Symmetries of noncommutative spaces and equivariant cohomology

Lucio Simone Cirio

Mathematical Physics Sector
International School for Advanced Studies
SISSA

Advisors: Prof. Ugo Bruzzo, Prof. G. Landi

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Oh, Dostoevskij a me ha dato più cose
di quante me ne abbia date Gauss.

(A. Einstein)

Contents

Introduction

As the title suggests, the main subject of this thesis is the study of symmetries of noncommutative spaces and related equivariant cohomologies. We focus on deformations of classical geometries coming from the action of some symmetry. A close relation between the deformation of the symmetry and the deformation of the space on which it acts is at the heart of our approach; we will use this idea to generate noncommutative geometries, and to define algebraic models for the equivariant cohomology of such actions.

Broadly speaking, action of symmetries on spaces have always played a central role both in mathematics and in physics. They often are the visible part of a more hidden and fundamental structure which describes and governs the system studied, and they frequently provide a beautiful bridge between physical phenomena and their mathematical formulations.

Historically they have been used as a guiding principle to formulate theories; in some sense, symmetries can be used to test the internal coherence of a model, the same way as experiments can be used to test its agreement with reality. From Maxwell's equations to special and general relativity, quantum mechanics and quantum field theory, arriving to standard model and string theory just to mention some of the most popular branches of theoretical physics, symmetries have been taken as the starting point of the comprehension and mathematical description of the physical world. At the same time an increasing number of mathematical tools useful to describe systems acted by some group of transformations was developed by mathematicians, starting from the the concept of covariance and equivariance, representation theory, conserved currents and conservation laws, leading to a more systematic and general approach to the study of the subject.

The other main topic of the thesis is nc geometry. Being a less known argument, we feel it is worth spending some word to explain in a very informal way what are nc spaces and where they come out.

The physical interest for noncommuting quantities, such as algebras of noncommuting operators, goes back to the early days of quantum mechanics. The replacement of the algebra of function on the phase space of a physical system by algebra of matrices was the first and crucial step toward the idea that physical observables in the microscopic world are described by nc algebras. Then some years later mathematicians rather remarkably realized that it is possible to describe a space X also by its commutative algebra of functions $C(X)$ and further algebraic structures. This new and prolific point of view may be summarized paraphrasing Manin's quote of Grothendieck: to do geometry we really do not need a space, all we need is a category of sheaves on this would-be space. Once we have expressed the geometry of the space X by algebraic data, we can relax all the assumptions on the commutativity and pretend these in general noncommutative algebraic structures are associated to some 'nc space'; this is for example the idea of Connes' spectral triples [Con94]. We then describe the geometry of nc spaces by studying the algebraic description of this would-be space. In this sense, as it is often said, the name 'geometry of nc spaces' is a bit misleading, since there are no spaces at all and what we do is algebra, not geometry.

In the thesis we focus our attention on symmetries of noncommutative spaces. The role played by symmetries is actually twofold, and it is reflected in the two main lines of research of the present work.

The first one is another example of how symmetries can be used as a source for new ideas and constructions. We start by considering a symmetry expressed by the action of some compact lie group G on a space X; we pass to an algebraic setting by looking at the associated action of the Lie algebra $\mathfrak{g}$, and more generally of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$, on the algebra of differential forms $\mathcal{A}(X)$ on X.

The action of Lie derivative, interior derivative and de Rham differential on $\mathcal{A}(X)$ is encoded in a representation of $\mathfrak{U}(\tilde{\mathfrak{g}})$, the enveloping algebra of a super or $\mathbb{Z}_2$ (to take into account the different grading of the three operators) Lie algebra $\tilde{\mathfrak{g}}$ naturally constructed from $\mathfrak{g}$. This leads to the notion of $\tilde{\mathfrak{g}}$-differential algebra ($\tilde{\mathfrak{g}}$-da for short, Def(1.1.2)), introduced (with a different terminology) long time ago by Cartan [Car50]; for a modern treatment of this approach a beautiful reference is [GS99].

A basic fact is that on $\tilde{\mathfrak{g}}$-da's the action of $\mathfrak{U}(\tilde{\mathfrak{g}})$ is by (graded) derivations. It is possible to generalize this property to a generic Hopf algebra $\mathcal{H}$ acting on an algebra $\mathcal{A}$, by requiring the action to be 'compatible' with the algebra structure of $\mathcal{A}$; this is the idea of covariant actions (Def(1.2.9)).

This concept provides the link between symmetries and spaces we mentioned at the beginning. Now we deform the symmetry, i.e. the Hopf algebra $\mathfrak{U}(\tilde{\mathfrak{g}})$, and we ask the covariance to be preserved: this forces the multiplicative structure of every $\tilde{\mathfrak{g}}$-da $\mathcal{A}$ to be deformed as well (Thm(1.2.17)), and we finally interpret the deformed algebra as differential forms on nc spaces. In this sense we can construct nc geometries by deforming classical symmetries.

We consider a particular class of Hopf algebra deformations, those coming from Drinfeld twists [Dri90a][Dri90b]; this choice, together with some assumption on the Lie algebra $\mathfrak{g}$, allows for quite explicit and manageable expressions for the deformations, and at the same time it is general enough to describe an interesting collection of nc spaces. Using the language of Drinfeld twists we can review isospectral deformations [CL01] (see also [CDV02]), as well as define noncommutative deformations of toric varieties [CLS] in the line of [Ing].

The second line of research is aimed to define and study algebraic models for equivariant cohomology of deformed symmetries acting on nc spaces. When a space X is acted on by some group G, besides ordinary (singular, cellular, de Rham) cohomology $H(X)$ a new tool which takes into account the action of the group is available as well; it is the equivariant cohomology ring $H_G(X)$. The exact definition depends on the class of spaces X belongs to; for our purposes we are interested in smooth differentiable manifolds acted by compact Lie groups.

In this setting equivariant cohomology may be thought as the de Rham cohomology of the orbit space X/G, but this picture works only for proper and free actions (otherwise the quotient presents singularities or fails to be Hausdorff). A way to introduce in full generality an equivariant cohomology which gives back $H(X/G)$ when the latter make sense is via the topological Borel model [Bor60]: $H_G(X)$ is defined to be the ordinary cohomology of $EG \times_G X$, where EG is the total space of the universal G-bundle (Def(2.1.1)).

It is often convenient to switch to an algebraic description of the Borel model, replacing the infinite dimensional space EG by a finitely generated algebra representing its differential forms: the Weil algebra $W_{\mathfrak{g}} = Sym(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*)$. It is possible to define a $\tilde{\mathfrak{g}}$-da structure on $W_{\mathfrak{g}}$, and in this way we arrive at the Weil model for equivariant cohomology, defined as the cohomology of the basic subcomplex of $W_{\mathfrak{g}} \otimes \mathcal{A}(X)$ (Def(2.1.8)). Another equivalent algebraic definition of $H_G(X)$, closer to the definition of de Rham cohomology of X, is formulated by introducing equivariant differential forms and then taking cohomology with respect to an equivariant differential operator d_G; this is known as the Cartan model (Def(2.1.11)).

These algebraic models are our starting point for the study of nc equivariant cohomology. Since the Weil algebra $W_{\mathfrak{g}}$ is the universal locally free (graded) commutative $\tilde{\mathfrak{g}}$-da, it seems natural to look for analogous objects in the categories of nc $\tilde{\mathfrak{g}}$-da's and deformed $\tilde{\mathfrak{g}}$-da's. To the first case belongs the nc Weil algebra $\mathcal{W}_{\mathfrak{g}}$ of Alekseev and Meinrenken [AM00], which they use to define a nc equivariant cohomology.

By realizing $\mathcal{W}_{\mathfrak{g}}$ as the super enveloping algebra $\mathfrak{U}(\tilde{\mathfrak{g}})$, we can adapt the construction of [AM00] to the specific class of deformations of the symmetry we are interested in. We study in particular the case of Drinfleld twists, so we define a twisted nc Weil algebra $\mathcal{W}_{\mathfrak{g}}^{(\chi)}$ (Def(2.3.1)) and twisted nc equivariant cohomology by both deformed Weil (Def(2.3.6)) as well as deformed Cartan (Def(2.3.15)) models. This construction of equivariant cohomology applies to the nc spaces obtained by Drinfeld twists, so in particular toric isospectral deformations and nc toric varieties.

Moreover we emphasize that one can use the same strategy to define nc equivariant cohomology of all nc spaces obtained by covariance from some deformed symmetry, just by deforming the nc Weil algebra $\mathcal{W}_{\mathfrak{g}}$ in a compatible manner we deformed the symmetry.

The thesis is structured in two chapters, which are devoted to describe the two lines of research mentioned above.

The first chapter contains four sections. In the first one after a short introduction to the notions of spectral triple and deformed spaces, we review how it is possible to study the action of Lie groups G on smooth manifolds in an algebraic language, formalizing the definition of $\tilde{\mathfrak{g}}$-differential algebras ($\tilde{\mathfrak{g}}$-da).

In the second section we introduce Hopf algebras and their actions as the algebraic generalization of symmetries. We define covariant actions and then we consider Hopf algebra deformations obtained using Drinfeld twists (Def(1.2.13) and Thm(1.2.14)); at the end of the section we make some remark on the generality of this kind of deformations.

In the third section we present an example of how it is possible to obtain nc spaces using Drinfel twists on Hopf algebras and their covariant actions. We review toric isospectral deformations as induced by a Drinfeld twist of the enveloping algebra of the torus, discussing also the different meaning of deformations of an algebra when it 'represents' a space and/or a symmetry. We finally consider a notion of deformed commutativity for nc algebras, relating this property to the quasitriangular structure of the twisted Hopf algebra which describes the symmetry responsible for the deformation (Def(1.3.6)), and prove that toric isospectral deformations satisfies this generalized definition of commutativity (Prop(1.3.8)).

In the fourth and last section we provide a second example of nc spaces obtained by Drinfeld twists; this time we consider toric varieties, so we first deform the algebraic torus (Def(1.4.1)) and then as usual we use its action to 'spread' the deformation to the whole space. We also propose a definition of nc toric varieties by a general construction which deforms the fan description, providing a local description and gluing morphisms in the nc setting. We then present examples and outline a homogeneous coordinate ring construction for the nc projective spaces (Thm(1.4.8), in analogy with [Cox95]), and a sheaf theory. The content of this section is work done in collaboration with G. Landi and R. Szabo [CLS].

The second chapter is organized in four sections. In the first one we review the classic definition of equivariant cohomology for smooth manifolds acted on by compact Lie groups. We present the topological Borel model, we introduce the Weil algebra $W_{\mathfrak{g}}$ and its $\tilde{\mathfrak{g}}$-da structure, and define the Weil model. We then construct a Kalkman map which links the Weil model to the equivalent Cartan model.

In the second section we recall a more abstract interpretation of the Weil algebra as the universal locally free $\tilde{\mathfrak{g}}$-da (Thm(2.19)); this result makes possible to define 'deformed' or 'generalized' Weil algebras when the category of $\tilde{\mathfrak{g}}$-da is enlarged. We review the construction of the nc Weil algebra of [AM00], relevant for the category of nc $\tilde{\mathfrak{g}}$-da, and show how this leads to a definition of nc equivariant cohomology (Def(2.2.8) and Def(2.2.12)). We end the section by discussing how to adapt the previous construction to deformed $\tilde{\mathfrak{g}}$-da, in order to define equivariant cohomology for deformed symmetries acting on nc spaces.

In the third section we apply the previous ideas to deformations coming from Drinfeld twists. We define a twisted nc Weil algebra $\mathcal{W}_{\mathfrak{g}}^{(\chi)}$ (Def(2.3.1)), realize its twisted $\tilde{\mathfrak{g}}$-da structure and shows it is possible to rephrase the Weil and Cartan models construction in this deformed setting (Def(2.3.1) and Def(2.3.15)). Moreover we explain how the Drinfeld twist affects the ring structure of the complexes defining the cohomology (Prop(2.3.7) and Prop(2.3.14)) and the relation of our construction with the nc cohomology of [AM00] (Prop(2.3.9)). The content of this section is original work appeared in [Cir].

In the last section we derive further results on twisted nc equivariant cohomology. We compute some examples, in particular nc spheres and homogeneous spaces, and prove that our cohomology satisfies a reduction property to the maximal torus (Thm(2.4.5)) similarly to classical equivariant cohomology; we end by discussing the consequences of this result. The contents of this section will appear in a revised and enlarged version of [Cir].

We present in the Conclusions a summary of the results achieved in this thesis and outline some interesting and open directions for future work on these topics. In particular we sketch applications for the construction of nc toric varieties and models for nc equivariant cohomology, and discuss a possible alternative definition of equivariant cohomology for algebras $\mathcal{A}_\chi$ deformed by the action of Drinfeld twisted Hopf algebras $\mathcal{H}^\chi$ by using cyclic homology of the crossed product algebra $\mathcal{A}_\chi \rtimes \mathcal{H}^\chi$.

Chapter 1

Deformations of symmetries and noncommutative spaces

In this chapter we will study symmetries of noncommutative spaces. First of all, we will begin with a brief discussion of what one means by a noncommutative space. This will be done without entering too deeply into the formalism, but at least motivating the general strategy behind the idea to express all the notions concerning spaces, group actions and symmetries with their algebraic counterparts.

Broadly speaking we will see that the right setting to describe the structure of a space acted on by some symmetry is the category of Hopf module algebras[1]. Once we have in this category a formulation of all the classical constructions we are interested in, we can start considering deformations of Hopf algebras and the effects of such deformations in the category of module algebras.

We remark that the role of deformed Hopf algebras, i.e. symmetries in the algebraic setting, is twofold. If we are given a noncommutative space, we must deform the symmetry in a 'compatible' way in order to have a well defined action; on the other hand, if we start by deforming a symmetry we can use it to 'generate' deformations of all the spaces where the symmetry acts. In a more sophisticated language we are saying that, in the category of Hopf module algebras, Hopf algebra deformations are related to module algebra deformations and vice versa.

The class of deformations we will mainly focus on come from Drinfeld twists; this kind of deformations are quite general (see the discussion about rigidity theorems at the end of the chapter), they allow for quite explicit descriptions and computations, and they are behind a large class of interesting noncommutative spaces. In the third and fourth section we will consider two classes of them, namely toric isospectral deformations and, using the same ideas with algebraic tori instead of compact ones, noncommutative toric varieties.

It worths saying however that the philosophy of what we are going to present

[1] A Hopf module algebra is an algebra carrying a Hopf algebra action compatible with the multiplicative structure, see Def(1.2.9).

here applies as well to other class of deformations, for examples Drinfeld-Jimbo or q-deformations. We will come back on this point in the second chapter, when we will describe models for equivariant cohomology of such deformed symmetries.

For simplicity we will use throughout the chapter the short notation 'nc' for noncommutative.

1.1 Symmetries in the algebraic setting

The interest in generalizing concepts of ordinary geometry to systems of non-commuting operators or nc algebras has both physical and mathematical origins.

Since the first years of quantum mechanics the idea of a nc phase space where the coordinates x and p no longer commute lead to the formulation of Moyal-type spaces; functions on such spaces are promoted to operators, and the coordinates x^μ (thought now as operators) satisfy commutation relations $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, with θ usually a real skewsymmetric constant matrix.

From the mathematical point of view, the celebrated theorems of Gelfand-Naimark and Serre-Swan showed that it is possible to describe locally compact Hausdorff spaces with commutative C^*-algebras, and complex vector bundles over compact Hausdorff spaces $\mathcal{M}$ with finitely generated projective modules over $C(\mathcal{M})$.

Following these equivalence of categories, it seems natural to look at nc C^*-algebras and finitely generated projective modules over nc algebras as the correct generalization for the idea of spaces and (sections of) vector bundles. Much more can be done if one consider extra structures, and an entire dictionary between topological and geometrical properties expressed in algebraic terms can be formulated. A promising definition of a nc geometry is contained in the notion of spectral triple [Con94].

Definition 1.1.1 *An even, real spectral triple, or K-cycle, is the assignment of the data $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$, where:*

1. *$\mathcal{A}$ is a pre-C^*-algebra;*
2. *$\mathcal{H}$ is a Hilbert space carrying a faithful representation of $\mathcal{A}$ in bounded operators;*
3. *D is a selfadjoint operator on $\mathcal{H}$ with compact resolvent;*
4. *J is an antilinear isometry of $\mathcal{H}$;*
5. *Γ is a selfadjoint unitary operator on $\mathcal{H}$ such that $\Gamma^2 = 1$.*

This quite abstract definition may become more intelligible looking at the way a compact Riemannian spin manifold $\mathcal{M}$ can be described via a spectral triple. The role of $\mathcal{A}$ is to represent the algebra of (a suitable class of sufficiently regular) functions on $\mathcal{M}$; then as Hilbert space $\mathcal{H}$ one takes square-integrable spinors $L^2(\mathcal{M}, S)$ and the faithful representation of $\mathcal{A}$ comes from pointwise multiplication of function on spinors. The operator D in this example is the Dirac operator $\mathcal{D}$ of the metric; note that the condition to have compact resolvent, which implies the discreteness of the spectrum, is satisfied due to the compactness of $\mathcal{M}$. Finally, if $\mathcal{M}$ has even dimension $2n$, Γ is the grading (or chirality) operator γ_{2n+1} and J is the complex conjugation operator.

To say that the data in Def(1.1.1) describe a geometry, further conditions are imposed; these reproduce properties of the geometry of $\mathcal{M}$ we want to carry on to nc setting. In this way we can still have at our disposal a definition of a metric dimension, coming from the infinitesimal order of $|D|^{-1}$, or the fact that D is a first order operator in the sense that $[[D, a], Jb^* J^\dagger] = 0$, or the existence of a volume form now described as a Hochschild cycle in $Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^0)$.

However we will not enter into a detailed description of the various properties and conditions satisfied and imposed on spectral triples, since our aim is to focus on the study of symmetries of quite concrete classes of nc spaces. Thus instead of considering general abstract nc algebras, the strategy is to start with a (compact, Riemannian, spin, ...) manifold $\mathcal{M}$, consider its commutative algebra of (continuous, smooth, ...) functions $\mathcal{A} = C(\mathcal{M})$ and deform the product in $\mathcal{A}$. These deformations usually depend on some real parameters $\theta^{\mu\nu}$, reduce to the classical case for $\theta \to 0$ and come from some extra structure, as for Poisson manifolds in the framework of deformation quantization, or as in the case of toric actions in the class of isospectral deformations. We will provide in the next sections a general scheme to produce deformed nc spaces $\mathcal{M}_\theta$ using deformations of symetries on $\mathcal{M}$.

Before dealing with deformations, we describe the setting we want to deform. Since in our nc setting one deforms algebraic structures, first of all we want to translate the language of actions of a compact Lie groups G on smooth compact Hausdorff manifolds $\mathcal{M}$ into a purely algebraic formalism. This ideas were first introduced in a seminal work of H. Cartan [Car50], and belong by now to the classical background of differential geometry; for a modern detailed treatment a good reference is [GS99]. Let $\mathcal{A} = \Omega^\bullet(\mathcal{M})$ be the graded-commutative algebra of differential forms on $\mathcal{M}$, and $\mathfrak{g}$ the Lie algebra of G with generators $\{e_a\}$ satisfying $[e_a, e_b] = f_{ab}$. A smooth action of G on $\mathcal{M}$ is a smooth transformation $\Phi : G \times \mathcal{M} \to \mathcal{M}$ such that denoting

$$\Phi_g : \mathcal{M} \to \mathcal{M} \qquad g \in G$$

we have a composition rule compatible with the group structure

$$\Phi_g \circ \Phi_h = \Phi_{gh}$$

This induces a pull-back action ρ on the algebra of differential forms by

$$\rho_g(\omega) := (\Phi_g^{-1})^* \omega \qquad g \in G, \quad \omega \in \mathcal{A}$$

which we will denote for simplicity as $g \triangleright \omega$. For each $\zeta \in \mathfrak{g}$ we denote by $\zeta^* \in \mathcal{X}(\mathcal{M})$ the vector field defined in $p \in \mathcal{M}$ by

$$\zeta^*(p) = \frac{d}{dt}(\exp\{-t\zeta\})(p)_{|t=0}$$

and call ζ^* the vector field generating the infinitesimal action of G along ζ. The Lie derivative along the vector field ζ^* is an even (degree zero) derivation of $\mathcal{A}$, i.e. it satisfies a Leibniz rule

$$L_\zeta(\omega\mu) = (L_\zeta\omega)\mu + \omega(L_\zeta\mu) \qquad \omega,\, \mu \in \mathcal{A}$$

The Lie derivatives along generators of $\mathfrak{g}$ have commutation relations

$$[L_{e_a}, L_{e_b}] = f_{ab}{}^c L_{e_c}$$

so they define a representation of $\mathfrak{g}$ on $\mathcal{A}$. Hence the algebraic analogue of a G action on $\mathcal{M}$ is a representation of $\mathfrak{g}$ on $\mathcal{A}$ by even derivations; note that by universality of the enveloping algebra this representation lifts to $\mathfrak{U}(\mathfrak{g})$ and the Leibniz rule of L_{e_a} is equivalent to the fact that e_a has primitive coproduct $\triangle(e_a) = e_a \otimes 1 + 1 \otimes e_a$ in $\mathfrak{U}(\mathfrak{g})$. We will come back on this point in the next section.

We then consider interior derivative i_ζ, defined as the odd (degree -1) derivation on $\mathcal{A}$ given by contraction with respect to ζ^*. So, for each $\zeta \in \mathfrak{g}$ we have $i_\zeta : \mathcal{A}^\bullet \to \mathcal{A}^{\bullet-1}$ and a graded Leibniz rule

$$i_\zeta(\omega\nu) = (i_\zeta\omega)\mu + (-1)^k\omega(i_\zeta\mu) \qquad \text{for } \omega \in \mathcal{A}^k,\, \nu \in \mathcal{A}$$

In the same way the (infinitesimal) action of G gives a representation of $\mathfrak{g}$ (and $\mathfrak{U}(\mathfrak{g})$) on $\mathcal{A}$, we look now for the algebraic analogue role of i_ζ.

Out of $\mathfrak{g}$ we can construct a super (or $\mathbb{Z}_2$-graded) Lie algebra $\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$ adding odd generators $\{\xi_a\}$ that span a second copy of $\mathfrak{g}$ as vector space, and putting relations (the brackets are compatible with the degrees)

$$[e_a, e_b] = f_{ab}{}^c e_c \qquad [\xi_a, \xi_b] = 0 \qquad [e_a, \xi_b] = f_{ab}{}^c \xi_c \tag{1.1}$$

The structure of $\bar{\mathfrak{g}}$ reflects the usual commutation relations of Lie and interior derivatives; indeed denoting with $L_a = L_{e_a}$ and similarly $i_b = i_{e_b}$ it is well known that

$$[L_a, L_b] = f_{ab}{}^c L_c \qquad [i_a, i_b] = 0 \qquad [L_a, i_b] = f_{ab}{}^c i_c \tag{1.2}$$

We can then say that L_a and i_a realize a representation of the super Lie algebra $\bar{\mathfrak{g}}$ on $\mathcal{A}$ as graded derivations; once again this representation lifts to the super enveloping algebra $\mathfrak{U}(\bar{\mathfrak{g}})$.

To conclude, let us consider also the De Rham differential $d : \mathcal{A}^\bullet \to \mathcal{A}^{\bullet+1}$ in this algebraic picture; this is not directly related to the action of some symmetry but a unified treatment of (L, i, d) will be relevant in the second chapter, when we will construct algebraic models for equivariant cohomology. We can add to $\bar{\mathfrak{g}}$ one more odd generator d, obtaining the super Lie algebra

$$\tilde{\mathfrak{g}} = \bar{\mathfrak{g}} \oplus \{d\} = \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(0)} \oplus \{d\}_{(1)} \tag{1.3}$$

with relations (1.1) completed with

$$[e_a, d] = 0 \qquad [\xi_a, d] = e_a \qquad [d, d] = 0 \tag{1.4}$$

The structure induced by (L, i, d) on the algebra of differential forms of a manifold acted by a Lie group may be summarized in the following general definition.

Definition 1.1.2 *An algebra $\mathcal{A}$ carrying a representation of the super Lie algebra $\tilde{\mathfrak{g}}$ by graded derivations will be called a $\tilde{\mathfrak{g}}$-differential algebra, or $\tilde{\mathfrak{g}}$-da for short.*

Note that this definition assumes no requirements on the commutativity of $\mathcal{A}$. In the following section deformations will touch only the $\bar{\mathfrak{g}}$-da structure, leading to the idea of deformed symmetries and covariant actions on nc spaces (algebras).

1.2 Deformations of symmetries by Drinfeld twists

Using the language of Def(1.1.2) we will consider a symmetry acting on a graded algebra $\mathcal{A}$ (representing a commutative or a nc geometry) as expressed by a $\tilde{\mathfrak{g}}$-da (or a $\mathfrak{g}$-da) structure on $\mathcal{A}$; equivalently, we may prefer to consider representations of associated enveloping algebras $\mathfrak{U}(\tilde{\mathfrak{g}})$ and $\mathfrak{U}(\mathfrak{g})$.

By deformation of a symmetry we thus mean a deformation of the Lie algebra structures $\tilde{\mathfrak{g}}$, $\mathfrak{g}$ or a deformation of the Hopf algebra structures of $\mathfrak{U}(\tilde{\mathfrak{g}})$, $\mathfrak{U}(\mathfrak{g})$. To the first case belong quantum Lie algebras, while in the second case one considers quantum enveloping algebras.

In both the approaches, and depending on the particular quantization considered, a general strategy is to relate the deformation of $\mathfrak{g}$ or $\mathfrak{U}(\mathfrak{g})$ to a deformation of the product in every $\mathfrak{g}$-da $\mathcal{A}$, and vice versa. When such a link between symmetries (i.e. Hopf or Lie algebras), spaces (i.e. $\mathfrak{g}$-da) and deformations is present, we will speak of covariant deformations or induced star products.

We can give a detailed presentations of this ideas picking up a particular class of deformations, the ones generated by Drinfeld twists in Hopf algebras [Dri90a, Dri90b]; we choose to work with Drinfeld twists because they provide the most natural setting to explicitly describe and study the nc geometries we are interested in, i.e. toric isospectral deformations and nc toric varieties.

Thus the following exposition will be focused on this specific, even if quite general, class of deformations. Of course for different kind of nc geometries, such as q-deformed spaces, the natural class of quantum enveloping algebras to study would be different (Drinfeld-Jimbo deformations); we will say something on these possible different choices when constructing Weil algebras and models for equivariant cohomology.

1.2.1 Hopf algebras and their actions

We start by recalling basic definitions in the theory of Hopf algebras; a standard reference for these topics is [Maj94], where the omitted proofs of the theorems of these introductory sections can be found. We will work with vector spaces, algebras and all others structures over the field $\mathbb{C}$.

Definition 1.2.1 *A coalgebra $(C, \triangle, \epsilon)$ is a vector space with a linear coproduct (or comultiplication) map $\triangle : C \to C \otimes C$ which is coassociative*

$$(\triangle \otimes id) \circ \triangle(c) = (id \otimes \triangle) \circ \triangle(c) \qquad \forall\, c \in C \tag{1.5}$$

and with a linear counit map $\epsilon : C \otimes \mathbb{C}$ satisfying

$$(\epsilon \otimes id) \circ \triangle(c) = c = (id \otimes \epsilon)\triangle(c) \tag{1.6}$$

The coassociativity and counitality conditions for $\triangle$ and ϵ may be obtained by reversing the arrows in the diagrammatic representation of associativity of the product and existence of the unit element in an algebra. We will make use of the Sweedler notation for the coproduct $\triangle(h) = h_{(1)} \otimes h_{(2)}$ with summation understood.

Definition 1.2.2 *A unital bialgebra $(\mathcal{H}, \cdot, \triangle, \epsilon)$ is a vector space which is both a unital algebra and a coalgebra in a compatible way. In symbols, we ask*

$$\triangle(hg) = \triangle(h)\triangle(g)\ , \qquad \triangle(1) = 1 \otimes 1\ , \qquad \epsilon(hg) = \epsilon(h)\epsilon(g) \qquad \epsilon(1) = 1 \tag{1.7}$$

Definition 1.2.3 *A Hopf algebra $(\mathcal{H}, \cdot, \triangle, S, \epsilon)$ is a unital bialgebra equipped with an antialgebra and anticoalgebra map $S : \mathcal{H} \to \mathcal{H}$ satisfying (we denote with $\cdot$ the product in $\mathcal{H}$)*

$$\cdot\,(S \otimes id) \circ \triangle(h) = \cdot\,(id \otimes S) \circ \triangle(h) = \epsilon(h) \qquad \forall h \in \mathcal{H}\,. \tag{1.8}$$

The standard examples of Hopf algebras are enveloping algebras $\mathfrak{U}(\mathfrak{g})$ and algebras of representable functions over a group $Fun(G)$.

Example 1.2.4 *Given a Lie algebra $\mathfrak{g}$ the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is defined to be the quotient of the tensor algebra*

$$\mathcal{T}(\mathfrak{g}) = \oplus_{n=0}^{\infty} \mathfrak{g}^{\otimes^n}$$

by the ideal generated by relations

$$x \otimes y - y \otimes x = [x, y] \qquad x, y \in \mathfrak{g}\,.$$

The product is induced by the tensor algebra. With coproduct, antipode and counity defined on $x \in \mathfrak{g}$ by

$$\triangle(x) = x \otimes 1 + 1 \otimes x \qquad S(x) = -x \qquad \epsilon(x) = 0 \tag{1.9}$$

and extended to the whole $\mathfrak{U}(\mathfrak{g})$ *by the rules*

$$\triangle(ab) = \triangle(a)\triangle(b) \qquad S(ab) = S(b)S(a) \qquad \epsilon(ab) = \epsilon(a)\epsilon(b) \qquad a, b \in \mathfrak{U}(\mathfrak{g})$$

it is easy to verify that $(\mathfrak{U}(\mathfrak{g}), \triangle, S, \epsilon)$ *is a Hopf algebra.*

Example 1.2.5 *Given a group* G*, let* $Fun(G)$ *be the algebra generated by the entries* p^{ij} *of finite dimensional representations of* G*. We write their evaluation as* $p^{ij}(g) = g^{ij}$*. Give* $Fun(G)$ *a commutative algebra structure* $p^{ij}p^{kl} = p^{kl}p^{ij}$*, while coproduct, antipode and counity are defined by*

$$\triangle(p^{ij}) = \sum_{\alpha} p^{i\alpha} \otimes p^{\alpha j} \qquad S(p^{ij})(g) = p^{ij}(g^{-1}) \qquad \epsilon(p^{ij}) = \delta^{ij} \tag{1.10}$$

and extended to generic elements by their algebra and antialgebra map properties. It is easy to verify that $(Fun(G), \triangle, S, \epsilon)$ *is a Hopf algebra.*

Note that $\mathfrak{U}(\mathfrak{g})$ is noncommutative (unless $\mathfrak{g}$ is abelian) but cocommutative, meaning that $\tau \circ \triangle = \triangle$, where τ is the flip map between the two copies of $\mathcal{H}$, while $Fun(G)$ is commutative but not cocommutative.

Definition 1.2.6 *Two Hopf algebras* $\mathcal{H}$ *and* $\mathcal{F}$ *are said (strictly) dually paired if there is a (nondegenerate) bilinear map* $\langle \quad , \quad \rangle : \mathcal{H} \otimes \mathcal{F} \to \mathbb{C}$ *satisfying*

$$\langle hg, a\rangle = \langle h \otimes g, \triangle(a)\rangle \qquad \langle 1, a\rangle = \epsilon(a) \qquad \langle S(h), a\rangle = \langle h, S(a)\rangle \tag{1.11}$$

$$\langle h, ab\rangle = \langle \triangle(h), a \otimes b\rangle \qquad \langle h, 1\rangle = \epsilon(h) \tag{1.12}$$

Example 1.2.7 *For a compact Lie group* G *the Hopf algebras* $\mathfrak{U}(\mathfrak{g})$ *and* $Fun(G)$ *are strictly dually paired, with*

$$\langle \zeta, a\rangle = \zeta(a)_{|1_G} \qquad \zeta \in \mathfrak{g},\ a \in Fun(G) \tag{1.13}$$

where in the rhs we mean the evaluation of the left invariant vector field associated to $\zeta \in \mathfrak{g}$ *acting on* $a \in Fun(G)$*.*

It is easy to prove that for dually paired Hopf algebras when the first is commutative the other is cocommutative and vice versa; this is for instance what happens with $\mathfrak{U}(\mathfrak{g})$ anf $Fun(G)$.

When one object has a property, it is often possible to relax the condition describing the property so that it holds only up to some 'cocycle'; the class of objects having such 'quasi'-property will be larger, but its behaviour will still be in some sense under control. A first example of this philosophy is given by quasitriangular Hopf algebras, where the condition to be cocommutative is relaxed up to conjugation by an element.

Definition 1.2.8 *A quasitriangular Hopf algebra is a pair* $(\mathcal{H}, \mathcal{R})$ *where* $\mathcal{H}$ *is a Hopf algebra and* $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ *is an invertible element in* $\mathcal{H} \otimes \mathcal{H}$ *obeying*

$$\tau \circ \triangle(h) = \mathcal{R}(\triangle h)\mathcal{R}^{-1} \tag{1.14}$$

$$(\triangle \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \qquad (id \otimes \triangle)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{1.15}$$

In (1.14) and in the following we use the notation $\mathcal{R}_{ij}$ to describe the element in $\mathcal{H}^{\otimes^n}$ who has $\mathcal{R}^{(1)}$ (resp $\mathcal{R}^{(2)}$) in the ith (resp jth) factor and 1 everywhere else. The first condition express the lack of cocommutativity of the coproduct, while the last two constraints on $\mathcal{R}$ comes from the coassociativity of the coproduct. A subclass of quasitriangular Hopf algebras, the most similar to cocommutative ones, satisfy the additional property $\mathcal{R}^{-1} = \mathcal{R}_{21}$, and they are called triangular.

We now consider actions, or representations, of Hopf algebras. By left action of $\mathcal{H}$ on a vector space V we mean a linear map $\rho : \mathcal{H} \otimes V \to V$ such that, denoted $\rho(h, v)$ with $\rho_h(v)$, we have

$$\rho_{hg}(v) = \rho_h(\rho_g(v)) \qquad \rho_1(v) = v \tag{1.16}$$

In the following we will use the short notation $\rho_h(v) = h \triangleright v$; there are of course analogue definitions for right actions. When $\mathcal{H}$ acts on something richer than a vector space, it is natural to ask for a compatibility of the action with the extra structures of the space acted. For actions over algebras, this compatibility condition is referred as covariance of the action.

Definition 1.2.9 *Let* $\mathcal{H}$ *be a Hopf algebra acting on a unital algebra* $\mathcal{A}$*. The action is said to be covariant if*

$$h \triangleright (ab) = \triangle(h) \triangleright (a \otimes b) = (h_{(1)} \triangleright a) \otimes (h_{(2)} \triangleright b) \qquad h \triangleright 1 = \epsilon(h) \tag{1.17}$$

When conditions (1.17) hold, we say that $\mathcal{A}$ *is a* $\mathcal{H}$*-module algebra.*

Three examples that are relevant to our interests are the following.

Example 1.2.10 *Let* $\mathcal{H}$ *be a Hopf algebra. A covariant action of* $\mathcal{H}$ *on itself is given by the adjoint action*

$$h \triangleright^{ad} g = ad_h(g) = h_{(1)} g S(h_{(2)}) \tag{1.18}$$

Note that when $\mathcal{H} = \mathfrak{U}(\mathfrak{g})$ *for some Lie algebra* $\mathfrak{g}$ *the adjoint action with respect* $x \in \mathfrak{g}$ *reduces to a bracket with* x

$$x \triangleright^{ad} h = ad_x(h) = xh - hx = [x, h] \qquad x \in \mathfrak{g},\ h \in \mathfrak{U}(\mathfrak{g})$$

Example 1.2.11 *Let* $\mathcal{H}$ *and* $\mathcal{F}$ *be two dually paired Hopf algebras. The left regular action of* $\mathcal{H}$ *on* $\mathcal{F}$*, defined by*

$$h \triangleright a = a_{(1)} \langle h, a_{(2)} \rangle \qquad h \in \mathcal{H}, a \in \mathcal{F} \tag{1.19}$$

is a covariant action and makes $\mathcal{F}$ *into a* $\mathcal{H}$*-module algebra.*

Example 1.2.12 *Let G be a Lie group acting on a manifold $\mathcal{M}$. We already discussed the action of $\mathfrak{g}$, $\tilde{\mathfrak{g}}$ and their enveloping algebras on $\mathcal{A} = \Omega^\bullet(\mathcal{M})$, referring to is as a $\mathfrak{g}$-da (resp $\tilde{\mathfrak{g}}$-da) structure (see Def(1.1.2)). We now notice that this action is covariant, and so the fact that (L, i, d) are (graded) derivations on $\mathcal{A}$ is equivalent to the fact that (e_a, ξ_a, d) have primitive coproduct $\triangle(x) = x \otimes 1 + 1 \otimes x$. Thus to be a $\tilde{\mathfrak{g}}$-da is equivalent to be a $\mathfrak{U}(\tilde{\mathfrak{g}})$-module algebra.*

The category of Hopf-module algebras, denoted by $\mathcal{H}$-mod or ${}_{\mathcal{H}}\mathscr{M}$, has a monoidal structure depending on the quasitriangular structure of $\mathcal{H}$; we will comment on this later (see Prop(1.3.4)).

Since Hopf algebras are algebras as well as coalgebras, everything we said about actions can be 'dualized' and put into a coaction language. A right coaction of a Hopf algebra $\mathcal{H}$ is a pair (β, V) where V is a vector space and $\beta : V \to V \otimes \mathcal{H}$ a linear map such that

$$(\beta \otimes id) \circ \beta = (id \otimes \beta) \circ \beta \qquad\qquad (id \otimes \epsilon) \circ \beta = id \tag{1.20}$$

We will use the shorthand notation $\beta(v) = v_{(\underline{-1})} \otimes v_{(0)}$; note that the index of the component living in the module is underlined. When the coaction is on an algebra $\mathcal{A}$, the covariance is expressed asking β to be an unital algebra morphism

$$\beta(ab) = \beta(a)\beta(b) \qquad\qquad \beta(1) = 1 \otimes 1 \tag{1.21}$$

In this case $\mathcal{A}$ is called a (right) $\mathcal{H}$-comodule algebra, and the corresponding category denoted by $\mathcal{A}^{\mathcal{H}}$. Once again there are analogue definitions for left coactions. Moreover, one could even consider actions and coactions of Hopf algebras on coalgebras, thus defining categories of $\mathcal{H}$-module coalgebras and $\mathcal{H}$-comodule coalgebras.

Let us finally remark that given dually paired Hopf algebras $\mathcal{H}$ and $\mathcal{F}$, every left $\mathcal{H}$-module algebra is automatically a right $\mathcal{F}$-comodule algebra and so on; roughly speaking the duality between $\mathcal{H}$ and $\mathcal{F}$ reflects into the exchange of left-right and action-coaction. The basic example of this phenomenon is for $\mathcal{H} = \mathfrak{U}(\mathfrak{g})$ and $\mathcal{F} = Fun(G)$; we can equally represent algebraically the action of the group G on the space $\mathcal{M}$ either via the action of $\mathfrak{U}(\mathfrak{g})$ and the $\mathfrak{U}(\mathfrak{g})$-module algebra structure of $\Omega(\mathcal{M})$, or via the coaction of $Fun(G)$ and the $Fun(G)$-comodule algebra structure of $\Omega(\mathcal{M})$.

1.2.2 Drinfeld twists

We come now to deformations. As previously announced, we consider deformations by Drinfeld twists [Dri90a, Dri90b]; given a Hopf algebra $\mathcal{H}$ this is a way to introduce a new Hopf algebra structure on the same $\mathcal{H}$ by using 2-(co)cycles. There are two dual definitions of Drinfeld twists, the first one deforming the coproduct structure while the second one deforming the product.

Definition 1.2.13 *Let $\mathcal{H}$ be an Hopf algebra. An element $\chi = \chi^{(1)} \otimes \chi^{(2)} \in \mathcal{H} \otimes \mathcal{H}$ is called a twist element for $\mathcal{H}$ if it satisfies the following properties:*

1. χ *is invertible*

2. $(1 \otimes \chi)(id \otimes \triangle)\chi = (\chi \otimes 1)(\triangle \otimes id)\chi$ *(cocycle condition)*

3. $(id \otimes \epsilon)\chi = (\epsilon \otimes id) = 1$ *(counitality)*

Theorem 1.2.14 *A twist element $\chi = \chi^{(1)} \otimes \chi^{(2)} \in \mathcal{H} \otimes \mathcal{H}$ defines a twisted Hopf algebra structure $\mathcal{H}^\chi$ with the same multiplication and counit, but new coproduct and antipode given by*

$$\triangle^\chi(h) = \chi\triangle(h)\chi^{-1} \quad , \quad S^\chi(h) = US(h)U^{-1} \quad \text{with} \quad U = \chi^{(1)}S\chi^{(2)} \tag{1.22}$$

When applied to quasitriangluar Hopf algebras $(\mathcal{H}, \mathcal{R})$ the twist deforms the quasi-triangular structure to $\mathcal{R}^\chi = \chi_{21}\mathcal{R}\chi^{-1}$.

We point out that the cocycle condition on χ is a sufficient condition for the coassociativity of $\triangle^\chi$, provided that we start from a coassociative $\triangle$. A more general theory of twists where this requirement is dropped out is well defined in the category of quasi-Hopf algebras [Dri90a][Dri90b] (or Drinfeld algebras in the teminology of [SS93]). We will come back on this at the end of this section, discussing rigidity theorems for deformations of enveloping algebras. Now the dual definition:

Definition 1.2.15 *Let $\mathcal{H}$ be an Hopf algebra. An element $\gamma : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ is called a Drinfeld twist element for $\mathcal{H}$ if it satisfies the following properties $\forall a, b, c \in \mathcal{H}$:*

1. γ *is invertible, i.e. there exists $\gamma^{-1} : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ such that*

$$\gamma(a_{(1)} \otimes b_{(1)})\gamma^{-1}(a_{(2)} \otimes b_{(2)}) = \gamma^{-1}(a_{(1)} \otimes b_{(1)})\gamma(a_{(2)} \otimes b_{(2)}) = \epsilon(a)\epsilon(b)$$

2. $\gamma(a_{(1)} \otimes b_{(1)})\gamma(a_{(2)}b_{(2)} \otimes c) = \gamma(b_{(1)} \otimes c_{(1)})\gamma(a \otimes b_{(2)}c_{(2)})$

3. $\gamma(a \otimes 1) = \gamma(1 \otimes a) = \epsilon(a)$

The second property is called the cycle condition, the third one the unitality condition.

Theorem 1.2.16 *A Drinfeld twist element γ for $\mathcal{H}$ defines a twisted Hopf algebra structure $\mathcal{H}_\gamma$ with the same coproduct and counity, but new algebra structure and antipode given by*

$$\begin{aligned} a \cdot_\gamma b &= \gamma(a_{(1)} \otimes b_{(1)})a_{(2)} \cdot b_{(2)}\gamma^{-1}(a_{(3)} \otimes b_{(3)}) \\ S_\gamma(a) &= U(a_{(1)})S(a_{(2)})U^{-1}(a_{(3)}) \quad \text{with} \quad U(a) = \gamma(a_{(1)} \otimes S(a_{(2)})) \end{aligned} \tag{1.23}$$

Considering the category of $\mathcal{H}$-module algebras, a deformation of $\mathcal{H}$ generates, by covariance of the action, a deformation of the algebra structure of every module algebra. So a Drinfeld twist in $\mathcal{H}$ provides a deformed product in each algebra acted covariantly by $\mathcal{H}$.

Theorem 1.2.17 *If $\mathcal{A}$ is a left $\mathcal{H}$-module algebra and χ a Drinfeld twist for $\mathcal{H}$, the deformed product*

$$a \cdot_\chi b := \cdot \left(\chi^{-1} \triangleright (a \otimes b)\right) \qquad \forall\, a, b \in \mathcal{A} \tag{1.24}$$

makes $A_\chi = (A, \cdot_\chi)$ into a left $\mathcal{H}^\chi$-module algebra with respect to the same action. If we consider a right action the formula for the deformed product in A will contain a χ acting from the right.

The analogue result for dual Drinfleld twists concerns a deformation in the algebra structure of $\mathcal{H}$-comodule algebras.

Theorem 1.2.18 *If A is a right $\mathcal{H}$-comodule algebra and γ a twist element for $\mathcal{H}$, the deformed product*

$$a \cdot_\gamma b = a_{(\underline{1})} b_{(\underline{2})} \gamma^{-1}(a_{(2)} \otimes b_{(2)}) \tag{1.25}$$

makes $A_\gamma = (A, \cdot_\gamma)$ into a right $\mathcal{H}_\gamma$-comodule algebra. If we consider a left coaction the formula for the deformed product in A will contain $\gamma(a_{(1)} \otimes b_{(1)})$.

Given two Hopf algebras $\mathcal{H}$ and $\mathcal{F}$ dually paired, to a twist element $\chi \in \mathcal{H} \otimes \mathcal{H}$ we can associate a dual twist element $\gamma : \mathcal{F} \otimes \mathcal{F} \to \mathbb{C}$ defined by

$$\gamma(a \otimes b) = \langle \chi, a \otimes b \rangle = \langle \chi^{(1)}, a \rangle \langle \chi^{(2)}, b \rangle \tag{1.26}$$

It should be clear that every deformation obtained using a twist χ of $\mathcal{H}$ can be described as well using the dual twist γ of $\mathcal{F}$ defined by (1.26).

There is also a nice cohomological classification of Drinfeld twists. An element $\chi \in \mathcal{H} \otimes \mathcal{H}$ is said to be a 2-coboundary if $\chi = \upsilon \otimes \upsilon$ for some $\upsilon \in \mathcal{H}$, i.e. it is a group-like element. One can define $\mathscr{H}^2(\mathcal{H}, \mathbb{C})$ to be the equivalence class of 2-cocycles on $\mathcal{H}$ modulo 2-coboundaries. The following theorem says that only the cohomology class of χ is non trivially involved in the process of twisting.

Theorem 1.2.19 *Let χ, ψ two twist elements for a Hopf algebra $\mathcal{H}$. The twisted Hopf algebras $\mathcal{H}^\chi$ and $\mathcal{H}^\psi$ are isomorphic via an inner automorphism if χ and ψ are cohomologous in $\mathscr{H}^2(\mathcal{H}, \mathbb{C})$. In particular a Drinfeld twist by a 2-coboundary may always be undone by an inner automorphism.*

Summarizing, up to now we have shown that the algebraic essence of the action of a symmetry $G \circlearrowleft \mathcal{M}$ lies in the $\mathfrak{U}(\mathfrak{g})$-module algebra structure of differential forms; next, we have described a class of deformation of Hopf algebras which generates deformations in every $\mathcal{H}$-module algebras.

The idea is then to apply Drinfeld twists to enveloping algebras to deform symmetries, and induce a nc deformation in the algebra of differential forms of the manifold acted on. In this way a deformation of a symmetry generates a nc geometry in each space where the symmetry acts. Looked the other way around, given a nc algebra we can use a Drinfeld twist to deform the enveloping algebra representing some symmetry in order to restore a Hopf-module algebra structure on the nc space.

So we focus on Drinfeld twists of enveloping algebras $\mathfrak{U}(\mathfrak{g})$. Even if all the result stated until now are valid for generic twist elements and in principle apply to every enveloping algebra, to have more explicit computations and a simplified theory we will restrict to the case of semisimple Lie algebras, in order to use a Cartan decomposition of $\mathfrak{g}$ with an abelian Cartan subalgebra $\mathfrak{h}$. Moreover we will use twist elements χ contained in $\mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{h}) \subset \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$; we may refer to this choice as the class of abelian Drinfeld twists, in the sense that $[\chi, \chi] = 0$. A general theory for Drinfeld twist deformations of enveloping algebras with non abelian twist elements could lead to very interesting results and deserves a detailed study in the future.

After these assumptions, let us fix the notations. Given a semisimple Lie algebra $\mathfrak{g}$ we fix a Cartan decomposition

$$\{H_i, E_r\} \qquad\qquad i = 1, \ldots, n, \quad r = (r_1, \ldots, r_n) \in \mathbb{Z}_n$$

where n is the rank of $\mathfrak{g}$, H_i are the generators of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and E_r are the roots element labelled by the n-dimensional root vector r. In this decomposition the structure constants are written as follows:

$$\begin{array}{rclcrcl} [H_i, H_j] & = & 0 & \quad & [H_i, E_r] & = & r_i E_r \\ [E_{-r}, E_r] & = & \sum_i r_i H_i & \quad & [E_r, E_s] & = & N_{r,s} E_{r+s} \end{array} \tag{1.27}$$

The explicit expression of $N_{r,s}$ is not needed in what follows, but it worths saying that it vanishes if $r + s$ is not a root vector.

Now we choose a twist element χ, depending on Cartan generators H_i. Since we want to use the Drinfeld twist as a source of 'quantization' or deformation, we want it to depend on some real parameter(s) θ and recover the classical enveloping algebra for $\theta \to 0$. Thus we are actually making a Drinfeld twist in the formal quantum enveloping algebra $\mathfrak{U}(\mathfrak{g})_{[[\theta]]}$. We will make use of the following twist element, firstly appeared in [Res90]:

$$\chi = \exp\{-\frac{i}{2}\theta^{kl} H_k \otimes H_l\} \qquad\qquad \chi \in (\mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{h}))_{[[\theta]]} \tag{1.28}$$

with θ a $p \times p$ real antisymmetric matrix, $p \leq n$ (i.e. we do not need to use the whole $\mathfrak{h}$ to generate the twist).

Using relations (1.27) and the expressions in Thm(1.2.14) for the twisted coproduct and antipode, we can describe explicitly the Hopf algebra structure of $\mathfrak{U}^\chi(\mathfrak{g})_{[[\theta]]}$.

Proposition 1.2.20 *Let χ be the twist element in* (1.28)*. The twisted coproduct $\triangle^\chi$ of $\mathfrak{U}^\chi(\mathfrak{g})_{[[\theta]]}$ on the basis $\{H_i, E_r\}$ of $\mathfrak{g}$ reads*

$$\triangle^\chi(H_i) = \triangle(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{1.29}$$

$$\triangle^\chi(E_r) = E_r \otimes \lambda_r^{-1} + \lambda_r \otimes E_r \tag{1.30}$$

where

$$\lambda_r = exp\,\{\frac{i}{2}\,\theta^{kl} r_k H_l\} \tag{1.31}$$

are untwisted group-like element (one for each root r) $\triangle^\chi(\lambda_r) = \triangle(\lambda_r) = \lambda_r \otimes \lambda_r$.

Proof: From $\triangle^\chi(X) = \chi\triangle(X)\chi^{-1}$ it is clear that whenever $[H_i, X] = 0$ the coproduct of X is not deformed. Thus (1.29) follows easily from the fact that $\mathfrak{h}$ is abelian. For (1.30) we compute

$$\exp\,\{-\frac{i}{2}\,\theta^{\mu\nu} H_\mu \otimes H_\nu\}(E_r \otimes 1 + 1 \otimes E_r)\exp\,\{\frac{i}{2}\,\theta^{\mu\nu} H_\mu \otimes H_\nu\}$$

at various order in θ, using

$$e^{tA}Be^{-tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!}[A, [A, \ldots [A, B]]]$$

At the first order we have

$$\begin{aligned} -\frac{i}{2}\theta^{\mu\nu}[H_\mu \otimes H_\nu, E_r \otimes 1 + 1 \otimes E_r] &= -\frac{i}{2}\theta^{\mu\nu}\left([H_\mu, E_r] \otimes H_\nu + H_\mu \otimes [H_\nu, E_r]\right) \\ &= -\frac{i}{2}\theta^{\mu\nu}\left(E_r \otimes r_\mu H_\nu + r_\nu H_\mu \otimes E_r\right) \end{aligned}$$

So the second order is

$$\begin{aligned} &(\frac{i}{2})^2\theta^{\mu\nu}\theta^{\rho\sigma}[H_\mu \otimes H_\nu, E_r \otimes r_\rho H_\sigma + r_\sigma H_\rho \otimes E_r] = \\ &\qquad = (\frac{i}{2})^2\theta^{\mu\nu}\theta^{\rho\sigma}\left([H_\mu, E_r] \otimes r_\rho H_\nu H_\sigma + r_\sigma H_\mu H_\nu \otimes [H_\nu, E_r]\right) = \\ &\qquad = (\frac{i}{2})^2\theta^{\mu\nu}\theta^{\rho\sigma}\left(E_r \otimes r_\mu r_\rho H_\nu H_\sigma + r_\sigma r_\nu H_\mu H_\rho \otimes E_r\right) \end{aligned}$$

It is clear that carrying on with higher orders the series gives (1.30). ■

Proposition 1.2.21 *Let χ be the twist element in (1.28). The element $U = \chi^{(1)}S\chi^{(2)}$ reduce to the identity so that the twisted antipode $S^\chi(h) = US(h)U^{-1}$ is equal to the untwisted one.*

Proof: We compute U at various order in θ. The order zero is trivially the identity; the first order is

$$-\frac{i}{2}\theta^{\mu\nu}H_\mu S(H_\nu) = \frac{i}{2}\theta^{\mu\nu}H_\mu H_\nu$$

and so it vanishes because $\theta^{\mu\nu}$ is antsymmetric by the exchange $\mu \leftrightarrow \nu$ while $H_\mu H_\nu$ is symmetric. The same happens to the second order

$$(\frac{i}{2})^2\theta^{\mu\nu}\theta^{\rho\sigma}H_\mu H_\rho S(H_\nu H_\sigma) = (\frac{i}{2})^2\theta^{\mu\nu}\theta^{\rho\sigma}H_\mu H_\rho H_\sigma H_\nu = 0$$

and it is evident that all higher orders are zero for the same reason. ■

Finally, the twisted quasitriangular structure (we start with $\mathcal{R} = 1 \otimes 1$ in $\mathfrak{U}(\mathfrak{g})$ since the enveloping algebra is cocommutative) is

$$\mathcal{R}^{\chi} = \chi_{21}\mathcal{R}\chi^{-1} = \chi^{-1}(1 \otimes 1)\chi^{-1} = \chi^{-2} \tag{1.32}$$

so the twisted enveloping algebra is triangular but no more cocommutative.

This completes the explicit computation of the Hopf algebra structure of $\mathfrak{U}^{\chi}(\mathfrak{g})_{[[\theta]]}$.

We end this section with a brief discussion on the relation between Drinfeld twists and other deformations of enveloping algebras; we refer to [Kas95][SS93] for a detailed treatment and the proofs. The theory of algebras and coalgebras deformations, and related cohomologies, is well defined in the setting of formal power series; the results we quickly present here are mainly due to Gerstenhaber, Schack, Shnider and Drinfeld.

To introduce quantum enveloping algebras several routes are possible: a first possibility is to consider deformations $\mathfrak{g}_\theta$ of the Lie algebra structure of $\mathfrak{g}$, basically defining structure constants on $\mathbb{C}_{[[\theta]]}$, so that $(\mathfrak{U}(\mathfrak{g}_\theta), \cdot_\theta, \triangle_\theta, \mathcal{R}_\theta)$ is the associated quantum enveloping algebra defined using the θ-deformed brackets in $\mathfrak{g}_\theta$. However a classical result in deformation theory, due to Gerstenhaber, states that if an algebra A has a vanishing second Hochschild cohomology group $H^2(A, A) = 0$, then any deformation A' is isomorphic to the θ-adic completion of the undeformed algebra, i.e., $A' \simeq A_{[[\theta]]}$; these algebras are called rigid. For example for semisimple Lie algebras rigidity is implied by the second Whitehead lemma, and so they do not have non-trivial deformations.

When $\mathfrak{g}$ is semisimple a standard deformation of its enveloping algebra is provided by the Drinfled-Jimbo quantum enveloping algebra $\mathfrak{U}_\theta(\mathfrak{g})$, defined as the topological algebra over $\mathbb{C}_{[[\theta]]}$ generated by Cartan and roots element $\{H_i, X_i, Y_i\}$ subjects to relations (a_{ij} is the rank$= n$ Cartan matrix and $D = (d_1 \dots d_n)$ the diagonal matrix of root lenght)

$$[H_i, H_j] = 0 \qquad\qquad [X_i, Y_j] = \delta_{ij}\frac{\sinh(\theta d_i H_i/2)}{\sinh(\theta d_i/2)} \tag{1.33}$$

$$[H_i, X_j] = a_{ij}X_j \qquad\qquad [H_i, Y_j] = -a_{ij}Y_j \tag{1.34}$$

plus the θ-quantized version of Serre relations between X_iX_j adn Y_iY_j for $i \neq j$.

Now, the rigidity of $\mathfrak{g}$ assures that there is an isomorphism of topological algebras

$$\alpha : \mathfrak{U}_\theta(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})_{[[\theta]]}$$

which transfers the Hopf algebra structure $\triangle_\theta, \epsilon_\theta, S_\theta$ of $\mathfrak{U}_\theta(\mathfrak{g})$ to $\mathfrak{U}(\mathfrak{g})_{[[\theta]]}$ by

$$\triangle' = (\alpha \otimes \alpha) \circ \triangle_\theta \circ \alpha^{-1} \quad , \quad \epsilon' = \epsilon_\theta \circ \alpha^{-1} \quad , \quad S' = \alpha \circ S_\theta \circ \alpha^{-1} \tag{1.35}$$

so that α becomes an isomorphism of Hopf algebras from $\mathfrak{U}_\theta(\mathfrak{g})$ to $\mathfrak{U}(\mathfrak{g})_{[[\theta]]}$ (with the primed Hopf algebra structure of (1.35)). Now, again for rigidity reasons the two coproducts $\triangle$ and $\triangle'$ in $\mathfrak{U}(\mathfrak{g})_{[[\theta]]}$ must be related by an inner automorphism: there should exist an invertible element $\chi \in (\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}))_{[[\theta]]}$ such that $\triangle'(h) = \chi \triangle(h) \chi^{-1}$. This χ quite often does not satisfy any cocycle condition, so it defines a generalized Drinfeld twist and $\mathfrak{U}^\chi(\mathfrak{g})_{[[\theta]]}$ is a quasi-Hopf algebra with a nontrivial coassociator Φ encoding basically all the information about the Drinfeld-Jimbo deformation.

So, at least for rigid Lie algebras, there is only one class (modulo isomorphism) of deformations possible. We can equivalently consider deformations involving Lie algebra generators and their relations, as in the spirit of $\mathfrak{U}_q(\mathfrak{g})$, or we can take (generalized) Drinfeld twists of $\mathfrak{U}(\mathfrak{g})_{[[\theta]]}$ in which the algebra structure is undeformed and the whole deformation is contained and limited to the coproduct and eventually a non trivial coassociator; in this case we preserve classical Lie brackets between generators on $\mathfrak{g}$ but the price to pay is ultimately to enlarge the category to quasi-Hopf algebras.

Finally, let us note that if one wants to avoid formal power series, a different approach is to realize directly a representation of the twisted universal enveloping algebra as unitary operators on some Hilbert space; this can indeed be a better strategy if one wants to study a covariant action in specific cases, as for example in [CL01][LvS07].

1.3 Toric isospectral deformations

In the previous section we fixed the class of Drinfeld twists χ we are interested in (1.28), and noted that they are generated by elements in the Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$. For this reason we called such χ abelian or toric Drinfeld twists. Then we showed (see Thm(1.2.17)) that as a consequence of the twist every $\mathfrak{U}(\mathfrak{g})$-module algebra deforms its product in order to preserve the covariance of the action. Following this strategy, it is clear we can induce a nc deformation in the algebra of funcions (or differential forms) of every manifold acted by some group of rank ≥ 2 (we need at least two toric generators to define a non trivial χ).

This is the setting of toric isospectral deformations [CL01][CDV02]. One starts with a compact Riemannian spin manifold $\mathcal{M}$ whose isometry[2] group has rank at least 2, and use the action of the compact torus $\mathbb{T}^n$ ($n \geq 2$) to construct a nc spectral triple $(C^\infty(\mathcal{M}_\theta), L^2(\mathcal{M}, S), D)$ by deforming the classical one; the name 'isospectral' refers to the fact that in the nc spectral triple only the algebra of functions and its

[2]In the construction of the deformed spectral triple it is relevant the fact that the symmetry is actually an isometry, since this assures the Dirac operator D commutes with the action. This fact however does not concern the deformation of the algebra $C^\infty(\mathcal{M})$, and so we can relax this request in the Drinfeld twist approach. Nevertheless note that the action of a compact Lie group G on a Riemannian manifold $(\mathcal{M}, g)$ can always turned into an isometry by averaging the metric g with respect to the action of the group.

representation on $\mathcal{H}$ is deformed, but not the Dirac operator D (and so its spectrum) which is still the classical one due to its invariance under the action.

1.3.1 The spectral triple of toric isospectral deformations

We review the construction of toric isospectral deformations in the language of spectral triples. Under the hypothesis of compactness of $\mathcal{M}$ and the presence of an isometry by $\mathbb{T}^n$ with $n \geq 2$, we can decompose the algebra of smooth functions $C^\infty(\mathcal{M}) = \bigoplus_{r \in (\mathbb{Z}^n)^*} C^\infty_r(\mathcal{M})$ in spectral subspaces labelled by weights r, such that every $f_r \in C^\infty_r(\mathcal{M})$ is an eigenfunction of the action. Representing elements of $\mathbb{T}^n$ as $e^{2\pi i t}$ with $t \in \mathbb{Z}^p$, the action σ on an eigenfunction f_r is given by a phase factor depending on r:

$$\sigma_t(f_r) = e^{2\pi i t \cdot r} f_r \qquad\qquad t \in \mathbb{Z}^n \, , \; r \in (\mathbb{Z}^n)^* \tag{1.36}$$

Taking a real $n \times n$ skew-symmetric matrix θ we can define a deformed product between eigenfunctions

$$f_r \times_\theta g_s := \exp\,[\frac{i}{2}\theta^{kl} r_k s_l] f_r g_s \tag{1.37}$$

and by linearity extend it on the whole of $C^\infty(\mathcal{M})$. We will call

$$C^\infty(\mathcal{M}_\theta) := (C^\infty(\mathcal{M}), \times_\theta) \tag{1.38}$$

the algebra of functions of the nc manifold $\mathcal{M}_\theta$. Clearly, $\mathbb{T}^n$-invariant functions form a commutative ideal in the nc algebra $C^\infty(\mathcal{M}_\theta)$.

The deformed product (1.37) is a sort of Moyal product, with the action of $\mathbb{R}^n$ replaced by the torus $\mathbb{T}^n$, i.e. considering periodic actions of $\mathbb{R}^n$. The idea to use actions (of $\mathbb{R}^n$) to produce strict deformation quantizations indeed appeared firstly in [Rie93].

Even if it is not directly related to our Drinfeld twist approach, we briefly complete the construction of the deformed spectral triple which describes in the sense of Def(1.1.1) the nc geometry $\mathcal{M}_\theta$, mainly following [LvS07]. Besides the nc algebra (1.38), the next ingredient is a Hilbert space $\mathcal{H}$ with a faithful representation π of the algebra by bounded operators. In the classical spectral triple we saw this is obtained by considering square integrable spinors and their module structure by pointwise multiplication by functions; now we can use the same Hilbert space, only deforming the module structure of spinors in a similar way we defined the nc product between functions.

Classically there is a double cover $c : \tilde{\mathbb{T}}^n \to \mathbb{T}^n$ and a representation of $\tilde{\mathbb{T}}^n$ on $L^2(\mathcal{M}, S)$ by unitary operators U_s, $s \in \tilde{\mathbb{T}}^n$, which commutes with the Dirac operator D and restricted to $\pi(C^\infty(\mathcal{M}))$ gives [DR03]

$$U_s \pi(f) U_s^{-1} = \pi(\sigma_{c(s)} f) \tag{1.39}$$

so that π intertwines the actions. As we did with functions, we can decompose (smooth) bounded operators T on $\mathcal{H}$ as well in spectral components with respect to the torus action, $T = \sum_{r\in(\mathbb{Z}^n)^*} T_r$, such that

$$U_s T_r U_s^{-1} = e^{2\pi rs} T_r \qquad\qquad s \in \tilde{\mathbb{T}}^n \tag{1.40}$$

Denoting with $(p_1, \dots, p_n)$ the infinitesimal generators of the action of $\tilde{\mathbb{T}}^n$ so that $U_s = \exp\,(2\pi isp)$ and using the same real skewsymmetric matrix θ of (1.37) we deform smooth bounded operators T putting

$$L_\theta(T) = \sum_r T_r \exp\,(2\pi i\theta^{kl} r_k p_l) \tag{1.41}$$

This sort of quantization map L_θ realizes a representation of the nc algebra $C^\infty(\mathcal{M}_\theta)$ as bounded operators, in the sense that

$$L_\theta(\pi(f \times_\theta g)) = L_\theta(\pi(f))L_\theta(\pi(g)) \tag{1.42}$$

It was then shown in [CL01] that $(C^\infty(\mathcal{M}_\theta), L^2(\mathcal{M}, S), D)$ with $\pi(f) := L_\theta(\pi(f))$ defines a nc spectral triple, known as the toric isospectral defomation of the classical Riemannian geometry of $\mathcal{M}$. An analogous treatment of toric isospectral deformations with a particular emphasis on deformed spheres and their symmetries may be found in [Var01], while the link with Drinfeld twists appeared in [Sit01].

We now express the previous deformation using Drinfeld twists. Since we supposed the compact Lie group G acting on $\mathcal{M}$ to have rank $n \geq 2$, we can use its Cartan generators $H_i \in \mathfrak{h} \subset \mathfrak{g}$ $(i = 1, \dots n)$ and the real skewsymmetric matrix θ to define a twist element $\chi \in (\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}))_{[[\theta]]}$ (the same of (1.28))

$$\chi = \exp\,\{-\frac{i}{2}\,\theta^{kl} H_k \otimes H_l\}$$

Note that we need at least rank 2 because with only one Cartan generator H the twist element χ reduces to a coboundary (define $K = \exp\{\frac{i}{2}\,\theta H\}$, then $\triangle(K) = K \otimes K = \chi$), so by Thm(1.2.19) it realizes a trivial deformation.

1.3.2 The same deformation by a Drinfeld twist

We now show how to recover the same nc spaces in the language of Drinfeld twists. We already computed the twisted Hopf algebra structure of $\mathfrak{U}^\chi(\mathfrak{g})_{[[\theta]]}$ in section(1.2.2); now following Thm(1.2.17) we describe the deformed product induced on the $\mathfrak{U}(\mathfrak{g})$-module algebra $\mathcal{A} = \Omega(\mathcal{M})$. As we did for functions, we decompose $\mathcal{A} = \oplus_r \mathcal{A}_r$ into spectral subspaces labelled by characters of the toric subgroup of G so that $H_k \triangleright \omega_r = r_k \omega_r$. On the spectral subspaces the induced deformed product is easily computed.

Proposition 1.3.1 *On spectral elements $\omega_r \in \mathcal{A}_r$ and $\omega_s \in \mathcal{A}_s$ the product induced from the Drinfeld twist of $\mathfrak{U}(\mathfrak{g})$ reads*

$$\omega_r \wedge_\theta \omega := \chi^{-1} \triangleright (\omega_r \otimes \omega_s) = exp\ \{\frac{1}{2}\theta^{\mu\nu} r_\mu s_\nu\} \omega_r \wedge \omega_s \qquad (1.43)$$

Proof: The result follows from a direct computation, using the explicit expression of χ and

$$\theta^{\mu\nu}(H_\mu \otimes H_\nu) \triangleright (\omega_r \otimes \omega_s) = \theta^{\mu\nu} r_\mu s_\nu (\omega_\mu \otimes \omega_\nu)$$

which use the spectral property of ω_r and ω_s. ■

We extend this product from spectral elements to the whole algebra $\mathcal{A}$ by linearity.

Definition 1.3.2 *The nc algebra $\mathcal{A}_\chi = (\mathcal{A}, \wedge_\theta)$ with product $\wedge_\theta$ defined in* (1.43) *is called the algebra of nc differential forms of the nc space $\mathcal{M}_\theta$.*

Considering the degree zero part of $\mathcal{A}_\theta$ we recover the algebra $C^\infty(\mathcal{M}_\theta)$ of (1.38). This shows it is possible to get toric isospectral deformations by Drinfeld twists.

We deformed the graded commutative wedge product $\wedge$ to obtain a nc product $\wedge_\theta$. A natural question is then if $\wedge_\theta$ satisfies some deformed graded commutativity. There is a positive answer to this question, provided we abstract the idea of what means for a product to be (graded) commutative by adapting the definition of commutativity to the category in which we consider the algebra.

We are interested in the category of (left) Hopf-module algebras, denoted ${}_{\mathcal{H}}\mathscr{M}$. To study some of its properties in a more efficient language, we present here some basic definition and facts on braided tensor categories.

Definition 1.3.3 *A braided monoidal or quasitensor category $(\mathscr{C}, \otimes, \Psi)$ is a monoidal category $(\mathscr{C}, \otimes)$ with a natural equivalence between the two functors $\otimes, \otimes^{op} : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ given by functorial isomorphisms (called braiding morphisms)*

$$\Psi_{V,W} : V \otimes W \to W \otimes V \qquad \forall V, W \in \mathscr{C} \qquad (1.44)$$

obeying hexagon conditions expressing compatibility of Ψ with the associative structure of $\otimes$ (for an explicit formulation see for example [Maj94](fig 9.4*, pg* 430*)). If in addition $\Psi^2 = id$ the category $(\mathscr{C}, \otimes, \Psi)$ is said to be a symmetric (or tensor) category.*

The relevant example for us is the tensor product of two Hopf-module algebras $\mathcal{A} \otimes \mathcal{B}$; it is still a Hopf-module algebra, with action defined by

$$h \triangleright (a \otimes b) = (h_{(1)} \triangleright a) \otimes (h_{(2)} \triangleright b) \qquad (1.45)$$

This means that ${}_{\mathcal{H}}\mathscr{M}$ is a monoidal category. The algebraic structure of $\mathcal{A} \otimes \mathcal{B}$ and the presence of a nontrivial braiding operator depend on the quasitriangular structure of $\mathcal{H}$, as it is shown in the following Theorem. For a proof, see for example [Maj94].

Proposition 1.3.4 *If $(\mathcal{H},\mathcal{R})$ is a quasitriangular Hopf algebra the category of left $\mathcal{H}$-module algebras ${}_{\mathcal{H}}\mathscr{M}$ is a braided monoidal category with braiding morphism*

$$\Psi_{\mathcal{A},\mathcal{B}}(a\otimes b)=(\mathcal{R}^{(2)}\triangleright b)\otimes(\mathcal{R}^{(1)}\triangleright a) \qquad \forall\, a\in\mathcal{A}, b\in\mathcal{B} \text{ and } \mathcal{A},\mathcal{B}\in {}_{\mathcal{H}}\mathscr{M} \quad (1.46)$$

Let us note that for $\mathcal{H}$ a cocommutative Hopf algebra, like the classical enveloping algebras, $\mathcal{R}=1\otimes 1$ and the braiding morphism is simply the flip operator, so that the category of module algebras is symmetric. In this case the ordinary tensor algebra structure of $\mathcal{A}\otimes\mathcal{B}$

$$(a_1\otimes b_1)\cdot(a_2\otimes b_2)=(a_1a_2)\otimes(b_1b_2)$$

is compatible with the action of $\mathcal{H}$. However in the general case, in order to get an algebra structure on $\mathcal{A}\otimes\mathcal{B}$ acted covariantly by $\mathcal{H}$, we have to take into account the quasitriangular structure.

Proposition 1.3.5 *If $(\mathcal{H},\mathcal{R})$ is a quasitriangular Hopf algebra and $\mathcal{A},\mathcal{B}\in {}_{\mathcal{H}}\mathscr{M}$, the braided tensor product $\mathcal{H}$-module algebra $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is the vector space $\mathcal{A}\otimes\mathcal{B}$ endowed with the product*

$$(a_1\otimes b_1)\cdot(a_2\otimes b_2):=a_1(\mathcal{R}^{(2)}\triangleright a_2)\otimes(\mathcal{R}^{(1)}\triangleright b_1)b_2 \qquad (1.47)$$

For a proof see again [Maj94].

We can now come back to the question about the 'deformed' graded commutativity of nc differential forms. All morphisms in a Hopf-module algebra intertwine the Hopf action; for example, the condition for the action to be covariant (Def(1.2.9)) may be restated by saying that the multiplication in the algebra $m:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ commutes with the Hopf action

$$m(h\triangleright(a\otimes b))=h\triangleright(m(a\otimes b)) \qquad a,b\in\mathcal{A},\ h\in\mathcal{H}$$

In the category of algebras (say over $\mathbb{C}$) a multiplication m is commutative if it commutes with the 'flip' operator $\tau: A\otimes A\to A\otimes A$ which exchange the first and second copy of A, $\tau(a\otimes b)=b\otimes a$. But for quasitriangular Hopf algebras $(\mathcal{H},\mathcal{R})$ the operator τ in not a morphism in the category ${}_{\mathcal{H}}\mathscr{M}$ (unless the Hopf algebra is cocommutative); the natural analogue of the flip map is the braiding morphism Ψ (1.46). Then the following definition is natural .

Definition 1.3.6 *In the category ${}_{\mathcal{H}}\mathscr{M}$ an algebra $\mathcal{A}$ is said to be braided commutative if its multiplication map $m:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ commutes with the braiding map $\Psi_{\mathcal{A},\mathcal{A}}$:*

$$m\circ\Psi_{\mathcal{A},\mathcal{A}}=m \qquad \Longleftrightarrow \qquad a\cdot b=(\mathcal{R}^{(2)}\triangleright b)\cdot(\mathcal{R}^{(1)}\triangleright a) \qquad (1.48)$$

Note that since the product on Hopf-module algebras is required to be compatible with the action of the Hopf algebra, the property to be commutative now depends on the Hopf algebra which acts; it could happen that an algebra is acted covariantly by two different Hopf algebras and it is braided commutative with respect the first one but not with respect the second one.

For graded algebras in ${}_{\mathcal{H}}\mathscr{M}$ also a definition of braided graded-commutativity is present; the request is

$$a \cdot b = (-1)^{|a||b|}(\mathcal{R}^{(2)} \triangleright b) \cdot (\mathcal{R}^{(1)} \triangleright a) \qquad\qquad a, b \in \mathcal{A} \tag{1.49}$$

So now the natural question is whether the algebra of nc differential forms $\mathcal{A}_\chi$ is braided graded-commutative. It is not hard to prove that the answer is affirmative.

Lemma 1.3.7 *Let $\mathcal{B}$ be a graded commutative algebra in ${}_{\mathcal{H}}\mathscr{M}$ and χ a twist element of the form* (1.28)*. Then*

$$b_1 \cdot_\chi b_2 := \cdot(\chi^{-1} \triangleright (b_1 \otimes b_2)) = (-1)^{|b_1||b_2|} \cdot (\chi \triangleright (b_2 \otimes b_1)) \tag{1.50}$$

Proof: By direct computation, starting from the rhs:

$$\begin{aligned}(-1)^{|b_1||b_2|}\left(\sum_n (-\frac{\theta^{\alpha\beta}}{2})^n \frac{1}{n!}\,(H^n_\alpha b_2)\cdot(H^n_\beta b_1)\right) &= \sum_n (-\frac{\theta^{\alpha\beta}}{2})^n \frac{1}{n!}\,(H^n_\beta b_1)\cdot(H^n_\alpha b_2) = \\ &= \sum_n (\frac{\theta^{\beta\alpha}}{2})^n \frac{1}{n!}\,(H^n_\beta b_1)\cdot(H^n_\alpha b_2) = \sum_n (\frac{\theta^{\alpha\beta}}{2})^n \frac{1}{n!}\,(H^n_\alpha b_1)\cdot(H^n_\beta b_2) = \\ &= \cdot\,(\chi^{-1} \triangleright (b_1 \otimes b_2)) = b_1 \cdot_\chi b_2 \qquad \blacksquare\end{aligned}$$

Proposition 1.3.8 *Let $\mathcal{A}_\chi$ be the algebra of nc differential forms deformed by the usual Drinfeld twist* (1.28) *in $\mathfrak{U}^\chi(\mathfrak{g})_{[[\theta]]}$. Then $\mathcal{A}_\chi$ is braided graded-commutative.*

Proof: The quasitriangular structure of $\mathfrak{U}(\mathfrak{g})_{[[\theta]]}$ is $\mathcal{R}^\chi = \chi^{-2}$. We compute the rhs of (1.49) with $\omega \in \mathcal{A}^n_\chi$ and $\nu \in \mathcal{A}^k_\chi$, and make use of the previous Lemma:

$$\begin{aligned}(-1)^{kn}((\mathcal{R}^\chi)^{(2)} \triangleright \nu) \wedge_\theta ((\mathcal{R}^\chi)^{(1)} \triangleright \omega) &= (-1)^{kn} \wedge \big((\mathcal{R}^\chi)^{(2)} \otimes (\mathcal{R}^\chi)^{(1)} \cdot \chi^{-1} \triangleright (\nu \otimes \omega)\big) = \\ &= (-1)^{kn} \wedge \big(\chi^2 \cdot \chi^{-1} \triangleright (\nu \otimes \omega)\big) = (-1)^{kn} \wedge (\chi \triangleright (\nu \otimes \omega)) = \\ &= \wedge\,(\chi^{-1} \triangleright (\omega \otimes \nu)) = \omega \wedge_\theta \nu \qquad \blacksquare\end{aligned}$$

We presented the result having in mind the deformed product in the algebra of differential forms, but it should be clear that the same conclusion applies to every (graded-)commutative algebra $\mathcal{A}$ deformed using a Drinfeld twist of the form (1.28) starting from a cocommutative Hopf algebra; in all these cases the deformed product in $\mathcal{A}_\chi$ turns out to be braided (graded-)commutative.

Let us consider an easy situation in which the difference between deforming a symmetry in the sense of Hopf algebra or a space in the sense of algebra is pointed out. We look at the noncommutative torus $\mathbb{T}^n_\theta$, the first and archetypal example of toric isospectral deformation. It is interesting that in this case the space acted by the symmetry and the symmetry itself are the same, the torus $\mathbb{T}^n$; the construction of $\mathbb{T}^n_\theta$ as isospectral deformation follows the general line described before, while in the Drinfeld twist approach we must pay some attention to distinguish between Hopf algebra and the Hopf-module algebra structure.

Indeed to deform the algebra of differential forms $\mathcal{A} = (\Omega(\mathbb{T}^n), \wedge)$ as usual we twist the enveloping algebra of the symmetry acting, i.e. $\mathfrak{U}(\mathfrak{t}^n)$, with the twist element (1.28). But since $\mathfrak{U}(\mathfrak{t}^n)$ is commutative, the twist has no effect and with no surprise we have $\mathfrak{U}(\mathfrak{t}^n) = \mathfrak{U}^\chi(\mathfrak{t}^n)$. However $\mathcal{A}$ is a $\mathfrak{U}(\mathfrak{t}^n)$-module algebra considering the left action of the torus on itself, so we can define on $\mathcal{A}$ the deformed product induced by χ and achieve the nc product of the isospectral deformation $\mathbb{T}^n_\theta$.

Thus we get a deformation of $\mathbb{T}^n$ into the noncommutative torus even if the Drinfeld twist on the enveloping algebra is trivial; this is not in contrast with the general theory, it only means that both $\mathcal{A} = (\Omega(\mathbb{T}^n), \wedge)$ and $\mathcal{A}_\chi = (\Omega(\mathbb{T}^n), \wedge_\theta)$ are $\mathfrak{U}(\mathfrak{t}^n)$-module algebras, i.e. the space of differential forms possesses two different algebraic structure (the first graded commutative and the second one braided graded-commutative) which are acted covariantly by $\mathfrak{U}(\mathfrak{t}^n)$.

A last remark about the braided graded-commutativity of $\mathcal{A}_\chi$: even if we said that the Drinfeld twist χ is trivial on $\mathfrak{U}(\mathfrak{t}^n)$ due to commutativity of the algebra, formally the twisted quasitriangular structure $\mathcal{R}^\chi = \chi^{-2}$ is different from the trivial one $\mathcal{R} = 1 \otimes 1$ of $\mathfrak{U}(\mathfrak{t}^n)$. Indeed the undeformed and cocommutative coproduct satisfies also

$$\tau \circ \triangle(h) = \mathcal{R}^\chi \triangle(h) (\mathcal{R}^\chi)^{-1} = \triangle(h)$$

which is a quite meaningless property due to the fact that $\triangle(h)$ and $\mathcal{R}^\chi$ commute, but which allows us to say that $\mathfrak{U}^\chi(\mathfrak{t}^n)$ is quasitriangular; since the action of $\mathcal{R}^\chi$ on differential forms (or in general on every other $\mathfrak{U}(\mathfrak{t}^n)$-module algebra) is not trivial, Def(1.3.6) applies and justifies the statement on the braided graded-commutativity of $\wedge_\theta$.

In the previous example the peculiar fact was that a trivial twist in the symmetry can nevertheless lead to a nontrivial deformation in the category ${}_{\mathcal{H}}\mathscr{M}$. Now we consider another situation in which the symmetry and the space are described by the same (or actually the dual) algebra, but again we point out the difference between deforming the former and the latter.

Let us take a compact semisimple Lie group G; we can try to use the action of G on itself to get a nc deformation of G in the language od Drinfeld twists, or isospectral deformations. Recalling Examples (1.2.10) and (1.2.11) both $\mathfrak{U}(\mathfrak{g})$ and $Fun(G)$ are $\mathfrak{U}(\mathfrak{g})$-module algebras (the first with respect the adjoint action, the second considering the left regular action), so we can apply the Drinfeld twist machinery; for instance this is what had been done for the torus $\mathbb{T}^n$.

The fundamental observation is that Thm(1.2.17) provides a way to deform the algebraic structure of $\mathfrak{U}(\mathfrak{g})$-module algebras, but does not assure that the deformation is compatible with further possible extra-structures of the module algebra.

It is true that $Fun(G)$ is acted covariantly by $\mathfrak{U}(\mathfrak{g})$ and we can use this action to define a nc product, but $Fun(G)$ is something more than a simple $\mathfrak{U}(\mathfrak{g})$-module algebra, it is a Hopf algebra itself; this is the algebraic counterpart of the fact that G is not only a manifold acted by a group, it as actually a group itself.

So the question now is if the deformation $Fun_{\chi}(G)$ of Thm(1.2.17) preserves (or at least deform) the Hopf algebra structure of $Fun(G)$ or not, and it is not too hard to see that the answer is negative; we loose for example the bialgebra structure, since we deform the product but not the coproduct.

What we get is then a deformation of G as a nc space, but in doing so we destroyed its group structure; for example the noncommutative torus $\mathbb{T}^n_\theta$ is a nc space but not a quantum group (in the sense of a nc Hopf algebra). If we were interested in a deformation of the whole group structure of G it would be not enough to deform the algebra structure in $Fun(G)$ alone, but we should apply the dual Drinfeld twist γ (see (1.26)) to the whole Hopf algebra, getting the twisted Hopf algebra $Fun_{\gamma}(G)$.

To summarize, it depends of what structures we want to deform and not destroy; once we decide which they are, this fixes the right category we are interested in, and then we have to use the deformation scheme of the relevant category. Our main interest was to deform manifold acted by symmetries, so we focus on the category of Hopf-module algebras and study a compatible way to deform both the space and the symmetry in order to do not leave the category.

A completely analogous phenomenon happens if we try to deform $\mathfrak{U}(\mathfrak{g})$ as a module algebra over itself via the adjoint action using Thm(1.2.17); we get a nc deformation of $\mathfrak{U}(\mathfrak{g})$ which is no more a Hopf algebra (of course it still is a Hopf-module algebra).

We conclude this section expressing explicitly the action of Lie derivative and interior derivative on the algebra of nc differential forms $\mathcal{A}_\chi$ of Def. (1.3.2). We expressed the action of these operators as the representation of the enveloping algebra $\mathfrak{U}(\tilde{\mathfrak{g}})$ (see (1.3) and Def(1.1.2)) and noted that the Leibniz rule satisfied by L_a and i_a was related to the primitive coproduct of even and odd generators e_a and ξ_a in $\mathfrak{U}(\tilde{\mathfrak{g}})$. We already computed explicitly the twisted Hopf algebra structure of $\mathfrak{U}^{\chi}(\mathfrak{g})$; now to include interior derivatives (the differential d will play a role only in the next chapter) we actually need to twist the super enveloping algebra $\mathfrak{U}(\tilde{\mathfrak{g}})$. This is a $\mathbb{Z}_2$-Hopf algebra, but the theory of Drinfeld twist and all related results generalize straightforwardly to the graded case. As usual one has only to ask that all (co)algebra morphisms are now $\mathbb{Z}_2$ graded (co)algebra morphisms and pay some attentions to the signs coming from anticommuting odd generators. In particular we note that the Drinfeld twist element for super Hopf algebras must be even; we will continue to use χ as defined in (1.28).

We use a Cartan decomposition of the odd part of $\bar{\mathfrak{g}}$, with odd generators $\{\xi_a\}$. For Lie superalgebras the Cartan subalgebra is defined as the maximal nilpotent subalgebra coinciding with its normalizer, and for $\bar{\mathfrak{g}}$ it is generated by $\{e_i, \xi_i\}$. Given that, note that odd elements cannot have a diagonal action so in the theory of root decomposition only the even part of the Cartan subalgebra it is usually considered. So we have Cartan-type odd generators ξ_i ($i = 1, \ldots, n$) and root-type odd generators ξ_r generating with d the odd part of $\tilde{\mathfrak{g}}$. The twisted coproduct and antipode on ξ_a are computed exactly as we did for the even generators in Prop(1.2.20) and (1.2.21).

Proposition 1.3.9 *The twisted coproduct on odd generators $\{\xi_i, \xi_r\}$ reads*

$$\triangle^{\chi}(\xi_i) = \triangle(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i \tag{1.51}$$

$$\triangle^{\chi}(\xi_r) = \xi_r \otimes \lambda_r^{-1} + \lambda_r \otimes \xi_r \tag{1.52}$$

The twisted antipode $S^{\chi}(\xi_a)$ is equal to the untwisted one, both for Cartan and root generators.

Proof: For the coproduct part, the proof is just alike to Prop(1.2.20); one computes explicitly the definition of $\triangle^{\chi}(\xi_a)$ and use commutation relations between ξ_a and H_i. For the antipode, we already showed in Prop(1.2.21) that the element U entering in the definition of S^{χ} for this class of Drinfeld twists is the identity, and so the antipode is undeformed regardless of whether it is computed on even or odd generators. ■

After the Drinfeld twist χ the algebra of nc differential forms carries a representation of the twisted enveloping algebra $\mathfrak{U}^{\chi}(\tilde{\mathfrak{g}})$: the twisted coproduct of e_a, ξ_a modifies the action of L_a, i_a on products.

Proposition 1.3.10 *The Lie derivative $L_a = L_{e_a}$ acts classically (as in the untwisted case) on single generators of $\mathcal{A}_{\chi}$. On the contrary on product of differential forms $\omega, \eta \in \mathcal{A}_{\chi}$ it satisfies the following deformed Leibniz rule*

$$L_{H_i}(\omega \wedge_\theta \eta) = (L_{H_i}\omega) \wedge_\theta \eta + \omega \wedge_\theta (L_{H_i}\eta) \tag{1.53}$$

$$L_{E_r}(\omega \wedge_\theta \eta) = (L_{E_r}\omega) \wedge_\theta (\lambda_r^{-1} \triangleright \eta) + (\lambda_r \triangleright \omega) \wedge_\theta (L_{E_r}\eta) \tag{1.54}$$

For this reason L_{E_r} is called a twisted derivation of the algebra $\mathcal{A}_{\chi}$ of degree 0.

Proof: We expressed the $L_a(\omega) = e_a \triangleright \omega$, with $e_a \in \mathfrak{g} \subset \mathfrak{U}(\mathfrak{g})$. After the Drinfeld twist the only change is to consider $e_a \in \mathfrak{U}^{\chi}(\mathfrak{g})$; this does not modify the action on single generators, while on product of differential forms by covariance L acts following the twisted coproduct of e_a. Thus for Cartan elements H_i due to (1.29) we still have a classical Leibniz rule, and this proves (1.53). For roots elements E_r using (1.30) we find

$$\begin{aligned} L_{E_r}(\omega \wedge_\theta \eta) &= E_r \triangleright (\omega \wedge_\theta \eta) = \wedge_\theta (\triangle^{\chi}(E_r) \triangleright (\omega \otimes \eta)) = \\ &= (E_r \triangleright \omega) \wedge_\theta (\lambda_r^{-1} \triangleright \eta) + (\lambda_r \triangleright \omega) \wedge_\theta (E_r \triangleright \eta) \\ &= (L_{E_r}\omega) \wedge_\theta (\lambda_r^{-1} \triangleright \eta) + (\lambda_r \triangleright \omega) \wedge_\theta (L_{E_r}\eta) \end{aligned}$$

Note that $\lambda_r \triangleright \omega$ involves only Lie derivatives along Cartan generators. ■

The same result holds for the interior derivative.

Proposition 1.3.11 *The interior derivative $i_a = i_{\xi_a}$ acts undeformed (as in the untwisted case) on single generators of $\mathcal{A}_\chi$. When it acts on products of differential forms it satisfies the following deformed graded Leibniz rule*

$$i_{\xi_i}(\omega \wedge_\theta \eta) = (i_{\xi_i}\omega) \wedge_\theta \eta + (-1)^{|\omega|}\omega \wedge_\theta (i_{\xi_i}\eta) \tag{1.55}$$

$$i_{\xi_r}(\omega \wedge_\theta \eta) = (i_{\xi_r}\omega) \wedge_\theta (\lambda_r^{-1} \triangleright \eta) + (-1)^{|\omega|}(\lambda_r \triangleright \omega) \wedge_\theta (i_{\xi_r}\eta) \tag{1.56}$$

For this reason i_{ξ_r} is called a twisted derivation of degree -1 of the algebra $\mathcal{A}_\chi$.

Proof: The proof is the same of Prop(1.3.10), now using the twisted coproduct structure of odd generators presented in Prop(1.3.9). ■

The differential d is completely undeformed, both on single generators and when acting on multiplications of differential forms, since it commutes with the generators of the twist H_i. One can also check directly from the definition of $\wedge_\theta$ that d satisfies the classical Leibniz rule.

Note that since the Drinfeld twist in $\mathfrak{U}(\tilde{\mathfrak{g}})$ does not change the Lie brackets in $\tilde{\mathfrak{g}}$, i.e. the Lie algebra structure of $\mathfrak{g}$ is undeformed, the twisted derivations (L, i, d) on nc differential forms still obey to the classical commutation relations (1.2).

1.4 Noncommutative toric varieties

As we have seen in the previous section toric isospectral deformations are nc geometries obtained by using the isometric action of a real torus $\mathbb{T}^n$ and its nc deformation $\mathbb{T}^n_\theta$. In this section we will extend these ideas to the algebraic torus $(\mathbb{C}^*)^n$, in order to obtain a similar deformation of toric algebraic varieties. This section refers to a work in progress [CLS].

After the definition of the noncommutative algebraic torus $(\mathbb{C}^*)^n_\theta$ we review the basic definitions and constructions of toric varieties following [Cox03] and we introduce the analogous nc deformations. We then provide some examples and focus on a general local description by nc homogeneous coordinate rings, in the spirit of [Cox95]; we also outline a sheaf theoretic approach more suitable for the study of instantons and bundles in this nc setting.

1.4.1 The noncommutative algebraic torus

The definition of the noncommutative real torus essentially relies on harmonic analysis and a choice of homomorphism of groups between the space of characters and the torus itself. This procedure may be applied to a generic locally compact abelian group G. We are ultimately interested in the case $G = (\mathbb{C}^*)^n$. Let $\mathcal{A}$ be the algebra of a suitable class of functions on G (with 'good' behaviour at infinity). The Fourier transform on G provides a decomposition of every function $f \in \mathcal{A}$ over a basis of functions $\{\chi_p\}_{p\in\widehat{G}}$ labelled by the characters of G. For every $p \in \widehat{G}$ and $g \in G$ we set $\chi_p(g) = \langle p, g\rangle$, where $\langle -, -\rangle : \widehat{G} \times G \to \mathbb{C}^*$ is the pairing between G and its Pontrjagin dual $\widehat{G}$. This defines the Fourier components $\widehat{f} : \widehat{G} \to \mathbb{C}^*$ of $f \in \mathcal{A}$ as

$$\widehat{f}(p) = \int_G f(g)\, \overline{\chi_p}(g)\ \mathrm{d}g$$

where $\mathrm{d}g$ denotes the invariant Haar measure of G.

In order to define a nc associative product on $\mathcal{A}$ it is enough to describe it on the basis $\{\chi_p\}_{p\in\widehat{G}}$ and then extend it to $\mathcal{A}$ by linearity. Given a homomorphism of groups $\Theta : \widehat{G} \to G$, we set

$$\chi_p \star_\Theta \chi_q := \chi_p \cdot \big(\Theta(p) \triangleright \chi_q\big) = \big\langle q\,,\, \Theta(p)\big\rangle\ \chi_{p+q} \tag{1.57}$$

For $G = \mathbb{R}^n$, the homomorphism Θ is a linear endomorphism on $\mathbb{R}^n$ defined by a real skew-symmetric $n \times n$ matrix and we get the Moyal product. For $G = \mathbb{T}^n$, we put $\Theta(p) = \exp(\frac{i}{2}\,\theta \cdot p)$ for $p \in \mathbb{Z}^n$ with θ again a real skew-symmetric $n \times n$ matrix, and we obtain the nc compact torus $\mathbb{T}^n_\theta$.

For $G = (\mathbb{C}^*)^n$, we proceed as follows. Let $L \cong \mathbb{Z}^n$ be a lattice of rank n. Let $L^* = \mathrm{Hom}_{\mathbb{Z}^n}(L, \mathbb{Z}^n)$ be the dual lattice and denote the canonical pairing between the lattices by $\langle -, -\rangle : L^* \times L \to \mathbb{Z}^n$. The dual lattice labels the characters $\{\chi_p\}_{p\in L^*}$ which provide a basis of functions on $T = L \otimes_{\mathbb{Z}^n} \mathbb{C}^* \cong (\mathbb{C}^*)^n$. Pick a $\mathbb{Z}^n$-basis $e_1, \dots, e_n$ of L, with corresponding dual basis $e_1^*, \dots, e_n^*$ for L^*. Set $p = \sum_i p_i\, e_i^* \in L^*$ and $t = \sum_i e_i \otimes t_i \in T$. Then the characters are given by $\chi_p(t) = t^p := t_1^{p_1} \cdot \ldots \cdot t_n^{p_n}$. The Fourier components in this case are

$$\widehat{f}(p) = \int_T f(t)\, \bar{t}^{\,p}\ \mathrm{d}^*t \tag{1.58}$$

with respect to the T-invariant measure $\mathrm{d}^*t = (\mathrm{d}t\, \mathrm{d}\bar{t}\,)/|t|^2$. Every function $f : T \to \mathbb{C}$ can be written in terms of its Fourier components via the power series expansion

$$f(t) = \sum_{p\in L^*} \widehat{f}(p)\ t^p\ .$$

The homomorphism Θ is defined by a complex skew-symmetric $n \times n$ matrix θ via $\Theta(p) = \exp(\frac{i}{2}\,\theta \cdot p)$. The real part of θ again describes the deformation of the compact real torus $\mathbb{T}^n \subset (\mathbb{C}^*)^n$, while the imaginary part applies to the 'dilatation' part given by $(\mathbb{R}^+)^n$, according to the decomposition

$$(\mathbb{C}^*)^n = \left(\mathbb{R}^+\right)^n \times \mathbb{T}^n \cong \mathbb{R}^n \times \mathbb{T}^n\ .$$

In this way we may think of the deformation of $(\mathbb{C}^*)^n$ as a simultaneous and independent deformation of $\mathbb{T}^n$ and $\mathbb{R}^n$ as given above. However, for concrete computations this prescription is not very useful, since the Moyal deformation affects $\log|t|$

for $t \in (\mathbb{C}^*)^n$ and thus leads to rather involved commutation relations. Note that eq. (1.58) with this decomposition of $(\mathbb{C}^*)^n$ is the Fourier transform with respect to the real torus and the Mellin transform[3] with respect to $(\mathbb{R}^+)^n$.

Regarding the torus as an algebraic variety, we will consider as algebra over $(\mathbb{C}^*)^n$ the Laurent polynomial ring in n variables $\mathbb{C}[t_1^\pm, \ldots, t_n^\pm]$. The monomials in this ring are the functions labelled by the characters $\chi_p(t) = t^p$ that we introduced above. The deformation of the product between such functions may be written explicitly as

$$z^p \star_\theta w^q = \exp\big(\frac{i}{2}\, p_i\, \theta^{ij}\, q_j\big)\; z^p \cdot w^q \tag{1.59}$$

where $z = \sum_i e_i \otimes z_i$, $w = \sum_i e_i \otimes w_i \in L \otimes_{\mathbb{Z}^n} \mathbb{C}^* = T$, and $p, q \in L^*$. The product (1.59) is extended linearly to all of $\mathbb{C}[t_1^\pm, \ldots, t_n^\pm]$.

Definition 1.4.1 *The algebra $\mathbb{C}[t_1^\pm, \ldots, t_n^\pm]$ with the product $\star_\theta$ is called the quantum Laurent algebra $\mathbb{C}_\theta[t_1^\pm, \ldots, t_n^\pm]$ and its elements are called quantum Laurent polynomials. It defines a nc variety denoted $(\mathbb{C}^*)^n_\theta$.*

Remember that θ is a complex matrix. Note that the T-action on $(\mathbb{C}^*)^n$ extends to an action on $(\mathbb{C}^*)^n_\theta$.

The deformed product (1.58) is constructed using the action of a group (the algebraic totus $(\mathbb{C}^*)^n$) over a space (the torus itself). We already discussed a similar situation for the compact torus in the previous section; we know we can obtain the same deformation, in this case the quantum Laurent algebra, by performing a twist in the (quantum) enveloping algebra of the group $(\mathbb{C}^*)^n$. This is simply the (formal power series) polynomial algebra in n commuting elements H_i, which are the infinitesimal generators of the group. As twist element we take as usual $\chi = \exp(\frac{i}{2}\, \theta^{ij}\, H_i \otimes H_j)$, but now θ has entries in $\mathbb{C}$. Then the usual Drinfeld deformed product (1.2.17) between monomials in $\mathcal{A} = \mathbb{C}[t_1^\pm, \ldots, t_n^\pm]$ coincides exactly with eq. (1.59).

The strategy of (toric) isospectral deformations is that once we have a nc deformation of the torus we can deform every topological space acted upon by it. For riemannian manifolds the isospectral condition means restricting to isometric actions. Using the algebraic torus $T \cong (\mathbb{C}^*)^n$ and its deformation constructed above, we will now proceed to deform toric algebraic varieties. We remark that toric isospectral deformations can be proven to be strict deformation quantizations in the sense of Rieffel [Rie93]; it is an open question if our deformation, which may be thought of as generated by $\mathbb{C}^n$ instead of Rieffel's $\mathbb{R}^n$, is of a similar nature.

1.4.2 Noncommutative deformations of toric varieties

Toric varieties X may be described in several equivalent ways. As complex varieties they come with an embedding of an algebraic torus, which is dense in X. In this

[3]The Mellin transform is the harmonic decomposition done with respect the multiplicative group of nonzero real numbers. Roughly speaking it is a Fourier analysis done with $\mathbb{R}^+$ instead of $\mathbb{R}$.

picture their geometry is represented by a set of combinatorial data, called fan, which describes the way $(\mathbb{C}^*)^n$ acts on X. As symplectic manifolds they come with a hamiltonian action of a real torus. The corresponding moment map, whose image is a convex polytope, provides the needed information about the structure of X. We will mainly use the fan picture. For a more exhaustive introduction to toric varieties, along with further definitions and terminology, see [Cox03].

Definition 1.4.2 *A toric variety X is an irreducible variety which contains the algebraic torus $(\mathbb{C}^*)^n$ as a Zariski open subset and the action of $(\mathbb{C}^*)^n$ on itself extends to an action on the whole of X.*

Basic examples are the complex spaces $\mathbb{C}^n$, the projective spaces $\mathbb{CP}^n$, the Grassmannians $Gr(k,n)$ and the weighted projective spaces $\mathbb{CP}^n[a_0,a_1,\dots,a_n]$.

In the following we will denote by $L_\mathbb{R} = L \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^n$ the real vector space obtained from a lattice L in $\mathbb{Z}^n$.

Definition 1.4.3 *A rational polyhedral cone $\sigma \subset L_\mathbb{R}$ is a cone $\sigma = \mathbb{R}^+ v_1 \oplus \cdots \oplus \mathbb{R}^+ v_s$ generated by finitely many elements $v_1,\dots,v_s \in L$. It is strongly convex if it does not contain any real line, $\sigma \cap (-\sigma) = \{0\}$.*

Definition 1.4.4 *For every strongly convex rational polyhedral cone $\sigma \subset L_\mathbb{R}$ of dimension n we define the dual cone*

$$\sigma^\vee = \left\{ m \in L^*_\mathbb{R} \;\middle|\; \langle m,u\rangle \geq 0 \quad \forall u \in \sigma \right\}$$

Given a strongly convex rational polyhedral cone σ, we will now show how to construct a normal affine toric variety $U[\sigma]$. The set $\sigma^\vee \cap L^*$ is a finitely generated semigroup under addition. Let $(m_1,\dots,m_l)$ be the generators of this semigroup, so that $\sigma^\vee \cap L^* \cong \mathbb{Z}^+ m_1 \oplus \cdots \oplus \mathbb{Z}^+ m_l$. Note that in general $\sigma^\vee$ is not strongly convex, so $l \geq n$. To each $m_a = \sum_i (m_a)_i\, e^*_i$ we associate a Laurent monomial in $\mathbb{C}[t_1^\pm,\dots,t_n^\pm]$ by $m_a \mapsto t^{m_a} = t_1^{(m_a)_1} \cdot \ldots \cdot t_n^{(m_a)_n}$. The product between two such elements is obtained from the corresponding sum of characters, $t^{m_a} \cdot t^{m_b} := t^{m_a+m_b}$. Thus the generators of $\sigma^\vee \cap L^*$ span a subring of $\mathbb{C}[t_1^\pm,\dots,t_n^\pm]$ which we denote by $\mathbb{C}[\sigma]$. The affine toric variety $U[\sigma]$ is defined to be the spectrum of $\mathbb{C}[\sigma]$, i.e. $\mathbb{C}[\sigma]$ is the coordinate ring of $U[\sigma]$.

The variety $U[\sigma]$ may also be described as an embedded subvariety in the complex plane $\mathbb{C}^l$. If $\sigma^\vee \cap L^*$ has l generators, consider the polynomial ring $\mathbb{C}[x_1,\dots,x_l]$ (one variable x_a for each m_a). Recall that the generators m_a are l rational vectors in $L^*_\mathbb{R}$, so there are exactly $l-n$ linear relations between them. Then we may quotient the ring $\mathbb{C}[x_1,\dots,x_l]$ by the ideal generated by the $l-n$ relations between the m_a's, realized as multiplicative relations among the variables x_a. If we denote the subspace of relations by $R[m_a] \subset \mathbb{C}[x_1,\dots,x_l]$, we get a realization of $U[\sigma]$ as the spectrum of the quotient algebra $\mathbb{C}[\sigma] = \mathbb{C}[x_1,\dots,x_l]/\langle R[m_a]\rangle$.

We obtain general toric varieties by gluing together affine toric varieties. This has a corresponding picture in terms of cones.

Definition 1.4.5 *A fan $\Sigma \subset L_{\mathbb{R}}$ is a finite collection of strongly convex rational polyhedral cones in $L_{\mathbb{R}}$, such that every face of every cone in Σ is also a cone in Σ, and for $\sigma, \tau \in \Sigma$ the intersection $\sigma \cap \tau$ is a face of each of them.*

To a fan Σ in $L_{\mathbb{R}}$ we associate a toric variety $X = X[\Sigma]$. The cones of Σ correspond to the open affine subvarieties of $X[\Sigma]$, and $U[\sigma]$ and $U[\tau]$ are glued together along their common open subset $U[\sigma \cap \tau]$. Various properties of $X[\Sigma]$, such as smoothness and compactness, may be stated entirely in terms of the fan structure Σ (see [Cox03] for more details).

Our definition of nc toric varieties will involve a one-parameter deformation $X[\Sigma] \to X_\theta[\Sigma]$ which makes use of the same fan structure Σ, deforming only the algebra structure of the coordinate ring of every strongly convex rational polyhedral cone of Σ. We have already defined the quantum Laurent algebra $\mathbb{C}_\theta[t_1^\pm, \ldots, t_n^\pm]$, which represents the nc algebraic torus $(\mathbb{C}^*)^n_\theta$. Since $(\mathbb{C}^*)^n$ is densely contained in every toric variety $X[\Sigma]$, we expect to have morphisms between the nc algebras corresponding to the nc varieties $X_\theta[\Sigma]$ and $\mathbb{C}_\theta[t_1^\pm, \ldots, t_n^\pm]$.

We begin by defining nc affine toric varieties. They are associated to a strongly convex rational polyhedral cone $\sigma \subset L_{\mathbb{R}}$, just as in the commutative case. However, now we use the complex skew-symmetric matrix θ to define a nc product in the ring $\mathbb{C}[\sigma]$, according to the group character relation

$$\chi_p \star_\theta \chi_q = \frac{i}{2}\,(p_i\,\theta^{ij}\,q_j)\ \chi_{p+q} \tag{1.60}$$

Thus if $(m_1, \ldots, m_l)$ are the generators of the semigroup $\sigma^\vee \cap L^*$ and t^{m_a} are the associated Laurent monomials, then the ring $\mathbb{C}_\theta[\sigma]$ is defined to be the subring of $\mathbb{C}_\theta[t_1^\pm, \ldots, t_n^\pm]$ generated by $\{t^{m_a}\}$ with product

$$t^{m_a} \star_\theta t^{m_b} := \exp\big(\frac{i}{2}\,(m_a)_i\,\theta^{ij}\,(m_b)_j\big)\ t^{m_a+m_b}\ . \tag{1.61}$$

This may be regarded as a deformation of the $\mathbb{C}$-algebra generated by the characters, but without deforming their group structure. It is for this reason that we will describe nc toric varieties by using the same fan of the corresponding commutative varieties. The nc affine variety corresponding to the algebra $\mathbb{C}_\theta[\sigma]$ is denoted $U_\theta[\sigma]$. It is a one-parameter deformation of $U[\sigma]$.

We have seen how affine toric varieties may also be regarded as subvarieties of complex spaces $\mathbb{C}^l$, via the quotient ring $\mathbb{C}[\sigma] = \mathbb{C}[x_1, \ldots, x_l]/\langle R[m_a]\rangle$. An analogous realization is possible for nc affine toric varieties. Remembering that in general $l \geq n$, the nc deformation of the polynomial algebra $\mathbb{C}[x_1, \ldots, x_l]$ is obtained from the multiplicative relations between the monomials t^{m_a}. If we denote $\theta'_{ab} := (m_a)_i\,\theta^{ij}\,(m_b)_j$, $a, b = 1, \ldots, l$, $i, j = 1, \ldots, n$ and $q'_{ab} = \exp(i\,\theta'_{ab})$, then the relation between Laurent monomials becomes

$$t^{m_a} \star_{\theta'} t^{m_b} := (q'_{ab})^{1/2}\ t^{m_a+m_b}\ . \tag{1.62}$$

As a consequence, the generators of the algebra of the affine variety obey

$$(q'_{ba})^{1/2}\ x_a \star_{\theta'} x_b = (q'_{ab})^{1/2}\ x_b \star_{\theta'} x_a$$

or equivalently

$$x_a \star_{\theta'} x_b = q'_{ab}\ x_b \star_{\theta'} x_a\ . \tag{1.63}$$

The relations (1.63) define the l-dimensional nc complex plane with coordinate algebra $\mathbb{C}_{\theta'}[x_1,\dots,x_l]$.

The $l-n$ relations among the generators of the dual cone $\{m_a\}$ are now expressed in the character algebra. The linear relations can always be brought to the form

$$\sum_{a=1}^{l} p_{s,a}\ m_a = \sum_{a=1}^{l} r_{s,a}\ m_a$$

for $s=1,\dots,l-n$, with non-negative integer coefficients $p_{s,a}$, $r_{s,a}$. For each s, one obtains from eq. (1.62) the additional relation

$$x_1^{p_{s,1}} \star_{\theta'} \cdots \star_{\theta'} x_l^{p_{s,l}} = \Big(\prod_{1\le a<b\le l} \big(q'_{ab}\big)^{p_{s,a}\,p_{s,b}-r_{s,a}\,r_{s,b}} \Big)\ x_1^{r_{s,1}} \star_{\theta'} \cdots \star_{\theta'} x_l^{r_{s,l}} \tag{1.64}$$

The subspace of relations (1.64) is denoted $R_{\theta'}[m_a]$. It is a one-parameter deformation of the subspace $R[m_a]$, which generates a two-sided ideal in $\mathbb{C}_{\theta'}[x_1,\dots,x_l]$. Thus we may realize $U_\theta[\sigma]$ either as the nc ring $\mathbb{C}_\theta[\sigma]$ or as the quotient $\mathbb{C}_{\theta'}[x_1,\dots,x_l]/\langle R_{\theta'}[m_a]\rangle$.

We obtain generic nc toric varieties $X_\theta[\Sigma]$ by gluing together nc affine toric varieties. If σ and σ' are two cones in the fan Σ which intersect along the face $\tau=\sigma\cap\sigma'$, then there are canonical morphisms between the associated nc algebras $\mathbb{C}_\theta[\sigma]\to\mathbb{C}_\theta[\tau]$ and $\mathbb{C}_\theta[\sigma']\to\mathbb{C}_\theta[\tau]$ induced by the inclusions $\tau\hookrightarrow\sigma$ and $\tau\hookrightarrow\sigma'$. The images of these morphisms in $\mathbb{C}_\theta[\tau]$ are related by an algebra automorphism which plays the role of a 'transition function' between $U_\theta[\sigma]$ and $U_\theta[\sigma']$.

We work out now some explicit examples of nc deformations of toric varieties. We set $q_{ij} := \exp\big(\frac{i}{2}\,\theta_{ij}\big)$ for $i<j$. When $n=2$ we write $q := \exp\big(\frac{i}{2}\,\theta\big)$ with $\theta=\theta^{12}=-\theta^{21}\in\mathbb{C}$. In the following we omit for brevity the star product $\star_\theta$ from the notation.

1.4.3 Examples

The complex Moyal plane

We begin with the simplest toric variety, the n-dimensional complex plane $\mathbb{C}^n$. Let us start from the embedding of the commutative torus $(\mathbb{C}^*)^n\hookrightarrow\mathbb{C}^n$ given by the log map

$$t_i \longmapsto z_i=\log t_i\ , \qquad i=1,\dots,n$$

so that the toric action on $\mathbb{C}^n$ is $\lambda_i \triangleright z_j = z_j + \delta_{ij} \log \lambda_j$ for a set of generators $\lambda_1, \dots, \lambda_n$ of the $(\mathbb{C}^*)^n$-action. Consider the one-parameter deformation $(\mathbb{C}^*_\theta)^n$ of the torus defined by the quantum Laurent algebra $\mathbb{C}_\theta[t_1^\pm, \dots, t_n^\pm]$ with generators $t_i^\pm$ and relations

$$t_i\, t_j = q_{ij}\ t_j\, t_i$$

An application of the Baker–Campbell–Hausdorff formula shows that the corresponding elements z_i obey the commutator relations

$$[z_i, z_j] = i\, \theta_{ij}$$

The algebra of polynomial functions $\mathbb{C}_\theta[z_1, \dots, z_n]$ over $\mathbb{C}$ generated by z_i, $i = 1, \dots, n$ subject to these relations defines a nc affine variety. It is called the complex Moyal space $\mathbb{C}^n_\theta$. For $n = 4$, this is the same as the nc variety $\mathbb{C}^4_\hbar$ defined in [KKO01, §3.4]. All algebras $\mathbb{C}_\theta[z_1, \dots, z_n]$ for $\theta = (\theta_{ij})$ nondegenerate are isomorphic, and hence the varieties $\mathbb{C}^n_\theta$ are the same for all nondegenerate θ. More generally, $\mathbb{C}^n_\theta$ and $\mathbb{C}^n_{\theta'}$ are isomorphic if and only if the matrices θ and θ' have the same rank.

Noncommutative projective plane $\mathbb{CP}^2$

The next example we consider is the projective plane $\mathbb{CP}^2$. It can be described by a fan Σ of the lattice $L \cong \mathbb{Z}^2$ of characters for the action of the algebraic torus $T = L \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^2$ on $\mathbb{CP}^2$. Let e_1, e_2 be a basis of L. Set $v_1 = e_1$, $v_2 = e_2$ and $v_3 = -e_1 - e_2$. These vectors generate the three one-dimensional cones $\tau_i = \mathbb{R}^+ v_i$ of Σ. The three maximal cones of Σ are generated by pairs of these vectors as

$$\sigma_i = \mathbb{R}^+ v_i \oplus \mathbb{R}^+ v_{i+1}\ , \qquad i = 1, 2, 3$$

(with the labels read mod 3) with $\sigma_i \cap \sigma_{i+1} = \tau_{i+1}$ and $\sigma_i \cap \sigma_j = \{0\}$ otherwise. The corresponding open affine subvarieties $U[\sigma_i]$ generate an open cover of $X[\Sigma] = \mathbb{CP}^2$. The zero cone is the triple overlap $\sigma_1 \cap \sigma_2 \cap \sigma_3 = \{0\}$.

We now go through the maximal cones and write out the relations among the generators of the subring $\mathbb{C}_\theta[\sigma_i] \subset \mathbb{C}_\theta[t_1^\pm, t_2^\pm]$. There are no relations $R[m_i]$ in this case, as each dual cone $\sigma_i^\vee$ is strongly convex and hence the generators of $\sigma_i^\vee \cap L^*$ are independent.

1. The generators of the semigroup $\sigma_1^\vee \cap L^*$ are $m_1 = e_1^*$ and $m_2 = e_2^*$. In this case $\theta' = \theta$ and the ring $\mathbb{C}_\theta[\sigma_1] = \mathbb{C}_\theta[x_1, x_2]$ is generated by the elements $x_i = t^{m_i} = t_i$ with the relations
$$x_1\, x_2 = q\ x_2\, x_1 \tag{1.65}$$
2. The semigroup $\sigma_2^\vee \cap L^*$ is generated by $m_1 = -e_1^*$ and $m_2 = e_2^* - e_1^*$. In this case $\theta' = -\theta$, and $\mathbb{C}_\theta[\sigma_2]$ is generated by the elements $x_1 = t^{m_1} = t_1^{-1}$ and $x_2 = t^{m_2} = t_1^{-1}\, t_2$ with the same relations (1.65).

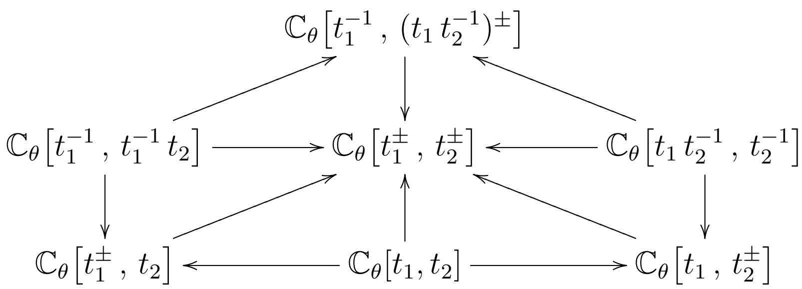
3. The semigroup $\sigma_3^\vee \cap L^*$ is generated by $m_1 = e_1^* - e_2^*$ and $m_2 = -e_2^*$. In this case $\theta' = \theta$, and $\mathbb{C}_\theta[\sigma_3]$ is generated by the elements $x_1 = t^{m_1} = t_1\, t_2^{-1}$ and $x_2 = t^{m_2} = t_2^{-1}$, again with the relations (1.65).

All three varieties $U_\theta[\sigma_i] \cong \mathbb{C}^2_\theta$ are thus copies of the two-dimensional complex Moyal plane.

We now glue the nc affine toric varieties together. Consider, for example, the face $\tau_1 = \sigma_3 \cap \sigma_1$. The semigroup $\tau_1^\vee \cap L^*$ is generated by $m_1 = e_1^*$, $m_2 = e_2^*$ and $m_3 = -e_2^* = -m_2$. The generators of the subring $\mathbb{C}_\theta[\tau_1] = \mathbb{C}_\theta[t_1, t_2^\pm]$ are the elements $y_1 = t_1$, $y_2 = t_2$ and $y_3 = t_2^{-1}$ with the relations

$$y_1\, y_2 = q\ y_2\, y_1 \ , \qquad y_1\, y_3 = q^{-1}\ y_3\, y_1 \ , \qquad y_2\, y_3 = 1 = y_3\, y_2 \ . \tag{1.66}$$

Recalling that $\mathbb{C}_\theta[\sigma_1] = \mathbb{C}_\theta[t_1, t_2]$ and $\mathbb{C}_\theta[\sigma_3] = \mathbb{C}_\theta[t_1\, t_2^{-1}, t_2^{-1}]$, it follows that the algebra morphisms $\mathbb{C}_\theta[\sigma_1] \to \mathbb{C}_\theta[\tau_1]$ and $\mathbb{C}_\theta[\sigma_3] \to \mathbb{C}_\theta[\tau_1]$ are both natural inclusions of subrings. Moreover, as subrings of $\mathbb{C}_\theta[\tau_1]$, there is a natural algebra automorphism $\mathbb{C}_\theta[\sigma_1] \to \mathbb{C}_\theta[\sigma_3]$ defined on generators by $(t_1, t_2) \mapsto (t_1\, t_2^{-1}, t_2^{-1})$. The other faces are similarly treated, and altogether the nc toric geometry of $\mathbb{CP}^2_\theta = X_\theta[\Sigma]$ can be assembled into a diagram of gluing morphisms



The nc affine variety associated to the zero cone is the spectrum of the full deformed character ring $\mathbb{C}_\theta[\{0\}] = \mathbb{C}_\theta[t_1^\pm, t_2^\pm]$.

Noncommutative $\mathbb{CP}^n_\theta$

The previous construction generalizes straightforwardly to the higher-dimensional projective spaces $\mathbb{CP}^n$, $n > 2$, regarded as toric varieties $X[\Sigma]$ generated by a fan Σ of the lattice $L \cong \mathbb{Z}^n$ of characters of $T = L \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$. Choose a basis $e_1, \dots, e_n$ of L. Set $v_i = e_i$ for $i = 1, \dots, n$ and $v_{n+1} = -e_1 - \cdots - e_n$, which generate the one-dimensional cones $\tau_i = \mathbb{R}^+ v_i$ of Σ. The $n+1$ maximal cones of Σ are given by

$$\sigma_i = \mathbb{R}^+ v_i \oplus \mathbb{R}^+ v_{i+1} \oplus \cdots \oplus \mathbb{R}^+ v_{i+n-1} \ , \qquad i = 1, \dots, n+1$$

(with indices understood mod $n+1$) with $\sigma_i \cap \sigma_{i+k} = \mathbb{R}^+ v_{i+k} \oplus \cdots \oplus \mathbb{R}^+ v_{i+n-1}$ a maximal cone of $\mathbb{CP}^{n-k} \hookrightarrow \mathbb{CP}^n$. There are of course many other overlaps, and hence cones, in this instance.

Again there are no relations and $\mathbb{C}[\sigma] = \mathbb{C}[x_1, \dots, x_n]$ for each maximal cone.

1. The generators of the semigroup $\sigma_1^\vee \cap L^*$ are $m_i = e_i^*$ for $i = 1, \dots, n$. The subring $\mathbb{C}_\theta[\sigma_1] \subset \mathbb{C}_\theta[t_1^\pm, \dots, t_n^\pm]$ is generated by the elements $x_i = t^{m_i} = t_i$ subject to the relations

$$x_i\, x_j = q_{ij}\ x_j\, x_i \ , \qquad i < j \ ,$$

and hence $U_\theta[\sigma_1] \cong \mathbb{C}^n_\theta$.

2. For $k \geq 2$, the semigroup $\sigma_k^\vee \cap L^*$ is generated by $m_i = e_i^* - e_k^*$ for $i \neq k$ and $m_k = -e_k^*$. The subring $\mathbb{C}_\theta[\sigma_k]$ in this case is generated by elements $x_i = t_i\, t_k^{-1}$, $i \neq k$ and $x_k = t_k^{-1}$ with relations

$$\begin{aligned} x_i\, x_k &= q_{ki}\ x_k\, x_i \ , \qquad && i \neq k \ , \\ x_i\, x_j &= q_{ij}\, q_{ik}\, q_{jk}\ x_j\, x_i \ , \qquad && k \neq i < j \end{aligned}$$

The faces can be treated analogously to the $n = 2$ case. It is quite long but straightforward to compute all the details, hence we will omit them.

Noncommutative orbifold

We can also deform singular toric varieties in our formalism. For illustration, let us consider the quotient singularity $\mathbb{C}^2/\mathbb{Z}_2$, where the cyclic group $\mathbb{Z}_2$ is generated by the action $(z_1, z_2) \mapsto (-z_1, -z_2)$ for $(z_1, z_2) \in \mathbb{C}^2$. The quotient can be described as the locus of the equation $x\, y - z^2 = 0$ in $\mathbb{C}^3$. The fan Σ of the lattice $L \cong \mathbb{Z}^2$ consists of a single cone $\sigma = \mathbb{R}^+ v_1 \oplus \mathbb{R}^+ v_2$, where $v_1 = e_1$ and $v_2 = e_1 + 2e_2$. The semigroup $\sigma^\vee \cap L^*$ is generated by $m_1 = 2e_1^* - e_2^*$, $m_2 = e_2^*$ and $m_3 = e_1^*$, so that $R[m_i]$ is generated by the single relation $m_1 + m_2 = 2m_3$. The coordinate algebra $\mathbb{C}_\theta[t_1, t_2]^{\mathbb{Z}_2}$ of the nc affine variety $X_\theta[\Sigma] = U_\theta[\sigma]$ is thus generated by $x = t_1^2\, t_2^{-1}$, $y = t_2$ and $z = t_1$ with the relations

$$x\, y = q^2\ y\, x \ , \qquad x\, z = q\ z\, x \ , \qquad y\, z = q^{-1}\ z\, y$$

and

$$x\, y - q\ z^2 = 0$$

The blow-up of the quotient singularity $\mathbb{C}^2/\mathbb{Z}_2$ is the total space of the holomorphic line bundle $\mathcal{O}_{\mathbb{CP}^1}(-2) \to \mathbb{CP}^1$, which defines a non-singular resolution. It is obtained by adding the vector $v_0 = e_1 + e_2$ to the fan Σ above. There are now two maximal cones $\sigma_+ = \mathbb{R}^+ v_1 \oplus \mathbb{R}^+ v_0$ and $\sigma_- = \mathbb{R}^+ v_0 \oplus \mathbb{R}^+ v_2$, with dual semigroups generated respectively by $m_1^\pm = \pm\, e_1^*$ and $m_2^\pm = e_2^* \mp e_1^*$. The coordinate algebras of the nc affine toric varieties $U_\theta[\sigma_\pm]$ are generated respectively by elements $u_\pm = t_1^{\pm 1}$, $v_\pm = t_1^{\mp 1}\, t_2$ subject to the relations

$$u_\pm\, v_\pm = q^{\pm 2}\ v_\pm\, u_\pm \ ,$$

and hence $U_\theta[\sigma_\pm] \cong \mathbb{C}^2_\theta$. The dual semigroup of the one-dimensional cone $\tau = \sigma_+ \cap \sigma_- = \mathbb{R}^+ v_0$ is generated by $m_1 = e_1^*$, $m_2 = e_1^* - e_2^*$ and $m_3 = e_2^* - e_1^* = -m_2$.

The generators of $\mathbb{C}_\theta[\tau]$ are the elements $y_1 = t_1$, $y_2 = t_1^{-1}\, t_2$ and $y_3 = t_1\, t_2^{-1}$ with the relations (1.66). The nc algebraic torus deformation of the resolution is thus described by the diagram of gluing morphisms given by

$$\begin{array}{ccc} & \mathbb{C}_\theta\big[t_1\,,\,(t_1^{-1}\,t_2)^{\pm}\big] & \\ \nearrow & & \nwarrow \\ \mathbb{C}_\theta\big[t_1\,,\,t_1^{-1}\,t_2\big] & & \mathbb{C}_\theta\big[t_1^{-1}\,,\,t_1\,t_2\big] \end{array}$$

where the first arrow is the natural subring inclusion and the second arrow is inclusion after the algebra automorphism $\mathbb{C}_\theta[\sigma_+] \to \mathbb{C}_\theta[\sigma_-]$ given by $(t_1, t_2) \mapsto (t_1^{-1}, t_2)$.

Noncommutative conifold

Next, we consider the threefold ordinary double point, or conifold singularity, defined by the locus of the equation $x\,y - z\,w = 0$ in $\mathbb{C}^4$. Its fan Σ in $L \cong \mathbb{Z}^3$ consists of a single maximal cone σ generated by $w_1 = e_1$, $w_2 = e_2$, $w_3 = e_1 + e_3$ and $w_4 = e_2 + e_3$. The dual cone $\sigma^\vee \cap L$ is generated by $m_1 = e_1$, $m_2 = e_2$, $m_3 = e_3$ and $m_4 = e_1 + e_2 - e_3$, so that $m_1 + m_2 = m_3 + m_4$. The generators of the coordinate algebra of the noncommutative conifold $X_\theta[\Sigma] = U_\theta[\sigma]$ are thus the elements $x = t_1$, $y = t_2$, $z = t_3$ and $w = t_1\, t_2\, t_3^{-1}$ subject to the relations[4]

$$x\,y = q^2\; y\,x \qquad x\,z = q^2\; z\,x \qquad x\,w = w\,x$$
$$y\,z = q^2\; z\,y \qquad y\,w = q^{-4}\; w\,y \qquad z\,w = q^{-4}\; w\,z$$

and

$$x\,y - q^3\; z\,w = 0$$

The crepant resolution[5] of the conifold singularity is the total space of the rank two holomorphic bundle $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1) \to \mathbb{CP}^1$, which is a non-singular toric Calabi-Yau threefold. The fan Σ of the lattice $L \cong \mathbb{Z}^3$ is defined by the vectors $v_1 = e_1 + e_2 + e_3$, $v_2 = e_1 + e_3$, $v_3 = e_1$ and $v_4 = e_1 + e_2$, the maximal cones $\sigma_1 = \mathbb{R}^+ v_1 \oplus \mathbb{R}^+ v_2 \oplus \mathbb{R}^+ v_3$ and $\sigma_2 = \mathbb{R}^+ v_1 \oplus \mathbb{R}^+ v_3 \oplus \mathbb{R}^+ v_4$, and their overlap $\tau = \sigma_1 \cap \sigma_2 = \mathbb{R}^+ v_1 \oplus \mathbb{R}^+ v_3$.

1. $\sigma_1^\vee \cap L^*$ is generated by $m_1 = e_2^*$, $m_2 = e_3^* - e_2^*$ and $m_3 = e_1^* - e_3^*$, so $\mathbb{C}_\theta[\sigma_1]$ is generated by $x = t_2$, $y = t_2^{-1}\, t_3$ and $z = t_1\, t_3^{-1}$ with the relations

$$x\,y = q^2\; y\,x\;, \qquad x\,z = q^{-2}\; z\,x\;, \qquad y\,z = q^2\; z\,y$$

[4] In this subsection, due to notational convenience, we will denote with q the coefficient $exp\,\{\frac{i}{2}\,\theta\}$, i.e. the square root of the previous notation.

[5] A crepant resolution (X, π) of a singular space Y is a resolution $\pi : X \to Y$ which moreover pullbacks the canonical bundle of the singular space K_Y isomorphically to K_X, i.e. $\pi^*(K_Y) \cong K_X$.

2. $\sigma_2^\vee \cap L^*$ is generated by $m_1 = e_3^*$, $m_2 = e_1^* - e_2^*$ and $m_3 = e_2^* - e_3^*$, so $\mathbb{C}_\theta[\sigma_2]$ is generated by $x = t_3$, $y = t_1\, t_2^{-1}$ and $z = t_2\, t_3^{-1}$ with the relations

$$x\, y = y\, x \ , \qquad x\, z = q^{-2}\ z\, x \ , \qquad y\, z = q^2\ z\, y$$

3. $\tau^\vee \cap L^*$ is generated by $m_1 = e_2^*$, $m_2 = e_1^* - e_2^*$, $m_3 = e_2^* + e_3^*$ and $m_4 = -e_2^* - e_3^* = -m_3$, so $\mathbb{C}_\theta[\tau]$ is generated by $y_1 = t_2$, $y_2 = t_1\, t_2^{-1}$, $y_3 = t_2\, t_3$ and $y_4 = t_2^{-1}\, t_3^{-1}$ with the relations

$$\begin{array}{lll} y_1\, y_2 = q^{-2}\ y_2\, y_1 & \qquad y_1\, y_3 = q^2\ y_3\, y_1 & \qquad y_1\, y_4 = q^{-2}\ y_4\, y_1 \\ y_2\, y_3 = q^2\ y_3\, y_2 & \qquad y_2\, y_4 = q^{-2}\ y_4\, y_2 & \qquad y_3\, y_4 = 1 = y_4\, y_3 \end{array}$$

The nc toric geometry is described by the diagram of gluing morphisms

$$\begin{array}{ccc} & \mathbb{C}_\theta\big[t_2\, ,\, t_1\, t_2^{-1}\, ,\, (t_2\, t_3)^{\pm}\big] & \\ \nearrow & & \nwarrow \\ \mathbb{C}_\theta\big[t_2\, ,\, t_2^{-1}\, t_3\, ,\, t_1\, t_3^{-1}\big] & & \mathbb{C}_\theta\big[t_3\, ,\, t_1\, t_2^{-1}\, ,\, t_2\, t_3^{-1}\big] \end{array}$$

where the second arrow is the subring inclusion and the first arrow is inclusion after the automorphism sending $t_3 \mapsto t_1$. Note the similarity with the gluing morphisms of the quotient singularity blow-up.

1.4.4 Homogeneous coordinate rings and sheaves

We recall now the construction of the homogeneous coordinate rings for toric varieties [Cox95], and discuss a generalization to the nc setting. Given a toric variety X described by a fan Σ, we assign to each 1-dimensional cone $\rho_i \in \Sigma$ a variable x_i. If the number of such 1-dimensional cones is $\Sigma(1)$, we define the homogeneous coordinate ring S to be the polynomial ring $S = \mathbb{C}[x_1, \dots, x_{\Sigma(1)}]$. A grading in S is defined using the induced toric action; more precisely, consider the group

$$G := \{(\mu_1 \dots, \mu_{\Sigma(1)}) \in (\mathbb{C}^*)^{\Sigma(1)} \text{ s.t. } \prod_{i=1}^{\Sigma(1)} \mu_i^{\langle m, n_i\rangle} = 1 \ \ \forall\, m \in \mathbb{Z}^n\}$$

where $\langle m, n_i\rangle$ is the usual scalar product in $\mathbb{Z}^n$ and $n_i = \rho_i \cap \mathbb{Z}^n$. Its action on S given by

$$(\mu_1 \dots, \mu_{\Sigma(1)}) \triangleright f(x_1, \dots, x_{\Sigma(1)}) = f(\mu_1 x_1, \dots, \mu_{\Sigma(1)} x_{\Sigma(1)})$$

We say that two monomials in S have the same degree if and only if G acts on them in the same way, i.e.

$$\deg(x_1^{a_1} \dots x_{\Sigma(1)}^{a_{\Sigma(1)}}) = \deg(x_1^{b_1} \dots x_{\Sigma(1)}^{b_{\Sigma(1)}}) \iff \exists\, m \in \mathbb{Z}^n \text{ s.t. } a_i = b_i + \langle n_i, m\rangle \ \ \forall i$$

The polynomial ring S with the above grading is referred as the homogeneous coordinate ring of the toric variety X.

It is possible to recover a local description of the variety by localizing S. We already know how to associate an open affine $U[\sigma]$ to each cone $\sigma \in \Sigma$; now, given such a σ consider the monomial $x^{\hat{\sigma}}$ in S defined by $x^{\hat{\sigma}} = \prod_{\rho_i \notin \sigma} x_i$ and the localization of S with respect $x^{\hat{\sigma}}$. In [Cox95] the following result is established.

Theorem 1.4.6 *Let X be a toric variety described by a fan Σ, with $\Sigma(1)$ the number of 1-dimensional cones. For each cone $\sigma \in \Sigma$ there is a ring isomorphism between the (ring associated to the) open affine $U[\sigma]$ and the degree zero part of the localization of S with respect the monomial $x^{\hat{\sigma}}$, $U[\sigma] \cong (S_{[x^{\hat{\sigma}}]})_{(0)}$.*

Let us focus on the example of projective spaces $\mathbb{CP}^n$, since this will be the case generalized to the nc setting. The fan Σ is generated by $n+1$ one dimensional cones $\{e_1, e_2, \ldots, e_{n+1}\}$ (each e_i for $i \neq n+1$ has components δ_{ki} in $\mathbb{Z}^n$, $k = 1, \ldots, n$ and $e_{n+1} = -\sum_{i=1}^n e_i$). Thus S is the ring of polynomials in $n+1$ variables $\mathbb{C}[x_1, \ldots, x_{n+1}]$; the group G is described by $\mu_1 = \mu_2 = \cdots = \mu_{n+1} = \mu$, hence $G \cong \mathbb{C}^*$, and the grading in S agrees with the usual polynomial grading. Consider now the maximal cones $\sigma_i \in \Sigma$ labelled by the missing generator, i.e. σ_i is generated by $\{e_k\}$ with $k \neq i$; they describe the usual affine open sets $U_i = U[\sigma_i] \subset \mathbb{CP}^n$ where the ith homogeneous coordinate is not zero, so that $U_i \cong \mathbb{C}^n$ and the isomorphism of Thm(1.4.6) is just $U_i \cong (S_{[x_i]})_{(0)} \simeq \mathbb{C}[x_1/x_i, \ldots, x_{n+1}/x_i]$.

We want to have a similar description and isomorphism for nc toric varieties X_θ; the fan Σ and the number of 1-dimensional cones agree with the ones of X, so the homogeneous coordinate ring is still a polynomial ring in $\Sigma(1)$ variables x_i, but now in general with nc relations $x_r x_s = Q_{rs} x_s x_r$ where $Q_{rs} = \exp\{i\,\Theta^{rs}\}$ and Θ is a skew-symmetric $\Sigma(1) \times \Sigma(1)$ complex matrix induced from the matrix θ deforming the torus $(\mathbb{C}^*)^n$ and hence the whole space X. On the other hand we know how to construct nc affine open sets $U_\theta[\sigma]$, so we have to check if the isomorphism of Thm(1.4.6) still holds in this nc deformation. In order to state the desired result, we first recall some basic notions of localization theory for noncommutative rings.

Given a unital commutative ring R such that $ab = 0$ implies that either a or b are zero (i.e. R is an integral domain), one usually localizes with respect to a subset $S \subset R$ closed under multiplication; for nc rings the existence of the localization is guaranteed for example by an Ore condition on the set S.

Definition 1.4.7 *Given a unital noncommutative ring R a left denominator set $S \subset R$ is a subset of R such that $\forall a, b \in R$ and $\forall s, t \in S$ the following conditions hold:*

1. *$st \in S$ (S is closed by multiplication)*
2. *$Sa \cap Rs \neq \{0\}$ (S is left permutable; known also as left Ore condition)*

3. if $as = 0$ there exists $u \in S$ such that $ua = 0$ (S is left invertible)

The last condition is automatically satisfied if R is a domain. A completely analogous definition holds for the right case. Given a left denominator set S one defines the localization ring $S^{-1}R$ as the equivalence classes in $S \times R$ by

$$(s_1, r_1) \sim (s_2, r_2) \Longleftrightarrow \exists\, t \in S \text{ s.t. } (s_1 r_2 - s_2 r_1)t = 0$$

As usual one thinks of the equivalence class $[(s, r)]$ as the 'fraction' $s^{-1}r$ and defines a ring structure on these equivalence classes. For the addition, the Ore condition applied to s_1 and s_2 means there are $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_1 = \tilde{r}s_2$; we can thus define

$$s_1^{-1}r_1 + s_2^{-1}r_2 := (\tilde{s}s_1)^{-1}(\tilde{s}r_1 + \tilde{r}r_2) \tag{1.67}$$

It is not difficult to prove that the definition does not depend on the representatives of the equivalence classes. For the multiplication, we use the Ore condition on s_2 and r_1 to introduce elements $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{r}s_2 = \tilde{s}r_1$; we then define

$$(s_1^{-1}r_1) \cdot (s_2^{-1}r_2) := (\tilde{s}s_1)^{-1}(\tilde{r}r_2) \tag{1.68}$$

and again this does not depend on the representatives.

We can now state the homogeneous coordinate ring isomorphism for nc projective spaces $\mathbb{CP}^n_\theta$. The notations are the one introduced above when describing the commutative case: the fan Σ is generated by vectors $\{e_i\}$ with $i = 1, \dots n+1$ with $e_{n+1} = -\sum_{i=1}^n e_i$, the $n+1$ maximal cones σ_i are labelled by the missing generator and to each of them is associated a nc affine open $U_\theta[\sigma_i] \cong \mathbb{C}^n_{\theta'}$ where $(\theta')^{ab} = (m^a)_i \theta^{ij} (m^b)_j$, denoting with m^a $(a = 1, \dots, n)$ the generators of the dual cone $\check{\sigma}_i$. The nc homogeneous coordinate ring is $S_\Theta = \mathbb{C}^{n+1}_\Theta[x_i, \dots, x_{n+1}]$ with the ordinary polynomial degree and relations $x_r x_s = Q_{rs} x_s x_r$ where $Q_{rs} = \exp\{i\,\Theta^{rs}\}$ and Θ is a $(n+1) \times (n+1)$ skew-symmetric complex matrix, obtained from θ just by adding a last column and row made of zeroes, i.e. $\Theta^{ab} = \theta^{ab}$ for $a, b = 1, \dots, n$ and $\Theta^{(n+1)k} = 0$. We show that this choice for Θ reproduces Thm(1.4.6) in the nc case.

Theorem 1.4.8 *Consider the noncommutative projective spaces $\mathbb{CP}^{n+1}_\theta$ and their noncommutative homogeneous coordinate rings S_Θ; for each maximal cone σ_i there is a ring homomorphism between the (ring associated to the) open affine set $U_\theta[\sigma_i]$ and the degree zero part of the (left) Ore localization of S_Θ with respect the monomial $x^{\hat{\sigma_i}} = x_i$, i.e. $\mathbb{C}^n_{\theta'}[z_1, \dots, z_n] \cong \big((S_\Theta)_{[x_i]}\big)_{(0)}$.*

<u>Proof</u>: We want to prove an isomorphism between two different nc polynomial rings, having the same number of generators n and with nc relations coming from respectively θ' and Θ. We consider first $\mathbb{C}^n_{\theta'}[z_1, \dots, z_n]$; the n generators of the dual cone $\check{\sigma}_i$ are $m^k = e_k - e_i$ for $k = 1, \dots n$, $k \neq i$ and $m^i = -e_i$, so that denoting with $q'_{ab} = \exp\{i(\theta')^{ab}\}$ (with q_{ab} the similar coefficients defined with θ) and recalling how

θ' is derived from θ we have relations $z_a z_b = q'_{ab} z_b z_a = q_{ab} q_{ia} q_{bi} z_b z_a$ for $a, b \neq i$ and $z_a z_i = q_{ia} z_i z_a$.

Now take $S_\Theta = \mathbb{C}^{n+1}_\Theta[x_1, \ldots, x_{n+1}]$; it is easy to verify that each monomial x_i generates a left denominator set in S_Θ so we can consider the left Ore localization $(S_\Theta)_{[x_i]}$ (see Def(1.4.7)). The degree zero part has generators $y_k = (x_i)^{-1} x_k$ for $k = 1, \ldots, n+1$ ($k \neq i$), and one can explicitly compute the nc relations between y's using the ones in S_Θ and the multiplication rule (1.68) in the localized ring, finding $y_a y_b = Q_{bi} Q_{ab} Q_{ia} y_b y_a$ for $a, b \neq n+1$ and $y_a y_{n+1} = Q_{ia} y_{n+1} y_a$ since $Q_{a(n+1)} = 1$ due to $\Theta^{a(n+1)} = 0$. The claimed isomorphism then is given by sending $z_a \mapsto y_a$ for $a \neq i$ and $z_i \mapsto y_{n+1}$. ■

We conclude this section by sketching a possible sheaf theory on nc toric varieties, following [Ing]. The idea is that the 'topology' of the nc space $X_\theta = X_\theta[\Sigma]$ is given by the cones in the fan Σ. The assignment $\sigma \mapsto \mathbb{C}_\theta[\sigma]$ of the nc algebra $U\theta[\sigma] = \mathbb{C}_\theta[\sigma]$ to every cone $\sigma \subset \Sigma$ (the toric open sets in the topology of X_θ) is viewed as the structure sheaf $\mathcal{O}_{X_\theta}$ of the nc toric variety X_θ. Besides their own interest, these ideas provide the necessary tools to study bundles and instantons (via deformed ADHM data) on $X_\theta[\Sigma]$; this part is still in progress [CLS].

We use the category $\mathsf{Open}(X_\theta)$ of toric open sets to define the category of sheaves on $X_\theta = X_\theta[\Sigma]$. We call a set of inclusions $(\sigma_i \hookrightarrow \sigma)_{i \in I}$ of cones a covering if $\sigma = \bigcup_{i \in I} \sigma_i$. Then $\mathsf{Open}(X_\theta)$ always contains a sufficiently fine open cover. The category $\mathsf{Open}(X_\theta)$ with the data of coverings forms a Grothendieck topology on X_θ.

Proposition 1.4.9 *The association $\sigma \mapsto \mathbb{C}_\theta[\sigma]$ defines a sheaf of rings $\mathcal{O}_{X_\theta}$ on $\mathsf{Open}(X_\theta)$.*

Proof: Let $(\sigma_i \hookrightarrow \sigma)_{i \in I}$ be a covering, i.e. $\sigma = \bigcup_{i \in I} \sigma_i$. Then $\mathbb{C}_\theta[\sigma] = \bigcap_{i \in I} \mathbb{C}_\theta[\sigma_i]$, where the intersection is well-defined since each ring $\mathbb{C}_\theta[\sigma_i]$ is contained in $\mathbb{C}_\theta[t_1^\pm, \ldots, t_n^\pm]$. Thus the sequence

$$0 \longrightarrow \mathbb{C}_\theta[\sigma] \longrightarrow \prod_{i \in I} \mathbb{C}_\theta[\sigma_i] \longrightarrow \prod_{i<j} \mathbb{C}_\theta[\sigma_i \cap \sigma_j]$$

is exact, and the result follows. ■

We now define $\mathsf{mod}(X_\theta)$ to be the category of sheaves of right $\mathcal{O}_{X_\theta}$-modules on $\mathsf{Open}(X_\theta)$. If Σ consists of a single cone σ, i.e. $X_\theta[\Sigma] = U_\theta[\sigma]$ is an affine variety, then

$$\mathsf{mod}(X_\theta) \cong \mathsf{mod}\big(\mathbb{C}_\theta[\sigma]\big)$$

We denote by $\widetilde{M}$ the sheaf associated to a module M. A sheaf of $\mathcal{O}_{X_\theta}$-modules is called quasi-coherent if its restriction to each affine open set $U[\sigma]$ is of the form $\widetilde{M}$

for some right $\mathbb{C}_\theta[\sigma]$-module M. It is called coherent if M is finitely-generated. Let $\mathsf{coh}(X_\theta)$ denote the category of quasi-coherent sheaves of $\mathcal{O}_{X_\theta}$-modules.

Given a cone σ in Σ, we write $\mathsf{coh}(\sigma)$ for the category of $\mathbb{C}_\theta[\sigma]$-modules. There are restriction functors

$$j_\sigma^\bullet \,:\, \mathsf{coh}(X_\theta) \ \longrightarrow\ \mathsf{coh}(\sigma)$$

Let $\mathsf{tor}(\sigma)$ be the full Serre subcategory[6] in $\mathscr{C}$ of $\mathsf{coh}(X_\theta)$ generated by objects E such that $j_\sigma^\bullet(E) = 0$. In [Ing] the following fundamental result is proven.

Proposition 1.4.10 *Let σ be a cone in Σ. Then the restriction functor $j_\sigma^\bullet : \mathsf{coh}(X_\theta) \to \mathsf{coh}(\sigma)$ is exact, and there is a natural equivalence of categories*

$$\mathsf{coh}(X_\theta) \big/ \mathsf{tor}(\sigma) \ \cong\ \mathsf{coh}(\sigma)$$

Each cone σ in the fan Σ gives a toric open set of $X_\theta[\Sigma]$. We will use Proposition 1.4.10 to reduce geometric problems in the category $\mathsf{coh}(X_\theta)$ to algebraic problems in the ring $\mathbb{C}_\theta[\sigma]$ via these localization functors. This gives an explicit description of the quotient category. The objects of $\mathsf{coh}(\sigma)$ are the same as those of $\mathsf{coh}(X_\theta)$. The morphisms are given by

$$\mathrm{Hom}_{\mathsf{coh}(\sigma)}(M,N) = \varinjlim_{M'}\ \mathrm{Hom}_{\mathsf{coh}(X_\theta)}(M',N)\ ,$$

where the limit is taken over all submodules $M' \subset M$ with $j_\sigma^\bullet(M/M'\,) = 0$.

For any pair of sheaves $E, F \in \mathsf{coh}(X_\theta)$, let $\mathrm{Ext}^p(E,F)$ be the p-th derived functor of the Hom-functor $\mathrm{Hom}(E,F) = \mathrm{Hom}_{\mathsf{coh}(X_\theta)}(E,F)$. For a sheaf $E \in \mathsf{coh}(X_\theta)$, we define

$$H^p(X_\theta, E) := \mathrm{Ext}^p(\mathcal{O}_{X_\theta}, E)$$

[6] A full subcategory is a subcategory $\mathscr{S} \subset \mathscr{C}$ that contains all the morphism between its objects (i.e. $\mathrm{Hom}_{\mathscr{S}}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$ for each pair of objects X, Y in $\mathscr{S}$). A Serre full subcategory $\mathscr{S}$ is a full subcategory of an abelian category $\mathcal{C}$ such that for each short exact sequence $0 \to M' \to M \to M'' \to 0$ the object M is in $\mathscr{S}$ if and only if both M' and M'' are in $\mathscr{S}$.

Chapter 2

Models for noncommutative equivariant cohomology

The subject of this second chapter will be to introduce algebraic models for the equivariant cohomology of noncommutative spaces acted by deformed symmetries. We will do it by following the classical ideas behind the definition of Weil and Cartan models, showing how to 'adjust' all the ingredients to the deformed case.

Classically, equivariant cohomology is a useful tool for studying actions of Lie groups G (or, infinitesimally, Lie algebras $\mathfrak{g}$) on manifolds $\mathcal{M}$. In some sense, it replaces the cohomology of the orbit space when the latter is not well defined, due to the fact that the group action in general does not need to be free.

It may be defined via topological models (the Borel construction) or via equivalent algebraic models (Weil and Cartan constructions) which are then our starting point in view of a deformed algebraic picture. The main algebraic notion used to pass from the Borel to the Weil model is the $\tilde{\mathfrak{g}}$-da structure we introduced in Def(1.1.2); each $\tilde{\mathfrak{g}}$-da admits a universal locally free object, depending on the category of algebras we are considering (i.e. commutative or not), called Weil algebra. Deformations of symmetries, at least in our framework, give deformed $\tilde{\mathfrak{g}}$-da structures and thus deformed universal objects, which we will call deformed Weil algebras.

In the first section we will review some background material on equivariant cohomology; as usual we will present it in a way it will be possible to adapt the classical notions to the deformed setting.

In the second section we will discuss the role played by Weil algebras and how they depend on the class of algebras and symmetries we consider; we will review as a first generalization the noncommutative Weil algebras introduced in [AM00][AM05].

In the third section we will come to the class of deformations we are interested in and their associated Weil algebras; we finally use these constructions to define noncommutative equivariant cohomology.

In the fourth section after few basic examples we will focus on a fundamental property of equivariant cohomology, i.e. its reduction to the maximal torus. We will discuss the existence of an analogue result for our noncommutative models and its deep effects on the general theory.

Also in this chapter when it does not create ambiguities we will use the abbreviation 'nc' for noncommutative.

2.1 Classical models

Let G be a compat Lie group acting on a topological space X. The action is said to be free if at each poit $x \in X$ the stabilizer

$$G_x = \{g \in G \text{ s.t. } g \cdot x = x\}$$

reduces to the identity group, i.e. there are no fixed points. Relaxing a bit the definition, locally free actions have discrete stabilizers. When the action is free the quotient space X/G is usually as nice as the space X itself; for this reason one can take its cohomology and compare with the one of X. We want that equivariant cohomology $H_G(X)$ for free actions agrees with the cohomology of the orbit space, $H_G(X) = H(X/G)$. In the class of smooth manifolds for $H(X/G)$ we can take the De Rham cohomology of the quotient manifold X/G.

So one looks for a definition of equivariant cohomology $H_G(X)$ which is well defined for general actions, but that reduces to $H(X/G)$ for free actions. We expect $H_G(X)$ to satisfy usual properties of every other cohomology theory, such as functoriality and homotopy invariance. Then the idea is to deform X into a homotopically equivalent space X' where the action is now free, and define $H_G(X) = H(X'/G)$. A possible way is to consider a contractible space E on which G acts freely, so that we can put $X' = X \times E$; of course at the end we have to prove that the definition does not depend on the choice of E.

Note that if G acts freely on X as well, the previous construction involving X' define a fibration on X/G with typical fiber E

$$(X \times E)/G \rightarrow X/G$$

and the cohomology of the total space $H((X \times E)/G)$ by aciclicity of the fiber is equal to the cohomology of the base $H(X/G)$. So at least for this class of examples we quickly showed the independence of $H_G(X)$ by E.

A natural choice for a space E having the requested properties is the total space of the universal G bundle

$$G \hookrightarrow EG \rightarrow BG$$

In this case we will denote X' by

$$X_G = (X \times EG)/G \tag{2.1}$$

This leads to the following topological definition of equivariant cohomology, known as Borel model; the original reference is [Bor60].

Definition 2.1.1 *The topological equivariant cohomology of a topological space X acted by a compact Lie group G is defined as the ordinary cohomology of the space X_G introduced in* (2.1)*:*

$$H_G(X) := H(X_G) = H((X \times EG)/G) \tag{2.2}$$

where EG is the total space of the universal G-bundle.

The problem of this definition is that EG is finite dimensional only for G discrete, hence for compact Lie groups we should say how to compute the ordinary cohomology of an infinite dimensional manifold. A good recipe to overcome this problem is to find a finitely generated algebraic model for the algebra of differential forms over EG; this is where the Weil algebra comes into the play. We present here a 'constructive' definition; a more abstract interpretation is postponed to the next section.

Definition 2.1.2 *The (classical) Weil algebra associated to a Lie group G is the tensor product (over the scalar field)*

$$W_{\mathfrak{g}} = Sym(\mathfrak{g}^*) \otimes \bigwedge(\mathfrak{g}^*) \tag{2.3}$$

where $\mathfrak{g}$ is the Lie agebra of G and $\mathfrak{g}^*$ its dual.

We are interested in the $\tilde{\mathfrak{g}}$-da structure of $W_{\mathfrak{g}}$, i.e. the definition of operators (L, i, d) on $W_{\mathfrak{g}}$. There are two equivalent presentations of $W_{\mathfrak{g}}$, the first one more appropriate to describe the action of the differential and to prove the aciclicity of the Weil algebra, while the second one is more useful to study its basic subcomplex.

Definition 2.1.3 *The Koszul complex of a n-dimensional vector space V is the tensor product between the symmetric and the exterior algebra of V*

$$\mathcal{K}_V = Sym(V) \otimes \bigwedge(V)$$

We assign to each element of $\bigwedge(V)$ its exterior degree, and to each element in $Sym^k(V)$ degree $2k$. The Koszul differential $d_{\mathcal{K}}$ is defined on generators

$$d_{\mathcal{K}}(v \otimes 1) = 0 \qquad\qquad d_{\mathcal{K}}(1 \otimes v) = v \otimes 1 \tag{2.4}$$

and then extended as a derivation on the whole $\mathcal{K}(V)$.

A standard way to prove that $\mathcal{K}(V)$ is acyclic is the following.

Proposition 2.1.4 *The Koszul complex $(\mathcal{K}(V), d_{\mathcal{K}})$ is acyclic, i.e. its cohomology is given by the scalar field.*

Proof: Let Q be the odd derivation of $\mathcal{K}$ defined on generators by

$$Q(1 \otimes v) = 0 \qquad\qquad Q(v \otimes 1) = 1 \otimes v$$

The (graded) commutator $[Q, d_{\mathcal{K}}]$ is an even derivation which on $Sym^k(V) \otimes \bigwedge^q(V)$ is given by $(k+q)id$, so Q is a homotopy for the complex. ■

The Weil algebra $W_{\mathfrak{g}}$ is the Koszul complex for $\mathfrak{g}^*$, with Weil differential d_W defined to be equal to the Koszul differential $d_{\mathcal{K}}$.

Definition 2.1.5 *Let $\{e_a\}$ be a basis for $\mathfrak{g}$. The set of Koszul generators of $W_{\mathfrak{g}}$ is given by*

$$e^a = e^a \otimes 1 \qquad\qquad \vartheta^a = 1 \otimes e^a \tag{2.5}$$

We then have $d_W(e^a) = 0$ and $d_W(\vartheta^a) = e^a$.

We describe the action of $\bar{\mathfrak{g}}$ on these generators, i.e. we define actions of Lie and interior derivatives along $\mathfrak{g}$ using the coadjoint action.

Definition 2.1.6 *The Lie derivative L_a is defined to be the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$, so on Koszul generators it is*

$$L_a(e^b) = -f_{ac}{}^b e^c \qquad\qquad L_a(\vartheta^b) = -f_{ac}{}^b \vartheta^c \tag{2.6}$$

The interior derivative i_a (compatible with $[i_a, d_W] = L_a$) is given by

$$i_a(e^b) = -f_{ac}{}^b \vartheta^c \qquad\qquad i_a(\vartheta^b) = \delta_a^b \tag{2.7}$$

They are extended by (graded) Leibniz rule on the whole Weil algebra. Note that L is of degree zero, i of degree -1 and the usual commutation relations among (L, i, d) are satisfied.

A different set of generators for $W_{\mathfrak{g}}$ is obtained by using horizontal (i.e. annihilated by interior derivatives) even elements.

Definition 2.1.7 *The set of horizontal generators for $W_{\mathfrak{g}}$ is $\{u^a, \vartheta^a\}$ where*

$$u^a := e^a + \frac{1}{2} f_{bc}{}^a \vartheta^b \vartheta^c \tag{2.8}$$

With basic computations one can find the action of (L, i, d) on horizontal generators; the new expressions are

$$\begin{array}{llll} L_a(u^b) & = -f_{ac}{}^b u^c & \qquad i_a(u^b) & = 0 \\ d_W(u^a) & = -f_{bc}{}^a \vartheta^b u^c & \qquad d_W(\vartheta^a) & = u^a - \frac{1}{2} f_{bc}{}^a \vartheta^b \vartheta^c \end{array} \tag{2.9}$$

so that even generators are killed by interior derivative, hence the name horizontal.

Given a commutative $\tilde{\mathfrak{g}}$-da $\mathcal{A}$ the tensor product $W_{\mathfrak{g}} \otimes \mathcal{A}$ is again a $\tilde{\mathfrak{g}}$-da with $L^{(tot)} = L \otimes 1 + 1 \otimes L$ and the same rule for i and d; this comes from the tensor structure of the category of $\mathfrak{U}(\tilde{\mathfrak{g}})$-module algebras and the fact that $\tilde{\mathfrak{g}}$ has primitive coproduct. The basic subcomplex of a $\tilde{\mathfrak{g}}$-da is the intersection between invariant and horizontal elements. We have now all the ingredients to define the Weil model for equivariant cohomology.

Definition 2.1.8 *The Weil model for the equivariant cohomology of a commutative $\tilde{\mathfrak{g}}$-da $\mathcal{A}$ is the cohomology of the basic subcomplex of $W_{\mathfrak{g}} \otimes \mathcal{A}$:*

$$H_G(\mathcal{A}) = \big((W_{\mathfrak{g}} \otimes \mathcal{A})^G_{hor},\ \delta = d_W \otimes 1 + 1 \otimes d\big) \tag{2.10}$$

The Weil model is the algebraic analogue of the Borel model with $\mathcal{A} = \Omega(X)$, $W_{\mathfrak{g}}$ playing the role of differential forms on EG and the basic subcomplex representing differential forms on the quotient space for free actions. A rigorous proof that topological and algebraic definitions are equivalent, a result known as the 'Equivariant de Rham Theorem', may be found for example in [GS99].

Another well known algebraic model for equivariant cohomology of $\tilde{\mathfrak{g}}$-da is the Cartan model; it defines equivariant cohomology in a 'de Rham' fashion as the cohomology of equivariant differential forms with respect to a 'completion' of the de Rham differential. We derive it as the image of an automorphism of the Weil complex $W_{\mathfrak{g}} \otimes \mathcal{A}$; the automorphism is usually referred as the Kalkman map [Kal93] and is defined as

$$\phi = \exp\,\{\vartheta^a \otimes i_a\} : W_{\mathfrak{g}} \otimes \mathcal{A} \longrightarrow W_{\mathfrak{g}} \otimes \mathcal{A} \tag{2.11}$$

The image via ϕ of the basic subcomplex of $W_{\mathfrak{g}} \otimes \mathcal{A}$, the relevant part for equivariant cohomology, is easily described.

Proposition 2.1.9 *The Kalkman map ϕ realizes an algebra isomorphism*

$$(W_{\mathfrak{g}} \otimes \mathcal{A})^G_{hor} \overset{\phi}{\simeq} (Sym(\mathfrak{g}^*) \otimes \mathcal{A})^G \tag{2.12}$$

Proof: The operator $\vartheta^a \otimes i_a$ is a nilpotent derivation, so its exponential is a finite sum with $\phi^{-1} = \exp\{-\vartheta^a \otimes i_a\}$; then ϕ is an automorphism. Let us show that it is equivariant: computing $\phi L^{tot} \phi^{-1} = \phi(L \otimes 1 + 1 \otimes L)\phi^{-1}$ by expanding the exponential, at the first order we have

$$[\vartheta^b \otimes i_b, L_a \otimes 1 + 1 \otimes L_a] = \vartheta^b \otimes (f_{ba}{}^c i_c) - L_a \vartheta^b \otimes i_b = -f_{ab}{}^c \vartheta^b \otimes i_c + f_{ac}{}^b \vartheta^c \otimes i_b = 0$$

and this show that L_a commutes with ϕ. Now the same calculation for i; at the first order we have

$$[\vartheta^b \otimes i_b, i_a \otimes 1 + 1 \otimes i_a] = -\,\delta^b_a \otimes i_b = -1 \otimes i_a$$

and the second order term vanishes

$$\frac{1}{2}\,[\vartheta^b \otimes i_b, -1 \otimes i_a] = -\vartheta^b \otimes [i_a, i_b] = 0$$

Summing the only nonzero contributions we obtain

$$\phi\, i_a\, \phi^{-1} = 1 \otimes i_a + i_a \otimes 1 - 1 \otimes i_a = i_a \otimes 1$$

Remembering that $(W_{\mathfrak{g}})_{hor} = Sym(\mathfrak{g}^*)$ (using horizontal generators in $W_{\mathfrak{g}}$) this completes the proof. ■

The algebra $(Sym(\mathfrak{g}^*) \otimes \mathcal{A})^G$ appearing in (2.12) will define the Cartan complex and is denoted by $C_G(\mathcal{A})$. The differential on $C_G(\mathcal{A})$ is induced from δ by the Kalkman map.

Proposition 2.1.10 *The Cartan differential $d_G = \phi\, \delta_{|bas}\phi^{-1}$ on $C_G(\mathcal{A})$ takes the form*

$$d_G = 1 \otimes d - u^a \otimes i_a \tag{2.13}$$

Proof: Note that $\phi_{|\,bas} = P_{hor} \otimes 1$ where P_{hor} is the projector of $W_{\mathfrak{g}}$ onto the horizontal subalgebra. Indeed on basic elements we have (we sum over latin indexes a but not over greek indexes α; indexes run from 1 to dim $\mathfrak{g}$)

$$\exp\{\vartheta^a \otimes i_a\} = \prod_{\alpha=1}(1 + \vartheta^\alpha \otimes i_\alpha) = \prod_\alpha (1 - \vartheta^\alpha i_\alpha \otimes 1) = \prod_\alpha (i_\alpha \vartheta^\alpha \otimes 1) = P_{hor} \otimes 1$$

We compute $(P_{hor} \otimes 1)(d_W \otimes 1 + 1 \otimes d)_{|bas}(P_{hor} \otimes 1)^{-1}$. The term $1 \otimes d$ commutes with $P_{hor} \otimes 1$; a better way to express the Weil differential is $d_W \otimes 1 = \vartheta^a L_a \otimes 1 + (u^a - \frac{1}{2} f_{bc}{}^a \vartheta^b \vartheta^c) i_a \otimes 1$; now all the terms involving ϑ's are killed by $P_{hor} \otimes 1$, the surviving $v^a i_a \otimes 1$ on the basic complex is equal to $-v^a \otimes i_a$ and which now commutes with $P_{hor} \otimes 1$. ■

We make now a remark on the relation between Weil, Cartan and BRST differentials [Kal93]. Denote by M_W the differential algebra $W_G \otimes \mathcal{A}$ with $\delta = d_W \otimes 1 + 1 \otimes d$; it is possible to define another differential on the same algebra, the BRST operator

$$\delta_{BRST} = \delta + \vartheta^a L_a^{\mathcal{A}} - u^a i_a^{\mathcal{A}} \tag{2.14}$$

We call M_{BRST} the differential algebra $(W_{\mathfrak{g}} \otimes \mathcal{A}, \delta^{BRST})$; for the physical interpretation of M_{BRST} see [Kal93]. The Kalkman map is a $\tilde{\mathfrak{g}}$-da isomorphism from M_W to M_{BRST}, i.e. it intertwines the two differential structures. When restricted to $(W_M)_{|bas}$ its image is the Cartan model, now seen as the G-invariant subcomplex of the BRST model M_{BRST}; then also the Cartan differential d_G is nothing but the restriction to the invariant subcomplex of the BRST differential δ_{BRST}.

Definition 2.1.11 *The Cartan model for the equivariant cohomology of a commutative $\tilde{\mathfrak{g}}$-da $\mathcal{A}$ is the cohomology of the Cartan complex $C_G(\mathcal{A})$:*

$$H_G(\mathcal{A}) = \Big((Sym(\mathfrak{g}^*) \otimes \mathcal{A})^G,\, d_G = 1 \otimes d - u^a \otimes i_a \Big) \tag{2.15}$$

We end the section by noting that the equivariant cohomology ring $H_G(\mathcal{A})$ has a module structure over the ring of invariant polynomials $(Sym(\mathfrak{g}^*))^G$, called the basic cohomology ring, which is also the equivariant cohomology ring of a point.

Indeed any homomorphism of $\tilde{\mathfrak{g}}$-da induces by functoriality an homomorphism between the corresponding equivariant cohomologies; any $\tilde{\mathfrak{g}}$-da $\mathcal{A}$ has the natural homomorphism $\mathbb{C} \to \mathcal{A}$, which then induces a $H_G(\mathbb{C}) = (Sym(\mathfrak{g}^*))^G$ module structure on $H_G(\mathcal{A})$. The differential d_G commutes with this module structure.

2.2 The role of Weil algebras and their deformations

In the previous section, reviewing the classical construction of equivariant cohomology, we introduced the Weil algebra $W_{\mathfrak{g}}$ (Def(2.1.2)) as a finitely generated algebraic model for differential forms over EG; this led to a rewriting of the topological Borel model, which due to the infinite-dimensionality of EG is quite subtle to deal with, into the more tractable Weil model.

In the spirit of nc geometry an even more appropriate way to think of the Weil algebra $W_{\mathfrak{g}}$ is as the universal locally free $\tilde{\mathfrak{g}}$-da for the category of commutative $\tilde{\mathfrak{g}}$-da $\mathcal{A}$. Indeed generalizing the same philosophy to deformed categories of $\tilde{\mathfrak{g}}$-differential algebras we can try to find the right universal object representing the deformed $\tilde{\mathfrak{g}}$-da structure, relating by covariance the deformation of a $\tilde{\mathfrak{g}}$-da $\mathcal{A}$ to the deformation of the Weil algebra.

The first step is then to describe a universal locally free nc $\tilde{\mathfrak{g}}$-da, without any reference to the origin of the noncommutativity or to the existence of a preferred class of deformations. This nc Weil algebra $\mathcal{W}_{\mathfrak{g}}$ was introduced by Alekseev and Meinrenken in [AM00], and used by the authors to define and study what they called nc equivariant cohomology [AMW00][AM05].

We will review in this section their construction of $\mathcal{W}_{\mathfrak{g}}$, pointing out some of its properties which will allows us to consider further models when a specific type of deformation leading to a nc $\tilde{\mathfrak{g}}$-da is chosen. The most important fact, somehow not explicitly exploited in the original work [AM00], will be to realize $\mathcal{W}_{\mathfrak{g}}$ as a (super) enveloping algebra, hence opening the door to different kind of deformations taken from the quantum enveloping algebras world.

Let us start again with the classical Weil algebra. We can describe $W_{\mathfrak{g}}$ in a more abstract way; a standard result (see for example [GS99]) is the following universal property.

Theorem 2.2.1 *The classical Weil algebra $W_{\mathfrak{g}}$ is the universal commutative locally free $\tilde{\mathfrak{g}}$-da.*

We recall that a $\tilde{\mathfrak{g}}$-da $\mathcal{A}$ is said to be locally free if it admits an algebraic connection, i.e. a linear map $\varpi : \mathfrak{g}^* \rightarrow \mathcal{A}$ satisfying

$$i_X(\varpi(\mu)) = \mu(X) \qquad L_X(\varpi(\mu)) = -\varpi(L_X\mu) \qquad \forall\, X \in \mathfrak{g},\, \mu \in \mathfrak{g}^* \tag{2.16}$$

The universality of $W_{\mathfrak{g}}$ means that for every commutative locally free $\tilde{\mathfrak{g}}$-da $\mathcal{A}$ there exists a unique $\tilde{\mathfrak{g}}$-da homomorphism $c^{\varpi} : W_{\mathfrak{g}} \rightarrow \mathcal{A}$ such that the following diagram commutes:

$$\begin{array}{ccc} W_{\mathfrak{g}} & \xrightarrow{c^{\varpi}} & \mathcal{A} \\ \uparrow & \nearrow_{\varpi} & \\ \mathfrak{g}^* & & \end{array} \tag{2.17}$$

An equivalent way to describe the algebraic connection is through the super vector space $E_{\mathfrak{g}^*} = (\mathfrak{g}^*)^{(ev)} \oplus (\mathfrak{g}^*)^{(odd)}$ equipped with a Koszul differential space structure (see Def(2.1.3) or [AM05] for more details). An algebraic connection on $\mathcal{A}$ is a $\tilde{\mathfrak{g}}$-differential space ($\tilde{\mathfrak{g}}$-ds for short) homomorphism

$$c : E_{\mathfrak{g}^*} \oplus \mathbb{C}\mathfrak{c} \rightarrow \mathcal{A} \tag{2.18}$$

where $\mathfrak{c}$ is an even generator, which maps to the unity of $\mathcal{A}$. Since $\mathcal{A}$ is commutative, the homomorphism (2.18) can be lifted to the (super) symmetric algebra $Sym(E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c})$; the Weil algebra W_G turns out to be the quotient

$$W_G = Sym(E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c})/\langle \mathfrak{c} - 1 \rangle \simeq Sym(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*) \tag{2.19}$$

These definitions and properties can be generalized by moving to the category of nc $\tilde{\mathfrak{g}}$-differential algebras, as Alekseev and Meinrenken did in [AM05].

Definition 2.2.2 *The noncommutative Weil algebra $\mathcal{W}_{\mathfrak{g}}$ is defined as the universal noncommutative locally free $\tilde{\mathfrak{g}}$-differential algebra.*

We still describe the algebraic connection as in (2.18), but note that now the homomorphism can no longer be lifted to the supersymmetric algebra but only to the tensor algebra, since $\mathcal{A}$ is nc. So now the analogue of (2.19) is

$$\mathcal{W}_{\mathfrak{g}} = \mathcal{T}(E_{\mathfrak{g}^*} \oplus \mathbb{C}\mathfrak{c})/ < \mathfrak{c} - 1 > \tag{2.20}$$

and in [AM05] it is shown that this expression satisfies the requirements of Def(2.2.2).

2.2.1 Noncommutative Weil model

If one makes an additional hypothesis, the nc Weil algebra $\mathcal{W}_\mathfrak{g}$ takes an even easier expression. We demand that $\mathfrak{g}$ is a quadratic Lie algebra, i.e. a Lie algebra carrying a nondegenerate *ad*-invariant quadratic form B which can be used to canonically identify $\mathfrak{g}$ with $\mathfrak{g}^*$. The most natural examples of quadratic Lie algebras are given by semisimple Lie algebras, taking the Killing forms as B; since already in the previous chapter we decided to restrict our attention to semisimple Lie algebras $\mathfrak{g}$ in order to have more explicit expressions for the Drinfeld twists, this additional hypothesis fits well in our setting and we shall use it from now on.

So let $(\mathfrak{g}, B)$ a quadratic Lie algebra. We construct a super Lie algebra $\bar{\mathfrak{g}}^B$ out of it.

Definition 2.2.3 *Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra. Fix a basis $\{e_a\}$ for $\mathfrak{g}$ and let ${f_{ab}}^c$ be the structure constants on this basis. The super Lie algebra $\bar{\mathfrak{g}}^B$ is defined as the super vector space $\mathfrak{g}^{(ev)} \oplus \mathfrak{g}^{(odd)} \oplus \mathbb{C}\mathfrak{c}$, with basis given by even elements $\{e_a, \mathfrak{c}\}$ and odd ones $\{\xi_a\}$, and brackets given by*

$$\begin{array}{llllll} [e_a, e_b] & = {f_{ab}}^c & [e_a, \xi_b] & = {f_{ab}}^c \xi_c & [\xi_a, \xi_b] & = B_{ab}\mathfrak{c} \\ [e_a, \mathfrak{c}] & = 0 & [\xi_a, \mathfrak{c}] & = 0 & & \end{array} \tag{2.21}$$

Identifying $\mathfrak{g} \simeq \mathfrak{g}^*$ (using B) we then have a (super) Lie algebra structure also on $E_{\mathfrak{g}^*} \oplus \mathbb{C}\mathfrak{c}$; looking back at the homomorphism (2.18), we note that now it is a Lie algebra homomorphism, where on $\mathcal{A}$ we use the commutator with respect to the noncommutative product, and so it lifts to the (super) enveloping algebra $\mathfrak{U}(\bar{\mathfrak{g}}^B)$.

We have just proved the following easy but nevertheless crucial result.

Proposition 2.2.4 *For quadratic Lie algebras $(\mathfrak{g}, B)$ the noncommutative Weil algebra $\mathcal{W}_\mathfrak{g}$ may be written as*

$$\mathcal{W}_\mathfrak{g} = \mathfrak{U}(\bar{\mathfrak{g}}^B)/\langle \mathfrak{c} - 1 \rangle \simeq \mathfrak{U}(\mathfrak{g}) \otimes Cl(\mathfrak{g}, B). \tag{2.22}$$

From now on we shall consider $\mathcal{W}_\mathfrak{g}$ as a super enveloping algebra; formally we are working in $\mathfrak{U}(\bar{\mathfrak{g}}^B)$ assuming implicitly every time $\mathfrak{c} = 1$. A remark is in order: the decomposition of $\mathcal{W}_\mathfrak{g}$ in the even part $\mathfrak{U}(\mathfrak{g})$ and an odd part $Cl(\mathfrak{g}, B)$ is by the time being only true as vector space isomorphism; to become an algebra isomorphism we have to pass to even generators which commute with odd ones: this will be done below.

Note finally that if the algebra $\mathcal{A}$ is commutative, then its Lie algebra structure is trivial, (2.18) lifts again to the symmetric algebra $Sym(\bar{\mathfrak{g}}^B)$ and we come back to the classical Weil algebra $W_\mathfrak{g}$.

We are interested in the $\tilde{\mathfrak{g}}$-da structure of the nc Weil algebra. The main difference with the classical Weil algebra is that the action of (L, i, d) on $\mathcal{W}_\mathfrak{g}$ may be realized by inner derivations.

Definition 2.2.5 *On a generic element $X \in \mathcal{W}_{\mathfrak{g}}$ the actions of L and i are given by*

$$L_a(X) := ad_{e_a}(X) \qquad\qquad i_a(X) := ad_{\xi_a}(X) \tag{2.23}$$

On generators one has

$$\begin{array}{ll} L_a(e_b) = [e_a, e_b] = f_{ab}{}^c e_c & i_a(e_b) = [\xi_a, e_b] = f_{ab}{}^c \xi_c \\ L_a(\xi_b) = [e_a, \xi_b] = f_{ab}{}^c \xi_c & i_a(\xi_b) = [\xi_a, \xi_b] = B_{ab}\mathfrak{c} \end{array} \tag{2.24}$$

Then L_a and i_a are derivations, and their action agrees with the commutator in the enveloping algebra. We wish to stress an easy fact, often at the origin of some confusion. The commutator for a fixed element in an enveloping algebra $\mathfrak{U}(\mathfrak{g})$ (i.e. with respect to the associative product of $\mathfrak{U}(\mathfrak{g})$) is always a derivation, thanks to the Jacobi identity of the Lie algebra structure of $(\mathfrak{U}(\mathfrak{g}), [\ ,\ ])$. On the contrary the adjoint action ad_Y for $Y \in \mathfrak{U}(\mathfrak{g})$ acts on products of element in $\mathfrak{U}(\mathfrak{g})$ as

$$ad_Y(X_1 X_2) = (ad_{Y_{(1)}} X_1)(ad_{Y_{(2)}} X_2) \qquad\qquad Y,\ X_1,\ X_2\ \in \mathfrak{U}(\mathfrak{g})$$

and so it is a derivation if and only if Y is primitive, i.e. $Y \in \mathfrak{g}$. When this happens one has $ad_Y(X) = [Y, X]$, while in general the two structures are different.

Definition 2.2.6 *The differential $d_{\mathcal{W}}$ on the noncommutative Weil algebra $\mathcal{W}_{\mathfrak{g}}$ is the Koszul differential $d_{\mathcal{W}}(e_a) = 0$, $d_{\mathcal{W}}(\xi_a) = e_a$, so that $(\mathcal{W}_{\mathfrak{g}}, d_{\mathcal{W}})$ is an acyclic differential algebra.*

Following the classical terminology (Def(2.1.3)) the set of generators $\{e_a, \xi_a\}$ of $\mathcal{W}_{\mathfrak{g}}$ will be called of the Koszul type. However it is often more convenient to use another set of generators, where the even ones are horizontal. This is obtained by the transformation

$$u_a := e_a + \frac{1}{2} f_a{}^{bc} \xi_b \xi_c \tag{2.25}$$

where we use B to raise and lower indexes. One can easily verify that $\{u_a, \xi_a\}$ is another set of generators for $\mathcal{W}_{\mathfrak{g}}$, with relations (compare with (2.21)):

$$[u_a, u_b] = f_{ab}{}^c u_c \qquad\qquad [u_a, \xi_b] = 0 \qquad\qquad [\xi_a, \xi_b] = B_{ab} \tag{2.26}$$

Note that u_a generators realize the same Lie algebra $\mathfrak{g}$ of $\{e_a\}$, but now decoupled from the odd part, so that using these generators we can write $\mathcal{W}_{\mathfrak{g}} \simeq \mathfrak{U}(\mathfrak{g}) \otimes Cl(\mathfrak{g}, B)$ as algebra isomorphism.

We skip the proof of the following elementary restatement of relations in Def(2.2.5).

Proposition 2.2.7 *The $\tilde{\mathfrak{g}}$-da structure, still given by adjoint action of generators $\{e_a, \xi_a\}$, now on $\{u_a, \xi_a\}$ reads:*

$$\begin{array}{rclrcl} L_a(u_b) & = & f_{ab}{}^c u_c & L_a(\xi_b) & = & f_{ab}^c \xi_c \\ i_a(u_b) & = & 0 & i_a(\xi_b) & = & B_{ab} \\ d_{\mathcal{W}}(u_a) & = & -f_a{}^{bc} \xi_b u_c & d_{\mathcal{W}}(\xi_a) & = & u_a - \frac{1}{2} f_a{}^{bc} \xi_b \xi_c \end{array} \tag{2.27}$$

The operator $d_{\mathcal{W}}$ may be expressed as an inner derivation as well: indeed it is given by the commutator with an element $\mathcal{D} \in (\mathcal{W}_{\mathfrak{g}}^{(3)})^G$. There are several ways (depending on the choice of generators used) one can write $\mathcal{D}$, and the simplest one for our calculations is

$$\mathcal{D} = \frac{1}{3}\xi^a e_a + \frac{2}{3}\xi^a u_a \tag{2.28}$$

For a generic element $X \in \mathcal{W}_{\mathfrak{g}}$ we can then write $d_{\mathcal{W}}(X) = [\mathcal{D}, X]$.

As pointed out in [AM00], this element $\mathcal{D}$ may be viewed as a Dirac operator; it is also related to Kostant's cubic Dirac operator [Kos99]. Recently it also appeared in [NT], where Dirac operators on quantum groups are discussed. Our construction of twisted nc Weil algebras may provide a natural framework where to look for these quantum Dirac operators.

Notice that $\mathcal{W}_{\mathfrak{g}}$ is a filtered differential algebra, with associated graded differential algebra the classical Weil algebra $W_{\mathfrak{g}}$; the $\tilde{\mathfrak{g}}$-da structure of $\mathcal{W}_{\mathfrak{g}}$ agrees with the classical one if we pass to $Gr(\mathcal{W}_{\mathfrak{g}})$.

Once a $\tilde{\mathfrak{g}}$-da structure has been defined on $\mathcal{W}_{\mathfrak{g}}$, we can consider its horizontal subalgebra and basic subcomplex; given any other $\tilde{\mathfrak{g}}$-da $\mathcal{A}$ the tensor product $\mathcal{W}_{\mathfrak{g}} \otimes \mathcal{A}$ gets a natural $\tilde{\mathfrak{g}}$-da structure by

$$L^{(tot)} = L\otimes 1+1\otimes L \qquad i^{(tot)} = i\otimes 1+1\otimes \qquad \delta^{(tot)} = d_{\mathcal{W}}\otimes 1+1\otimes d \tag{2.29}$$

which comes from the general $L_X^{(tot)} = L_{X_{(1)}} \otimes L_{X_{(2)}}$ when X is in $\mathfrak{g}$ and hence it has primitive coproduct. Following the classical construction, i.e. defining equivariant cohomology as the cohomology of the basic subcomplex of $W_{\mathfrak{g}} \otimes \mathcal{A}$, a nc Weil model is now defined.

Definition 2.2.8 *[AM00] The Weil model for the equivariant cohomology of a noncommutative $\tilde{\mathfrak{g}}$-differential algebra $\mathcal{A}$ is the cohomology of the complex*

$$\mathcal{H}_G(\mathcal{A}) = \big((\mathcal{W}_{\mathfrak{g}} \otimes \mathcal{A})^G_{(hor)},\, \delta^{(tot)} = d_{\mathcal{W}} \otimes 1 + 1 \otimes d\big) \tag{2.30}$$

We stress that this model does apply to nc algebras, but the request is that $\mathfrak{U}(\tilde{\mathfrak{g}})$ is represented on $\mathcal{A}$ by derivations satisfying a classical undeformed Leibniz rule. So we cannot use it for nc spaces where Lie derivative and interior derivatives are deformed derivations, like the example of toric isospectral deformations. We will show in the next section how to modify the previous construction in order to obtain models which apply to these class of nc spaces (or actually algebras).

2.2.2 Noncommutative Cartan model

There is a noncommutative analogue of the Kalkman map, leading to a nc Cartan model. The map (2.11) expressed using generators of $\mathcal{W}_{\mathfrak{g}}$ now reads

$$\Phi = \exp\left\{\xi^a \otimes i_a\right\} : \mathcal{W}_{\mathfrak{g}} \otimes \mathcal{A} \longrightarrow \mathcal{W}_{\mathfrak{g}} \otimes \mathcal{A} \tag{2.31}$$

By a proof which is completely similar to the classical case, in [AM00] it is shown how Φ intertwines the action of $L^{(tot)}$ and $i^{(tot)}$, leading to the following result.

Proposition 2.2.9 *The noncommutative Kalkman map Φ defines a vector space isomorphism*

$$(\mathcal{W}_{\mathfrak{g}} \otimes \mathcal{A})^G_{hor} \overset{\Phi}{\simeq} (\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{A})^G \tag{2.32}$$

Proof: Since $\xi^a \otimes i_a$ is nilpotent, as a vector space map Φ is given by a finite sum and it is clearly invertible with $\Phi^{-1} = \exp\{-\xi^a \otimes i_a\}$. The equivariance follows from

$$[\xi^a \otimes i_a, L_b \otimes 1 + 1 \otimes L_b] = -(L_b \xi^a) \otimes i_a + \xi^a \otimes [i_a, L_b] = -f_b{}^{ac} \xi_c \otimes i_a + f_a{}^{bc} \xi^a \otimes i_c = 0$$

The commutator with $i^{(tot)}$ is computed order by order as well. The first one reads

$$[\xi^a \otimes i_a, i_b \otimes 1 + 1 \otimes i_b] = [\xi^a, i_b] \otimes i_a + \xi^a \otimes [i_a, i_b] = -\delta^a_b \otimes i_b$$

so that $Ad_\Phi i^{(tot)}_a = i_a \otimes 1$. ■

Note that as in the classical case the restriction of the Kalkman map to the basic subcomplex agrees with the horizontal projector

$$\Phi_{|bas} = P_{hor} \otimes 1 = \prod_\beta i_\beta \xi^\beta$$

The main difference between the classical and the nc Kalkman map is that $\xi^a \otimes i_a$ is no longer a derivation; for this reason Φ is not an algebra homomorphism, and the natural algebra structure on $(\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{A})^G$ does not agree with the one induced by Φ. Before looking at the algebra structure of the image of the Kalkman map we describe the induced differential.

Proposition 2.2.10 *[AM00] The noncommutative Cartan differential d_G induced from $\delta^{(tot)} = d_{\mathcal{W}} \otimes 1 + 1 \otimes d$ by the Kalkman map Φ via $d_G = \Phi(d_{\mathcal{W}} \otimes 1 + 1 \otimes d)_{|bas} \Phi^{-1}$ takes the following expression*

$$d_G = 1 \otimes d - \frac{1}{2}(u^a_{(L)} + u^a_{(R)}) \otimes i_a + \frac{1}{24} f^{abc} (1 \otimes i_a i_b i_c) \tag{2.33}$$

where with $u^a_{(L)}$ (resp. $u^a_{(R)}$) we denote the left (resp. right) multiplication for u^a. In particular d_G commutes with L and squares to zero on $(\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{A})^G$.

Proof: The result could be proved by explicit computation of $\Phi(d_{\mathcal{W}} \otimes 1 + 1 \otimes d)\Phi^{-1}$ expanding the Kalkman map at the various order. A shorter way is to use the symbol map $\sigma : Cl(\mathfrak{g}, B) \rightarrow \wedge(\mathfrak{g})$ to identify $\mathcal{W}_{\mathfrak{g}} \simeq \mathfrak{U}(\mathfrak{g}) \otimes \wedge(\mathfrak{g})$ and, denoting by y_a the odd generators of the exterior algebra, express the Weil differential $d_{\mathcal{W}}$ as [AM00](Prop(3.7))

$$d_{\mathcal{W}} = y^a L_a \otimes 1 + \left(\frac{u^a_{(L)} + u^a_{(R)}}{2} - \frac{1}{2} f^{abc} y_b y_c \right) i_a - \frac{1}{24} f^{abc} i_a i_b i_c$$

Using $\Phi_{|bas} = P_{hor} \otimes 1$ the Cartan differential is defined by

$$d_G \circ (P_{hor} \otimes 1) = (P_{hor} \otimes 1) \circ (d_{\mathcal{W}} \otimes 1 + 1 \otimes d)$$

The differential $1 \otimes d$ commutes with $P_{hor} \otimes 1$, while acting on $d_{\mathcal{W}}$ the projector P_{hor} kills all the terms involving y_a's. Since we are applying these operators to the basic subcomplex, we may replace $i_a \otimes 1$ by $-1 \otimes i_a$, and the latter commutes with $P_{hor} \otimes 1$. The equivariance of d_G and the fact that on $(\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{A})^G$ it squares to zero follow from the equivariance of δ and Φ and the fact that $\delta^2 = 0$. ■

We denote the complex $((\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{A})^G, d_G)$ by $\mathcal{C}_G(\mathcal{A})$. Its ring structure is induced by the Kalkman map; by definition on $u_i \otimes a_i \in (\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{A})^G$ we have

$$(u_1 \otimes a_1) \odot (u_2 \otimes a_2) := \Phi\left(\Phi^{-1}(u_1 \otimes a_1) \cdot_{\mathcal{W}_{\mathfrak{g}} \otimes \mathcal{A}} \Phi^{-1}(u_2 \otimes a_2)\right) \tag{2.34}$$

Proposition 2.2.11 *[AM00] The ring structure of* $\mathcal{C}_G(\mathcal{A})$ *defined in* (2.34) *takes the explicit form*

$$(u_1 \otimes a_1) \odot (u_2 \otimes a_2) = (u_1 u_2) \otimes \cdot_{\mathcal{A}} \left(exp\ \{B^{rs} i_r \otimes i_s\}(a_1 \otimes a_2)\right) \tag{2.35}$$

Note that d_G *is a derivation of* $\odot$.

<u>Proof</u>: It is once more time useful to use the symbol map $\sigma : Cl(\mathfrak{g}, B) \rightarrow \wedge(\mathfrak{g})$ and represent $\mathcal{W}_{\mathfrak{g}}$ as $\mathfrak{U}(\mathfrak{g}) \otimes \wedge(\mathfrak{g})$. The multiplicative structure of $\wedge(\mathfrak{g})$ induced by $Cl(\mathfrak{g}, B)$ involves the extra operator $\exp\{B^{rs} i_r \otimes i_s\}$ to be followed by wedge product [AM00](Lemma 3.1). Since i_a acts as zero on $\mathfrak{U}(\mathfrak{g})$ and bringing this extra operator on the $\mathcal{A}$-side of $\mathcal{C}_G(\mathcal{A})$, eventually paying a sign since we are on the basic subcomplex, we obtain an expression which now commutes with $P_{hor} \otimes 1$. The fact that d_G is a derivation of $\odot$ follows directly from the definitions of d_G and $\odot$, both induced by Φ, since δ is a derivation of the algebra structure of $\mathcal{W}_{\mathfrak{g}} \otimes \mathcal{A}$. ■

Definition 2.2.12 *The Cartan model for the equivariant cohomology of a noncommutative* $\tilde{\mathfrak{g}}$*-differential algebra* $\mathcal{A}$ *is the cohomology of the complex* $(\mathcal{C}_G(\mathcal{A}), d_G)$*:*

$$\mathcal{H}_G(\mathcal{A}) = \left((\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{A})^G,\ d_G = 1 \otimes d - \frac{1}{2}(u^a_{(L)} + u^a_{(R)}) \otimes i_a + \frac{1}{24} f^{abc} \otimes i_a i_b i_c \right) \tag{2.36}$$

The ring structure $\odot$ *of* $\mathcal{C}_G(\mathcal{A})$ *is given in* (2.35).

Note that for abelian groups the Cartan model reduces to the classical one; in the non abelian case this ring structure is not compatible with a possible pre-existing grading on $\mathcal{A}$, since for $a_1 \in \mathcal{A}^r$ and $a_2 \in \mathcal{A}^s$ the product $(1 \otimes a_1) \odot (1 \otimes a_2)$ does not belong to $\mathcal{A}^{r+s}$ due to the extra terms involving interior derivatives. The only structure left in $\mathcal{C}_G(\mathcal{A})$ is a double filtration; its associated graded differential module is a double graded differential model and agrees with the classical Cartan model.

2.2.3 A general strategy for algebraic models

We propose in this section a general approach towards a definition of algebraic models for nc cohomology which makes a crucial use of (deformed or generalized) Weil algebras.

The class of nc geometries where the deformation of the classical space comes from the action of some deformed symmetry is our natural setting. Of this kind are for example the geometries described in the previous chapter, such as toric isospectral deformations or nc toric varieties, all of them collected in the unique class of Drinfeld twist deformations. Another family of deformations comes from Drinfeld-Jimbo quantum enveloping algebras and covariant homogeneous spaces for q-deformed symmetries. In this second approach a lot of differences with the Drinfeld twist case occurs, and the same will happens considering further deformations. But one thing is preserved: the idea that the deformation of the symmetries and the deformation of the spaces are related.

This 'covariance' principle has a specific formulation depending on the class of deformations considered; for example for Drinfeld twists we expressed it in Thm(1.2.17), but in general we may say that to maintain a link between spaces and symmetries the right strategy is do not leave the category of Hopf-module algebras.

We showed so far that a classical space acted by some Lie group is described by an algebra (the algebra of differential forms) in the category of $\mathfrak{U}(\tilde{\mathfrak{g}})$-module algebras; whatever the noncommutative deformation, this will mean that we move into the category of quantum enveloping algebras (here in a broadly sense, to be more precise we should know which deformation occurred in particular) and their module algebras.

Let us call this category $\tilde{\mathfrak{g}}$-deformed differential algebras. We described in this section the abstract definition of Weil algebra as the universal locally free $\tilde{\mathfrak{g}}$-differential algebra. It should be clear why in every category of $\tilde{\mathfrak{g}}$-deformed differential algebras we suggest to call the locally free [1] universal object the 'deformed' (or generalized) Weil algebra; for quadratic Lie algebras a good candidate to hold this role is the (super) quantum enveloping algebra of $\bar{\mathfrak{g}}^B$.

Once a deformed Weil algebra is considered, the construction of a Weil model and the definition of equivariant cohomology follows without problems. We will describe this strategy applied to Drinfeld twists deformations in the next two section; we remand a similar approach on q-deformed symmetries for future works.

2.3 Models for Drinfeld twist deformations

In this section we consider the class of deformations given by Drinfeld twists. We show how it is possible to adapt the construction of Alekseev and Meinrenken [AM00] to

[1] A generalized notion of locally free algebra is needed, if the original one is meaningless in the deformed category considered. See for example the discussion in the next section for the Drinfeld twist case.

construct algebraic models for the equivariant cohomology of nc algebras $\mathcal{A}_\chi$ deformed by a Drinfeld twist.

We will deform the nc Weil algebra $\mathcal{W}_\mathfrak{g}$ with the same χ; we stress that this strategy uses the realization of $\mathcal{W}_\mathfrak{g}$ as an enveloping algebra, and this description is possible when $\mathfrak{g}$ is a quadratic Lie algebra. The definition of Weil and Cartan models will follow as usual from the cohomology of the appropriate basic subcomplexes.

We will point out the various analogies with the nc models of Alekseev and Meinrenken, as well as the differences coming from the Drinfeld twist, which will mainly concern deformed algebraic structure in both models and in the cohomology rings.

The exposition follows quite closely [Cir]. However we present here some relevant improvements (in particular we make explicit the braided nature of the category of twisted $\tilde{\mathfrak{g}}$-da, crucial for the Weil model) and we discuss some new material; this contents will appear soon in a revised version.

2.3.1 Twisted noncommutative Weil model

So now we consider, instead of a generic nc $\tilde{\mathfrak{g}}$-da $\mathcal{A}$, an algebra $\mathcal{A}_\chi$ where noncommutativity is realized via a Drinfel twist as in Thm(1.2.17); we know then that its $\tilde{\mathfrak{g}}$-differential structure is given by twisted derivations like the ones defined in Prop(1.3.10) and Prop(1.3.11) for $\mathcal{A}_\chi = \Omega(\mathcal{M}_\theta)$. In what follows everything is done having in mind this example, but the construction really applies to a generic twisted nc algebra.

Dealing with this kind of algebras, which we will call twisted $\tilde{\mathfrak{g}}$-da's, it seems natural to look for a Weil algebra with a twisted $\tilde{\mathfrak{g}}$-da structure as well, due to its universal role in the category of Hopf-module algebras. This goal can be reached starting from the nc Weil algebra $\mathcal{W}_\mathfrak{g}$, since we know how to twist an enveloping algebra, and how the twist modifies the adjoint action.

A last remark before starting: we stress again the fact that the construction of twisted nc Weil algebras applies to every quadratic Lie algebra (for which $\mathcal{W}_\mathfrak{g}$ can be realized as a super enveloping algebra), and it makes sense also in the case an explicit form of the twist element χ is unknown. All we need is the existence of the twist, and in this we can get help from the above mentioned rigidity and uniqueness theorems of Drinfeld and others; so it is a quite general procedure. The same holds true for the construction of the equivariant cohomology models. Obviously if one needs to deal with explicit expressions and computations, like the ones presented here, an explicit form of χ is crucial; in what follows we will continue to use the expression of χ given in (1.28).

Definition 2.3.1 *Let $\mathfrak{g}$ be a quadratic Lie algebra, and $\mathcal{A}_\chi$ a noncommutative algebra associated to a Drinfeld twist χ. The twisted noncommutative Weil algebra $\mathcal{W}_\mathfrak{g}^{(\chi)}$ is defined as the Drinfeld twist of $\mathcal{W}_\mathfrak{g}$ by the same χ, now viewed as an element in $(\mathcal{W}_\mathfrak{g} \otimes \mathcal{W}_\mathfrak{g})^{(ev)}_{[[\theta]]}$.*

We have to say that in general the generators of the twist χ need not to belong to $\mathfrak{g}$. The relevant Lie algebra in this case is the product between $\mathfrak{t}^p$, the generators of χ, and $\mathfrak{g}$, the symmetry whose action is relevant for equivariant cohomology. If we call this algebra $\mathfrak{g}'$, we are actually twisting $\mathfrak{U}(\tilde{\mathfrak{g}'})_{[[\theta]]}$. Of course the interesting case is when $\mathfrak{t}$ and $\mathfrak{g}$ have non trivial commutation relations, otherwise the twist is trivial. In what follows we will directly assume that $\mathfrak{g}$ contains the generators of the twist.

We then repeat the construction of the previous section to describe the $\tilde{\mathfrak{g}}$-da structure of the Weil algebra now in the twisted case $\mathfrak{U}^\chi(\tilde{\mathfrak{g}})$. According to the notation introduced above we start with even and odd generators $\{e_i, e_r, \xi_i, \xi_r\}$ distinguishing between Cartan (index i) and root (index r) elements of $\tilde{\mathfrak{g}}$. We already computed the twisted coproduct of the even subalgebra (see Prop(1.2.20)) and of odd generators ξ_a (see Prop(1.3.9)); we repeat here for convenience the results:

$$\begin{aligned} \triangle^\chi(e_i) &= e_i \otimes 1 + 1 \otimes e_i & \qquad \triangle^\chi(e_r) &= e_r \otimes \lambda_r^{-1} + \lambda_r \otimes e_r \\ \triangle^\chi(\xi_i) &= \xi_i \otimes 1 + 1 \otimes \xi_i & \qquad \triangle^\chi(\xi_r) &= \xi_r \otimes \lambda_r^{-1} + \lambda_r \otimes \xi_r \end{aligned}$$

Recall also that, as showed in Prop(1.2.21), for this class of Drinfeld twist elements χ (in particular with commuting generators H such that $S(H) = -H$ and saturated with a skewsymmetric matrix θ) the antipode is undeformed.

The $\tilde{\mathfrak{g}}$-da structure of the nc Weil algebra $\mathcal{W}_\mathfrak{g}$ has been realized by the adjoint action with respect to even generators (for the Lie derivative), odd generators (for the interior derivative) and by commutation with a fixed element in the center (for the differential). We use the same approach for $\mathcal{W}_\mathfrak{g}^{(\chi)}$, the only difference is that now the Weil algebra is a twisted enveloping algebra; the deformed coproduct of the generators and the definition of adjoint action on super Hopf algebra

$$ad_Y(X) = \sum (-1)^{|X||(Y)_{(2)}|}(Y)_{(1)} X (S(Y)_{(2)}) \tag{2.37}$$

motivate the following definition.

Definition 2.3.2 *The action of L and i on $\mathcal{W}_\mathfrak{g}^{(\chi)} = \mathfrak{U}^\chi(\tilde{\mathfrak{g}})$ is given by the adjoint action with respect to even and odd generators. In particular $L_i = ad_{e_i}$ and $i_i = ad_{\xi_i}$ are the same as in the untwisted case. On the contrary for roots elements the operators L_r and i_r are modified on a single generator as well:*

$$\begin{aligned} L_r(X) &= ad^\chi_{e_r}(X) = e_r X \lambda_r - \lambda_r X e_r \\ i_r(X) &= ad^\chi_{\xi_r}(X) = \xi_r X \lambda_r + (-1)^{|X|} \lambda_r X \xi_r \end{aligned} \tag{2.38}$$

Expressing explicitly this action on $\{e_a, \xi_a\}$ we have (one should compare with (2.24)):

$$\begin{aligned} L_j(e_a) &= f_{ja}{}^b e_b & \qquad L_j(\xi_a) &= f_{ja}{}^b \xi_b \\ L_r(e_i) &= e_r e_i \lambda_r - \lambda_r e_i e_r & \qquad L_r(\xi_i) &= e_r \xi_i \lambda_r - \lambda_r \xi_i e_r \\ &= -r_i \lambda_r e_r & &= -r_i \lambda_r \xi_r \\ L_r(e_s) &= e_r e_s \lambda_r - \lambda_r e_s e_r & \qquad L_r(\xi_s) &= e_r \xi_s \lambda_r - \lambda_r \xi_s e_r \end{aligned} \tag{2.39}$$

$$\begin{array}{rlrl} i_j(e_a) = & f_{ja}{}^b \xi_b & i_j(\xi_a) = & B_{ja} = \delta_{ja} \\ i_r(e_i) = & \xi_r e_i \lambda_r - \lambda_r e_i \xi_r & i_r(\xi_i) = & \xi_r \xi_i \lambda_r + \lambda_r \xi_i \xi_r \\ = & -r_i \lambda_r \xi_r & = & \lambda_r B_{ri} = 0 \\ i_r(e_s) = & \xi_r e_s \lambda_r - \lambda_r e_s \xi_r & i_r(\xi_s) = & \xi_r \xi_s \lambda_r + \lambda_r \xi_s \xi_r \end{array} \tag{2.40}$$

where we use i, j for Cartan indexes, r, s for roots indexes and a, b for generic indexes. On products one just applies the usual rule for the adjoint action

$$ad_Y(X_1 X_2) = (ad_{Y_{(1)}} X_1)(ad_{Y_{(2)}} X_2) \tag{2.41}$$

which shows that L_r and i_r are twisted derivations.

Due to the presence of the λ_r terms coming from twisted coproducts, the classical set of generators $\{e_a, \xi_a\}$ is no more closed under the action of L, i. This is not a big problem, but there is however another set of generators (which we will call quantum generators for their relation to quantum Lie algebras, see below) which seems to be more natural when dealing with the twist.

Definition 2.3.3 *We take as new set of generators of* $\mathcal{W}_{\mathfrak{g}}^{(\chi)}$ *the elements*

$$X_a := \lambda_a e_a \qquad \eta_a := \lambda_a \xi_a \tag{2.42}$$

Recall from (1.31) *that for* $a = i$ *we have* $\lambda_i = 1$, *so* $X_i = e_i$. *We define also coefficients*

$$q_{rs} := exp\ \{\frac{i}{2}\theta^{kl} r_k s_l\} \tag{2.43}$$

with properties $q_{sr} = q_{rs}^{-1}$ *and* $q_{rs} = 1$ *if* $r = -s$; *we also set* $q_{ab} = 1$ *if at least one index is of Cartan type (due to the vanishing of the correspondent root vector).*

The following relations, easily proved by direct computation, will be very useful:

$$\begin{array}{rlrl} \lambda_r \lambda_s & = \lambda_{r+s} & \lambda_r \lambda_s & = \lambda_s \lambda_r \\ \lambda_r e_s & = q_{rs} e_s \lambda_r & \lambda_r \xi_s & = q_{rs} \xi_s \lambda_r \\ L_{\lambda_r} e_s & = q_{rs} e_s & L_{\lambda_r} \xi_s & = q_{rs} \xi_r \end{array} \tag{2.44}$$

and since all λ_r's commute with each other, the same equalities hold for X_r and η_r.

Using the definition of the adjoint action, the previous relations (2.44) and the commutation rules between $\{e_a, \xi_a\}$ in $\mathcal{W}_{\mathfrak{g}}^{(\chi)}$ we can by straightforward computations express the twisted $\tilde{\mathfrak{g}}$-da structure on quantum generators.

Proposition 2.3.4 *The action of* L *and* i *on quantum generators* $\{X_a, \eta_a\}$ *of* $\mathcal{W}_{\mathfrak{g}}^{(\chi)}$ *is*

$$\begin{array}{rlrl} L_a X_b & = f_{ab}{}^c X_c & i_a X_b & = f_{ab}{}^c \eta_c \\ L_a \eta_b & = f_{ab}{}^c \eta_c & i_a \eta_b & = B_{ab} \end{array} \tag{2.45}$$

Note that this is exactly the same action we have in the classical case (2.24). The difference however is that we act on quantum generators with classical generators: $L_a X_b = ad_{e_a} X_b \neq ad_{X_a} X_b$.

We make a quick digression on the meaning of quantum generators and their link with quantum Lie algebras, even if this is not directly related to the construction of equivariant cohomology.

The fact that the base of the Lie algebra $\{e_a, \xi_a\}$ is not closed under the (twisted) adjoint action is a typical feature of quantized enveloping algebras $\mathfrak{U}_q(\mathfrak{g})$, where the deformation (say of the Drinfeld-Jimbo type) involves also the Lie algebra structure of $\mathfrak{g}$ (while with the Drinfeld twist we change only the Hopf-algebra structure on the enveloping algebra, but not the Lie bracket in $\mathfrak{g}$). Since $\mathfrak{g}$ can be viewed as the closed ad-submodule of $\mathfrak{U}(\mathfrak{g})$ where the adjoint action is given by Lie bracket, one can try to recover a Lie algebra inside $\mathfrak{U}_q(\mathfrak{g})$ by defining the quantum Lie algebra $\mathfrak{g}_q$ as a closed ad-submodule of $\mathfrak{U}_q(\mathfrak{g})$ with quantum Lie bracket given by the adjoint action of $\mathfrak{U}_q(\mathfrak{g})$. Linearity still holds, skew-symmetry becomes q-skew-symmetry and the Jacobi identity generalizes to a braided identity [DG97].

The deformation of the coproduct in $\mathfrak{U}^\chi(\mathfrak{g})$ leads to a deformation of the adjoint action, even if the brackets $[e_a, e_b]$ are unchanged; thus $ad_{e_r}(e_s)$ is no more equal to $[e_r, e_s]$. However $\{X_a\}$ are generators of a closed ad-submodule (see (2.44)), so we can define quantum Lie brackets $[\ ,\ ]_{(\chi)}$ using the twisted adjoint action, obtaining a quantum Lie algebra structure $\mathfrak{g}_\chi$:

$$\begin{aligned}
&[X_i, X_j]_{(\chi)} &&:= ad^\chi_{X_i} X_j = 0 \\
&[X_i, X_r]_{(\chi)} &&:= ad^\chi_{X_i} X_r = r_i X_r = -[X_r, X_i]_{(\chi)} \\
&[X_{-r}, X_r]_{(\chi)} &&:= ad^\chi_{X_{-r}} X_r = \sum r_i X_i = [X_r, X_{-r}]_{(\chi)} \\
&[X_r, X_s]_{(\chi)} &&:= ad^\chi_{X_r} X_s = q_{rs} f_{rs}{}^{r+s} X_{r+s} \\
&[X_s, X_r]_{(\chi)} &&:= ad^\chi_{X_s} X_r = q_{sr} f_{sr}{}^{r+s} X_{r+s} = -(q_{rs})^{-1} f_{rs}{}^{r+s} X_{r+s}
\end{aligned} \tag{2.46}$$

The q-antisymmetry is explicit only in the $[X_r, X_s]_{(\chi)}$ brackets since $q_{ab} \neq 1$ if and only if both indexes are root type. The same result holds also for the odd part of $\bar{\mathfrak{g}}$, so we may consider $\{X_a, \eta_a, \mathfrak{c}\}$ a base for the quantum (super) Lie algebra inside $\mathfrak{U}^\chi(\bar{\mathfrak{g}})$.

The last observation is that $\triangle_\chi X_r = X_r \otimes 1 + \lambda_r^2 \otimes X_r$, so if we want $\mathfrak{g}_\chi$ to be closed also under the coproduct, we may consider mixed generators $\{\Lambda_j, X_r\}$ where the Cartan-type generators are defined as group-like elements $\Lambda_j := \exp\{\frac{i}{2}\theta^{jl} H_l\}$. Now $\{\Lambda_j, X_r, \mathfrak{c}\}$ describe a different quantum Lie algebra $\mathfrak{g}'_\chi$, due to the presence of group-like elements; the structure of $\mathfrak{g}_\chi$ is recovered taking the first order terms in θ of the commutators involving Λ_j's (this is a standard procedure in quantum enveloping algebras).

Remembering eq. (2.25) and the terminology introduced there, also in the twisted nc Weil algebra it is useful to pass to horizontal generators.

Definition 2.3.5 *The quantum horizontal generators defined by*

$$K_a := \lambda_a u_a = \lambda_a(e_a + \frac{1}{2}f_a{}^{bc}\xi_b\xi_c) = X_a - \frac{1}{2}\eta^b ad_{X_b}(\eta_a) \tag{2.47}$$

are in the kernel of the twisted interior derivative

$$i_a K_b = ad^{\chi}_{\xi_a}(\lambda_b u_b) = \xi_a \lambda_b u_b \lambda_a - \lambda_a \lambda_b u_b \xi_a = 0 \tag{2.48}$$

and their transformation under L_a is given by

$$L_a K_b = ad^{\chi}_{e_a}(\lambda_b u_b) = e_a \lambda_b u_b \lambda_a - \lambda_a \lambda_b u_b e_a = f_{ab}{}^c K_c \tag{2.49}$$

The last thing to describe is the action of the differential $d_{\mathcal{W}}$. Recall that in $\mathcal{W}_{\mathfrak{g}}$ we had $d_{\mathcal{W}}(X) = [\mathcal{D}, X]$, and this is still true in $\mathcal{W}^{(\chi)}_{\mathfrak{g}}$. In fact

$$\mathcal{D} = \frac{1}{3}\xi^a e_a + \frac{2}{3}\xi^a u_a = \frac{1}{3}\eta^a X_a + \frac{2}{3}\eta^a K_a$$

Moreover $d_{\mathcal{W}}$ being a commutator, the Jacobi identity assures it is an untwisted derivation. This is not surprising: the twisted $\tilde{\mathfrak{g}}$-da structure of an algebra does not change the action of the differential. Note that $\eta^a = \lambda_a^{-1}\xi^a$ and $d_{\mathcal{W}}\lambda_a = [\mathcal{D}, \lambda_a] = 0$. For even generators we have

$$d_{\mathcal{W}}(K_a) = \lambda_a d_W(u_a) = -f_a{}^{bc}\lambda_a \xi_b u_c = -f_a{}^{bc}\lambda_b\lambda_c\xi_b u_c = -q_{ab}f_a{}^{bc}\eta_b K_c \tag{2.50}$$

where if we want to raise the index of η we have to take in account the λ inside η

$$-q_{ab}f_a{}^{bc}\eta_b K_c = -q_{ba}f_{ab}{}^c\eta^b K_c \tag{2.51}$$

and for odd generators

$$d_{\mathcal{W}}(\eta_a) = \lambda_a e_a = \lambda_a(u_a - \frac{1}{2}f_a{}^{bc}\xi_b\xi_c) = K_a - \frac{1}{2}q_{ba}f_{ab}{}^c\eta^b\eta_b \tag{2.52}$$

We have found all the relations which define a twisted $\tilde{\mathfrak{g}}$-da structure on $\mathcal{W}^{(\chi)}_{\mathfrak{g}}$. The difference with the untwisted case is that the elements $\{K_a, \eta_a\}$ generate the whole algebra $\mathfrak{U}^{\chi}(\tilde{\mathfrak{g}})$, but $L_a = ad^{\chi}_{e_a} \neq ad^{\chi}_{K_a}$.

Once we have the twisted $\tilde{\mathfrak{g}}$-da $\mathcal{W}^{(\chi)}_{\mathfrak{g}}$ we can define a Weil complex for the equivariant cohomology of any twisted $\tilde{\mathfrak{g}}$-da $\mathcal{A}_{\chi}$; nc differential forms $\Omega(\mathcal{M}_{\theta})$ provide a natural example on which the theory applies. This involves the tensor product between the two twisted $\tilde{\mathfrak{g}}$-da's $\mathcal{W}^{(\chi)}_{\mathfrak{g}}$ and $\mathcal{A}_{\chi}$. We already showed that this construction depends on the quasitriangular structure of $\mathfrak{U}^{\chi}(\mathfrak{g})$ (see Prop(1.3.5)).

The relevant Hopf algebra for the definition of a Weil model is the twisted enveloping algebra $\mathfrak{U}^{\chi}(\tilde{\mathfrak{g}})$, with deformed $\mathcal{R}$ matrix (see Thm(1.2.14))

$$\mathcal{R}^{\chi} = \chi_{21}\mathcal{R}\chi^{(-1)} \qquad \text{(with } \chi_{21} = \chi^{(2)} \otimes \chi^{(1)}\text{)}$$

Since the original $\mathcal{R}$ matrix of $\mathfrak{U}(\tilde{\mathfrak{g}})$ is trivial we have the simple expression

$$\mathcal{R}^{\chi} = \chi^{-2} = \exp\{i\theta^{kl} H_k \otimes H_l\} \tag{2.53}$$

Note by the way that the twisted Hopf algebra is however triangular, hence its module algebra category is symmetric (even if with non trivial braiding morphism).

Now we can construct, following the usual definition, the twisted nc Weil model; the relevant difference from the Weil model of [AM00] is that now the $\mathcal{W}_{\mathfrak{g}}^{(\chi)}$ and $\mathcal{A}_\chi$ are both in the braided monoidal category of $\mathfrak{U}^{\chi}(\tilde{\mathfrak{g}})$-module algebras.

Definition 2.3.6 *The Weil model for the twisted noncommutative equivariant cohomology of a twisted noncommutative $\tilde{\mathfrak{g}}$-da $\mathcal{A}_\chi$ is the cohomology of the complex*

$$\mathcal{H}_G^{\chi}(\mathcal{A}_\chi) = \big((\mathcal{W}_{\mathfrak{g}}^{(\chi)} \widehat{\otimes} \mathcal{A}_\chi)_{bas},\, \delta = d_{\mathcal{W}} \otimes 1 + 1 \otimes d \big) \tag{2.54}$$

The hat over the tensor product is put to remind the braided multiplicative structure, according to Prop(1.3.5).

The basic subcomplex is taken with respect to L^{tot} and i^{tot}; these operators act on $\mathcal{W}_{\mathfrak{g}}^{(\chi)} \widehat{\otimes} \mathcal{A}_\chi$ with the covariant rule $L_X^{tot} = L_{X_{(1)}} \otimes L_{X_{(2)}}$ using the twisted coproduct. Thus we have

$$L_{H_i}^{tot} = L_{H_1} \otimes 1 + 1 \otimes L_{H_i} \qquad L_{E_r}^{tot} = L_{E_r} \otimes \lambda_r^{-1} + \lambda_r \otimes L_{E_r}$$
$$i_{H_i}^{tot} = i_{H_i} \otimes 1 + 1 \otimes i_{H_i} \qquad i_{E_r}^{tot} = i_{E_r} \otimes \lambda_r^{-1} + \lambda_r \otimes i_{E_r}$$

We can use the G-invariance to explicitly compute the effect of the braiding on the multiplicative structure of the Weil model.

Proposition 2.3.7 *Let $\mathcal{A}$ be a graded commutative algebra, and $\mathcal{A}_\chi$ its deformation induced from a Drinfeld twist χ. The multiplication in the Weil complex, according to the general formula* (1.47)*, reads*

$$(u_1 \otimes \nu_1) \cdot (u_2 \otimes b_2) = (-1)^{|\nu_1||\nu_2|} u_1 u_2 \otimes \nu_2 \cdot_\chi \nu_1 \tag{2.55}$$

Proof: By direct computation, applying Lemma(1.3.7) to the left hand side and using G-invariance:

$$\sum_n \frac{(i\theta^{\alpha\beta})^n}{n!} u_1 (H_\beta^n u_2) \otimes (H_\alpha^n \nu_1) \cdot_\chi \nu_2 =$$
$$= \sum_n \frac{(-i\theta^{\alpha\beta})^n}{n!} u_1 u_2 \otimes (H_\alpha^n \nu_1) \cdot_\chi (H_\beta^n \nu_2) = u_1 u_2 \otimes \cdot (\chi^2 \chi^{-1} \triangleright \nu_1 \otimes \nu_2) =$$
$$= u_1 u_2 \otimes \cdot (\chi \triangleright \nu_1 \otimes \nu_2) = (-1)^{|\nu_1||\nu_2|} u_1 u_2 \otimes \nu_2 \cdot_\chi \nu_1 \qquad \blacksquare$$

We want to compare the twisted basic subcomplex $(\mathcal{W}_{\mathfrak{g}}^{(\chi)} \widehat{\otimes} \mathcal{A}_\chi)_{bas}$ with the one of [AM00]. According to the philosophy of Drinfeld twist deformations, namely to preserve the vector space structure and to deform only the algebra structure of $\tilde{\mathfrak{g}}$-da's, we find that they are isomorphic roughly speaking as 'vector spaces'; the precise statement, since we are comparing quantities depending on formal series in θ, involves topologically free $\mathbb{C}_{[[\theta]]}$ modules, or θ-adic vector spaces.

Proposition 2.3.8 *There is an isomorphism of (graded) topologically free* $\mathbb{C}_{[[\theta]]}$ *modules*

$$(\mathcal{W}_{\mathfrak{g}}^{(\chi)}\widehat{\otimes}\mathcal{A}_{\chi})_{bas} \simeq ((\mathcal{W}_{\mathfrak{g}} \otimes \mathcal{A})_{bas})_{[[\theta]]}$$

Proof: We first show the inclusion $((\mathcal{W}_{\mathfrak{g}} \otimes \mathcal{A})_{bas})_{[[\theta]]} \subset (\mathcal{W}_{\mathfrak{g}}^{(\chi)}\widehat{\otimes}\mathcal{A}_{\chi})_{bas}$. Consider

$$u \otimes \nu \in ((\mathcal{W}_{\mathfrak{g}} \otimes \mathcal{A})_{bas})_{[[\theta]]} \Rightarrow (L \otimes 1 + 1 \otimes L)(u \otimes \nu) = 0$$

The $\mathfrak{g}$ invariance property applied to powers of toric generators gives

$$H_{\alpha}^{n} u \otimes \nu = (-1)^{n} u \otimes H_{\alpha}^{n} \nu$$

and in particular

$$\lambda_r u \otimes \nu = u \otimes \lambda_r^{-1} \nu$$

This can be used to compute

$$(L_r \otimes \lambda_r^{-1} + \lambda_r \otimes L_r)(u \otimes \nu) = (L_r \lambda_r \otimes 1 - \lambda_r L_r \otimes 1)(u \otimes \nu) = ([L_r, \lambda_r] \otimes 1)(u \otimes \nu) = 0$$

A similar short calculation (just writing i_r instead of L_r) gives the analogous result for i_r as well; so we showed that $u \otimes \nu \in (\mathcal{W}_{\mathfrak{g}}^{(\chi)}\widehat{\otimes}\mathcal{A}_{\chi})_{bas}$. For the opposite inclusion, take now $v \otimes \eta \in (\mathcal{W}_{\mathfrak{g}}^{(\chi)}\widehat{\otimes}\mathcal{A}_{\chi})_{bas}$; this implies $(L_r \otimes \lambda_r^{-1} + \lambda_r \otimes L_r)(v \otimes \eta) = 0$ and in particular again $\lambda_r v \otimes \eta = v \otimes \lambda_r^{-1} \eta$. We use these two equalities to compute

$$L_r v \otimes \eta = L_r \lambda_r^{-1} v \otimes \lambda_r^{-1} \eta = -(1 \otimes L_r \lambda_r)(1 \otimes \lambda_r)(v \otimes \eta) = -v \otimes L_r \eta$$

Substituting again L_r with i_r we easily find the same result i_r, and this prove that $v \otimes \eta \in ((\mathcal{W}_{\mathfrak{g}} \otimes \mathcal{A})_{bas})_{[[\theta]]}$. The linearity of the operators with respect to formal series in θ and the compatibility of the eventual grading (coming from $\mathcal{A}$) with the $\mathbb{C}_{[[\theta]]}$-module structure complete the proof. ∎

The previous result easily generalizes to the associated equivariant cohomologies, since the differentials for both the complexes are the same.

Proposition 2.3.9 *There is an isomorphism of (graded) topologically free modules*

$$\mathcal{H}_{G}^{\chi}(\mathcal{A}_{\chi}) \simeq \mathcal{H}_{G}(\mathcal{A})_{[[\theta]]} \tag{2.56}$$

Proof: Since both $\mathcal{H}^{\chi}_{G}(\mathcal{A}_{\chi})$ and $(\mathcal{H}_{G}(\mathcal{A}))_{[[\theta]]}$ are defined starting from the respective basic subcomplexes with the same $\mathbb{C}_{[[\theta]]}$-linear differential $\delta = d_{\mathcal{W}} \otimes +1 \otimes d$ the isomorphism of Prop(2.3.8) lifts to the cohomologies. ■

Roughly speaking we are saying that our twisted equivariant cohomology is equal to the trivial formal series extension of the nc cohomology of Alekseev and Meinrenken, as 'vector space' over $\mathbb{C}_{[[\theta]]}$ (i.e. as topologically free $\mathbb{C}_{[[\theta]]}$-module). This is not extremely surprising, since we expect the deformation coming from the Drinfeld twist to be visible only at the ring structure level.

2.3.2 Twisted noncommutative Cartan model

We introduce a twisted nc Kalkman map to obtain a Cartan model for $\mathcal{H}^{\chi}_{G}(\mathcal{A}_{\chi})$; basically once more we need to twist the construction of Alekseev and Meinrenken [AM00]. Their nc Kalkman map $\Phi = \exp\{\xi^a \otimes i_a\}$ satisfies

$$\Phi L^{(tot)} \Phi^{-1} = L^{(tot)} \qquad\qquad \Phi i^{(tot)} \Phi^{-1} = i \otimes 1$$

with untwisted $L^{(tot)}$ and $i^{(tot)}$; we want a map Φ^{χ} which reproduces the same relations with twisted operators.

Definition 2.3.10 *The twisted noncommutative Kalkman map*

$$\Phi^{\chi} : \mathcal{W}^{(\chi)}_{\mathfrak{g}} \widehat{\otimes} \mathcal{A}_{\chi} \to \mathcal{W}^{(\chi)}_{\mathfrak{g}} \widehat{\otimes} \mathcal{A}_{\chi}$$

is the conjugation by the twist element χ of the noncommutative Kalkman map Φ

$$\Phi^{\chi} = \chi \Phi \chi^{-1} \qquad\qquad \textit{with } \Phi = exp\,\{\xi^a \otimes i_a\} \tag{2.57}$$

We prove now that Φ^{χ} gives the desired intertwining relations with twisted $L^{(tot)}$ and $i^{(tot)}$; this leads to the following isomorphism which maps $(\mathcal{W}^{(\chi)}_{\mathfrak{g}} \widehat{\otimes} \mathcal{A}_{\chi})_{bas}$ into what will define the Cartan model.

Proposition 2.3.11 *There is an isomorphism of $\mathbb{C}_{[[\theta]]}$-vector spaces (formally topological free $\mathbb{C}_{[[\theta]]}$-modules)*

$$(\mathcal{W}^{(\chi)}_{\mathfrak{g}} \widehat{\otimes} \mathcal{A}_{\chi})_{bas} \overset{\Phi^{\chi}}{\simeq} (\mathcal{W}^{(\chi)}_{\mathfrak{g}} \widehat{\otimes} \mathcal{A}_{\chi})^{G}_{i_a \otimes \lambda_a^{-1}} = (\mathcal{W}^{(\chi)}_{\mathfrak{g}} \widehat{\otimes} \mathcal{A}_{\chi})^{G}_{i_a \otimes 1} \tag{2.58}$$

Proof: First note that Φ^{χ} is invertible with $(\Phi^{\chi})^{-1} = \chi \Phi^{-1} \chi^{-1}$. We then prove equivariance; we see that the χ coming from the twisted coproduct cancels with the χ in Φ^{χ}:

$$\begin{aligned}\Phi^{\chi} L^{(tot)}_{r} (\Phi^{\chi})^{-1} &= (\chi \Phi \chi^{-1})(\chi \triangle(u_r) \chi^{-1})(\chi \Phi^{-1} \chi^{-1}) = \chi(\Phi \triangle(u_r) \Phi^{-1})\chi^{-1} = \\ &= \chi \triangle(u_r) \chi^{-1} = L^{(tot)}_{r}\end{aligned}$$

where we used the equivariance of Φ with respect to the untwisted $L^{(tot)}$. A similar computation for $i^{(tot)}$ gives

$$\Phi^\chi i_r^{(tot)}(\Phi^\chi){-}1 = (\chi\Phi\chi^{-1})(\chi\triangle(\xi_r)\chi^{-1})(\chi\Phi^{-1}\chi^{-1}) = \chi(\Phi\triangle(\xi_r)\Phi^{-1})\chi^{-1} = \\ = \chi(i_r\otimes 1)\chi^{-1} = i_r\otimes\lambda_r^{-1}$$

The last equality comes easily from the computation of $\chi(i_r\otimes 1)\chi^{-1}$ expanding χ at various orders in θ. Finally we get the right hand side of (2.58) using $\lambda_a\otimes\lambda_a = 1\otimes 1$ on basic elements. ■

In the untwisted setting since $(\mathcal{W}_{\mathfrak{g}})_{hor}\simeq\mathfrak{U}(\mathfrak{g})$ we arrive to the definition of the Cartan complex as $(\mathfrak{U}(\mathfrak{g})\otimes\mathcal{A})^G$. For the twisted case notice that

$$(\mathcal{W}_{\mathfrak{g}}^{(\chi)})_{hor} = \{K_a\} \neq \mathfrak{U}^\chi(\mathfrak{g})$$

that is the horizontal subalgebra of $\mathcal{W}_{\mathfrak{g}}^{(\chi)}$ is spanned by quantum horizontal generators K_a (see Def(2.3.5)) which do not describe any enveloping algebra. We will use the following notation to refer to the image of Φ^χ:

$$\mathcal{C}_G^\chi(\mathcal{A}_\chi) = (\mathcal{W}_{\mathfrak{g}}^{(\chi)}\widehat{\otimes}\mathcal{A}_\chi)^G_{i_a\otimes 1} = (\{K_a\}\otimes\mathcal{A}_\chi)^G \qquad (2.59)$$

Next we want to describe the induced differential and multiplicative structure on $\mathcal{C}_G^\chi(\mathcal{A}_\chi)$.

Definition 2.3.12 *The twisted noncommutative Cartan differential d_G^χ on $\mathcal{C}_G^\chi(\mathcal{A}_\chi)$ is defined as the differential induced by the Kalkman map Φ^χ as*

$$d_G^\chi = \Phi^\chi(d_{\mathcal{W}}\otimes 1 + 1\otimes d)(\Phi^\chi)^{-1} \qquad (2.60)$$

There is a large class of Drinfeld twists the Cartan differential is insensitive to. A sufficient condition for the equality $d_G^\chi = d_G$, as we are going to prove, is that χ acts as identity on $\mathcal{C}_G^\chi(\mathcal{A}_\chi)$; this in turn is true in particular for every χ depending antisymmetrically by commuting generators $H_i\in\mathfrak{g}$, as it easy to check. Note that this is exactly the class of Drinfeld twists relevant for isospectral deformations.

Proposition 2.3.13 *The differential d_G^χ is the twist of the noncommutative Cartan differential d_G (2.33), $d_G^\chi = \chi d_G\chi^{-1}$. In particular, when χ acts as the identity on $\mathcal{C}_G^\chi(\mathcal{A}_\chi)$ we have $d_G^\chi = d_G$.*

<u>Proof</u>: The first statement follows directly from (2.60), using $[d_G,\chi]=0$ on $\mathcal{C}_G^\chi(\mathcal{A}_\chi)$, which comes from $[d_G, L_a]=0$; the second part is evident. ■

Since so far we discussed Drinfeld twists elements of the type (1.28) which satisfies the above conditions, in the following we will use $d_G^\chi = d_G$.

The last thing to compute is the multiplicative structure induced in the Cartan complex $(\mathcal{C}_G^\chi(\mathcal{A}_\chi), d_G)$; this is determined by Φ^χ following (2.34). A nice expression is obtained under the following assumption, which is natural if we think of $\mathcal{A}$ as the algebra of differential forms.

Proposition 2.3.14 *Let assume $(\mathcal{A}, \cdot)$ is graded-commutative and let $(\mathcal{A}_\chi, \cdot_\chi)$ be its Drinfeld twist deformation. The multiplication in the Cartan complex $\mathcal{C}_G^\chi(\mathcal{A}_\chi)$ is given, for $u_i \otimes \nu_i \in \mathcal{C}_G^\chi(\mathcal{A}_\chi)$, by*

$$(u_1 \otimes \nu_1) \odot_\chi (u_2 \otimes \nu_2) = u_1 u_2 \otimes (-1)^{|\nu_1||\nu_2|} \cdot_\chi \left(exp\{\frac{1}{2} B^{ab} i_a \otimes i_b\}(\nu_2 \otimes \nu_1)\right) \quad (2.61)$$

Proof: Remember that in the twisted Weil model $(\mathcal{W}_\mathfrak{g}^{(\chi)} \otimes \mathcal{A}_\chi)_{bas}$ the spaces $\mathcal{W}_\mathfrak{g}^{(\chi)}$ and $\mathcal{W}_\mathfrak{g}$ have the same algebra structure; moreover we showed that the twisted basic subcomplex is isomorphic to the untwisted one (see Prop(2.3.8)). Hence we can use again (as in the proof of Prop(2.2.11)) the formula relating Clifford and wedge products in the odd part of $\mathcal{W}_\mathfrak{g}^{(\chi)}$ [AM00](Lemma 3.1)

$$\xi_1 \cdot_{Cl} \xi_2 = \wedge \left(\exp\{-\frac{1}{2} B^{ab} i_a \otimes i_b\}(\xi_1 \otimes \xi_2)\right)$$

However note that i_a is the untwisted interior derivative, as well as $\wedge$ is the undeformed product (the twist only deforms the Hopf algebra structure of $\mathcal{W}_\mathfrak{g}^{(\chi)}$, not its algebra structure). But thanks to Prop(2.3.8) we can nevertheless pass the exponential factor from $\mathcal{W}_\mathfrak{g}^{(\chi)}$ to $\mathcal{A}_\chi$ on the twisted basic complex as well, so that the remaining part of $\cdot_{\mathcal{W}_\mathfrak{g}^{(\chi)}}$ commutes with $(\Phi^\chi)_{bas} = (\Phi)_{bas} = P_{hor} \otimes 1$. The effect of the braiding on the multiplicative stricture of $(\mathcal{W}_\mathfrak{g}^{(\chi)} \otimes \mathcal{A}_\chi)_{bas}$ is reduced to (2.55), so for the moment we have on $u_i \otimes \nu_i \in (\mathfrak{U}^\chi(\mathfrak{g}) \otimes \mathcal{A}_\chi)^G$ the multiplication rule

$$(u_1 \otimes \nu_1) \odot_\chi (u_2 \otimes \nu_2) = u_1 u_2 \otimes (-1)^{|\nu_1||\nu_2|} \exp\{\frac{1}{2} B^{ab} i_a \otimes i_b\}(\nu_2 \cdot_\chi \nu_1)$$

In the previous formula the interior product in the exponential are untwisted, since they came from the undeformed Clifford product of the Weil algebra; however using $(\triangle \xi_a)\chi^{-1} = \chi^{-1}(\triangle^\chi \xi_a)$ to replace $\cdot_\chi$ by the exponential we get the claimed expression in (2.61) where now the i_a operators are the twisted derivations which act covariantly on $\mathcal{A}_\chi$. ■

Note that for $\mathcal{A}_\chi = \Omega(\mathcal{M}_\theta)$ the deformed product $\cdot_\chi$ is the deformed wedge product $\wedge_\theta$, hence the induced multiplication on the Cartan model acts like a deformed Clifford product on $\Omega(\mathcal{M}_\theta)$; moreover note that this deformed Clifford product has arguments ν_1 and ν_2 switched, as a consequence of the braided product in the Weil model. Indeed this switch is the only reminiscence of the braided structure of the Weil model, due to the non-cocommutative structure of the twisted Weil algebra (compare (2.34) with (2.61)); the other contribution from the Drinfeld twist is of course in the deformed product $\cdot_\chi$ of $\mathcal{A}_\chi$.

As in the untwisted case, this ring structure is not compatible with a possible grading in $\mathcal{A}$ and gives the twisted nc Cartan model a filtered double complex structure, opposed to the graded double complex structure of the classical Cartan model.

Finally, for $\theta \to 0$ we get back the product of the untwisted model (2.34).

Definition 2.3.15 *The Cartan model for the equivariant cohomology of a twisted noncommutative* $\tilde{\mathfrak{g}}$*-da* $\mathcal{A}_\chi$ *is the cohomology of the complex* $(\mathcal{C}^\chi_G(\mathcal{A}_\chi), d_G)$*:*

$$\mathcal{H}^\chi_G(\mathcal{A}_\chi) = \big((\{K_a\} \otimes \mathcal{A}_\chi)^G,\, d_G\big) \tag{2.62}$$

The explicit expression of d_G *is given in* (2.33)*; the ring structure* $\odot_\chi$ *of* $\mathcal{C}^\chi_G(\mathcal{A}_\chi)$ *in* (2.61)*.*

We presented here the definition of Cartan model for the class of twists and twisted algebras relevant for isospectral deformations; the general case might involve a deformed Cartan differential d^χ_G (see Prop(2.3.13)) and, if $\mathcal{A}$ is not graded commutative, the initial multiplicative structure in the Weil model is of the form (1.47) instead of the simplified (2.55).

2.4 More on twisted noncommutative equivariant cohomology

In the previous section we defined algebraic models (Weil and Cartan) for the equivariant cohomology of twisted $\tilde{\mathfrak{g}}$-differential algebras $\mathcal{A}_\chi$. The construction basically relies on an appropriate Drinfeld twist of the nc models of Alekseev and Meinrenken [AM00].

We already met several situations in which Drinfeld twists reveal their nature: usually they generate a sort of 'mild' deformation, a lot of classical results can be adapted to the deformed setting, and non trivial changes appear only looking at the algebra (for quantities acted) or bialgebra (for the symmetry acting) structures. Therefore we expect that some of the basic properties of classical, or even better noncommutative (in the sense of [AM00]) equivariant cohomology will still hold in the twisted case, or at least they will have a quite natural appropriate equivalent formulation.

The material presented in this section shows that indeed this is often true; we investigate the most natural and basic examples of equivariant cohomology, starting from the single point, going on with trivial actions and finally homogeneous spaces. We will find that much of what is true in the classical case can be correctly restated in the twisted models.

We also discuss a key property of equivariant cohomology: its reduction to the maximal torus $H_G(\mathcal{M}) \simeq (H_T(\mathcal{M}))^W$ where $T \subset G$ and W is the Weyl group of T. This equality says that basically we can always reduce everything to the study of equivariant cohomology with respect to abelian groups, eventually restricting our attention to some invariant part of it. The fact that Drinfeld twists do not affect abelian symmetries (or at least they do only in a very trivial way, see the discussion on the nc torus in Sec.(1.3.2)) should be a clear evidence that a similar reduction property becomes extremely powerful (and 'to be honest' a bit 'destructive') for twisted nc equivariant cohomology. We prove that, in the appropriate language, a similar reduction does exist, and make some comments on this.

2.4.1 Examples

The noncommutative equivariant cohomology of a point

Let us begin with a very special case, that is $\mathcal{A} = \mathbb{C}$; this is the algebraic equivalent of the equivariant cohomology of a point. Despite its simplicity, in equivariant cohomology the result is not completely trivial; indeed the ring which describes $\mathcal{H}^{\chi}_{G}(\mathbb{C})$ is called the 'basic cohomology ring' for twisted nc equivariant cohomology, in the sense that by functoriality every $\mathcal{H}^{\chi}_{G}(\mathcal{A}_{\chi})$ has a natural $\mathcal{H}^{\chi}_{G}(\mathbb{C})$-module structure coming from the algebra inclusion $\mathbb{C} \to \mathcal{A}_{\chi}$. To confirm its importance, let us for example recall that classically the torsion with respect to this module structure plays a crucial role in the localization theorems of Borel, Berline-Vergne and Atiyah-Bott.

Since every $\tilde{\mathfrak{g}}$-da module structure on $\mathcal{A} = \mathbb{C}$ is necessarily trivial, without much surprise the Drinfeld twist does not deform the algebra and we have $\mathbb{C}_{\chi} = \mathbb{C}$. Let us just apply the definition of the Weil model:

$$\mathcal{H}^{\chi}_{G}(\mathbb{C}) = H\left((\mathcal{W}^{(\chi)}_{\mathfrak{g}} \otimes \mathbb{C})_{bas}, d_{\mathcal{W}} \otimes 1\right) = H((\mathcal{W}^{(\chi)}_{\mathfrak{g}})_{bas}, d_{\mathcal{W}}) = (\mathcal{W}^{(\chi)}_{\mathfrak{g}})_{bas} \qquad (2.63)$$

The last equality is due to $(d_{\mathcal{W}})_{|bas} = 0$. So we found that the basic cohomology ring for twisted nc equivariant cohomology is the basic subcomplex of the twisted Weil algebra, $(\mathcal{W}^{(\chi)}_{\mathfrak{g}})_{bas}$. The next step is to get a more explicit expression of this ring, and to compare it with the basic rings of noncommutative and classic equivariant cohomology.

For the nc Weil algebra $\mathcal{W}_{\mathfrak{g}} = \mathfrak{U}(\bar{\mathfrak{g}}^B)$ the basic subcomplex consist of elements which commute with even generators (G-invariance) and odd generators as well (horizontality); in other words, it is the center on the super enveloping algebra $\mathfrak{U}(\bar{\mathfrak{g}}^B)$. Passing to horizontal generators we are left with G-invariant elements of $\mathfrak{U}(\mathfrak{g})$, or again the center; this ring is isomorphic, via Duflo map, to the ring of G-invariant polynomial over $\mathfrak{g}$. At the end we have $(\mathcal{W}_{\mathfrak{g}})_{bas} \simeq (\mathfrak{U}(\mathfrak{g}))^G \simeq Sym(\mathfrak{g})^G$, and the latter is the basic cohomology ring of classical equivariant cohomology.

In the twisted Weil algebra $\mathcal{W}^{(\chi)}_{\mathfrak{g}}$ the action of L and i is no more given by the commutator with even and odd generators of $\bar{\mathfrak{g}}$, but by twisted adjoint action, which is deformed even on single generators; so there is no evident reason why the basic subcomplex should agree with the center. The following shows nevertheless that is true.

Proposition 2.4.1 *The basic subcomplex of the twisted Weil algebra $\mathcal{W}^{(\chi)}_{\mathfrak{g}}$ is isomorphic as a ring to $(\mathcal{W}_{\mathfrak{g}})_{bas} \simeq \mathfrak{U}(\mathfrak{g})^G$.*

Proof: We prove separately the two opposite inclusions; note that the two basic subcomplexes are subalgebras of the same algebra $\mathcal{W}_{\mathfrak{g}} \simeq \mathcal{W}^{(\chi)}_{\mathfrak{g}}$. Let us start with

$X \in (\mathcal{W}_{\mathfrak{g}})^G_{hor}$; thus $[X, e_a] = [X, \lambda_a] = 0$ by (untwisted) G-invariance. But then

$$L^{\chi}_a(X) = ad^{\chi}_{e_a}(X) = e_a X \lambda_a - \lambda_a X e_a = \lambda_a(e_a X - X e_a) = 0$$

and similarly

$$i^{\chi}_a(X) = ad^{\chi}_{\xi_a}(X) = \xi_a X \lambda_a - \lambda_a X \xi_a = \lambda_a(\xi_a X - X \xi_a) = 0$$

and so $X \in (\mathcal{W}^{(\chi)}_{\mathfrak{g}})^G_{hor}$. On the other hand, take now $Y \in (\mathcal{W}^{(\chi)}_{\mathfrak{g}})^G_{hor}$; on Cartan generators the twisted adjoint action still agrees with the commutator, so $[H_i, Y] = 0$ and then $[\lambda_a, Y] = 0$. But then

$$ad^{\chi}_{e_a} Y = 0 = e_a Y \lambda_a - \lambda_a Y e_a = \lambda_a(e_a Y - Y e_a)$$

implies the untwisted $ad_{e_a}(Y) = [e_a, Y] = 0$; the same for

$$ad^{\chi}_{\xi_a}(Y) = 0 = \xi_a Y \lambda_a - \lambda_a Y \xi_a = \lambda_a(\xi_a Y - Y \lambda_a)$$

which gives the untwisted $ad_{\xi_a} Y = [\xi_a, Y] = 0$. So $Y \in (\mathcal{W}_{\mathfrak{g}})^G_{bas}$. The linearity follows from the one of operators L and i; the ring structures are the same because they descend from the isomorphic algebra structures of $\mathcal{W}_{\mathfrak{g}} \simeq \mathcal{W}^{(\chi)}_{\mathfrak{g}}$. ■

As conclusion we can say that classical, noncommutative and twisted noncommutative equivariant cohomologies have the same basic cohomology ring $Sym(\mathfrak{g}^*) \simeq \mathfrak{U}(\mathfrak{g})^G$ (we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ since we are considering quadratic Lie algebras).

Trivial actions

The next easy example we consider is when the action is trivial; algebraically this means that the $\tilde{\mathfrak{g}}$-da structure is degenerate, that is, L and i are identically zero. This is another 'limit' situation in which the Drinfeld twist deformation of every such $\tilde{\mathfrak{g}}$-da is absent, since its generators do not act on the algebra. Again directly applying the Weil model definition we find

$$\begin{aligned} \mathcal{H}^{\chi}_G(\mathcal{A}) &= H((\mathcal{W}^{(\chi)}_{\mathfrak{g}} \otimes \mathcal{A})^G_{hor}, d_{\mathcal{W}} \otimes 1 + 1 \otimes d) = \\ &= H((\mathcal{W}^{(\chi)}_{\mathfrak{g}})^G_{hor} \otimes \mathcal{A}, d_{\mathcal{W}} \otimes 1 + 1 \otimes d) = (\mathcal{W}^{(\chi)}_{\mathfrak{g}})^G_{hor} \otimes H(\mathcal{A}) = \\ &= (\mathfrak{U}(\mathfrak{g})^G) \otimes H(\mathcal{A}) \end{aligned} \tag{2.64}$$

Thus also in this case the three different models for equivariant cohomology collapse to the same; again no surprise since now not the 'space' but the action was trivial, and there were no room for any significant deformation. The only interesting remark is that the $(\mathfrak{U}(\mathfrak{g}))^G$-module structure of $\mathcal{H}^{\chi}_G(\mathcal{A})$ is given by multiplication on the left factor of the tensor product, so that there is no torsion. This is a very special example of a more general class of spaces we are going to describe later for which this phenomenon always takes place; they are called equivariantly formal spaces.

Homogeneous spaces

We now come to homogeneous spaces. Classically they are defined as the quotient of a (Lie) group G by a left (or right) action of a closed subgroup $K \subset G$; the action is free, so the quotient is a smooth manifold $X = G/K$ on which G still acts transitively, but now with nontrivial isotropy group (by transitivity all isotropy groups are conjugated). We will recall a classical result which leads to a very easy computation of $H_G(G/K)$, and we will extend this idea to twisted nc equivariant cohomology.

There is of course plenty of interesting homogeneous spaces, like flag manifolds, grasmannians, projective spaces, spheres and so on; we will focus on general results which apply to all of them, but if one prefers to have a specific example in mind, especially in the twisted picture, we suggest the easiest non trivial Drinfeld-twisted sphere S^4_θ acted by $\mathfrak{U}^\chi(\mathfrak{so}(5))$ and realized as the subalgebra of $Fun_\gamma(SO(4))$-coinvariants inside $Fun_\gamma(SO(5))$ (with γ the dual Drinfeld twist of χ, see (1.26)).

The classical picture starts by considering commuting actions of two Lie groups K_1 and K_2. If we define $G = K_1 \times K_2$ its Weil algebra decomposes in $W_\mathfrak{g} = W_{\mathfrak{k}_1} \otimes W_{\mathfrak{k}_2}$ with $[\mathfrak{k}_1, \mathfrak{k}_2] = 0$ by commutativity of the actions. Then every $\tilde{\mathfrak{g}}$-da algebra $\mathcal{A}$ can be thought separately as a $\tilde{\mathfrak{k}}_{1,2}$-da and the basic subcomplex can be factorized in both ways

$$\mathcal{A}_{bas\,\mathfrak{g}} = (\mathcal{A}_{bas\,\mathfrak{k}_1})_{bas\,\mathfrak{k}_2} = (\mathcal{A}_{bas\,\mathfrak{k}_2})_{bas\,\mathfrak{k}_1} \tag{2.65}$$

Proposition 2.4.2 *Under the previous assumptions and notations, if $\mathcal{A}$ is also both $\mathfrak{k}_1$ and $\mathfrak{k}_2$ locally free we have*

$$H_G(\mathcal{A}) = H_{K_1}(\mathcal{A}_{bas\,\mathfrak{k}_2}) = H_{K_2}(\mathcal{A}_{bas\,\mathfrak{k}_1}) \tag{2.66}$$

Proof: We simply follow the Weil model definition for equivariant cohomology and make use of the commutativity between the two locally free K_1 and K_2 actions. Remember that for a locally free action $H_G(\mathcal{A}) = H(\mathcal{A}_{bas})$. To prove the first assertion:

$$\begin{aligned} H_G(\mathcal{A}) &= H((W_\mathfrak{g} \otimes \mathcal{A})_{bas\,\mathfrak{g}}, \delta) = H((W_{\mathfrak{k}_1} \otimes W_{\mathfrak{k}_2} \otimes \mathcal{A})_{bas\,\mathfrak{g}}, \delta) = \\ &= H(((W_{\mathfrak{k}_1} \otimes W_{\mathfrak{k}_2} \otimes \mathcal{A})_{bas\,\mathfrak{k}_1})_{bas\,\mathfrak{k}_2}, \delta) = H((W_{\mathfrak{k}_1} \otimes \mathcal{A}_{\mathfrak{k}_2})_{bas\,\mathfrak{k}_1}, \delta) = \\ &= H_{K_1}(\mathcal{A}_{bas\,\mathfrak{k}_2}) \end{aligned}$$

For the other assertion just repeat the previous proof switching $\mathfrak{k}_1$ and $\mathfrak{k}_2$. ■

This easy fact is very useful for computing equivariant cohomology of homogeneous spaces $H_G(G/K)$. Indeed take on G the two free actions of K and G itself by multiplication; we make them commute by considering K acting from the right and G from the left, or vice versa. The hypothesis of Prop(2.4.2) are satisfied, so we quickly have

$$H_G(G/K) = H_K(G\backslash G) = H_K(\{pt\}) = Sym(\mathfrak{k}^*)^K \tag{2.67}$$

We want to find a similar result for twisted $\tilde{\mathfrak{g}}$-da. The definition of commuting actions makes perfectly sense in the twisted setting; we require that the two twisted $\tilde{\mathfrak{k}}_{1,2}$-da structure commute. This is an easy consequence of the commutation of the classical algebras, provided the generators of the twists commute with each other (for example using a unique abelian twist for both algebras, which is the most common situation). The assumption of the local freeness of the action is a bit trickier; we need to restate this notion in order to include also twisted nc algebras.

In (2.16) we expressed this property by means of an algebraic connection, which can be seen as a basic element $\varpi \in (\mathcal{A}^1 \otimes \mathfrak{g})_{bas}$ (note that the conditions in (2.16) states exactly this fact). Using the twisted Weil algebra[2], a natural generalization is then to ask for the presence of a basic element $\varpi^\chi \in ((\mathcal{A}_\chi)_{(1)} \otimes (\mathcal{W}^{(\chi)}_{\mathfrak{g}})_{(2)})_{bas}$. [3] Looking at Prop 2.3.8 it is clear that if $\mathcal{A}$ is a locally free $\tilde{\mathfrak{g}}$-da then $\mathcal{A}_\chi$ is a locally free twisted $\tilde{\mathfrak{g}}$-da.

So we can apply Prop(2.4.2) also to Drinfeld twist deformations of homogeneous spaces, since all the hypotheses are still satisfied. The appropriate statement involves Drinfeld twists on function algebras over classical groups; we already discussed there are two kinds of (in general different) deformations on $Fun(G)$ induced by a Drinfeld twist χ.

The first one consists in deforming $Fun(G)$ simply as a $\mathfrak{U}(\mathfrak{g})$-module algebra, and we will denote it by $Fun_\chi(G)$; the second one consist in deforming $Fun(G)$ as Hopf algebra using the dual Drinfeld twist γ associated to χ (see (1.26)), and we will denote is ad $Fun_\gamma(G)$.

To have a nice algebraic description of G/K it is better to use the dual Drinfeld twist deformation. In this way we have a (possibly reduced) Drinfeld twist on $\mathfrak{U}(\mathfrak{k})$ as well, or equivalently a (dually) Drinfeld twisted Hopf algebra $Fun_\gamma(K)$; this one coacts on $Fun_\gamma(G)$, and the subalgebra of coinvariants $(Fun_\gamma(G))^{coK} \subset Fun_\gamma(G)$ represents the deformation of the homogeneous space G/K.

The equivalent result of (2.67) for nc homogeneous spaces is

$$\mathcal{H}^\chi_G((Fun_\gamma(G))^{coK}) = \mathcal{H}^\chi_K((Fun_\gamma(G))^{coG}) = \mathcal{H}^\chi_K(\mathbb{C}) = \mathfrak{U}(\mathfrak{k})^K \qquad (2.68)$$

As an explicit example, we can apply (2.68) to nc spheres S^n_θ. For simplicity let us consider S^4_θ; it can be constructed as a toric isospectral deformation of the classical

[2] We are stating the locally freeness condition by already using the twisted nc Weil algebra $\mathcal{W}^{(\chi)}_{\mathfrak{g}}$; in principle we should first define what we mean for locally free $\tilde{\mathfrak{g}}$-da and then show that $\mathcal{W}^{(\chi)}_{\mathfrak{g}}$ is the universal object in the category. Once this is done, we can restate the property in a shorter way using directly $\mathcal{W}^{(\chi)}_{\mathfrak{g}}$; this is what we are doing here for simplicity.

[3] Here the lower indexes are filtration degrees. This is to mimic the classical definition, since now we do not have any more the possibility to refer to any degree on $\mathcal{A}_\chi$ ($\mathcal{A}_\chi$ is not graded), nor to elements in $\mathfrak{g}$ in $\mathcal{W}^{(\chi)}_{\mathfrak{g}}$ (they are not closed under the action of L and i. The filtration is naturally defined as follows: for $\mathcal{A}$ graded we put $\mathcal{A}_{(p)} = \oplus_{i\leq p}\mathcal{A}^i$, while for $\mathcal{W}^{(\chi)}_{\mathfrak{g}}$ we use the usual filtration of enveloping algebras, with the only remark that even generators $\{u_a\}$ have filtration degree 2 and odd generators $\{\xi_a\}$ filtration degree 1.

sphere S^4 twisting the $\mathbb{T}^2$ symmetry acting on it. Equivalently, to stress the fact that it is a homogeneous space we can think of it as the $Fun_\gamma(SO(4))$-coinvariant subalgebra of $Fun_\gamma(SO(5))$. On S^4_θ we have the action of the twisted symmetry $\mathfrak{U}^\chi(so(5))$; the action of course is not free since the twisted Hopf subalgebra $\mathfrak{U}^\chi(so(4))$ acts trivially. The equivariant cohomology of this twisted action is defined using the twisted Weil (or Cartan) models introduced in the previous section, and it may be computed using (2.68). We find

$$\mathcal{H}^\chi_{\mathfrak{so}(5)}(S^4_\theta) = \mathfrak{U}^\chi(\mathfrak{so}(4))^{SO(4)} = \mathfrak{U}(\mathfrak{so}(4))^{SO(4)} \simeq Sym(\mathfrak{so}(4))^{SO(4)} \simeq Sym(\mathfrak{t}^2)^W \quad (2.69)$$

where the last equality is given by Chevalley's theorem $Sym(\mathfrak{g})^G \simeq Sym(\mathfrak{t})^W$ for W the Weyl group.

Thus we see that we need to deform the classical algebraic models for $H_G(\mathcal{A})$ in order to adapt them to nc spaces and to have a consistent definition of equivariant cohomology on nc spaces, or twisted $\tilde{\mathfrak{g}}$-da's; however at the end for a large class of examples the cohomology rings agree with the classical ones.

2.4.2 Reduction to the maximal torus

In this section we study the reduction of twisted nc equivariant cohomology to the maximal torus $T \subset G$. The two main ingredients in the algebraic proof of the isomorphism $H_G(X) = H_T(X)^W$ (W is the Weyl group of T) for classical equivariant cohomology are the functoriality of $H_G(X)$ with respect to reduction to subgroups $P \subset G$, and spectral sequences arguments.

In order to reproduce a similar result and proof for the noncommutative (and then twisted) case we first need to work out the functorial properties of $\mathcal{H}_G(\mathcal{A})$; since in both nc and twisted case Weil and Cartan models are built using the Lie algebra $\mathfrak{g}$ contrary to the classical case which make use of the dual $\mathfrak{g}^*$, it is not obvious that for every subgroup $P \subset G$ we have a morphism of Cartan complexes $\mathcal{C}_P(\mathcal{A}) \to \mathcal{C}_G(\mathcal{A})$. And in fact we will show the existence of such a morphism for the specific choice $K = N(T)$, the normalizer of the maximal torus, by using a generalized Harish-Chandra projection map [AM05]; once we have constructed the morphism, the rest of the proof follows quite easily.

We start with a quick review of the classical reduction, referring to [GS99] for the full details. The Cartan complex $C_G(\mathcal{A})$ may be seen as a double $\mathbb{Z}$-graded complex $C^{p,q}(\mathcal{A}) = (Sym^p(\mathfrak{g}^*) \otimes \mathcal{A}^{q-p})^G$ with differentials $\delta_1 = -v^a \otimes i_a$ and $\delta_2 = 1 \otimes d$ of grading $(1,0)$ and $(0,1)$ respectively; the cohomology of the total complex with respect to $d_G = \delta_1 + \delta_2$ is the classical equivariant cohomology. This is the usual setting to construct a spectral sequence converging to $H_G(\mathcal{A})$ with $E_1^{p,q}$ term (for G compact and connected) given by $Sym^p(\mathfrak{g}^*) \otimes H^{q-p}(\mathcal{A})$. We can get the desired isomorphism $H_G(X) \cong H_T(X)^W$ by looking at a different spectral sequence having the same E_1 term. For each closed subgroup $P \subset G$ we get a morphism between

Cartan complexes $C_G(\mathcal{A}) \to C_P(\mathcal{A})$ and hence between E_1 terms; whenever P is such that $Sym(\mathfrak{g}^*)^G \cong Sym(\mathfrak{p}^*)^P$ we have an isomorphism at the E_1 step compatibly with the differentials, thus at every following step and in particular $H_G(\mathcal{A}) \cong H_P(\mathcal{A})$. We can use this result with $P = N(T)$, the normalizer of the maximal torus.

Theorem 2.4.3 *Let G be a compact connected Lie group and $\mathcal{A}$ a $\tilde{\mathfrak{g}}$-da. There is a ring homomorphism $H_G(\mathcal{A}) \cong H_T(\mathcal{A})^W$ where $T \subset G$ is the maximal torus in G and W its Weil group $N(T)/T$.*

Proof: The Weil group $W = P/T = N(T)/T$ is finite, thus $\mathfrak{p} \cong \mathfrak{t}$ and $Sym(\mathfrak{p}^*)^P \cong Sym(\mathfrak{t}^*)^P \cong Sym(\mathfrak{t}^*)^W$ since T acts trivially on itself. Then by Chevalley's theorem $Sym(\mathfrak{g}^*)^G \cong Sym(\mathfrak{t}^*)^W$, so as discussed before $H_G(\mathcal{A}) \cong H_{N(T)}(\mathcal{A})$. To conclude we have to prove that $H_{N(T)}(\mathcal{A}) \cong H_T(\mathcal{A})^W$; the inclusion $T \hookrightarrow P = N(T)$ induces a morphism $Sym(\mathfrak{p}^*) \otimes \mathcal{A} \to Sym(\mathfrak{t}^*) \otimes \mathcal{A}$ and taking the P-invariant subcomplexes we get a morphism $C_P(\mathcal{A}) \to C_T(\mathcal{A})^W$ and so on at each stage of the spectral sequences. In particular we obtain a morphism between equivariant cohomologies $H_P(\mathcal{A}) \to H_T(\mathcal{A})^W$; but note that at the E_1 step the morphism is indeed an isomorphism, since we already showed $Sym(\mathfrak{p}^*)^P \cong Sym(\mathfrak{t}^*)^W$, and so the previous morphism between cohomologies is an isomorphism as well. ∎

This result allows to reduce the computation of classical equivariant cohomology for generic compact Lie groups G to abelian groups. Another important feature of $H_G(X)$ is its $Sym(\mathfrak{g}^*)^G$-module structure, with the torsion part playing a central role in localization theorems. We proved that the E_1 term of the spectral sequence converging to $H_G(X)$ is $Sym(\mathfrak{g}^*)^G \otimes H(\mathcal{A})$; at this stage the module structure is simply given by left multiplication, so E_1 is a free $Sym(\mathfrak{g}^*)^G$-module. This already implies that if $H(\mathcal{A})$ is finite dimensional, the equivariant cohomology ring $H_G(\mathcal{A})$ is finitely generated as $Sym(\mathfrak{g}^*)^G$-module. When the spectral sequence collapses at this stage, the algebra $\mathcal{A}$ is called equivariantly formal. The definition comes from [GKR98] (using the language of with G-spaces X rather then $\tilde{\mathfrak{g}}$-da's $\mathcal{A}$), where sufficient conditions for the collapsing are studied. In this case since $E_\infty \cong E_1$ we have that $H_G(\mathcal{A})$ is a free $Sym(\mathfrak{g}^*)^G$-module. We can also express the ordinary cohomology in term of equivariant cohomology by tensoring the E_1 term by the trivial $Sym(\mathfrak{g})^*$-module $\mathbb{C}$, obtaining $H(\mathcal{A}) = \mathbb{C} \otimes_{Sym(\mathfrak{g}^*)} H_G(\mathcal{A})$.

We now come to nc equivariant cohomology. Given a closed subgroup $P \subset G$ we have a Lie algebra homomorphism $\mathfrak{p} \to \mathfrak{g}$ which may be lifted to the enveloping algebras and nc Weil algebras, but in general does not intertwines the differentials and most unpleasantly goes in the opposite direction we are interested in to reduce equivariant cohomology. We have to look for a $\tilde{\mathfrak{p}}$-da (or at least $\tilde{\mathfrak{p}}$-ds, i.e. $\tilde{\mathfrak{p}}$-differential space) homomorphism $\mathcal{W}_{\mathfrak{g}} \to \mathcal{W}_{\mathfrak{p}}$ which then may be used to get a morphism between the nc Cartan complexes $\mathcal{C}_G(\mathcal{A}) \to \mathcal{C}_P(\mathcal{A})$. This homomorphism can be constructed for a very special choice of subgroup P, namely for $P = N(T)$, which is exactly the case we need. We refer to [AM05](Section 7) for the details on the construction. It is shown that for a quadratic Lie algebra $\mathfrak{g}$ with quadratic subalgebra $\mathfrak{p}$ and orthogonal complement $\mathfrak{p}^\perp$ it is possible to define a 'generalized' Harish-Chandra projection[4] $k_{\mathcal{W}} : \mathcal{W}_\mathfrak{g} \rightarrow \mathcal{W}_\mathfrak{p}$ which is a $\tilde{\mathfrak{p}}$-ds homomorphism and becomes a $\tilde{\mathfrak{p}}$-da homomorphism between the basic subcomplexes $\mathfrak{U}(\mathfrak{g})^G \rightarrow \mathfrak{U}(\mathfrak{p})^P$. Moreover this construction reduces to the classical Harish-Chandra map up to $\mathfrak{p}$-chain homotopy [AM05](Thm7.2) and then looking at the basic subcomplexes (where the differential is zero) we find the commutative diagram of $\tilde{\mathfrak{p}}$-da's [AM05](Thm7.3)

$$\begin{array}{ccc} Sym(\mathfrak{g})^G & \longrightarrow & \mathfrak{U}(\mathfrak{g})^G \\ {\scriptstyle k_{Sym}}\downarrow & & \downarrow{\scriptstyle (k_{\mathcal{W}})_{|bas}} \\ Sym(\mathfrak{p})^P & \longrightarrow & \mathfrak{U}(\mathfrak{p})^P \end{array} \tag{2.70}$$

where horizontal maps are Duflo algebra isomorphism.[5] For $P = N(T)$ by Chevalley theorem the map $k_{Sym} : Sym(\mathfrak{g})^G \rightarrow Sym(\mathfrak{t})^W$ is an algebra isomorphism as well. This is the morphism we need to prove the reduction of nc equivariant cohomology. We note that this result, even if not explicitly stated, is already contained in [AM00] when the authors prove the ring isomorphism $H_G(\mathcal{A}) \cong \mathcal{H}_G(\mathcal{A})$ induced by the quantization map $Q_\mathfrak{g} : W_\mathfrak{g} \rightarrow \mathcal{W}_\mathfrak{g}$. We prefer to give here a direct proof based on morphisms between Cartan complexes and spectral sequences since this approach will be generalized to our twisted nc equivariant cohomology.

Theorem 2.4.4 *The ring isomorphism of Thm(2.4.3) holds also between noncommutative equivariant cohomology rings; for every noncommutative $\tilde{\mathfrak{g}}$-da $\mathcal{A}$ and compact connected Lie group G the reduction reads $\mathcal{H}_G(\mathcal{A}) \cong \mathcal{H}_T(\mathcal{A})^W$.*

Proof: As for the classical reduction, the proof is based on the presence of a morphism between Cartan complexes and a comparison between the two associated spectral sequences. The setting is now the following: the nc Cartan model $\mathcal{C}_G(\mathcal{A}) = (\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{A})^G$ is looked as a double filtered differential complex. On one side we have the standard increasing filtration of the enveloping algebra $\mathfrak{U}(\mathfrak{g})_{(0)} \subset \mathfrak{U}(\mathfrak{g})_{(1)} \subset \mathfrak{U}(\mathfrak{g})_{(2)} \ldots$; on the other side, supposing $\mathcal{A}$ is a finitely generated graded algebra we have an increasing filtration $\mathcal{A}_{(p)} = \oplus_{i \leq p} \mathcal{A}^i$; note that this double filtration on $\mathcal{C}_G(\mathcal{A})$ is compatible with the ring structure (2.35) (contrary to the grading of $\mathcal{A}$, which is not

[4]The Harish-Chandra homomorphism is classically constructed between enveloping algebras, i.e. for $\mathfrak{p} \subset \mathfrak{g}$ it is a map $\mathfrak{U}(\mathfrak{g}) \rightarrow \mathfrak{U}(\mathfrak{p})$ which then descend to an analogue map between symmetric algebras. For details on the construction and properties of the maps see e.g. [Kna02]. Note that sometimes in the literature the map between enveloping algebras is already mentioned as 'generalized' Harish-Chandra map. Here with 'generalized' we mean its analogue map between nc Weil algebras.

[5]The Duflo map is a vector space isomorphism $Sym(\mathfrak{g}) \rightarrow \mathfrak{U}(\mathfrak{g})$ which moreover restricts to an algebra isomorphism between G-invariants. For its construction and relations with the PBW map see [Duf77]

compatible with the induced product on $\mathcal{C}_G(\mathcal{A})$). The operators

$$\delta_1 = \Phi(d^{\mathcal{W}_\mathfrak{g}} \otimes 1)\Phi^{-1} = -\frac{1}{2}(u^a_{(L)} + u^a_{(R)}) \otimes i_a + \frac{1}{24} f^{abc} \otimes i_a i_b i_c$$

and

$$\delta_2 = \Phi(1 \otimes d)\Phi^{-1} = 1 \otimes d$$

square to zero (since their counterpart on the Weil complex do), and then anticommute since their sum is the nc Cartan differential d_G; they are the differentials of the double complex, with filtration degree respectively $(1,0)$ and $(0,1)$. The cohomology of the total complex with respect to $d_G = \delta_1 + \delta_2$ is the nc equivariant cohomology ring $\mathcal{H}_G(\mathcal{A})$; the filtration of $\mathcal{C}_G(\mathcal{A})$ induces a filtration on the cohomology. We can compute its graded associated module $Gr(\mathcal{H}_G(\mathcal{A}))$ by a spectral sequence with E_0 term given by the graded associated module of the nc Cartan model $Gr(\mathcal{C}_G(\mathcal{A})) = C_G(\mathcal{A})$; this is the spectral sequence we already introduced before. Note that the differentials δ_1 and δ_2 map to the ordinary differentials of the Cartan complex $-\frac{1}{2}v^a \otimes i_a$ and $1 \otimes d$. Now let us consider the inclusion $P = N(T) \subset G$ and the Harish-Chandra projection map $k_\mathcal{W} : \mathcal{W}_\mathfrak{g} \to \mathcal{W}_\mathfrak{p}$. This induces a $\tilde{\mathfrak{p}}$-ds morphism between the Weil complexes $(\mathcal{W}_\mathfrak{g}\widehat{\otimes}\mathcal{A})_{bas} \to (\mathcal{W}_\mathfrak{p}\widehat{\otimes}\mathcal{A})_{bas}$ and by Kalkman map a $\tilde{\mathfrak{p}}$-ds morphism between nc Cartan models $\mathcal{C}_G(\mathcal{A}) \to \mathcal{C}_P(\mathcal{A})$ compatible with the filtrations; commuting with differentials, it also lifts to cohomology giving a morphism of filtered rings $\mathcal{H}_G(\mathcal{A}) \to \mathcal{H}_P(\mathcal{A})$. By going to the graded associated modules and computing the E_1 term of the spectral sequence we get a $\tilde{\mathfrak{p}}$-ds morphism $Sym(\mathfrak{g})^G \otimes H(\mathcal{A}) \to Sym(\mathfrak{t})^W \otimes H(\mathcal{A})$ (see (2.70) and [AM05](Thm7.3)). Now this is a $\tilde{\mathfrak{p}}$-da isomorphism, and it induces $\tilde{\mathfrak{p}}$-da isomorphisms at every further step of the spectral sequence. The isomorphism between $Gr(\mathcal{H}_G(\mathcal{A}))$ and $Gr(\mathcal{H}_P(\mathcal{A})$ implies that the morphism $\mathcal{H}_G(\mathcal{A}) \to \mathcal{H}_P(\mathcal{A})$ introduced before is in fact a ring isomorphism. As in the classical case, the last step is to show $\mathcal{H}_P(\mathcal{A}) \cong \mathcal{H}_T(\mathcal{A})^W$; this easily follows from the morphism $\mathcal{C}_P(\mathcal{A}) \to \mathcal{C}_T(\mathcal{A})$ (note that $\mathfrak{p} \cong \mathfrak{t}$ so the previous morphism is just group action reduction) and a completely similar spectral sequence argument. ■

We finally note another equivalent proofs of Thm(2.4.4) may be obtained by a different construction of the morphism $\mathcal{C}_G(\mathcal{A}) \to \mathcal{C}_P(\mathcal{A})$ via a diagram

$$(\mathfrak{U}(\mathfrak{p}) \otimes \mathcal{A})^P \longrightarrow ((\mathfrak{U}(\mathfrak{g}) \otimes Cl(\mathfrak{p}^\perp)) \otimes \mathcal{A})^P \longleftarrow (\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{A})^G \qquad (2.71)$$

Considering the spectral sequence associated to these three Cartan models (the cohomology of the middle complex is a sort of 'relative' equivariant cohomology $\mathcal{H}_{G,P}(\mathcal{A})$ of G with respect to P, see [AM05](Section6)) it is possible to prove an isomorphism between the image of the left and right E_1 terms inside the E_1 term of the middle complex [AM05](Thm6.4). This isomorphism is referred as a version of Vogan's conjecture for quadratic Lie algebras.

We finally consider twisted nc equivariant cohomology. It is a natural question to ask if our model satisfies a reduction property as well; an easy nevertheless crucial fact is that Drinfeld twists act trivially on abelian symmetries. This will allow us to basically use the same proof of Thm(2.4.4); moreover for the same reason when restricted to the maximal torus T, twisted nc equivariant cohomology $\mathcal{H}^{\chi}_{T}(\mathcal{A}_{\chi})$ agrees with $\mathcal{H}_{T}(\mathcal{A}_{\chi})$.

Theorem 2.4.5 *Let G be a compact connected Lie group, and $\mathcal{A}_{\chi}$ a twisted $\tilde{\mathfrak{g}}$-da. There is a ring homomorphism $\mathcal{H}^{\chi}_{G}(\mathcal{A}_{\chi}) \cong \mathcal{H}^{\chi}_{T}(\mathcal{A}_{\chi})^{W}$ where $T \subset G$ is the maximal torus in G and W its Weil group $N(T)/T$.*

Proof: We can use the generalized Harish-Chandra projection also for twisted nc Weil algebras, since for $P = N(T)$ as $\tilde{\mathfrak{p}}$-da's $\mathcal{W}_{\mathfrak{g}} \cong \mathcal{W}^{(\chi)}_{\mathfrak{g}}$. The twisted nc Cartan model $\mathcal{C}^{\chi}_{G}(\mathcal{A}_{\chi})$ is a double filtered differential complex similarly to $\mathcal{C}_{G}(\mathcal{A})$, and we can consider the spectral sequence constructed from its graded associated module. At the E_1 step as usual we are left with the basic part of $Gr(\mathcal{W}^{(\chi)}_{\mathfrak{g}})$ tensored with $H(\mathcal{A}_{\chi})$; since $(\mathcal{W}^{(\chi)}_{\mathfrak{g}})_{|bas} \cong (\mathcal{W}_{\mathfrak{g}})_{|bas}$ (see Thm(2.4.1)) any effect of the twist is now present only in the cohomology of $\mathcal{A}_{\chi}$. Then the isomorphism between the E_1 terms of $\mathcal{C}^{\chi}_{G}(\mathcal{A}_{\chi})$ and $\mathcal{C}^{\chi}_{P}(\mathcal{A}_{\chi})$ follows as in the proof of Thm(2.4.4). The same happens for the last part of the proof, when going from $P = N(T)$ to T. ■

This result shows one more time that deformations coming from Drinfeld twists do not affect much of the classical setting. The definition of a twisted nc equivariant cohomology was needed dealing with algebras which carry a twisted action of a symmetry, and this in turn is what happens for covariant actions of Drinfeld twisted Hopf algebras. However the possibility to reduce the cohomology to the maximal torus part leaves the only contribution coming from the Drinfeld twist in the deformed ring structure of $\mathcal{H}^{\chi}(\mathcal{A}_{\chi})$, while the vector space and $Sym(\mathfrak{g})^{G}$-module structures are undeformed.

The 'useful' side of this quite classical behaviour is that for what concerns this class of deformations, a lot of techniques of equivariant cohomology may be lifted with an appropriate and careful rephrasing to the nc setting. On the contrary, if we are interested in purely new phenomena which do not admit a classical counterpart, it seems we have to enlarge the class of deformations considered, either taking Drinfeld twists χ which do not satisfy the 2-cocycle condition or moving to other class of deformations.

Conclusions

We present here a summary of our work, emphasizing the general ideas which can be used to investigate further aspects of the theory so far developed.

Our starting motivation and interest was to study symmetries of noncommutative spaces, and in particular their associated equivariant cohomologies. The very first question is what do we mean by a symmetry acting on a nc space. There is a natural answer to this question if we consider nc spaces X_θ realized as deformations of classical spaces X; indeed very often the deformation directly comes from an action of some symmetry on X, so that it is possible to deform in a 'compatible' way the symmetry and have a deformed action on X_θ as well.

This strategy is made more precise by rephrasing everything in an algebraic framework, following the philosophy of nc geometry. We then realize that the appropriate setting is the category of $\tilde{\mathfrak{g}}$-da's, or more in general Hopf-module algebras; Hopf algebras play the role of symmetry, module algebras represent the space where the symmetry acts and the module structure is the mathematical expression for the covariance of the action. This link between symmetries and spaces is what allows us to deform a classical symmetry in order to define an action on a deformed nc space. But we can remarkably use covariance in the opposite direction as well: if we start by deforming a Hopf algebra $\mathcal{H}$ we automatically get an induced deformation on every $\mathcal{H}$-module algebra.

This is the twofold role played by covariance: we can use symmetries to generate nc spaces, forcing the module algebras to 'adapt' to the deformation of the Hopf algebras, or we can start with a nc space and force symmetries to 'adjust' their actions in order to respect the structure of the nc space. The first case has been developed in the first chapter, and led us to recover toric isospectral deformations as well as to define nc toric varieties. The second situation has been studied in the second chapter, and culminated in the definition of deformed Weil algebras and models for nc equivariant cohomology of such nc spaces.

In this thesis we considered for both theories the class of deformations coming from Drinfeld twists, but we wish to stress that, once the appropriate category of Hopf-module algebra has been found in a compatible way with the deformation chosen, the same strategy in both directions may be applied in full generality.

Some examples of deformations which could be used are:

- Drinfeld-Jimbo Hopf algebra deformations or q-deformed quantum groups;
- Drinfeld twists which do not satisfy the 2-cocycle condition, thus leading to quasi-Hopf algebras;
- Drinfeld twists whose generators are root elements instead of Cartan ones;
- the procedure sketched in Sect(1.4.1) on the definition of a nc product on a generic abelian locally compact group G by harmonic analysis and the choice of a homomorphism of groups between G and its Pontrjagin dual $\hat{G}$;
- for nonabelian compact groups G a similar idea could be stated using techniques of noncommutative harmonic analysis and a generalized definition of Pontrjagin duality.

We now discuss in more detail some topics emerged in the thesis, pointing out interesting aspects which could deserve a further study.

Noncommutative toric varieties have been formulated by using a deformed fan description, which naturally provide a local description of the nc spaces by open affine sets associated to cones and represented by deformed coordinate rings.

From one side we can go on with this local description by developing a sheaf theory on such spaces (as it has been sketched at the end of Sec(1.4.4)) with the long-term motivation of defining a deformed ADHM description of sheaves and related nc instantons, in the spirit of [KKO01].

On the other hand, at least for homogeneous spaces G/K (like for example flag varieties, grassmannians and projective spaces), we could also find a global description of the deformed nc toric varieties by looking at coinvariants subalgebras of the Drinfeld twisted Hopf algebras $Fun_\gamma(G)$ (using the dual Drinfeld twist γ of Def(1.2.15)) as the algebraic analogue of the quotient space. It would be interesting then to compare this global description with the general local one provided by the deformed fan picture or by the localized homogeneous coordinate ring, once the latter formalism is generalized from $\mathbb{CP}^n$ (Thm(1.4.8)) to a larger class of nc toric varieties.

For what concerns noncommutative equivariant cohomology, the results shown in the present work seem to suggest that at least for Drinfeld twist deformations the theory does not present significant differences or novelties with respect to the classical undeformed theory.

A way to check if Drinfeld twist deformations are the only reason why twisted nc equivariant cohomology behaves is a so quite classical manner, or rather it is an intrinsic property of the proposed Weil and Cartan models, is to apply the same construction to other class of deformations, as we already discussed. However an argument which supports the hypothesis that this quite classical behaviour is generated by Drinfeld twists comes from another possible definition of nc equivariant cohomology. We only sketch it, referring to the references for all the missing details.

The algebraic analogue of the de Rham cohomology of a smooth manifold X is the (periodic) cyclic homology of its algebra of functions $C^\infty(X)$, by the Hochschild-Kostant-Rosenberg theorem proved by Connes for the smooth setting [Con94]. In [BG94] an equivariant version of this result is formulated, by proving an isomorphism between equivariant periodic cyclic homology of $C^\infty(X)$ and the cohomology of global equivariant differential forms on X. The latter are global sections of a sheaf over the group G and reproduce ordinary equivariant differential forms by restricting to the contribution over the identity $e \in G$; for this reason this theory is also known as 'delocalized' equivariant cohomology. An equivalent way to compute equivariant cyclic cohomology is to consider ordinary homology of the crossed product algebra $C^\infty(G \times X)$ [Bry87]. After the usual formulation in an algebraic setting, considering the category of Hopf-module algebras we are eventually interested in the cyclic or periodic homology of crossed product algebras $\mathcal{H} \ltimes \mathcal{A}$; this description is ready to be generalized to Drinfeld twist deformations, since it makes perfectly sense to consider the twisted crossed product algebra $\mathcal{H}^\chi \ltimes \mathcal{A}_\chi$ and to take cyclic homology. This could provide an alternative definition of twisted nc equivariant cohomology, but unfortunately it does not bring to anything new, since $\mathcal{H} \ltimes \mathcal{A} \cong \mathcal{H}^\chi \ltimes \mathcal{A}_\chi$ as algebras for χ satisfying the 2-cocycle condition [BPO00].

On the other hand, looking at the bright side of the story, the classical behaviour of Drinfleld twists could mean that we have at our disposal a lot of powerful tools from of equivariant cohomology also when dealing with nc spaces (realized from Drinfeld twists). One of the original motivations for the study of equivariant cohomology for nc spaces was the question if equivariant localization techniques could have been applied to explicitly compute invariants on nc instantons and nc Yang-Mills theories as it happens for the commutative theories.

Considering isospectral deformations, a measure and integration theory is present also in the nc setting. If the 'direction' $\xi \in \mathfrak{g}$ on which we want to localize commutes with the generators of the noncommutativity (i.e. with the generators of the Drinfeld twist) the answer seems to be positive, since once we restrict our attention from G to the S^1 generated by ξ the twisted nc equivariant cohomology 'reduces' to the classical one (apart from the noncommutativity coming from the ordinary cohomology ring). For the general case a close analysis of the relation between the localization element ξ and the Drinfeld twist is needed, but the isomorphism between the $\mathfrak{U}(\mathfrak{g})^G \cong Sym(\mathfrak{g}^*)^G$-module structures of $\mathcal{H}^\chi(\mathcal{A}_\chi)$ and $H_G(\mathcal{A})$ (via Thm(2.4.5) and the isomorphism $\mathcal{H}_G(\mathcal{A}) \cong H_G(\mathcal{A})$ of [AM00]) provides a good reason to conjecture that the localization holds.

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