MODIFIED ACTIONS FOR GRAVITY: THEORY AND PHENOMENOLOGY

Thesis submitted for the degree of
"Doctor Philosophiæ"

October 2007

CANDIDATE

SUPERVISORS

Thomas P. Sotiriou

Stefano Liberati and John Miller

International School for Advanced Studies
Via Beirut 2-4, 34014 Trieste, Italy.
E-mail: *sotiriou@sissa.it*

The important thing is not to stop questioning.

Albert Einstein

Abstract

This thesis is devoted to the study of gravitational theories which can be seen as modifications or generalisations of General Relativity. The motivation for considering such theories, stemming from Cosmology, High Energy Physics and Astrophysics is thoroughly discussed (cosmological problems, dark energy and dark matter problems, the lack of success so far in obtaining a successful formulation for Quantum Gravity). The basic principles which a gravitational theory should follow, and their geometrical interpretation, are analysed in a broad perspective which highlights the basic assumptions of General Relativity and suggests possible modifications which might be made. A number of such possible modifications are presented, focusing on certain specific classes of theories: scalar-tensor theories, metric $f(R)$ theories, Palatini $f(R)$ theories, metric-affine $f(R)$ theories and Gauss–Bonnet theories. The characteristics of these theories are fully explored and attention is payed to issues of dynamical equivalence between them. Also, cosmological phenomenology within the realm of each of the theories is discussed and it is shown that they can potentially address the well-known cosmological problems. A number of viability criteria are presented: cosmological observations, Solar System tests, stability criteria, existence of exact solutions for common vacuum or matter configurations *etc*. Finally, future perspectives in the field of modified gravity are discussed and the possibility for going beyond a trial-and-error approach to modified gravity is explored.

Collaborations

The research presented in this thesis was mainly conducted in SISSA-International School for Advanced Studies between November 2004 and October 2007. This thesis is the result of the authors own work, as well as the outcome of scientific collaborations stated below, except where explicit reference is made to the results of others.

The content of this thesis is based on the following research papers published in refereed Journals or refereed conference proceedings:

1. **"The nearly Newtonian regime in Non-Linear Theories of Gravity"**
   T. P. Sotiriou
   Gen. Rel. Grav. **38** 1407 (2006) [arXiv:gr-qc/0507027]

2. **"Unification of inflation and cosmic acceleration in the Palatini formalism"**
   T. P. Sotiriou
   Phys. Rev. D **73**, 063515 (2006) [arXiv:gr-qc/0509029]

3. **"Constraining f(R) gravity in the Palatini formalism"**
   T. P. Sotiriou
   Class. Quant. Grav. **23**, 1253 (2006) [arXiv:gr-qc/0512017]

4. **"Metric-affine f(R) theories of gravity"**
   T. P. Sotiriou and S. Liberati
   Ann. Phys. **322**, 935 (2007) [arXiv:gr-qc/0604006]

5. **"f(R) gravity and scalar-tensor theory"**
   T. P. Sotiriou
   Class. Quant. Grav. **23**, 5117 (2006) [arXiv:gr-qc/0604028]

6. **"The metric-affine formalism of $f(R)$ gravity"**
   T. P. Sotiriou and S. Liberati
   J. Phys. Conf. Ser. **68**, 012022 (2007) [arXiv:gr-qc/0611040]
   Talk given (by T. P. S.) at the 12th Conference on Recent Developments in Gravity (NEB XII), Nafplio, Greece, 29 Jun-2 Jul 2006

7. **"Curvature scalar instability in f(R) gravity."**
   T. P. Sotiriou,
   Phys. Lett. B **645**, 389 (2007) [arXiv:gr-qc/0611107]

8. **"The significance of matter coupling in $f(R)$ gravity"**
   T. P. Sotiriou, Proceedings of the 11th Marcel Grossman Meeting in press
   Talk given at the 11th Marcel Grossman Meeting, Berlin, Germany, 23-29 Jul 2006

9. **"Post-Newtonian expansion for Gauss-Bonnet gravity."**
   T. P. Sotiriou and E. Barausse,
   Phys. Rev. D **75**, 084007 (2007) [arxiv:gr-qc/0612065]

10. **"A no-go theorem for polytropic spheres in Palatini f(R) gravity."**
    E. Barausse, T. P. Sotiriou and J. C. Miller,
    Submitted to Phys. Rev. Lett. [arXiv:gr-qc/0703132]

11. **"Theory of gravitation theories: a no-progress report."**
    T. P. Sotiriou, V. Faraoni and S. Liberati,
    Submitted to Int. J. Mod. Phys. D [arXiv:0707.2748 [gr-qc]]
    Invited paper in the Special Issue: Invited Papers and Selected Essays from the Annual Essay Competition of the Gravity Research Foundation for the Year 2007

Notation

An attempt has been made to keep the basic notation as standard as possible. However, the use of non-metric connections did require the use of some non-standard notation. The following list will hopefully be a useful tool for clarifying these non-standard notation. In general, the notation, standard or not, is always defined at its first occurence in the text and in all places that ambiguities may arise, irrespectivelly of whether it has been included in this guide. The signature of the metric is assumed to be $(-,+,+,+)$ and the speed of light c is taken to be equal to 1 throughout this thesis. In order to lighten the notation, in some cases a coordinate system is used in which $G = c = 1$, where G is Newton's gravitational constant. However, for clarity G is not set to be equal to 1 throught the text.

$g^{\mu\nu}$:	Lorentzian metric
g:	Determinant of $g^{\mu\nu}$
$\Gamma^{\lambda}{}_{\mu\nu}$:	General Affine Connection
$\{^{\lambda}{}_{\mu\nu}\}$:	Levi–Civita Connection
∇_{μ}:	Covariant derivative with respect to $\{^{\lambda}{}_{\mu\nu}\}$
$\bar{\nabla}_{\mu}$:	Covariant derivative with respect to $\Gamma^{\lambda}{}_{\mu\nu}$
$(\mu\nu)$:	Symmetrization over the indices μ and ν
$[\mu\nu]$:	Anti-symmetrization over the indices μ and ν
$Q_{\mu\nu\lambda}$:	Non-metricity $(\equiv -\bar{\nabla}_{\mu} g_{\nu\lambda})$
$S_{\mu\nu}{}^{\lambda}$:	Cartan torsion tensor $(\equiv \Gamma^{\lambda}{}_{[\mu\nu]})$
$R^{\lambda}{}_{\sigma\mu\nu}$:	Riemann tensor of $g_{\mu\nu}$
$R_{\mu\nu}$:	Ricci tensor of $g_{\mu\nu}$ $(\equiv R^{\sigma}{}_{\mu\sigma\nu})$
R:	Ricci scalar of $g_{\mu\nu}$ $(\equiv g^{\mu\nu} R_{\mu\nu})$
$\mathcal{R}^{\lambda}{}_{\sigma\mu\nu}$:	Riemann tensor constructed with $\Gamma^{\lambda}{}_{\mu\nu}$
$\mathcal{R}_{\mu\nu}$:	$\equiv \mathcal{R}^{\sigma}{}_{\mu\sigma\nu}$
$\mathcal{R}$:	$\equiv g^{\mu\nu}\mathcal{R}_{\mu\nu}$
S_M:	Matter action
$T_{\mu\nu}$:	Stress-energy tensor $\left(\equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}\right)$
$\Delta^{\lambda}{}_{\mu\nu}$:	Hypermomentum $\left(\equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta \Gamma^{\lambda}{}_{\mu\nu}}\right)$
ϕ:	Scalar field (generic)
ψ:	Matter fields (collectively)

Preface

The terms “modified gravity” and “alternative theory of gravity” have become standard terminology for theories proposed for describing the gravitational interaction which differ from the most conventional one, General Relativity. Modified or alternative theories of gravity have a long history. The first attempts date back to the 1920s, soon after the introduction of Einstein’s theory. Interest in this research field, which was initially driven by curiosity or a desire to challenge the then newly introduced General Theory of Relativity, has subsequently varied depending on circumstances, responding to the appearance of new motivations. However, there has been more or less continuous activity in this subject over the last 85 years.

When the research presented in this thesis began, interest in modified gravity was already at a high point and it has continued increasing further until the present day. This recent stimulus has mostly been due to combined motivation coming from the well-known cosmological problems related to the accelerated expansion of the universe and the feedback from High Energy Physics.

Due to the above, and even though the main scope of this thesis is to present the research conducted by the author during the period November 2004 - October 2007, a significant effort has been made so that this thesis can also serve as a guide for readers who have recently developed an interest in this field. To this end, special attention has been paid to giving a coherent presentation of the motivation for considering alternative theories of gravity as well as to giving a very general analysis of the foundations of gravitation theory. Also, an effort has been made to present the theories discussed thoroughly, so that readers less familiar with this subject can be introduced to them before gradually moving on to their more complicated characteristics and applications.

The outline of this thesis is as follows: In the Introduction, several open issues related to gravity are discussed, including the cosmological problems related to dark matter and dark energy, and the search for a theory of Quantum Gravity. Through the presentation of a historical timeline of the passage from Newtonian gravity to General Relativity, and a comparison with the current status of the latter in the light of the problems just mentioned, the motivations for considering alternative theories of gravity are introduced. Chapter 2 is devoted to the basic principles which gravitation theories should follow. The Dicke framework, the various forms of the Equivalence Principle and the so-called metric postulates are critically reviewed and the assumptions that lead to General Relativity are examined.

Additionally, the ways of relaxing these assumptions are explored together with the resulting theories. In Chapter 3, we focus on specific theories: scalar-tensor theory, metric, Palatini and metric-affine $f(R)$ gravity and Gauss–Bonnet gravity, and their theoretical characteristics are thoroughly presented. Chapter 4 contains a discussion about the possible dynamical equivalence between these theories, while in Chapter 5 their cosmological phenomenology is presented. Attention is paid to their ability to address the well-known cosmological problems and to their cosmological viability. Chapter 6 is devoted to the study of the weak and strong gravity regimes in these modified theories of gravity. The Newtonian and post-Newtonian limits, stability issues, non-vacuum solutions, *etc.* are discussed as criteria for the viability of these theories. Finally, Chapter 7 contains the conclusions of this work, as well as suggestions and remarks about future work in the field of modified gravity.

A number of people have contributed in this thesis in various ways. First and foremost, I would like to thank my PhD advisors, Stefano Liberati and John Miller, for their constant support during the course of this work. It is difficult for me to imagine having better advisors than Stefano and John, to whom I am truly grateful, not only for their guidance but also for standing by me in all my choices and for the impressive amount of patience they have exhibited during the course of these three years. Special thanks to John for his untiring correction of my spelling, grammar and (ab)use of the English language.

I cannot thank enough my collaborators Enrico Barausse and Valerio Faraoni, not only for their hard work on our common projects, but also for numerous hours of conversation and debate mostly, but definitely not exclusively, on scientific issues. It has really been a pleasure for me to collaborate with them. I am also very grateful to Matt Visser and Salvatore Capozziello, my thesis examiners, for undertaking the task of reviewing this manuscript and for their invaluable suggestions.

During the course of this research I have benefited from systematic or occasional but always stimulating discussions with a number of people, besides those already mentioned. Aware of the fact that I am running the risk of forgetting a few — and apologising in advance for that — I will attempt to name them here and express my gratitude: Sebastiano Sonego, Tomi Koivisto, Gonzalo Olmo, Mihalis Dafermos, Sergei Odintsov, Mauro Francaviglia, Gianluca Allemandi, Carlo Baccigalupi, Francesca Perrotta and Urbano Franca.

SISSA has provided an ideal environment for conducting my research over the last three years. I would like to thank all of my colleagues for contributing to that. I will have numerous lunches, dinners and unforgettable get-togethers in Trieste to remember. Special thanks to my office mate Federico Stivoli, for all the fun we had while sharing a room in SISSA.

I am also grateful to the friends and loved ones that visited me for shorter or longer periods during my stay in Trieste. Their presence definitely made these three years more memorable. Last, but definitely not least, I would like to thank my family for their love and their help in finding my way through life. If it was not for their continuous and untiring support it would have been impossible for me to start, let alone finish, this PhD.

Trieste, 23/10/2007 **Thomas P. Sotiriou**

Contents

Chapter 1

Introduction

1.1 General Relativity is the theory of gravity, isn't it?

It is remarkable that gravity is probably the fundamental interaction which still remains the most enigmatic, even though it is so related with phenomena experienced in everyday life and is the one most easily conceived of without any sophisticated knowledge. As a matter of fact, the gravitational interaction was the first one to be put under the microscope of experimental investigation, obviously due to exactly the simplicity of constructing a suitable experimental apparatus.

Galileo Galilei was the first to introduce pendulums and inclined planes to the study of terrestrial gravity at the end of the 16th century. It seems that gravity played an important role in the development of Galileo's ideas about the necessity of experiment in the study of science, which had a great impact on modern scientific thinking. However, it was not until 1665, when Sir Isaac Newton introduced the now renowned "inverse-square gravitational force law", that terrestrial gravity was actually united with celestial gravity in a single theory. Newton's theory made correct predictions for a variety of phenomena at different scales, including both terrestrial experiments and planetary motion.

Obviously, Newton's contribution to gravity — quite apart from his enormous contribution to physics overall — is not restricted to the expression of the inverse square law. Much attention should be paid to the conceptual basis of his gravitational theory, which incorporates two key ideas: i) The idea of absolute space, *i.e.* the view of space as a fixed, unaffected structure; a rigid arena in which physical phenomena take place. ii) The idea of what was later called the Weak Equivalence Principle which, expressed in the language of Newtonian theory, states that the inertial and the gravitational mass coincide.

Asking whether Newton's theory, or any other physical theory for that matter, is right or wrong, would be ill-posed to begin with, since any consistent theory is apparently "right". A more appropriate way to pose the question would be to ask how suitable is this theory for describing the physical world or, even better, how large a portion of the physical world is sufficiently described by this theory. Also, one could ask how unique the specific theory is for the description of the relevant phenomena. It was obvious in the first 20 years after the introduction of Newtonian gravity that it did manage to explain all of the aspects of gravity known at that time. However, all of the questions above were posed sooner or later.

In 1855, Urbain Le Verrier observed a 35 arc-second excess precession of Mercury's orbit and later on, in 1882, Simon Newcomb measured this precession more accurately to be 43 arc-seconds. This experimental fact was not predicted by Newton's theory. It should be noted that Le Verrier initially tried to explain the precession within the context of Newtonian gravity, attributing it to the existence of another, yet unobserved, planet whose orbit lies within that of Mercury. He was apparently influenced by the fact that examining the distortion of the planetary orbit of Uranus in 1846 had led him and his collaborator, John Couch Adams, to the discovery of Neptune and the accurate prediction of its position and momenta. However, this innermost planet was never found.

On the other hand, in 1893 Ernst Mach stated what was later called by Albert Einstein "Mach's principle". This is the first constructive attack on Newton's idea of absolute space after the 17th century debate between Gottfried Wilhelm Leibniz and Samuel Clarke (Clarke was acting as Newton's spokesman) on the same subject, known as the Leibniz–Clarke Correspondence. Mach's idea can be considered as rather vague in its initial formulation and it was essentially brought into mainstream physics later on by Einstein along the following lines: "...inertia originates in a kind of interaction between bodies...". This is obviously in contradiction with Newton's ideas, according to which inertia was always relative to the absolute frame of space. There exists also a later, probably clearer interpretation of Mach's Principle, which, however, also differs in substance. This was given by Dicke: "The gravitational constant should be a function of the mass distribution in the universe". This is different from Newton's idea of the gravitational constant as being universal and unchanging. Now Newton's basic axioms were being reconsidered.

But it was not until 1905, when Albert Einstein completed Special Relativity, that Newtonian gravity would have to face a serious challenge. Einstein's new theory, which managed to explain a series of phenomena related to non-gravitational physics, appeared to be incompatible with Newtonian gravity. Relative motion and all the linked concepts had gone well beyond the ideas of Galileo and Newton and it seemed that Special Relativity should somehow be generalised to include non-inertial frames. In 1907, Einstein introduced the equivalence between gravitation and inertia and successfully used it to predict the gravitational redshift. Finally, in 1915, he completed the theory of General Relativity, a generalisation of Special Relativity which included gravity. Remarkably, the theory matched perfectly the experimental result for the precession of Mercury's orbit, as well as other experimental findings like the Lense-Thirring gravitomagnetic precession (1918) and the gravitational deflection of light by the Sun, as measured in 1919 during a Solar eclipse by Arthur Eddington.

General Relativity overthrew Newtonian gravity and continues to be up to now an extremely successful and well-accepted theory for gravitational phenomena. As mentioned before, and as often happens with physical theories, Newtonian gravity did not lose its appeal to scientists. It was realised, of course, that it is of limited validity compared to General Relativity, but it is still sufficient for most applications related to gravity. What is more, at a certain limit of gravitational field strength and velocities, General Relativity inevitably reduces to Newtonian gravity. Newton's equations for gravity might have been generalised and some of the axioms of his theory may have been abandoned, like the notion of an absolute frame, but some of the cornerstones of his theory still exist in the foundations of General Relativity, the most prominent example being the Equivalence Principle, in a more suitable formulation of course.

This brief chronological review, besides its historical interest, is outlined here also for a practical reason. General Relativity is bound to face the same questions as were faced by Newtonian gravity and many would agree that it is actually facing them now. In the forthcoming sections, experimental facts and theoretical problems will be presented which justify that this is indeed the case. Remarkably, there exists a striking similarity to the problems which Newtonian gravity faced, *i.e.* difficulty in explaining particular observations, incompatibility with other well established theories and lack of uniqueness. This is the reason behind the question mark in the title of this section.

1.2 A high-energy theory of gravity?

Many will agree that modern physics is based on two great pillars: General Relativity and Quantum Field Theory. Each of these two theories has been very successful in its own arena of physical phenomena: General Relativity in describing gravitating systems and non-inertial frames from a classical viewpoint or on large enough scales, and Quantum Field Theory in revealing the mysteries of high energy or small scale regimes where a classical description breaks down. However, Quantum Field Theory assumes that spacetime is flat and even its extensions, such as Quantum Field Theory in curved space time, consider spacetime as a rigid arena inhabited by quantum fields. General Relativity, on the other hand, does not take into account the quantum nature of mater. Therefore, it comes naturally to ask what happens if a strong gravitational field is present at small, essentially quantum, scales? How do quantum fields behave in the presence of gravity? To what extend are these amazing theories compatible?

Let us try to pose the problem more rigorously. Firstly, what needs to be clarified is that there is no precise proof that gravity should have some quantum representation at high energies or small scales, or even that it will retain its nature as an interaction. The gravitational interaction is so weak compared with other interactions that the characteristic scale under which one would expect to experience non-classical effects relevant to gravity, the Planck scale, is 10^{-33} cm. Such a scale is not of course accessible by any current experiment and it is doubtful whether it will ever be accessible to future experiments either[1]. However, there are a number of reasons for which one would prefer to fit together General Relativity and Quantum Field Theory [1, 2]. Let us list some of the most prominent ones here and leave the discussion about how to address them for the next section.

1.2.1 Searching for the unknown

Curiosity is probably the motivation leading scientific research. From this perspective it would be at least unusual if the gravity research community was so easily willing to abandon any attempt to describe the regime where both quantum and gravitational effects are important. The fact that the Planck scale seems currently experimentally inaccessible does not in any way imply that it is physically irrelevant. On the contrary, one can easily name some very important open issues of contemporary physics that are related to the Planck scale.

A particular example is the Big Bang scenario in which the universe inevitably goes though an era in which its dimensions are smaller than the Planck scale (Planck era). On the other hand, spacetime in General Relativity is a continuum and so in principle all scales are relevant. From this perspective, in order to derive conclusions about the nature of spacetime one has to answer the question of what happens on very small scales.

1.2.2 Intrinsic limits in General Relativity and Quantum Field Theory

The predictions of a theory can place limits on the extent of its ability to describe the physical world. General Relativity is believed by some to be no exception to this rule. Surprisingly, this problem is related to one of the most standard processes in a gravitational theory: gravitational collapse. Studying gravitational collapse is not easy since generating solutions to Einstein's field equations can be a tedious procedure. We only have a few exact solutions to hand and numerical or approximate solutions are often the only resort. However, fortunately, this does not prevent one from making general arguments about the ultimate fate of a collapsing object.

This was made possible after the proof of the Penrose–Hawking singularity theorems [3, 4]. These theorems state that a generic spacetime cannot remain regular beyond a finite proper time, since gravitational collapse (or time reversal of cosmological expansion) will inevitably lead to spacetime singularities. In a strict interpretation, the presence of a singularity is inferred by geodesic incompleteness, *i.e.* the inability of an observer travelling along a geodesic to extend this geodesic for an infinite time as measured by his clock. In practical terms this can be loosely interpreted to mean that an observer free-falling in a gravitational field will "hit" a singularity in a finite time and Einstein's equation cannot then predict what hap-

[1]This does not imply, of course, that imprints of Quantum Gravity phenomenology cannot be found in lower energy experiments.

pens next. Such singularities seem to be present in the centre of black holes. In the Big Bang scenario, the universe itself emerges out of such a singularity.

Wheeler has compared the problem of gravitational collapse in General Relativity with the collapse of the classical Rutherford atom due to radiation [5]. This raises hopes that principles of quantum mechanics may resolve the problem of singularities in General Relativity, as happened for the Rutherford model. In a more general perspective, it is reasonable to hope that quantization can help to overcome these intrinsic limits of General Relativity.

On the other hand, it is not only General Relativity that has an intrinsic limit. Quantum Field Theory presents some disturbing ultraviolet divergences. Such divergences, caused by the fact that integrals corresponding to the Feynman diagrams diverge due to very high energy contributions — hence the name ultraviolet — are discretely removed by a process called renormalization. These divergences are attributed to the perturbative nature of the quantization process and the renormalization procedure is somehow unappealing and probably not so fundamental, since it appears to cure them in a way that can easily be considered as non-rigorous from a mathematical viewpoint. A non-perturbative approach is believed to be free of such divergences and there is hope that Quantum Gravity may allow that (for early results see [6, 7, 8, 9, 10]).

1.2.3 A conceptual clash

Every theory is based on a series of conceptual assumption and General Relativity and Quantum Field Theory are no exceptions. On the other hand, for two theories to work in a complementary way to each other and fit well together, one would expect an agreement between their conceptual bases. This is not necessarily the case here.

There are two main points of tension between General Relativity and Quantum Field Theory. The first has to do with the concept of time: Time is given and not dynamical in Quantum Field Theory and this is closely related to the fact that spacetime is considered as a fixed arena where phenomena take place, much like Newtonian mechanics. On the other hand, General Relativity considers spacetime as being dynamical, with time alone not being such a relevant concept. It is more of a theory describing relations between different events in spacetime than a theory that describes evolution over some running parameter. One could go further and seek for the connection between what is mentioned here and the differences between gauge invariance as a symmetry of Quantum Field Theory and diffeomorphism invariance as a symmetry of General Relativity.

The second conceptual issue has to do with Heisenberg's uncertainty principle in Quantum Theory which is absent in General Relativity as a classical theory. It is interesting to note that General Relativity, a theory in which background independence is a key concept, actually introduces spacetime as an exact and fully detailed record of the past, the present and the future. Everything would be fixed for a super-observer that could look at this 4-dimensional space from a fifth dimension.

On the other hand, Quantum Field Theory, a background dependent theory, manages to include a degree of uncertainty for the position of any event in spacetime.

Having a precise mathematical structure for a physical theory is always important, but getting answers to conceptual issues is always the main motivation for studying physics in the first place. Trying to attain a quantum theory of gravity could lead to such answers.

1.2.4 The vision for unification

Apart from strictly scientific reasons for trying to make a match between Quantum Field Theory and General Relativity, there is also a long-standing intellectual desire, maybe of a philosophical nature or stemming from physical intuition, to bring the fundamental interactions to a unification. This was the vision of Einstein himself in his late years. His perspective was that a geometric description might be the solution. Nowdays most of the scientists active in this field would disagree with this approach to unification and there is much debate about whether the geometric interpretation or a field theory interpretation of General Relativity is actually preferable — Steven Weinberg for example even claimed in [11] that “no-one” takes a geometric viewpoint of gravity “seriously”. However, very few would argue that such a unification should not be one of the major goals of modern physics. An elegant theory leading to a much deeper understanding of both gravity and the quantum world could be the reward for achieving this.

1.3 The Cosmological and Astrophysical riddles

1.3.1 Cosmology in a nutshell

Taking things in chronological order, we started by discussing the possible shortcomings of General Relativity on very small scales, as those were the first to appear in the literature. However, if there is one scale for which gravity is by far of the utmost importance, this is surely the cosmic scale. Given the fact that other interactions are short-range and that at cosmological scales we expect matter characteristics related to them to have “averaged out” — for example we do not expect that the universe has an overall charge — gravity should be the force which rules cosmic evolution. Let us see briefly how this comes about by considering Einstein’s equations combined with our more obvious assumptions about the main characteristics of the observable universe.

Even though matter is not equally distributed through space and by simple browsing through the sky one can observe distinct structures such as stars and galaxies, if attention is focused on larger scales the universe appears as if it was made by patching together multiple copies of the same pattern, *i.e.* a suitably large elementary volume around the Earth and another elementary volume of the same size elsewhere will have little difference. This suitable scale is actually $\approx 10^8$ light years, slightly larger than the typical size of a cluster of galaxies. In Cosmology one wants to deal with scales larger than that and to describe the universe as a whole. Therefore, as far as Cosmology is concerned the universe can be very well described as homogeneous and isotropic.

To make the above statement useful from a quantitative point of view, we have to turn it into an idealized assumption about the matter and geometry of the Universe. Note that the universe is assumed to be spatially homogeneous and isotropic at each instant of cosmic time. In more rigorous terms, we are talking about homogeneity on each one of a set of 3-dimensional space-like hypersurfaces. For the matter, we assume a perfect fluid description and these spacelike hypersurfaces are defined in terms of a family of fundamental observers who are comoving with this perfect fluid and who can synchronise their comoving clocks so as to measure the universal cosmic time. The matter content of the universe is then just described by two parameters, a uniform density ρ and a uniform pressure p, as if the matter in stars and atoms is scattered through space. For the geometry we idealize the curvature of space to be everywhere the same.

Let us proceed by imposing these assumption on the equation describing gravity and very briefly review the derivation of the equations governing the dynamics of the universe, namely the Friedmann equations. We refer the reader to standard textbooks for a more detailed discussion of the precise geometric definitions of homogeneity and isotropy and their implications for the form of the metric (*e.g.* [11]). Additionally, for what comes next, the reader is assumed to be acquainted with the basics of General Relativity, some of which will also be reviewed in the next chapter.

Einstein's equation has the following form

$$G_{\mu\nu} = 8\,\pi\,G\,T_{\mu\nu}, \tag{1.1}$$

where

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu} \tag{1.2}$$

is the Einstein tensor and $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar of the metric $g_{\mu\nu}$. G is the gravitational constant and $T_{\mu\nu}$ is the matter stress-energy tensor. Under the assumptions of homogeneity and isotropy, the metric can take the form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\right], \tag{1.3}$$

known as the Friedmann-Lemaître-Robertson-Walker metric (FLRW). $k = -1, 0, 1$ according to whether the universe is hyperspherical (closed), spatially flat, or hyperbolic (open) and $a(t)$ is called the scale factor. Inserting this metric into eq. (1.1) and taking into account that for a perfect fluid

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + p\,g^{\mu\nu}, \tag{1.4}$$

where u^μ denotes the four-velocity of an observer comoving with the fluid and ρ and p are the energy density and the pressure of the fluid, one gets the following equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{k}{a^2}, \tag{1.5}$$

$$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G}{3}\left(\rho + 3p\right), \tag{1.6}$$

where an overdot denotes differentiation with respect to coordinate time t.

Eqs. (1.5) and (1.6) are called the Friedmann equations. By imposing homogeneity and isotropy as characteristics of the universe that remain unchanged with time on suitably large scales we have implicitly restricted any evolution to affect only one remaining characteristic: its size. This is the reason why the Friedmann equations are equations for the scale factor, $a(t)$, which is a measure of the evolution of the size of any length scale in the universe. Eq. (1.5), being an equation in $\dot{a}$, tells us about the velocity of the expansion or contraction, whereas eq. (1.6), which involves $\ddot{a}$, tells us about the acceleration of the expansion or the contraction. According to the Big Bang scenario, the universe starts expanding with some initial velocity. Setting aside the contribution of the k-term for the moment, eq. (1.5) implies that the universe will continue to expand as long as there is matter in it. Let us also take into consideration the contribution of the k-term, which measures the spatial curvature and in which k takes the values $-1, 0, 1$. If $k = 0$ the spatial part of the metric (1.3) reduces to a flat metric expressed in spherical coordinates. Therefore, the universe is spatially flat and eq. (1.5) implies that it has to become infinite, with ρ approaching zero, in order for the expansion to halt. On the other hand, if $k = 1$ the expansion can halt at a finite density at which the matter contribution is balanced by the k-term. Therefore, at a finite time the universe will stop expanding and will re-collapse. Finally for $k = -1$ one can see that even if matter is completely dissolved, the k-term will continue to "pump" the expansion which means that the latter can never halt and the universe will expand forever.

Let us now focus on eq. (1.6) which, as already mentioned, governs the acceleration of the expansion. Notice that k does not appear in this equation, *i.e.* the acceleration does not depend on the characteristics of the spatial curvature. Eq. (1.6) reveals what would be expected by simple intuition: that gravity is always an attractive force. Let us see this in detail. The Newtonian analogue of eq. (1.6) would be

$$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G}{3}\rho, \tag{1.7}$$

where ρ denotes the matter density. Due to the minus sign on the right hand side and the positivity of the density, this equation implies that the expansion will always be slowed by gravity.

The presence of the pressure term in eq. (1.6) is simply due to the fact that in General Relativity, it is not simply matter that gravitates but actually energy and therefore the pressure should be included. For what could be called ordinary matter (*e.g.* radiation, dust, perfect fluids, *etc*.) the pressure can be expected to be positive, as with the density. More precisely, one could ask that the matter satisfies the four energy conditions [12]:

1. Null Energy Condition: $\rho + p \geq 0$,
2. Weak Energy Condition: $\rho \geq 0, \quad \rho + p \geq 0$,
3. Strong Energy Condition: $\rho + p \geq 0, \quad \rho + 3p \geq 0$,
4. Dominant Energy Condition: $\rho \geq |p|$.

We give these conditions here in terms of the components of the stress-energy tensor of a perfect fluid but they can be found in a more generic form in [12]. Therefore, once positivity of the pressure or the validity of the Strong Energy Condition is assumed, gravity remains always an attractive force also in General Relativity [2].

To sum up, even without attempting to solve the Friedmann equations, we have already arrived at a well-established conclusion: Once we assume, according to the Big Bang scenario, that the universe is expanding, then, according to General Relativity and with ordinary matter considerations, this expansion should always be decelerated. Is this what actually happens though?

1.3.2 The first need for acceleration

We derived the Friedmann equations using two assumptions: homogeneity and isotropy of the universe. Both assumptions seem very reasonable considering how the universe appears to be today. However, there are always the questions of why does the universe appear to be this way and how did it arrive at its present form through its evolution. More importantly though, one has to consider whether the description of the universe by the Big Bang model and the Friedmann equations is self-consistent and agrees not only with a rough picture of the universe but also with the more precise current picture of it.

Let us put the problem in more rigorous terms. First of all one needs to clarify what is meant by "universe". Given that the speed of light (and consequently of any signal carrying information) is finite and adopting the Big Bang scenario, not every region of spacetime is accessible to us. The age of the universe sets an upper limit for the largest distance from which a point in space may have received information. This is what is called a "particle horizon" and its size changes with time. What we refer to as the universe is the part of the universe causally connected to us — the part inside our particle horizon. What happens outside this region is inaccessible to us but more importantly it does not affect us, at least not directly. However, it is possible to have two regions that are both accessible and causally connected to us, or to some other observer, but are not causally connected with

[2]When quantum effects are taken into account, one or more of the energy conditions can be violated, even though a suitably averaged version may still be satisfied. However, there are even classical fields that can violate the energy conditions, as we will see latter on.

each other. They just have to be inside our particle horizon without being inside each other's particle horizons. It is intuitive that regions that are causally connected can be homogeneous — they have had the time to interact. However, homogeneity of regions which are not causally connected would have to be attributed to some initial homogeneity of the universe since local interactions cannot be effective for producing this.

The picture of the universe that we observe is indeed homogeneous and isotropic on scales larger than we would expect based on our calculation regarding its age and causality. This problem was first posed in the late 1960s and has been known as the horizon problem [11, 13]. One could look to solve it by assuming that the universe is perhaps much older and this is why in the past the horizon problem has also been reformulated in the form of a question: how did the universe grow to be so old? However, this would require the age of the universe to differ by orders of magnitude from the value estimated by observations. So the homogeneity of the universe, at least at first sight and as long as we believe in the cosmological model at hand, appears to be built into the initial conditions.

Another problem, which is similar and appeared at the same time, is the flatness problem. To pose it rigorously let us return to the Friedmann equations and more specifically to eq. (1.5). The Hubble parameter H is defined as $H = \dot{a}/a$. We can use it to define what is called the critical density

$$\rho_c = \frac{3\,H^2}{8\,\pi\,G}, \tag{1.8}$$

which is the density which would make the 3-geometry flat. Finally, we can use the critical density in order to create the dimensionless fractions

$$\Omega = \frac{\rho}{\rho_c}, \tag{1.9}$$

$$\Omega_k = -\frac{k}{a^2 H^2}. \tag{1.10}$$

It is easy to verify from eq. (1.5) that

$$\Omega + \Omega_k = 1. \tag{1.11}$$

As dimensionless quantities, Ω and Ω_k are measurable, and by the 1970s it was already known that the current value of Ω appears to be very close to 1 (see for example [14]). Extrapolating into the past reveals that Ω would have had to be even closer to 1, making the contribution of Ω_k, and consequently of the k-term in eq. (1.5), exponentially small.

The name "flatness problem" can be slightly misleading and therefore it needs to be clarified that the value of k obviously remains unaffected by the evolution. To avoid misconceptions it is therefore better to formulate the flatness problems in terms of Ω itself. The fact that Ω seems to be taking a value so close to the critical one at early times is not a consequence of the evolution and once more, as happened with the horizon problem, it appears as a strange coincidence which can only be attributed to some fine tuning of the initial conditions.

But is it reasonable to assume that the universe started in such a homogeneous state, even at scales that where not causally connected, or that its density was dramatically close to its critical value without any apparent reason? Even if the universe started with extremely small inhomogeneities it would still not present such a homogeneous picture currently. Even if shortcomings like the horizon and flatness problems do not constitute logical inconsistencies of the standard cosmological model but rather indicate that the present state of the universe depends critically on some initial state, this is definitely a feature that many consider undesirable.

So, by the 1970s Cosmology was facing new challenges. Early attempts to address these problems involved implementing a recurring or oscillatory behaviour for the universe and therefore were departing from the standard ideas of cosmological evolution [15, 16, 17]. This problem also triggered Charles W. Misner to propose the "Mixmaster Universe" (Bianchi type IX metric), in which a chaotic behaviour was supposed to ultimately lead to statistical homogeneity and isotropy [18]. However, all of these ideas have proved to be non-viable descriptions of the observed universe.

A possible solution came in the early 1980s when Alan Guth proposed that a period of exponential expansion could be the answer [19]. The main idea is quite simple: an exponential increase of the scale factor $a(t)$ implies that the Hubble parameter H remains constant. On the other hand, one can define the Hubble radius $c/H(t)$ which, roughly speaking, is a measure of the radius of the observable universe at a certain time t. Then, when $a(t)$ increases exponentially, the Hubble radius remains constant, whereas any physical length scale increases exponentially in size. This implies that in a short period of time, any lengthscale which could, for example, be the distance between two initially causally connected observers, can become larger than the Hubble radius. So, if the universe passed through a phase of very rapid expansion, then the part of it that we can observe today may have been significantly smaller at early times than what one would naively calculate using the Friedmann equations. If this period lasted long enough, then the observed universe could have been small enough to be causally connected at the very early stage of its evolution. This rapid expansion would also drive Ω_k to zero and consequently Ω to 1 today, due to the very large value that the scale factor $a(t)$ would currently have, compared to its initial value. Additionally, such a procedure is very efficient in smoothing out inhomogeneities, since the physical wavelength of a perturbation can rapidly grow to be larger than the Hubble radius. Thus, both of the problems mentioned above seem to be effectively addressed.

Guth was not the only person who proposed the idea of an accelerated phase and some will argue he was not even the first. Contemporarily with him, Alexei Starobinski had proposed that an exponential expansion could be triggered by quantum corrections to gravity and provide a mechanism to replace the initial singularity [20]. There are also earlier proposals whose spirit is very similar to that of Guth, such as those by Demosthenes Kazanas [21], Katsuhiko Sato [22] and

Robert Brout *et al.* [23]. However, Guth's name is the one most related with these idea since he was the first to provide a coherent and complete picture on how an exponential expansion could address the cosmological problems mentioned above. This period of accelerated expansion is known as inflation, a terminology borrowed from economics due to the apparent similarity between the growth of the scale factor in Cosmology and the growth of prices during an inflationary period. To be more precise, one defines as inflation any period in the cosmic evolution for which

$$\ddot{a} > 0. \tag{1.12}$$

However, a more detailed discussion reveals that an exponential expansion, or at least quasi-exponential since what is really needed is that the physical scales increase much more rapidly than the Hubble radius increases, is not something trivial to achieve. As discussed in the previous section, it does not appear to be easy to trigger such an era in the evolution of the universe, since accelerated expansion seems impossible according to eq. (1.6), as long as both the density and the pressure remain positive. In other words, satisfying eq. (1.12) requires

$$(\rho + 3p) < 0 \Rightarrow \rho < -3p, \tag{1.13}$$

and assuming that the energy density cannot be negative, inflation can only be achieved if the overall pressure of the ideal fluid which we are using to describe the universe becomes negative. In more technical terms, eq. (1.13) implies the violation of the Strong Energy Condition [12].

It does not seem possible for any kind of baryonic matter to satisfy eq. (1.13), which directly implies that a period of accelerated expansion in the universe evolution can only be achieved within the framework of General Relativity if some new form of matter field with special characteristics is introduced. Before presenting any scenario of this sort though, let us resort to observations to convince ourselves about whether such a cosmological era is indeed necessary.

1.3.3 Cosmological and Astronomical Observations

In reviewing the early theoretical shortcomings of the Big Bang evolutionary model of the universe we have seen indications for an inflationary era. The best way to confirm those indications is probably to resort to the observational data at hand for having a verification. Fortunately, there are currently very powerful and precise observations that allow us to look back to very early times.

A typical example is the Cosmic Microwave Background Radiation (CMBR). In the early universe, baryons, photons and electrons formed a hot plasma, in which the mean free path of a photon was very short due to constant interactions of the photons with the plasma through Thomson scattering. However, due to the expansion of the universe and the subsequent decrease of temperature, it subsequently became energetically favourable for electrons to combine with protons to form hydrogen atoms (recombination). This allowed photons to travel freely through space. This decoupling of photons from matter is believed to have taken place at a redshift of $z \sim 1088$, when the age of the universe was about $380,000$ years old or approximately 13.7 billion years ago. The photons which left the last scattering surface at that time, then travelled freely through space and have continued cooling since then. In 1965 Penzias and Wilson noticed that a Dicke radiometer which they were intending to use for radio astronomy observations and satellite communication experiments had an excess 3.5K antenna temperature which they could not account for. They had, in fact, detected the CMBR, which actually had already been theoretically predicted in 1948 by George Gamow. The measurement of the CMBR, apart from giving Penzias and Wilson a Nobel prize publication [24], was also to become the number one verification of the Big Bang model.

Later measurements showed that the CMBR has a black body spectrum corresponding to approximately 2.7 K and verifies the high degree of isotropy of the universe. However, it was soon realized that attention should be focused not on the overall isotropy, but on the small anisotropies present in the CMBR, which reveal density fluctuations [25, 26]. This triggered a numbered of experiments, such as COBE, Toco, BOOMERanG and MAXIMA [27, 28, 29, 30, 31, 32]. The most recent one is the Wilkinson Microwave Anisotropy Probe (WMAP) [33] and there are also new experiments planned for the near future, such as the Planck mission [34].

The density fluctuations indicated by the small anisotropies in the temperature of CMBR are believed to act as seeds for gravitational collapse, leading to gravitationally bound objects which constitute the large scale matter structures currently present in the universe [35]. This allows us to build a coherent scenario about how these structures where formed and to explain the current small scale inhomogeneities and anisotropies. Besides the CMBR, which gives information about the initial anisotropies, one can resort to galaxy surveys for complementary information. Current surveys determining the distribution of galaxies include the 2 degree Field Galaxy Redshift Survey (2dF GRS) [36] and the ongoing Sloan Digital Sky Survey (SDSS) [37]. There are also other methods used to measure the density variations such as gravitational lensing [38] and X-ray measurements [39].

Besides the CMBR and Large Scale Structure surveys, another class of observations that appears to be of special interest in Cosmology are those of type Ia supernovae. These exploding stellar objects are believed to be approximately standard candles, *i.e.* astronomical objects with known luminosity and absolute magnitude. Therefore, they can be used to reveal distances, leading to the possibility of forming a redshift-distance relation and thereby measuring the expansion of the universe at different redshifts. For this purpose, there are a number of supernova surveys [40, 41, 42].

But let us return to how we can use the outcome of the experimental measurements mentioned above in order to infer whether a period of accelerated expansion has occurred. The most recent CMBR dataset is that of the Three-Year WMAP Observations [43] and results are derived using combined WMAP data and data from supernova and galaxy surveys in many cases. To begin with, let us focus on the value of Ω_k. The WMAP data (combined with Supernova Legacy Survey data [41]) indicates that

$$\Omega_k = -0.015^{+0.020}_{-0.016}, \tag{1.14}$$

i.e. that Ω is very close to unity and the universe appears to be spatially flat, while the power spectrum of the CMBR appears to be consistent with gaussianity and adiabaticity [44, 45]. Both of these facts are in perfect agreement with the predictions of the inflationary paradigm.

In fact, even though the theoretical issues mentioned in the previous paragraph (*i.e.* the horizon and the flatness problem) were the motivations for introducing the inflationary paradigm, it is the possibility of relating large scale structure formation with initial quantum fluctuations that appears today as the major advantage of inflation [46]. Even if one would choose to dismiss, or find another way to address, problems related to the initial conditions, it is very difficult to construct any other theory which could successfully explain the presence of over-densities with values suitable for leading to the present picture of our universe at smaller scales [35]. Therefore, even though it might be premature to claim that the inflationary paradigm has been experimentally verified, it seems that the evidence for there having been a period of accelerated expansion of the universe in the past is very compelling.

However, observational data hold more surprises. Even though Ω is measured to be very close to unity, the contribution of matter to it, Ω_m, is only of the order of 24%. Therefore, there seems to be some unknown form of energy in the universe, often called *dark energy*. What is more, observations indicate that, if one tries to model dark energy as a perfect fluid with equation of state $p = w\rho$ then

$$w_{de} = -1.06^{+0.13}_{-0.08}, \tag{1.15}$$

so that dark energy appears to satisfy eq. (1.13). Since it is the dominant energy component today, this implies that the universe should be undergoing an accelerated expansion currently as well. This is also what was found earlier using supernova surveys [40].

As is well known, between the two periods of acceleration (inflation and the current era) the other conventional eras of evolutionary Cosmology should take place. This means that inflation should be followed by Big Bang Nucleosynthesis (BBN), referring to the production of nuclei other than hydrogen. There are very strict bounds on the abundances of primordial light elements, such as deuterium, helium and lithium, coming from observations [47] which do not seem to allow significant deviations from the standard cosmological model [48]. This implies that BBN most probably took place during an era of radiation domination, *i.e.* a period in which radiation was the most important contribution to the energy density. On the other hand, the formation of matter structures requires that the radiation dominated era is followed by a matter dominated era. The transition, from radiation domination to matter domination, comes naturally since the matter energy density is inversely proportional to the volume and, therefore, proportional to a^{-3}, whereas the radiation energy density is proportional to a^{-4} and so it decreases faster than the matter energy density as the universe expands.

To sum up, our current picture of the evolution of the universe as inferred from observations comprises a pre-inflationary (probably quantum gravitational) era followed by an inflationary era, a radiation dominated era, a matter dominated era and then a second era of accelerated expansion which is currently taking place. Such an evolution departs seriously from the one expected if one just takes into account General Relativity and conventional matter and therefore appears to be quite unorthodox.

But puzzling observations do not seem to stop here. As mentioned before, Ω_m accounts for approximately 24% of the energy density of the universe. However, one also has to ask how much of this 24% is actually ordinary baryonic matter. Observations indicate that the contribution of baryons to that, Ω_b, is of the order of $\Omega_b \sim 0.04$ leaving some 20% of the total energy content of the universe and some 83% of the matter content to be accounted for by some unknown unobserved form of matter, called *dark matter*. Differently from dark energy, dark matter has the gravitational characteristics of ordinary matter (hence the name) and does not violate the Strong Energy Condition. However, it is not directly observed since it appears to interact very weakly if at all.

The first indications for the existence of dark matter did not come from Cosmology. Historically, it was Fritz Zwicky who first posed the “missing mass” question for the Coma cluster of galaxies [49, 50] in 1933. After applying the virial theorem in order to compute the mass of the cluster needed to account for the motion of the galaxies near to its edges, he compared this with the mass obtained from galaxy counts and the total brightness of the cluster. The virial mass turned out to be larger by a factor of almost 400.

Later, in 1959, Kahn and Waljter were the first to propose the presence of dark matter in individual galaxies [51]. However, it was in the 1970s that the first compelling evidence for the existence of dark matter came about: the rotation curves of galaxies, *i.e.* the velocity curves of stars as functions of the radius, did not appear to have the expected shapes. The velocities, instead of decreasing at large distances as expected from Keplerian dynamics and the fact that most of the visible mass of a galaxy is located near to its centre, appeared to be flat [52, 53, 54]. As long as Keplerian dynamics are considered correct, this implies that there should be more matter than just the luminous matter, and this additional matter should have a different distribution within the galaxy (dark matter halo).

Much work has been done in the last 35 years to analyse the problem of dark matter in astrophysical environments (for recent reviews see [55, 56, 57]) and there are also recent findings, such as the observations related to the Bullet Cluster, that deserve a special mention[3]. The main conclusion that can be drawn is that some form of dark matter is present in galaxies and clusters of galaxies. What is more,

[3]Weak lensing observations of the Bullet cluster (1E0657-558), which is actually a unique cluster merger, appear to provide direct evidence for the existence of dark matter [58].

taking also into account the fact that dark matter appears to greatly dominate over ordinary baryonic matter at cosmological scales, it is not surprising that current models of structure formation consider it as a main ingredient (*e.g.* [59]).

1.3.4 The Cosmological Constant and its problems

We have just seen some of the main characteristics of the universe as inferred from observations. Let us now set aside for the moment the discussion of the earlier epochs of the universe and inflation and concentrate on the characteristic of the universe as it appears today: it is probably spatially flat ($\Omega_k \sim 0$), expanding in a accelerated manner as confirmed both from supernova surveys and WMAP, and its matter energy composition consists of approximately 76% dark energy, 20% dark matter and only 4% ordinary baryonic matter. One has to admit that this picture is not only surprising but maybe even embarrassing, since it is not at all close to what one would have expected based on the standard cosmological model and what is more it reveals that almost 96% of the energy content of the universe has a composition which is unknown to us.

In any case, let us see which is the simplest model that agrees with the observational data. To begin with, we need to find a simple explanation for the accelerated expansion. The first physicist to consider a universe which exhibits an accelerated expansion was probably Willem de Sitter [60]. A de Sitter space is the maximally symmetric, simply-connected, Lorentzian manifold with constant positive curvature. It may be regarded as the Lorentzian analogue of an n-sphere in n dimensions. However, the de Sitter spacetime is not a solution of the Einstein equations, unless one adds a cosmological constant Λ to them, *i.e.* adds on the left hand side of eq. (1.1) the term $\Lambda g_{\mu\nu}$.

Such a term was not included initially by Einstein, even though this is technically possible since, according to the reasoning which he gave for arriving at the gravitational field equations, the left hand side has to be a second rank tensor constructed from the Ricci tensor and the metric, which is divergence free. Clearly, the presence of a cosmological constant does not contradict these requirements. In fact, Einstein was the first to introduce the cosmological constant, thinking that it would allow him to derive a solution of the field equations describing a static universe [61]. The idea of a static universe was then rapidly abandoned however when Hubble discovered that the universe is expanding and Einstein appears to have changed his mind about the cosmological constant: Gamow quotes in his autobiography, My World Line (1970): “Much later, when I was discussing cosmological problems with Einstein, he remarked that the introduction of the cosmological term was the biggest blunder of his life” and Pais quotes a 1923 letter of Einstein to Weyl with his reaction to the discovery of the expansion of the universe: “If there is no quasi-static world, then away with the cosmological term!” [62].

In any case, once the cosmological term is included in the Einstein equations, de Sitter space becomes a solution. Actually, the de Sitter metric can be brought into the form of the FLRW metric in eq. (1.3) with the scale factor and the Hubble parameter given by

$$a(t) = e^{H\,t}, \tag{1.16}$$

$$H^2 = \frac{8\,\pi\,G}{3}\,\Lambda. \tag{1.17}$$

This is sometimes referred to as the de Sitter universe and it can be seen that it is expanding exponentially.

The de Sitter solution is a vacuum solution. However, if we allow the cosmological term to be present in the field equations, the Friedmann equations (1.5) and (1.6) will be modified so as to include the de Sitter spacetime as a solution:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi\,G\,\rho+\Lambda}{3}-\frac{k}{a^2}, \tag{1.18}$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3}-\frac{4\,\pi\,G}{3}\,(\rho+3p)\,. \tag{1.19}$$

From eq. (1.19) one infers that the universe can now enter a phase of accelerated expansion once the cosmological constant term dominates over the matter term on the right hand side. This is bound to happen since the value of the cosmological constant stays unchanged during the evolution, whereas the matter density decreases like a^3. In other words, the universe is bound to approach a de Sitter space asymptotically in time.

On the other hand Ω in eq. (1.11) can now be split in two different contributions, $\Omega_\Lambda = \Lambda/(3\,H^2)$ and Ω_m, so that eq. (1.11) takes the form

$$\Omega_m+\Omega_\Lambda+\Omega_k = 1. \tag{1.20}$$

In this sense, the observations presented previously can be interpreted to mean that $\Omega_\Lambda \sim 0.72$ and the cosmological constant can account for the mysterious dark energy responsible for the current accelerated expansion. One should not fail to notice that Ω_m does not only refer to baryons. As mentioned before, it also includes dark matter, which is actually the dominant contribution. Currently, dark matter is mostly treated as being cold and not baryonic, since these characteristics appear to be in good accordance with the data. This implies that, apart from the gravitational interaction, it does not have other interactions — or at least that it interacts extremely weakly — and can be regarded as collisionless dust, with an effective equation of state $p = 0$ (we will return to the distinction between cold and hot dark matter shortly).

We have sketched our way to what is referred to as the Λ Cold Dark Matter or ΛCDM model. This is a phenomenological model which is sometimes also called the concordance model of Big Bang Cosmology, since it is more of an empirical fit to the data. It is the simplest model that can fit the cosmic microwave background observations as well as large scale structure observations and supernova observations of the accelerating expansion of the universe with a remarkable agreement (see for instance [43]). As a phenomenological model, however, it gives no insight about the nature of dark matter, or the reason for the presence of the cosmological constant, neither does it justify the value of the latter.

While it seems easy to convince someone that an answer is indeed required to the question "what exactly is dark matter and why is it almost 9 times more abundant than ordinary matter", the presence of the cosmological constant in the field equations might not be so disturbing for some. Therefore, let us for the moment put aside the dark matter problem — we will return to it shortly — and consider how natural it is to try to explain the dark energy problem by a cosmological constant (see [63, 64, 65, 66] for reviews).

It has already been mentioned that there is absolutely no reason to discard the presence of a cosmological constant in the field equations from a gravitational and mathematical perspective. Nonetheless, it is also reasonable to assume that there should be a theoretical motivation for including it — after all there are numerous modifications that could be made to the left hand side of the gravitational field equation and still lead to a consistent theory from a mathematical perspective and we are not aware of any other theory that includes more than one fundamental constant. On the other hand, it is easy to see that the cosmological term can be moved to the right hand side of the field equations with the opposite sign and be regarded as some sort of matter term. It can then be put into the form of a stress-energy tensor $T^{\mu}_{\nu} = \mathrm{diag}(\Lambda, -\Lambda, -\Lambda, -\Lambda)$, *i.e.* resembling a perfect fluid with equation of state $p = -\rho$ or $w = -1$. Notice the very good agreement with the value of w_{de} inferred from observations (eq. (1.15)), which explains the success of the ΛCDM model.

Once the cosmological constant term is considered to be a matter term, a natural explanation for it seems to arise: the cosmological constant can represent the vacuum energy associated with the matter fields. One should not be surprised that empty space has a non-zero energy density once, apart from General Relativity, field theory is also taken into consideration. Actually, Local Lorentz Invariance implies that the expectation value of the stress energy tensor in vacuum is

$$\langle T_{\mu\nu}\rangle = -\langle\rho\rangle g_{\mu\nu}, \tag{1.21}$$

and $\langle\rho\rangle$ is generically non-zero. To demonstrate this, we can take the simple example of a scalar field [67]. Its energy density will be

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla_{\mathrm{sp}}\phi)^2 + V(\phi), \tag{1.22}$$

where ∇_{sp} denotes the spatial gradient and $V(\phi)$ is the potential. The energy density will become constant for any constant value $\phi = \phi_0$ and there is no reason to believe that for $\phi = \phi_0$, $V(\phi_0)$ should be zero. One could in general assume that there is some principle or symmetry that dictates it, but nothing like this has been found up to now. So in general one should expect that matter fields have a non-vanishing vacuum energy, *i.e.* that $\langle\rho\rangle$ is non-zero.

Within this perspective, effectively there should be a cosmological constant in the field equations, given by

$$\Lambda = 8\,\pi\, G\langle\rho\rangle. \tag{1.23}$$

One could, therefore, think to use the Standard Model of particle physics in order to estimate its value. Unfortunately, however, $\langle\rho\rangle$ actually diverges due to the contribution of very high-frequency modes. No reliable exact calculation can be made but it is easy to make a rough estimate once a cutoff is considered (see for instance [11, 67]). Taking the cutoff to be the Planck scale ($M_{\rm Plank} = 10^{18}$ GeV), which is a typical scale at which the validity of classical gravity is becoming questionable, the outcome is

$$\rho_\Lambda \sim (10^{27}\,\mathrm{eV})^4. \tag{1.24}$$

On the other hand, observations indicate that

$$\rho_\Lambda \sim (10^{-3}\,\mathrm{eV})^4. \tag{1.25}$$

Obviously the discrepancy between these two estimates is very large for being attributed to any rough approximation. There is a difference of 120-orders-of-magnitude, which is large enough to be considered embarrassing. One could validly claim that we should not be comparing energy densities but mass scales by considering a mass scale for the vacuum implicitly defined through $\rho_\Lambda = M_\Lambda^4$. However, this will not really make a difference, since a 30-orders-of-magnitude discrepancy in mass scale hardly makes a good estimate. This constitutes the so-called *cosmological constant problem*.

Unfortunately, this is not the only problem related to the cosmological constant. The other known problem goes under the name of *the coincidence problem*. It is apparent from the data that $\Omega_\Lambda \sim 0.72$ and $\Omega_m \sim 0.28$ have comparable values today. However, as the universe expands their fractional contributions change rapidly since

$$\frac{\Omega_\Lambda}{\Omega_m} = \frac{\rho_\Lambda}{\rho_m} \propto a^3. \tag{1.26}$$

Since Λ is a constant, ρ_Λ should once have been negligible compared to the energy densities of both matter and radiation and, as dictated by eq. (1.26), it will come to dominate completely at some point in the late time universe. However, the striking fact is that the period of transition between matter domination and cosmological constant domination is very short compared to cosmological time scales[4]. The puzzle is, therefore, why we live precisely in this very special era [67]. Obviously, the transition from matter domination to cosmological constant domination, or, alternatively stated, from deceleration to acceleration, would happen eventually. The question is, why now?

To sum up, including a cosmological constant in the field equations appears as an easy way to address issues like the late time accelerated expansion but unfortunately it comes with a price: the cosmological constant and coincidence problems.

[4]Note that in the presence of a cosmological constant there is an infinite future in which Λ is dominating.

We will return to this discussion from this point later on but for the moment let us close the present section with an overall comment about the ΛCDM model. Its value should definitely not be underestimated. In spite of any potential problems that it may have, it is still a remarkable fit to observational data while at the same time being elegantly simple. One should always bear in mind how useful a simple empirical fit to the data may be. On the other hand, the ΛCDM model should also not be over-estimated. Being a phenomenological model, with poor theoretical motivation at the moment, one should not necessarily expect to discover in it some fundamental secrets of nature.

1.4 Is there a way out?

In the previous sections, some of the most prominent problems of contemporary physics were presented. As one would expect, since these questions were initially posed, many attempts to address one or more of them have been pursued. These problems may be viewed as being unrelated to each other, or grouped in different categories at will. For instance, one could follow a broad research field grouping, much like the one attempted in the previous section, dividing them into problems related with Cosmology and problems related with high energy physics, or group them according to whether they refer to unexplained observations or theoretical shortcomings. In any case there is one common denominator in all of these problems. They are all somehow related to gravity.

The way in which one chooses to group or divide these problems proposes a natural path to follow for their solution. In this section let us very briefly review some of the most well-known and conventional solutions proposed in the literature, which mainly assume that all or at least most of these issues are unrelated. Then we can proceed to argue why and how the appearance of so many yet to be explained puzzles related to gravity and General Relativity may imply that there is something wrong with our current understanding of the gravitational interaction even at a classical level, resembling the historically recorded transition from Newtonian gravity to General Relativity described in section 1.1. With that we will conclude this introductory chapter.

1.4.1 Scalar fields as matter fields in Cosmology

We have already discussed the need for an inflationary period in the early universe. However, we have not yet attempted to trace the cause of such an accelerated expansion. Since the presence of a cosmological constant could in principle account for that, one is tempted to explore this possibility, as in the case of late time acceleration. Unfortunately, this simple solution is bound not to work for a very simple reason: once the cosmological constant dominates over matter there is no way for matter to dominate again. Inflation has to end at some point, as already mentioned, so that Big Bang Nucleosynthesis and structure formation can take place. Our presence in the universe is all the evidence one needs for that. Therefore, one is forced to seek other, dynamical solutions to this problem.

As long as one is convinced that gravity is well described by General Relativity, the only option left is to assume that it is a matter field that is responsible for inflation. However, this matter field should have a rather unusual property: its effective equation of state should satisfy eq. (1.13), *i.e.* it should have a negative pressure and actually violate the Strong Energy Condition. Fortunately, matter fields with this property do exist. A typical simple example is a scalar field ϕ.

A scalar field minimally coupled to gravity, satisfies the Klein–Gordon equation

$$\nabla^2\phi + V'(\phi) = 0, \tag{1.27}$$

where ∇_μ denotes the covariant derivative, $\nabla^2 \equiv \nabla^\mu\nabla_\mu$, $V(\phi)$ is the potential and the prime denotes partial differentiation with respect to the argument. Assuming that the scalar field is homogeneous and therefore $\phi \equiv \phi(t)$ we can write its energy density and pressure as

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{1.28}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{1.29}$$

while, in a FLRW spacetime, eq. (1.27) takes the following form:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{1.30}$$

It is now apparent that if $\dot{\phi}^2 < V(\phi)$ then the pressure is indeed negative. In fact $w_\phi = p_\phi/\rho_\phi$ approaches -1 when $\dot{\phi}^2 \ll V(\phi)$.

In general a scalar field that leads to inflation is referred to as the *inflaton*. Since we invoked such a field instead of a cosmological constant, claiming that in this way we can successfully end inflation, let us see how this is achieved. Assuming that the scalar dominates over both matter and radiation and neglecting for the moment the spatial curvature term for simplicity, eq. (1.5) takes the form

$$H^2 \approx \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{1.31}$$

If, together with the condition $\dot{\phi}^2 < V(\phi)$, we require that $\ddot{\phi}$ is negligible in eq. (1.30) then eqs. (1.31) and (1.30) reduce to

$$H^2 \approx \frac{8\pi G}{3}V(\phi), \tag{1.32}$$

$$3H\dot{\phi} \approx -V'(\phi). \tag{1.33}$$

This constitutes the *slow-roll approximation* since the potential terms are dominant with respect to the kinetic terms, causing the scalar to roll slowly from one value to another. To be more rigorous, one can define two slow-roll parameters

$$\epsilon(\phi) = 4\,\pi\,G\left(\frac{V'}{V}\right)^2, \tag{1.34}$$

$$\eta(\phi) = 8\,\pi\,G\frac{V''}{V}, \tag{1.35}$$

for which the conditions $\epsilon(\phi) \ll 1$ and $\eta(\phi) \ll 1$ are necessary in order for the slow-roll approximation to hold [68, 69]. Note that these are not sufficient conditions since they only restrict the form of the potential. One also has to make sure that eq. (1.33) is satisfied. In any case, what we want to focus on at this point is that one can start with a scalar that initially satisfies the slow-roll conditions but, after some period, ϕ can be driven to such a value so as to violate them. A typical example is that of $V(\phi) = m^2\phi^2/2$, where these conditions are satisfied as long as $\phi^2 > 16\,\pi\,G$ but, as ϕ approaches the minimum of the potential, a point will be reached where $\phi^2 > 16\,\pi\,G$ will cease to hold. Once the slow-roll conditions are violated, inflation can be naturally driven to an end since $\dot{\phi}^2$ can begin to dominate again in eq. (1.29).

However, just ending inflation is not enough. After such an era the universe would be a cold and empty place unable to evolve dynamically to anything close to the picture which we observe today. A viable model for inflation should include a mechanism that will allow the universe to return to the standard Big Bang scenario. This mechanism is called reheating and consists mainly of three processes: a period of non-inflationary scalar field dynamics, after the slow-roll approximation has ceased to be valid, the creation and decay of inflaton particles and the thermalization of the products of this decay [35]. Reheating is an extensive and intricate subject and analyzing it goes beyond the scope of this introduction. We refer the reader to [70, 71, 72, 73, 74, 75, 76] for more information.

On the same grounds, we will refrain here from mentioning specific models for inflation and from discussing subtleties with using inflation in order to address problems of initial conditions such as those stated in paragraph 1.3.2. We refer the reader to the literature for further reading [75, 76, 77, 78, 79].

Before closing this paragraph, it should be mentioned that scalar fields can be used to account for the late-time accelerated expansion of the universe in the same way as the inflaton is used in inflationary models. Since, however, this subject overlaps with the subject of dark energy, we will discuss it in the next sub-section which is dedicated to the dark energy problem.

1.4.2 The dark energy problem

We have already seen that there seems to be compelling observational evidence that the universe is currently undergoing an accelerated expansion and we have also discussed the problems that arise if a cosmological constant is considered to be responsible for this acceleration within the framework of the ΛCDM model. Based on that, one can classify the attempts to address the problem of finding a mechanism that will account for the late-time accelerated expansion in two categories: those that try to find direct solutions to the cosmological constant and the coincidence problems and consequently attempt to provide an appealing theoretical explanation for the presence and the value of the cosmological constant, and those that abandon the idea of the cosmological constant altogether and attempt to find alternative ways to explain the acceleration.

Let us state two of the main approaches followed to solve the cosmological constant problem directly:

The first approach resorts to High Energy Physics. The general idea is simple and can be summed up in the question: Are we counting properly? This refers to the quite naive calculation mentioned previously, according to which the energy density of the cosmological constant as calculated theoretically should be 10^{120} times larger than its observed value. Even though the question is simple and reasonable, giving a precise answer to it is actually very complicated since, as mentioned already, little is known about how to make an exact calculation of the vacuum energy of quantum fields. There are indications coming from contemporary particle physics theories, such as supersymmetry (SUSY), which imply that one can be led to different values for the energy density of vacuum from the one mentioned before (eq. (1.24)). For instance, since no superpartners of known particles have been discovered in accelerators, one can assume that supersymmetry was broken at some scale of the order of 10^3GeV or higher. If this is the case, one would expect that

$$\rho_\Lambda \sim M_{\rm SUSY}^4 \geq (10^{12}{\rm eV})^4. \tag{1.36}$$

This calculation gives an estimate for the energy density of the vacuum which is 60 orders of magnitude smaller than the one presented previously in eq. (1.24). However, the value estimated here is still 60 orders of magnitude larger than the one inferred from observations (eq. (1.25)). Other estimates with or without a reference to supersymmetry or based on string theory or loop quantum gravity exist. One example is the approach of Ref. [80] where an attempt is made to use our knowledge from condensed matter systems in order to explain the value of the cosmological constant. We will not, however, list further examples here but refer the reader to [63, 65] and references therein for more details. In any case, the general flavour is that it is very difficult to avoid the cosmological constant problem by following such approaches without making some fine tuning within the fundamental theory used to perform the calculation for the energy density of vacuum. Also, such approaches mostly fail to address the second problem related to the cosmological constant: the coincidence problem.

The second direct approach for solving problems related to the cosmological constant has a long history and was given the name "anthropic principle" by Brandon Carter [81, 82, 83]. Unfortunately, the anthropic principle leaves a lot of room for different formulations or even misinterpretations. Following [63] we can identify at least three versions, starting from a very mild one, that probably no one really disagrees with but is not very useful for answering questions, stating essentially that our mere existence can potentially serve as an experimental tool. The second version on the other hand is a rather strong one, stating that the laws of nature are by themselves incomplete and become complete only if one adds the requirement that conditions should allow intelligent life to arise, for only in the presence of intelligent life does science become meaningful. It is apparent that such a formulation places intelligent life or science at the centre of attention as far as the universe is concerned. From this perspective one cannot help but notice that the anthropic principle becomes reminiscent of the Ptolemaic model. Additionally, to quote Weinberg: "...although science is clearly impossible without scientists, it is not clear that the universe is impossible without science". The third and most moderate version of the anthropic principle, known as the "weak anthropic principle" states essentially that observers will only observe conditions which allow for observers. This version is the one mostly discussed from a scientific perspective and even though it might seem tautological, it acquires a meaning if one invokes probability theory.

To be more concrete, as opposed to the second stronger formulation, the weak anthropic principle does not assume some sort of conspiracy of nature aimed at creating intelligent life. It merely states that, since the existence of intelligent observes requires certain conditions, it is not possible for them in practice to observe any other conditions, something that introduces a bias in any probabilistic analysis. This, of course, requires one extra assumption: that parts of the universe, either in space or time, might indeed be in alternative conditions. Unfortunately we cannot conclude at this point whether this last statement is true. Assuming that it is, one could put constrains on the value of the cosmological constant by requiring that it should be small enough for galaxies to form as in [84] and arrive at the conclusion that the currently observed value of the cosmological constant is by no means unlikely. Some modern theories do allow such alternative states of the universe to co-exist (multiverse), and for this reason it has recently been argued that the anthropic principle could even be placed on firm ground by using the ideas of string theory for the "anthropic or string landscape", consisting of a large number of different false vacua [85]. However, admitting that there are limits on our ability to unambiguously and directly explain the observable universe inevitably comes with a disappointment. It is for this reason that many physicists would refrain from using the anthropic principle or at least they would consider it only as a last resort, when all other possibilities have failed.

Let us now proceed to the indirect ways of solving problems related with the cosmological constant. As already mentioned, the main approach of this kind is to dismiss the cosmological constant completely and assume that there is some form of dynamical dark energy. In this sense, dark energy and vacuum energy are not related and therefore the cosmological constant problem ceases to exist, at least in the strict formulation given above. However, this comes with a cost: as mentioned previously, observational data seem to be in very good agreement with having a cosmological constant, therefore implying that any form of dynamical dark energy should be able to mimic a cosmological constant very precisely at present times.

This is not something easy to achieve. In order to be clearer and also to have the possibility to discuss how well dynamical forms of dark energy can address the cosmological constant and coincidence problems, let us use an example.

Given the discussion presented earlier about inflation, it should be clear by now that if a matter field is to account for accelerated expansion, it should have a special characteristic: negative pressure or more precisely $p \leq -\rho/3$. Once again, as in the inflationary paradigm, the obvious candidate is a scalar field. When such a field is used to represent dark energy it is usually called *quintessence* [86, 87, 88, 89, 90, 91, 92, 93, 94]. Quintessence is one of the simplest and probably the most common alternative to the cosmological constant.

If the scalar field is taken to be spatially homogeneous, its equation of motion in an FLRW spacetime will be given by eq. (1.30) and its energy density and pressure will be given by eqs. (1.28) and (1.29) respectively, just like the inflaton. As dictated by observations through eq. (1.15), a viable candidate for dark energy should have an effective equation of state with w very close to minus one. In the previous section it was mentioned that this can be achieved for a scalar field if the condition $\dot{\phi}^2 \ll V(\phi)$ holds. This should not be confused with the slow-roll condition for inflation, which just requires that $\dot{\phi}^2 < V(\phi)$ and also places a constraint for $\ddot{\phi}$. However, there is a similarity in the spirit of the two conditions, namely that in both cases the scalar field is required, roughly speaking, to be slowly-varying. It is worth mentioning that the condition $\dot{\phi}^2 \ll V(\phi)$ effectively restricts the form of the potential V.

Let us see how well quintessence can address the cosmological constant problem. One has to bear in mind that the value given in eq. (1.25) for the energy density of the cosmological constant now becomes the current value of the energy density of the scalar ρ_ϕ. Since we have asked that the potential terms should be very dominant with respect to the kinetic terms, this value for the energy density effectively constrains the current value of the potential. What is more, the equation of motion for the scalar field, eq. (1.30) is that of a damped oscillator, $3H\dot{\phi}$ being the friction term. This implies that, for ϕ to be rolling slowly enough so that $\dot{\phi}^2 \ll V(\phi)$ could be satisfied, then $H \sim \sqrt{V''(\phi)}$. Consequently, this means that the current value of $V''(\phi)$ should be that of the observed cosmological constant or, taking also into account that $\sqrt{V''(\phi)}$ represents the effective mass of the scalar m_ϕ, that

$$m_\phi \sim 10^{-33} \text{ eV}. \tag{1.37}$$

Such a small value for the mass of the scalar field raises doubts about whether quintessence really solves the cosmological constant problem or actually just transfers it from the domain of Cosmology to the domain of particle physics. The reason for this is that the scalar fields usually present in quantum field theory have masses many orders of magnitude larger than that given in eq. (1.37) and, hence, this poses a naturalness question (see [65] for more details). For instance, one of the well-known problems in particle physics, the hierarchy problem, concerns explaining why the Higgs field appears to have a mass of 10^{11} eV which is much smaller that the grand unification/Planck scale, 10^{25}-10^{28} eV. As commented in [67], one can then imagine how hard it could be to explain the existence of a field with a mass equal to 10^{-33} eV. In all fairness to quintessence, however, it should be stated that the current value of the energy density of dark energy (or vacuum, depending on the approach) is an observational fact, and so it does not seem possible to completely dismiss this number in some way. All that is left to do, therefore, is to put the cosmological constant problem on new grounds that will hopefully be more suitable for explaining it.

One should not forget, however, also the coincidence problem. There are attempts to address it within the context of quintessence mainly based on what is referred to as tracker models [95, 96, 97, 98, 99, 100, 101]. These are specific models of quintessence whose special characteristic is that the energy density of the scalar parallels that of matter or radiation for a part of the evolution which is significant enough so as to remove the coincidence problem. What is interesting is that these models do not in general require specific initial conditions, which means that the coincidence problem is not just turned into an initial conditions fine-tuning problem. Of course, the dependence of such approaches on the parameters of the potential remains inevitable.

It is also worth mentioning that ϕ should give rise to some force, which judging from its mass should be long-range, if the scalar couples to ordinary matter. From a particle physics point of view, one could expect that this is indeed the case, even if those interactions would have to be seriously suppressed by powers of the Planck scale [102, 103]. However, current limits based on experiments concerning a fifth-force or time dependence of coupling constants, appear to be several orders of magnitude lower than this expectation [102, 103]. This implies that, if quintessence really exists, then there should be a mechanism — probably a symmetry — that suppresses these couplings.

Yet another possibility for addressing the cosmological constant problems, or more precisely for dismissing them, comes when one adopts the approach that the accelerated expansion as inferred by observations is not due to some new physics but is actually due to a misinterpretation or an abuse of the underlying model being used. The Big Bang model is based on certain assumptions, such as homogeneity and isotropy, and apparently all calculations made rely on these assumptions. Even though at present one cannot claim that there is compelling evidence for this, it could be, for example, that the role of inhomogeneities is underestimated in the standard cosmological model and a more detailed model may provide a natural solution to the problem of dark energy, even by changing our interpretation of current observations (for instance see [104] and references therein).

1.4.3 The dark matter problem

As we have already seen, the presence of dark matter is indirectly inferred from observations through its gravitational interaction. Therefore, if one accepts that General Relativity describes gravity correctly, then an explanation for the nature of dark matter as some form of matter yet to be observed in the universe or in the laboratory should be given. Note that dark matter is used here generically to mean matter that does not emit light. So, to begin with, its nature could be either baryonic and non-baryonic. The candidates for baryonic dark matter are mostly quite conventional astrophysical objects such as brown dwarfs, massive black holes and cold diffuse gas. However, there is precise evidence from observations that only a small fraction of dark matter can be baryonic (see for example [43] and [105, 106] for reviews). Therefore, the real puzzle regards the nature of non-baryonic dark matter.

One can separate the candidates into two major categories: hot dark matter, *i.e.* non-baryonic particles which move (ultra-)relativistically, and cold dark matter *i.e.* non-baryonic particles which move non-relativistically. The most prominent candidate for hot dark matter is the the neutrino. However, studies of the cosmic microwave background, numerical simulations and other astrophysical observations indicate that dark matter has clumped to form some structures on rather small scales and therefore it cannot consist mainly of particles with high velocities, since this clumping would then have been suppressed (see for example [107, 108] and references in [106]). For this reason, and because of its simplicity, cold dark matter currently gives the favoured picture.

There are many cold dark matter candidates and so we will refrain from listing them all or discussing their properties in detail here and address the reader to the literature [106]. The most commonly considered ones are the axion and a number of weakly interacted massive particles (WIMPs) naturally predicted in supersymmetry theories, such as the neutralino, the sneutrino, the gravitino, the axino *etc*. There are a number of experiments aiming for direct and indirect detection of dark matter and some of them, such as the DAMA/NaI experiment [109], even claim to have already achieved that (see [110] for a full list of dark matter detection experiments and [105] for a review of experimental searches for dark matter). Great hope is also being placed on the Large Hadron Collider (LHC) [111], which is due to start operating shortly, to constrain the parameter space of particles arising from supersymmetric theories. Finally, the improvement of cosmological and astrophysical observations obviously plays a crucial role. Let us close by saying that the general flavour or expectation seems to be that one of the proposed candidates will soon be detected and that the relevant dark matter scenario will be verified. Of course expectations are not always fulfilled and it is best to be prepared for surprises.

1.4.4 OK, Quantum Gravity, but how?

In Section 1.2 we discussed some of the more prominent motivations for seeking a high energy theory of gravity which would allow a matching between General Relativity and Quantum Field Theory. These triggered research in this direction at a very early stage and already in the 1950s serious efforts were being made towards what is referred to as Quantum Gravity. Early attempts followed the conventional approach of trying to quantize the gravitational field in ways similar to the quantization of Electromagnetism, which had resulted in Quantum Electrodynamics (QED). This led to influential papers about the canonical formulation of General Relativity [112, 113]. However, it was soon realized that the obvious quantization techniques could not work, since General Relativity is not renormalizable as is the case with Quantum Electrodynamics [114]. In simple terms, this means that if one attempts to treat gravity as another particle field and to assign a gravity particle to it (graviton) then the sum of the interactions of the graviton diverges. This would not be a problem if these divergences were few enough to be removable via the technique called renormalization and this is indeed what happens in Quantum Electrodynamics, as also mentioned in Section 1.2. Unfortunately, this is not the case for General Relativity and renormalization cannot lead to sensible and finite results.

It was later shown that a renormalizable gravitation theory — although not a unitary one — could be constructed, but only at the price of admitting corrections to General Relativity [114, 115]. Views on renormalization have changed since then and more modern ideas have been introduced such as the concept of effective field theories. These are approximate theories with the following characteristic: according to the length-scale, they take into account only the relevant degrees of freedom. Degrees of freedom which are only relevant to shorter length-scales and higher energies and are, therefore, responsible for divergences, are ignored. A systematic way to integrate out short-distance degrees of freedom is given by the renormalization group (see [116] for an introduction to these concepts).

In any case, quantizing gravity has proved to be a more difficult task than initially expected and quantum corrections seem to appear, introducing deviations away from General Relativity [117, 118, 119]. Contemporary research is mainly focused on two directions: String Theory and Loop Quantum Gravity. Analysing the basis of either of these two approaches would go beyond the scope of this introduction and so we will only make a short mention of them. We refer the reader to [120, 121, 122] and [123, 124, 125, 126, 127] for text books and topical reviews in String Theory and Loop Quantum Gravity respectively.

String Theory attempts to explain fundamental physics and unify all interactions under the key assumption that the building blocks are not point particles but one dimensional objects called strings. There are five different versions of String Theory, namely Type I, Type IIA, Type IIB and two types of Heterotic String Theory. M-Theory is a proposed theory under development that attempts to unify all of the above types. A simplified version of the idea behind String Theory would be that its fundamental constituents, strings, vibrate at resonant frequencies. Different strings have different resonances and this is what determines their nature and results in the discrimination between different forces.

Loop Quantum Gravity follows a more direct approach to the quantization of gravity. It is close to the picture of canonical quantization and relies on a non-perturbative method called loop quantization. One of its main disadvantages is that it is not yet clear whether it can become a theory that can include the description of matter as well or whether it is just a quantum theory of gravitation.

It is worth mentioning that a common problem with these two approaches is that, at the moment, they do not make any experimentally testable predictions which are different from those already know from the standard model of particle physics. As far as gravity is concerned, String Theory appears to introduce deviations from General Relativity (see for example [128, 129, 130]), whereas, the classical limit of Loop Quantum gravity is still under investigation.

1.4.5 Gravity on the stand

In this introductory chapter, an attempt has been made to pose clearly a series of open questions related, in one way or the other, to gravity and to discuss some of the most common approaches currently being pursued for their solution. This brings us to the main question motivating the research presented in this thesis: could all or at least some of the problems mentioned earlier be somehow related and is the fact that General Relativity is now facing so many challenges indicative of a need for some new gravitational physics, even at a classical level?

Let us be more analytic. In Section 1.1 we presented a brief chronological review of some landmarks in the passage from Newtonian Gravity to General Relativity. One could find striking similarities with what has happened in the last decades with General Relativity itself. For instance, the cosmological and astrophysical observations which are interpreted as indicating the existence of dark matter and/or dark energy could be compared with Le Verrier's observation of the excess precession of Mercury's orbit. Remarkably, the first attempt to explain this phenomenon, was exactly the suggestion that an extra unseen — and therefore dark, in a way — planet orbited the Sun inside Mercury's orbit. The basic motivation behind this attempt, much like the contemporary proposals for matter fields to describe dark matter and dark energy, was to solve the problem within the context of an otherwise successful theory, instead of questioning the theory itself. Another example one could give, is the theoretical problems faced by Newtonian gravity once Special Relativity was established. The desire for a unified description of coordinate frames, inertial or not, and the need for a gravitational theory that is in good accordance with the conceptual basis of Special Relativity (*e.g.* Lorentz invariance) does not seem to be very far from the current desire for a unified description of forces and the need to resolve the conceptual clash between General Relativity and Quantum Field Theory.

The idea of looking for an alternative theory to describe the gravitational interaction is obviously not new. We already mentioned previously that attempts to unify gravity with quantum theory have included such considerations in the form of making quantum corrections to the gravitational field equations (or to the action, from a field theory perspective). Such corrections became effective at small scales or high energies. Additionally, many attempts have been made to modify General Relativity on both small and large scales, in order to address specific problems, such as those discussed earlier. Since we will refer to such modification extensively in the forthcoming chapters we will refrain from listing them here to avoid repetition. At present we will confine ourselves to giving two very early examples of such attempts which were not triggered so much by a theoretical or observational need for a new theory, but by another important issue in our opinion: the desire to test the uniqueness of General Relativity as the only viable gravitational theory and the need to verify its conceptual basis.

Sir Arthur Stanley Eddington, the very man who performed the deflection of light experiment during the Solar eclipse of 1919 which was one of the early experimental verifications of General Relativity, was one of first people to question whether Einstein's theory was the unique theory that could describe gravity [131]. Eddington tried to develop alternative theories sharing the same conceptual basis with General Relativity, most probably for the sake of theoretical completeness, since at the time there was no apparent reason coming from observations. Robert Dicke was also one of the pioneers in exploring the conceptual basis of General Relativity and questioning Einstein's equivalence principle. He reformulated Mach's principle and together with Carl Brans developed an alternative theory, known as Brans–Dicke theory [132, 133]. Part of the value of Dicke's work lies on the fact that it helped people to understand that we do not know as much as we thought about the basic assumptions of General Relativity, a subject that we will discuss shortly.

Even though the idea of an alternative theory for gravitation is not new, a new perspective about it has emerged quite recently. The quantum corrections predicted in the 1960s were expected to appear only at small scales. On the other hand, Eddington's modification or Brans–Dicke theory were initially pursued as a conceptual alternative of General Relativity and had phenomenological effects on large scales as well. Now, due to both the shortcomings of Quantum Gravity and the puzzling cosmological and astrophysical observations, these ideas have stopped being considered unrelated. It seem worthwhile to consider the possibility of developing a gravitation theory that will be in agreement with observations and at the same time will be closer to the theories that emerge as a classical limit of our current approaches to Quantum Gravity, especially since it has been understood that quantum corrections might have an effect on large scale phenomenology as well.

Unfortunately, constructing a viable alternative to General Relativity with the above characteristics is far from being an easy task since there are numerous theoretical and observational restrictions. Two main paths have been followed towards achieving this goal: proposing phenomenological models tailored to fit observations, with the hope that they will soon gain some theoretical motivation from high energy physics and current Quantum Gravity candidates, and developing ideas for Quantum Gravity, with the hope that they will eventually give the answer in the form of an effective gravitational theory through their classical limit which will account for unexplained observations. In this thesis a different approach will be followed in an attempt to combine and complement these two. At least according to the author's opinion, we seem to be still at too early a stage in the development of our ideas about Quantum Gravity to be able to give precise answers about the type and form of the expected quantum corrections to General Relativity. Current observations still leave scope for a wide range of different phenomenological models and so it seems a good idea to attempt exploring the limits of classical gravity by combining theory and observations. In a sense, this approach lies somewhere in the middle between the more conventional approaches mentioned before. Instead of starting from something known in order to extrapolate to the unknown, we attempt here to jump directly into the unknown, hoping that we will find an answer.

To this end, we will examine different theories of gravity, trying to determine how far one can go from General Relativity. These theories have been chosen in such a way as to present a resemblance with the low energy effective actions of contemporary candidates for Quantum Gravity in a quest to study the phenomenology of the induced corrections. Their choice has also been motivated by a desire to fit recent unexplained observations. However, it should be stressed that both of these criteria have been used in a loose manner, since the main scope of this study is to explore the limits of alternative theories of gravity and hopefully shed some light on the strength and validity of the several assumptions underlying General Relativity. In that sense, many of the theories which we will consider can be regarded as toy theories or straw-man theories. The main motivation comes from the fear that we may not know as much as we think or as much as needed to be known before making the key steps pursued in the last 50 years in gravitational physics; and from the hope that a better understanding of classical gravity might have a lot to offer in this direction.

As a conclusion to this introduction it is worth saying the following: it is probably too early to conclude whether it is General Relativity that needs to be modified or replaced by some other gravitational theory or whether other solutions to the problems presented in this chapter, such as those mentioned earlier, will eventually give the required answers. However, in scientific research, pursuing more than one possible solution to a problem has always been the wisest and most rewarding choice; not only because there is an already explored alternative when one of the proposed solutions fails, but also due to the fact that trial and error is one of the most efficient ways to get a deeper understanding of a physical theory. Exploring alternative theories of gravity, although having some disadvantages such as complexity, also presents a serious advantage: it is bound to be fruitful even if it leads to the conclusion that General Relativity is the only correct theory for gravitation, as it will have helped us both to understand General Relativity better and to secure our faith in it.

Chapter 2

Foundations of Gravitation Theory

2.1 Viability criteria and the various forms of the Equivalence Principle

2.1.1 Viability and the Dicke framework

Even though it took only 4 years for having the first experimental verification of General Relativity to appear — Eddington's measurement of light deflection in 1919 — Einstein's theory did not become the object of systematic and accurate experimental testing until the early 1960s. In fact, it was only in 1960 that the gravitational redshift of light was successfully measured by Pound and Rebka [134] even though this test was proposed by Einstein in 1907 and it is considered one of the three classical tests of General Relativity, together with the perihelion shift of Mercury and light deflection. After that, a number of new experimental tests were performed based on effects which were either new or which had been discovered earlier but their verification was not technologically possible at the time. Examples range from the Lense–Thirring effect [135], the Nordtvedt effect [136] and Shapiro time delay [137] to the Nobel Prize discovery of the binary pulsar by Taylor and Hulse [138] which led to the first indirect evidence for the existence of gravitational waves (for a historical review see Chapter 1 of [139]).

However, it was soon realised and first proposed by Schiff and Dicke referring to the redshift experiments [140, 141], that gravitational experiments do not necessarily test General Relativity, since, instead of testing the validity of specific field equations, experiments test the validity of principles, such as the equivalence principle. Contemporarily, a number of alternative theories of gravitation had been developed, many of which shared some of the principles of General Relativity and were therefore indistinguishable as far as some of the tests were concerned. This triggered the development of powerful tools for distinguishing and testing theories, the most commonly used of which is the Parametrized Post-Newtonian

(PPN) expansion, pioneered by Nordtvedt and extended by Nordtvedt and Will [142, 143, 144, 145].

The idea that experiments actually test principles and not specific theories highlights the importance of exploring the conceptual basis of a gravitation theory. In fact, it would be very helpful to provide a framework for analysing gravitation theories and experiments and deriving general conclusions about the viability criteria of the theories. This would provide a starting point for constructing gravitation theories which are not obviously non-viable for theoretical or experimental reasons. Dicke was one of the pioneers in this direction and presented what was later known as the Dicke Framework [146]. We will focus on this for the rest of the present section.

Following [139] we identify the two mathematical assumptions of the Dicke Framework as being:

- Spacetime is a 4-dimensional manifold, with each point in the manifold corresponding to a physical event.

- The equations of gravity and the mathematical entities in them are to be expressed in a form that is independent of the coordinates used, *i.e.* in covariant form.

A comment is due for each of these statements. Regarding the first one, it should be stressed that it does not presuppose that the manifold has either a metric or an affine connection, since one would prefer to arrive at this as a conclusion from experiments. Regarding the second one, it is important to bear in mind that non-covariant equations can in many cases be written in a covariant form if a number of covariant constraints are imposed. Such constraints introduce absolute structures into the theory (*e.g.* preferred coordinate frames) and therefore, even though coordinate invariance is justified, the theory does not really become background independent (see [147] for an interesting discussion). From this viewpoint, requiring only covariance of the field equations is not very restrictive.

Dicke also proposed two further assumptions related to those just presented: that gravity should be associated with one or more fields of tensorial character (scalars, vectors, tensors) and that the field equation governing the dynamics of gravity should be derivable from an invariant action via a stationary action principle. The first of these two assumptions appears as an almost direct consequence of the two previous ones, whereas the second seems less fundamental, at least at a classical level, and should not be imposed lightheartedly because it may lead to unnecessary confinement of acceptable theories.

The assumptions of Dicke's framework are probably the minimal unbiased assumptions that one can start with in order to develop a gravitation theory. There are also other fundamental criteria which a gravitation theory should satisfy in order to be viable. From a theoretical viewpoint there are the two basic requirements of all theories, *i.e.*

1. completeness: The theory should be able to analyse from “first principles” the outcome of any experiment,

2. self-consistency: predictions should be unique and independent of the calculation method.

From an experimental viewpoint there are two more very basic requirements:

1. The theory should be relativistic, *i.e.* should reduce to Special Relativity when gravity is “turned off” (and at low energies).

2. The theory should have the correct Newtonian limit, *i.e.* in the the limit of weak gravitational fields and slow motion it should reproduce Newton’s laws.

Both of these requirements are based on the fact that Special Relativity and Newtonian Gravity are extremely well tested theories — at least in regimes in which we theoretically expect them to be valid — and therefore any gravitation theory should be able to reproduce them in the suitable limit (see also [139] for more details).

One would like to combine with the above requirements also experiments that aim directly at testing gravity in its full glory in order to confine acceptable theories. We intend to do so in what follows.

2.1.2 Equivalence Principle(s)

In an abstract (and loose) sense a theory can usually be thought of as a set of axioms from which one can derive logical statements. When it comes to a physical theory one should also add that the statements of the theory should be able to predict the outcome of experiments that fall with its purview. However, it is common to think of General Relativity or other gravitation theories as a set of field equations (or an action). A complete and coherent axiomatic formulation of Einstein’s theory, or any other gravitation theory, is still pending; the viability criteria presented above are a step in this direction, but when referring to an axiomatic formulation one needs to go further. What is needed here is to formulate “principles”. One of these is the general covariance principle included in the Dicke Framework[1].

A principle that has triggered much more discussion and is probably much less understood is the Equivalence Principle, or more precisely each of its various formulations. As we have already mentioned in the Introduction, a formulation of the Equivalence Principle was already incorporated in Newtonian gravity. Newton pointed out in *Principia* that the “mass” of any body — meaning the quantity that regulates its response to an applied force — and the “weight” of the body — the property regulating its response to gravitation — should be equal. The terms “inertial mass” and “passive gravitational mass” where later introduced by Bondi [148] to distinguish the quantities present in Newton’s second law of motion

$$\vec{F} = m_I \vec{a}, \tag{2.1}$$

[1]We will return to the issue of the axiomatic formulation of gravitation theories in Chapter 7.

where $\vec{F}$ is the force 3-vector and $\vec{a}$ is the 3-acceleration, and Newton's gravitation law

$$\vec{F} = m_P \vec{g}, \tag{2.2}$$

where $\vec{g}$ is the gravitational acceleration 3-vector. In terms of m_I and m_P, the Equivalence Principle in Newtonian theory can be rigorously expressed as

$$m_I = m_p. \tag{2.3}$$

Einstein, by the use of gedanken experiments such as the famous free falling elevator one, realised that a free falling observer does not feel the effects of gravity and saw in a reformulation of the Equivalence Principle the foundations for generalizing Special Relativity and developing a theory to describe both non-inertial frames and gravity. The meaning of mass in such a theory is questionable and so the Equivalence principle should be expressed in terms of some more fundamental concept. Free-fall comes to the rescue. Eq. (2.3) within the framework of Newtonian gravity, implies in practice that all bodies experience the same acceleration when they are in free-fall, irrespective of composition. An expression of some kind of universality of free fall should therefore be sought for when attempting to reformulate the Newtonian version of the Equivalence Principle.

We will not attempt to review such endeavours historically. We focus directly on the several current forms of the Equivalence Principle. These are[2]:

Weak Equivalence Principle (WEP): If an uncharged test body is placed at an initial event in spacetime and given an initial velocity there, then its subsequent trajectory will be independent of its internal structure and composition.

Einstein Equivalence Principle (EEP): (i) the WEP is valid, (ii) the outcome of any local non-gravitational test experiment is independent of the velocity of the freely-falling apparatus (Local Lorentz Invariance-LLI) and (iii) the outcome of any local non-gravitational test experiment is independent of where and when in the universe it is performed (Local Position Invariance-LPI).

Strong Equivalence Principle (SEP): (i) the WEP is valid for self-gravitating bodies as well as for test bodies, (ii) the outcome of any local test experiment is independent of the velocity of the freely-falling apparatus (Local Lorentz Invariance-LLI) and (iii) the outcome of any local test experiment is independent of where and when in the universe it is performed (Local Position Invariance-LPI).

In order for these definitions to be clear, the following clarifications are needed [139]: An uncharged test body is an electrically neutral body that has negligible self-gravitation as estimated using Newtonian theory (it does not contribute to the dynamics of the gravitational field) and it is small enough in size so that its couplings to inhomogeneities in the external fields can be ignored. A local non-gravitational test experiment is defined to be any experiment which is performed in

[2]We follow the definitions given in [139]. The reader should be cautious since several different formulations exist in the literature and the terminology can be misleading (*e.g.* some authors refer to the EEP as WEP or to the SEP as EEP).

a freely falling laboratory which is shielded and small enough in size for inhomogeneities in the external fields to be ignored throughout its volume. Additionally, self-gravitating effects in this laboratory should be negligible.

Let us now focus on the differences between the WEP, EEP and SEP. The WEP implies that spacetime is endowed with a family of preferred trajectories which are the world lines of freely falling test bodies. Note that the existence of a metric is not suggested by the WEP. Even if an external assumption for the existence of the metric is made though, the geodesics of this metric do not necessarily coincide with the free-fall trajectories as far the WEP is concerned.

The EEP adds two more statement to the WEP: Local Lorentz Invariance and Local Position Invariance. A freely-falling observer carries a local frame in which test bodies have unaccelerated motions. According to the requirements of the LLI, the outcomes of non-gravitational experiments are independent of the velocity of the freely-falling frame and therefore if two such frames located at the same event $\mathcal{P}$ have different velocities, this should not affect the predictions for identical non-gravitational experiments. Local Position Invariance requires that the above should hold for all spacetime points. Therefore, roughly speaking, in local freely falling frames the theory should reduce to Special Relativity.

This implies that there should be at least one second rank tensor field which reduces, in the local freely falling frame, to a metric conformal with the Minkowski one. The freedom of having an arbitrary conformal factor is due to the fact that the EEP does not forbid a conformal rescaling in order to arrive at special-relativistic expressions for the physical laws in the local freely-falling frame. Note however, that while one could think of allowing each specific matter field to be coupled to a different one of these conformally related second rank tensors, the conformal factors relating these tensors can at most differ by a multiplicative constant if the couplings to different matter fields are to be turned into constants under a conformal rescaling as the LPI requires (this highlights the relation between the LPI and varying coupling constants)[3]. We can then conclude that rescaling coupling constants and performing a conformal transformation leads to a metric $g_{\mu\nu}$ which, in every freely falling local frame reduces (locally) to the Minkowski metric $\eta_{\mu\nu}$.

It should be stressed that all conformal metrics $\phi g_{\mu\nu}$ (ϕ being the conformal factor) can be used to write down the equations or the action of the theory. $g_{\mu\nu}$ is only special in the following sense: Since at each event $\mathcal{P}$ there exist local frames called local Lorentz frames, one can find suitable coordinates in which at $\mathcal{P}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}\left(\sum_{\alpha} |x^{\alpha} - x^{\alpha}(\mathcal{P})|^2\right), \tag{2.4}$$

and $\partial g_{\mu\nu}/\partial x^{\alpha} = 0$. In local Lorentz frames, the geodesics of the metric $g_{\mu\nu}$ are straight lines. Free-fall trajectories are straight lines in a local freely-falling frame.

[3] This does not exclude the possibility of having a second metric tensor in the theory as long as this metric does not couple to the matter (this then leads to theories of the bi-metric kind).

Identifying the two frames we realize that the geodesics of $g_{\mu\nu}$ coincide with free-fall trajectories. Put in other words, the EEP requires the existence of a family of conformal metrics, one of which should have geodesics which coincide with free-fall trajectories.

Finally, let as focus on the SEP. The SEP extends the validity of the WEP to self-gravitating bodies and the validity of the EEP to local gravitational experiments. Note that the Newtonian Equivalence principle also did not make a distinction between test bodies and self gravitating bodies. Extending the validity of LLI and LPI to local gravitational experiments is also a quite strong requirement. For the time being there is no theory other than General Relativity that satisfies the SEP. However, there is no explicit proof that the SEP leads uniquely to General Relativity.

Let us attempt to argue heuristically that this is indeed the case. First we have to understand how local gravitational experiments are influenced by the form of the theory. Following [139] we can consider a local freely-falling frame that is small enough for inhomogeneities in the external gravitational fields to be neglected throughout its volume, but is large enough to encompass a system of gravitating matter and its associated gravitational fields. We do not assume here that the metric is the only gravitational field. In order to solve the field equations and determine the behaviour of the system, we need to impose boundary conditions, *i.e.* determine the values of the fields, gravitational or not, on the boundary of our local frame. These values will generically depend on the behaviour of the fields far from the local frame which we are considering.

Since the EEP is anyway included in the SEP, let as assume that the EEP is indeed valid. LLI and LPI imply that the outcome of local non-gravitational experiments should be unaffected by the boundary values of gravitational fields other than the metric, since these are sensitive to the position or velocity of the frame, depending on their nature (see also [139]). Therefore, in a representation in which $g_{\mu\nu}$ is taken to be the metric, any gravitational fields other than the metric should not be coupled to matter directly due to the EEP (recall the freedom to use conformal metrics).

Let us now suppose that the theory indeed includes gravitational fields other than the metric that are not directly coupled to the matter. If one tries to solve the field equations and determine the outcome of gravitational experiments, then the boundary values of these fields will influence the result. This directly implies that the SEP cannot be satisfied.

All that is left is to consider theories in which the only gravitational field is the metric. In the local frame which we are considering it is always possible to find a coordinate system in which $g_{\mu\nu}$ reduces to $\eta_{\mu\nu}$ and $\partial g_{\mu\nu}/\partial x^\alpha = 0$ at the boundary between the local system and its surroundings (*cf.* eq. (2.4)) [139]. Therefore, if the field equation for the metric contains derivatives of the metric of not higher than second order, then the outcome of any experiment, gravitational or non-gravitational, is independent of the surroundings and is therefore independent of the position and velocity of the frame.

However, this is not the case if the field equation for the metric is of higher differential order. The boundary values of the second or higher derivatives of the metric cannot be "trivialized" in any coordinate system and the outcome of gravitational experiments becomes sensitive to the position and velocity of the local frame. Therefore, theories that include higher order derivatives of the metric, such as fourth-order gravity, do not satisfy the SEP [4].

We conclude that theories which can satisfy the SEP should not include gravitational fields other than the metric and that the differential order of the field equations should be at most second order. As it stands, this discussion does not prove that General Relativity is the only theory that satisfies the SEP. However, if one adds some of the viability arguments listed in the previous section, then the candidate list reduces significantly. For instance, if one requires that the theory should come from an action, it is quite straightforward to show that the Einstein–Hilbert action, modulo surface terms or topological invariants (see Section 3.5.1), is the only generally covariant action that depends only on the metric and leads to second order field equations under metric variation. There is therefore strong evidence to believe that the validity of the SEP leads to General Relativity. It should be stressed that in principle it is possible to build up other theories that satisfy the SEP but up to this point only one is actually known: Nordström's conformally-flat scalar theory, which dates back to 1913 [149]. However, even this theory is not viable since it predicts no deflection of light.

Before closing this section, we return to one of the initial motivations for discussing the principles which a viable gravitation theory should satisfy: the fact that experiments do not always test theories, but more often they test principles. There are specific tests for each version of the Equivalence Principle. The basis of the WEP is the universality of free-fall, *i.e.* the requirement that different (test) bodies should experience the same acceleration in an external gravitational field irrespective of their composition. Experiments testing the WEP attempt to measure the fractional difference in acceleration between two bodies, leading to what is called the "Eötvös ratio", named after the classic torsion balance measurements of Eötvös [150]. There are many very sophisticated experiments trying to measure violations of the WEP with accuracies close to 10^{-13} and hoping to reach 10^{-17} soon. We address the reader to [151] and references therein for details.

To test the EEP one has to test, apart from the WEP, Local Lorentz Invariance and Local Position Invariance. LLI is a principle already embodied in Special Relativity. From this perspective, questioning it would affect not only gravitation theories, but also most of modern physics in general. However, a violation of LLI would not necessarily constitute a menace for physics as we know it. It can just be a manifestation of new "beyond Einstein" physics related, for instance, to Quantum Gravity phenomenology. For more information on testing LLI, we refer the reader to [151] and references therein, and especially to the thorough review of Mattingly [152]. As far as LPI is concerned, there are two crucial tests: gravitational red-

[4] This contradicts what is claimed in [139].

shift experiments (*e.g.* measurement of clock frequencies at different spacetime locations) and measurements of possible variations of non-gravitational coupling constants. One should stress at this point that LPI also refers to the position in time. See [151] for a thorough presentation of relevant experiments.

Finally, there are tests related to the SEP. Recall that the SEP extends the validity of the EEP to gravitational experiments as well. Amongst the most common experimental tests of the SEP are measurements of the possible variation of the gravitational constant, preferred-location and preferred-frame effects in the locally measured gravitational constant, possible violations of the WEP for gravitating bodies, *etc*. [151].

2.1.3 Metric Postulates

So far we have stated a number of principles which it is reasonable to assume that all viable gravitational theories should satisfy. Experiments will show the extent to which these assumptions are valid by placing constraints on the possible violations of the principles concerned. However, this is not quite the end of the story. From a practical perspective, it is not at all straightforward to understand whether a specific theory does satisfy these principles. More precisely, if one is given an action or a set of field equations, usually a series of tedious manipulations will have to be performed before concluding that the EEP, for instance, is valid within the framework of the theory represented by them.

The reverse problem is also of interest: Given that we have a series of principles which our theories have to satisfy, can we turn them into practical constraints on their general form? Could we identify some mathematically, and not abstractly, formulated characteristics which a candidate theory should have in order to comply with the principles described above? An attempt in this direction was made in 1971 by Thorne and Will with the introduction of the so-called metric postulates [153]. We have already described, in the previous chapter, how the validity of the EEP implies the existence of the metric (a member of a family of conformal metrics) whose geodesics coincide with the free-fall trajectories. This is encapsulated in Thorne and Will's metric postulates:

1. there exists a metric $g_{\mu\nu}$ (second rank non-degenerate tensor),
2. $\nabla_\mu T^{\mu\nu} = 0$ where ∇_μ is the covariant derivative defined with the Levi–Civita connection of this metric and $T_{\mu\nu}$ is the stress-energy tensor of non-gravitational (matter) fields.

Note that geodesic motion can be derived using the second metric postulate [154]. Theories that satisfy the metric postulates are referred to as *metric theories*.

The metric postulates have proved to be very useful. They are part of the foundation for the Parametrized Post-Newtonian expansion which has been extensively used to constrain alternative theories of gravity. They do, however, have a major disadvantage. As pointed out also by the authors of [153], any metric theory can perfectly well be given a representation that appears to violate the metric postulates (recall, for instance, that $g_{\mu\nu}$ is a member of a family of conformal metrics and that there is no *a priori* reason why this metric should be used to write down the field equations). On top of that, one can add that there are some ambiguities in the definition of quantities related to the metric postulates. For example, what exactly is the precise definition of the stress energy tensor and which fields are included in it? What exactly is the difference between gravitational and non-gravitational fields?

Let us not elaborate more on these issues here since they will become much more apparent once we study some alternative theories of gravity in the next chapter and discuss the equivalence between theories. Therefore, it is preferable to return to this issue later on. We close the present section by pointing out that the metric postulates are, at the moment, the closest thing we have to a guide about where to start when constructing alternative theories of gravity.

2.2 Geometric description of spacetime

It should be clear from the discussion of the previous sections that there is very strong motivation for assuming that gravity is related to spacetime geometry and that any reasonable theory for the gravitational interaction is most likely to include a metric. Therefore it is useful before going further to take a moment to recall the basics of the geometric description of a 4-dimensional manifold. This is by no means a rigorous introduction to the differential geometry of 4-dimensional manifolds but merely a collection of some basic definitions which will prove useful later on and in which a physicist's perspective is probably apparent.

Let us start by considering a 4-dimensional manifold with a connection, $\Gamma^{\lambda}{}_{\mu\nu}$, and a symmetric metric $g_{\mu\nu} (= g_{\nu\mu})$. By definition the metric allows us to measure distances. We assume that this metric is non-degenerate and therefore invertible. Consequently it can be used to raise and lower indices. The connection is related to parallel transport and therefore defines the covariant derivative. The definition is the following:

$$\bar{\nabla}_{\mu} A^{\nu}{}_{\sigma} = \partial_{\mu} A^{\nu}{}_{\sigma} + \Gamma^{\nu}{}_{\alpha\mu} A^{\alpha}{}_{\sigma} - \Gamma^{\alpha}{}_{\mu\sigma} A^{\nu}{}_{\alpha}. \tag{2.5}$$

We give this here even though it may be considered trivial, since several different conventions exist in the literature. Additionally one has to be careful about the position of the indices when the connection is not symmetric.

Notice that we use $\bar{\nabla}_{\mu}$ to denote the covariant derivative here because we have not yet related $\Gamma^{\lambda}{}_{\mu\nu}$ in any way to the metric. This would be an extra assumption that is not needed at this stage. This connection is not assumed to be the Levi–Civita connection of $g_{\mu\nu}$ and the symbol ∇_{μ} is reserved for the covariant derivative defined with the latter.

Using the connection, one can construct the Riemann tensor:

$$\mathcal{R}^{\mu}{}_{\nu\sigma\lambda} = -\partial_{\lambda} \Gamma^{\mu}{}_{\nu\sigma} + \partial_{\sigma} \Gamma^{\mu}{}_{\nu\lambda} + \Gamma^{\mu}{}_{\alpha\sigma} \Gamma^{\alpha}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\alpha\lambda} \Gamma^{\alpha}{}_{\nu\sigma} \,. \tag{2.6}$$

which has no dependence on the metric. Notice that the Riemann tensor here has only one obvious symmetry; it is antisymmetric in the last two indices. The rest of the standard symmetries are not present for an arbitrary connection [155].

Since we do not assume here any relation between the metric and the connections, the former is not necessarily covariantly conserved. The failure of the connection to preserve the metric is usually measured by the non-metricity tensor:

$$Q_{\mu\nu\lambda} \equiv -\bar{\nabla}_\mu g_{\nu\lambda}. \tag{2.7}$$

The trace of the non-metricity tensor with respect to its last two (symmetric) indices is called the Weyl vector:

$$Q_\mu \equiv \frac{1}{4} Q_{\mu\nu}{}^{\nu}. \tag{2.8}$$

At the same time, the connection is not necessarily symmetric. The antisymmetric part of the connection is often called the Cartan torsion tensor:

$$S_{\mu\nu}{}^{\lambda} \equiv \Gamma^{\lambda}{}_{[\mu\nu]}. \tag{2.9}$$

One has to be careful when deriving the Ricci tensor in this case, since only some of the standard symmetry properties of the Riemann tensor hold here. A straightforward contraction leads, in fact, to two Ricci tensors [155]:

$$\mathcal{R}_{\mu\nu} \equiv \mathcal{R}^{\sigma}{}_{\mu\sigma\nu} = -\mathcal{R}^{\sigma}{}_{\mu\nu\sigma}, \qquad \mathcal{R}'_{\mu\nu} \equiv \mathcal{R}^{\sigma}{}_{\sigma\mu\nu}. \tag{2.10}$$

The first one is the usual Ricci tensor given by

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\lambda}. \tag{2.11}$$

The second is given by the following equation

$$\mathcal{R}'_{\mu\nu} = -\partial_\nu \Gamma^{\alpha}{}_{\alpha\mu} + \partial_\mu \Gamma^{\alpha}{}_{\alpha\nu}. \tag{2.12}$$

For a symmetric connection, this tensor is equal to the antisymmetric part of $\mathcal{R}_{\mu\nu}$. Fully contracting both tensors with the metric to get a scalar gives, for $\mathcal{R}_{\mu\nu}$

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu} \tag{2.13}$$

which is the Ricci scalar, and for $\mathcal{R}'_{\mu\nu}$

$$\mathcal{R}' = g^{\mu\nu}\mathcal{R}'_{\mu\nu} = 0, \tag{2.14}$$

since the metric is symmetric and $R'_{\mu\nu}$ is antisymmetric. Therefore the Ricci scalar is uniquely defined by eq. (2.13).

We have considered so far second rank tensors that one gets from a contraction of the Riemann tensor without using the metric, *i.e.* tensors independent of the metric. There is a third second rank tensor which can be built from the Riemann tensor [156]: $\mathcal{R}''_{\mu\nu} \equiv \mathcal{R}_{\mu}{}^{\sigma}{}_{\sigma\nu} = g^{\sigma\alpha}g_{\mu\beta}\mathcal{R}^{\beta}{}_{\alpha\sigma\nu}$. This tensor, however, depends on the metric. A further contraction with the metric will give $\mathcal{R}'' = g^{\mu\nu}\mathcal{R}''_{\mu\nu} = -\mathcal{R}$, and so even if we consider this tensor, the Ricci scalar is in practice uniquely defined.

2.3 General Relativity through its assumptions

What is really distinguishing General Relativity from other candidate theories for gravitation? In Section 2.1 this problem was approached from a conceptual perspective and the discussion evolved around several principles that can be formed to describe the key features of the gravitational interaction. Even though this is indeed the most fundamental and therefore the most noble way to address this problem, a rigorous axiomatic formulation of General Relativity is still pending, as already mentioned. The next best thing that one can do is to list the assumptions that uniquely lead to Einstein's theory and distinguish it from alternative theories once the geometrical nature of gravity is itself assumed.

We have already argued why it is very reasonable to describe gravity as a geometric phenomenon and why a metric is most likely to be present in the gravity sector. We have also already presented the tools needed for such a description in the previous section. However, even if the metric postulates are adopted, General Relativity is not the only theory that satisfies them and there are extra restrictions that should be imposed in order to be led uniquely to this theory. Let us present these here as they come about in the derivation of the field equations.

General Relativity is a classical theory and therefore no reference to an action is really physically required; one could just stay with the field equations. However, the Lagrangian formulation of the theory has its merits. Besides its elegance, there are at least two more reasons it has now become standard:

- At the quantum level the action indeed acquires a physical meaning and one expects that a more fundamental theory for gravity (or including gravity), will give an effective low energy gravitational action at a suitable limit.

- It is much easier to compare alternative gravity theories through their actions rather than by their field equations, since the latter are far more complicated. Also, it seems that in many cases we have a better grasp of the physics as described through the action (couplings, kinetic and dynamical terms *etc*.).

For the above reasons we will follow the Lagrangian formulation here. However, we will be keeping track of the analogy with the geometric derivation of the field equations of General Relativity, initially used by Einstein, and comment on it whenever necessary. In Einstein's derivation the analogy with the Poisson equation, which describes the dynamics of Newtonian gravity, plays a significant role. Such an approach can be found in many textbooks (for instance [157]).

Let us start with what is probably the most basic assumption of General Relativity: that the affine connection ${\Gamma^\lambda}_{\mu\nu}$ is the Levi–Civita connection, *i.e.*

$$ {\Gamma^\lambda}_{\mu\nu} = \{{}^{\lambda}{}_{\mu\nu}\}. \tag{2.15} $$

This assumption is actually dual, since it requires both the metric to be covariantly conserved,

$$ \bar{\nabla}_\lambda g_{\mu\nu} = 0, \tag{2.16} $$

and the connection to be symmetric,

$$\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu}. \tag{2.17}$$

Assumption (2.16) can also be written in terms of the non-metricity as $Q_{\mu\nu\lambda} = 0$, while assumption (2.17) can be written in terms of the Cartan torsion tensor as $S_{\mu\nu}{}^{\lambda} = 0$. General Relativity assumes that there is neither torsion nor non-metricity.

Given these assumptions, the Riemann tensor will turn out to be antisymmetric also with respect to the first two indices as well as symmetric in an exchange of the first and the second pairs. Therefore, one can construct only one second rank tensor from straightforward contraction, *i.e.* without using the metric. This is the well-known Ricci tensor, $R_{\mu\nu}$ (we use $\mathcal{R}_{\mu\nu}$ for the Ricci tensor constructed with an independent connection). A full contraction with the metric will then lead to the Ricci scalar, R in the usual way.

Before writing down an action for General Relativity, we need to refer to another key assumption. *General Relativity assumes that no fields other than the metric mediate the gravitational interaction*. Any field other than the metric is considered to be matter and should be included in the matter action. Therefore the general structure of the action should include a Lagrangian for gravity which depends only on the metric and a Lagrangian for the matter which depends on the matter fields. In terms of the field equations, this requirement can be put in the following terms: the left hand side should depend only on the metric and the right hand side should depend only on the matter fields, at least if we want our equations to have a form similar to the Poisson equation.

For the matter Lagrangian we have one basic requirement: We want its variation with respect to the metric to lead to the the matter stress-energy tensor, since this is what we expect to have on the right hand side of the field equations. Therefore, we define

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{2.18}$$

where $\delta/\delta g^{\mu\nu}$ is a functional derivative with respect to the metric,

$$S_M = \int d^4x\sqrt{-g}L_M(g_{\mu\nu},\psi), \tag{2.19}$$

is the matter action, g denotes the determinant of the metric $g_{\mu\nu}$, L_M is the matter Lagrangian and ψ collectively denotes the matter fields. In a sense, here we just draw our insight from the analogy with the Poisson equation and the fact that in Special Relativity the stress-energy tensor is the analogue of the matter density in Newtonian theory.

Let us now go one step further and examine the form of the gravitational Lagrangian. Hilbert, to whom we owe the Lagrangian formulation of General Relativity, recognised two requirements. Firstly, the Lagrangian should be a generally covariant scalar if it is to lead to covariant equations (equations of tensors). This depicts the requirements that the field equations are to be independent of the coordinates. Secondly, the Lagrangian should depend only on the metric and its first derivatives and not on any higher order derivatives, so that metric variation will lead to a second order differential equation. This requirement comes from the fact that we do not know of any other theory which has higher order field equations.

However, there was an obstacle to Hilbert's requirements: There is no generally covariant scalar that one can construct with only the metric and its first derivatives. The first derivatives of the metric are not covariant objects and no combination can be formed using them that turns out to be covariant. The simplest generally covariant scalar that one can construct is the Ricci scalar which depends also on the second derivatives of the metric. This was Hilbert's motivation for defining the gravitational action for General Relativity as:

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R. \tag{2.20}$$

The coefficient $(16\pi G)^{-1}$ is chosen with some anticipation since at this stage any constant would do and one has to resort to the Newtonian limit in order to calculate its value.

Let us now see how one derives the field equations from the action (2.20). We shall not discuss this procedure in detail however, since it is a standard text-book calculation (see *e.g.* [12]). The variation of the action (2.20) with respect to the metric gives

$$\delta S_{EH} = \frac{1}{16\pi G} \left[\int_U d^4x \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} - 2 \int_{\delta U} d^3x \sqrt{|h|}\, \delta K \right], \tag{2.21}$$

where U denotes the volume, δU denotes the boundary of U, and K is, as usual, the trace of the extrinsic curvature of that boundary [12].

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} \tag{2.22}$$

is the Einstein tensor.

The second term in eq. (2.21) is a surface term. Assuming that $g_{\mu\nu}$ is fixed on the boundary does not imply, however, that this term goes to zero. That would require also the first derivatives of the metric to be fixed on the boundary which is not an option since the number of degrees of freedom of the metric is all that we are allowed to fix [12, 158, 159]. Note that ignoring the surface term is not an alternative here; it just means that we are implicitly fixing the first derivatives. This implies that trying to apply the stationary action principle to the action (2.20), or to the sum of this with action (2.19), in order to derive field equations is unfeasible due to the presence of the non-vanishing surface term. In fact, S_{EH} would not even be functionally differentiable at the solutions of the field equations even if those were attainable. This is the price which we pay for having allowed the action to include second derivatives of the metric in order to maintain the requirement of general covariance.

Note that these terms turned out to be of a certain form: they can be combined to give a surface term which does not really affect the differential order of the field equations, since $G_{\mu\nu}$ does indeed include only up to second derivatives of the metric. This is a special and remarkable characteristic of the Einstein–Hilbert action, S_{EH}, and it is not shared by other actions including second derivatives of the metric. In fact, even before the variation, S_{EH} can be split into a bulk part and a surface term (see [160] for an explicit calculation). The bulk Lagrangian, however, is not a generally covariant scalar. Therefore, one can find non-covariant actions which lead to a variation (2.21) without the unwanted surface term. A typical example is the action proposed by Schrödinger [155]:

$$S_{\rm Scr} = \frac{1}{16\pi G}\int_U d^4x\sqrt{-g}g^{\mu\nu}\left(\{^{\alpha}{}_{\beta\mu}\}\{^{\beta}{}_{\alpha\mu}\} - \{^{\alpha}{}_{\mu\nu}\}\{^{\beta}{}_{\alpha\beta}\}\right), \tag{2.23}$$

In fact, Einstein was one of the first to realize that the gravitational action does not necessarily have to be built out of a generally covariant scalar in order to lead to covariant equations [161]. Of course this does not mean that a non-covariant action would be physically meaningful as an object, since it is reasonable to require than an action carrying some physical meaning should still be coordinate independent or, better yet, diffeomorphism invariant.

Therefore in order to properly derive the Einstein equations, one has to redefine the gravitational action in such a way that no surface term will be present after the variation and at the same time covariance is preserved. Note that since the surface term is actually a total variation of a surface action this is not that hard to do. Starting from the action

$$S'_{EH} = S_{EH} + \frac{1}{8\pi\, G}\int_{\delta U} d^3x\sqrt{|h|}\, K, \tag{2.24}$$

variation with respect to the metric gives

$$\delta S'_{EH} = \frac{1}{16\pi\, G}\int_U d^4x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}. \tag{2.25}$$

Adding the variation of the matter action and applying the stationary action principle, one can straightforwardly derive the Einstein equations

$$G_{\mu\nu} = 8\pi\, G\, T_{\mu\nu}. \tag{2.26}$$

Using S'_{EH}, one has a cancellation of the surface term and hence a clean derivation of the Einstein field equations. This action is usually referred to as the “healed” Einstein–Hilbert action.

It is worth commenting that the surface term in the “healed” Einstein–Hilbert action is more than a trick in order to find a way to combine covariance and well defined variation. It has turned out to have interesting properties since, for instance, it is related to black hole entropy (for a detailed discussion of the role and nature of the surface term see *e.g.* [162]). One also has to mention that even though S'_{EH}

is manifestly covariant, it is not foliation independent, since the presence of the surface term requires the choice of a preferred foliation. Therefore, the action (2.24) cannot be considered really background independent (which is the actual physical property usually enforced by requiring diffeomorphism invariance) (see, for instance, the relevant discussion in [147]).

Let us conclude the derivation of the Einstein equations by mentioning that their left hand side can be derived without reference to an action principle, based on the following arguments: It has to be a divergence free second rank tensor in order to match the right hand side and it has to depend only on the metric and its first and second derivatives. The Einstein tensor is an obvious choice (even though not the only one). It is worth mentioning at this point that one could easily add a cosmological constant Λ to the field equation, either by adding the term $-\Lambda\, g_{\mu\nu}$ on the right hand side of eq. (2.26), or by subtracting 2Λ from the Einstein–Hilbert Lagrangian.

Before closing this section we can sum up the assumptions used to arrive at General Relativity within the framework of metric theories of gravitation:

1. ${\Gamma^\lambda}_{\mu\nu} = {\Gamma^\lambda}_{\nu\mu}$ or ${S_{\mu\nu}}^\lambda = 0$. Spacetime is torsion-less.
2. $\bar{\nabla}_\lambda g_{\mu\nu} = 0$ or $Q_{\mu\nu\lambda} = 0$. The connection is a metric one.
3. No fields other than the metric mediate the gravitational interaction.
4. The field equations should be second order partial differential equations.
5. The field equations should be covariant (or the action should be diffeomorphism invariant).

2.4 Relaxing the assumptions

Having listed the assumptions that lead to General Relativity, one may wonder what a theory which relaxes one or more of these assumptions would look like. Before going further, let us clarify that the assumptions listed in the previous section lead to General Relativity only once one has already adopted some of the viability criteria presented in Section 2.1. For example, we started the discussion presented in this section presupposing the existence of a metric and the dynamical nature of spacetime. Therefore, one should not overestimate the value of the discussion presented in the previous section: it sums up some of the key features of General Relativity but it does not necessarily trace their root.

Relaxing some of the assumptions listed above leads, for instance, to much more drastic departures from General Relativity than others. It is easy to argue that covariance of the field equations is not an assumption that can be cast aside as easily as the absence of any extra field mediating gravity. Indeed, for the reasons discussed in Section 2.1, we will consider covariance as being a very basic principle and will not attempt the relax this assumption in the rest of this thesis. Let us, therefore, concern ourselves here with the relaxation of the following assumptions: those related to the symmetry and the metricity of the connection, the requirement for second order field equations and the absence of any extra field mediating the gravitational interaction.

2.4.1 The Palatini formalism

It is obvious that if one does not specify a relation between the metric $g^{\mu\nu}$ and the connection $\Gamma^\lambda{}_{\mu\nu}$ then this connection can be regarded as an independent field. Therefore, any theory with this characteristic would be drastically different from General Relativity. There is, however, one more possibility: relaxing the assumptions related to the connection but at the same time ending up with General Relativity by deriving them as consequences of the field equations. It is exactly this possibility that we will explore here. It can be found in standard text books under the name of the Palatini formalism (*e.g.* [12, 163]) even though it was Einstein and not Palatini who introduced it [164].

Let us assume that the connection is indeed symmetric and eq. (2.17) holds, but abandon the covariant conservation of the metric, *i.e.* assumption (2.16). We therefore start with a symmetric metric and an independent symmetric connection. We now have two fields describing gravity and we want to construct an action for our theory.

In the process of deriving the Einstein–Hilbert action, eq. (2.20), we considered only R, motivated initially by wanting the resulting field equations to be second order differential equations. This requirement comes from the fact that all other theories besides gravity are described by such field equations. We can build our action here using the same requirement. We need a generally covariant scalar that depends only on our fundamental fields, the metric and the connections, and on their first derivatives at most. Therefore the obvious choice is the Ricci scalar $\mathcal{R}$ [eq. (2.11)].

This is clearly not the only choice. In fact $\mathcal{R}$ does not even include the first derivatives of the metric. It also does not include terms quadratic in the first derivatives of the connection. In this sense, our choice just comes from analogy with the standard Einstein–Hilbert action and is, in practice, a choice of convenience since, as we are about to find out, it will give the desired result.

As far as the matter action is concerned we do not want to abandon the metric postulates. This implies that the matter action will depend only on the metric and not on the independent connection $\Gamma^\lambda{}_{\mu\nu}$. However, if our theory is to be a metric theory of gravity, even though it includes an independent connection, then this connection is not, by definition, carrying its usual physical meaning [165, 166, 167]. It does not define parallel transport or the covariant derivative. The reader should not be surprised by that. In the matter action there can be covariant derivatives and the only way to avoid having a matter action generically independent of $\Gamma^\lambda{}_{\mu\nu}$ is to assume that it is the Levi–Civita connection of the metric that is used for the definition of the covariant derivative. We will analyse this fact extensively later on.

For the moment, let us stress once more that the underlying geometry is indeed *a priori* pseudo-Riemannian. It is worth noticing that this make our choice for the gravitational action even more *ad hoc* since $\mathcal{R}$ will now not really be related to the curvature of spacetime from a geometrical perspective.

In any case the total action will be

$$S_{\rm p} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\mathcal{R} + S_M(g^{\mu\nu},\psi). \tag{2.27}$$

the variation of the action (2.27) should now be performed with respect to both the metric and the connections (or the covariant derivatives) separately. An independent variation with respect to the metric and the connection is called Palatini variation. Note that this should not be confused with the term Palatini formalism, which refers not only to the Palatini variation, but also to having the matter action being independent of the connection.

The easiest way to proceed with the independent variation is to follow [12] and express the Γs, as a sum of the Levi–Civita connections of the metric, $g_{\mu\nu}$, and a tensor field $C^{\lambda}{}_{\mu\nu}$. Variation with respect to the Γs (or the covariant derivative) will then be equivalent to the variation of $C^{\lambda}{}_{\mu\nu}$. On the boundary, $g_{\mu\nu}$ and $C^{\lambda}{}_{\mu\nu}$ will be fixed and we get the following:

$$\begin{aligned} 0 \;=\; & -\frac{1}{8\pi G}\int d^4x\sqrt{-g}g^{\mu\nu}\nabla_{[\mu}\delta C^{\lambda}{}_{\lambda]\nu} + \\ & + \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(C^{\nu\sigma}{}_{\sigma}\delta^{\mu}{}_{\lambda} + C^{\sigma}{}_{\sigma\lambda}g^{\mu\nu} - 2C^{\nu}{}_{\lambda}{}^{\mu}\right)\delta C^{\lambda}{}_{\mu\nu} + \\ & + \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}\,g_{\mu\nu} - 8\pi G\,T_{\mu\nu}\right)\delta g^{\mu\nu}. \end{aligned} \tag{2.28}$$

We see immediately that the first term in eq. (2.28) is again a surface term. This time, however, it is exactly zero since now $\delta C^{\lambda}{}_{\mu\nu} = 0$ on the boundary as $C^{\lambda}{}_{\mu\nu}$ is fixed there. This is, in a sense, an advantage with respect to the metric formalism since no "healing" of the action is required.

Coming back to (2.28) and considering that the independent variations with respect to the metric and with respect to $C^{\lambda}{}_{\mu\nu}$ should vanish separately, we see now that requiring the second term to vanish corresponds to the condition

$$C^{\lambda}{}_{\mu\nu} = 0, \tag{2.29}$$

or

$$\Gamma^{\lambda}{}_{\mu\nu} = \{^{\lambda}{}_{\mu\nu}\}, \tag{2.30}$$

i.e., the Γs have to be the Levi–Civita connections of the metric. So, in the end, the last term leads to the standard Einstein equations given that now, due to eq. (2.30), $\mathcal{R}_{\mu\nu} = R_{\mu\nu}$. Note that the above results remain unchanged if a cosmological constant is added to the action as the resulting equations will then be just the standard Einstein equations with a non-vanishing cosmological constant.

It should be stressed that eq. (2.30) is now a dynamical equation and therefore not a definition, so the Palatini formalism leads to General Relativity without the metricity condition being an external assumption. However, this comes at a price. Our choice for the action is much more *ad hoc* and the physical meaning of the independent connection is obscure since, as we argued, it is not present in the matter action and it is not the one defining parallel transport.

One might decide to allow $\Gamma^{\lambda}{}_{\mu\nu}$ to be present in the matter action and to define the covariant derivative. Even if we start from the same gravitational action, the resulting theory in this case will not be General Relativity [168, 165, 166, 167]. We will return and fully analyse these issues in the next chapter.

2.4.2 Higher order field equations

There is yet another way to deviate from General Relativity without including gravitational fields other than the metric: one can abandon the assumption of having second order field equations and allow the action to depend on higher derivatives of the metric. Taking into account the general covariance requirement, what one does in order to raise the differential degree of the field equations is to add higher order curvature invariants in the gravitational action, for instance $R_{\mu\nu}R^{\mu\nu}$.

Higher order theories of this short are not new. In fact, they date back to 1919 [169, 131] and there have been many periods in which they have received increased interest, including in the last few years. Since we intend to refer to such theories extensively in the forthcoming chapters, we will refrain from saying more here, hoping for the reader's patience.

2.4.3 Extra fields mediating gravity

Up to this point we have only referred to theories where the metric is the only gravitational field[5]. One can consider having other fields mediating the gravitational interaction in some way. Let us stress once more that the terms "gravitational" and "non-gravitational" field are quite ambiguous. Even though we will attempt to clarify this issue at a later stage (Chapter 7), it is important to state what is meant here when we refer to extra fields describing gravity. The term is used, in a loose sense, to refer to any field that can somehow participate in the dynamics of gravity. This could be a field directly describing part of the spacetime geometry, or a field that intervenes passively in the generation of the spacetime geometry by the matter fields. In a Lagrangian formalism such a field is mostly expected to be coupled non-minimally to the metric (otherwise the standard lore is to consider it as a matter field).

Several theories including fields other than the metric have been proposed. Most of them have been ruled out by experiments and can now be considered obsolete. We will avoid referring to such theories unless they constituted a crucial

[5]Even in the Palatini formalism presented in Section 2.4.1, the final outcome was General Relativity.

step towards a more modern theory or may seriously contribute to a better understanding of some subtle issues of contemporary gravitation research. We address the reader interested in the history of such theories to [139] for a more complete list of reference.

We will proceed with our discussion, classifying theories according to the nature of the extra gravitational field (scalar field, vector field *etc*.). However, it should be mentioned that one could also perform a classification according to the dynamics of the field. Note that a non-dynamical field can introduce preferred frame effects in a theory, leading to violation of LLI and/or LPI without necessarily violating general covariance[6].

Scalar fields

In Newtonian gravity the gravitational field is represented by a scalar. Therefore, it should not come as a surprise that one of the early attempts to create a relativistic gravitation theory is indeed a generalisation of Newtonian gravity which preserves the scalar gravitational field as the key field related to gravity. This is Nordstöm's theory, which is actually a predecessor of General Relativity as it was first introduced in 1913 [149]. Apart from its pedagogical value, Nordström's theory can now be considered obsolete. Additionally, it is not a theory that besides the metric includes also a scalar, but more of a scalar theory of gravity.

The study of theories which in addition to the metric include also a scalar field was mainly stimulated by the works of Jordan in 1955 [170] and Brans and Dicke in 1961, leading to the development of what was later called (Jordan–)Brans–Dicke theory. Generalisations of this theory are now called scalar-tensor theories of gravity. We will study such theories in some detail in the next chapter.

Vector fields

As in the case of scalars, also here the first gravitation theory including a vector field came before General Relativity; it was sketched by Hermann Minkowski in 1908. Details about the general form and characteristics of a theory which includes a dynamical vector field in addition to the metric can be found in [139, 144, 171]. We want to concentrate here on two theories that attract significant attention at present.

The first is Tensor-Vector-Scalar gravity (TeVeS), proposed by Jacob Bekenstein in 2004 [172]. Bekenstein's theory includes, besides the metric, not only a vector, but also a scalar field. This theory was tailored to be a relativistic extension of Milgrom's modified Newtonian dynamics (MOND) [173, 174, 175]. MOND suggests a modification of Newton's law of universal gravitation in order to account for the unexpected shape of the rotational curves of galaxies without the need

[6]Non covariant expressions can easily be brought into a covariant form by imposing a list of covariantly expressed constraints via, for example, a Lagrange multiplier (see also [147]).

for dark matter (see Section 1.3.3). TeVeS reduces to MOND instead of standard Newtonian gravity in what is usually called the Newtonian limit.

The second theory which we want to consider is the so called Einstein-Aether theory, proposed by Jacobson and Mattingly [176, 177]. This theory includes a dynamical vector field as well as the metric, but no scalar field. Note that the Lagrangian of the vector field in TeVeS is a special case of the more general Lagrangian of Einstein-Aether theory. The word aether in Einstein–Aether theory refers to some preferred frame. This frame is to be determined by some yet unknown physics which may lead to Lorentz symmetry violations. Such violations can leave an imprint not only on non-gravitational physics, but also on gravity itself and this is exactly the gap which Einstein–Aether theory is hoping to fill. The role of the aether is played by the vector field. Even though the field is dynamical and the theory is fully covariant, the vector is set to be of unitary length *a priori*. This is an implicit violation of background independence and introduces preferred frame effects. It should be noted that Bekenstein's theory also shares this characteristic, even though the fact that the vector field is not coupled to the matter prevents detection of the preferred frame (at least classically).

Tensor fields

Apart from scalar and vector fields, one could also consider including tensor fields in the mediation of the gravitational interaction. Most of the theories developed under this perspective include an extra second rank tensor field, which actually serves as a second metric. The most well known of these theories is Rosen's bimetric theory, which, in addition to the spacetime metric, also includes a flat, non-dynamical metric [178, 179, 180]. Clearly, the presence of the flat, non-dynamical metric implies the existence of some prior geometry and, therefore, the theory is not background independent. Most of the current interest in bimetric theories comes from what is called "variable speed of light Cosmology" which is proposed as an alternative way to approach the problems usually address by the inflationary paradigm [181, 182, 183, 184]. In brief, the relation between a variable speed of light and the existence of a second metric can be explained as follows: The causal propagation of electromagnetic waves is determined by the metric present in Maxwell's equations. Therefore, if one introduces a metric different from that describing the geometry and uses this metric in Maxwell's equations, the outcome will be a theory in which the light speed will not be determined by the spacetime metric.

Affine connections

Finally, let us consider the case of gravitation theories that include affine connections that are not necessarily related to the metric. Before going further, it is worth commenting that even in the early 1920s there was an ongoing discussion about whether it is the metric or the connection that should be considered as being the principal field related to gravity (see *e.g.* [185]). In 1924 Eddington presented a purely affine version of General Relativity in vacuum [131]. In Eddington's theory the metric came about as a derived quantity. Later on, Schödinger generalized Eddington's theory to include a non-symmetric metric [186], therefore arriving at a purely affine version of Einstein–Straus theory which was introduced as a unification of gravity and electromagnetism [187] (see also [188, 189] for a recent review). Purely affine theories of gravity do not now receive much attention, most probably due to the difficulties that arise when one attempts to add matter (however see [190] for some proposals).

A more conventional approach is to consider theories where both a metric and a connection are present but are, at least to some degree, independent. By far the most well-known theory of this sort is Einstein–Cartan(–Sciama–Kibble) theory [191, 192, 193, 194, 195, 196]. This theory assumes that a connection and a metric describe the geometry. The metric is symmetric and covariantly conserved by the connection (vanishing non-metricity). However, the connection is not necessarily symmetric (and therefore it is not the Levi–Civita connection of the metric). The spacetime associated with this theory is called a Riemann-Cartan spacetime. One of the main advantages of the theory is that it allows torsion and relates its presence with the spin of matter. In fact, one could argue that if General Relativity were to be extended to microphysics, spin angular momentum should somehow become a source of the gravitational field, much like standard macroscopic angular momentum [196].

Einstein–Cartan(–Sciama–Kibble) theory is not the only theory that includes an independent connection. For instance, one can decide to abandon the metricity assumption as well and allow the connection to be completely independent of the metric. This generically leads to metric-affine theories of gravity. However, as we will devote a large portion of the next chapter to theories with such characteristics and to the interaction between spin and gravity, we refrain from mentioning more here and refer the reader to Section 3.6.

Chapter 3

Modified actions and field equations for gravity

3.1 Introduction

Having discussed the more general aspects of gravitation theory and briefly reviewed or mentioned some early proposed alternatives to General Relativity, we will concentrate now on a number of specific gravitation theories that have received attention lately. We begin by devoting this chapter to the exploration of their theoretical aspects. In the following chapters, their phenomenological aspects will be studied as well.

The theories considered can come from an action as can many of the interesting theories of gravity. We concentrate on theories which include a scalar field as an extra field mediating the gravitational interaction (such as scalar-tensor theories), theories whose action includes higher order curvature invariants and some specific combinations of these two cases (*e.g.* Gauss–Bonnet gravity). We also extensively consider theories with a connection which is independent of the metric.

The actions of these theories are presented and in many cases their resemblance with effective low-energy actions coming from more fundamental theories is briefly discussed. We also present the derivation of the field equations through the application of a suitable variational principle and analyse the basic characteristics of the theory, as expressed through the field equations.

3.2 Scalar-Tensor theory of gravity

3.2.1 A predecessor: Brans–Dicke theory

As already discussed in the previous chapters, Dicke has been one of the pioneers in the discussion of the conceptual basis of gravitation theories. In 1961, motivated by Mach's Principle — which, according to Dicke, can take the clearer formulation "the gravitational constant should be a function of the mass distribution of the universe" — he introduced, together with his student Carl Brans, what is now called Brans–Dicke Theory [133]. This theory includes, apart from the metric, also a scalar field in the mediation of the gravitational interaction and it was based on earlier works by Pascual Jordan [170] among others.

The action for Brans–Dicke theory is

$$S_{BD} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega_0}{\phi} (\partial_\mu \phi \partial^\mu \phi) \right] + S_M(g_{\mu\nu}, \psi), \tag{3.1}$$

where ϕ is a scalar field and ω_0 is called the Brans–Dicke parameter. Note that ϕ is not present in the matter action, *i.e.* it is not coupled to the matter, but it is non-minimally coupled to gravity. Note also that G is, as usual, Newton's gravitational constant.

It is apparent from the action (3.1) why Brans–Dicke theory can be considered as a theory with a varying gravitational constant, since one can always define an effective gravitational "constant", or better an effective gravitational coupling

$$G_{\text{eff}} = \frac{G}{\phi}. \tag{3.2}$$

Therefore, the theory can indeed be thought as a manifestation of Dicke's formulation of Mach's Principle.

Brans–Dicke theory has only one extra free parameter with respect to General Relativity, ω_0. This is a characteristic many would consider as a merit for an alternative theory of gravity, since it makes it easy to test and constrain or even rule out the theory. Indeed, using the standard post-Newtonian expansion [139] one can utilize Solar System tests to derive a bound for ω_0 (see for example [197]):

$$|\omega_0| > 40\,000. \tag{3.3}$$

This unusually large value is hardly appealing since one expects dimensionless coupling parameters to be of order unity. Thus, Brans–Dicke theory is no longer considered a viable alternative to General Relativity but serves as a model theory within a more general class of theories including a scalar field.

3.2.2 Action and field equations

Brans–Dicke theory can be straightforwardly generalised into what is called a scalar-tensor theory of gravity. A general form for the action of such theories is

$$S_{ST} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} (\partial_\mu \phi \partial^\mu \phi) - V(\phi) \right] + S_M(g_{\mu\nu}, \psi), \tag{3.4}$$

where $V(\phi)$ is the potential of the scalar field ϕ and $\omega(\phi)$ is some function of ϕ. Note that by setting $\omega(\phi) = \omega_0$ we derive the action

$$S_{BDV} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega_0}{\phi} (\partial_\mu \phi \partial^\mu \phi) - V(\phi) \right] + S_M(g_{\mu\nu}, \psi). \tag{3.5}$$

If we also exclude the potential term $V(\phi)$, then we return to the action (3.1).

The theory described by action (3.5) is a Brans–Dicke theory with a potential for the scalar and is sometimes referred to in the literature as a scalar-tensor theory and sometimes simply as Brans–Dicke theory. Even though, strictly speaking, the theory introduced by Brans and Dicke did not include a potential, we will reserve the term scalar-tensor theories for more general theories described by the action (3.4) and in what comes next we will be referring to the action (3.5) as Brans–Dicke theory with a potential or simply Brans–Dicke theory.

It is worth clarifying here that Brans–Dicke theory and any other version of scalar-tensor theory are metric theories of gravity: the scalar field is not coupled directly to the matter and so matter responds only to the metric. The role of the scalar field is just to intervene in the generation of the spacetime curvature associated with the metric [139].

Note also that the bound on ω_0 mentioned earlier for Brans–Dicke theory without a potential is still applicable in the presence of a potential or even for a general scalar-tensor theory in the form

$$|\omega(\phi_0)| > 40\,000, \tag{3.6}$$

where ϕ_0 is the present value of the scalar. However, for this constraint to be applicable, the effective mass of the scalar field should be low or, as commonly said, the potential should be light ($\partial^2 V/\partial\phi^2$ evaluated at ϕ_0 plays the role of an effective mass). If the potential is heavy, then the scalar field becomes very short-ranged and the bound is not applicable.

Since scalar-tensor theory is one of the most widely-studied alternatives to General Relativity and there are standard text books analysing its characteristics [198, 199], we will not go much further in our discussion of it here. Before deriving the field equations from the action, let us just comment that non-minimally coupled scalar fields are present in the low energy effective action of more fundamental theories, such as String Theory (*e.g.* dilaton), and a potential might be expected to be present after supersymmetry breaking. We address the reader to the literatures for more details [198, 199].

We can now proceed to vary the action (3.4) to derive the field equations. Independent variation with respect to the metric and the scalar field gives

$$\begin{aligned} G_{\mu\nu} = \frac{8\pi G}{\phi}T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2}\Big(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi\Big)+ \\ +\frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) - \frac{V}{2\phi}g_{\mu\nu}, \end{aligned} \tag{3.7}$$

$$\Box\phi = -\frac{\phi}{2\omega}\left(R - V'\right) - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi\left(\frac{\omega'(\phi)}{\omega(\phi)} - \frac{1}{\phi}\right), \tag{3.8}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$ is the stress-energy tensor, ∇ denotes covariant differentiation, $\Box \equiv \nabla^\mu\nabla_\mu$ and a prime denotes differentiation with respect to the argument. One can take the trace of eq. (3.7) and use the result to replace R in eq. (3.8) to derive

$$(2\omega(\phi)+3)\Box\phi = 8\pi\, G\, T - \omega'(\phi)\nabla^\lambda\phi\nabla_\lambda\phi + \phi V' - 2V, \tag{3.9}$$

where $T \equiv g^{\mu\nu}T_{\mu\nu}$ is the the trace of the stress energy tensor. Note that eq. (3.8) implies a coupling between the scalar field and the metric but no coupling with matter, as expected, so we should not be misled by the presence of matter in eq. (3.9): the field ϕ acts back on matter only through the geometry.

By setting $\omega(\phi) = \omega_0$ or by varying the action (3.5) directly, we can get the simpler field equations for Brans–Dicke theory with a potential:

$$\begin{aligned} G_{\mu\nu} = \frac{8\,\pi\, G}{\phi}T_{\mu\nu} \quad &+ \quad \frac{\omega_0}{\phi^2}\Big(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi\Big) + \\ &+ \quad \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) - \frac{V}{2\phi}g_{\mu\nu}, \end{aligned} \tag{3.10}$$

$$\frac{2\omega_0}{\phi}\Box\phi + R - \frac{\omega_0}{\phi^2}\nabla^\mu\phi\nabla_\mu\phi - V' = 0. \tag{3.11}$$

In Brans–Dicke theory, eq. (3.9) takes the simpler form

$$(2\omega_0+3)\Box\phi = 8\pi\, G\, T + \phi V' - 2V. \tag{3.12}$$

Let us close this section with a warning about the effective gravitational coupling. As we said in the previous section, for Brans–Dicke theory one can define $G_{\rm eff}$ through eq. (3.2). In the same way as in eq. (3.2) one can define the effective gravitational coupling for any scalar-tensor theory. However, it should be stressed that this is not going to be the coupling as measured by a Cavendish experiment. The latter would be [200]

$$G^{(*)}_{\rm eff} = \frac{G}{\phi}\frac{2\omega\,\phi+2}{2\omega\,\phi+3}. \tag{3.13}$$

The reason for this difference is quite straightforward: $G_{\rm eff}$ is, in practice, the inverse of the coefficient of R as read from the action, whereas $G^{(*)}_{\rm eff}$ is the quantity of dimensions ${\rm cm}^3\,{\rm g}^{-1}\,{\rm s}^{-2}$ which appears in Newton's second law in a two body problem, such as a Cavendish experiment. These two quantities are not generically the same.

3.3 $f(R)$ gravity in the metric formalism

3.3.1 The action

We have already briefly discussed in the previous chapter the possibility of including higher order curvature invariants in the gravitational action. Attempts towards this direction were first examined by Weyl and Eddington in 1919 and 1922 respectively [169, 131], mainly on the basis of theoretical completeness. It is easy to understand that complicating the action, and consequently the field equations, with no apparent reason is not so appealing. For instance, the degree of the field equations will become higher than second and we currently are unaware of any other physical theory with such characteristics.

However, starting from the early 1960s, there appeared indications that complicating the gravitational action might indeed have its merits. As discussed in the Introduction, General Relativity is not renormalisable and therefore cannot be conventionally quantized. In 1962, Utiwama and De Witt showed that renormalisation at one-loop demands that the Einstein–Hilbert action should be supplemented by higher order curvature terms [114]. Later on, Stelle showed that higher order actions are indeed renormalisable (but not unitary) [115]. More recent results show that when quantum corrections or String Theory are taken into account, the effective low energy gravitational action admits higher order curvature invariants [117, 118, 119].

Even though initially the relevance of such terms in the action was considered to be restricted to very strong gravity phenomena and they were expected to be strongly suppressed by small couplings, this perspective has recently changed as discussed in the Introduction. The main reason for this was the motivation provided by the cosmological problems such as the dark energy problem, the late-time accelerated expansion of the universe, the cosmological constant problems *etc.* (see Chapter 1).

Higher order actions may include various curvature invariants, such as R^2, $R_{\mu\nu}R^{\mu\nu}$ *etc.*, but for orientation purposes one can consider an action of the form

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R). \tag{3.14}$$

The appealing feature of such an action is that it combines mathematical simplicity and a fair amount of generality. For example, viewing f as a series expansion of f, *i.e.*

$$f(R) = \cdots + \frac{\alpha_2}{R^2} + \frac{\alpha_1}{R} - 2\Lambda + R + \frac{R^2}{\beta_2} + \frac{R^3}{\beta_3}\cdots, \tag{3.15}$$

where the α_i and β_j coefficients have the appropriate dimensions, we see that the action includes a number of phenomenologically interesting terms.

$f(R)$ actions where first rigorously studied by Buchdahl [201]. We will proceed to derive the field equations for such actions here. We will discuss the cosmological implications and the way in which such theories can address the cosmological problems in the next chapter.

3.3.2 Field equations

Adding a matter action, the total action for $f(R)$ gravity takes the form

$$S_{met} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu}, \psi). \tag{3.16}$$

Variation with respect to the metric gives, after some manipulations,

$$\begin{aligned}
\delta S_{met} &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[f'(R)R_{\mu\nu}-\frac{1}{2}f(R)g_{\mu\nu}- \\
&\quad -\nabla_\mu\nabla_\nu f'(R)+g_{\mu\nu}\Box f'-8\pi G T_{\mu\nu}\Big]\delta g^{\mu\nu}- \\
&\quad -\frac{1}{8\pi G}\int_{\delta U} d^3x\sqrt{|h|}\,f'(R)\,\delta K. \qquad (3.17)
\end{aligned}$$

The integral in the last line represents a surface term. However, unlike the variation of the Einstein–Hilbert action, this surface term is not the total variation of a quantity, due to the presence of $f'(R)$. This implies that it is not possible to “heal” the action just by subtracting some surface term before making the variation.

Formally speaking, we cannot derive the field equations from this variation by applying the stationary action principle before finding a way to treat the surface term. However, the action includes higher order derivatives of the metric and therefore it is possible to fix more degrees of freedom on the boundary than those of the metric itself. It has to be stressed at this point that there are several auxiliary variables which one can fix in order to set the surface term to zero. Additionally, the choice of the auxiliary variable is not void of physical meaning, even though this might not be obvious at a classical level, since it will be relevant for the Hamiltonian formulation of the theory.

There is no unique prescription for making the fixing in the literature so far. The situation gets even more complicated if one takes into account that arbitrary surface terms could also be added into the action in order to allow different fixings to lead to a well define variation (see also [202] for a discussion on the surface term of $f(R)$ gravity). Therefore, non-rigorous as it may be, the standard approach at this stage is to neglect the surface term, silently assuming that a suitable fixing has been chosen, and go directly to the field equations

$$f'(R)R_{\mu\nu}-\frac{1}{2}f(R)g_{\mu\nu}-\nabla_\mu\nabla_\nu f'(R)+g_{\mu\nu}\Box f'=8\pi G T_{\mu\nu}. \qquad (3.18)$$

This mathematical jump might seem worrying and certainly gives no insight for the choice of auxiliary variables, which would be necessary for a Hamiltonian formulation or a canonical quantisation. However, the field equations (3.18) would be unaffected by the fixing chosen and from a purely classical perspective the field equations are all that one needs.

Eqs. (3.18) are obviously fourth order partial differential equations in the metric. Notice, however, that the fourth order terms — the last two on the left hand side — vanish when $f'(R)$ is a constant, *i.e.* for an action which is linear in R. Thus, it is straightforward for these equations to reduce to the Einstein equation once $f(R)=R$.

It is also worth noticing that the trace of eq. (3.18)

$$f'(R)R-2f(R)+3\Box f'=8\pi G T, \qquad (3.19)$$

where $T = g^{\mu\nu}T_{\mu\nu}$, relates R with T differentially and not algebraically as in General Relativity, where $R = -8\pi GT$. This is already an indication that the field equations of $f(R)$ theories will admit more solutions than Einstein's theory. As an example, we can mention here that Birkhoff's theorem, stating that the Schwarzschild solution is the unique spherically symmetric vacuum solution, no longer holds in metric $f(R)$ gravity. Without going into the details of the calculation, let us stress that $T = 0$ no longer implies that $R = 0$, or is even constant.

Another important aspect of such theories has to do with their maximally symmetric solutions. The functional form of f is what will affect whether the maximally symmetric solution will be Minkowski, de Sitter or anti-de Sitter. To see this, let us recall that maximally symmetric solutions lead to a constant Ricci scalar. For $R = \text{constant}$ and $T_{\mu\nu} = 0$ eq. (3.19) reduces to

$$f'(R)R - 2f(R) = 0, \tag{3.20}$$

which, for a given f, is an algebraic equation in R. If $R = 0$ is a root of this equation and one takes this root, then eq. (3.18) reduces to $R_{\mu\nu} = 0$ and the maximally symmetric solution is Minkowski spacetime. On the other hand, if the root of eq. (3.20) is $R = C$, where C is a constant, then eq. (3.18) reduces to $R_{\mu\nu} = C/4 g_{\mu\nu}$ and the maximally symmetric solution is de Sitter or anti-de Sitter depending on the sign of C, just as in General Relativity with a cosmological constant.

3.4 $f(R)$ gravity in the Palatini formalism

3.4.1 The action

In Section 2.4.1, we showed how Einstein's equation can be derived using, instead of the standard metric variation of the Einstein–Hilbert action, the Palatini formalism, *i.e.* an independent variation with respect to the metric and an independent connection (Palatini variation) of an action with gravitational Lagrangian $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$, where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor constructed with the independent connection, and a matter action independent of the connection. Recall the importance of this last assumption, of the independence of the matter action and the connection, as it is crucial for the derivation and is a main characteristic of the Palatini formalism, which as we argued in Section 2.4.1 has consequences for the physical meaning of the independent connection: namely, this connection does not define parallel transport and the geometry is actually pseudo-Riemannian.

One can generalise the action in exactly the same way that the Einstein–Hilbert action was generalised in the previous section:

$$S_{pal} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(\mathcal{R}) + S_M(g_{\mu\nu}, \psi). \tag{3.21}$$

The motivation for studying such actions is, in practice, the same as in metric $f(R)$ gravity, and so we will not repeat it here. Applying the Palatini variation to the action (3.21) leads to what is called $f(R)$ gravity in the Palatini formalism or simply Palatini $f(R)$ gravity. Even though f is really a function of $\mathcal{R}$ and not R in this case, the term $f(R)$ gravity is used as a generic terminology to refer to a theory whose Lagrangian is a general function of some Ricci scalar.

3.4.2 Field equations

Let us proceed to derive the field equations for Palatini $f(R)$ gravity. The variation with respect to the metric is quite straightforward, since $\mathcal{R}_{\mu\nu}$ does not depend on it. However, the variation with respect to the connection is more intricate, since it requires $\delta\mathcal{R}_{\mu\nu}$. Taking into account the definition and symmetries of $\mathcal{R}_{\mu\nu}$, and after some manipulations, it can be shown that [155]

$$\delta\mathcal{R}_{\mu\nu} \;=\; \bar{\nabla}_\lambda \delta\Gamma^\lambda{}_{\mu\nu} - \bar{\nabla}_\nu \delta\Gamma^\lambda{}_{\mu\lambda}. \tag{3.22}$$

We remind to the reader $\bar{\nabla}_\lambda$ denotes the covariant derivative defined with the independent connection.

The variation of the matter action with respect to the independent connection is zero since we do not allow the matter action to depend on $\Gamma^\lambda{}_{\mu\nu}$. On the other hand, by definition

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{3.23}$$

Using eq. (3.22), the variation of the gravitational part of the action takes the form

$$\begin{aligned}\delta S_{pal} \;=\; & \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} - 8\pi G T_{\mu\nu}\right)\delta g^{\mu\nu} + \\ & + \frac{1}{16\pi G}\int d^4x\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\left(\bar{\nabla}_\lambda\delta\Gamma^\lambda{}_{\mu\nu} - \bar{\nabla}_\nu\delta\Gamma^\lambda{}_{\mu\lambda}\right).\end{aligned} \tag{3.24}$$

Integrating by parts the terms in the second line and taking into account that on the boundary $\delta\Gamma^\lambda{}_{\mu\nu} = 0$ and therefore surface terms linear in $\delta\Gamma^\lambda{}_{\mu\nu}$ vanish, we get

$$\begin{aligned}\delta S_{pal} \;=\; & \frac{1}{16\pi G}\int d^4x\Bigg\{\sqrt{-g}\left(f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} - 8\pi G T_{\mu\nu}\right)\delta g^{\mu\nu} + \\ & + \left[-\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right)\delta^\nu_\lambda\right]\delta\Gamma^\lambda{}_{\mu\nu}\Bigg\}.\end{aligned} \tag{3.25}$$

Applying the stationary action principle, straightforwardly leads to the equations

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3.26}$$

$$-\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\sigma(\mu}\right)\delta^{\nu)}_\lambda = 0, \tag{3.27}$$

where indices inside parentheses are symmetrised. Taking the trace of eq. (3.27), it can be easily shown that

$$\bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\sigma\mu}\right)=0, \tag{3.28}$$

which implies that we can bring the field equations into the more economic form

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)}-\frac{1}{2}f(\mathcal{R})g_{\mu\nu}=8\pi G T_{\mu\nu}, \tag{3.29}$$
$$\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right)=0, \tag{3.30}$$

3.4.3 Manipulations of the field equations

Let us explore the characteristics of eqs. (3.29) and (3.30). Taking the trace of eq. (3.29), we get

$$f'(\mathcal{R})\mathcal{R}-2f(\mathcal{R})=8\pi G T. \tag{3.31}$$

For a given f, this is an algebraic equation in $\mathcal{R}$. For all cases for which $T=0$, which includes vacuum and electrovacuum, $\mathcal{R}$ will therefore be a constant and a root of the equation

$$f'(\mathcal{R})\mathcal{R}-2f(\mathcal{R})=0. \tag{3.32}$$

We will not consider cases for which this equation has no roots since it can be shown that the field equations are then not consistent [203]. Therefore choices of f that lead to this behaviour should simply be avoided. Eq. (3.32) can also be identically satisfied if $f(\mathcal{R})\propto\mathcal{R}^2$. This very particular choice for f leads to a conformally invariant theory [203]. As is apparent from eq. (3.31), if $f(\mathcal{R})\propto\mathcal{R}^2$ then only conformally invariant matter, for which $T=0$ identically, can be coupled to gravity. Matter is not generically conformally invariant though and so this particular choice of f is not suitable for a low energy theory of gravity. We will, therefore, neglect it for now and return to it in a later section.

Let us now consider eq. (3.30). Notice that if we define a metric conformal to $g_{\mu\nu}$ to be

$$h_{\mu\nu}=f'(\mathcal{R})g_{\mu\nu}, \tag{3.33}$$

then this equation becomes the definition of the Levi–Civita connection of $h_{\mu\nu}$. In this way, one in practice solves eq. (3.30) and can then express the independent connection as

$$\Gamma^\lambda{}_{\mu\nu}=h^{\lambda\sigma}\left(\partial_\mu h_{\nu\sigma}+\partial_\nu h_{\mu\sigma}-\partial_\sigma h_{\mu\nu}\right), \tag{3.34}$$

or equivalently in terms of $g_{\mu\nu}$

$$\Gamma^\lambda{}_{\mu\nu}=\frac{1}{f'(\mathcal{R})}g^{\lambda\sigma}\left(\partial_\mu\left(f'(\mathcal{R})g_{\nu\sigma}\right)+\partial_\nu\left(f'(\mathcal{R})g_{\mu\sigma}\right)-\partial_\sigma\left(f'(\mathcal{R})g_{\mu\nu}\right)\right), \tag{3.35}$$

Given that eq. (3.31) relates $\mathcal{R}$ algebraically with T, and since we have an explicit expression for $\Gamma^\lambda{}_{\mu\nu}$ in terms of $\mathcal{R}$ and $g^{\mu\nu}$, we can in principle eliminate the independent connection from the field equations and express them only in terms of the metric and the matter fields. In fact, taking into account how the Ricci tensor transforms under conformal transformations, we can write

$$\begin{aligned}\mathcal{R}_{\mu\nu} = R_{\mu\nu} \quad + \quad & \frac{3}{2}\frac{1}{(f'(\mathcal{R}))^2}\left(\nabla_\mu f'(\mathcal{R})\right)\left(\nabla_\nu f'(\mathcal{R})\right) - \\ & -\frac{1}{f'(\mathcal{R})}\left(\nabla_\mu\nabla_\nu - \frac{1}{2}g_{\mu\nu}\Box\right)f'(\mathcal{R}).\end{aligned} \tag{3.36}$$

Contracting with $g^{\mu\nu}$ we get,

$$\mathcal{R} = R + \frac{3}{2(f'(\mathcal{R}))^2}\left(\nabla_\mu f'(\mathcal{R})\right)\left(\nabla^\mu f'(\mathcal{R})\right) + \frac{3}{f'(\mathcal{R})}\Box f'(\mathcal{R}). \tag{3.37}$$

Note the difference between $\mathcal{R}$ and the Ricci scalar of $h_{\mu\nu}$ due to the fact that $g_{\mu\nu}$ is used here for the contraction of $\mathcal{R}_{\mu\nu}$.

Replacing eqs. (3.36) and (3.37) in eq. (3.29), and after some easy manipulations, we get

$$\begin{aligned}G_{\mu\nu} = & \frac{8\pi G}{f'}T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(\mathcal{R} - \frac{f}{f'}\right) + \frac{1}{f'}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)f' - \\ & - \frac{3}{2}\frac{1}{f'^2}\left((\nabla_\mu f')(\nabla_\nu f') - \frac{1}{2}g_{\mu\nu}(\nabla f')^2\right).\end{aligned} \tag{3.38}$$

Notice that, assuming we know the root of eq. (3.31), $\mathcal{R} = \mathcal{R}(T)$ and we have completely eliminated the presence of the independent connection. Therefore, we have successfully reduced the number of field equations to one and at the same time both side of eq. (3.38) depend only on the metric and the matter fields. In a sense the theory has been brought to the form of General Relativity with a modified source.

We can now straightforwardly deduce the following:

- When $f(\mathcal{R}) = \mathcal{R}$, the theory reduces to General Relativity, as discussed in Section 2.4.1.

- For matter fields for which $T = 0$, due to eq. (3.32) $\mathcal{R}$ and consequently $f(\mathcal{R})$ and $f'(\mathcal{R})$ are constants and the theory reduces to General Relativity with a cosmological constant and a modified coupling constant G/f'. If we denote the value of $\mathcal{R}$ when $T = 0$ as $\mathcal{R}_0$, then the value of the cosmological constant is
$$\frac{1}{2}\left(\mathcal{R}_0 - \frac{f(\mathcal{R}_0)}{f'(\mathcal{R}_0)}\right) = \frac{\mathcal{R}_0}{4}, \tag{3.39}$$
where we have used eq. (3.32). Besides vacuum, $T = 0$ also for electromagnetic fields, radiation, and any other conformally invariant type of matter.

- In the general case $T \neq 0$, the modified source on the right hand side includes derivatives of the stress-energy tensor, unlike in General Relativity.

These are implicit in the last two terms of eq. (3.38), since f' is in practice a function of T, given that $f' = f'(\mathcal{R})$ and $\mathcal{R} = \mathcal{R}(T)$[1].

The last observation is a crucial characteristic of Palatini $f(R)$ gravity. We will return to this later on and discuss its implications. We will also reconsider the possible representations of the field equations and the action of Palatini $f(R)$ gravity in Chapter 4.

3.5 Other actions including higher-order curvature invariants

3.5.1 Metric formalism

Generalising the Einstein–Hilbert action into an $f(R)$ action is a minimal modification that one can pursue in order to include higher curvature invariants. In fact, there are a number of invariants that one can construct from the metric, which are not included in an $f(R)$ action. One can, for instance, contract the Ricci or the Riemann tensor with itself to form $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$. Other combinations are also allowed, such as $R^{\mu\nu\lambda\sigma}R_{\mu\nu}R_{\lambda\sigma}$ or invariants formed with other tensors, such as the Weyl tensor. All of these invariants can be considered as combinations of contractions of the Riemann tensor one or more times with itself and the metric.

A specific choice is the Gauss–Bonnet invariant

$$\mathcal{G} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\kappa\lambda}R_{\mu\nu\kappa\lambda}. \tag{3.40}$$

$\mathcal{G}$ apart from being an invariant in the sense used here, *i.e.* being a generally covariant scalar, is also a topological invariant in four dimensions. This means that it is related through the Gauss–Bonnet formula to the Euler characteristic of the 4-dimensional manifold, which characterises the topology. Also, from Gauss's theorem, the variation of the scalar density $\sqrt{-g}\mathcal{G}$ with respect the metric is a total divergence. Therefore, adding $\mathcal{G}$ to the Einstein–Hilbert action will not contribute to the field equations and additionally a suitable surface term can be found to eliminate the total divergence [204].

Due to the above, it is possible to write the most general action which is linear in second order curvature invariants as [205]:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R + aR^2 + bR^{\mu\nu}R_{\mu\nu}\right), \tag{3.41}$$

where the coefficients a and b should have suitable dimensions. Including an $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ term is equivalent to altering those coefficients, since one can always add a Gauss–Bonnet term with a suitable coefficient in order to eliminate

[1]Note that, apart from special cases such as a perfect fluid, $T_{\mu\nu}$ and consequently T already include first derivatives of the matter fields, given that the matter action has such a dependence. This implies that the right hand side of eq. (3.38) will include at least second derivatives of the matter fields, and possibly up to third derivatives.

$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$. The theory described by this action is referred to as fourth-order gravity, since it leads to fourth order equations. Numerous papers have been devoted to the study of fourth-order gravity. Instead of listing them here, we refer the reader to some historical reviews [206, 207].

Notice that one can also choose to include invariants involving derivatives of the curvature terms, such as $R\Box R$. The differential order of the field equations is increased as one adds higher derivative terms in the action. The rule of the thumb is that for every one order increase in the action one gets a two order increase in the field equations. Thus, the R term leads to second order equations, the R^2 term or more general $f(R)$ actions lead to fourth order equations and the $R\Box R$ and $R\Box^2 R$ terms lead to sixth and eighth order equations respectively [208, 209, 210, 211].

Following the example of $f(R)$ gravity, one can also choose to include arbitrary functions of some of the above invariants in the action. For instance, actions of the form $f(R, R^{\mu\nu}R_{\mu\nu})$ can be considered. A comment is due at this point: even though $\mathcal{G}$ is a topological invariant and does not contribute to the field equations if included in the action, the presence of functions of $\mathcal{G}$ in the action will influence the dynamics. For example the action

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi\, G} + f(\mathcal{G})\right), \tag{3.42}$$

does not lead to the Einstein equations [212]. We will discuss actions that include the Gauss–Bonnet invariant more extensively towards the end of this chapter.

3.5.2 Palatini formalism

As in the metric formalism, one can generalise the action to include higher order curvature invariants also in the Palatini formalism. Not much work has been done in this direction. In this section we shall focus mainly on two aspects of such generalisations: the role of $\mathcal{G}$ and the effect of such generalisations on the field equations.

As we have mentioned, in the Palatini formalism the geometry of spacetime is pseudo-Riemannian, due to the fact that the independent connection $\Gamma^{\lambda}{}_{\mu\nu}$ is not present in the matter action and does not define parallel transport. This implies that $\mathcal{G}$, as defined in eq. (3.40), is still the topological invariant related to the Euler characteristic. To make this discussion clearer, let us consider what happens if $\mathcal{G}$ is added to action (2.27) of Section 2.4.1. The variation with respect to the metric will remain unchanged, since $\delta(\sqrt{-g}\mathcal{G})$ contributes only by a surface term that can be removed as mentioned earlier. Variation with respect to the connection will also remain unchanged, since $\mathcal{G}$ does not depend on $\Gamma^{\lambda}{}_{\mu\nu}$ but is constructed only using the metric.

One should not confuse $\mathcal{G}$ with the combination $\mathcal{R}^2 - 4\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} + \mathcal{R}^{\mu\nu\kappa\lambda}\mathcal{R}_{\mu\nu\kappa\lambda}$ which is not a topological invariant. This implies that $\mathcal{R}^{\mu\nu\kappa\lambda}\mathcal{R}_{\mu\nu\kappa\lambda}$ in the action cannot be eliminated in favour of $\mathcal{R}^2$ and $\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}$ terms as in the metric formalism. In the Palatini formalism one is more interested in including in the action invariants such as $\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}$ and $\mathcal{R}^{\mu\nu\kappa\lambda}\mathcal{R}_{\mu\nu\kappa\lambda}$ which are constructed using the independent connection as well and not terms like $R^{\mu\nu}R_{\mu\nu}$ which depend only on the metric.

Let us see how the presence of such invariants will affect the field equations. Consider the action

$$S_{\mathrm{p}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(\mathcal{R} + a\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}\right) + S_M(g^{\mu\nu},\psi) \tag{3.43}$$

where a should be chosen so as to have proper dimensions. Since we have already computed the variation of the rest of the action, let us focus on the $\sqrt{-g}\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}$ part. This gives

$$\begin{aligned}\delta(\sqrt{-g}\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}) &= -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}\delta g^{\alpha\beta} + \sqrt{-g}\delta(\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}) = \\ &= -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}\delta g^{\alpha\beta} + 2\sqrt{-g}\mathcal{R}^{\mu}_{\alpha}\mathcal{R}_{\mu\beta}\delta g^{\alpha\beta} \\ &\quad +2\sqrt{-g}\mathcal{R}^{\mu\nu}\delta\mathcal{R}_{\mu\nu}.\end{aligned} \tag{3.44}$$

Using eq. (3.22) and the variations (3.24) and (3.25) for $f(\mathcal{R}) = \mathcal{R}$ as a guide, we can straightforwardly derive the field equations

$$\mathcal{R}_{(\mu\nu)} + 2a\mathcal{R}^{\sigma}_{\mu}\mathcal{R}_{\sigma\nu} - \frac{1}{2}\left(\mathcal{R} + a\mathcal{R}^{\sigma\lambda}\mathcal{R}_{\sigma\lambda}\right)g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3.45}$$

$$\bar{\nabla}_{\lambda}\left(\sqrt{-g}\left(g^{\mu\nu} + 2a\mathcal{R}^{\mu\nu}\right)\right) = 0. \tag{3.46}$$

Comparing these equations with eqs. (3.29) and (3.30), the following comment is due: Eq. (3.30) is in practice an algebraic equation in ${\Gamma^{\lambda}}_{\mu\nu}$ since $\bar{\nabla}$ is linear in the connection and no derivatives of ${\Gamma^{\lambda}}_{\mu\nu}$ are present. This is why we were able to solve for ${\Gamma^{\lambda}}_{\mu\nu}$, eliminate it and rewrite the equations easily in the form of eq. (3.38). This is not the case here because $\mathcal{R}^{\mu\nu}$ depends on the derivatives of the connection. Therefore, eq. (3.46) is a differential equation relating ${\Gamma^{\lambda}}_{\mu\nu}$ and $g_{\mu\nu}$ and we can conclude that including a higher order term, such as $\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}$, induces more dynamics in the theory.

3.6 Metric-affine gravity

3.6.1 The significance of coupling the connection to matter

We have mentioned several times that in the Palatini formalism the independent connection ${\Gamma^{\lambda}}_{\mu\nu}$ is not present in the matter action and that this makes the theory a metric theory of gravity and the geometry pseudo-Riemannian. In fact, Palatini $f(R)$ gravity satisfies the metric postulates, since it can be shown that the stress energy tensor of matter is indeed divergence-free with respect to the Levi–Civita connection of the metric [213]. This should have been expected from the fact that the only field coupled to matter is the metric $g_{\mu\nu}$.

How physical is it though to include an independent connection in the theory without coupling it to the matter fields? Usually the affine connection defines parallel transport and the covariant derivative. The matter action includes covariant derivatives of the matter fields and consequently couplings between the fields and the connection in the general cases. Some known exceptions to this rule are scalar fields (since in the case of a scalar, a covariant derivative reduces to a partial one) and the Electromagnetic field, due to the specific structure of its action having its roots in gauge invariance (we will discuss this in detail shortly). Therefore, the assumption

$$\frac{\delta S_M}{\delta \Gamma^{\lambda}{}_{\mu\nu}} = 0 \tag{3.47}$$

has physical implications [167]. It either implies that the matter action includes only specific matter fields — an implausibly limiting option for a gravitation theory — or that $\Gamma^{\lambda}{}_{\mu\nu}$ is not the affine connection with which we define parallel transport and the covariant derivative, as we have been stressing in the previous sections.

Of course, one is allowed to add an affinity as an extra field, even if this affinity does not have the usual geometric interpretation and it is the Levi–Civita connection of the metric that plays this role. This is what happens in the Palatini formalism. However, it is interesting to explore what actually happens if the independent connection is given its usual geometric characteristics, *i.e.* if it is $\Gamma^{\lambda}{}_{\mu\nu}$ that defines parallel transport and therefore is coupled to the matter. The matter action will then be $S_M(g^{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \psi)$ and its variation with respect to the connection will no longer vanish.

Such a theory is a metric-affine theory of gravity. Besides the standard motivation for alternative theories of gravity, from High Energy Physics and Cosmology (mentioned in the Introduction and discussed previously in this chapter for other theories), metric-affine gravity has one more appealing characteristic: the connection can be left non-symmetric and the theory can naturally include torsion. This implies that the theory can be coupled in a more natural way to some matter fields, such as fermions (Dirac fields). Note that the stress energy tensor of a Dirac field is not symmetric by definition and this is something that poses an extra difficulty when one attempts to couple such fields to General Relativity. In fact, one might expect that at some intermediate or high energy regime, the spin of particles might interact with the geometry and torsion can naturally arise [214] [*c.f.* with Section 2.4.3 and Einstein–Cartan Theory [196]]. Metric-affine gravity, unlike General Relativity, allows for this to happen.

There are a number of early works in which the metric and the parallel transport defining connection are considered as being, to some degree, independent (see for instance [215, 216, 168, 217] and references therein). In many cases, including Einstein–Cartan theory, some part of the connection is related to the metric (*e.g.* the non-metricity) [196]. We will consider the case where $\Gamma^{\lambda}{}_{\mu\nu}$ is left completely unconstrained and is determined by the field equations. This approach was first presented in [168] for an action linear in $\mathcal{R}$. We will generalize it here for $f(\mathcal{R})$

actions [165, 218]. Before going any further, it should be noted that the metric-affine approach has also been widely used in order to interpret gravity as a gauge theory (see, for example, [219] for a study of $f(R)$ actions and [214] for a thorough review).

3.6.2 The action

Let us construct the action which we will be using step by step. To begin with, we have already specified that the matter action will have the general form $S_M = S_M\left(g^{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \psi\right)$. We can then concentrate on the gravitational action. We can once more use the requirement for having second order differential field equations, as with the Einstein–Hilbert action, and combine it with that of having a Lagrangian which is a generally covariant scalar. Again $\mathcal{R}$ is an obvious choice but not the only one, unlike in purely metric theories. Remember that in the case of the Palatini formalism, we commented that the choice of the action was to a large extent ad hoc.

For instance, besides invariants built combining the metric and the independent connection, one might be tempted to use also invariants that depend only on the metric. Using R, *i.e.* the scalar curvature related to the metric alone, would still lead to second order field equations. Another option can arise if the connections are of such a form that one can define a second metric, $h_{\mu\nu}$, that is covariantly conserved, *i.e.* the metric of which the Γs are the Levi–Civita connections (note that this is not necessarily true for a general connection [220], and so it would lead to a less general theory). Then we could use this metric to contract the Riemann tensor and derive the Ricci scalar $R(h)$, which is actually the scalar curvature of the metric $h_{\mu\nu}$. Going even further we could even use one of the two metrics, $g_{\mu\nu}$ or $h_{\mu\nu}$, to go from the Riemann tensor to the Ricci tensor and the other to derive the Ricci scalar from the Ricci tensor. The question that arises is whether not using these other scalar quantities in the action constitutes a further assumption, which is not needed in the purely metric formulation.

From the mathematical point of view, we could use any of the Ricci scalars defined above. However we think that for any possible choice other than $\mathcal{R}$, there are good physical reasons for discarding it. In fact, when constructing a metric-affine theory, one assumes that the spacetime is fully described by two independent geometrical objects, the metric and the connection. The metric defines the chronological structure, the connection defines the affine structure of the manifold. This manifold is not chosen to be pseudo-Riemannian (at least initially). One can always mathematically consider two pseudo-Riemannian manifolds, one described by the metric $g_{\mu\nu}$ and the other by the metric $h_{\mu\nu}$ (if it exists), but these separate manifolds are not relevant for the spacetime in which the theory acts. Therefore, quantities related to them, such as their scalar curvatures, should not be used in the action of a theory living on the non-Riemannian manifold under consideration. Also, using quantities derived by contracting once with one metric and once with the other, should also be avoided. There is only one metric that determines how distances are measured in our spacetime and this is $g_{\mu\nu}$. This is the metric that is used to evaluate inner products and therefore it is the one that should be used to raise or lower indices and perform contractions.

Since $\mathcal{R}$ does not depend on derivatives higher than first order in either the metric or the connection, as already mentioned in section 2.4.1, there is no reason *a priori* to restrict ourselves to an action linear in $\mathcal{R}$. Therefore, it is equally "natural" to consider an $f(\mathcal{R})$ action:

$$S_{ma} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(\mathcal{R}) + S_M(g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \psi). \tag{3.48}$$

Choosing an action linear in R, like (2.27), must be considered as a simplifying choice in metric-affine gravity, unlike in purely metric theories where an action linear in R is the only one that leads to second order equations [2].

Even the $f(\mathcal{R})$ action is a simplicity choice in metric-affine gravity and it is not the most general action that would lead to second order equations. Apart from not including the first derivatives of the metric, an $f(\mathcal{R})$ action also does not include terms quadratic in the first derivative of the connection. Moreover, if the connection is not symmetric there is an extra tensor available for constructing invariants: the Cartan torsion tensor (eq. (2.9)). We will comment on possible generalisations of the action in the next section, as this issue will prove to be crucial in metric-affine gravity.

3.6.3 Field equations

Since we assume that the metric and the connection are fully independent, we do not intend to make any assumptions about non-metricity and torsion. Therefore, the connection will not be taken to be symmetric or covariantly conserved by the connection. The definitions presented in section 2.2 will be used extensively here. Let us attempt to derive field equations from the action (3.48).

The variation

If we denote the gravitational part of the action as

$$S_{\text{grav}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(\mathcal{R}) \tag{3.49}$$

the least action principle gives

$$0 = \delta S_{ma} = \delta S_{\text{grav}} + \delta S_M, \tag{3.50}$$

[2] We are confining ourselves to Lagrangians that are functions of the Ricci scalar only. In a more general setting, one should mention that Gauss–Bonnet type Lagrangians lead to second order field equations as well.

and the variation of the gravitational part gives

$$\begin{aligned}
\delta S_{\rm grav} &= \frac{1}{16\pi\, G}\int d^4x\, \delta\left(\sqrt{-g} f(\mathcal{R})\right) = \\
&= \frac{1}{16\pi\, G}\int d^4x \left(f(\mathcal{R})\delta\sqrt{-g} + \sqrt{-g}f'(\mathcal{R})\delta\mathcal{R}\right) \\
&= \frac{1}{16\pi\, G}\int d^4x \left(f(\mathcal{R})\delta\sqrt{-g} + \sqrt{-g}f'(\mathcal{R})\delta\left(g^{\mu\nu}\mathcal{R}_{\mu\nu}\right)\right) \\
&= \frac{1}{16\pi\, G}\int d^4x \sqrt{-g}\left(f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu}\right)\delta g^{\mu\nu} + \\
&\quad + \frac{1}{16\pi\, G}\int d^4x \sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}\delta\mathcal{R}_{\mu\nu}, \qquad (3.51)
\end{aligned}$$

where we have used the symmetry of the metric ($\delta g^{\mu\nu}\mathcal{R}_{\mu\nu} = \delta g^{\mu\nu}\mathcal{R}_{(\mu\nu)}$).

To complete this variation, we need to evaluate the quantity $\delta\mathcal{R}_{(\mu\nu)}$. $\mathcal{R}_{\mu\nu}$ depends only on the connections and so we can already see that the second term of the last line of eq. (3.51) will be the one related to the variation with respect to ${\Gamma^\lambda}_{\mu\nu}$. We cannot use eq. (3.22) here since this was derived under the assumption that ${\Gamma^\lambda}_{\mu\nu}$ is symmetric. Taking into account the definition of the Ricci tensor, eq. (2.11), one can generalise eq. (3.22) for a non-symmetric connection:

$$\delta\mathcal{R}_{\mu\nu} = \bar{\nabla}_\lambda \delta{\Gamma^\lambda}_{\mu\nu} - \bar{\nabla}_\nu \delta{\Gamma^\lambda}_{\mu\lambda} + 2{\Gamma^\sigma}_{[\nu\lambda]}\delta{\Gamma^\lambda}_{\mu\sigma}. \qquad (3.52)$$

Using eq. (3.52), the variation of the gravitational part of the action takes the form

$$\begin{aligned}
\delta S_G &= \frac{1}{2\kappa}\int d^4x \sqrt{-g}\left(f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu}\right)\delta g^{\mu\nu} + \\
&\quad + \frac{1}{2\kappa}\int d^4x \sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}\left(\bar{\nabla}_\lambda \delta{\Gamma^\lambda}_{\mu\nu} - \bar{\nabla}_\nu \delta{\Gamma^\lambda}_{\mu\lambda}\right) + \\
&\quad + \frac{1}{2\kappa}\int d^4x\, 2\sqrt{-g} f'(\mathcal{R}) g^{\mu\sigma}{\Gamma^\nu}_{[\sigma\lambda]}\delta{\Gamma^\lambda}_{\mu\nu}. \qquad (3.53)
\end{aligned}$$

Integrating the terms in the second line by parts, we get

$$\begin{aligned}
\delta S_G &= \frac{1}{2\kappa}\int d^4x \sqrt{-g}\left(f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu}\right)\delta g^{\mu\nu} + \qquad (3.54) \\
&\quad + \frac{1}{2\kappa}\int d^4x \Big[- \bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right)\delta^\nu_\lambda \\
&\quad + 2\sqrt{-g}f'(\mathcal{R})\left(g^{\mu\nu}{\Gamma^\sigma}_{[\lambda\sigma]} - g^{\mu\rho}{\Gamma^\sigma}_{[\rho\sigma]}\delta^\nu_\lambda + g^{\mu\sigma}{\Gamma^\nu}_{[\sigma\lambda]}\right)\Big]\delta{\Gamma^\lambda}_{\mu\nu} + \mathrm{ST},
\end{aligned}$$

where ST stands for "Surface Terms". These terms are total divergences linear in $\delta{\Gamma^\lambda}_{\mu\nu}$. Being total divergences, we can turn their integral over the volume into an integral over the boundary surface. Since $\delta{\Gamma^\lambda}_{\mu\nu} = 0$ on the boundary, they will then vanish. [Note that the first two terms in the last line of eq. (3.54) came from the integration by parts of the second line of (3.53). This is because differentiation by parts and integration of covariant derivatives becomes non-trivial in the presence of a non-symmetric connection (for more information on this, see chapter 2 and p. 109 of Ref. [155]).] This concludes the variation of the gravitational part of the action.

We now have to consider the variation of the matter action. Since $S_M = S_M(g_{\mu\nu}, \Gamma^\lambda{}_{\mu\nu}, \psi)$, we have

$$\delta S_M = \frac{\delta S_M}{\delta g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\delta S_M}{\delta \Gamma^\lambda{}_{\mu\nu}}\delta\Gamma^\lambda{}_{\mu\nu}. \tag{3.55}$$

We can define the stress-energy tensor in the usual way

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{3.56}$$

We also define a new tensor, which we shall call (following the nomenclature of [168]) the "hypermomentum", as

$$\Delta_\lambda{}^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta \Gamma^\lambda{}_{\mu\nu}}, \tag{3.57}$$

i.e. the variation of the matter action with respect to the connections. Therefore, the variation of the matter action will be

$$\delta S_M = -\frac{1}{2}\int d^4x\sqrt{-g}\left[T_{\mu\nu}\delta g^{\mu\nu} + \Delta_\lambda{}^{\mu\nu}\delta\Gamma^\lambda{}_{\mu\nu}\right]. \tag{3.58}$$

Even though $\Gamma^\lambda{}_{\mu\nu}$ is not a tensor, this does not mean that $\Delta_\lambda{}^{\mu\nu}$ is not a tensor. $\delta\Gamma^\lambda{}_{\mu\nu}$ is a tensor and, therefore, so is $\Delta_\lambda{}^{\mu\nu}$.

Note also that the vanishing of $\Delta_\lambda{}^{\mu\nu}$ would imply independence of the matter action from the connections. As we discussed, this would be contrary to the spirit of metric-affine gravity if it happened for any field and the theory would reduce to $f(R)$ gravity in the Palatini formalism. There are, however, specific fields that have this attribute; the most common example is the scalar field. There will therefore be certain sorts of matter field, as we will see later on, where metric-affine $f(R)$ gravity and $f(R)$ gravity in the Palatini formalism will give equivalent physical predictions, without of course being equivalent theories overall. For instance, if we consider a massive vector field or a Dirac field, the matter action is no longer independent of the connection and $\Delta_\lambda{}^{\mu\nu}$ does not vanish.

Projective invariance and consistent field equations

We are now ready to derive the field equations using the variation of the gravitational and matter actions. This can be achieved simply by summing the variations (3.54) and (3.58) and applying the least action principle. We obtain

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.59}$$

and

$$\frac{1}{\sqrt{-g}}\Big[-\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right)+\bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right)\delta^\nu{}_\lambda\Big]+ \tag{3.60}$$
$$+\,2f'(\mathcal{R})\left(g^{\mu\nu}\Gamma^\sigma{}_{[\lambda\sigma]}-g^{\mu\rho}\Gamma^\sigma{}_{[\rho\sigma]}\delta^\nu{}_\lambda+g^{\mu\sigma}\Gamma^\nu{}_{[\sigma\lambda]}\right)=\kappa\Delta_\lambda{}^{\mu\nu}.$$

We can also use the Cartan torsion tensor, eq. (2.9), to re-express eq. (3.60) and highlight the presence of torsion:

$$\frac{1}{\sqrt{-g}}\Big[-\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right)+\bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right)\delta^\nu{}_\lambda\Big]+ \tag{3.61}$$
$$+\,2f'(\mathcal{R})\left(g^{\mu\nu}S_{\lambda\sigma}{}^\sigma-g^{\mu\rho}S_{\rho\sigma}{}^\sigma\delta^\nu{}_\lambda+g^{\mu\sigma}S_{\sigma\lambda}{}^\nu\right)=\kappa\Delta_\lambda{}^{\mu\nu}.$$

A careful look at the above equation reveals that if we take the trace on λ and μ we get

$$0=\kappa\Delta_\mu{}^{\mu\nu}, \tag{3.62}$$

since the left hand side is traceless. One can interpret this as a constraint on the form of $\Delta_\lambda{}^{\mu\nu}$, meaning that the matter action has to be chosen in such a way that its variation with respect to the connections leads to a traceless tensor. However, it is easy to understand that this is not satisfactory since there exist common forms of matter which do not have this attribute. Therefore the field equations which we have derived are inconsistent. This problem is not new; it was pointed out for the simple case of the Einstein–Hilbert action long ago [168, 155, 221]. Its roots can be traced in the form of the action itself and in the fact that in metric-affine gravity $\Gamma^\lambda{}_{\mu\nu}$ has no *a priori* dependence on the metric.

Let us consider the projective transformation

$$\Gamma^\lambda{}_{\mu\nu}\to\Gamma^\lambda{}_{\mu\nu}+\delta^\lambda{}_\mu\xi_\nu, \tag{3.63}$$

where ξ_ν is an arbitrary covariant vector field. One can easily show that the Ricci tensor will correspondingly transform like

$$\mathcal{R}_{\mu\nu}\to\mathcal{R}_{\mu\nu}-2\partial_{[\mu}\xi_{\nu]}. \tag{3.64}$$

However, given that the metric is symmetric, this implies that the curvature scalar does not change

$$\mathcal{R}\to\mathcal{R}, \tag{3.65}$$

i.e. $\mathcal{R}$ is invariant under projective transformations. Hence the Einstein–Hilbert action or any other action built from a function of $\mathcal{R}$, such as the one used here, is projective invariant in metric-affine gravity. However, the matter action is not generically projective invariant and this is the cause of the inconsistency in the field equations.

The conclusion that we have to draw is that when we want to consider a theory with a symmetric metric and an independent general connection, an action that depends only on the scalar curvature is not suitable. The way to bypass this problem is then obvious: we have to drop one of the assumptions just listed. The first option is to abandon the requirement of having a symmetric metric, since in this case $\mathcal{R}$, and consequently the gravitational action, would not be projectively invariant (see eq. (3.64)). For the Einstein–Hilbert Lagrangian this would lead to the well known Einstein–Straus theory [155], and using an $f(R)$ Lagrangian would lead to a generalisation of it. This theory, even though it leads to fully consistent field equations, is characterised by the fact that, in vacuum, neither non-metricity nor torsion vanish [155]. In particular, this implies that torsion in the Einstein–Strauss theory is not just introduced by matter fields but is intrinsic to gravity and can propagate. Although logically possible, such an option does not seem very well motivated from a physical point of view, as one would more naturally expect any "twirling" of spacetime to be somehow directly induced by the interaction with matter. Additionally, there is no experimental evidence so far of propagating torsion. Note that the effects of non-propagating torsion appear only in the presence of the matter inducing it and therefore they are significantly harder to detect. We shall therefore not pursue a route that allows for propagating torsion any further. Instead we will consider the alternative solutions to our problem.

The second path towards a consistent theory is to modify the action by adding some extra terms. These terms should be chosen in such a way so as to break projective invariance. There were proposals in this direction in the past, based on the study of an action linear in R (see [217] and references therein). As an example, we can mention the proposal of [215]: adding to the Lagrangian the term $g^{\mu\nu}\partial_\mu\Gamma^{\sigma}{}_{[\nu\sigma]}$. Such a choice leads to a fully consistent theory and is mathematically very interesting. However, we find it difficult to physically motivate the presence of this term in the gravitational action. Much more physically justified, instead, are corrections of the type $\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}$, $\mathcal{R}^{\alpha\beta\mu\nu}\mathcal{R}_{\alpha\beta\mu\nu}$ *etc*. In fact, as we have already mentioned, such terms might very naturally be present in the gravitational action if we consider it as an effective, low energy, classical action coming from a more fundamental theory [117, 118, 119, 128, 129, 130]. We shall not discuss such modifications in detail here, since this goes beyond the scope of this study; however, we will make some comments. It is easy to verify, working for example with the simplest term $\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}$, that such modifications will in general lead to consistent field equations. One should also mention that from a field theory point of view one could choose to include all of the terms of the same order in some variable. As we have already mentioned, an $f(\mathcal{R})$ action does not include first derivatives of the metric and, what is more, there are a number of terms which one could consider that can be constructed with combinations of the derivatives of the connection, especially now that the latter is not symmetric.

However, any of the additions discussed above will generically lead to a theory with the same attribute as Einstein–Straus theory, *i.e.* in vacuum, torsion will not generically vanish. One might imagine that a certain combination of higher order curvature invariants would lead to a theory with vanishing torsion in vacuum. To find such a theory would certainly be very interesting but is beyond the scope of the present investigation[3]. In conclusion, this route generically leads to theories where again the presence of torsion seems to be an unmotivated complication rather than a physical feature.

With no prescription for how to form a more general gravitational action which can lead to a physically attractive theory, we are left with only one alternative: to find a way of deriving consistent field equations with the action at hand. To understand how this is possible, we should re-examine the meaning of projective invariance. This is very similar to gauge invariance in Electromagnetism (EM). It tells us that the corresponding field, in this case the connections $\Gamma^{\lambda}{}_{\mu\nu}$, can be determined from the field equations up to a projective transformation (eq. (3.63)). Breaking this invariance can therefore come by fixing some degrees of freedom of the field, similarly to gauge fixing. The number of degrees of freedom which we need to fix is obviously the number of the components of the four-vector used for the transformation, *i.e.* simply four. In practice, this means that we should start by assuming that the connection is not the most general which one can construct, but satisfies some constraints. Instead of placing an unphysical constraint on the action of the matter fields, as dictated by eqs. (3.61) and (3.62), we can actually make a statement about spacetime properties. This is equivalent to saying that the matter fields can have all of the possible degrees of freedom but that the spacetime has some rigidity and cannot respond to some of them. (We shall come back to this point again later on. Let us just say that this is, for example, what happens in General Relativity when one assumes that there is no torsion and no non-metricity.)

We now have to choose the degrees of freedom of the connections that we need to fix. Since there are four of these, our procedure will be equivalent to fixing a four-vector. We can again let the studies of the Einstein–Hilbert action [217] lead the way. The proposal of Hehl *et al.* [217] was to fix part of the non-metricity, namely the Weyl vector Q_μ (eq. (2.8)). The easiest way to do this is by adding to the action a term containing a Lagrange multiplier A^μ, which has the form

$$S_{LM} = \int d^4x\sqrt{-g}A^\mu Q_\mu. \tag{3.66}$$

This way, one does not need to redo the variation of the rest of the action, but instead, only to evaluate the variation of the extra term. Varying with respect to the

[3] One could even imagine proposing the absence of torsion in vacuum as a possible criterion in order to select a suitable combination of high energy (strong gravity) corrections to our $f(R)$ action.

metric, the connections and A respectively, we get the new field equations

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu} + \frac{\kappa}{4\sqrt{-g}}\partial_\sigma(\sqrt{-g}A^\sigma)g_{\mu\nu}, \quad (3.67)$$

$$\frac{1}{\sqrt{-g}}\Big[-\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right)\delta^\nu{}_\lambda\Big] +$$
$$+2f'(\mathcal{R})\left(g^{\mu\nu}S_{\lambda\sigma}{}^\sigma - g^{\mu\rho}S_{\rho\sigma}{}^\sigma\delta^\nu{}_\lambda + g^{\mu\sigma}S_{\sigma\lambda}{}^\nu\right) =$$
$$= \kappa\left(\Delta_\lambda{}^{\mu\nu} - \frac{1}{4}\delta^\mu{}_\lambda A^\nu\right), \quad (3.68)$$

$$Q_\mu = 0. \quad (3.69)$$

Taking the trace of eq. (3.68) gives

$$A^\nu = \Delta_\mu{}^{\mu\nu}, \quad (3.70)$$

which is the consistency criterion, *i.e.* it gives the value which we should choose for A^ν so that the equations are consistent. This procedure obviously works when $f(R)$ is a linear function as shown in [217]. However, we will demonstrate here that it is not equally appealing in any other case.

Consider the simple case where no matter is present and let us search for the solution of the field equations for which the torsion vanishes, *i.e.*

$$S_{\sigma\lambda}{}^\nu = 0. \quad (3.71)$$

In this case eqs. (3.68) and (3.70) give

$$\frac{1}{\sqrt{-g}}\Big[-\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right)\delta^\nu{}_\lambda\Big] = 0, \quad (3.72)$$

which is no different from eq. (3.27) which we derived for Palatini $f(R)$ gravity. Therefore, once more by contracting the indices ν and λ and replacing the result back in the equation, we get

$$\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) = 0. \quad (3.73)$$

This equation implies that one can define a metric $h_{\mu\nu}$ such that

$$h_{\mu\nu} = f'(\mathcal{R})g_{\mu\nu}, \quad (3.74)$$

which is covariantly conserved by the connections $\Gamma^\lambda{}_{\mu\nu}$. Now notice the following: $h_{\mu\nu}$ has zero non-metricity by definition, leading to

$$\bar{\nabla}_\lambda h_{\mu\nu} = 0. \quad (3.75)$$

A contraction with the metric will give

$$4\frac{1}{f'(\mathcal{R})}\partial_\lambda f'(\mathcal{R}) + g^{\mu\nu}f'(\mathcal{R})\bar{\nabla}_\lambda g_{\mu\nu} = 0 \quad (3.76)$$

Now remember that eq. (3.69) forces the vanishing of the Weyl vector $Q_\lambda \equiv g^{\mu\nu}\bar{\nabla}_\lambda g_{\mu\nu}$. Therefore the above equation implies that

$$\frac{1}{f'(\mathcal{R})}\partial_\lambda f'(\mathcal{R}) = 0, \tag{3.77}$$

i.e. that $f'(\mathcal{R})$ is just a constant. If $f(\mathcal{R})$ is taken to be linear in $\mathcal{R}$, everything is consistent, but this is not the case if one considers a more general $f(\mathcal{R})$ action[4].

The above exercise clearly shows that there exist no solutions of the field equations under our assumptions whenever $f(\mathcal{R})$ is non-linear, *i.e.* there is no vacuum solution with vanishing torsion. The reason for this is simply that part of the non-metricity in our theory is due to the form of the action. Therefore, constraining the non-metricity in any way turns out to be a constraint on the form of the Lagrangian itself, unless the rest of the unconstrained part of the connection, torsion, can help to cancel out the non-metricity induced by $f(\mathcal{R})$. This indicates that if we want to consider an action more general than the Einstein–Hilbert one, we should definitely avoid placing such kinds of constraint.

One could add that in a true metric-affine theory of gravity, the connection and the metric are assumed to be completely independent fields, related only by the field equations. Therefore, imposing a constraint that includes both the metric and the connection, such as a metricity condition, seems to be contradicting the very spirit of the theory, since it gives an *a priori* relation between the two quantities.

The above not only demonstrate the unappealing features of the procedure adopted in [217] but also makes it clear that the four degrees of freedom which we have to fix are related to torsion. This implies that the torsionless version of the theory should be fully consistent without fixing any degrees of freedom. Let us now verify that. We can go back to the variation of the action in eq. (3.54) and force the connection to be symmetric. This gives

$$\begin{aligned} \delta S_G \;=\; & \frac{1}{2\kappa}\int d^4x \Bigg[\sqrt{-g}\left(f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu}\right)\delta g^{\mu\nu} + \\ & + \left[-\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\sigma(\mu}\right)\delta^{\nu)}{}_\lambda\right]\delta\Gamma^\lambda{}_{\mu\nu}\Bigg], \end{aligned} \tag{3.78}$$

and so the corresponding field equations are

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.79}$$

$$\frac{1}{\sqrt{-g}}\left[-\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\sigma(\mu}\right)\delta^{\nu)}{}_\lambda\right] = \kappa\Delta_\lambda{}^{(\mu\nu)}. \tag{3.80}$$

[4] In [165] a miscalculation (eq. (57)) led to an erroneous claim that torsion vanishes in vacuum in this version of the theory. This is not true, but the result concerning whether a non metricity condition should be forced still holds as shown by the current discussion.

where ${\Delta_\lambda}^{\mu\nu}$ is also symmetrized due to the symmetry of the connection. One can easily verify that these equations are fully consistent. They are the field equations of $f(R)$ metric-affine gravity without torsion.

Turning back to our problem, we need to fix four degrees of freedom of the torsion tensor in order to make the version of the theory with torsion physically meaningful. A prescription has been given in [221] for a linear action and we shall see that it will work for our more general Lagrangian too. This prescription is to set the vector $S_\mu = {S_{\sigma\mu}}^\sigma$ equal to zero. Note that this does not mean that ${\Gamma_{\mu\sigma}}^\sigma$ should vanish but merely that ${\Gamma_{\mu\sigma}}^\sigma = {\Gamma_{\sigma\mu}}^\sigma$. We shall again use a Lagrange multiplier, B^μ, for this purpose. The additional term in the action will be

$$S_{LM} = \int d^4x\sqrt{-g}B^\mu S_\mu. \tag{3.81}$$

It should be clear that the addition of this term does not imply that we are changing the action, since it is simply a mathematical trick to avoid doing the variation of the initial action under the assumption that $S_\mu = 0$. The new field equations which we get from the variation with respect to the metric, the connections and B^μ are

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.82}$$

$$\begin{aligned}\frac{1}{\sqrt{-g}}\Big[& -\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right){\delta^\nu}_\lambda\Big] + \\ & +2f'(\mathcal{R})\left(g^{\mu\nu}{S_{\lambda\sigma}}^\sigma - g^{\mu\rho}{S_{\rho\sigma}}^\sigma{\delta^\nu}_\lambda + g^{\mu\sigma}{S_{\sigma\lambda}}^\nu\right) = \\ & = \kappa({\Delta_\lambda}^{\mu\nu} - B^{[\mu}{\delta^{\nu]}}_\lambda),\end{aligned} \tag{3.83}$$

$${S_{\mu\sigma}}^\sigma = 0, \tag{3.84}$$

respectively. Using the third equation, we can simplify the second one to become

$$\begin{aligned}\frac{1}{\sqrt{-g}}\Big[& -\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right){\delta^\nu}_\lambda\Big] + \\ & +2f'(\mathcal{R})g^{\mu\sigma}{S_{\sigma\lambda}}^\nu = \kappa({\Delta_\lambda}^{\mu\nu} - B^{[\nu}{\delta^{\mu]}}_\lambda).\end{aligned} \tag{3.85}$$

Taking the trace over μ and λ gives

$$B^\mu = \frac{2}{3}{\Delta_\sigma}^{\sigma\mu}. \tag{3.86}$$

Therefore the final form of the field equations is

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.87}$$

$$\begin{aligned}\frac{1}{\sqrt{-g}}\Big[& -\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right){\delta^\nu}_\lambda\Big] + \\ & +2f'(\mathcal{R})g^{\mu\sigma}{S_{\sigma\lambda}}^\nu = \kappa({\Delta_\lambda}^{\mu\nu} - \frac{2}{3}{\Delta_\sigma}^{\sigma[\nu}{\delta^{\mu]}}_\lambda),\end{aligned} \tag{3.88}$$

$${S_{\mu\sigma}}^\sigma = 0. \tag{3.89}$$

These equations have no consistency problems and are the ones which we will be using from now on.

So, in the end, we see that we can solve the inconsistency problem of the unconstrained field equations by imposing a certain rigidness on spacetime, in the sense that spacetime is allowed to twirl due to its interaction with the matter fields but only in a way that keeps $S_\mu = 0$. This is not, of course, the most general case that one can think of but as we demonstrated here, it is indeed the most general within the framework of $f(R)$ gravity.

We are now ready to investigate further the role of matter in determining the properties of spacetime. In particular, we shall investigate the physical meaning of the hypermomentum $\Delta_\lambda{}^{\mu\nu}$ and discuss specific examples of matter actions so as to gain a better understanding of the gravity-matter relation in the theories under scrutiny here.

3.6.4 Matter actions

In the previous section, we derived the field equations for the gravitational field in the presence of matter. We considered both the case where torsion was allowed (eqs. (3.87), (3.88) and (3.89)) and the torsionless version of the same theory (eqs. (3.79) and (3.80)). Observe that the first equation in both sets is the same, namely eqs. (3.79) and (3.87). The second one in each set is the one that has an explicit dependence on $\Delta_\lambda{}^{\mu\nu}$, the quantity that is derived when varying the matter action with respect to the connection, which has no analogue in General Relativity. We shall now consider separately more specific forms of the matter action.

Matter action independent of the connection

Let us start by examining the simple case where the quantity $\Delta_\lambda{}^{\mu\nu}$ is zero, *i.e.* S_M is independent of the connection. In this case eq. (3.88) takes the form

$$\frac{1}{\sqrt{-g}}\left[-\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right)+\bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right)\delta^\nu{}_\lambda\right]+ \\ +2f'(\mathcal{R})g^{\mu\sigma}S_{\sigma\lambda}{}^{\nu}=0. \qquad (3.90)$$

Contracting the indices ν and λ and using eq. (3.89), this gives

$$\bar{\nabla}_\sigma\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}\right)=0. \qquad (3.91)$$

Using this result, eq. (3.90) takes the form

$$-\frac{1}{\sqrt{-g}}\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right)+2f'(\mathcal{R})g^{\mu\sigma}S_{\sigma\lambda}{}^{\nu}=0. \qquad (3.92)$$

Taking the antisymmetric part of this equation with respect to the indices μ and ν leads to

$$g^{\sigma[\mu}S_{\sigma\lambda}{}^{\nu]}=0, \qquad (3.93)$$

which can be written as

$$S_{\mu\lambda\nu} = S_{\nu\lambda\mu}. \tag{3.94}$$

This indicates that the Cartan torsion tensor must be symmetric with respect to the first and third indices. However, by definition, it is also antisymmetric in the first two indices.

It is easy to prove that any third rank tensor with symmetric and antisymmetric pairs of indices, vanishes: Take the tensor $M_{\mu\nu\lambda}$ which is symmetric in its first and third index ($M_{\mu\nu\lambda} = M_{\lambda\nu\mu}$) and antisymmetric in the first and second index ($M_{\mu\nu\lambda} = -M_{\nu\mu\lambda}$). Exploiting these symmetries we can write

$$M_{\mu\nu\lambda} = M_{\lambda\nu\mu} = -M_{\nu\lambda\mu} = -M_{\mu\lambda\nu} = M_{\lambda\mu\nu} = M_{\nu\mu\lambda} = -M_{\mu\nu\lambda}.$$

Therefore, $M_{\mu\nu\lambda} = 0$.

Consequently, eq. (3.94) leads to

$$S_{\sigma\lambda}{}^{\nu} = 0, \tag{3.95}$$

and torsion vanishes. The connection is now fully symmetric and the field equations are

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.96}$$

$$\bar{\nabla}_\lambda\left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) = 0. \tag{3.97}$$

Note that these are the same equations that one derives for a theory in which the matter action is assumed *a priori* to be independent of the connection, *i.e.* for Palatini $f(R)$ gravity and eqs. (3.29) and (3.30). It should be stressed, however, that here the independence of the matter action from the connection is due to the fact that we have chosen to consider matter fields with this property and not to a general characteristic of the theory, as in Palatini $f(R)$ gravity. We will discuss shortly which matter fields have this property and what is the form of the field equations when matter fields without this property are present.

Returning to the field equations, we see that eq. (3.97) implies that one can define a metric $h_{\mu\nu}$ such that

$$h_{\mu\nu} = f'(\mathcal{R})g_{\mu\nu}, \tag{3.98}$$

which is covariantly conserved by the connections $\Gamma^\lambda{}_{\mu\nu}$. This metric is, of course, symmetric since it is conformal to $g_{\mu\nu}$, and so the connections should be symmetric as well. In other words, it has been shown that $\Delta_\lambda{}^{\mu\nu} = 0$ leads to a symmetric connection, which means that there is no torsion when the matter action does not depend on the connection. This is an important aspect of this class of metric-affine theories of gravity. It shows that *metric-affine $f(R)$ gravity allows the presence of torsion but does not force it. Torsion is merely introduced by specific forms of matter*, those for which the matter action has a dependence on the connections. Therefore, as "matter tells spacetime how to curve", matter will also tell spacetime how to twirl. Notice also that the non-metricity does not vanish. This is because, as we also saw previously, part of the non-metricity is introduced by the form of the Lagrangian, *i.e.* $f(\mathcal{R})$ actions lead generically to theories with intrinsic non-metricity.

It is interesting to note the special nature of the particular case in which the $f(\mathcal{R})$ Lagrangian is actually linear in $\mathcal{R}$, *i.e.*

$$f(\mathcal{R}) = \mathcal{R} - 2\Lambda. \tag{3.99}$$

Then eq. (3.96) gives

$$\mathcal{R}_{(\mu\nu)} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.100}$$

and eq. (3.97) gives

$$\Gamma^\lambda{}_{\mu\nu} = \{^\lambda{}_{\mu\nu}\}, \tag{3.101}$$

i.e. the Γs turn out to be the Levi–Civita connections of the metric and so the theory actually reduces to standard General Relativity which, from this point of view, can now be considered as a sub-case of a metric-affine theory.

Vacuum

Having explored the case where $\Delta_\lambda{}^{\mu\nu} = 0$, it is easy to consider the vacuum case, where also $T_{\mu\nu} = 0$. The field equations in this case take the form

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = 0, \tag{3.102}$$

$$\bar{\nabla}_\lambda \left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) = 0. \tag{3.103}$$

We do not need to make any manipulations to investigate the nature of these equations. They coincide with the equations of Palatini $f(R)$ gravity in vacuum and therefore we can just follow the step of section 3.4.3 setting $T_{\mu\nu}$ to zero in order to realize that the theory reduces to General Relativity with a cosmological constant.

However, for the sake of clarity, let us repeat some of the steps. Contracting eq. (3.102) we get

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 0. \tag{3.104}$$

This is an algebraic equation for $\mathcal{R}$ once $f(\mathcal{R})$ has been specified. In general, we expect this equation to have a number of solutions,

$$R = c_i, \quad i = 1, 2, \dots \tag{3.105}$$

where the c_i are constants. As already mentioned in section 3.4.3, there is also a possibility that eq. (3.104) (eq. (3.31) in section 3.4.3) will have no real solutions or will be satisfied for any $\mathcal{R}$ (which happens for $f(\mathcal{R}) = a\mathcal{R}^2$, where a is an arbitrary constant) but since such cases mainly seem uninteresting or are burdened with serious difficulties when matter is also considered, we shall not study them here (see section 3.4.3 and [203]).

Let us, therefore, return to the case where eq. (3.104) has the solutions given in eq. (3.105). In this case, since $\mathcal{R}$ is a constant, $f'(\mathcal{R})$ is also a constant and eq. (3.103) becomes

$$\bar{\nabla}_\lambda \left(\sqrt{-g} g^{\mu\nu}\right) = 0. \tag{3.106}$$

This is the metricity condition for the affine connections, $\Gamma^\lambda{}_{\mu\nu}$. Therefore, the affine connections now become the Levi–Civita connections of the metric, $g_{\mu\nu}$,

$$\Gamma^\lambda{}_{\mu\nu} = \{^\lambda{}_{\mu\nu}\}, \tag{3.107}$$

and $\mathcal{R}_{\mu\nu} = R_{\mu\nu}$. Eq. (3.102) can be re-written in the form

$$R_{\mu\nu} - \frac{1}{4} c_i g_{\mu\nu} = 0, \tag{3.108}$$

which is exactly the Einstein field equation with a cosmological constant.

Therefore, in the end we see that a general $f(R)$ theory of gravity in vacuum, studied within the framework of metric-affine variation, will lead to the Einstein equation with a cosmological constant. This is not the case if one uses the metric variational principle as, in this case, one ends up with fourth order field equations, *i.e.* with a significant departure from the standard Einstein equations (see for example section 3.3.2 or [201]). Another important feature that deserves to be commented upon is the following: Contrary to the spirit of General Relativity where the cosmological constant has a unique value, here the cosmological constant is also allowed to have different values, c_i, corresponding to different solutions of eq. (3.104). So, in vacuum, the action (3.48) is in a sense equivalent to a whole set of Einstein–Hilbert actions [222] (or, more precisely, actions of the form (2.24) plus a cosmological constant).

Matter action dependent on the connection

We now focus on the more general case in which $\Delta_\lambda{}^{\mu\nu} \neq 0$ and therefore the matter action includes matter fields coupled to the connection. We can find two interesting sub-cases here. These are when $\Delta_\lambda{}^{\mu\nu}$ is either fully symmetric or fully antisymmetric in the indices μ and ν. As before, the equation under investigation will be eq. (3.88). We shall split it here into its symmetric and antisymmetric parts in the indices μ and ν:

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\left[-\bar{\nabla}_\lambda \left(\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}\right) + \bar{\nabla}_\sigma \left(\sqrt{-g} f'(\mathcal{R}) g^{\sigma(\mu}\right) \delta^{\nu)}{}_\lambda\right] + \\ &\qquad +2 f'(\mathcal{R}) g^{\sigma(\mu} S_{\sigma\lambda}{}^{\nu)} = \kappa \Delta_\lambda{}^{(\mu\nu)}, \end{aligned} \tag{3.109}$$

$$\begin{aligned} &\frac{1}{\sqrt{-g}} \bar{\nabla}_\sigma \left(\sqrt{-g} f'(\mathcal{R}) g^{\sigma[\mu}\right) \delta^{\nu]}{}_\lambda + 2 f'(\mathcal{R}) g^{\sigma[\mu} S_{\sigma\lambda}{}^{\nu]} = \\ &\qquad = \kappa (\Delta_\lambda{}^{[\mu\nu]} - \frac{2}{3} \Delta_\sigma{}^{\sigma[\nu} \delta^{\mu]}{}_\lambda). \end{aligned} \tag{3.110}$$

Let us assume now that

$$\Delta_\lambda^{\ [\mu\nu]} = 0, \tag{3.111}$$

and take the trace of either of the above equations. This leads to

$$3\bar{\nabla}_\sigma \left(\sqrt{-g} f'(\mathcal{R}) g^{\sigma\mu}\right) = 2\sqrt{-g}\kappa\Delta_\sigma^{\ \sigma\mu}. \tag{3.112}$$

Using this and eq. (3.111), eq. (3.110) takes the form

$$g^{\sigma[\mu} S_{\sigma\lambda}^{\ \ \ \nu]} = 0, \tag{3.113}$$

which is the same as eq. (3.93) which we have shown leads to

$$S_{\sigma\lambda}^{\ \ \ \nu} = 0. \tag{3.114}$$

Then, once again, the torsion tensor vanishes and we drop to the system of equations

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.115}$$

$$\frac{1}{\sqrt{-g}}\Big[- \bar{\nabla}_\lambda \left(\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}\right) + \bar{\nabla}_\sigma \left(\sqrt{-g} f'(\mathcal{R}) g^{\sigma(\mu}\right) \delta^{\nu)}{}_\lambda \Big] = \kappa\Delta_\lambda^{\ (\mu\nu)}. \tag{3.116}$$

which are the same as eqs. (3.79) and (3.80) *i.e.* the equations for the torsionless version of the theory. This indicates that any torsion is actually introduced by the antisymmetric part of $\Delta_\lambda^{\ \mu\nu}$.

We can now examine the opposite case where it is the symmetric part of $\Delta_\lambda^{\ \mu\nu}$ that vanishes. Then

$$\Delta_\lambda^{\ (\mu\nu)} = 0, \tag{3.117}$$

and taking the trace of either eq. (3.109) or eq. (3.110) straightforwardly gives

$$\bar{\nabla}_\sigma \left(\sqrt{-g} f'(\mathcal{R}) g^{\sigma\mu}\right) = 0. \tag{3.118}$$

Therefore, eqs. (3.109) and (3.110) take the form

$$-\frac{1}{\sqrt{-g}}\bar{\nabla}_\lambda \left(\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}\right) + 2 f'(\mathcal{R}) g^{\sigma(\mu} S_{\sigma\lambda}^{\ \ \ \nu)} = 0, \tag{3.119}$$

$$2 f'(\mathcal{R}) g^{\sigma[\mu} S_{\sigma\lambda}^{\ \ \ \nu]} = \kappa(\Delta_\lambda^{\ [\mu\nu]} - \frac{2}{3}\Delta_\sigma^{\ \sigma[\nu}\delta^{\mu]}{}_\lambda). \tag{3.120}$$

Taking into account the general expression for the covariant derivative of a tensor density

$$\bar{\nabla}_\lambda(\sqrt{-g} J^{\alpha\ldots}_{\beta\ldots}) = \sqrt{-g}\bar{\nabla}_\lambda(J^{\alpha\ldots}_{\beta\ldots}) - \sqrt{-g}\Gamma^\sigma{}_{\sigma\lambda} J^{\alpha\ldots}_{\beta\ldots}, \tag{3.121}$$

and the fact that $\Gamma^\sigma{}_{\sigma\lambda} = \Gamma^\sigma{}_{\lambda\sigma}$ by eq. (3.89), one can easily show that eq. (3.119) can be written as

$$\hat{\nabla}_\lambda \left(\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}\right) = 0, \tag{3.122}$$

where $\hat{\nabla}_\lambda$ denotes the covariant derivative defined with the symmetric part of the connection. This equation tells us that, as before, we can define a symmetric metric

$$h_{\mu\nu} = f'(\mathcal{R}) g_{\mu\nu}, \tag{3.123}$$

which is now covariantly conserved by the symmetric part of connections, $\Gamma^\lambda{}_{(\mu\nu)}$. If $f(\mathcal{R})$ is linear in $\mathcal{R}$, $h_{\mu\nu}$ and $g_{\mu\nu}$ coincide, of course. Additionally, eq. (3.120) shows that the torsion is fully introduced by the matter fields. Therefore we can conclude that when $\Delta_\lambda{}^{\mu\nu}$ is fully antisymmetric, there is torsion, but the only non-metricity present is that introduced by the form of the gravitational Lagrangian, *i.e.* matter introduces no extra non-metricity.

We can then conclude that, in the metric-affine framework discussed here, matter can induce both non-metricity and torsion: the symmetric part of $\Delta_\lambda{}^{\mu\nu}$ introduces non-metricity, the antisymmetric part is instead responsible for introducing torsion. While some non-metricity is generically induced also by the $f(\mathcal{R})$ Lagrangian (with the relevant exception of the linear case), torsion is only a product of the presence of matter.

Specific matter fields

Having studied the implications of a vanishing or non vanishing $\Delta_\lambda{}^{\mu\nu}$, we now want to discuss these properties in terms of specific fields. Since $\Delta_\lambda{}^{\mu\nu}$ is the result of the variation of the matter action with respect to the connection, we will need the matter actions of the fields in curved spacetime for this purpose. In purely metric theories one knows that any covariant equation, and hence also the action, can be written in a local inertial frame by assuming that the metric is flat and the connections vanish, turning the covariant derivatives into partial ones. Therefore, one can expect that the inverse procedure, which is called the minimal coupling principle, should hold as well and can be used to provide us with the matter action in curved spacetime starting from its expression in a local inertial frame. This expectation is based on the following conjecture: *The components of the gravitational field should be used in the matter action on a necessity basis*. The root of this conjecture can be traced to requiring minimal coupling between the gravitational field and the matter fields (hence the name "minimal coupling principle"). In General Relativity this conjecture can be stated for practical purposes in the following form: *the metric should be used in the matter action only for contracting indices and constructing the terms that need to be added in order to write a viable covariant matter action*. This implies that the connections should appear in this action only inside covariant derivatives and never alone which is, of course, perfectly reasonable since, first of all, they are not independent fields and, secondly, they are not tensors themselves and so they have no place in a covariant expression. At the same time, other terms that would vanish in flat spacetime like, for example, contractions of the curvature tensor with the fields or their derivatives, should be avoided.

The previous statements are not applicable in metric affine gravity for several reasons: the connections now are independent fields and, what is more, if they are not symmetric, there is a tensor that one can construct via their linear combination: the Cartan torsion tensor. Additionally, going to some local inertial frame in metric-affine gravity is a two-step procedure in which one has to separately impose that the metric is flat and that the connections vanish. However, the critical point is that when inverting this procedure one should keep in mind that there might be dependences on the connections in the equations other than those in the covariant derivatives. *The standard minimal coupling principle will therefore not, in general, give the correct answer in metric-affine gravity theories.*

The above discussion can be well understood through a simple example, using the electromagnetic field. In order to compute the hypermomentum ${\Delta_\lambda}^{\mu\nu}$ of the electromagnetic field, we need to start from the action

$$S_{EM} = -\frac{1}{4}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu}, \tag{3.124}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor. As we know, in the absence of gravity this tensor is defined as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3.125}$$

where A_μ is the electromagnetic four-potential. If we naively followed the minimal coupling principle and simply replaced the partial derivatives with covariant ones, the definition of the electromagnetic field tensor would take the form:

$$F_{\mu\nu} \equiv \bar{\nabla}_\mu A_\nu - \bar{\nabla}_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu - 2\,{\Gamma^\sigma}_{[\mu\nu]}A_\sigma, \tag{3.126}$$

and one can easily verify that it would then no longer be gauge invariant, *i.e.* invariant under redefinition of the four potential of the form $A_\mu \to A_\mu + \partial_\mu\phi$, where ϕ is a scalar quantity. Gauge invariance, however, is a critical aspect of the electromagnetic field since it is related to the conservation of charge and the fact that the electric and magnetic fields are actually measurable quantities. Therefore breaking gauge invariance cannot lead to a viable theory. One could assume that the problem lies in the fact that the connection is not symmetric, *i.e.* torsion is allowed, since it is the antisymmetric part of the connection that prevents gauge invariance of eq. (3.126), and hence it might seem that standard electromagnetism is incompatible with torsion. This explanation was given for example in [223] (see also references therein for other discussions following the same line). We do not agree with either this approach or its conclusion: As we said, the problem is actually much simpler but also more fundamental and lies in the assumption that the minimal coupling principle still holds in metric-affine gravity.

In order to demonstrate this point, let us turn our attention to the definition of the electromagnetic field tensor in the language of differential forms. This is

$$\mathbf{F} \equiv \mathbf{dA}, \tag{3.127}$$

where $\mathbf{d}$ is the standard exterior derivative [163]. Remember that the exterior derivative is related to Gauss's theorem which allows us to go from an integral over the volume to an integral over the boundary surface of this volume. Now notice that the volume element has no dependence on the connection and is the same as that of General Relativity, $\sqrt{-g}\, d^4x$. This implies that the definition of the exterior derivative should remain unchanged when expressed in terms of partial derivatives. Partial derivatives on the other hand are defined in the same way in this theory as in General Relativity. Therefore, from the definition (3.127) we understand that $F_{\mu\nu}$ should be given in terms of the partial derivatives by the following equation

$$F_{\mu\nu} \equiv \mathrm{d}A = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3.128}$$

which is the same as eq. (3.125) and respects gauge invariance. The expression in terms of the partial derivatives may not look covariant but can easily be written in a manifestly covariant form:

$$\begin{aligned} F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu &= \bar{\nabla}_\mu A_\nu - \bar{\nabla}_\nu A_\mu + 2\,\Gamma^{\sigma}{}_{[\mu\nu]} A_\sigma = \\ &= \bar{\nabla}_\mu A_\nu - \bar{\nabla}_\nu A_\mu + 2\, S_{\mu\nu}{}^{\sigma} A_\sigma. \end{aligned} \tag{3.129}$$

Besides, the expression for $\mathbf{F}$ in terms of the exterior derivative is covariant anyway.

It is now obvious that the minimal coupling principle was leading us to the wrong expression, causing a series of misconceptions. However, we are still in need of a prescription that will allow us to derive the matter actions in curved spacetime. Notice that if we require gravity and matter to be minimally coupled, then the physical basis of the conjecture that *the components of the gravitational field should be used in the matter action on a necessity basis* still holds, since its validity is not related to any of the assumptions of General Relativity. Thus, we can use it to express a metric-affine minimal coupling principle: *The metric should be used in the matter action only for contracting indices and the connection should be used only in order to construct the extra terms that must be added in order to write a viable covariant matter action*. The analogy with the statement used in General Relativity is obvious, and differences lie in the different character of the connections in the two theories. One can easily verify that the matter action of the electromagnetic field which we derived earlier can be straightforwardly constructed using this metric–affine minimal coupling principle.

We would like to stress once more that both the metric–affine minimal coupling principle presented above and the standard one, are based on the requirement that the gravitational field should be minimally coupled to the matter. One could, of course, choose to construct a theory without such a requirement and allow non-minimal coupling[5]. This can be done both in metric-affine gravity and in General

[5]Note that if one considers the possible actions for classical gravity as effective ones —obtained as the low energy limit of some more fundamental high energy theory —then it is natural to imagine that the form of the coupling (minimal or some specific type of non-minimal) might cease to be a free choice (see *e.g.* Chapter 7 of [199] for an enlightening discussion). However, one could still expect that non-minimal coupling terms will be suppressed at low energies by appropriate powers of the scale associated with the fundamental theory (Planck scale, string scale, *etc.*) and in this sense the use of a minimal coupling principle at low energies could be justified.

Relativity. Clearly, in metric-affine gravity one has more options when it comes to non-minimal coupling, since besides curvature terms, also terms containing the Cartan torsion tensor can be used. However, it is easy to see that the number of viable coupling terms is strongly reduced by the symmetry of the metric (which also implies symmetry of the stress-energy tensor) and by the constraints of the theory, *e.g.* the vanishing of the trace of ${S_{\mu\nu}}^{\sigma}$ when considering $f(\mathcal{R})$ actions.

Allowing non-minimal coupling between gravity and matter in a gravitation theory drastically changes the corresponding phenomenology and there might be interesting prospects for such attempts in metric-affine gravity. For the rest of this thesis, however, we will continue to assume minimal coupling between gravity and matter, since this is the most conventional option.

Let us now return to the electromagnetic field. Now that we have a suitable expression for the electromagnetic field tensor, we can proceed to derive the field equations for electrovacuum. Notice that $F_{\mu\nu}$ has no real dependence on the connections and so we can straightforwardly write

$$ {\Delta_{\lambda}}^{\mu\nu} = 0. \tag{3.130} $$

The stress-energy tensor $T_{\mu\nu}$ can be evaluated using eq. (3.56) and has the standard form

$$ T_{\mu\nu} = {F_{\mu}}^{\sigma} F_{\sigma\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}. \tag{3.131} $$

With the use of eqs. (3.96) and(3.97), we can write the field equations as

$$ f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \kappa {F_{\mu}}^{\sigma} F_{\sigma\nu} - \frac{\kappa}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}, \tag{3.132} $$

$$ \bar{\nabla}_{\lambda} \left(\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu} \right) = 0. \tag{3.133} $$

We can use, however, the fact that the stress-energy tensor of the electromagnetic field is traceless. If we take the trace of eq. (3.132) we get

$$ f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 0, \tag{3.134} $$

which, as we discussed previously for the vacuum case, is an algebraic equation in $\mathcal{R}$ once $f(\mathcal{R})$ has been specified. Solving it will give a number of roots (see also the discussion after eq. (3.105))

$$ \mathcal{R} = c_i, \quad i = 1, 2, \ldots \tag{3.135} $$

and $f(c_i)$ and $f'(c_i)$ will be constants. Therefore eq. (3.133) implies that the metric is covariantly conserved by the covariant derivative defined using the connection and so

$$ {\Gamma^{\lambda}}_{\mu\nu} = \{{}^{\lambda}{}_{\mu\nu}\}, \tag{3.136} $$

and we are left with the following field equation:

$$ \mathcal{R}_{\mu\nu} - \frac{1}{4} c_i g_{\mu\nu} = \kappa' {F_{\mu}}^{\sigma} F_{\sigma\nu} - \frac{\kappa'}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}, \tag{3.137} $$

which is the Einstein equation for electrovacuum with a cosmological constant and a modified "coupling constant" $\kappa' = \kappa/f'(c_i)$. The rescaling of κ should not mislead us into thinking that either the gravitational constant, G, or the fine structure constant, α, change in any way. It just affects the strength of the "coupling" between gravity and the electromagnetic field, *i.e.* how much curvature is induced per unit energy of the electromagnetic field. The values of the cosmological constant and κ' depend on the functional form of $f(\mathcal{R})$ and therefore they are fixed once one selects an action. For example, $f(\mathcal{R}) = \mathcal{R}$ or $f(\mathcal{R}) = a\mathcal{R}^2 + \mathcal{R}$ both lead to $c_i = 0$ and $\kappa' = \kappa$ and the resulting theory will be indistinguishable from General Relativity. For more general forms of $f(\mathcal{R})$, the theory is still formally equivalent to General Relativity but note that the modification of κ should, at least theoretically, be subject to experiment. If such an experiment is technically possible it might help us place bounds on the form of the action.

As already mentioned, a vanishing $\Delta_\lambda{}^{\mu\nu}$ implies that there is no dependence of the matter action on the connections, or equivalently on the covariant derivative. As we just saw, the electromagnetic field, and consequently any other gauge field, has this attribute. The same is true for a scalar field, as the covariant derivatives of a scalar are reduced to partial derivatives. Therefore, neither of these fields will introduce torsion or extra non-metricity. For the electromagnetic field specifically, the fact that the trace of its stress energy tensor is zero leads to the Einstein field equations, since the non-metricity introduced by the form of the Lagrangian has to vanish as well. For the scalar field, whose stress energy tensor does not have a vanishing trace, this will not happen. The field equations can be derived straightforwardly by replacing the usual stress energy tensor of a scalar field in eqs. (3.96) and (3.97).

Let us now turn to matter fields for which $\Delta_\lambda{}^{\mu\nu}$ does not vanish. In principle, any massive vector field or tensor field should have an action with an explicit dependence on the connection, leading to a non vanishing $\Delta_\lambda{}^{\mu\nu}$. A typical example would be the Dirac field. The Dirac Lagrangian has an explicit dependence on the covariant derivative, and therefore an explicit dependence on the connections. Additionally, there are no viability criteria, unlike in the case of the electromagnetic field, that will force us to include extra terms proportional to the Cartan torsion tensor which will cancel out the presence of the antisymmetric part of the connection. Therefore, the procedure for deriving the matter action is straightforward (see [196] for the full form of the action[6]). We can infer from the above that a Dirac

[6] Note that the result of [196] is for a theory that has, by definition, vanishing non-metricity (U_4 theory). However, the form of the matter action is the same once the proper covariant derivative is used. For discussions about the matter actions in theories with torsion see also [224, 225]. Note that, even though the results obtained here are in complete agreement with the ones presented in those works, in many cases the reasoning differs since there is no attempt there to formulate a metric-affine minimal coupling principle. The standard minimal coupling principle is used there in cases where it provides the correct results, while it is noted that it does not apply to specific cases, such as the electromagnetic field. For each of these cases, individual arguments are used in order to derive the matter action in curved spacetime. The underlying physics in the two approaches is the same, but we believe that the idea of a metric-affine minimal coupling principle is an essential concept since,

field will potentially introduce both torsion and non-metricity. Note that the fields which cannot introduce torsion will also not "feel" it, since they are not coupled to the Cartan tensor, and so photons or scalar particles will not be affected by torsion even if other matter fields produce it.

It is also interesting to study matter configurations in which matter is treated macroscopically, the most common being that of a perfect fluid. Let us here consider separately the cases where torsion is allowed in the theory and where it is not included. In the latter case, the consideration of the perfect fluid is identical to standard General Relativity. Since the matter action can be described by three scalars, the energy density, the pressure and the velocity potential (see for example [226, 227]), the action has no dependence on the covariant derivative and so $\Delta_\lambda{}^{\mu\nu}$ will vanish. When torsion is allowed, there are two distinct cases depending on the microscopic properties of the fluid. If a perfect fluid is used to effectively describe particles whose corresponding field description does not introduce torsion, then no difference from the previous case arises. If, on the other hand, the fluid is composed of particles whose field description *can* introduce torsion, then their spin has to be taken into account (see [196] and references therein). There will however be an averaging over volume of the quantities describing the matter, and if one assumes that the spin is randomly oriented and not polarized, then it should average to zero. This description can be applied in physical situations such as gravitational collapse or Cosmology. The fact that the expectation value of the spin will be zero will lead to a vanishing expectation value for the torsion tensor. However, fluctuations around the expectation value will affect the geometry leading to corrections to the field equations which will depend on the energy density of the specific species of particle. Since the torsion tensor is coupled to the hypermomentum through the gravitational constant (eq. (3.88)), the effect of these fluctuations will be suppressed by a Planck mass squared. Therefore we can conclude that for Cosmology, and especially for late times where the energy density is small, the standard perfect fluid description might serve as an adequate approximation.

It is remarkable that the two matter descriptions most commonly used in Cosmology, the perfect fluid and the scalar field, lead to a vanishing $\Delta_\lambda{}^{\mu\nu}$ for a symmetric connection. It is also noticeable that in our framework, even if torsion is allowed, the results remain unchanged for the perfect fluid case, apart from small corrections which should be negligible. It would be interesting to consider also the case of a an imperfect fluid (*i.e.* to allow also viscosity, heat flow, *etc*.), which is certainly relevant for some observationally interesting systems in relativistic Astrophysics. As in the case of a perfect fluid, if we consider particles with a spin and allow torsion, the standard imperfect fluid description will not be exact. Note however, that even in the simpler case of *a priori* symmetric connections, we do not expect the matter action to be independent of such connections (in contrast with the perfect fluid case). This could lead to a non-vanishing $\Delta_\lambda{}^{\mu\nu}$ and consequently to

besides its elegance and the analogy with the standard minimal coupling principle, it leaves no room for exceptions.

some non-metricity, which might lead to interesting deviations away from General Relativity results.

Discussion

It has been shown that when the variation of the matter action leads to a tensor symmetric in its last two indices, then torsion vanishes. When the same tensor is antisymmetric, matter introduces only torsion and not non-metricity. Matter fields whose matter action is independent of the connection cannot introduce either torsion or non-metricity. As already mentioned, since torsion is absent in vacuum and in some specific matter configurations but present in all other cases, we can infer that it is actually introduced by matter. By considering for which kind of fields torsion vanishes and for which it does not, we can arrive at a very interesting conclusion. Torsion is zero in vacuum and in the presence of a scalar field or an electromagnetic field. It does not necessarily vanish, however, in the presence of a Dirac field or other vector and tensor fields. This shows a correspondence between torsion and the presence of fields that describe particles with spin. We are, therefore, led to the idea that particles with spin seem to be the sources of torsion. Of course a photon, the particle associated with the electromagnetic field, is a spin one particle. However, in Quantum Field Theory a photon is not really characterized by its spin but actually by its helicity. It is remarkable that this exceptional nature of the photon seems to be present also here, since the electromagnetic field is unable to introduce torsion.

The study of the electromagnetic field turned out to be very helpful, since it demonstrated that the usual minimal coupling principle does not hold in metric-affine gravity. However, as we showed, one can still express a metric-affine minimal coupling principle based on the spirit of minimal coupling between gravity and matter.

We have also discussed the case where matter is treated macroscopically. As already mentioned, a perfect fluid cannot introduce any extra non-metricity for a symmetric connection. When torsion is allowed, the concept of a perfect fluid has to be generalized if one wants to include particles with spin, but also in this case only small contributions to torsion will be introduced which will be negligible in most cases. On the other hand, for both symmetric and general connections, we suspect that there might be larger deviations from General Relativity when a seriously imperfect fluid is considered. However, for many applications in Cosmology and Astrophysics, a perfect fluid description is taken as being a good approximation. Moreover, many of the experimental tests passed by General Relativity are related to either vacuum or to environments where matter can be more or less accurately described as a perfect fluid. This means that a metric-affine theory could be in total accordance with these tests when the Einstein–Hilbert action and possibly many of its extensions are used.

However, the possible relevance of imperfect fluid matter in some yet to be accurately observed astrophysical systems (such as accretion flows or compact objects [228]) leaves open the possibility for future discrimination between the class of theories discussed here and standard General Relativity. In physical systems where matter cannot necessarily be described accurately enough by a perfect fluid, one might hope to see deviations from the standard behaviour predicted by General Relativity. Even starting with the standard Einstein–Hilbert action, torsion and non-metricity should affect the dynamics and might make them deviate noticeably from the standard ones. This deviation could persist even in a nearly-Newtonian regime. It could be interesting to study this in the context of galactic dynamics since in this case the effects may be important and may even make some contribution in relation to the unexpected behaviour of galactic rotation curves. Of course, until a thorough and quantitative study is performed, all of the above remains at the level of speculations, even though they seem qualitatively interesting.

It is important to note that our attempt to include torsion showed that this cannot be done in the context of $f(R)$ gravity unless one fixes some degrees of freedom of the connection as mentioned earlier. The other possibility that was discussed here was to modify the action by adding some higher order curvature invariant. As we said, it is very difficult to find a prescription for an action of this form that will lead to a physically meaningful theory of gravitation with torsion since the simple case will have unwanted attributes. This is the reason why we did not pursue this here. Note however, that we already know that rotating test particles do not follow geodesics. Therefore, it would be reasonable to assume that, since macroscopic angular momentum interacts with the geometry, intrinsic angular momentum (spin) should interact with the geometry as well. This property should become more important at small length-scales or high energies. Therefore, it seems remarkable that an attempt to include torsion and at the same time avoid placing *a priori* constraints on the connection, leads to the conclusion that the action should be supplemented with higher order curvature invariants, which is in total agreement with the predictions coming from quantum corrections, String Theory and M-theory.

To conclude this section, we would like to stress once more that metric-affine $f(R)$ gravity reduces to General Relativity, or a theory very close to it, in most of the cases relevant to known experimental tests (vacuum, electrovacuum, *etc.*) and yet is phenomenologically much richer. This may help to address some of the puzzles of physics related to gravity.

3.7 Gauss–Bonnet gravity

3.7.1 The action

In the course of this chapter we have studied $f(R)$ theories of gravity extensively and we have briefly considered scalar-tensor theory and theories whose action includes higher-order curvature invariants, such as $R_{\mu\nu}R^{\mu\nu}$. Since an action may include such invariants, one is tempted to consider the option that a scalar field might not only be coupled to the Ricci scalar, as in scalar-tensor theory, but also to higher order terms. A theory with a scalar field and more general couplings would be quite complicated and difficult to handle though. Therefore, besides the general motivation for pursuing alternative theories of gravity coming from puzzles related to Cosmology and Quantum Gravity, one would like to have some motivation for specific couplings in order to go further.

Indeed there are motivations from String Theory to believe that scalar fields might be coupled to the Gauss–Bonnet invariant $\mathcal{G}$, as defined in eq. (3.40). To be more precise, one expects to find two types of scalar field in the low energy effective action of gravity coming from heterotic String Theory: moduli, ϕ, which are related to the size and shape of the internal compactification manifold, and the dilaton, σ, which plays the role of the string loop expansion parameter. There are reasons to believe that moduli generally couple to curvature squared terms [229, 230] but that moduli-dependent higher loop contributions, such as terms cubic or higher order in the Riemann tensor, vanish leaving a coupling with a Gauss–Bonnet term to be of specific interest [229, 230, 231]. On the other hand, the dilaton usually couples to the the Ricci scalar, as in scalar-tensor theory[7]. However, there are claims that the dilaton might evolve in such a way so as to settle to a constant [232, 233]. Under these assumptions, the effective low energy gravitational action takes the form

$$S_{GB} = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{\lambda}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + f(\phi)\mathcal{G}\right] + S_M(g^{\mu\nu},\psi), \tag{3.138}$$

where λ is $+1$ for a canonical scalar field and -1 for a phantom field.

It is straightforward to generalize this action in order to include a kinetic term and a coupling with $\mathcal{G}$ for the dilaton σ [8]. One can also allow a coupling for the dilaton to R and/or matter, if, of course, the dilaton is not assumed to settle to a constant as claimed in [232, 233]. However, considering these claims and the complications which such couplings would introduce, we will concern ourselves here with the action (3.138), which in any case can work as an excellent starting point for studying couplings between a scalar and the Gauss–Bonnet invariant.

A theory described by the action (3.138) is usually called Gauss–Bonnet gravity. Note, however, that the term Gauss–Bonnet gravity is sometimes used to refer to other theories in 4 or more dimensions including in some way the Gauss–Bonnet invariant in the gravitational action and, therefore, care should be taken to avoid confusion. We will be using this terminology here strictly referring to the action (3.138). Before going further and deriving the field equations, it is also worth mentioning that Gauss–Bonnet gravity has been shown to have many appealing features when it comes to singularities and cosmological applications (*e.g.* [231, 234]) and therefore part of the motivation for its study comes from that. We will discuss such

[7] A conformal transformation of the metric can be used in order to find a representation of the theory in which the coupling with the Ricci scalar is avoided and a coupling to matter is introduced (Jordan to Einstein frame). We will discuss this extensively in the forthcoming chapters.

[8] In scalar-tensor theory, the scalar is in many cases considered to be the dilaton. Even though we denote the dilaton here by σ and the moduli by ϕ, in section 3.2 we used ϕ for the scalar since this is standard notation for a general scalar field.

applications of the theory in the next chapter.

3.7.2 The field equations

We proceed here with the variation of the action. Variation with respect to the metric $g_{\mu\nu}$ quite straightforwardly leads to the equation

$$\begin{aligned}&\frac{1}{\kappa^2}G^{\mu\nu}-\frac{1}{2}g^{\mu\nu}f(\phi)\mathcal{G}+2f(\phi)RR^{\mu\nu}-2\nabla^{\mu}\nabla^{\nu}\left(f(\phi)R\right)+\\&+2g^{\mu\nu}\nabla^{2}\left(f(\phi)R\right)-8f(\phi)R^{\mu}_{\ \rho}R^{\nu\rho}+4\nabla_{\rho}\nabla^{\mu}\left(f(\phi)R^{\nu\rho}\right)+\\&+4\nabla_{\rho}\nabla^{\nu}\left(f(\phi)R^{\mu\rho}\right)-4\nabla^{2}\left(f(\phi)R^{\mu\nu}\right)-4g^{\mu\nu}\nabla_{\rho}\nabla_{\sigma}\left(f(\phi)R^{\rho\sigma}\right)+\\&+2f(\phi)R^{\mu\rho\sigma\tau}R^{\nu}_{\ \rho\sigma\tau}-4\nabla_{\rho}\nabla_{\sigma}\left(f(\phi)R^{\mu\rho\sigma\nu}\right)=T^{\mu\nu}+T^{\mu\nu}_{\phi},\end{aligned}\tag{3.139}$$

where we have defined

$$T^{\mu\nu}_{\phi}=\lambda\left(\frac{1}{2}\partial^{\mu}\phi\partial^{\nu}\phi-\frac{1}{4}g^{\mu\nu}\partial_{\rho}\phi\partial^{\rho}\phi\right)-\frac{1}{2}g^{\mu\nu}V(\phi).\tag{3.140}$$

Note that, as stressed in section 3.5.1, $\mathcal{G}$ is a topological invariant and the variation of the term $\sqrt{-g}\mathcal{G}$ is a total divergence not contributing to the field equations (formally a suitable surface term should be added in the action in order to cancel the total divergence). However, the term $\phi\sqrt{-g}\mathcal{G}$ will contribute in the field equations for the metric since $\phi\delta\sqrt{-g}\mathcal{G}$ will no longer be a surface term but can only be turned into one after an integration by parts.

Following [234], we can use the following relations coming from the Bianchi identities:

$$\nabla^{\rho}R_{\rho\tau\mu\nu}=\nabla_{\mu}R_{\nu\tau}-\nabla_{\nu}R_{\mu\tau}\,,\tag{3.141}$$

$$\nabla^{\rho}R_{\rho\mu}=\frac{1}{2}\nabla_{\mu}R\,,\tag{3.142}$$

$$\nabla_{\rho}\nabla_{\sigma}R^{\mu\rho\nu\sigma}=\nabla^{2}R^{\mu\nu}-\frac{1}{2}\nabla^{\mu}\nabla^{\nu}R+R^{\mu\rho\nu\sigma}R_{\rho\sigma}-R^{\mu}_{\ \rho}R^{\nu\rho},\tag{3.143}$$

$$\nabla_{\rho}\nabla^{(\mu}R^{\nu)\rho}=\frac{1}{2}\nabla^{(\mu}\nabla^{\nu)}R-R^{\mu\rho\nu\sigma}R_{\rho\sigma}+R^{\mu}_{\ \rho}R^{\nu\rho}\,,\tag{3.144}$$

$$\nabla_{\rho}\nabla_{\sigma}R^{\rho\sigma}=\frac{1}{2}\Box R\,,\tag{3.145}$$

in order to obtain from eq. (3.139) the equation

$$\begin{aligned}&\frac{1}{\kappa^2}G^{\mu\nu}-\frac{1}{2}g^{\mu\nu}f(\phi)\mathcal{G}+2f(\phi)RR^{\mu\nu}+4f(\phi)R^{\mu}_{\ \rho}R^{\nu\rho}+\\&+2f(\phi)R^{\mu\rho\sigma\tau}R^{\nu}_{\ \rho\sigma\tau}-4f(\phi)R^{\mu\rho\sigma\nu}R_{\rho\sigma}=T^{\mu\nu}+T^{\mu\nu}_{\phi}+T^{\mu\nu}_{f},\end{aligned}\tag{3.146}$$

where

$$\begin{aligned}T^{\mu\nu}_{f}=\;&2\left(\nabla^{\mu}\nabla^{\nu}f(\phi)\right)R-2g^{\mu\nu}\left(\nabla^{2}f(\phi)\right)R-\\&-4\left(\nabla_{\rho}\nabla^{\mu}f(\phi)\right)R^{\nu\rho}-4\left(\nabla_{\rho}\nabla^{\nu}f(\phi)\right)R^{\mu\rho}+\\&+4\left(\nabla^{2}f(\phi)\right)R^{\mu\nu}+4g^{\mu\nu}\left(\nabla_{\rho}\nabla_{\sigma}f(\phi)\right)R^{\rho\sigma}-\\&-4\left(\nabla_{\rho}\nabla_{\sigma}f(\phi)\right)R^{\mu\rho\nu\sigma}\end{aligned}\tag{3.147}$$

We know that the Gauss–Bonnet term in the action is topologically invariant and therefore for $f(\phi) =$ constant the field equations should be unmodified with respect to General Relativity. Thus, the terms proportional to $f(\phi)$ without derivatives should cancel out leading to the identity

$$g^{\mu\nu}\mathcal{G} = 4RR^{\mu\nu} - 8R^{\mu}_{\ \rho}R^{\nu\rho} + 4R^{\mu\rho\sigma\tau}R^{\nu}_{\ \rho\sigma\tau} - 8R^{\mu\rho\sigma\nu}R_{\rho\sigma}. \tag{3.148}$$

Now eq. (3.146) can be simply written as

$$G^{\mu\nu} = \kappa^2\left[T^{\mu\nu} + T^{\mu\nu}_{\phi} + T^{\mu\nu}_{f}\right]. \tag{3.149}$$

On the other hand, variation of the action with respect to ϕ gives

$$\lambda\nabla^2\phi - V'(\phi) + f'(\phi)\mathcal{G} = 0, \tag{3.150}$$

and eqs. (3.149) and (3.150) constitute the field equations of the theory.

A comment is due at this point concerning the conservation of energy-momentum. The matter action for Gauss–Bonnet gravity is built out of a generally covariant scalar and the matter is minimally coupled to the metric and not coupled to the scalar field ϕ. Therefore, Gauss–Bonnet gravity is a metric theory of gravity and $T_{\mu\nu}$ is divergence free. Also, one can add that the action (3.138) is manifestly diffeomorphism invariant, being constructed with a generally covariant scalar. It is trivial to use diffeomorphism invariance to derive that $\nabla_\mu T^{\mu\nu} = 0$.

However, in General Relativity the fact that $T^{\mu\nu}$ is divergence free follows also as a consequence of the field equations due to the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ (eq. (3.142)). Therefore, one expects that $\nabla_\mu T^{\mu\nu} = 0$ should be derivable also from a combination of the field equations (3.149) and (3.150), the Bianchi identity and probably some generalization of the Bianchi identity. As an exercise, we will prove that this is indeed the case.

For $\nabla_\mu T^{\mu\nu} = 0$ to hold, and given that $\nabla_\mu G^{\mu\nu} = 0$ is a mathematical identity (eq. (3.142)), one needs

$$\nabla_\mu T^{\mu\nu}_{\phi} + \nabla_\mu T^{\mu\nu}_{f} = 0. \tag{3.151}$$

Let us examine these terms separately. For the first one, a straightforward calculation together with the use of the identity $\nabla_\mu\nabla_\nu\psi = \nabla_\nu\nabla_\mu\psi$ for any scalar ψ, gives

$$\nabla_\mu T^{\mu\nu}_{\phi} = \frac{1}{2}\left(\lambda\nabla^2\phi - V'(\phi)\right)\nabla^\nu\phi. \tag{3.152}$$

Calculating $\nabla_\mu T^{\mu\nu}_{f}$ is not, unfortunately, equally straightforward but is rather a tedious calculation, so we will not present it here in detail. We will, however, sketch the steps so that the reader can easily reproduce it. The first step is to take into account that for an arbitrary vector V^μ

$$\nabla_\beta\nabla_\alpha V^\mu - \nabla_\alpha\nabla_\beta V^\mu = R^{\mu}_{\ \nu\beta\alpha}V^\nu, \tag{3.153}$$

and that for an arbitrary scalar ψ

$$(\nabla^2\nabla_\nu - \nabla_\nu\nabla^2)\psi = R_{\mu\nu}\nabla^\mu\psi, \tag{3.154}$$

and one can then deduce, after some manipulations, that

$$\begin{aligned}\nabla_\mu T_f^{\mu\nu} &= \left(2RR^{\mu\nu} - 4R^\mu{}_\rho R^{\nu\rho} - 4R^{\mu\rho\sigma\nu}R_{\rho\sigma}\right)\nabla_\mu f(\phi) - \\ &\quad -4(\nabla_\mu\nabla_\rho\nabla_\sigma f)R^{\mu\rho\nu\sigma},\end{aligned} \tag{3.155}$$

where the identity (3.142) has also been used extensively to replace $\nabla_\mu R^{\mu\nu}$ at all occurrences and the symmetries of the Riemann tensor have been used as well. Now note that from eq. (3.153), with a suitable contraction with the Riemann tensor, one gets

$$R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau}\nabla_\mu f = -R^{\nu\sigma\mu\rho}(\nabla_\mu\nabla_\rho\nabla_\sigma f - \nabla_\rho\nabla_\mu\nabla_\sigma f), \tag{3.156}$$

where some relabelling of the dummy indices has also taken place. Since the Riemann tensor is antisymmetric in its last two indices and symmetric in the exchange of pairs of indices, we can write

$$\begin{aligned}R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau}\nabla_\mu f &= -2R^{\nu\sigma\mu\rho}\nabla_\mu\nabla_\rho\nabla_\sigma f = \\ &= -2R^{\mu\rho\nu\sigma}\nabla_\mu\nabla_\rho\nabla_\sigma f.\end{aligned} \tag{3.157}$$

We can then use this to substitute for the last term in eq. (3.155), giving

$$\begin{aligned}\nabla_\mu T_f^{\mu\nu} &= (2RR^{\mu\nu} - 4R^\mu{}_\rho R^{\nu\rho} + 2R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau} - \\ &\quad -4R^{\mu\rho\sigma\nu}R_{\rho\sigma})\nabla_\mu f(\phi).\end{aligned} \tag{3.158}$$

Finally, we re-write this equation in a more economical form, taking advantage of the identity (3.148):

$$\nabla_\mu T_f^{\mu\nu} = \frac{1}{2}\mathcal{G}\nabla^\nu f. \tag{3.159}$$

Note that eq. (3.159) is just a mathematical identity.

We can now substitute eqs. (3.152) and (3.159) into eq. (3.151). This gives

$$\frac{1}{2}\left(\lambda\nabla^2\phi - V'(\phi) + f'(\phi)\mathcal{G}\right)\nabla^\nu\phi = 0. \tag{3.160}$$

Obviously this equation is trivially satisfied if and only if the scalar field satisfies its field equation, namely eq. (3.60). Therefore, the matter stress-energy tensor is divergence-free on shell, *i.e.* when ϕ satisfies its field equation. Note that in General Relativity one can consider the matter stress-energy tensor as being divergence-free as a consequence of the Bianchi identity, whereas in this case eq. (3.151) is not a mathematical identity, as just demonstrated, but requires knowledge of the dynamics of the scalar field. Therefore we will avoid calling it a generalised Bianchi identity, even though this is often done for similar equations in the literature. In a sense, one could refer to the combination of eq. (3.159) with the Bianchi identity as the generalised Bianchi identity.

Chapter 4

Redefinition of variables and equivalence of theories

4.1 Dynamical Equivalence

In the previous chapters, we have presented a number of alternative theories of gravity. A reasonable question to ask is how different these theories really are. Indeed, as we will see shortly, some of the theories which we have considered so far can be cast into the form of others, once suitable redefinitions of the fields are utilized.

There is no unique prescription for redefining the fields of a theory. Some of the most common redefinitions are renormalizations and conformal transformations. Additionally, one can utilize auxiliary fields in order to re-write the action or the field equations of a theory. Before getting into this issue though, some clarifying remarks are needed.

It is important to mention that, at least within a classical perspective like the one followed here, two theories are considered to be dynamically equivalent if, under a suitable redefinition of the gravitational and matter fields, one can make their field equations coincide. The same statement can be made at the level of the action. Dynamically equivalent theories give exactly the same results in describing a dynamical system to which the theories are applicable. There are clearly advantages in exploring the dynamical equivalence between theories: we can use results already derived for one theory in another equivalent theory.

The term dynamical equivalence can be considered misleading in classical gravity. Within a classical perspective, a theory is fully described by a set of field equations. When we are referring to gravitation theories, these equations will be describing the dynamics of gravitating systems. Therefore, two dynamically equivalent theories can actually be considered just different representations of the same theory.

The issue of distinguishing between truly different theories and different representations of the same theory (or dynamically equivalent theories) is an intricate one. It has serious implications and has been the cause of many misconceptions in the past, especially when conformal transformations are used in order to redefine the fields (*e.g.* the Jordan and Einstein frames in scalar-tensor theory). Since many of its aspects can be more easily appreciated once a complete discussion of the alternative theories of gravity has already been presented, we have decided to allow this discussion to extend over two different chapters, the current one and Chapter 7.

In the current chapter we will approach the problem only from an operational viewpoint and consider only the theories presented in the previous chapter. We will, therefore, merely analyse specific field redefinitions that are necessary to show the dynamical equivalence between some of these theories. The use of conformal redefinitions of the metric will be avoided in order to simplify the discussion and we will confine ourselves to deriving results that are needed in the forthcoming chapters.

We will return to this subject again in Chapter 7, where we intend to discuss the role of conformal transformations and redefinition of fields in gravitation theories, and to analyse extensively the implications which these have for our understanding of the underlying theory and our ability to propose alternative gravity theories, hopefully clarifying some common and longstanding misconceptions concerning these issues.

4.2 $f(R)$ gravity and Brans–Dicke theory

4.2.1 Redefinition of variables

The dynamical equivalence between $f(R)$ gravity and scalar-tensor theory, or more specifically Brans–Dicke theory with a potential for the scalar, has been considered by many authors (see, for instance, [235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 166]). Let us follow the lines of [166] in order to see how it comes about[1]. We will work at the level of the action but the same approach can be used to work directly at the level of the field equations. We begin with metric $f(R)$ gravity. For the convenience of the reader, we re-write here the action (3.16):

$$S_{met} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu},\psi). \tag{4.1}$$

One can introduce a new field χ and write a dynamically equivalent action [236]:

$$S_{met} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\big(f(\chi)+f'(\chi)(R-\chi)\big) + S_M(g_{\mu\nu},\psi). \tag{4.2}$$

Variation with respect to χ leads to the equation $\chi = R$ if $f''(\chi) \neq 0$, which reproduces action (3.16). Redefining the field χ by $\phi = f'(\chi)$ and setting

$$V(\phi) = \chi(\phi)\phi - f(\chi(\phi)), \tag{4.3}$$

[1]Note that there are minor differences between the terminology of Ref. [166] and the one used here.

the action takes the form

$$S_{met} = \frac{1}{16\pi\, G} \int d^4x \sqrt{-g}\, (\phi R - V(\phi)) + S_M(g_{\mu\nu}, \psi). \tag{4.4}$$

Comparison with the action (3.5) reveals that action (4.4) is the action of a Brans–Dicke theory with Brans–Dicke parameter $\omega_0 = 0$ [2]. Therefore, as has been observed long ago, metric $f(R)$ theories are dynamically equivalent to a class of Brans–Dicke theories with vanishing kinetic term [236, 240].

Let us set aside Palatini $f(R)$ gravity for the moment, and consider directly metric-affine $f(R)$ gravity. For simplicity, we will assume that the independent connection is symmetric (torsion-less theory) since, as we will argue later on, the results of this section will be completely unaffected by this choice. Once more, we re-write the action for this theory here for the convenience of the reader:

$$S_{ma} = \frac{1}{16\pi\, G} \int d^4x \sqrt{-g} f(\mathcal{R}) + S_M(g_{\mu\nu}, \Gamma^\lambda{}_{\mu\nu}, \psi). \tag{4.5}$$

Following the same steps as before, we can introduce the scalar field χ and then redefine it in terms of ϕ. The action takes the form:

$$S_{ma} = \frac{1}{16\pi\, G} \int d^4x \sqrt{-g}\, (\phi\mathcal{R} - V(\phi)) + S_M(g_{\mu\nu}, \Gamma^\lambda{}_{\mu\nu}, \psi). \tag{4.6}$$

Even though the gravitational part of this action is formally the same as that of action (4.4), this action is not a Brans–Dicke action with $\omega_0 = 0$ for two reasons: Firstly, the matter action depends on the connection, unlike Brans–Dicke theory, and secondly $\mathcal{R}$ is not the Ricci scalar of the metric $g_{\mu\nu}$. Therefore, there is no equivalence between Brans–Dicke theory and the general case of $f(R)$ theories in which the connections are independent of the metric. The reason is that, unlike Brans–Dicke theory, the theory described by the action (4.5) is not a metric theory. The matter action is coupled to the connection as well, which in this case is an independent field. This makes the theory a metric-affine theory of gravity, as has been discussed extensively in the previous chapter.

Let us examine what will happen if we force the matter action to be independent of the connection, as is usually done in the literature [242, 246]. Essentially, by forbidding the coupling between the matter fields and the connection, we reduced the action to (3.21)

$$S_{pal} = \frac{1}{16\pi\, G} \int d^4x \sqrt{-g} f(\mathcal{R}) + S_M(g_{\mu\nu}, \psi) \tag{4.7}$$

and the theory to Palatini $f(R)$ gravity. The field equations of the theory are eqs. (3.29) and (3.30) and as already mentioned, the latter implies that the connections are the Levi–Civita connections of the metric $h_{\mu\nu} = f'(\mathcal{R}) g_{\mu\nu}$ (see Section 3.4.3). Using the redefinition which we introduced to relate the actions (4.7) and

[2] Action (4.4) is also known as the O'Hanlon action [247].

(4.6), we can express the relation between the two conformal metrics simply as $h_{\mu\nu} = \phi g_{\mu\nu}$. Then, using eq. (3.37), we can express $\mathcal{R}$ in terms of R and ϕ:

$$\mathcal{R} = R + \frac{3}{2\phi^2}\nabla_\mu\phi\nabla^\mu\phi - \frac{3}{\phi}\Box\phi. \tag{4.8}$$

Putting this into the action (4.6), the latter takes the form:

$$S_{pal} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(\phi R + \frac{3}{2\phi}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) + S_M(g_{\mu\nu},\psi), \tag{4.9}$$

where we have neglected a total divergence. The matter action now has no dependence on ${\Gamma^\lambda}_{\mu\nu}$ since this was our initial requirement. Therefore, this is indeed the action of a Brans–Dicke theory with Brans–Dicke parameter $\omega_0 = -3/2$.

The equations that one derives from the action (4.9) are eqs. (3.10) and (3.11) once ω_0 is set to be $-3/2$. Note that, for $\omega_0 = -3/2$ and once we set $\phi \equiv f'(\mathcal{R})$, eqs. (3.10) and (3.11) can be combined to give eq. (3.38), which we derived in Section 3.4.3 after simple mathematical manipulations and without any reference to Brans–Dicke theory. Additionally, it is worth mentioning that, for $\omega_0 = -3/2$, eq. (3.12) reduces to

$$\kappa T + \phi V'(\phi) - 2V(\phi) = 0, \tag{4.10}$$

which is an algebraic equation linking ϕ and T for a given potential. One can then verify again that in vacuum, where $T = 0$, ϕ will have to be a constant and so the theory reduces to Einstein gravity with a cosmological constant, this time determined by the value of ϕ.

4.2.2 Physical Implications and special cases

In order to obtain the equivalence between Brans–Dicke theory with $\omega_0 = -3/2$ and metric-affine $f(R)$ gravity, we had to force the matter action to be independent of the connections, i.e. to reduce the theory to Palatini $f(R)$ gravity. This led to the fact that the connections became the Levi–Civita connections of the metric $h_{\mu\nu} = \phi g_{\mu\nu}$, which allowed us to eliminate the dependence of the action on the connections. We can construct a theory where the matter action would be allowed to depend on the connections, but the connections would be assumed to be the Levi–Civita connections of a metric conformal to $g_{\mu\nu}$ *a priori*. One may be misled into thinking that such a theory could be cast into the form of a Brans–Dicke theory, since in this case the dependence of the action on the connections can indeed be eliminated. No mathematical calculations are required to show that this is not so. The gravitational part of the action would, of course, turn out to be the same as that of (4.9) if ϕ (its square root to be precise) is used to represent the conformal factor. Notice, however, that since the matter action initially had a dependence on the independent connection, after eliminating the connection in favour of scalar ϕ, the matter will have a dependence not only on the metric but also on ϕ because the connection will be function of both the metric and ϕ. Therefore, the scalar field would be coupled to matter directly and in a non-trivial way, unlike Brans–Dicke theory or scalar–tensor theory in general.

The above discussion demonstrates that it is the coupling of the connections to matter that really prevents the action (4.5) from being dynamically equivalent to (3.1). One cannot achieve such equivalence by constraining the connection. The only exception is if the conformal factor relating $g_{\mu\nu}$ and the metric that is compatible with the connection, is a constant. In this case the theory will just reduce to metric $f(R)$ gravity and, as mentioned before, it will be equivalent to a Brans–Dicke theory with $\omega_0 = 0$.

The fact that $f(R)$ gravity in the Palatini formalism is equivalent to a class of Brans–Dicke theories when the matter action is independent of the connection, demonstrates clearly that the former is intrinsically a metric theory. This, as mentioned in the previous chapter, should have been expected since the matter is coupled only to the metric. Even though $\Gamma^\lambda{}_{\mu\nu}$ is not a scalar, the theory actually has only one extra scalar degree of freedom with respect to General Relativity [3]. The independent connection representation of the theory just prevents us from seeing this directly, because the action is written in this frame in terms of what turns out to be an unfortunate choice of variables. On the other hand, if one wants to construct a metric-affine theory of gravity, matter should be coupled to the connection as also claimed in [165]. In this case, any dynamical equivalence with Brans–Dicke theory breaks down. This clarifies once more why we have reserved the term “metric–affine $f(R)$ theory of gravity” for these theories, in order to distinguish them from those for which there is no coupling between the matter and the connection and which are usually referred to in the literature as $f(R)$ theories of gravity in the Palatini formalism.

It is also important to mention that $f(R)$ theories of gravity in the metric formalism and in the Palatini formalism are dynamically equivalent to different classes of Brans–Dicke theories. This implies that they cannot be dynamically equivalent to each other, *i.e.* no redefinition of variables or manipulation can be found that will bring a Palatini $f(R)$ theory into the form of some metric $f(R)$ theory. Therefore, these theories will give different physical predictions. The same is, of course, true for metric-affine $f(R)$ theories of gravity as well, since they cannot be cast into the form of a Brans–Dicke theory. There is, however, an exception: metric–affine $f(R)$ gravity will reduce to Palatini $f(R)$ gravity in vacuum, or in any other case where the only matter fields present are by definition independent of the connection such as scalar fields, the electromagnetic field or a perfect fluid [165]. Therefore, even though there is no equivalence between metric-affine $f(R)$ gravity and Brans–Dicke theories with $\omega_0 = -3/2$, their phenomenology will be identical in many interesting cases, including cosmological applications.

We have mentioned that the results will remain unchanged if we allow the connection present in the action (4.5) to be non-symmetric. Let us now justify this:

[3] *cf.* [213] where similar conclusions about the role of the independent connection in Palatini $f(R)$ gravity are derived by examining energy conservation.

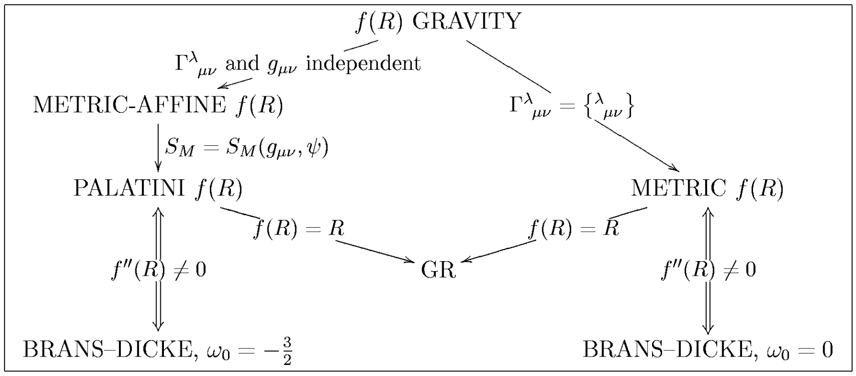

Figure 4.1: Schematic diagram relating various versions of $f(R)$ gravity and Brans–Dicke theory, stating the various assumptions needed in passing from one to another.

in the case of $f(R)$ gravity where the matter is independent of the connection, it is true since the non-symmetric part of the connection vanishes even if no such assumption is made *a priori*, and the field equations of the corresponding theory are identical to eqs. (3.29) and (3.30) (see Section 3.6 and [165]). On the other hand, when studying the case where matter is coupled to the connection, we did not have to use the symmetry of the connections, neither did we have to use the field equations, but we worked at the level of the action. We have summed up the results presented so far in this section in the schematic diagram of fig. 4.1 [166].

It should be mentioned that Brans–Dicke theory with $\omega_0 = -3/2$ has not received very much attention (see however [248]). The reason is that when Brans–Dicke theory was first introduced, only the kinetic term of the scalar field was present in the action. Therefore, choosing $\omega_0 = -3/2$ would lead to an ill-posed theory, since only matter described by a stress-energy tensor with a vanishing trace could be coupled to the theory. This can be understood by examining eq. (4.10) in the absence of terms including the potential. However, once the potential of the scalar field is considered in the action, no inconsistency occurs. Note that a Brans–Dicke gravitational action with $\omega_0 = -3/2$ and no potential term is conformally invariant and dynamically equivalent to Conformal Relativity (or Hoyle-Narlikar theory) [249]. The action of Conformal Relativity [250, 251, 252] has the form

$$S_{CR} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,\Psi\left(\frac{1}{6}\Psi R - \Box\Psi\right). \tag{4.11}$$

A field redefinition $\Psi^2 = 6\phi$ will give

$$S_{CR} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(\phi R + \frac{3}{2\phi}\nabla_\mu\phi\nabla^\mu\phi\right), \tag{4.12}$$

where a total divergence has been discarded. The dynamical equivalence is therefore straightforward.

The case of a vanishing potential has no analogue in $f(R)$ gravity. Using eq. (4.3) and remembering that $\phi = f'(\chi)$ and on shell $\mathcal{R} = \chi$, one can easily verify that setting $V(\phi) = 0$ will lead to the the following equation for $f(\mathcal{R})$:

$$f'(\mathcal{R})\mathcal{R} - f(\mathcal{R}) = 0. \tag{4.13}$$

This equation can be identically satisfied only for $f(\mathcal{R}) = \mathcal{R}$. However, to go from the $f(R)$ representation to the Brans–Dicke representation, one assumes $f''(\mathcal{R}) \neq 0$, so no $f(\mathcal{R})$ Lagrangian can lead to a Brans–Dicke theory with a vanishing potential. It is remarkable that this ill-posed case does not exist in Palatini $f(R)$ gravity. There is, however, a conformally invariant gravitational action in this context as well. One has to choose $f(\mathcal{R}) = a\mathcal{R}^2$, where a is some constant [235]. In this case the potential of the equivalent Brans–Dicke theory will be $V(\phi) = \phi^2/4a$. For this potential, all terms apart from the one containing T in eq. (4.10) will again vanish, as would happen for a vanishing potential. The correspondence can easily be generalised for n-dimensional manifolds, where $n \geq 2$. For the gravitational action to be conformally invariant in the context of $f(R)$ gravity, one should choose $f(\mathcal{R}) = a\mathcal{R}^{n/2}$ [203]. The corresponding potential can be computed using eq. (4.3) and has the form

$$V(\phi) = \left(\frac{n}{2} - 1\right) a \left(\frac{2\phi}{na}\right)^{n/(n-2)} \tag{4.14}$$

Eq. (3.12) will generalize for n dimensions in the following way:

$$(n-2)\left(\omega_0 + \frac{n-1}{n-2}\right)\Box\phi = \kappa T + \left(\frac{n}{2} - 1\right)\left(\phi V' - \frac{n}{n-2}V\right), \tag{4.15}$$

implying that for n dimensions the special case which we are examining corresponds to $\omega_0 = -(n-1)/(n-2)$. This indicates that a Brans–Dicke gravitational action with $\omega_0 = -(n-1)/(n-2)$ and a potential $V(\phi) = b\phi^{n/(n-2)}$, where b is some constant, will be conformally invariant in an n-dimensional manifold. As an example we can examine the 4-dimensional action:

$$S_{n=4} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(\phi R + \frac{3}{2\phi}\nabla_\mu\phi\nabla^\mu\phi - b\phi^2\right). \tag{4.16}$$

Using the redefinition $\Psi^2 = 6\phi$ as before, we can bring the action to the following form

$$S_{n=4} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Psi\left(\frac{1}{6}\Psi R - \Box\Psi - b\frac{\Psi^3}{36}\right), \tag{4.17}$$

which is a generalization of the action (4.11). It is easy to verify that this specific potential will not break conformal invariance. Under the conformal transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, the root of the determinant will transform as $\sqrt{-g} \to \Omega^4\sqrt{-g}$ and so, with an appropriate redefinition of the scalar field $\tilde{\Psi} = \Omega^{-1}\Psi$, the action will return to the form of (4.17).

4.2.3 Higher-order curvature invariants

We have seen that both metric and Palatini $f(R)$ gravity acquire a Brans–Dicke representation. It is interesting to examine whether we can eliminate higher-order curvature invariants in favour of scalar fields in the action as well. As has been discussed in Section 3.5.1, in the metric formalism, the presence of a general function of the scalar curvature or of quadratic terms such as $R^{\mu\nu}R_{\mu\nu}$, leads to fourth order equations, whereas an $R\Box R$ term or an $R\Box^2 R$ term lead to sixth and eighth order equations respectively [208, 209, 210, 211]. As can be found in the literature, theories including any of the above terms can be rewritten as a multi-scalar-tensor theory, *i.e.* these terms can be eliminated in favour of one or more scalars. The number of scalar fields needed is directly related to the order of the field equations: for fourth order equations, as in metric $f(R)$ gravity, one needs just one scalar, for sixth order equations one needs two scalars, and for every further two orders one more scalar has to be introduced [209, 253].

In Palatini $f(R)$ gravity, however, things are not equally straightforward. To understand this, one has to recall that, in order to bring a Palatini $f(R)$ action into the form of a Brans–Dicke action, we used the solution of the second field equation, eq. (3.30). More specifically, due to this equation we were able to introduce the conformal metric $h_{\mu\nu}$ and therefore relate quantities constructed with the initially independent connection, such as $\mathcal{R}$, with purely metric quantities, such as R. Adding a higher order curvature term in the action will inevitably modify eq. (3.30). In Section 3.5.1, we have given the simple example of adding an $\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}$ term to an otherwise linear action in $\mathcal{R}$, in order to demonstrate how the presence of this term introduces more dynamics for the independent connection. Indeed, this simple example can be used here as well: eq. (3.45) is no longer an algebraic equation in $\Gamma^{\lambda}{}_{\mu\nu}$, as is eq. (3.30), and therefore it cannot be trivially solved in order to express this connection, and consequently the quantities constructed with it, in terms of the metric $g_{\mu\nu}$. Thus, a dynamical equivalence with some scalar-tensor theory is neither straightforward nor guaranteed.

4.3 Why $f(R)$ gravity then?

Since $f(R)$ gravity in both the metric formalism and the Palatini formalism can acquire a scalar-tensor theory representation, one might be led to ask two questions: firstly, why should we consider the $f(R)$ representation and not just work with the scalar-tensor one, and secondly, why, since we know quite a lot about scalar-tensor theory, should we consider $f(R)$ gravity unexplored or interesting?

The answer to the first question is quite straightforward. There is actually no reason to prefer either of the two representations *a priori* — at least as far as classical gravity is concerned. There can be applications where the $f(R)$ representation can be more convenient and applications where the scalar-tensor representation is more convenient. One should probably mention that habit affects our taste and, therefore, an $f(R)$ representation seems to appear more appealing to relativists due to its more apparent geometrical nature, whereas the scalar-tensor representation seems to be more appealing to particle physicists. This issue can have theoretical implications. To give an example: if $f(R)$ gravity is considered as a step towards a more complicated theory, which generalisation would be more straightforward will depend on the chosen representation. This issue will be addressed more extensively and in more general terms in Chapter 7.

Whether $f(R)$ theories of gravity are unexplored and interesting or just an already-studied subcase of scalar tensor theory, is a more practical question that certainly deserves a direct answer. It is indeed true that scalar-tensor theory and, more precisely, Brans–Dicke theory are well-studied theories which have been extensively used in many applications, including Cosmology. However, the specific choices $\omega_0 = 0, -3/2$ for the Brans–Dicke parameter are quite exceptional. We have already mention in Section 4.2.2 why the $\omega_0 = -3/2$ case has not been studied in the past. It is also worthwhile mentioning that most calculations which are done for a general value of ω_0 in the literature actually exclude $\omega_0 = -3/2$, mainly because they are done in such a way that the combination $2\omega_0 + 3$ appears in a denominator (see Chapter 6 for details and examples). As far as the $\omega_0 = 0$ case is concerned, one can probably speculate that it is the apparent absence of the kinetic term for the scalar in the action which did not seem appealing and prevented the study of this theory. In any case, the conclusion is that the theories in the Brans–Dicke class that correspond to metric and Palatini $f(R)$ gravity had not yet been explored before the recent re-introduction of $f(R)$ gravity and, as will also become apparent later, several of their special characteristics when compared with more standard Brans–Dicke theories were revealed though studies of $f(R)$ gravity.

4.4 Gauss–Bonnet gravity and $f(\mathcal{G})$ gravity

In Section 3.7 we introduced Gauss–Bonnet gravity through the action (3.138) which we repeat here for convenience

$$S_{GB} = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{\lambda}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + f(\phi)\mathcal{G}\right] + S_M(g^{\mu\nu}, \psi). \tag{4.18}$$

This theory includes a specific combination of higher-order curvature invariants: the Gauss–Bonnet term $\mathcal{G}$ (see also Section 3.5.1). As has been discussed, the Gauss–Bonnet term is a topological invariant and the variation of the density $\sqrt{-g}\mathcal{G}$ leads to a total divergence, therefore not contributing to the field equations. However, in Gauss–Bonnet gravity this term is coupled to the scalar field ϕ and, therefore, does contribute to the field equations. One could think along the following lines: a conformal redefinition of the metric, together with a suitable redefinition of the scalar field could potentially decouple the transformed Gauss–Bonnet term from the redefined scalar, therefore allowing us to omit its presence. However, this idea cannot work in practice, simply because $\sqrt{-g}\mathcal{G}$ transforms under conformal redefinition of the metric $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ as

$$\begin{aligned}\sqrt{-g}\mathcal{G} \to \sqrt{-g}\mathcal{G} &+ \sqrt{-g}\big[8R^{\mu\nu}\left(\nabla_\mu \ln\Omega \nabla_\nu \ln\Omega - \nabla_\mu\nabla_\nu \ln\Omega\right) \\ &+ 8(\nabla^2 \ln\Omega)^2 - 8(\nabla_\mu\nabla_\nu \ln\Omega)^2 \\ &+ 8(\nabla^2 \ln\Omega)(\nabla_\mu \ln\Omega)^2 - 4R(\nabla^2 \ln\Omega) \\ &+ 16(\nabla_\mu \ln\Omega\nabla_\nu \ln\Omega)(\nabla^\mu\nabla^\nu \ln\Omega)\big]. \end{aligned} \tag{4.19}$$

Even though extra terms containing derivatives of the conformal factor will appear after the conformal transformation, no factor appears in front of the Gauss–Bonnet term. Therefore, it is not possible to eliminate the coupling with the scalar field, and consequently the presence of the Gauss–Bonnet term, by means of a conformal transformation. We conclude that Gauss–Bonnet gravity cannot be rewritten as a scalar-tensor theory.

However, there is a specific subcase of the action (4.18) that can be cast into the form of another theory already present in the literature, namely that given by the action

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi\, G} + f(\mathcal{G})\right), \tag{4.20}$$

which has already been mentioned in Section 3.5.1 as action (3.42) [212]. The function $f(\mathcal{G})$ is a general function of the Gauss–Bonnet term. Following Ref. [212], one can introduce two auxiliary scalar fields A and B, in order to re-write the action (4.20) as

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi\, G} + B\left(\mathcal{G} - A\right) + f(A)\right). \tag{4.21}$$

Variation with respect to B leads to $A = \mathcal{G}$ and so action (4.20) is then recovered. Variation with respect to A leads to

$$B = f'(A). \tag{4.22}$$

Replacing this back in eq. (4.21) gives

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi\, G} + f'(A)\left(\mathcal{G} - A\right) + f(A)\right). \tag{4.23}$$

Simply redefining

$$\phi = A, \tag{4.24}$$

$$V(\phi) = Af'(A) - f(A), \tag{4.25}$$

leads to

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi\, G} - V(\phi) + f(\phi)\mathcal{G}\right] + S_M(g^{\mu\nu}, \psi). \tag{4.26}$$

Clearly, this is action (4.18) for $\lambda = 0$, *i.e.* when the scalar field has no kinetic term. Therefore, for the specific case of $\lambda = 0$, Gauss–Bonnet gravity is dynamically equivalent to a theory described by action (4.20), which includes a general function of the Gauss–Bonnet term in addition to the standard Einstein–Hilbert action [212].

Chapter 5

Cosmology in modified gravity

5.1 Introduction

During the course of the 90 years that have passed since the introduction of General Relativity by Einstein, the study and development of alternative theories of gravity has always been pursued in parallel, even though there have been periods of intense effort and periods of slower development, depending on the contemporary motivation. We have extensively discussed the motivations for modifying gravity and it would not be wise to attempt to rank them according to importance. However, one could still observe that the current stimulus in this subject area is mainly powered by observational cosmology, simply because the cosmological riddles are the most recent of the problems which alternative gravity aims to address. Therefore, it is important to consider the cosmological phenomenology of the theories presented in the previous chapter and to explore how well they can address issues such as the late time accelerated expansion of the universe, the nature of dark energy *etc*.

Irrespective of the theory of gravity, the main assumptions of cosmology remain the same since they are not related to the dynamics but to the symmetries that we expect the universe to exhibit when one focuses on large scale evolution and ignores small scale inhomogeneities. Therefore, the arguments for homogeneity and isotropy presented in Section 1.3.1 are still valid and we can use the Friedmann-Lemaître-Robertson-Walker metric as a global description of spacetime:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\right]. \tag{5.1}$$

We remind the reader that $k = -1, 0, 1$ according to whether the universe is hyperspherical (closed), spatially flat, or hyperbolic (open) and that $a(t)$ is called the scale factor. Part of the standard approach, which we follow here as well, is to use a perfect fluid description for the matter for which

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p\, g^{\mu\nu}, \tag{5.2}$$

where u^μ denotes the four-velocity of an observer comoving with the fluid and ρ and p are the energy density and the pressure of the fluid respectively. Once a gravity theory is chosen, one can insert the FLRW ansatz (5.1) and the stress-energy tensor (5.2) into the field equations of the theory and derive equations governing the evolution of the scale factor $a(t)$. These are generalizations of the Friedmann equations (1.5) and (1.6) and we will occasionally refer to them as the "generalised Friedmann equations".

Note that the value of k is an external parameter. As in many other works in the literature, for what follows we will choose $k = 0$, *i.e.* we focus on a spatially flat universe. This choice in made in order to simplify the equations and should be viewed sceptically. It is sometimes claimed in the literature that such a choice is favoured by the data. However, this is not entirely correct. Even though the data (*e.g.* [43]) indicate that the current value of Ω_k is very close to zero (see eq. (1.14) and the related discussion in Section 1.3.3) it should be stressed that this does not really reveal the value of k itself. Since

$$\Omega_k = -\frac{k}{a^2 H^2}, \tag{5.3}$$

the current value of Ω_k is sensitive of the current value of $a(t)$, *i.e.* the amount of expansion the universe has undergone after the Big Bang. A significant amount of expansion can easily drive Ω_k very close to zero. The success of the inflationary paradigm is exactly that it explains the flatness problem — how did the universe become so flat (see Section 1.3.2) — in a dynamical way, allowing us to avoid having to fine tune the parameter k (having $k = 0$ is statistically very exceptional).

The above having been said, choosing $k = 0$ for simplicity is not a dramatic departure from generality when it come to late time cosmology. If it is viewed as an approximation and not as a choice of an initial condition, then one can say that, since Ω_k as inferred from observations is very close to zero at current times, the terms related to k will be subdominant in the Friedmann or generalised Friedmann equations and therefore one could choose to discard them by setting $k = 0$, without great loss of accuracy. In any case, results derived under the assumption that $k = 0$ should be considered preliminary until the influence of the spatial curvature is precisely determined, since there are indications that even a very small value of Ω_k may have an effect on them (see, for instance, Ref. [254]).

Having set the ground, we are now ready to explore the cosmological implication of a number of alternative theories of gravitation. Taking into account that cosmology in scalar-tensor theory has already been extensively studied [199], we focus on the various versions of $f(R)$ gravity and on Gauss–Bonnet gravity. A study limited to these theories cannot be considered as exhaustive and many more modifications of the gravitational actions are possible. However, from a phenomenological point of view, such theories can work as useful examples for understanding how modifications of the gravitational action can help us to address the well-known cosmological problems, since they include a number of interesting terms in the gravitational action.

5.2 $f(R)$ gravity in the metric formalism

5.2.1 Generalised Friedmann equations

We start with $f(R)$ gravity in the metric formalism. We present in this section the modified Friedmann equations for such theories, which date back to Buchdahl's paper [201]. The procedure for deriving these equations is actually quite straightforward since the components of $R_{\mu\nu}$ for the ansatz (5.1) can be easily found in textbooks. The time-time component of the field equations (3.18) gives the modified Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{3f'(R)}\left\{\frac{1}{2}\left[f(R) - Rf'(R)\right] - 3\left(\frac{\dot{a}}{a}\right)\dot{R}f''(R)\right\} = \frac{1}{3}\kappa\rho, \qquad (5.4)$$

and the space-space components gives

$$\begin{aligned} 2\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{f'(R)}\Big\{2\left(\frac{\dot{a}}{a}\right)\dot{R}f''(R) + \ddot{R}f''(R) + \\ + \dot{R}^2 f'''(R) - \frac{1}{2}\left[f(R) - Rf'(R)\right]\Big\} = -\kappa p, \end{aligned} \qquad (5.5)$$

where $\kappa = 8\pi G$. R is given by

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = 6\left(\dot{H} + 2H^2\right), \qquad (5.6)$$

and $H = \dot{a}/a$ as usual. A combination of eqs. (5.4) and (5.5) gives

$$\begin{aligned} \left(\frac{\ddot{a}}{a}\right) + \frac{1}{2f'(R)}\Big\{\left(\frac{\dot{a}}{a}\right)\dot{R}f''(R) + \ddot{R}f''(R) + \\ + \dot{R}^2 f'''(R) - \frac{1}{3}\left[f(R) - Rf'(R)\right]\Big\} = -\frac{\kappa}{6}\left[\rho + 3p\right]. \end{aligned} \qquad (5.7)$$

Setting $f(R) = R$, one has $f'(R) = 1$ and $f'' = f''' = 0$ and eqs. (5.4) and (5.7) reduce to the standard Friedmann equations (1.5) and (1.6).

5.2.2 $1/R$ terms and late time cosmology

The first attempt to consider $f(R)$ theories of gravity as a way to explain late-time cosmological acceleration was probably Ref. [255]. The main objective is to have modified gravity account for dark energy and consequently explain the late time accelerated expansion of the universe. One of the easiest ways to see how this comes about is the following [255]: If we define the quantities

$$\rho_{de} = \frac{\kappa^{-1}}{f'(R)}\left\{\frac{1}{2}\left[f(R) - Rf'(R)\right] - 3\left(\frac{\dot{a}}{a}\right)\dot{R}f''(R)\right\}, \tag{5.8}$$

$$p_{de} = \frac{\kappa^{-1}}{f'(R)}\left\{2\left(\frac{\dot{a}}{a}\right)\dot{R}f''(R) + \ddot{R}f''(R) + \right.$$
$$\left. + \dot{R}^2 f'''(R) - \frac{1}{2}\left[f(R) - Rf'(R)\right]\right\}, \tag{5.9}$$

and use them to re-write eqs. (5.4) and (5.7) we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\kappa\rho_{\rm tot}, \tag{5.10}$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{\kappa}{6}\left[\rho_{\rm tot} + 3p_{\rm tot}\right], \tag{5.11}$$

where

$$\rho_{\rm tot} = \rho + \rho_{de}, \tag{5.12}$$

$$p_{\rm tot} = p + p_{de}. \tag{5.13}$$

Through some simple redefinitions, we have brought the equations governing the cosmological dynamics into the form of the standard Friedmann equations. Additionally, the terms related to high order terms present in the action are now conveniently denoted as ρ_{de} and p_{de}, since these terms are playing here the role of dark energy. Therefore, the whole theory, for what regards cosmology, has been brought into the form of General Relativity with some kind of dark energy, whose nature can actually be attributed to a modification of gravity.

Now the important question is: What is the effective equation of state relating ρ_{de} and p_{de}? Viewed as functions of R, these quantities are obviously related and the effective equation of state will depend on the functional form of $f(R)$. Therefore, without going into more details, one can expect that a convenient choice for $f(R)$ can lead to a suitable value of w_{de} ($p_{de} = w_{de}\rho_{de}$), so that the modified theory of gravity can account for the observations indicating late time accelerated expansion.

We will give two simple examples that can be found in the literature: Firstly, one can consider the function f to be of the form $f(R) \propto R^n$. It is quite straightforward to calculate w_{de} as a function of n if the scale factor is assumed to be a generic power law $a(t) = a_0(t/t_0)^\alpha$ [255]. The result is

$$w_{de} = -\frac{6n^2 - 7n - 1}{6n^2 - 9n + 3}, \tag{5.14}$$

for $n \neq 1$, and α is given is terms of n as

$$\alpha = \frac{-2n^2 + 3n - 1}{n - 2}. \tag{5.15}$$

A suitable choice of n can lead to a desired value for w_{de}. The second example which we will refer to is a model of the form $f(R) = R - \mu^{2(n+1)}/R^n$, where μ is a suitably chosen parameter [256]. In these case w_{de} can again be written as a function of n [256]:

$$w_{de} = -1 + \frac{2(n+2)}{3(2n+1)(n+1)}. \tag{5.16}$$

The most typical model within this class is that with $n = 1$ [256], in which case $w_{de} = -2/3$. Note that in this class of models, a positive n implies the presence of a term inversely proportional to R in the action, contrary to the situation for the R^n models.

We have chosen to discuss metric $f(R)$ gravity and late time acceleration in terms of its representation as introducing a form of effective dark energy, since this approach is simple and has a direct relation with the usual approach to cosmological problems. Obviously, there is more to say about the cosmological dynamics of metric $f(R)$ gravity, such as making a complete dynamical analysis of the equations governing the cosmic evolution, or making a precise study of specific models and their cosmological behaviour. For the moment, however, let us mention that our goal here is merely to demonstrate how simple modifications of gravity can address the dark energy problem. Note also that we have not chosen the examples according to their overall viability as gravitation theories — we will discuss these issues later on — but according to simplicity.

5.2.3 More general models and cosmological constraints

We have seen qualitatively how simple metric $f(R)$ gravity models, and especially those including an inverse power of R, can be used to solve the cosmological puzzle of the late time accelerated expansion. Clearly one can consider much more general functions f. These do not have to be polynomials necessarily. However, taking f to be a polynomial with positive and/or negative powers of R has certain advantages. Besides simplicity, one can argue that choosing f to be a polynomial is a practical way to include in the action some phenomenologically interesting terms which might be of leading order in the Taylor expansion of an effective Lagrangian [*c.f.* eq. (3.15)].

However, before referring to more general models we should mention that specific models of what is now called metric $f(R)$ gravity were not initially introduced in cosmology in order to account for the phenomenology related to the later stages of its evolution: Alexei Starobinski, who, as mentioned in the introduction, was one of the pioneers of the idea of inflation, had first proposed a scenario of this sort for giving a gravity driven inflationary period [20]. The model, which was actually presented before the more conventional models based on scalar fields, included an R^2 term in the gravitational Lagrangian. The presence of this term is able to drive the universe to an accelerated expansion which takes place at early times. We address the reader to the literature for more details (*e.g.* [20, 257, 239]).

It seems reasonable to expect that a single model, including both positive and negative powers of R, would be able to lead to both an early time inflationary period and a late time accelerated expansion. This was indeed shown in [258]. Another interesting class of models are those containing a $\ln R$ term [259]. See also [260, 261] for reviews of metric $f(R)$ gravity and cosmology and [262] for a discussion of the cosmological dynamics of R^n models.

For a model of $f(R)$ gravity to be considered successful from a cosmological perspective, however, it is not enough to have the correct early or late time behaviour in a qualitative sense. There are a number of precise tests related to cosmological observations that any gravity theory should pass[1]. For instance, there have been studies of the constraints imposed on specific models of metric $f(R)$ gravity by Big Bang Nucleosynthesis and local fifth-force experiments [263] and attempts to explore the details of cosmological perturbations [264]. The stability of the de-Sitter solution, which is supposed to be a late time attractor for models with late time acceleration, has been considered [265, 266, 267], as well as the process of producing the baryon asymmetry in the universe, Baryogenesis [268]. To avoid getting into technical details let us just say that, even though some of these studies, such as the one related to Baryogenesis, show that metric $f(R)$ gravity does not lead to significant deviations away from the standard picture, the overall impression is that simple models are unlikely to produce the cosmological dynamics and also agree in detail with all of the other observations. There is, of course, ongoing research on this (*e.g.* [269, 270]).

An issue that requires a special mention is the question raised in [271] about whether metric $f(R)$ gravity can lead to cosmological models which include both a standard matter dominated era and a phase of accelerated expansion. According to [271] all $f(R)$ theories that behave as a power of R at large or small R will have a matter era in which the scale factor will scale like $t^{1/2}$ instead of the standard law $t^{2/3}$, making the theory grossly inconsistent with observations. This issue has raised a lively debate [272, 273]. The outcome is that R^n and $R-\mu^{2(n+1)}/R^n$ models do indeed lead to an unacceptable behaviour during matter domination [274], but there can be more complicated models that do not have this unappealing characteristic. In fact, a scheme has been developed to reconstruct the action for metric $f(R)$ gravity from a desired cosmological evolution as inferred from observation [275, 276].

Let us close this section by stressing that $f(R)$ actions, and especially R^n and $R-\mu^{2(n+1)}/R^n$ models, should be considered as toy theories. Even from a dimensional analysis or leading order point of view, it is hard to consider such actions as exact effective low energy actions. An action including an R^2 term, for example, is most likely to include an $R_{\mu\nu}R^{\mu\nu}$ term as well. From this view point, $f(R)$ gravity is just a preliminary step that one takes in order to explore

[1] In addition, viable gravity theories should at the same time pass also tests relevant to other scales, such as the scales of the Solar System and compact objects. It is not an easy task to construct a theory that fulfils all of these requirements simultaneously. We will however discuss this issue later.

the possibilities which are offered by modifications of the gravitational actions. Attempts to study the cosmology of more general actions including higher order curvature invariants have been made (*e.g.* [277]) and there is hope that some of the shortcomings of metric $f(R)$ gravity may not be there for more complete theories.

5.3 $f(R)$ gravity in the Palatini formalism

5.3.1 Generalised Friedmann equations

Let us now consider Palatini $f(R)$ gravity. The action of the theory is (3.21) and the field equations are eqs. (3.29) and (3.30). We start by deriving the generalised Friedmann equation. Using the FLRW metric (5.1) we need to compute the components of $\mathcal{R}_{\mu\nu}$, which is the Ricci tensor constructed with the independent connection (see eq. (2.11)). Since what we know is an ansatz for the metric, it is practical to work with metric quantities and, therefore, eqs. (3.35), (3.36) and (3.37) can be used to arrive to the result. The non-vanishing components of $\mathcal{R}_{\mu\nu}$ are

$$\mathcal{R}_{00} = -3\frac{\ddot{a}}{a} + \frac{3}{2}(f')^{-2}(\partial_0 f')^2 - \frac{3}{2}(f')^{-1}\nabla_0\nabla_0 f', \tag{5.17}$$

$$\mathcal{R}_{ij} = [a\ddot{a} + 2\dot{a}^2 + (f')^{-1}\left\{{\lambda \atop \mu\nu}\right\}\partial_0 f' + \frac{a^2}{2}(f')^{-1}\nabla_0\nabla_0 f']\delta_{ij}, \tag{5.18}$$

where the subscript 0 denotes the time component and we remind the reader that ∇ is the covariant derivative associated with $g_{\mu\nu}$. Combining eqs. (5.17) and (5.18) with eq. (3.29) one quite straightforwardly arrives at the generalised Friedmann equation (*e.g.* [278])

$$\left(H + \frac{1}{2}\frac{\dot{f}'}{f'}\right)^2 = \frac{1}{6}\frac{\kappa(\rho + 3p)}{f'} + \frac{1}{6}\frac{f}{f'}, \tag{5.19}$$

where the overdot denotes differentiation with respect to coordinate time. Note that when f is linear, $f' = 1$ and, therefore, $\dot{f}' = 0$. Taking into account eq. (3.31), one can easily show that in this case eq. (5.19) reduces to the standard Friedmann equation.

5.3.2 A toy model as an example

Having derived the generalised Friedmann equation, we can now go ahead and study the cosmological evolution in Palatini $f(R)$ gravity. The first thing that we would like to check is which cosmological eras can take place in general. It is required that any model should lead to a radiation dominated era followed by a matter dominated era. In addition to this, a model which draws its motivation from late time cosmology should provide a resolution for the accelerated expansion of the universe. In fact, it was shown in [279] that models which include in the action a $1/\mathcal{R}$ term in addition to the more standard $\mathcal{R}$ term do indeed have this property. Several studies of this issued followed [280, 281, 282, 283].

It is also interesting to consider whether specific choices for the Lagrangian can lead to early time inflation without the need for a field introduced for this purpose. In the case of metric $f(R)$ gravity, this was indeed the case. In Palatini $f(R)$ gravity things are quite different. Models which include an $\mathcal{R}^2$ term have been studied and it has been shown that the presence of this term cannot lead to an inflationary period [278, 284, 285]. This, however, as will also become clearer later, is actually due to the special nature of an $\mathcal{R}^2$ term within the framework of Palatini $f(R)$ gravity [282]. An $\mathcal{R}^2$ term, as already mentioned, gives a zero contribution on the left hand side of eq. (3.31) and we will see that this makes this term quite ineffective as far as inflation is concerned.

Let us explore all of the above in more detail through an example. Consider the specific model in which f is given by

$$f(\mathcal{R})=\frac{\mathcal{R}^3}{\beta^2}+\mathcal{R}-\frac{\epsilon^2}{3\mathcal{R}}, \tag{5.20}$$

where ϵ and β are for the moment some parameters, on which we will try to put constraints later. Our choice of the form of the Lagrangian is based on the interesting phenomenology which it will lead to. When f is chosen to have the form given in eq. (5.20), in vacuum eq. (3.31) gives

$$\mathcal{R}^4-\beta^2\mathcal{R}^2+\epsilon^2\beta^2=0. \tag{5.21}$$

Note that even if we included an $\mathcal{R}^2$ term in eq. (5.20), eq. (5.21) would remain unchanged due to the form of eq. (3.31). Thus, even though we have avoided including this term for the sake of simplicity, there is no reason to believe that this will seriously affect our results in any way. One can easily solve eq. (5.21) to get

$$\mathcal{R}^2=\frac{\beta^2}{2}\left[1\pm\sqrt{1-4\left(\epsilon/\beta\right)^2}\right]. \tag{5.22}$$

If $\epsilon \ll \beta$, this corresponds to two de Sitter and two anti-de Sitter solutions for $\mathcal{R}$. Here we will consider the two de Sitter solutions, namely:

$$\mathcal{R}_1\sim\beta,\quad \mathcal{R}_2\sim\epsilon. \tag{5.23}$$

If we further assume that ϵ is sufficiently small and β is sufficiently large, then since the expansion rate of the de Sitter universe scales like the square root of the scalar curvature, $\mathcal{R}_1$ can act as the seed for an early-time inflation and $\mathcal{R}_2$ as the seed for a late-time accelerated expansion.

To see this explicitly, let us consider FLRW cosmology in more detail. At very early times, we expect the matter to be fully relativistic. Denoting by ρ_r and p_r the energy density and pressure, the equation of state will be $p_r=\rho_r/3$. Thus $T=0$ and eq. (3.31) will reduce to eq. (5.21) and have the solution given by eqs. (5.23).

If we ask for the curvature to be large, we infer that $\mathcal{R} = \mathcal{R}_1 = \beta$. Therefore, the universe will undergo a de Sitter phase which can account for the early-time inflation. As usual, conservation of energy implies $\rho_r \sim a^{-4}$. On the other hand, $\mathcal{R}$ and consequently $f(\mathcal{R})$ are now large constants, whereas $\dot{f}' = 0$ since $f'(\mathcal{R})$ is a constant as well. Therefore, it is easy to verify that the last term on the right hand side of eq. (5.19) will quickly dominate, with H being given by

$$H \sim \sqrt{\beta/12}. \tag{5.24}$$

In this sense, an inflationary period can occur in Palatini $f(R)$ gravity. Whether this scenario is realistic or not will be explored shortly.

Let us now consider the matter and radiation dominated eras. When the temperature is low enough, we expect some matter components to be non-relativistic. As an idealisation we can assume that the matter has two components. Radiation, for which $p_r = \rho_r/3$, and non-relativistic matter, for which the pressure $p_m = 0$ (dust) and the energy density is denoted by ρ_m. Eq. (3.31) takes the following form:

$$\frac{\mathcal{R}^3}{\beta^2} - \mathcal{R} + \frac{\epsilon^2}{\mathcal{R}} = -\kappa^2 \rho_m. \tag{5.25}$$

Energy conservation requires that

$$\dot{\rho}_m + 3H\rho_m = 0. \tag{5.26}$$

Using eqs. (5.25) and (5.26), it is easy to show after some mathematical manipulations that

$$\dot{\mathcal{R}} = \frac{3H\mathcal{R}\left(\mathcal{R}^2 - \frac{\mathcal{R}^4}{\beta^2} - \epsilon^2\right)}{\left(\frac{3\mathcal{R}^4}{\beta^2} - \mathcal{R}^2 - \epsilon^2\right)}. \tag{5.27}$$

The modified Friedmann equation (5.19) takes the form

$$H^2 = \frac{2\kappa^2\rho + \Lambda_{\rm eff}}{6\left(\frac{3\mathcal{R}^2}{\beta^2} + 1 + \frac{\epsilon^2}{3\mathcal{R}^2}\right)\left(1 + \frac{3}{2}A\right)^2} \tag{5.28}$$

where $\rho = \rho_r + \rho_m$,

$$A = \frac{\left(\frac{6\mathcal{R}^4}{\beta^2} - \frac{2}{3}\epsilon^2\right)\left(\mathcal{R}^2 - \frac{\mathcal{R}^4}{\beta^2} - \epsilon^2\right)}{\left(\frac{3\mathcal{R}^4}{\beta^2} - \mathcal{R}^2 - \epsilon^2\right)\left(\frac{3\mathcal{R}^4}{\beta^2} + \mathcal{R}^2 + \frac{\epsilon^2}{3}\right)}, \tag{5.29}$$

$$\Lambda_{\rm eff} = 2\left(\frac{\mathcal{R}^3}{\beta^2} + \frac{\epsilon^2}{3\mathcal{R}}\right), \tag{5.30}$$

and we have used eq. (5.25) and the equation of state for the relativistic component of the cosmological fluid.

Now $\mathcal{R}$ is no longer a constant. Its value is given by the root of eq. (5.25). Assuming that $\mathcal{R}$ is now less than β and significantly larger than ϵ, eqs. (5.27) and (5.28) give

$$\dot{\mathcal{R}} \sim -3H\mathcal{R}, \qquad H^2 \sim \frac{\mathcal{R}^3}{3\beta^2} \tag{5.31}$$

Thus, it is easy to see that

$$\mathcal{R} \sim t^{-2/3}, \qquad a(t) \sim t^{2/9}. \tag{5.32}$$

From eqs. (5.32), one concludes that

$$\rho_r \sim t^{-8/9}, \qquad \Lambda_{\text{eff}} \sim t^{-2}. \tag{5.33}$$

i.e. Λ_{eff} decreases much faster than the energy density of relativistic matter. Hence, the universe will soon enter a radiation dominated era characterized by a very low value of $\mathcal{R}$ (and consequently Λ_{eff}).

We next investigate the behaviour of the modified Friedmann equation (5.28). Λ_{eff} at this stage of the evolution will be negligible compared to $\kappa\rho$ and $A \sim 0$ to a very good approximation. On the other hand, $f' = 3\mathcal{R}^2/\beta^2 + 1 + \epsilon^2/(3\mathcal{R}^2)$ will tend to 1 provided that ϵ is small enough. Therefore, eq. (5.28) reduces to

$$H^2 \sim \kappa^2\rho, \tag{5.34}$$

where, as before, $\rho = \rho_r + \rho_m$. Eq. (5.34) resembles standard cosmology. It is reasonable, therefore to assume that everything can continue as expected, *i.e.* radiation dominated era, Big Bang Nucleosynthesis (BBN), and matter dominated era.

Finally, we consider what will happen at late times. At some point we expect matter to become subdominant with respect to Λ_{eff} due to the increase of the scale factor. Note that Λ_{eff} asymptotically reaches $2\epsilon/3$ as matter dilutes. Thus at late times we can arrive at the picture where $\rho \sim 0$ and the universe will therefore again enter a de Sitter phase of accelerated expansion qualitatively similar to that indicated by current observations.

What needs to be stressed here is that the analysis of this section gives a very rough description of the cosmological dynamics of Palatini $f(R)$ gravity. Additionally, the model used is chosen *ad hoc*, just because it leads to some interesting phenomenology from a demonstrative point of view. In the next section, we will proceed to check the validity of what has been presented here in more detail and for more generic choices of f.

5.3.3 Constraining positive and negative powers of $\mathcal{R}$

Even though a simple model like the one described by eq. (5.20) was helpful for understanding the basic features of cosmology in Palatini $f(R)$ gravity, one would like to consider more generic choices for f. At the same time it is important to go beyond the qualitative results and use the numerous observations to get quantitative ones. Such a study was performed in [286]. Assuming that the gravitational action includes, besides the standard linear term, a term inversely proportional to $\mathcal{R}$, the authors used four different sets of cosmological data to constrain it. These are the Supernova Type Ia gold set [40], the CMBR shift parameter [287], the baryon oscillation length scale [42] and the linear growth factor at the 2dF Galaxy Redshift Survey effective redshift [288, 289]. However, as stated in the conclusions of [286], the restricted form of $f(\mathcal{R})$, including only a term inversely proportional to $\mathcal{R}$, prevents the study from being exhaustive.

In [283] the cosmological behaviour of more general models of Palatini $f(R)$ gravity was studied and the results of [286] where generalised. Following [283] we consider here a general model. We leave the function f unspecified and try to derive results independent of its form as long as this is possible. Since such a general study is a quite tedious analytical task given the complexity of the functions involved and the non-linearity of the equations, we also adopt the following representation for f, which is suitable for our purposes, whenever needed:

$$f(\mathcal{R}) = \frac{1}{\epsilon_1^{d-1}}\mathcal{R}^d + \mathcal{R} - \frac{\epsilon_2^{b+1}}{\mathcal{R}^b}, \tag{5.35}$$

with $\epsilon_1, \epsilon_2 > 0$, $d > 1$ and $b \geq 0$; $b = 0$ corresponds to the ΛCDM model when $\epsilon_1 \to \infty$. The dimensions of ϵ_1 and ϵ_2 are $(\text{eV})^2$. Part of our task in this section will be to constrain the value of ϵ_1, given that for the value of ϵ_2 no extended discussion is really necessary. In fact, in order for a model to be able to lead to late-time accelerated expansion consistent with the current observations, ϵ_2 should by roughly of the order of 10^{-67} $(\text{eV})^2$ [279].

First of all, let us see how a model with a general function f would behave in vacuum, or whenever $T = 0$ (radiation, *etc.*). If we define

$$\mathcal{F}(\mathcal{R}) \equiv f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}), \tag{5.36}$$

then eq. (3.31) becomes

$$\mathcal{F}(\mathcal{R}) = \kappa T, \tag{5.37}$$

and for $T = 0$,

$$\mathcal{F}(\mathcal{R}) = 0. \tag{5.38}$$

Eq. (5.38) is an algebraic equation which, in general, will have a number of roots, $\mathcal{R}_n$. Our notation implies that $\mathcal{R}$ is positive in the presence of ordinary matter and so here we will consider the positive solutions (*i.e.* the positive roots). Each of these solutions corresponds to a de Sitter expansion, since $\mathcal{R}$ is constant. If one wants to explain the late-time accelerated expansion one of these solutions, say $\mathcal{R}_2$, will have to be small. If in addition to this we also want our model to drive an early-time inflation, there should be a second solution, $\mathcal{R}_1$ corresponding to a larger value of $\mathcal{R}$.

For example, introducing in eq. (3.31) the ansatz given for f in eq. (5.35), one gets

$$\frac{d-2}{\epsilon_1^{d-1}}\mathcal{R}^{d+b} - \mathcal{R}^{b+1} + (b+2)\epsilon_2^{b+1} = 0. \tag{5.39}$$

If $\epsilon_1 \gg \epsilon_2$ and $d > 2$ then this equation has two obvious solutions

$$\mathcal{R}_1 \sim \epsilon_1, \qquad \mathcal{R}_2 \sim \epsilon_2. \tag{5.40}$$

These solutions can act as seeds for a de Sitter expansion, since the expansion rate of the de Sitter universe scales like the square root of the scalar curvature. Notice that the rest of the solutions of eq. (5.38), for $\mathcal{R} < \mathcal{R}_1$ and $\mathcal{R} > \mathcal{R}_2$ will not be relevant here. During the evolution we do not expect, as will become even more obvious later on, that $\mathcal{R}$ will exceed $\mathcal{R}_2$ or become smaller than $\mathcal{R}_1$.

Before going further, it is worth mentioning that one can use eq. (3.31) together with the conservation of energy to express $\dot{\mathcal{R}}$ as a function of $\mathcal{R}$:

$$\dot{\mathcal{R}} = -\frac{3H(\mathcal{R}f' - 2f)}{\mathcal{R}f'' - f'}. \tag{5.41}$$

Using eq. (5.41) to re-express $\dot{f}'(= f''\dot{\mathcal{R}})$ and assuming that the universe is filled with dust ($p = 0$) and radiation ($p = \rho/3$), eq. (5.19) gives, after some mathematical manipulation,

$$H^2 = \frac{1}{6f'}\frac{2\kappa\rho + \mathcal{R}f' - f}{\left(1 - \frac{3}{2}\frac{f''(\mathcal{R}f'-2f)}{f'(\mathcal{R}f''-f')}\right)^2}, \tag{5.42}$$

where again $\rho = \rho_r + \rho_m$.

Early times

Following the lines of Section 5.3.2, we can make the following observation. Since we expect the matter to be fully relativistic at very early times, in this regime $T = 0$ and consequently $\mathcal{R}$ is constant. This implies that the second term on the left hand side of eq. (5.19) vanishes. Additionally, conservation of energy requires that the first term on the right hand side of the same equation scales like $a(t)^{-4}$, which means that the second term on the same side, depending only on the constant curvature, will soon dominate if $f(\mathcal{R})$ is large enough for this to happen before matter becomes non relativistic. $\mathcal{R}$ can either be equal to $\mathcal{R}_1$ or to $\mathcal{R}_2$. Since we want $\mathcal{R}_2$ to be the value that will provide the late time acceleration, f should be chosen in such a way that $f(\mathcal{R}_2)$ will become dominant only at late times when the energy densities of both matter and radiation have dropped significantly. Therefore, if we want to have an early inflationary era we have to choose the larger solution $\mathcal{R}_1$, and f should have a form that allows $f(\mathcal{R}_1)$ to dominate with respect to radiation at very early times. The Hubble parameter will then be given by

$$H \sim \sqrt{\frac{f(\mathcal{R}_1)}{6f'(\mathcal{R}_1)}}, \tag{5.43}$$

As an example, we can use the ansatz given in eq. (5.35). The modified Friedmann equation is then

$$H \sim \sqrt{\frac{\epsilon_1}{3(d+1)}}, \tag{5.44}$$

and the universe undergoes a de Sitter expansion which can account for the early-time inflation.

Sooner or later this inflationary expansion will lead to a decrease of the temperature and some portion of the matter will become non relativistic. This straightforwardly implies that $\mathcal{R}$ will stop being constant and will have to evolve. Recall that in Palatini $f(R)$ gravity, the field equation for the connection implies the existence of a metric $h_{\mu\nu}$, which is conformal to $g_{\mu\nu}$ (see Sections 3.4.3 and eq. (3.33)). Then $f'(\mathcal{R})$ plays the role of the conformal factor relating these two metrics, and therefore we do not consider sign changes to be feasible throughout the evolution of the universe. We also know that in a certain range of values of $\mathcal{R}$ it should be close to one. This is the case because there should be a range of values of $\mathcal{R}$, for which $f(\mathcal{R})$ behaves essentially like $\mathcal{R}$, *i.e.* our theory should be well approximated by standard General Relativity in order for us to be able to derive the correct Newtonian limit (see [290] and Section 6.2). Together with $T \leq 0$, the above implies the following:

$$f'(\mathcal{R}) > 0, \quad \mathcal{F} < 0, \qquad \forall \quad \mathcal{R}_2 < \mathcal{R} < \mathcal{R}_1. \tag{5.45}$$

Since $\mathcal{F}$ is a continuous function keeping the same sign in this interval and $\mathcal{F}(\mathcal{R}_1) = \mathcal{F}(\mathcal{R}_2) = 0$, there should be a value for $\mathcal{R}$, say $\mathcal{R}_e$, where $\mathcal{F}'(\mathcal{R}_e)$=0, i.e. an extremum. Eq. (5.37) implies that the time evolution of $\mathcal{R}$ is given by

$$\dot{\mathcal{R}} = \kappa\dot{T}/\mathcal{F}'(\mathcal{R}). \tag{5.46}$$

Differentiating eq. (5.36), we get

$$\mathcal{F}' = f''\mathcal{R} - f'. \tag{5.47}$$

Using the fact that $\dot{f}' = f''\dot{\mathcal{R}}$, and using eq. (5.47) to express f'' in terms of $\mathcal{F}'$, f' and $\mathcal{R}$, one can easily show that

$$\frac{\dot{f}'}{f'} = \frac{\mathcal{F}' + f'}{\mathcal{R}f'}\dot{\mathcal{R}}. \tag{5.48}$$

The constraints given in eq. (5.45) imply that for $\mathcal{R}_e < \mathcal{R} < \mathcal{R}_1$, $\mathcal{F}' > 0$. An easy way to understand this is to remember that $\mathcal{F}$ is negative in that interval but zero at $\mathcal{R} = \mathcal{R}_1$ and so it should be an increasing function (see also fig. 5.1). Since, f' and $\mathcal{R}$ are also positive, then what determines the sign of $\dot{f}'/f'$ in the neighbourhood of $\mathcal{R}_e$ is the sign of $\dot{\mathcal{R}}$.

Let us see what will happen if we require $\mathcal{R}$ to decrease, i.e. $\dot{\mathcal{R}} < 0$. Eq. (5.41) implies that as $\mathcal{R} \to \mathcal{R}_e$, $\dot{\mathcal{R}} \to -\infty$ if $\dot{T} \neq 0$, since $\mathcal{F}'(\mathcal{R}_e) = 0$. Therefore, $\dot{f}'/f' \to -\infty$ and using eq. (5.19) we can infer that $H \to \infty$. Physically, the

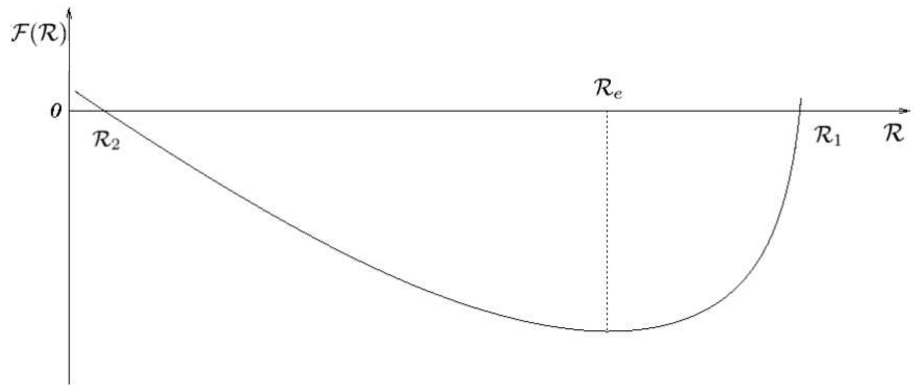

Figure 5.1: The behaviour of a general function $\mathcal{F}(\mathcal{R})$ over the interval $\mathcal{R}_1 > \mathcal{R} > \mathcal{R}_2$. $\mathcal{R}_e$ denotes the value of $\mathcal{R}$ where $\mathcal{F}$ has a minimum. From this graph one can easily see that $\mathcal{F}'$ is positive when $\mathcal{R}_e < \mathcal{R} < \mathcal{R}_1$ and negative when $\mathcal{R}_2 < \mathcal{R} < \mathcal{R}_e$.

above implies the following: $\mathcal{R}$ has no way to decrease to a value less than $\mathcal{R}_e$ without giving the universe an infinite expansion. In practice, any attempt for $\mathcal{R}$ to approach $\mathcal{R}_e$ would lead to a dramatically rapid expansion until non-relativistic matter fully dilutes and $\mathcal{R}$ settles back to $\mathcal{R}_1$. Thus, once the curvature terms in the modified Friedmann equation dominate the evolution, there is no turning back to matter domination through a continuous process. The two vacuum solutions $\mathcal{R}_1$ and $\mathcal{R}_2$ seem to be somehow disconnected in the evolution and $\mathcal{R}$ has to remain in the region close to only one of them. This is a general statement independent of the form of matter that is present since T was left unspecified in its derivation. So, even though, as shown in [282], including positive powers of $\mathcal{R}$ in the action can lead to early-time inflation, there seems to be no graceful exit from it. The only alternative left would be to consider that due to some other physical and non-classical process, the equation presented here ceases to be valid for some time interval, which, however seems highly implausible.

Since it seems impossible to provide an exit from this gravity driven inflation it seems reasonable to check whether we can at least totally avoid it. If we choose as our initial solution $\mathcal{R}_2$ instead of $\mathcal{R}_1$ then the curvature terms will not dominate as long as $\mathcal{R}$ is constant. However, there is still one subtle point. At some stage during the evolution the energy density of non-relativistic matter will have to rise sooner or later, forcing $\mathcal{R}$ to change its value. It is also reasonable to assume that if inflation is not driven by curvature, we will have to adopt a more standard approach to guarantee that it will happen, like an inflaton field. It is obvious, however, keeping in mind the previous discussion, that one would want $\mathcal{R}$ to be always less than $\mathcal{R}_e$ and this will impose a constraint which will depend on the functional form of f. For example, if one assumes that f is described by the ansatz given in eq. (5.35) then, considering ordinary matter, $\mathcal{R} < \mathcal{R}_e$ at all times implies that $\epsilon_1 \gg \kappa\rho_m$ at all times. Let us also consider the case of a slow-rolling inflaton field, ϕ. Then, if we denote its energy density by ρ_ϕ and its pressure by p_ϕ we have, as usual,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{5.49}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{5.50}$$

where $V(\phi)$ is the scalar field potential. During the period when ϕ dominates the evolution, $T = \dot{\phi}^2 - 4V(\phi)$ and since slow-roll implies that $\dot{\phi}^2 \ll V(\phi)$, $T \approx -4V(\phi)$. Therefore, if we want $\mathcal{R} < \mathcal{R}_e$, so that inflation proceeds as usual, then $\epsilon_1 \gg V(\phi)$ at all times.

If $\mathcal{R}$ is less than $\mathcal{R}_e$ for all values of T then it is easy to verify that everything will evolve naturally after the end of inflation. For $\mathcal{R}_2 < \mathcal{R} < \mathcal{R}_e$, $\mathcal{F}' < 0$ and as non-relativistic matter dilutes, $\dot{\rho}_m < 0$, and so from eq. (5.46) we see that $\dot{\mathcal{R}} < 0$ so that $\mathcal{R}$ will have to decrease to reach the value $\mathcal{R}_2$ asymptotically.

The above discussion is not relevant, of course, if the only term with a positive power present in the action, besides $\mathcal{R}$ itself, is $\mathcal{R}^2$. In this case, due to the form of $\mathcal{F}$, this term does not appear in eq. (5.37). This specific case has been studied in [285]. One thing that is worth commenting on, before closing this discussion, is the following. The constraint $f' > 0$ (see eq. (5.45)), which is implied by the fact that f' plays the role of the conformal factor relating the metrics $g_{\mu\nu}$ and $h_{\mu\nu}$, can, depending on the form of f, impose a further constraint on the value of the constants in front of the positive power terms. In [284] inflation driven by an inflaton field was studied in the presence of an $\mathcal{R}^2$ term. The authors derived a constraint for the constant appearing in front of the $\mathcal{R}^2$ term in the action by requiring that the square of the Hubble parameter should be positive during the kinetic dominated phase. This constraint is exactly what would one derive by requiring f' to be always positive.

Big Bang Nucleosynthesis

Let us now turn our attention to the next cosmological era, radiation domination and Big Bang Nucleosynthesis (BBN). Current observations indicate that the standard cosmological model can fit the data related to the primordial abundances of light elements. On the other hand, how a modified gravity model like the one discussed here would fit those data has not been yet worked out. However, there is little room for modifying the behaviour of the Friedmann equation during BBN and it seems reasonable to ask that the model under investigation should resemble standard cosmology during these cosmological eras [48]. This implies that eq. (5.42) should be similar to the standard Friedmann equation

$$H^2 = \frac{1}{3}\kappa\rho. \tag{5.51}$$

By comparing eqs. (5.42) and (5.51), one sees that during BBN

$$f' \sim 1 \tag{5.52}$$

$$1-\frac{3}{2}\frac{f''(\mathcal{R}f'-2f)}{f'(\mathcal{R}f''-f')}\sim 1 \tag{5.53}$$

$$\mathcal{R}f'-3f\sim 0 \tag{5.54}$$

To make the picture clearer, we give the explicit expressions for f' and f'' when f is given by eq. (5.35):

$$f' = d\frac{\mathcal{R}^{d-1}}{\epsilon_1^{d-1}}+1+b\frac{\epsilon_2^{b+1}}{\mathcal{R}^{b+1}}, \tag{5.55}$$

$$f'' = d(d-1)\frac{\mathcal{R}^{d-2}}{\epsilon_1^{d-1}}-b(b+1)\frac{\epsilon_2^{b+1}}{\mathcal{R}^{b+2}}. \tag{5.56}$$

Let us for the moment assume that the term inversely proportional to $\mathcal{R}$ is not present. In order for condition (5.52) to be fulfilled $\epsilon_1 \gg \mathcal{R}_{BBN}$. This is the natural constraint on the value of ϵ_1 imposed when one asks for the model to have almost identical behaviour to the standard one during BBN. Once ϵ_1 is chosen to have a large enough value, all three constraints (5.52), (5.53) and (5.54) are easily fulfilled and the modified Friedmann equation (5.42) becomes identical to the standard one, eq. (5.51), for the relevant values of $\mathcal{R}$. The above constraint can be viewed as a sufficient constraint for the model to be viable but not as a necessary one. However, one could also claim that, even if the modifications in the Friedmann equation do not necessarily have to be negligible, they should at least lead to second order corrections and not affect the leading order. This implies that ϵ_1 should definitely be larger than $\mathcal{R}_{BBN}$.

We have, however, neglected the presence of the term inversely proportional to $\mathcal{R}$. In the absence of positive powers, this term should be negligible during BBN since $\mathcal{R}_{BBN}$ is much larger than ϵ_2. This picture may change if we consider the full version of the model. Eq. (3.104) can take the following form

$$\mathcal{R}f'-2f=-\kappa\rho_m^0(1+z)^3, \tag{5.57}$$

where ρ_m^0 is the present value of the energy density of non-relativistic matter and z is the redshift. We have assumed here that $a_0=1$. Using eq. (5.57) one can derive how $\mathcal{R}$ will scale with the redshift. If $\epsilon_1 \gg \mathcal{R}_{BBN}$, then for all of the evolution of the universe after BBN, $\mathcal{R}$ scales almost like $(1+z)^3$. This indicates that, since BBN takes place at a very high redshift, $\mathcal{R}_{BBN}$ is indeed much larger than ϵ_2. If, however, one assumes that ϵ_1 is large enough to alter the behaviour of eq. (5.57), then $\mathcal{R}$ will have a milder scaling with the redshift, meaning that $\mathcal{R}_{BBN}$ can get very close to ϵ_2. Then the three constraints (5.52), (5.53) and (5.54) might not be fulfilled not only due to the presence of the term with the positive power of $\mathcal{R}$ greater than 1 but also because of the term with the negative power. This will be a secondary effect related to the term with a positive power greater than 1, as shown earlier. It will be avoided if again $\epsilon_1 \gg \mathcal{R}_{BBN}$ and will be subdominant if ϵ_1 is just smaller than $\mathcal{R}_{BBN}$. Unfortunately, since the value of $\mathcal{R}_{BBN}$ is very model dependent, it is difficult to turn this constraint into a numerical one.

5.3.4 Late times

Now let us check the behaviour of the modified Friedmann equation at late times. The scalar curvature $\mathcal{R}$ decreases with time to reach a value close to ϵ_2. Therefore the conditions (5.52), (5.53) and (5.54) will at some point cease to hold because of the term involving the negative power of $\mathcal{R}$. Any contribution of the term involving the positive power of $\mathcal{R}$ greater than 1 will be negligible for two reasons. Firstly, since the value of ϵ_1 should be such that these terms are already negligible during BBN, it is safe to assume that they will remain so throughout the rest of the evolution of the universe. The same results can be inferred by using the constraints derived in Section 5.3.3.

As we mentioned earlier, satisfying the present bounds related to the primordial abundancies of light elements according to BBN is straightforward if the modified Friedmann equation of the model does not deviate significantly from the standard one during the BBN epoch. This was the condition which we imposed to derive the constraints just presented. However, one cannot completely discard the possibility that a modified gravity model whose modified Friedmann equation does deviate slightly but significantly from the standard one during BBN could still be viable: it is not yet clear how a modification of gravity will then influence the light element abundancies and BBN as a whole. Under this perspective, the constraints presented here are suffiecient for a viable model: if the modified Friedmann equation reduces to the standard one with high precision during BBN, one is assured that the current bounds on light element abundancies will be both unaffected and satisfied. However, these constraints cannot yet be considered as being an absolutely necessary condition for the model to be viable, until a more detailed study of the effect of a modification of gravity on BBN is performed. In any case, even if one assumes that there is some slight contribution from the terms being discussed in the modified Friedmann equation during BBN, such a contribution should become weaker at later times.

The constraints coming from the early-time behaviour are necessary but it is difficult to turn them into numerical ones. At the same time one can always claim that the early time evolution of the universe is not very well established and there might still be room for new physics there affecting these constraints. However, let us anticipate that the Newtonian limit of the theory will also provide constraints which will turn out to be in agreement with those derived here and actually more stringent (see Section 6.2). The range of values of $\mathcal{R}$ which is of interest for late-time observations is between the value of $\mathcal{R}$ at decoupling $\mathcal{R}_{dec}$ and the current value of $\mathcal{R}$, $\mathcal{R}_0$. For these values, it is safe to consider that [283]

$$f \sim \mathcal{R} - \frac{\epsilon_2^{b+1}}{\mathcal{R}^b}, \tag{5.58}$$

$$f' \sim 1 + b\frac{\epsilon_2^{b+1}}{\mathcal{R}^{b+1}}, \tag{5.59}$$

$$f'' \sim -b(b+1)\frac{\epsilon_2^{b+1}}{\mathcal{R}^{b+2}}, \tag{5.60}$$

with extremely high accuracy for all times after decoupling. It is easy to see that the modified Friedmann equation of the model described in (5.35) will be identical at late times to that of a model with no positive powers of the curvature greater than 1 ($\epsilon_1 \to \infty$).

In [286] the authors consider f to be of the form

$$f(\mathcal{R}) = \mathcal{R}\left(1+\alpha\left(\frac{\mathcal{R}}{H_0^2}\right)^{\beta-1}\right), \tag{5.61}$$

where α and β are dimensionless parameters, with $\beta < 1$ (note that in our notation $\mathcal{R}$ is positive). This representation of the function f is very useful when one wants to constrain some dimensionless parameter. Comparing it with our ansatz, eq. (5.35), we get $d = \delta + 1$, $b = -\beta$ and $\epsilon_1 \to \infty$, since in eq. (5.61) there is no positive power of $\mathcal{R}$ greater than 1. In order to constrain the values of α and β they use a rather extensive list of cosmological observations. The first quantity which they consider is the CMBR shift parameter [287, 291, 292] which in a spatially flat universe is given by

$$\mathscr{R} = \sqrt{\Omega_m H_0^2}\int_0^{z_{dec}} \frac{d\tilde{z}}{H(\tilde{z})}, \tag{5.62}$$

where z_{dec} is the redshift at decoupling and $\Omega_m \equiv \kappa\rho_m^0/(3H_0^2)$. When expressed in terms of the scalar curvature, eq. (5.62) becomes

$$\begin{aligned}
\mathscr{R} &= \sqrt{\Omega_m H_0^2}\int_0^{z_{dec}} \frac{dz}{H(z)} \\
&= \sqrt{\Omega_m H_0^2}\int_{\mathcal{R}_{dec}}^{\mathcal{R}_0} \frac{a'(\mathcal{R})}{a(\mathcal{R})^2}\frac{d\mathcal{R}}{H(\mathcal{R})} \\
&= \frac{1}{3^{4/3}}\left(\Omega_m H_0^2\right)^{1/6}\int_{\mathcal{R}_0}^{\mathcal{R}_{dec}} \frac{\mathcal{R}f''-f'}{(\mathcal{R}f'-2f)^{2/3}}\frac{d\mathcal{R}}{H(\mathcal{R})}.
\end{aligned} \tag{5.63}$$

Using the values for z_{dec} and $\mathscr{R}$ obtained with WMAP [43], namely $z_{dec} = 1088^{+1}_{-2}$ and $\mathscr{R} = 1.716 \pm 0.062$, they find that the best fit model is $(\alpha,\beta) = (-8.4, -0.27)$. The “Gold data set” of Supernovae is also used [40]. What is important for this analysis is the expression for the luminosity distance, which in terms of $\mathcal{R}$ is

$$\begin{aligned}
D_L(z) &= (1+z)\int_0^z \frac{d\tilde{z}}{H(\tilde{z})} \\
&= \sqrt{\Omega_m H_0^2}\frac{1}{a(\mathcal{R})}\int_{\mathcal{R}}^{\mathcal{R}_0} \frac{a'(\mathcal{R})}{a(\mathcal{R})^2} \\
&= \frac{1}{3}\sqrt{\Omega_m H_0^2}\left(\mathcal{R}f'-2f\right)^{1/3}\times \\
&\quad \int_{\mathcal{R}_0}^{\mathcal{R}_{dec}} \frac{\mathcal{R}f''-f'}{(\mathcal{R}f'-2f)^{2/3}}\frac{d\mathcal{R}}{H(\mathcal{R})}.
\end{aligned} \tag{5.64}$$

Marginalizing over the Hubble parameter h, the authors again constrain α and β and the best fit model is $(\alpha, \beta) = (-10.0, -0.51)$. Another independent observation which they use is that of the imprint of the primordial baryon-photon acoustic oscillations on the matter power spectrum. The dimensionless quantity A [293, 294, 295, 296, 297],

$$A = \sqrt{\Omega_m} E(z_1)^{-1/3} \left[\frac{1}{z_1} \int_0^{z_1} \frac{dz}{E(z)} \right]^{2/3}, \tag{5.65}$$

where $E(z) = H(z)/H_0$, can act as a "standard ruler". The data from the Sloan Digital Sky Survey [42] provide a value for A, namely

$$A = D_v(z = 0.35) \frac{\sqrt{\Omega_m H_0^2}}{0.35c} = 0.469 \pm 0.017, \tag{5.66}$$

where

$$D_v(z) = \left[D_M(z)^2 \frac{cz}{H(z)} \right]^{1/3}, \tag{5.67}$$

and $D_M(z)$ is the comoving angular diameter distance. The best fit model using this value is $(\alpha, \beta) = (-1.1, 0.57)$. Finally, in [286] these three sets of data are combined to give a best fit for $(\alpha, \beta) = (-3.6, 0.09)$.

The above observations are potentially very useful, of course, in studying the viability of a model like (5.61). Two comments are due:

Firstly, as shown here, the modified Friedmann equation of a more general model like (5.35), which also includes positive powers of $\mathcal{R}$ greater than 1, is effectively identical to that of (5.61) at late times. Therefore, it is expected that the results of [286] will remain unaffected by the inclusion of positive powers of the scalar curvature greater than 1, since these terms have to satisfy the constraints derived in this section. Additionally, one can conclude that the late evolution of the universe is *not* affected by the positive powers of the scalar curvature greater than 1 present in the action. This can be rephrased in two interesting ways: *The results of observational tests relevant to the late-time evolution of the universe are insensitive to the inclusion of additional positive powers of R* or *observational tests relevant to the late-time evolution cannot constrain the presence of additional positive powers of R in the gravitational action*. The second expression implies that such tests are not sufficient to judge the overall form of the gravitational action.

Secondly, it is worth commenting on the result of [286]. The best fit model for the combination of the different data sets suggests that their exponent β is equal 0.09 (see eq. (5.61)) and therefore favours the ΛCDM model, being well within the 1σ contour. However, one gets different values for β when the different data sets are considered individually. For the SNe data the best fit model has $\beta = -0.51$ and the baryon oscillations $\beta = 0.57$, both disfavouring the ΛCDM model, but also being mutually contradictory. The CMBR shift parameter gives $\beta = -0.27$ which again is significantly different from the other two values. Of course, one might expect that the combination of the data will give the most trustworthy result. However, this is not necessarily true, since one could regard the very wide discrepancies in the value of β coming from different observations as an indication that more accurate data are needed to derive any safe conclusion. It would also be interesting to study to what extent a model with $\beta = -1$, which is the model commonly used in the literature, can individually fit the current data and to compare the results with other models built to explain the current accelerated expansion, such as quintessence, scalar-tensor theories, *etc*. One should bear in mind that the ΛCDM model has always been the best fit so far. However, the motivation for creating alternative models does not come from observations but from our inability to solve the theoretical problems that arise if one adopts the standard picture (coincidence problem, *etc*.).

Before closing this section, let us briefly mention the fourth scheme used in [286] in order to obtain constraints: large scale structure and growth of perturbations. The results derived there through this scheme do not actually improve the constraints obtained with the three schemes already mentioned. As the authors of [286] correctly state, a more detailed analysis should be performed along the lines of [298], in which a more standard linearized perturbation analysis is performed. In fact there are a a number of works which place constraints on Palatini $f(R)$ gravity using perturbation analysis, the matter power spectrum, large scale structure *etc*. The general conclusion is that the models which are very close to the ΛCDM model are the only ones that could satisfy the relevant constraints. It is, therefore, very difficult to construct viable alternative models. Simple models such as those used above are very much disfavoured by the observations. We address the reader to the literature for more details [298, 299, 300, 301, 302, 303].

5.4 Metric-affine $f(R)$ gravity and cosmology

Instead of presenting here a detailed study of the cosmological aspects of metric-affine $f(R)$ theories of gravity, we wish to remind the reader that the main difference from Palatini $f(R)$ gravity is the following: In the Palatini formalism, the matter action is assumed to be independent of the connections whereas in the metric-affine formalism no such assumption is made. More precisely, as we have argued, such an *a priori* assumption is against the spirit of metric-affine gravity. However, as was explained in Section 3.6, whenever the only matter fields considered are perfect fluids, electromagnetic fields or scalar fields, metric-affine $f(R)$ gravity does reduce to Palatini $f(R)$ gravity, as the matter action of those fields are independent of the connection without this having to be imposed as an external assumption. In cosmology, these are indeed the only fields considered. This implies that the main features of cosmology in metric-affine $f(R)$ gravity will not be different from those of Palatini $f(R)$ gravity.

Therefore, no detailed study is needed and the reader may refer to the previous section. However, what is mentioned here has to be approached with caution. In

Section 3.6.4 we have already commented on the difficulties that arise when one attempts to adopt macroscopic descriptions of matter, such as perfect fluids, when spin and torsion are taken into account. The definition of a perfect fluid might have to be generalised and the details of cosmological evolution could be affected (see 3.6.4 for details). A more detailed analysis of cosmology in metric-affine $f(R)$ gravity is still pending.

5.5 Gauss–Bonnet gravity

5.5.1 Generalised Friedmann equations

In Section 3.7 we introduced Gauss–Bonnet gravity, derived the field equations and studied some of their characteristics. We have also made a reference to the motivation from heterotic String Theory for the study of such actions. As is the case for many alternative theories of gravity, in Gauss–Bonnet gravity there is also strong motivation from cosmology. More specifically there have been works showing that Gauss–Bonnet gravity can address the dark energy problem without the need for any exotic matter components [234, 304, 305]. Additionally, such a theory can have other interesting characteristics in relation to cosmological phenomenology including early time inflation [306, 307, 308] and avoidance of future and past singularities [231, 234, 309].

Let us briefly derive the modified Friedmann equations for Gauss–Bonnet gravity (see also [234, 306]). Recall that the action of the theory is given by eq. (3.138) and the field equations for the metric and the scalar field are given by eqs. (3.149) and (3.150) respectively. Using the flat ($k = 0$) form of the FLRW metric, eq. (5.1), and assuming that the scalar only depends on time, one gets from the time-time and space-space components of eq. (3.149) respectively (after some manipulations involving also the definitions for the quantities $T_\phi^{\mu\nu}$ and $T_f^{\mu\nu}$)

$$H^2 = \frac{1}{3}\kappa\left(\rho + \frac{\lambda}{2}\dot{\phi}^2 + V(\phi) - 24\dot{\phi}f'(\phi)H^3\right), \tag{5.68}$$

$$\left(\frac{\ddot{a}}{a}\right) \equiv \dot{H} + H^2 = -\frac{\kappa}{6}(\rho + 3p) - \frac{\kappa}{6}\Big(2\lambda\dot{\phi}^2 - 2V(\phi) + +24H^3\dot{\phi}f'(\phi) + 24\frac{\partial}{\partial t}\left(H^2\dot{f}\right)\Big). \tag{5.69}$$

The Gauss–Bonnet invariant can easily be expressed in terms of H and its time derivative as

$$\mathcal{G} = 24H^2\left(\dot{H} + H^2\right). \tag{5.70}$$

The equation of motion for the scalar takes the form

$$\lambda\left(\ddot{\phi} + 3H\dot{\phi}\right) + V'(\phi) - 24f'(\phi)H^2\left(\dot{H} + H^2\right) = 0. \tag{5.71}$$

Eqs. (5.68), (5.69) and (5.71) govern the cosmological dynamics. Note that the potential $V(\phi)$ and the coupling between the scalar and the Gauss–Bonnet term $f(\phi)$ are left unspecified at this stage. Besides providing initial conditions, one needs to choose the functional form of $V(\phi)$ and $f(\phi)$ in order to solve the equations.

5.5.2 Gauss–Bonnet gravity as dark energy

Let us see how Gauss–Bonnet gravity can account for dark energy. We will proceed as in Section 5.2.2 and attempt to bring the modified Friedmann equations into the form for standard matter plus a dark energy component. It is not difficult to see that if one defines

$$\rho_{de} = \frac{\lambda}{2}\dot{\phi}^2 + V(\phi) - 24\dot{\phi}f'(\phi)H^3, \tag{5.72}$$

$$p_{de} = \frac{\lambda}{2}\dot{\phi}^2 - V(\phi) + 8\frac{\partial}{\partial t}\left(H^2\dot{f}\right) + 16H^3\dot{\phi}f'(\phi), \tag{5.73}$$

then eqs. (5.68) and (5.69) can be written in the form

$$H^2 = \frac{1}{3}\kappa\left(\rho_{\rm tot}\right), \tag{5.74}$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{\kappa}{6}\left(\rho_{\rm tot} + 3p_{\rm tot}\right), \tag{5.75}$$

where

$$\rho_{\rm tot} = \rho + \rho_{de}, \tag{5.76}$$

$$p_{\rm tot} = p + p_{de}. \tag{5.77}$$

Clearly, eqs. (5.74) and (5.75) are formally the same as those which one would derive in General Relativity once the presence of a dark energy component is assumed. As usual, we can define an effective equation of state parameter $w_{de} \equiv p_{de}/\rho_{de}$. Recall that, if we assume that the scalar field dominates the evolution, then

$$w_{de} = \frac{2q-1}{3}, \tag{5.78}$$

where $q \equiv -a\ddot{a}/\dot{a}^2$ is the deceleration parameter.

Since the role of dark energy is to provide the late-time accelerated expansion, let us focus on how this can be achieved within the framework of Gauss–Bonnet gravity. Due to the resemblance of the theory to General Relativity with a minimally coupled scalar field, we know that if $\dot{f} = 0$ (or $f = 0$), the minimal condition for having acceleration, $\rho_{de}+3p_{de} < 0$, is satisfied when $V(\phi) > \lambda\dot{\phi}^2$. However, in our case $\dot{f} \neq 0$ and its sign is important for determining the behaviour of the scale factor. One can show that when $\dot{f} < 0$ (which will generically hold for a canonical scalar, $\lambda > 0$), the acceleration condition is indeed satisfied if $V(\phi) > \lambda\dot{\phi}^2$.

Note that the role of f can actually be more active and in principle it can even lead to an era of acceleration even if there is no potential term. The effective equation of state depends on both f and V. Choosing these functions appropriately, one can control how much effect each of them will have on the cosmic evolution. Common well motivated choices for f and V are exponentials, *i.e.* $V = V_0 e^{-a\kappa\phi}$ and $f = f_0 e^{b\kappa\phi}$ where a, b and f_0 are of order unity while V_0 is as small as the energy density of the cosmological constant in order to guarantee that the theory will fit observations related to the late-time cosmological expansion (*e.g.* [234]). In such models, the acceleration is mainly due to the potential terms.

In the same way that Gauss–Bonnet gravity can lead to a late-time acceleration, it can also lead to an early-time inflationary period. As already mentioned, the theory is very similar to General Relativity plus a minimally coupled scalar field, such as the inflaton field usually used to drive inflation. Of course, the coupling of the scalar field to the Gauss–Bonnet term does lead to qualitative differences. For instance, the slow roll variables should be redefined, the generation of perturbations will differ *etc*. [306]. We will not go further in examining specific models or analysing the cosmological features of Gauss–Bonnet gravity. We refer the reader to the relevant literature instead [234, 305, 306, 307, 308].

However, before closing it is important to refer to the confrontation with cosmological observations. Current literature on the subject includes studies of the impact of the Gauss–Bonnet coupling in relation to constraints coming from the Cosmic Microwave Background, galaxy distributions, large scale structure and supernovae as well as from studies of cosmological perturbations related to the Cosmic Microwave Background and the matter power spectrum [310, 311]. Remarkably, the theory seems to be in good agreement with the data as far as these studies are concerned. However, as reported in [311], constraints from baryon oscillations and nucleosynthesis appear to disfavour simple models. A scheme with which one can reconstruct the action from the expansion history has been developed in [312]. Finally, note that more general actions which include couplings between the dilaton and the Gauss–Bonnet term have also been studied (*e.g.* [307, 308]).

Chapter 6

Weak and strong gravity regimes in Modified gravity

6.1 Introduction

Up to now we have studied the theoretical basis of several theories of gravity and have examined their cosmological features. However, we have not yet referred to a number of other important issues for any theory of gravitation.

To begin with, we have not considered the Newtonian limit of any of the theories mentioned. This is obviously a crucial issue, since any theory of gravity should reduce to Newtonian gravity at a suitable limit. The validity of Newton's theory in a weak gravity regime and at certain length scales is hardly questionable. Additionally, deviations from it as gravity becomes stronger are well constrained by Solar System tests and the post-Newtonian expansion [139] is a powerful tool for judging the viability of a theory. Constraints coming from Cosmology are important but in most cases constraints coming from Solar System tests with the use of the post-Newtonian approximation are more stringent. One should take into account that alternative theories of gravity motivated by cosmological problems are tailored to fit cosmological observations to some extent. Difficulties start to arise when a theory is required to perform well in Cosmology and, at the same time, to comply with the bounds imposed by Solar System tests.

Another aspect of the theories under investigation which has not been discussed so far regards other solutions of their field equations, apart from the cosmological ones. Even though many of the characteristics of a gravitation theory can be inferred from the form of its action or of its field equations and without any reference to specific solutions, the study of specific solutions always adds to our insight. It is important, for instance, that one should check whether solutions which describe any configuration of physical interest do exist and do have properties which agree with observations. In addition to this, it is not only the weak gravity regime that can provide constraints for alternative theories of gravity. Binary system tests (see [313] and references therein) or other strong gravity tests (*e.g.* [314]) offer the opportunity to test gravity beyond the weak field regime.

We will attempt to cover this gap in the present chapter. To this end, we will study the Newtonian limit and the post-Newtonian expansion of $f(R)$ gravity in both the metric formalism and the Palatini formalism, as well as in Gauss–Bonnet gravity. We will also address other issues related to the weak field regime of $f(R)$ theories of gravity as well as referring to vacuum solutions in these theories. For what regards non-vacuum solutions and the strong gravity regime the progress in the literature is much smaller. We restrict ourselves to discussing non-vacuum solutions in Palatini $f(R)$ gravity which, apart from the interest which they have within the framework of this theory, also serve as a very good example to demonstrate how studying matter configurations in the strong gravity regime can help to constrain or even rule out theories.

It should be mentioned that our discussion of the strong and weak gravity regimes of alternative theories of gravity in this chapter is far from exhaustive. One could also consider other theories but, more importantly, there are aspects of the theories under investigation that we will not be extensively referring to here. In some cases, such as vacuum [315, 316, 317] and non vacuum [318] solutions in metric $f(R)$ gravity, the reader can refer to the literature for more details. However, studies of most of the subjects which we will not refer to here are still pending. To name a few: the Newtonian and Post Newtonian limits in metric-affine $f(R)$ gravity have not yet been considered and not much attention has yet been paid to vacuum and non-vacuum solutions in Gauss–Bonnet gravity and metric-affine $f(R)$ gravity, or to strong gravity tests in either of these two theories.

6.2 The Nearly Newtonian regime in $f(R)$ gravity

6.2.1 Metric $f(R)$ gravity

Within the context of metric $f(R)$ gravity, the subjects of the Newtonian limit, the post-Newtonian expansion and confrontation with Solar System experiments, have long been debated. A large number of papers have been published and a lot of subtleties have been revealed. [319, 320, 321, 246, 322, 290, 323, 324, 325, 326, 327, 328, 329]. Since it is not possible to extensively review all of the works in this subject, we will attempt to focus on the major points and guide the reader through the literature.

As we have seen in Chapter 4, metric $f(R)$ gravity is dynamically equivalent to a Brans–Dicke theory with a potential $V(\phi)$ and Brans–Dicke parameter $\omega_0 = 0$. Solar System constraints on Brans–Dicke theories are quite well known (see Section 3.2.2 and [197]) and, therefore, it is natural to exploit this equivalence in order to derive such constraints for metric $f(R)$ gravity. This is indeed what was done in [319].

The PPN parameter γ is given in terms of ω_0 as [139]

$$\gamma = \frac{\omega_0 + 1}{\omega_0 + 2}. \tag{6.1}$$

Thus, in our case where $\omega_0 = 0$ one gets $\gamma = 1/2$. Obviously this value is far below the current bound, $|\omega_0| > 40\,000$. However, this bound only applies for very light scalar fields, *i.e.* scalars with very small effective masses. A large mass for the scalar implies that the force mediated by it would be short range and would not affect the results of Solar System experiments. In Brans–Dicke theory the square of the effective mass of the scalar is given by the second derivative of its potential evaluated at the minimum (extremum).

When one expresses metric $f(R)$ gravity as a Brans–Dicke theory, the functional form of the potential depends on f. In [319] attention was focused on the model of [256] for which

$$f(R) = R - \frac{\mu^4}{R}, \tag{6.2}$$

and $\mu \sim 10^{-42}$ GeV. The effective mass of the potential was evaluated for $R \sim H_0^2 \sim \mu$ and it was found to be of the order of μ^2. This is clearly a very small value and therefore the conclusion of [319] was that this model is ruled out. Additionally, even though it is possible to construct sophisticated models in order to make the scalar heavy (see for example [258]), this requires significant fine tuning of the parameters and in general models with $1/R$ terms will lead to a very small mass for the scalar.

As already mentioned, however, the effective mass is given by the second derivative of the potential at the extremum, *i.e.* at a point where the first derivative of the potential vanishes. It was pointed out in [246, 322] that, even though this is indeed the case for a general Brans–Dicke theory [139, 330, 331], having the scalar satisfying the extremum condition cannot be trivially assumed for the $\omega_0 = 0$ case. The equivalence with metric $f(R)$ gravity requires that the Jordan frame potential $V(\phi)$ is given by [246]

$$V(\phi) = Rf' - f \tag{6.3}$$

and that

$$V'(\phi) = R. \tag{6.4}$$

Since in a post-Newtonian expansion $R = R_0 + \sigma(t, x)$, where R_0 is the background value and $\sigma(t, x)$ denotes the local deviation from this value, the extremum condition is not generically satisfied (neither R_0 nor $\sigma(t, x)$ have to vanish). In this sense, $\omega_0 = 0$ Brans–Dicke theory constitutes an exception and standard results related to the post-Newtonian expansion should not be trusted according to [246]. In the same paper the post-Newtonian expansion was re-developed. However, the results were not qualitatively different from those of [319] and the cosmologically interesting $f(R)$ models with $1/R$ terms were again ruled out.

Contemporarily with [319], Dick considered the Newtonian limit of metric $f(R)$ gravity without resorting to the equivalent Brans–Dicke theory [320]. The approach was based on the more standard linearized perturbative expansion. However, this expansion was performed around a de Sitter background, since this is the generic maximally symmetric solution for metric $f(R)$ gravity. Again, attention was focused on the $1/R$ models and, again, these were ruled out.

Such a perturbative treatment requires a Taylor expansion to be made for $f(R)$ and $f'(R)$ around their background values. This is easy to see, since the field equations, eq. (3.18), involve these functions. As pointed out in [290], even though the results of [320] might well be correct, the convergence of these expansions was not checked and relevant higher orders were truncated *ad hoc*. For example, since

$$f(R) = f(R_0) + f'(R_0)R_1 + \frac{1}{2}f''(R_0)R_1^2 + \ldots \tag{6.5}$$

if we use the model of eq. (6.2) we get

$$f(R) = f(R_0) + \left(1 + \frac{\mu^4}{R_0^2}\right) R_1 - \frac{1}{2}\frac{2\mu^4}{R_0^3}R_1^2 + \ldots \tag{6.6}$$

where now $R_0 = \mu^2$. It is then easy to see that the second term on the right hand side of the above equation is of the order of R_1, whereas the third term is of the order of R_1^2/a. Therefore, in order to truncate before the third term, one needs $R_1 \gg R_1^2/a$ or

$$\mu^2 \gg R_1. \tag{6.7}$$

The evolution of R is governed by the trace of the field equation which for this model takes the form

$$-R + \frac{3\mu^4}{R} - \frac{6\mu^4}{R^3}\nabla^2 R + \frac{18\mu^4}{R^4}\nabla^\mu R \nabla_\mu R = 8\,\pi\, G\, T, \tag{6.8}$$

where we have denoted $8\,\pi\, G$ by κ. It is therefore not straightforward to judge whether the condition (6.7) is indeed satisfied.

The same issue is relevant also for the approach of [246] where the equivalent Brans–Dicke theory is used, since one has to expand the potential of the scalar field $V(\phi)$ around a background value ϕ_0 in order to arrive the post-Newtonian expansion. Since $V(\phi)$ is given in terms of $f(R)$ by eq. (6.3), it is reasonable to assume that any problematic behaviour in the expansion of $f(R)$ and $f'(R)$ might be inherited by the expansion of $V(\phi)$. Let us stress that this is not to say that the results of [320, 246] are necessarily incorrect, but merely that a more detailed and rigorous approach is required.

Another point is that part of the debate about the Newtonian and post-Newtonian limits of metric $f(R)$ gravity was based on a quite common misconception: that the existence of the Schwarzschild–de Sitter solution in vacuum guarantees that the Solar System tests will be passed (see for instance [332, 316]). To be more explicit, let us consider the trace of the field equations, eq. (3.19):

$$f'(R)R - 2f(R) + 3\Box f'(R) = 8\,\pi\, G\, T, \tag{6.9}$$

In vacuum $T = 0$. If we search for solutions for which the Ricci scalar is constant, then $\Box f'(R) = 0$ and the equation reduces to

$$f'(R)R - 2f(R) = 0, \tag{6.10}$$

where R is now a constant. Eq. (6.10) then becomes an algebraic equation. We can use this equation to write the field equation (3.18) in vacuum as

$$R_{\mu\nu} - \frac{C}{4} g_{\mu\nu} = 0, \tag{6.11}$$

where C is the constant value of R (see also the last paragraph of Section 3.3.2). Since eq. (6.11) is formally the same as the field equation of General Relativity with a cosmological constant in vacuum, we know that, according to the sign of C, the static spherically symmetric solutions will be Schwarzschild–de Sitter or Schwarzschild–anti-de Sitter. The mere existence of these solutions does not, however, have any implication for the Newtonian and post-Newtonian limits. As was correctly pointed out in [326], these are not the unique spherically symmetric static solutions (R does not have to be a constant) and in order to find the correct vacuum solution for the exterior of a spherically symmetric star, one has to search for the solution that can be properly matched to the interior. The results of [326] support the findings of [319].

One more issue that was presented as a drawback for the use of the equivalent Brans–Dicke theory for deriving Solar System constraints was that of [333]. The claim there was that, since the equivalence between the two theories enforces the requirement $f'' \neq 0$ and, on the other hand, in the weak field regime $f'' \to 0$, the equivalence should break down at this limit and bounds derived though the equivalent Brans–Dicke theory should not be considered trustworthy. This claim was later retracted in [334]. In fact, as pointed out also in [335], the condition $f'' \neq 0$ is not needed if the equivalence between the two theories is shown at the level of the field equations, instead of using the action, and it then constitutes a superfluous condition.

Later works appear to resolve some of the issues raised earlier concerning the validity of the results of [319, 320, 246]. In [327], the approach of [326] was followed and the results were extended to more general models. The outcome was that if one properly takes into account the matching with an interior solution, then only very special models which are very close to General Relativity with a cosmological constant can pass the Solar System tests and at the same time give interesting late time cosmological phenomenology. In [328], the derivation of constraints from the Solar System tests by means of using the equivalent Brans–Dicke theory was considered again and attention was paid to ensuring that the Taylor expansions of $f(R)$ and $f'(R)$ were well defined and dominated by terms that are linear in deviations away from $R = R_0$ (as proposed in [290]). Again the outcome was that the results of [319, 246] are indeed valid. However, different opinions are still present [329].

To summarise: after a long debate, it seems that most of the models of metric $f(R)$ gravity that have been proposed as solutions to the dark energy problem and which therefore include $1/R$ terms, do not have correct Newtonian and post-Newtonian limits. Exceptions to this do exist, but significant fine tuning is required, to the extent that it can be characterised as unnatural. It should be mentioned that this is not the case for some other models which lead to interesting early time cosmological phenomenology, such as the Starobinski model ($f(R) = R + \epsilon R^2$) [20].

6.2.2 Palatini $f(R)$ gravity

The Newtonian and post-Newtonian limits of Palatini $f(R)$ gravity have also been a matter of some debate [336, 337, 338, 290, 339, 340], similarly to metric $f(R)$ gravity. Again, most of the attention has focused on models with terms inversely proportional to the scalar curvature, since these are the cosmologically motivated ones. The two first results in this direction were in clear contradiction. In [336], Meng and Wang claimed that all models with inverse powers of the scalar curvature in the action give a correct Newtonian limit. On the other hand, in [337] it was claimed that this is not true and that there are constraints on the form of the Lagrangian. However, it was shown in [290] that both of these results suffered from a serious problem.

Let us see this in more detail. In [336] and [337], the authors expand around de Sitter in order to derive the Newtonian limit. We can write

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1, \tag{6.12}$$

where $\mathcal{R}_0$ is the scalar curvature of the background de Sitter spacetime and $\mathcal{R}_1$ is the correction to $\mathcal{R}_0$, including all possible terms, with $\mathcal{R}_1/\mathcal{R}_0$ being considered as being a small quantity. We will need to calculate $f(\mathcal{R}_0+\mathcal{R}_1)$ and $f'(\mathcal{R}_0+\mathcal{R}_1)$. The usual approach is to Taylor expand around $\mathcal{R} = \mathcal{R}_0$ and keep only the leading order terms in $\mathcal{R}_1$ but we will show that this cannot be done in the present context because $\mathcal{R}_1/\mathcal{R}_0$ is not small.

Take as an example the model of [256] studied by Vollick [279] in the Palatini formalism. Then

$$f(\mathcal{R}) = \mathcal{R} - \frac{\epsilon_2^2}{\mathcal{R}}, \tag{6.13}$$

and $\epsilon_2 \sim 10^{-67}(\text{eV})^2 \sim 10^{-53}\text{m}^{-2}$. Expanding, we get

$$f(\mathcal{R}) = f(\mathcal{R}_0) + f'(\mathcal{R}_0)\mathcal{R}_1 + \frac{1}{2}f''(\mathcal{R}_0)\mathcal{R}_1^2 + \dots \tag{6.14}$$

and, using (6.13), we get

$$f(\mathcal{R}) = f(\mathcal{R}_0) + \left(1 + \frac{\epsilon_2^2}{\mathcal{R}_0^2}\right)\mathcal{R}_1 - \frac{1}{2}\frac{2\epsilon_2^2}{\mathcal{R}_0^3}\mathcal{R}_1^2 + \dots \tag{6.15}$$

where now $\mathcal{R}_0 = \epsilon_2$. It is then easy to see that the second term on the right hand side of the above equation is of the order of $\mathcal{R}_1$, whereas the third term is of the order of $\mathcal{R}_1^2/\epsilon_2$. Therefore, in order to truncate before the third term, one needs $\mathcal{R}_1 \gg \mathcal{R}_1^2/\epsilon_2$ or

$$\epsilon_2 \gg \mathcal{R}_1. \tag{6.16}$$

Note that this is not any exceptional constraint. $\mathcal{R}_0 \sim \epsilon_2$ and so this is the usual condition for being able to truncate non linear terms in a Taylor expansion.

Let us now return to eq. (3.31). For the model of eq. (6.13) this gives

$$\mathcal{R} = \frac{1}{2}\left(-8\,\pi\,G\,T \pm \sqrt{(8\pi\,G)^2\,T^2 + 12\epsilon_2^2}\right). \tag{6.17}$$

When discussing whether a theory has a good Newtonian limit, we are in practice checking whether the field equations reduce to a high precision to the Poisson equation under the following assumptions: energy densities should be small enough so that there are no strong gravity effects, and velocities related to the motion of the matter should be negligible compared to the velocity of light. At the same time, energy densities should be high enough so that the system under investigation can be considered gravitationally bound[1].

It is clear from eq. (6.17) that the value of $\mathcal{R}$, and consequently $\mathcal{R}_1$, is algebraicly related to T. This already implies that whether or not the condition (6.16) is satisfied will critically depend on the value of the energy density. To demonstrate this, let us pick a typical example of a density satisfying the weak field limit criteria: the mean density of the Solar System, $\rho \sim 10^{-11}$gr/cm^3. For this value $|\epsilon_2/8\pi\,G\,T| \sim 10^{-21}$, where $T \sim -\rho$. The "physical" branch of the solution given in eq. (6.17) seems to be the one with the plus sign in front of the square root. In fact, given that $T < 0$, on this branch it is ensured that the matter leads to a standard positive curvature in a strong gravity regime. Then

$$\mathcal{R} \sim -8\pi\,G\,T - \frac{3\epsilon_2^2}{8\pi\,G\,T} \tag{6.18}$$

and $\mathcal{R}_1 \sim -8\pi\,G\,T \sim 8\pi\,G\,\rho$. Thus $\epsilon_2/\mathcal{R}_1 \sim 10^{-21}$ and it is now evident that condition (9) does not hold for some typical densities that could be related to the Newtonian limit.

Note that the situation does not improve even if we choose the "unphysical" branch of eq. (6.17) which has a minus sign in front of the square root. In fact, in this case $\mathcal{R}_1 \sim \epsilon_2[3\epsilon_2/(8\pi\,G\,T) + \sqrt{3}]$ and so the correction to the background curvature is of the order ϵ_2 and not much smaller than that, as required in order to truncate before the higher order terms in the expansion eq. (6.15).

In [337], this fact was overlooked and only linear terms in $\mathcal{R}_1$ were kept in the expansion of $f(\mathcal{R})$ and $f'(\mathcal{R})$ around $\mathcal{R}_0$. In [336] even though it is noticed in the final stages of the analysis and is actually used in order to neglect some terms, the authors do not take it into account properly from the beginning, keeping again only first order terms (eq. (11) of [336] for example).

[1]For example, in General Relativity with a cosmological constant one could consider, even on non-cosmological scales, densities low enough so that the correction coming from the cosmological constant dominates with respect to the matter density in the Poisson equation. This, of course, would not imply that this model does not have a correct Newtonian limit

An alternative way to see the dependence of the weak field limit on the energy density is the following. We already know (see Section 3.4.3) that in Palatini $f(R)$ gravity the connection is the Levi–Civita connection of the metric

$$h_{\mu\nu} = f'(\mathcal{R})g_{\mu\nu}. \tag{6.19}$$

For the model given by eq. (6.13) then, and if we define $\epsilon = \epsilon_2^2/\mathcal{R}^2$, eq. (3.29) takes the form

$$(1+\epsilon)\mathcal{R}_{\mu\nu} - \frac{1}{2}(1-\epsilon)\mathcal{R}g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{6.20}$$

and

$$h_{\mu\nu} = (1+\epsilon)g_{\mu\nu}. \tag{6.21}$$

Due to eq. (3.31), ϵ depends only on T. Combining eqs. (6.21) and (6.20) we get

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}h_{\mu\nu} + \epsilon\left(\frac{\mathcal{R}}{1+\epsilon}h_{\mu\nu} + \mathcal{R}_{\mu\nu}\right) = 8\pi G T_{\mu\nu}. \tag{6.22}$$

Note that up to this point no approximation or truncation has been used. We have merely expressed the left hand side of the field equation for the metric in terms of quantities depending only on the $h_{\mu\nu}$ metric, which is conformal to $g_{\mu\nu}$. However, using eq. (6.17) and (6.18) we see that $\epsilon \sim 10^{-42}$ if we consider the mean density of the Solar System as before and is even smaller for higher densities. Therefore the two metrics are practically indistinguishable in such cases, due to eq. (6.21). Thus we can use the h metric to derive the Newtonian limit.

As usual, it is expected that a suitable coordinate system can be found in which

$$h_{\mu\nu} = \eta_{\mu\nu} + h^1_{\mu\nu}, \qquad |h^1| \ll 1, \tag{6.23}$$

where $h^1_{\mu\nu}$ denotes the correction with respect to the Minkowski metric. Then the first two terms of eq. (6.22) will give the standard Newtonian limit and the last two terms will give a negligible contribution, since they are suppressed by the ϵ coefficient. A deviation of the order of 10^{-42} is far below the accuracy of any known experiment. In fact, one can consider densities several orders of magnitude smaller and still get corrections which will be far below experimental accuracies.

A critical point is that we assumed here that the metric is flat plus a small correction instead of de Sitter plus a small correction. Note, however, that we are not claiming that we are expanding around the background or any corresponding maximally symmetric spacetime. We are merely asking for the matter to account for the deviation from flatness, which is the basic concept related to the Newtonian limit. In any case, de Sitter is essentially identical to Minkowski for the densities discussed, and the important corrections to the metric come from the local matter, not from considerations of the universe as a whole.

According to the above, the Lagrangian of eq. (6.13) can give a perfectly good Newtonian limit for some typical weak-field-limit densities. The approach can be extended to more general Lagrangians. Indeed, for a general function f, eq. (6.22) will be

$$\mathcal{R}_{\mu\nu}-\frac{1}{2}\mathcal{R}h_{\mu\nu}+(f'-1)\left(\mathcal{R}_{\mu\nu}-\frac{\mathcal{R}}{2f'}h_{\mu\nu}\right)=8\pi\, G\, T_{\mu\nu}. \tag{6.24}$$

Since due to eq. (3.31) $\mathcal{R}$ and consequently $f'(\mathcal{R})$ are functions of the energy density, the deviation of f' from 1 will always depend on it. This dependence of the weak field limit on the energy density is a novel characteristic of Palatini $f(R)$ gravity [2].

Similar things can be said if the problem is approach via the equivalent Brans–Dicke theory. This was studied in [246]. Note that the usual bounds coming from Solar System experiments do not apply in the $\omega_0 = -3/2$ case, which is equivalent to Palatini $f(R)$ gravity. This is because the standard treatment of the post-Newtonian expansion of Brans–Dicke theory, which one uses to arrive at such bounds, is critically based on the assumption that $\omega_0 \neq -3/2$ and the term $(2\omega_0+3)$ frequently appears as a denominator. Making this assumption is not necessary, of course, in order to derive a post-Newtonian expansion, but is a convenience choice, which allows for this otherwise general treatment. Therefore, a different approach, such as the one followed in [246], was indeed required for the $\omega_0 = -3/2$ case . Following the standard assumptions of a post-Newtonian expansion around a background specified by a cosmological solution [139], the following relations were derived for the post-Newtonian limit

$$-\frac{1}{2}\nabla^2\left[h^1_{00}-\Omega(T)\right] = \frac{8\pi\, G\,\rho - V(\phi)}{2\phi}, \tag{6.25}$$

$$-\frac{1}{2}\nabla^2\left[h^1_{ij}+\delta_{ij}\Omega(T)\right] = \left[\frac{8\pi\, G\,\rho + V(\phi)}{2\phi}\right], \tag{6.26}$$

where V is the potential of the scalar field ϕ and $\Omega(T)\equiv \log[\phi/\phi_0]$. The subscript 0 in ϕ_0, and in any other quantity from here on, denotes that it is evaluated at $T=0$. Note at this point that normalization by ϕ_0 in this definition is not required. In [246], the constant $\log(\phi_0)$ was just added inside the brackets on the left hand side of eq. (6.25) (and subtracted in eq. (6.26)) using the fact that the latter remains unchanged. Thus we are not going to use it here and will refer to $\Omega(T)$ just as $\Omega(T)=\log[\phi]$.

The solutions of eqs. (6.25) and (6.26) are

$$h^1_{00}(t,x) = 2G_{\text{eff}}\frac{M_0}{r}+\frac{V_0}{6\phi_0}r^2+\Omega(T), \tag{6.27}$$

$$h^1_{ij}(t,x) = \left[2\gamma G_{\text{eff}}\frac{M_0}{r}-\frac{V_0}{6\phi_0}r^2-\Omega(T)\right]\delta_{ij}, \tag{6.28}$$

[2] This discussion clarifi es why in Section 5.3.3 we required that at least in some regime $f' \to 1$. Additionally, it is now apparent that if one selects the model of eq. (5.35) and assumes a typical value for the density, then stringent constraints on the value of ϵ_1 can be placed in the spirit of [283].

where $M_0 \equiv \phi_0 \int d^3x' \rho(t,x')/\phi$. The effective Newton constant $G_{\rm eff}$ and the post-Newtonian parameter γ are defined as

$$G_{\rm eff} = \frac{G}{\phi_0}\left(1+\frac{M_V}{M_0}\right), \tag{6.29}$$

$$\gamma = \frac{M_0 - M_V}{M_0 + M_V}, \tag{6.30}$$

where $M_V \equiv (8\pi\, G)^{-1}\phi_0 \int d^3x' \, [V_0/\phi_0 - V(\phi)/\phi]$.

Even though we agree with the approach followed to derive eqs. (6.27) and (6.28) and on their validity, we disagree with the line of reasoning used by the author to argue that models with inverse powers of the scalar curvature do not have a good Newtonian limit. We will demonstrate this using, once again, the model of eq. (6.13).

As stated in different words in [246], if we define the Newtonian mass as $M_N \equiv \int d^3x' \rho(t,x')$, the requirement for a theory to have a good Newtonian limit is that $G_{\rm eff} M_0$ is equal to GM_N, where N denotes Newtonian and $\gamma \sim 1$ to very high precision. Additionally, the second term on the right hand side of both eq. (6.27) and eq. (6.28) should be negligible, since it acts as a term coming from a cosmological constant. $\Omega(T)$ should also be small and have a negligible dependence on T. The above have to be true for the range of densities relevant to the Newtonian limit, as discussed before. Using the equation that related V and ϕ with $\mathcal{R}$ (see Chapter 4)

$$\phi = f', \tag{6.31}$$

$$V(\phi) = \mathcal{R} f' - f, \tag{6.32}$$

one can easily show that

$$\phi = 1 + \frac{\epsilon_2^2}{\mathcal{R}^2}, \tag{6.33}$$

$$V(\phi) = 16\pi\, G\, \epsilon_2 \sqrt{\phi - 1}. \tag{6.34}$$

Additionally, for $T = 0$, $\mathcal{R} = \sqrt{3}a$ and so

$$\phi_0 = 4/3, \tag{6.35}$$

$$V_0 = 16\pi\, G\, \epsilon_2/\sqrt{3}. \tag{6.36}$$

For the densities which we are considering, we can use the parameter ϵ defined above. Then

$$V(\phi) = 16\pi\, G\, \frac{\epsilon_2^2}{\mathcal{R}} = 16\pi\, G\, \epsilon_2 \sqrt{\epsilon}, \tag{6.37}$$

and $M_V \sim \epsilon_2$. It is easy to see, using eq. (6.33), (6.35), (6.36) and (6.37), that

$$G_{\rm eff} \approx \frac{G}{\phi_0}, \tag{6.38}$$

$$\gamma \approx 1, \tag{6.39}$$

and $\phi \approx 1$ plus corrections of order ϵ_2 or smaller, which is well beyond the limit of any experiment.

Additionally

$$\Omega(T) \equiv \log[\phi] = \log[1+\epsilon] \approx \log\left[1 + \frac{\epsilon_2^2}{(8\,\pi\,G)^2 T^2}\right]. \tag{6.40}$$

V_0 is of the order of ϵ_2, which is a perfectly acceptable value, and $\Omega(T)$ is negligible at the densities being considered and decreases even more when the density increases. Therefore, our previous results are valid and theories including inverse powers of the scalar curvature can have a correct Newtonian limit in the Palatini formalism for a specific density range.

This result contradicts that reported in [246], even though the approach followed there seems to be satisfactory. The main reason for this problem seems to be the following. In [246] the fact that $\Omega(T)$ should have a mild dependence on T is used to obtain a constraint for the dependence of ϕ on T (eq. (26) of [246]). Following a number of steps, this constraint is turned into a constraint on the functional form of $f(R)$ (eq. (37) of [246]) and from this a conclusion is derived about its possible nonlinearity. We disagree with this line of thought. Such inequalities constrain merely the value of the relevant quantity at the point where it is evaluated and not its true functional form. One could probably use them to make some assumptions about the leading order term but not to exclude any terms of a different form, as long as they are negligible with respect to the leading order for the relevant values of R. This, for example, is the case for the model discussed above. Any constraint placed by the Newtonian limit has to hold over a certain range of relevant densities (and consequently curvatures), and not for all densities as implied in [246].

However, the dependence of the outcome of the Newtonian limit on the energy density is not only surprising but also problematic. Even though, according to the above, we can expect that inside or outside a cloud of matter of a typical weak-field density, gravity may behave in the same way as in Newtonian gravity, this is definitely not the end of the story. As correctly pointed out in [246] the dependence on the energy density, and especially that coming from $\Omega(T)$, signals a problem. One has to take into account that matter can also come as a perturbation. Indeed this is the case for Solar System tests (light deflection, Shapiro time delay, *etc*.) which do not necessarily examine gravitationally bound systems but are essentially vacuum tests in which the presence of matter (*e.g.* Solar winds) has to be taken into account as a correction [139]. Therefore the relevant densities can be many orders of magnitude smaller than those discussed above. In addition to this, in eqs. (6.27) and (6.28) $\Omega(T)$ is algebraically related to the metric, which implies that the metric depends directly on the density and not on some integral over it, as would be expected. This creates doubts about how the theory would behave if a very weak point source (approximated by a delta function) is taken into account as a perturbation.

Due to the above, we can conclude the following: Even though it can be shown that, for some typical energy densities, an acceptable weak field limit can be recovered from Palatini $f(R)$ gravity, this provides no guarantee that the theory passes Solar System tests. Additionally, the direct dependence of the outcome on the density, signals the existence of a deeper problem. In Section 6.5 this problem will become apparent through a completely different approach, so we will refrain from saying more here.

Before closing, it should be mentioned that, similarly to the metric formalism, also in Palatini $f(R)$ gravity the existence of the Schwarzschild–de Sitter solution as a vacuum spherically symmetric solution has triggered some confusion concerning Solar System tests. As shown in Section 3.4.3, Palatini $f(R)$ gravity reduces in vacuum to General Relativity with a cosmological constant. This implies that this theory retains a useful characteristic of GR: the exterior spherically symmetric solution is unique (*Birkhoff's theorem*) [3]. Depending to the sign of the effective cosmological constant, the solutions are either Schwarzschild–de Sitter or Schwarzschild–anti-de Sitter. This was interpreted in [339, 340] as an indication that the only parameter that can be constrained is the effective cosmological constant and therefore models that are cosmologically interesting, for which this parameter is very small, trivially satisfy Solar System tests. However, even though the uniqueness of the solution implies that here we will not face problems like those discussed in the previous section for metric $f(R)$ gravity (concerning which exterior solution can properly match an interior one, *etc.*), this claim is still incorrect. It should be clarified that the existence of a spherically symmetric vacuum solution solution, irrespective of its uniqueness, is not enough to guarantee a good Newtonian limit for the theory. For instance, the Schwarzschild–de Sitter solution has two free parameters. One of them can be associated with the effective cosmological constant in a straightforward manner. However, it is not clear how the other parameter, which in General Relativity is identified as the mass of the object in the Newtonian regime, is related to the internal structure of the object in Palatini $f(R)$ gravity. Assuming that it represents the mass defined in the usual way is not enough of course. The essence of deriving the Newtonian limit of the theory is exactly in deriving an explicit relation for this quantity and showing that it agrees with the Newtonian expression.

6.3 Curvature scalar instability in $f(R)$ gravity

Besides the post-Newtonian limit, there is another problem related to the weak field regime of metric $f(R)$ gravity which was pointed out soon after the introduction of the model with a $1/R$ term [256]: an instability in the equation governing the dynamics of the scalar curvature R was discovered by Dolgov and Kawasaki [341] in the presence of matter for the specific model where $f(R) = R - \mu^4/R$, with μ

[3] This does not hold for metric $f(R)$ gravity, as discussed in the previous section.

being a constant. This instability is not just a special characteristic of this model but occurs in a more general class of models [342].

Let us briefly review the results of [341, 342]. By contracting eq. (3.18) one gets

$$3\Box f'(R) + f'(R)R - 2f(R) = 8\pi GT, \tag{6.41}$$

where $T = g^{\mu\nu}T_{\mu\nu}$. Following [342], we can write $f(R) = R + \epsilon\varphi(R)$, where ϵ is a constant. If we consider a small region in a weak field regime within matter, we can assume that $g_{ab} = \eta_{ab} + h_{ab}$ and $R = -8\pi GT + R_1$, where η_{ab} is the Minkowski metric and $|R_1/(8\pi GT)| \ll 1$. In this approximation, and to first order in R_1, eq. (6.41) gives

$$\begin{aligned} &\ddot{R}_1 - \nabla^2 R_1 - \frac{16\pi G\,\varphi'''}{\varphi''}\left(\dot{T}\dot{R}_1 - \vec{\nabla}T\cdot\vec{\nabla}R_1\right) \\ &+ \frac{1}{3\varphi''}\left(\frac{1}{\epsilon} - \varphi'\right)R_1 = 8\pi G\ddot{T} - 8\pi G\,\nabla^2 T - \frac{(8\pi GT\varphi' + \varphi)}{3\varphi''}, \end{aligned} \tag{6.42}$$

where an over-dot denotes differentiation with respect to time, while $\vec{\nabla}$ and ∇^2 denote the gradient and Laplacian operators respectively in Euclidean three-dimensional space.

The instability occurs if $\varphi'' = f''(R) < 0$ and ϵ is very small, since the coefficient of the last term on the left hand side of eq. (6.42) is the square of an effective mass (notice the resemblence with a damped harmonic oscilator). As already mentioned in [342], it can be considered as an instability in the gravity sector. Because of this, and since it appears in the equations governing the dynamics of the curvature scalar, we refer to it as the "curvature scalar instability". Theories with $f''(R) > 0$ will be stable irrespective of the value of ϵ. However, for several models that lead to the desired cosmological dynamics at late times, ϵ is indeed very small and $f''(R)$ is indeed negative. A typical example is the model of [256], where $\varphi(R) = -\mu^4/R$, with $\mu \sim 10^{-33}$eV, and the time-scale for the instability to occur is of the order of 10^{-26} s [341].

All of the above is with reference to the metric formalism. Let us now consider the Palatini formalism. Following the lines of [343], we will argue that such an instability cannot occur in this case irrespective of the form of the Lagrangian. Contracting eq. (3.29) gives eq. (3.31), which we repeat here for the convenience of the reader:

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = 8\pi GT. \tag{6.43}$$

Recall that $\mathcal{R}$ is not the Ricci scalar of the metric. In Section 3.4.3, we derived eq. (3.37) in which R in expressed in terms of $\mathcal{R}$:

$$R = \mathcal{R} - \frac{3}{2[f'(\mathcal{R})]^2}\nabla_\mu f'(\mathcal{R})\nabla^\mu f'(\mathcal{R}) + \frac{3}{f'(\mathcal{R})}\Box f'(\mathcal{R}). \tag{6.44}$$

Now notice that eq. (6.43) is an algebraic equation in $\mathcal{R}$ for a given $f(\mathcal{R})$, which will have solutions of the form $\mathcal{R} = \theta(T)$, where θ is some function. As has been mentioned several times before, we are not interested in cases in which eq. (6.43) has no solutions or is identically satisfied ($f(\mathcal{R}) \propto \mathcal{R}^2$), since these do not constitute viable choices for a low-energy gravitational theory [164, 165].

We can now write eqs. (6.44) as

$$\begin{aligned} R = \theta(T) - \frac{3}{2[f'(\theta(T))]^2}\nabla_\mu f'(\theta(T))\nabla^\mu f'(\theta(T)) + \\ + \frac{3}{f'(\theta(T))}\Box f'(\theta(T)), \end{aligned} \tag{6.45}$$

or alternatively $R = W(T)$, where $W(T)$ is a function of T. This clearly demonstrates that the Ricci scalar of the metric can be expressed directly as a function of the trace of the stress-energy tensor. In fact, eq. (6.45) is a straightforward generalization of the contracted Einstein equation, $R = -8\pi GT$. From the form of eq. (6.45), it is clear that no instability can occur in this case, since R carries no dynamics in eq. (6.45), unlike eq. (6.41).

Let us now analyse where this difference between the two formalisms stems from. By generalizing the Lagrangian from R or $\mathcal{R}$ one inevitably adds a scalar degree of freedom [166]. However, as mentioned in Chapter 4, this degree of freedom seems to be of a different nature in the two versions of the theory. In the metric version, it is dynamical and therefore care should be taken to ensure stability, whereas in the Palatini version it is non-dynamical. This is related to the fact that the Palatini formalism leads to second order field equations in the metric whereas the metric formalism leads to fourth order field equations, but it can also be easily seen by using the equivalence of $f(R)$ gravity and scalar-tensor theory (see Chapter 4 and references therein).

As we have seen, the Brans–Dicke action equivalent to metric $f(R)$ gravity is

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(\phi R - V(\phi)\right) + S_M(g_{\mu\nu}, \psi), \tag{6.46}$$

with $\omega_0 = 0$. In the Palatini formalism, the action will be formally the same apart from the fact the R will become $\mathcal{R}$, but in this case it will not be a scalar-tensor theory with $\omega_0 = 0$ since $\mathcal{R}$ is not the Ricci scalar of the metric [166]. However, if we use eq. (6.44) and $\phi = f'(\mathcal{R})$, we get

$$S_{pal} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(\phi R + \frac{3}{2\phi}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) + S_M(g_{\mu\nu}, \psi), \tag{6.47}$$

which is indeed a scalar-tensor theory, but with $\omega_0 = -3/2$.

The field equation of the scalar field in scalar-tensor theory is

$$(2\omega_0 + 3)\Box\phi = 8\pi GT + \phi V' - 2V. \tag{6.48}$$

Note that ϕ is the extra degree of freedom of $f(R)$ gravity, with respect to General Relativity. Using eq. (6.48), it is obvious that ϕ satisfies the field equations

$$3\Box\phi + 2V(\phi) - \phi V'(\phi) = 8\pi GT, \tag{6.49}$$

$$2V(\phi) - \phi V(\phi) = 8\pi GT, \tag{6.50}$$

in the metric and Palatini formalisms respectively. This demonstrates that ϕ is indeed dynamical in the metric formalism, whereas it is not dynamical in the Palatini formalism, as mentioned above. At this point, it is worth mentioning that one should not be misled into judging the dynamics of a non-minimally coupled field by the presence or absence of a kinetic term in the action. There are no kinetic terms for ϕ in action (6.46) but it is still dynamical. Exactly the opposite holds for the Palatini formalism. The reason for this is that both fields are coupled not only to the metric, but also to its derivatives. Therefore, when varying the action with respect to the metric and then integrating by parts in order to "free" $\delta g^{\mu\nu}$, terms including derivatives of the scalar field are bound to appear. Therefore, in the metric formalism, even though there are no apparent kinetic terms for ϕ in the action, there will be kinetic terms in the field equations. For Palatini $f(R)$ gravity, $\omega_0 = -3/2$ and this is the remarkable case where these kinetic terms exactly cancel out the ones coming from the kinetic part of the action.

To conclude, the curvature scalar instability discovered by Dolgov and Kawasaki for metric $f(R)$ gravity places an additional constraint on the form of the Lagrangian, whereas it is not present in the Palatini formalism, irrespective of the functional form of f. It should be stressed, however, that even though this instability does not occur in Palatini $f(R)$ gravity, other types of instability might well be present. For example, judging from the form of eq. (6.45), it is not difficult to imagine that specific forms of f could lead to a blow-up of the scalar curvature for small density perturbations around a stable matter configuration. Such instabilities would be, of course, of a different nature. This issue seems to be directly related to the problems with the weak field limit of the theory discussed in the previous section and it will be fully clarified in Section 6.5.

6.4 Post-Newtonian expansion of Gauss–Bonnet gravity

In Section 3.7 we presented the action and field equations of Gauss–Bonnet gravity and in Section 5.5 we studied its cosmological applications. In order to confront the theory with Solar System observations, as we have already done for metric and Palatini $f(R)$ gravity, one needs the Post-Newtonian Parametrized expansion of the theory. This is the issue that will concern us in this section and we will approach it along the lines of Ref. [344]. We will not consider the exceptional case, where $\lambda = 0$ and the scalar field has no kinetic term in the action. Such actions are dynamically equivalent to an action with a general function of $\mathcal{G}$ added to the Ricci scalar (see Section 4.4) and their Newtonian limit has been considered in [345].

We begin by bringing the field equations of the theory, namely eqs. (3.149) and (3.150), into a form more suitable for our purposes. Taking the trace of eq. (3.149) and using the definitions for the quantities $T_\phi^{\mu\nu}$ and $T_f^{\mu\nu}$ given in eqs. (3.140) and (3.147), we get

$$R = 8\,\pi\,G\left[-T - T^\phi + 2(\Box f(\phi))R - 4(\nabla^\rho\nabla^\sigma f(\phi))R_{\rho\sigma}\right], \tag{6.51}$$

where $T = g^{\mu\nu}T_{\mu\nu}$ and $T^{\phi} = g^{\mu\nu}T^{\phi}_{\mu\nu}$. Replacing eq. (6.51) back in eq. (3.149), the latter becomes:

$$\begin{aligned} R_{\mu\nu} \;=\;& 8\,\pi\,G\Big[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T + \frac{1}{2}\lambda\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}V(\phi) + \\ & +2(\nabla_\mu\nabla_\nu f(\phi))R - g_{\mu\nu}(\Box f(\phi))R - \\ & -4(\nabla^\rho\nabla_\mu f(\phi))R_{\nu\rho} - 4(\nabla^\rho\nabla_\nu f(\phi))R_{\mu\rho} + \\ & +4(\Box f(\phi))R_{\mu\nu} + 2g_{\mu\nu}(\nabla^\rho\nabla^\sigma f(\phi))R_{\rho\sigma} - \\ & -4(\nabla^\rho\nabla^\sigma f(\phi))R_{\mu\rho\nu\sigma}\Big] \end{aligned} \tag{6.52}$$

Following the standard approach for post-Newtonian expansions (see [139]), we choose a system of coordinates in which the metric can be perturbatively expanded around Minkowski spacetime. We write the metric and the scalar field as

$$\phi = \phi_0 + \delta\phi, \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{6.53}$$

where the value of ϕ_0 is determined by the cosmological solution. The perturbed field equations are

$$\begin{aligned} &\lambda[\Box_{\rm flat}\delta\phi + (\delta\Box)\delta\phi] - V''(\phi_0)\delta\phi - \frac{1}{2}V'''(\phi_0)(\delta\phi)^2 \\ &\qquad + f'(\phi_0)\mathcal{G} = \mathcal{O}(\delta\phi^3, \delta\phi(h_{\mu\nu})^2, h_{\mu\nu}\dot{\phi}_0, h_{\mu\nu}\ddot{\phi}_0), \end{aligned} \tag{6.54}$$

$$\begin{aligned} R_{00} = 8\,\pi\,G\Big\{&T_{00} + \frac{1}{2}T - \frac{1}{2}h_{00}T + \frac{1}{2}\lambda\partial_0\delta\phi\partial_0\delta\phi + \frac{1}{2}\lambda\,\dot{\phi}_0^2 - \frac{1}{2}V(\phi_0) \\ & + \frac{1}{2}V'(\phi_0)\delta\phi\,(-1 + h_{00}) + f'(\phi_0)\Big[2(\partial_0\partial_0\delta\phi)R \\ & + (\Box_{\rm flat}\delta\phi)R - 8(\partial^\rho\partial_0\delta\phi)R_{0\rho} + 4(\Box_{\rm flat}\delta\phi)R_{00} \\ & - 2(\partial^\rho\partial^\sigma\delta\phi)R_{\rho\sigma} - 4(\partial^\rho\partial^\sigma\delta\phi)R_{0\rho0\sigma}\Big]\Big\} \\ & + \mathcal{O}(\delta\phi^2 h_{\mu\nu}, \delta\phi^3, \dot{\phi}_0\delta\phi, \ddot{\phi}_0 h_{\mu\nu}, V(\phi_0)h_{00}), \end{aligned} \tag{6.55}$$

$$R_{0i} = 8\,\pi\,GT_{0i} + \mathcal{O}(\delta\phi h_{\mu\nu}, \delta\phi^2, Th_{0i}, \dot{\phi}_0\delta\phi, \ddot{\phi}_0 h_{\mu\nu}, V(\phi_0)h_{0i}), \tag{6.56}$$

$$\begin{aligned} R_{ij} = 8\,\pi\,G\Big[&T_{ij} + \frac{1}{2}\delta_{ij}\left(-T + V'(\phi_0)\delta\phi + V(\phi_0)\right)\Big] \\ & + \mathcal{O}(\delta\phi h_{\mu\nu}, \delta\phi^2, Th_{ij}, \ddot{\phi}_0 h_{\mu\nu}, V(\phi_0)h_{ij}), \end{aligned} \tag{6.57}$$

where $\Box_{\rm flat}$ denotes the D'Alembertian of flat spacetime. Notice that, as usually done in scalar-tensor theories [330, 331], we have neglected all of the terms containing derivatives of ϕ_0 multiplying perturbed quantities (e.g. $\dot{\phi}_0\delta\phi$). This is due to the fact that ϕ_0 changes on cosmological timescales and consequently one expects that it remains practically constant during local experiments. Therefore its time derivatives can be neglected as far as Solar System tests are concerned.

This can easily be verified by some order-of-magnitude analysis. Take for instance Eq. (6.55): the terms containing a time derivative of ϕ_0 multiplying a perturbation are $\mathcal{O}(\ddot{f}(\phi_0)h_{\mu\nu}/(r^2 M_p^2))$ and $\mathcal{O}(\dot{\phi}_0\delta\dot{\phi}/M_p^2)$, where $\dot{\phi}_0 \sim H_0 M_p$

and $\ddot{f} \sim H_0^2$ ($M_p = (8\,\pi\, G)^{-1/2}$ is the Planck mass and H_0 the present Hubble constant) and $h_{00} \sim h_{ij} \sim r\delta\phi \sim h_{0i}/v \sim r^2\delta\dot{\phi}/v \sim GM_\odot/r$ (r is the distance from the Sun, $M_\odot$ is the Solar mass and $v = \sqrt{GM_odot/r}$). On the other hand, the $\mathcal{O}(v^4)$ post-Newtonian correction to R_{00} is $\sim (GM_\odot)^2/r^4 \sim 10^{-55}\mathcal{O}(\ddot{f}(\phi_0)h_{\mu\nu}/(r^2M_p^2), \dot{\phi}_0\delta\dot{\phi}/M_p^2)$ even if r is taken as large as 1000 AU. Therefore, the corrections coming from terms containing time derivatives of ϕ_0 multiplying perturbations are at least 55 orders of magnitude smaller than the post-Newtonian corrections, and neglecting these terms cannot affect our results in any way. A similar treatment applies to the terms containing the potential V multiplying perturbed quantities (e.g.$V(\phi_0)h_{00}$): in order to give a reasonable description of Cosmology, $V(\phi_0)$ should be of the same order as the energy density of the cosmological constant and these terms cannot therefore lead to any observable deviations at Solar System scales.

In the perturbed field equations, $V(\phi_0)$ and $\frac{1}{2}\dot{\phi}^2$ are also present without multiplying perturbations. We will adopt a different treatment for these simple $V(\phi_0)$ and $\frac{1}{2}\dot{\phi}^2$ terms: since they need to be of the same order as the energy density of the cosmological constant, they will not lead to any observational consequences as far as Solar System tests are concerned (see [346] and references therein). For the sake of the argument, we will keep track of them but, due to their small values, we can treat them as $\mathcal{O}(v^4)$ quantities following [346]. They will therefore not appear in the $\mathcal{O}(v^2)$ equations. As far as terms related to $V'(\phi_0)$ are concerned, we intend to just keep track of them for the time being and discuss their contribution later on.

Up to now, we have just perturbed the field equations. The further step needed to arrive at a post-Newtonian expansion is to expand the perturbations of the metric and the scalar field in post-Newtonian orders, *i.e.* orders in the velocity v. The Parametrized Post-Newtonian expansion requires that we expand ϕ and h_{00} to $\mathcal{O}(v^4)$, h_{ij} to $\mathcal{O}(v^2)$ and h_{0i} to $\mathcal{O}(v^3)$. Therefore, we write

$$\delta\phi = {}_2\delta\phi + {}_4\delta\phi \dots \tag{6.58}$$
$$h_{00} = {}_2h_{00} + {}_4h_{00} \dots \tag{6.59}$$
$$h_{ij} = {}_2h_{ij} + \dots \tag{6.60}$$
$$h_{0i} = {}_3h_{0i} + \dots \tag{6.61}$$

where the subscript denotes the order in the velocity, *i.e.* quantities with a subscript $_2$ are $\mathcal{O}(v^2)$, quantities with a subscript $_3$ are $\mathcal{O}(v^3)$, *etc.*

We can now write the field equations for each post-Newtonian order. To derive the parametrized post-Newtonian metric, we need to solve these equations at each order and then use our results to solve to the next order, and successively repeat the process. We start from the field equation for the scalar, eq. (6.54). To order $\mathcal{O}(v^2)$ this gives

$$\lambda\nabla^2({}_2\delta\phi) - V''(\phi_0)_2\delta\phi = 0\ : \tag{6.62}$$

where $\nabla^2 \equiv \delta_{ij}\partial_i\partial_j$. Note that, since the metric is flat in the background, $\mathcal{G} = \mathcal{O}(v^4)$. This explains why we do not get any contribution from the coupling with

$\mathcal{G}$ in eq. (6.62). We want ϕ to take its cosmological value at distances far away from the sources. This is equivalent to saying that the perturbations due to the matter present in the Solar System should vanish at cosmological distances, and this can be achieved by imposing asymptotic flatness for the solution of eq. (6.62), *i.e.* ${}_2\delta\phi \to 0$ for $r \to \infty$. This implies that

$$ {}_2\delta\phi = 0. \tag{6.63} $$

Now we turn our attention to eqs. (6.55), (6.56) and (6.57). To order $\mathcal{O}(v^2)$ for the components 00 and ij and $\mathcal{O}(v^3)$ for the components $0i$, and after applying the standard gauge conditions

$$ h^{\mu}_{i,\mu} - \frac{1}{2} h^{\mu}_{\mu,i} = 0 \,, \qquad h^{\mu}_{0,\mu} - \frac{1}{2} h^{\mu}_{\mu,0} = \frac{1}{2} h^{0}_{0,0} \,, \tag{6.64} $$

the field equations for the metric take the form

$$ \begin{aligned} -\nabla^2({}_2h_{00}) &= 8\,\pi\, G\rho & (6.65)\\ -\nabla^2({}_2h_{ij}) &= 8\,\pi\, G\rho\delta_{ij} & (6.66)\\ \frac{1}{2}\left(\nabla^2({}_3h_{0i}) + \frac{1}{2}({}_2h_{00,j0})\right) &= 8\,\pi\, G\rho v^i & (6.67) \end{aligned} $$

which, remarkably, is exactly the same as in General Relativity [139]. The well-known solutions are

$$ \begin{aligned} {}_2h_{00} &= 2U, & (6.68)\\ {}_2h_{ij} &= 2U\delta_{ij}, & (6.69)\\ {}_3h_{0i} &= -\frac{7}{2}V_i - \frac{1}{2}W_i & (6.70) \end{aligned} $$

where, following [139], we define the post-Newtonian potentials

$$ \begin{aligned} U &= G\int d^3x'\,\frac{\rho(x',t)}{|x-x'|}, & (6.71)\\ V_i &= G\int d^3x'\,\frac{\rho(x',t)v_i(x',t)}{|x-x'|}, & (6.72)\\ W_i &= G\int d^3x'\,\frac{\rho(x',t)v^k(x',t)(x-x')_k(x-x')_i}{|x-x'|^3}. & (6.73) \end{aligned} $$

We can already see that the theory has no deviations away from General Relativity at order $\mathcal{O}(v^3)$: in particular it gives the correct Newtonian limit. It is now easy to go one step further and write down the perturbed equations that we need to $\mathcal{O}(v^4)$. For the scalar field, using ${}_2\delta\phi = 0$, we get

$$ \lambda\nabla^2({}_4\delta\phi) - V''(\phi_0)\,{}_4\delta\phi + f'(\phi_0)\,{}_4\mathcal{G} = 0 \,, \tag{6.74} $$

with

$$
\begin{aligned}
{}_4\mathcal{G} \;=\;& ({}_2h_{00,ij})({}_2h_{00,ij}) - ({}_2h_{00,ii})({}_2h_{00,jj}) + ({}_2h_{ij,ij})^2 + \\
&+({}_2h_{ij,kl})({}_2h_{ij,kl}) - ({}_2h_{ij,kk})({}_2h_{ij,kk}) - \\
&-2({}_2h_{ij,kl})({}_2h_{il,jk}) + ({}_2h_{ij,kl})({}_2h_{kl,ij}) \,, \qquad (6.75)
\end{aligned}
$$

where we have again applied the gauge conditions (6.64). Using eqs. (6.68) and (6.69), eq. (6.75) becomes

$$
{}_4\mathcal{G} = 8\, U_{,kl}U_{,kl} - 8\,(U_{,kk})^2\ . \qquad (6.76)
$$

The solution of eq. (6.74) is therefore

$$
{}_4\delta\phi = \frac{f'(\phi_0)}{4\,\pi}\int d^3x' \frac{{}_4\mathcal{G}(x',t)}{|x-x'|} e^{-\sqrt{V''(\phi_0)}|x-x'|} \qquad (6.77)
$$

The time-time component of the perturbed field equations for the metric to $\mathcal{O}(v^4)$ is

$$
\begin{aligned}
{}_4R_{00} \;=\;& 8\,\pi\, G\Big[({}_4T_{00}) + \frac{1}{2}({}_4T) - \frac{1}{2}({}_2h_{00})({}_2T) \\
&-\frac{1}{2}V'(\phi_0)({}_4\delta\phi) - \frac{1}{2}V(\phi_0) + \frac{1}{2}\lambda\dot{\phi}_0^2\Big], \qquad (6.78)
\end{aligned}
$$

where we have already used the fact that ${}_2\delta\phi = 0$. Note also that no contribution coming from the coupling between ϕ and the curvature terms in eq. (3.149) is present in the above equations. This was to have been expected since in eq. (3.149) these terms always have the structure of two derivatives of ϕ times a curvature term, and so, due to the fact that in the background the metric is flat and ϕ_0 is slowly varying, they can only contribute to orders higher than $\mathcal{O}(v^4)$.

Let us discuss the contribution of the term proportional to $V'(\phi_0)$. Using eqs. (6.77) and (6.76), we can write this term as an integral over the sources times a dimensionless coefficient $8\,\pi\, GV'(\phi_0)f'(\phi_0)$. One can argue that $V'(\phi)$ should be practically zero as far as the post-Newtonian expansion is concerned [330, 331]. This is equivalent to saying that the cosmological solution corresponds to a minimum of the potential. Even though such assumptions are not exact, they are accurate enough for our purposes. Note that even in cases where V does not have a minimum, well-motivated models usually introduce exponential forms for the potential and the coupling function, *i*.e. $V = V_0 e^{-a\kappa\phi}$ and $f = f_0 e^{b\kappa\phi}$ where $\kappa^2 = 8\,\pi\, G$, a, b and f_0 are of order unity while V_0 is as small as the energy density of the cosmological constant in order to guarantee that the theory will fit observations related to the late-time cosmological expansion. This implies that, since $G \sim 1/M_p^2$, then $8\pi\, GV'(\phi_0)f'(\phi_0)$ is dimensionless and of the order of the now renowned 10^{-123}. Therefore, we will not take the term proportional to $V'(\phi_0)$ into account for what comes next. We will return to this issue shortly in order to discuss how this choice affects the generality of our results.

We can use the solutions for ${}_2h_{00}$ and ${}_2h_{ij}$, the gauge conditions (6.64) and the standard post-Newtonian parametrization for matter [139] to write eq. (6.78) as

$$-\nabla^2({}_4h_{00}+2U^2-8\mathbf{\Phi}_2)=8\,\pi\,G\Big[2\rho\left(v^2-U+\frac{1}{2}\Pi-\frac{3p}{2\rho}\right) \\ -V(\phi_0)+\frac{1}{2}\lambda\dot{\phi}_0^2\Big], \tag{6.79}$$

where Π is the specific energy density (the ratio of the energy density to the rest-mass density) [139] and

$$\Phi_2=G\int d^3x'\frac{\rho(x',t)U(x',t)}{|x-x'|}. \tag{6.80}$$

The solution to this equation is

$${}_4h_{00}=2U^2+4\Phi_1+4\Phi_2+2\Phi_3+6\Phi_4+\frac{8\,\pi\,G}{6}\left(V(\phi_0)-\frac{1}{2}\lambda\dot{\phi}_0^2\right)|x|^2, \tag{6.81}$$

where

$$\Phi_1 = G\int d^3x'\frac{\rho(x',t)v(x',t)^2}{|x-x'|}, \tag{6.82}$$

$$\Phi_3 = G\int d^3x'\frac{\rho(x',t)\Pi(x',t)v(x',t)^2}{|x-x'|}, \tag{6.83}$$

$$\Phi_4 = G\int d^3x'\frac{p(x',t)}{|x-x'|}. \tag{6.84}$$

Therefore the metric, expanded in post-Newtonian orders, is

$$g_{00} = -1+2U-2U^2+4\Phi_1+4\Phi_2+ \\ +2\Phi_3+6\Phi_4+\frac{8\,\pi\,G}{6}\left(V(\phi_0)-\frac{1}{2}\lambda\dot{\phi}_0^2\right)|x|^2, \tag{6.85}$$

$$g_{0j} = -\frac{7}{2}V_i-\frac{1}{2}W_i, \tag{6.86}$$

$$g_{ij} = (1+2U)\delta_{ij}, \tag{6.87}$$

which, apart from the term related to $V(\phi_0)-1/2\lambda\dot{\phi}_0^2$, is exactly the result that one obtains for General Relativity. This term corresponds to the standard correction normally arising from a cosmological constant and, since $V(\phi_0)-1/2\lambda\dot{\phi}_0^2$ should indeed be of the same order as the energy density of the cosmological constant, the contribution of this term is negligible on Solar System scales. Since the metric is written in the standard PPN gauge, one can read off the PPN parameters [139]. The only non-vanishing ones are γ and β, which are equal to 1. Therefore, the theory discussed here seems to be indistinguishable from General Relativity at the post-Newtonian order.

The above implies that a gravitational theory with a scalar field coupled to the Gauss–Bonnet invariant trivially satisfies the constraints imposed on the post-Newtonian parameters by Solar System tests, if the reasonable assumptions that we made for the values of $V(\phi_0)$, $V'(\phi_0)$ and $f'(\phi_0)$ hold. This appears to be due to the fact that the terms arising in the field equations for the metric from the coupling between the scalar field and $\mathcal{G}$ in the action always have the structure of two derivatives of f times a curvature term. Such terms do not contribute to the post-Newtonian expansion to $\mathcal{O}(v^4)$. This is not the case for other possible couplings of a scalar to a quadratic curvature term, such as ϕR^2. Remarkably, the characteristic structure of such terms can be traced back to the special nature of $\mathcal{G}$, *i.e.* to the fact that it is a topological invariant in four dimensions.

We now return to discuss how strongly our result depends on the assumption that $V(\phi_0)$ and $V'(\phi_0)$ are reasonably small so as to give a negligible contribution in the PPN expansion. This assumption stems from the fact that $V(\phi_0)$ will play the role of an effective cosmological constant if the theory is to account for the late-time accelerated expansion of the universe and should therefore be of the relevant order of magnitude. Additionally we expect that $V'(\phi_0)$ will also be small enough so that its contribution can be considered negligible, based on the fact that either the field approaches a minimum at late times, or the potential is of the form $V = V_0 e^{-a\kappa\phi}$, where a is of order unity, and therefore $V'(\phi_0) \sim \kappa V(\phi_0)$. The above should be true for all models that lead to a reasonable cosmological phenomenology, once the latter is attributed mainly to presence of the potential $V(\phi)$.

An alternative which one could consider is to attribute the cosmological phenomenology to the coupling function $f(\phi)$. However, it is important to stress that the values of $f'(\phi_0)$ and $f''(\phi_0)$ should be suitable in order for the post-Newtonian expansion to remain trustworthy. From eq. (6.77) we see that non-trivial corrections will indeed be present at post-post-Newtonian orders and, even though such corrections are normally subdominant, if $f'(\phi_0)$ or $f''(\phi_0)$ is sufficiently large it can become crucial for the viability of the theory. This was first observed in [347] where the same theory, but without a potential V, was confronted with Solar System observations, considering a nearly Schwarzschild metric as an approximation. As mentioned before, the potential plays the role of an effective cosmological constant if one wants a theory that leads to a late-time accelerated expansion as in [234, 304, 306, 307, 308, 309, 310, 311, 312]. If this potential is not present, it is the coupling $f(\phi)$ between the scalar field and the Gauss–Bonnet term that will have to account for this phenomenology. In that case, it turns out that $f''(\phi_0)$ has to be of the same order as the inverse of the cosmological constant, and this is enough to make the post-Newtonian approximation break down. Fortunately, models with a potential do not suffer from this problem and, in fact, f is usually assumed to be of the form $f = f_0 e^{b\kappa\phi}$ where both f_0 and b are of order unity. Therefore, as shown here and also predicted in [347], reasonable models with a potential will pass the Solar System tests.

There is yet another possibility: to consider models in which the presence of both the potential and the coupling will somehow be responsible for the cosmological phenomenology. In this sense, the assumptions which we made here for $V(\phi_0)$, $V'(\phi_0)$ and $f'(\phi_0)$ can become more loose and the resulting model would not straightforwardly satisfy the Solar System constraints. This possibility has very recently been considered in Ref. [348] and relevant constraints have been derived. In any case, it is striking that, according to the analysis which we have presented here, it is very easy to propose well-motivated models of Gauss–Bonnet gravity which are practically indistinguishable from General Relativity on Solar System scales. It is also worth commenting that if the coupling function $f(\phi)$ is set to a constant, the action (3.138) simply describes General Relativity with a minimally coupled scalar field or, in other words, quintessence. This implies that as long as the coupling is undetectable on Solar System scales, the theory also cannot be distinguished from any successful quintessence model on these scales.

Finally, let us discuss the possibility of including a second scalar field in the action, coupled to the Gauss–Bonnet invariant, which could, for example, be the dilaton. If this second scalar field is not coupled to matter or to the Ricci scalar, then it can be treated using the same approach as above. If the coupling functions with the Gauss–Bonnet invariant and with the potential, if present, have similar properties to those discussed above, we expect our result to remain unaffected. Of course, there is also the possibility that the dilaton is coupled to matter. This goes beyond the scope of the present discussion since in this case the theory would be phenomenologically different not only regarding Solar System tests, but also for other aspects such as cosmological phenomenology, covariant conservation of matter, the equivalence principle (*e.g.* see ref. [349]) *etc*.

6.5 Non-vacuum solutions in Palatini $f(R)$ gravity

We have already established that, in vacuum, Palatini $f(R)$ gravity reduces to General Relativity with an effective cosmological constant and that, consequently, vacuum spherically symmetric solutions will be Schwarzschild–de Sitter or Schwarzschild–anti-de Sitter. However, one would like to go further than that and derive solutions in the presence of matter as well. The first and simplest step in this direction is to consider solutions with a high degree of symmetry. Indeed spherically symmetric static solutions are quite realistic when it comes to the description of stars and compact objects.

In this section we will consider static spherically symmetric solutions in Palatini $f(R)$ gravity, in the presence of matter. Examining such solutions, apart from the usual interest in them as descriptions of stars, will turn out to be crucial for the understanding of the theory, as will become clear later on. In fact, we will see that serious doubts concerning the viability of the theory will be raised [351].

The standard procedure for arriving at a full solution describing the spacetime inside and outside a static spherically symmetric object is to separately find an exterior solution and an interior solution and then match them together using appropriate junctions condition on the matching surface (Israel junction conditions)

[163]. In General Relativity, in order to determine the interior solution one needs, apart from the field equations, also the Tolman-Oppenheimer-Volkoff (TOV) hydrostatic equilibrium equation (see *e.g.* [157]).

In [350] a generalisation of the TOV equation for Palatini $f(R)$ gravity was derived. Let us briefly review the derivation of this generalized TOV equation and then proceed along the lines of [351] to discuss the solutions. As shown in Section 3.4.3, after suitable manipulations the field equations of Palatini $f(R)$ gravity can be rewritten as a single one (eq. (3.38))

$$\begin{aligned} G_{\mu\nu} = & \frac{8\pi}{F}T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(\mathcal{R} - \frac{f}{F}\right) + \frac{1}{F}\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)F - \\ & - \frac{3}{2}\frac{1}{F^2}\left((\nabla_\mu F)(\nabla_\nu F) - \frac{1}{2}g_{\mu\nu}(\nabla F)^2\right), \end{aligned} \tag{6.88}$$

where ∇_μ is the covariant derivative with respect to the Levi–Civita connection of $g_{\mu\nu}$, $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ and $F = \partial f/\partial\mathcal{R}$.

Using the static spherically symmetric ansatz

$$ds^2 \equiv -e^{A(r)}\mathrm{d}t^2 + e^{B(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{6.89}$$

in eq. (6.88), considering perfect-fluid matter with $T_{\mu\nu} = (\rho + p)u^\mu u^\nu + pg_{\mu\nu}$ (where ρ is the energy density, p is the pressure and u^μ is the fluid 4-velocity) and representing d/dr with a prime [4], one arrives at the equations

$$A' = \frac{-1}{1+\gamma}\left(\frac{1-e^B}{r} - \frac{e^B}{F}8\pi G r p + \frac{\alpha}{r}\right), \tag{6.90}$$

$$B' = \frac{1}{1+\gamma}\left(\frac{1-e^B}{r} + \frac{e^B}{F}8\pi G r \rho + \frac{\alpha+\beta}{r}\right), \tag{6.91}$$

$$\alpha \equiv r^2\left(\frac{3}{4}\left(\frac{F'}{F}\right)^2 + \frac{2F'}{rF} + \frac{e^B}{2}\left(\mathcal{R} - \frac{f}{F}\right)\right), \tag{6.92}$$

$$\beta \equiv r^2\left(\frac{F''}{F} - \frac{3}{2}\left(\frac{F'}{F}\right)^2\right), \qquad \gamma \equiv \frac{rF'}{2F}. \tag{6.93}$$

Making the definition $m_{\mathrm{tot}}(r) \equiv r(1 - e^{-B})/2$ and using Euler's equation,

$$p' = -\frac{A'}{2}(p + \rho), \tag{6.94}$$

[4] In this section we modify our standard notation and instead of using a prime to denote differentiation with respect to the argument of the function, we use it to denote differentiation with respect to the radial coordinate. This signifi cantly lightens the notation.

one gets the generalised TOV equations [350]:

$$
\begin{aligned}
p' = & -\frac{1}{1+\gamma}\frac{(\rho+p)}{r(r-2m_{\rm tot})}\times \\
& \times\left(m_{\rm tot}+\frac{4\pi r^3 p}{F}-\frac{\alpha}{2}(r-2m_{\rm tot})\right),
\end{aligned} \tag{6.95}
$$

$$
m'_{\rm tot} = \frac{1}{1+\gamma}\left(\frac{4\pi r^2\rho}{F}+\frac{\alpha+\beta}{2}-\frac{m_{\rm tot}}{r}(\alpha+\beta-\gamma)\right). \tag{6.96}
$$

In order to determine the interior solution, one needs, besides eqs. (6.95) and (6.96), to have information about the microphysics of the matter configuration under investigation. In the case of a perfect fluid this is effectively given by an equation of state (EOS). A one-parameter EOS relates the pressure directly to the energy density, *i.e.* $p = p(\rho)$. This is the case which we will consider here.

Equations (6.95) and (6.96) are implicit, their right-hand sides effectively including through F' and F'' both first and second derivatives of the pressure, *e.g.* $F' = d/dr\,[F(\mathcal{R}(T))] = (dF/d\mathcal{R})\,(d\mathcal{R}/dT)\,(dT/dp)\,p'$. Therefore, they are difficult to solve so as to derive an interior solution. We therefore first put them into an explicit form, which allows us not only to solve them numerically, but also to study their behaviour at the stellar surface where the matching with the exterior solution occurs.

Multiplying eq. (6.95) by dF/dp and using the definitions of α and γ, we get a quadratic equation in F' whose solution is

$$
F' = \frac{-4rF(\mathcal{C}-F)(r-2m_{\rm tot})+D\sqrt{2\Delta}}{r^2(3\mathcal{C}-4F)(r-2m_{\rm tot})} \tag{6.97}
$$

where $D = \pm 1$ and where we have defined

$$
\mathcal{C} = \frac{dF}{dp}(p+\rho) = \frac{dF}{d\rho}\frac{d\rho}{dp}(p+\rho), \tag{6.98}
$$

$$
\begin{aligned}
\Delta = & Fr^2(r-2m_{\rm tot})\left[8F(\mathcal{C}-F)^2(r-2m_{\rm tot})\, -\right. \\
& \left. -\,\mathcal{C}(4F-3\mathcal{C})\left((16\pi p - F\mathcal{R}+f)r^3+4Fm_{\rm tot}\right)\right].
\end{aligned} \tag{6.99}
$$

We will now focus on polytropic EOSs given by $p = k{\rho_0}^\Gamma$, where ρ_0 is the rest-mass density and k and Γ are constants, noting that this can be rewritten as $\rho = (p/k)^{1/\Gamma} + p/(\Gamma-1)$, giving a direct link between p and ρ. In eq. (6.98), we have written $\mathcal{C}$ in terms of $dF/d\rho$ because this is finite at the stellar surface ($r = r_{\rm out}$ where $p = \rho = 0$). In fact, $dF/d\rho = (dF/d\mathcal{R})\,(d\mathcal{R}/dT)\,(3dp/d\rho - 1)$, where $dF/d\mathcal{R}$ and $d\mathcal{R}/dT$ are in general finite even when $T = 3p - \rho$ goes to zero and $dp/d\rho \to 0$ for $p \to 0$. This can be easily checked, for instance, for the $\mathcal{R}^2$ or $1/\mathcal{R}$ models and it appears to be quite a general characteristic that only very special models (and definitely none of the cosmologically interesting ones) might be able to escape. Note also that while $d\rho/dp$ diverges when $p \to 0$, the product

$(p+\rho)\, d\rho/dp$ goes to zero for $p \to 0$ if $\Gamma < 2$. Therefore, for a polytrope with $\Gamma < 2$, $\mathcal{C} = 0$ at the surface.

We now consider the matching between the interior and exterior solutions. For the latter, the general solution is that of General Relativity plus a cosmological constant. Here, the value of the cosmological constant is equal to $\mathcal{R}_0/4$, where $\mathcal{R}_0$ is the vacuum value of $\mathcal{R}$ (see Section 3.4.3), *i.e.*

$$\exp(-B(r)) = \ell \exp(A(r)) = 1 - 2m/r - \mathcal{R}_0 r^2/12, \tag{6.100}$$

where ℓ and m are integration constants to be fixed by requiring continuity of the metric coefficients across the surface. Using the definition of $m_{\rm tot}(r)$ this gives, in the exterior,

$$m_{\rm tot}(r) = m + \frac{r^3}{24}\mathcal{R}_0 \ . \tag{6.101}$$

Besides continuity of the metric, the junction conditions also require continuity of A', since $\rho = 0$ at the matching surface and, therefore, no surface layer approach can be adopted. For the exterior, at the surface one has

$$A'(r_{\rm out}) = \frac{2\left(r_{\rm out}^3 \mathcal{R}_0 - 12m\right)}{r_{\rm out}\left(\mathcal{R}_0 r_{\rm out}^3 - 12 r_{\rm out} + 24m\right)} \ . \tag{6.102}$$

and this must be matched to the value of $A'(r_{\rm out})$ calculated for the interior solution using eq. (6.90). For this we need $F'(r_{\rm out})$. Evaluating eq. (6.97) at the surface, where $\mathcal{C} = p = 0$ and R, F and f take their constant vacuum values $\mathcal{R}_0$, F_0 and $f_0 = F_0\mathcal{R}_0/2$, we get

$$F'(r_{\rm out}) = -\frac{(1+\widetilde{D})F_0}{r_{\rm out}} \ , \tag{6.103}$$

where $\widetilde{D} = D\,\mathrm{sign}(r_{\rm out} - 2m_{\rm tot})$. Note that, differently from GR, one cannot prove here that $r_{\rm out} > 2m_{\rm tot}$ from eq. (6.95) because p' is not necessarily positive, although one might expect $r_{\rm out} > 2m_{\rm tot}$ in sensible solutions.

In any case, it is easy to see that $\widetilde{D} = 1$ does not give a satisfactory solution, since it implies $\gamma = -1$ at the surface [*cf.* eq. (6.93)] giving $A' \to \infty$ for $r \to r_{\rm out}^-$ [see eq. (6.90)]. Since F' has a discontinuity for $\widetilde{D} = 1$ ($F' \to -2F_0/r_{\rm out}$ when $r \to r_{\rm out}^-$, $F' = 0$ when $r > r_{\rm out}$) Dirac deltas will appear on the right-hand side of eq. (3.38) due to the presence of second derivatives of F and one could hope that they might cancel out with the Dirac deltas arising in the field equations due to the discontinuity of A'. However, the discontinuity in A' is an infinite one and therefore the Dirac deltas arising on the left-hand side of eq. (3.38) can never be cancelled by those on the right-hand side. As already mentioned, one cannot attempt to use here a surface layer approach to avoid discontinuities, because $\rho = 0$ on the surface for a polytrope. In addition, even if it were possible to add a surface layer, the infinite discontinuity of A' would require this layer to have an infinite surface density. We conclude, then, that $\widetilde{D} = 1$ cannot give a satisfactory solution. For $\widetilde{D} = -1$, on the other hand, $F'(r_{\rm out}) = 0$ for $r \to r_{\rm out}^-$ giving the correct interior value of A'

required for matching to eq. (6.102) and making F' continuous across the surface. We will concentrate only on this case in the following.

In order to study the behaviour of $m_{\rm tot}$ at the surface, we need first to derive an explicit expression for F''. If we take the derivative of eq. (6.97), F'' appears on the left-hand side and also on the right-hand side [through $m'_{\rm tot}$, calculated from eq. (6.96) and the definition of β, eq. (6.93)], giving a linear equation in F''. The solution to this, evaluated at the surface, is

$$F''(r_{\rm out}) = \frac{\left(\mathcal{R}_0 r_{\rm out}^3 - 8m_{\rm out}\right)\mathcal{C}'}{8r_{\rm out}(r_{\rm out} - 2m_{\rm out})} \tag{6.104}$$

Evaluating α, β and γ at the surface using $F' = 0$ and F'' given by eq. (6.104), and inserting into eq. (6.96) gives

$$m'_{\rm tot}(r_{\rm out}) = \frac{2F_0\mathcal{R}_0 r_{\rm out}^2 + \left(r_{\rm out}^3\mathcal{R}_0 - 8m_{\rm tot}\right)\mathcal{C}'}{16F_0}\,. \tag{6.105}$$

For $1 < \Gamma < 3/2$, $\mathcal{C}' = d\mathcal{C}/dp\,p' \propto d\mathcal{C}/dp\,(p+\rho) \to 0$ at the surface so that expression (6.105) is finite and it gives continuity of $m'_{\rm tot}$ across the surface [*cf.* eq. (6.101)]. However, for $3/2 < \Gamma < 2$, $\mathcal{C}' \to \infty$ as the surface is approached, provided that $dF/d\mathcal{R}(\mathcal{R}_0) \neq 0$ and $d\mathcal{R}/dT(T_0) \neq 0$ (note that these conditions are satisfied by generic forms of $f(\mathcal{R})$, *i.e.* whenever an $\mathcal{R}^2$ term or a term inversely proportional to $\mathcal{R}$ is present). While $m_{\rm tot}$ keeps finite (as can be shown using the fact that $p' = 0$ at the surface), the divergence of $m'_{\rm tot}$ drives to infinity the Riemann tensor of the metric, $R_{\mu\nu\sigma\lambda}$, and curvature invariants, such as R or $R^{\mu\nu\sigma\lambda}R_{\mu\nu\sigma\lambda}$, as can easily be checked[5]. This singular behaviour would give rise to unphysical phenomena, such as infinite tidal forces at the surface [*cf.* the geodesic deviation equation] which would destroy anything present there.

We can then conclude that no physically relevant solution exists for any polytropic EOS with $3/2 < \Gamma < 2$. Certainly, it is clear that polytropes give only simplified toy models for stars and one would like to use a more accurate description of the interior structure. As an example, we can consider neutron stars, in which case one has a much more complicated dependence of pressure on density, taking account of variations of composition (see, for example, Ref. [353] and references therein). The behaviour of the EOS in the outer layers would be critical for the behaviour of $m'_{\rm tot}$ at the surface in the non-GR case. However, while there are indeed cases where a reasonable solution would be attainable (for instance when $p \propto \rho_0$), one can argue that the viability of a gravity theory should not depend on details such as this and that a real difficulty has been identified.

Setting aside the surface singularity issue, we next focus in neutron stars in order to investigate the interior solution. For such stars we do have more physical descriptions of the interior than a polytrope, a typical example being the FPS EOS, as given in [353], which we use here. As can be seen from eq. (3.38), the metric

[5] This seems to have been missed in Ref. [352].

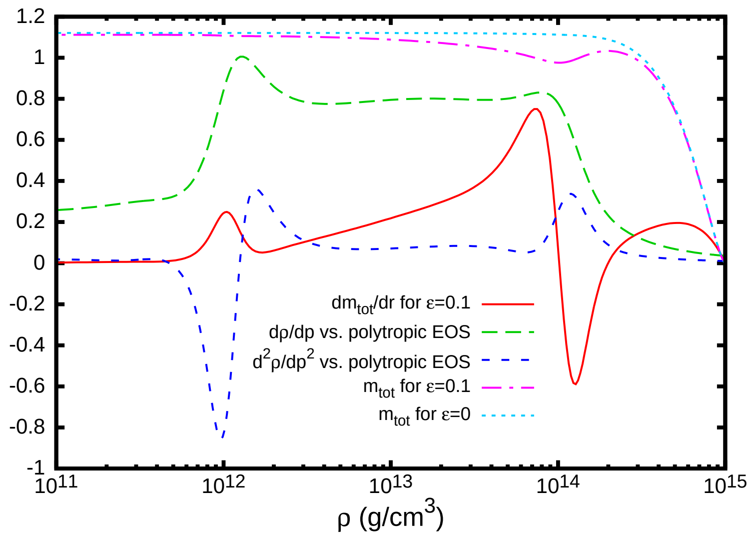

Figure 6.1: Profiles of $m_{\rm tot}$ (measured in units of $M_\odot$) and of other associated quantities plotted against density in the interior of neutron-star models with central density $10^{15}{\rm g/cm^3}$ and $p' = 0$ in the centre as required by local flatness. We have used an analytic fit to the FPS EOS [353] and the gravity theory given by $f(\mathcal{R}) = \mathcal{R} + \epsilon\mathcal{R}^2$. The dot-dashed purple line shows $m_{\rm tot}$ as calculated with $\epsilon = 0.1$ and the dotted cyan line shows the equivalent curve in GR ($\epsilon = 0$); the solid red line shows $dm_{\rm tot}/dr$ (in $M_\odot/{\rm km}$) for $\epsilon = 0.1$. Note the bumps in the $dm_{\rm tot}/dr$ curve which correspond to rapid composition changes in the EOS (the corresponding features in the $m_{\rm tot}$ curve for $\epsilon = 0.1$ are less apparent but a noticeable dip is seen at around $\rho = 10^{14}{\rm g/cm^3}$). To make evident the influence of composition changes, we also show comparisons between the FPS EOS and a corresponding polytrope (with $\Gamma = 4/3$ and $\kappa = 10^{15}$ cgs): the green long-dashed curve and blue short-dashed curve show $0.1 \times (d\rho/dp)_{\rm FPS}/(d\rho/dp)_{\rm polytrope}$ and $0.01 \times (d^2\rho/dp^2)_{\rm FPS}/(d^2\rho/dp^2)_{\rm polytrope}$, respectively.

coefficients will be sensitive to derivatives of the matter fields, since $\mathcal{R}$ is a function of T [6]. This can be seen in Fig. 6.1: For $f(\mathcal{R}) = \mathcal{R} + \epsilon\mathcal{R}^2$, $m_{\rm tot}$, which in GR has a smooth profile, now develops peculiar features when $d\rho/dp$ and $d^2\rho/dp^2$ change rapidly in going from the core to the inner crust and from the inner crust to the outer crust. If $m_{\rm tot}$ were plotted against the radius, these features would look much more abrupt, because they occur in a small range of radii close to the surface. While $m_{\rm tot}$ does not represent a real mass in the interior, such a strong dependence of the metric on the derivatives of the matter field is not very plausible and could

[6]The unusual behaviour of this class of theories has been mentioned in a different context in Ref. [354]. However, we disagree with the claims made there about the violation of the equivalence principle, because they seem to be based on an ill-posed identifi cation of the metric whose geodesics should coincide with free-fall trajectories.

have dramatic consequences.

In our example for the neutron star interior we have chosen $f(\mathcal{R}) = \mathcal{R} + \epsilon\mathcal{R}^2$, even though most interesting models, at least from a cosmological perspective, include a $1/\mathcal{R}$ term. The reason for this is that, since $\mathcal{R}$ is algebraically related to the energy density, a $1/\mathcal{R}$ term is not going to produce large deviations from General Relativity in the interiors of compact objects. Therefore, an $\mathcal{R}^2$ term was definitely more suitable for the specific example considered here. To this one can add that an $\mathcal{R}^2$ term should generically be present in the action if $f(\mathcal{R})$ is taken to be some power series representing the effective low-energy action of a more fundamental theory, even if the $1/\mathcal{R}$ is greatly dominant at cosmological scales. It should be stressed, in any case, that a $1/\mathcal{R}$ term will have similar effects in the interior to those for an $\mathcal{R}^2$ term but they will be more prominent in diffuse objects, where the $\mathcal{R}^2$ term will be quite ineffective. This can actually be even more critical, since the gravitational behaviour of more diffuse objects is even more well established than that of compact objects.

In our attempt to determine and study non-vacuum static spherically symmetric solutions, we have then found two unappealing characteristics of Palatini $f(R)$ gravity as applied to stellar models, each of which arises because of the dependence of the metric on higher order derivatives of the matter field. First: whether or not a regular matching can be made to the exterior solution depends crucially on the microphysics, through the EOS, with polytropic EOSs having $3/2 < \Gamma < 2$ being ruled out for generic $f(\mathcal{R})$. Second: even if an EOS does allow for a regular solution at the surface, the interior metric depends on the first and second derivatives of the density with respect to the pressure, giving a problematic behaviour. While polytropic EOSs are highly idealised, we note that $\Gamma = 5/3$, corresponding to an isentropic monatomic gas or a degenerate non-relativistic particle gas, falls within the range not giving a regular solution. The demonstration that the gravity theory is unable to provide a consistent description for this perfectly physical sort of matter configuration strongly suggests that it is not suitable for being considered as a viable alternative to GR.

Since the problems discussed here arise due to the dependence of the metric on higher order derivatives of the matter fields, one can expect that they will also appear in other gravity theories having these characteristics. Any theory having a representation in which the field equations include second derivatives of the metric and higher than first derivatives of the matter fields will face similar problems because having a higher differential order in the metric than in the matter field is what guarantees that the metric depends in a cumulative way on the matter. If this is not the case then the metric loses its immunity to rapid changes in matter gradients since it is directly related to them instead of being an integral over them. This is the same issue that was pointed out in Section 6.2.2, where the post-Newtonian corrections to the metric were found to depend directly on T instead of being an integral over the sources [eqs. (6.27) and (6.28) and the related discussion about the role of $\Omega(T)$] and in Section 6.3, where R was found to be very sensitive to matter perturbations.

Such shortcomings should be expected for any theory which includes fields other than the metric for describing the gravitational interaction (*e.g.* scalar fields) which are algebraically related to matter rather than being dynamically coupled. In this case one can always solve the field equations of the extra field and insert the solution into the field equation for the metric, inducing a dependence of the metric on higher derivatives of the matter fields. A typical example of such a theory is a scalar-tensor theory with Brans–Dicke parameter $\omega = -3/2$, which is anyway an equivalent representation of Palatini $f(R)$ gravity (see Chapter 4). One should mention that this problem could probably be addressed in Palatini $f(R)$ gravity by adding higher order curvature invariants in the action (*e.g.* $f(\mathcal{R}, \mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu})$), since this would introduce more dynamics and break the non-dynamical coupling between matter and the extra gravitational degrees of freedom.

The results presented in this section can be interpreted as a no-go theorem for theories including higher order derivatives of the matter fields in one of their possible representations, such as Palatini $f(R)$ gravity or $\omega = -3/2$ scalar-tensor theory.

6.6 Conclusions

To summarise: in this chapter we have discussed viability constraints related to the weak and strong gravity regimes for metric and Palatini $f(R)$ gravity and for Gauss–Bonnet gravity. It has been shown that such constraints can act in a complementary manner to the cosmological constraints discussed in the previous chapter. Additionally, given that most of the models considered in the literature are actually tailored to fit cosmological observations, non-cosmological constraints, such as those mentioned here, are crucial for establishing the overall viability of alternative theories of gravity.

Specifically, we have seen that the post-Newtonian limit and stability considerations severely constrain the parameter space of metric $f(R)$ gravity models. In the case of Palatini $f(R)$ gravity, even though issues of stability similar to those present for metric $f(R)$ gravity do not appear, the post-Newtonian limit provides serious indications of non-viability for most models. However, the crucial problem with this theory, its inability to give reasonable solutions for common matter configurations, signalling an incompleteness, becomes apparent when one considers non-vacuum solutions. Finally, well motivated models in Gauss–Bonnet gravity seem to be indistinguishable from General Relativity as far as the Solar System tests are concerned. These last two results highlight, in different ways, the importance of going beyond the standard weak-field-limit tests when trying to constrain alternative theories of gravity.

Chapter 7

Future perspectives and conclusions

7.1 Brief summary

Before concluding this thesis or discussing future perspectives of the work presented here, let us attempt to summarize in this section some of the results presented so far. The motivation of this thesis has been thoroughly discussed in Chapter 1 and a general discussion about modifications of gravity was laid out in Chapter 2. In Chapter 3, a number of specific model theories were introduced and in Chapter 4 the relation between them was explored. Chapters 5 and 6 focused on the cosmological and astrophysical aspects of these theories and on their viability.

As mentioned in the Introduction, these theories were introduced as tools that could help us to examine how much and in which ways one can deviate from General Relativity. Our intention was neither to tailor a model within the framework of any of these theories that would fit the data adequately nor to pick out a specific well-motivated low-energy effective action from some fundamental theory and to confront it with observations. The task which we undertook was to consider theories that were easy to handle, each of them deviating from the framework of General Relativity in a distinct way, and to exploit them in order to get a deeper understanding of the difficulties and limitations of modified gravity. In the light of this, it is probably preferable to provide here a qualitative summary of our results which summarizes the lessons learned from this procedure, instead of repeating in detail the results already presented in the previous chapters.

Starting from the theoretical side, one of the clear outcomes of this thesis is that generalizing the Einstein–Hilbert action to include higher-order curvature invariants is not such a straightforward procedure as it might seem. Even when considering the simplest of generalisations: an $f(R)$ action as studied here, two distinct classes of theory arise depending on the variational principle which one decides to apply. The metric variational principle leads to fourth order equations for the metric, whereas the Palatini variational principle, which treats the connection as an independent variable, leads to second order equations for the metric and an algebraic equation relating the metric and the connections. Remarkably, both approaches lead to General Relativity for the Einstein–Hilbert action. Additionally, we saw that allowing the independent connection to couple to the matter in order to restore its geometrical meaning — that of defining parallel transport and the covariant derivative — led again to a distinct class of theories: metric-affine $f(R)$ theories of gravity, which present an enriched phenomenology since the independence of the connection allows for torsion and non-metricity. In practice, metric-affine $f(R)$ gravity appears to comprise a very general class of theories from which metric $f(R)$ gravity, Palatini $f(R)$ gravity and General Relativity can come about after making a number of simplifying assumptions.

As discussed in Chapter 4 some of the theories presented here can acquire different representations. For instance, metric and Palatini $f(R)$ theories of gravity can be rewritten as Brans–Dicke theories with Brans–Dicke parameters $\omega_0 = 0$ and $\omega_0 = -3/2$ respectively. This equivalence between theories has proved fruitful for clarifying their characteristics. For example, the equivalence between Palatini $f(R)$ gravity and $\omega_0 = -3/2$ Brans–Dicke theory served as a straightforward demonstration of the fact that even though the former theory has an independent connection, it is intrinsically a metric theory of gravity. As we will see in the next section, where we will resume the discussion of theories and their representations, there is much more to be said about this issue.

The discussion about the cosmological and astrophysical aspects of the theories examined here and the confrontation of the theories with cosmological, astrophysical and Solar System observations hopefully clarified that it is very difficult to construct a simple viable model in an alternative theory of gravity. Mainly using metric and Palatini $f(R)$ gravity as toy theories, it was demonstrated that observations which are relevant to different scales provide different constraints for the model examined and that simplistic models which provide an adequate description of the phenomenology related to one scale are easily ruled out when the experimental bounds related to a different scale are taken into account. Solar System tests and bounds from Large Scale Structure perturbations, appear to be very difficult to satisfy with a single theory and, in most cases, constrain the parameter space of the theory unnaturally close to the ΛCDM model.

One might ask how discouraging is the fact that simple models fail to be viable? Indeed an Ockham's razor approach strongly disfavours very complicated models. On the other hand, it should be stressed that in order to explain with an alternative gravitation theory, phenomenology that General Relativity cannot explain without the inclusion of new mysterious matter components, one will inevitably have to add to this theory more complexity and more dynamics. Even though simplicity should not be given away lightheartedly, the best theory is always the simplest one among those that do account for the observations.

Allowing for more dynamics in a gravitational theory, however, has proved to be a far from easy task during the course of this work. Even if the theory is tailored to fit cosmological observations and pass Solar System tests, we saw that problems related to stability can very easily appear. A typical example is the curvature scalar instability in metric $f(R)$ gravity discussed in Section 6.3. On the other hand, in Palatini $f(R)$ gravity, in which the equations are not fourth order in the metric, this instability is not present.

As just mentioned, in order to account for the phenomenology remaining unexplained by General Relativity, one inevitably needs to add more dynamics to the theory. The fact that this dynamics was not added in terms of the metric in Palatini $f(R)$ gravity, did help with issues of stability and simplify the field equations, but this came at a very high price as we saw in Section 6.5. The extra dynamics were implicitly added in the matter part of the theory, even if this is not at all obvious in the standard formulation, and this has dramatic consequences for commulativity. This last example also highlights the importance of going beyond applications to Cosmology and the Solar System when testing alternative theories of gravity.

7.2 What comes next?

7.2.1 Towards a theory of gravitation theories?

Clearly this thesis is far from being an exhaustive study of the theories considered: scalar-tensor theory, $f(R)$ gravity and Gauss–Bonnet gravity. One could, therefore, list here a number of proposals for future work on these theories, some of which have indeed already been mentioned in the previous chapters. For instance, metric-affine gravity is the least studied of the theories considered here and several of its aspects are completely obscure, such as exact solutions, post-Newtonian expansions and Solar System tests, cosmological phenomenology, structure formation, *etc*. Exact solutions have also not been studied in Gauss–Bonnet gravity and there is definitely more to be said about this issue in metric and Palatini $f(R)$ gravity as well.

Instead of continuing this list, which indeed can get quite long, we prefer to follow a different perspective here. We remind the reader once more that all of the above theories should be viewed mainly as toy or straw-man theories used to provide a better understanding of gravity. A more elaborate plan could be, therefore, to go beyond such approaches and this is what we would like to consider here.

Going beyond a trial-and-error approach to modified gravity has very important advantages. From the theoretical side, one has to bear in mind that what we are aiming for is really a better understanding of the conceptual basis of gravity. Even though proposing an alternative theory that violates one of the assumptions of General Relativity and examining whether it is viable or not is a straightforward procedure, it is definitely not the most efficient one. It is a complicated procedure and in many cases it can be misleading, since more than one characteristic of the specific theory can often influence the result. On the other hand, we hope that the reader will be convinced by now that there are already a very large number of alternative theories of gravity in the literature (viable or not) and it is not always clear how much we have managed to learn by studying them.

The benefits at the experimental level are even more clear. Past experience has taught us that experiments test principles and not theories (*e.g.* weak equivalence principle tests such as the gravitational redshift tests [355], which were initially regarded as tests of General Relativity). This directly indicates that the most efficient approach, from an experimental point of view, is to boot-strap our way to a theory starting from the principles which it should satisfy. This would save us a lot of the effort required in bringing the theory to a form suitable for confrontation with observations. The Parametrized Post-Newtonian expansion for $f(R)$ gravity, presented earlier, serves as an ideal example with all of its complications.

Of course, the above hardly constitutes an easy project. In some sense what is being discussed here is essentially the need for an axiomatic formulation of gravitation theories in general. Even in the simplest of these theories, General Relativity, such an axiomatic formulation is not yet available. One could, of course, ask how useful a collection of axioms would be for a theory like General Relativity, when we already know the field equations and the action. Indeed, knowledge of either of these suffices to fully describe the dynamics of the theory, at least at the classical level, which makes the absence of an axiomatic formulation less significant as far as practical purposes are concerned. However, as soon as one moves to even the simplest generalizations of Einstein's theory, such as scalar-tensor gravity for instance, the problem becomes acute as argued above.

A set of axioms could help us to understand the theory in depth and provide a better insight for finding solutions to long standing problems, the most prominent being that of Quantum Gravity. For example, it could help us to determine the fundamental classical properties which one expects to recover in the classical limit and to recognise which of them should break down at the quantum level. Even more, if emergent gravity scenarios are considered (*i.e.* scenarios in which the metric and the affine-connections are collective variables and General Relativity would be a sort of hydrodynamics emergent from more fundamental constituents) such a set of axioms could provide much needed guidance for reconstructing the microscopic system at the origin of classical gravitation, for example by constraining its microscopic properties so as to reproduce the emergent physical features encoded in these axioms.

However, with such a large number of alternative theories of gravity, how can we characterise the way in which they differ from General Relativity, group them, or obtain some insight into which of them are preferable to others? Even if we are far from having a coherent and strict axiomatic formulation, at least a set of principles would definitely prove useful towards this end, as well as for analyzing experimental results to assess the viability of alternative theories.

Already in the 1970s there were attempts to present a set of ground rules, sometimes referred to as a "theory of gravitation theories", which gravitation theories should satisfy in order to be considered viable in principle and, therefore, interesting enough to deserve further investigation. However, no real progress seems to have been made in this direction over the last thirty years, even though the subject of alternative gravity theories has been an active one. It is important to understand the practical reasons for this lack of progress if we wish to proceed beyond the trial-and-error approach that is mostly being used in current research on modified gravity. Hopefully, this exploration, largely based on [356], will also give as a byproduct some interesting clarifications of some common misconceptions (regarding the WEP, equivalence of theories, *etc.*) and serve as a motivational point of reference for future work [1].

7.2.2 From principles to practice and vice-versa

We have argued why it would be interesting to utilise some theory-independent observations to enunciate general viability criteria as a set of theoretical principles that can help us to distinguish potentially viable theories from theories which are ill-posed from the very beginning. As already mentioned, providing a strict axiomatic formulation is hardly an easy goal[2], but one could hope to give at least some set of physical viability principles, even if the latter are not necessarilly at the level of axioms. It is clear that in order to be useful such statements need to be formulated in a theory-independent way and should be amenable to experimental tests so that we could select at least among classes of gravitational theories by suitable observations/experiments.The best example in this direction so far is the Equivalence Principle in its various versions, *i.e.* the Weak Equivalence Principle (WEP), the Einstein Equivalence Principle (EEP) and the Strong Equivalence Principle (SEP) [139]. We have already discussed extensively in Section 2.1.2 the three forms of the equivalence principle as well as their implications for a gravitation theory, such as the existence of a metric and of local Lorentz frames, the coupling of the metric to matter fields *etc*. Therefore, let us just recall the following important remarks and refer the reader back to Section 2.1.2 for more details:

The WEP only says that there exist some preferred trajectories, the free fall trajectories, that test particles will follow and that these curves are the same independently of the mass and internal composition of the particles that follow them (universality of free fall). The WEP does not imply, by itself, that there exists a metric, geodesics, *etc.* — this comes about only through the EEP by combining the WEP with the requirements of Local Lorentz invariance (LLI) and Local Position Invariance (LPI). The same is true for the covariance of the field equations. As far as the SEP is concerned, the main thrust consists of extending the validity of the WEP to self-gravitating bodies and the applicability of LLI and LPI to gravitational experiments, in contrast to the EEP. As mentioned in Section 2.1.2, even though there are experimental tests for all of the EPs, the most stringent ones are those for the WEP and the EEP.

Let us stress that there are at least three subtle points in relation to the use and

[1] In what follows purely classical physics will be considered. The issue of the compatibility between the Equivalence Principle(s) and quantum mechanics, although rich in facets and consequences (see *e.g.* [357, 358, 359, 360, 361, 362]) is beyond the scope of this discussion.

[2] See, however, Refs. [363, 364, 365, 366, 367] for an attempt towards an axiomatic formulation of gravitational theories from a more mathematically-minded point of view.

meaning of the EP formulations, the first one concerning the relation between the SEP and General Relativity. While there are claims that the SEP holds only for General Relativity [139], no proof of this statement has so far been given. Indeed, it would be a crucial step forward to pinpoint a one-to-one association between GR and the SEP but it is easy to realize that it is difficult to relate directly and uniquely a qualitative statement, such as the SEP, to a quantitative one, namely Einstein's equations. The second subtle point is the reference to test particles in all of the EP formulations. Clearly, no true test particles exist, hence the question is: how do we know how "small" a particle should be in order for it to be considered as a test particle (*i.e.* so that its gravitational field can be neglected)? The answer is likely to be theory-dependent[3], and there is no guarantee that a theory cannot be concocted in which the WEP is valid in principle but, in practice, experiments would show a violation because, within the framework of the theory, a "small" particle is not close enough to being a test particle. Of course, such a theory would not be viable but this would not be obvious when we refer to the WEP only from a theoretical perspective (*e.g.* if we calculate free fall trajectories and compare with geodesics). A third subtlety, which we shall come back to later, is related to the fact that sometimes the same theory can appear to either satisfy or not satisfy some version of the EP depending on which variables are used for describing it, an example being the contrast between the Jordan and Einstein frames in scalar-tensor theories of gravity.

Taking all of the above into consideration, it seems that the main problem with all forms of the equivalence principle is that they are of little practical value. As principles they are by definition qualitative and not quantitative. However, quantitative statements are what is needed in practice.

To this end, Thorne and Will [153] proposed the metric theories postulates, which were presented in Section 2.1.3. Essentially, the metric postulates require the existence of a metric $g_{\mu\nu}$ and that the matter stress-energy tensor $T_{\mu\nu}$ should be divergence free with respect to the covariant derivative defined with the Levi–Civita connection of this metric. We have already thoroughly discussed in Sections 2.1.2 and 2.1.3 how the metric postulates encapsulate the validity of the EEP, a key point being that $\nabla_\mu T^{\mu\nu} = 0$ leads to geodesic motion for test particles [154].

Appealing as they may seem, however, the metric postulates lack clarity. As pointed out also by the authors of Ref. [153], any metric theory can perfectly well be given a representation that appears to violate the metric postulates (recall, for instance, that $g_{\mu\nu}$ is a member of a family of conformal metrics and that there is no *a priori* reason why this particular metric should be used to write down the field equations)[4]. One of our goals here is to demonstrate this problem, and also some other prominent ambiguities that we have already very briefly stated in Section 2.1.3, and to trace their roots.

[3] See [368] and references therein for the case of General Relativity.

[4] See also Ref. [369] for an earlier criticism of the need for a metric and, indirectly, of the metric postulates.

What precisely is the definition of stress-energy tensor?

In order to answer this question one could refer to an action. This would be a significant restriction to begin with though, since it would add to the EEP the prerequisite that a reasonable theory has to come from an action. Even so, this would not solve the problem: one could claim that $T_{\mu\nu} \equiv -(2/\sqrt{-g})\delta S_M/\delta g^{\mu\nu}$ but then how is the matter action S_M defined? Claiming that it is the action from which the field equations for matter are derived is not sufficient since it does not provide any insight about the presence of the gravitational fields in S_M. Invoking a minimal coupling argument, on the other hand, is strongly theory-dependent (which coupling is really minimal in a theory with extra fields or an independent connection? [165]). Furthermore, whether a matter field couples minimally or non-minimally to gravity or to matter should be decided by experiments. Since a non-minimal coupling could be present and yet evade experimental detection (as proposed in string theories [349]), it seems prudent to allow for it in the action and in the theory.

Setting actions aside and resorting to the correspondence with the stress-energy tensor of Special Relativity does not help either. There is always more than one tensor that one can construct which will reduce to the special-relativistic stress-energy tensor when gravity is "switched off" and it is not clear what "switched off" exactly means when extra fields describing gravity (scalar or vector) are present in the theory together with the metric tensor.

Finally, mixing the two tentative definitions described above makes the situation even worse: one can easily imagine theories in which

$$T_{\mu\nu} \equiv -(2/\sqrt{-g})\delta S_M/\delta g^{\mu\nu} \tag{7.1}$$

does not reduce to the special-relativistic stress-energy tensor in some limit. Are these theories necessarily non-metric? This point highlights also another important question: are the metric postulates necessary or sufficient conditions for the validity of the EEP? Concrete examples are provided in Sections 7.2.4, 7.2.5 and 7.2.6.

What does "non-gravitational field" mean?

There is no precise definition of "gravitational" and "non-gravitational" field. One could say that a field which is non-minimally coupled to the metric is gravitational whereas all others are matter fields. This definition does not appear to be rigorous or sufficient though and it is shown in the following that it strongly depends on the perspective and terminology that one chooses.

Consider, for example, a scalar field ϕ non-minimally coupled to the Ricci curvature in $\lambda\phi^4$ theory, as described by the action

$$S = \int d^4x\,\sqrt{-g}\,\left[\left(\frac{1}{16\,\pi\,G} - \xi\phi^2\right)R - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - V(\phi)\right]\,. \tag{7.2}$$

If one begins with a classical scalar field minimally coupled to the curvature (*i.e.* $\xi = 0$) in the potential $V(\phi) = \lambda\phi^4$ and quantizes it, one finds that first loop corrections prescribe a non-minimal coupling term (*i.e.* $\xi \neq 0$) if the theory is to be renormalizable, thus obtaining the "improved energy-momentum tensor" of Callan, Coleman, and Jackiw [370] (see also [371]). Does quantization change the character of this scalar field from "non-gravitational" to "gravitational"? Formally, the resulting theory is a scalar-tensor theory according to every definition of such theories that one finds in the literature (*e.g.* [199, 139, 330, 372, 373, 198]) but many authors consider ϕ to be a non-gravitational field, and certainly this is the point of view of the authors of Ref. [370] (in which ϕ is regarded as a matter field to be quantized) and of most particle physicists.

7.2.3 Theories and representations

We have already discussed in Chapter 4 the fact that theories can acquire more than one representation. We used the term "dynamical equivalence" there, in order to refer to the fact that two theories can describe the same dynamics. Within a classical perspective, however, a gravitation theory is indeed a description of the dynamics of a a gravitating system and in this sense, as also mentioned in Chapter 4, when one refers to two dynamically equivalent theories what is actually meant is two different representations of the same theory.

As will be demonstrated later, many misconceptions arise when a theory is identified with one of its representations and other representations are implicitly treated as different theories. Even though this might seem to be a very abstract point, to avoid confusion, one would like to provide precise definitions of the words "theory" and "representation". It is not trivial to do this, however. For the term "theory", even if one looks at a popular internet dictionary, a number of possible definitions can be found [374]:

1. An unproven conjecture.
2. An expectation of what should happen, barring unforeseen circumstances.
3. A coherent statement or set of statements that attempts to explain observed phenomena.
4. A logical structure that enables one to deduce the possible results of every experiment that falls within its purview.
5. A field of study attempting to exhaustively describe a particular class of constructs.
6. A set of axioms together with all statements derivable from them.

Definitions (1) and (2) are not what is meant for scientific theories. On the other hand, (3) and (4) seem to be complementary statements describing the use of the word "theory" in natural sciences, whereas (5) and (6) have mathematical and logical bases respectively. In a loose sense, a more complete definition for the word "theory" in the context of physics would probably come from a combination of (4) and (6), in order to combine the reference to experiments in (4) and the mathematical rigour of (6). An attempt in this direction could be:

Definition 1 *Theory: A coherent logical structure, preferably expressed through a set of axioms together with all statements derivable from them, plus a set of rules for their physical interpretation, that enables one to deduce the possible results of every experiment that falls within its purview.*[5]

Note that no reference is made to whether there is agreement between the predictions of the theory and actual experiments. This is a further step which could be included in the characterization of a theory. There could be criteria according to which the theory is successful or not according to how large a class of observations is explained by it and the level of accuracy obtained (see for example [377]). Additionally, one could consider simplicity as a merit and characterize a theory according to the number of assumptions on which it is based (Ockham's razor). However, all of the above should not be included in the definition itself of the word "theory".

Physical theories should have a mathematical representation. This requires the introduction of physical variables (functions or fields) in terms of which the axioms can be encoded in mathematical relations. We attempt to give a definition:

Definition 2 *Representation (of a theory): A finite collection of equations interrelating the physical variables which are used to describe the elements of a theory and assimilate its axioms.*

The reference to equations can be restrictive, since one may claim that in many cases a theory could be fully represented by an action. At the same time it is obvious that any representation of a theory is far from being unique. Therefore, one might prefer to modify the above definition as follows:

Definition 3 *Representation (of a theory): A non-unique choice of physical variables between which, in a prescribed way, one can form inter-relational expressions that assimilate the axioms of the theory and can be used in order to deduce derivable statements.*

It is worth stressing here that when choosing a representation for a theory it is essential to provide also a set of rules for the physical interpretation of the variables involved in it. This is needed for formulating the axioms (*i.e.* the physical statements) of the theory in terms of these variables. It should also be noted that these rules come as extra information not *a priori* contained in the mathematical formalism. Furthermore, once they are consistently used to interpret the variables

[5] One might argue that when a theory is defined as a set of axioms, as suggested above, it is doomed to face the implications of Gödel's incompleteness theorems. However, it is neither clear if such theorems are applicable to physical theories, nor how physically relevant they would be even if they were applicable [375, 376].

of the latter, they would allow to consistently predict the outcome of experiments in any alternative representation (we shall come back to this point and discuss an example later on in Section 7.2.4).

All of the above definitions are, of course, tentative or even naive ones and others can be found that are more precise and comprehensive. However, they are good enough to make the following point: the arbitrariness that inevitably exists in choosing the physical variables is bound to affect the representation. More specifically, it will affect the clarity with which the axioms or principles of the theory appear in each representation. Therefore, there will be some representations in which it will be obvious that a certain principle is satisfied and others in which it will be more intricate to see that. However, it is clear that the theory is one and the same and that the axioms or principles are independent of the representation. One may consider it a worthy goal to express theories in a representation-invariant language. However, it should be borne in mind that this is exactly what axiomatic formulation is all about and there is probably no way to do this once reference to a set of physical variables has been made. In a sense, the loss of quantitative statements is the price which one has to pay in order to avoid representation dependence.

7.2.4 Example no. 1: Scalar-tensor gravity

In order to make the discussion of the previous sections clearer, let us use scalar-tensor theories of gravitation as an example. As in most current theories, scalar-tensor theories were not originally introduced as collections of axioms but directly through a representation. Instead of using the conventional notation found in the literature, which we have also used when discussing scalar-tensor theories in Section 3.2 and in the rest of this thesis, we will here write the action using the notation of Ref. [243] (see also Ref. [378]):

$$S = S^{(g)} + S^{(m)}\left[e^{2\alpha(\phi)}g_{ab}, \psi^{(m)}\right] , \tag{7.3}$$

where

$$S^{(g)} = \int d^4x \sqrt{-g}\left[\frac{A(\phi)}{16\pi G}R - \frac{B(\phi)}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right] \tag{7.4}$$

and $\psi^{(m)}$ collectively denotes the matter fields. Some of the unspecified functions A, B, V, and α in this notation can be fixed without loss of generality, *i.e.* without choosing a particular theory from within the class, and this is the way in which one is led to the action of a scalar-tensor theory in the more standard notation of Section 3.2. However, this would come at the expense of fixing the representation, which is exactly what we intend to analyse here. Therefore, the present notation is indeed the most convenient for our purposes.

Let us first see how action (7.3) comes about from first principles. As already discussed in Section 2.1.2, following Will's book [139] one can argue that the EEP can only be satisfied if some metric exists and the matter fields are coupled to it not necessarily minimally but through a non-constant scalar, *i.e.* they can be coupled to a quantity $\phi g_{\mu\nu}$, where ϕ is some scalar. However, this coupling should be universal in the sense that all fields should couple to ϕ in the same way [6]. Therefore, the most general form of the matter action will have a dependence on $\phi g_{\mu\nu}$. Of course, one can always choose to write ϕ as $e^{2\alpha(\phi)}$, where ϕ is a dynamical field.

Now the rest of the action should depend on ϕ, the metric and their derivatives. No real principle leads directly to the action above. However, one could impose that the resulting field equations should be of second order both in the metric and in the scalar field and utilize diffeomorphism invariance arguments to arrive at this action. Then, (2.27) is the most general scalar-tensor action that one can write, once no fields other than ϕ and the metric are considered, and no couplings other than a non-minimal coupling of the scalar to the curvature is allowed.

We now return to the role of the four yet-to-be-defined functions $A(\phi)$, $B(\phi)$, $V(\phi)$, $\alpha(\phi)$ and examine whether there are redundancies. As we have already said, the action (2.27) describes a *class* of theories, not a single theory. Specifying some of the four functions will specialize it to a specific theory within that class. However, one can already see that this action is formally invariant under arbitrary conformal transformations $\tilde{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu}$. In fact, it can be recast into its initial form simply by redefining the undetermined functions $A(\phi), B(\phi), V(\phi), \alpha(\phi)$ after making the conformal transformation. This implies that any one of the functions $A(\phi)$, $B(\phi)$, $V(\phi)$ and $e^{2\alpha(\phi)}$ can be set to a (non-vanishing) constant by means of making a suitable choice for $\Omega(\phi)$. Additionally, the scalar field ϕ can be conveniently redefined so as to set yet another of these functions to be a constant. Therefore, we conclude that setting two of these functions to be constants (or just unity) is merely making a choice of representation and has nothing to do with the content of the theory. In fact, it does not even select a theory within the class.

This has a precise physical meaning: it demonstrates our ability to choose our clocks and rods at will [132]. One could decide not to allow this in a theory (irrespectively of how natural that would be). Therefore, it constitutes a very basic physical assumption which can be described as an axiom.

Let us now turn our attention to the matter fields $\psi^{(m)}$: the way in which we have written the action implies that we have already chosen a representation for them. However, it should be clear that we could always redefine the matter fields at will. For example, one could set $\tilde{\psi} = \Omega^s\psi^{(m)}$ where s is a conveniently selected conformal weight [12] so that, after making a conformal transformation, the matter action will be

$$S^{(m)} = S^{(m)}\left[\tilde{g}_{ab}, \tilde{\psi}\right] \ . \tag{7.5}$$

The tilde is used here in order to distinguish between the physical variables in the two representations. We can now make use of the freedom discussed above to fix two of the four functions of the field at will and set $A = B = 1$. Then the action

[6] This is not the case in supergravity and string theories, in which gravivector and graviscalar fields can couple differently to particles with different quark content [379, 380].

(2.27) will formally become that of General Relativity with a scalar field minimally coupled to gravity.

However this theory is not actually General Relativity, since now $\tilde{\psi} = \tilde{\psi}(\phi)$ which essentially means that we have allowed the masses of elementary particles and the coupling constants to vary with ϕ and consequently with the position in spacetime. From a physical perspective, this is translated into our ability to choose whether it will be our clocks and rods that are unchanged in time and space or instead the outcome of our measurements [132] (which, remember, can always be expressed as dimensionless constants or dimensionless ratios, since even the measurement of a dimensional quantity such as *e.g.* a mass, is nothing more than a comparison with a fixed standard unit having the same dimensions). We will return to this issue again in Section 7.2.4.

To summarize, we can in practice choose two of the four functions in the action (7.3) without specifying the theory. In addition, we can even fix a third function at the expense of allowing the matter fields $\psi^{(m)}$ to depend explicitly on ϕ, which leads to varying fundamental units [132]. Once either of these two options is chosen, the representation is completely fixed and any further fixing of the remaining function or functions leads to a specific theory within the class. On the other hand, by choosing any two functions and allowing for redefinitions of the metric and the scalar field, it is possible to fully specify the theory and still leave the representation completely arbitrary.

It is now obvious that each representation might display different characteristics of the theory and care should be taken in order not to be misled into representation-biased conclusions, exactly as happens with different coordinate systems. This highlights the importance of distinguishing between different *theories* and different *representations*.

This situation is very similar to a gauge theory in which one must be careful to derive only gauge-independent results. Every gauge is an admissible "representation" of the theory, but only gauge-invariant quantities should be computed for comparison with experiment. In the case of scalar-tensor gravity however, it is not clear what a "gauge" is and how one should identify the analogue of "gauge-independent" quantities.

Alternative theories and alternative representations:
Jordan and Einstein frames

Let us now go one step further and focus on specific scalar-tensor theories. With $\psi^{(m)}$ representing the matter fields and choosing $\alpha = 0$ and $A(\phi) = \phi$, we fully fix the representation. Let us now suppose that all of the other functions are known. The action then takes the form

$$S = S^{(g)} + S^{(m)} \left[g_{ab}, \psi^{(m)} \right] \ , \tag{7.6}$$

where

$$S^{(g)} = \int d^4x\,\sqrt{-g}\left[\frac{\phi}{16\pi G}R - \frac{B(\phi)}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right]\,, \tag{7.7}$$

and it is apparent that $T_{\mu\nu} \equiv -(2/\sqrt{-g})\delta S^{(m)}/\delta g^{\mu\nu}$ is divergence-free with respect to the metric $g_{\mu\nu}$ and, therefore, the metric postulates are satisfied.

Now we take a representation where $A = B = 1$ and the action takes the form

$$S = S^{(g)} + S^{(m)}\left[e^{2\tilde{\alpha}(\phi)}\tilde{g}_{ab}, \psi^{(m)}\right]\,, \tag{7.8}$$

where

$$S^{(g)} = \int d^4x\,\sqrt{-\tilde{g}}\left[\frac{1}{16\pi G}\tilde{R} - \frac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_a\phi\tilde{\nabla}_b\phi - \tilde{V}(\phi)\right]. \tag{7.9}$$

As we have argued, for any (non-pathological) choice of B and V in the action (7.6), there exists some conformal factor $\Omega(\phi)$, relating $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, and some suitable redefinition of the scalar ϕ to the scalar $\tilde{\phi}$, which brings action (7.6) into the form of action (7.8), therefore relating B and V with $\tilde{V}$ and $\tilde{\alpha}$. Actions (7.6) and (7.8) are just different representations of the same theory after all, assuming that B and V or $\tilde{V}$ and $\tilde{\alpha}$ are known.

According to the most frequently used terminology, the first representation is called the *Jordan frame* and the second the *Einstein frame* and the way in which we have just introduce them should make it very clear that they are just alternative, but physically equivalent, representations of the same theory. (Furthermore, infinitely many conformal frames are possible, corresponding to the freedom in choosing the conformal factor.)

Let us note, however, that if one defines the stress-energy tensor in the Einstein frame as $\tilde{T}_{\mu\nu} \equiv -(2/\sqrt{-\tilde{g}})\delta S^{(m)}/\delta\tilde{g}^{\mu\nu}$, one can show that it is *not* divergence-free with respect to the Levi–Civita connections of the metric $\tilde{g}_{\mu\nu}$. In fact, the transformation property of the matter stress-energy tensor under the conformal transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2\,g_{\mu\nu}$ is $\tilde{T}_{\mu\nu} = \Omega^s\,T_{\mu\nu}$, where the appropriate conformal weight in four spacetime dimensions is $s = -6$ [12]. The Jordan frame covariant conservation equation $\nabla^\beta T_{\alpha\beta} = 0$ is therefore mapped into the Einstein frame equation

$$\tilde{\nabla}_\alpha\tilde{T}^{\alpha\beta} = -\tilde{T}\;\frac{\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\Omega}{\Omega}\,, \tag{7.10}$$

which highlights the fact that the Einstein frame energy-momentum tensor for matter is not covariantly conserved unless it describes conformally invariant matter with vanishing trace T which is not, of course, the general case.

In summary, we see that while the actions (7.6) and (7.8) are just different representations of the same theory, the metric postulates and the EEP are obviously satisfied in terms of the variables of the Jordan frame, whereas, at least judging naively from eq. (7.10), one could be led to the conclusion that the the EEP is not satisfied by the variables of the Einstein frame representation. However this is obviously paradoxical as we have seen that the general form of the scalar-tensor action (7.3) can be derived from the EEP.

The point is that an experiment is not sensitive to the representation, and hence in the case of the action (7.3) it will not show any violation of the EEP. The EEP will *not* be violated in *any* chosen representation of the theory. A common misconception is that people speak about violation of the EEP or the WEP in the Einstein frame simply implying that $\tilde{g}_{\mu\nu}$ is not the metric whose geodesics coincide with free fall trajectories. Even though this is correct, it does not imply a violation of the WEP or the EEP simply because all that these principles require is that there should exist *some* metric whose geodesics coincide with free fall trajectories, and indeed we do have one, namely $g_{\mu\nu}$, the metric tensor of the Jordan frame. The fact of whether or not one chooses to represent the theory with respect to this metric is not relevant.

To go another step further, let us study free fall trajectories in the Einstein frame. Considering a dust fluid with stress-energy tensor $\tilde{T}_{\alpha\beta} = \tilde{\rho}\,\tilde{u}_\alpha\tilde{u}_\beta$, eq. (7.10) becomes

$$\tilde{\nabla}_\alpha\left(\tilde{\rho}\,\tilde{u}^\alpha\tilde{u}^\beta\right) = \tilde{\rho}\,\frac{\tilde{g}^{\alpha\beta}\,\tilde{\nabla}_\alpha\Omega}{\Omega}\;. \tag{7.11}$$

By projecting this equation onto the 3-space orthogonal to $\tilde{u}^\mu$ by means of the operator $\tilde{h}^\mu_\nu$ defined by $\tilde{g}_{\mu\nu} = -\tilde{u}_\mu\tilde{u}_\nu + \tilde{h}_{\mu\nu}$ and satisfying $\tilde{h}^\alpha_\beta\,\tilde{u}^\beta = 0$, one obtains

$$\tilde{a}^\gamma \equiv \tilde{h}^\gamma_\beta\tilde{u}^\alpha\tilde{\nabla}_\alpha\tilde{u}^\beta = \delta^{\gamma\alpha}\frac{\partial_\alpha\Omega(\phi)}{\Omega(\phi)}\;. \tag{7.12}$$

The term on the right hand side of eq. (7.12), which would have been zero if the latter was the standard geodesic equation, can be seen as arising due to the gradient of the scalar field field ϕ, or due to the variation of the particle mass $\tilde{m} = \Omega^{-1}\,m$ along its trajectory, or due to the variation with position in spacetime of the Einstein frame unit of mass $\tilde{m}_u = \Omega^{-1}\,m_u$ (where m_u is the constant unit of mass in the Jordan frame) — see Ref. [381] for an extensive discussion.

Massive particles in the Einstein frame are *always* subject to a force proportional to $\nabla^\mu\phi$, hence there are no massive test particles in this representation of the theory. From this perspective, the formulation of the EEP "(massive) test particles follow (timelike) geodesics" is neither satisfied nor violated: it is simply empty. Clearly, the popular formulation of the EEP in terms of the metric postulates is representation-dependent.

In this sense, the metric $g_{\mu\nu}$ certainly has a distinguished status with respect to any other conformal metric, including $\tilde{g}_{\mu\nu}$. However, it is a matter of taste and sometimes misleading to call a representation physical or non-physical. The fact that it is better highlighted in the Jordan frame that the theory under discussion satisfies the EEP, does not make this frame preferable, in the same sense that the Local Lorentz coordinate frame is not a preferred one. The Einstein frame is much more suitable for other applications, *e.g.* finding new exact solutions by using mappings from the conformal frame, or computing the spectrum of density perturbations during inflation in the early universe.

Let us now concentrate on the ambiguities related to the metric postulates mentioned in Sections 3.56 and 7.2.2. One should already be convinced that these postulates should be generalized to include the phrase "there exists a representation in which". But apart from that, there are additional problems. For example, in the Jordan frame ϕ couples explicitly to the Ricci scalar. One could, therefore, say that ϕ is a gravitational field and not a matter field. In the Einstein frame, however, ϕ is not coupled to the Ricci scalar—it is actually minimally coupled to gravity and non-minimally coupled to matter. Can one then consider it as being a matter field? If this is the case then maybe one should define the stress-energy tensor differently from before and include the ϕ terms in the matter action, *i.e.* define

$$\begin{aligned} \bar{S}^{(m)} &= \int d^4x\, \sqrt{-\tilde{g}} \left[-\frac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \tilde{V}(\tilde{\phi})\right] \\ &+ S^{(m)}\left[e^{2\tilde{\alpha}(\tilde{\phi})}\tilde{g}_{ab}, \psi^{(m)}\right] \end{aligned} \tag{7.13}$$

and

$$\bar{T}_{\mu\nu} \equiv -(2/\sqrt{-\tilde{g}})\delta\bar{S}^{(m)}/\delta\tilde{g}^{\mu\nu}. \tag{7.14}$$

In this case though, $\bar{T}_{\mu\nu}$ will indeed be divergence-free with respect to $\tilde{g}_{\mu\nu}$! The easiest way to see this is to consider the field equations that one derives from the action (7.8) through a variation with respect to $\tilde{g}_{\mu\nu}$ with the redefinitions in eqs. (7.8) and (7.14) taken into account. This gives

$$\tilde{G}_{\mu\nu} = \kappa\bar{T}_{\mu\nu}, \tag{7.15}$$

where $\tilde{G}_{\mu\nu}$ is the Einstein tensor of the metric $\tilde{g}_{\mu\nu}$. The contracted Bianchi identity $\tilde{\nabla}_\mu\tilde{G}^{\mu\nu} = 0$ directly implies that $\tilde{\nabla}_\mu\bar{T}^{\mu\nu} = 0$.

Does this solve the problem, and was the fact that it was not apparent that the EEP is not violated in the Einstein frame just due to a wrong choice of the stress-energy tensor? Unfortunately, this is not the case. First of all, $\tilde{g}^{\mu\nu}$ is still not the metric whose geodesics coincide with free fall trajectories, as shown earlier. Secondly, $\bar{T}_{\mu\nu}$ has the following form

$$\bar{T}_{\mu\nu} = \tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\nabla}^\sigma\tilde{\phi}\tilde{\nabla}_\sigma\tilde{\phi} - \tilde{g}_{\mu\nu}\,\tilde{V}(\tilde{\phi}) + \tilde{T}_{\mu\nu}, \tag{7.16}$$

with $\tilde{T}_{\mu\nu}$ depending on $\tilde{\phi}$ as well as on the matter, and it will not reduce to the special-relativistic stress-energy tensor for the matter field $\psi^{(m)}$ if $\tilde{g}_{\mu\nu}$ is taken to be flat. The same is true for the action $\bar{S}^{(m)}$. Both of these features are due to the fact that $\bar{T}_{\mu\nu}$ includes a non-minimal coupling between the matter fields $\psi^{(m)}$ and the scalar field ϕ. Actually, setting $\tilde{g}_{\mu\nu}$ equal to the Minkowski metric does not correspond to choosing the Local Lorentz frame: that would be the one in which $g_{\mu\nu}$ is flat to second order (see Section 2.1.2).

The moral of this is that one can find quantities that indeed formally satisfy the metric postulates but these quantities are not necessarily physically meaningful. There are great ambiguities, as mentioned before, in defining the stress-energy tensor or in judging whether a field is gravitational or just a matter field, that in practice make the metric postulates useless outside of a specific representation (and how does one know, in general, when given an action, whether it is in this representation, *i.e.* whether the quantities of the representation are the ones to be used directly to check the validity of the metric postulates or whether a representation change is necessary before doing this?).

Matter or geometry? An ambiguity

We already saw that treating ϕ as a matter field merely because it is minimally coupled to gravity and including it in the stress-energy tensor did not help in clarifying the ambiguities of the metric postulates. Since, however, this did not answer the question of whether a field should be considered as gravitational ("geometric") or as non-gravitational ("matter"), let us try to get some further insight into this.

Consider again, as an example, scalar-tensor gravity. Choosing $A(\phi) = 8\pi\, G\, \phi$ and α to be a constant, the action (7.3) can be written as

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[\frac{\phi R}{2} - \frac{B(\phi)}{2}\, g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right. \\ &\quad \left. +\alpha_\psi \mathcal{L}^{(\psi)}\left(g_{\mu\nu}, \psi^{(m)}\right)\right] \;, \end{aligned} \tag{7.17}$$

where α_ψ is the coupling constant between gravity and the specific matter field $\psi^{(m)}$ described by the Lagrangian density $\mathcal{L}^{(\psi)}$. This representation is in the Jordan frame and it is no different from that of the action (7.6), apart from the fact that we have not specified the value of the coupling constant to be 1.

It is common practice to say that the Brans–Dicke scalar field ϕ is gravitational, *i.e.* that it describes gravity together with the metric $g_{\mu\nu}$ [139, 330, 132, 133]. Indeed, $1/\phi$ plays the role of a (variable) gravitational coupling. However, this interpretation only holds in the Jordan frame. As discussed earlier, the conformal transformation to the Einstein frame $g_{\alpha\beta} \to \tilde{g}_{\alpha\beta} = \Omega^2\, g_{\alpha\beta}$ with $\Omega = \sqrt{G\,\phi}$, together with the scalar field redefinition

$$d\tilde{\phi} = \sqrt{\frac{2\omega(\phi)+3}{16\pi G}}\,\frac{d\phi}{\phi} \tag{7.18}$$

casts the action into the form

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2} - \frac{1}{2}\,\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi} - \tilde{V}\left(\tilde{\phi}\right) + \tilde{\alpha}_\psi\mathcal{L}^{(\psi)}\right] \;, \tag{7.19}$$

where

$$\tilde{V}\left(\tilde{\phi}\right) = \frac{V\left[\phi\left(\tilde{\phi}\right)\right]}{\phi^2\left(\tilde{\phi}\right)} \tag{7.20}$$

and

$$\tilde{\alpha}_{\psi}\left(\tilde{\phi}\right)=\frac{\alpha_{\psi}}{\phi^{2}\left(\tilde{\phi}\right)} \,. \tag{7.21}$$

The "new" scalar field $\tilde{\phi}$ is now minimally coupled to the Einstein frame Ricci scalar $\tilde{R}$ and has canonical kinetic energy: *a priori*, nothing forbids one to interpret $\tilde{\phi}$ as being a "matter field". The only memory of its gravitational origin as seen from the Jordan frame is in the fact that now $\tilde{\phi}$ couples non-minimally to matter, as described by the varying coupling $\tilde{\alpha}_{\psi}(\tilde{\phi})$. However, by itself this coupling only describes an interaction between $\tilde{\phi}$ and the "true" matter field $\psi^{(m)}$. One could, for example, take $\psi^{(m)}$ to be the Maxwell field and consider an axion field that couples explicitly to it, obtaining an action similar to (7.19) in which case it would not be possible to discriminate between this axion field and a putative "geometrical" field on the basis of its non-minimal coupling. Even worse, this "anomalous" coupling of $\tilde{\phi}$ to matter is lost if one considers only the gravitational sector of the theory by dropping $\mathcal{L}^{(\psi)}$ from the discussion. This is the situation, for example, if the scalar $\tilde{\phi}$ is taken to dominate the dynamics of an early, inflationary, universe or a late, quintessence-dominated, universe.

More generally, the distinction between gravity and matter ("gravitational" versus "non-gravitational") becomes blurred in any change of representation involving a conformal transformation of the metric $g_{\mu\nu}\rightarrow\tilde{g}_{\mu\nu}=\Omega^{2}\,g_{\mu\nu}$. The transformation property of the Ricci tensor is [12, 382]

$$\begin{aligned}\tilde{R}_{\alpha\beta} \;=\;& R_{\alpha\beta}-2\nabla_{\alpha}\nabla_{\beta}\left(\ln\Omega\right)-g_{\alpha\beta}g^{\gamma\delta}\nabla_{\gamma}\nabla_{\delta}\left(\ln\Omega\right)\\ &+2\left(\nabla_{\alpha}\ln\Omega\right)\left(\nabla_{\beta}\ln\Omega\right)-2g_{\alpha\beta}g^{\gamma\delta}\left(\nabla_{\gamma}\ln\Omega\right)\left(\nabla_{\delta}\ln\Omega\right)\,.\end{aligned} \tag{7.22}$$

The conformal transformation maps a vacuum solution in the Jordan frame (*i.e.* one with $R_{\alpha\beta}=0$) into a non-vacuum solution in the Einstein frame ($\tilde{R}_{\alpha\beta}\neq 0$). The conformal factor Ω, which was a purely "geometrical" field in the Jordan frame, is now playing the role of a form of "matter" in the Einstein frame.

A possible way of keeping track of the gravitational nature of Ω is by remembering that the Einstein frame units of time, length, and mass are not constant but scale according to $\tilde{t}_u=\Omega\,t_u$, $\tilde{l}_u=\Omega\,l_u$, and $\tilde{m}_u=\Omega^{-1}\,m_u$, respectively (where t_u, l_u, and m_u are the corresponding constant units in the Jordan frame) [132]. However, one would not know this prescription by looking only at the Einstein frame action (7.19) unless the prescription for the units is made part of the theory (*i.e.* by carrying extra information additional to that given by the action!). In practice, even when the action (7.19) is explicitly obtained from the Jordan frame representation, the variation of the units with Ω (and therefore with the spacetime location) is most often forgotten in the literature [381] hence leading to the study of a different theory with respect to that expressed by the action (7.17).

Going back to the distinction between material and gravitational fields, an alternative possibility to distinguish between "matter" and "geometry" would seem to arise by labeling as "matter fields" only those described by a stress-energy tensor that satisfies some energy condition. In fact, a conformally transformed field that originates from Jordan frame geometry does not, in general, satisfy any energy condition. The "effective stress-energy tensor" of the field Ω derived from eq. (7.22) does not have the canonical structure quadratic in the first derivatives of the field but contains instead terms that are linear in the second derivatives. Because of this structure, the stress-energy tensor Ω violates the energy conditions. While it would seem that labelling as "matter fields" those that satisfy the weak or null energy condition could eliminate the ambiguity, this is not the case. As we have previously seen, one can always redefine the scalar field in such a way that it is minimally coupled to gravity and has canonical kinetic energy (this is precisely the purpose of the field redefinition (7.18)). Keeping track of the transformation of units in what amounts to a full specification of the representation adopted (action plus information on how the units scale with the scalar field) could help making the property of satisfying energy conditions frame-invariant, but at the cost of extra "structure" in defining a given theory.

As a conclusion, the concept of vacuum versus non-vacuum, or of "matter field" versus "gravitational field" is representation-dependent. One might be prepared to accept *a priori* and without any real physical justification that one representation should be chosen in which the fields are to be characterized as gravitational or non-gravitational and might be willing to carry this extra "baggage" in any other representation in the way described above. Even so, a solution to the problem which would be as tidy as one would like, is still not provided.

These considerations, as well those discussed at the previous sub-section, elucidate a more general point: it is not only the mathematical formalism associated with a theory that is important, but the theory must also include a set of rules to interpret physically the mathematical laws. As an example from the classical mechanics of point particles, consider two coupled harmonic oscillators described by the Lagrangian

$$L = \frac{\dot{q}_1^2}{2} + \frac{\dot{q}_2^2}{2} - \frac{q_1^2}{2} - \frac{q_2^2}{2} + \alpha\, q_1 q_2 \; . \tag{7.23}$$

A different representation of this physical system is obtained by using normal coordinates $Q_1\left(q_1, q_2\right), Q_2\left(q_1, q_2\right)$, in terms of which the Lagrangian (7.23) becomes

$$L = \frac{\dot{Q}_1^2}{2} + \frac{\dot{Q}_2^2}{2} - \frac{Q_1^2}{2} - \frac{Q_2^2}{2} \; . \tag{7.24}$$

Taken at face value, this Lagrangian describes a different physical system, but we know that the mathematical expression (7.24) is not all there is to the theory: the *interpretation* of q_1 and q_2 as the degrees of freedom of the two original oscillators prevents viewing Q_1 and Q_2 as the physically measurable quantities. In addition to the equations of motion, a set of interpretive rules constitutes a fundamental part of a theory. Without such rules it is not only impossible to connect the results derived through the mathematical formalism to a physical phenomenology but one would not even be able to distinguish alternative theories from alternative representations of the same theory. Note however, that once the interpretative rules are assigned to the variables in a given representation they do allow to predict the outcome of experiments in any other given representation of the theory (if consistently applied), hence assuring the physical equivalence of the possible representations.

While the above comments hold in general for any physical theory, it must however be stressed that gravitation theories are one of those cases in which the problem is more acute. In fact, while the physical interpretation of the variables is clear in simple systems, such as the example of the two coupled oscillators discussed above, the physical content of complex theories (like quantum mechanics or gravitation theories) is far less intuitive. Indeed, for what regards gravity, what we actually know more about is the phenomelogy of the system instead of the system itself. Therefore, it is often difficult, or even arbitrary, to formulate explicit interpretive rules, which should nevertheless be provided in order to completely specify the theory.

7.2.5 Example no. 2: $f(R)$ gravity

To highlight further the ambiguity concerning whether a field is a gravitational or matter field, s well as to demostrate how the problems discussed here can actually go beyond representations that just involve conformal redefinitions of the metric, let us examine one further example: that of $f(R)$ gravity in the metric and Palatini formalisms. We have already extensively discussed these theories and in Chapter 4 we have established that they can acquire the representation of a Brans–Dicke theory. Metric $f(R)$ gravity can be re-written as Brans–Dicke theory with Brans–Dicke parameter $\omega_0 = 0$. In terms of the action 7.3, this corresponds to the choice $A = \phi$, $B = 0$, $\alpha = 0$. Palatini $f(R)$ gravity, on the other hand, can be re-written as an $\omega_0 = -3/2$ Brans–Dicke theory, corresponding to the choice $A = \phi$, $B = -3/2$, $\alpha = 0$ when one refers to the action (7.3)

Note that the general representation used in the action (7.3) is actually not as general as one might expect, since we have just shown that theories described by this action can even, with suitable choices of the parameters, acquire completely different non-conformal representations. One can, in principle, add at will auxiliary fields, such as the scalar field used above, in order to change the representation of a theory and these fields need not necessarily be scalar fields. Therefore, all of the problems described so far are not specific to conformal representations. In this $f(R)$ representation, the scalar ϕ is not even there, so how can one decide whether it is a gravitational field or a matter field? For the case of metric $f(R)$ gravity, the scalar field was eliminated without introducing any other field and the metric became the only field describing gravity. On the other hand, in the Palatini formalism the outcome is even more surprising if one considers that the scalar field was replaced with an independent connection which, theoretically speaking, could have forty degrees of freedom assuming that it is symmetric but in practice has only one!

7.2.6 Example no. 3: Einstein–Cartan–Sciama–Kibble theory

Our final example is Einstein–Cartan–Sciama–Kibble theory. In this theory, one starts with a metric and an independent connection which is not symmetric but has zero non-metricity. We will not present extensive calculations and details here but, instead, address the reader to Ref. [196] for a thorough review. What we would like to focus on is the fact that, since the theory has an independent connection, one usually arrives at the field equations through independent variations with respect to the metric and the connections. Additionally, since the matter action depends on both the metric and the connections, its variation will lead to two objects describing the matter fields: the stress-energy tensor $T_{\mu\nu}$, which comes from varying the matter action with respect to the metric as usual, and the hypermomentum $\Delta^{\lambda}{}_{\mu\nu}$, which comes from varying the matter action with respect to the independent connections.

In this theory, $T_{\mu\nu}$ is not divergence-free with respect to either the covariant derivative defined with the Levi–Civita connection or with respect to the one defined with the independent connection. It also does not reduce to the special-relativistic stress-energy tensor in the suitable limit. However, it can be shown that a suitable non-trivial combination of $T_{\mu\nu}$ and $\Delta^{\lambda}{}_{\mu\nu}$ does lead to a tensor that indeed has the latter property [196]. What is more, a third connection can be defined which leads to a covariant derivative with respect to which this tensor is divergence-free [196]! This is sufficient to guarantee that the EEP is satisfied. Does this make Einstein–Cartan theory a metric theory? And how useful are the metric postulates for discussing violations of the EEP if, in order to show that they are satisfied, one will already have demonstrated geodesic motion or LLI on the way?

7.2.7 Discussion

We have attempted to shed some light on the differences between different theories and different representations of the same theory and to reveal the important role played by a representation in our understanding of a theory. For doing this, several examples have been presented which hopefully highlight this issue. It has been argued that certain conclusions about a theory which may be drawn in a straightforward manner in one representation, might require serious effort when a different representation is used and vice-versa. Additionally, care should be taken as certain representations may be completely inconvenient or even misleading for specific applications.

It is worth commenting at this point, that the literature is seriously biased towards particular representations and this bias is not always a result of the convenience of certain representations in a specific application, but often is a mere outcome of habit. It is common, for instance, to bring alternative theories of gravity into a General-Relativity-like representation due to its familiar form, even if this might be misleading when it comes to getting a deeper understanding of the theory.

This seemingly inevitable representation-dependent formulation of our gravitation theories has already been the cause of several misconceptions. What is more, one can very easily recognise a representation bias in the definition of commonly used quantities, such as the stress-energy tensor. Notions such as that of vacuum and the possibility of distinguishing between gravitational fields and matter fields are also representation-dependent. This is often overlooked due to the fact that one is very accustomed to the representation-dependent definitions given in the literature. On the other hand, representation-free definitions do not exist.

Note that even though the relevant literature focuses almost completely on conformal frames, the problems discussed here are not restricted to conformal representations. Even if conformally invariant theories were considered, nothing forbids the existence of other non-conformal representations of these theories in which the action or the field equations will not, of course, be invariant. This might imply that creating conformally invariant theories is not the answer to this issue. After all, even though measurable quantities are always dimensionless ratios and are therefore conformally invariant, matter is not generically conformally invariant and, therefore, neither can (classical) physics be conformally invariant, at least when its laws are written in terms of the fields representing this matter.

The issue discussed here seems to have its roots in a more fundamental problem: the fact that in order to describe a theory in mathematical terms, a non-unique set of variables has to be chosen. Such variables will always correspond to just one of the possible representations of the theory. Therefore, even though *abstract statements such as the EEP are representation-independent*, attempts to turn such statements into *quantitative mathematical relations that are of practical use, such as the metric postulate, turn out to be severely representation-dependent*.

The comparison between a choice of representation and a choice of coordinate system is practically unavoidable. Indeed, consider classical mechanics: one can choose a coordinate system in order to write down an action describing some system. However, such an action can be written in a coordinate invariant way. In classical field theory one has to choose a set of fields — a representation — in order to write down the action. From a certain viewpoint, these fields can be considered as generalized coordinates. Therefore, one could expect that there should be some representation-independent way to describe the theory. However, up to this point no real progress has been made on this issue.

The representation dependence of quantitative statements acts in such a way that, instead of merely selecting viable theories for us, they actually predispose us to choose theories which, in a specific representation, appear more physically meaningful than others irrespective of whether this is indeed the case. The same problem is bound to appear if one attempts to generalise a theory but is biased towards a specific representation, since certain generalisations might falsely appear as being more "physical" than others in this representation. This effectively answers the question of why most of our current theories of gravitation eventually turn out to be just different representations of the same theory or class of theories. Scalar-tensor theories and theories which include higher order curvature invariants, such as $f(R)$ gravity or fourth order gravity, are typical examples.

Even though this discussion might at some level appear to be purely philosophical, the practical implications of representation dependence should not be underestimated. For instance, how can we formulate theories that relate matter/energy and gravity if we do not have a clear distinction between the two, or if we cannot even conclude whether such a distinction should be made? Should we then aim to avoid any statement based on a sharp separation between the matter and gravity sectors?

7.3 Concluding remarks

To conclude, even though some significant progress has been made with developing alternative gravitation theories, one cannot help but notice that it is still unclear how to relate principles and experiments in practice, in order to form simple theoretical viability criteria which are expressed mathematically. Our inability to express these criteria and also several of our very basic definitions in a representation-invariant way seems to have played a crucial role in the lack of development of a theory of gravitation theories. This is a critical obstacle to overcome if we want to go beyond a trial-and-error approach in developing alternative gravitation theories.

It is the author's opinion that such an approach should be one of the main future goals in the field of modified gravity. This is not to say, of course, that efforts to propose or use individual theories, such as $f(R)$ gravity or Gauss–Bonnet gravity, in order to deepen our understanding about the gravitational interaction should be abandoned or have less attention paid to them. Such theories have proved to be excellent tools for this cause so far, and there are still a lot of unexplored corners of the theories mentioned in this thesis, as well as in other alternative theories of gravity.

The motivation for modified gravity coming from High Energy Physics, Cosmology and Astrophysics is definitely strong. Even though modifying gravity might not be the only way to address the problems mentioned in Chapter 1, it is our hope that the reader is by now convinced that it should at least be considered very seriously as one of the possible solutions and, therefore, given appropriate attention. The path to the final answer is probably long. However, this has never been a good enough reason for scientists to be discouraged.

If I have ever made any valuable discoveries, it has been owing more to patient attention, than to any other talent.

Isaac Newton

Bibliography

[1] D. R. Brill and R. H. Gowdy, *Rep. Prog. Phys.* **33**, 413 (1970).

[2] C. J. Isham, in *Quantum Gravity 2: A Second Oxford Symposium*, edited by C. J. Isham, R. Penrose and D. W. Sciama, (Clarendon Press, Oxford, 1981).

[3] S. W. Hawking, *Proc. R. Soc. A*, **300**, 187 (1967).

[4] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, (Cambridge University Press, Cambridge, 1973).

[5] J. A. Wheeler, in *Relativity, Groups and Topology*, edited by B. S. DeWitt and C. M. DeWitt, (Gordon and Breach, New York, 1964).

[6] W. Pauli, *Theory of Relativity*, (Pergamon Press, London, 1967).

[7] S. Deser, *Gen. Rel. Grav.* **1**, 181 (1970).

[8] I. B. Khriplovich, *Sov. J. Nucl. Phys.* **3**, 415 (1966).

[9] B. S. DeWitt, *Phys. Rev. Lett.* **13**, 114 (1964).

[10] C. J. Isham, Abdus Salam and J. Strathdee, *Phys. Rev. D* **3**, 1805 (1971).

[11] S. Weinberg, *Gravitation and Cosmology*, (John Wiley & Sons, United States of America, 1972).

[12] R. M. Wald, *General Relativity*, (University of Chicago Press, United States of America, 1984).

[13] C. W. Misner, *Astrophys. J.* **151**, 431 (1968).

[14] R. H. Dicke and P. J. E. Peebles, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel, (Cambridge University Press, Cambridge, 1979).

[15] G. Lemaître, *Ann. Soc. Sci. Bruxelles A* **47**, 49 (1933).

[16] G. Lemaître, *Gen. Rel. Grav.* **29**, 641 (1997).

[17] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology*, (Clarendon Press, Oxford, 1934).

[18] C. W. Misner, *Phys. Rev. Lett.* **22**, 1071 (1969).

[19] A. H. Guth, *Phys. Rev. D* **23**, 347 (1981).

[20] A. A. Starobinsky, *Phys. Lett. B* **91**, 99 (1980).

[21] D. Kazanas, *Astrophys. J.* **241**, L59 (1980).

[22] K. Sato, *Phys. Lett. B* **33**, 66 (1981).

[23] R. Brout, F. Englert and E. Gunzig, *Gen. Rel. Grav.* **10**, 1 (1979).

[24] A. A. Penzias and R. W. Wilson, *Astrophys. J.* **142**, 419 (1965).

[25] P. J. E. Peebles and J. T. Yu, *Astrophys. J.* **162**, 815 (1970).

[26] R. A. Sunyaev, in *Large Scale Structure of the Universe*, edited by M. S. Longair and J. Einasto, (Dordrecht, Reidel, 1978).

[27] G. F. Smoot *et al.*, *Astrophysic. J.* **396**, L1 (1992).

[28] C. L. Bennett *et al.*, *Astrophys. J.* **464**, L1 (1996).

[29] A. D. Miller *et al.*, *Astrophys. J.* **524**, L1 (1999).

[30] P. de Bernardis *et al.*, *Nature* **404**, 955 (2000).

[31] S. Hanany *et al.*, *Astrophys. J.* **545**, L5 (2000).

[32] W. Hu and M. White, *Astrophys. J.* **471**, 30 (1996).

[33] http://map.gsfc.nasa.gov/

[34] http://www.rssd.esa.int/index.php?project=Planck

[35] A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large Scale Structure*, (Cambridge University Press, Cambridge, 2000).

[36] http://www.mso.anu.edu.au/2dFGRS

[37] http://www.sdss.org/

[38] A. Refregier, *Ann. Rev. Astron. Astrophys.* **41**, 645 (2003).

[39] S. W. Allen *et al.*, *Mon. Not. Roy. Astr. Soc.* **342**, 287 (2003).

[40] A. G. Riess *et al.* [Supernova Search Team Collaboration], *Astrophys. J.* **607**, 665 (2004).

[41] P. Astier *et al.*, *Astron. Astroph.* **447**, 31 (2006).

[42] D. J. Eisenstein *et al.* [SDSS Collaboration], *Astrophys. J.* **633**, 560 (2005).

[43] D. N. Spergel *et al.*, *Astrophys. J. Suppl.* **148**, 175 (2003).

[44] E. Komatsu *et al.*, *Astrophys. J. Suppl.* **148**, 119 (2003).

[45] H. V. Peiris *et al.*, *Astrophys. J. Suppl.* **148**, 213 (2003).

[46] V. F. Mukhanov, *preprint* arXiv: astro-ph/0303077.

[47] S. Burles, K. M. Nollett and M. S. Turner, *Astrophys. J. Lett.* **552**, L1 (2001).

[48] S. M. Carroll and M. Kaplinghat, *Phys. Rev. D* **65**, 063507 (2002).

[49] F. Zwicky, *Helvetica Physica Acta* **6**, 110 (1933).

[50] F. Zwicky, *Astrophys. J.* **86**, 217 (1937).

[51] F. Kahn and L. Waltjer, *Astrophys. J.* **130**, 705 (1959).

[52] V. Rubin and W. K. Ford Jr, *Astrophys. J.* **159**, 379 (1970).

[53] V. Rubin, W. K. Ford Jr, N. Thonnard, *Astrophys. J.* **238**, 471 (1980).

[54] A. Bosma, *The distribution and kinematics of neutral hydrogen in spiral galaxies of various morphological types*, PhD Thesis, (Rejksuniversiteit Groningen, 1978).

[55] A. Bosma, *preprint* arXiv: astro-ph/9812015.

[56] J. R. Ellis, *preprint* arXiv: astro-ph/0204059.

[57] B. Moore, *preprint* arXiv: astro-ph/0103100.

[58] D. Clowe, M. Bradac, A. H. Gonzalez, M. Markevitch, S. W. Randall, C. Jones and D. Zaritsky, *preprint* arXiv: astro-ph/0608407.

[59] V. Springel *et al.*, *Nature* **435**, 629 (2005).

[60] W. de Sitter, *Mon. Not. R. Astron. Soc.* **78**, 3 (1917).

[61] A. Einstein, *Sitzungber. Preuss. Akad. Wiss. Phys.-Math. Kl.* 142 (1917).

[62] A. Pais, *'Subtle is the Lord...': The Science and the Life of Albert Einstein*, (Oxford University Press, New York, 1982).

[63] S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989).

[64] S. M. Carroll, W. H. Press, E. W. Turner, *Annu. Rev. Astron. Astrophys.* **30**, 499 (1992).

[65] S. M. Carroll, *Living Rev. Relativity* **4**, 1 (2001).

[66] P. J. E. Peebles and B. Ratra, *Rev. Mod. Phys.* **75**, 559 (2003).

[67] S. M. Carroll, *preprint* arXiv: astro-ph/0107571.

[68] A. R. Liddle and D. H. Lyth, *Phys. Lett. B* **291**, 391 (1992).

[69] A. R. Liddle and D. H. Lyth, *Phys. Rep.* **231**, 1 (1993).

[70] L. F. Abbott, E. Farhi and M. B. Wise, *Phys. Lett. B* **117**, 29 (1982).

[71] A. Albrecht, P. J. Steinhardt, M. S. Turner and F. Wilczek, *Phys. Rev. Lett.* **48**, 1437 (1982).

[72] Y. Shtanov, J. H. Traschen and R. H. Brandenberger, *Phys. Rev. D* **51**, 5438 (1995).

[73] L. Kofman, A. D. Linde and A. A. Starobinsky, *Phys. Rev. Lett.* **73**, 3195 (1994).

[74] L. Kofman, A. D. Linde and A. A. Starobinsky, *Phys. Rev. D* **56**, 3258 (1997).

[75] A. Linde, *Particle Physics and Inflationary Cosmology*, (Harwood Academic Publishers, Switzerland, 1990).

[76] E. W. Kold and M. S. Turner, *The Early universe*, (Addison-Wesley, California, 1990).

[77] V. Mukhanov, *Physical Foundations of Cosmology*, (Cambridge University Press, Cambridge, 2005).

[78] P. J. E. Peebles, *Principles of Physical Cosmology*, (Princeton University Press, Princeton,, 1993).

[79] D. H. Lyth, *preprint* arXiv: hep-th/0311040.

[80] G. E. Volovik, *Int. J. Mod. Phys. D* **15**, 1987 (2006).

[81] B. Carter, in *International Astronomical Union Symposium 63: Confrontation of Cosmological Theories with Observational Data*, edited by M. S. Lorgair, (Dordrecht, Reidel, 1974).

[82] B. Carter, in *The Constants of Physics, Proceedings of a Royal Society Discussion Meeting, 1983*, edited by W. H. McCrea and M. J. Rees, (Cambrige University Press, Cambridge, 1983).

[83] J. D. Barrow and F. J. Tipler, *The Anthropic Cosmological Principle*, (Clarendon Press, Oxford, 1986).

[84] S. Weinberg, *Phys. Rev. Lett.* **59**, 2607 (1987).

[85] L. Susskind, *preprint* arXiv: hep-th/0302219.

[86] P. J. Peebles and R. Ratra, *Astrophys. J.* **325**, L17 (1988).

[87] R. Ratra and P. J. Peebles, *Phys. Rev. D* **37**, 3406 (1988).

[88] C. Wetterich, *Nucl. Phys. B* **302**, 668 (1988).

[89] J. P. Ostriker and P. J. Steinhardt, *Nature* **377**, 600 (1995).

[90] R. R. Caldwell, R. Dave and P. J. Steinhardt, *Phys. Rev. Lett.* **80**, 1582 (1998).

[91] S. M. Carroll, *Phys. Rev. Lett.* **81**, 3067 (1998).

[92] N. A. Bahcall, J. P. Ostriker, S. Perlmutter and P. J. Steinhardt, *Science* **284**, 1481 (1999).

[93] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, *Phys. Rev. Lett.* **85**, 4483 (2000).

[94] L. M. Wang, R. R. Caldwell, J. P. Ostriker and P. J. Steinhardt, *Astrophys. J.* **530**, 17 (2000).

[95] E. J. Copeland, A. R. Liddle and D. Wands, *Phys. Rev. D* **57**, 4686 (1998).

[96] P. G. Ferreira and M. Joyce, *Phys. Rev. D* **58**, 023503 (1998).

[97] I. Zlatev, L. Wang and P. J. Steinhardt, *Phys, Rev. Lett.* **82**, 896 (1999).

[98] A. R. Liddle and R. J. Scherrer, *Phys. Rev. D* **59**, 023509 (1999).

[99] P. J. Steinhardt, L. Wang and I. Zlatev, *Phys. Rev. D* **59**, 123504 (1999).

[100] I. Zlatev and P. J. Steinhardt, *Phys. Lett. B* **459**, 570 (1999).

[101] V. Sahni and L. Wang, *Phys. Rev. D* **62**, 103517 (2000).

[102] S. M. Carroll, *Phys. Rev. Lett.* **81**, 3067 (1998).

[103] R. Horvat, *Mod. Phys. Lett. A* **14**, 2245 (1999).

[104] T. Buchert, *preprint* arXiv: gr-qc/0612166.

[105] T. J. Sumner, *Living Rev. Relativity* **5**, 4 (2002).

[106] G. Bertone, D. Hooper and J. Silk, *Phys. Rept.* **405**, 279 (2005).

[107] S. D. M. White, C. Frenk and M. Davis, *Astrophys. J. Lett.* **274**, L1 (1983).

[108] X. Wang, M. Tegmark and M. Zaldarriaga, *Phys. Rev. D* **65**, 123001 (2002).

[109] R. Bernabei *et al.*, *Riv. Nuovo Cim.* **26N1**, 1 (2003).

[110] http://lpsc.in2p3.fr/mayet/dm.html

[111] http://lhc.web.cern.ch/lhc/

[112] R. Arnowitt, S. Deser and C. W. Misner, *Phys. Rev.* **117**, 1595 (1960).

[113] R. Arnowitt, S. Deser and C. W Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten, (John Wiley and Sons, New York, 1962).

[114] R. Utiyama and B. S. De Witt, *J. Math. Phys.* **3**, 608 (1962).

[115] K. S. Stelle, *Phys. Rev. D* **16**, 953 (1977).

[116] B. Delamotte, *Am. J. Phys.* **72**, 170 (2004).

[117] I. L. Buchbinder, S. D. Odintsov and I. L. Shapiro, *Effective Action in Quantum Gravity*, (IOP Publishing, Bristol, 1992).

[118] N. D. Birrell and P. C. W. Davies, *Quantum Fields In Curved Space*, (Cambridge University Press, Cambridge, 1982)

[119] G. A. Vilkovisky, *Class. Quant. Grav.* bf 9, 895 (1992).

[120] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, (Cambridge University Press, Cambridge, 1987).

[121] J. Polchinski, *String Theory*, (Cambridge University Press, New York, 1998).

[122] M. J. Duff, *Int. J. Mod. Phys. A* **11**, 5623 (1996).

[123] C. Rovelli, *Living Rev. Relativity* **1**, 1 (1998).

[124] T. Thiemann, *Lect. Notes Phys.* **631**, 41 (2003).

[125] A. Ashtekar and J. Lewandowski, *Class. Quant. Grav.* **21**, R53 (2004).

[126] A. Ashtekar, *Lectures on Non-Perturbative Canonical Gravity*, (World Scientific, Singapore, 1991).

[127] C. Rovelli, *Quantum Gravity*, (Cambridge University Press, New York, 2004).

[128] M. Gasperini and G. Veneziano, *Phys. Lett. B* **277**, 256 (1992).

[129] S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **576**, 5 (2003).

[130] D. V. Vassilevich, *Phys. Rept.* **388**, 279 (2003).

[131] A. S. Eddington, *The Mathematical Theory of Relativity*, (Cambridge University Press, Cambridge, 1923).

[132] R. H. Dicke, *Phys. Rev.* **125**, 2163 (1962).

[133] C. Brans and R. H. Dicke, *Phys. Rev.* **124**, 925 (1961).

[134] R. V. Pound and G. A. Rebka Jr., *Phys. Rev. Lett.* **4**, 337 (1960).

[135] J. Lense and H. Thirring, *Physik. Zeitschr.* **19**, 156 (1918).

[136] K. Nordtvedt Jr., *Phys. Rev.* **169**, 1014 (1968).

[137] I. I. Shapiro, *Phys. Rev. Lett.* **13**, 789 (1964).

[138] R. A. Hulse and J. H. Taylor, *Astrophys. J.* **195**, L51 (1975).

[139] C. M. Will, *Theory and experiment in gravitational physics*, (Cambridge University Press, New York, 1981).

[140] L. I. Schiff, *Am. J. Phys.* **28**, 340 (1960).

[141] R. H. Dicke, *Am. J. Phys.* **28**, 344 (1960).

[142] K. Nordtvedt Jr., *Phys. Rev.* **169**, 1017 (1968).

[143] C. M. Will, *Astrophys. J.* **163**, 611 (1971).

[144] C. M. Will and K. Nordtvedt Jr., *Astrophys. J.* **177**, 757 (1972).

[145] C. M. Will, *Astrophys. J.* **185**, 31 (1973).

[146] R. H. Dicke, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt, (Gordon and Breach, New York, 1964).

[147] D. Giulini, *preprint* arXiv: gr-qc/0603087.

[148] H. Bondi, *Rev. Mod. Phys.* **29**, 423 (1957).

[149] G. Nordström, *Ann. Phys.* **42**, 533 (1913).

[150] R. V. Eötvös, V. Pekár and E. Fekete, *Ann. Phys.* **68**, 11 (1922).

[151] C. M. Will, *Living Rev. Relativity* **9**, 3 (2006).

[152] D. Mattingly, *Living Rev. Relativity* **8**, 5 (2005).

[153] K. S. Thorne and C. M. Will, *Astrophys. J.* **163**, 595 (1971).

[154] V. Fock, *The Theory of Space, Time and Gravitation*, (Pergamon Press, New York, 1964.

[155] E. Schrödinger, *Space-Time Structure*, (Cambridge University Press, Cambridge, 1963).

[156] M. Tsamparlis, *J. Math. Phys.* **19**, 555 (1977).

[157] B. F. Schutz, *A First Course in General Relativity*, (Cambridge University Press, Cambridge, 1985).

[158] S. W. Hawking, *General Relativity: An Einstein Centenary Survey*, edited by S. Hawking and W. Israel, (Cambridge University Press, Cambridge, 1979).

[159] J. W. York, *Phys. Rev. Lett.* **28**, 1082 (1972).

[160] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, (Pergamon Press, Oxford, 1975).

[161] A. Einstein, Doc. 31 in *The Collected Papers of Albert Einstein, Vol. 6, 1914-1917*, A. J. Kox, M. J. Klein and R. Schulmann (eds.), (Princeton University Press, New Jersey, 1996).

[162] T. Padmanabhan, *Phys. Rept.* **406**, 49 (2005).

[163] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (W. H. Freeman and Co., San Francisco, 1973).

[164] M. Ferraris, M. Francaviglia and C. Reina, *Gen. Rel. Grav.* **14**, 243 (1982).

[165] T. P. Sotiriou and S. Liberati, *Ann. Phys.* **322**, 935 (2007).

[166] T. P. Sotiriou, *Class. Quant. Grav.* **23**, 5117 (2006).

[167] T. P. Sotiriou, *preprint* arXiv: gr-qc/0611158.

[168] F. W. Hehl and G. D. Kerling, *Gen. Rel. Grav.* **9**, 691 (1978).

[169] H. Weyl, *Ann. Phys.* **59**, 101 (1919).

[170] P. Jordan, *Schwerkraft und Weltall*, (Vieweg, Braunschweig, 1955).

[171] R. W. Hellings and K. Nordtvedt Jr., *Phys. Rev. D* **7**, 3593 (1973).

[172] J. D. Bekenstein, *Phys. Rev. D* **70**, 083509 (2004) [Erratum-ibid. D **71** (2005) 069901].

[173] M. Milgrom, *Astroph. J.* **270**, 365 (1983).

[174] M. Milgrom, *Astroph. J.* **270**, 371 (1983).

[175] M. Milgrom, *Astroph. J.* **270**, 384 (1983).

[176] T. Jacobson and D. Mattingly, *Phys. Rev. D* **64**, 024028 (2001).

[177] C. Eling, T. Jacobson and D. Mattingly, *preprint* arXiv: gr-qc/0410001.

[178] N. Rosen, *J. Gen. Rel. and Grav.* **4**, 435 (1973).

[179] N. Rosen, *Ann. Phys.* **84**, 455 (1974).

[180] N. Rosen, *Astroph. J.* **211**, 357 (1977).

[181] M. A. Clayton and J. W. Moffat, *Phys. Lett. B* **460**, 263 (1999).

[182] J. D. Barrow and J. Magueijo, *Class. Quant. Grav.* **16**. 1435 (1999).

[183] J. D. Barrow, *Phys. Rev. D* **59** 043515 (1999).

[184] J. D. Barrow and J. Magueijo, *Phys. Lett. B* **443**, 104 (1998).

[185] H. Weyl, *Space, Time, Matter*, (Methuen, London, 1922).

[186] E. Scrödinger, *Proc. R. Ir. Acad. A* **51**, 163 (1947).

[187] A. Einstein and E. G. Straus, *Ann. Math.* **47**, 731 (1946).

[188] N. J. Poplawski, *preprint* arXiv: gr-qc/0612193.

[189] N. J. Poplawski, *preprint* arXiv: gr-qc/0701176.

[190] J. Kijowski and R. Werpachowski, *Rept. Math. Phys.* **59**, 1 (2007).

[191] É. Cartan, *C. R. Acad. Sci.* (Paris) **174**, 593 (1922).

[192] É. Cartan, *Ann. Ec. Norm. Sup.* **40**, 325 (1923).

[193] É. Cartan, *Ann. Ec. Norm. Sup.* **41**, 1 (1924).

[194] D. W. Sciama, *Rev. Mod. Phys.* **36**, 463 (1964).

[195] T. W. B. Kibble, *J. Math. Phys.* **2**, 212 (1961).

[196] F. W. Hehl, P. von der Heyde, G. D. Kerlick and J. M. Nester, *Rev. Mod. Phys.* **48**, 393 (1976).

[197] B. Bertotti, L. Iess and P. Tortora, *Nature* **425**, 374 (2003).

[198] Y. Fujii and K. Maeda, *The Scalar-Tensor Theory of Gravitation* (Cambridge University Press, Cambridge, 2003).

[199] V. Faraoni, *Cosmology in Scalar-Tensor Gravity*, (Kluwer Academic, Dordrecht, 2004).

[200] K. Nordvedt, *Phys. Rev.* **169**, 1017 (1968).

[201] H. A. Buchdahl, *Mon. Not. Roy. Astr. Soc.* **150**, 1 (1970).

[202] J. D. Barrow and M. Madsen, *Nucl. Phys. B* **323**, 242 (1989).

[203] M. Ferraris, M. Francaviglia and I. Volovich, *preprint* arXiv: gr-qc/9303007.

[204] T. S. Bunch, *J. Phys. A: Math. Gen.* **14**, L139 (1981).

[205] C. Lanczos, *Ann. Math.* **39**, 842 (1938).

[206] R. Schimming and H.-J. Schmidt, *NTM Schriftenr. Gesch. Naturw. Tech. Med.* **27**, 41 (1990).

[207] H.-J. Schmidt, *preprint* arXiv: gr-qc/0602017.

[208] T. V. Ruzmaikina and A. A. Ruzmaikin, *JETP* **30**, 372 (1970).

[209] S. Gottlöber, H.-J. Schmidt and A. A. Starobinski, *Class. Quant. Grav.* **7**, 893 (1990).

[210] L. Amendola, A. Battaglia-Mayer, S. Capozziello, S. Gottlöber, V. Müller, F. Occhionero and H.-J. Schmidt, *Class. Quant. Grav.* **10**, L43 (1993).

[211] A. Battaglia-Mayer and H.-J. Schmidt, *Class. Quant. Grav.* **10**, 2441 (1993).

[212] S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **631**, 1 (2005).

[213] T. Koivisto, *Class. Quant. Grav.* **23**, 4289 (2006).

[214] F. W. Hehl, J. D. McCrea, E. W. Mielke and Y. Ne'eman, *Phys. Rept.* **258**, 1 (1995).

[215] A. Papapetrou and J. Stackel, *Gen. Rel. Grav.* **9**, 1075 (1978).

[216] G. Kunstatter, *Gen. Rel. Grav.* **12**, 373 (1979).

[217] F. W. Hehl, E. A. Lord and L. L. Smalley, *Gen. Rel. Grav.* **13**, 1037 (1981).

[218] T. P. Sotiriou and S. Liberati, *preprint* arXiv: gr-qc/0611040.

[219] G. F. Rubilar, *Class. Quant. Grav.* **15**, 239 (1998).

[220] G. S. Hall and D. P. Lonie, *J. Phys. A* **39** 2995 (2006).

[221] V. D. Sandberg, *Phys. Rev. D* **12**, 3013 (1975).

[222] G. Magnano, *preprint* arXiv: gr-qc/9511027.

[223] P. Majumbar and S. Gupta, *Class. Quant. Grav.* **16**, L89 (1999).

[224] I. M. Benn, T. Dereli and R. W. Tucker, *Phys. Lett. B* **96**, 100 (1980).

[225] S. M. Carroll and G. B. Field, *Phys. Rev. D* **50**, 3867 (1994).

[226] A. M. J. Schakel, *Mod. Phys. Lett.* **10**, 999 (1996).

[227] M. Stone, *Phys. Rev. E* **62**, 1341 (2000).

[228] I. P. Novikov and K. S. Thorne, in *Black Holes*, edited by C. DeWitt and B. S. DeWitt, (Gordon and Breach, New York, 1973).

[229] I. Antoniadis, E. Gava and K. S. Narain, *Phys. Lett. B* **283**, 209 (1992).

[230] A. A. Tseytlin, *Nucl. Phys. B* **467**, 383 (1996).

[231] I. Antoniadis, J. Rizos and K. Tamvakis, *Nucl. Phys. B* **415**,497 (1994).

[232] I. Antoniadis, E. Gava and K. S. Narain, *Nucl. Phys. B* **383**, 93 (1992).

[233] T. Damour and A. M. Polyakov, *Nucl. Phys. B* **423**, 532 (1994).

[234] S. Nojiri, S. D. Odintsov and M. Sasaki, *Phys. Rev. D* **71**, 123509 (2005).

[235] P. W. Higgs, *Nuovo Cimento* **11**, 816 (1959).

[236] P. Teyssandier and P. Tourrenc, *J. Math. Phys.* **24**, 2793 (1983).

[237] B. Whitt, *Phys. Lett. B* **575**, 176 (1984).

[238] K. Maeda, *Phys. Rev. D* **39**, 3159 (1989).

[239] J. D. Barrow and S. Cotsakis, *Phys. Lett. B* **214**, 515 (1994).

[240] D. Wands, *Class. Quant. Grav.* **11**, 269 (1994).

[241] G. Magnano and L. M. Sokolowski, *Phys. Rev. D* **50**, 5039 (1994).

[242] E. E. Flanagan, *Phys. Rev. Lett.* **92**, 071101 (2004).

[243] E. E. Flanagan, *Class. Quant. Grav.* **21**, 3817 (2004).

[244] G. J. Olmo and W. Komp, *preprint* arXiv: gr-qc/0403092.

[245] G. Allemandi, M. Capone, S. Capozziello and M. Francaviglia, *Gen. Rel. Grav.* **38**, 33 (2006).

[246] G. J. Olmo, *Phys. Rev. D* **72** (2005), 083505.

[247] J. O'Hanlon, *Phys. Rev. Lett.* **29**, 137 (1972).

[248] J. C. Fabris, S. V. B. Gonçalves and R. de Sá Ribeiro, *Grav. Cosmol.* **12**, 49 (2006).

[249] D. Blaschke and M. P. Dąbrowski, *preprint* arXiv: hep-th/0407078.

[250] F. Hoyle and J. V. Narlikar, *Proc. Roy. Soc. A* **282**, 191 (1964).

[251] J. D. Bekenstein, *Ann. Phys. (NY)* **82**, 535 (1974).

[252] J. V. Narlikar, *Introduction to Cosmology*, (Jonas and Bartlett Publishers, Inc. Portola Valley, 1983).

[253] S. Capozziello, R. de Ritis and A. A. Marino, *Gen. Rel. Grav.* **30**, 1247 (1998).

[254] C. Clarkson, M. Cortes and B. A. Bassett, *preprint* arXiv: astro-ph/0702670.

[255] S. Capozziello, S. Carloni and A. Troisi, *preprint* arXiv: astro-ph/0303041.

[256] S. M. Carroll, V. Duvvuri, M. Trodden and M. Turner, *Phys. Rev. D* **70**, 043528 (2004).

[257] M. B. Mijić, M. S. Morris and W.-M. Suen, *Phys. Rev. D* **34**, 2934 (1986).

[258] S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **68**, 123512 (2003).

[259] S. Nojiri and S. D. Odintsov, *Gen. Rel. Grav.* **36**, 1765 (2004).

[260] S. Nojiri and S. D. Odintsov, *Int. J. Geom. Meth. Mod. Phys.* **4**, 115 (2007).

[261] S. Capozziello and M. Francaviglia, *preprint* arXiv: 0706.1146 [astro-ph].

[262] S. Carloni, P. K. S. Dunsby, S. Capozziello and A. Troisi, *Class. Quant. Grav.* **22**, 4839 (2005).

[263] A. W. Brookfield, C. van de Bruck and L. M. H. Hall, *Phys. Rev. D* **74**, 064028 (2006).

[264] R. Bean, D. Bernat, L. Pogosian, A. Silvestri and M. Trodden, *Phys. Rev. D* **75**, 064020 (2007).

[265] V. Faraoni, *Phys. Rev. D* **72**, 061501 (2005).

[266] V. Faraoni, *Phys. Rev. D* **72**, 124005 (2005).

[267] G. Cognola, M. Gastaldi and S. Zerbini, *preprint* arXiv: gr-qc/0701138.

[268] G. Lambiase and G. Scarpetta, *Phys. Rev. D* **74**, 087504 (2006).

[269] Y. S. Song, H. Peiris and W. Hu, *preprint* arXiv: 0706.2399 [astro-ph].

[270] A. De Felice, P. Mukherjee and Y. Wang, *preprint* arXiv: 0706.1197 [astro-ph].

[271] L. Amendola, D. Polarski and S. Tsujikawa, *Phys. Rev. Lett.* **98**, 131302 (2007).

[272] S. Capozziello, S. Nojiri, S. D. Odintsov and A. Troisi, *Phys. Lett. B* **639**, 135 (2006).

[273] L. Amendola, D. Polarski and S. Tsujikawa, *preprint* arXiv: astro-ph/0605384.

[274] L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa, *Phys. Rev. D* **75**, 083504 (2007).

[275] S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **74**, 086005 (2006).

[276] S. Nojiri and S. D. Odintsov, *preprint* arXiv: hep-th/0610164.

[277] S. M. Carroll, A. De Felice, V. Duvvuri, D. A. Easson, M. Trodden and M. S. Turner, *Phys. Rev. D* **71**, 063513 (2005).

[278] X. H. Meng and P. Wang, *preprint* arXiv: astro-ph/0308284.

[279] D. N. Vollick, *Phys. Rev. D* **68**, 063510 (2003).

[280] G. Allemandi, A. Borowiec and M. Francaviglia, *Phys. Rev. D* **70**, 043524 (2004).

[281] G. Allemandi, A. Borowiec, M. Francaviglia and S. D. Odintsov, em Phys. Rev. D **72**,063505 (2005).

[282] T. P. Sotiriou, *Phys. Rev. D* **73**, 063515 (2006).

[283] T. P. Sotiriou, *Class. Quant. Grav.* **23**, 1253 (2006).

[284] X. H. Meng and P. Wang, *Class. Quant. Grav.* **21**, 2029 (2004).

[285] X. H. Meng and P. Wang, *Class. Quant. Grav.* **22**, 23 (2005).

[286] M. Amarzguioui, O. Elgaroy, D. F. Mota and T. Multamaki, *Astron. Astrophys.* **454**, 707 (2006).

[287] J. R. Bond, G. Efstathiou and M. Tegmark, *Mon. Not. R. Astr. Soc.* **291**, L33 (1997).

[288] E. Hawkins *et al.*, *Mon. Not. R. Astr. Soc.* **346**, 78 (2003).

[289] Y. Wang and M. Tegmark, *Phys. Rev. Lett.* **92**, 241302 (2004).

[290] T. P. Sotiriou, *Gen. Rel. Grav.* **38**, 1407 (2006).

[291] A. Melchiorri, L. Mersini, C. J. Ödman and M. Trodden, *Phys. Rev. D* **68**, 043509 (2003).

[292] C. J. Ödman, A. Melchiorri, M. P. Hobson and A. N. Lasenby, *Phys. Rev. D* **67**, 083511 (2003).

[293] E. V. Linder, *preprint* arXiv: astro-ph/0507308.

[294] E. V. Linder, *Phys. Rev. D* **68**, 083504 (2003).

[295] D. J. Eisenstein and M. J. White, *Phys. Rev. D* **70**, 103523 (2004).

[296] W. Hu and N. Sugiyama, *Astrophys. J.* **471**, 542 (1996).

[297] D. J. Eisenstein and W. Hu, *Astrophys. J.* **496**, 605 (1998).

[298] T. Koivisto and H. Kurki-Suonio, *Class. Quant. Grav.* **23**, 2355 (2006).

[299] T. Koivisto, *Phys. Rev. D* **73**, 083517 (2006).

[300] B. Li and M. C. Chu, *Phys. Rev. D* **74**, 104010 (2006).

[301] B. Li, K. C. Chan and M. C. Chu, *preprint* arXiv: astro-ph/0610794.

[302] S. Fay, R. Tavakol and S. Tsujikawa, *Phys. Rev. D* **75**, 063509 (2007).

[303] K. Uddin, J. E. Lidsey and R. Tavakol, *preprint* arXiv: 0705.0232 [gr-qc].

[304] S. Nojiri, S. D. Odintsov and M. Sami, *Phys. Rev. D* **74**, 046004 (2006).

[305] B. M. Leith and I. P. Neupane, *preprint* arXiv: hep-th/0702002.

[306] B. M. N. Carter and I. P. Neupane, *J. Cosmol. Astropart. Phys.* **0606**, 004 (2006).

[307] I. P. Neupane, *Class. Quant. Grav.* **23**, 7493 (2006).

[308] I. P. Neupane, *preprint* arXiv: hep-th/0605265.

[309] S. Tsujikawa, *Ann. Phys.* **15**, 302 (2006).

[310] T. Koivisto and D. F. Mota, *Phys. Rev. D* **75**, 023518 (2007).

[311] T. Koivisto and D. F. Mota, *Phys. Lett. B* **644**, 104 (2007).

[312] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, *Phys. Rev. D* **75**, 086002 (2007).

[313] T. Damour, *preprint* arXiv: 0704.0749 [gr-qc].

[314] T. Damour and G. Esposito-Farese, *Phys. Rev. Lett.* **70**, 2220 (1993).

[315] T. Clifton, *Class. Quant. Grav.* **23**, 7445 (2006).

[316] T. Multamaki and I. Vilja, *Phys. Rev. D* **74**, 064022 (2006).

[317] S. Capozziello, A. Stabile and A. Troisi, *Class. Quant. Grav.* **24**, 2153 (2007).

[318] T. Multamaki and I. Vilja, *preprint* arXiv: astro-ph/0612775.

[319] T. Chiba, *Phys. Lett. B* **575**, 1 (2003).

[320] R. Dick, *Gen. Rel. Grav.* **36**, 217 (2004).

[321] M. E. Soussa and R. P. Woodard, *Gen. Rel. Grav.* **36**, 855 (2004).

[322] G. J. Olmo, *Phys. Rev. Lett.* **95**, 261102 (2005).

[323] I. Navarro and K. Van Acoleyen, *Phys. Lett. B* **622**, 1 (2005).

[324] S. Capozziello and A. Troisi, *Phys. Rev. D* **72**, 044022 (2005).

[325] S. Capozziello, A. Stabile and A. Troisi, *Mod. Phys. Lett. A* **21**, 2291 (2006).

[326] A. L. Erickcek, T. L. Smith and M. Kamionkowski, *Phys. Rev. D* **74**, 121501 (2006).

[327] X. H. Jin, D. J. Liu and X. Z. Li, *preprint* arXiv: astro-ph/0610854.

[328] T. Chiba, T. L. Smith and A. L. Erickcek, *preprint* arXiv: astro-ph/0611867.

[329] P. J. Zhang, *preprint* arXiv:astro-ph/0701662.

[330] R. V. Wagoner, *Phys. Rev. D* **1**, 3209 (1970)

[331] P. J. Steinhardt and C. M. Will, *Phys. Rev. D* **52**, 628 (1995).

[332] A. Rajaraman, *preprint* arXiv: astro-ph/0311160.

[333] V. Faraoni, *Phys. Rev. D* **74**, 023529 (2006).

[334] V. Faraoni, *Phys. Rev. D* **75**, 067302 (2007).

[335] G. J. Olmo, *Phys. Rev. D* **75**, 023511 (2007).

[336] X. Meng and P. Wang, *Gen. Rel. Grav.* **36**, 1947 (2004).

[337] A. E. Domínguez and D. E. Barraco, *Phys. Rev. D* **70**, 043505 (2004).

[338] G. Allemandi, M. Francaviglia, M. L. Ruggiero and A. Tartaglia, *Gen. Rel. Grav.* **37**, 1891 (2005).

[339] G. Allemandi and M. L. Ruggiero, *preprint* arXiv: astro-ph/0610661.

[340] M. L. Ruggiero and L. Iorio, *J. Cosmol. Astropart. Phys.* **0701**, 010 (2007).

[341] A. D. Dolgov and M. Kawasaki, *Phys. Lett. B* **573**, 1 (2003).

[342] V. Faraoni, *Phys. Rev. D* **74**, 104017 (2006).

[343] T. P. Sotiriou, *Phys. Lett. B* **645**, 389 (2007).

[344] T. P. Sotiriou and E. Barausse, *Phys. Rev. D* **75**, 084007 (2007).

[345] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, *Phys. Rev. D* **73**, 084007 (2006).

[346] P. Jetzer and M. Sereno, Phys. Rev. D **73**, 044015 (2006).

[347] G. Esposito-Farese, *AIP Conf. Proc.* **736**, 35 (2004).

[348] L. Amendola, C. Charmousis and S. C. Davis, *preprint* arXiv: 0704.0175 [astro-ph].

[349] T. R. Taylor and G. Veneziano, *Phys. Lett. B* **213**, 450 (1988).

[350] K. Kainulainen, V. Reijonen and D. Sunhede, *preprint* arXiv: gr-qc/0611132.

[351] E. Barausse, T. P. Sotiriou and J. C. Miller, *preprint* arXiv: gr-qc/0703132.

[352] D. E. Barraco and V. H. Hamity, *Phys. Rev. D* **62**, 044027 (2000).

[353] P. Haensel and A. Y. Potekhin, *Astron. Astroph.* **428**, 191 (2004).

[354] G. J. Olmo, *Phys. Rev. Lett.* **98**, 061101 (2007).

[355] R. V. Pound and G. A. Rebka Jr., *Phys. Rev. Lett.* **4**, 337 (1960).

[356] T. P. Sotiriou, V. Faraoni and S. Liberati, *preprint* arXiv: 0707.2748 [gr-qc].

[357] J. Audretsch, *Phys. Rev. D* **27**, 2872 (1983).

[358] J. Audretsch and C. Lämmerzhal, *J. Math. Phys.* **32**, 2099 (1991).

[359] S. Sonego, *Phys. Lett. A* **208**, 1 (1995).

[360] S. Sonego and H. Westman, *Class. Quant. Grav.* **21**, 433 (2004).

[361] M. M. Ali, A. S. Majumdar, D. Home and A. K. Pan, *Class. Quant. Grav.* **23**, 6493 (2006).

[362] S. Huerfano, S. Sahu, M. Socolovsky, *preprint* arXiv: quant-ph/0606172.

[363] J. Ehlers, F. A. E. Pirani and A. Schild, in *General Relativity, Papers in Honor of J. L. Synge*, edited by O'Raifeartaigh, (Oxford University Press, Oxford, 1972).

[364] J. Ehlers and A. Schild, *Comm. Math. Phys.* **32**, 119 (1973).

[365] R. A. Coleman and H. Korte, *J. Math. Phys.* **25**, 3513 (1984).

[366] H. Ewen and H.-J. Schmidt, *J. Math. Phys.* **30**, 1480 (1989).

[367] M. Castagnino and A. Ordonez, *Rend. Mat. Serie VII*, **9**, 299 (1989).

[368] R. Geroch and P. S. Jang, *J. Math. Phys.* **16**, 65 (1975).

[369] J. L. Anderson, *preprint* arXiv: gr-qc/9912051.

[370] C. G. Callan, S. Coleman and R. Jackiw, *Ann. Phys. (NY)* **59**, 42 (1970).

[371] N. A. Chernikov and E. A. Tagirov, *Ann. Inst. H. Poincarè* **A9**, 109 (1968).

[372] P. G. Bergmann, *Int. J. Theor. Phys.* **1**, 25 (1968).

[373] K. Nordvedt, *Astrophys. J.* **161** 1059 (1970).

[374] http://en.wiktionary.org/wiki/theory

[375] T. Franzen, *Gödel's Theorem: An Incomplete Guide to Its Use and Abuse*, (Addison-Wesley, Massachussets, 2005).

[376] J. D. Barrow, *preprint* arXiv: physics/0612253.

[377] S. W. Hawking, *A Brief History of Time*, (Bantam Books, Toronto, 1988).

[378] I. L. Shapiro and H. Takata, *Phys. Rev. D* **52**, 2162 (1995).

[379] J. Sherk, *Phys. Lett. B* **88**, 265 (1979).

[380] M. Gasperini, *Phys. Lett. B* **470**, 67 (1999).

[381] V. Faraoni and S. Nadeau, *Phys. Rev. D* **75**, 023501 (2007).

[382] J. L. Synge, *Relativity: The General Theory*, (North Holland, Amsterdam, 1955).