γ-convergence for bulk and surface energies.

Definition 4.5.1 (unilateral minimizers). *Let $f : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function and let $g : \Omega \times S^{N-1} \to [0, +\infty[$ be a Borel function satisfying the growth estimates (4.21) and (4.22). We say that the pair (u, K) with $u \in SBV^p(\Omega)$ and K rectifiable set in Ω is a unilateral minimizer with respect to f and g if $S(u) \tilde{\subseteq} K$, and*

$$\int_\Omega f(x, \nabla u(x))\, dx \le \int_\Omega f(x, \nabla v(x))\, dx + \int_{H\setminus K} g(x, \nu),$$

for all pairs (v, H) with $v \in SBV^p(\Omega)$, H rectifiable set in Ω such that $S(v) \tilde{\subseteq} H$ and $K \tilde{\subseteq} H$.

As in the previous sections, let $f_n : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function and let $g_n : \Omega \times S^{N-1} \to [0, +\infty[$ be a Borel function satisfying the growth estimates (4.21) and (4.22).

Let us assume that the functionals $(\mathcal{F}_n(\cdot, A))_{n\in\mathbb{N}}$ and $(\mathcal{G}_n(\cdot, A))_{n\in\mathbb{N}}$ defined in (4.25) and (4.27) Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot, A)$ and $\mathcal{G}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$ respectively. Let f be the density of $\mathcal{F}$ according to Proposition 4.2.1 and let g be the density of $\mathcal{G}$ according to Proposition 4.2.2.

The main result of the chapter is the following stability result for unilateral minimality properties under σ-convergence of rectifiable sets (see Definition 4.4.1), and Γ-convergence of bulk and surface energies.

Theorem 4.5.2. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^p(\Omega)$ with $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$, and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω with $\mathcal{H}^{N-1}(K_n) \le C$ and such that K_n σ-converges in Ω to K.*

Let us assume that the pair $(u_n, K_n)_{n\in\mathbb{N}}$ is a unilateral minimizer for f_n and g_n. Then (u, K) is a unilateral minimizer for f and g. Moreover we have

$$\lim_{n\to+\infty} \int_\Omega f_n(x, \nabla u_n(x))\, dx = \int_\Omega f(x, \nabla u(x))\, dx. \tag{4.85}$$

Proof. By Theorem 4.3.1 we have that the functionals

$$\mathcal{E}_n(u) := \begin{cases} \int_\Omega f_n(x, \nabla u(x))\, dx + \int_{S(u)\setminus K_n} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x) & u \in SBV^p(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converge with respect to the strong topology of $L^1(\Omega)$ to the functional

$$\mathcal{E}(u) := \begin{cases} \int_\Omega f(x, \nabla u(x))\, dx + \int_{S(u)} g^-(x, \nu)\, d\mathcal{H}^{N-1}(x) & u \in SBV^p(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where f and g^- are defined in (4.31) and (4.35) respectively, with $g^- \le g$.

By Proposition 4.4.6 we have $S(u) \tilde{\subseteq} K$, so that u is admissible for K, while by Proposition 4.4.5 we have that

$$g^-(x, \nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K.$$

Then the unilateral minimality of the pair (u, K) easily follows. In fact, by Γ-convergence we have that u is a minimizer for $\mathcal{E}$ and $\mathcal{E}_n(u_n) \to \mathcal{E}(u)$. Then for all pairs (v, H) with $S(v) \tilde{\subseteq} H$ and $K \tilde{\subseteq} H$ we have

$$\begin{aligned} \int_\Omega f(x, \nabla u(x))\, dx = \mathcal{E}(u) \le \mathcal{E}(v) &= \int_\Omega f(x, \nabla v(x))\, dx + \int_{S(v)} g^-(x, \nu)\, d\mathcal{H}^{N-1} \\ &= \int_\Omega f(x, \nabla v(x))\, dx + \int_{S(v)\setminus K} g^-(x, \nu) \le \int_\Omega f(x, \nabla v(x))\, dx + \int_{H\setminus K} g(x, \nu), \end{aligned}$$

so that the unilateral minimality property holds. The convergence of bulk energies (4.85) is given by the convergence $\mathcal{E}_n(u_n) \to \mathcal{E}(u)$. □

Remark 4.5.3 (stability under σ^p-convergence). In the case of fixed bulk and surface energies f and g, Dal Maso, Francfort and Toader [35] proved the stability of the unilateral minimality property under σ^p-convergence for the rectifiable sets K_n (see the section of preliminaries for the definition). This result readily follows by Theorem 4.5.2. In fact by Corollary 4.4.7 we have that if K_n σ^p-converges in Ω to $\tilde{K}$, then $\tilde{K}$ is contained in the σ-limit of $(K_n)_{n\in\mathbb{N}}$. Since $S(u) \tilde{\subseteq} \tilde{K}$, we get that the unilateral minimality of the pair $(u, \tilde{K})$ is implied by the unilateral minimality of (u, K).

As mentioned in the Introduction, a method for proving stability of unilateral minimality properties nearer to the approach of [35] would be to prove a generalization of the Transfer of Jump Sets by Francfort and Larsen [47, Theorem 2.1] to the case of varying energies. The following theorem based on the arguments of Section 4.3 provides such a generalization.

Theorem 4.5.4 (Transfer of Jump Sets). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω with $\mathcal{H}^{N-1}(K_n) \le C$ and K_n σ-converging in Ω to K. For every $v \in SBV^p(\Omega)$ there exists $(v_n)_{n\in\mathbb{N}}$ sequence in $SBV^p(\Omega)$ with $v_n \rightharpoonup v$ weakly in $SBV^p(\Omega)$ and such that*

$$\lim_{n\to+\infty} \int_\Omega f_n(x, \nabla v_n(x))\, dx = \int_\Omega f(x, \nabla v(x))\, dx$$

and

$$\limsup_{n\to+\infty} \int_{S(v_n)\setminus K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) \le \int_{S(v)\setminus K} g(x,\nu)\, d\mathcal{H}^{N-1}(x).$$

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a recovering sequence for v with respect to $(\mathcal{E}_n)_{n\in\mathbb{N}}$ defined in (4.23). By growth estimates on f_n and g_n, and since $\mathcal{H}^{N-1}(K_n) \le C$, we get $v_n \rightharpoonup v$ weakly in $SBV^p(\Omega)$. Since no interaction between bulk and surface energies occurs in view of Theorem 4.3.1, we get that

$$\lim_{n\to+\infty} \int_\Omega f_n(x, \nabla v_n(x))\, dx = \int_\Omega f(x, \nabla v(x))\, dx$$

and

$$\lim_{n\to+\infty} \int_{S(v_n)\setminus K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1} = \int_{S(v)} g^-(x,\nu)\, d\mathcal{H}^{N-1} \le \int_{S(v)\setminus K} g(x,\nu)\, d\mathcal{H}^{N-1}$$

because $g^- = 0$ on K, and $g^- \le g$. □

4.6 Stability under boundary conditions

In view of the application of Section 4.7, we need a stability result for unilateral minimality properties with boundary conditions.

In order to set the problem, let us consider $\partial_D\Omega \subseteq \partial\Omega$, let $f_n : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function satisfying the growth estimate (4.21), and let $g_n : \overline{\Omega} \times S^{N-1} \to [0, +\infty[$ be a Borel function satisfying the growth estimate (4.22). We consider unilateral minimality properties of the form

$$\int_\Omega f_n(x, \nabla u_n)\, dx \le \int_\Omega f_n(x, \nabla v)\, dx + \int_{H\setminus K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) \tag{4.86}$$

for every $v \in SBV^p(\Omega)$ and for every rectifiable set H in $\overline{\Omega}$ such that $S^{\psi_n}(v) \tilde{\subseteq} H$. Here $(K_n)_{n\in\mathbb{N}}$ is a sequence of rectifiable sets in $\overline{\Omega}$ with $\mathcal{H}^{N-1}(K_n) \le C$, $(u_n)_{n\in\mathbb{N}}$ is a sequence in $SBV^p(\Omega)$ with $S^{\psi_n}(u_n) \tilde{\subseteq} K_n$, $\psi_n \in W^{1,p}(\Omega)$ with $\psi_n \to \psi$ strongly in $W^{1,p}(\Omega)$, and $S^{\psi_n}(\cdot)$ is defined in (4.20).

In order to treat $S^{\psi_n}(\cdot)$ as an internal jump and in order to recover the surface energy on $\partial_D\Omega$ for the minimality property in the limit, let us consider an open bounded set Ω' such that $\overline{\Omega} \subset \Omega'$ and let us consider $g'_n : \Omega' \times S^{N-1} \to [0, +\infty[$ such that

$$g'_n(x,\nu) := \begin{cases} g_n(x,\nu) & \text{if } x \in \overline{\Omega}, \\ \beta + 1 & \text{otherwise.} \end{cases} \tag{4.87}$$

Let us consider the functionals $\mathcal{G}'_n : P(\Omega') \times \mathcal{A}(\Omega') \to [0, +\infty]$ defined by

$$\mathcal{G}'_n(v, A) := \int_{S(v)\cap A} g'_n(x,\nu)\, d\mathcal{H}^{N-1}(x)$$

and let $\mathcal{G}' : P(\Omega') \times \mathcal{A}(\Omega') \to [0, +\infty]$ be their Γ-limit in the strong topology of $L^1(\Omega')$, which according to Proposition 4.2.3 is of the form

$$\mathcal{G}'(v, A) := \int_{S(v)\cap A} g'(x,\nu)\, d\mathcal{H}^{N-1}(x) \tag{4.88}$$

We clearly have $g'(x,\nu) = g(x,\nu)$ for $x \in \Omega$, where g is the surface energy density defined in (4.33), while it turns out that (see Remark 4.6.2) the surface energy given by the restriction of g' to $\partial\Omega \times S^{N-1}$ is completely determined by the functions g_n.

Let us set

$$f'_n(x,\xi) := \begin{cases} f_n(x,\xi) & \text{if } x \in \Omega, \\ \alpha|\xi|^p & \text{otherwise,} \end{cases} \tag{4.89}$$

and let f' be the energy density of the Γ-limit of the functionals on $W^{1,p}(\Omega')$ associated to f'_n according to Proposition 4.2.1. We easily have that

$$f'(x,\xi) := \begin{cases} f(x,\xi) & \text{if } x \in \Omega, \\ \alpha|\xi|^p & \text{otherwise.} \end{cases} \tag{4.90}$$

Since Ω is Lipschitz, we can assume using an extension operator that $\psi_n, \psi \in W^{1,p}(\mathbb{R}^N)$ and $\psi_n \to \psi$ strongly in $W^{1,p}(\mathbb{R}^N)$.

Before stating our stability result, we need the following Γ-convergence result.

Lemma 4.6.1. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in $\overline{\Omega}$ such that $\mathcal{H}^{N-1}(K_n) \le C$. Let us assume that the functionals*

$$\mathcal{E}'_n(v) := \begin{cases} \int_{\Omega'} f'_n(x, \nabla v(x))\, dx + \int_{S(v)\setminus K_n} g'_n(x,\nu)\, d\mathcal{H}^{N-1}(x) & \textit{if } v \in SBV^p(\Omega'), \\ +\infty & \textit{otherwise} \end{cases} \tag{4.91}$$

Γ-converge in the strong topology of $L^1(\Omega')$ according to Theorem 4.3.1 to

$$\mathcal{E}'(v) := \begin{cases} \int_{\Omega'} f'(x, \nabla v(x))\, dx + \int_{S(v)} g'^{-}(x,\nu)\, d\mathcal{H}^{N-1}(x) & \textit{if } v \in SBV^p(\Omega'), \\ +\infty & \textit{otherwise.} \end{cases} \tag{4.92}$$

Then we have that the functionals

$$\tilde{\mathcal{E}}'_n(v) := \begin{cases} \mathcal{E}'_n(v) & \textit{if } v = \psi_n \textit{ on } \Omega' \setminus \overline{\Omega}, \\ +\infty & \textit{otherwise} \end{cases} \tag{4.93}$$

Γ-converge in the strong topology of $L^1(\Omega')$ to

$$\tilde{\mathcal{E}}'(v) := \begin{cases} \mathcal{E}'(v) & \text{if } v = \psi \text{ on } \Omega' \setminus \overline{\Omega}, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.94}$$

Proof. Let $v \in SBV^p(\Omega')$ with $v = \psi$ on $\Omega' \setminus \overline{\Omega}$, and let $(v_n)_{n\in\mathbb{N}}$ be a recovering sequence for v with respect to the functionals $\mathcal{E}'_n$. We have that

$$\nabla v_n \to \nabla \psi \qquad \text{strongly in } L^p(\Omega' \setminus \overline{\Omega}; \mathbb{R}^N), \tag{4.95}$$

and

$$\mathcal{H}^{N-1}(S(v_n) \cap (\Omega' \setminus \overline{\Omega})) \to 0. \tag{4.96}$$

In fact we have that for all $U \in \mathcal{A}(\Omega')$ such that $\overline{U} \subseteq \Omega' \setminus \overline{\Omega}$ and $\mathcal{E}'(v, \partial U) = 0$

$$\nabla v_n \to \nabla \psi \qquad \text{strongly in } L^p(U; \mathbb{R}^N), \tag{4.97}$$

and

$$\mathcal{H}^{N-1}(S(v_n) \cap U) \to 0. \tag{4.98}$$

Let $\varepsilon > 0$ and let us consider an open set $V \in \mathcal{A}(\Omega')$ such that $\partial\Omega \subseteq V$, $\mathcal{E}'(v, \partial V) = 0$, $\int_{V\cap\Omega} |a_1|\, dx < \varepsilon$ (a_1 is defined in (4.21)),

$$\int_V f'(x, \nabla v(x))\, dx < \varepsilon \quad \text{and} \quad \int_V f'(x, \nabla\psi(x))\, dx < \varepsilon. \tag{4.99}$$

Then for n large (no interaction between bulk and surface part occurs) we have

$$\int_V f'_n(x, \nabla v_n(x))\, dx < \varepsilon. \tag{4.100}$$

Notice that

$$\begin{aligned} \int_{\Omega'\setminus\overline{\Omega}} |\nabla v_n - \nabla\psi|^p\, dx &= \int_{\Omega'\setminus(\Omega\cup V)} |\nabla v_n - \nabla\psi|^p\, dx + \int_{V\setminus\overline{\Omega}} |\nabla v_n - \nabla\psi|^p\, dx \\ &\leq \int_{\Omega'\setminus(\Omega\cup V)} |\nabla v_n - \nabla\psi|^p\, dx + \frac{2^{p-1}}{\alpha} \int_V f'_n(x, \nabla v_n(x)) + f'(x, \nabla\psi(x))\, dx + \frac{2^{p-1}}{\alpha} \int_{V\cap\Omega} 2|a_1|\, dx. \end{aligned}$$

Since $\nabla v_n \to \nabla\psi$ strongly in $L^p(\Omega' \setminus (\Omega \cup V); \mathbb{R}^N)$, because of (4.99) and (4.100), and since ε is arbitrary, we get that (4.95) holds.

Let us come to (4.96). Up to a subsequence we have

$$\mu_n := \mathcal{H}^{N-1} \llcorner (S(v_n) \cap (\Omega' \setminus \overline{\Omega})) \stackrel{*}{\rightharpoonup} \mu \quad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega').$$

In view of (4.98), in order to prove (4.96) it is sufficient to show that $\mu(\partial\Omega) = 0$. Let us assume by contradiction that $\mu(\partial\Omega) \neq 0$: then there exists a cube Q_ρ of center $x \in \partial\Omega$ and edge 2ρ such that $\mathcal{E}'(v, \partial Q_\rho) = 0$ and

$$\mu(Q_\rho) > \sigma > 0. \tag{4.101}$$

Up to a translation we may assume that $x = 0$, and moreover we can assume that

$$\Omega \cap Q_\rho = \{(x', y) : x' \in (-\rho, \rho), y \in (-\rho, h(x'))\},$$

where (x', y) is a suitable orthogonal coordinate system and h is a Lipschitz function. Let $\eta > 0$ be such that setting

$$V_\eta := \{(x', y) : x' \in (-\rho, \rho),\ y \in (h(x') - \eta, h(x') + \eta)\}$$

we have $V_\eta \subseteq Q_\rho$, and $\mathcal{E}'(v, \partial V_\eta) = 0$. Let us set

$$V_\eta^- := \{(x', y) \in V_\eta : y < h(x')\} \quad \text{and} \quad V_\eta^+ := \{(x', y) \in V_\eta : y > h(x')\}.$$

By (4.101) we have that for n large

$$\mathcal{H}^{N-1}(S(v_n) \cap V_\eta^+) > \sigma. \tag{4.102}$$

Let $\hat{v}$ be the function defined on V_η obtained reflecting $v_{|V_\eta^+}$ to V_η^-: more precisely let us set

$$\hat{v} = \begin{cases} v(x', y) & \text{if } (x', y) \in V_\eta^+, \\ v(x', 2h(x') - y) & \text{if } (x', y) \in V_\eta^-. \end{cases}$$

We clearly have $v \in W^{1,p}(V_\eta)$. Let $\hat{v}_n$ be obtained in the same way from $(v_n)_{|V_\eta^+}$. Let us consider

$$w_n := v_n + \hat{v} - \hat{v}_n.$$

We have $w_n \rightharpoonup v$ weakly in $SBV^p(V_\eta)$ so that by lower semicontinuity given by Proposition 4.3.3 we get

$$\int_{S(v) \cap V_\eta} g'^-(x, \nu)\, d\mathcal{H}^{N-1}(x) \le \liminf_{n \to +\infty} \int_{(S(w_n) \setminus K_n) \cap V_\eta} g_n'(x, \nu)\, d\mathcal{H}^{N-1}(x). \tag{4.103}$$

On the other hand, since $\mathcal{E}'(v, \partial V_\eta) = 0$, we have that v_n is a recovering sequence for v in V_η. In particular we get that

$$\int_{S(v) \cap V_\eta} g'^-(x, \nu)\, d\mathcal{H}^{N-1}(x) = \lim_{n \to +\infty} \int_{(S(v_n) \setminus K_n) \cap V_\eta} g_n'(x, \nu)\, d\mathcal{H}^{N-1}(x). \tag{4.104}$$

Formulas (4.103) and (4.104) give a contradiction because for n large by (4.102) and since $K_n \tilde{\subseteq} \overline{\Omega}$ and $S(w_n) \tilde{\subseteq} \overline{\Omega} \cap Q_\rho$ (recall that $g_n'(x, \nu) = \beta + 1$ for $x \in \Omega' \setminus \overline{\Omega}$)

$$\int_{(S(v_n) \setminus K_n) \cap V_\eta} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x) - \int_{(S(w_n) \setminus K_n) \cap V_\eta} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x) > \sigma.$$

We conclude that (4.96) holds.

We are now in a position to prove the Γ-limsup inequality for $\tilde{\mathcal{E}}_n'$ and $\tilde{\mathcal{E}}'$ (the Γ-liminf is immediate from the Γ-convergence of $\mathcal{E}_n'$ to $\mathcal{E}'$ and the fact that the constraint is closed under the strong topology of $L^1(\Omega)$). Let $\varepsilon > 0$, and let $U \in \mathcal{A}(\Omega')$ be such that $\partial\Omega \subseteq U$, $\mathcal{E}'(v, \partial U) = 0$, and

$$\int_U f(x, \nabla v)\, dx < \varepsilon. \tag{4.105}$$

In view of (4.95) and (4.96) we can find $\varphi_n \in SBV^p(\Omega')$ such that $\varphi_n = \psi_n - v_n$ on $\Omega' \setminus \overline{\Omega}$, $\varphi_n = 0$ on $\Omega \setminus U$ and

$$\begin{aligned} &\varphi_n \to 0 \quad \text{strongly in } L^1(\Omega'), \\ &\nabla\varphi_n \to 0 \quad \text{strongly in } L^p(\Omega'; \mathbb{R}^N), \\ &\mathcal{H}^{N-1}(S(\varphi_n)) \to 0. \end{aligned}$$

Let us consider

$$\tilde{v}_n := v_n + \varphi_n.$$

We have $\tilde{v}_n = \psi_n$ on $\Omega' \setminus \overline{\Omega}$. Moreover

$$\limsup_{n\to+\infty} \int_{S(\tilde{v}_n)\setminus K_n} g_n'(x,\nu)\, d\mathcal{H}^{N-1} = \limsup_{n\to+\infty} \int_{S(v_n)\setminus K_n} g_n'(x,\nu)\, d\mathcal{H}^{N-1},$$

and using the growth estimate on f_n'

$$\limsup_{n\to+\infty} \left| \int_{\Omega'} f_n'(x,\nabla\tilde{v}_n(x))\,dx - \int_{\Omega'} f_n'(x,\nabla v_n(x))\,dx \right| \le \limsup_{n\to+\infty} \int_{U\cap\Omega} f_n(x,\nabla\tilde{v}_n(x)) + f_n(x,\nabla v_n(x))\,dx$$

$$\le \limsup_{n\to+\infty} \int_U a_2(x)\,dx + \left(\frac{2^{p-1}}{\alpha}+1\right)\int_U f_n(x,\nabla v_n(x))\,dx + \frac{2^{p-1}}{\alpha}\int_U |a_1|\,dx + 2^{p-1}\int_U |\nabla\varphi_n|^p\,dx.$$

By (4.105) we get

$$\limsup_{n\to+\infty} \int_U f_n(x,\nabla v_n(x))\,dx < \varepsilon.$$

Then we conclude

$$\limsup_{n\to+\infty} \left| \int_{\Omega'} f_n'(x,\nabla\tilde{v}_n(x))\,dx - \int_{\Omega'} f_n'(x,\nabla v_n(x))\,dx \right| \le e(\varepsilon)$$

with $e(\varepsilon) \to 0$ as $\varepsilon \to 0$. We deduce that

$$\limsup_{n\to+\infty} \tilde{\mathcal{E}}'(\tilde{v}_n) \le \tilde{\mathcal{E}}'(v) + e(\varepsilon),$$

with $e(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since ε is arbitrary, using a diagonal argument we have that the Γ-limsup inequality is proved. □

Remark 4.6.2. In view of Lemma 4.6.1 we can prove that the surface energy determined by the restriction of g' to $\partial\Omega$ is actually independent of the choice of Ω' and of the constant value c' of g_n' on $\Omega' \setminus \overline{\Omega}$ provided that $c' > \beta$. In fact g' is the density of the surface energy of the Γ-limit in the strong topology of $L^1(\Omega)$ of the functionals on $SBV^p(\Omega')$ defined as

$$\hat{\mathcal{E}}_n'(v) := \int_{\Omega'} f_n'(x,\nabla v(x))\,dx + \int_{S(v)} g_n'(x,\nu)\,d\mathcal{H}^{N-1}(x).$$

Following the proof of Lemma 4.6.1 (for the functionals $\mathcal{E}_n'$ with $K_n = \emptyset$), if $v = \psi$ outside $\overline{\Omega}$, we can find $(v_n)_{n\in\mathbb{N}}$ recovering sequence for v with respect to $(\hat{\mathcal{E}}_n', \Omega', c')$ such that $v_n = \psi_n$ outside $\overline{\Omega}$. Then if Ω'' is an open set such that $\overline{\Omega} \subseteq \Omega''$ we have that $(v_n)_{|\Omega'\cap\Omega''}$ is a recovering sequence also for $(\hat{\mathcal{E}}_n', \Omega''\cap\Omega', c'')$, and we have

$$\int_{S(v)} g'(x,\nu)\,d\mathcal{H}^{N-1} = \lim_{n\to+\infty} \int_{S(v_n)} g_n(x,\nu)\,d\mathcal{H}^{N-1}.$$

We deduce that the surface energy given by the restriction of g' to $\overline{\Omega}\times S^{N-1}$ is determined only by the $g_n : \overline{\Omega}\times S^{N-1} \to [0,+\infty]$.

The stability result for unilateral minimality properties with boundary conditions under σ-convergence in $\overline{\Omega}$ for rectifiable sets (see Definition 4.4.10) and Γ-convergence of bulk and surface energies is the following.

Theorem 4.6.3. *Let $\psi_n \in W^{1,p}(\Omega)$ with $\psi_n \to \psi$ strongly in $W^{1,p}(\Omega)$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^p(\Omega)$ with $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$, and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in $\overline{\Omega}$ with $\mathcal{H}^{N-1}(K_n) \leq C$, such that K_n σ-converges in $\overline{\Omega}$ to K, and $S^{\psi_n}(u_n) \tilde{\subseteq} K_n$.*

Let us assume that the pair (u_n, K_n) satisfies the unilateral minimality property (4.86) with respect to f_n, g_n and ψ_n. Then (u, K) satisfies the unilateral minimality property with respect to f, g and ψ, where f is defined in (4.31) and g is the restriction of g' defined in (4.88) to $\overline{\Omega} \times S^{N-1}$. Moreover we have

$$\lim_{n\to+\infty} \int_\Omega f_n(x, \nabla u_n(x))\,dx = \int_\Omega f(x, \nabla u(x))\,dx. \tag{4.106}$$

Proof. Since the boundary datum ψ_n is imposed just on $\partial_D\Omega$, we can consider $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$ as part of the cracks, that is we can replace in the unilateral minimality properties K_n with $K_n' := K_n \cup \partial_N\Omega$.

It is easy to prove that K_n' σ-converges in $\overline{\Omega}$ to $K \cup \partial_N\Omega$. Then the proof follows that of Theorem 4.5.2 employing the functionals $(\tilde{\mathcal{E}}_n')_{n\in\mathbb{N}}$ defined in Lemma 4.6.1 with K_n' in place of K_n. □

4.7 Quasistatic evolution of cracks in composite materials

The aim of this section is to apply the stability results of Section 4.6 to the study the asymptotic behavior of crack evolutions relative to varying bulk and surface energies f_n and g_n. As mentioned in the Introduction, this problem is inspired by the problem of crack propagation in composite materials. We restrict our analysis to the case of antiplanar shear, where the elastic body is an infinite cylinder.

Let us recall the result of Dal Maso, Francfort and Toader [35] about quasistatic crack evolution in nonlinear elasticity: it is a very general existence and approximation result concerning a variational theory crack propagation inspired by the variational model introduced by Francfort and Marigo in [48]. As already said, we consider the antiplanar case and for simplicity we neglect body and traction forces, and so we adapt the mathematical tools employed in [35] to this scalar setting.

As in the previous sections, let $\Omega \subset \mathbb{R}^N$ (which, for $N = 2$ represents a section of the cylindrical hyperelastic body) be an open bounded set with Lipschitz boundary. The family of admissible cracks is the class of rectifiable subsets of $\overline{\Omega}$, while the class of admissible displacements is given by the functional space $SBV^p(\Omega)$, where $1 < p < +\infty$. Let $\partial_D\Omega$ be a subset of $\partial\Omega$. Given $\psi \in W^{1,p}(\Omega)$, we say that the displacement u is admissible for the fracture K and the boundary datum ψ and we write $u \in AD(\psi, K)$ if

$$S(v) \tilde{\subseteq} K \qquad \text{and} \qquad v = \psi \qquad \text{on } \partial_D\Omega \setminus K.$$

This can be summarized by the notation $S^\psi(u) \tilde{\subseteq} K$, where $S^\psi(\cdot)$ is defined in (4.20).

Let $f(x,\xi) : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function which is convex and C^1 in ξ for a.e. $x \in \Omega$, and satisfies the growth estimate

$$a_1(x) + \alpha|\xi|^p \leq f(x,\xi) \leq a_2(x) + \beta|\xi|^p, \tag{4.107}$$

where $a_1, a_2 \in L^1(\Omega)$ and $\alpha, \beta > 0$. Let moreover $g : \overline{\Omega} \times S^{N-1} \to [0, +\infty[$ be a Borel function such that

$$\alpha \leq g(x,\nu) \leq \beta. \tag{4.108}$$

The total energy of a configuration (u, K) is given by

$$\mathcal{E}(u,K) := \int_\Omega f(x, \nabla u(x))\,dx + \int_K g(x,\nu)\,d\mathcal{H}^{N-1}(x).$$

We will usually refer to the first term as *bulk energy* of u and we write

$$\mathcal{E}^b(u) := \int_\Omega f(x, \nabla u(x))\, dx, \tag{4.109}$$

while we will refer to the second term as *surface energy* of K and we write

$$\mathcal{E}^s(K) := \int_K g(x,\nu) d\mathcal{H}^{N-1}(x). \tag{4.110}$$

Let us consider now a time dependent boundary datum $\psi \in W^{1,1}([0,T]; W^{1,p}(\Omega))$ (i.e. the function $t \to \psi(t)$ is absolutely continuous from $[0,T]$ to the Banach space $W^{1,p}(\Omega)$, with summable time derivative, see for instance [14]), such that for all $t \in [0,T]$

$$\|\psi(t)\|_{L^\infty(\Omega)} \le C. \tag{4.111}$$

In [35] Dal Maso, Francfort and Toader proved the esistence of an *irreversible quasistatic crack evolution* in Ω relative to the boundary displacement ψ, i.e. the existence of a map $t \to (u(t), K(t))$ where $u(t) \in AD(\psi(t), K(t))$, $\|u(t)\|_{L^\infty(\Omega)} \le \|\psi(t)\|_\infty$ and such that the following three properties hold:

(1) *irreversibility:* $K(t_1) \tilde{\subseteq} K(t_2)$ for all $0 \le t_1 \le t_2 \le T$;

(2) *static equilibrium:* $\mathcal{E}(u(0), K(0)) \le \mathcal{E}(v,K)$ for all (v,K) such that $v \in AD(\psi(0), K)$, and

$$\mathcal{E}(u(t), K(t)) \le \mathcal{E}(v,K) \qquad \text{for all } K(t) \tilde{\subseteq} K,\ v \in AD(\psi(t), K);$$

(3) *nondissipativity:* the function $t \to \mathcal{E}(u(t), K(t))$ is absolutely continuous and

$$\frac{d}{dt}\mathcal{E}(u(t), K(t)) = \int_\Omega \nabla_\xi f(x, \nabla u(t)) \nabla \dot{\psi}(t)\, dx,$$

where $\dot{\psi}$ denotes the time derivative of $t \to \psi(t)$.

For every $n \in \mathbb{N}$ let us consider admissible bulk and surface energies $f_n : \Omega \times \mathbb{R}^N \to \mathbb{R}$ and $g_n : \Omega \times S^{N-1} \to [0, +\infty[$ for the model of [35] satisfying the growth estimates (4.107) and (4.108) uniformly in n. Let us moreover assume that f_n is such that for a.e. $x \in \Omega$ and for all $M \ge 0$

$$|\nabla_\xi f_n(x, \xi_n^1) - \nabla_\xi f_n(x, \xi_n^2)| \to 0 \tag{4.112}$$

for all ξ_n^1, ξ_n^2 such that $|\xi|_n^1 \le M$ $|\xi|_n^2 \le M$ and $|\xi_n^1 - \xi_n^2| \to 0$. We denote by $\mathcal{E}_n$, $\mathcal{E}_n^b$ and $\mathcal{E}_n^s$ the total, bulk and surface energies associated to f_n and g_n.

Let f and g be the effective energies associated to f_n and g_n in the sense of Theorem 4.6.3, i.e. let f be given by Proposition 4.2.1 and let g be the restriction to $\overline{\Omega} \times S^{N-1}$ of the function g' defined in (4.88). Notice that by Theorem 4.1.5 we have that the function $f(x,\cdot)$ is C^1: as it is also convex in ξ and satisfies the growth estimate (4.107), we have that f and g are admissible bulk and surface energies for the model of [35].

Let $t \to \psi_n(t)$ be a sequence of admissible time dependent boundary displacements such that

$$\psi_n \to \psi \qquad \text{strongly in } W^{1,1}\Big([0,T], H^1(\Omega)\Big).$$

Let $t \to (u_n(t), K_n(t))$ be a quasistatic evolution for the boundary datum ψ_n relative to the energies f_n and g_n according to [35]. The main result of this section is the following Theorem which asserts that the σ-limit in $\overline{\Omega}$ of $K_n(t)$ (see Definition 4.4.10) still determines a quasistatic crack growth with respect to the energies f and g.

Theorem 4.7.1. *There exists a quasistatic crack growth $t \to (u(t), K(t))$ relative to the energies f and g and the boundary datum ψ such that up to a subsequence (not rabelled) the following hold:*

(1) for all $t \in [0,T]$

$$K_n(t) \ \sigma\text{-converges in } \overline{\Omega} \text{ to } K(t),$$

and there exists a further subsequence n_k (depending possibly on t) such that

$$u_{n_k}(t) \rightharpoonup u(t) \qquad \text{weakly in } SBV^p(\Omega);$$

(2) for every $t \in [0,1]$ we have convergence of total energies

$$\mathcal{E}_n(u_n(t), K_n(t)) \to \mathcal{E}(u(t), K(t)),$$

and in particular separate convergence for bulk and surface energies, i.e.

$$\mathcal{E}^b_n(u_n(t)) \to \mathcal{E}^b(u(t)) \quad \text{and} \quad \mathcal{E}^s(K_n(t)) \to \mathcal{E}^s(K(t)).$$

Proof. Notice that by nondissipativity for $t \to (u_n(t), K_n(t))$ and by growth estimates on f_n and g_n we have that there exists a constant C such that for all $t \in [0,T]$ and for all $n \in \mathbb{N}$

$$\|\nabla u_n(t)\|^p + \mathcal{H}^{N-1}(K_n(t)) + \|u_n(t)\|_{L^\infty(\Omega)} \le C. \tag{4.113}$$

We divide the proof in several steps.

Step 1: Compactness for the cracks. In view of (4.113), using a variant of Helly's theorem (see for instance [36, Theorem 6.3] for the case of Hausdorff converging compact sets), we can find a subsequence (not rabelled) of $(K_n(\cdot))_{n\in\mathbb{N}}$ and an increasing map $t \to K(t)$ such that for all $t \in [0,T]$

$$K_n(t) \ \sigma\text{-converges in } \overline{\Omega} \text{ to } K(t). \tag{4.114}$$

Step 2: Compactness for the displacements. Notice that the sequence $(u_n(t))_{n\in\mathbb{N}}$ is relatively compact in $SBV^p(\Omega)$ by (4.113). We now want to select a particular limit point of this sequence. With this aim, let us consider

$$\vartheta_n(t) := \int_\Omega \nabla_\xi f_n(x, \nabla u_n(t)) \nabla \dot\psi_n(t)\, dx,$$

and let us set

$$\vartheta(t) := \limsup_{n\to+\infty} \vartheta_n(t). \tag{4.115}$$

Let us see that there exists $u(t) \in SBV^p(\Omega)$ such that

$$\vartheta(t) = \int_\Omega \nabla_\xi f(x, \nabla u(t)) \nabla \dot\psi(t)\, dx \tag{4.116}$$

and

$$u_{n_k}(t) \rightharpoonup u(t) \qquad \text{weakly in } SBV^p(\Omega) \tag{4.117}$$

for a suitable subsequence n_k depending on t. In fact let us consider a subsequence n_k such that

$$\vartheta(t) = \lim_{k\to+\infty} \int_\Omega \nabla_\xi f(x, \nabla u_{n_k}(t)) \nabla \dot\psi_{n_k}(t)\, dx,$$

and

$$u_{n_k}(t) \rightharpoonup u \qquad \text{weakly in } SBV^p(\Omega).$$

By static equilibrium for $(u_n(t), K_n(t))$ we have that

$$\int_\Omega f_{n_k}(x, \nabla u_{n_k}(t))\,dx \le \int_\Omega f_{n_k}(x, \nabla v(x))\,dx + \int_{H\setminus K_{n_k}(t)} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x)$$

for all $v \in AD(\psi_{n_k}(t), H)$ with $K_{n_k}(t) \tilde{\subseteq} H$. Then by Theorem 4.6.3 we get that

$$\int_\Omega f(x, \nabla u)\,dx \le \int_\Omega f(x, \nabla v(x))\,dx + \int_{H\setminus K(t)} g(x,\nu)\,d\mathcal{H}^{N-1}(x) \tag{4.118}$$

for all $v \in AD(\psi(t), H)$ with $K(t) \tilde{\subseteq} H$ and

$$\int_\Omega f_{n_k}(x, \nabla u_{n_k}(t))\,dx \to \int_\Omega f(x, \nabla u)\,dx. \tag{4.119}$$

We claim that

$$\lim_{k\to+\infty} \int_\Omega \nabla_\xi f_{n_k}(x, \nabla u_{n_k}(t))\nabla\Phi\,dx = \int_\Omega \nabla_\xi f(x,\nabla u)\nabla\Phi\,dx \tag{4.120}$$

for all $\Phi \in W^{1,p}(\Omega)$. This has been done in [35, Lemma 4.11] in the case of fixed bulk energy, and our proof is just a variant based on the Γ-convergence results of Section 4.3 and on assuption (4.112) which permit to deal with varying energies. Let us consider $s_j \searrow 0$ and $k_j \to +\infty$: up to a further subsequence for k_j we can assume that

$$\int_\Omega \frac{f(x,\nabla u(x) + s_j\nabla\Phi(x)) - f(x,\nabla u(x))}{s_j}\,dx - \frac{1}{j} \le \int_\Omega \nabla_\xi f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t) + \tilde{s}_j\nabla\Phi)\nabla\Phi\,dx$$

where $\tilde{s}_j \in [0, s_j]$. This comes from lower semicontinuity for bulk energies under Γ-convergence given by Proposition 4.3.3, and by Lagrange's Theorem. By Lemma 4.1.4 we have

$$\liminf_{j\to+\infty} \int_\Omega \nabla_\xi f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t) + \tilde{s}_j\nabla\Phi)\nabla\Phi\,dx = \liminf_{j\to+\infty} \int_\Omega \nabla_\xi f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t))\nabla\Phi\,dx,$$

so that we get

$$\int_\Omega \nabla_\xi f(x,\nabla u)\nabla\Phi\,dx \le \liminf_{j\to+\infty} \int_\Omega \nabla_\xi f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t))\nabla\Phi\,dx.$$

Changing Φ with $-\Phi$, we get that (4.120) is proved: setting $u(t) := u$ we deduce that (4.116) and (4.117) hold.

Step 3: Conclusion. Let us consider $t \to (u(t), K(t))$ with $u(t)$ and $K(t)$ defined in Step 2 and Step 1 respectively. In order to see that $t \to (u(t), K(t))$ is a quasistatic crack evolution we have to check the admissibility condition $u(t) \in AD(\psi(t), K(t))$ for all t, and the properties ofirreversibility, static equilibrium and nondissipativity conditions with respect to f and g.

As for admissibility, this is guaranteed by (4.114) and (4.117) which ensures that $S^{\psi(t)}(u(t)) \tilde{\subseteq} K(t)$. *Irreversibility* is given by construction in Step 1, and *static equilibrium* comes from (4.118) for $t \in (0,T]$, and by Lemma 4.6.1 (where we take $K_n = \emptyset$) for $t = 0$. As for *nondissipativity*, we have that static equilibrium implies that (see [35]) for all $t \in [0,T]$

$$\mathcal{E}(u(t), K(t)) \ge \mathcal{E}(u(0), K(0)) + \int_0^t \int_\Omega \nabla_\xi f(x, \nabla u(\tau))\nabla\dot{\psi}(\tau)\,dx\,d\tau.$$

On the other hand by lower semicontinuity given by Proposition 4.3.3 and by Proposition 4.4.9 (applied to g' from which g is obtained by restriction) we have for all $t \in [0,T]$

$$\mathcal{E}(u(t),K(t)) \leq \liminf_{n\to+\infty} \mathcal{E}_n(u_n(t),K_n(t)),$$

and by Γ-convergence given by Lemma 4.6.1 (where we take $K_n = \emptyset$)

$$\mathcal{E}(u(0),K(0)) = \lim_{n\to+\infty} \mathcal{E}_n(u_n(0),K_n(0)).$$

Hence we get for all $t \in [0,T]$ (applying also Fatou's Lemma in the limsup version)

$$\begin{aligned}\mathcal{E}(u(t),K(t)) &\leq \liminf_{n\to+\infty} \mathcal{E}_n(u_n(t),K_n(t)) \leq \limsup_{n\to+\infty} \mathcal{E}_n(u_n(t),K_n(t)) \\ &= \limsup_{n\to+\infty} \mathcal{E}_n(u_n(0),K_n(0)) + \int_0^t \vartheta_n(s)\,ds \leq \mathcal{E}(u(0),K(0)) + \int_0^t \vartheta(s)\,ds \\ &= \mathcal{E}(u(0),K(0)) + \int_0^t \int_\Omega \nabla_\xi f(x,\nabla u(\tau))\nabla\dot{\psi}(\tau)\,dx\,d\tau \leq \mathcal{E}(u(t),K(t)),\end{aligned}$$

so that we get

$$\mathcal{E}(u(t),K(t)) = \mathcal{E}(u(0),K(0)) + \int_0^t \int_\Omega \nabla_\xi f(x,\nabla u(\tau))\nabla\dot{\psi}(\tau)\,dx\,d\tau$$

and

$$\lim_{n\to+\infty} \mathcal{E}_n(u_n(t),K_n(t)) = \mathcal{E}(u(t),K(t)). \tag{4.121}$$

Finally by lower semicontinuity for the bulk and surface energies under weak convergence for the displacements and σ-convergence in $\overline{\Omega}$ for the cracks, we conclude that

$$\lim_{n\to+\infty} \mathcal{E}_n^b(u_n(t)) = \mathcal{E}^b(u(t)),$$

and

$$\lim_{n\to+\infty} \mathcal{E}_n^s(K_n(t)) = \mathcal{E}^s(K(t)),$$

so that the theorem is proved. □

Chapter 5

Finite element approximation of the model by Francfort-Larsen

5.1 Introduction

In this chapter [1] We propose a discontinuous finite element approximation for the model of quasi-static growth of crack proposed by Francfort and Larsen in [47]. We use a suitable finite element method and we give a rigorous proof of its convergence to a quasi-static evolution in the sense of Francfort and Larsen. We restrict our analysis to a two dimensional setting considering only a polygonal reference configuration $\Omega \subseteq \mathbb{R}^2$.

The discretization of the domain Ω is carried out, following [62] (see also [63]), considering two parameters $\varepsilon > 0$ and $a \in]0, \frac{1}{2}[$. We consider a regular triangulation $\mathbf{R}_\varepsilon$ of size ε of Ω, i.e. we assume that there exist two constants c_1 and c_2 so that every triangle $T \in \mathbf{R}_\varepsilon$ contains a ball of radius $c_1\varepsilon$ and is contained in a ball of radius $c_2\varepsilon$. In order to treat the boundary data, we assume also that $\partial_D\Omega$ is composed of edges of $\mathbf{R}_\varepsilon$. On each edge $[x, y]$ of $\mathbf{R}_\varepsilon$ we consider a point z such that $z = tx + (1-t)y$ with $t \in [a, 1-a]$. These points are called *adaptive vertices.* Connecting together the adaptive vertices, we divide every $T \in \mathbf{R}_\varepsilon$ into four triangles. We take the new triangulation $\mathbf{T}$ obtained after this division as the discretization of Ω. The family of all such triangulations is denoted by $\mathcal{T}_{\varepsilon,a}(\Omega)$.

The discretization of the energy functional is obtained restricting the total energy to the family of functions u which are affine on the triangles of some triangulation $\mathbf{T}(u) \in \mathcal{T}_{\varepsilon,a}(\Omega)$ and are allowed to jump across the edges of $\mathbf{T}(u)$. We indicate this space by $\mathcal{A}_{\varepsilon,a}(\Omega)$. The boundary data is assumed to belong to the space $\mathcal{AF}_\varepsilon(\Omega)$ of continuous functions which are affine on every triangle $T \in \mathbf{R}_\varepsilon$.

Given the boundary data $g \in W^{1,1}([0,1], H^1(\Omega))$ with $g(t) \in \mathcal{AF}_\varepsilon(\Omega)$ for all $t \in [0,1]$, we divide $[0,1]$ into subintervals $[t_i^\delta, t_{i+1}^\delta]$ of size $\delta > 0$ for $i = 0, \dots, N_\delta$, we set $g_i^\delta = g(t_i^\delta)$, and for all $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$ we indicate by $S_D^{g_i^\delta}(u)$ the edges of the triangulation $\mathbf{T}(u)$ contained in $\partial_D\Omega$ on which $u \neq g_i^\delta$. Using a variational argument we construct a *discrete evolution* $\{u_{\varepsilon,a}^{\delta,i} : i = 0, \dots, N_\delta\}$ such that $u_{\varepsilon,a}^{\delta,i} \in \mathcal{A}_{\varepsilon,a}(\Omega)$ for all $i = 0, \dots, N_\delta$, and such that considering the *discrete fracture*

$$\Gamma_{\varepsilon,a}^{\delta,i} := \bigcup_{r=0}^{i} [S(u_{\varepsilon,a}^{\delta,r}) \cup S_D^{g_r^\delta}(u_{\varepsilon,a}^{\delta,r})],$$

[1]The results of this chapter are contained in Giacomini-Ponsiglione [50]

the following *unilateral minimality property* holds:

$$\int_\Omega |\nabla u^{\delta,i}_{\varepsilon,a}|^2\,dx \leq \int_\Omega |\nabla v|^2\,dx + \mathcal{H}^1\left(\left(S(v) \cup S^{g_i^\delta}_D(v)\right) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right). \tag{5.1}$$

Moreover we get suitable estimates for the discrete total energy

$$\mathcal{E}^{\delta,i}_{\varepsilon,a} := \|\nabla u^{\delta,i}_{\varepsilon,a}\|^2_{L^2(\Omega;\mathbb{R}^2)} + \mathcal{H}^1\left(\Gamma^{\delta,i}_{\varepsilon,a}\right).$$

In order to perform the asymptotic analysis of the *discrete evolution* $\{u^{\delta,i}_{\varepsilon,a} : i = 0, \dots, N_\delta\}$, we make the piecewise constant interpolation in time $u^\delta_{\varepsilon,a}(t) = u^{\delta,i}_{\varepsilon,a}$ and $\Gamma^\delta_{\varepsilon,a}(t) = \Gamma^{\delta,i}_{\varepsilon,a}$ for all $t^\delta_i \leq t < t^\delta_{i+1}$. The main result of the chapter is the following theorem.

Theorem 5.1.1. *Let $g \in W^{1,1}([0,1], H^1(\Omega))$ be such that $\|g(t)\|_\infty \leq C$ for all $t \in [0,1]$ and let $g_\varepsilon \in W^{1,1}([0,1], H^1(\Omega))$ be such that $\|g_\varepsilon(t)\|_\infty \leq C$, $g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega)$ for all $t \in [0,1]$ and*

$$g_\varepsilon \to g \quad \text{strongly in } W^{1,1}([0,1], H^1(\Omega)). \tag{5.2}$$

Given the discrete evolution $\{t \to u^\delta_{\varepsilon,a}(t)\}$ relative to the boundary data g_ε, let $\Gamma^\delta_{\varepsilon,a}$ and $\mathcal{E}^\delta_{\varepsilon,a}$ be the associated fracture and total energy.

Then there exist $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, and a quasi-static evolution in the sense of [47] $\{t \to (u(t), \Gamma(t)),\ t \in [0,1]\}$ relative to the boundary data g, such that setting $u_n := u^{\delta_n}_{\varepsilon_n,a_n}$, $\Gamma_n := \Gamma^{\delta_n}_{\varepsilon_n,a_n}$, $\mathcal{E}_n := \mathcal{E}^{\delta_n}_{\varepsilon_n,a_n}$, the following hold:

(a) if $\mathcal{N}$ is the set of discontinuities of $\mathcal{H}^1(\Gamma(\cdot))$, for all $t \in [0,1] \setminus \mathcal{N}$ we have

$$\nabla u_n(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega;\mathbb{R}^2) \tag{5.3}$$

and

$$\lim_n \mathcal{H}^1(\Gamma_n(t)) = \mathcal{H}^1(\Gamma(t)); \tag{5.4}$$

(b) for all $t \in [0,1]$ we have

$$\lim_n \mathcal{E}_n(t) = \mathcal{E}(t). \tag{5.5}$$

We conclude that we have the convergence of the total energy at each time $t \in [0,1]$, and the separate convergence of bulk and surface energy for all $t \in [0,1]$ except a countable set.

In order to prove Theorem 5.1.1, we proceed in two steps. Firstly, we fix a and let $\delta \to 0$ and $\varepsilon \to 0$. We obtain an evolution $\{t \to u_a(t) : t \in [0,1]\}$ such that $\nabla u^\delta_{\varepsilon,a}(t) \to \nabla u_a(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ for all t up to a countable set and such that the following minimality property holds: for all $v \in SBV(\Omega)$

$$\int_\Omega |\nabla u_a(t)|^2\,dx \leq \int_\Omega |\nabla v|^2\,dx + \mu(a)\mathcal{H}^1\left((S(v) \cup (\partial_D\Omega \cap \{v \neq g(t)\})) \setminus \Gamma_a(t)\right), \tag{5.6}$$

where $\mu :]0, \frac{1}{2}[\to]0, +\infty[$ is a function independent of ε and δ, such that $\mu \geq 1$, $\lim_{a \to 0} \mu(a) = 1$ and $\Gamma_a(t) := \bigcup_{s \leq t, s \in D} S(u_a(s)) \cup (\partial_D\Omega \cap \{u_a(s) \neq g(s)\})$. The minimality property (5.6) takes into account possible anisotropies that could be generated as δ and $\varepsilon \to 0$: in fact, since a is fixed, we have that the angles of the triangles in $\mathcal{T}_{\varepsilon,a}(\Omega)$ are between fixed values (determined by a), and so fractures with certain directions cannot be approximated in length. In the second step, we let $a \to 0$ and determine from $\{t \to u_a(t) : t \in [0,1]\}$ a quasi-static evolution $\{t \to u(t) : t \in [0,1]\}$ in the sense of Francfort and Larsen. Then, using a diagonal argument, we find sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, and $a_n \to 0$ satisfying Theorem 5.1.1.

The main difficulties arise in the first part of our analysis, namely when $\delta, \varepsilon \to 0$. The convergence $u^\delta_{\varepsilon,a}(t) \to u_a(t)$ in $SBV(\Omega)$ for $t \in D \subseteq [0,1]$ countable and dense is easily obtained by means of Ambrosio's Compactness Theorem. The minimality property (5.6) derives from its discrete version (5.1) using a variant of Lemma 1.2 of [47]: given $v \in SBV(\Omega)$, we construct $v^\delta_{\varepsilon,a} \in \mathcal{A}_{\varepsilon,a}(\Omega)$ such that

$$\nabla v^\delta_{\varepsilon,a} \to \nabla v \quad \text{strongly in } L^2(\Omega;\mathbb{R}^2) \tag{5.7}$$

and

$$\limsup_{\delta,\varepsilon\to 0} \mathcal{H}^1 \left[(S(v^\delta_{\varepsilon,a}) \cup S_D^{g^\delta_\varepsilon(t)}(v^\delta_{\varepsilon,a})) \setminus \Gamma^\delta_{\varepsilon,a}(t) \right] \le \mu(a)\mathcal{H}^1 \left[(S(v) \cup (\partial_D\Omega \cap \{v \neq g(t)\})) \setminus \Gamma_a(t) \right], \tag{5.8}$$

where $g^\delta_\varepsilon(t) := g_\varepsilon(t^\delta_i)$ for $t^\delta_i \le t < t^\delta_{i+1}$. The main difference with respect to Lemma 1.2 of [47] is that we have to find the approximating functions $v^\delta_{\varepsilon,a}$ in the finite element space $\mathcal{A}_{\varepsilon,a}(\Omega)$. This can be regarded as an interpolation problem, so we try to construct triangulations $\mathbf{T}_\varepsilon \in \mathcal{T}_{\varepsilon,a}(\Omega)$ adapted to v in order to obtain (5.7) and (5.8). In all the geometric operations involved, we need to avoid degeneration of the triangles of $\mathbf{T}(u^\delta_{\varepsilon,a}(t))$ which is guaranteed from the fact that a is constant: this is the principal reason to keep a fixed in the first step. A second difficulty arises when $u_a(\cdot)$ is extended from D to the entire interval $[0,1]$: indeed it is no longer clear whether $\nabla u^\delta_{\varepsilon,a}(t) \to \nabla u_a(t)$ for $t \notin D$. Since the space $\mathcal{A}_{\varepsilon,a}(\Omega)$ is not a vector space, we cannot provide an estimate on $\|\nabla u^\delta_{\varepsilon,a}(t) - \nabla u^\delta_{\varepsilon,a}(s)\|$ with $s \in D$ and $s < t$: we thus cannot expect to recover the convergence at time t from the convergence at time s. We overcome this difficulty observing that $\nabla u^\delta_{\varepsilon,a}(t) \to \nabla \tilde{u}_a$ with $\tilde{u}_a$ satisfying a minimality property similar to (5.6) and then proving $\nabla \tilde{u}_a = \nabla u_a(t)$ by a uniqueness argument for the gradients of the solutions.

The plan of the chapter is the following. In Section 5.2 we give the basic definitions and prove some auxiliary results. In Section 5.3, we prove the existence of a discrete evolution. In Section 5.4 we prove the convergence of the discrete evolution to a quasi-static evolution of brittle fractures in the sense of Francfort and Larsen. The proof of minimality property (5.6) requires a careful analysis to which is dedicated Section 5.5. In Section 5.6 we show that the arguments of Section 5.4 can be used to improve the convergence results for the discrete in time approximation considered in [47].

5.2 Preliminaries

Triangulations. Let $\Omega \subseteq \mathbb{R}^2$ be a polygonal set and let us fix two positive constants $0 < c_1 < c_2$. By a *regular triangulation* of Ω of size ε we intend a finite family of (closed) triangles T_i such that $\overline{\Omega} = \bigcup_i T_i$, $T_i \cap T_j$ is either empty or equal to a common edge or to a common vertex, and each T_i contains a ball of diameter $c_1\varepsilon$ and is contained in a ball of diameter $c_2\varepsilon$.

We indicate by $\mathcal{R}_\varepsilon(\Omega)$ the family of all regular triangulations of Ω of size ε. It turns out that there exist $0 < \vartheta_1 < \vartheta_2 < \pi$ such that for all T belonging to a triangulation $\mathbf{T} \in \mathcal{R}_\varepsilon(\Omega)$, the inner angles of T are between ϑ_1 and ϑ_2. Moreover, every edge of T has length greater than $c_1\varepsilon$ and lower than $c_2\varepsilon$.

Let us fix a triangulation $\mathbf{R}_\varepsilon \in \mathcal{R}_\varepsilon(\Omega)$ for all $\varepsilon > 0$ and let $a \in]0, \frac{1}{2}[$. Let us consider a new triangulation $\mathbf{T}$ nested in $\mathbf{R}_\varepsilon$ obtained dividing each $T \in \mathbf{R}_\varepsilon$ into four triangles taking over every edge $[x,y]$ of T a knot z which satisfies

$$z = tx + (1-t)y, \qquad t \in [a, 1-a].$$

We will call these new vertices *adaptive*, the triangles obtained joining these points *adaptive triangles*, and their edges *adaptive edges* (see Fig.1).

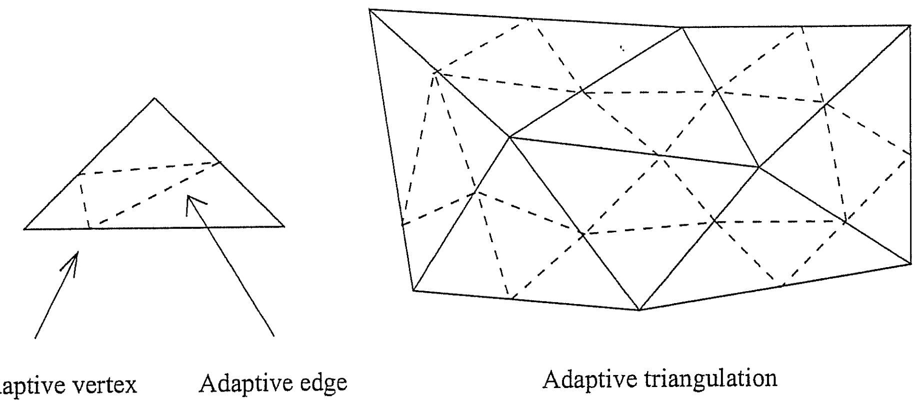

Fig. 1

We denote by $\mathcal{T}_{\varepsilon,a}(\Omega)$ the set of all triangulations $\mathbf{T}$ constructed in this way. Note that for all $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$ there exists $0 < c_1^a < c_2^a < +\infty$ such that every $T_i \in \mathbf{T}$ contains a ball of diameter $c_1^a\varepsilon$ and is contained in a ball of diameter $c_2^a\varepsilon$. Then there exist $0 < \vartheta_1^a < \vartheta_2^a < \pi$ such that for all triangles T belonging to a triangulation $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$, the inner angles of T are between ϑ_1^a and ϑ_2^a. Moreover, every edge of T has length greater than $c_1^a\varepsilon$ and lower than $c_2^a\varepsilon$.

We will often use the following *interpolation estimate* (see [26, Theorem 3.1.5]). If $u \in W^{2,2}(\Omega)$ and $T \in \mathbf{R}_\varepsilon$, let u_T denote the affine interpolation of u on T. We have that there exists K depending only on c_1, c_2 such that

$$\|u_T - u\|_{W^{1,2}(T)} \leq K\varepsilon\|u\|_{W^{2,2}(T)}. \tag{5.9}$$

Estimate (5.9) holds also for $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$: in this case K depends on a.

Some elementary constructions. The following lemmas will be used in Section 4.

Lemma 5.2.1. *Let $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$, and let $l \subseteq \Omega$ be a segment with extremes p, q belonging to edges of $\mathbf{T}$. There exists a polyhedral curve Γ with extremal points p and q (see Fig.2) such that Γ is contained in the union of the edges of those $T \in \mathbf{T}$ with $T \cap l \neq \emptyset$, and such that the following properties hold:*

(1) $\Gamma = \gamma_p \cup \gamma \cup \gamma_q$, where γ is union of edges of $\mathbf{T}$ and γ_p, γ_q are segments containing p and q respectively, and each one is contained in an edge of $\mathbf{T}$;

(2) there exists a constant c independent of ϵ (but depending on a) such that

$$\mathcal{H}^1(\Gamma) \leq c\,\mathcal{H}^1(l).$$

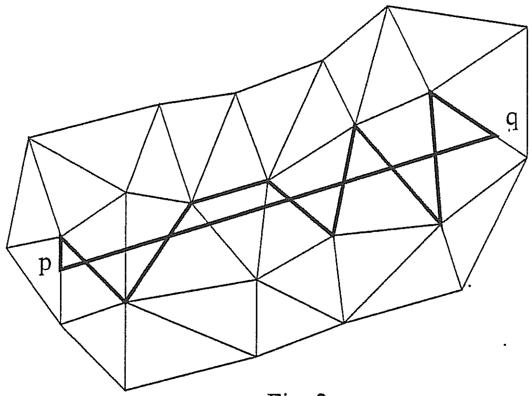

Fig. 2

Proof. Let $\{T_1, \dots T_k\}$ be the family of triangles in $\mathbf{T}$ such that the intersection with l is a segment with positive length. For every integer $1 \le i \le k$, let $l_i := T_i \cap l$. If l_i is an edge of T_i, we set $D_i = T_i$. Otherwise let D_i be a connected component of $T_i \setminus l_i$ such that $|D_i| \le \frac{1}{2}|T_i|$. We claim that there exists a constant $c > 0$ independent of ϵ such that

$$\mathcal{H}^1(\partial D_i) \le c\mathcal{H}^1(l_i). \tag{5.10}$$

We have to analyze two possibilities, namely D_i is a triangle, or D_i is a trapezoid. Suppose that D_i is a triangle and that m_i is an edge of D_i. Let α be the angle of D_i opposite to l_i. It is easy to prove that $\mathcal{H}^1(l_i) \ge \mathcal{H}^1(m_i)\sin\alpha$, and so

$$\mathcal{H}^1(l_i) \ge \frac{1}{3}\sin\alpha\mathcal{H}^1(\partial D_i).$$

Since $\vartheta_1^a \le \alpha \le \vartheta_2^a$, $\sin\alpha$ is uniformly bounded from below, and hence inequality (5.10) follows. If D_i is a trapezoid, since $|D_i| \le \frac{1}{2}|T_i|$, it follows that $T_i \setminus D_i$ is a triangle such that its edges different from l_i have length greater than $\frac{1}{2}c_1^a\varepsilon$. Let α be the inner angle of $T_i \setminus D_i$ opposite to l_i. We have that

$$\mathcal{H}^1(l_i) \ge \frac{1}{2}\sin\alpha c_1^a\varepsilon \ge \frac{1}{2}\sin\alpha\frac{c_1^a}{c_2^a}\frac{1}{4}\mathcal{H}^1(\partial D_i).$$

Since $\vartheta_1^a \le \alpha \le \vartheta_2^a$, inequality (5.10) follows.

By (5.10), we deduce that

$$\mathcal{H}^1\left(\bigcup_{i=1}^k \partial D_i\right) \le c\mathcal{H}^1(l);$$

moreover, since $\bigcup_{i=1}^k(\partial D_i \setminus (l_i \cap \mathrm{int}(T_i)))$ is arcwise connected and contains p, q, we conclude that there exists a curve $\Gamma \subseteq \bigcup_{i=1}^k \partial D_i$ which satisfies the thesis. □

Lemma 5.2.2. *There exists a constant $c > 0$ such that for every segment $l \subseteq \Omega$ there exists ε_0 with the following property: for every $\varepsilon \le \varepsilon_0$, setting $\mathcal{R}(l) := \{T \in \mathbf{R}_\varepsilon : T \cap l \ne \emptyset\}$, we have*

$$\mathcal{H}^1(\partial\mathcal{R}(l)) \le c\mathcal{H}^1(l).$$

Proof. Let $\mathcal{N}_\varepsilon(l) := \{x \in \Omega : \operatorname{dist}(x, l) \leq c_2\varepsilon\}$. We have that $|\mathcal{N}_\varepsilon(l)| = \mathcal{H}^1(l)c_2\varepsilon + \pi c_2^2\varepsilon^2$, and hence there exists a positive constant ε_0 such that, for every $\varepsilon \leq \varepsilon_0$, we have that

$$|\mathcal{N}_\varepsilon(l)| \leq 2\mathcal{H}^1(l)c_2\varepsilon.$$

We have that $\mathcal{R}(l) \subseteq \mathcal{N}_\varepsilon(l)$, and

$$\sharp\mathcal{R}(l) \leq \frac{4}{c_1^2\pi^2}\frac{|\mathcal{N}_\varepsilon(l)|}{\varepsilon^2},$$

where $\sharp\mathcal{R}(l)$ denotes the number of triangles of $\mathcal{R}(l)$. Then, we have

$$\mathcal{H}^1(\partial\mathcal{R}(l)) \leq 3c_2\varepsilon\,\sharp\mathcal{R}(l) \leq 3c_2\varepsilon\frac{4}{c_1^2\pi^2}\frac{|\mathcal{N}_\varepsilon(l)|}{\varepsilon^2} \leq 3c_2^2\frac{4}{c_1^2\pi^2}2\mathcal{H}^1(l),$$

and so the proof is concluded. □

A density result. Let $A \subseteq \mathbb{R}^2$ be open. We say that $K \subseteq A$ is polygonal (with respect to A), if it is the intersection of A with the union of a finite number of closed segments. The following density result is proved in [30].

Theorem 5.2.3. *Assume that ∂A is locally Lipschitz, and let $u \in SBV(A)$ such that $u \in L^2(A)$, $\nabla u \in L^2(\Omega;\mathbb{R}^2)$, and $\mathcal{H}^1(S(u)) < +\infty$. For every $\varepsilon > 0$, there exists a function $v \in SBV(A)$ such that*

(a) $S(v)$ is essentially closed, i.e., $\mathcal{H}^1(\overline{S(v)} \setminus S(v)) = 0$;

(b) $\overline{S(v)}$ is a polyhedral set;

(c) $v \in W^{k,\infty}(A \setminus \overline{S(v)})$ for every $k \in \mathbb{N}$;

(d) $||v - u||_{L^2(A)} < \varepsilon$;

(e) $||\nabla v - \nabla u||_{L^2(A;\mathbb{R}^2)} < \varepsilon$;

(f) $|\mathcal{H}^1(S(v)) - \mathcal{H}^1(S(u))| < \varepsilon$.

Let $\partial_D\Omega$ be a relatively open subset of $\partial\Omega$ composed of edges lying in $\partial\Omega$. Let us consider Ω_D polygonal open bounded subset of $\mathbb{R}^2$ such that $\Omega_D \cap \Omega = \emptyset$ and $\partial\Omega \cap \partial\Omega_D = \partial_D\Omega$ up to a finite number of vertices. We set $\Omega' := \Omega \cup \Omega_D \cup \partial_D\Omega$. In Section 4, we will use the following result.

Proposition 5.2.4. *Given $u \in SBV(\Omega')$ with $u = 0$ on $\Omega' \setminus \overline{\Omega}$ and $\mathcal{H}^{N-1}(S(u)) < +\infty$, there exists $u_h \in SBV(\Omega')$ such that*

(a) $u_h = 0$ in $\Omega' \setminus \overline{\Omega}$;

(b) $S(u_h)$ is polyhedral, $\overline{S(u_h)} \subseteq \Omega$ and $u_h \in W^{k,\infty}(\Omega' \setminus \overline{S(u_h)})$ for all k;

(c) $u_h \to u$ strongly in $L^2(\Omega')$ and $\nabla u_h \to \nabla u$ strongly in $L^2(\Omega'; \mathbb{R}^2)$;

(d) for all A open subset of Ω' with $\mathcal{H}^1(\partial A \cap S(u)) = 0$, we have

$$\lim_h \mathcal{H}^1(A \cap S(u_h)) = \mathcal{H}^1(A \cap S(u)).$$

Proof. Using a partition of unity, we may prove the result in the case $\Omega :=]-1,1[\times]0,1[$, $\Omega' :=]-1,1[\times]-1,1[$, and $\partial_D\Omega :=]-1,1[\times\{0\}$. We set $w_h(x,y) := u(x, y-h)$, and let φ_h be a cut off function with $\varphi_h = 1$ on $]-1,1[\times]-1,\frac{h}{3}[$, $\varphi_h = 0$ on $]-1,1[\times]\frac{h}{2},1[$, and $||\nabla\varphi_h||_\infty \leq \frac{7}{h}$. Let us set $v_h := (1-\varphi_h)w_h$. We have that $v_h = 0$ in $\Omega' \setminus \overline{\Omega}$; moreover we have

$$\nabla v_h = (1-\varphi_h)\nabla w_h - \nabla\varphi_h w_h.$$

Since $\nabla\varphi_h w_h \to 0$ strongly in $L^2(\Omega'; \mathbb{R}^2)$, we have $\nabla v_h \to \nabla u$ strongly in $L^2(\Omega'; \mathbb{R}^N)$. Finally, for all A open subset of Ω' with $\mathcal{H}^1(\partial A \cap S(u)) = 0$, we have

$$\lim_h \mathcal{H}^1(A \cap S(v_h)) = \mathcal{H}^1(A \cap S(u)).$$

In order to conclude the proof, let us apply Theorem 5.2.3 obtaining $\tilde{v}_h$ with polyhedral jumps in Ω such that $\tilde{v}_h \in W^{k,\infty}(\Omega' \setminus \overline{S(\tilde{v}_h)})$, $||w_h - \tilde{v}_h||_{L^2(\Omega)} + ||\nabla w_h - \nabla\tilde{v}_h||_{L^2(\Omega;\mathbb{R}^2)} \leq h^2$ and $|\mathcal{H}^{N-1}(S(w_h)) - \mathcal{H}^{N-1}(S(\tilde{v}_h))| \leq h$. If we set $u_h := \varphi_h g + (1-\varphi_h)\tilde{v}_h$, we obtain the thesis. □

5.3 The discontinuous finite element approximation

In this section we construct a discrete approximation of quasi-static evolution of brittle fractures in linearly elastic bodies: the discretization is done both in space and time.

From now on we suppose that Ω is a polygonal open bounded subset of $\mathbb{R}^2$, and that $\partial_D\Omega \subseteq \partial\Omega$ is open in the relative topology. For all $\varepsilon > 0$, we fix a triangulation $\mathbf{R}_\varepsilon \in \mathcal{R}_\varepsilon(\Omega)$, and suppose that $\partial_D\Omega$ is composed of edges of $\mathbf{R}_\varepsilon$ for all ε; we indicate the family of these edges by $\mathbf{S}_\varepsilon$.

We consider the following discontinuous finite element space. We indicate by $\mathcal{A}_{\varepsilon,a}(\Omega)$ the set of all u such that there exists a triangulation $\mathbf{T}(u) \in \mathcal{T}_{\varepsilon,a}(\Omega)$ nested in $\mathbf{R}_\varepsilon$ with u affine on every $T \in \mathbf{T}(u)$. For every $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$, we write $\|\nabla u\|$ for the L^2-norm of ∇u and we indicate by $S(u)$ the family of edges of $\mathbf{T}(u)$ inside Ω across which u is discontinuous. Notice that $u \in SBV(\Omega)$ and that the notation is consistent with the usual one employed in the theory of functions with bounded variation. Let us also denote by $\mathcal{AF}_\varepsilon(\Omega)$ the set of affine functions in Ω with respect to the triangulation $\mathbf{R}_\varepsilon$. Finally, given any $g \in \mathcal{AF}_\varepsilon(\Omega)$, for all $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$ we set

$$S^g_D(u) := \{\zeta \in \mathbf{S}_\varepsilon : u \neq g \text{ on } \zeta\}, \tag{5.11}$$

that is $S^g_D(u)$ denotes the edges at which the boundary condition is not satisfied. Moreover we set

$$S^g(u) := S(u) \cup S^g_D(u) \tag{5.12}$$

Let now consider $g \in W^{1,1}([0,1]; H^1(\Omega))$ with $g(t) \in \mathcal{AF}_\varepsilon(\Omega)$ for all $t \in [0,1]$. Let $\delta > 0$ and let N_δ be the largest integer such that $\delta(N_\delta - 1) < 1$; for $0 \leq i \leq N_\delta - 1$ we set $t^\delta_i := i\delta$, $t^\delta_{N_\delta} := 1$ and $g^\delta_i := g(t^\delta_i)$. The following proposition holds.

Proposition 5.3.1. *Let $\varepsilon > 0$, $a \in]0, \frac{1}{2}]$ and $\delta > 0$ be fixed. Then for all $i = 0, \ldots, N_\delta$ there exists $u^{\delta,i}_{\varepsilon,a} \in \mathcal{A}_{\varepsilon,a}(\Omega)$ such that, setting*

$$\Gamma^{\delta,i}_{\varepsilon,a} := \bigcup_{r=0}^{i} S^{g^\delta_r}(u^{\delta,r}_{\varepsilon,a}),$$

the following hold:

(a) $\|u^{\delta,i}_{\varepsilon,a}\|_\infty \leq \|g^\delta_i\|_\infty$;

(b) for all $v \in \mathcal{A}_{\varepsilon,a}(\Omega)$ we have

$$\|\nabla u^{\delta,0}_{\varepsilon,a}\|^2 + \mathcal{H}^1\left(S^{g^\delta_0}(u^{\delta,0}_{\varepsilon,a})\right) \leq \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g^\delta_0}(v)\right), \tag{5.13}$$

and

$$\|\nabla u^{\delta,i}_{\varepsilon,a}\|^2 \leq \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g^\delta_i}(v) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right). \tag{5.14}$$

Proof. The proof is carried out through a variational argument. Let $u^{\delta,0}_{\varepsilon,a}$ be a minimum of the following problem

$$\min\left\{\|\nabla u\|^2 + \mathcal{H}^1(S^{g^\delta_0}(u))\right\}. \tag{5.15}$$

We set $\Gamma^{\delta,0}_{\varepsilon,a} := S^{g^\delta_0}(u^{\delta,0}_{\varepsilon,a})$. Recursively, supposing to have constructed $u^{\delta,i-1}_{\varepsilon,a}$ and $\Gamma^{\delta,i-1}_{\varepsilon,a}$, let $u^{\delta,i}_{\varepsilon,a}$ be a minimum for

$$\min\left\{\|\nabla u\|^2 + \mathcal{H}^1\left(S^{g^\delta_i}(u) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right)\right\}. \tag{5.16}$$

We set $\Gamma^{\delta,i}_{\varepsilon,a} := S^{g^\delta_i}(u^{\delta,i}_{\varepsilon,a}) \cup \Gamma^{\delta,i-1}_{\varepsilon,a}$. We claim that problems (5.15) and (5.16) admit a solution $u^{\delta,i}_{\varepsilon,a}$ such that $\|u^{\delta,i}_{\varepsilon,a}\|_\infty \leq \|g^\delta_i\|_\infty$ for all $i = 0, \ldots, N_\delta$. We prove the claim for problem (5.16), the other case being similar. Let (u_n) be a minimizing sequence for problem (5.16): since g^δ_i is an admissible test function, we deduce that for n large

$$\|\nabla u_n\|^2 + \mathcal{H}^1\left(S^{g^\delta_i}(u_n) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right) \leq \|\nabla g^\delta_i\|^2 + 1.$$

Moreover, we may modify u_n in the following way. If π denotes the projection in $\mathbb{R}$ over the interval $I := [-\|g^\delta_i\|_\infty, \|g^\delta_i\|_\infty]$, let $\tilde{u}_n \in \mathcal{A}_{\varepsilon,a}(\Omega)$ be defined on each $T \in \mathbf{T}(u_n)$ as the affine interpolation of the values $(\pi(u_n(x_1)),\ \pi(u_n(x_2)),\ \pi(u_n(x_3))$, where x_1, x_2 and x_3 are the vertices of T. Note that by construction we have for all n

$$\|\tilde{u}_n\|_\infty \leq \|g^\delta_i\|_\infty, \qquad \|\nabla \tilde{u}_n\| \leq \|\nabla u_n\|, \qquad S^{g^\delta_i}(\tilde{u}_n) \subseteq S^{g^\delta_i}(u_n),$$

so that $(\tilde{u}_n)$ is a minimizing sequence for problem (5.16). We conclude that it is not restrictive to assume $\|u_n\|_\infty \leq \|g^\delta_i\|_\infty$.

Since $\mathbf{T}(u_n) \in \mathcal{T}_{\varepsilon,a}(\Omega)$, we have that the number of elements of $\mathbf{T}(u_n)$ is uniformly bounded. Up to a subsequence, we may suppose that there exists an integer k such that $\mathbf{T}(u_n)$ has exactly k elements $T^1_n, \ldots, T^k_n$. Using a diagonal argument we may suppose that, up to a further subsequence, there exists $\mathbf{T} = \{T^1, \ldots, T^k\} \in \mathcal{T}_{\varepsilon,a}(\Omega)$ such that $T^i_n \to T^i$ in the Hausdorff metric for all $i = 1, \ldots, k$. Let us consider $T^i \in \mathbf{T}$, and let $\tilde{T}^i$ be contained in the interior of T^i. For n large enough, $\tilde{T}^i$ is contained in the interior of T_n^i and $(u_n)_{|\tilde{T}^i}$ is affine with $\int_{\tilde{T}^i} |\nabla u_n|^2\,dx \le C$ with C independent of n. We deduce that there exists a function u^i affine on $\tilde{T}^i$ such that up to a subsequence $u_n \to u$ uniformly on $\tilde{T}^i$. Since $\tilde{T}^i$ is arbitrary, it turns out that u^i is actually defined on T^i and

$$\int_{T^i} |\nabla u^i|^2\,dx \le \liminf_n \int_{T_n^i} |\nabla u_n|^2\,dx.$$

Let $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$ such that $u = u^i$ on T^i for every $i = 1, \dots, k$: we have

$$\|\nabla u\|^2 \le \liminf_n \|\nabla u_n\|^2.$$

On the other hand, it is easy to see that $S^{g_i^\delta}(u)$ is contained in the Hausdorff limit of $S^{g_i^\delta}(u_n)$, and that

$$\mathcal{H}^1\left(S^{g_i^\delta}(u) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right) \le \liminf_n \mathcal{H}^1\left(S^{g_i^\delta}(u_n) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right).$$

We conclude that u is a minimum point for the problem (5.16) with $\|u\|_\infty \le \|g_i^\delta\|_\infty$. We have that point (a) is proved.

Concerning point (b), by construction we get (5.13); for $i \ge 1$ we have

$$\|\nabla u_{\varepsilon,a}^{\delta,i}\|^2 + \mathcal{H}^1\left(S^{g_i^\delta}(u_{\varepsilon,a}^{\delta,i}) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right) \le \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g_i^\delta}(v) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right)$$

for all $v \in \mathcal{A}_{\varepsilon,a}(\Omega)$, so that

$$\|\nabla u_{\varepsilon,a}^{\delta,i}\|^2 \le \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g_i^\delta}(v) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right),$$

and this proves point (b). □

Remark 5.3.2. For technical reasons due to the asymptotic analysis of the discrete evolution $u_{\varepsilon,a}^{\delta,i}$ when $\delta \to 0$, $\varepsilon \to 0$ and $a \to 0$, we define $u_{\varepsilon,a}^{\delta,i}$ from $u_{\varepsilon,a}^{\delta,i-1}$ through problem (5.16) without requiring that the adaptive vertices determining $\Gamma_{\varepsilon,a}^{\delta,i-1}$ remain fixed. We just penalize their possible changes if they are used to create new fracture: in fact in this case, the surface energy increases at each change of a quantity at least of order $a\varepsilon$. As a consequence, during the step by step minimization, it could happen that some triangles $T \in \mathcal{T}_{\varepsilon,a}(\Omega)$ contain the fracture $\Gamma_{\varepsilon,a}^{\delta,i}$ in their interior. This is in contrast with the interpretation of the triangles as elementary blocks for the elasticity problem, but being this situation penalized in the minimization process, we expect that it occurs rarely.

The following estimate is essential for the study of asymptotic behavior of the discrete evolution.

Proposition 5.3.3. *If $(u_{\varepsilon,a}^{\delta,i}, \Gamma_{\varepsilon,a}^{\delta,i})$ for $i = 0, \dots, N_\delta$ satisfies condition (b) of Proposition 5.3.1, setting $\mathcal{E}_{\varepsilon,a}^{\delta,i} := \|\nabla u_{\varepsilon,a}^{\delta,i}\|^2 + \mathcal{H}^1(\Gamma_{\varepsilon,a}^{\delta,i})$, we have for $0 \le j \le i \le N_\delta$*

$$\mathcal{E}_{\varepsilon,a}^{\delta,i} \le \mathcal{E}_{\varepsilon,a}^{\delta,j} + 2\sum_{r=j}^{i-1} \int_{t_r^\delta}^{t_{r+1}^\delta} \int_\Omega \nabla u_{\varepsilon,a}^{\delta,r} \nabla \dot{g}(\tau)\,dx\,d\tau + o^\delta, \tag{5.17}$$

where

$$o^\delta := \left[\max_{r=0,\dots,N_\delta-1} \int_{t_r^\delta}^{t_{r+1}^\delta} \|\dot{g}(\tau)\|_{H^1(\Omega)}\,d\tau\right] \int_0^1 \|\dot{g}(\tau)\|_{H^1(\Omega)}\,d\tau. \tag{5.18}$$

Proof. For all $0 \leq j \leq N_\delta - 1$, by construction of $u_{\varepsilon,a}^{\delta,j+1}$ we have that

$$\|\nabla u_{\varepsilon,a}^{\delta,j+1}\|^2 + \mathcal{H}^1\left(S^{g_{j+1}^\delta}(u_{\varepsilon,a}^{\delta,j+1}) \setminus \Gamma_{\varepsilon,a}^{\delta,j}\right) \leq \|\nabla u_{\varepsilon,a}^{\delta,j} + \nabla(g_{j+1}^\delta - g_j^\delta)\|^2 =$$
$$= \|\nabla u_{\varepsilon,a}^{\delta,j}\|^2 + 2\int_\Omega \nabla u_{\varepsilon,a}^{\delta,j} \nabla(g_{j+1}^\delta - g_j^\delta)\, dx + \|\nabla(g_{j+1}^\delta - g_j^\delta)\|^2.$$

Notice that

$$\nabla(g_{j+1}^\delta - g_j^\delta) = \int_{t_j^\delta}^{t_{j+1}^\delta} \nabla \dot{g}(\tau)\, d\tau,$$

so that

$$\|\nabla u_{\varepsilon,a}^{\delta,j+1}\|^2 + \mathcal{H}^1\left(S^{g_{j+1}^\delta}(u_{\varepsilon,a}^{\delta,j+1}) \setminus \Gamma_{\varepsilon,a}^{\delta,j}\right) \leq$$
$$\leq \|\nabla u_{\varepsilon,a}^{\delta,j}\|^2 + 2\int_{t_j^\delta}^{t_{j+1}^\delta} \int_\Omega \nabla u_{\varepsilon,a}^{\delta,j} \nabla \dot{g}(\tau)\, dx\, d\tau + e(\delta)\int_{t_j^\delta}^{t_{j+1}^\delta} \|\dot{g}(\tau)\|_{H^1(\Omega)}\, d\tau, \quad (5.19)$$

where

$$e(\delta) := \max_{r=0,\ldots,N_\delta-1} \int_{t_r^\delta}^{t_{r+1}^\delta} \|\dot{g}(\tau)\|_{H^1(\Omega)}\, d\tau.$$

From (5.19), we obtain that for all $0 \leq j \leq i \leq N_\delta$

$$\|\nabla u_{\varepsilon,a}^{\delta,i}\|^2 + \mathcal{H}^1(\Gamma_{\varepsilon,a}^{\delta,i}) \leq \|\nabla u_{\varepsilon,a}^{\delta,j}\|^2 + \mathcal{H}^1(\Gamma_{\varepsilon,a}^{\delta,j}) +$$
$$+ 2\sum_{r=j}^{i-1} \int_{t_r^\delta}^{t_{r+1}^\delta} \int_\Omega \nabla u_{\varepsilon,a}^{\delta,r} \nabla \dot{g}(\tau)\, dx\, d\tau + e(\delta)\int_{t_j^\delta}^{t_i^\delta} \|\dot{g}(\tau)\|_{H^1(\Omega)}\, d\tau,$$

and so the proof of point (c) is complete choosing

$$o^\delta := e(\delta)\int_0^1 \|\dot{g}(\tau)\|_{H^1(\Omega)}\, d\tau.$$

□

5.4 The convergence result

This section is devoted to the proof of Theorem 5.1.1. As in Section 5.3, let Ω be a polygonal open bounded subset of $\mathbb{R}^2$, and let $\partial_D\Omega \subseteq \partial\Omega$ be open in the relative topology. For all $\varepsilon > 0$, let $\mathbf{R}_\varepsilon \in \mathcal{R}_\varepsilon(\Omega)$ be a regular triangulation of Ω such that $\partial_D\Omega$ is composed of edges of $\mathbf{R}_\varepsilon$. As in the previous section, let $\mathcal{AF}_\varepsilon(\Omega)$ be the family of continuous piecewise affine functions with respect to $\mathbf{R}_\varepsilon$, and let $\mathcal{A}_{\varepsilon,a}(\Omega)$ be the family of functions which are affine on the triangles of some triangulation $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$ nested in $\mathbf{R}_\varepsilon$ and can jump across the edges of $\mathbf{T}$.

In the following, it will be useful to treat points at which the boundary condition is violated (see (5.11)) as internal jumps. Thus we consider Ω_D polygonal open bounded subset of $\mathbb{R}^2$ such that $\Omega_D \cap \Omega = \emptyset$ and $\partial\Omega \cap \partial\Omega_D = \partial_D\Omega$ up to a finite number of points; we set $\Omega' := \Omega \cup \Omega_D \cup \partial_D\Omega$. Given $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$ and $g \in \mathcal{AF}_\varepsilon(\Omega)$, we may extend g to a function of $H^1(\Omega')$ and u to a function $\tilde{u} \in SBV(\Omega')$ setting $\tilde{u} = g$ on Ω_D. In this way, recalling (5.12), we have

$$S^g(u) = S(\tilde{u}),$$

so that the violation of the boundary condition of u can be read in the set of jumps of $\tilde{u}$. Analogously, given $u \in SBV(\Omega)$ and $g \in H^1(\Omega)$, we set

$$S^g(u) := S(u) \cup \{x \in \partial_D\Omega \,:\, \gamma(u)(x) \neq \gamma(g)(x)\}. \tag{5.20}$$

where γ denotes the trace operator on $\partial\Omega$. We may assume $g \in H^1(\Omega')$ using an extension operator. We can then consider $\tilde{u} \in SBV(\Omega')$ such that $\tilde{u} = u$ on Ω, and $\tilde{u} = g$ on Ω_D. In this way we have

$$S^g(u) = S(\tilde{u}) \quad \text{up to a set of } \mathcal{H}^1\text{-measure } 0.$$

Let us consider $g \in W^{1,1}([0,1], H^1(\Omega))$ such that $\|g(t)\|_\infty \leq C$ for all $t \in [0,1]$ and let $g_\varepsilon \in W^{1,1}([0,1], H^1(\Omega))$ be such that $g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega)$ for all $t \in [0,1]$,

$$\|g_\varepsilon(t)\|_\infty \leq C \tag{5.21}$$

for all $t \in [0,1]$, and for $\varepsilon \to 0$

$$g_\varepsilon \to g \quad \text{strongly in } W^{1,1}([0,1], H^1(\Omega)). \tag{5.22}$$

We indicate by $\{u^{\delta,i}_{\varepsilon,a},\ i = 0, \dots, N_\delta\}$ the discrete evolution relative to the boundary data g_ε given by Proposition 5.3.1, and we denote by $\mathcal{E}^{\delta,i}_{\varepsilon,a}$ its total energy as in Proposition 5.3.3.

We assume that $g(\cdot)$ and $g_\varepsilon(\cdot)$ are defined in $H^1(\Omega')$ (we still denote these extensions by $g(\cdot)$ and $g_h(\cdot)$), in such a way that (5.21) and (5.22) hold in Ω'. Let us moreover set $g^\delta_\varepsilon(t) := g_\varepsilon(t^\delta_i)$ for all $t^\delta_i \leq t < t^\delta_{i+1}$ with $i = 0, \dots, N_\delta - 1$ and $g^\delta_\varepsilon(1) := g_\varepsilon(1)$.

Let us make the following piecewise constant interpolation in time:

$$u^\delta_{\varepsilon,a}(t) := u^{\delta,i}_{\varepsilon,a} \quad \text{for } t^\delta_i \leq t < t^\delta_{i+1} \quad i = 0, \dots, N_\delta - 1,$$

and $u^\delta_{\varepsilon,a}(1) := u^{\delta,N_\delta}_{\varepsilon,a}$. For all $t \in [0,1]$ we define the *discrete fracture* at time t as

$$\Gamma^\delta_{\varepsilon,a}(t) := \bigcup_{s \leq t} S^{g^\delta_\varepsilon(s)}(u^\delta_{\varepsilon,a}(s)),$$

and the *discrete total energy* at time t as

$$\mathcal{E}^\delta_{\varepsilon,a}(t) := \|\nabla u^\delta_{\varepsilon,a}(t)\|^2 + \mathcal{H}^1\left(\Gamma^\delta_{\varepsilon,a}(t)\right).$$

We have for all $t \in [0,1]$

$$\|u^\delta_{\varepsilon,a}(t)\|_\infty \leq \|g^\delta_\varepsilon(t)\|_\infty. \tag{5.23}$$

Moreover for all $v \in \mathcal{A}_{\varepsilon,a}(\Omega)$ we have

$$\|\nabla u^\delta_{\varepsilon,a}(0)\|^2 + \mathcal{H}^1\left(S^{g^\delta_\varepsilon(0)}(u^\delta_{\varepsilon,a}(0))\right) \leq \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g^\delta_\varepsilon(0)}(v)\right), \tag{5.24}$$

and for all $t \in]0,1]$ and for all $v \in \mathcal{A}_{\varepsilon,a}(\Omega)$

$$\|\nabla u^\delta_{\varepsilon,a}(t)\|^2 \leq \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g^\delta_\varepsilon(t)}(v) \setminus \Gamma^\delta_{\varepsilon,a}(t)\right). \tag{5.25}$$

Finally for all $0 \leq s \leq t \leq 1$ we have

$$\mathcal{E}^\delta_{\varepsilon,a}(t) \leq \mathcal{E}^\delta_{\varepsilon,a}(s) + 2\int_{s^\delta_i}^{t^\delta_i} \int_\Omega \nabla u^\delta_{\varepsilon,a}(\tau) \nabla \dot{g}_\varepsilon(\tau)\, dx\, d\tau + o^\delta_\varepsilon, \tag{5.26}$$

where $t_i^\delta \leq t < t_{i+1}^\delta$, $s_i^\delta \leq s < s_{i+1}^\delta$ and

$$o_\varepsilon^\delta := \left[\max_{r=0,\dots,N_\delta-1} \int_{t_r^\delta}^{t_{r+1}^\delta} \|\dot g_\varepsilon(\tau)\|_{H^1(\Omega)}\,d\tau\right] \int_0^1 \|\dot g_\varepsilon(\tau)\|_{H^1(\Omega)}. \tag{5.27}$$

For $s=0$ we obtain the following estimate from above for the discrete total energy

$$\mathcal{E}_{\varepsilon,a}^\delta(t) \leq \mathcal{E}_{\varepsilon,a}^\delta(0) + 2\int_0^{t_i^\delta}\int_\Omega \nabla u_{\varepsilon,a}^\delta(\tau)\nabla \dot g_\varepsilon(\tau)\,dx\,d\tau + o_\varepsilon^\delta, \tag{5.28}$$

where $t_i^\delta \leq t < t_{i+1}^\delta$.

We study the behavior of the evolution $\{t \to u_{\varepsilon,a}^\delta(t),\ t \in [0,1]\}$ varying the parameters in the following way. We let firstly $\varepsilon \to 0$ and $\delta \to 0$ obtaining an evolution $\{t \to u_a(t),\ t \in [0,1]\}$ relative to the boundary data g with the minimality property (5.36); then we let $a \to 0$ obtaining a quasi-static evolution of brittle fractures $\{t \to u(t)\,, t \in [0,1]\}$ relative to the boundary data g. Finally, by a diagonal argument we deal with (δ, ε, a) at the same time.

In order to develop this program, we need some compactness, and so we derive a bound for the total energy $\mathcal{E}_{\varepsilon,a}^\delta$. By (5.14), we have that for all $t \in [0,1]$

$$\|\nabla u_{\varepsilon,a}^\delta(t)\| \leq \|\nabla g_\varepsilon^\delta(t)\| \leq \tilde C$$

with $\tilde C$ independent of δ, ε and t. We deduce for all $t \in [0,1]$

$$\mathcal{E}_{\varepsilon,a}^\delta(t) \leq \mathcal{E}_{\varepsilon,a}^\delta(0) + 2\tilde C^2 + o_\varepsilon^\delta$$

Notice that $\mathcal{E}_{\varepsilon,a}^\delta(0)$ is uniformly bounded as δ, ε vary. Moreover, by (5.23) and since $\|g_\varepsilon(t)\|_\infty \leq C$ for all $t \in [0,1]$, we have that $u_{\varepsilon,a}^\delta(t)$ is uniformly bounded in $L^\infty(\Omega)$ independently of δ, ε and a. Taking into account (5.22), we conclude that there exists C' independent of δ, ε, a such that for all $t \in [0,1]$

$$\mathcal{E}_{\varepsilon,a}^\delta(t) + \|u_{\varepsilon,a}^\delta(t)\|_\infty \leq C'. \tag{5.29}$$

Formula (5.29) gives the desired compactness in order to perform the asymptotic analysis of the discrete evolution.

Let now consider $\delta_n \to 0$ and $\varepsilon_n \to 0$: by (5.22) we have

$$o_{\varepsilon_n}^{\delta_n} \to 0, \tag{5.30}$$

where $o_{\varepsilon_n}^{\delta_n}$ is defined in (5.27). By Helly's theorem on monotone functions, we may suppose that there exists an increasing function λ_a such that (up to a subsequence) for all $t \in [0,1]$

$$\lambda_{n,a}(t) := \mathcal{H}^1\left(\bigcup_{s\leq t} S^{g_{\varepsilon_n}^{\delta_n}(s)}(u_{\varepsilon_n,a}^{\delta_n}(s))\right) \to \lambda_a(t). \tag{5.31}$$

Let us fix $D \subseteq [0,1]$ countable and dense with $0 \in D$.

Lemma 5.4.1. *For all $t \in D$ there exists $u_a(t) \in SBV(\Omega)$ such that up to a subsequence independent of t*

$$u_{\varepsilon_n,a}^{\delta_n}(t) \to u_a(t) \quad \text{in } SBV(\Omega).$$

Moreover for all $t \in D$ we have

$$\|\nabla u_a(t)\|^2 + \mathcal{H}^1\left(S^{g(t)}(u_a(t))\right) + \|u_a(t)\|_\infty \leq C'. \tag{5.32}$$

Proof. Let us consider $t \in D$. By (5.29), we can apply Ambrosio's Compactness Theorem 3.8.1 obtaining $u \in SBV(\Omega)$ such that, up to a subsequence, $u^{\delta_n}_{\varepsilon_n,a}(t) \to u$ in $SBV(\Omega)$. Let us set $u_a(t) := u$. Using a diagonal argument, we deduce that there exists a subsequence of $(\delta_n, \varepsilon_n)$ (which we still denote by $(\delta_n, \varepsilon_n)$) such that $u^{\delta_n}_{\varepsilon_n,a}(t) \to u_a(t)$ in $SBV(\Omega)$ for all $t \in D$. In order to obtain inequality (5.32), we extend $u^{\delta_n}_{\varepsilon_n,a}(t)$ and $u_a(t)$ to Ω' setting $u^{\delta_n}_{\varepsilon_n,a}(t) := g^{\delta_n}_{\varepsilon_n}(t)$ and $u_a(t) := g(t)$ on Ω_D; since $g^{\delta_n}_{\varepsilon_n}(t) \to g(t)$ on Ω_D strongly in $H^1(\Omega_D)$, we have that $u^{\delta_n}_{\varepsilon_n,a}(t) \to u_a(t)$ in $SBV(\Omega')$, so that we can apply Ambrosio's Theorem, and derive (5.32) from (5.29). $\square$

The following result is essential for the sequel: its proof is postponed to Section 5.5.

Proposition 5.4.2. *Let $t \in D$. For all $v \in SBV(\Omega)$ we have*

$$\|\nabla u_a(t)\|^2 \le \|\nabla v\|^2 + \mu(a)\mathcal{H}^1(S^{g(t)}(v) \setminus \bigcup_{s \le t, s \in D} S^{g(s)}(u_a(s))), \tag{5.33}$$

where $\mu :]0, \frac{1}{2}[\to]0, +\infty[$ is such that $\lim_{a\to 0} \mu(a) = 1$. Moreover, $\nabla u^{\delta_n}_{\varepsilon_n,a}(t) \to \nabla u_a(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$.

We now extend the evolution $\{t \to u_a(t) : t \in D\}$ to the entire interval $[0, 1]$. Let us set for all $t \in [0, 1]$

$$\Gamma_a(t) := \bigcup_{s \le t, s \in D} S^{g(s)}(u_a(s)).$$

Lemma 5.4.3. *For every $t \in [0, 1]$ there exists $u_a(t) \in SBV(\Omega)$ such that the following hold:*

(a) for all $t \in [0, 1]$

$$S^{g(t)}(u_a(t)) \subseteq \Gamma_a(t) \quad \textit{up to a set of } \mathcal{H}^1\textit{-measure } 0, \tag{5.34}$$

and

$$\|\nabla u_a(t)\|^2 + \mathcal{H}^1(S^{g(t)}(u_a(t))) + \|u_a(t)\|_\infty \le C'; \tag{5.35}$$

(b) for all $v \in SBV(\Omega)$

$$\|\nabla u_a(t)\|^2 \le \|\nabla v\|^2 + \mu(a)\mathcal{H}^1\left(S^{g(t)}(v) \setminus \Gamma_a(t)\right); \tag{5.36}$$

(c) ∇u_a is left continuous in $[0, 1] \setminus D$ with respect to the strong topology of $L^2(\Omega; \mathbb{R}^2)$;

(d) for all $t \in [0, 1] \setminus \mathcal{N}_a$ we have that

$$\nabla u^{\delta_n}_{\varepsilon_n,a}(t) \to \nabla u_a(t) \quad \textit{strongly in } L^2(\Omega, \mathbb{R}^2),$$

where $\mathcal{N}_a$ is the set of discontinuities of the function λ_a defined in (5.31).

Proof. Let $t \in [0, 1] \setminus D$ and let $t_n \in D$ with $t_n \nearrow t$. By (5.32), we can apply Ambrosio's Theorem to the sequence $(u_a(t_n))$ obtaining $u \in SBV(\Omega)$ such that, up to a subsequence, $u_a(t_n) \to u$ in $SBV(\Omega)$. Let us set $u_a(t) := u$. Let us extend $u_a(t_n)$ and $u_a(t)$ to Ω' setting $u_a(t_n) := g(t_n)$ and $u_a(t) := g(t)$ on Ω_D: we have $u_a(t_n) \to u_a(t)$ in $SBV(\Omega')$. Since $\mathcal{H}^1 \llcorner S(u_a(t_n)) \le \mathcal{H}^1 \llcorner \Gamma_a(t)$ for all n, as a consequence of Ambrosio's Theorem, we deduce that $\mathcal{H}^1 \llcorner S(u_a(t)) \leq \mathcal{H}^1 \llcorner \Gamma_a(t)$. This means $\mathcal{H}^1 \llcorner S^{g(t)}(u_a(t)) \leq \mathcal{H}^1 \llcorner \Gamma_a(t)$, so that (5.34) holds. Moreover, for all $v \in SBV(\Omega)$, by (5.33) we may write

$$\|\nabla u_a(t_n)\|^2 \leq \|\nabla v - \nabla g(t) + \nabla g(t_n)\|^2 + \mu(a)\mathcal{H}^1\left(S^{g(t)}(v) \setminus \Gamma_a(t_n)\right) \leq$$
$$\leq \|\nabla v - \nabla g(t) + \nabla g(t_n)\|^2 + \mu(a)\mathcal{H}^1\left(S^{g(t)}(v) \setminus \Gamma_a(t)\right) + \mu(a)\mathcal{H}^1\left(\Gamma_a(t) \setminus \Gamma_a(t_n)\right), \quad (5.37)$$

so that, since by definition of $\Gamma_a(t)$ we have $\mathcal{H}^1(\Gamma_a(t) \setminus \Gamma_a(t_n)) \to 0$, we obtain that (5.36) holds; choosing $v = u_a(t)$ and taking the limsup in (5.37), we obtain that

$$\limsup_n \|\nabla u_a(t_n)\|^2 \leq \|\nabla u_a(t)\|^2,$$

and so the convergence $\nabla u_a(t_n) \to \nabla u_a(t)$ is strong in $L^2(\Omega, \mathbb{R}^2)$. Notice that $\nabla u_a(t)$ is uniquely determined by (5.34) and (5.36) since the gradient of the solutions of the minimum problem

$$\min\left\{\|\nabla u\|^2 \,:\, S^{g(t)}(u) \subseteq \Gamma_a(t) \text{ up to a set of } \mathcal{H}^1\text{-measure } 0\right\}$$

is unique by the strict convexity of the functional: we conclude that $\nabla u_a(t)$ is well defined. The same arguments prove that ∇u_a is left continuous at all $t \in [0,1] \setminus D$. Finally (5.35) is a direct consequence of (5.32) and of Ambrosio's Theorem, and so points (a), (b), (c) are proved.

Let us come to point (d). Let us consider $u^{\delta_n}_{\varepsilon_n,a}(t)$ with $t \notin \mathcal{N}_a$; we may suppose that $t \notin D$, since otherwise the result has already been established. By Proposition 5.4.2 with $D' := D \cup \{t\}$ in place of D, we have that, up to a subsequence, $u^{\delta_n}_{\varepsilon_n,a}(t) \to u$ in $SBV(\Omega)$ such that

$$\|\nabla u\|^2 \leq \|\nabla v\|^2 + \mu(a)\mathcal{H}^1\left(S^{g(t)}(v) \setminus (\Gamma_a(t) \cup S^{g(t)}(u))\right)$$

for all $v \in SBV(\Omega)$ and $\nabla u^{\delta_n}_{\varepsilon_n,a}(t) \to \nabla u$ strongly in $L^2(\Omega;\mathbb{R}^2)$. Let $s < t$ with $s \in D$; by the minimality of $u^{\delta_n}_{\varepsilon_n,a}(s)$ and by (5.29) we have

$$\|\nabla u^{\delta_n}_{\varepsilon_n,a}(s)\|^2 \leq \|\nabla u^{\delta_n}_{\varepsilon_n,a}(t) - \nabla g^{\delta_n}_{\varepsilon_n}(t) + \nabla g^{\delta_n}_{\varepsilon_n}(s)\|^2 + \lambda_{n,a}(t) - \lambda_{n,a}(s)$$
$$\leq \|\nabla u^{\delta_n}_{\varepsilon_n,a}(t)\|^2 + 2\sqrt{C'}\|\nabla g^{\delta_n}_{\varepsilon_n}(t) - \nabla g^{\delta_n}_{\varepsilon_n}(s)\| + \|\nabla g^{\delta_n}_{\varepsilon_n}(t) - \nabla g^{\delta_n}_{\varepsilon_n}(s)\|^2 + \lambda_{n,a}(t) - \lambda_{n,a}(s).$$

Passing to the limit for $n \to +\infty$, recalling that $g^{\delta_n}_{\varepsilon_n}(\tau) \to g(\tau)$ strongly in $H^1(\Omega)$ for all $\tau \in [0,1]$, we deduce

$$\|\nabla u_a(s)\|^2 \leq \|\nabla u\|^2 + 2\sqrt{C'}\|\nabla g(t) - \nabla g(s)\| + \|\nabla g(t) - \nabla g(s)\|^2 + \lambda_a(t) - \lambda_a(s),$$

so that, since t is a point of continuity for λ_a, ∇u_a is left continuous at t, and g is absolutely continuous, we get for $s \to t$

$$\|\nabla u_a(t)\|^2 \leq \|\nabla u\|^2.$$

We conclude that $u_a(t)$ is a solution of

$$\min\{\|\nabla v\|^2 \,:\, S^{g(t)}(v) \subseteq \Gamma_a(t) \cup S^{g(t)}(u) \text{ up to a set of } \mathcal{H}^1\text{-measure } 0\},$$

so that $\nabla u = \nabla u_a(t)$ by uniqueness of the gradient of the solution. We deduce that $\nabla u^{\delta_n}_{\varepsilon_n,a}(t) \to \nabla u_a(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$, and so the proof is complete. □

We can now let $a \to 0$.

Lemma 5.4.4. *There exists $a_n \to 0$ such that, for all $t \in D$, $u_{a_n}(t) \to u(t)$ in $SBV(\Omega)$ for some $u(t) \in SBV(\Omega)$ such that for all $v \in SBV(\Omega)$ we have*

$$\|\nabla u(t)\|^2 \leq \|\nabla v\|^2 + \mathcal{H}^1(S^{g(t)}(v) \setminus \bigcup_{s \leq t, s \in D} S^{g(s)}(u(s))). \tag{5.38}$$

Moreover, $\nabla u_{a_n}(t) \to \nabla u(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$ and

$$\|\nabla u(t)\|^2 + \mathcal{H}^1(S^{g(t)}(u(t))) + \|u(t)\|_\infty \leq C'. \tag{5.39}$$

Proof. By (5.35), applying Ambrosio's Theorem to the extensions of $u_a(t)$ to Ω' by setting $u_a(t) := g(t)$ on Ω_D, and using a diagonal argument, we find a sequence $a_n \to 0$ such that, for all $t \in D$, $u_{a_n}(t) \to u(t)$ in $SBV(\Omega)$ for some $u(t) \in SBV(\Omega)$ such that (5.39) holds.

We now prove that $u(t)$ satisfies property (5.38). Let $v \in SBV(\Omega)$. Let us fix $t_1 \leq t_2 \leq \ldots \leq t_k = t$ with $t_i \in D$. We extend v and $u_{a_n}(t_i)$ to Ω' setting $v := g(t)$ and $u_{a_n}(t_i) := g(t_i)$ on Ω_D respectively. Since $u_{a_n}(t_i) \to u(t_i)$ in $SBV(\Omega')$ for all $i = 1, \ldots, k$, by Theorem 3.8.4 there exists $v_n \in SBV(\Omega')$ with $v_n = g(t)$ on Ω_D such that $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega'; \mathbb{R}^2)$ and

$$\limsup_n \mathcal{H}^1\left(S(v_n) \setminus \bigcup_{i=1}^k S(u_{a_n}(t_i))\right) \leq \mathcal{H}^1\left(S(v) \setminus \bigcup_{i=1}^k S(u(t_i))\right). \tag{5.40}$$

By (5.33) we obtain

$$\|\nabla u_{a_n}(t)\|^2 \leq \|\nabla v_n\|^2 + \mu(a_n)\mathcal{H}^1\left(S(v_n) \setminus \bigcup_{i=1}^k S(u_{a_n}(t_i))\right), \tag{5.41}$$

so that passing to the limit for $n \to +\infty$ and recalling that $\mu(a) \to 1$ as $a \to 0$, we obtain

$$\|\nabla u(t)\|^2 \leq \|\nabla v\|^2 + \mathcal{H}^1\left(S(v) \setminus \bigcup_{i=1}^k S(u(t_i))\right).$$

Thus we get

$$\|\nabla u(t)\|^2 \leq \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g(t)}(v) \setminus \bigcup_{i=1}^k S^{g(t_i)}(u(t_i))\right).$$

Since $t_1, \ldots, t_k$ are arbitrary, we obtain (5.38). Choosing $v = u(t)$, taking the limsup in (5.41) and using (5.40), we obtain $\nabla u_{a_n}(t) \to \nabla u(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$. □

In order to deal with δ, ε and a at the same time, we need the following lemma.

Lemma 5.4.5. *Let $\{u(t) : t \in D\}$ be as in Lemma 5.4.4. There exist $\delta_n \to 0$, $\varepsilon_n \to 0$, and $a_n \to 0$ such for all $t \in D$ we have*

$$u^{\delta_n}_{\varepsilon_n, a_n}(t) \to u(t) \quad \text{in } SBV(\Omega).$$

Moreover, for all n there exists $\mathcal{B}_n \subseteq [0,1]$ with $|\mathcal{B}_n| < 2^{-n}$ such that for all $t \in [0,1] \setminus \mathcal{B}_n$

$$\|\nabla u^{\delta_n}_{\varepsilon_n, a_n}(t) - \nabla u_{a_n}(t)\| \leq \frac{1}{n}. \tag{5.42}$$

Finally, we have that for all $v \in SBV(\Omega)$

$$\|\nabla u(0)\|^2 + \mathcal{H}^1\left(S^{g(0)}(u(0))\right) \leq \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g(0)}(v)\right) \tag{5.43}$$

and

$$\mathcal{E}^{\delta_n}_{\varepsilon_n,a_n}(0) \to \|\nabla u(0)\|^2 + \mathcal{H}^1\left(S^{g(0)}(u(0))\right). \tag{5.44}$$

Proof. Let (a_n) be the sequence determined by Lemma 5.4.4. By Lemma 5.4.1, for all n there exists $(\delta^n_m, \varepsilon^n_m)$ such that for all $t \in D$ and $m \to +\infty$ we have

$$u^{\delta^n_m}_{\varepsilon^n_m,a_n}(t) \to u_{a_n}(t) \quad \text{in } SBV(\Omega),$$

and

$$\nabla u^{\delta^n_m}_{\varepsilon^n_m,a_n}(t) \to \nabla u_{a_n}(t) \quad \text{strongly in } L^2(\Omega;\mathbb{R}^2).$$

Moreover by Lemma 5.4.3 we have that $\nabla u^{\delta^n_m}_{\varepsilon^n_m,a_n} \to \nabla u_{a_n}$ quasi-uniformly on $[0,1]$ as $m \to +\infty$. Let $\mathcal{B}_n \subseteq [0,1]$ with $|\mathcal{B}_n| < 2^{-n}$ such that $\nabla u^{\delta^n_m}_{\varepsilon^n_m,a_n} \to \nabla u_{a_n}$ uniformly on $[0,1] \setminus \mathcal{B}_n$ as $m \to +\infty$. We now perform the following diagonal argument. Let $D = \{t_n,\, n \geq 1\}$. Choose m_1 such that

$$\|\nabla u^{\delta^1_{m_1}}_{\varepsilon^1_{m_1},a_1}(t_1) - \nabla u_{a_1}(t_1)\| + \|u^{\delta^1_{m_1}}_{\varepsilon^1_{m_1},a_1}(t_1) - u_{a_1}(t_1)\| \leq 1,$$

and

$$\|\nabla u^{\delta^1_{m_1}}_{\varepsilon^1_{m_1},a_1}(t) - \nabla u_{a_1}(t)\| \leq 1 \quad \text{for all } t \in [0,1] \setminus \mathcal{B}_1 .$$

Let m_n be such that

$$\|\nabla u^{\delta^n_{m_n}}_{\varepsilon^n_{m_n},a_n}(t_j) - \nabla u_{a_n}(t_j)\| + \|u^{\delta^n_{m_n}}_{\varepsilon^n_{m_n},a_n}(t_j) - u_{a_n}(t_j)\| \leq \frac{1}{n} \quad \text{for all } j = 1, \dots, n$$

and

$$\|\nabla u^{\delta^n_{m_n}}_{\varepsilon^n_{m_n},a_n}(t) - \nabla u_{a_n}(t)\| \leq \frac{1}{n} \quad \text{for all } t \in [0,1] \setminus \mathcal{B}_n .$$

We may suppose that $\delta^n_{m_n} \to 0$, $\varepsilon^n_{m_n} \to 0$. Then $(\delta^n_{m_n}, \varepsilon^n_{m_n}, a_n)$ is the sequence which satisfies the thesis. In fact by construction and taking into account (5.29), for all $t \in D$ we have $u^{\delta^n_{m_n}}_{\varepsilon^n_{m_n},a_n}(t) \to u(t)$ in $SBV(\Omega)$; moreover the set $\mathcal{B}_n$ satisfies (5.42). Notice that $u^{\delta^n_{m_n}}_{\varepsilon^n_{m_n},a^n_{m_n}}(0)$ satisfies (5.24) and so (5.43) and (5.44) follow by the Γ-convergence result of [62]. $\square$

Let $(\delta_n, \varepsilon_n, a_n)$ be the sequence determined by Lemma 5.4.5. For all $t \in [0,1]$ let us set

$$\lambda_n(t) := \mathcal{H}^1\left(\Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)\right).$$

By Helly's theorem, we may suppose that there exist two increasing functions λ and η such that up to a subsequence

$$\lambda_n \to \lambda \quad \text{pointwise in } [0,1];$$

and

$$\lambda_{a_n} \to \eta \quad \text{pointwise in } [0,1]; \tag{5.45}$$

where λ_{a_n} is defined as in (5.31). We now extend the evolution $\{t \to u(t) : t \in D\}$ to the entire interval $[0,1]$. Let us set for all $t \in [0,1]$

$$\Gamma(t) := \bigcup_{s \leq t, s \in D} S^{g(s)}(u(s)),$$

and let $\mathcal{N}$ be the set of discontinuities of $\mathcal{H}^1(\Gamma(\cdot))$. Notice that for all $t \in [0,1]$

$$\mathcal{H}^1(\Gamma(t)) \leq \lambda(t). \tag{5.46}$$

In fact if $t \in D$, let $t_1 \leq t_2 \leq \ldots \leq t_k = t$ with $t_i \in D$, consider $w_n \in SBV(\Omega'; \mathbb{R}^k)$ defined as

$$w_n(x) := (u^{\delta_n}_{\varepsilon_n, a_n}(t_1)(x), \ldots, u^{\delta_n}_{\varepsilon_n, a_n}(t_k)(x)),$$

where we assume that $u^{\delta_n}_{\varepsilon_n, a_n}(t_i) = g^{\delta_n}_{\varepsilon_n}(t_i)$ on Ω_D. We have $w_n \to w := (u(t_1), \ldots, u(t_k))$ in $SBV(\Omega'; \mathbb{R}^k)$, where $u(t_i) = g(t_i)$ on Ω_D. Note that for all n we have $S(w_n) = \bigcup_{i=1}^k S(u^{\delta_n}_{\varepsilon_n, a_n}(t_i))$ so that

$$\mathcal{H}^1(S(w_n)) \leq \lambda_n(t).$$

Passing to the limit for $n \to +\infty$ and applying Ambrosio's Theorem we get

$$\mathcal{H}^1\left(\bigcup_{i=1}^k S(u(t_i))\right) = \mathcal{H}^1(S(w)) \leq \liminf_n \mathcal{H}^1(S(w_n)) \leq \lambda(t);$$

we thus have

$$\mathcal{H}^1\left(\bigcup_{i=1}^k S^{g(t_i)}(u(t_i))\right) = \mathcal{H}^1(S(w)) \leq \lambda(t)$$

and taking the sup over all $t_1, \ldots, t_k$, we obtain (5.46) in D. The case $t \notin D$ follows since $\mathcal{H}^1(\Gamma(\cdot))$ is left continuous by definition.

Lemma 5.4.6. *For every $t \in [0,1]$ there exists $u(t) \in SBV(\Omega)$ such that the following hold:*

(a) for all $t \in [0,1]$

$$S^{g(t)}(u(t)) \subseteq \Gamma(t) \quad \text{up to a set of } \mathcal{H}^1\text{-measure } 0, \tag{5.47}$$

and for all $t \in [0,1]$ and for all $v \in SBV(\Omega)$

$$\|\nabla u(t)\|^2 \leq \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g(t)}(v) \setminus \Gamma(t)\right); \tag{5.48}$$

(b) ∇u is continuous in $[0,1] \setminus (D \cup \mathcal{N})$ with respect to the strong topology of $L^2(\Omega; \mathbb{R}^2)$;

(c) if $\tilde{\mathcal{N}}$ is the set of discontinuities of the function η defined in (5.45), for all $t \in [0,1] \setminus \tilde{\mathcal{N}}$ we have that

$$\nabla u_{a_n}(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega, \mathbb{R}^2).$$

Finally

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau, \tag{5.49}$$

where

$$\mathcal{E}(t) := \|\nabla u(t)\|^2 + \mathcal{H}^1(\Gamma(t)).$$

Proof. The definition of $u(t)$ is carried out as in Lemma 5.4.3 considering $t \in [0,1] \setminus D$, $t_n \in D$ with $t_n \nearrow t$, and the limit (up to a subsequence) of $u(t_n)$ in $SBV(\Omega)$: (5.47) and (5.48) hold, so that point (a) is proved. It turns out that $\nabla u(t)$ is uniquely determined and that it is left continuous in $[0,1] \setminus D$. Let us consider $t \in [0,1] \setminus (D \cup \mathcal{N})$, and let $t_n \searrow t$. By Ambrosio's Theorem, we have that there exists $u \in SBV(\Omega)$ with such that, up to a subsequence, $u(t_n) \to u$ in $SBV(\Omega)$. Since t is a continuity point of $\mathcal{H}^1(\Gamma(\cdot))$, we deduce that $S^{g(t)}(u) \subseteq \Gamma(t)$ up to a set of $\mathcal{H}^1$-measure 0. Moreover by the minimality property for $u(t_n)$ and the fact $\Gamma(t) \subseteq \Gamma(t_n)$, we have that for all $v \in SBV(\Omega)$ with

$$\|\nabla u(t_n)\|^2 \leq \|\nabla v - \nabla g(t) + \nabla g(t_n)\|^2 + \mathcal{H}^1\left(S^{g(t)}(v) \setminus \Gamma(t_n)\right) \leq$$
$$\leq \|\nabla v - \nabla g(t) + \nabla g(t_n)\|^2 + \mathcal{H}^1\left(S^{g(t)}(v) \setminus \Gamma(t)\right),$$

and so we deduce that (5.48) holds with u in place of $u(t)$, and that $\nabla u(t_n) \to \nabla u$ strongly in $L^2(\Omega;\mathbb{R}^2)$. We obtain by uniqueness that $\nabla u = \nabla u(t)$, and so $\nabla u(\cdot)$ is continuous in $[0,1] \setminus (D \cup \mathcal{N})$ and this proves point (b). Point (c) follows in the same way of point (d) of Lemma 5.4.3.

Let us come to the proof of (5.49). Given $t \in [0,1]$ and $k > 0$, let $s_i^k := \frac{i}{k}t$ for all $i = 0, \ldots, k$. Let us set $u^k(s) := u(s_{i+1}^k)$ for $s_i^k < s \leq s_{i+1}^k$. By (5.48), comparing $u(s_i^k)$ with $u(s_{i+1}^k) - g(s_{i+1}^k) + g(s_i^k)$, it is easy to see that

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u^k(\tau)\nabla \dot{g}(\tau)\, d\tau\, dx + o_k,$$

where $o_k \to 0$ as $k \to +\infty$. Since ∇u is continuous with respect to the strong topology of $L^2(\Omega;\mathbb{R}^2)$ in $[0,1]$ up to a countable set, passing to the limit for $k \to +\infty$ we deduce (5.49). □

We are now ready to prove the main result of the chapter.

PROOF OF THEOREM 5.1.1. Let D be a countable and dense set in $[0,1]$ such that $0 \in D$, and let $(\delta_n, \varepsilon_n, a_n)$ and $\{t \to u(t) \in SBV(\Omega) \,:\, t \in [0,1]\}$ be the sequence and the evolution determined in Lemma 5.4.5 and Lemma 5.4.6. Let us set

$$u_n := u^{\delta_n}_{\varepsilon_n,a_n}, \qquad \Gamma_n := \Gamma^{\delta_n}_{\varepsilon_n,a_n}, \qquad \mathcal{E}_n := \mathcal{E}^{\delta_n}_{\varepsilon_n,a_n}.$$

Let $\overline{\mathcal{N}}$ be the union of the sets of discontinuities of η and $\mathcal{H}^1(\Gamma(\cdot))$, where η is defined in (5.45). Let $\mathcal{B} := \bigcap_{k=1}^{+\infty}\bigcup_{h=k}^{\infty}\mathcal{B}_h$, where $\mathcal{B}_h$ are as in Lemma 5.4.5; since $|\bigcup_{h=k}^{\infty}\mathcal{B}_h| < 2^{-k+1}$, we have $|\mathcal{B}| = 0$. For all $t \in [0,1] \setminus (\mathcal{B} \cup \overline{\mathcal{N}})$ we claim that

$$\nabla u_n(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega;\mathbb{R}^2). \tag{5.50}$$

In fact, since $t \notin \bigcup_{h=k}^{\infty}\mathcal{B}_h$ for some k, by Lemma 5.4.5 we have

$$\lim_n \|\nabla u^{\delta_n}_{\varepsilon_n,a_n}(t) - \nabla u_{a_n}(t)\| = 0;$$

for $t \notin \overline{\mathcal{N}}$, by Lemma 5.4.6 we have that $\nabla u_{a_n}(t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ and so (5.50) holds.

Since $g_{\varepsilon_n} \to g$ strongly in $W^{1,1}([0,1];H^1(\Omega))$, we deduce that for a.e. $\tau \in [0,1]$

$$\nabla \dot{g}_{\varepsilon_n}(\tau) \to \nabla \dot{g}(\tau) \quad \text{strongly in } L^2(\Omega;\mathbb{R}^2).$$

Since $\mathcal{E}_n(0) \to \mathcal{E}(0)$ by (5.44) and $o^{\delta_n}_{\varepsilon_n} \to 0$, by semicontinuity of the energy and by (5.28) we have that for all $t \in D$

$$\mathcal{E}(t) \leq \liminf_n \mathcal{E}_n(t) \leq \limsup_n \mathcal{E}_n(t) \leq \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau)\nabla \dot{g}(\tau)\, dx\, d\tau. \tag{5.51}$$

In view of (5.49), we conclude that for all $t \in D$

$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau)\nabla \dot{g}(\tau)\, dx\, d\tau,$$

and since $\nabla u(\cdot)$ and $\mathcal{H}^1(\Gamma(\cdot))$ are left continuous at $t \notin D$ and so $\mathcal{E}(\cdot)$ is, we conclude that the equality holds for all $t \in [0,1]$. As a consequence $\{t \to u(t), t \in [0,1]\}$ is a quasi-static evolution of brittle fractures. Let us prove that (5.51) is indeed true for all $t \in [0,1]$. In fact, if $t \notin D$, it is sufficient to prove

$$\liminf_n \mathcal{E}_n(t) \geq \mathcal{E}(t). \tag{5.52}$$

Considering $s \geq t$ with $s \in D$, by (5.26) we have

$$\mathcal{E}_n(s) \leq \mathcal{E}_n(t) + \int_{t_{j_n}^{\delta_n}}^{s_{j_n}^{\delta_n}} \int_\Omega \nabla u_n(\tau) \nabla \dot{g}_{\varepsilon_n}(\tau)\, dx\, d\tau + o_{\varepsilon_n}^{\delta_n} \quad t_{j_n}^{\delta_n} \leq t < t_{j_n+1}^{\delta_n},\ s_{j_n}^{\delta_n} \leq s < s_{j_n+1}^{\delta_n},$$

so that

$$\liminf_n \mathcal{E}_n(t) \geq \mathcal{E}(s) - \int_t^s \int_\Omega \nabla u(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau.$$

Letting $s \searrow t$, since $\mathcal{E}(\cdot)$ is continuous, we have (5.52) holds. By (5.51) we deduce that $\mathcal{E}_n(t) \to \mathcal{E}(t)$ for all $t \in [0,1]$, so that point (b) is proved.

We now come to point (a). Since $\lambda(t) \geq \mathcal{H}^1(\Gamma(t))$ for all $t \in [0,1]$, by (5.50) and point (b), we deduce that $\lambda = \mathcal{H}^1(\Gamma(\cdot))$ in $[0,1]$ up to a set of measure 0. Since they are increasing functions, we conclude that λ and $\mathcal{H}^1(\Gamma(\cdot))$ share the same set of continuity points $[0,1] \setminus \mathcal{N}$, and that $\lambda = \mathcal{H}^1(\Gamma(\cdot))$ on $[0,1] \setminus \mathcal{N}$. In view of (5.50), point (a) is thus established for all t except $t \in (\mathcal{B} \cup \overline{\mathcal{N}}) \setminus \mathcal{N}$. In order to treat this case, we use the following argument. Considering the measures $\mu_n := \mathcal{H}^1 \llcorner \Gamma_n(t)$, we have that, up to a subsequence, $\mu_n \stackrel{*}{\rightharpoonup} \mu$ weakly-star in the sense of measures, and as a consequence of Ambrosio's Theorem we have $\mathcal{H}^1 \llcorner \Gamma(t) \leq \mu$ as measures. Since $t \notin \mathcal{N}$ we have $\mu_n(\mathbb{R}^2) \to \mathcal{H}^1(\Gamma(t))$, and so we deduce $\mathcal{H}^1 \llcorner \Gamma(t) = \mu$. Let us consider now $u_n(t)$; we have up to a subsequence $u_n(t) \to u$ in $SBV(\Omega)$ for some $u \in SBV(\Omega)$. Setting $u_n(t) := g_{\varepsilon_n}^{\delta_n}(t)$ and $u := g(t)$ on Ω_D, we have $u_n(t) \to u$ in $SBV(\Omega')$, and as a consequence of Ambrosio's Theorem, we get that $\mathcal{H}^1 \llcorner S^{g(t)}(u) \leq \mu = \mathcal{H}^1 \llcorner \Gamma(t)$, that is $S^{g(t)}(u) \subseteq \Gamma(t)$. By Theorem 3.8.4, we deduce that u is a minimum for

$$\min\{\|\nabla v\|^2 : S^{g(t)}(v) \subseteq \Gamma(t) \text{ up to a set of } \mathcal{H}^1\text{-measure } 0\},$$

and by uniqueness of the gradient we get that $\nabla u = \nabla u(t)$, so that the proof is concluded. □

5.5 Piecewise Affine Transfer of Jump and Proof of Proposition 5.4.2

The proof of Proposition 5.4.2 is based on the following proposition, which is a variant of Theorem 3.8.4 in the context of piecewise affine approximation.

Proposition 5.5.1. *Given $\varepsilon_n \to 0$, let $g_n^r \in H^1(\Omega)$ be such that $g_n^r \in \mathcal{AF}_{\varepsilon_n}(\Omega)$ and $g_n^r \to g^r$ strongly in $H^1(\Omega)$ for all $r = 0, \dots, i$. If $u_n^r \in \mathcal{A}_{\varepsilon_n,a}(\Omega)$ is such that $u_n^r \to u^r$ in $SBV(\Omega)$ for $r = 0, \dots, i$, then for all $v \in SBV(\Omega)$ with $\mathcal{H}^1\left(S^{g^i}(v)\right) < +\infty$ and $\nabla v \in L^2(\Omega; \mathbb{R}^2)$, there exists $v_n \in \mathcal{A}_{\varepsilon_n,a}(\Omega)$ such that $v_n \to v$ strongly in $L^1(\Omega)$, $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega;\mathbb{R}^2)$ and*

$$\limsup_n \mathcal{H}^1\left(S^{g_n^i}(v_n) \setminus \bigcup_{r=0}^{i} S^{g_n^r}(u_n^r)\right) \leq \mu(a) \mathcal{H}^1\left(S^{g(t)}(v) \setminus \bigcup_{r=0}^{i} S^{g^r}(u^r)\right), \tag{5.53}$$

where $\mu :]0; \frac{1}{2}[\to \mathbb{R}$ with $\lim_{a \to 0^+} \mu(a) = 1$.

In view of Proposition 5.5.1, we can now prove Proposition 5.4.2.

Proof of Proposition 5.4.2. Notice that, in order to prove (5.33), it is sufficient to prove the existence of $\mu :]0; \frac{1}{2}[\to \mathbb{R}$ with $\lim_{a \to 0^+} \mu(a) = 1$ such that, given $t \in D$, for every $0 = t_0 \leq \ldots \leq t_r \leq \ldots \leq t_i = t$, $t_r \in D$, for all $v \in SBV(\Omega)$ we have

$$\|\nabla u_a(t)\|^2 \leq \|\nabla v\|^2 + \mu(a)\mathcal{H}^1\left(S^{g(t)}(v) \setminus \bigcup_{r=0}^{i} S^{g(t_r)}(u_a(t_r))\right). \tag{5.54}$$

In fact, taking the sup over all possible $t_0, \ldots, t_i$, we get (5.33).

We apply Proposition 5.5.1 considering $g_n^r := g_{\varepsilon_n}^{\delta_n}(t_r)$, $g^r := g(t_r)$, $u_n^r := u_{\varepsilon_n,a}^{\delta_n}(t_r)$, and $u^r := u_a(t_r)$ for $r = 0, \ldots, i$. There exists $\mu :]0; \frac{1}{2}[\to \mathbb{R}$ with $\lim_{a \to 0^+} \mu(a) = 1$ such that for $v \in SBV(\Omega)$, there exists $v_n \in \mathcal{A}_{\varepsilon_n,a}(\Omega)$ with $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega; \mathbb{R}^2)$ and

$$\limsup_n \mathcal{H}^1\left(S^{g_{\varepsilon_n}^{\delta_n}(t)}(v_n) \setminus \bigcup_{r=0}^{i} S^{g_{\varepsilon_n}^{\delta_n}(t_r)}(u_{\varepsilon_n,a}^{\delta_n}(t_r))\right) \leq \mu(a)\mathcal{H}^1\left(S^{g(t)}(v) \setminus \bigcup_{r=0}^{i} S^{g(t_r)}(u_a(t_r))\right),$$

Comparing $u_{\varepsilon_n,a}^{\delta_n}(t)$ and v_n by means of (5.25), we obtain

$$\begin{aligned}\|\nabla u_{\varepsilon_n,a}^{\delta_n}(t)\|^2 &\leq \|\nabla v_n\|^2 + \mathcal{H}^1\left(S^{g_{\varepsilon_n}^{\delta_n}(t)}(v_n) \setminus \Gamma_{\varepsilon_n,a}^{\delta_n}(t)\right) \leq \\ &\leq \|\nabla v_n\|^2 + \mathcal{H}^1\left(S^{g_{\varepsilon_n}^{\delta_n}(t)}(v_n) \setminus \bigcup_{r=0}^{i} S^{g_{\varepsilon_n}^{\delta_n}(t_r)}(u_{\varepsilon_n,a}^{\delta_n}(t_r))\right),\end{aligned} \tag{5.55}$$

so that, passing to the limit for $n \to +\infty$, we obtain that (5.54) holds. Moreover, we have that choosing $v = u_a(t)$, and taking the limsup in (5.55), we get that $\nabla u_{\varepsilon_n,a}^{\delta_n}(t) \to \nabla u_a(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$. □

The rest of the section is devoted to the proof of Proposition 5.5.1. It will be convenient, as in Section 5.4, to consider Ω_D polygonal open bounded subset of $\mathbb{R}^2$ such that $\Omega_D \cap \Omega = \emptyset$ and $\partial\Omega \cap \partial\Omega_D = \partial_D\Omega$ up to a finite number of vertices; we set $\Omega' := \Omega \cup \Omega_D \cup \partial_D\Omega$. We suppose that $\mathbf{R}_\varepsilon$ can be extended to a regular triangulation of Ω' which we still indicate by $\mathbf{R}_\varepsilon$.

We need several preliminary results. Let us set $z_n^r := u_n^r - g_n^r$, and let us extend z_n^r to zero on Ω_D. Similarly, we set $z^r := u^r - g^r$, and we extend z^r to zero on Ω_D.

Let $\sigma > 0$, and let C be the set of corners of $\partial_D\Omega$. Let us fix $G \subseteq \mathbb{R}$ countable and dense: we recall that for all $r = 0, \ldots, i$ we have up to a set of $\mathcal{H}^1$-measure zero

$$S(z^r) = \bigcup_{c_1, c_2 \in G} \partial^* E_{c_1}(r) \cap \partial^* E_{c_2}(r),$$

where $E_c(r) := \{x \in \Omega' : z^r(x) > c\}$ and ∂^* denotes the essential boundary (see [7]). Let us consider

$$J_j := \{x \in \bigcup_{r=0}^{i} S(z^r) \setminus C : (z^l)^+(x) - (z^l)^-(x) > \frac{1}{j} \text{ for some } l = 0, \ldots, i\},$$

with j so large that $\mathcal{H}^1(\bigcup_{r=0}^{i} S(z^r) \setminus J_j) \leq \sigma$. Let U be a neighborhood of $\bigcup_{r=0}^{i} S(z^r)$ such that $|U| \leq \frac{\sigma}{j^2}$. Following [47, Theorem 2.1] (see Fig.3), we can find a finite disjoint collection of closed cubes $\{Q_k\}_{k=1,\ldots,K}$ with center $x_k \in J_j$, edge of length $2r_k$ and oriented as the normal $\nu(x_k)$ to $S(z^{r(k)})$ at x_k, such that

$\bigcup_{k=1}^{K} Q_k \subseteq U$ and $\mathcal{H}^1(J_j \setminus \bigcup_{k=1}^{K} Q_k) \leq \sigma$. Moreover for all $k = 1, \ldots, K$ there exists $r(k) \in \{0, \ldots, i\}$ and $c_1(r(k)), c_2(r(k)) > 0$ such that

$$\mathcal{H}^1\left(\left[\bigcup_{r=0}^{i} S(z^r) \setminus S(z^{r(k)})\right] \cap Q_k\right) \leq \sigma r_k,$$

and the following hold

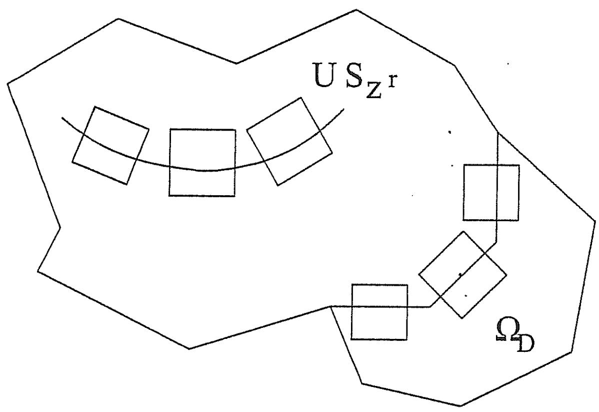

Fig. 3

(a) if $x_k \in \Omega$ then $Q_k \subseteq \Omega$, and if $x_k \in \partial_D \Omega$ then $Q_k \cap \partial_D \Omega = H_k$, where H_k denotes the intersection of Q_k with the straight line through x_k orthogonal to $\nu(x_k)$;

(b) $\mathcal{H}^1(S(z^{r(k)}) \cap \partial Q_k) = 0$;

(c) $r_k \leq c\mathcal{H}^1(S(z^{r(k)}) \cap Q_k)$ for some $c > 0$;

(d) $(z^{r(k)})^-(x) < c_1(r(k)) < c_2(r(k)) < (z^{r(k)})^+(x)$ and $c_2(r(k)) - c_1(r(k)) \geq \frac{1}{2j}$;

(e) $\mathcal{H}^1([S(z^{r(k)}) \setminus \partial^* E_{c_s(r(k))}(r(k))] \cap Q_k) \leq \sigma r_k$ for $s = 1, 2$;

(f) if $s = 1, 2$, $\mathcal{H}^1(\{y \in \partial^* E_{c_s(r(k))}(r(k)) \cap Q_k \,:\, \mathrm{dist}(y, H_k) \geq \frac{\sigma}{2} r_k\}) < \sigma r_k$;

(g) if $Q_k^+ := \{x \in Q_k \,|\, x \cdot \nu(x_k) > 0\}$ and $s = 1, 2$

$$\|1_{E_{c_s(r(k))}(r(k)) \cap Q_k} - 1_{Q_k^+}\|_{L^1(\Omega')} \leq \sigma^2 r_k^2; \tag{5.56}$$

(h) $\mathcal{H}^1((S_v \setminus S(z^{r(k)})) \cap Q_k) < \sigma r_k$ and $\mathcal{H}^1(S(v) \cap \partial Q_k) = 0$.

Let us indicate by R_k the intersection of Q_k with the strip centered in H_k with width $2\sigma r_k$, and let us set $V_k^\pm := \{x_k \pm r_k e(x_k) + s\nu(x_k) : \; s \in \mathbb{R}\} \cap R_k$, where $e(x_k)$ is such that $\{e(x_k), \nu(x_k)\}$ is an orthonormal base of $\mathbb{R}^2$ with the same orientation of the canonical one.

For all $B \subseteq \Omega'$, let us set

$$\mathcal{R}_n(B) := \{T \in \mathbf{R}_{\varepsilon_n} \,:\, T \cap B \neq \emptyset\}, \qquad \mathcal{T}_n^k(B) := \{T \in \mathbf{T}(z_n^{r(k)}) \,:\, T \cap B \neq \emptyset\}.$$

In the following, we will often indicate with the same symbol a family of triangles and their support in $\mathbb{R}^2$, being clear from the context in which sense has to be intended. We will consider $z_n^{r(k)}$ defined pointwise in $\Omega' \setminus \overline{S}_{z_n^{r(k)}}$ and so the upper levels of $z_n^{r(k)}$ are intended as subsets of $\Omega' \setminus \overline{S}_{z_n^{r(k)}}$.

Lemma 5.5.2. *For all $k = 1, \ldots, K$ there exists $c_n^k \in [c_1(r(k)), c_2(r(k))]$ such that, setting $E_n^k := \{x \in \mathcal{R}_n(Q_k) \,:\, z_n^{r(k)}(x) > c_n^k\}$, we have*

$$\limsup_n \sum_{k=1}^K \mathcal{H}^1\left((\partial_{\mathcal{R}_n(Q_k)} E_n^k) \setminus S(z_n^{r(k)})\right) = o_\sigma, \tag{5.57}$$

and

$$\limsup_n \|1_{E_n^k} - 1_{Q_k^+}\|_{L^1(\Omega')} \leq \sigma^2 r_k^2, \tag{5.58}$$

where $\partial_{\mathcal{R}_n(Q_k)}$ denotes the boundary operator in $\mathcal{R}_n(Q_k)$, and $o_\sigma \to 0$ as $\sigma \to 0$.

Proof. Note that for n large we have $\bigcup_{k=1}^K \mathcal{R}_n(Q_k) \subseteq U$, so that $|\bigcup_{k=1}^K \mathcal{R}_n(Q_k)| \leq \frac{\sigma}{j^2}$. By Hölder inequality and since $\|\nabla z_n^r\| \leq C'$ for all $r = 0, \ldots, i$, it follows that

$$\sum_{r=0}^i \int_{\{\bigcup_k \mathcal{R}_n(Q_k) : r(k) = r\}} |\nabla z_n^r| \, dx \leq \sum_{r=0}^i \|\nabla z_n^r\| \frac{\sqrt{\sigma}}{j} \leq (i+1) C' \frac{\sqrt{\sigma}}{j}.$$

Following [47, Theorem 2.1], we can apply coarea-formula for BV-functions (see [7]) taking into account that $z_n^{r(k)}$ belongs to $SBV(\Omega')$ so that the singular part of the derivative is carried only by $S(z_n^{r(k)})$: since for n large the $\mathcal{R}_n(Q_k)$'s are disjoint, we obtain

$$\sum_{k=1}^K \int_{\mathbb{R}} \mathcal{H}^1\Big((\partial E_{c,n}(r(k)) \cap \mathcal{R}_n(Q_k)) \setminus S(z_n^{r(k)}) \Big) \, dc \leq (i+1) C' \frac{\sqrt{\sigma}}{j}, \tag{5.59}$$

where $E_{c,n}(r(k)) := \{x \in \Omega' \setminus \overline{S}_{z_n^{r(k)}} \,:\, z_n^{r(k)}(x) > c\}$, and so

$$\sum_{k=1}^K \int_{c_1(r(k))}^{c_2(r(k))} \mathcal{H}^1\Big((\partial E_{c,n}(r(k)) \cap \mathcal{R}_n(Q_k)) \setminus S(z_n^{r(k)}) \Big) \, dc \leq (i+1) C' \frac{\sqrt{\sigma}}{j}.$$

Notice that we can use the topological boundary instead of the reduced boundary of $E_{c,n}(r(k))$ in (5.59) since $z_n^{r(k)}$ is piecewise affine, and so $\partial E_{c,n}(r(k)) \setminus \partial^* E_{c,n}(r(k)) \neq \emptyset$ just for a finite number of c's. By the Mean Value Theorem we have that there exist $c_n^k \in [c_1(r(k)), c_2(r(k))]$ such that

$$\sum_{k=1}^{K} \mathcal{H}^1\Big(\big(\partial E_{c_n^k,n}(r(k)) \cap \mathcal{R}_n(Q_k)\big) \setminus S(z_n^{r(k)})\Big) \leq 2iC'\sqrt{\sigma},$$

and taking the limsup for $n \to +\infty$, we get (5.57). Let us come to (5.58). Since

$$E_{c_2(r(k)),n}(r(k)) \subseteq E_{c_n^k,n}(r(k)) \subseteq E_{c_1(r(k)),n}(r(k)),$$

by (4.77) we have that for n large

$$\|1_{E_{c_n^k,n}(r(k))\cap Q_k} - 1_{Q_k^+}\|_{L^1(\Omega')} \leq \sigma^2 r_k^2,$$

and so, since $|\mathcal{R}_n(Q_k) \setminus Q_k| \to 0$, we conclude that (5.58) holds. □

Fix $k \in \{1, \dots, K\}$, and let us consider the family $\mathcal{T}_n^k(E_n^k)$. Let us modify this family in the following way. Let $T \in \mathcal{T}_n^k(E_n^k)$; we keep it if $|T \cap E_n^k| > \frac{1}{2}|T|$, and we erase it otherwise. Let $E_n^{k,+}$ be this new family of triangles, and let $E_n^{k,-}$ be its complement in $\mathcal{T}_n^k(\mathcal{R}_n(Q_k))$.

Lemma 5.5.3. *For all $k = 1, \dots, K$ we have*

$$\limsup_n \sum_{k=1}^{K} \mathcal{H}^1\left(\partial_{\mathcal{R}_n(Q_k)} E_n^{k,+} \setminus S(z_n^{r(k)})\right) = o_\sigma, \tag{5.60}$$

and

$$\limsup_n \|1_{E_n^{k,+}} - 1_{Q_k^+}\|_1 \leq 4\sigma^2 r_k^2, \tag{5.61}$$

where $o_\sigma \to 0$ as $\sigma \to 0$.

Proof. Let $T \in \mathcal{T}_n^k(E_n^k)$. Since $z_n^{r(k)}$ is affine on T, it follows that $T \cap E_n^k$ is either a triangle with at least two edges contained in the edges of T or a trapezoid with three edges contained in the edges of T. Let $l(T)$ be the edge inside T where $z_n^{r(k)} = c_n^k$, where c_n^k is the value determining E_n^k (we consider $l(T) = \emptyset$ if $\mathrm{int}(T) \subseteq E_n^k$). In the case $T \in E_n^{k,+}$ as in the case $T \in E_n^{k,-}$, since the angles of the triangles of $\mathbf{T}(z_n^{r(k)})$ are uniformly bounded away from 0 and from π, arguing as in Lemma 5.2.1, we deduce that keeping or erasing T, we increase $\partial_{\mathcal{R}_n(Q_k)} E_n^k$ of a quantity which is less than $c\mathcal{H}^1(l(T))$ with c independent of ε_n. Then we have

$$\sum_{k=1}^{K} \mathcal{H}^1(\partial_{\mathcal{R}_n(Q_k)} E_n^{k,+} \setminus \partial_{\mathcal{R}_n(Q_k)} E_n^k) \leq \sum_{k=1}^{K} \sum_{T \in \mathcal{T}_n^k(E_n^k)} c\mathcal{H}^1(l(T)) \leq c \sum_{k=1}^{K} \mathcal{H}^1(\partial_{\mathcal{R}_n(Q_k)} E_n^k \setminus S(z_n^{r(k)})),$$

so that taking the limsup for $n \to +\infty$ and in view of (5.57) we deduce that (5.60) holds.

Let us come to (5.61). Note that $|\mathcal{T}_n^k(\partial Q_k^+)| \to 0$ as $n \to +\infty$. Then if $A_n^{k,+} := \{T \in \mathbf{T}(z_n^{r(k)}) : T \subseteq \mathrm{int}(Q_k^+)\}$, for n large we have

$$|Q_k^+ \setminus E_n^{k,+}| \leq |A_n^{k,+} \setminus E_n^{k,+}| + |\mathcal{T}_n^k(\partial Q_k^+)| \leq 2|Q_k^+ \setminus E_n^k| + |\mathcal{T}_n^k(\partial Q_k^+)|,$$

where the last inequality follows by construction of $E_n^{k,+}$. Taking the limsup for $n \to +\infty$, in view of (5.58) we get

$$\limsup_n |Q_k^+ \setminus E_n^{k,+}| \leq 2\sigma^2 r_k^2.$$

The inequality $\limsup_n |E_n^{k,+} \setminus Q_k^+| \leq 2\sigma^2 r_k^2$ follows analogously. □

For all $k = 1, \dots, K$ and $s \in \mathbb{R}$, let us set

$$H_k(s) := \{x + s\nu(x_k),\ x \in H_k\}.$$

Lemma 5.5.4. *There exist $s_n^+ \in]\frac{\sigma}{4}r_k, \frac{\sigma}{2}r_k[$ and $s_n^- \in]-\frac{\sigma}{2}r_k, -\frac{\sigma}{4}r_k[$ such that, setting $H_n^{k,+} := H_k(s_n^+)$ and $H_n^{k,-} := H_k(s_n^-)$ we have for n large enough*

$$\mathcal{H}^1(H_n^{k,+} \setminus E_n^{k,+}) \le 20\sigma r_k, \quad \mathcal{H}^1(H_n^{k,-} \cap E_n^{k,+}) \le 20\sigma r_k.$$

Proof. By (5.61) we can write for n large

$$\int_{\frac{\sigma}{4}r_k}^{\frac{\sigma}{2}r_k} \mathcal{H}^1(H_k(s) \setminus E_n^{k,+})\, ds \le 5\sigma^2 r_k^2,$$

so that we get $s_n^+ \in]\frac{\sigma}{4}r_k, \frac{\sigma}{2}r_k[$ with

$$\mathcal{H}^1(H_k(s_n^+) \setminus E_n^{k,+}) \le 20\sigma r_k.$$

Similarly we can reason for s_n^-. □

Let $S_n^{k,+}$ be the straight line containing $H_n^{k,+}$: up to replacing $H_n^{k,+}$ by the connected component of $S_n^{k,+} \cap \mathcal{R}_n(Q_k)$ to which it belongs, we may suppose that $H_n^{k,+} \setminus E_n^{k,+}$ is a finite union of segments l_j^+ with extremes A_j and B_j belonging to the edges of the triangles of $\mathcal{T}_n^k(\mathcal{R}_n(Q_k))$ such that for n large

$$\mathcal{H}^1(H_n^{k,+} \setminus E_n^{k,+}) = \mathcal{H}^1\left(\bigcup_{j=1}^m l_j^+\right) \le 20\sigma r_k.$$

By Lemma 5.2.1, for all j there exists a curve L_j^+ inside the edges of the triangles of $\mathcal{T}_n^k(\mathcal{R}_n(Q_k))$ joining A_j and B_j and such that

$$\mathcal{H}^1(L_j^+) \le c\mathcal{H}^1(l_j^+), \tag{5.62}$$

with c independent of ε_n. Let us set

$$\gamma_n^{k,+} := L_1^+ \cup B_1A_2 \cup L_2^+ \cup \dots \cup B_{m-1}A_m \cup L_m^+.$$

Similarly, let us construct $\gamma_n^{k,-}$ relative to $H_n^{k,-} \cap E_n^+$. Note that for n large enough $\gamma_n^{k,+} \cap H_k(\sigma) = \emptyset$, $\gamma_n^{k,-} \cap H_k(-\sigma) = \emptyset$, and $\gamma_n^{k,+} \cap \gamma_n^{k,-} = \emptyset$. Let us consider the connected component $\mathcal{C}_k^+$ of $\mathcal{R}_n(Q_k) \setminus \gamma_n^{k,+}$ containing $H_k(\sigma)$. Similarly, let us consider the connected component $\mathcal{C}_k^-$ of $\mathcal{R}_n(Q_k) \setminus \gamma_n^{k,-}$ containing $H_k(-\sigma)$. For n large enough, by (5.62)

$$\mathcal{H}^1\left(\partial_{\mathcal{R}_n(Q_k)}\mathcal{C}_k^+ \setminus \bigcup_{i=1}^{m-1} B_iA_{i+1}\right) \le c\sum_{j=1}^m \mathcal{H}^1(l_j^+) \le 20c\sigma r_k. \tag{5.63}$$

A similar estimate holds for $\partial_{\mathcal{R}_n(Q_k)}\mathcal{C}_k^-$.

Let $\tilde{E}_n^{k,+}$ be the family of triangles obtained adding to $E_n^{k,+}$ those $T \in E_n^{k,-}$ such that $T \subseteq \mathcal{C}_k^+$, and subtracting those $T \in E_n^{k,+}$ such that $T \subseteq \mathcal{C}_k^-$. Let $\tilde{E}_n^{k,-}$ be the complement of $\tilde{E}_n^{k,+}$ in $\mathcal{T}_n^k(\mathcal{R}_n(Q_k))$.

We claim that there exists $C > 0$ independent of n such that for all $k = 1, \dots, K$ and for n large

$$\mathcal{H}^1\left(\partial_{\mathcal{R}_n(Q_k)}\tilde{E}_n^{k,+} \setminus \partial_{\mathcal{R}_n(Q_k)}E_n^{k,+}\right) \le C\sigma r_k. \tag{5.64}$$

In fact, let ζ be an edge of $\partial_{\mathcal{R}_n(Q_k)}\tilde{E}_n^{k,+} \setminus \partial_{\mathcal{R}_n(Q_k)}E_n^{k,+}$, that is ζ belongs to a triangle T that has been changed in the operation above described. Let us assume for instance that $T \in E_n^{k,-}$ and $T \subseteq C_k^+$. If T' is such that $T \cap T' = \zeta$, then $T' \in E_n^{k,-}$: in fact if by contradiction $T' \in E_n^{k,+}$, then $T' \in \tilde{E}_n^{k,+}$ and so we would have $\zeta \notin \partial_{\mathcal{R}_n(Q_k)}\tilde{E}_n^{k,+}$ which is absurd. Similarly we get $T' \not\subseteq C_k^+$. This means that $\zeta \subseteq \partial_{\mathcal{R}_n(Q_k)}C_k^+$, and since the horizontal edges of $\gamma_n^{k,+}$ intersect by construction only elements of $E_n^{k,+}$, we deduce that $\zeta \subseteq \partial_{\mathcal{R}_n(Q_k)}C_k^+ \setminus (\cup_{i=1}^m A_iB_i)$, and by (5.63) we conclude that (5.64) holds.

We can summarize the previous results as follows.

Lemma 5.5.5. *For all $k = 1, \dots, K$ there exist two families $\tilde{E}_n^{k,+}$ and $\tilde{E}_n^{k,-}$ of triangles with $\mathcal{T}_n^k(\mathcal{R}_n(Q_k)) = \tilde{E}_n^{k,+} \cup \tilde{E}_n^{k,-}$, $Q_k^+ \setminus R_k \subseteq \tilde{E}_n^{k,+}$ and $Q_k^- \setminus R_k \subseteq \tilde{E}_n^{k,-}$, and such that*

$$\limsup_n \sum_{k=1}^K \mathcal{H}^1\left(\partial_{\mathcal{R}_n(Q_k)}\tilde{E}_n^{k,+} \setminus S(z_n^{r(k)})\right) = o_\sigma, \tag{5.65}$$

where $o_\sigma \to 0$ as $\sigma \to 0$. Moreover, in the case $x_k \in \partial_D\Omega$, we can modify $\tilde{E}_n^{k,+}$ or $\tilde{E}_n^{k,-}$ in such a way that $\tilde{E}_n^{k,+} \subseteq \Omega$ or $\tilde{E}_n^{k,-} \subseteq \Omega$.

Proof. We have that (5.65) follows from (5.60) and (5.64), and the fact that $\sum_{k=1}^K r_k \le c$, with c independent of σ. Let us consider the case $x_k \in \partial_D\Omega$ with $Q_k^+ \setminus R_k \subseteq \Omega$ (the other case being similar). From (5.65) we have that for n large $\sum_{k=1}^K \mathcal{H}^1\left(\partial_{\mathcal{R}_n(Q_k)}\tilde{E}_n^{k,+} \cap Q_k^-\right) \le o_\sigma$ because $z_n^{r(k)} = R_{\varepsilon_n} g_{h_n}(r(k))$ on Q_k^- and so there are no jumps in Q_k^-. We can thus redefine $\tilde{E}_n^{k,+}$ subtracting those triangles that are in Q_k^- obtaining again (5.65). $\square$

We are now in position to prove Proposition 5.5.1.

Proof of Proposition 5.5.1. We work in the context of Ω'. For all $v \in SBV(\Omega')$ with $v = g^i$ on Ω_D, $\mathcal{H}^1(S(v)) < +\infty$ and $\nabla v \in L^2(\Omega'; \mathbb{R}^2)$, we have to construct $v_n \in SBV(\Omega')$ such that $v_n = g_n^i$ on Ω_D, $(v_n)_{|\Omega} \in \mathcal{A}_{\varepsilon_n,a}(\Omega)$, $v_n \to v$ strongly in $L^1(\Omega')$, $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega'; \mathbb{R}^2)$ and

$$\limsup_n \mathcal{H}^1\left(S(v_n) \setminus \bigcup_{r=0}^i S(u_n^r)\right) \le \mu(a)\mathcal{H}^1\left(S(v) \setminus \bigcup_{r=0}^i S(u^r)\right), \tag{5.66}$$

where we suppose that u_n^r and u^r are extended to Ω' setting $u_n^r := g_n^r$, and $u^r := g^r$ on Ω_D respectively.

We set $v = g^i + w$, where $w \in SBV(\Omega')$ with $w = 0$ on Ω_D. By density, it is sufficient to consider the case $w \in L^\infty(\Omega')$. Up to reducing U, we may assume that $\|\nabla g^i\|_{L^2(U;\mathbb{R}^2)} < \sigma$ and $\|\nabla w\|_{L^2(U;\mathbb{R}^2)} < \sigma$. Let R'_k be a rectangle centered in x_k, oriented as R_k, and such that $\overline{R'_k} \subset \mathrm{int}R_k$ and $\mathcal{H}^1(S(w) \cap (R_k \setminus R'_k)) < \sigma r_k$. We claim that there exists $w_\sigma \in SBV(\Omega')$ with $w_\sigma = w$ on $\bigcup_{k=1}^K R'_k$ and $w_\sigma = 0$ in Ω_D such that

(1) $\|w - w_\sigma\| + \|\nabla w - \nabla w_\sigma\| \le \sigma$;

(2) $\mathcal{H}^1(S(w_\sigma) \cap (Q_k \setminus R'_k)) \le o_\sigma r_k$, with $o_\sigma \to 0$ as $\sigma \to 0$;

(3) $\mathcal{H}^1(S(w_\sigma) \setminus \bigcup_{k=1}^K R_k) \le \mathcal{H}^1(S_w \setminus \bigcup_{k=1}^K R_k) + \sigma$;

(4) $S(w_\sigma) \setminus \bigcup_{k=1}^K R_k$ is union of disjoint segments with closure contained in $\Omega \setminus \bigcup_{k=1}^K R_k$;

(5) w_σ is of class $W^{2,\infty}$ on $\Omega \setminus \left(\bigcup_{k=1}^K R_k \cup \overline{S(w_\sigma)}\right)$.

In fact, by Proposition 5.2.4, there exists $w_m \in SBV(\Omega')$ with $w_m = 0$ in $\Omega' \setminus \overline{\Omega}$ such that $w_m \to w$ strongly in $L^2(\Omega')$, $\nabla w_m \to \nabla w$ strongly in $L^2(\Omega';\mathbb{R}^2)$, S_{w_m} is polyhedral with $\overline{S_{w_m}} \subseteq \Omega$, w_m is of class $W^{2,\infty}$ on $\Omega \setminus \left(\bigcup_{k=1}^K R_k \cup \overline{S_{w_m}}\right)$, and $\lim_m \mathcal{H}^1(A \cap S_{w_m}) = \mathcal{H}^1(A \cap S_w)$ for all A open subset of Ω' with $\mathcal{H}^1(\partial A \cap S_w) = 0$. It is not restrictive to assume that $\mathcal{H}^1(S_w \cap \partial R_k) = 0$ and $\mathcal{H}^1(S_{w_m} \cap \partial R_k) = 0$ for all m. Let ψ_k be a smooth function such that $0 \le \psi_k \le 1$, $\psi_k = 1$ on R'_k and $\psi_k = 0$ outside R_k. Setting $\psi := \sum_{k=1}^K \psi_k$, let us consider $\tilde{w}_m := \psi w + (1-\psi) w_m$. Note that $\tilde{w}_m \to w$ strongly in $L^2(\Omega')$, $\nabla \tilde{w}_m \to \nabla w$ strongly in $L^2(\Omega';\mathbb{R}^2)$, $\tilde{w}_m = 0$ in Ω_D. Moreover, by capacity arguments, we may assume that $S_{\tilde{w}_m} \setminus \bigcup_{k=1}^K R_k$ is a finite union of disjoint segments with closure contained in $\Omega \setminus \bigcup_{k=1}^K R_k$. Finally, for $m \to +\infty$, we have

$$\mathcal{H}^1(S_{\tilde{w}_m} \setminus \bigcup_{k=1}^K R_k) \to \mathcal{H}^1(S_w \setminus \bigcup_{k=1}^K R_k)),$$

$$\mathcal{H}^1(S_{\tilde{w}_m} \cap \bigcup_{k=1}^K (Q_k \setminus R_k)) \to \mathcal{H}^1(S_w \cap \bigcup_{k=1}^K (Q_k \setminus R_k))$$

and $\limsup_m \mathcal{H}^1(S_{\tilde{w}_m} \cap (R_k \setminus R'_k)) \le 2\mathcal{H}^1(S_w \cap (R_k \setminus R'_k)) \le 2\sigma r_k$. Then we can take $w_\sigma := \tilde{w}_m$ for m large enough.

Let $S(w_\sigma) \setminus \bigcup_{k=1}^K Q_k := \bigcup_{j=1}^m l_j$, where, by capacity arguments, we can always assume that l_j are disjoint segments with closure contained in $\Omega \setminus \bigcup_{k=1}^K Q_k$. We define a triangulation $\mathbf{T}_n \in \mathcal{T}_{\varepsilon_n,a}(\Omega')$ specifying its adaptive vertices as follows. Let us consider the families $\mathcal{R}_n(Q_k)$ and $\mathcal{R}_n(l_j)$ for $k = 1, \dots, K$ and $j = 1, \dots, m$. Note that for n large enough, $\mathcal{R}_n(Q_{k_1}) \cap \mathcal{R}_n(Q_{k_2}) = \emptyset$ for $k_1 \neq k_2$, $\mathcal{R}_n(l_{j_1}) \cap \mathcal{R}_n(l_{j_2}) = \emptyset$ for $j_1 \neq j_2$, and $\mathcal{R}_n(Q_k) \cap \mathcal{R}_n(l_j) = \emptyset$ for every k, j. We consider inside the triangles of $\mathcal{R}_n(Q_k)$ the adaptive vertices of $\mathbf{T}(z_n^{r(k)})$. Passing to $\mathcal{R}_n(l_j)$, by density arguments it is not restrictive to assume that l_j does not pass through the vertices of $\mathbf{R}_{\varepsilon_n}$ and that its extremes belong to the edges of $\mathbf{R}_{\varepsilon_n}$. Let $\zeta := [x, y]$ be an edge of $\mathcal{R}_n(l_j)$ such that $l_j \cap \zeta = \{P\}$. Proceeding as in [62], we take as adaptive vertex of ζ the projection of P on $\{tx + (1-t)y : t \in [a, (1-a)]\}$. Connecting these adaptive vertices, we obtain an *interpolating* polyhedral curve $\tilde{l}_j$ with

$$\mathcal{H}^1(\tilde{l}_j) \le \mu(a)\mathcal{H}^1(l_j), \tag{5.67}$$

where μ is an increasing function such that $\lim_{a\to 0} \mu(a) = 1$. Finally, in the remaining edges, we can consider any admissible adaptive vertex, for example the middle point.

Let us define $w_n \in SBV(\Omega')$ in the following way. For all Q_k, let w_n be equal to w_σ on $\mathcal{R}_n(Q_k) \setminus R_k$, equal to the reflection of $w_{\sigma|Q_k^+ \setminus R_k}$ with respect to $H_k(\sigma)$ on $\tilde{E}_n^{k,+} \cap R_k$ and equal to the reflection of $w_{\sigma|Q_k^- \setminus R_k}$ with respect to $H_k(-\sigma)$ on $\tilde{E}_n^{k,-} \cap R_k$, where $\tilde{E}_n^{k,\pm}$ are defined as in Lemma 5.5.5. On the other elements of $\mathbf{T}_n$, let us set $w_n = w_\sigma$. Notice that $w_n = 0$ on Ω_D and that inside each $\mathcal{R}_n(Q_k)$, all the discontinuities of w_n are contained in $\partial_{\mathcal{R}_n(Q_k)} \tilde{E}_n^{k,+} \cup V_k \cup P_{w_\sigma}^k$, where $P_{w_\sigma}^k$ is the union of the polyhedral jumps of w_σ in $\mathcal{R}_n(Q_k)$ and of their reflected version with respect to $H_k(\pm\sigma)$. By Lemma 5.5.5 and since

$\sum_{k=1}^K \mathcal{H}^1(V_k \cup P^k_{w_\sigma}) \leq o_\sigma$ with $o_\sigma \to 0$ as $\sigma \to 0$, and $\mathcal{H}^1\left(\bigcup_{r=0}^i S(z^r) \setminus \bigcup_{k=1}^K Q_k\right) \leq 2\sigma$, we have that

$$\limsup_n \mathcal{H}^1\left(S_{w_n} \setminus \bigcup_{r=0}^i S_{z_n^r}\right) \leq \mathcal{H}^1\left(S(w_\sigma) \setminus \bigcup_{k=1}^K Q_k\right) + \limsup_n \mathcal{H}^1\left((S_{w_n} \setminus \bigcup_{r=0}^i S_{z_n^r}) \cap \mathcal{R}_n(Q_k)\right) \leq$$
$$\leq \mathcal{H}^1\left(S(w_\sigma) \setminus \bigcup_{r=0}^i S(z^r)\right) + \mathcal{H}^1\left(\bigcup_{r=0}^i S(z^r) \setminus \bigcup_{k=1}^K Q_k\right) + o_\sigma \leq \mathcal{H}^1\left(S(w_\sigma) \setminus \bigcup_{r=0}^i S(z^r)\right) + o_\sigma,$$

and since $\|\nabla w_\sigma\|_{L^2(U;\mathbb{R}^2)} \leq o_\sigma$ we get for n large

$$\|\nabla w_n\|^2_{L^2(\bigcup_{k=1}^K \mathcal{R}_n(Q_k))} \leq o_\sigma. \tag{5.68}$$

We now want to define an interpolation $\tilde{w}_n$ of w_n on $\mathbf{T}_n$. Firstly, we set $\tilde{w}_n = 0$ on all regular triangles of Ω_D. Passing to the triangles in $\mathcal{R}_n(Q_k)$ (see fig.4), by Lemma 5.2.2, we know that for n large enough, we have

$$\mathcal{H}^1(\partial\mathcal{R}_n(V_k)) \leq c\mathcal{H}^1(V_k), \quad \mathcal{H}^1(\partial\mathcal{R}_n(P^k_{w_\sigma})) \leq c\mathcal{H}^1(P^k_{w_\sigma}),$$

with c independent of n. If $T \in \mathcal{R}_n(V_k) \cup \mathcal{R}_n(P^k_{w_\sigma})$, we set $\tilde{w}_n = 0$ on T; otherwise, we define $\tilde{w}_n$ on T as the affine interpolation of w_n.

Since $\nabla\tilde{w}_n$ is uniformly bounded on $\mathcal{R}_n(H_k(\pm\sigma))$, $|\mathcal{R}_n(H_k(\pm\sigma))| \to 0$ and since w_n is uniformly bounded in $W^{2,\infty}$ on the triangles contained in $\mathcal{R}_n(Q_k) \setminus \mathcal{R}_n(V_k \cup P^k_{w_\sigma} \cup H_k(\pm\sigma))$ we have by the interpolation estimate (5.9) and by (5.68)

$$\limsup_n \|\nabla\tilde{w}_n\|^2_{L^2(\bigcup_{k=1}^K \mathcal{R}_n(Q_k))} \leq o_\sigma. \tag{5.69}$$

Moreover we have

$$\limsup_n \sum_{k=1}^K \mathcal{H}^1\left(\left(S_{\tilde{w}_n} \setminus \bigcup_{r=0}^k S_{z_n^r}\right) \cap \mathcal{R}_n(Q_k)\right) \leq o_\sigma. \tag{5.70}$$

Let us come to the triangles not belonging to $\mathcal{R}_n(Q_k)$ for $k = 1, \ldots, K$. For all $j = 1, \ldots, m$, we denote by $\hat{\mathcal{R}}_n(l_j)$ the family of regular triangles that have edges in common with triangles of $\mathcal{R}_n(l_j)$. For n large we have that $\hat{\mathcal{R}}_n(l_{j_1}) \cap \hat{\mathcal{R}}_n(l_{j_2}) = \emptyset$ for $j_1 \neq j_2$. On every regular triangle $T \notin \bigcup_{k=1}^K \mathcal{R}_n(Q_k) \cup \bigcup_{j=1}^m \hat{\mathcal{R}}_n(l_j)$, we define $\tilde{w}_n$ as the affine interpolation of w_σ. Since w_σ is of class $W^{2,\infty}$ on T and T is regular, we obtain by the interpolation estimate (5.9)

$$\|\tilde{w}_n - w_\sigma\|^2_{W^{1,2}(T)} \leq K\varepsilon_n\|w_\sigma\|_{W^{2,\infty}}. \tag{5.71}$$

Let us consider now those triangles that are contained in the elements of $\bigcup_{j=1}^m \hat{\mathcal{R}}_n(l_j)$. Following [62], we can define $\tilde{w}_n$ on every T in such a way that $\tilde{w}_n$ admits discontinuities only on $\bar{l}_j$, and $\|\nabla\tilde{w}_n\|_{L^\infty(T)} \leq \|\nabla w_\sigma\|_\infty$. Since $|\hat{\mathcal{R}}_n(l_j)| \to 0$ as $n \to \infty$, we deduce that

$$\lim_n \|\nabla\tilde{w}_n\|^2_{L^2(\hat{\mathcal{R}}_n(l_j))} = 0. \tag{5.72}$$

Moreover by (5.67) and since $\mathcal{H}^1\left(\bigcup_{r=0}^i S(z^r) \setminus \bigcup_{k=1}^K Q_k\right) \leq 2\sigma$, we have

$$\mathcal{H}^1\left(S_{\tilde{w}_n} \cap \bigcup_{j=1}^m \hat{\mathcal{R}}_n(l_j)\right) \leq \mu(a)\mathcal{H}^1\left(S(w_\sigma) \setminus \bigcup_{i=1}^k S(z^r)\right) + o_\sigma, \tag{5.73}$$

where $o_\sigma \to 0$ as $\sigma \to 0$.

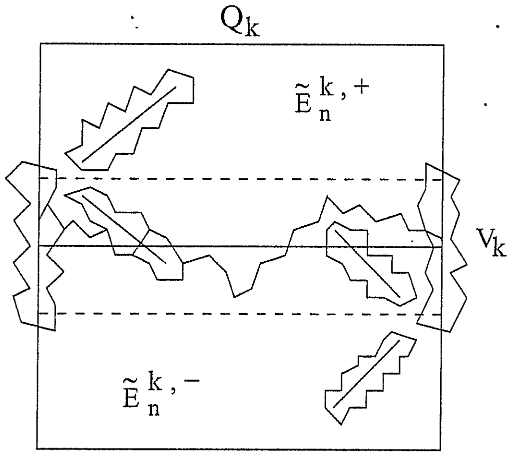

Fig. 4

We are now ready to conclude. Let us consider $\hat{w}_n \in \mathcal{A}_{\varepsilon_n,a}(\Omega)$ defined as $\hat{w}_n := g_n^i + \tilde{w}_n$. We have $\hat{w}_n \to g^i + w_\sigma$ strongly in $L^2(\Omega')$. By (5.69), (5.71), (5.72) we get

$$\limsup_n \|\nabla \hat{w}_n\|^2 \le \|\nabla g^i + \nabla w_\sigma\|^2 + o_\sigma,$$

while by (5.70) and (5.73) we have

$$\limsup_n \mathcal{H}^1 \left(S_{\hat{w}_n} \setminus \bigcup_{r=0}^{i} S_{z_n^r} \right) \le \mu(a)\mathcal{H}^1 \left(S_{(w_\sigma)} \setminus \bigcup_{r=0}^{i} S(z^r) \right) + o_\sigma.$$

Letting now $\sigma \to 0$, using a diagonal argument, we conclude that Proposition 5.5.1 holds. $\square$

5.6 Revisiting the approximation by Francfort and Larsen

In this section we show how the arguments of Section 5.4 may be used to deal with the discrete in time approximation of quasi-static growth of brittle fractures proposed by Francfort and Larsen in [47]. More precisely, we prove that there is strong convergence of the gradient of the displacement (in particular convergence of the bulk energy) and convergence of the surface energy at all times of continuity of the length of the crack; moreover there is convergence of the total energy at any time.

We briefly recall the notation employed in [47]. Let I_∞ be countable and dense in $[0,1]$, and let $I_n := \{0 = t_0^n \le \ldots \le t_n^n = 1\}$ such that (I_n) is an increasing sequence of sets whose union is I_∞. Let $\Omega \subseteq \mathbb{R}^N$ be a Lipschitz bounded domain, and let $\partial\Omega = \partial\Omega_f^c \cup \partial\Omega_f$, where $\partial\Omega_f^c$ is open in the relative topology. Let $\Omega' \subseteq \mathbb{R}^N$ be open and such that $\overline{\Omega} \subseteq \Omega'$, and let $g \in W^{1,1}([0,1]; H^1(\Omega'))$. At any time t_k^n, Francfort and Larsen consider u_k^n minimizer of

$$\int_\Omega |\nabla v|^2\,dx + \mathcal{H}^{N-1}\left(S(v) \setminus \left[\bigcup_{0\le j\le k-1} S(u_j^n) \cup \partial\Omega_f\right]\right)$$

in $\{v \in SBV(\Omega') : v = g(t_k^n) \text{ in } \Omega' \setminus \overline{\Omega}\}$. Setting $u^n(t) := u_k^n$ for $t \in [t_k^n, t_{k+1}^n[$, and $\Gamma^n(t) := \bigcup_{s\le t, s\in I_n} S(u^n(s)) \cup \partial\Omega_f$, they prove that

$$\mathcal{E}^n(t) \le \mathcal{E}^n(0) + 2\int_0^{t_k^n}\int_\Omega \nabla u^n(\tau)\nabla\dot g(\tau)\,dx\,d\tau + o_n, \qquad t \in [t_k^n, t_{k+1}^n[, \tag{5.74}$$

where $\mathcal{E}^n(t) := \int_\Omega |\nabla u^n(t)|^2\,dx + \mathcal{H}^{N-1}(\Gamma^n(t))$ and $o_n \to 0$ as $n \to +\infty$. Using Theorem 3.8.4, they obtain a subsequence of $(u^n(\cdot))$, still denoted by the same symbol, such that $u^n(t) \to u(t)$ in $SBV(\Omega')$ and $\nabla u^n(t) \to \nabla u(t)$ strongly in $L^2(\Omega';\mathbb{R}^N)$ for all $t \in I_\infty$, with $u(t)$ a minimizer of

$$\int_\Omega |\nabla v|^2\,dx + \mathcal{H}^{N-1}(S(v) \setminus \Gamma(t)),$$

where $\Gamma(t) := \bigcup_{s\in I_\infty, s\le t} S(u(s)) \cup \partial\Omega_f$. The evolution $\{t \to u(t),\ t \in I_\infty\}$ is extended to the whole $[0,1]$ using the approximation from the left in time.

We can now use the arguments of Section 5.4. Following Lemma 5.4.6, it turns out that for all $t \in [0,1]$

$$\mathcal{E}(t) \ge \mathcal{E}(0) + 2\int_0^t\int_\Omega \nabla u(\tau)\nabla\dot g(\tau)\,dx\,d\tau. \tag{5.75}$$

Moreover, by the Transfer of Jump and the uniqueness argument of Lemma 5.4.3, we have that $\nabla u^n(t) \to \nabla u(t)$ strongly in $L^2(\Omega';\mathbb{R}^N)$ for all $t \notin \mathcal{N}$, where $\mathcal{N}$ is the (at most countable) set of discontinuities of the pointwise limit λ of $\mathcal{H}^{N-1}(\Gamma(\cdot))$ (which exists up to a further subsequence by Helly's Theorem). Then we pass to the limit in (5.74) obtaining

$$\mathcal{E}(t) \le \mathcal{E}(0) + 2\int_0^t\int_\Omega \nabla u(\tau)\nabla\dot g(\tau)\,dx\,d\tau;$$

moreover, following the proof of Theorem 5.1.1, we have that for all $t \in [0,1]$

$$\mathcal{E}(t) \le \liminf_n \mathcal{E}_n(t) \le \limsup_n \mathcal{E}_n(t) = \mathcal{E}(0) + 2\int_0^t\int_\Omega \nabla u(\tau)\nabla\dot g(\tau)\,dx\,d\tau,$$

and taking into account (5.75) we get the convergence of the total energy at any time. Since $\nabla u^n(t) \to \nabla u(t)$ strongly in $L^2(\Omega';\mathbb{R}^N)$ for every $t \in I_\infty$, we deduce that $\lambda = \mathcal{H}^{N-1}(\Gamma(\cdot))$ on I_∞, so that the convergence of the surface energy holds in I_∞. The extension to the continuity times for $\mathcal{H}^{N-1}(\Gamma(\cdot))$ follows like in the final part of the proof of Theorem 5.1.1.

Chapter 6

Approximation of the model by Dal Maso-Francfort-Toader

6.1 Introduction

The aim of this chapter is to provide a discontinuous finite element approximation of a model of quasistatic growth of brittle fractures in nonlinear elasticity recently proposed in [35] by Dal Maso, Francfort and Toader.

Let us recall their model (a complete description is given in the Preliminaries to part II). The authors consider the case of nonlinear elasticity, and take into account possible volume and traction forces applied to the elastic body. Let us assume that the elastic body has a reference configuration given by $\Omega \subseteq \mathbb{R}^N$ open, bounded and with Lipschitz boundary. Let $\partial_D\Omega \subseteq \partial\Omega$ be open in the relative topology, and let $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. Let $\Omega_B \subseteq \Omega$, and let $\partial_S\Omega \subseteq \partial_N\Omega$ be such that $\overline{\Omega}_B \cap \partial_S\Omega = \emptyset$. Ω_B is the *brittle part* of Ω, and $\partial_S\Omega$ is the part of the boundary where traction forces are supposed to act. A crack is given by any rectifiable set in $\overline{\Omega}_B$ with finite $(N-1)$ Hausdorff measure. Given a boundary deformation g on $\partial_D\Omega$ and a crack Γ, the family of all admissible deformation of Ω is given by the set $AD(g,\Gamma)$ of all function $u \in GSBV(\Omega;\mathbb{R}^N)$ such that $S(u) \subseteq \Gamma$ and $u = g$ on $\partial_D\Omega \setminus \Gamma$. Here $S(u)$ denotes the set of jumps of u, and the equality $u = g$ is intended in the sense of traces. Requiring $u = g$ only on $\partial_D\Omega \setminus \Gamma$ means that the deformation is assumed not to be transmitted through the fracture. The bulk energy considered in [35] is of the form

$$\int_\Omega W(x, \nabla u(x))\, dx,$$

where $W(x,\xi)$ is quasiconvex in ξ, and satisfies suitable regularity and growth assumptions (see (3.66) and (3.67)). Moreover the time dependent body and traction forces are supposed to be conservative with work given by

$$-\int_{\Omega\setminus\Gamma} F(t,x,u(x))\, dx - \int_{\partial_S\Omega} G(t,x,u(x))\, d\mathcal{H}^{N-1}(x),$$

where F and G satisfy suitable regularity and growth conditions (see the Preliminaries of part II). Finally the work made to produce the crack Γ is given by

$$\mathcal{E}^s(\Gamma) := \int_\Gamma k(x,\nu(x))\, d\mathcal{H}^{N-1}(x),$$

where $\nu(x)$ is the normal to Γ at x, and $k(x,\nu)$ satisfies standard hypotheses which guarantee lower semicontinuity (see the Preliminaries of part II). Clearly, W, F, G and k depend on the material. Let us set

$$\mathcal{E}^b(t)(u) := \int_\Omega W(x, \nabla u(x))\,dx - \int_{\Omega\setminus\Gamma} F(t,x,u(x))\,dx - \int_{\partial_S\Omega} G(t,x,u(x))\,d\mathcal{H}^{N-1}(x),$$

and

$$\mathcal{E}(t)(u,\Gamma) := \mathcal{E}^b(t)(u) + \mathcal{E}^s(\Gamma). \tag{6.1}$$

Given a boundary deformation $g(t)$ with $t \in [0,T]$ and a preexisting crack Γ_0, a quasistatic crack growth relative to g and Γ_0 is a map $\{t \to (u(t),\Gamma(t)) : t \in [0,T]\}$ which satisfies properties (a), (b), (c), (d), (e) of Theorem 3.8.8.

In this chapter we discretize the model using a suitable finite element method and prove its convergence to this notion of quasistatic crack growth. We restrict our analysis to a two dimensional setting considering only a polygonal reference configuration $\Omega \subseteq \mathbb{R}^2$.

The discretization of the domain Ω is carried out as in [50] employing *adaptive triangulations* introduced by M. Negri in [62] (see also [63]). Let us fix two parameters $\varepsilon > 0$ and $a \in]0, \frac{1}{2}[$. We consider a regular triangulation $\mathbf{R}_\varepsilon$ of size ε of Ω, i.e. we assume that there exist two constants c_1 and c_2 so that every triangle $T \in \mathbf{R}_\varepsilon$ contains a ball of diameter $c_1\varepsilon$ and is contained in a ball of diameter $c_2\varepsilon$. In order to treat the boundary data, we assume also that $\partial_D\Omega$ is composed of edges of $\mathbf{R}_\varepsilon$. On each edge $[x,y]$ of $\mathbf{R}_\varepsilon$ we consider a point z such that $z = tx + (1-t)y$ with $t \in [a, 1-a]$. These points are called *adaptive vertices*. Connecting together the adaptive vertices, we divide every $T \in \mathbf{R}_\varepsilon$ into four triangles. We take the new triangulation $\mathbf{T}$ obtained after this division as the discretization of Ω. The family of all such triangulations will be denoted by $\mathcal{T}_{\varepsilon,a}(\Omega)$.

The discretization of the energy functional is obtained restricting the total energy (6.1) to the family of functions u which are affine on the triangles of some triangulation $\mathbf{T}(u) \in \mathcal{T}_{\varepsilon,a}(\Omega)$ and are allowed to jump across the edges of $\mathbf{T}(u)$ contained in $\overline{\Omega}_B$. We indicate this space by $\mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$. The boundary data is assumed to belong to the space $\mathcal{AF}_\varepsilon(\Omega)$ of continuous functions which are affine on every triangle $T \in \mathbf{R}_\varepsilon$.

Let us consider a boundary datum $g_\varepsilon \in W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2))$ with $g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega)$ for all $t \in [0,T]$ (p, q are related to the growth assumptions on W, F, G) and an initial crack $\Gamma^0_{\varepsilon,a}$ (see Section 6.5). For all $u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ we indicate by $S(u)$ the edges of the triangulation $\mathbf{T}(u)$ across which u jumps, while we denote by $S^{g_\varepsilon(t)}_D(u)$ the edges of the triangulation $\mathbf{T}(u)$ contained in $\partial_D\Omega$ on which $u \neq g_\varepsilon(t)$. Let us divide $[0,1]$ into subintervals $[t^\delta_i, t^\delta_{i+1}]$ of size $\delta > 0$ for $i = 0, \dots, N_\delta$. Using a variational argument (Proposition 6.5.1), we construct a *discrete* (in time and space) *evolution* $\{(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a}) : i = 0, \dots, N_\delta\}$ such that for all $i = 0, \dots, N_\delta$ we have $u^{\delta,i}_{\varepsilon,a} \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$,

$$\Gamma^{\delta,i}_{\varepsilon,a} := \bigcup_{r=0}^{i} [S(u^{\delta,r}_{\varepsilon,a}) \cup S^{g_\varepsilon(t^\delta_r)}_D(u^{\delta,r}_{\varepsilon,a})],$$

and the following *unilateral minimality property* holds: for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$

$$\mathcal{E}^b(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}) \le \mathcal{E}^b(t^\delta_i)(v) + \mathcal{E}^s\left((S(v) \cup S^{g_\varepsilon(t^\delta_i)}_D(v)) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right). \tag{6.2}$$

In order to perform the asymptotic analysis of the *discrete evolution* $\{(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a}) : i = 0, \dots, N_\delta\}$ we make the piecewise constant interpolation in time $u^\delta_{\varepsilon,a}(t) = u^{\delta,i}_{\varepsilon,a}$ and $\Gamma^\delta_{\varepsilon,a}(t) = \Gamma^{\delta,i}_{\varepsilon,a}$ for all $t^\delta_i \le t < t^\delta_{i+1}$. Let us suppose that

$$g_\varepsilon \to g \qquad \text{strongly in } W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2))$$

(where on $W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$ we take the norm $\|u\| := \|u\|_{W^{1,p}(\Omega;\mathbb{R}^2)} + \|u\|_{L^q(\Omega;\mathbb{R}^2)}$), and that $\Gamma^0_{\varepsilon,a}$ approximate an initial crack Γ^0 in the sense of Proposition 6.4.1.

The main result of the chapter (Theorem 6.6.1) states that there exist a quasistatic evolution $\{t \to (u(t),\Gamma(t)) : t \in [0,T]\}$ in the sense of [35] relative to the boundary deformation g and the preexisting crack Γ^0 and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, such that setting

$$u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t), \qquad \Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t),$$

for all $t \in [0,T]$ the following facts hold:

(a) $(u_n(t))_{n\in\mathbb{N}}$ is weakly precompact in $GSBV^p_q(\Omega;\mathbb{R}^2)$, and every accumulation point $\tilde{u}(t)$ is such that $\tilde{u}(t) \in AD(g(t),\Gamma(t))$, and $(\tilde{u}(t),\Gamma(t))$ satisfy the static equilibrium (2); moreover there exists a subsequence $(\delta_{n_k},\varepsilon_{n_k},a_{n_k})_{k\in\mathbb{N}}$ of $(\delta_n,\varepsilon_n,a_n)_{n\in\mathbb{N}}$ (depending on t) such that

$$u_{n_k}(t) \rightharpoonup u(t) \quad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

(b) convergence of the total energy holds, and more precisely elastic and surface energies converge separately, that is

$$\mathcal{E}^b(t)(u_n(t)) \to \mathcal{E}^b(t)(u(t)), \qquad \mathcal{E}^s(\Gamma_n(t)) \to \mathcal{E}^s(\Gamma(t)).$$

By point (a), the approximation of the deformation $u(t)$ is available only up to a subsequence depending on t: this is due to the possible non uniqueness of the minimum energy deformation associated to $\Gamma(t)$. In the case $\mathcal{E}^b(t)(u)$ is strictly convex, it turns out that the deformation $u(t)$ is uniquely determined, and we prove that (Theorem 6.7.1)

$$\nabla u_n(t) \to \nabla u(t) \qquad \text{strongly in } L^p(\Omega;\mathcal{M}^{2\times 2}),$$

and

$$u_n(t) \to u(t) \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2).$$

In order to find the fracture $\Gamma(t)$ in the limit, in Lemma 6.6.2 and Lemma 6.6.4 we adapt to the context of finite elements the notion of σ^p- convergence of sets formulated in [35]. This is the key tool to obtain the convergence of elastic and surface energies at all times $t \in [0,T]$ (while in [50] this was available only at the continuity points of $\mathcal{H}^1(\Gamma(t))$). In order to infer the static equilibrium of $\Gamma(t)$ from that of $\Gamma_n(t)$, we employ a generalization of the piecewise affine transfer of jumps [50, Proposition 5.1] (see Proposition 6.3.2).

The chapter is organized as follows. In Section 3.1 we introduce the basic notation, and some tools employed throughout the chapter. In Section 6.3 we introduce the finite element space, and in Section 6.4 we prove an approximation result for a preexisting crack configuration. In Section 6.5 we prove the existence of a discrete evolution, and in Section 6.6 we prove the main approximation result (Theorem 6.6.1). In Section 6.7 we treat the case of strictly convex total energy.

6.2 Notation and Preliminaries

In this section we introduce the main notations and the preliminary results employed in the Part II of this thesis.

SBV and GSBV spaces. Let A be an open subset of $\mathbb{R}^n$, and let $u : A \to \mathbb{R}^m$ be a measurable function. Given $x \in A$, we say that $\tilde{u}(x)$ is the *approximate limit* of u at x, and we write $\tilde{u}(x) = \operatorname{ap}\lim_{y\to x} u(y)$, if for every $\varepsilon > 0$

$$\lim_{r\to 0} r^{-n}\mathcal{L}^n \left(\{y \in B_r(x) \,:\, |u(y) - \tilde{u}(x)| > \varepsilon\}\right) = 0.$$

Here $B_r(x)$ denotes the ball of center x and radius r. We indicate by $S(u)$ the set of points where the approximate limit of u does not exist. We say that the matrix $m \times n$ $\nabla u(x)$ is the approximate gradient of u at x if

$$\operatorname{ap}\lim_{y\to x} \frac{u(y) - u(x) - \nabla u(x)(y-x)}{|y-x|} = 0.$$

We say that $u \in BV(A;\mathbb{R}^m)$ if $u \in L^1(A;\mathbb{R}^m)$, and its distributional derivative Du is a vector-valued Radon measure on A. In this case, it turns out that $S(u)$ is rectifiable, that is there exists a sequence $(M_i)_{i\in\mathbb{N}}$ of C^1-manifolds such that $S(u) \subseteq \bigcup_i M_i$ up to a set of $\mathcal{H}^{n-1}$-measure zero; as a consequence $S(u)$ admits a normal ν_x for $\mathcal{H}^{n-1}$-almost every $x \in S(u)$. Moreover the approximate gradient $\nabla u(x)$ exists for a.e. $x \in A$, and ∇u is the density of the absolutely continuous part of Du.

We say that $u \in SBV(A;\mathbb{R}^m)$ if $u \in BV(A;\mathbb{R}^m)$ and the singular part $D^s u$ of its distributional derivative Du is concentrated on $S(u)$. The space $SBV(A;\mathbb{R}^m)$ is called the space of $\mathbb{R}^m$-valued *special functions of bounded variation*. For more details, the reader is referred to [7].

6.3 The finite element space and the transfer of jump

Let $\Omega \subseteq \mathbb{R}^2$ be a polygonal set and let us fix two positive constants $0 < c_1 < c_2 < +\infty$. By a *regular triangulation* of Ω of size ε we intend a finite family of (closed) triangles T_i such that $\overline{\Omega} = \bigcup_i T_i$, $T_i \cap T_j$ is either empty or equal to a common edge or to a common vertex, and each T_i contains a ball of diameter $c_1\varepsilon$ and is contained in a ball of diameter $c_2\varepsilon$.

We indicate by $\mathcal{R}_\varepsilon(\Omega)$ the family of all regular triangulations of Ω of size ε. It turns out that there exist $0 < \vartheta_1 < \vartheta_2 < \pi$ such that for all triangle T belonging to a regular triangulation $\mathbf{T} \in \mathcal{R}_\varepsilon(\Omega)$, the inner angles of T are between ϑ_1 and ϑ_2. Moreover, every edge of T has length greater than $c_1\varepsilon$ and lower than $c_2\varepsilon$.

Let $\varepsilon > 0$, $\mathbf{R}_\varepsilon \in \mathcal{R}_\varepsilon(\Omega)$, and let $a \in]0, \frac{1}{2}[$. Let us consider a triangulation $\mathbf{T}$ nested in $\mathbf{R}_\varepsilon$ obtained dividing each triangle $T \in \mathbf{R}_\varepsilon$ into four triangles taking over every edge $[x,y]$ of T a knot z which satisfies

$$z = tx + (1-t)y, \qquad t \in [a, 1-a].$$

We will call these vertices *adaptive*, the triangles obtained gluing these points *adaptive triangles*, and their edges *adaptive edges*. We denote by $\mathcal{T}_{\varepsilon,a}(\Omega)$ the set of triangulations $\mathbf{T}$ constructed in this way. Note that for all $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$ there exist $0 < c_1^a < c_2^a < +\infty$ such that every $T_i \in \mathbf{T}$ contains a ball of diameter $c_1^a\varepsilon$ and is contained in a ball of diameter $c_2^a\varepsilon$. Then there exist $0 < \vartheta_1^a < \vartheta_2^a < \pi$ such that for all T belonging to $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$, the inner angles of T are between ϑ_1^a and ϑ_2^a. Moreover, every edge of T has length greater than $c_1^a\varepsilon$ and lower than $c_2^a\varepsilon$.

From now on for all $\varepsilon > 0$ we fix $\mathbf{R}_\varepsilon \in \mathcal{R}_\varepsilon(\Omega)$. We suppose that the brittle part Ω_B and the region Ω_S introduced before for the model of quasistatic growth of fractures are composed of triangles of $\mathbf{R}_\varepsilon$ for all ε. Moreover we suppose that $\partial_D\Omega$ and $\partial_S\Omega$ are composed of edges of $\mathbf{R}_\varepsilon$ for all ε up to a finite number of points.

We consider the following discontinuous finite element space. We indicate by $\mathcal{A}_{\varepsilon,a}(\Omega)$ the set of all $u : \Omega \to \mathbb{R}^2$ such that there exists a triangulation $\mathbf{T}(u) \in \mathcal{T}_{\varepsilon,a}(\Omega)$ nested in $\mathbf{R}_\varepsilon$ with u affine on every

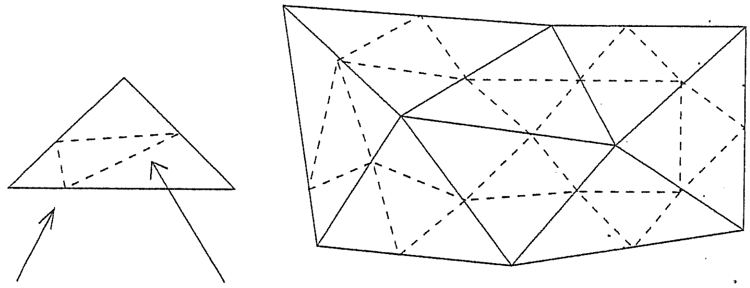

Figure 6.1:

triangle $T \in \mathbf{T}(u)$. For every $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$, we indicate by $S(u)$ the family of edges of $\mathbf{T}(u)$ inside Ω across which u is discontinuous. Notice that $u \in SBV(\Omega;\mathbb{R}^2)$ and that the notation is consistent with the usual one employed in the theory of functions of bounded variation. Let us set

$$\mathcal{AF}_\varepsilon(\Omega) := \{u : \Omega \to \mathbb{R}^2 \,:\, u \text{ is continuous and affine on each triangle } T \in \mathbf{R}_\varepsilon\}. \tag{6.3}$$

The discretization of the problem will be carried out using the space

$$\mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2) := \{u \in \mathcal{A}_{\varepsilon,a}(\Omega) \,:\, S(u) \subseteq \overline{\Omega}_B\}. \tag{6.4}$$

Given any $g \in \mathcal{AF}_\varepsilon(\Omega)$, for every $u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ let

$$S^g_D(u) := \{x \in \partial_D\Omega \,:\, u(x) \neq g(x)\}, \tag{6.5}$$

that is $S^g_D(u)$ denotes the set of edges of $\partial_D\Omega$ at which the boundary condition is not satisfied. For every $u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$, let us also set

$$S^g(u) := S(u) \cup S^g_D(u). \tag{6.6}$$

An essential tool in the approximation result of this chapter is Proposition 6.3.2 which generalizes the piecewise affine transfer of jump [50, Proposition 5.1] to the case of vector valued functions with bulk energy $\mathcal{E}^b$ and surface energy $\mathcal{E}^s$ of the form (3.81) and (3.63) respectively.

In order to deal with the surface energy $\mathcal{E}^s$ we will need the following geometric construction. Let $S \subseteq \Omega$ be a segment and let us suppose that S intersects the edges of $\mathbf{R}_\varepsilon$ at most in one point for all $\varepsilon > 0$. Let $a \in]0, \frac{1}{2}[$, and let $P = S \cap \zeta$, where $\zeta = [x, y]$ is an edge of $\mathbf{R}_\varepsilon$: we indicate with $\pi_a(P)$ the projection of P on the segment $\{tx + (1-t)y \,:\, t \in [a, 1-a]\}$. The *interpolating curve* $S_{\varepsilon,a}$ of S in $\mathbf{R}_\varepsilon$ with parameter a is given connecting all the $\pi_a(P)$'s belonging to the same triangle of $\mathbf{R}_\varepsilon$ (see Figure 2).

Lemma 6.3.1. *Under the previous assumptions, there exists a function $\eta(a)$ independent of S with $\eta(a) \to 0$ as $a \to 0$ such that*

$$\limsup_{\varepsilon\to 0} |\mathcal{E}^s(S_{\varepsilon,a}) - \mathcal{E}^s(S)| \le \eta(a)\mathcal{E}^s(S),$$

where $\mathcal{E}^s$ is defined in (3.63).

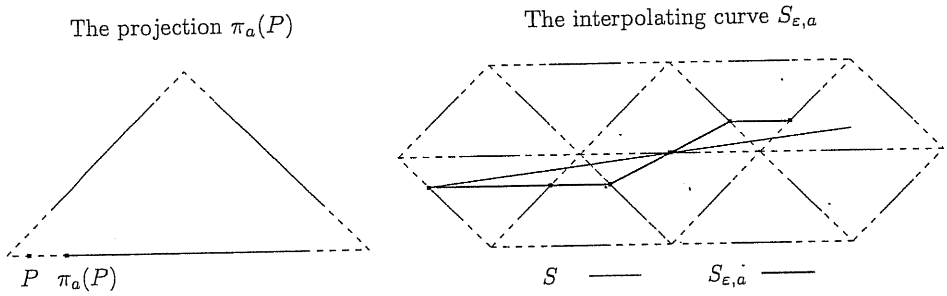

Figure 6.2:

Proof. By (3.64), we have that there exist ω and $K_3 > 0$ such that for all $x_1, x_2 \in \overline{\Omega}$ and $|\nu_1| = |\nu_2| = 1$

$$|k(x_1, \nu_1) - k(x_2, \nu_2)| \leq \omega(|x_1 - x_2|) + K_3|\nu_1 - \nu_2|,$$

where $\omega :]0, +\infty[\to]0, +\infty[$ is a decreasing function such that $\omega(s) \to 0$ as $s \to 0$. Let $T \in \mathbf{R}_\varepsilon$ be such that $T \cap S \neq \emptyset$, and let us choose $x_T \in T \cap S$ and $x_T^{\varepsilon,a} \in T \cap S_{\varepsilon,a}$. Let $c_2 > 0$ denote the characteristic constant of $\mathbf{R}_\varepsilon$ such that every $T \in \mathbf{R}_\varepsilon$ is contained in a ball of diameter $c_2\varepsilon$. Then we have

$$\begin{aligned}
&\left| \int_{S_{\varepsilon,a} \cap T} k(x, \nu_T^{\varepsilon,a})\, d\mathcal{H}^1 - \int_{S \cap T} k(x, \nu_T)\, d\mathcal{H}^1 \right| \\
&\leq \left| \int_{S_{\varepsilon,a} \cap T} k(x_T^{\varepsilon,a}, \nu_T^{\varepsilon,a})\, d\mathcal{H}^1 - \int_{S \cap T} k(x_T, \nu_T)\, d\mathcal{H}^1 \right| + \omega(c_2\varepsilon)\mathcal{H}^1(S_{\varepsilon,a} \cap T) + \omega(c_2\varepsilon)\mathcal{H}^1(S \cap T) \\
&\leq \left| k(x_T^{\varepsilon,a}, \nu_T^{\varepsilon,a})\mathcal{H}^1(S_{\varepsilon,a} \cap T) - k(x_T, \nu_T)\mathcal{H}^1(S \cap T) \right| + \omega(c_2\varepsilon) \left[\mathcal{H}^1(S_{\varepsilon,a} \cap T) + \mathcal{H}^1(S \cap T) \right],
\end{aligned}$$

where $\nu_T^{\varepsilon,a}, \nu_T$ are the (constant) normal to $S_{\varepsilon,a} \cap T$ and $S \cap T$ respectively. We have

$$\begin{aligned}
&\left| k(x_T^{\varepsilon,a}, \nu_T^{\varepsilon,a})\mathcal{H}^1(S_{\varepsilon,a} \cap T) - k(x_T, \nu_T)\mathcal{H}^1(S \cap T) \right| \\
&\leq k(x_T^{\varepsilon,a}, \nu_T^{\varepsilon,a}) \left| \mathcal{H}^1(S_{\varepsilon,a} \cap T) - \mathcal{H}^1(S \cap T) \right| + \left| k(x_T^{\varepsilon,a}, \nu_T^{\varepsilon,a}) - k(x_T, \nu_T) \right| \mathcal{H}^1(S \cap T) \\
&\leq K_2 \left| \mathcal{H}^1(S_{\varepsilon,a} \cap T) - \mathcal{H}^1(S \cap T) \right| + \omega(|x_T^{\varepsilon,a} - x_T|)\mathcal{H}^1(S \cap T) + K_3|\nu_T^{\varepsilon,a} - \nu_T|\mathcal{H}^1(S \cap T),
\end{aligned}$$

where K_2 is defined in (3.64). We are now ready to conclude: in fact, following [62, Lemma 5.2.2], we can choose the orientation of $\nu_T^{\varepsilon,a}$ in such a way that

$$|\nu_T^{\varepsilon,a} - \nu_T|\mathcal{H}^1(S \cap T) \leq D_2 a\varepsilon, \qquad \left| \mathcal{H}^1(S_{\varepsilon,a} \cap T) - \mathcal{H}^1(S \cap T) \right| \leq D_1 a\varepsilon,$$

with $D_1, D_2 > 0$ independent of T, ε, a. Then, summing up the preceding inequalities, recalling that the number of triangles of $\mathbf{R}_\varepsilon$ intersecting S is less than $\tilde{c}\varepsilon^{-1}\mathcal{H}^1(S)$ for ε small enough, with $\tilde{c}$ independent of S and ε (see for example [50, Lemma 2.5] we obtain

$$\limsup_{\varepsilon \to 0} |\mathcal{E}^s(S_{\varepsilon,a}) - \mathcal{E}^s(S)| \leq \rho(a)\mathcal{H}^1(S),$$

where $\rho(a) := \tilde{c}(K_2 D_1 + K_3 D_2)a$. In view of (3.64), we conclude that

$$\limsup_{\varepsilon \to 0} |\mathcal{E}^s(S_{\varepsilon,a}) - \mathcal{E}^s(S)| \leq K_1^{-1}\rho(a)\mathcal{E}^s(S),$$

and so the proof is concluded choosing $\eta(a) := K_1^{-1}\rho(a)$. □

For all $u \in GSBV^p_q(\Omega;\mathbb{R}^2)$ and for all $g \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$, let us set

$$S^g(u) := S(u) \cup \{x \in \partial_D\Omega \,:\, u(x) \neq g(x)\}, \tag{6.7}$$

where the inequality is intended in the sense of traces. We are now in a position to state the piecewise affine transfer of jump proposition in our setting.

Proposition 6.3.2. *Let $a \in]0, \frac{1}{2}[$, and for all $i = 1, \dots, m$ let*

$$u^i_\varepsilon \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2), \qquad u^i \in GSBV^p_q(\Omega;\mathbb{R}^2)$$

be such that

$$u^i_\varepsilon \rightharpoonup u^i \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

Let moreover $g^i_\varepsilon, h_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega)$, $g^i, h \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$ be such that

$$g^i_\varepsilon \to g^i, \quad h_\varepsilon \to h \qquad \text{strongly in } W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2).$$

Then for every $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$, there exists $v_\varepsilon \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ such that

$$\nabla v_\varepsilon \to \nabla v \qquad \text{strongly in } L^p(\Omega;\mathcal{M}^{2\times 2}),$$

$$v_\varepsilon \to v \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2),$$

and such that

$$\limsup_{\varepsilon\to 0} \mathcal{E}^s\left(S^{h_\varepsilon}(v_\varepsilon) \setminus \bigcup_{i=1}^m S^{g^i_\varepsilon}(u^i_\varepsilon)\right) \leq \mu(a)\mathcal{E}^s\left(S^h(v) \setminus \bigcup_{i=1}^m S^{g^i}(u^i)\right),$$

where $\mu(a)$ depends only on a, $\mu(a) \to 1$ as $a \to 0$, and $\mathcal{E}^s$ is defined in (3.63). In particular for all $t \in [0,T]$ and for all $t_\varepsilon \to t$ we have

$$\mathcal{E}^b(t_\varepsilon)(v_\varepsilon) \to \mathcal{E}^b(t)(v),$$

where $\mathcal{E}^b$ is defined in (3.81).

The proof of Proposition 6.3.2 can be obtained from that of [50, Proposition 5.1] where the same result is proved in the scalar valued setting of SBV functions with

$$\mathcal{E}^b(t)(v) = \|\nabla v\|^2 \quad \text{and} \quad \mathcal{E}^s(\Gamma) = \mathcal{H}^1(\Gamma),$$

taking into account the following modifications. We can consider v scalar valued since vector valued maps can be easily dealt componentwise. Even if the surface energy is of the form (3.63), by using the density result of [31] we can still restrict ourselves to the case in which v has piecewise linear jumps outside a suitable open set U such that

$$|U| < \sigma \qquad \text{and} \qquad \mathcal{H}^1\left(\bigcup_{i=1}^m S^{g^i}(u^i) \setminus U\right) < \sigma,$$

where σ is an arbitrarily small constant. In order to approximate the piecewise linear jumps, we use Lemma 6.3.1. Finally, we are not assuming $p = 2$, and this prevents us from considering the piecewise jumps as union of disjoint segments: we overcome this difficulty choosing $v_\varepsilon = 0$ in the regular triangles which contain the intersection points, and then interpolating v outside as in [50, Proposition 5.1].

6.4 Preexisting cracks and their approximation

In Section 6.6, we will need to approximate the surface energy of a given preexisting crack Γ^0. We take the initial crack in the class

$$\begin{aligned}\mathbf{\Gamma}(\Omega) := \{\Gamma \tilde{\subseteq} \overline{\Omega}_B \,:\, \mathcal{H}^1(\Gamma) < +\infty,\, \Gamma = S^h(z) \\ \text{for some } h \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2) \text{ and } z \in GSBV_q^p(\Omega;\mathbb{R}^2)\}.\end{aligned} \tag{6.8}$$

Notice that it is not restrictive to assume $h \equiv 0$. We take as discretization of $\mathbf{\Gamma}(\Omega)$ the following class

$$\mathbf{\Gamma}_{\varepsilon,a}(\Omega) := \{\Gamma \tilde{\subseteq} \overline{\Omega}_B \,:\, \mathcal{H}^1(\Gamma) < +\infty,\, \Gamma = S^0(z) \text{ for some } z \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)\}. \tag{6.9}$$

We have the following approximation result.

Proposition 6.4.1. *Let $\Gamma^0 \in \mathbf{\Gamma}(\Omega)$. Then for every $\varepsilon > 0$ and $a \in]0, \frac{1}{2}[$ there exists $\Gamma^0_{\varepsilon,a} \in \mathbf{\Gamma}_{\varepsilon,a}(\Omega)$ such that*

$$\lim_{\varepsilon,a\to 0} \mathcal{E}^s(\Gamma^0_{\varepsilon,a}) = \mathcal{E}^s(\Gamma^0),$$

where $\mathcal{E}^s$ is defined in (3.63).

Moreover let $g_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega)$, $g \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$ be such that as $\varepsilon \to 0$

$$g_\varepsilon \to g \text{ strongly in } W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2),$$

and let us consider

$$F_{\varepsilon,a}(v) := \begin{cases} \mathcal{E}^b(0)(v) + \mathcal{E}^s\left(S^{g_\varepsilon}(v) \setminus \Gamma^0_{\varepsilon,a}\right) & \text{if } v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2), \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^2), \end{cases}$$

and

$$F(v) := \begin{cases} \mathcal{E}^b(0)(v) + \mathcal{E}^s\left(S^{g}(v) \setminus \Gamma^0\right) & \text{if } v \in GSBV^p_q(\Omega;\mathbb{R}^2),\, S(v) \tilde{\subseteq} \overline{\Omega}_B, \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^2), \end{cases}$$

where $\mathcal{E}^b$ is defined in (3.81). *Then the family $(F_{\varepsilon,a})$ Γ-converges to F in the strong topology of $L^1(\Omega;\mathbb{R}^2)$ as $\varepsilon \to 0$ and $a \to 0$.*

Proof. Let us consider $\Gamma^0 \in \mathbf{\Gamma}(\Omega)$ with $\Gamma^0 = S^0(z)$ for some $z \in GSBV^p_q(\Omega;\mathbb{R}^2)$. Then by Proposition 6.3.2 for every $\varepsilon > 0$ and $a \in (0, \frac{1}{2})$, there exists $\tilde{z}_{\varepsilon,a} \in \mathcal{A}_{\varepsilon,a}(\Omega)$ such that for $\varepsilon \to 0$ and for all a

$$\begin{aligned} \nabla \tilde{z}_{\varepsilon,a} \to \nabla z \qquad & \text{strongly in } L^p(\Omega;\mathcal{M}^{2\times 2}), \\ \tilde{z}_{\varepsilon,a} \to z \qquad & \text{strongly in } L^q(\Omega;\mathbb{R}^2), \end{aligned}$$

and

$$\limsup_{\varepsilon\to 0} \mathcal{E}^s(S^0(\tilde{z}_{\varepsilon,a})) \leq \mu(a)\mathcal{E}^s(S^0(z))$$

with $\mu(a) \to 1$ as $a \to 0$, where $\mathcal{E}^s$ is defined in (3.63). Let $a_i \searrow 0$, and let $\varepsilon_i \searrow 0$ be such that for all $\varepsilon \leq \varepsilon_i$

$$\mathcal{E}^s(S^0(\tilde{z}_{\varepsilon,a_i})) \leq \mu(a_i)\mathcal{E}^s(S^0(z)) + a_i,$$

and

$$\|\nabla \tilde{z}_{\varepsilon,a_i} - \nabla z\|_{L^p(\Omega;\mathcal{M}^{2\times 2})} \leq a_i, \qquad \|\tilde{z}_{\varepsilon,a_i} - z\|_{L^q(\Omega;\mathbb{R}^2)} \leq a_i.$$

Setting

$$z_{\varepsilon,a} := \begin{cases} \tilde{z}_{\varepsilon,a_i} & \varepsilon_{i+1} < \varepsilon \leq \varepsilon_i, \ a \leq a_i, \\ \tilde{z}_{\varepsilon,a_{j-1}} & \varepsilon_{i+1} < \varepsilon \leq \varepsilon_i, \ a_j < a \leq a_{j-1}, \ j \leq i, \end{cases}$$

we have that

$$\lim_{\varepsilon,a \to 0} \nabla z_{\varepsilon,a} = \nabla z \qquad \text{strongly in } L^p(\Omega; \mathcal{M}^{2\times 2}),$$

$$\lim_{\varepsilon,a \to 0} z_{\varepsilon,a} = z \qquad \text{strongly in } L^q(\Omega; \mathbb{R}^2),$$

and

$$\limsup_{\varepsilon,a \to 0} \mathcal{E}^s(S^0(z_{\varepsilon,a})) \leq \mathcal{E}^s(S^0(z)).$$

Since by Theorem 3.8.7 we have $\mathcal{E}^s(S^0(z_{\varepsilon,a})) \leq \liminf_{\varepsilon,a\to 0} \mathcal{E}^s(S^0(z_{\varepsilon,a}))$, we conclude that

$$\lim_{\varepsilon,a \to 0} \mathcal{E}^s(S^0(z_{\varepsilon,a})) = \mathcal{E}^s(S^0(z)).$$

Let us set for every ε, a

$$\Gamma^0_{\varepsilon,a} := S^0(z_{\varepsilon,a}).$$

We have that

$$\lim_{\varepsilon,a \to 0} \mathcal{E}^s(\Gamma^0_{\varepsilon,a}) = \mathcal{E}^s(\Gamma^0).$$

Let us come to the second part of the proof. Let us consider $(\varepsilon_n, a_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$ and $a_n \to 0$. If we prove that $(F_{\varepsilon_n,a_n})_{n\in\mathbb{N}}$ Γ-converges to F in the strong topology of $L^1(\Omega;\mathbb{R}^2)$, the proposition is proved since the sequence is arbitrary. Since we can reason up to subsequences, it is not restrictive to assume $a_n \searrow 0$.

Let us start with the Γ-limsup inequality considering $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$, with $S(v) \subseteq \overline{\Omega}_B$. For any n fixed, by Proposition 6.3.2 there exists $\tilde{v}_{\varepsilon,a_n} \in \mathcal{A}^B_{\varepsilon,a_n}(\Omega;\mathbb{R}^2)$ such that for $\varepsilon \to 0$

$$\nabla \tilde{v}_{\varepsilon,a_n} \to \nabla v \qquad \text{strongly in } L^p(\Omega; \mathcal{M}^{2\times 2}),$$

$$\tilde{v}_{\varepsilon,a_n} \to v \qquad \text{strongly in } L^q(\Omega; \mathbb{R}^2),$$

and such that

$$\limsup_{\varepsilon \to 0} \mathcal{E}^s(S^{g_\varepsilon}(\tilde{v}_{\varepsilon,a_n}) \setminus \Gamma^0_{\varepsilon,a_n}) \leq \mu(a_n)\mathcal{E}^s(S^{g_\varepsilon}(v) \setminus \Gamma^0)$$

with $\mu(a) \to 1$ as $a \to 0$. For every $m \in \mathbb{N}$ let ε^m be such that for all $\varepsilon \leq \varepsilon^m$

$$\mathcal{E}^s(S^{g_\varepsilon}(\tilde{v}_{\varepsilon,a_m}) \setminus \Gamma^0_{\varepsilon,a_m}) \leq \mu(a_m)\mathcal{E}^s(S^g(v) \setminus \Gamma^0) + a_m,$$

and

$$\|\nabla \tilde{v}_{\varepsilon,a_m} - \nabla v\|_{L^p(\Omega;\mathcal{M}^{2\times 2})} \leq a_m, \qquad \|\tilde{v}_{\varepsilon,a_m} - v\|_{L^q(\Omega;\mathbb{R}^2)} \leq a_m.$$

We can assume $\varepsilon^m \searrow 0$. Setting

$$v_{\varepsilon_n,a_n} := \begin{cases} \tilde{v}_{\varepsilon_n,a_m} & \varepsilon^{m+1} < \varepsilon_n \leq \varepsilon^m, \ n \geq m, \\ \tilde{v}_{\varepsilon_n,a_n} & \varepsilon^{m+1} < \varepsilon_n \leq \varepsilon^m, \ n < m, \end{cases}$$

we have that

$$\lim_n \nabla v_{\varepsilon_n,a_n} = \nabla v \quad \text{strongly in } L^p(\Omega; \mathcal{M}^{2\times 2}),$$

$$\lim_n v_{\varepsilon_n,a_n} = v \quad \text{strongly in } L^q(\Omega;\mathbb{R}^2),$$

and

$$\limsup_n \mathcal{E}^s(S^{g_\varepsilon}(v_{\varepsilon_n,a_n}) \setminus \Gamma^0_{\varepsilon,a}) \le \mathcal{E}^s(S^g(v) \setminus \Gamma^0).$$

Then we get

$$\begin{aligned}\limsup_n F_{\varepsilon_n,a_n}(v_{\varepsilon_n,a_n}) &\le \limsup_n \mathcal{E}^b(0)(v_{\varepsilon_n,a_n}) + \limsup_n \mathcal{E}^s(S^{g_{\varepsilon_n}}(v_{\varepsilon_n,a_n}) \setminus \Gamma^0_{\varepsilon_n,a_n}) \\ &\le \mathcal{E}^b(0)(v) + \mathcal{E}^s(S^g(v) \setminus \Gamma^0) = F(v),\end{aligned}$$

so that the Γ-limsup inequality holds.

Let us come to the Γ-liminf inequality. Let $v_n, v \in L^1(\Omega;\mathbb{R}^2)$ be such that $v_n \to v$ strongly in $L^1(\Omega;\mathbb{R}^2)$ and $\liminf_n F_{\varepsilon_n,a_n}(v_n) < +\infty$. By Theorem 3.8.7, we have $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$ and

$$\mathcal{E}^b(0)(v) \le \liminf_n \mathcal{E}^b(0)(v_n).$$

Let us consider Ω_D polygonal such that $\Omega_D \cap \Omega = \emptyset$, and $\partial\Omega_D \cap \partial\Omega = \partial_D\Omega$ up to a finite number of points, and let us set

$$\Omega' := \Omega \cup \Omega_D \cup \partial_D\Omega.$$

Let us extend g_{ε_n} and g to $W^{1,p}(\Omega';\mathbb{R}^2) \cap L^q(\Omega';\mathbb{R}^2)$ in such a way that $g_{\varepsilon_n} \to g$ strongly in $W^{1,p}(\Omega';\mathbb{R}^2) \cap L^q(\Omega';\mathbb{R}^2)$, and let us also extend v_n, v to Ω' setting $v_n = g_{\varepsilon_n}$ and $v = g$ on Ω_D. We indicate these extensions with w_n and w respectively. Notice that $w_n, w \in GSBV^p_q(\Omega;\mathbb{R}^2)$, and that $S^{g_{\varepsilon_n}}(v_n) = S(w_n)$ and $S^{g_\varepsilon}(v) = S(w)$. Let us also set $z_{\varepsilon_n,a_n} = z = 0$ on Ω_D, where z_{ε_n,a_n} and z are such that $\Gamma^0_{\varepsilon_n,a_n} = S^0(z_{\varepsilon_n,a_n})$ and $\Gamma^0 = S^0(z)$. We indicate these extension by $\zeta_{\varepsilon_n,a_n}$ and ζ respectively: we have $\zeta_{\varepsilon_n,a_n}, \zeta \in GSBV^p_q(\Omega;\mathbb{R}^2)$ and $\Gamma^0_{\varepsilon_n,a_n} = S(\zeta_{\varepsilon_n,a_n})$ and $\Gamma^0 = S(\zeta)$. Then for every $\eta > 0$ we have by Theorem 3.8.7

$$\mathcal{E}^s(S(w + \eta\zeta)) \le \liminf_n \mathcal{E}^s(S(w_n + \eta\zeta_{\varepsilon_n,a_n})).$$

Since for a.e. $\eta > 0$ we have $S(w+\eta\zeta) = S(w) \cup S(\zeta)$ and $S(w_n + \eta\zeta_{\varepsilon_n,a_n}) = S(w_n) \cup S(\zeta_{\varepsilon_n,a_n})$, we deduce that

$$\mathcal{E}^s(S^g(v) \cup \Gamma^0) \le \liminf_n \mathcal{E}^s(S^{g_{\varepsilon_n}}(v_n) \cup \Gamma^0_{\varepsilon_n,a_n}).$$

Since by assumption $\mathcal{E}^s(\Gamma^0_{\varepsilon_n,a_n}) \to \mathcal{E}^s(\Gamma^0)$, we conclude that

$$\mathcal{E}^s(S^g(v) \setminus \Gamma^0) \le \liminf_n \mathcal{E}^s(S^{g_{\varepsilon_n}}(v_n) \setminus \Gamma^0_{\varepsilon_n,a_n}).$$

We deduce that

$$\mathcal{E}^b(0)(v) + \mathcal{E}^s(S^g(v) \setminus \Gamma^0) \le \liminf_n \left[\mathcal{E}^b(0)(v_n) + \mathcal{E}^s(S^{g_{\varepsilon_n}}(v_n) \setminus \Gamma^0_{\varepsilon_n,a_n})\right]$$

that is

$$F(v) \le \liminf_n F_{\varepsilon_n,a_n}(v_n).$$

The Γ-liminf inequality holds, and so the proof is concluded. □

6.5 The discontinuous finite element approximation

In this section we construct a discrete approximation of the quasistatic evolution of brittle fractures proposed in [35] and described in the Preliminaries: the discretization is done both in space and time. Let us consider

$$g_\varepsilon \in W^{1,1}([0,T]; W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)), \quad g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega) \text{ for all } t \in [0,T],$$

where $\mathcal{AF}_\varepsilon(\Omega)$ is defined in (6.3). Let $\delta > 0$, and let N_δ be the largest integer such that $\delta(N_\delta - 1) < T$; we set $t_i^\delta := i\delta$ for $0 \le i \le N_\delta - 1$, $t_{N_\delta}^\delta := T$ and $g_\varepsilon^{\delta,i} := g_\varepsilon(t_i^\delta)$. Let $\Gamma^0 \in \mathbf{\Gamma}_{\varepsilon,a}(\Omega)$ be a preexisting crack in Ω, where $\mathbf{\Gamma}_{\varepsilon,a}(\Omega)$ is defined in (6.9).

Proposition 6.5.1. *Let $\varepsilon > 0$, $a \in]0,\frac{1}{2}[$ and $\delta > 0$ be fixed. Then for all $i = 0,\dots,N_\delta$ there exists $u_{\varepsilon,a}^{\delta,i} \in \mathcal{A}_{\varepsilon,a}^B(\Omega;\mathbb{R}^2)$ such that, setting*

$$\Gamma_{\varepsilon,a}^{\delta,i} := \Gamma^0 \cup \bigcup_{r=0}^{i} S^{g^{\delta,r}}(u_{\varepsilon,a}^{\delta,r}),$$

we have for all $v \in \mathcal{A}_{\varepsilon,a}^B(\Omega;\mathbb{R}^2)$

$$\mathcal{E}^b(0)(u_{\varepsilon,a}^{\delta,0}) + \mathcal{E}^s\left(S^{g^{\delta,0}}(u_{\varepsilon,a}^{\delta,0}) \setminus \Gamma^0\right) \le \mathcal{E}^b(0)(v) + \mathcal{E}^s\left(S^{g^{\delta,0}}(v) \setminus \Gamma^0\right), \tag{6.10}$$

and for $1 \le i \le N_\delta$

$$\mathcal{E}^b(t_i^\delta)(u_{\varepsilon,a}^{\delta,i}) + \mathcal{E}^s\left(S^{g^{\delta,i}}(u_{\varepsilon,a}^{\delta,i}) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right) \le \mathcal{E}^b(t_i^\delta)(v) + \mathcal{E}^s\left(S^{g^{\delta,i}}(v) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right). \tag{6.11}$$

Proof. Let $u_{\varepsilon,a}^{\delta,0}$ be a minimum of the following problem

$$\min_{u \in \mathcal{A}_{,a}^B(\Omega;\mathbb{R}^2)} \left\{ \mathcal{E}^b(0)(u) + \mathcal{E}^s(S^{g^{\delta,0}}(u) \setminus \Gamma^0) \right\}. \tag{6.12}$$

We set $\Gamma_{\varepsilon,a}^{\delta,0} := \Gamma^0 \cup S^{g^{\delta,0}}(u_{\varepsilon,a}^{\delta,0})$. Recursively, supposing to have constructed $u_{\varepsilon,a}^{\delta,i-1}$ and $\Gamma_{\varepsilon,a}^{\delta,i-1}$, let $u_{\varepsilon,a}^{\delta,i}$ be a minimum for

$$\min_{u \in \mathcal{A}_{,a}^B(\Omega;\mathbb{R}^2)} \left\{ \mathcal{E}^b(t_i^\delta)(u) + \mathcal{E}^s(S^{g^{\delta,i}}(u) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}) \right\}. \tag{6.13}$$

We set $\Gamma_{\varepsilon,a}^{\delta,i} := \Gamma_{\varepsilon,a}^{\delta,i-1} \cup S^{g^{\delta,i}}(u_{\varepsilon,a}^{\delta,i})$. It is clear by construction that (6.10) and (6.11) hold.

Let us prove that problem (6.13) admits a solution, problem (6.12) being similar. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for problem (6.13): since $g_\varepsilon^{\delta,i}$ is an admissible test function, we deduce that for n large

$$\mathcal{E}^b(t_i^\delta)(u_n) + \mathcal{E}^s\left(S^{g^{\delta,i}}(u_n) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right) \le \mathcal{E}^b(t_i^\delta)(g_\varepsilon^{\delta,i}) + 1.$$

By the lower estimate on the elastic energy (3.82), we deduce that for n large

$$\alpha_0^{\mathcal{E}}\left(\|\nabla u_n\|_p^p + \|u_n\|_q^q\right) + \mathcal{E}^s\left(S^{g^{\delta,i}}(u_n) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right) \le \mathcal{E}^b(t_i^\delta)(g_\varepsilon^{\delta,i}) + 1 + \beta_0^{\mathcal{E}}. \tag{6.14}$$

Up to a subsequence, we have that the adaptive vertices z_n^i converge to some adaptive vertices z^i, and also the values of the deformation u_n at every vertex are converging. In the end we find interpolating these values an admissible deformation $u \in \mathcal{A}_{\varepsilon,a}^B(\Omega;\mathbb{R}^2)$. Since the functional in the minimization problem (6.13) is lower semicontinuous with respect to the position of the vertices and the values of the deformation, we conclude that u is a minimum point for the problem. $\square$

The following estimate on the total energy is essential in order to study the asymptotic behavior of the discrete evolution as $\delta \to 0$, $\varepsilon \to 0$ and $a \to 0$. Let us set $u^\delta_{\varepsilon,a}(t) := u^{\delta,i}_{\varepsilon,a}$ for all $t^\delta_i \le t < t^\delta_{i+1}$ and $i = 0, \dots, N_\delta - 1$, $u^\delta_{\varepsilon,a}(T) = u^{\delta,N_\delta}_{\varepsilon,a}$.

Proposition 6.5.2. *For all $0 \le j \le i \le N_\delta$ we have*

$$\begin{aligned}\mathcal{E}(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a}) \le \mathcal{E}(t^\delta_j)(u^{\delta,j}_{\varepsilon,a}, \Gamma^{\delta,j}_{\varepsilon,a}) + \int_{t^\delta_j}^{t^\delta_i} \langle \partial \mathcal{W}(\nabla u^\delta_{\varepsilon,a}(\tau)), \nabla \dot g_\varepsilon(\tau)\rangle \, d\tau \\ - \int_{t^\delta_j}^{t^\delta_i} \dot{\mathcal{F}}(\tau)(u^\delta_{\varepsilon,a}(\tau))\, d\tau - \int_{t^\delta_j}^{t^\delta_i} \langle \partial \mathcal{F}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle \, d\tau \\ - \int_{t^\delta_j}^{t^\delta_i} \dot{\mathcal{G}}(\tau)(u^\delta_{\varepsilon,a}(\tau))\, d\tau - \int_{t^\delta_j}^{t^\delta_i} \langle \partial \mathcal{G}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle \, d\tau + e^\delta_{\varepsilon,a}, \quad (6.15)\end{aligned}$$

where $e^\delta_{\varepsilon,a} \to 0$ as $\delta \to 0$ uniformly in ε and a.

Proof. By the minimality property (5.14), comparing $u^{\delta,i}_{\varepsilon,a}$ with $u^{\delta,i-1}_{\varepsilon,a} - g^{\delta,i-1}_\varepsilon + g^{\delta,i}_\varepsilon$ we get

$$\begin{aligned}\mathcal{W}(\nabla u^{\delta,i}_{\varepsilon,a}) - \mathcal{F}(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}) - \mathcal{G}(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}) + \mathcal{E}^s(S^{g^\delta_i}(u^{\delta,i}_{\varepsilon,a}) \setminus \Gamma^\delta_{i-1}) \\ \le \mathcal{W}(\nabla u^{\delta,i-1}_{\varepsilon,a} - \nabla g^{\delta,i-1}_\varepsilon + \nabla g^{\delta,i}_\varepsilon) - \mathcal{F}(t^\delta_i)(u^{\delta,i-1}_{\varepsilon,a} - g^{\delta,i-1}_\varepsilon + g^{\delta,i}_\varepsilon) \\ - \mathcal{G}(t^\delta_i)(u^{\delta,i-1}_{\varepsilon,a} - g^{\delta,i-1}_\varepsilon + g^{\delta,i}_\varepsilon). \quad (6.16)\end{aligned}$$

We have

$$\begin{aligned}\mathcal{W}(\nabla u^{\delta,i-1}_{\varepsilon,a} - \nabla g^{\delta,i-1}_\varepsilon + \nabla g^{\delta,i}_\varepsilon) = \mathcal{W}(\nabla u^{\delta,i-1}_{\varepsilon,a}) \\ + \langle \partial \mathcal{W}(\nabla u^{\delta,i-1}_{\varepsilon,a} + \vartheta^{\delta,i-1}_{\varepsilon,a}(\nabla g^{\delta,i}_\varepsilon - \nabla g^{\delta,i-1}_\varepsilon)), \nabla g^{\delta,i}_\varepsilon - \nabla g^{\delta,i-1}_\varepsilon\rangle \\ = \mathcal{W}(\nabla u^{\delta,i-1}_{\varepsilon,a}) + \int_{t^\delta_{i-1}}^{t^\delta_i} \langle \partial \mathcal{W}(\nabla u^\delta_{\varepsilon,a}(\tau) + v^\delta_{\varepsilon,a}(\tau)), \nabla \dot g_\varepsilon(\tau)\rangle \, d\tau, \quad (6.17)\end{aligned}$$

where $\vartheta^{\delta,i-1}_{\varepsilon,a} \in]0,1[$ and $v^\delta_{\varepsilon,a}(\tau) := \vartheta^{\delta,i-1}_{\varepsilon,a}(\nabla g^{\delta,i}_\varepsilon - \nabla g^{\delta,i-1}_\varepsilon)$ for all $\tau \in [t^\delta_{i-1}, t^\delta_i[$.

Similarly we obtain

$$\mathcal{F}(t^\delta_i)(u^{\delta,i-1}_{\varepsilon,a} - g^{\delta,i-1}_\varepsilon + g^{\delta,i}_\varepsilon) = \mathcal{F}(t^\delta_i)(u^{\delta,i-1}_{\varepsilon,a}) + \int_{t^\delta_{i-1}}^{t^\delta_i} \langle \partial \mathcal{F}(t^\delta_i)(u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle \, d\tau, \quad (6.18)$$

and

$$\mathcal{G}(t^\delta_i)(u^{\delta,i-1}_{\varepsilon,a} - g^{\delta,i-1}_\varepsilon + g^{\delta,i}_\varepsilon) = \mathcal{G}(t^\delta_i)(u^{\delta,i-1}_{\varepsilon,a}) + \int_{t^\delta_{i-1}}^{t^\delta_i} \langle \partial \mathcal{G}(t^\delta_i)(u^\delta_{\varepsilon,a}(\tau) + z^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle \, d\tau, \quad (6.19)$$

where $w^\delta_{\varepsilon,a}(\tau) := \lambda^{\delta,i-1}_{\varepsilon,a}(g^{\delta,i}_\varepsilon - g^{\delta,i-1}_\varepsilon)$, $z^\delta_{\varepsilon,a}(\tau) := \nu^{\delta,i-1}_{\varepsilon,a}(g^{\delta,i}_\varepsilon - g^{\delta,i-1}_\varepsilon)$ for all $\tau \in [t^\delta_{i-1}, t^\delta_i[$, and $\lambda^{\delta,i-1}_{\varepsilon,a}, \nu^{\delta,i-1}_{\varepsilon,a} \in]0,1[$.

Since by (3.72) we have for $\tau \in [t^\delta_{i-1}, t^\delta_i[$

$$\begin{aligned}\langle \partial \mathcal{F}(t^\delta_i)(u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle - \langle \partial \mathcal{F}(\tau)(u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle \\ = \int_\tau^{t^\delta_i} \langle \partial \dot{\mathcal{F}}(s)(u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle \, ds\end{aligned}$$

we get by (3.75)

$$\begin{aligned}
&\left|\langle \partial\mathcal{F}(t_i^\delta)(u_{\varepsilon,a}^\delta(\tau)+w_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle - \langle \partial\mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau)+w_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle\right| \\
&\qquad\le \int_\tau^{t_i^\delta} |\langle \partial\dot{\mathcal{F}}(s)(u_{\varepsilon,a}^\delta(\tau)+w_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle|\,ds \\
&\qquad\le \int_\tau^{t_i^\delta} \left[\alpha_4^{\mathcal{F}}(s)\|u_{\varepsilon,a}^\delta(\tau)+w_{\varepsilon,a}^\delta(\tau)\|_{\dot q}^{\dot q-1} + \beta_4^{\mathcal{F}}(s)\right]\|\dot g_\varepsilon(\tau)\|_{\dot q}\,ds \le \gamma_{\mathcal{F}}^{\delta,\varepsilon,a}\|\dot g_\varepsilon(\tau)\|_{\dot q}, \qquad (6.20)
\end{aligned}$$

where

$$\gamma_{\mathcal{F}}^{\delta,\varepsilon,a} := \max_{1\le i\le N_\delta}\left(\|u_{\varepsilon,a}^{\delta,i-1}+\lambda_{\varepsilon,a}^{\delta,i-1}(g_\varepsilon^{\delta,i}-g_\varepsilon^{\delta,i-1})\|_{\dot q-1}^{\dot q}\int_{t_{i-1}^\delta}^{t_i^\delta}\alpha_4^{\mathcal{F}}(s)\,ds + \int_{t_{i-1}^\delta}^{t_i^\delta}\beta_4^{\mathcal{F}}(s)\,ds\right).$$

Similarly we obtain

$$\left|\langle \partial\mathcal{G}(t_i^\delta)(u_{\varepsilon,a}^\delta(\tau)+z_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle - \langle \partial\mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau)+z_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle\right| \le \gamma_{\mathcal{G}}^{\delta,\varepsilon,a}\|\dot g_\varepsilon(\tau)\|_{r,\partial_S\Omega}, \qquad (6.21)$$

where

$$\gamma_{\mathcal{G}}^{\delta,\varepsilon,a} := \max_{1\le i\le N_\delta}\left(\|u_{\varepsilon,a}^{\delta,i-1}+\nu_{\varepsilon,a}^{\delta,i-1}(g_\varepsilon^{\delta,i}-g_\varepsilon^{\delta,i-1})\|_{r,\partial_S\Omega}^{r-1}\int_{t_{i-1}^\delta}^{t_i^\delta}a_4^{\mathcal{G}}(s)\,ds + \int_{t_{i-1}^\delta}^{t_i^\delta}b_4^{\mathcal{G}}(s)\,ds\right).$$

From (6.16), taking into account (6.17), (6.18), (6.19), (6.20), (6.21), we have

$$\begin{aligned}
\mathcal{E}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i},\Gamma_{\varepsilon,a}^{\delta,i}) &\le \mathcal{E}(t_{i-1}^\delta)(u_{\varepsilon,a}^{\delta,i-1},\Gamma_{\varepsilon,a}^{\delta,i-1}) + \int_{t_{i-1}^\delta}^{t_i^\delta}\langle \partial\mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau)+v_{\varepsilon,a}^\delta(\tau)), \nabla\dot g_\varepsilon(\tau)\rangle\,d\tau \\
&- \int_{t_{i-1}^\delta}^{t_i^\delta}\dot{\mathcal{F}}(\tau)(u_{\varepsilon,a}^\delta(\tau))\,d\tau - \int_{t_{i-1}^\delta}^{t_i^\delta}\langle \partial\mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau)+w_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle\,d\tau \\
&- \int_{t_{i-1}^\delta}^{t_i^\delta}\dot{\mathcal{G}}(\tau)(u_{\varepsilon,a}^\delta(\tau))\,d\tau - \int_{t_{i-1}^\delta}^{t_i^\delta}\langle \partial\mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau)+z_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle\,d\tau \\
&+ \gamma_{\mathcal{F}}^{\delta,\varepsilon,a}\int_{t_{i-1}^\delta}^{t_i^\delta}\|\dot g_\varepsilon(\tau)\|_{\dot q}\,d\tau + \gamma_{\mathcal{G}}^{\delta,\varepsilon,a}\int_{t_{i-1}^\delta}^{t_i^\delta}\|\dot g_\varepsilon(\tau)\|_{r,\partial_S\Omega}\,d\tau. \qquad (6.22)
\end{aligned}$$

Taking now $0 \le j \le i \le N_\delta$, summing in (6.22) from t_j^δ to t_i^δ, we obtain

$$\begin{aligned}
\mathcal{E}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i},\Gamma_{\varepsilon,a}^{\delta,i}) &\le \mathcal{E}(t_j^\delta)(u_{\varepsilon,a}^{\delta,j},\Gamma_{\varepsilon,a}^{\delta,j}) + \int_{t_j^\delta}^{t_i^\delta}\langle \partial\mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau)+v_{\varepsilon,a}^\delta(\tau)), \nabla\dot g_\varepsilon(\tau)\rangle\,d\tau \\
&- \int_{t_j^\delta}^{t_i^\delta}\dot{\mathcal{F}}(\tau)(u_{\varepsilon,a}^\delta(\tau))\,d\tau - \int_{t_j^\delta}^{t_i^\delta}\langle \partial\mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau)+w_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle\,d\tau \\
&- \int_{t_j^\delta}^{t_i^\delta}\dot{\mathcal{G}}(\tau)(u_{\varepsilon,a}^\delta(\tau))\,d\tau - \int_{t_j^\delta}^{t_i^\delta}\langle \partial\mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau)+z_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle\,d\tau \\
&+ \gamma_{\mathcal{F}}^{\delta,\varepsilon,a}\int_{t_j^\delta}^{t_i^\delta}\|\dot g_\varepsilon(\tau)\|_{\dot q}\,d\tau + \gamma_{\mathcal{G}}^{\delta,\varepsilon,a}\int_{t_j^\delta}^{t_i^\delta}\|\dot g_\varepsilon(\tau)\|_{r,\partial_S\Omega}\,d\tau. \qquad (6.23)
\end{aligned}$$

Setting

$$\begin{aligned} e^{\delta}_{\varepsilon,a} := \int_0^1 & |\langle \partial\mathcal{W}(\nabla u^{\delta}_{\varepsilon,a}(\tau) + v^{\delta}_{\varepsilon,a}(\tau)), \nabla \dot{g}_{\varepsilon}(\tau)\rangle - \langle \partial\mathcal{W}(\nabla u^{\delta}_{\varepsilon,a}(\tau)), \nabla\dot{g}_{\varepsilon}(\tau)\rangle|\, d\tau \\ &+ \int_0^1 |\langle \partial\mathcal{F}(\tau)(u^{\delta}_{\varepsilon,a}(\tau) + w^{\delta}_{\varepsilon,a}(\tau)), \dot{g}_{\varepsilon}(\tau)\rangle - \langle \partial\mathcal{F}(\tau)(u^{\delta}_{\varepsilon,a}(\tau)), \dot{g}_{\varepsilon}(\tau)\rangle|\, d\tau \\ &+ \int_0^1 |\langle \partial\mathcal{G}(\tau)(u^{\delta}_{\varepsilon,a}(\tau) + z^{\delta}_{\varepsilon,a}(\tau)), \dot{g}_{\varepsilon}(\tau)\rangle - \langle \partial\mathcal{G}(\tau)(u^{\delta}_{\varepsilon,a}(\tau)), \dot{g}_{\varepsilon}(\tau)\rangle|\, d\tau \\ &\qquad + \gamma^{\delta,\varepsilon,a}_{\mathcal{F}} \int_0^1 \|\dot{g}_{\varepsilon}(\tau)\|_{q}\, d\tau + \gamma^{\delta,\varepsilon,a}_{\mathcal{G}} \int_0^1 \|\dot{g}_{\varepsilon}(\tau)\|_{r,\partial_S\Omega}\, d\tau, \end{aligned} \tag{6.24}$$

from (6.23) we formally obtain (6.15). Let us prove that $e^{\delta}_{\varepsilon,a} \to 0$ as $\delta \to 0$ uniformly in ε and a. By (6.11), comparing $u^{\delta,i}_{\varepsilon,a}$ with $g^{\delta,i}_{\varepsilon}$, and taking into account (3.82), we get for all $i = 1, \dots, N_{\delta}$,

$$\|\nabla u^{\delta,i}_{\varepsilon,a}\|_p + \|u^{\delta,i}_{\varepsilon,a}\|_q \le C',$$

where

$$C' := \frac{1}{\alpha_0^{\mathcal{E}}} \max_{i=0,\dots,N_\delta} \left(\mathcal{E}^b(t_i^{\delta})(g^{\delta,i}_{\varepsilon}) + \beta_0^{\mathcal{E}} \right).$$

Since Ω_S is Lipschitz, there exists $K_S > 0$ depending only on p, q such that

$$\|u\|_{p,\Omega_S} \le K_S(\|\nabla u\|_{p,\Omega_S} + \|u\|_{q,\Omega_S})$$

for all $u \in W^{1,p}(\Omega_S;\mathbb{R}^2) \cap L^q(\Omega_S;\mathbb{R}^2)$. Taking into account (3.77), we obtain

$$\|u^{\delta,i}_{\varepsilon,a}\|_{r,\partial_S\Omega} \le C''$$

for some C'' independent of δ. Since $g_\varepsilon \in W^{1,1}([0,T]; W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2))$, we obtain that for all $\tau \in [0,T]$ as $\delta \to 0$

$$v^{\delta}_{\varepsilon,a}(\tau) \to 0 \text{ strongly in } L^p(\Omega; \mathcal{M}^{2\times 2}),$$

$$w^{\delta}_{\varepsilon,a}(\tau) \to 0 \text{ strongly in } L^q(\Omega; \mathbb{R}^2),$$

$$z^{\delta}_{\varepsilon,a}(\tau) \to 0 \text{ strongly in } L^r(\partial_S\Omega; \mathbb{R}^2).$$

Moreover $\gamma^{\delta,\varepsilon,a}_{\mathcal{F}} \to 0$ and $\gamma^{\delta,\varepsilon,a}_{\mathcal{G}} \to 0$ as $\delta \to 0$. Finally, by [35, Lemma 4.9], as $\delta \to 0$ we have that for all $\tau \in [0,T]$

$$|\langle \partial\mathcal{W}(\nabla u^{\delta}_{\varepsilon,a}(\tau) + v^{\delta}_{\varepsilon,a}(\tau)), \nabla \dot{g}_{\varepsilon}(\tau)\rangle - \langle \partial\mathcal{W}(\nabla u^{\delta}_{\varepsilon,a}(\tau)), \nabla\dot{g}_{\varepsilon}(\tau)\rangle| \to 0,$$

$$|\langle \partial\mathcal{F}(\tau)(u^{\delta}_{\varepsilon,a}(\tau) + w^{\delta}_{\varepsilon,a}(\tau)), \dot{g}_{\varepsilon}(\tau)\rangle - \langle \partial\mathcal{F}(\tau)(u^{\delta}_{\varepsilon,a}(\tau)), \dot{g}_{\varepsilon}(\tau)\rangle| \to 0,$$

$$|\langle \partial\mathcal{G}(\tau)(u^{\delta}_{\varepsilon,a}(\tau) + z^{\delta}_{\varepsilon,a}(\tau)), \dot{g}_{\varepsilon}(\tau)\rangle - \langle \partial\mathcal{G}(\tau)(u^{\delta}_{\varepsilon,a}(\tau)), \dot{g}_{\varepsilon}(\tau)\rangle| \to 0,$$

uniformly in ε, a. By the Dominated Convergence Theorem, we conclude that $e^{\delta}_{\varepsilon,a} \to 0$ as $\delta \to 0$ uniformly in ε and a, and the proof is finished. $\square$

6.6 The approximation result

In this section we study the asymptotic behavior of the discrete evolution obtained in Section 6.5. Let us consider a given initial crack $\Gamma^0 \in \mathbf{\Gamma}(\Omega)$ where $\mathbf{\Gamma}(\Omega)$ is defined as in (6.8), and a boundary deformation $g \in W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2))$. Let $\Gamma^0_{\varepsilon,a} \in \mathbf{\Gamma}_{\varepsilon,a}(\Omega)$ be an approximation of Γ^0 in the sense of Proposition 6.4.1, and let us consider

$$g_\varepsilon \in W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)),$$

such that

$$g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega) \text{ for all } t \in [0,T],$$

and such that for $\varepsilon \to 0$

$$g_\varepsilon \to g \quad \text{strongly in } W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)).$$

Let

$$\{(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a}),\ i = 0, \dots, N_\delta\}$$

be the discrete evolution relative to the initial crack $\Gamma^0_{\varepsilon,a}$ and boundary data g_ε given by Proposition 6.5.1. We make the following piecewise constant interpolation in time:

$$u^\delta_{\varepsilon,a}(t) := u^{\delta,i}_{\varepsilon,a}, \qquad \Gamma^\delta_{\varepsilon,a}(t) := \Gamma^{\delta,i}_{\varepsilon,a}, \qquad g^\delta_\varepsilon(t) := g_\varepsilon(t^\delta_i) \qquad \text{for } t^\delta_i \le t < t^\delta_{i+1}, \tag{6.25}$$

$i = 0, \dots, N_\delta - 1$, and $u^\delta_{\varepsilon,a}(T) := u^{\delta,N_\delta}_{\varepsilon,a}$, $\Gamma^\delta_{\varepsilon,a}(T) := \Gamma^{\delta,N_\delta}_{\varepsilon,a}$, $g^\delta_\varepsilon(T) := g_\varepsilon(T)$.

By Proposition 6.5.2, for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ we have

$$\mathcal{E}^b(0)(u^\delta_{\varepsilon,a}(0)) + \mathcal{E}^s\left(S^{g^\delta(0)}(u^\delta_{\varepsilon,a}(0)) \setminus \Gamma^0_{\varepsilon,a}\right) \le \mathcal{E}^b(0)(v) + \mathcal{E}^s\left(S^{g^\delta(0)}(v) \setminus \Gamma^0_{\varepsilon,a}\right),$$

and for all $t \in [t^\delta_i, t^\delta_{i+1}[$ and for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$

$$\mathcal{E}^b(t^\delta_i)(u^\delta_{\varepsilon,a}(t)) \le \mathcal{E}^b(t^\delta_i)(v) + \mathcal{E}^s\left(S^{g^\delta(t)}(v) \setminus \Gamma^\delta_{\varepsilon,a}(t)\right). \tag{6.26}$$

Here $\mathcal{E}^b$ and $\mathcal{E}^s$ are defined in (3.81) and (3.63) respectively. Finally for all $0 \le s \le t \le 1$ we have

$$\begin{aligned}\mathcal{E}(t^\delta_i)(u^\delta_{\varepsilon,a}(t), \Gamma^\delta_{\varepsilon,a}(t)) \le{}& \mathcal{E}(s^\delta_j)(u^\delta_{\varepsilon,a}(s), \Gamma^\delta_{\varepsilon,a}(s)) + \int_{s^\delta_j}^{t^\delta_i} \langle \partial\mathcal{W}(\nabla u^\delta_{\varepsilon,a}(\tau)), \nabla \dot g_\varepsilon(\tau)\rangle\, d\tau \\ &- \int_{s^\delta_j}^{t^\delta_i} \dot{\mathcal{F}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) - \int_{s^\delta_j}^{t^\delta_i} \langle \partial\mathcal{F}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle\, d\tau \\ &- \int_{s^\delta_j}^{t^\delta_i} \dot{\mathcal{G}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) - \int_{s^\delta_j}^{t^\delta_i} \langle \partial\mathcal{G}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle\, d\tau + e^\delta_{\varepsilon,a},\end{aligned} \tag{6.27}$$

where $s^\delta_j \le s < s^\delta_{j+1}$ and $t^\delta_i \le t < t^\delta_{i+1}$, $e^\delta_{\varepsilon,a}$ is defined as in (6.24), and $\mathcal{E}(t)(u,\Gamma)$ is as in (3.80). Recall that $e^\delta_{\varepsilon,a} \to 0$ as $\delta \to 0$ uniformly in ε, a.

Comparing $u^\delta_{\varepsilon,a}(t)$ with $g^\delta_\varepsilon(t)$ by (6.26), and in view of (3.69), (3.73), (3.74), (3.78), (3.79), (6.10) and (3.64), by (6.27) with $s = 0$ we deduce that there exists $C' \in]0, +\infty[$ such that for all t, δ, ε and a

$$\|\nabla u^\delta_{\varepsilon,a}(t)\|_p + \|u^\delta_{\varepsilon,a}(t)\|_q + \mathcal{H}^1(\Gamma^\delta_{\varepsilon,a}(t)) \le C'. \tag{6.28}$$

By the time dependence of $\mathcal{E}^b(\cdot,\cdot)$, in view of (6.28), by (6.26) and (6.27) we have that there exists $o^\delta_{\varepsilon,a} \to 0$ as $\delta, \varepsilon \to 0$ uniformly in a such that for all $t \in [0,T]$ and for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$

$$\mathcal{E}^b(t)(u^\delta_{\varepsilon,a}(t)) \leq \mathcal{E}^b(t)(v) + \mathcal{E}^s\left(S^{g^\delta(t)}(v) \setminus \Gamma^\delta_{\varepsilon,a}(t)\right) + o^\delta_{\varepsilon,a}, \tag{6.29}$$

and for all $0 \leq s \leq t \leq T$

$$\begin{aligned}\mathcal{E}(t)(u^\delta_{\varepsilon,a}(t),\Gamma^\delta_{\varepsilon,a}(t)) \leq& \mathcal{E}(s)(u^\delta_{\varepsilon,a}(s),\Gamma^\delta_{\varepsilon,a}(s)) + \int_s^t \langle \partial\mathcal{W}(\nabla u^\delta_{\varepsilon,a}(\tau)), \nabla\dot{g}_\varepsilon(\tau)\rangle\, d\tau \\ &- \int_s^t \dot{\mathcal{F}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) - \int_s^t \langle \partial\mathcal{F}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot{g}_\varepsilon(\tau)\rangle\, d\tau \\ &- \int_s^t \dot{\mathcal{G}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) - \int_s^t \langle \partial\mathcal{G}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot{g}_\varepsilon(\tau)\rangle\, d\tau + o^\delta_{\varepsilon,a}.\end{aligned} \tag{6.30}$$

Inequality (6.28) gives a natural precompactness of $(u^\delta_{\varepsilon,a}(t))$ in $GSBV^p_q(\Omega;\mathbb{R}^2)$. The main result of the chapter is the following.

Theorem 6.6.1. *Let $\delta > 0$, $\varepsilon > 0$, $a \in]0,\frac{1}{2}[$, and let $\{t \to (u^\delta_{\varepsilon,a}(t),\Gamma^\delta_{\varepsilon,a}(t)) : t \in [0,T]\}$ be the discrete evolution given by (6.25) relative to the initial crack $\Gamma^0_{\varepsilon,a}$ and the boundary data g_ε. Then there exist a quasistatic evolution $\{t \to (u(t),\Gamma(t))\}$ in the sense of Theorem 3.8.8 and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, such that setting $u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t)$ and $\Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)$, for all $t \in [0,T]$ the following facts hold.*

(a) For every $t \in [0,T]$, $(u_n(t))_{n\in\mathbb{N}}$ is weakly precompact in $GSBV^p_q(\Omega;\mathbb{R}^2)$, and every accumulation point $\tilde{u}(t)$ is such that $S^{g(t)}(\tilde{u}(t)) \tilde{\subseteq} \Gamma(t)$,

$$\mathcal{E}^b(t)(\tilde{u}(t)) \leq \mathcal{E}^b(t)(v) + \mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma(t)\right) \tag{6.31}$$

for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$, and

$$\mathcal{E}^b(t)(u_n(t)) \to \mathcal{E}^b(t)(\tilde{u}(t)).$$

Moreover there exists a subsequence of $(\delta_n,\varepsilon_n,a_n)_{n\in\mathbb{N}}$ (depending on t) such that

$$u_n(t) \rightharpoonup u(t) \quad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

(b) For every $t \in [0,T]$ we have

$$\mathcal{E}(t)(u_n(t),\Gamma_n(t)) \to \mathcal{E}(t)(u(t),\Gamma(t)); \tag{6.32}$$

more precisely elastic and surface energies converge separately, that is

$$\mathcal{E}^b(t)(u_n(t)) \to \mathcal{E}^b(t)(u(t)), \qquad \mathcal{E}^s(\Gamma_n(t)) \to \mathcal{E}^s(\Gamma(t)). \tag{6.33}$$

For the proof of Theorem 6.6.1 we need two preliminary steps. First of all, we fix a and study the asymptotic for $\delta, \varepsilon \to 0$ (Lemma 6.6.2), and then we let $a \to 0$ using a diagonal argument (Lemma 6.6.4).

Lemma 6.6.2. *Let a be fixed, $t \in [0,T]$, and let $\delta_n \to 0$ and $\varepsilon_n \to 0$. There exists $\Gamma_a(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N \Omega)$ and a subsequence of $(\delta_n, \varepsilon_n)_{n\in\mathbb{N}}$ (which we denote with the same symbol), such that the following facts hold:*

(a) if $w_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ is such that $S^{g^{\delta_n}_n(t)}(w_n) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a}(t)$ and

$$w_n \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2),$$

then we have

$$S^{g(t)}(w) \tilde{\subseteq} \Gamma_a(t);$$

(b) there exists $\mu(a)$ with $\mu(a) \to 1$ as $a \to 0$ such that for every accumulation point $u_a(t)$ of $(u^{\delta_n}_{\varepsilon_n,a}(t))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV^p_q(\Omega;\mathbb{R}^2)$ and for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$, we have

$$\mathcal{E}^b(t)(u_a(t)) \le \mathcal{E}^b(t)(v) + \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(t)\right); \tag{6.34}$$

moreover

$$\lim_n \mathcal{E}^b(t)(u^{\delta_n}_{\varepsilon_n,a}(t)) = \mathcal{E}^b(t)(u_a(t)); \tag{6.35}$$

(c) we have

$$\mathcal{E}^s(\Gamma_a(t)) \le \liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a}(t)).$$

Proof. We now perform a variant of [35, Theorem 4.7]. Let $(\varphi_k)_{k\in\mathbb{N}} \subseteq L^1(\Omega;\mathbb{R}^2)$ be dense in $L^1(\Omega;\mathbb{R}^2)$. For every φ_k and for every $m \in \mathbb{N}$, let $v^{n,a}_{k,m}(t)$ be a minimum of the problem

$$\min\{\|\nabla v\|_p + \|v\|_q + m\|v - \varphi_k\|_1 \ : \ v \in V^n_a\},$$

where

$$V^n_a := \{v \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2),\ S^{g^{\delta_n}_n(t)}(v) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\}.$$

Since by (6.28) we have $\mathcal{H}^1(\Gamma^{\delta_n}_{\varepsilon_n,a}(t)) \le C'$, by Theorem 3.8.2 there exists a subsequence of $(\delta_n, \varepsilon_n)_{n\in\mathbb{N}}$ (which we denote with the same symbol) such that $v^{n,a}_{k,m}(t)$ weakly converges to some $v^a_{k,m}(t) \in GSBV^p_q(\Omega;\mathbb{R}^2)$ as $n \to +\infty$ for all $k, m \in \mathbb{N}$. We set

$$\Gamma_a(t) := \bigcup_{k,m} S^{g(t)}(v^a_{k,m}(t)). \tag{6.36}$$

Let us see that $\Gamma_a(t)$ satisfies all the properties of the lemma. Clearly $\Gamma_a(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$ and point (c) is a consequence of Theorem 3.8.2. In particular by (6.28) we have that

$$\mathcal{H}^1(\Gamma_a(t)) \le C'. \tag{6.37}$$

Let us come to point (a). Let $w_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ be such that $S^{g^{\delta_n}_n(t)}(w_n) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a}(t)$ and

$$w_n \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

We claim that there exists $k_m \to +\infty$ such that

$$v^a_{k_m,m}(t) \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2). \tag{6.38}$$

Then since $S^{g(t)}(v^a_{k_m,m}(t)) \subseteq \Gamma_a(t)$ for all m and in view of (6.37), we deduce that $S^{g(t)}(w) \tilde{\subseteq} \Gamma_a(t)$. Let us prove (6.38). Fixed $m \in \mathbb{N}$, let us choose k_m in such a way that

$$m\|w - \varphi_{k_m}\|_1 \to 0.$$

By minimality of $v^{n,a}_{k_m,m}(t)$ we have for all n

$$\|\nabla v^{n,a}_{k_m,m}(t)\|_p + \|v^{n,a}_{k_m,m}(t)\|_q + m\|v^{n,a}_{k_m,m}(t) - \varphi_{k_m}\|_1 \le \|\nabla w_n\|_p + \|w_n\|_q + m\|w_n - \varphi_{k_m}\|_1.$$

Passing to the limit in n, by lower semicontinuity we get for some $C \ge 0$

$$\|\nabla v^a_{k_m,m}(t)\|_p + \|v^a_{k_m,m}(t)\|_q + m\|v^a_{k_m,m}(t) - \varphi_{k_m}\|_1 \le C + m\|w - \varphi_{k_m}\|_1.$$

We deduce for $m \to +\infty$

$$\|v^a_{k_m,m}(t) - \varphi_{k_m}\|_1 \to 0,$$

which together with $\|\varphi_{k_m} - w\|_1 \to 0$ implies that

$$v^a_{k_m,m}(t) \to w \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^2).$$

Since

$$\|\nabla v^a_{k_m,m}(t)\|_p + \|v^a_{k_m,m}(t)\|_q \le C + m\|w - \varphi_{k_m}\|_1 \le C + 1$$

for m large, we have that $v^a_{k_m,m}(t) \rightharpoonup w$ weakly in $GSBV^p_q(\Omega;\mathbb{R}^2)$, and this proves (6.38).

Finally, let us come to point (b). Let $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \subseteq \overline{\Omega}_B$, and let us fix $k_1, \dots, k_s$ and $m_1, \dots, m_r$ in $\mathbb{N}$. By Proposition 6.3.2, there exists $v_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ such that

$$\lim_n \mathcal{E}^b(t)(v_n) = \mathcal{E}^b(t)(v)$$

and

$$\begin{aligned}\limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_n(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) &\le \limsup_n \mathcal{E}^s(S^{g^{\delta_n}_n(t)}(v_n) \setminus \bigcup_{i\le s,\, j\le r} S(v^{n,a}_{k_i,m_j}))\\ &\le \mu(a)\mathcal{E}^s(S^{g(t)}(v) \setminus \bigcup_{i\le s,\, j\le r} S(v^a_{k_i,m_j})),\end{aligned}$$

where $\mu(a) \to 1$ as $a \to 0$. Since the k_i's and the m_j's are arbitrary, we obtain that

$$\limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_n(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) \le \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(t)\right). \tag{6.39}$$

Let us suppose that $u^{\delta_n}_{\varepsilon_n,a}(t) \rightharpoonup u_a(t)$ weakly in $GSBV^p_q(\Omega;\mathbb{R}^2)$ along a suitable subsequence which we indicate by the same symbol. By the minimality property (6.29), comparing $u^{\delta_n}_{\varepsilon_n,a}(t)$ with v_n we get

$$\mathcal{E}^b(t)(u^{\delta_n}_{\varepsilon_n,a}(t)) \le \mathcal{E}^b(t)(v_n) + \mathcal{E}^s\left(S^{g^{\delta_n}_n(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) + o_n, \tag{6.40}$$

with $o_n \to 0$ as $n \to +\infty$. Then we have

$$\begin{aligned}\mathcal{E}^b(t)(u_a(t)) &\le \liminf_n \mathcal{E}^b(t)(u^{\delta_n}_{\varepsilon_n,a}(t)) \le \limsup_n \left(\mathcal{E}^b(t)(v_n) + \mathcal{E}^s\left(S^{g^{\delta_n}_n(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right)\right)\\ &\le \mathcal{E}^b(t)(v) + \limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_n(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) \le \mathcal{E}^b(t)(v) + \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(t)\right),\end{aligned}$$

that is (6.34) holds. Choosing $v = u_a(t)$, passing to the limsup in (6.40), and taking into account (6.39) we obtain that

$$\limsup_n \mathcal{E}^b(t)(u^{\delta_n}_{\varepsilon_n,a}(t)) \leq \mathcal{E}^b(t)(u_a(t)).$$

Since by (6.34) $\mathcal{E}^b(t)(u_a(t))$ is independent of the accumulation point $u_a(t)$, we conclude that (6.35) holds. □

Remark 6.6.3. Using Lemma 6.6.2, it is possible to construct an increasing family $\{t \to \Gamma_a(t) : t \in [0,T]\}$ and a subsequence of $(\delta_n, \varepsilon_n)_{n\in\mathbb{N}}$ such that points (a), (b) and (c) of Lemma 6.6.2 hold for every $t \in [0,T]$. This evolution $\{t \to \Gamma_a(t) : t \in [0,T]\}$ can be considered as an approximate quasistatic evolution, in the sense that it satisfies irreversibility, but it satisfies static equilibrium and nondissipativity up to a small error due to the fact that a is kept fixed. The presence of $\mu(a)$ in the minimality property (6.34) takes into account the anisotropy in the approximation of the surface energy: in fact, since a is kept fixed, the adaptive edges of the triangulations $\mathcal{T}_{\varepsilon,a}(\Omega)$ cannot recover all the possible directions. Notice that $\mu(a) = 1 + Ca$, where C depends only on the coercivity constants of the surface energy and on the range of the angles $\vartheta_1 \leq \vartheta \leq \vartheta_2$ defining the regular triangolations $\mathcal{R}_\varepsilon(\Omega)$. Using the minimality property (6.34) and following [35, Lemma 7.1] (estimate from below of the total energy) we can obtain the nondissipativity condition up to a small error, that is

$$\left|\mathcal{E}(t)(u_a(t), \Gamma_a(t)) - \mathcal{E}(0)(u_a(0), \Gamma_a(0)) - \int_0^t \vartheta_a(s)\,ds\right| \leq \tilde{C} \sup_{t\in[0,T]} \|g(t)\|_{W^{1,p}(\Omega;\mathbb{R}^2)\cap L^q(\Omega;\mathbb{R}^2)} a,$$

where $\tilde{C}$ is an explicit constant depending on the coercivity constants of the bulk and surface energies and on the range of the angles of the regular triangolations $\mathcal{R}_\varepsilon(\Omega)$, and ϑ_a is defined as in (3.83).

Using the arguments of Lemma 6.6.4, it can be proved that $(u_a(t), \Gamma_a(t))$ approaches along a suitable $a_n \to 0$ a quasistatic crack growth $(u(t), \Gamma(t))$ providing in particular an approximation of the bulk and surface energies at any time, i.e. for all $t \in [0,T]$

$$\mathcal{E}^b(t)(u_{a_n}(t)) \to \mathcal{E}^b(t)(u(t)) \quad \text{and} \quad \mathcal{E}^s(\Gamma_{a_n}(t)) \to \mathcal{E}^s(\Gamma(t)).$$

However it seems difficult to obtain by this approach an explicit estimate for $|\mathcal{E}(t)(u_{a_n}(t), \Gamma_{a_n}(t)) - \mathcal{E}(t)(u(t), \Gamma(t))|$ in term of a_n.

The construction of $\{t \to \Gamma_a(t) : t \in [0,T]\}$ is the following. If $D \subseteq [0,T]$ is countable and dense, by Lemma 6.6.2 and using a diagonalization argument, we can find a subsequence of $(\delta_n, \varepsilon_n)_{n\in\mathbb{N}}$ and an increasing family $\Gamma_a(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$, $t \in D$, such that points (a), (b) and (c) hold for every $t \in D$. Let us set for every $t \in [0,T]$

$$\Gamma_a^+(t) := \bigcap_{s\geq t, s\in D} \Gamma_a(s).$$

Clearly $\{t \to \Gamma_a^+(t) : t \in [0,T]\}$ is increasing, in the sense that $\Gamma_a(s) \tilde{\subseteq} \Gamma_a(t)$ for all $s \leq t$. As a consequence, the set J of discontinuity points of $\mathcal{H}^1(\Gamma_a^+(t))$ is at most countable. We can extract a further subsequence of $(\delta_n, \varepsilon_n)_{n\in\mathbb{N}}$ such that $\Gamma_a(t)$ is determined also for all $t \in J$ (notice that $\Gamma_a(t) \tilde{\subseteq} \Gamma_a^+(t)$). For all $t \in [0,T] \setminus (D \cup J)$ we set $\Gamma_a(t) := \Gamma_a^+(t)$. We have that $\Gamma_a(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$ and $\{t \to \Gamma_a(t) : t \in [0,T]\}$ is increasing.

For $t \in D \cup J$, $\Gamma_a(t)$ satisfies by construction points (a), (b) and (c) of Lemma 6.6.2. Let us consider the case $t \in [0,T] \setminus (D \cup J)$.

Concerning point (a), we have that $S^{g(t)}(w) \tilde{\subseteq} \Gamma_a(s)$ for every $s \in D \cap [t,T]$, so that passing to the intersection we get $S^{g(t)}(u_a(t)) \tilde{\subseteq} \Gamma_a(t)$.

As for point (b), considering $s \in D \cap [0,t[$, for every $v \in GSBV_q^p(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$, we have that there exists $v_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ such that

$$\lim_n \mathcal{E}^b(t)(v_n) = \mathcal{E}^b(t)(v),$$

and

$$\limsup_n \mathcal{E}^s\left(S^{g_n^{\delta_n}(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) \le \limsup_n \mathcal{E}^s\left(S^{g_n^{\delta_n}(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(s)\right) \le \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(s)\right).$$

Then by minimality property (6.29) and passing to the limit in n we have

$$\mathcal{E}^b(t)(u(t)) \le \mathcal{E}^b(t)(v) + \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(s)\right).$$

Letting $s \to t$ we get that (6.34) holds. Reasoning as in Lemma 6.6.2, we get that also (6.35) holds.

Finally, coming to point (c), we have that for all $s \in D \cap [0,t[$

$$\liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a}(t)) \ge \liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a}(s)) \ge \mathcal{E}^s(\Gamma_a(s)),$$

so that letting $s \nearrow t$, and recalling that t is a continuity point for $\mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a}(\cdot))$, we obtain that the lower semicontinuity holds.

We can now let $a \to 0$.

Lemma 6.6.4. *There exist a map $\{t \to \Gamma(t) \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega), t \in [0,T]\}$ and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$ such that the following facts hold:*

(a) $\Gamma^0 \tilde{\subseteq} \Gamma(s) \tilde{\subseteq} \Gamma(t)$ for all $0 \le s \le t \le T$;

(b) for all $t \in [0,T]$, if $w_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ with $S^{g_n^{\delta_n}(t)}(w_n) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)$ is such that

$$w_n \rightharpoonup w \qquad \text{weakly in } GSBV_q^p(\Omega;\mathbb{R}^2),$$

then we have

$$S^{g(t)}(w) \tilde{\subseteq} \Gamma(t);$$

(c) for all $t \in [0,T]$ and for every accumulation point $u(t)$ of $(u^{\delta_n}_{\varepsilon_n,a_n}(t))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV_q^p(\Omega;\mathbb{R}^2)$ and for all $v \in GSBV_q^p(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$, we have

$$\mathcal{E}^b(t)(u(t)) \le \mathcal{E}^b(t)(v) + \mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma(t)\right), \tag{6.41}$$

and

$$\mathcal{E}^b(t)(u(t)) = \lim_n \mathcal{E}^b(t)(u^{\delta_n}_{\varepsilon_n,a_n}(t)); \tag{6.42}$$

(d) for all $t \in [0,T]$ *we have*

$$\mathcal{E}^s(\Gamma(t)) \le \liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)). \tag{6.43}$$

Proof. Let us consider $\delta_h \to 0$ and $\varepsilon_h \to 0$. Given $a \in]0, \frac{1}{2}[$ and $t \in [0,T]$, let $\Gamma_a(t)$ be the rectifiable set given by Lemma 6.6.2. Recall that by (6.36) we have

$$\Gamma_a(t) = \bigcup_{k,m} S^{g(t)}(v^a_{k,m}(t)),$$

where $v^a_{k,m}(t)$ is the weak limit in $GSBV^p_q(\Omega;\mathbb{R}^2)$ along a suitable subsequence depending on a of a minimum $v^{h,a}_{k,m}(t)$ of the problem

$$\min\{\|\nabla v\|_p + \|v\|_q + m\|v - \varphi_k\|_1 \,:\, v \in V^h_a(t)\}, \tag{6.44}$$

where $(\varphi_k)_{k\in\mathbb{N}} \subseteq L^1(\Omega;\mathbb{R}^2)$ is dense in $L^1(\Omega;\mathbb{R}^2)$ and

$$V^h_a(t) := \{v \in \mathcal{A}^B_{\varepsilon,a}(\Omega';\mathbb{R}^2), S^{g^{\delta_h}_h(t)}(v) \subseteq \Gamma^{\delta_h}_{\varepsilon_h,a}(t)\}.$$

Let $a_n \to 0$, and let $D := \{t_j \,:\, j \in \mathbb{N}\} \subseteq [0,T]$ be countable and dense with $0 \in D$. Using a diagonal argument, up to a subsequence of $(\delta_h, \varepsilon_h)_{h\in\mathbb{N}}$, we may suppose that for all $t \in D$ and for all n

$$v^{h,a_n}_{k,m}(t) \rightharpoonup v^{a_n}_{k,m}(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

Moreover, we may assume that for all $t \in D$ and for all n

$$u^{\delta_h}_{\varepsilon_h,a_n}(t) \rightharpoonup u_{a_n}(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2)$$

with

$$\mathcal{E}^b(t)(u^{\delta_h}_{\varepsilon_h,a_n}(t)) \to \mathcal{E}^b(t)(u_{a_n}(t)).$$

By Lemma 6.6.2, we have that $u_{a_n}(t)$ satisfies the minimality property (6.34).

Up to a subsequence of $(a_n)_{n\in\mathbb{N}}$, we may suppose that for all k, m and $t \in D$ we have

$$v^{a_n}_{k,m}(t) \rightharpoonup v_{k,m}(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2), \tag{6.45}$$

and

$$u_{a_n}(t) \rightharpoonup u(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2). \tag{6.46}$$

For all $t \in D$, let us set

$$\Gamma(t) := \bigcup_{k,m} S^{g(t)}(v_{k,m}(t)). \tag{6.47}$$

By Proposition 6.3.2, in view of the minimality property (6.34) and taking into account that $\mu(a_n) \to 1$, we have that for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$

$$\mathcal{E}^b(t)(u(t)) \le \mathcal{E}^b(t)(v) + \mathcal{E}^s(S^{g(t)}(v) \setminus \Gamma(t)), \tag{6.48}$$

and as a consequence, we obtain

$$\mathcal{E}^b(t)(u_{a_n}(t)) \to \mathcal{E}^b(t)(u(t)).$$

We now perform the following diagonal argument. Choose $\delta_{h_0}, \varepsilon_{h_0}$ in such a way that

$$\|v^{h_0,a_0}_{0,0}(t_0) - v^{a_0}_{0,0}(t_0)\|_1 + \|u^{\delta_{h_0}}_{\varepsilon_{h_0},a_0}(t_0) - u_{a_0}(t_0)\|_1 + |\mathcal{E}^b(t_0)(u^{\delta_{h_0}}_{\varepsilon_{h_0},a_0}(t_0)) - \mathcal{E}^b(t_0)(u_{a_0}(t_0))| \le 1.$$

Supposing to have constructed $\delta_{h_n}, \varepsilon_{h_n}$, we choose $\delta_{h_{n+1}}, \varepsilon_{h_{n+1}}$ in such a way that for all $k \leq n+1$, $m \leq n+1$ and for all t_i with $1 \leq i \leq n+1$ we have

$$\|v_{k,m}^{h_{n+1},a_{n+1}}(t_i) - v_{k,m}^{a_{n+1}}(t_i)\|_1 + \|u^{\delta_{h_{n+1}}}_{\varepsilon_{h_{n+1}},a_{n+1}}(t_i) - u_{a_{n+1}}(t_i)\|_1 + |\mathcal{E}^b(t_i)(u^{\delta_{h_{n+1}}}_{\varepsilon_{h_{n+1}},a_{n+1}}(t_i)) - \mathcal{E}^b(t_i)(u_{a_{n+1}}(t_i))| \leq \frac{1}{n+1}.$$

Let us set $\delta_n := \delta_{h_n}$ and $\varepsilon_n := \varepsilon_{h_n}$, and let us prove that $\Gamma(t)$ defined in (6.47) satisfies the properties of the Lemma. We have immediately that $\Gamma(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N \Omega)$.

Concerning point (d), notice that

$$\Gamma^{\delta_n}_{\varepsilon_n,a_n}(t) = \bigcup_{m,k} S^{g^{\delta_n}_n(t)}(v_{k,m}^{h_n,a_n}(t)), \qquad \Gamma(t) = \bigcup_{m,k} S^{g(t)}(v_{k,m}(t)),$$

and that for all k, m

$$v_{k,m}^{h_n,a_n}(t) \rightharpoonup v_{k,m}(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2);$$

then (6.43) is a consequence of Theorem 3.8.2. In particular, by (6.28), we get that

$$\mathcal{H}^1(\Gamma(t)) \leq C'. \tag{6.49}$$

Let us come to point (b). Let $w_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ with $S^{g^{\delta_n}_n(t)}(w_n) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)$ be such that

$$w_n \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

For every $m \in \mathbb{N}$, let us choose k_m in such a way that

$$m\|w - \varphi_{k_m}\|_1 \to 0.$$

By minimality of $v_{k_m,m}^{h_n,a_n}(t)$ we have for all n

$$\|\nabla v_{k_m,m}^{h_n,a_n}(t)\|_p + \|v_{k_m,m}^{h_n,a_n}(t)\|_q + m\|v_{k_m,m}^{h_n,a_n}(t) - \varphi_{k_m}\|_1 \leq \|\nabla w_n\|_p + \|w_n\|_q + m\|w_n - \varphi_{k_m}\|_1.$$

By construction of h_n, and in view of (6.45), we have

$$v_{k_m,m}^{h_n,a_n}(t) \rightharpoonup v_{k_m,m}(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

Then passing to the limit in n, by lower semicontinuity we get for some $C \geq 0$

$$\|\nabla v_{k_m,m}(t)\|_p + \|v_{k_m,m}(t)\|_q + m\|v_{k_m,m}(t) - \varphi_{k_m}\|_1 \leq C + m\|w - \varphi_{k_m}\|_1.$$

We deduce for $m \to +\infty$

$$\|v_{k_m,m}(t) - \varphi_{k_m}\|_1 \to 0,$$

which together with $\|\varphi_{k_m} - w\|_1 \to 0$ implies that

$$v_{k_m,m}(t) \to w \quad \text{strongly in } L^1(\Omega;\mathbb{R}^2).$$

Since

$$\|\nabla v_{k_m,m}(t)\|_p + \|v_{k_m,m}(t)\|_q \leq C + m\|w - \varphi_{k_m}\|_1 \leq C + 1$$

for m large, we have that

$$v_{k_m,m}(t) \rightharpoonup w \qquad \text{weakly in } GSBV_q^p(\Omega;\mathbb{R}^2).$$

Since $S^{g(t)}(v_{k_m,m}(t)) \subseteq \Gamma(t)$ for all m, and since $\mathcal{H}^1(\Gamma(t)) < C'$, we deduce that $S^{g(t)}(w) \tilde{\subseteq} \Gamma(t)$.

Coming to point (c), we have that (6.42) holds by construction. Moreover (6.41) holds in view of (6.48) and by the fact that $u^{\delta_n}_{\varepsilon_n,a_n}(t)$ weakly converges in $GSBV_q^p(\Omega;\mathbb{R}^2)$ to $u(t)$ defined in (6.46).

In order to prove point (a), notice that if $s \le t$ with $s,t \in D$, we have for all $k,m \in \mathbb{N}$ that

$$S^{g^{\delta_n}_n(t)}(v^{h_n,a_n}_{k,m}(s) + g^{\delta_n}_{\varepsilon_n}(t) - g^{\delta_n}_{\varepsilon_n}(s)) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a_n}(s) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t),$$

and

$$v^{h_n,a_n}_{k,m}(s) + g^{\delta_n}_{\varepsilon_n}(t) - g^{\delta_n}_{\varepsilon_n}(s) \rightharpoonup v_{k,m}(s) + g(t) - g(s) \qquad \text{weakly in } GSBV_q^p(\Omega;\mathbb{R}^2),$$

where $v^{h,a}_{k,m}(s)$ and $v_{k,m}(s)$ are defined in (6.44) and (6.45). By point (b) we deduce that

$$S^{g(t)}(v_{k,m}(s) + g(t) - g(s)) = S^{g(s)}(v_{k,m}(s)) \tilde{\subseteq} \Gamma(t).$$

Then by the definition of $\Gamma(s)$ we get $\Gamma(s) \tilde{\subseteq} \Gamma(t)$. Finally, by the same argument, we deduce $\Gamma^0 \tilde{\subseteq} \Gamma(s)$.

In order to deal with all $t \in [0,T]$, we proceed as in Remark 6.6.3. For all $t \in [0,T] \setminus D$ let us set

$$\Gamma^+(t) := \bigcap_{s \ge t, s \in D} \Gamma(s).$$

Clearly $\Gamma^+(t) \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ and satisfies point (a), so that the set J of discontinuity points of $\mathcal{H}^1(\Gamma^+(\cdot))$ is at most countable. We can then extract a further subsequence of $(\delta_n,\varepsilon_n,a_n)_{n\in\mathbb{N}}$ such that $\Gamma(t)$ is determined also for all $t \in J \setminus D$ (notice that $\Gamma(t) \tilde{\subseteq} \Gamma^+(t)$). For all $t \in [0,T] \setminus (D \cup J)$ we set $\Gamma(t) := \Gamma^+(t)$. We have that $\Gamma(t) \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ and that $\Gamma(t)$ satisfies point (a). Let us see that $\Gamma(t)$ satisfies also points (b), (c) and (d) also for $t \in [0,T] \setminus (D \cup J)$.

Concerning point (b), for every accumulation point $u(t)$ of $(u^{\delta_n}_{\varepsilon_n,a_n}(t))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV_q^p(\Omega;\mathbb{R}^2)$, by the first part of the proof, we have that $S^{g(t)}(u(t)) \tilde{\subseteq} \Gamma(s)$ for all $s \in D$ with $s \ge t$, so that passing to the intersection, we get that $S^{g(t)}(u(t)) \tilde{\subseteq} \Gamma(t)$.

Let us come to point (c). Let

$$u_j(t) := u^{\delta_{n_j}}_{\varepsilon_{n_j},a_{n_j}}(t) \rightharpoonup u(t) \qquad \text{weakly in } GSBV_q^p(\Omega;\mathbb{R}^2)$$

along a subsequence $n_j \nearrow +\infty$. Let us set $\Gamma_j := \Gamma^{\delta_{n_j}}_{\varepsilon_{n_j},a_{n_j}}$ and $g_j := g^{\delta_{n_j}}_{\varepsilon_{n_j}}$. Up to a further subsequence there exists $s_j \in D$ with $s_j \nearrow t$, and such that setting $u_j(s_j) := u^{\delta_{n_j}}_{\varepsilon_{n_j},a_{n_j}}(s_j)$, we have

$$\|u_j(s_j) - u(s_j)\|_1 + |\mathcal{E}^b(s_j)(u_j(s_j)) - \mathcal{E}^b(s_j)(u(s_j))| \to 0. \tag{6.50}$$

We have that there exists $u^*(t) \in GSBV_q^p(\Omega;\mathbb{R}^2)$ such that up to a subsequence

$$u(s_j) \rightharpoonup u^*(t) \qquad \text{weakly in } GSBV_q^p(\Omega;\mathbb{R}^2).$$

By the minimality property (6.41) of $u(s_j)$, for all $v \in GSBV_q^p(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$, we have that

$$\mathcal{E}^b(s_j)(u(s_j)) \le \mathcal{E}^b(s_j)(v - g(t) + g(s_j)) + \mathcal{E}^s(S^{g(t)}(v) \setminus \Gamma(s_j)).$$

Passing to the limit in j we have that for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$

$$\mathcal{E}^b(t)(u^*(t)) \le \mathcal{E}^b(t)(v) + \mathcal{E}^s(S^{g(t)}(v) \setminus \Gamma(t)). \tag{6.51}$$

As a consequence of the stability of this unilateral minimality property, it follows that

$$\mathcal{E}^b(s_j)(u(s_j)) \to \mathcal{E}^b(t)(u^*(t)).$$

By (6.50) we get

$$u_j(s_j) \rightharpoonup u^*(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2),$$

and

$$\mathcal{E}^b(s_j)(u_j(s_j)) \to \mathcal{E}^b(t)(u^*(t)). \tag{6.52}$$

By (6.29), comparing $u_j(t)$ with $u_j(s_j) - g_j(s_j) + g_j(t)$, taking into account that

$$S^{g_j(s_j)}(u_j(s_j)) \subseteq \Gamma_j(s_j) \subseteq \Gamma_j(t),$$

we obtain

$$\mathcal{E}^b(t)(u_j(t)) \le \mathcal{E}^b(s_j)(u_j(s_j)) + o_j$$

where $o_j \to 0$ as $j \to +\infty$. Passing to the limit in j we have by (6.52)

$$\mathcal{E}^b(t)(u(t)) \le \liminf_j \mathcal{E}^b(t)(u_j(t)) \le \limsup_j \mathcal{E}^b(t)(u_j(t)) \le \mathcal{E}^b(t)(u^*(t)).$$

By (6.51) we deduce that (6.41) holds. Moreover we have that $\mathcal{E}^b(t)(u(t)) = \mathcal{E}^b(t)(u^*(t))$ and that $\mathcal{E}^b(t)(u(t))$ is independent of the accumulation point $u(t)$. Then we deduce that (6.42) holds.

Finally, concerning point (d), we have that for all $s \in D \cap [0,t[$

$$\liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)) \ge \liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a_n}(s)) \ge \mathcal{E}^s(\Gamma(s)),$$

so that letting $s \nearrow t$ we obtain (6.43). The proof is now complete. $\square$

We can now prove Theorem 6.6.1.

PROOF OF THEOREM 6.6.1. Let $(\delta_n, \varepsilon_n, a_n)_{n\in\mathbb{N}}$ and $\{t \to \Gamma(t) \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega), t \in [0,T]\}$ be given by Lemma 6.6.4. For all $t \in [0,T]$, let us set

$$u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t), \qquad \Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t).$$

Let us see that it is possible to choose an accumulation point $u(t) \in GSBV^p_q(\Omega;\mathbb{R}^2)$ of $(u_n(t))_{n\in\mathbb{N}}$ such that $\{t \to (u(t),\Gamma(t)) : t \in [0,T]\}$ is a quasistatic growth of brittle fractures in the sense of Dal Maso-Francfort-Toader. Let us set

$$\begin{aligned}\vartheta_n(s) := \langle \partial\mathcal{W}(\nabla u_n(s)), \nabla \dot g_{\varepsilon_n}(s)\rangle - \dot{\mathcal{F}}(s)(u_n(s)) - \langle \partial\mathcal{F}(s)(u_n(s)), \dot g_{\varepsilon_n}(s)\rangle \\ - \dot{\mathcal{G}}(s)(u_n(s)) - \langle \partial\mathcal{G}(s)(u_n(s)), \dot g_{\varepsilon_n}(s)\rangle.\end{aligned}$$

By growth conditions of $\mathcal{W}, \mathcal{F}, \mathcal{G}$ and by (6.28) we have that there exists $\psi \in L^1(0,T)$ such that $\vartheta_n(s) \le \psi(s)$ for all n. Let us consider

$$\vartheta(s) := \limsup_n \vartheta_n(s).$$

By [35, Theorem 5.5 and Lemma 4.11], for every $s \in [0,T]$ there exists $u(s)$ accumulation point of $(u_n(s))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV^p_q(\Omega;\mathbb{R}^2)$ such that

$$\vartheta(s) := \langle \partial\mathcal{W}(\nabla u(s)), \nabla\dot{g}(s)\rangle - \dot{\mathcal{F}}(s)(u(s)) - \langle\partial\mathcal{F}(s)(u(s)), \dot{g}(s)\rangle - \dot{\mathcal{G}}(s)(u(s)) - \langle\partial\mathcal{G}(s)(u(s)), \dot{g}(s)\rangle.$$

Applying Fatou's Lemma (in the limsup version) to (6.30) with $s = 0$, we have that

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \limsup_n \mathcal{E}(0)(u_n(0),\Gamma_n(0)) + \int_0^t \vartheta(s)\,ds.$$

By Proposition 6.4.1, we have that $\limsup_n \mathcal{E}(0)(u_n(0),\Gamma_n(0)) = \mathcal{E}(0)(u(0),\Gamma(0))$, so that we get

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \mathcal{E}(0)(u(0),\Gamma(0)) + \int_0^t \vartheta(s)\,ds.$$

Moreover, again by [35, Theorem 3.13],

$$\mathcal{E}(t)(u(t),\Gamma(t)) \ge \mathcal{E}(0)(u(0),\Gamma(0)) + \int_0^t \vartheta(s)\,ds,$$

so that

$$\mathcal{E}(t)(u(t),\Gamma(t)) = \mathcal{E}(0)(u(0),\Gamma(0)) + \int_0^t \vartheta(s)\,ds. \tag{6.53}$$

We deduce that $\{t \to (u(t),\Gamma(t)) \,:\, t \in [0,T]\}$ is a quasistatic growth of brittle fractures: in fact by Lemma 6.6.4 we get that $\Gamma(\cdot)$ is increasing, and for $t \in [0,T]$ $(u(t),\Gamma(t)) \in AD(g(t))$ and the static equilibrium holds; moreover the nondissipativity condition is given by (6.53).

Let us see that points (a) and (b) of Theorem 6.6.1 holds. By (6.28), $(u_n(t))_{n\in\mathbb{N}}$ is weakly precompact in $GSBV^p_q(\Omega;\mathbb{R}^2)$ for all $t\in[0,T]$. Moreover by Lemma 6.6.4 every accumulation point $\tilde{u}(t)$ of $(u_n(t))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV^p_q(\Omega;\mathbb{R}^2)$ is such that $S^{g(t)}(\tilde{u}(t)) \subseteq \Gamma(t)$ and the minimality property (6.31) holds. Moreover we have

$$\mathcal{E}^b(t)(\tilde{u}(t)) = \lim_n \mathcal{E}^b(t)(u_n(t)).$$

Since $\mathcal{E}^b(t)(\tilde{u}(t))$ is independent of the particular accumulation point $\tilde{u}(t)$, we have that point (a) is proved.

Let us come to point (b). Taking into account (6.42) and (6.43), for all $t \in [0,T]$ we have

$$E(t) \le \liminf_n E_n(t) \le \limsup_n E_n(t) \le E(0) + \int_0^t \vartheta(s)\,ds = E(t),$$

so that (6.32) holds. Moreover we deduce that separate convergence of elastic and surface energies holds at any time, so that (6.33) is proved. The proof is now concluded. □

6.7 The strictly convex case

In this section we assume that the function $W(x,\xi)$ is strictly convex in ξ for a.e. $x \in \Omega$ and that the function $F(t,x,z)$ is strictly convex in z for all $t\in[0,T]$ and for a.e. $x\in\Omega$: as a consequence, the elastic energy $\mathcal{E}^b(t,v)$ is strictly convex in v for all $t \in [0,T]$, and a stronger approximation result is available.

Theorem 6.7.1. *Let $g \in W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2))$ and let*

$$g_\varepsilon \in W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)), \qquad g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega) \quad \text{for all } t \in [0,T]$$

be such that for $\varepsilon \to 0$

$$g_\varepsilon \to g \qquad \text{strongly in } W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)).$$

Let $\Gamma^0 \in \mathbf{\Gamma}(\Omega)$ be an initial crack and let $\Gamma^0_{\varepsilon,a}$ be its approximation in the sense of Proposition 6.4.1. Let us suppose that

$$\begin{aligned} &W(x,\cdot) \text{ is strictly convex for a.e. } x \in \Omega, \\ &F(t,x,\cdot) \text{ is strictly convex for a.e. } (t,x) \in [0,T] \times \Omega. \end{aligned} \tag{6.54}$$

Given $\delta > 0$, $\varepsilon > 0$, $a \in]0, \frac{1}{2}[$, let $\{t \to (u^\delta_{\varepsilon,a}(t), \Gamma^\delta_{\varepsilon,a}(t)) : t \in [0,T]\}$ be the piecewise constant interpolation of the discrete evolution given by Proposition 6.5.1 relative to the initial crack $\Gamma^0_{\varepsilon,a}$ and the boundary data g_ε. Then there exists a quasistatic evolution $\{t \to (u(t), \Gamma(t)) : t \in [0,T]\}$ relative to the initial crack Γ^0 and the boundary data g in the sense of Theorem 3.8.8, and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, such that setting

$$u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t), \qquad \Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t),$$

for all $t \in [0,T]$ the following facts hold:

(a) $\nabla u_n(t) \to \nabla u(t)$ strongly in $L^p(\Omega;\mathcal{M}^{2\times 2})$ and $u_n(t) \to u(t)$ strongly in $L^q(\Omega;\mathbb{R}^2)$;

(b) $\mathcal{E}(t)(u_n(t), \Gamma_n(t)) \to \mathcal{E}(t)(u(t), \Gamma(t))$, and in particular elastic and surface energies converge separately, that is

$$\mathcal{E}^b(t)(u_n(t)) \to \mathcal{E}^b(t)(u(t)), \qquad \mathcal{E}^s(\Gamma_n(t)) \to \mathcal{E}^s(\Gamma(t)).$$

Proof. Let us consider the sequence $(\delta_n, \varepsilon_n, a_n)_{n\in\mathbb{N}}$ and the quasistatic growth of brittle fractures $\{t \to (u(t), \Gamma(t)) : t \in [0,T]\}$ given in Theorem 6.6.1. Under assumptions (6.54), we have that $u(t)$ is uniquely determined, because by (6.31) $u(t)$ minimizes

$$\min\{\mathcal{E}^b(t)(v) : v \in GSBV^p_q(\Omega;\mathbb{R}^2),\ S^{g(t)}(v) \subseteq \Gamma(t)\},$$

and $\mathcal{E}^b(t)(\cdot)$ is strictly convex. We conclude by point (a) of Theorem 6.6.1 that $u_n(t) \rightharpoonup u(t)$ weakly in $GSBV^p_q(\Omega;\mathbb{R}^2)$. Point (b) is a direct consequence of Theorem 6.6.1. By the convergence of the elastic energy, we deduce that

$$\begin{aligned} \lim_n \int_\Omega W(x, \nabla u_n(t))\,dx &= \int_\Omega W(x, \nabla u(t))\,dx, \\ \lim_n \int_\Omega F(t,x,u_n(t))\,dx &= \int_\Omega F(t,x,u(t))\,dx. \end{aligned}$$

By the assumption on the strict convexity of W and F we deduce by [15]

$$\nabla u_n(t) \to \nabla u(t) \qquad \text{strongly in } L^p(\Omega;\mathcal{M}^{2\times 2}),$$

and

$$u_n(t) \to u(t) \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2).$$

Point (a) is now proved, and the proof is concluded. $\square$

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