Scuola Internazionale Superiore di Studi Avanzati - Trieste

Correspondences in String Field Theory
(The Importance of Being Noncommutative...)

Thesis submitted for the degree of

Doctor Philosophiæ

Candidate:
Davide Mamone

Supervisor:
Prof. Loriano Bonora

Academic year 2002/03

SISSA – Via Beirut 2-4 – 34014 TRIESTE – ITALY

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A Silvia,
che mi ha indicato dove si trova davvero il Graal
e soprattutto mi ha fatto capire cosa in realta' sia...

"Ditemi un motivo per cui dovrei lavorare per voi!"
"Perche' avrebbe accesso a conoscenze che teniamo segrete.
Roba come Teoria delle Superstringhe..."
(Matt Damon a colloquio con un agente dell'NSA
in "Will Hunting, Genio ribelle")

"In realta' l'universo e' un'unica stringa annodata su se stessa..."
(Silvia Maschio, Conversazione privata)

"O Tempora, O Mores..."
(Cicerone, Catilinarie)

Contents

Introduction

"Is string theory the theory of everything?"
"What is the true vacuum of string theory?"
"How can we give a nonperturbative definition of string theory?"
At the moment, nobody can truly answer these questions.
Without an off-shell definition, string theory is a theory *in fieri*, a strange collection of Feynmann rules endowed with a huge, beautiful symmetry, a mythological, charming monster with many heads which are capable to talk each other in a mysterious language made of dualities, correspondences but whose body remains invisible to us. Every head is connected through his neck to the central body which should have his feet on our universe, which is the true vacuum of the true "theory of everything". If string theory is the theory of everything, then we already know which is its vacuum: our universe is the background of the monster. One of the problems is that we do not have a dynamical mechanism which choose for us our universe among millions of possible backgrounds. We need a formulation of string theory endowed with such mechanism. Experimental predictability is mainly a theoretical problem before to be technological and the fact the string theory would be able to predict General Relativity, if it would not already exist, is not enough.
In the last seven years our knowledge about nonperturbative aspects was enormously increased since the discovery of D-branes and the role played by them, powerful dualities, M-theory and decoupling limits in which string theory can be described with a gauge theory (AdS/CFT correspondence, pp-waves, noncommutative field theories...). In fact, these important developments hide our ignorance about the true vacuum and nonperturbative structure and suggest, at the same time, an underlying symmetry between open and closed strings, the latter related to some kind of quantum effect on the open string side.

String Field Theory (SFT) is an attempt to find a second quantized version of string theory. Witten's String Field Theory, proposed by Edward Witten in 1986

[27], describes open strings interactions in terms of a Chern-Simons-like action in which the one-form is replaced by the string field, the external derivative by the BRST operator and the wedge product by the Witten star, which consist of gluing open strings in an associative but noncommutative way. In the last three years, renewed attention has been paid to this theory, because it was powerful enough to verify Sen's conjectures and describe tachyon condensation.
Sen suggested in 1999 that open bosonic string theory should be seen as a theory which describes a D25-brane. Indeed, open strings move across the entire space and at the same time they have to end on a D-brane. Hence, this D25-brane fills the entire space. The existence of the open string tachyon, hence, is not a problem for bosonic string theory but just the signal of an incorrect choice of the vacuum: it is a consequence of the instability of such D25-brane which is the perturbative vacuum but also decays. Then, Sen argued that the decay of the D25-brane should correspond to the condensation of the tachyon. In particular, three conjectures have been made by Ashoke Sen [20, 21, 22], about how this decay process takes place:

1) the difference in energy between the maximum at the origin, corresponding to the negative mass^2 of the tachyon, and the perturbatively stable vacuum should be equal to the mass of the D25–brane.

2) Starting from the D25–brane, lower dimensional branes are realized as soliton configurations of the tachyon and other fields.

After tachyon condensation the D25–brane completely disappeared, and then

3) In the perturbatively stable vacuum there are no physical open string excitations: only *closed* strings are there.

According to third conjecture, closed strings will arise as excitations of the stable vacuum. This would give, in principle, the hope to have a complete description of string theory, open strings as well as of closed ones.

Unfortunately, we do not know how to solve the equation of motion of Witten's theory. Rastelli, Sen and Zwiebach, in 2000 supposed to know one of such solutions and proposed a shifted and string field redefined version of Witten's theory called Vacuum String Field Theory [36]. Vacuum String Field Theory (VSFT) is the guessed form of String Field Theory at the closed string vacuum. VSFT is more simple then Witten's SFT but it is not a simplified version of that in the sense that we should be able to recover in some way the original theory if VSFT truly represents the Witten's one formulated on the stable vacuum, as it seems. Indeed, D-branes arise as solitonic solutions of this theory and this interpretation is confirmed by the correct ratio of tensions between different dimensional solutions. Essentially, the main difference is encoded in the kinetic operator which is purely ghost instead to be the BRST charge. This leads to a factorized form of the solutions, a splitting of the equation of motion in ghost and matter sector and it turns out that the matter equation is simply a projection condition under the star noncommutative multiplication. In this sense, D-branes are projectors of this noncommutative algebra and can be seen as a sort of noncommutative solitons of this theory. In another framework respect to SFT, open string with a B field turned on, Harvey et al. [59] showed that taking the effective action for the tachyon field, which is the action of a scalar field theory on a noncommutative space, D-branes can be seen as the solitonic solutions of such theory, called GMS solitons We will see in chapter 5 that there is a natural way to incorporate these two interpretations of D-branes as noncommutative solitons, turning on a B field in VSFT.

This thesis is organized as follows.
Chapter 1 is an introduction to String Field Theory and its use to describe tachyon condensation. Recent reviews of the subject can be found in [24, 25]. Chapter 2 is a review of the works of Rastelli, Sen and Zwiebach that defined VSFT, [36, 37, 38]. Chapter 4, a review of solitons in noncommutative field theory [61].

The original part of this thesis is contained in chapters 3, 5 and 6 which refer to the three main results we obtained.

The first one [78], chapter 3, concerns the definition of the multiplication operation in SFT, which is noncommutative. There are three different type of star products, one matter type and two ghosts. They differ in the Neumann coefficient which define the star product. We will show that such coefficients for the three stars are related to each other in a very simple way: a $SL(2,R)$-like map connects the matter ones with the so called reduced ghost ones; the same map but with an extra minus sign connects the reduceds with the twisted ghosts and, finally, the twisted ghost ones are equal up tp a minus sign to the matter ones. We enphasize the these two ghost star products are different although they give rise to the same solution of equation of motion in VSFT, source of confusion in the past.

The second one [75, 76, 77], chapter 5, concerns the possibility to find solutions of VSFT if a B field is switched on, the differences between VSFT with or without the B field and the definition of a new infinite class of solutions that we called "Ancestors" because in the low energy limit they give rise to all so called GMS solitons, which we review in chapter 4.
In particular we find that B field behaves as a natural regulator: in [43] it was shown that the geometry of the lower-dimensional lump states is singular at the string level

because the midpoint of the string is confined on the brane and that is singular also at the low-energy level because in this limit you must introduce an ad hoc regulator by hand. In [76], starting from the lump solution with the B field, we showed that at high energy the string midpoint is no more confined on the brane and at low-energy the lump (representing a D-brane) becomes the simplest GMS soliton, using the Seiberg-Witten limit [6] that gives a noncommutative field theory from a string theory when a B field is turned on. This gave the inspiration to write down the Ancestors solutions. In particular, we pointed out a precise isomorphism which seem to be hidden between such solitonic solutions in VSFT and in noncommutative field theory.

The third one [80, 81], chapter 6, concerns relations among the small "zoo" of projectors of the star algebra, which we review in chapter 2. They play an important role in the theory because they are solutions of matter equation of motion and/or define the star algebra of string fields. It turns out that they can be rewritten in a general form involving a matrix U which, case by case, is nothing but the null matrix, the identity matrix, the twist matrix or even and odd powers of the fundamental matrix S which define the D25-brane, the so called "Sliver". In particular, we speculate the possibility to obtain such "general form" using a suitable resummation of the Ancestors.

We can summarize saying that the first is a correspondence between the matter and the ghosts (note the plural) noncommutative structure, the second is a correspondence at the same time between B and not B regime and between SFT and noncommutative field theory and, finally, the third is a correspondence among the relevant actors playng in the game of star algebra.

These are the correspondences we mean in the title of this thesis.
Of course, it is crucial to have noncommutativity.

Chapter 1

An introduction to String Field Theory

1.1 Witten's Open Bosonic String Field Theory

The open bosonic SFT action proposed by E. Witten is

$$S(\Phi) = -\frac{1}{g_o^2}\left[\frac{1}{2}\langle\,\Phi, Q\Phi\rangle + \frac{1}{3}\langle\,\Phi, \Phi * \Phi\rangle\right] \tag{1.1}$$

where Φ is the string field, Q the kinetic operator, $*$ an associative but noncommutative product (Witten's star-product), $\langle\,\cdot,\cdot\rangle$ an inner product and g_o is the open string coupling constant. The string field is defined as the most general state living in the Hilbert space $\mathcal{H}$ of the first-quantized open string theory:

$$\begin{aligned} |\Phi\rangle &= \Big(\phi(x) + A_\mu(x)\alpha^\mu_{-1} + B_{\mu\nu}(x)\alpha^\mu_{-1}\alpha^\mu_{-1} + \dots\Big)c_1|\Omega\rangle \\ &= \int d^{26}k\Big(\phi(k) + A_\mu(k)\alpha^\mu_{-1} + B_{\mu\nu}(k)\alpha^\mu_{-1}\alpha^\mu_{-1} + \dots\Big)c_1|k\rangle \end{aligned} \tag{1.2}$$

where $|k\rangle = e^{ik\cdot X(0)}|0\rangle$. $|\Omega\rangle = c_1|0\rangle$ is the ghost number 1 vacuum. It is defined by

$$\begin{aligned} \alpha^\mu_n\,|\Omega\rangle &= 0 \qquad (n>0) \\ b_n\,|\Omega\rangle &= 0 \qquad (n\geq 0) \\ c_n\,|\Omega\rangle &= 0 \qquad (n>0) \\ k^\mu\,|\Omega\rangle &= 0 \end{aligned} \tag{1.3}$$

$|0\rangle$ is the SL(2,$\mathbb{R}$) invariant vacuum, x and k are the center-of-mass coordinate and momentum and the functions in front of the basis states are spacetime fields. The kinetic operator Q, the $*$ product and the inner product $\langle\cdot,\cdot\rangle$ act on $\mathcal{H}$ in the following way

$$\begin{aligned}
Q &: \mathcal{H} \to \mathcal{H} \\
* &: \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \\
\langle\cdot,\cdot\rangle &: \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}
\end{aligned} \tag{1.4}$$

The kinetic operator Q is defined to satisfy the following identities

$$\begin{aligned}
Q^2 A &= 0 \\
Q(A * B) &= (QA) * B + (-1)^A A * (QB) \\
\langle QA, B\rangle &= -(-1)^A \langle A, QB\rangle
\end{aligned} \tag{1.5}$$

In the properties above $(-1)^A$ is $+1$ when A is ghost number even, and -1 when A is ghost number odd. The kinetic operator is immediately chosen as the BRST operator Q_B: it satisfies all the properties (1.5) and the variation of the free (quadratic) part only of (1.1), with $Q = Q_B$ is nothing but the physical state condition on the first-quantized string theory states:

$$Q_B|\Phi\rangle = 0\ . \tag{1.6}$$

Inner product and $*$ product should satisfy

$$\begin{aligned}
\langle A, B\rangle &= (-1)^{AB}\langle B, A\rangle \\
\langle A, B * C\rangle &= \langle A * B, C\rangle \\
A * (B * C) &= (A * B) * C
\end{aligned} \tag{1.7}$$

Since $S(\Phi)$ should obviously be a real number, we will impose a reality condition on the string field. The quadratic part of (1.1) is the *free part* of the action and represents an evolving on shell string state from $\tau = -\infty$ to $\tau = +\infty$. The cubic term is the interaction vertex of three string states.

Using the properties (1.5), (1.7) and the ghost numbers assignments, is easy to see that the action (1.1) is invariant under the gauge transformation

$$\delta\Phi = Q\Lambda + \Phi * \Lambda - \Lambda * \Phi \tag{1.8}$$

where Λ is a ghost number zero string field. Variation of the action (1.1) gives the field equation of motion

$$Q\Phi + \Phi * \Phi = 0 \tag{1.9}$$

From the second of the equations (1.7) we have

$$\langle A, B * C \rangle = (-1)^{A(B+C)} \langle B, C * A \rangle \tag{1.10}$$

and, since the string fields are all of ghost number 1, the above equation states that the cubic term $\langle \Phi, \Phi * \Phi \rangle$ in the action (1.1) is cyclic in the permutations of the fields. In a similar way is possible to show that

$$\langle \Phi_1, Q\Phi_2 \rangle = \langle \Phi_2, Q\Phi_1 \rangle \tag{1.11}$$

We will define the Witten's $*$-product in two different but equivalent ways. The first one, in the original formulation of Witten's String Field Theory, written in the "string functionals" formalism, instead of "string state", allows us to understand in which sense the interaction between two open strings is given by the gluing of the right half of the first one with the left half of the second one (or viceversa). The second one, in the CFT and oscillator formalism, will be more useful for our calculation later on. Before to do that, let us have a look of the formal correspondence between Witten's SFT action and the Cher-Simons action which is

$$S(A) = \frac{1}{2} \int_M A \wedge dA + \frac{1}{3} \int_M A \wedge A \wedge A \tag{1.12}$$

where M is a 3-manifold and A a 1-form. This action is invariant under the gauge transformation

$$\delta A = d\epsilon + A \wedge \epsilon - \epsilon \wedge A \tag{1.13}$$

where ϵ is a 0–form. We can write the correspondence between Witten's Cubic SFT and Chern-Simons theory with the help of the following 'dictionary':

Chern-Simons	Witten's open SFT
differential form	state in CFT
wedge product $\wedge$	$*$ product
degree of a differential form	ghost number of a state
gauge state A	string field Φ with ghost number 1
gauge parameter ϵ	state in the CFT with ghost number 0
exterior derivative d	BRST operator Q
integration $\int$	Witten's integration

1.2 $*$ as gluing of strings

In Witten's original approach is more manifest the interpretation of the $*$ product as 'gluing of strings'. In particular the starting point [27] of SFT dealt with *string functionals* $\Phi[X(\sigma), c(\sigma), b(\sigma)]$ defined as the Schrödinger representation of the first quantized string field $|\Phi\rangle$

$$\Phi[X(\sigma), c(\sigma), b(\sigma)] \equiv \langle X(\sigma), c(\sigma), b(\sigma)|\Phi\rangle \tag{1.14}$$

The $*$ product is defined by

$$(\Phi_1 * \Phi_2)(X_0(\sigma)) = \int \mathcal{D}X_1(\sigma)\mathcal{D}X_2(\sigma)\, \Phi_1(X_1(\sigma))\, \Phi_2(X_2(\sigma)) \tag{1.15}$$
$$\prod_{0\leq\sigma\leq\pi/2} \delta(X_2(\sigma) - X_1(\pi-\sigma))\, \delta(X_1(\sigma) - X_0(\sigma))\, \delta(X_0(\pi-\sigma) - X_2(\pi-\sigma)) \ ,$$

and the integration by

$$\int \Phi = \int \mathcal{D}X(\sigma)\Phi(X(\sigma)) \prod_{0\leq\sigma\leq\pi/2} \delta(X(\sigma) - X(\pi-\sigma)) \tag{1.16}$$

The definition (1.15) of the $*$ product should be interpreted in the following way: the functional $(\Phi_1 * \Phi_2)$ of the strings coordinates X_0 is given by gluing the left half of the first string with the right half of the second string (first δ function in (1.15)), and then imposing that the remaining halves of the strings X_1 and X_2 constitutes the whole X_0 string. The integration (1.16) means to take a string and then obtaining a number from it collapsing the two halves of the string on each other. With the two definitions (1.15, 1.16) the 3-string interaction vertex can be written as

$$\begin{aligned}\langle \Phi_1, \Phi_2 * \Phi_3 \rangle &= \int \Phi_1 * \Phi_2 * \Phi_3 \\ &= \int \mathcal{D}X_1(\sigma)\mathcal{D}X_2(\sigma)\mathcal{D}X_3(\sigma)\Phi_1(X_1(\sigma))\,\Phi_2(X_2(\sigma))\,\Phi_3(X_3(\sigma)) \\ &\quad \prod_{0\leq\sigma\leq\pi/2} \delta(X_2(\sigma) - X_1(\pi-\sigma))\,\delta(X_3(\sigma) - X_2(\pi-\sigma))\,\delta(X_1(\sigma) - X_3(\pi-\sigma))\end{aligned} \tag{1.17}$$

Bpz conjugation, that reverses the σ orientation on the boundary of the unit disk, has the Schrödinger representation

$$\langle \mathrm{bpz}\,(\Phi)|X(\sigma)\rangle = \Phi[X(\pi-\sigma)] \tag{1.18}$$

and the reality condition on $\langle\Phi|$ translates into

$$\Phi[X(\sigma)] = \Phi^*[X(\pi-\sigma)] \tag{1.19}$$

Using the representation of the identity

$$1 = \int \mathcal{D}X(\sigma)|X(\sigma)\rangle\langle X(\sigma)| \tag{1.20}$$

we have for the quadratic term

$$\begin{aligned}\langle \Phi_1, Q\Phi_2 \rangle &= \langle \mathrm{bpz}\,(\Phi_1)|Q\Phi_2\rangle \\ &= \int \mathcal{D}X(\sigma)\,\langle \mathrm{bpz}\,(\Phi_1)|X(\sigma)\rangle\,\langle X(\sigma)|Q\Phi_2\rangle \\ &= \int \mathcal{D}X(\sigma)\,\Phi_1(X(\pi-\sigma))\,Q\Phi_2(X(\sigma)) \\ &= \int \mathcal{D}X_1(\sigma)\mathcal{D}X_2(\sigma)\,\Phi_1(X_1(\sigma))\,Q\Phi_2(X_2(\sigma)) \prod_{0\leq\sigma\leq\pi/2} \delta(X_2(\sigma) - X_1(\pi-\sigma))\end{aligned}$$

The cubic term is

$$\begin{aligned}\langle \Phi_1, \Phi_2 * \Phi_3 \rangle &= \langle \mathrm{bpz}\,(\Phi_1)|\Phi_2 * \Phi_3\rangle \\ &= \int \mathcal{D}X\,\langle \mathrm{bpz}\,(\Phi_1)|X(\sigma)\rangle\langle X(\sigma)|\Phi_2 * \Phi_3\rangle \\ &= \int \mathcal{D}X_1(\sigma)\,\mathcal{D}X_2(\sigma)\mathcal{D}X_3(\sigma)\;\Phi_1(X_1(\sigma))\,\Phi_2(X_2(\sigma))\,\Phi_3(X_3(\sigma)) \\ &\quad \prod_{0\leq\sigma\leq\pi/2} \delta(X_2(\sigma) - X_1(\pi-\sigma))\,\delta(X_3(\sigma) - X_2(\pi-\sigma))\,\delta(X_1(\sigma) - X_3(\pi-\sigma))\end{aligned}$$

where we used the reality condition on $\Phi[X(\sigma)]$. The above equation is obviously equal to (1.17).

1.3 $*$-product and SFT in the CFT language

The action (1.1) can also be written by defining two states

$$\langle V_2| \in \mathcal{H}^* \otimes \mathcal{H}^* \tag{1.21}$$

and

$$\langle V_3| \in \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H}^* \tag{1.22}$$

such that

$$S(\Phi) = -\frac{1}{g_o^2}\left[\frac{1}{2}\,{}_{(12)}\langle V_2||\Phi\rangle_{(1)}|Q\Phi\rangle_{(2)} + \frac{1}{3}\,{}_{(123)}\langle V_3||\Phi\rangle_{(1)}|\Phi\rangle_{(2)}|\Phi\rangle_{(3)}\right] \tag{1.23}$$

where the pedices are introduced to distinguish explicitly among the different copies of the string Fock space referred to different strings. Our final task will be the explicit determination of $\langle V_2|$ and $\langle V_3|$. This will be done using the two dimensional conformal field theory (CFT) structure that underlines string theory.

First of all we need a recipe to define $\mathcal{H}^*$, the dual of the Hilbert space of first-quantized string states. This is done by means of the internal product defined through **bpz** conjugation [19]:

$$\begin{aligned} &\text{bpz} : \mathcal{H} \to \mathcal{H}^* \\ &\text{bpz}\,|A\rangle = \langle\,\text{bpz}\,(A)| \end{aligned} \tag{1.24}$$

To define bpz conjugation consider a primary field $\phi(z)$ of dimension d with mode expansion

$$\phi(z) = \sum_{n=-\infty}^{\infty} \frac{\phi_n}{z^{n+d}}\,, \qquad \phi_n = \oint \frac{dz}{2\pi i} z^{n+d-1}\phi(z)\,. \tag{1.25}$$

By the state-operator correspondence, $\phi(z)$ creates in the far past ($\tau \to -\infty, z \to 0$) the state

$$|\phi\rangle = \lim_{z\to 0}\phi(z)|0\rangle \tag{1.26}$$

We define

$$\langle \text{bpz}\,(\phi)| = \langle 0| \lim_{z\to 0} \phi\Big(-\frac{1}{z}\Big) \tag{1.27}$$

The state $\langle 0|$ is the left (out) vacuum defined as the time evoluted $|0\rangle$ at $\tau \to \infty$ ($z \to \infty$). The transformation

$$\mathcal{I} \,:\, z \mapsto -\frac{1}{z} \tag{1.28}$$

is a SL(2,$\mathbb{C}$) transformation which sends the origin to infinity while taking the unit circle to itself. On the modes ϕ_n the inversion $\mathcal{I}$ acts as

$$\begin{aligned} \text{bpz}\,(\phi_n) &= \oint \frac{dz}{2\pi i} z^{n+d-1} \mathcal{I}\circ\phi(z) \\ &= \oint \frac{dz}{2\pi i} z^{n+d-1} \Big(\frac{1}{z^2}\Big)^h \phi\Big(-\frac{1}{z}\Big) \\ &= \oint \frac{dz}{2\pi i} z^{n-d-1} \sum_m \phi_m (-1)^{m+d} z^{m+d} \\ &= (-1)^{-n+d} \phi_{-n} \end{aligned} \tag{1.29}$$

Bpz conjugated of known operators are

$$\begin{aligned} \text{bpz}\,(L_n) &= (-1)^n L_{-n} \\ \text{bpz}\,(\alpha^\mu_{-n}) &= (-1)^{n+1} \alpha^\mu_n \end{aligned}$$

Equipped with bpz conjugation we can now discuss the reality condition on $|\Phi\rangle$. Hermitian conjugation (hc) just transforms a bra into a ket. Bpz and hc conjugation are used together to define complex conjugation:

$$|A^*\rangle = \text{bpz}^{\,-1} \circ \text{hc}\,|A\rangle = \text{hc}^{\,-1} \circ \text{bpz}\,|A\rangle \tag{1.30}$$

Reality condition is

$$|A^*\rangle = |A\rangle \;\;\rightarrow\;\; \text{hc}\,|A\rangle = \text{bpz}\,|A\rangle \tag{1.31}$$

This condition ensures the reality of the fields $\phi, A_\mu, B_{\mu\nu}, \ldots$ in the expansion (1.2).

The importance of using bpz conjugation instead of hc one, is in the conformal field theory character of the former. What we want to do now is indeed to give a conformal field theory prescription for calculating the vertices $\langle V_2|$ and $\langle V_3|$, and to be able to do actual computations.

Comparing the two forms (1.1) and (1.23) of the string field action it also follows that

$$\langle A, B\rangle = \langle \mathrm{bpz}(A)|B\rangle = {}_{(12)}\langle V_2||A\rangle_{(1)}|B\rangle_{(2)} \tag{1.32}$$

and

$$\langle A, B * C\rangle = \langle \mathrm{bpz}\,(A)|B * C\rangle = {}_{(123)}\langle V_3||A\rangle_{(1)}|B\rangle_{(2)}|C\rangle_{(3)}, \tag{1.33}$$

this means that the vertex $|V_3\rangle$ realizes the $*$ product:

$$(|A\rangle * |B\rangle)_3 = {}_1\langle \mathrm{bpz}\,(A)|_2\langle \mathrm{bpz}\,(B)||V_3\rangle_{123} \tag{1.34}$$

We now have a CFT definition of $\langle \cdot, \cdot\rangle$ in terms of bpz conjugation. What we want is a correspondent cft definition of the $*$ product. It is useful to define the $*$ product through the interaction term of the action. Consider three generic string states A, B and C , and their corresponding vertex operators $\mathcal{O}_A(z), \mathcal{O}_B(z), \mathcal{O}_C(z)$. We define three conformal transformations $f_i(z), i = 1, 2, 3$ such that

$$\langle A, B * C\rangle \equiv \left\langle f_1^D \circ \mathcal{O}_A(0)\, f_2^D \circ \mathcal{O}_B(0)\, f_3^D \circ \mathcal{O}_C(0)\right\rangle_D \tag{1.35}$$

There is a crucial conceptual difference between the two sides of the above equation. The left-hand side is an inner product in the product space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ of the 3 Fock spaces $\mathcal{H}_i$, while the right-hand side is a correlation function of a single string conformal field theory in the z upper half plane. The index D indicates that the conformal transformations and the correlation function are defined on the disk (that is conformally equivalent to the upper half plane).

Remember the time evolution of a single free string when the worldsheet is parametrized by the upper half plane: in $z = 0$ the string starts to evolve, the real positive axis being the boundary $\sigma = \pi$ and the negative one the boundary $\sigma = 0$. The front of the evolving string is represented by half circumferences centered in the origin, all the points belonging to the same radius being points at the same time. The intersection of the front line of the string with the imaginary positive axis is the midpoint Q of the string ($\sigma = \pi/2$).

At this point we could write 3-string vertex of the general form

$$\langle T^2 h\,[\mathcal{O}_A]\, Th\,[\mathcal{O}_B]\, h\,[\mathcal{O}_C]\rangle \tag{1.36}$$

with T an $SL(2,\mathbb{C})$ transformation such that $T^3 = 1$ but with $h(z)$ a completely arbitrary map of the unit circle into the complex plane. There are two very simple choice of $h(z)$. The first is $h(z) = z$ which carries the unit circle into the complex plane unchanged and which lead to a SFT vertex which generalizes the Caneschi-Schwimmer-Veneziano dual model vertex [8]. The second is the map which carries the unit circle into a wedge covering a 120 degrees angle. This choice lead to the Witten vertex. The basic idea for defining the conformal transformations f_i is then to map three upper-half disks into a single disk representing the interaction vertex of the three strings. The operation of $*$ product is then interpreted as a gluing of two string worldsheets. We start with three upper-half disks parametrized by their own local coordinates z_i. On each half disk we perform the following coordinate transformation

$$h : z_i \mapsto \zeta = \frac{1 + iz_i}{1 + iz_1} \tag{1.37}$$

This transformation maps the mid-string point Q of each string in the center $\zeta = 0$ of the unit disk, and the open string boundaries to the boundary of the unit disk. Then we shrink the half disks obtained by a factor $2/3$, and rotate the first one counterclockwise by a $2\pi/3$ angle, and the third one clockwise by a $2\pi/3$ angle. The three transformations are

$$\begin{aligned} f_1(z_1) &= e^{-\frac{2\pi i}{3}}\left(\frac{1+iz_1}{1-iz_1}\right)^{\frac{2}{3}} \\ f_2(z_2) &= \left(\frac{1+iz_2}{1-iz_2}\right)^{\frac{2}{3}} \\ f_3(z_3) &= e^{\frac{2\pi i}{3}}\left(\frac{1+iz_3}{1-iz_3}\right)^{\frac{2}{3}} \end{aligned} \tag{1.38}$$

The *global* disk is now constructed gluing together the three world sheets: for instance the right part ($\pi/2 \leq \sigma \leq \pi$) of the front line of the first string is glued with the left part ($0 \leq \sigma \leq \pi/2$) of the of the front line of the second string and so forth. Cyclicity of the cubic term in the action (1.1) is now manifest by construction. The open string worldsheet is also represented by the upper half plane; the SL(2,$\mathbb{C}$) transformation that sends the disk in the upper half plane is

$$h^{-1} : \zeta \mapsto z = -i\frac{\zeta - 1}{\zeta + 1} \tag{1.39}$$

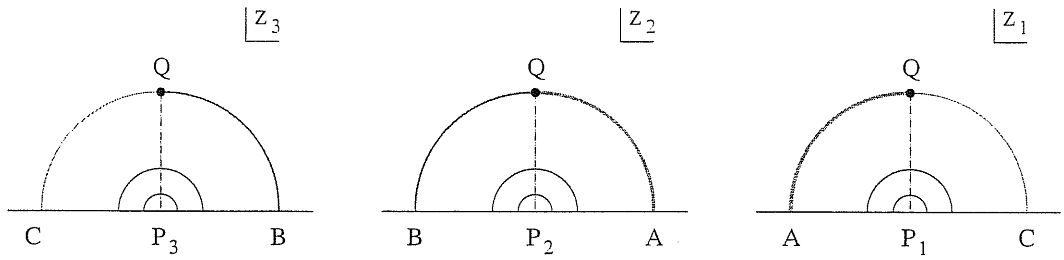

Figure 1.1: Representation of the cubic vertex as the gluing of 3 half–disks.

Of course, since the correlator in (1.35) is SL(2,$\mathbb{C}$) invariant, computing it on the disk or on the plane gives the same result.

It is now straightforward to define an arbitrary n-point vertex through the transformations

$$f_k(z_k) = e^{\frac{2\pi i}{n}(k-1)}\Big(\frac{1+iz_k}{1-iz_k}\Big)^{\frac{2}{n}}, \quad 1 \le k \le n \tag{1.40}$$

Each f_k maps an upper half disk to a $2\pi/n$ wedge, and n such wedges gather to make a unit disk.

Also the CFT description of the quadratic term can be encoded in this formulation, with the $n=2$ case of (1.41). Writing explicitly f_1 and f_2

$$h^{-1}f_1(z_1) \;=\; h^{-1}\Big(\frac{1+iz_1}{1-iz_1}\Big) \;=\; z_1 \;=\; I(z_1) \tag{1.41}$$

$$h^{-1}f_2(z_2) \;=\; h^{-1}\Big(-\frac{1+iz_2}{1-iz_2}\Big) \;=\; -\frac{1}{z_2} \;\equiv\; \mathcal{I}(z_2) \tag{1.42}$$

The quadratic term becomes

$$\begin{aligned}\langle\, \Phi, Q\Phi\rangle &= \langle f_2\circ\Phi(0)\, f_1\circ Q\Phi(0)\rangle\\ &= \big\langle h^{-1}\circ f_2\circ\Phi(0)\, h^{-1}\circ f_1\circ Q\Phi(0)\big\rangle\\ &= \langle \mathcal{I}\circ\Phi(0)\, Q\Phi(0)\rangle\end{aligned} \tag{1.43}$$

The complete action, rewritten in terms of CFT correlators, is

$$S = -\frac{1}{g_o^2}\left[\frac{1}{2}\Big\langle \mathcal{I}\circ\Phi(0)\, Q\Phi(0)\Big\rangle + \frac{1}{3}\Big\langle f_1\circ\Phi(0)\, f_2\circ\Phi(0)\, f_3\circ\Phi(0)\Big\rangle\right] \tag{1.44}$$

The last step is the explicit expression of $\langle V_3|$ and $\langle V_2|$ in terms of the transformations f_i and $\mathcal{I}$. First, we will give a simple derivation of them; in the next two

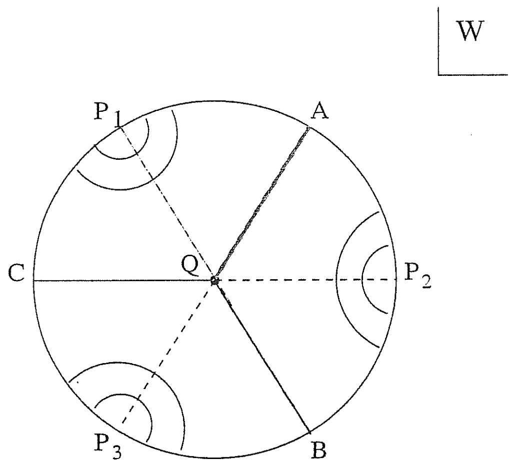

Figure 1.2: Representation of the cubic vertex as a 3–punctured unit disk.

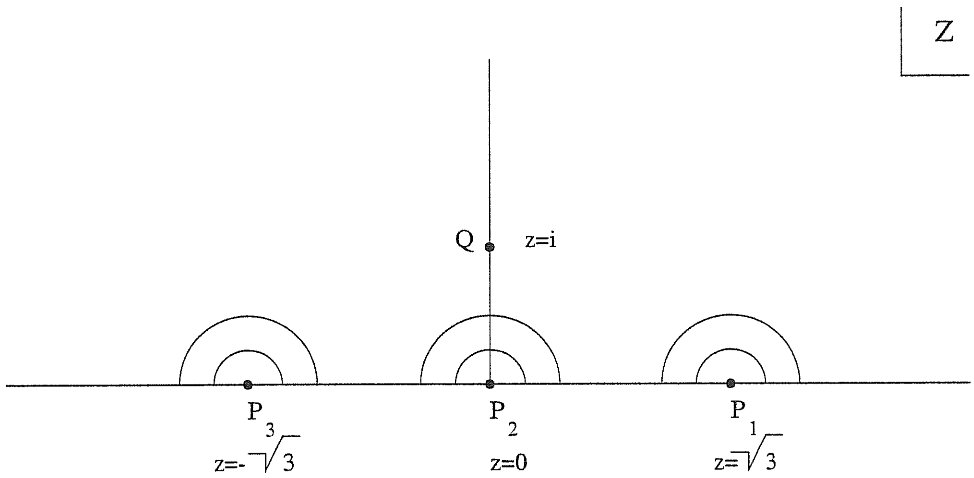

Figure 1.3: Representation of the cubic vertex as the upper–half plane with 3 punctures on the real axis.

sections, we will derive them again in such a way that allows us to take into account an ambiguity in the ghost sector. We start with $\langle V_3|$. The ansatz for this vertex is

$$\begin{aligned}
\langle V_3| &= \int d^{26}p_{(1)} d^{26}p_{(2)} d^{26}p_{(3)} \, \langle\tilde{0},p|_1 \otimes \langle\tilde{0},p|_2 \otimes \langle\tilde{0},p|_3 \\
&\times \exp\Big(-\frac{1}{2}\sum_{r,s}\sum_{n,m\geq 0} \eta_{\mu\nu}\, \alpha_m^{(r)\mu}\, N_{mn}^{rs}\, \alpha_n^{(s)\nu}\Big) \\
&\times \exp\Big(\sum_{r,s}\sum_{\substack{m\geq 0\\ n\geq 1}} b_m^{(r)}\, X_{mn}^{rs}\, c_n^{(s)}\Big)
\end{aligned} \tag{1.45}$$

where $\langle\tilde{0},p|_i = \langle p|_i \otimes \langle 0|_i c_{-1}c_0$. We want to find explicit expressions for the coefficients N_{mn}^{rs} and X_{mn}^{rs}, known as Neumann coefficients, in terms of the functions f_i defining the vertex. We begin the derivation considering the matter sector and setting the momenta to zero $(m, n > 0)$; consider the expression

$$\mathcal{M} = \langle V_3|\, i\partial X^{(r)}(z)\, i\partial X^{(s)}(w)\, |\Omega\rangle_{(1)}|\Omega\rangle_{(2)}|\Omega\rangle_{(3)} \tag{1.46}$$

We compute it first using the contractions of the conformal fields $i\partial X(z)$, and then the oscillator form (1.46) of the vertex. $\mathcal{M}$ is rewritten as

$$\begin{aligned}
\mathcal{M} &= \langle V_3|\, i\partial X^{(r)}(z)\, i\partial X^{(s)}(w)\, c^{(1)}(0)\, c^{(2)}(0)\, c^{(3)}(0)\, |0\rangle_{(1)}|0\rangle_{(2)}|0\rangle_{(3)} \\
&= \Big\langle f_r \circ \Big(i\partial X(z)c(0)\Big)\, f_s \circ \Big(i\partial X(w)c(0)\Big)\, f_t \circ c(0)\Big\rangle
\end{aligned} \tag{1.47}$$

where $t \neq r, s$. The ghost part gives a constant that we will not have to calculate explicitly: $\langle f_r \circ c(0)\, f_s \circ c(0)\, f_t \circ c(0)\rangle \equiv \mathcal{N}$. Being $i\partial X$ a primary field it transforms as $f \circ i\partial X(z) = i\partial X(f(z))\frac{df}{dz}$, and we have

$$\begin{aligned}
\mathcal{M} &= \mathcal{N}\, f_r'(z) f_s'(w) \Big\langle\, i\partial X(f_r(z))\, i\partial X(f_s(w))\, \Big\rangle \\
&= \mathcal{N}\, \frac{f_r'(z) f_s'(w)}{(f_r(z) - f_s(w))^2}
\end{aligned} \tag{1.48}$$

Using the oscillator form of the vertex

$$\begin{aligned}
\mathcal{M} &= \sum_{m,n} z^{m-1} w^{n-1} \langle V_3|\alpha_{-m}^{(r)}\alpha_{-m}^{(s)} c_1^{(1)}|0\rangle_{(1)} c_1^{(2)}|0\rangle_{(2)} c_1^{(3)}|0\rangle_{(3)} \\
&= -\mathcal{N}\sum_{m,n} z^{m-1} w^{n-1}\, mn N_{mn}^{rs}
\end{aligned} \tag{1.49}$$

Comparing the equations (1.49) and (1.50) we have

$$\frac{f_r'(z)f_s'(w)}{(f_r(z)-f_s(w))^2} = -\sum_{m,n} z^{m-1}w^{n-1}\, mnN_{mn}^{rs} \tag{1.50}$$

that means

$$N_{mn}^{rs} = -\frac{1}{mn}\oint_0 \frac{dz}{2\pi i}\oint_0 \frac{dz}{2\pi i}\frac{1}{z^m w^n}\frac{f_r'(z)f_s'(w)}{(f_r(z)-f_s(w))^2} \tag{1.51}$$

Taking into account also the momenta we have to remember the transformation law for the exponentials $f \circ \exp(ip \cdot X(z)) = |f'(z)|^{p^2/2} \exp(ip \cdot X(f(z)))$ and the fact that the operators $i\partial X$ can also contract with factors $ip \cdot X$ in the exponentials. For the part of the vertex bilinear in the momenta we choose, as states in (1.35), A and B as tachyonic states of momenta p_r, p_s and C as a tachyonic state of zero momentum. Consider the expression $\mathcal{M}_{00}$

$$\mathcal{M}_{00} \;=\; \langle V_3|\, e^{ip_1\cdot X(0)}\, e^{ip_2\cdot X(0)}\, |\Omega\rangle_{(1)}|\Omega\rangle_{(2)}|\Omega\rangle_{(3)} \tag{1.52}$$

Evaluated through the oscillator form of the vertex is

$$\begin{aligned}\mathcal{M}_{00} \;=\; & \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\, \langle\tilde{0},p|_1\langle\tilde{0},p|_2\langle\tilde{0},p|_3 \\ & \exp\Big(-\frac{1}{2}\sum_{u,v} p^u\, N_{00}^{uv}\, p^v\Big) c_1^{(1)}|0,p_r\rangle_{(1)} c_1^{(2)}|0,p_s\rangle_{(2)} c_1^{(3)}|0,0\rangle_{(3)}\end{aligned} \tag{1.53}$$

Calculated as a correlation function it has the form

$$\begin{aligned}\mathcal{M}_{00} \;&=\; \Big\langle f_r\circ\Big(e^{ip_r\cdot X(0)}c(0)\Big)\, f_s\circ\Big(e^{ip_s X(0)}c(0)\Big)\, f_t\circ c(0)\Big\rangle \\ &=\; \mathcal{N}\,|f_r'(0)|^{p_r^2/2}\,|f_s'(0)|^{p_s^2/2}\,\exp\left(p_r\cdot p_s\,\log|f_r(0)-f_s(0)|\,\right)\end{aligned} \tag{1.54}$$

It follows that N_{00} is

$$N_{00}^{rs} = \begin{cases} \log|f_r'(0)| & r=s \\ \log|f_r(0)-f_s(0)| + \frac{1}{2}\log|f_r'(0)f_s'(0)| & r\neq s\end{cases} \tag{1.55}$$

In a similar way it is possible to show that

$$N_{0m}^{rs} = -\oint_0 \frac{dw}{2\pi i}\frac{1}{w^m}\frac{\log|f_r'(0)|^{1/2}f_s'(w)}{(f_r(0)-f_s(w))} \tag{1.56}$$

For the ghost sector the coefficients X^{rs}_{mn} are computed equating two different ways of calculating the expression

$$\mathcal{G} = \langle V_3|\, b^{(s)}(z)\, c^{(r)}(w)\, c_1^{(1)}|0\rangle_{(1)} c_1^{(2)}|0\rangle_{(2)} c_1^{(3)}|0\rangle_{(3)} \tag{1.57}$$

Using the mode expansion for ghost and antighost and interpreting $\mathcal{G}$ as a correlator we find

$$\begin{aligned}\mathcal{G} &= \Big\langle f_s \circ b(z)\, f_r \circ c(w)\, f_1 \circ c(0)\, f_2 \circ c(0)\, f_3 \circ c(0) \Big\rangle \qquad (1.58)\\ &= \frac{(f'_s(z))^2}{f'_r(w)} \frac{1}{f'_1(0) f'_2(0) f'_3(0)} \Big\langle\, b(f_s(z))\, c(f_r(w))\, c(f_1(0))\, c(f_2(0))\, c(f_3(0)) \Big\rangle\end{aligned}$$

The simpler way to calculate this correlator is to see its singular structure and derive its normalization from a special configuration. There must be zeroes when any pair of c fields approach to each other. This will give a factor $(f_1(0) - f_2(0))(f_1(0) - f_3(0))(f_2(0) - f_3(0))$ as for $\mathcal{N}$. There are also poles when the antighost approaches any ghost. These considerations imply that

$$\mathcal{G} = \mathcal{N} \frac{(f'_s(z))^2}{f'_r(w)} \frac{1}{f_s(z) - f_r(w)} \frac{\prod_{i=1}^3 (f_r(w) - f_i(0))}{\prod_{j=1}^3 (f_s(z) - f_j(0))} \tag{1.59}$$

Using instead the vertex (1.46) we find

$$\begin{aligned}\mathcal{G} &= \sum_{m,n} \frac{1}{z^{-n+2}} \frac{1}{w^{-m-1}} \langle V_3|\, b^{(s)}_{-n}\, c^{(r)}_{-m} c_1^{(1)}|0\rangle_{(1)} c_1^{(2)}|0\rangle_{(2)} c_1^{(3)}|0\rangle_{(3)}\\ &= \mathcal{N} \sum_{m,n} z^{-n+2} w^{-m-1}\, X^{rs}_{mn} \qquad (1.60)\end{aligned}$$

Comparing the two expressions (1.60) and (1.61) we have

$$X^{rs}_{mn} = \oint \frac{dz}{2\pi i} \frac{1}{z^{n-1}} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+2}} \frac{(f'_s(z))^2}{f'_r(w)} \frac{1}{f_s(z) - f_r(w)} \frac{\prod_{i=1}^3 (f_r(w) - f_i(0))}{\prod_{j=1}^3 (f_s(z) - f_j(0))} \tag{1.61}$$

The explicit expression for $\langle V_2|$ is found by calculating the product $\langle\, i\partial X(z), i\partial X(w)\rangle$:

$$\langle\, i\partial X(z), i\partial X(w)\rangle = -\langle \mathcal{I} \circ \partial X(z)\, \partial X(w)\rangle = -\Big\langle\, \partial X\Big(-\frac{1}{z}\Big)\, \partial X(w) \Big\rangle \tag{1.62}$$

Remembering the transformation law (1.30) of the oscillators α_n under the inversion $\mathcal{I}$, the form of the vertex $\langle V_2|$ is

$$\begin{aligned}\langle V_2| &= \int d^{26}p^{(1)} d^{26}p^{(2)}\, \langle \tilde{0}, p|_1 \otimes \langle \tilde{0}, p|_2\, \delta(p^{(1)} + p^{(2)})\\ &\times \exp\left(-\frac{1}{n}\alpha_n^{(1)} C_{nm} \alpha_m^{(2)} - c_n^{(1)} C_{nm} b_m^{(2)} - c_n^{(2)} C_{nm} b_m^{(1)}\right) \qquad (1.63)\end{aligned}$$

where

$$C_{nm} = \delta_{nm}(-1)^n \tag{1.64}$$

is the twist matrix.

Let us be more precise, specially on the ghost side.

1.4 Three strings vertex and matter Neumann coefficients

The three strings vertex [27, 32, 33] of the Open String Field Theory is given by

$$|V_3\rangle = \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\delta^{26}(p_{(1)} + p_{(2)} + p_{(3)}) \exp(-E)\,|0,p\rangle_{123} \tag{1.65}$$

where

$$E = \sum_{a,b=1}^{3}\left(\frac{1}{2}\sum_{m,n\geq 1}\eta_{\mu\nu}a_m^{(a)\mu\dagger}V_{mn}^{ab}a_n^{(b)\nu\dagger} + \sum_{n\geq 1}\eta_{\mu\nu}p_{(a)}^{\mu}V_{0n}^{ab}a_n^{(b)\nu\dagger} + \frac{1}{2}\eta_{\mu\nu}p_{(a)}^{\mu}V_{00}^{ab}p_{(b)}^{\nu}\right) \tag{1.66}$$

Summation over the Lorentz indices $\mu, \nu = 0, \ldots, 25$ is understood and η denotes the flat Lorentz metric. The operators $a_m^{(a)\mu}, a_m^{(a)\mu\dagger}$ denote the non–zero modes matter oscillators of the a–th string, which satisfy

$$[a_m^{(a)\mu}, a_n^{(b)\nu\dagger}] = \eta^{\mu\nu}\delta_{mn}\delta^{ab}, \qquad m,n \geq 1 \tag{1.67}$$

$p_{(r)}$ is the momentum of the a–th string and $|0,p\rangle_{123} \equiv |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$ is the tensor product of the Fock vacuum states relative to the three strings. $|p_{(a)}\rangle$ is annihilated by the annihilation operators $a_m^{(a)\mu}$ and it is eigenstate of the momentum operator $\hat{p}_{(a)}^{\mu}$ with eigenvalue $p_{(a)}^{\mu}$. The normalization is

$$\langle p_{(a)}|\, p'_{(b)}\rangle = \delta_{ab}\delta^{26}(p + p') \tag{1.68}$$

The symbols $V_{nm}^{ab}, V_{0m}^{ab}, V_{00}^{ab}$ will denote the coefficients computed in [32, 33]. We will use them in the notation of Appendix A and B of [37] and refer to them as the *standard* ones. The notation V_{MN}^{rs} for them will also be used at times (with $M(N)$ denoting the couple $\{0,m\}$ $(\{0,n\})$) .

An important ingredient in the following are the bpz transformation properties of the oscillators

$$bpz(a_n^{(a)\mu}) = (-1)^{n+1} a_{-n}^{(a)\mu} \tag{1.69}$$

Our purpose here is to discuss the definition and the properties of the three strings vertex by exploiting as far as possible the definition given in [2] for the Neumann coefficients. Remembering the description of the star product given in the previous section, the latter is obtained in the following way. Let us consider three unit semidisks in the upper half z_a ($a = 1, 2, 3$) plane. Each one represents the string freely propagating in semicircles from the origin (world-sheet time $\tau = -\infty$) to the unit circle $|z_a| = 1$ ($\tau = 0$), where the interaction is supposed to take place. We map each unit semidisk to a 120° wedge of the complex plane via the following conformal maps:

$$f_a(z_a) = \alpha^{2-a} f(z_a)\,,\; a = 1, 2, 3 \tag{1.70}$$

where

$$f(z) = \left(\frac{1+iz}{1-iz}\right)^{\frac{2}{3}} \tag{1.71}$$

Here $\alpha = e^{\frac{2\pi i}{3}}$ is one of the three third roots of unity. In this way the three semidisks are mapped to nonoverlapping (except at the interaction points $z_a = i$) regions in such a way as to fill up a unit disk centered at the origin.

The interaction vertex is defined by a correlation function on the disk in the following way

$$\int \psi * \phi * \chi = \langle f_1 \circ \psi(0)\, f_2 \circ \phi(0)\, f_3 \circ \chi(0)\rangle = \langle V_{123}|\psi\rangle_1 |\phi\rangle_2 |\chi\rangle_3 \tag{1.72}$$

Now we consider the string propagator at two generic points of this disk. The Neumann coefficients N^{ab}_{NM} are nothing but the Fourier modes of the propagator with respect to the original coordinates z_a. We shall see that such Neumann coefficients are related in a simple way to the standard three strings vertex coefficients.

Due to the qualitative difference between the $\alpha_{n>0}$ oscillators and the zero modes p, the Neumann coefficients involving the latter will be treated separately.

1.4.1 Non zero modes

The Neumann coefficients N^{ab}_{mn} are given by [2]

$$N^{ab}_{mn} = \langle V_{123}|\alpha^{(a)}_{-n}\alpha^{(b)}_{-m}|0\rangle_{123} = -\frac{1}{nm}\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m} f'_a(z)\frac{1}{(f_a(z)-f_b(w))^2} f'_b(w) \tag{1.73}$$

where the contour integrals are understood around the origin. It is easy to check that

$$\begin{aligned} N^{ab}_{mn} &= N^{ba}_{nm} \\ N^{ab}_{mn} &= (-1)^{n+m} N^{ba}_{mn} \\ N^{ab}_{mn} &= N^{a+1,b+1}_{mn} \end{aligned} \tag{1.74}$$

In the last equation the upper indices are defined mod 3.

Let us consider the decomposition

$$N^{ab}_{mn} = \frac{1}{3\sqrt{nm}}\Big(E_{nm} + \bar{\alpha}^{a-b} U_{nm} + \alpha^{a-b} \bar{U}_{nm}\Big) \tag{1.75}$$

After some algebra one gets

$$\begin{aligned} E_{nm} &= \frac{-1}{\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} \Big(\frac{1}{(1+zw)^2} + \frac{1}{(z-w)^2} \Big) \qquad (1.76) \\ U_{nm} &= \frac{-1}{3\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} \Big(\frac{f^2(w)}{f^2(z)} + 2\frac{f(z)}{f(w)} \Big) \Big(\frac{1}{(1+zw)^2} + \frac{1}{(z-w)^2} \Big) \\ \bar{U}_{nm} &= \frac{-1}{3\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} \Big(\frac{f^2(z)}{f^2(w)} + 2\frac{f(w)}{f(z)} \Big) \Big(\frac{1}{(1+zw)^2} + \frac{1}{(z-w)^2} \Big) \end{aligned}$$

By changing $z \to -z$ and $w \to -w$, it is easy to show that

$$(-1)^n U_{nm} (-1)^m = \bar{U}_{nm}, \qquad \text{or} \qquad CU = \bar{U}C, \qquad C_{nm} = (-1)^n \delta_{nm} \tag{1.77}$$

In the second part of this equation we have introduced a matrix notation which we will use throughout the paper.

The integrals can be directly computed in terms of the Taylor coefficients of f. The result is

$$\begin{aligned} E_{nm} &= (-1)^n \delta_{nm} \qquad (1.78) \\ U_{nm} &= \frac{1}{3\sqrt{nm}} \sum_{l=1}^{m} l \Big[(-1)^n B_{n-l} B_{m-l} + 2 b_{n-l} b_{m-l} (-1)^m \\ &\quad - (-1)^{n+l} B_{n+l} B_{m-l} - 2 b_{n+l} b_{m-l} (-1)^{m+l} \Big] \qquad (1.79) \\ \bar{U}_{nm} &= (-1)^{n+m} U_{nm} \qquad (1.80) \end{aligned}$$

where we have set

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} b_k z^k \\ f^2(z) &= \sum_{k=0}^{\infty} B_k z^k, \qquad \text{i.e.} \qquad B_k = \sum_{p=0}^{k} b_p b_{k-p} \end{aligned} \tag{1.81}$$

Eqs.(3.11, 3.12, 3.13) are obtained by expanding the relevant integrands in powers of z, w and correspond to the pole contributions around the origin. We notice that the above integrands have poles also outside the origin, but these poles either are not in the vicinity of the origin of the z and w plane, or, like the poles at $z = w$, simply give vanishing contributions.

One can use this representation for (3.12, 3.13) to make computer calculations. For instance it is easy to show that the equations

$$\sum_{k=1}^{\infty} U_{nk} U_{km} = \delta_{nm}, \qquad \sum_{k=1}^{\infty} \bar{U}_{nk} \bar{U}_{km} = \delta_{nm} \tag{1.82}$$

are satisfied to any desired order of approximation. Each identity follows from the other by using (1.78). In the same way it is also easy to make the identification

$$V_{nm}^{ab} = (-1)^{n+m} \sqrt{nm}\, N_{nm}^{ab} \tag{1.83}$$

of the Neumann coefficients with the standard three strings vertex coefficients[1]. Using (3.15), together with the decomposition (5.29), it is easy to establish the commutativity relation (written in matrix notation)

$$[CV^{ab}, CV^{a'b'}] = 0 \tag{1.84}$$

for any a, b, a', b'. This relation is fundamental for the next developments.

We do not have a simple analytic proof of eq.(3.15). However, due to the identification (3.16), we can take advantage of the proofs contained in [32] to claim that (3.15) are true.

1.4.2 Zero modes

The Neumann coefficients involving one zero mode are given by

$$N_{0m}^{ab} = -\frac{1}{m} \oint \frac{dw}{2\pi i} \frac{1}{w^m} f_b'(w) \frac{1}{f_a(0) - f_b(w)} \tag{1.85}$$

In this case too we make the decomposition

$$N_{0m}^{ab} = \frac{1}{3}\left(E_m + \bar{\alpha}^{a-b} U_m + \alpha^{a-b} \bar{U}_m\right) \tag{1.86}$$

[1] The factor of $(-1)^{n+m}$ in (3.16) arises from the fact that the original definition of the Neumann coefficients (3.7) in [2] refers to the bra three strings vertex $\langle V_3|$, rather than to the ket vertex like in (3.1); therefore the two definitions differ by a bpz operation.

where $E, U, \bar{U}$ can be given, after some algebra, the explicit expression

$$\begin{aligned} E_n &= \frac{4i}{m}\oint \frac{dw}{2\pi i}\frac{1}{w^n}\frac{1}{1+w^2}\frac{f^3(w)}{1-f^3(w)} = \frac{2i^m}{m} \\ U_n &= \frac{-4i}{n}\oint \frac{dw}{2\pi i}\frac{1}{w^n}\frac{1}{1+w^2}\frac{f^2(w)}{1-f^3(w)}\frac{1}{(z-w)^2} = \frac{\alpha_m}{m} \\ \bar{U}_n &= (-1)^n\, U_n = (-1)^m \frac{\alpha_m}{m} \end{aligned} \tag{1.87}$$

The numbers α_n are Taylor coefficients

$$\sqrt{f(z)} = \sum_0^\infty \alpha_n z^n$$

They are related to the A_n coefficients of Appendix B of [37] (see also [32]) as follows: $\alpha_n = A_n$ for n even and $\alpha_n = iA_n$ for n odd. N_{0n}^{ab} are not related in a simple way as (3.16) to the corresponding three strings vertex coefficients. The reason is that the latter satisfy the conditions

$$\sum_{a=1}^{3} V_{0n}^{ab} = 0 \tag{1.88}$$

These constraints fix the invariance $V_{0n}^{ab} \to V_{0n}^{ab} + B_n^b$, where B_n^b are arbitrary numbers, an invariance which arises in the vertex (3.1) due to momentum conservation. For the Neumann coefficients N_{0n}^{ab} we have instead

$$\sum_{a=1}^{3} V_{0n}^{ab} = E_n \tag{1.89}$$

It is thus natural to define

$$\tilde{N}_{0n}^{ab} = N_{0n}^{ab} - \frac{1}{3}E_n \tag{1.90}$$

Now one can easily verify that[2]

$$V_{0n}^{ab} = -\sqrt{2n}\,\tilde{N}_{0n}^{ab} \tag{1.91}$$

It is somewhat surprising that in this relation we do not meet the factor $(-1)^n$, which we would expect on the basis of the *bpz* conjugation (see footnote after eq.(3.16)). However eq.(1.92) is also naturally requested by the integrable structure found in [?]. The absence of the $(-1)^n$ factor corresponds to the exchange $V_{0n}^{12} \leftrightarrow V_{0n}^{21}$. This

[2]The $\sqrt{2}$ factor is there because in [37] the $\alpha' = 1$ convention is used

exchange does not seem to affect in any significant way the results obtained so far in this field.

Before we end this section we would like to recall one of the most surprising and mysterious aspects of SFT, that is its underlying integrable structure: the matter Neumann coefficients obey the Hirota equations of the dispersionless Toda lattice hierarchy. This was explained in [?] following a suggestion of [?]. On the basis of these equations the matter Neumann coefficients with nonzero labels can be expressed in terms of the remaining ones. This fact can have far-reaching consequences for SFT at higher genus [?].

1.5 Ghost three strings vertex and bc Neumann coefficients

The three strings vertex for the ghost part is more complicated than the matter part due to the zero modes of the c field. As we will see, the latter generate an ambiguity in the definition of the Neumann coefficients. Such an ambiguity can however be exploited to formulate and solve in a compact form the problem of finding solutions to eq.(3.57).

1.5.1 Neumann coefficients: definitions and properties

To start with we define, in the ghost sector, the vacuum states $|\hat{0}\rangle$ and $|\dot{0}\rangle$ as follows

$$|\hat{0}\rangle = c_0 c_1 |0\rangle, \qquad |\dot{0}\rangle = c_1 |0\rangle \tag{1.92}$$

where $|0\rangle$ is the usual $SL(2,\mathbb{R})$ invariant vacuum. Using bpz conjugation

$$c_n \to (-1)^{n+1} c_{-n}, \qquad b_n \to (-1)^{n-2} b_{-n}, \qquad |0\rangle \to \langle 0| \tag{1.93}$$

one can define conjugate states.

The three strings interaction vertex is defined, as usual, as a squeezed operator acting on three copies of the bc Hilbert space

$$\langle \tilde{V}_3| = {}_1\langle \hat{0}|\, {}_2\langle \hat{0}|\, {}_3\langle \hat{0}| e^{\prime} \qquad = \sum_{a,b=1}^{3} \sum_{n,m}^{\infty} c_n^{(a)} {}_{nm}^{ab} b_m^{(b)} \tag{1.94}$$

Under bpz conjugation

$$|\tilde{V}_3\rangle = e^{'} |\hat{0}\rangle_1 |\hat{0}\rangle_2 |\hat{0}\rangle_3, \qquad ' = -\sum_{a,b=1}^{3} \sum_{n,m}^{\infty} (-1)^{n+m} c_n^{(a)\dagger} {}_{nm}^{ab} b_m^{(b)\dagger} \tag{1.95}$$

In eqs.(3.21, 3.22) we have not specified the lower bound of the m, n summation. This point will be clarified below.
The Neumann coefficients ${}^{ab}_{nm}$ are given by the contraction of the bc oscillators on the unit disk (constructed out of three unit semidisks, as explained in section 3). They represent Fourier components of the $SL(2,\mathbb{R})$ invariant bc propagator (i.e. the propagator in which the zero mode have been inserted at fixed points ζ_i, $i = 1, 2, 3$):

$$\langle b(z)c(w)\rangle = \frac{1}{z-w}\prod_{i=1}^{3}\frac{w-\zeta_i}{z-\zeta_i} \tag{1.96}$$

Taking into account the conformal properties of the b, c fields we get

$$\begin{aligned} {}^{ab}_{nm} &= \langle \tilde{V}_{123}| b^{(a)}_{-n} c^{(b)}_{-m} |\dot{0}\rangle_{123} \\ &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{n-1}}\frac{1}{w^{m+2}}(f_a'(z))^2 \frac{-1}{f_a(z)-f_b(w)}\prod_{i=1}^{3}\frac{f_a(w)-\zeta_i}{f_b(z)-\zeta_i}(f_b'(w)) \end{aligned} \tag{1.97}$$

It is straightforward to check that

$$\widetilde{N}^{ab}_{nm} = \widetilde{N}^{a+1,b+1}_{nm} \tag{1.98}$$

and (by letting $z \to -z$, $w \to -w$)

$$\widetilde{N}^{ab}_{nm} = (-1)^{n+m}\widetilde{N}^{ba}_{nm} \tag{1.99}$$

Now we choose $\zeta_i = f_i(0) = \alpha^{2-i}$ so that the product factor in (3.24) nicely simplifies as follows

$$\prod_{i=1}^{3}\frac{f_a(w)-f_i(0)}{f_b(z)-f_i(0)} = \frac{f^3(w)-1}{f^3(z)-1}, \qquad \forall\, a, b = 1, 2, 3 \tag{1.100}$$

Now, as in the matter case, we consider the decomposition

$$\widetilde{N}^{ab}_{nm} = \frac{1}{3}(\widetilde{E}_{nm} + \bar{\alpha}^{a-b}\widetilde{U}_{nm} + \alpha^{a-b}\widetilde{\bar{U}}_{nm}) \tag{1.101}$$

where

$$\begin{aligned} \widetilde{E}_{nm} &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\mathcal{N}_{nm}(z,w)\mathcal{A}(z,w) \\ \widetilde{U}_{nm} &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\mathcal{N}_{nm}(z,w)\mathcal{U}(z,w) \\ \widetilde{\bar{U}}_{nm} &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\mathcal{N}_{nm}(z,w)\bar{\mathcal{U}}(z,w) \end{aligned} \tag{1.102}$$

and

$$
\begin{aligned}
\mathcal{A}(z,w) &= \frac{3f(z)f(w)}{f^3(z)-f^3(w)}\\
\mathcal{U}(z,w) &= \frac{3f^2(z)}{f^3(z)-f^3(w)}\\
\bar{\mathcal{U}}(z,w) &= \frac{3f^2(w)}{f^3(z)-f^3(w)}\\
\mathcal{N}_{nm}(z,w) &= \frac{1}{z^{n-1}}\frac{1}{w^{m+2}}(f'(z))^2(f'(z))^{-1}\frac{f^3(w)-1}{f^3(z)-1}
\end{aligned}
$$

After some elementary algebra, using $f'(z) = \frac{4i}{3}\frac{1}{1+z^2}f(z)$, one finds

$$
\begin{aligned}
\widetilde{E}_{nm} &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\Big(\frac{1}{1+zw}-\frac{w}{w-z}\Big)\\
\widetilde{U}_{nm} &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\frac{f(z)}{f(w)}\Big(\frac{1}{1+zw}-\frac{w}{w-z}\Big)\\
\bar{}_{nm} &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\frac{f(w)}{f(z)}\Big(\frac{1}{1+zw}-\frac{w}{w-z}\Big)
\end{aligned}
\qquad (1.103)
$$

Using the property $f(-z) = (f(z))^{-1}$, one can easily prove that

$$
\bar{\widetilde{U}}_{nm} = (-1)^{n+m}\widetilde{U}_{nm} \qquad (1.104)
$$

1.5.2 Computation of the coefficients

In this section we explicitly compute the above integrals. We shall see that the presence of the three c zero modes induces an ambiguity in the $(0,0)$, $(-1,1)$, $(1,-1)$ components of the Neumann coefficients. This in turn arises from the ambiguity in the radial ordering of the integration variables z, w. While the result does not depend on what variable we integrate first, it does depend in general on whether $|z| > |w|$ or $|z| < |w|$.

If we choose $|z| > |w|$ we get

$$
\widetilde{E}^{(1)}_{nm} = \theta(n)\theta(m)(-1)^n\delta_{nm} + \delta_{n,0}\delta_{m,0} + \delta_{n,-1}\delta_{m,1} \qquad (1.105)
$$

while, if we choose $|z| < |w|$, we obtain

$$
\widetilde{E}^{(2)}_{nm} = \theta(n)\theta(m)(-1)^n\delta_{nm} - \delta_{n,1}\delta_{m,-1} \qquad (1.106)
$$

where $\theta(n) = 1$ for $n > 0$, $\theta(n) = 0$ for $n \leq 0$. We see that the result is ambiguous for the components $(0,0)$, $(-1,1)$, $(1,-1)$.

To compute $_{nm}$ we expand $f(z)$ for small z, as in section 3,

$$f(z) \quad = \quad \sum_{k=0}^{\infty} b_k z^k$$

Since $f^{-1}(z) = f(-z)$ we get the relation

$$\sum_{k=0}^{n} (-1)^k b_k b_{n-k} = \delta_{n,0} \tag{1.107}$$

which is identically satisfied for n odd, while for n even it can be also rewritten as

$$b_n^2 = -2 \sum_{k=1}^{n} (-1)^k b_{n-k} b_{n+k} \tag{1.108}$$

Taking $|z| > |w|$ and integrating in z first, one gets

$$\widetilde{U}_{nm}^{(1a)} = \delta_{n+m} + (-1)^m \sum_{l=1}^{n} (b_{n-l} b_{m-l} - (-1)^l b_{n-l} b_{m+l}) \tag{1.109}$$

If, instead, we integrate in w first

$$\widetilde{U}_{nm}^{(1b)} = (-1)^m b_n b_m + (-1)^m \sum_{l=1}^{m} (b_{n-l} b_{m-l} + (-1)^l b_{n+l} b_{m-l}) \tag{1.110}$$

One can check that, due to (1.109),

$$\widetilde{U}_{nm}^{(1a)} \quad = \quad \widetilde{U}_{nm}^{(1b)} \equiv \widetilde{U}_{nm}^{(1)} \tag{1.111}$$

Now we take $|z| < |w|$ and get similarly

$$\widetilde{U}_{nm}^{(2a)} = (-1)^m \sum_{l=1}^{n} (b_{n-l} b_{m-l} - (-1)^l b_{n-l} b_{m+l})$$

$$\widetilde{U}_{nm}^{(2b)} = -\delta_{n+m} + (-1)^m b_n b_m + (-1)^m \sum_{l=1}^{m} (b_{n-l} b_{m-l} + (-1)^l b_{n+l} b_{m-l})$$

Again, due to (1.109)

$$\widetilde{U}_{nm}^{(2a)} \quad = \quad \widetilde{U}_{nm}^{(2b)} = \widetilde{U}_{nm}^{(2)} \tag{1.112}$$

Comparing $^{(1)}$ with $^{(2)}$, we see once more that the ambiguity only concerns the $(0,0)$, $(-1,1)$, $(1,-1)$ components. Using (3.27) we define

$${}_{nm}^{ab,(1,2)} \quad = \quad \frac{1}{3}(\widetilde{E}_{nm}^{(1,2)} + \bar{\alpha}_{nm}^{(a-b)(1,2)} + \alpha^{a-b}(-1)^{n+m} \widetilde{U}_{nm}^{(1,2)})$$

The above ambiguity propagates also to these coefficients, but only when $a = b$. For later reference it is useful to notice that

$$\begin{array}{lllll} \widetilde{N}^{ab,(1,2)}_{-1,m} = 0, & \text{except} & \text{perhaps} & \text{for} & a = b, \quad m = 1 \\ \widetilde{N}^{ab,(1,2)}_{0,m} = 0, & \text{except} & \text{perhaps} & \text{for} & a = b \quad m = 0 \\ \widetilde{N}^{ab,(1,2)}_{n,1} = 0, & \text{except} & \text{perhaps} & \text{for} & a = b \quad n = 0 \end{array} \tag{1.113}$$

It is important to notice that the above ambiguity does not come out of the blue, but is consistent with the general identification proposed in [2]

$$\widetilde{N}^{ab}_{nm} = \langle \tilde{V}_3 | b^{(a)}_{-n} c^{(b)}_{-m} | \dot{0} \rangle_1 | \dot{0} \rangle_2 | \dot{0} \rangle_3 \tag{1.114}$$

It is easy to see that the expression in the RHS is not bpz covariant when (m, n) take values $(0, 0)$, $(-1, 1)$, $(1, -1)$ and the lower bound of the m, n summation in the vertex (see above) is -1. Such bpz noncovariance corresponds exactly to the ambiguity we have come across in the explicit evaluation of the Neumann coefficients.

1.5.3 Two alternatives

It is clear that we are free to fix the ambiguity the way we wish, provided the convention we choose is consistent with bpz conjugation. We consider here two possible choices. The first consists in setting to zero all the components of the Neumann coefficients which are ambiguous, i.e. the $(0, 0)$, $(-1, 1)$, $(1, -1)$ ones. This leads to a definition of the vertex (3.21) in which the summation over n starts from 1 while the summation over m starts from 0. In this way any ambiguity is eliminated and the Neumann coefficients are bpz covariant. This is the preferred choice in the literature, [?, 41, 14, 44, 45]. In particular, it has led in [?] to a successful comparison of the operator formulation with a twisted conformal field theory one.

We would like, now, to make some comments about this first choice, with the purpose of stressing the difference with the alternative one we will discuss next. In particular we would like to emphasize some aspects of the BRST cohomology in VSFT. In VSFT the BRST operator is conjectured [?, 14] to take the form

$$\mathcal{Q} = c_0 + \sum_{n=1}^{\infty} f_n (c_n + (-1)^n c_{-n}) \tag{1.115}$$

It is easy to show that the vertex is BRST invariant, i.e.

$$\sum_{a=1}^{3} \mathcal{Q}^{(a)} | \widetilde{V}_3 \rangle = 0 \tag{1.116}$$

Due to

$$\{\mathcal{Q}, b_0\} = 1 \tag{1.117}$$

it follows that the cohomology of $\mathcal{Q}$ is trivial. As was noted in [44], this implies that the subset of the string field algebra that solves (3.57) is the direct sum of $\mathcal{Q}$–closed states and b_0–closed states (i.e. states in the Siegel gauge).

$$|\Psi\rangle = \mathcal{Q}|\lambda\rangle + b_0|\chi\rangle \tag{1.118}$$

As a consequence of the BRST invariance of the vertex it follows that the star product of a BRST-exact state with any other is identically zero. This implies that the VSFT equation equation of motion can determine only the Siegel-gauge part of the solution.

For this reason previous calculations were done with the use of the *reduced* vertex [14, ?] which consists of Neumann coefficients starting from the (1,1) component. The unreduced star product can be recovered by the midpoint insertion of $\mathcal{Q} = \frac{1}{2i}(c(i) - c(-i))$ as

$$|\psi * \phi\rangle = \mathcal{Q}|\psi *_{b_0} \phi\rangle \tag{1.119}$$

where $*_{b_0}$ is the reduced product.

In the alternative treatment given below, thanks to the enlargement of the Fock space, we compute the star product, and hence solve (3.57) without any gauge choice and any explicit midpoint insertion.

Motivated by the advantages it offers in the search of solutions to (3.57), we propose therefore a second option. It consists in fixing the ambiguity by setting

$$\widetilde{N}^{aa}_{-1,1} = \widetilde{N}^{aa}_{1,-1} = 0, \qquad \widetilde{N}^{aa}_{0,0} = 1. \tag{1.120}$$

If we do so we get a fundamental identity, valid for $\widetilde{U}_{nm} \equiv \widetilde{U}^{(1)}_{nm}$ (for $n, m \geq 0$),

$$\sum_{k=0} \widetilde{U}_{nk}\widetilde{U}_{km} = \delta_{nm} \tag{1.121}$$

Defining

$$\widetilde{X}^{ab} = C\widetilde{V}^{ab}, \tag{1.122}$$

eq.(3.33) entails

$$[\widetilde{X}^{ab}, \widetilde{X}^{a'b'}] = 0 \tag{1.123}$$

One can prove eq.(3.33) numerically. By using a cutoff in the summation one can approximate the result to any desired order (although the convergence with increasing cutoff is less rapid than in the corresponding matter case, see section

3.1). Alternatively one can notice that, at least for $n, m \neq 0$, the same conclusion can be derived analytically by using the results in the literature [33].

The next subsection is devoted to working out some remarkable consequences of eq.(3.33).

1.5.4 Matrix structure

Once the convention (3.30) is chosen, we recognize that all the matrices $(\widetilde{E}, \widetilde{U}, \bar{\widetilde{U}})$ have the $(0,0)$ component equal to 1, all the other entries of the upper row equal to 0, and a generally non vanishing zeroth column. More precisely

$$\begin{array}{lll} \widetilde{U}_{00} = \widetilde{E}_{00} = 1 & & \\ \widetilde{U}_{n0} = b_n \quad \widetilde{E}_{n0} = 0, & & \widetilde{U}_{0n} = \widetilde{E}_{0n} = \delta_{n,0} \\ \widetilde{U}_{nm} \neq 0, \quad n, m > 0 & & \end{array} \tag{1.124}$$

This particular structure makes this kind of matrices simple to handle under a generic analytic map f. In order to see this, let us inaugurate a new notation, which we will use in this and the next section. We recall that the labels M, N indicate the couple $(0, m), (0, n)$. Given a matrix M, let us distinguish between the 'big' matrix M_{MN} denoted by the calligraphic symbol $\mathcal{M}$ and the 'small' matrix M_{mn} denoted by the plain symbol M. Accordingly, we will denote by Y_ε a matrix of the form (1.125), $\vec{y} = (y_1, y_2, ...)$ will denote the nonvanishing column vector and Y the 'small' matrix

$$Y_{\varepsilon NM} = \delta_{N0}\delta_{M0} + y_n\delta_{M0} + Y_{mn}, \tag{1.125}$$

or, symbolically, $Y_\varepsilon = (1, \vec{y}, Y)$.

Then, using a formal Taylor expansion for f, one can show that

$$f[Y_\varepsilon]_{NM} = f[1]\delta_{N0}\delta_{M0} + \left(\frac{f[1] - f[Y]}{1 - Y}\vec{y}\right)_n \delta_{M0} + f[Y]_{mn} \tag{1.126}$$

Now let us define

$$\begin{aligned} Y_\varepsilon &\equiv \widetilde{X}^{11} \\ Y_{\varepsilon+} &\equiv \widetilde{X}^{12} \qquad (1.127) \\ Y_{\varepsilon-} &\equiv \widetilde{X}^{21} \qquad (1.128) \end{aligned}$$

These three matrices have the above form. Using (3.33) one can prove the following properties (which are well–known for the 'small' matrices)

$$Y_\varepsilon + Y_{\varepsilon+} + Y_{\varepsilon-} = 1$$

$$
\begin{aligned}
Y_\varepsilon^2 + Y_{\varepsilon+}^2 + Y_{\varepsilon-}^2 &= 1 \\
Y_{\varepsilon+}Y_{\varepsilon-} &= Y_\varepsilon^2 - Y_\varepsilon \\
[Y_\varepsilon, Y_{\varepsilon\pm}] &= 0 \\
[Y_{\varepsilon+}, Y_{\varepsilon-}] &= 0
\end{aligned}
\tag{1.129}
$$

Using (6.71, 1.127) we immediately obtain (we point out that, in particular for Y_ε, $y_{2n} = \frac{2}{3}\, b_{2n}$, $y_{2n+1} = 0$ and $Y_{nm} = \widetilde{X}_{nm}$ for $n, m > 0$)

$$
\begin{aligned}
Y + Y_+ + Y_- &= 1 \\
\vec{y} + \vec{y}_+ + \vec{y}_- &= 0 \\
Y^2 + Y_+^2 + Y_-^2 &= 1 \\
(1+Y)\vec{y} + Y_+\vec{y}_+ + Y_-\vec{y}_- &= 0 \\
Y_+Y_- &= Y^2 - Y \\
[Y, Y_\pm] &= 0 \\
[Y_+, Y_-] &= 0 \\
Y_+\vec{y}_- = Y\vec{y} &= Y_-\vec{y}_+ \\
-Y_\pm\vec{y} &= (1-Y)\vec{y}_\pm
\end{aligned}
\tag{1.130}
$$

These properties were shown in various papers, see [14, 45]. Here they are simply consequences of (1.130), and therefore of (3.33). In particular we note that the properties of the 'big' matrices are isomorphic to those of the 'small' ones. This fact allows us to work directly with the big matrices, handling at the same time both zero and not zero modes.

1.5.5 Enlarged Fock space

We have seen in the last subsection the great advantages of introducing the convention (3.30). Here we explain how to incorporate this convention in an enlargement of the bc system's Fock space. In fact, in order for eq.(1.115) to be consistent, a modification in the RHS of this equation is in order. This can be done by, so to speak, 'blowing up' the zero mode sector. We therefore enlarge the original Fock space, while warning that our procedure may be far from unique. For each string, we split the modes b_0 and c_0. In other words we introduce two additional couple of conjugate anticommuting creation and annihilation operators $\eta_0, \eta_0^\dagger$ and $\xi_0, \xi_0^\dagger$

$$
\{\xi_0, \eta_0\} = 1, \qquad \{\xi_0^\dagger, \eta_0^\dagger\} = 1 \tag{1.131}
$$

with the following rules on the vacuum

$$\xi_0|0\rangle = 0, \qquad \langle 0|\xi_0^\dagger = 0 \tag{1.132}$$

$$\eta_0^\dagger|0\rangle = 0, \qquad \langle 0|\eta_0 = 0 \tag{1.133}$$

while $\xi_0^\dagger, \eta_0$ acting on $|0\rangle$ create new states. The *bpz* conjugation properties are defined by

$$bpz(\eta_0) = -\eta_0^\dagger, \qquad bpz(\xi_0) = \xi_0^\dagger \tag{1.134}$$

The reason for this difference is that η_0 (ξ_0) is meant to be of the same type as c_0 (b_0). The anticommutation relation of c_0 and b_0 remain the same

$$\{c_0, b_0\} = 1 \tag{1.135}$$

All the other anticommutators among these operators and with the other *bc* oscillators are required to vanish. In the enlarged Fock space all the objects we have defined so far may get slightly changed. In particular the three strings vertex (3.21,3.22) is now defined by

$$\widetilde{E}'_{(en)} = \sum_{n\geq 1, m\geq 0}^{\infty} c_n^{(a)\dagger} \widetilde{V}_{n0}^{(ab)} b_m^{(b)\dagger} - \eta_0^{(a)} b_0^{(a)} \tag{1.136}$$

With this redefinition of the vertex any ambiguity is eliminated, and one can return to the original Fock space by introducing suitable constraints.

1.6 Sen's conjectures

Bosonic String Theory is affected by the presence of the tachyon. D-branes in the open bosonic theory are unstable objects and this unstability is related to the tachyonic mode of the string ending on the brane. One can reject completely the bosonic theory as a definitely unrealistic theory or can recall another famous example in theoretical physics in which unstability and a "tachyon" field take place at the same time: the Higgs mechanism in Standard Model. There, what is unstable is the vacuum which is not the correct one, and what is tachyonic is the Higgs field, which condenses. The analogy with String Theory is the following: we know String Theory on the perturbative vacuum of the open string, which can be seen as the D25-brane. So the tachyon should condense. Ashoke Sen made three conjectures about the tachyon condensation [20, 21, 22].

1. The difference in the potential between the unstable vacuum and the perturbatively stable vacuum should be the mass of the D25-brane.

2. Lower-dimensional D–branes should be realized as soliton configurations of the tachyon and other string fields.

3. The perturbatively stable vacuum should correspond to the closed string vacuum. In particular, there should be no physical open strings excitations around this vacuum.

Why String Field Theory is good for checking such conjectures? Because we are talking about off-shell phenomena and we simply need an off-shell formulation of string theory. Sen showed that the tachyon potential has a universal form which is independent of the details of the theory describing the D–brane [22], and he also related, in the formalism of SFT, the open string coupling constant g_o to the D–brane tension [22]. This 'universality' of the tachyon potential means that we can choose the easiest background for the theory that we want describing the tachyon potential. In particular, using SFT, one can take the conformal background to be the Boundary Conformal Field Theory (BCFT) of any bosonic Dp-brane, with the flat 26 dimensional Minkowski space being just the space filling D25–brane. The study of Sen's conjectures becomes then the study of the fields at zero momentum, living at the bottom of the tachyonic potential. The simplest of these states is the zero momentum tachyon state. The tachyon state at zero momentum is $tc_1|0\rangle$ where t is a constant. It belongs to the subspace $\mathcal{H}_t$ of the whole string Fock space $\mathcal{H}$ defined as the space of states of ghost number one obtained by acting on $|0\rangle$ with oscillators b_n, c_n and matter Virasoro generators L_n. The subspace $\mathcal{H}_t$ of $\mathcal{H}$ is a background independent subspace having the property that we can consistently set the component of the string field along $\mathcal{H} - \mathcal{H}_1$ to zero in looking for a solution of the equation of motion. $\mathcal{H}_1$ is background independent for the simple reason that there is no room in this theory for containing information on the boundary CFT which describes any brane. Since the fields in $\mathcal{H}_t$ have zero momenta, and hence are independent of the coordinates on the D–brane world-volume, the integration in the string field action over x gives the $(p+1)$-dimensional volume factor V_{p+1}, so we have

$$S(T) = V_{p+1}\mathcal{L}(T) = -V_{p+1}U(T) \tag{1.137}$$

where we defined the tachyon potential as the negative of the lagrangian.

The string field $|T\rangle = T(0)|0\rangle$ includes an infinite collection of variables corresponding to the coefficients of expansion of a state in $\mathcal{H}_t$ in some basis. The tension

τ_{25} of the D25–brane in terms of the open string coupling constant g_0 is

$$\tau_{25} = \frac{1}{2\pi^2 g_o^2} \tag{1.138}$$

In [22] was shown by Sen that, for a generic Dp-brane, the tachyon potential on it has the universal form

$$U(T) = 2\pi^2 M \left[\frac{1}{2}\langle \mathcal{I} \circ T(0)\, QT(0)\rangle + \frac{1}{3}\langle f_1 \circ T(0)\, f_2 \circ T(0)\, f_3 \circ T(0)\rangle\right] \tag{1.139}$$

where $M = V_p \tau_p$. If we consider the zero momentum tachyon $tc_1|0\rangle$, then the kinetic term of the potential is

$$\begin{aligned}\langle T, QT\rangle &= t^2\langle 0|c_{-1}Qc_1|0\rangle \\ &= t^2\langle 0|c_{-1}c_0 L_0 c_1|0\rangle \\ &= -t^2\langle 0|c_{-1}c_0c_1|0\rangle \\ &= -t^2\end{aligned} \tag{1.140}$$

while the cubic one, calculated in the upper half plane,

$$\langle T, T * T\rangle = t^3\langle f_1^H \circ c(0) f_2^H \circ c(0) f_3^H \circ c(0)\rangle_H \tag{1.141}$$

where $f_i^H = h^{-1} \circ f_i^D$. The field $c(z)$ is primary of dimension -1, so we have

$$f \circ c(0) = \frac{c(f(0))}{f'(0)} \tag{1.142}$$

From equations (1.39) we find

$$f_1^H \circ c(0) = \frac{c(f_1^H(0))}{f_1^{H\prime}(0)} = \frac{c(\sqrt{3})}{8/3} \tag{1.143}$$

In the same way, we obtain

$$\begin{aligned}\langle T, T * T\rangle &= t^3\left\langle \frac{c(\sqrt{3})}{8/3}\,\frac{c(0)}{2/3}\,\frac{c(-\sqrt{3})}{8/3}\right\rangle_H \\ &= \frac{3^3}{2^7} t^3 \langle c(\sqrt{3})\, c(0)\, c(-\sqrt{3})\rangle_H \\ &= \left(\frac{3\sqrt{3}}{4}\right)^3 t^3 \\ &\equiv K^3 t^3\end{aligned} \tag{1.144}$$

Putting into eq.(1.140) we get the first approximation to the tachyon potential

$$f^{(0)}(t) \equiv \frac{U(T = tc(z))}{M} = 2\pi^2\Big(-\frac{1}{2}t^2 + \frac{1}{3}\Big(\frac{3\sqrt{3}}{4}\Big)^3 t^3\Big) \tag{1.145}$$

This has a local minimum at

$$t = t_c = \Big(\frac{4}{3\sqrt{3}}\Big)^3 \simeq 0.456 \tag{1.146}$$

where

$$f(t_c) \simeq -0.684 \tag{1.147}$$

Following the first Sen's conjecture

$$U(T_0) + \tau_p = \tau_p(1 + f(T_0)) = 0 \tag{1.148}$$

we see that the tachyon state alone satisfies the condition (1.149) as much as 68% of the conjectured value. The tachyon field is said to be of level zero. The level l of a state is related to the L_0 eigenvalue as

$$l = L_0 + 1 \tag{1.149}$$

More generally, one truncates the string field $|T\rangle$ at a finite number of terms. Of course, more space-time components one keeps, better is the approximation. More precisely, a (m, n) approximation means to keep all fields up to level m and all interactions up to level n. The level of an interaction is defined to be the sum of the levels of all fields entering into it. What we just did is the $(0, 0)$ approximation. A simplification is given by choosing the Feynman-Siegel gauge

$$b_0|T\rangle = 0 \tag{1.150}$$

The tachyon field up to level two will be

$$|T\rangle = tc_1|0\rangle + uc_{-1}|0\rangle + vL_2c_1|0\rangle \tag{1.151}$$

At level (2,4) we have a stationary point at $t_c \simeq -0.541, u_c \simeq -0.173, v_0 \simeq 0.051$ which gives 0.948% of the exact answer. Higher level calculations give a consistency check with very high accuracy [40].

Chapter 2

Vacuum String Field Theory

2.1 SFT at the "true" vacuum

The main problem in Open Bosonic SFT is that we don't know how to solve the equation of motion. Rastelli, Sen and Zwiebach [36, 37] proposed a new formulation of SFT, a simplified version, conjectured to be the SFT written at the "true" vacuum, the nonperturbative vacuum, the minimum of the tachyon potential, instead of the usual perturbative one, which should correspond to a maximum of the potential, the D25 brane, thus unstable. This formulation is know as Vacuum String Field Theory. The advantage of this theory is that we are able to find exact solutions of it which describe D25 brane (the "Sliver"), and lower dimensional ones, which correct tensions expected for such Dbranes. In this chapter we will review the basics of this theory and the method to find solutions of it.

Let us suppose to know Φ_0, the string field configuration describing the tachyon vacuum, a solution of the classical field equations following from the action in (1.1):

$$Q\Phi_0 + \Phi_0 * \Phi_0 = 0\,. \tag{2.1}$$

If we indicate with $\widetilde{\Phi} = \Phi - \Phi_0$ the shifted open string field, then the cubic string field theory action expanded around the tachyon vacuum takes the form:

$$S(\Phi_0 + \widetilde{\Phi}) = S(\Phi_0) - \frac{1}{g_o^2}\left[\frac{1}{2}\langle\,\widetilde{\Phi}\,,\,\widehat{Q}\,\widetilde{\Phi}\rangle + \frac{1}{3}\langle\,\widetilde{\Phi}\,,\,\widetilde{\Phi} * \widetilde{\Phi}\rangle\right]. \tag{2.2}$$

where $S(\Phi_0)$ is a constant, which, following the Sen's conjectures, equals the mass M of the D-brane and minus the potential energy $V(\Phi_0) = -S(\Phi_0)$ associated to this string field configuration. The kinetic operator $\widehat{Q}$ is given in terms of Q and Φ_0 as:

$$\widehat{Q}\widetilde{\Phi} = Q\widetilde{\Phi} + \Phi_0 * \widetilde{\Phi} + \widetilde{\Phi} * \Phi_0 \,. \tag{2.3}$$

More generally, on arbitrary string fields one would define

$$\widehat{Q}A = QA + \Phi_0 * A - (-1)^A A * \Phi_0 \,. \tag{2.4}$$

The consistency of the action (2.162) is guaranteed from the consistency of the one in (1.1). Since neither the inner product nor the star multiplication have changed, the identities in (1.7) still hold. One can readily check that the identities in (1.5) hold when Q is replaced by $\widehat{Q}$. Just as (1.1) is invariant under the gauge trasformations (1.8), the action in (2.162) is invariant under $\delta\widetilde{\Phi} = \widehat{Q}\Lambda + \widetilde{\Phi} * \Lambda - \Lambda * \widetilde{\Phi}$ for any Grassmann-even ghost-number zero state Λ.

Since the energy density of the brane represents a positive cosmological constant, it is natural to add the constant $-M = -S(\Phi_0)$ to (1.1). This will cancel the $S(\Phi_0)$ term in (2.162), and will make manifest the expected zero energy density in the final vacuum without D–brane. For the analysis around this final vacuum it suffices therefore to study the action

$$S_0(\widetilde{\Phi}) \equiv -\frac{1}{g_o^2}\left[\frac{1}{2}\langle\,\widetilde{\Phi}\,,\,\widehat{Q}\,\widetilde{\Phi}\rangle + \frac{1}{3}\langle\,\widetilde{\Phi}\,,\,\widetilde{\Phi} * \widetilde{\Phi}\rangle\right]. \tag{2.5}$$

If we had a closed form solution Φ_0 available, the problem of formulating SFT around the tachyon vacuum would be significantly simplified, as we would only have to understand the properties of the new kinetic operator $\widehat{Q}$ in (2.4). In particular we would like to confirm that its cohomology vanishes in accordance with the expectation that all conventional open string excitations disappear in the tachyon vacuum. Even if we knew Φ_0 explicitly and constructed $S_0(\widetilde{\Phi})$ using eq.(2.5), this may not be the most convenient form of the action. Typically a nontrivial field redefinition is necessary to bring the shifted SFT action to the canonical form representing the new background. In fact, in some cases, such as in the formulation of open SFT for D-branes with various values of magnetic fields, it is simple to formulate the various SFT's directly, but the nontrivial classical solution relating theories with different magnetic fields are not known. This suggests that if a simple form exists for the SFT action around the tachyon vacuum it might be easier to guess it than to derive it.

In proposing a simple form of the tachyon action, we have in mind field redefinitions of the action in (2.5) that leave the cubic term invariant but simplify the operator $\widehat{Q}$ in (2.4) by transforming it into a simpler operator $\mathcal{Q}$. To this end we consider homogeneous field redefinitions of the type

$$\widetilde{\Phi} = e^{K}\,\Psi\,, \tag{2.6}$$

where K is a ghost number zero Grassmann even operator. In addition, we require

$$\begin{aligned} K(A*B) &= (KA)*B + A*(KB)\,, \\ \langle\, KA, B\,\rangle &= -\langle A, KB\rangle\,. \end{aligned} \tag{2.7}$$

These properties guarantee that the form of the cubic term is unchanged and that after the field redefinition the action takes the form

$$\mathcal{S}(\Psi) \equiv -\frac{1}{g_0^2}\left[\frac{1}{2}\langle\,\Psi\,,\,\mathcal{Q}\,\Psi\rangle + \frac{1}{3}\langle\,\Psi, \Psi * \Psi\rangle\right], \tag{2.8}$$

where

$$\mathcal{Q} = e^{-K}\widehat{Q}e^{K}\,. \tag{2.9}$$

Again, gauge invariance only requires:

$$\begin{aligned} &\mathcal{Q}^2 = 0, \\ &\mathcal{Q}(A*B) = (\mathcal{Q}A)*B + (-1)^A A*(\mathcal{Q}B)\,, \\ &\langle\,\mathcal{Q}A, B\,\rangle = -(-1)^A\langle A, \mathcal{Q}B\rangle\,. \end{aligned} \tag{2.10}$$

These identities hold by virtue of (2.7) and (2.9). We will proceed here postulating a $\mathcal{Q}$ that satisfies these identities as well as other conditions, since, lacking knowledge of Φ_0, the above field redefinitions cannot be attempted.

The choice of $\mathcal{Q}$ will be required to satisfy the following properties:

- The operator $\mathcal{Q}$ must be of ghost number one and must satisfy the conditions (2.10) that guarantee gauge invariance of the string action.
- The operator $\mathcal{Q}$ must have vanishing cohomology.
- The operator $\mathcal{Q}$ must be universal, namely, it must be possible to write without reference to the brane boundary conformal field theory.

We can satisfy the three requirements by letting $\mathcal{Q}$ be constructed purely from ghost operators. In particular we claim that the ghost number one operators

$$\mathcal{C}_n \equiv c_n + (-1)^n\, c_{-n}\,, \quad n = 0, 1, 2, \cdots \tag{2.11}$$

satisfy the properties

$$\begin{aligned}
&\mathcal{C}_n\mathcal{C}_n = 0, \\
&\mathcal{C}_n(A * B) = (\mathcal{C}_nA) * B + (-1)^A A * (\mathcal{C}_nB), \\
&\langle\, \mathcal{C}_nA, B\,\rangle = -(-1)^A \langle A, \mathcal{C}_nB\rangle\,.
\end{aligned} \tag{2.12}$$

The first property is manifest. The last property follows because under BPZ conjugation $c_n \to (-1)^{n+1}c_{-n}$. The second property follows from the conservation law

$$\langle V_3|(\mathcal{C}_n^{(1)} + \mathcal{C}_n^{(2)} + \mathcal{C}_n^{(3)}) = 0\,, \tag{2.13}$$

on the three string vertex [35].

Each of the operators $\mathcal{C}_n$ has vanishing cohomology since for each n the operator $\mathcal{B}_n = \frac{1}{2}(b_n + (-1)^n\, b_{-n})$ satisfies $\{\mathcal{C}_n, \mathcal{B}_n\} = 1$. It then follows that whenever $\mathcal{C}_n\psi = 0$, we have $\psi = \{\mathcal{C}_n, \mathcal{B}_n\}\psi = \mathcal{C}_n(\mathcal{B}_n\psi)$, showing that ψ is $\mathcal{C}_n$ trivial. Finally, since they are built from ghost oscillators, all $\mathcal{C}_n$'s are manifestly universal.

It is clear from the structure of the conditions (2.10) that they are satisfied for the general choice:

$$\mathcal{Q} = \sum_{n=0}^{\infty} a_n\, \mathcal{C}_n\,, \tag{2.14}$$

where the a_n's are constant coefficients.

There may be other choices of $\mathcal{Q}$ satisfying all the requirements stated above. Fortunately, the future analysis will not require the knowledge of the detailed form of $\mathcal{Q}$, as long as it does not involve any matter operators. To this end, it will be useful to note that since $\mathcal{Q}$ does not involve matter operators, we can fix the gauge by choosing a gauge fixing condition that also does not involve any matter operator. In such a gauge, the propagator will factor into a non-trivial operator in the ghost sector, and the identity operator in the matter sector.

2.1.1 Factorization of the matter and ghost sector

If (2.8) really describes the string field theory around the tachyon vacuum, then the equations of motion of this field theory:

$$\mathcal{Q}\Psi = -\Psi * \Psi\,, \tag{2.15}$$

must have a space-time independent solution describing the D25–brane, and also lump solutions of all codimensions describing lower dimensional D-branes. We shall look for solutions of the form:

$$\Psi = \Psi_m \otimes \Psi_g\,, \tag{2.16}$$

where Ψ_g denotes a state obtained by acting with the ghost oscillators on the SL(2,$\mathbb{R}$) invariant vacuum of the ghost CFT, and Ψ_m is a state obtained by acting with matter oscillators on the SL(2,$\mathbb{R}$) invariant vacuum of the matter CFT. Let us denote by $*^g$ and $*^m$ the star product in the ghost and matter sector respectively. Since Qis made purely of ghost operators, eq.(2.15) factorizes as

$$\mathcal{Q}\Psi_g = -\Psi_g *^g \Psi_g \,, \tag{2.17}$$

and

$$\Psi_m = \Psi_m *^m \Psi_m \,. \tag{2.18}$$

In looking for the solutions describing D-branes of various dimensions we shall assume that Ψ_g remains the same for all solutions, whereas Ψ_m is different for different D-branes. Given two static solutions Ψ_m and Ψ'_m, the ratio of the energy associated is obtained by taking the ratio of the actions associated with the two solutions. For a string field configuration satisfying the equation of motion (2.15), the action (2.8) is given by

$$\mathcal{S}|_\Psi = -\frac{1}{6g_0^2}\,\langle\,\Psi\,,\,\mathcal{Q}\,\Psi\,\rangle\,. \tag{2.19}$$

Thus with the ansatz (2.16) the action takes the form:

$$\mathcal{S}|_\Psi = -\frac{1}{6g_0^2}\,\langle\,\Psi_g\,|\,\mathcal{Q}\,\Psi_g\,\rangle_g\,\langle\Psi_m|\Psi_m\rangle_m \equiv K\langle\Psi_m|\Psi_m\rangle_m\,, \tag{2.20}$$

where $\langle\,|\,\rangle_g$ and $\langle\,|\,\rangle_m$ denote BPZ inner products in ghost and matter sectors respectively. $K = -(6g_0^2)^{-1}\langle\Psi_g|\mathcal{Q}\Psi_g\rangle_g$ is a constant factor calculated from the ghost sector which remains the same for different solutions. Thus we see that the ratio of the action associated with the two solutions is

$$\frac{\mathcal{S}|_{\Psi'}}{\mathcal{S}|_\Psi} = \frac{\langle\Psi'_m|\Psi'_m\rangle_m}{\langle\Psi_m|\Psi_m\rangle_m}\,. \tag{2.21}$$

It is worthwhile to notice that the ghost part drops out of this calculation.

What is more important, is that we already know two solutions to the equation of motion for the matter sector (2.18): they are the matter part of identity state $|I\rangle$ and sliver state $|\Xi\rangle$. It is important to say that the sliver state is interpreted as the D25–brane, the unstable vacuum of open strings [72]. Roughly speaking what we do is to 'build' on the not yet found closed string vacuum a state that is the old perturbative vacuum. Furthermore, from the sliver we can construct lump solutions of arbitrary co-dimension with the correct ratios of tensions of lower dimensional Dp–branes. We will give the description of the sliver state through the operator formalism first proposed by Kostelecky and Potting [34]. In the operator formalism, computations are algebraically and involves infinite dimensional matrices and their determinants: we will see that is also possible to have some analytical exact results. The advantage respect to CFT formalism is a less abstract approach. In the next section we will writethe solution to $\Psi_m *^m \Psi_m = \Psi_m$ given by Kostelecky and Potting in the form of a squezeed state: an exponential of bilinears of the string creation operators acting on the vacuum. Okuda [54] showed analitically that the squeezed state $|\Psi_m\rangle$ is indeed equal to the sliver $|\Xi\rangle$.

2.2 Matter solutions: $*_m$ Projectors

2.2.1 The space filling D25–brane

The three string vertex [27, 32, 33] of the Open String Field Theory is given by

$$|V_3\rangle = \int d^{26}p_{(1)} d^{26}p_{(2)} d^{26}p_{(3)}\, \delta^{26}(p_{(1)} + p_{(2)} + p_{(3)})\, \exp(-E)\, |0, p\rangle_{123} \qquad (2.22)$$

where

$$E = \sum_{r,s=1}^{3} \left(\frac{1}{2} \sum_{m,n\geq 1} \eta_{\mu\nu} a_m^{(r)\mu\dagger}\, V_{mn}^{rs}\, a_n^{(s)\nu\dagger} + \sum_{n\geq 1} \eta_{\mu\nu} p_{(r)}^{\mu}\, V_{0n}^{rs}\, a_n^{(s)\nu\dagger} + \frac{1}{2} \eta_{\mu\nu} p_{(r)}^{\mu}\, V_{00}^{rs}\, p_{(s)}^{\nu} \right) \qquad (2.23)$$

Summation over the Lorentz indices $\mu, \nu = 0, \ldots, 25$ is understood and η denotes the flat Lorentz metric and the operators $a_m^{(r)\mu}, a_m^{(r)\mu\dagger}$ denote the non–zero modes matter oscillators of the r–th string, which satisfy

$$[a_m^{(r)\mu}, a_n^{(s)\nu\dagger}] = \eta^{\mu\nu}\delta_{mn}\delta^{rs}, \qquad m, n \geq 1 \qquad (2.24)$$

$p_{(r)}$ is the momentum of the r–th string and $|0, p\rangle_{123} \equiv |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$ is the tensor product of the Fock vacuum states relative to the three strings. $|p_{(r)}\rangle$ is annihilated by the annihilation operators $a_m^{(r)\mu}$ and is eigenstate of the momentum operator $\hat{p}_{(r)}^{\mu}$ with eigenvalue $p_{(r)}^{\mu}$. The normalization is

$$\langle p_{(r)} | \, p'_{(s)} \rangle = \delta_{rs} \delta^{26}(p + p')$$

The coefficients V^{rs}_{MN} have been computed in [32, 33]: capital indeces mean the inclusion also of the zero modes (see Appendix B.

Some appreciation of the properties reviewed in Appendix C is useful. Equation (C.15), in particular, gives

$$V^{r\,s} = \frac{1}{3}(C + \omega^{s-r}U + \omega^{r-s}\bar{U}), \tag{2.25}$$

where $\omega = e^{2\pi i/3}$, U and C are regarded as matrices with indices running over $m, n \geq 1$,

$$C_{mn} = (-1)^m \delta_{mn}, \quad m, n \geq 1, \tag{2.26}$$

and U satisfies (C.17)

$$\bar{U} \equiv U^* = CUC, \qquad U^2 = \bar{U}^2 = 1, \qquad U^\dagger = U, \quad \bar{U}^\dagger = \bar{U}. \tag{2.27}$$

The superscripts r, s are defined mod(3), and (2.25) manifestly implements the cyclicity property $V^{rs} = V^{(r+1)(s+1)}$. Also note the transposition property $(V^{rs})^T = V^{sr}$. Finally, eqs.(2.25), (2.27) allow one to show that

$$[CV^{rs}, CV^{r's'}] = 0 \quad \forall \quad r, s, r', s', \tag{2.28}$$

and

$$\begin{aligned} (CV^{12})(CV^{21}) = (CV^{21})(CV^{12}) = (CV^{11})^2 - CV^{11}, \\ (CV^{12})^3 + (CV^{21})^3 = 2(CV^{11})^3 - 3(CV^{11})^2 + 1. \end{aligned} \tag{2.29}$$

We are looking for a space-time independent solution of eq.(2.18). The strategy of Kostelecky and Potting [34] is to take a trial solution of the form:

$$|\Psi_m\rangle = \mathcal{N}^{26} \exp\Big(-\frac{1}{2}\eta_{\mu\nu} \sum_{m,n\geq 1} S_{mn}\, a_m^{\mu\dagger} a_n^{\nu\dagger}\Big)|0\rangle, \tag{2.30}$$

where $|0\rangle$ is the SL(2,$\mathbb{R}$) invariant vacuum of the matter CFT, $\mathcal{N}$ is a normalization constant, and S_{mn} is an infinite dimensional matrix with indices m, n running from 1 to ∞. We will assume S_{mn} to be twist invariant, that is

$$CSC = S. \tag{2.31}$$

This corresponds to hermitianity of the string state $|\Psi_m\rangle$.

Now, we will construct the star product of the trial solution 2.30 with itself. Since we are looking for the space filling D25 brane which is, obviously, a space-time translational invariant solution, the three string vertex (3.1) that we will use will be reduced to the momenta independent part

$$E = \frac{1}{2}\sum_{r,s} \eta_{\mu\nu} a^{(r)\mu\dagger} \cdot V^{rs} \cdot a^{(s)\nu\dagger} \,, \tag{2.32}$$

Defining

$$\Sigma = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{pmatrix}, \tag{2.33}$$

and

$$\chi^{\mu T} = (\, a^{(3)\mu\dagger} V^{31} , \quad a^{(3)\mu\dagger} V^{32} \,), \qquad \chi^{\mu} = \begin{pmatrix} V^{13} a^{(3)\mu\dagger} \\ V^{23} a^{(3)\mu\dagger} \end{pmatrix}, \tag{2.34}$$

and using the general formula [34]

$$\begin{aligned} &\langle 0|\exp\left(\lambda_i a_i - \frac{1}{2} P_{ij} a_i a_j\right) \exp\left(\mu_i a_i^\dagger - \frac{1}{2} Q_{ij} a_i^\dagger a_j^\dagger\right)|0\rangle \\ &= \det(K)^{-1/2} \exp\left(\mu^T K^{-1}\lambda - \frac{1}{2}\lambda^T Q\, K^{-1}\lambda - \frac{1}{2}\mu^T K^{-1} P\mu\right), \; K \equiv 1 - PQ\,, \end{aligned} \tag{2.35}$$

in which a_i is the list of oscillators $(a_m^{(1)}, a_m^{(2)})$ with $m \geq 1$ and we have identified P with Σ, Q with $\mathcal{V}$, μ with χ and set λ to 0, we finally obtain

$$\begin{aligned} |\Psi_m * \Psi_m\rangle_3 &= \mathcal{N}^{52} \det\{(1 - \Sigma\mathcal{V})^{-1/2}\}^{26} \\ &\times \exp\Big[-\frac{1}{2}\, \eta_{\mu\nu}\{\chi^{\mu T}[(1-\Sigma\mathcal{V})^{-1}\Sigma]\chi^{\nu} + a^{(3)\mu\dagger} \cdot V^{33} \cdot a^{(3)\nu\dagger}\}\Big]|0\rangle_3 \,. \end{aligned} \tag{2.36}$$

Demanding that the exponents in $|\Psi_m\rangle$ and $|\Psi_m * \Psi_m\rangle$, match, we obtain

$$S = V^{11} + (V^{12}, V^{21})(1 - \Sigma\mathcal{V})^{-1}\Sigma \begin{pmatrix} V^{21} \\ V^{12} \end{pmatrix}, \tag{2.37}$$

Now, we multiply (2.37) by C and rewrite it as

$$T = X + (M^{12}, M^{21})(1 - \Sigma\mathcal{V})^{-1} \begin{pmatrix} TM^{21} \\ TM^{12} \end{pmatrix}, \tag{2.38}$$

in terms of

$$T \equiv CS = SC, \qquad M^{rs} \equiv CV^{rs}, \tag{2.39}$$

$$X = M^{11} = CV^{11}. \tag{2.40}$$

Assuming

$$[CS, CV^{rs}] = 0 \quad \forall \quad r, s. \tag{2.41}$$

and because of (2.28) we can manipulate the equation as if T and M^{rs} are numbers rather than infinite dimensional matrices.

Since

$$\begin{aligned}(1-\Sigma\mathcal{V})^{-1} &= \begin{pmatrix} 1-TX & -TM^{12} \\ -TM^{21} & 1-TX \end{pmatrix}^{-1} \\ &= ((1-TX)^2 - T^2M^{12}M^{21})^{-1} \begin{pmatrix} 1-TX & TM^{12} \\ TM^{21} & 1-TX \end{pmatrix}.\end{aligned} \tag{2.42}$$

and using $M^{12}M^{21} = X^2 - X$ that comes from (2.29), we can write

$$\det(1-\Sigma\mathcal{V}) = \det(1 - 2TX + T^2X), \tag{2.43}$$

and then eliminating M^{12} and M^{21} in favor of X, we finally get:

$$(T-1)(XT^2 - (1+X)T + X) = 0. \tag{2.44}$$

This gives three solutions for S: the first one $T = 1$ gives the identity state $|I_m\rangle$, the second one has diverging eigenvalues, so we are interested in the third one

$$S = CT, \qquad T = \frac{1}{2X}(1 + X - \sqrt{(1+3X)(1-X)}). \tag{2.45}$$

The result is coherent with the assumption of twist invariance that we made (2.31) and (2.41). Indeed, since CS is a function of X, and since $X(\equiv CV^{11})$ commutes with CV^{rs}, CS also commutes with CV^{rs}. Furthermore, since V^{11} is twist invariant, so is X. It then follows that the inverse of X and any polynomial in X are twist invariant. Therefore T and S are twist invariant.

Demanding that the normalization factors in $|\Psi_m\rangle$ and $|\Psi_m * \Psi_m\rangle$ match gives

$$\mathcal{N} = \det(1-\Sigma\mathcal{V})^{1/2} = (\det(1-X)\det(1+T))^{1/2}, \tag{2.46}$$

where we have used eqn.(2.43) and simplified it further using (2.44). Thus the solution is given by

$$|\Psi_m\rangle = \{\det(1-X)^{1/2}\det(1+T)^{1/2}\}^{26}\exp\Big(-\frac{1}{2}\,\eta_{\mu\nu}\sum_{m,n\geq 1} S_{mn}\,a_m^{\mu\dagger}a_n^{\nu\dagger}\Big)|0\rangle\,. \quad (2.47)$$

This is the matter part of the state found in [34] after suitable correction to the normalization factor. From eq.(2.20) we see that the value of the action associated with this solution has the form:

$$\mathcal{S}|_{\Psi} = K\mathcal{N}^{52}\,\langle 0|\exp\Big(-\frac{1}{2}\eta_{\mu'\nu'}\sum_{m',n'\geq 1} S_{m'n'}a^{\mu'}_{m'}a^{\nu'}_{n'}\Big)\exp\Big(-\frac{1}{2}\eta_{\mu\nu}\sum_{m,n\geq 1} S_{mn}a^{\mu\dagger}_m a^{\nu\dagger}_n\Big)|0\rangle\,.$$

By evaluating the matrix element using eq.(2.35), and using the normalization:

$$\langle 0|0\rangle = \delta^{(26)}(0) = \frac{V^{(26)}}{(2\pi)^{26}}\,, \quad (2.48)$$

where $V^{(26)}$ is the volume of the 26-dimensional space-time, we get the value of the action to be

$$\begin{aligned}\mathcal{S}|_{\Psi} &= K\,\frac{V^{(26)}}{(2\pi)^{26}}\,\mathcal{N}^{52}\{\det(1-S^2)^{-1/2}\}^{26}\\ &= K\,\frac{V^{(26)}}{(2\pi)^{26}}\,\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\}^{26}\,. \end{aligned}\quad (2.49)$$

In arriving at the right hand side of eq.(2.49) we have made use of eqs.(2.45) and (2.46). Thus the tension of the D25–brane is given by

$$\mathcal{T}_{25} = K\,\frac{1}{(2\pi)^{26}}\,\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\}^{26}\,. \quad (2.50)$$

2.2.2 Lower dimensional branes

The solution (2.47) representing the D25–brane has a factorized form, 26 factors, each involving the oscillators associated with a given space-time direction. In order to construct a solution of codimension k representing a D$(25-k)$-brane, we need to replace k of the factors associated with directions transverse to the D-brane by a different set of solutions, but the factors associated with directions tangential to the D-brane remains the same. Suppose we are interested in a D$(25-k)$-brane solution. Let us denote by $x^{\bar{\mu}}$ $(0\leq\bar{\mu}\leq(25-k))$ the directions tangential to the brane and by x^{α} $((26-k)\leq\alpha\leq 25)$ the directions transverse to the brane. We now use the representation of the vertex in the zero mode oscillator basis for the directions x^α, as given in Appendix C. For this we define, for each string,

$$a_0^\alpha = \frac{1}{2}\sqrt{b}\,\hat{p}^\alpha - \frac{1}{\sqrt{b}}i\hat{x}^\alpha, \qquad a_0^{\alpha\dagger} = \frac{1}{2}\sqrt{b}\,\hat{p}^\alpha + \frac{1}{\sqrt{b}}i\hat{x}^\alpha, \tag{2.51}$$

where b is an arbitrary constant (a length, from a dimensional point of view) and $\hat{x}^\alpha$ and $\hat{p}^\alpha$ are the zero mode coordinate and momentum operators associated with the direction x^α. We also denote by $|\Omega_b\rangle$ the normalized state which is annihilated by all the annihilation operators a_0^α, and by $|\Omega_b\rangle_{123}$ the direct product of the vacuum $|\Omega_b\rangle$ for each of the three strings.

The relation between the momentum basis and the new oscillator basis is given by (for each string)

$$|\{p^\alpha\}\rangle = (2\pi/b)^{-k/4}\exp\Big[-\frac{b}{4}p^\alpha p^\alpha + \sqrt{b}a_0^{\alpha\dagger}p^\alpha - \frac{1}{2}a_0^{\alpha\dagger}a_0^{\alpha\dagger}\Big]|\Omega_b\rangle\,. \tag{2.52}$$

In the above equation $\{p^\alpha\}$ label momentum eigenvalues. Substituting eq.(2.52) into eq.(3.2), and integrating over $p_{(i)}^\alpha$, we can express the three string vertex as

$$\begin{aligned}
|V_3\rangle \;=\; & \int d^{26-k}p_{(1)}d^{26-k}p_{(2)}d^{26-k}p_{(3)}\delta^{(26-k)}(p_{(1)}+p_{(2)}+p_{(3)}) \\
& \exp\Big(-\frac{1}{2}\sum_{\substack{r,s\\ m,n\geq 1}}\eta_{\bar{\mu}\bar{\nu}}a_m^{(r)\bar{\mu}\dagger}V_{mn}^{rs}a_n^{(s)\bar{\nu}\dagger} - \sum_{\substack{r,s\\ n\geq 1}}\eta_{\bar{\mu}\bar{\nu}}p_{(r)}^{\bar{\mu}}V_{0n}^{rs}a_n^{(s)\bar{\nu}\dagger} - \frac{1}{2}\sum_r \eta_{\bar{\mu}\bar{\nu}}p_{(r)}^{\bar{\mu}}V_{00}^{rr}p_{(r)}^{\bar{\nu}}\Big)|0,p\rangle_{123} \\
& \otimes\Big(\frac{\sqrt{3}}{(2\pi b^3)^{1/4}}(V_{00}^{rr}+\frac{b}{2})\Big)^{-k}\exp\Big(-\frac{1}{2}\sum_{\substack{r,s\\ M,N\geq 0}}a_M^{(r)\alpha\dagger}V_{MN}^{\prime rs}a_N^{(s)\alpha\dagger}\Big)|\Omega_b\rangle_{123}\,.
\end{aligned} \tag{2.53}$$

In this expression the sums over $\bar{\mu},\bar{\nu}$ run from 0 to $(25-k)$, and sum over α runs from $(26-k)$ to 25. Note that in the last line the sums over M, N run from 0 to ∞. The coefficients $V_{MN}^{\prime rs}$ have been given in terms of V_{mn}^{rs} in Appendix in C eq.(C.7).

In Appendix C it is shown that $V^{\prime rs}$, regarded as matrices with indices running from 0 to ∞, satisfy

$$V^{\prime\, r\, s} = \frac{1}{3}(C' + \omega^{s-r}U' + \omega^{r-s}\bar{U}')\,, \tag{2.54}$$

where we have dropped the explicit b dependence from the notation, $C'_{MN} = (-1)^M\delta_{MN}$ with indices M, N now running from 0 to ∞, and U', $\bar{U}' \equiv U'^*$ viewed as matrices with $M, N \geq 0$ satisfy the relations:

$$\bar{U}' = C'U'C', \qquad U'^2 = \bar{U}'^2 = 1, \qquad U'^\dagger = U'\,. \tag{2.55}$$

We note now the complete analogy with equations (2.25) and (2.27) [34]. It follows also that the V' matrices, together with C' will satisfy equations exactly analogous to (2.28), (2.29). Thus we can construct a solution of the equations of motion (2.18) in an identical manner with the unprimed quantities replaced by the primed quantities. Taking into account the extra normalization factor appearing in the last line of eq.(2.53), we get the following form of the solution of eq.(2.18):

$$\begin{aligned} |\Psi'_m\rangle &= \{\det(1-X)^{1/2}\det(1+T)^{1/2}\}^{26-k}\exp(-\frac{1}{2}\eta_{\bar\mu\bar\nu}\sum_{m,n\geq 1} S_{mn}a_m^{\bar\mu\dagger}a_n^{\bar\nu\dagger})|0\rangle \\ &\otimes\Big(\frac{\sqrt{3}}{(2\pi b^3)^{1/4}}(V_{00}^{rr}+\frac{b}{2})\Big)^k\{\det(1-X')^{1/2}\det(1+T')^{1/2}\}^k \\ &\quad \exp\Big(-\frac{1}{2}\sum_{M,N\geq 0} S'_{MN}a_M^{\alpha\dagger}a_N^{\alpha\dagger}\Big)|\Omega_b\rangle\,, \end{aligned} \tag{2.56}$$

where

$$S' = C'T'\,, \qquad T' = \frac{1}{2X'}(1+X'-\sqrt{(1+3X')(1-X')})\,, \tag{2.57}$$

$$X' = C'V'^{11}\,. \tag{2.58}$$

Using eq.(2.20) we can calculate the value of the action associated with this solution. It is given by an equation analogous to (2.49):

$$\begin{aligned} \mathcal{S}_{\Psi'} &= K\frac{V^{(26-k)}}{(2\pi)^{26-k}}\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\}^{26-k} \\ &\times\Big(\frac{3}{(2\pi b^3)^{1/2}}(V_{00}^{rr}+\frac{b}{2})^2\Big)^k\{\det(1-X')^{3/4}\det(1+3X')^{1/4}\}^k\,, \end{aligned} \tag{2.59}$$

where $V^{(26-k)}$ is the D$(25-k)$–brane world-volume. This gives the tension of the D$(25-k)$–brane to be

$$\begin{aligned} \tau_{25-k} &= K\frac{1}{(2\pi)^{26-k}}\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\}^{26-k} \\ &\times\Big(\frac{3}{(2\pi b^3)^{1/2}}(V_{00}^{rr}+\frac{b}{2})^2\Big)^k\{\det(1-X')^{3/4}\det(1+3X')^{1/4}\}^k\,. \end{aligned} \tag{2.60}$$

Clearly for $k=0$ this agrees with (2.50). From eq.(2.60) we get

$$\frac{\tau_{24-k}}{2\pi\tau_{25-k}} = \frac{3}{\sqrt{2\pi b^3}}\left(V_{00}^{rr}+\frac{b}{2}\right)^2\frac{\{\det(1-X')^{3/4}\det(1+3X')^{1/4}\}}{\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\}} \equiv R\,. \tag{2.61}$$

Okuyama [52] proved in an analitycal way that $R = 1$. Therefore VSFT describes the correct ratios of Dp–brane:

$$\frac{\tau_{24-k}}{2\pi\tau_{25-k}} = 1 \,.$$

We will see in the next chapter that this is also true if a B field is switched on [76]. The detailed calculations are given in Appendix E.

2.3 Multiple D–branes

In [38] it was shown that it is possible to construct solutions to the projection equation (2.17) representing multiple D–branes. We briefly review this construction. In the next section we will show that these kind of solutions are particular cases of an infinite class of solutions that we call "Ancestors" because they reduce to all GMS solitons in the low energy limit.

Before to give the ansatz for the projector analog but inequivalent to the sliver, we need to define the projectors

$$\begin{aligned} \rho_1 &= \frac{1}{(1+T)(1-X)}\left[M^{12}(1-TX) + T(M^{21})^2\right], \\ \rho_2 &= \frac{1}{(1+T)(1-X)}\left[M^{21}(1-TX) + T(M^{12})^2\right], \end{aligned} \tag{2.62}$$

Using the algebraic properties of the matrices M and X:

$$\begin{aligned} & X + M^{12} + M^{21} = 1\,, \\ & M^{12}M^{21} = X^2 - X\,, \\ & (M^{12})^2 + (M^{21})^2 = 1 - X^2\,, \\ & (M^{12})^3 + (M^{21})^3 = 2X^3 - 3X^2 + 1 = (1-X)^2(1+2X)\,, \end{aligned} \tag{2.63}$$

and some useful combination of them

$$(M^{12} - M^{21})^2 = (1-X)(1+3X)\,. \tag{2.64}$$

$$\frac{1-TX}{1-X} = \frac{1}{1-T}\,, \quad \frac{1-T}{1+T} = \sqrt{\frac{1-X}{1+3X}}\,, \quad \frac{X}{1-X} = \frac{T}{(1-T)^2}\,. \tag{2.65}$$

it is easy to see that

$$\rho_1^T = \rho_1\,, \quad \rho_2^T = \rho_2\,, \quad C\rho_1 C = \rho_2\,, \tag{2.66}$$

and

$$\begin{aligned} \rho_1 + \rho_2 &= 1\,, \\ \rho_1 - \rho_2 &= \frac{M^{12} - M^{21}}{\sqrt{(1-X)(1+3X)}}\,. \end{aligned} \tag{2.67}$$

From eq.(2.64) we see that the square of the second right hand side is the unit matrix. Thus $(\rho_1 - \rho_2)^2 = 1$ and

$$\rho_1 \rho_2 = 0\,. \tag{2.68}$$

Multiplying the first equation in (2.67) by ρ_1 and by ρ_2 we find

$$\rho_1 \rho_1 = \rho_1\,, \quad \rho_2 \rho_2 = \rho_2\,. \tag{2.69}$$

This shows that ρ_1 and ρ_2 are projection operators into orthogonal subspaces, and the C exchanges these two subspaces.

For future purposes we call the projector $|\Lambda_1\rangle$. The general ansatz is:

$$|\Lambda_1\rangle = \Big(- \xi \cdot a^\dagger \zeta \cdot a^\dagger + k \Big) |\Xi\rangle \tag{2.70}$$

where ζ and ξ are infinite dimensional vectors such that

$$\rho_1 \xi = 0\,, \quad \rho_2 \xi = \xi\,, \tag{2.71}$$

and $\zeta = C\xi$ in order to guarantee the Hermiticity of $|\Lambda_1\rangle$. Since $C\rho_1 C = \rho_2$ we have also

$$\rho_2 \zeta = 0\,, \quad \rho_2 \zeta = \zeta\,. \tag{2.72}$$

$|\Lambda_1\rangle$ satisfies the following properties

1. $|\Lambda_1\rangle * |\Xi\rangle = 0$

2. $|\Lambda_1\rangle * |\Lambda_1\rangle = |\Lambda_1\rangle$

3. $\langle\Lambda_1|\Lambda_1\rangle = \langle\Xi|\Xi\rangle$

The first property will be used to fix k, the second will normalize ξ, and the third will follow from the first two.

The product $|\Lambda_1\rangle * |\Xi\rangle$ is nothing but a sliver times a sliver with two oscillators acting on it. See Appendix for details of such calculations. (It is enough to apply the differential operator $\frac{\partial^2}{\partial\beta_{1m\mu}\partial\beta_{1n\nu}}$ on both sides of eq.(A.12) and then setting β_1 and β_2 to zero). The result is

$$|\Lambda_1 * \Xi\rangle = |\Xi * \Lambda_1\rangle = (k + \xi^T(\mathcal{V}\mathcal{K}^{-1})_{11}\zeta)|\Xi\rangle\,, \tag{2.73}$$

Since we want $|\Lambda_1\rangle * |\Xi\rangle = 0$, it must be

$$k = -\xi^T(\mathcal{V}\mathcal{K}^{-1})_{11}\tilde{\xi} = -\xi^T T(1-T^2)^{-1}\xi\,, \tag{2.74}$$

where the last equation in (2.65) was used to simplify the expression for $(\mathcal{V}\mathcal{K}^{-1})_{11}$.

We calculate $|\Lambda_1\rangle * |\Lambda_1\rangle$ in a similar way getting:

$$\begin{aligned} |\Lambda_1\rangle * |\Lambda_1\rangle \;=\; & -(\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\zeta)\,\xi\cdot a^\dagger\,\zeta\cdot a^\dagger|\Xi\rangle \\ & +\Big((\xi^T(\mathcal{V}\mathcal{K}^{-1})_{11}\zeta)(\xi^T(\mathcal{V}\mathcal{K}^{-1})_{22}\zeta) + (\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\zeta)(\zeta^T(\mathcal{V}\mathcal{K}^{-1})_{12}\xi) - \kappa^2\Big)|\Xi\rangle\,. \end{aligned} \tag{2.75}$$

Using (A.10), the last equation in (2.65), and (2.74) one finds that

$$\zeta^T(\mathcal{V}\mathcal{K}^{-1})_{12}\xi = -\xi^T T(1-T^2)^{-1}\xi = k\,. \tag{2.76}$$

Furthermore $(\mathcal{V}\mathcal{K}^{-1})_{11} = (\mathcal{V}\mathcal{K}^{-1})_{22}$. Using this and eqs.(2.74), (2.76), we see that eq.(2.75) can be written as

$$|\Lambda_1\rangle * |\Lambda_1\rangle = (\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\zeta)\Big(-\xi\cdot a^\dagger\,\zeta\cdot a^\dagger + k\Big)|\Xi\rangle\,. \tag{2.77}$$

So the problem to have

$$|\Lambda_1\rangle * |\Lambda_1\rangle = |\Lambda_1\rangle\,. \tag{2.78}$$

reduces to normalize the vector ξ such that

$$\xi^T\,(\mathcal{V}\mathcal{K}^{-1})_{12}\,\zeta = 1\,, \tag{2.79}$$

The normalization condition eq.(2.79) can be simplified using the first equation in (2.65) to obtain:

$$\xi^T(1-T^2)^{-1}\xi = 1\,. \tag{2.80}$$

In order to show that the new solution $|\Lambda_1\rangle$ also represents a single D25–brane we can calculate the tension associated with this solution and try to verify that it agrees with the tension of the brane described by the sliver. Since the tension of the brane associated to a given state is proportional to the BPZ norm of the state [37], all we need to show is that $\langle\Lambda_1|\Lambda_1\rangle$ is equal to $\langle\Xi|\Xi\rangle$. This is a straightforward calculation using the formula (2.35), that, rewritten for the present purposes is

$$\begin{aligned}
&\langle 0|\exp\Big(-\frac{1}{2}a\cdot Sa+\lambda\cdot a\Big)\exp\Big(-\frac{1}{2}a\cdot Sa^\dagger+\beta\cdot a\Big)|0\rangle \qquad (2.81)\\
=\;&\det(1-S^2)^{-1}\exp\Big(\beta^T\cdot(1-S^2)^{-1}\cdot\lambda-\frac{1}{2}\beta^T\cdot S(1-S^2)^{-1}\cdot\beta\\
&-\frac{1}{2}\lambda^T\cdot S(1-S^2)^{-1}\cdot\lambda\Big)\,,
\end{aligned}$$

The result is

$$\langle\Lambda_1|\Lambda_1\rangle = \langle\Xi|\Xi\rangle\,. \tag{2.82}$$

Thus the solution described by $|\Lambda_1\rangle$ has the same tension as the solution described by $|\Xi\rangle$.

In a similar way:

$$\langle\Xi|\Lambda_1\rangle = 0\,. \tag{2.83}$$

The BPZ norm of $|\Xi\rangle+|\Lambda_1\rangle$ is $2\langle\Xi|\Xi\rangle$. This shows that $|\Xi\rangle+|\Lambda_1\rangle$ represents a configuration with twice the tension of a single D25–brane.

Consider now another projector $|\Lambda_1'\rangle$ built just as $|\Lambda_1\rangle$ but using a vector ξ':

$$|\Lambda_1'\rangle = \Big(-\xi'\cdot a^\dagger\,\zeta'\cdot a^\dagger + k'\Big)|\Xi\rangle\,, \tag{2.84}$$

with

$$\rho_1\xi' = 0, \qquad \rho_2\xi' = \xi'\,, \tag{2.85}$$

k' given as

$$k' = -\xi'^T T(1-T^2)^{-1}\xi' , \tag{2.86}$$

and normalization fixed by

$$\xi'^T(1-T^2)^{-1}\xi' = 1 . \tag{2.87}$$

Thus $|\Lambda_1'\rangle$ is a projector orthogonal to $|\Lambda_1\rangle$. We now want to find the condition under which $|\Lambda_1'\rangle$ projects into a subspace orthogonal to $|\Lambda_1\rangle$ as well, *i.e.* the condition under which $|\Lambda_1\rangle * |\Lambda_1'\rangle$ vanishes. We can compute $|\Lambda_1\rangle * |\Lambda_1'\rangle$ in a manner identical to the one used in computing $|\Lambda_1\rangle * |\Lambda_1\rangle$ and find that it vanishes if:

$$\xi^T(1-T^2)^{-1}\xi' = 0 . \tag{2.88}$$

Since this equation is symmetric in ξ and ξ', it is clear that $|\Lambda_1'\rangle * |\Lambda_1\rangle$ also vanishes when eq.(2.88) is satisfied. Given eqs.(2.87) and (2.88) we also have:

$$\langle\Lambda_1'|\Lambda_1'\rangle = 1, \quad \langle\Lambda_1|\Lambda_1'\rangle = \langle\Xi|\Lambda_1'\rangle = 0 . \tag{2.89}$$

Thus $|\Xi\rangle + |\Lambda_1\rangle + |\Lambda_1'\rangle$ describes a solution with three D25–branes. To conclude, this method allows us to generate solutions with arbitrary number of D25–branes. One can ask if such configuration of D-branes correspond to separated or coincident D-branes or if there is a generalization of it. In chapter 4 we will see an answer to the second question and a suggestion to the first one.

2.4 Surface states and star projectors

We will see that projectos of the star algebra will play a fundamental role in Vacuum String Field Theory. In order to define and to describe some of them in a conformal field theory point of view, we introduce the definition of *surface states*. We mainly refer to [63]. These surface states are Riemann surfaces whose boundary consists of a parametrized open string and a piece with open string boundary conditions. Thinkin of a surface state as a string field, it is possible to define a geometric operation which corresponds to star product in SFT: one glues the right-half of the open string in the first surface to the left-half of the open string in the second surface, and the surface state corresponding to the glued surface is the desired product. Usually, multiplication of a surface state to itself leads to a surface state that looks different from the initial state. This is the reason why it is not trivial to find projectors. Let us define the thing more precisely.

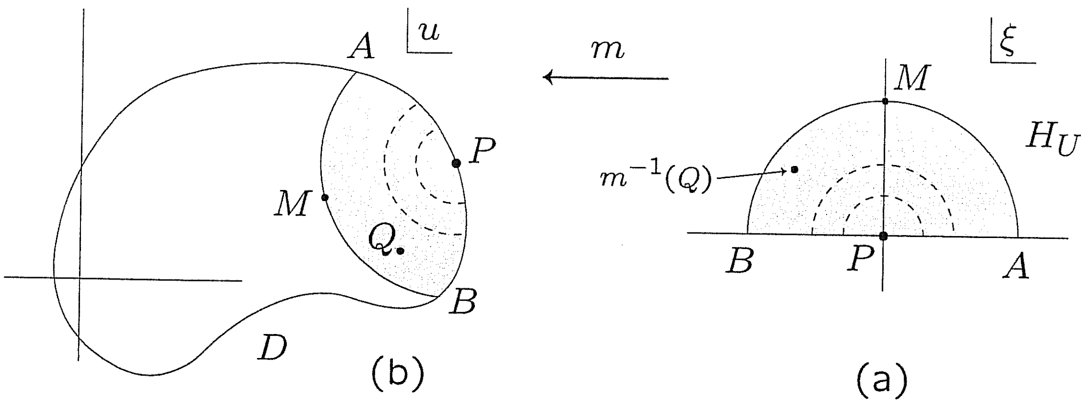

Figure 2.1: D is the punctured disk and P the puncture. The coordinate is defined through a map m from a canonical half disk H_U to the disk. M is the midpoint of the string and the arcs AM and MB in D are the left and right half part. (figure taken from [26]).

A surface state $\langle\Sigma|$ is a state related with a Riemann surface Σ with the topology of a disk D, with a marked point P, the puncture, lying on the boundary of D, and a local coordinate around it. Local coordinate at a puncture is obtained from an analytic map m taking a canonical half-disk H_U defined as

$$H_U : \{|\xi| \leq 1, \mathrm{Im}(\xi) \geq 0\} \tag{2.90}$$

into D, where $\xi = 0$ maps to the puncture P, and the image of the real segment $\{|\xi| \leq 1, \Im(\xi) = 0\}$ lies on the boundary of D. The coordinate ξ of the half disk is called the local coordinate. Using any global coordinate u on the disk D, the map m can be described by some analytic function s:

$$u = s(\xi), \quad u(P) = s(0) \tag{2.91}$$

Given a BCFT with state space $\mathcal{H}$, the state $\langle\Sigma| \in \mathcal{H}^*$ associated to the surface Σ is defined as follows. For any local operator $\phi(\xi)$, with associated state $|\phi\rangle = \lim_{\xi\to 0}\phi(\xi)|0\rangle$ we set

$$\langle\Sigma|\phi\rangle = \langle s \circ \phi(0)\rangle_D \tag{2.92}$$

where $\langle\ \rangle_D$ is the correlation function on D and $s \circ \phi(0)$ is the transform of the operator by the map $s(\xi)$. The gluing of surfaces, confomal analog of star product of string field states, requires a well defined full map of the half disk H_U into the disk D.

2.4.1 Operator representation of surface states

Let us see the representation of surface states in terms of Virasoro operators acting on the SL(2,R) invariant vacuum. We will see an explicit example later on treating wedge states. We can write the surface state $\Sigma|$ as

$$\langle\Sigma| = \langle 0|U_f \equiv \langle 0|\exp\left(\sum_{n=2}^{\infty} v_n^{(f)} L_n\right), \tag{2.93}$$

where the coefficients $v_n^{(f)}$ are determined by the condition that the vector field

$$v(\xi) = \sum_{n=2}^{\infty} v_n^{(f)} \xi^{n+1}, \tag{2.94}$$

exponentiates to f,

$$\exp\left(v(\xi)_\xi\right)\xi = f(\xi). \tag{2.95}$$

We now consider the one-parameter family of maps

$$f_\beta(\xi) = \exp\left(\beta\, v(\xi)\frac{\partial}{\partial\xi}\right)\xi. \tag{2.96}$$

This gives

$$\frac{d}{d\beta} f_\beta(\xi) = v(f_\beta(\xi)). \tag{2.97}$$

Solution, taking into account the boundary condition $f_{\beta=0}(\xi) = \xi$, gives:

$$f_\beta(\xi) = g^{-1}(\beta + g(\xi)), \tag{2.98}$$

where

$$g'(\xi) = \frac{1}{v(\xi)}. \tag{2.99}$$

Thus

$$f(\xi) = g^{-1}(1 + g(\xi)). \tag{2.100}$$

Equations (2.99) and (2.100) give $f(\xi)$ if $v(\xi)$ is known. They determine $v(\xi)$ in terms of $f(\xi)$, but eqn. (2.100) is in general hard to solve for g. When a solution for $v(\xi)$ is available, eqn. (2.93) gives the operator expression for $|\Sigma\rangle$.

2.4.2 Oscillator representation of surface states

We consider the matter part of the state and the oscillators will be associated to free scalar fields of the Boundary CFT we consider. If a_m, $a_m^\dagger$ denote the annihilation and creation operators we have:

$$|\Sigma\rangle = \exp\left(-\frac{1}{2}\sum_{m,n=1}^{\infty} a_m^\dagger V_{mn}^f a_n^\dagger\right)|0\rangle\,. \tag{2.101}$$

and

$$V_{mn}^f = \frac{(-1)^{m+n+1}}{\sqrt{mn}}\oint_0 \frac{dw}{2\pi i}\oint_0 \frac{dz}{2\pi i}\,\frac{1}{z^m w^n}\,\frac{f'(z)f'(w)}{(f(z)-f(w))^2}\,. \tag{2.102}$$

Both w and z integration contours are circles around the origin, inside the unit circle and with the w contour outside the z contour.

The crucial point is that when the vector field $v(\xi)$, generating the conformal map $f(\xi)$, is known we can put the integral expression for the matrix V^f of Neumann coefficients in another form. We consider the matrix $V(\beta)$ associated to the family of maps (2.96), and rewrite (2.102) as

$$V_{mn}(\beta) \equiv V_{mn}^{f_\beta} = \frac{(-1)^{m+n+1}}{\sqrt{mn}}\oint_0 \frac{dw}{2\pi i}\oint_0 \frac{dz}{2\pi i}\,\frac{1}{z^m w^n}\,\frac{\partial}{\partial z}\frac{\partial}{\partial w}\log(f_\beta(z)-f_\beta(w)) \tag{2.103}$$

Taking a derivative with respect to the parameter β,

$$\begin{aligned}\frac{d}{d\beta}V_{mn}(\beta) &= \frac{(-1)^{m+n+1}}{\sqrt{mn}}\oint_0 \frac{dw}{2\pi i}\oint_0 \frac{dz}{2\pi i}\,\frac{1}{z^m w^n}\,\frac{\partial}{\partial z}\frac{\partial}{\partial w}\frac{\partial}{\partial \beta}\log(f_\beta(z)-f_\beta(w)) \qquad (2.104)\\ &= \frac{(-1)^{m+n+1}}{\sqrt{mn}}\oint_0 \frac{dw}{2\pi i}\oint_0 \frac{dz}{2\pi i}\,\frac{1}{z^m w^n}\,\frac{\partial}{\partial z}\frac{\partial}{\partial w}\left(\frac{v(f_\beta(z))-v(f_\beta(w))}{f_\beta(z)-f_\beta(w)}\right)\,,\end{aligned}$$

where we have exchanged the order of derivatives and used (2.97). Integrating by parts in z and w:

$$\frac{d}{d\beta}V_{mn}(\beta) = (-1)^{m+n+1}\sqrt{mn}\oint_0 \frac{dw}{2\pi i}\oint_0 \frac{dz}{2\pi i}\,\frac{1}{z^{m+1}w^{n+1}}\,\frac{v(f_\beta(z))-v(f_\beta(w))}{f_\beta(z)-f_\beta(w)} \tag{2.105}$$

Neumann coefficients $V_{mn}(\beta=1)$ can be calculated integrating over β.

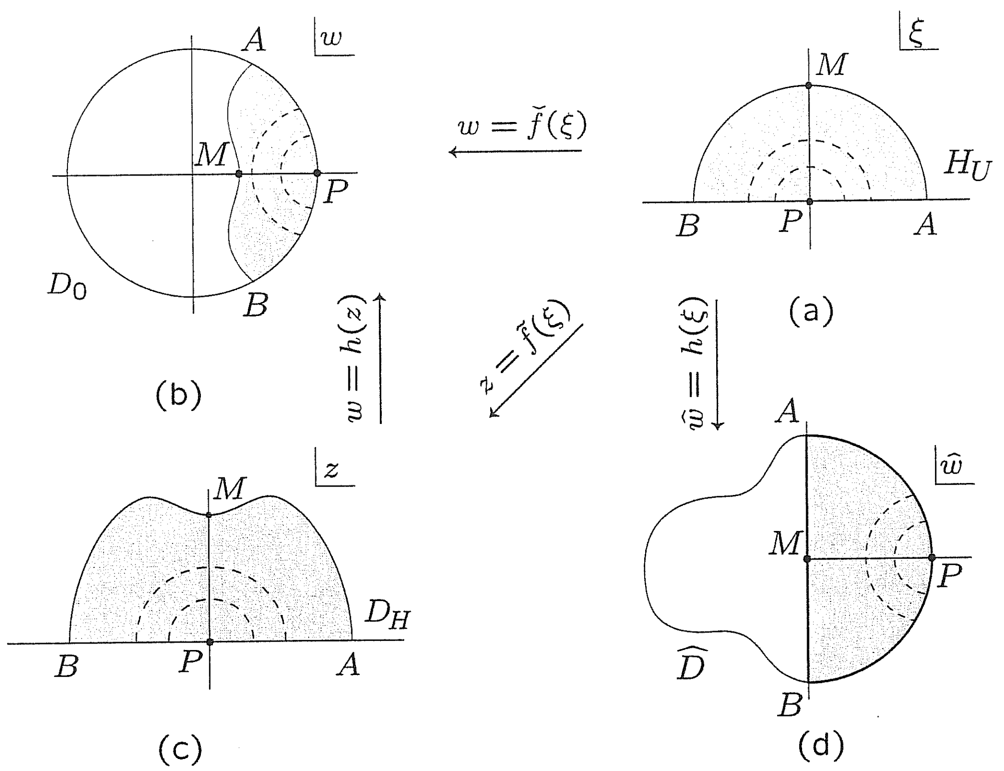

Figure 2.2: As above, D is the punctured disk, M the string midpoint and the arcs AM and MB the left half and the right half of the open string. (figure taken from [26]

2.5 Wedge states

An interesting class of surface states are the *wedge states*. The identity, the $SL(2, R)$ vacuum and the so called sliver, the state obtained star-multiplying an infinite number of vacuum, are particular wedge states. In this case, the Riemann surface is an angular sector of the unit disk, with the left-half and the right-half of the open string being the two radial segments, and the unit radius arc having the open string boundary conditions. A wedge state is thus defined by the angle at the open string midpoint, and this angle simply adds under star multiplication, as intuition suggests using the gluing prescription with angles. The identity string field is the wedge state of zero angle, and the sliver is the wedge state of infinite angle.

They are defined by the map

$$w_n = \check{f}_n(\xi) \equiv (h(\xi))^{2/n} = \left(\frac{1+i\xi}{1-i\xi}\right)^{2/n} \tag{2.106}$$

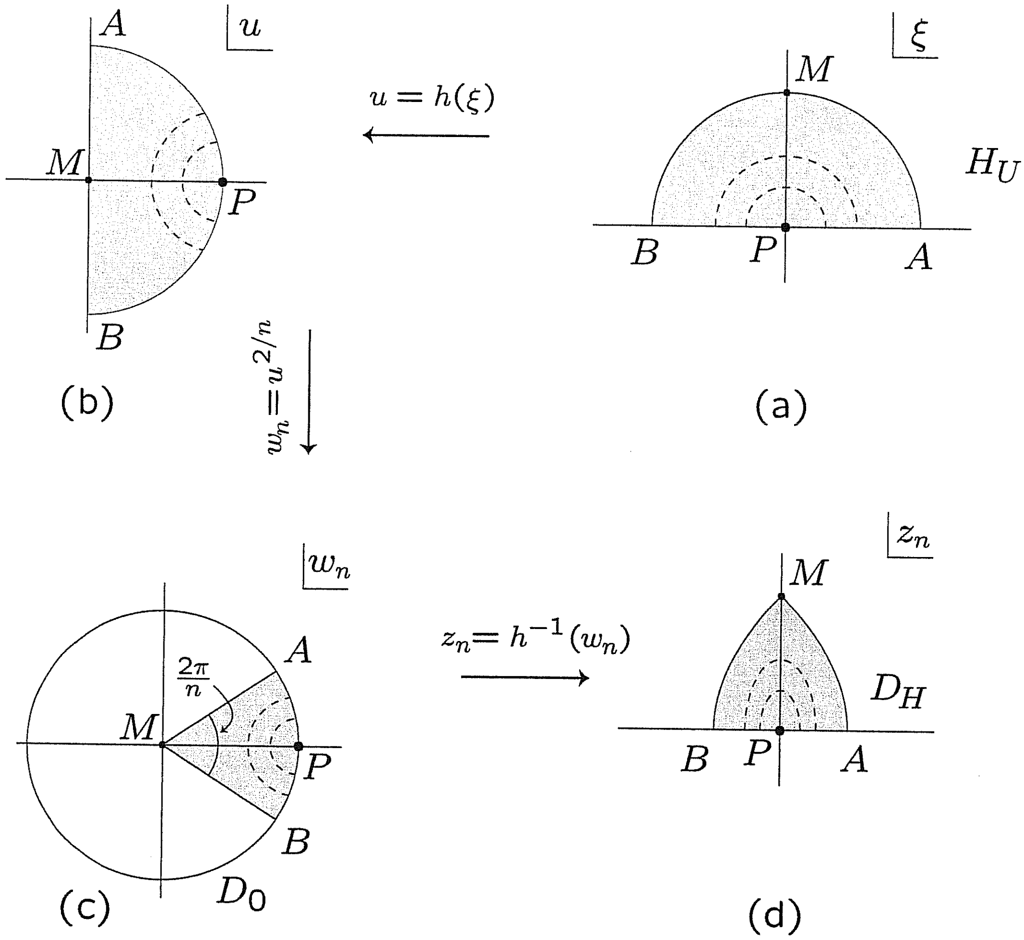

Figure 2.3: wedge mapping

that sends the upper half-disk H_U into a wedge with the angle at $w_n = 0$ equal to $2\pi/n$. The transformation (2.106) can be rewritten as

$$w_n = \exp\left(i\frac{4}{n}\tan^{-1}(\xi)\right) \tag{2.107}$$

We define $\langle n|$ such that

$$\langle n|\phi\rangle \equiv \langle \check{f}_n \circ \phi(0)\rangle_{D_0} \tag{2.108}$$

The state that we obtain for $n = 1$ is the identity state, that in the coordinates w_n is the full unit disk D_0 with a cut on the negative real axis. The left-half and the right-half of the string coincides along this cut. The $n = 2$ is the vacuum state, and, in the w_n plane, the image of H_U covers the right half of the full unit disk D_0 in the w_n plane. The $n \to \infty$ limit is the sliver. It is an infinitely thin sliver of the disk D_0 around the positive real axis. In the next section we will see that $n = 2$ is indeed the vacuum state and that the limit $n \to \infty$ gives rise to a well-defined state.

We describe now $|n\rangle$ taking back the wedge on the upper half plane. We define

$$z_n = h^{-1}(w_n) = i\frac{1 - w_n}{1 + w_n} = \tan\left(-\frac{i}{2}\ln w_n\right) \tag{2.109}$$

Putting together (2.107) and (2.109) we have

$$z_n = \tan\left(\frac{2}{n}\tan^{-1}(\xi)\right) \equiv \tilde{f}_n(\xi) \tag{2.110}$$

and

$$\langle n|\phi\rangle = \langle \tilde{f}_n \circ \phi(0)\rangle_{D_H} \tag{2.111}$$

The two description of the sliver (2.108) and (2.111)seems to be singular, in the sense that the maps $\check{f}_n(\xi)$ and $\tilde{f}_n(\xi)$ are singular in the $n \to \infty$ limit. This apparent singular behaviour is solved by noticing the SL(2,$\mathbb{R}$) invariance of the correlation functions on the upper half plane. Given any SL(2,$\mathbb{R}$) map $R(z)$ we have the relation

$$\langle \prod_i \mathcal{O}_i(x_i)\rangle_{D_H} = \langle \prod_i R \circ \mathcal{O}_i(x_i)\rangle_{D_H} \tag{2.112}$$

for any set of operators $\mathcal{O}_i$ and with D_H denoting the upper half plane. Since the sliver $|\Xi\rangle$ is defined through a correlation function, we can set

$$R_n(z) = \frac{n}{2}z \tag{2.113}$$

so that

$$\langle \Xi | \phi \rangle = \langle f \circ \phi(0) \rangle_{D_H} \tag{2.114}$$

where

$$\begin{aligned} f(\xi) &= \lim_{n\to\infty} R_n \circ \tilde{f}_n(\xi) \\ &= \lim_{n\to\infty} \frac{n}{2} \tan\left(\frac{2}{n}\tan^{-1}(\xi)\right) = \tan^{-1}\xi \end{aligned} \tag{2.115}$$

Since this map is non-singular at $\xi = 0$, we have a finite expression for $\langle \Xi | \phi \rangle$ for any state $|\phi\rangle$. We need now a prescription for $*$ multiplicating the surface states. $*$ multiplication is better understood in a third representation of the punctured disk D, where D itself is mapped into a disk $\widehat{D}$ having the special property that the local coordinate patch, *i.e.* the image of H_U in $\widehat{D}$, is nothing else that the vertical half-disk. This is done by taking, for $\xi \in H_U$

$$\hat{w} = h(\xi) = \frac{1 + i\xi}{1 - i\xi} \tag{2.116}$$

It is clear that in this representation the remaining part of $\widehat{D}$ may take a complicated form. Using eqs. (2.106) and (2.116) we see that

$$\widehat{w}_n = (w_n)^{n/2} \tag{2.117}$$

Under this map the unit disk D_0 in the w_n-coordinates is mapped to a cone in the $\widehat{w}_n$ coordinate, subtending an angle $n\pi$ at the origin $\widehat{w}_n = 0$. The disk D_0 mapped in this way represents then a wedge $|n\rangle$. We can give now the prescription for the $*$ product. Les us consider directly wedge states and remove the local coordinate patch from the disk D_0 in the w_n coordinate: the left over region becomes a sector of angle $\pi(n-1)$. If we denote by $|\mathcal{R}_\alpha\rangle$ a sector state arising from a sector of angle α, we have the identification of sector states with wedge states

$$|n\rangle = |\mathcal{R}_{\pi(n-1)}\rangle \tag{2.118}$$

We declare that the operation of $*$ multiplication of two wedge states $|m\rangle * |n\rangle$ is given by gluing together the two sector states $|\mathcal{R}_{\pi(m-1)}\rangle$ and $|\mathcal{R}_{\pi(n-1)}\rangle$ identifying the left-hand side of the string front of $|m\rangle$ with the right-hand side of the string front of $|n\rangle$. With this prescription we obtain the rule

$$|\mathcal{R}_a\rangle * |\mathcal{R}_b\rangle = |\mathcal{R}_{a+b}\rangle \tag{2.119}$$

that means

$$|m\rangle * |n\rangle = |m+n-1\rangle \tag{2.120}$$

The sliver state ($n \to \infty$) is a projector under the $*$ product:

$$|\mathcal{R}_\infty\rangle * |\mathcal{R}_\infty\rangle = |\mathcal{R}_\infty\rangle \tag{2.121}$$

Let us give the operator representation of wedge states. We consider $U = U(f_n)$ depending only on matter Virasoro generators L_n and ghost fields b and c such that $\langle n| = \langle 0|U$. Since a primary field of conformal weight d transforms under finite conformal transformation f as

$$f \circ \phi(z) = (f'(z))^d \phi(f(z)) \tag{2.122}$$

we can write

$$(f'(z))^d \phi(f(z)) = U_f \phi(z) U_f^{-1} \tag{2.123}$$

with

$$U_f = \exp[v_0 L_0] \exp\left[\sum_{n\geq 1} v_n L_n\right] \tag{2.124}$$

The coefficients v_n can be determined recursively from the Taylor expansion of f, by requiring

$$\begin{aligned} e^{v_0} &= f'(0) \\ \exp\left[\sum_{n\geq 1} v_n z^{n+1}\partial_z\right] &= (f'(0))^{-1} f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \end{aligned} \tag{2.125}$$

For instance, for the first coefficients one finds

$$v_1 = a_2\,, \quad v_2 = -a_2^2 + a_3\,, \quad v_3 = \frac{3}{2}a_2^2 - \frac{5}{2}a_2 a_3 + a_4 \tag{2.126}$$

One can determine eqs.(2.125) in the following way. Using the commutation relation

$$[L_m, \phi_n] = ((d-1)m - n)\phi_{m+n} \tag{2.127}$$

we have

$$U_f \phi(z) U_f^{-1} = \exp[v(z)\partial_z + dv'(z)]\phi(z) \tag{2.128}$$

where the function $v(z)$ is such that

$$e^{v(z)\partial_z} z = f(z) \tag{2.129}$$

Choosing $\tilde{f}_n(z) = \tan\left(\frac{2}{n}\tan^{-1}(z)\right)$ to define the wedge states $|n\rangle$, we have

$$\begin{aligned} |n\rangle &= \exp\left[-\frac{n^2-4}{3n^2}L_{-2} + \frac{n^4-16}{30n^4}L_{-4} - \frac{(n^2-4)(176+128n^2+11n^4)}{1890n^6}L_{-6}\right. \\ &\quad \left. + \frac{(n^2-4)(n^2+4)(16+32n^2+n^4)}{1260n^8}L_{-8} + \cdots\right]|0\rangle \end{aligned} \tag{2.130}$$

Among them, it is possible to recognise, for particular value of n:
the Identity State ($n = 1$) (see [58]):

$$\begin{aligned} |I\rangle &\equiv |1\rangle \\ &= \exp\left[L_{-2} - \frac{1}{2}L_{-4} + \frac{1}{2}L_{-6} - \frac{7}{12}L_{-8} + \cdots\right]|0\rangle \end{aligned} \tag{2.131}$$

the Vacuum State ($n = 2$):

$$|0\rangle = |2\rangle \tag{2.132}$$

and the Sliver State ($n \to \infty$):

$$\begin{aligned} |\Xi\rangle &\equiv |\infty\rangle \\ &= \exp\left[-\frac{1}{3}L_{-2} + \frac{1}{30}L_{-4} - \frac{11}{1890}L_{-6} + \frac{1}{1260}L_{-8} + \cdots\right]|0\rangle \end{aligned} \tag{2.133}$$

2.6 The Butterfly State

The butterfly state is defined by a map from ξ to the upper half z plane

$$z = \frac{\xi}{\sqrt{1+\xi^2}} \equiv f_B(\xi)\,, \tag{2.134}$$

more precisely the surface state $|B\rangle$ that defines the butterfly is such:

$$\langle B|\phi\rangle = \langle f_B \circ \phi(0)\rangle_{UHP}\,. \tag{2.135}$$

In the ξ-plane we have the half-disk: the circumference is the string, the point $\xi = i$ the midpoint (second case of figure 2.4). In the z-coordinate, the open string

$|\xi| = 1, \Im(\xi) \geq 0$ is mapped to the hyperbola $x^2 - y^2 = \frac{1}{2}$ with $z = x + iy$ (first of figure 2.4). The fact that $z(\xi = i) = \infty$ means that the open string midpoint coincides with the boundary of the disk.

There is a general analysis in [63] about surface state and projectors of star product: a generic surface state in which the midpoint of the string touches the boundary is a projector under $*$-product. So the butterfly it is. This general analysis is based on a geometric formulation of star-product as a gluing of surface states in a precise way. We will not enter in such details. If we use (2.115) to recognize that (2.134) can be rewritten as

$$z = \sin(\tan^{-1}(\xi)) = \sin \widehat{z} \tag{2.136}$$

we can invert the previous equation to write

$$\widehat{z} = \sin^{-1} z \,. \tag{2.137}$$

This is the transformation that maps the full upper half z-plane into the region $|\Re(\widehat{z})| \leq \pi/2\,, \Im(\widehat{z}) \geq 0$. (See third case of figure 2.4). The vertical lines $\Re(\widehat{z}) = \pm\pi/2$ are images of the boundary. Even though the surface occupies a portion of the $\widehat{z}$-plane the boundary reaches the point at infinity, and so does the midpoint.

We have

$$d\widehat{z} = \frac{dz}{\sqrt{(1-z)(1+z)}} \tag{2.138}$$

The real line in the z-plane is mapped into a polygon in the $\widehat{z}$ presentation, where the turning points are $z = \pm 1$ and the turning angles are both $\pi/2$.

Finally, we give the $\widehat{w}$ presentation (fig. 2.4(d)). Using (??) the region $|\Re(\widehat{z})| \leq \pi/2\,, \Im(\widehat{z}) \geq 0$ of the $\widehat{z}$ presentation turns into the full disk with a pair of cuts into the $\widehat{w}$ origin from $\widehat{w} = -1$. Indeed the boundary of the surface is the arc $e^{i\theta}$ with $0 < \theta < \pi$ together with the line going from $\widehat{w} = -1$ to $\widehat{w} = 0$, plus the backwards line from $\widehat{w} = 0$ to $\widehat{w} = -1$ plus the arc $e^{i\theta}$ with $-\pi < \theta < 0$.

2.6.1 Operator representation of the butterfly state

We can represent the butterfly $|B_t\rangle$ in the operator formalism. We will use the "reguleted" butterfly $|B_t\rangle$ defined by the map

$$\widehat{z} = \tan^{-1}\Big(\frac{z}{\sqrt{1 - z^2t^2}}\Big)\,, \tag{2.139}$$

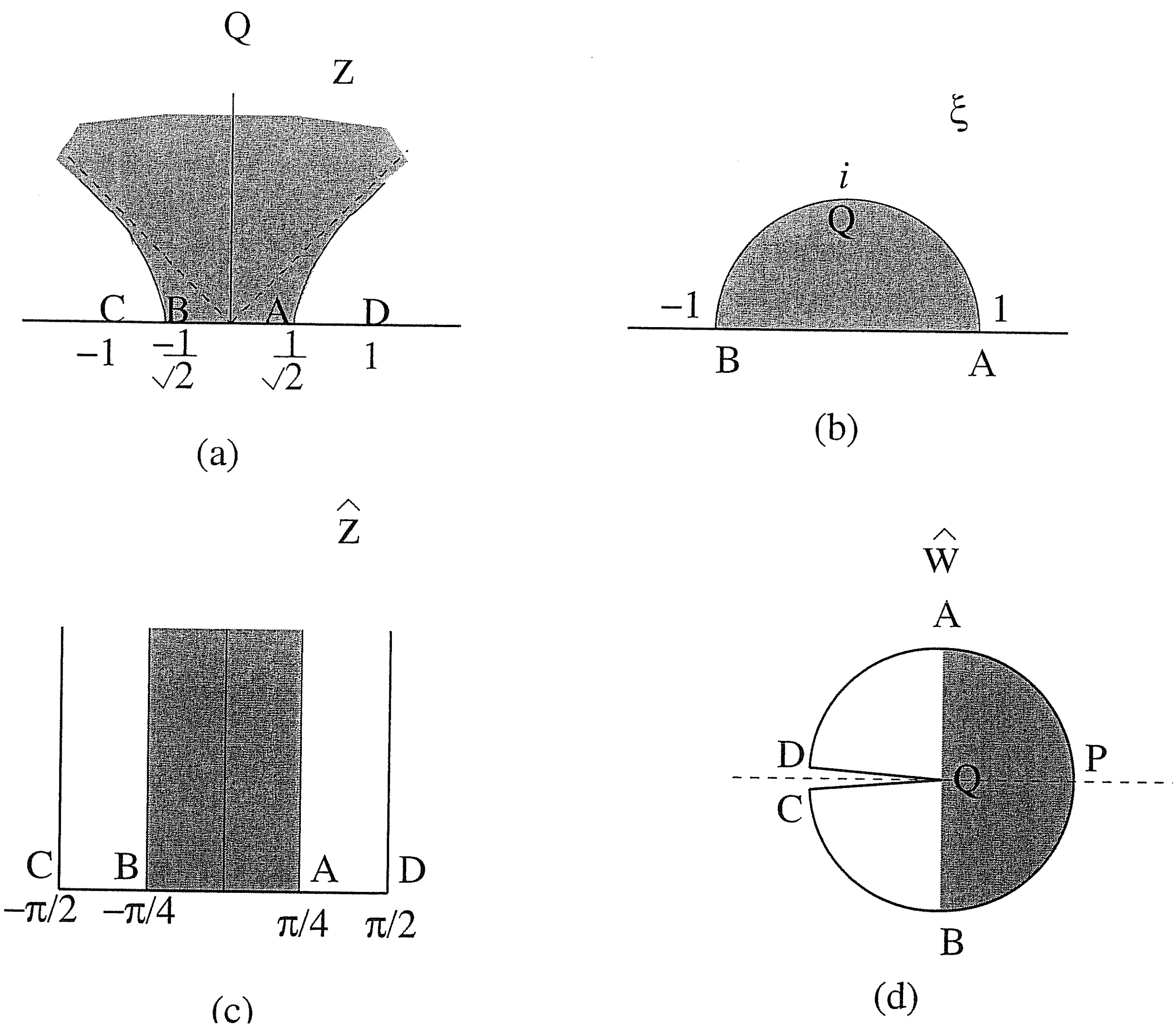

Figure 2.4: Representation of butterfly state in various coordinate systems.

or

$$z = \frac{\tan \widehat{z}}{\sqrt{1 + t^2 \tan^2 \widehat{z}}} = \frac{\xi}{\sqrt{1 + t^2 \xi^2}} \,. \tag{2.140}$$

The regulator parameter t must therefore satisfy $t < 1$. (See [63] for details). Clearly when $t = 1$ in (2.140) we recover the butterfly as defined in (2.134). We have

$$z = f_t(\xi) = \frac{\xi}{\sqrt{1+t^2\xi^2}} = \exp\left(v_t(\xi)\frac{\partial}{\partial\xi}\right)\xi\,. \tag{2.141}$$

Eqs. (2.100), (2.99) give

$$v_t(\xi) = -t^2\,\xi^3/2\,. \tag{2.142}$$

Eq. (2.93), (2.95) now gives:

$$|_t\rangle = \exp\left(-\frac{t^2}{2}L_{-2}\right)|0\rangle\,. \tag{2.143}$$

Virasoro conservation laws allows us to derive an interesting property of the butterfly state,

$$K_2|\rangle = 0\,, \tag{2.144}$$

where $K_2 = L_2 - L_{-2}$. In the global UHP the vector field

$$\tilde{v}_2(z) = 2z - \frac{1}{z}\,, \tag{2.145}$$

is holomorphic everywhere except for the pole at the puncture $z = 0$. This implies

$$\left\langle \oint dz\,\tilde{T}(z)\,\tilde{v}_2(z) f\circ\phi(0)\right\rangle_{UHP} = 0\,, \tag{2.146}$$

for any state $|\phi\rangle$. In local coordinate ξ, we have

$$\left\langle f\circ\left(\oint d\xi\,T(\xi)(\xi^3 - \xi^{-1})\phi(0)\right)\right\rangle_{UHP} = 0\,. \tag{2.147}$$

This gives

$$|K_2|\phi\rangle = 0 \tag{2.148}$$

which means (2.144).

2.6.2 Oscillator representation of the butterfly state

Let us represent the matter part of the regulated butterfly state in the oscillator representation. Take $\beta \equiv t^2$,

$$v(\xi) = -\frac{\xi^3}{2}\,, \qquad f_\beta(\xi) = \frac{\xi}{\sqrt{1+\beta\xi^2}}\,. \tag{2.149}$$

Equ. (2.104) gives

$$\begin{aligned}
\frac{d}{d\beta}V^B_{mn}(\beta) &= (-1)^{m+n}\frac{\sqrt{mn}}{2}\oint_0 \frac{dw}{2\pi i}\oint_0 \frac{dz}{2\pi i}\frac{1}{z^{m+1}w^{n+1}}\frac{f_\beta(z)^3 - f_\beta(w)^3}{f_\beta(z) - f_\beta(w)} \qquad (2.150)\\
&= (-1)^{m+n}\frac{\sqrt{mn}}{2}\oint_0 \frac{dw}{2\pi i}\frac{f_\beta(w)}{w^{m+1}}\oint_0 \frac{dz}{2\pi i}\frac{f_\beta(z)}{z^{m+1}} = (-1)^{m+n}\frac{\sqrt{mn}}{2}x_m x_n\,,
\end{aligned}$$

where

$$\begin{aligned}
x_m = \oint_0 \frac{dw}{2\pi i}\frac{f_\beta(w)}{w^{m+1}} &= (-\beta)^{\frac{m-1}{2}}\frac{\Gamma[\frac{m}{2}]}{\sqrt{\pi}\Gamma[\frac{m+1}{2}]} \quad \text{for } m \text{ odd}\,, \qquad (2.151)\\
&= 0 \quad \text{for } m \text{ even}\,.
\end{aligned}$$

Integrating (2.150) with the initial condition $V(\beta = 0) = 0$, we find the Neumann coefficients of the regulated butterfly ($\beta \to t^2$):

$$\begin{aligned}
V^B_{mn}(t) &= -(-1)^{\frac{m+n}{2}}\frac{\sqrt{mn}}{m+n}\frac{\Gamma[\frac{m}{2}]\Gamma[\frac{n}{2}]}{\pi\Gamma[\frac{m+1}{2}]\Gamma[\frac{n+1}{2}]}t^{m+n}\,, \quad \text{for } m \text{ and } n \text{ odd}\,, \qquad (2.152)\\
&= 0\,, \quad \text{for } m \text{ or } n \text{ even}\,.
\end{aligned}$$

2.7 The Nothing State

The nothing state is defined by the relation:

$$\langle \mathbb{N}|\phi\rangle = \langle f_{\mathbb{N}} \circ \phi(0)\rangle_{UHP} \qquad (2.153)$$

with

$$f_{\mathbb{N}(\xi)} = \frac{\xi}{\xi^2 + 1} \qquad (2.154)$$

Under the map $\widehat{w}(\xi) = \left(\frac{1+i\xi}{1-i\xi}\right.$ *theupperhalfz*=f$_{\mathbb{N}(\xi)}$ plane gets mapped to the vertical half-disk $\widehat{\Sigma}$ as shown in Fig.2.5. Since the boundary along the vertical line passes through the string midpoint, which is at the origin of the $\widehat{w}$-plane, this state satisfies the criterion of being a projector of the $*$-algebra [63].

- The map $f_{\mathbb{N}}(\xi)$ defining the nothing state and that one defining the identity $f(\xi)$ are related by

$$f_{\mathbb{N}}(\xi) = -if(i\xi)\,. \qquad (2.155)$$

So the operator expressions of the identity and of the nothing state are related by the replacement $L_{-2n} \leftrightarrow (-)^n L_{-2n}$. Changing the sign of L_{-2} in (3.3) of [58], we immediately have

$$\begin{aligned} |\mathbb{N} &= \left(\prod_{n=2}^{\infty} \exp\left\{ -\frac{2}{2^n} L_{-2^n} \right\} \right) e^{-L_{-2}} |0 \\ &= \ldots \exp(-\frac{2}{2^3} L_{-2^3}) \exp(-\frac{2}{2^2} L_{-2^2}) \exp(-L_{-2}) |0 \,. \end{aligned} \tag{2.156}$$

- V^f_{mn} computed using (2.102), (2.154) turns out to be equal to δ_{mn}. Thus the oscillator representation of the matter part of the nothing state is given by:

$$|\mathbb{N}\rangle_m = \exp\left(-\frac{1}{2} \sum_{m,n=1}^{\infty} a_n^\dagger a_n^\dagger \right) |0\rangle \,. \tag{2.157}$$

- The nothing state is annihilated by all even operators,

$$K_{2n} |\mathbb{N}\rangle = 0 \qquad \forall n \,, \tag{2.158}$$

where $K_{2n} = L_{2n} - L_{-2n}$, is a sort of reparametrization of the cubic vertex of SFT. In a similar way to the butterfly case, the globally defined vector fields in eqs. (2.144)-(2.148)

$$\tilde{v}_2(z) = -\frac{1}{z} + 4z \,, \quad \tilde{v}_4(z) = -\frac{1}{z^3} + \frac{6}{z} - 8z \,, \tag{2.159}$$

implies

$$K_2 |\mathbb{N}\rangle = 0 \,, \quad K_4 |\mathbb{N}\rangle = 0 \,. \tag{2.160}$$

More generally, the commutation relations

$$[K_m \,, K_n] = (m-n) K_{m+n} - (-1)^n (m+n) K_{m-n} \tag{2.161}$$

imply (2.158) for all n. On the other hand, the identity string field is annihilated by all, even and odd, vertex reparametrizations.

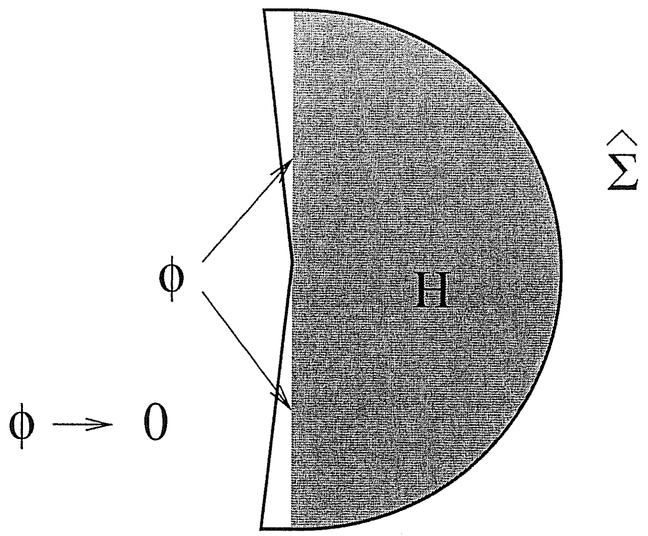

Figure 2.5: The geometric view of the nothing state: the full disk $\widehat{\Sigma}$ minus the local coordinate patch, which fill the whole disk. Thus the riemann surface collapses to nothing.

2.8 General butterflies

The generalized butterfly state $|B_\alpha\rangle$ is defined through a generalization of eq. (2.134) to

$$z = \frac{1}{\alpha}\sin(\alpha \tan^{-1}\xi) \equiv f_\alpha(\xi)\,. \tag{2.162}$$

It follows that eq. (2.136) is generalized to

$$z = \frac{1}{\alpha}\sin(\alpha\widehat{z})\,. \tag{2.163}$$

Looking at eqs. (2.136) and (2.163) it is possible to see that the generalized butterfly is a sort of rescaling of the $\widehat{z}$ coordinate by a factor α. In the upper half plane, it can be seen as the region $-\frac{\pi}{2\alpha} < \Re(\widehat{z}) \leq \frac{\pi}{2\alpha}$ (third case of figure 2.4 corresponds to $\alpha = 2$). Let us call this region $\mathbb{C}_\alpha$.

From eq. (2.162) we can see that the map $f_\alpha(\xi)$ is singular at the string mid-point $\xi = i$. In particular the midpoint is sent to $i\infty$ and hence touches the boundary of the upper half z-plane. There is a general analysis in [63] about surface state and projectors of star product: a generic surface state in which the midpoint of the string touches the boundary is a projector under $*$-product. So it is the generalized butterfly. Note that the case $\alpha = 1$:

$$f_{\alpha=1} = \frac{\xi}{\sqrt{1+\xi^2}} \,. \tag{2.164}$$

so $|B_{\alpha=1}\rangle$ is nothing but the butterfly while for $\alpha = 0$ we have :

$$f_{\alpha=0} = \tan^{-1}\xi \,. \tag{2.165}$$

so $|B_{\alpha=0}\rangle$ is the sliver.
$|_\alpha\rangle$ is a family of projectors, interpolating between the butterfly and the sliver. For $\alpha = 2$ we have the map

$$f_{\alpha=2} = \frac{\xi}{1+\xi^2} \,. \tag{2.166}$$

which corresponds to the 'nothing' state.

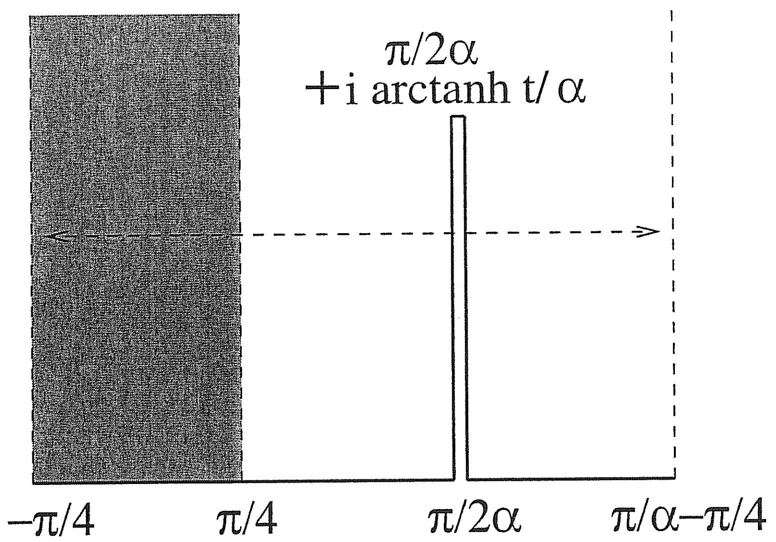

Figure 2.6: The geometry of $\mathbb{C}_{\alpha,t}$ in the complex $\widehat{z}$ plane. The shaded region denotes the local coordinate patch, and the lines $\Re(\widehat{z}) = -\pi/4$, $\Re(\widehat{z}) = \pi/\alpha - \pi/4$ are identified.

We can regularize the singularity at the midpoint and define the regularized butterfly by generalizing defrbut to

$$z = f_{\alpha,t}(\xi) = \frac{1}{\alpha}\frac{\tan(\alpha\tan^{-1}\xi)}{\sqrt{1+t^2\tan^2(\alpha\tan^{-1}\xi)}} = \frac{1}{\alpha}\frac{\tan(\alpha\widehat{z})}{\sqrt{1+t^2\tan^2(\alpha\widehat{z})}} \,. \tag{2.167}$$

In the $\widehat{z}$ plane we get

$$\langle B_{\alpha,t}|\phi\rangle = \langle f^{(0)} \circ \phi(0)\rangle_{\mathbb{C}_{\alpha,t}}, \tag{2.168}$$

where $\mathbb{C}_{\alpha,t}$ is the image of the upper half z plane in the $\widehat{z}$ coordinate system and $f^{(0)}(\xi) = \tan^{-1}\xi$. Note that the local coordinate patch always occupies the same region $|\Re(\widehat{z})| \leq \frac{\pi}{4}$, $\Im(\widehat{z}) \geq 0$, since $\widehat{z} = \tan^{-1}\xi$.

Note that for $\alpha = 2$ the region of $\mathbb{C}_{\alpha,t}$ outside the local coordinate patch collapses to nothing. For this reason we call the associated surface state the 'nothing' state.

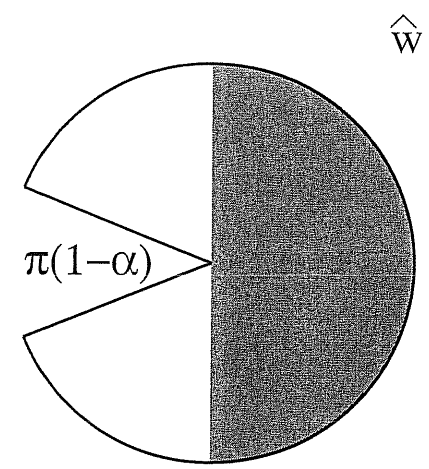

Figure 2.7: The image of $\mathbb{C}_\alpha$ in the complex $\widehat{w} = e^{2i\widehat{z}}$ plane. The shaded region denotes the local coordinate patch.

2.9 Other Projectors and Star Subalgebras

There are other projectors whose Virasoro representation is as simple as that of the butterfly and there are subalgebras of surface states that generalize the commutative wedge state subalgebra.

2.9.1 Butterfly-like projectors

Consider the vector fields

$$v_{(n)}(\xi) = -\frac{\beta}{n}\,\xi^{n+1}, \tag{2.169}$$

which generate the diffeomorphisms [?]

$$z = f_{(n)}(\xi) = \exp\Big(v_{(n)}(\xi)\frac{\partial}{\partial\xi}\Big)\,\xi = \frac{\xi}{(1+\beta\,\xi^n)^{1/n}}\,. \tag{2.170}$$

The associated surface states are

$$|B_n(\beta)\rangle = \exp\Big(-\frac{\beta}{n}(-1)^n L_{-n}\Big)|0\rangle\,. \tag{2.171}$$

For even n one can readily implement the projector condition $f(\xi = \pm i) = \infty$ by a choice of the parameter β. Indeed, this condition fixes

$$\beta = -(-)^{n/2}\,, \qquad n \text{ even}\,. \tag{2.172}$$

We therefore obtain candidate projectors

$$|P_{2m}\rangle = \exp\Big((-1)^m\frac{1}{2m}L_{-2m}\Big)\;|0\rangle\,. \tag{2.173}$$

The case $m=1$ is the canonical butterfly, and the next projectors are

$$\exp\Big(\frac{1}{4}L_{-4}\Big)|0\rangle\,,\quad \exp\Big(-\frac{1}{6}L_{-6}\Big)|0\rangle\,,\quad \exp\Big(\frac{1}{8}L_{-8}\Big)|0\rangle\;\;\cdots \tag{2.174}$$

and so on. These projectors obey the conservation law

$$K_{2m}|P_{2m}\rangle = 0\,, \tag{2.175}$$

which is the obvious generalization of PK and can be proven in the same way considering the global vector fields

$$\tilde{v}_{2m}(z) = 2(-1)^{m+1}z - z^{-2m+1}\,. \tag{2.176}$$

2.9.2 Subalgebras of surface states

The family of wedge states $|_r$ we saw, defined by the maps

$$z = \frac{r}{2}\tan\Big(\frac{2}{r}\arctan(\xi)\Big)\,, \tag{2.177}$$

obeys $K_1|_r = 0$, for all values of the parameter r, $1 \leq r \leq \infty$. The wedge states interpolate between the identity $|_1 \equiv |$ and the sliver, $|_\infty \equiv |\Xi$. There is also a family of states all annihilated by K_2, interpolating between the identity and the butterfly, and also containing the nothing state. They are defined by the maps

$$z = g^{(2)}_\mu(\xi) = \frac{1}{\sqrt{4\mu}}\left[1-\left(\frac{1-\xi^2}{1+\xi^2}\right)^{2\mu}\right]^{\frac{1}{2}}\,. \tag{2.178}$$

For $\mu = -1$ we recover the identity, for $\mu = 1/2$ the canonical butterfly and for $\mu = 1$ the nothing state. The condition $g_\mu^{(2)}(\pm i) = \infty$ is satisfied for $\mu \geq 0$, so all the states with $\mu \geq 0$ are candidate projectors.

The key ingredient in such subalgebras is to belong to the kernel of K_n which is a derivation of the star algebra and so every such family will be closed under the star product. More generally, for any given integer n, let us consider the map $z = g^{(n)}(\xi)$ that defines a surface state annihilated by K_n. The general form of $g^{(n)}(\xi)$ is found requiring that the vector field

$$\tilde{v}_n(z) = \frac{dz}{d\xi}(\xi^{n+1} - (-1)^n \xi^{-n+1}), \tag{2.179}$$

is globally defined in the Upper Half Plane, taking into account that fact that $g^{(n)}(\xi)$ must have a regular Taylor expansion in $\xi = 0$ with $g^{(n)}(0) = 0$, $\frac{dg^{(n)}(0)}{d\xi} = 1$. Since $\tilde{v}_n(z)$ must have a pole of order $(n-1)$ at $z = 0$. the form for such a vector field is

$$\tilde{v}_n(z) = \frac{P_{n+1}(z)}{z^{n-1}}, \tag{2.180}$$

where $P_{n+1}(z)$ is a polynomial of order $(n+1)$. The order of P_{n+1} is fixed by the requirement that the vector field is regular at infinity, $\lim_{z\to\infty} z^{-2}\tilde{v}_n(z) = \text{const}$. The differential equations are different in even and odd n cases:

$$\begin{aligned} \frac{1}{2n} d\ln\left(\frac{1-\xi^n}{1+\xi^n}\right) &= \frac{z^{n-1}}{P_{n+1}(z)} dz \qquad \text{for } n \text{ even}, \\ \frac{1}{n} d\arctan(\xi^n) &= \frac{z^{n-1}}{P_{n+1}(z)} dz \qquad \text{for } n \text{ odd}. \end{aligned} \tag{2.181}$$

Demanding that the surface state is twist even requires that z be an odd function of ξ, and this restricts the polynomial P_{n+1} to contain only even powers of z. For $n = 1$, the most general twist even solution is the family of maps defining the wedge states.

Chapter 3

Star democracy in open SFT

The aim of this chapter is to write down the relations among the three different star products used in OSFT, the matter one and the two ghost, clarifyng, in particular, the difference between reduced and twisted ghost star product.

Open Bosonic String Theory has matter and ghost sector: the first one is made by embedding coordinates in the 26 dimensional target space, the second one is a bc system with conformal weights $(2, -1)$.
Vacuum String Field Theory is formally identical to Witten's one, except for the kinetic operator, which is no more the usual BRST charge, but a c-midpoint insertion. In such a theory the ghost sector is completely decoupled from the matter, hence solutions can be found in a factorized form *matter* $\otimes$ *ghost*. Such solutions were obtained following two parallel methods: one which is algebraic and is based on the oscillator expansion of the string field [34, 36, 37], the other is based on the Boundary Conformal Theory which describes the original unstable D–brane configuration [39, 63] .
The two approaches, although very different in the formalism, were shown to lead to the same results, first numerically by level expansion analysis, and then analytically in [54, 45] by making use of the continuous basis of the star product.
Moreover, if for the matter the correspondence between algebraic and conformal approach is quite clear and actually relies on the isomorphism between CFT fields and their Fourier transformed oscillators, the correspondence in the ghost sector is more subtle since it compares, on the one hand projectors squeezed states in Siegel gauge build up with oscillators of the bc system [14] and, on the other hand, projectors surface states in a bc CFT, twisted by one unit of ghost current [39] in order to have a star product preserving the ghost number.

Since, when restricted to Siegel gauge, we can define the reduced star product which is also ghost number preserving, it would be natural to identify the reduced star product on the algebraic side with the twisted star product on the BCFT side but this is actually not correct, as we will show. The two product are different: they have different Neumann coefficients although they define the same sliver-projectors. The picture which arise at the end of our analysis is that of three star products matter, ghost and twisted ghost, each of them can be defined independently from the underlying CFT, and all sliver–like projectors have the same Neumann coefficients, up to a minus sign. It is interesting to note that such equality of solutions is implied in a bijective way by the Gross-Jevicki relation [33] which connects the ghost Neumann coefficients to the matter ones. Morover the relevant structures of the matter star product (at least at zero momentum) are completely encoded in the Neumann coefficients of the twisted bc system. This fact puts the bc CFT and its twisted variant, in an equivalent position with respect to the matter.

The chapter is organized as follows. We briefly recall definitions and properties of the matter and ghost product from a conformal point of view, as done in [79], then we perform the twist as in [39] and define properly the twisted star product. Then, we determine the relations that connect the three vertices in the game. In particular we show that the twisted CFT defines, up to a sign, the same coefficients as the matter.
Finally, we derive the algebraic expression of the twisted sliver and identify it with the sliver–like state in Siegel gauge, through the Gross–Jevicki relation [33].

3.1 The three stars

In this section we briefly review the construction of the interaction vertex of matter and ghost sector. Details will be almost skipped and can be found in [79]

3.1.1 Matter star

The matter part of the three strings vertex [27, 32, 33] is given by

$$|V_3\rangle = \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\delta^{26}(p_{(1)} + p_{(2)} + p_{(3)}) \exp(-E)\, |0,p\rangle_{123} \tag{3.1}$$

where

$$E = \sum_{a,b=1}^{3}\left(\frac{1}{2}\sum_{m,n\geq 1}\eta_{\mu\nu}a_m^{(a)\mu\dagger}V_{mn}^{ab}a_n^{(b)\nu\dagger} + \sum_{n\geq 1}\eta_{\mu\nu}p_{(a)}^{\mu}V_{0n}^{ab}a_n^{(b)\nu\dagger} + \frac{1}{2}\eta_{\mu\nu}p_{(a)}^{\mu}V_{00}^{ab}p_{(b)}^{\nu}\right) \quad (3.2)$$

Summation over the Lorentz indices $\mu, \nu = 0, \ldots, 25$ is understood and η denotes the flat Lorentz metric. The operators $a_m^{(a)\mu}, a_m^{(a)\mu\dagger}$ denote the non–zero modes matter oscillators of the a–th string, which satisfy

$$[a_m^{(a)\mu}, a_n^{(b)\nu\dagger}] = \eta^{\mu\nu}\delta_{mn}\delta^{ab}, \qquad m, n \geq 1 \quad (3.3)$$

$p_{(r)}$ is the momentum of the a–th string and $|0, p\rangle_{123} \equiv |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$ is the tensor product of the Fock vacuum states relative to the three strings. $|p_{(a)}\rangle$ is annihilated by the annihilation operators $a_m^{(a)\mu}$ and it is eigenstate of the momentum operator $\hat{p}_{(a)}^{\mu}$ with eigenvalue $p_{(a)}^{\mu}$. The normalization is

$$\langle p_{(a)}|\, p'_{(b)}\rangle = \delta_{ab}\delta^{26}(p + p')$$

The conformal definition of the vertex starts with the gluing functions

$$f_a(z_a) = \alpha^{2-a} f(z_a)\,,\ a = 1, 2, 3 \quad (3.4)$$

where

$$\begin{aligned} f(z) &= \left(\frac{1 + iz}{1 - iz}\right)^{\frac{2}{3}} \\ \alpha &= e^{\frac{2\pi i}{3}} \end{aligned} \quad (3.5)$$

The interaction vertex is defined by a correlation function on the disk in the following way

$$\int \psi * \phi * \chi = \langle f_1 \circ \psi(0)\, f_2 \circ \phi(0)\, f_3 \circ \chi(0)\rangle = \langle V_3|\psi\rangle_1|\phi\rangle_2|\chi\rangle_3 \quad (3.6)$$

Now we consider the string propagator at two generic points of this disk. The Neumann coefficients N_{nm}^{ab} are nothing but the Fourier modes of the propagator with respect to the original coordinates z_a. We shall see that such Neumann coefficients are related in a simple way to the standard three strings vertex coefficients. Here we will deal only with the zero momentum vertex, which is the one which is strictly connected to the (twisted) ghost vertex.

The Neumann coefficients N_{mn}^{ab} are given by [2]

$$N_{mn}^{ab} = \langle V_3|\alpha_{-n}^{(a)}\alpha_{-m}^{(b)}|0\rangle_{123} = -\frac{1}{nm}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m}f'_a(z)\frac{1}{(f_a(z) - f_b(w))^2}f'_b(w) \quad (3.7)$$

where the contour integrals are understood around the origin. It is easy to check that

$$\begin{aligned} N^{ab}_{mn} &= N^{ba}_{nm} \\ N^{ab}_{mn} &= (-1)^{n+m} N^{ba}_{mn} \\ N^{ab}_{mn} &= N^{a+1,b+1}_{mn} \end{aligned} \tag{3.8}$$

In the last equation the upper indices are defined mod 3.

Let us consider the decomposition

$$N^{ab}_{nm} = \frac{1}{3\sqrt{nm}}\Big(C_{nm} + \bar{\alpha}^{a-b} U_{nm} + \alpha^{a-b}\bar{U}_{nm}\Big) \tag{3.9}$$

After some algebra one gets

$$\begin{aligned} C_{nm} &= \frac{-1}{\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} \Big(\frac{1}{(1+zw)^2} + \frac{1}{(z-w)^2}\Big) \qquad (3.10) \\ U_{nm} &= \frac{-1}{3\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} \Big(\frac{f^2(w)}{f^2(z)} + 2\frac{f(z)}{f(w)}\Big)\Big(\frac{1}{(1+zw)^2} + \frac{1}{(z-w)^2}\Big) \\ \bar{U}_{nm} &= \frac{-1}{3\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} \Big(\frac{f^2(z)}{f^2(w)} + 2\frac{f(w)}{f(z)}\Big)\Big(\frac{1}{(1+zw)^2} + \frac{1}{(z-w)^2}\Big) \end{aligned}$$

The integrals can be directly computed in terms of the Taylor coefficients of f. The result is

$$\begin{aligned} C_{nm} &= (-1)^n \delta_{nm} && (3.11) \\ U_{nm} &= \frac{1}{3\sqrt{nm}} \sum_{l=1}^{m} l\Big[(-1)^n B_{n-l} B_{m-l} + 2 b_{n-l} b_{m-l} (-1)^m \\ &\quad -(-1)^{n+l} B_{n+l} B_{m-l} - 2 b_{n+l} b_{m-l} (-1)^{m+l}\Big] && (3.12) \\ \bar{U}_{nm} &= (-1)^{n+m} U_{nm} && (3.13) \end{aligned}$$

where we have set

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} b_k z^k \\ f^2(z) &= \sum_{k=0}^{\infty} B_k z^k, \qquad \text{i.e.} \qquad B_k = \sum_{p=0}^{k} b_p b_{k-p} \end{aligned} \tag{3.14}$$

Using the integral representation (3.11) one can prove [79]

$$\sum_{k=1}^{\infty} U_{nk} U_{km} = \delta_{nm}, \qquad \sum_{k=1}^{\infty} \bar{U}_{nk} \bar{U}_{km} = \delta_{nm} \tag{3.15}$$

In order to make contact with the standard notations (for example [36]) we define[1]

$$V^{ab}_{nm} = (-1)^{n+m}\sqrt{nm}\, N^{ab}_{nm} \tag{3.16}$$

and

$$\begin{aligned} M &= CV^{11} \\ M_+ &= CV^{12} \\ M_- &= CV^{21} \end{aligned} \tag{3.17}$$

Using (3.15), together with the decomposition (5.29), it is easy to establish the following linear and non linear relations (written in matrix notation).

$$\begin{gathered} M + M_+ + M_- = 1 \\ M^2 + M_+^2 + M_-^2 = 1 \\ M_+^3 + M_-^3 = 2M^3 - 3M^2 + 1 \\ M_+ M_- = M^2 - M \\ [M, M_\pm] = 0 \\ [M_+, M_-] = 0 \end{gathered} \tag{3.18}$$

3.1.2 Ghost star

To start with we define, in the ghost sector, the vacuum states $|\hat{0}\rangle$ and $|\dot{0}\rangle$ as follows

$$|\hat{0}\rangle = c_0 c_1 |0\rangle, \qquad |\dot{0}\rangle = c_1|0\rangle \tag{3.19}$$

where $|0\rangle$ is the usual $SL(2,\mathbb{R})$ invariant vacuum. Using *bpz* conjugation

$$c_n \to (-1)^{n+1} c_{-n}, \qquad b_n \to (-1)^{n-2} b_{-n}, \qquad |0\rangle \to \langle 0| \tag{3.20}$$

one can define conjugate states.

The three strings interaction vertex is defined, as usual, as a squeezed operator acting on three copies of the bc Hilbert space

$$\langle \tilde{V}_3| = {}_1\langle\hat{0}|\, {}_2\langle\hat{0}|\, {}_3\langle\hat{0}| e^{\widetilde{E}}, \qquad =\textstyle\sum_{a,b=1}^3 \sum_{n,m}^\infty c_n^{(a)} \widetilde{N}^{ab}_{nm} b_m^{(b)} \tag{3.21}$$

Under *bpz* conjugation

$$|\tilde{V}_3\rangle = e^{\widetilde{E}'} |\hat{0}\rangle_1 |\hat{0}\rangle_2 |\hat{0}\rangle_3, \qquad \widetilde{E}' = -\sum_{a,b=1}^{3}\sum_{n,m}^{\infty} (-1)^{n+m} c_n^{(a)\dagger} \widetilde{N}^{ab}_{nm} b_m^{(b)\dagger} \tag{3.22}$$

[1]The factor $(-1)^{n+m}$ is there because these coefficients refer to the Ket vertex $|V_3\rangle$, so *bpz* is needed.

To make the propagator $SL(2,\mathbb{R})$ we have to insert three c zero modes at points ξ_i [2]

$$\langle b(z)c(w)\rangle = \frac{1}{z-w}\prod_{i=1}^{3}\frac{w-\zeta_i}{z-\zeta_i} \tag{3.23}$$

So we get

$$\begin{aligned} \widetilde{N}^{ab}_{nm} &= \langle\tilde{V}_3|b^{(a)}_{-n}c^{(b)}_{-m}|\dot{0}\rangle_{123} \qquad (3.24)\\ &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n-1}}\frac{1}{w^{m+2}}(f'_a(z))^2\frac{-1}{f_a(z)-f_b(w)}\prod_{i=1}^{3}\frac{f_b(w)-f_i(0)}{f_a(z)-f_i(0)}(f'_b(w))^{-1} \end{aligned}$$

It is straightforward to check that

$$\widetilde{N}^{ab}_{nm} = \widetilde{N}^{a+1,b+1}_{nm} \tag{3.25}$$

and (by letting $z\to -z$, $w\to -w$)

$$\widetilde{N}^{ab}_{nm} = (-1)^{n+m}\widetilde{N}^{ba}_{nm} \tag{3.26}$$

As in the matter case, we consider the decomposition

$$\widetilde{N}^{ab}_{nm} = \frac{1}{3}(\widetilde{C}_{nm} + \bar{\alpha}^{a-b}\widetilde{U}_{nm} + \alpha^{a-b}\bar{\widetilde{U}}_{nm}) \tag{3.27}$$

After some algebra one finds

$$\begin{aligned} \widetilde{C}_{nm} &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)\\ \widetilde{U}_{nm} &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\frac{f(z)}{f(w)}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right)\\ \bar{\widetilde{U}}_{nm} &= \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{w^{m+1}}\frac{f(w)}{f(z)}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right) \end{aligned} \tag{3.28}$$

It is easy to show that

$$\bar{\widetilde{U}}_{nm} = (-1)^{n+m}\widetilde{U}_{nm} \tag{3.29}$$

As discussed in detail in [79] the evaluation of these integrals is sensible to radial ordering in the $(n,-n)$, components. We fix the ambiguity by setting

$$\widetilde{N}^{aa}_{-1,1} = \widetilde{N}^{aa}_{1,-1} = 0, \qquad \widetilde{N}^{aa}_{0,0} = 1. \tag{3.30}$$

Which corresponds to

$$\widetilde{C}_{NM} = (-1)^N \delta_{NM} \quad N, M \geq 0 \tag{3.31}$$

$$\widetilde{U}_{NM} = (-1)^M b_N b_M + (-1)^M \sum_{l=1}^{M} (b_{N-l} b_{M-l} + (-1)^l b_{N+l} b_{M-l}) \tag{3.32}$$

where the b_n's have been defined in (3.14). The reason for this is that we get the fundamental identity

$$\sum_{K=0} \widetilde{U}_{NKKM} = \delta_{NM} \tag{3.33}$$

As for the matter case we will consider from now on the coefficients of the Ket vertex

$$\mathcal{V}^{ab}_{NM} = -(-1)^{n+m} \widetilde{N}^{ab}_{NM} \tag{3.34}$$

3.1.3 The twisted star

In [39] another type of star-product is considered. It represents the gluing condition in a twisted conformal field theory of the ghost system. The twist is done by subtracting to the stress tensor one unit of derivative of the ghost current

$$T'(z) = T(z) - \partial j_{gh}(z) \tag{3.35}$$

This redefinition changes the conformal weight of the bc fields from (2,-1) to (1,0). It follows that the background charge is shifted from -3 to -1.As a consequence, in order not to have vanishing correlation functions, we have to fix only one c zero-mode. In particular, the $SL(2,R)$–invariant propagator of the bc system is

$$\langle b(z) c(w) \rangle' = \frac{1}{z-w} \frac{w-\xi}{z-\xi} \tag{3.36}$$

where ξ is one fixed point.
In [39] it was shown that the usual product can be obtained from the twisted one by inserting a $n_{gh} = 1$–operator at the midpoint which, on singular states like the sliver, can be identified with a c–midpoint insertion. This implies that, on such singular projectors, the twisted product can be identified with the reduced one.
The twisted ghost Neumann coefficients are then defined to be

$$\begin{aligned}\widetilde{N}'^{ab}_{nm} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} f'_a(z) \frac{-1}{f_a(z) - f_b(w)} \frac{f_b(w)}{f_a(z)} \\ &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{4i}{3} \frac{1}{1+z^2} \frac{\bar{\alpha}^b f(w)}{\bar{\alpha}^a f(z) - \bar{\alpha}^b f(w)}\end{aligned} \tag{3.37}$$

As in (3.24) these coefficients refer to the Bra vertex, the corresponding coefficients for the Ket vertex are

$$\widetilde{V}'_{nm} ab = -(-1)^{n+m} \widetilde{N}'^{ab}_{nm} \tag{3.38}$$

We will see in the next section how to compute such coefficients using previous results. This will lead to interesting connections with the other star-products.

3.2 Relations among the stars

In this section we will show how the stars products defined above are related to each other. In particular we will show the explicit relations which connect all the Neumann coefficients in the game, so at the end the three star star products are homeomorphic and in this sense can be considered equivalent.

3.2.1 Twisted ghosts vs Matter

The commuting matter Neumann coefficients which appear in (3.18) are given by

$$M^{ab}_{nm} = -\frac{(-1)^m}{\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} f'_a(z) \frac{1}{(f_a(z) - f_b(w))^2} f'_b(w) \tag{3.39}$$

We can rewrite them as

$$\begin{aligned}M^{ab}_{nm} &= -\frac{(-1)^m}{\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^m} f'_a(z) \partial_w \frac{1}{f_a(z) - f_b(w)} \\ &= -(-1)^m \sqrt{\frac{m}{n}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{f'_a(z)}{f_a(z) - f_b(w)}\end{aligned} \tag{3.40}$$

where we have integrated by part to respect the variable w. Now, recalling

$$f'_a(z) = \frac{4i}{3} \frac{1}{1+z^2} \alpha^{2-a} f(z) \ , \tag{3.41}$$

we obtain

$$M^{ab}_{nm} = -(-1)^m \sqrt{\frac{m}{n}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{4i}{3} \frac{1}{1+z^2} \frac{\bar{\alpha}^a f(z)}{\bar{\alpha}^a f(z) - \bar{\alpha}^b f(w)} \tag{3.42}$$

Let us now consider the corrisponding twisted ghost Neumann coefficients

$$
\begin{aligned}
Y'^{ab}_{nm} &= (C\widetilde{V}'^{ab})_{nm} \\
&= (-1)^m \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{f'_a(z)}{(f_a(z) - f_b(w))} \frac{f_b(w)}{f_a(z)} \\
&= (-1)^m \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{4i}{3} \frac{1}{1+z^2} \frac{\bar{\alpha}^b f(w)}{(\bar{\alpha}^a f(z) - \bar{\alpha}^b f(w))}
\end{aligned} \tag{3.43}
$$

This coefficients are not symmetric if we exchange n with m, however we can easily symmetrize them by the use of the matrix $E_{nm} = \sqrt{n}\delta_{nm}$

$$
Y'^{ab} \to E^{-1} Y'^{ab} E \tag{3.44}
$$

It is now easy to show the following

$$
\begin{aligned}
(E^{-1}Y'^{ab}E)_{nm} + M^{ab}_{nm} &= (-1)^m \sqrt{\frac{m}{n}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{4i}{3} \frac{1}{1+z^2} \frac{(\bar{\alpha}^b f(w) - \bar{\alpha}^a f(z))}{(\bar{\alpha}^a f(z) - \bar{\alpha}^b f(w))} \\
&= -(-1)^m \sqrt{\frac{m}{n}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n} \frac{1}{w^{m+1}} \frac{4i}{3} \frac{1}{1+z^2} = 0
\end{aligned} \tag{3.45}
$$

the last equality holding since there are no poles for $n, m \geq 1$.
So we obtain

$$
E^{-1} Y'^{ab} E = -M^{ab} \tag{3.46}
$$

a remarkable relation between twisted ghost and matter vertices, which is the same relation that holds in the four-string vertex between the non-twisted ghost and the matter Neumann coefficients [33]. This relation proves also that the ghost integral is independent of the background charge, for $n, m \geq 1$: the matter integral, indeed, can be seen as the ghost integral without the background charge[2]. As a consequence of the relation with the matter coefficients we can derive all the relevant properties of the twisted ghost Neumann coefficients, by simply taking the matter results and change the sign in odd powers.

$$
\begin{aligned}
Y' + Y'_+ + Y'_- &= -1 \\
Y'^2 + Y'^2_+ + Y'^2_- &= 1 \\
Y'^3_+ + Y'^3_- &= 2Y'^3 + 3Y'^2 - 1 \\
Y'_+ Y'_- &= Y'^2 + Y' \\
[Y', Y'_\pm] &= 0 \\
[Y'_+, Y'_-] &= 0
\end{aligned} \tag{3.47}
$$

[2]The independence of the background charge is also crucial to prove $'^{ab} = C'^{ba}C$

3.2.2 Twisted vs Reduced

The relation between the twisted and non-twisted ghost Neumann coefficients can be now obtained using the previous relation

$$Y' = -EME^{-1} \tag{3.48}$$

and the Gross-Jevicky relation [33][3]

$$Y = E\frac{-M}{1+2M}E^{-1} \tag{3.51}$$

between matter and non-twisted ghosts. So, finally, we have

$$Y = \frac{Y'}{1-2Y'} \tag{3.52}$$

or

$$Y' = \frac{Y}{1+2Y} \tag{3.53}$$

The Witten's action represents open string theory about the trivial unstable vacuum $|\Psi_0\rangle = c_1|0\rangle$. Vacuum string field theory (VSFT) is instead a version of Witten's open SFT which is conjectured to correspond to the minimum of the tachyon potential. As explained in the introduction at the minimum of the tachyon potential a dramatic change occurs in the theory, which, corresponding to the new vacuum, is expected to represent closed string theory rather that the open string theory we started with. In particular, this theory should host tachyonic lumps representing unstable D–branes of any dimension less than 25, beside the original D25–brane. Unfortunately we have been so far unable to find an exact classical solution, say $|\Phi_0\rangle$, representing the new vacuum. One can nevertheless guess the form taken by the theory at the new minimum, see [?]. The VSFT action has the same form as Witten's one, where the new string field is still denoted by Ψ, the $*$ product is the same as in the previous theory, while the BRST operator Q is

[3]This relation, as noted in [70] contains the map

$$P(z) = \frac{-z}{1+2z} \tag{3.49}$$

which is a PSL(2,R) transformation that squares to itself

$$P \circ P(z) = z \tag{3.50}$$

replaced by a new one, usually denoted $\mathcal{Q}$, which is characterized by universality and vanishing cohomology. Relying on such general arguments, one can even deduce a precise form of $\mathcal{Q}$ ([?],[41], see also [14, 44, 45, 46, 47, 48] and [38, 50, ?, 51, 42, 49, 64]),

$$\mathcal{Q} = c_0 + (-1)^n \sum_{n>0} (c_{2n} + c_{-2n}) \tag{3.54}$$

Now, the equation of motion of VSFT is

$$\mathcal{Q}\Psi = -\Psi * \Psi \tag{3.55}$$

and nonperturbative solutions are looked for in the factorized form

$$\Psi = \Psi_m \otimes \Psi_g \tag{3.56}$$

where Ψ_g and Ψ_m depend purely on ghost and matter degrees of freedom, respectively. Then eq.(3.55) splits into

$$\mathcal{Q}\Psi_g = -\Psi_g * \Psi_g \tag{3.57}$$

$$\Psi_m = \Psi_m * \Psi_m \tag{3.58}$$

We will see later on how to compute solutions to both equations. A solution to eq.(3.57) was calculated in [39, 14]. Various solutions of the matter part have been found in the literature, [?, 41, 38, 50, 34, 63].

3.3 Solving the ghost equation of motion in VSFT

We are now ready to deal with the problem of finding a solution to (3.57)

$$\mathcal{Q}|\psi\rangle + |\psi\rangle * |\psi\rangle = 0 \tag{3.59}$$

Since now we are operating in an enlarged the Fock space, $\mathcal{Q}$ must be modified, with respect to the conjectured form of the BRST operator (1.116) in VSFT, in the following way

$$\mathcal{Q} \to \mathcal{Q}_{(en)} = c_0 - \eta_0 + \eta_0^\dagger + \sum_{n=1}^{\infty} f_n (c_n + (-1)^n c_{-n}) \tag{3.60}$$

we see that the vanishing of f_n for n odd is consistent since $\vec{y}$ has no odd components, while for n even we have

$$y_{2n} = \sum_{k=1}^{\infty} \frac{2}{3}(-1)^k \left(\delta_{2n,2k} - \widetilde{U}_{2n,2k}\right) \tag{3.74}$$

The second sum is evaluated with the use of the integral representation of $\mathfrak{U}$ (3.28)

$$\begin{aligned}
\widetilde{U}_{2n,2k} &= \frac{2}{3}\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{2n+1}}\sum_{k=1}^{\infty}(-1)^k \frac{1}{w^{2k+1}}\frac{f(z)}{f(w)}\Big(\frac{1}{1+zw} - \frac{w}{w-z}\Big)\\
&= -\frac{2}{3}\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{2n+1}}\frac{1}{w}\frac{1}{1+w^2}\frac{f(z)}{f(w)}\Big(\frac{1}{1+zw} - \frac{w}{w-z}\Big)\\
&= -\frac{2}{3}\oint \frac{dz}{2\pi i}\frac{1}{z^{2n+1}}f(z)\Big(1 - \frac{1}{f(z)}\frac{1}{1+z^2}\Big)\\
&= -\frac{2}{3}b_{2n} + \frac{2}{3}\sum_{k=1}^{\infty}(-1)^k\delta_{2n,2k}
\end{aligned} \tag{3.75}$$

The δ-piece cancels with the one in (3.74), while the remaining one is precisely y_{2n}.

The derivation in (3.75) requires some comments. In passing from the first to the second line we use $\sum_{k=1}^{\infty}(-1)^k\frac{1}{w^{2k+1}} = -\frac{1}{w}\frac{1}{1+w^2}$, which converges for $|w| > 1$. Therefore, in order to make sense of the operation, we have to move the w contour outside the circle of radius one. This we can do provided we introduce a regulator to avoid the collapsing of the contour with the branch points of $f(w)$, which are located at $w = \pm i$. With the help of a regulator we move them far enough and eventually we will move them back to their original position. Now we can fully rely on the integrand in the second line of (3.75). Next we start moving the w contour back to its original position around the origin. In so doing we meet two poles (those referring to the $\frac{1}{1+w^2}$ factor), but it is easy to see that their contribution neatly vanishes due to the last factor in the integrand. The remaining contributions come from the poles at $w = z$ and at $w = 0$. Their evaluation leads to the third line in (3.75). The rest is obvious.

As a result of this calculation we find that eq.(3.71) becomes

$$\mathcal{Q}_{(en)}|\dot{S}_{(en)}\rangle + |\hat{S}_{(en)}\rangle = (c_0 - \eta_0)|\dot{S}_{(en)}\rangle \tag{3.76}$$

Finally, as a last step, we return to the original Fock space. This can be done by imposing the condition $c_0 - \eta_0 = 0$ on the states, i.e. by considering all the states that differ by the application of $c_0 - \eta_0$ as equivalent. Then the RHS of eq.(3.76) is in the class of 0. We also notice that this constraint implies that $c_0, \eta_0, \eta_0^\dagger$ effectively collapse to a unique mode c_0. It also implies that the relative conjugate modes $b_0, \xi_0, \xi_0^\dagger$ also collapse to b_0.

Let us collect the results. In the original Fock space the three string vertex is defined by

$$\widetilde{E}' = \sum_{n\geq 1, M\geq 0}^{\infty} c_n^{(a)\,\dagger} \widetilde{V}_{nM}^{(ab)} b_M^{(b)\dagger} \tag{3.77}$$

eqs.(3.63,3.64) becomes

$$|\hat{S}\rangle = \mathcal{N} \exp\left(\sum_{n,m\geq 1} c_n^\dagger S_{nm} b_m^\dagger + \sum_{n\geq 1} c_n^\dagger S_{n0} b_0\right)|\hat{0}\rangle \tag{3.78}$$

$$|\dot{S}\rangle = \mathcal{N} \exp\left(\sum_{n,m\geq 1} c_n^\dagger S_{nm} b_m^\dagger\right)|\dot{0}\rangle \tag{3.79}$$

It is now easy to prove, as a check, that

$$\mathcal{Q}|\dot{S}\rangle + |\hat{S}\rangle = 0 \tag{3.80}$$

where

$$\mathcal{Q} = c_0 + \sum_{n=1}^{\infty} (-1)^n (c_{2n} + c_{-2n}) \tag{3.81}$$

The above computation proves in a very direct way that the BRST operator is nothing but the midpoint insertion ($z = i$) of the operator $\frac{1}{2i}(c(z) - c(\bar{z}))$ [?]. A different proof of this identification, which makes use the continuous basis of the $*$–algebra [63]), was given in [45].

3.4 Slivers

In this section we review the algebraic derivation of the sliver state in matter and ghost sector. Then we compute algebraically the slivers in the twisted ghost sector and show how identity of such states is implied by the relations (3.52) between the Neumann coefficients in the game.

3.4.1 Matter sliver and Ghost solution

The projection equation in the matter sector

$$|\psi\rangle_m = |\psi\rangle_m *_m |\psi\rangle_m \tag{3.82}$$

can be solved as in [34, 37], by the ansatz

$$|\psi\rangle_m = \mathcal{N}_m \exp\left(\sum_{n,m\geq 1} a_n^\dagger S_{nm} a_m^\dagger\right)|0\rangle \tag{3.83}$$

$$S = CSC \tag{3.84}$$

where

$$T = CS = \frac{1}{2M}\left(1 + M - \sqrt{(1-M)(1+3M)}\right) \tag{3.85}$$

The ghost equation of motion is

$$\mathcal{Q}|\psi\rangle_g + |\psi\rangle_g *_g |\psi\rangle_g = 0 \tag{3.86}$$

This equation is easy to solve if we use big matrices in order to handle at the same time both zero and non zero modes (see [79]). The relevant results are

$$|\psi\rangle_g = \tilde{\mathcal{N}}_g \exp\left(\sum_{n,m\geq 1} c_n^\dagger{}_{nm} b_m^\dagger\right)|\dot{0}\rangle \tag{3.87}$$

$$= C = \frac{1}{2Y}\left(1 + Y - \sqrt{(1-Y)(1+3Y)}\right) \tag{3.88}$$

$$\mathcal{Q} = c_0 + \vec{f}\cdot(\vec{c} + C\vec{c}^\dagger) \tag{3.89}$$

$$\vec{f} = \frac{\vec{y}}{1-Y} \tag{3.90}$$

Using the integral representations (3.28) one can actually prove that $\mathcal{Q}$ is a midpoint insertion [79, 45]

$$\mathcal{Q} = c_0 + \sum_{n=1}^{\infty}(-1)^n(c_{2n} + c_{-2n}) = \frac{1}{2i}\left(c(i) - c(-i)\right) \tag{3.91}$$

3.4.2 The twisted sliver in the algebraic approach

We have seen that the Neumann coefficients of the star product in the twisted CFT coincides to (minus) the matter ones at zero momentum. This implies that we can solve the algebraic equation for projectors, as for the usual ghost star product, but now using the linear and non linear relations (3.47).
So we impose the projector equation

$$|S\rangle' = |S\rangle' *' |S\rangle' \tag{3.92}$$

with the ansatz

$$|S\rangle' = \mathcal{N} \exp\left(\sum_{n,m\geq 1} c_n^\dagger S'_{nm} b_m^\dagger\right)|0'\rangle \tag{3.93}$$

we can safely follow the way of the non twisted case [14, 79] and arrive at the equation

T'=CS'= Y' +(Y'₊, $Y'_-)\frac{1}{1-\Sigma'\mathcal{V}'}\Sigma'\begin{pmatrix} Y'_- \\ Y'_+ \end{pmatrix}$ $where\Sigma' = \begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix}$, $\mathcal{V}' = \begin{pmatrix} Y' & Y'_+ \\ Y'_- & Y' \end{pmatrix}$.ı

1)$(-Y'T'^2+(1-Y')T'-Y') = 0$(3.94)which, apart from the trivial solution $T' = -1$, gives[5]

$$T' = CS' = \frac{1}{2Y'}\left(1 - Y' - \sqrt{(1+Y')(1-3Y')}\right) \tag{3.95}$$

It is interesting to compare it with the algebraic projector w.r.t. the reduced product

$$= \frac{1}{2Y}\left(1 + Y - \sqrt{(1-Y)(1+3Y)}\right) \tag{3.96}$$

The equality of this two solutions holds if and only if the following relation between twisted and non twisted Neumann coefficients is obeyed

$$Y' = \frac{Y}{1+2Y} \tag{3.97}$$

which is exactly (3.52). This shows that equality of solution in VSFT is equivalent to the statement (3.51) which, on the other hand, have its explanation via the 4-string vertex [33].

[5] As usual we choose the square root branch cut which doesn't have divergence as $Y' \to 0$.

Chapter 4

Solitons in noncommutative field theory, string theory with B field and D-branes

The third of Sen's conjectures says that lower dimensional D-branes, lower respect the D25 brane, are solitonic excitations of the tachyon potential. So, in principle, one could take the effective action of bosonic string theory for the tachyon, obtained integrating out massive string field and look for solitonic solutions and the check if such solutions exist and share some aspects of D-brane physics. Unfortunatly, the tachyon is a scalar field and there are no non trivial solitonic solutions of scalar field theory. But things are different in the noncommutative case. This is the crucial point. Gopakumar, Minwalla and Strominger found an infinite class of these noncommutative solitons, the so called GMS solitons that will play an important role in the next chapters of thus tesis, and Harvey, Kraus, Larsen and Martinec proposed a description of D-branes as noncmmutative solitons, achieved turning on a B field in the string theory action. We will see that the inclusion of the B field in Vacuum String Field Theory allows to unify, in a natural way, these two description of D-branes as solitons of noncommutative theories.

4.1 Open Strings in a Constant B Field

Let us review the link between string theory in the presence of a B field and non-commutative field theories, following [5, 6].

Let us consider an open string ending on a Dp-brane in the presence of a constant Neveu-Schwarz B field. Since the components of B outside the brane can always be gauged away the rank r of B will be $r < p+1$. The worldsheet action is

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\Big[g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu + 2\pi\alpha'\epsilon^{ab}B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\Big] \\ &= \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \frac{1}{2}\oint_{\partial\Sigma} B_{\mu\nu}X^\mu \partial_t X^\nu\ , \end{aligned} \tag{4.1}$$

where Σ is the string worldsheet that we take with Lorentz signature, and ∂_t is the derivative tangent to the worldsheet boundary $\partial\Sigma$. Since the term proportional to $B_{\mu\nu}$ can be written as total derivative term, it does not affect the equation of motion but the boundary condition, which reads as

$$g_{\mu\nu}\partial_n X^\nu + 2\pi\alpha' B_{\mu\nu}\partial_t X^\mu\Big|_{\partial\Sigma} = 0\ , \tag{4.2}$$

where ∂_n is the normal derivative to $\partial\Sigma$. The new dimensionless parameter $\alpha' B_{\mu\nu}$ determines which type of boundary condition, Neumann or Dirichlet, is dominant in (4.2). If $\alpha' B_{\mu\nu} \gg g_{\mu\nu}$ along the spatial directions of the brane, the boundary conditions become Dirichlet. Indeed, in this limit, the second term in (4.2) dominates, and, with B being invertible, (4.2) reduces to $\partial_t X^j = 0$. For $B = 0$ we recover Neumann boundary conditions in (4.2).

It is important to remark that the presence of a B field has a physical effect only along the brane and not outside it. This is because outside the brane there is always the possibility of making a 'gauge transformation' $B \to B + d\Lambda$ that sets the condition $B = 0$. Along the brane lives also the $U(1)$ gauge field of the string endpoints. It is described by the action

$$S(A) = \frac{1}{2\pi\alpha'}\oint_{\partial\Sigma} d\tau A_\mu(X)\partial_\tau X^\mu = \frac{-1}{4\pi\alpha'}\int_\Sigma d^2\sigma\epsilon^{ab}F_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\ . \tag{4.3}$$

Therefore for open strings B and F always appear together in the combination $\mathcal{F} = B - F$, and the 'gauge transformation' above, when performed, has now physical effects on the $U(1)$ gauge field. The combination $\mathcal{F} = B - F$ is indeed invariant under both gauge transformations for the one-form gauge field A

$$A \to A + d\Lambda, \quad B \to B \tag{4.4}$$

and for the two-form gauge field B

$$B \to B + d\Lambda, \quad A \to A + \Lambda\ . \tag{4.5}$$

From now on we will restrict our analysis to the case of the presence of the B field only.

Boundary condition can be rewritten in the convenient form

$$E_{\mu\nu}\partial_- X^\nu = (E^T)_{\mu\nu}\partial_+ X^\nu \tag{4.6}$$

where $E_{\mu\nu} \equiv g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu}$, and $\partial_\pm$ are derivatives with respect to the light cone variables $\sigma^\pm = \tau \pm \sigma$. The string coordinates $X^\mu(\sigma,\tau)$ satisfying the boundary condition (4.6) have the following expansion:

$$\begin{aligned} X^\mu &= x_0^\mu + \alpha'\Big[(E^{-1})^{\mu\nu}\, g_{\nu\rho}\, p^\rho\, \sigma^- + (E^{-1T})^{\mu\nu}\, g_{\nu\rho}\, p^\rho\, \sigma^+\Big] \\ &+ i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\Big[(E^{-1})^{\mu\nu}\, g_{\nu\rho}\, \alpha_n^\rho\, e^{-in\sigma^-} + (E^{-1T})^{\mu\nu} g_{\nu\rho}\, \alpha_n^\rho\, e^{-i\sigma^+}\Big] \end{aligned} \tag{4.7}$$

Let us perform a Wick rotation on the worldsheet in order to map it onto the upper half plane: $\tau \to -i\tau$, $z = e^{\tau + i\sigma}$, $\bar z = e^{\tau - i\sigma}$ $(0 \le \sigma \le \pi)$. The boundary conditions become

$$E_{\mu\nu}\partial_{\bar z} X^\nu = (E^T)_{\mu\nu}\partial_z X^\nu \tag{4.8}$$

that are imposed on the real axis $z = \bar z$. The propagator is

$$\begin{aligned} \langle X^\mu(z,\bar z) X^\nu(z',\bar z')\rangle &= -\alpha'\Big[g^{\mu\nu}\ln|z-z'| - g^{\mu\nu}\ln|z-\bar z'| \\ &+ G^{\mu\nu}\ln|z-\bar z'|^2 + \frac{1}{2\pi\alpha'}\theta^{\mu\nu}\ln\frac{z-\bar z'}{\bar z - z'} + D^{\mu\nu}\Big] \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} G^{\mu\nu} &= (E^{-1}\, g\, (E^T)^{-1})^{\mu\nu} \\ &= \left(\frac{1}{g + 2\pi\alpha' B}\, g\, \frac{1}{g - 2\pi\alpha' B}\right)^{\mu\nu} \\ \theta^{\mu\nu} &= (2\pi\alpha')^2 (E^{-1}\, B\, (E^T)^{-1})^{\mu\nu} \\ &= -(2\pi\alpha')^2\left(\frac{1}{g + 2\pi\alpha' B}\, B\, \frac{1}{g - 2\pi\alpha' B}\right)^{\mu\nu} . \end{aligned} \tag{4.10}$$

On the upper half plane the mode expansion (4.7) becomes

$$\begin{aligned} X^{\mu\nu} &= x_0^\mu + \alpha'\Big[(E^{-1})^{\mu\nu} g_{\nu\rho} p^\rho \ln\bar z + (E^{-1T})^{\mu\nu} g_{\nu\rho} p^\rho \ln z\Big] \\ &+ i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\Big[(E^{-1})^{\mu\nu} g_{\nu\rho}\alpha_n^\rho \bar z^{-n} + (E^{-1T})^{\mu\nu} g_{\nu\rho}\alpha_n^\rho z^{-n}\Big] . \end{aligned} \tag{4.11}$$

The indices of p^ρ and α_n^ρ were lowered by the metric $g_{\mu\nu}$ and not by the metric $G_{\mu\nu}$. From the definition of the propagator we can read the commutation rules for α_n^ρ, x_0, and p:

$$\langle X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\rangle \equiv R(X^\mu(z,\bar{z})X^\nu(z',\bar{z}')) - N(X^\mu(z,\bar{z})X^\nu(z',\bar{z}')) \tag{4.12}$$

where R and N stand for the radial ordering and the normal ordering respectively. We assume that the normal ordering prescription for the product of x_0^μ with p_ν is $: x_0^\mu p_\nu :\equiv x_0^\mu p_\nu$. The vacuum is then defined as

$$p_\mu|0\rangle = \alpha_n^\mu|0\rangle = 0 \quad (n>0), \quad \langle 0|\alpha_n^\mu = 0 \quad (n<0) \tag{4.13}$$

From standard calculations of two dimensional conformal field theory, one can obtain the commutators

$$[\alpha_m^\mu, \alpha_n^\nu] = n\delta_{m+n}G^{\mu\nu} \qquad [x_0^\mu, p_\nu] = \delta_\nu^\mu \tag{4.14}$$

where, as usual, we set $\alpha_{0,\mu} \equiv \sqrt{2\alpha'}p_\mu$. The constant $D^{\mu\nu}$ is written as $\alpha' D^{\mu\nu} = -\langle 0|X_0^\mu X_0^\nu|0\rangle$, that is equivalent to set the normal ordering for $x_0^\mu x_0^\nu$ as

$$: x_0^\mu x_0^\nu := x_0^\mu x_0^\nu + \alpha' D^{\mu\nu} \tag{4.15}$$

If we fix, as in [6], $D^{\mu\nu}$ as $\alpha' D^{\mu\nu} = -\frac{1}{2}\theta^{\mu\nu}$, the coordinates x_0^μ become noncommutative :

$$[x_0^\mu, x_0^\nu] = i\theta^{\mu\nu} \tag{4.16}$$

The center of mass coordinates

$$\hat{x}_0^\mu \equiv x_0^\mu + \frac{1}{2}\theta^{\mu\nu}p_\nu \tag{4.17}$$

still commute among each other. In order to understand the physical meaning of the parameters $G^{\mu\nu}$ and $\theta^{\mu\nu}$ let us restrict the analysis to the boundary of the worldsheet. On the boundary the propagator is

$$\langle X^\mu(\tau)X^\nu(\tau')\rangle = -\alpha' G^{\mu\nu}\ln(\tau-\tau')^2 + \frac{i}{2}\theta^{\mu\nu}\epsilon(\tau-\tau') \tag{4.18}$$

where $\epsilon(\tau)$ is the function that is 1 or -1 for positive or negative $(\tau-\tau')$. $G^{\mu\nu}$ is the effective open metric seen by the open strings. We will then refer to $g_{\mu\nu}$ as to the closed string metric and to $G_{\mu\nu}$ as to the open string metric. To understand the physical interpretation of $\theta^{\mu\nu}$ is useful to rewrite the string mode expansion in the following way:

$$\begin{aligned} X^\mu(\sigma,\tau) &= \hat{x}_0 + 2\alpha'\Big(G^{\mu\nu}\tau + \frac{1}{2\pi\alpha'}\theta^{\mu\nu}(\sigma-\pi/2)\Big)p_\nu \\ &+i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}e^{-in\tau}\Big[G^{\mu\nu}\cos(n\sigma) - i\frac{1}{2\pi\alpha'}\theta^{\mu\nu}\sin(n\sigma)\Big]\alpha_{n,\nu} \end{aligned} \tag{4.19}$$

The endpoints of the string become noncommutative [1]:

$$[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = \begin{cases} i\theta^{\mu\nu} & (\sigma=\sigma'=0) \\ -i\theta^{\mu\nu} & (\sigma=\sigma'=\pi) \\ 0 & (\text{otherwise}) \end{cases}$$

and, since the the brane is by definition the place where the string endpoints are forced to belong, the world volume of the brane itself becomes a noncommutative space.

We want now to perform the low energy limit on the string amplitudes with B field. Without B field this limit is done by taking $\alpha' \to 0$. With B field we have to add to this condition also the request that the natural open strings parameters G and θ are kept fixed. By looking at the eqs.(4.10) we see that both the requests can be satisfied by imposing

$$\alpha' \to 0 \tag{4.21}$$

in such a way that

$$\begin{aligned} &\alpha' B_{\mu\nu} \sim \epsilon \to 0 \\ &g_{\mu\nu} \sim \epsilon^2 \to 0 \quad \text{for} \quad \mu,\nu = 1,\ldots,r \end{aligned} \tag{4.22}$$

with B fixed. In this way eqs.(4.10) become

$$G^{\mu\nu} = -\frac{1}{(2\pi\alpha')^2}\left(\frac{1}{B}g\frac{1}{B}\right)^{\mu\nu} \tag{4.23}$$

$$G_{\mu\nu} = -(2\pi\alpha')^2(B\,g^{-1}\,B)_{\mu\nu} \tag{4.24}$$

$$\theta^{\mu\nu} = \left(\frac{1}{B}\right)^{\mu\nu} \tag{4.25}$$

[1] We have used the formula:

$$\sum_{n=1}^{\infty}\frac{2}{n}\sin(n(\sigma+\sigma')) = \begin{cases} \pi-\sigma-\sigma' & (\sigma+\sigma'\neq 0, 2\pi) \\ 0 & (\sigma+\sigma' = 0, 2\pi) \end{cases} \tag{4.20}$$

As one can immediately see, G and θ are finite in this limit. The conditions (4.22) guarantee that $\alpha' B_{\mu\nu} \gg g_{\mu\nu}$, and the boundary conditions (4.2) become more and more Dirichlet. There exists in the literature another form of this limit that is particularly used in the analysis of noncommutative solitons. It is characterized by $\alpha' B_{\mu\nu} \to \infty$ with $g_{\mu\nu}$ held fixed. α' can then be taken finite or can be sent to 0 in such a way still $\alpha' B_{\mu\nu} \to \infty$. In either form of the limit

$$\theta^{\mu\nu} = \left(\frac{1}{B}\right)^{\mu\nu}, \tag{4.26}$$

In order to avoid confusion we will always refer to the form (4.22) of the low energy limit.

In the low energy limit the boundary propagator becomes

$$\langle X^i(\tau) X^j(\tau') \rangle = \frac{i}{2}\theta^{ij}\epsilon(\tau - \tau') \tag{4.27}$$

and the action (4.1) reduces to

$$S \to -\frac{i}{2}\int_{\partial\Sigma} B_{\mu\nu} X^\mu \partial_t X^\nu \tag{4.28}$$

This action, regarded as a one-dimensional action, describes the motion of a charged particle in a large magnetic field. Indeed the action for such (nonrelativistic) point particle is

$$S = \int dt \left(\frac{1}{2} m \dot{x}^i \dot{x}^i + e B_{ij} x^i \dot{x}^j\right) \tag{4.29}$$

The conjugate momentum Π_i to x^i is

$$\Pi_i = m \dot{x}_i + e B_{ij} x^j \tag{4.30}$$

In the limit where the energy $\omega \ll e|B|/m$, the canonical commutation relations become simply

$$[x^i, x^j] = i(B^{-1})^{ij}\frac{m}{e} \tag{4.31}$$

Thus at energies much less than the cyclotron frequency $e|B|/m$, when one is in the lowest Landau level, one effectively has noncommuting coordinates.

We need to determine the expression of G_s in terms of the closed string variables g, B and g_s. The constant term in the effective lagrangian will do this job. For slowly varying fields, the effective Lagrangian is the Dirac-Born-Infeld lagrangian

$$\mathcal{L}_{DBI} = \frac{1}{g_s (2\pi)^2 (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(g + 2\pi\alpha'(B - F))} \tag{4.32}$$

The constant part is

$$\mathcal{L}(F=0) = \frac{1}{g_s(2\pi)^2(\alpha')^{\frac{p+1}{2}}}\sqrt{\det(g+2\pi\alpha' B)} \tag{4.33}$$

On the other side we know that when we describe the effective action in terms of open string quantities the whole θ dependence is contained in the $\star$ product. In this description the correspondent of (4.32) is

$$\mathcal{L}(\widehat{F}) = \frac{1}{G_s(2\pi)^2(\alpha')^{\frac{p+1}{2}}}\sqrt{\det(G+2\pi\alpha'\widehat{F})} \tag{4.34}$$

and the constant term is

$$\mathcal{L}(\widehat{F}=0) = \frac{1}{G_s(2\pi)^2(\alpha')^{\frac{p+1}{2}}}\sqrt{\det G} \tag{4.35}$$

Equating the two constant parts (4.33) and (4.35)

$$G_s = g_s\left(\frac{\det G}{\det(g+2\pi\alpha' B)}\right)^{\frac{1}{2}} \tag{4.36}$$

that in the $\alpha' \to 0$ limit becomes

$$G_s = g_s \det(2\pi\alpha' B g^{-1})^{\frac{1}{2}} \tag{4.37}$$

where the determinant is calculated only in the dimensions with nonzero B field.

4.2 Noncommutative field theory

In the presence of a constant NSNS B-field, the low energy effective action of the open strings attached to D-branes can be represented by a Yang-Mills theory defined on a noncommutative spacetime endowed with a Moyal bracket (see [5, 6] and references therein).

This holds at a semiclassical level, tree amplitudes computed in the string theory and on the field theory side, but also several calculations at one loop of this type have been carried out [?, ?, 7, ?, 9].

A noncommutative space is a space with the following commutation relations:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}\,. \tag{4.38}$$

We need to define the multiplication law on this space. which is induced from (4.38) through the so called Weyl-Moyal correspondence: the product of operators and the product of the corresponding functions do share the same Fourier transform:

$$\hat{\Phi}(\hat{x}) \longleftrightarrow \Phi(x) \ ;$$

$$\begin{aligned} \hat{\Phi}(\hat{x}) &= \int_\alpha e^{i\alpha\hat{x}} \ \phi(\alpha) \ d\alpha \\ \phi(\alpha) &= \int e^{-i\alpha x} \ \Phi(x) \ dx \,, \end{aligned} \tag{4.39}$$

where α and x are real variables. Then,

$$\begin{aligned} \hat{\Phi}_1(\hat{x}) \ \hat{\Phi}_2(\hat{x}) &= \int\int_{\alpha\,\beta} e^{i\alpha\hat{x}} \ \phi(\alpha) \ e^{i\beta\hat{x}} \ \phi(\beta) \ d\alpha \ d\beta \\ &= \int\int_{\alpha\,\beta} e^{i(\alpha+\beta)\hat{x}-\frac{1}{2}\alpha_\mu\beta_\nu[\hat{x}_\mu,\hat{x}_\nu]} \ \phi_1(\alpha) \ \phi_2(\beta) \ d\alpha \ d\beta \ , \end{aligned} \tag{4.40}$$

and hence,

$$\hat{\Phi}_1(\hat{x}) \ \hat{\Phi}_2(\hat{x}) \longleftrightarrow \Big(\Phi_1 \star \Phi_2\Big)(x) \ , \tag{4.41}$$

with

$$\Big(\Phi_1 \star \Phi_2\Big)(x) \equiv \Big[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\eta_\nu}} \ \Phi_1(x+\xi) \ \Phi_2(x+\eta)\Big]_{\xi=\eta=0} \ . \tag{4.42}$$

This means that we have to modify the usual multiplication to the so called star product. It is easy to check that the Moyal bracket, the commutator in which the product is modified with a star product, of two coordinates x_μ and x_ν gives the right commutation relations,

$$[x_\mu, x_\nu]_{MB} \ = \ i\theta_{\mu\nu} \tag{4.43}$$

Some properties of the star product algebra are:

- Star product between exponentials:

$$\begin{aligned} e^{ikx} \star e^{iqx} &= e^{i(k+q)x} e^{-\frac{i}{2}(k\theta q)} \text{, where} \\ k\theta p &\equiv k^\mu p^\nu \theta_{\mu\nu} \end{aligned} \tag{4.44}$$

- Momentum space representation:

  Let $\tilde{f}(k)$ and $\tilde{g}(k)$ be the Fourier components of f and g. Then using (4.44)

$$(f \star g)(x) = \int d^4k\, d^4q\, \tilde{f}(k)\, \tilde{g}(q)\, e^{-\frac{i}{2}(k\theta q)}\, e^{i(k+q)x}\,. \tag{4.45}$$

- Associativity:

$$\Big[(f \star g) \star h\Big](x) = \Big[f \star (g \star h)\Big](x)\,, \tag{4.46}$$

  which can be checked in momentum space.

$$\begin{aligned} \text{rhs} &= \int d^4k\, d^4q\, d^4p\, \tilde{f}(k)\, \tilde{g}(q)\, \tilde{h}(p)\, e^{-\frac{i}{2}(k\theta q)}\, e^{-\frac{i}{2}((k+q)\theta p)}\, e^{i(k+q+p)x}\,, \quad \text{and} \\ \text{lhs} &= \int d^4k\, d^4q\, d^4p\, \tilde{f}(k)\, \tilde{g}(q)\, \tilde{h}(p)\, e^{-\frac{i}{2}(q\theta p)} e^{-\frac{i}{2}(k\theta(q+p))}\, e^{i(k+q+p)x}\,. \end{aligned} \tag{4.47}$$

- Star products under integral sign:

$$\int (f \star g)(x)\, d^4x = \int (g \star f)(x)\, d^4x = \int (f \cdot g)(x)\, d^4x\,. \tag{4.48}$$

  Using (4.45) we can immediately perform the integration over x which will give a $\delta^4(k+q)$. Due to the antisymmetry of θ the exponent vanishes and so:

$$\begin{aligned} \int (f \star g)(x)\, d^4x &= \int d^4k\, \tilde{f}(k)\tilde{g}(-k) \\ &= \int (f \cdot g)(x)\, d^4x \end{aligned} \tag{4.49}$$

- Cyclic property:

$$\int (f_1 \star f_2 \star \ldots f_n)(x)\, d^4x = \int (f_n \star f_1 \star \ldots f_{n-1})(x)\, d^4x\,. \tag{4.50}$$

4.2.1 Oscillator vertex for the Moyal star product

We have met two different noncommutative star products until now: the Witten's star in SFT and the Moyal's one in a generic noncommutative field theory. We will see that, actually, althought these two star's appear very different, they are intimately related. Indeed, the Witten's star can be written, on a particular basis, as an infinite continous Moyal star. For future purposes and in order to understand better the difference between Moyal and ordinary pointwise multiplication of functions, we will show an oscillator form three-vertex for the ordinary and the Moyal product, following [66]. The pointwise, usual multiplicaton between two functions $f(x)$ and $g(x)$, can be written as

$$fg(x) = \langle x|\langle f|\langle g|V3\rangle \tag{4.51}$$

where

$$f(x) = \langle x|f\rangle \tag{4.52}$$

$$\langle x| = \frac{1}{\pi^{1/4}}\langle 0|exp\left(-1/2x^2 + \sqrt{2}iax + 1/2aa\right) \tag{4.53}$$

$$|V3\rangle = \left(\frac{2}{3\sqrt{\pi}}\right)^{1/2} exp\left(\frac{1}{6}\left(a_1^\dagger a_1^\dagger + a_2^\dagger a_2^\dagger + a_3^\dagger a_3^\dagger\right) - \frac{2}{3}\left(a_1^\dagger a_2^\dagger + a_2^\dagger a_3^\dagger + a_3^\dagger a_1^\dagger\right)\right)|0\rangle \tag{4.54}$$

The Moyal star product is encoded in the three-vertex

$$|V_3(\theta)\rangle_{123} = \frac{2}{3\sqrt{\pi}}\frac{1}{1+\frac{\theta^2}{12}} \exp\left[-\frac{1}{2}\frac{\theta^2-4}{\theta^2+12}(a_1^\dagger a_1^\dagger + b_1^\dagger b_1^\dagger + cyclic)\right. \tag{4.55}$$

$$-\frac{8}{\theta^2+12}(a_1^\dagger a_2^\dagger + b_1^\dagger b_2^\dagger + cyclic) \tag{4.56}$$

$$\left.-\frac{4i\theta}{\theta^2+12}(a_1^\dagger b_2^\dagger - b_1^\dagger a_2^\dagger + cyclic)\right]|0\rangle_{123}. \tag{4.57}$$

(4.58)

which reduces to the previous one, of course, when $\theta --> 0$, that is when the space become commutative.

4.3 Noncommutative solitons

As we already anticipated, the crucial point in our discussion is that scalar theory without noncommutativity does not have any lump solutions. This is actually true for any bounded potential in spatial dimension greater than one (Derrick's theorem), and follows from a simple scaling argument [17, 18]. We consider the energy of the field configurations $\phi_\lambda(x) = \phi_0(\lambda x)$ with $\phi_0(x)$ an extremum of the energy functional

$$\begin{aligned} E(\lambda) &= \frac{1}{g^2}\int d^Dx\left(\frac{1}{2}(\partial\phi_0(\lambda x))^2 + V(\phi_0(\lambda x))\right) \\ &= \frac{1}{g^2}\int d^Dx\left(\frac{1}{2}\lambda^{2-D}(\partial\phi_0(x))^2 + \lambda^{-D}V(\phi_0(x))\right) \end{aligned} \tag{4.59}$$

Since $\phi_0(x)$ is an extremum, we require $\partial_\lambda E(\lambda)|_{\lambda=1} = 0$. This means

$$\int d^Dx\left(\frac{1}{2}(D-2)(\partial\phi_0(x))^2 + DV(\phi_0(x))\right) = 0 \tag{4.60}$$

For spatial dimension $D \geq 2$, for a potential bounded from below by zero, the only way this relation can hold is that the kinetic and the potential terms separately vanish. There are therefore no nontrivial configurations. This argument fails if one includes higher derivative terms. Instead, for $D = 2$, only the potential energy should be zero.

On the other hand, if the space become noncommutative an infinite number of classical solutions of scalar field theory can be found [56, ?] We consider a theory of a single scalar field in $2+1$ dimensions with noncommutativity in the two spatial directions. We parametrize the spatial $\mathbb{R}^2$ by complex coordinates $z, \bar{z}$. The energy functional is

$$E = \frac{1}{g^2}\int d^2z(\partial_z\phi\partial_{\bar{z}}\phi + V(\phi)_\star), \tag{4.61}$$

where $d^2z = dxdy$. Fields are multiplied using the Moyal star product, that in complex coordinates is

$$(f\star g)(z\bar{z}) = e^{\frac{\theta}{2}(\partial_z\partial_{\bar{z}'} - \partial_z\partial_{\bar{z}'})}f(z,\bar{z})g(z',\bar{z}')\Big|_{z=z'} \tag{4.62}$$

The limit of large noncommutativity is useful to simplify the search for finite energy solution of (4.61) but it is not necessary. For simplicity, let us take $\theta \to \infty$. This is exactly the low energy limit we introduced before, seen from the point of view of the field theory. If we now rescale the coordinates $z \to z\sqrt{\theta}$, $\bar{z} \to \bar{z}\sqrt{\theta}$, the commutation relations will not depend on θ and the energy functional becomes

$$E = \frac{1}{g^2}\int d^2z(\partial_z\phi\partial_{\bar{z}}\phi + \theta V(\phi)_\star) \tag{4.63}$$

In the limit $\theta \to \infty$ we can neglect the kinetic term and consider just the potential in (4.63), and the energy

$$E = \frac{\theta}{g^2} \int d^2 z V(\phi)_\star \tag{4.64}$$

is extremized by solving the equation

$$\frac{\partial V}{\partial \phi} = 0 \tag{4.65}$$

We will consider a polynomial potential

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \sum_{j=3}^{r} \frac{b_j}{j} \phi^j \tag{4.66}$$

where the product among the fields is the Moyal one.

If $V(\phi)$ were the potential in a commutative scalar field theory, the only solutions would be the constant configurations

$$\phi = \lambda_i \tag{4.67}$$

where $\lambda_i \in \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ are the various real extrema of the function $V(x)$. For $V(\phi)$ as in (4.66), λ_i are the real roots of the equation $m^2 x + \sum_{j=3}^{r} b_j x^{j-1} = 0$. The first non-trivial solution to (4.65) can be constructed starting from a function ϕ_0 that satisfies

$$(\phi_0 \star \phi_0)(x) = \phi_0(x) \tag{4.68}$$

For such a function the two following relation hold

$$\phi_0^n(x) = \phi(x), \qquad f(a\phi_0(x)) = f(a)\,\phi_0(x) \tag{4.69}$$

We can solve (4.65) with $\lambda_i \phi_0(x)$ where λ_i is an extremum of $V(x)$. The problem of solving (4.65) becomes then that one of finding projectors under the star product, a function f that squares to itself under $\star$ product. Such a function is

$$\psi(r) = 2e^{-r^2} \tag{4.70}$$

where $r^2 = x^2 + y^2$. Going to momentum space:

$$\tilde{\psi}(k) = \int d^2 x\, \psi(x)\, e^{ik\cdot x} = 2\pi e^{-k^2/4} \tag{4.71}$$

and

$$\begin{aligned}\left(\tilde{\psi} \star \tilde{\psi}\right)(p) &= 4\pi^2 \int \frac{d^2k}{(2\pi)^2} \tilde{\psi}(k)\, \tilde{\psi}(p-k)\, e^{\frac{i}{2}\epsilon_{\mu\nu}k^\mu (p-k)^\nu} \\ &= 2\pi\, e^{-p^2/4} \end{aligned} \tag{4.72}$$

Going back to coordinate space

$$(\psi \star \psi)(r) = 2e^{-r^2} = \psi(r) \tag{4.73}$$

and $\lambda_i \psi(x)$ solves (4.65).

General solutions of (4.65) can be found recalling the definition of the Weyl-Moyal correspondence adapted to the present case. Given a C^∞ function $f(p,q)$ on $\mathbb{R}^2$ we assign to it an operator $O_f(\hat{p},\hat{q}) \in \mathcal{H}$:

$$O_f(\hat{q},\hat{p}) = \frac{1}{(2\pi^2)} \int d^2k\, \tilde{f}(k)\, e^{-i(k_q \hat{q} + k_p \hat{p})} \tag{4.74}$$

where

$$\tilde{f}(k) = \int d^2x\, e^{i(k_q q + k_p p)}\, f(q,p)\ , \qquad [\hat{q},\hat{p}] = i \tag{4.75}$$

The map given by (4.74) can also be inverted. Using

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-i(k_q \hat{q} + k_p \hat{p})}\, e^{i(k'_q \hat{q} + k'_p \hat{p})} = 2\pi\, \delta(k_q - k'_q)\delta(k_p - k'_p) \tag{4.76}$$

we can project $\tilde{f}$ in (4.74) and then perform the Fourier transform to find

$$f(q,p) = \int dk'_p e^{-ipk'_p} \langle q + k'_p/2 | O_f(\hat{q},\hat{p}) | q - k'_p/2 \rangle \tag{4.77}$$

Remember that the Moyal product has been transformed into ordinary operator product:

$$O_{f\star g} = O_f \cdot O_g \tag{4.78}$$

A useful identity relates the integral of the phase space function to the trace of its Weyl transform

$$\begin{aligned}\int dqdp f(p,q) &= \int dqdpdk'_p\, e^{-ipk'_p} \langle q + k'_p/2 | O_f(\hat{q},\hat{p}) | q - k'_p/2 \rangle \\ &= \int dqdk'_p 2\pi \delta(k'_p) \langle q + k'_p/2 | O_f(\hat{q},\hat{p}) | q - k'_p/2 \rangle \\ &= 2\pi \int dq \langle q | O_f | q \rangle \\ &= 2\pi \mathrm{Tr}_{\mathcal{H}} O_f \end{aligned} \tag{4.79}$$

In order to solve any algebraic equation involving the star product, it is thus sufficient to determine all operator solutions to the equation in $\mathcal{H}$. The functions on phase space corresponding to each of these operators may then be read off from (4.77).

It is easy to see that $O = \lambda_i P$ is a solution to $V'(O) = 0$ if P is an arbitrary projection operator on some subspace of $\mathcal{H}$ and if λ_i is an extremum of $V(x)$. The energy of this solution is, using (4.79),

$$E = \frac{2\pi\theta}{g^2}\mathrm{Tr}V(O_\phi) = \frac{2\pi\theta}{g^2}V(\lambda_i)\mathrm{Tr}P \tag{4.80}$$

Thus the energy is finite if P is projector onto a finite dimensional subspace of $\mathcal{H}$. In fact, the most general solution to (4.65) has the form

$$O = \sum_j a_j P_j \tag{4.81}$$

where $\{P_j\}$ are mutually orthogonal projectors onto one dimensional subspaces

$$P_i P_j = \delta_{ij} P_j, \qquad \mathrm{Tr}_{\mathcal{H}} P_i = 1 \tag{4.82}$$

with a_j taking values in the set $\{\lambda_i\}$ of real extrema of $V(x)$. We have an infinite number of solutions of the form λP. To see what they mean, let us choose a particular basis in $\mathcal{H}$. Let $|n\rangle$ represent the energy eigenstates of the one dimensional harmonic oscillator whose creation and annihilation operators are defined by

$$a = \frac{\widehat{q} + i\widehat{p}}{\sqrt{2}}, \quad a^\dagger = \frac{\widehat{q} - i\widehat{p}}{\sqrt{2}} \tag{4.83}$$

where $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Any operator may be written as a linear combination of the basis operators $|m\rangle\langle n|$, that may be expressed in terms of a and $a^\dagger$ as

$$|m\rangle\langle n| =: \frac{(a^\dagger)^m}{\sqrt{m!}} e^{-a^\dagger a} \frac{a^n}{\sqrt{n!}} : \tag{4.84}$$

where double dots denote normal ordering. We will first describe operators of the form (4.81) that correspond to radially symmetric functions in space. As $a^\dagger a \approx r^2/2$, operators corresponding to radially symmetric wave functions are functions of $a^\dagger a$. From (4.84), the only such operators are linear combinations of the diagonal projection operators $|n\rangle\langle n| = \frac{1}{n!} : a^{\dagger n} e^{-a^\dagger a} a^n :$. Hence all radially symmetric solutions of (4.65) correspond to operators of the form $O = \sum_n a_n |n\rangle\langle n|$, where the numbers a_n can take any values in the set $\{\lambda_i\}$.

We now translate these operator solutions back to field space. From the Baker-Campbell-Hausdorff formula

$$e^{-i(k_q\hat{q}+k_p\hat{p})} = e^{-i\left(k_{\bar{z}}a+k_z a^\dagger\right)} = e^{-\frac{k^2}{4}} : e^{-i\left(k_{\bar{z}}a+k_z a^\dagger\right)} :, \tag{4.85}$$

where

$$k_z = \frac{k_x + ik_y}{\sqrt{2}}, \quad k_{\bar{z}} = \frac{k_x - ik_y}{\sqrt{2}}, \quad k^2 = 2k_z k_{\bar{z}}.$$

Any operator O expressed as a normal ordered function of a and $a^\dagger$, $f_N(a, a^\dagger)$, can be rewritten in Weyl ordered form as follows. By definition,

$$O =: f_N(a, a^\dagger) := \frac{1}{(2\pi)^2}\int d^2k \tilde{f}_N(k) : e^{-i\left(k_{\bar{z}}a+k_z a^\dagger\right)} : . \tag{4.86}$$

Using (4.85), (4.86) may be rewritten as

$$O = \frac{1}{(2\pi)^2}\int d^2k \tilde{f}_N(k) e^{\frac{k^2}{4}} e^{-i\left(k_{\bar{z}}a+k_z a^\dagger\right)}. \tag{4.87}$$

Thus, the momentum space function $\tilde{f}$ associated with the operator O, is

$$\tilde{f}(k) = e^{\frac{k^2}{4}} \tilde{f}_N(k) . \tag{4.88}$$

For the operator $O_n = |n\rangle\langle n|$ we find, using (4.85) and (4.86), that the corresponding normal ordered function $\tilde{\phi}_N^{(n)}(k) = 2\pi e^{\frac{-k^2}{2}} L_n(\frac{k^2}{2})$. (4.88) then becomes

$$|n\rangle\langle n| = \frac{1}{(2\pi)}\int d^2k e^{\frac{-k^2}{4}} L_n\left(\frac{k^2}{2}\right) e^{-i\left(k_{\bar{z}}a+k_z a^\dagger\right)} \tag{4.89}$$

where $L_n(x)$ is the n^{th} Laguerre polynomial. The field $\phi_n(x, y)$ that corresponds to the operator $O_n = |n\rangle\langle n|$ is, therefore,

$$\phi_n(r^2 = x^2 + y^2) = \frac{1}{(2\pi)}\int d^2k e^{\frac{-k^2}{4}} L_n\left(\frac{k^2}{2}\right) e^{-ik.x} = 2(-1)^n e^{-r^2} L_n(2r^2). \tag{4.90}$$

Note that $\phi_0(r^2)$ is precisely the gaussian solution found in the previous section.

To conclude, (4.65) has an infinite number of solutions given by

$$\sum_{n=0}^{\infty} a_n \phi_n(r^2) \tag{4.91}$$

where $\phi_n(r^2)$ is given by (4.90) and each a_n takes values in $\{\lambda_i\}$. We refer to them as GMS solitons (Gopakumar, Minwalla, Strominger).

4.4 D–branes as noncommutative solitons

The GMS solitons are very useful in the description of D–branes first proposed by Harvey, Kraus, Larsen and Martinec [59]. In the bosonic string theory there are D-branes of all dimensions that are however unstable because they have a tachyon on their world volume. In particular, the space filling D25–brane is unstable, and reflects the instability of the bosonic open string in 26 dimensions. Following Sen's conjectures, making in other words some general assumption about the tachyon potential we will see how the lower dimensional D-branes can be interpreted as solitonic excitations of the tachyon potential.

The effective action for the tachyon field, obtained by integrating out the massive string fields, is expected to take the form

$$S = \frac{C}{g_s} \int d^{26}\sqrt{g}\left(\frac{1}{2}f(T)g^{\mu\nu}\partial_\mu T\partial_\nu T - U(T) + \cdots\right) \tag{4.92}$$

where the dots stand for omitted higher derivative terms and terms involving the massless modes. The potential $U(T)$ is a general potential having an unstable extremum at $T = T_*$ (the unstable vacuum), and a minimum that we choose at $T = 0$. The constant $C = g_s\tau_{25}$ is independent of g_s. With these conventions Sen's conjecture requires $U(T = T_*) = 1$: in this way $S(T = T_*) = \tau_{25}V_{26}$. Always according Sen's conjecture the whole action should vanish at the local minimum $U(T = 0) = 0$.

Let us turn on a B field in two spatial directions of the theory, say $x_{1,2}$. In the presence of a B field the action becomes

$$S = \frac{C}{G_s} \int d^{26}\sqrt{G}\left(\frac{1}{2}f(T)G^{\mu\nu}\partial_\mu T\partial_\nu T - U(T) + \cdots\right)_\star \tag{4.93}$$

The advantage of taking the limit of large B field, as we know, is that derivative terms can be neglected. The soliton solutions of (4.93) are then exactly the noncommutative solitons we described earlier. The simplest noncommutative soliton solution to (4.93) is

$$T = T_*\phi_0(r^2) \equiv T_*\,2e^{-r^2/\theta} \tag{4.94}$$

where $r^2 = x_1^2 + x_2^2$ and the dependence on θ has been reintroduced. This is a codimension two object and a candidate for the D23–brane. Let us see the energy of such an object. In the large B field limit the action is

$$S = -\frac{C}{G_s} \int d^{26}\sqrt{G}\,U(T) \tag{4.95}$$

Inserting $T = T_* \phi_0(r^2)$ we have

$$S = -\frac{C\,U(T_*)}{G_s} \int d^{24}x \int d^2x \sqrt{G}\,\phi_0(r) = -\frac{(2\pi\theta)\,C\,U(T_*)}{G_s} \int d^{24}x\,\sqrt{G} \quad (4.96)$$

Using the relation (4.36) between G_s and g_s, that for large B field is

$$G_s = \frac{g_s\sqrt{G}}{2\pi\alpha' B\sqrt{g}}\,, \quad (4.97)$$

and keeping in mind that $\theta = 1/B$, and $U(T_*) = 1$, we have

$$S = -(2\pi)^2\alpha'\,\frac{C}{g_s} \int d^{24}x\sqrt{g} = -(2\pi)^2\alpha'\,\tau_{25}V_{24} \quad (4.98)$$

The tension of the soliton is then

$$\tau_{\text{soliton}} \;=\; (2\pi)^2\alpha'\tau_{25} \;\equiv\; \tau_{23} \quad (4.99)$$

that is exactly the right tension of a D23–brane. The only information we needed to obtain the energy of the noncommutative soliton is the value of U at the extremum T_* that is a part of the potential that we have some information about from Sen's conjecture. Using noncommutativity in additional spatial directions, it is also possible to obtain branes of all even codimension as noncommutative solitons, all of them with the right tensions.

It is important to say that the correct tensions of such soliton is not the only result that matchs with D-brane physics and to remark that the limit of large noncommutativity is not necessary. Harvey et al., indeed, developed a solution generating technique that allow them to find solutions for any value of θ and adding also gauge fields into the action they proved that a n-rank solution found in this way corresponds to a configuration with a tension n times that of a single D-brane and in which a $U(n)$ group is preserved: in other words, n coincident D-branes. Without enter into the subject, we can say that this solution generating technique makes use of partial isometry symmetry of the theory. For future purposes, let us see briefly what we mean with partial isometry. Given a lagrangian invariant under a group of unitary transformations on Hilbert space $U(\mathcal{H})$, then

$$\phi \to U\phi U^\dagger \quad (4.100)$$

with

$$UU^\dagger = U^\dagger U = 1 \quad (4.101)$$

leaves the action invariant and takes solutions of the equation of motion to solutions since

$$\frac{\partial V}{\partial \phi} \to U \left(\frac{\partial V}{\partial \phi} \right) U^\dagger \tag{4.102}$$

But, in order to show that solutions transform into solutions, it is only necessary to use $U^\dagger U = 1$, since it is required that fields transform covariantly.Operators U such that are called isometries because they preserve metric on Hilbert space:

$$\langle \chi | \psi \rangle \to \langle \chi | U^\dagger U | \psi \rangle = \langle \chi | \psi \rangle \tag{4.103}$$

If it is also true that $UU^\dagger = 1$ then U is unitary. In a finite dimensional Hilbert space it is always true but not in a infinite dimensional one. Thus if we find a nonunitary isometry it will still map solutions to solutions, but these solutions will not be related by the global symmetry (or gauge symmetry if we add gauge fields) of the action. So, you will obtain new solutions. A typical example of nonunitary isometry is the shift operator

$$S : |\mathrm{n}\rangle \to |\mathrm{n}{+}1\rangle \quad , \quad S = \sum_{n=0}^{\infty} |\mathrm{n}{+}1\rangle\langle \mathrm{n}|$$

$$\tag{4.105}$$

Obviously,

$$S^\dagger S = 1 \tag{4.106}$$

but

$$SS^\dagger = 1 - P = 1 - |0\rangle\langle 0| \tag{4.107}$$

More generally, $U = S^n$ is a nonunitary isometry and

$$UU^\dagger = 1 - P_n \tag{4.108}$$

with

$$P_n = \sum_{k=0}^{n-1} |\mathrm{k}\rangle\langle \mathrm{k}|$$

(4.109)

To apply the solution generating technique we can start with the trivial constant solution $\phi = \lambda_i I$ with I the identity operator and transforming with $U = S^n$ we obtain the new solution

$$\phi = S^n \lambda_i I S^{\dagger n} = \lambda_i \left(1 - P_n\right) \tag{4.110}$$

This solution will describe a finite energy excitation above the vacuum (λ_i is a global minumum of the potential).

Chapter 5

Vacuum String Field Theory with B Field

We have seen in the previous chapter that if you turn on a B field in open string theory, D-branes can be interpreted as noncommutative solitons of the scalar effective field theory. We have also seen that there is another completely different treatment, Vacuum SFT, in which D-branes are described as nonperturbative, solitonic solutions. In the latter case, however, we are considering *exact* solutions, and we are dealing with string fields, objects containing all string modes, not only the tachyon. In a certain sense, we have two ways to describe Dbranes as noncommutative solitons, one involving the inclusion of the B field and thus noncommutativity, the other related to SFT. One may ask if there is a connection between these noncommutative descriptions of Dbranes. Sugino [28] and Kawano and Takahashi [29] proved that when a B field is turned on, the kinetic term of the SFT action (1.1) is modified only by changing the closed string metric $g_{\mu\nu}$ with the open one $G_{\mu\nu}$, while the three string vertex changes being multiplied by the (cyclically invariant) noncommutative phase factor. Witten [30] and Schnabl [31] proved that a Moyal structure emerges from Witten's star product in the low energy limit.

What are the effects of the B field on the nonperturbative structure of SFT? Beeing the solutions nonperturbative, this problem deserves attention. In this chaper we will repeat the method to find exact solutions to VSFT equation of motion: they can be written down. Wedge-like states and orthogonal projectors will be defined in the presence of a B field. An interesting effect is that a B field can be precious tool to regularize some of the several singularities that arise in VSFT. In particular, without the B field the midpoint of the string is confined on the Dbrane while in presence of the B field is not. We will see that the B field behaves as a regulator which allows us to make in a very natural way the low energy limit without introduce one by hand [43]. But what is really important is that this study gave the inspiration to find a new class of solutions of VSFT which in the low energy limit reduces to all GMS solitons and that could be useful to understand better the full structure of the theory.

First we will write down the SFT vertex in the presence of a constant B field. This result was first found by Sugino [28] and Kawano and Takahashi [29], using the overlap conditions as in [32, 33]. We will give here an alternative derivation of their result, based on the LeClair, Peskin and Preitshopf construction of the vertex $\langle V_3|$. Then we will construct squeezed states solutions with B field, as much as done in the previous chapter. We will show analogies and differences with the $B=0$ case by constructing wedge-like states and orthogonal projectors, and investigating the behaviour of the string midpoint with $B \neq 0$. Finally we construct a series of orthogonal projectors that in the low energy limit give exactly the GMS solitons.

5.1 String Field Theory with B field

It is useful to recall the form of the propagator and of the string field expansion when the B field is turned on. They are

$$\begin{aligned}\langle X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\rangle &= -\alpha'\Big[g^{\mu\nu}\ln|z-z'| - g^{\mu\nu}\ln|z-\bar{z}'| \\ &\quad + G^{\mu\nu}\ln|z-\bar{z}'|^2 + \frac{1}{2\pi\alpha'}\theta^{\mu\nu}\ln\frac{z-\bar{z}'}{\bar{z}-z'} + D^{\mu\nu}\Big]\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}X^\mu &= x_0^\mu + \alpha'\Big[(E^{-1})^{\mu\nu}g_{\nu\rho}p^\rho \ln\bar{z} + (E^{-1T})^{\mu\nu}g_{\nu\rho}p^\rho\ln z\Big] \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\Big[(E^{-1})^{\mu\nu}g_{\nu\rho}\alpha_n^\rho\bar{z}^{-n} + (E^{-1T})^{\mu\nu}g_{\nu\rho}\alpha_n^\rho z^{-n}\Big]\end{aligned} \tag{5.2}$$

where $E_{\mu\nu} = g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu}$. The two and three string vertices were defined as correlation functions of string fields operators computed on the upper half plane. For instance the three string vertex was

$$\langle A, B * C\rangle = \langle\, f_1^D \circ \mathcal{O}_A(0)\, f_2^D \circ \mathcal{O}_B(0)\, f_3^D \circ \mathcal{O}_C(0)\,\rangle_D \tag{5.3}$$

Inverting eq.(5.2) to obtain α_{-n}^μ as function of $X^\mu(z)$ we find

$$\alpha_{-n}^\mu = \oint \frac{dz}{2\pi i} z^{-n}(E)^{\mu\nu}g_{\nu\rho}\partial_z X^\rho(z) \tag{5.4}$$

The operators $\mathcal{O}$ are products of polynomials in the creation operators α_{-n}^{μ} with exponentials $e^{ip\cdot X}$. Under conformal transformations the latter change as

$$f_i[\alpha_{-n}^{\mu}] = \oint \frac{dz}{2\pi i} z^{-n}(h_1'(z))(E^{-1})^{\mu\nu} g_{\nu\rho}\partial_z X^{\rho}(f_i(z))$$

$$f_i[e^{ip\cdot X(0)}] = |f_i'(0)|^{p^2/2}\, e^{ip\cdot X(f_i(0))}$$

The contraction of any two α_{-n}^{μ} is then

$$f_i[\ldots\alpha_{-m}^{\mu}\ldots]f_j[\ldots\alpha_{-n}^{\nu}\ldots]$$

$$= \oint \frac{dz}{2\pi i} z^{-n}(f_i'(z)) \oint \frac{dw}{2\pi i} w^{-m}(f_j'(w))$$

$$\cdot\,(E^{-1})^{\mu\nu} g_{\nu\rho}(E^{-1})^{\nu\eta} g_{\eta\sigma}\langle \partial X^{\rho}(f_i(z))\partial X^{\sigma}(f_j(w))\rangle$$

$$= \oint \frac{dz}{2\pi i} z^{-n}(f_i'(z)) \oint \frac{dw}{2\pi i} w^{-m}(f_j'(w))(E^{-1})^{\mu\nu} g_{\nu\rho}(E^{-1})^{\nu\eta} g_{\eta\sigma}\frac{-g^{\rho\sigma}}{(f_i(z)-f_j(w))^2}$$

$$= \oint \frac{dz}{2\pi i} z^{-n}(f_i'(z)) \oint \frac{dw}{2\pi i} w^{-m}(f_j'(w))\frac{-G^{\mu\nu}}{(f_i(z)-f_j(w))^2} \tag{5.5}$$

where the correlation function $\langle\, \partial X^{\mu}\partial X^{\nu}\,\rangle$ is obtained deriving eq.(5.1). We see that the modification induced by the B field on the part of the vertex with indices $m, n > 0$ is completely taken into account by substituting the *closed string metric* $g_{\mu\nu}$ with the *open string metric* $G_{\mu\nu}$. The same modification occurs in the kinetic term. Things are different when both the mode indices of the Neumann coefficients N_{MN} are zero: N_{00}. This corresponds to the contraction of two exponentials that are both forced to belong to the real axis. To do this contraction we need then the propagator for points belonging to the boundary of the worldsheet. It was given in (4.18) and is

$$\langle X^{\mu}(\tau)X^{\nu}(\tau')\rangle = -\alpha' G^{\mu\nu}\ln(\tau-\tau')^2 + \frac{i}{2}\theta^{\mu\nu}\epsilon(\tau-\tau') \tag{5.6}$$

With this propagator the matrix element of exponentials becomes, up to normalization factors

$$\left\langle \prod_i e^{ip_i\cdot X(f_i(0))} \right\rangle$$

$$= \exp\Big[\sum_{i<j} p_i^{\mu} G_{\mu\nu} p_j^{\nu} \log|f_i(0)-f_j(0)|\Big]\exp\Big[\frac{i}{2}\sum_{i<j} p_{\mu}\theta^{\mu\nu}p_{\nu}\,\epsilon(f_i(0)-f_j(0))\Big]$$

The B modified $|V_3\rangle$ is then

$$\begin{aligned}|V_3\rangle &= \delta(p^{(1)}+p^{(2)}+p^{(3)})|\Omega_1\rangle\otimes|\Omega_3\rangle\otimes|\Omega_3\rangle\\ &\times\ \exp\left(\sum_{M,N=0}^{\infty}\frac{1}{2}G_{\mu\nu}\alpha_{-n}^{(r)\mu}N_{NM}^{rs}\alpha_{-m}^{(s)\nu}+\sum_{m=0,n=1}^{\infty}c_{-n}^{(r)}X_{mn}^{rs}b_{-m}^{(s)}-\frac{i}{2}\theta_{\mu\nu}p^{(1)\mu}p^{(2)\nu}\right)\end{aligned}$$

5.2 The interaction vertex with B field

Our next goal is to find the form of the coefficients V_{MN}^{rs} when a constant B field is switched on. We start from the simplest case, i.e. when B is nonvanishing in the two space directions, say the 24–th and 25–th ones. Let us denote these directions with the Lorentz indices α and β. Then, as we saw in the first chapter, in these two directions we have a new effective metric $G_{\alpha\beta}$, the open string metric, as well as an effective antisymmetric parameter $\theta_{\alpha\beta}$, given by

$$\begin{aligned}G^{\alpha\beta} &= \left(\frac{1}{\eta+2\pi\alpha' B}\eta\frac{1}{\eta-2\pi\alpha' B}\right)^{\alpha\beta},\\ \theta^{\alpha\beta} &= -(2\pi\alpha')^2\left(\frac{1}{\eta+2\pi\alpha' B}B\frac{1}{\eta-2\pi\alpha' B}\right)^{\alpha\beta}\end{aligned}$$

The three string vertex is modified only in the 24-th and 25-th direction, which, in view of the subsequent D-brane interpretation, we call the transverse directions. We split the three string vertex into the tensor product of the perpendicular part and the parallel part

$$|V_3\rangle = |V_{3,\perp}\rangle\otimes|V_{3,\|}\rangle \tag{5.7}$$

The parallel part is the same as in the ordinary case and will not be re-discussed here. On the contrary we will describe in detail the perpendicular part of the vertex. We rewrite the exponent E as $E=E_{\|}+E_{\perp}$, according to the above splitting. $E_{\perp}$ will be modified as follows

$$\begin{aligned}E_{\perp}\ \to\ E'_{\perp} = \sum_{r,s=1}^{3}\Bigg(&\frac{1}{2}\sum_{m,n\geq 1}G_{\alpha\beta}a_m^{(r)\alpha\dagger}V_{mn}^{rs}a_n^{(s)\beta\dagger}+\sum_{n\geq 1}G_{\alpha\beta}p_{(r)}^{\alpha}V_{0n}^{rs}a_n^{(s)\beta\dagger}\\ &+\frac{1}{2}G_{\alpha\beta}p_{(r)}^{\alpha}V_{00}^{rs}p_{(s)}^{\beta}+\frac{i}{2}\sum_{r<s}p_{\alpha}^{(r)}\theta^{\alpha\beta}p_{\beta}^{(s)}\Bigg)\end{aligned} \tag{5.8}$$

where we set $\alpha'=1$. Next, as far as the zero modes are concerned, we pass from the momentum to the oscillator basis, [32, 33]. As before we define

$$a_0^{(r)\alpha}=\frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha}-i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha},\qquad a_0^{(r)\alpha\dagger}=\frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha}+i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \tag{5.9}$$

where $\hat{p}^{(r)\alpha}, \hat{x}^{(r)\alpha}$ are the zero momentum and position operator of the r–th string, and we have kept the 'gauge' parameter b of ref.[37] ($b \sim \alpha'$). It is understood that $p^{(r)\alpha} = G^{\alpha\beta}p^{(r)}_\beta$. We have

$$[a_0^{(r)\alpha}, a_0^{(s)\beta\dagger}] = G^{\alpha\beta}\delta^{rs} \tag{5.10}$$

Denoting by $|\Omega_{b,\theta}\rangle$ the oscillator vacuum ($a_0^\alpha|\Omega_{b,\theta}\rangle = 0$), the relation between the momentum basis and the oscillator basis is defined by

$$|p^{24}\rangle_{123} \otimes |p^{25}\rangle_{123} \equiv |\{p^\alpha\}\rangle_{123} =$$

$$\left(\frac{b}{2\pi\sqrt{\det G}}\right)^{\frac{3}{2}} \exp\left[\sum_{r=1}^{3}\left(-\frac{b}{4}p^{(r)}_\alpha G^{\alpha\beta}p^{(r)}_\beta + \sqrt{b}a_0^{(r)\alpha\dagger}p^{(r)}_\alpha - \frac{1}{2}a_0^{(r)\alpha\dagger}G_{\alpha\beta}a_0^{(r)\beta\dagger}\right)\right]|\Omega_{b,\theta}\rangle$$

Now we insert this equation inside $E'_\perp$ and try to eliminate the momenta along the perpendicular directions by integrating them out. To this end we rewrite $E'_\perp$ in the following way and, for simplicity, drop all the labels α, β and r, s:

$$E'_\perp = \frac{1}{2}\sum_{m,n\geq 1} a_m^\dagger G V_{mn} a_n^\dagger + \sum_{n\geq 1} pV_{0n}a_n^\dagger + \frac{1}{2}p\left[G^{-1}(V_{00} + \frac{b}{2}) + \frac{i}{2}\theta\epsilon\chi\right]p - \sqrt{b}pa_0^\dagger + \frac{1}{2}a_0^\dagger G a_0^\dagger$$

where we have set $\theta^{\alpha\beta} = \epsilon^{\alpha\beta}\theta$ and introduced the matrices ϵ with entries $\epsilon^{\alpha\beta}$ and χ with entries

$$\chi^{rs} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{5.11}$$

At this point we impose momentum conservation. There are three distinct ways to do that and eventually one has to (multiplicatively) symmetrize with respect to them. Let us start by setting $p_3 = -p_1 - p_2$ in $E'_\perp$ and obtain an expression of the form

$$p\, X_{00}\, p + \sum_{N\geq 0} p\, Y_{0N}\, a_N^\dagger + \sum_{M,N\geq 0} a_M^\dagger\, Z_{MN}\, a_N^\dagger \tag{5.12}$$

where, in particular, X_{00} is given by

$$X_{00}^{\alpha\beta,rs} = G^{\alpha\beta}\,(V_{00} + \frac{b}{2})\,\eta^{rs} + i\frac{\theta}{4}\,\epsilon^{\alpha\beta}\,\epsilon^{rs} \tag{5.13}$$

Here the indices r, s take only the values 1,2, and

$$\eta = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.14}$$

Now, as usual, we redefine p so as eliminate the linear term in (5.12). At this point we can easily perform the Gaussian integration over $p_{(1)}, p_{(2)}$, while the remnant of (5.12) will be expressed in terms of the inverse of X_{00}:

$$\left(X_{00}^{-1}\right)^{\alpha\beta,rs} = \frac{2A^{-1}}{4a^2+3}\left(\frac{3}{2}\, G^{\alpha\beta}\,(\eta^{-1})^{rs} - 2i\,a\,\hat{\epsilon}^{\alpha\beta}\,\epsilon^{rs}\right) \tag{5.15}$$

where

$$A = V_{00} + \frac{b}{2}, \qquad a = \frac{\theta}{4A}\sqrt{\mathrm{Det}G}, \qquad \epsilon^{\alpha\beta} = \sqrt{\mathrm{Det}G}\,\hat{\epsilon}^{\alpha\beta} \tag{5.16}$$

Let us use henceforth for the B field the explicit form

$$B_{\alpha\beta} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \tag{5.17}$$

so that

$$\mathrm{DetG} = \left(1 + (2\pi B)^2\right)^2, \qquad \theta\sqrt{\mathrm{DetG}} = -(2\pi)^2 B, \qquad a = -\frac{\pi^2}{A}B \tag{5.18}$$

Now one has to symmetrize with respect to the three possibilities of imposing the momentum conservation. Remembering the factors due to integration over the momenta and collecting the results one gets for the three string vertex in the presence of a B field

$$|V_3\rangle' = |V_{3,\perp}\rangle' \otimes |V_{3,\|}\rangle \tag{5.19}$$

$|V_{3,\|}\rangle$ is the same as in the ordinary case (without B field), while

$$|V_{3,\perp}\rangle' = K_2\, e^{-E'}|\tilde{0}\rangle \tag{5.20}$$

with

$$K_2 = \frac{\sqrt{2\pi b^3}}{A^2(4a^2+3)}(\mathrm{Det}G)^{1/4}, \tag{5.21}$$

$$E' = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{M,N\geq 0} a_M^{(r)\alpha\dagger}\mathcal{V}^{rs}_{\alpha\beta,MN}a_N^{(s)\beta\dagger} \tag{5.22}$$

and $|\tilde{0}\rangle = |0\rangle \otimes |\Omega_{b,\theta}\rangle$. The coefficients $\mathcal{V}^{\alpha\beta,rs}_{MN}$ are given by

$$\mathcal{V}^{\alpha\beta,rs}_{00} = G^{\alpha\beta}\delta^{rs} - \frac{2A^{-1}b}{4a^2+3}\left(G^{\alpha\beta}\phi^{rs} - ia\hat{\epsilon}^{\alpha\beta}\chi^{rs}\right) \tag{5.23}$$

$$\mathcal{V}^{\alpha\beta,rs}_{0n} = \frac{2A^{-1}\sqrt{b}}{4a^2+3}\sum_{t=1}^{3}\left(G^{\alpha\beta}\phi^{rt} - ia\hat{\epsilon}^{\alpha\beta}\chi^{rt}\right)V^{ts}_{0n} \tag{5.24}$$

$$\mathcal{V}^{\alpha\beta,rs}_{mn} = G^{\alpha\beta}V^{rs}_{mn} + \frac{2A^{-1}}{4a^2+3}\sum_{t,v=1}^{3}V^{rv}_{m0}\left(G^{\alpha\beta}\phi^{vt} - ia\hat{\epsilon}^{\alpha\beta}\chi^{vt}\right)V^{ts}_{0n} \tag{5.25}$$

where, by definition, $V_{0n}^{rs} = V_{n0}^{sr}$, and

$$\phi = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \tag{5.26}$$

while the matrix χ has been defined above (5.11). These two matrices satisfy the algebra

$$\chi^2 = -2\phi, \qquad \phi\chi = \chi\phi = \frac{3}{2}\chi, \qquad \phi^2 = \frac{3}{2}\phi \tag{5.27}$$

To end this section we would like to notice that the above results can be easily extended to the case in which the transverse directions are more than two (i.e. the 24–th and 25–th ones) and even. The canonical form of the transverse B field is

$$B_{\alpha\beta} = \begin{pmatrix} \begin{matrix} 0 & B_1 \\ -B_1 & 0 \end{matrix} & 0 & \dots \\ 0 & \begin{matrix} 0 & B_2 \\ -B_2 & 0 \end{matrix} & \dots \\ \dots & \dots & \dots \end{pmatrix} \tag{5.28}$$

It is not hard to see that each couple of conjugate transverse directions under this decomposition, can be treated in a completely independent way. The result is that each couple of directions $(26-i, 25-i)$, corresponding to the eigenvalue B_i, will be characterized by the same formulas (5.23, 5.24, 5.25) above with B replaced by B_i. The properties of the new coefficients $\mathcal{V}_{MN}^{\alpha\beta,rs}$ are reported in Appendix D

5.3 Lump solutions with B field

A squeezed state in the present context is written as

$$|S\rangle = |S_\perp\rangle \otimes |S_{||}\rangle \tag{5.29}$$

where $|S_{||}\rangle$ has the ordinary form that we presented in the previous chapter, and is treated in the usual way, while

$$\langle S_\perp| = \mathcal{N}^2 \langle\tilde{0}| \exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^\alpha\, \tilde{\mathcal{S}}_{\alpha\beta,MN}\, a_N^\beta\right) \tag{5.30}$$

$$|S_\perp\rangle = \mathcal{N}^2 \exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha\dagger}\, \mathcal{S}_{\alpha\beta,MN}\, a_N^{\beta\dagger}\right) |\tilde{0}\rangle \tag{5.31}$$

where $|\tilde{0}\rangle = |\Omega_{b,\theta}\rangle \otimes |0\rangle$. Here we have written down both bra and ket in order to stress the difference with the $B = 0$ case, which stems from the fact that, in view of (D.26), we assume $C'\mathcal{S}^{\alpha\beta}C' = (\mathcal{S}^{\alpha\beta})^* = \mathcal{S}^{\beta\alpha}$. The $*$ product of two such states, labeled $_1$ and $_2$, is carried out in the same way as in the ordinary case, see Chapter 2. Therefore we limit ourselves to writing down the result

$$|S'_\perp\rangle = |S_{1,\perp}\rangle * |S_{2,\perp}\rangle = \frac{K_2\,(\mathcal{N}_1\mathcal{N}_2)^2}{\mathrm{DET}(\mathbf{I} - \Sigma\mathcal{V})^{1/2}} \exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha\dagger}\mathcal{S}'_{\alpha\beta,MN}a_N^{\beta\dagger}\right)|\tilde{0}\rangle \tag{5.32}$$

where, in matrix notation which includes both the indices N, M and α, β,

$$\mathcal{S}' = \mathcal{V}^{11} + (\mathcal{V}^{12}, \mathcal{V}^{21})(\mathbf{I} - \Sigma\mathcal{V})^{-1}\Sigma\begin{pmatrix}\mathcal{V}^{21}\\ \mathcal{V}^{12}\end{pmatrix} \tag{5.33}$$

In RHS of these equations

$$\Sigma = \begin{pmatrix}\tilde{\mathcal{S}}_1 & 0\\ 0 & \tilde{\mathcal{S}}_2\end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix}\mathcal{V}^{11} & \mathcal{V}^{12}\\ \mathcal{V}^{21} & \mathcal{V}^{22}\end{pmatrix}, \tag{5.34}$$

and $\mathbf{I}^{\alpha,rs}_{\beta,MN} = \delta^\alpha_\beta\,\delta_{MN}\,\delta^{rs}$, $r,s = 1,2$. DET is the determinant with respect to all indices. In order to avoid confusion we remind the reader that we work with three kind of indices: $r,s = 1,2,3$ for the three strings, $\alpha,\beta = 24,25$ for the space-time direction where the B field is switched on, and $m,n = 1,\ldots\infty$ for the string modes. We adopt the following notation for different identity operators:

$$\begin{aligned} \mathbf{I}^{\alpha,rs}_{\beta,MN} &= \delta^\alpha_\beta\,\delta_{MN}\,\delta^{rs}\\ \mathbb{I}^{\alpha}_{\beta,MN} &= \delta^\alpha_\beta\,\delta_{MN}\\ \mathbf{1}^\alpha_\beta &= \delta^\alpha_\beta \end{aligned} \tag{5.35}$$

To reach the form (5.33) one has to use cyclicity of $\mathcal{V}^{rs}$ (see Appendix D). The expression of $\mathcal{S}'$ is in fact a series, therefore some kind of condition on the coefficients $\mathcal{S}_i$ must be satisfied in order for it to make sense. The squeezed states $\mathcal{S}$ satisfying this condition form a subalgebra of the algebra defined by the $*$ product.

Let us now discuss the squeezed state solution of the equation $|\Psi\rangle * |\Psi\rangle = |\Psi\rangle$ in the matter sector. In order for this to be satisfied with the above states $|S\rangle$, we must first impose

$$\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}' \equiv \mathcal{S}$$

and then suitably normalize the resulting state. Then (5.33) becomes an equation for $\mathcal{S}$, i.e.

$$\tilde{\mathcal{S}} = \mathcal{V}^{11} + (\mathcal{V}^{12}, \mathcal{V}^{21})(\mathbf{I} - \Sigma\mathcal{V})^{-1}\Sigma\begin{pmatrix}\mathcal{V}^{21}\\ \mathcal{V}^{12}\end{pmatrix} \tag{5.36}$$

where $\Sigma, \mathcal{V}$ are the same as above with $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}$. Eq.(5.36) has an obvious (formal) solution by iteration. However we saw that it is possible to obtain the solution in compact form by 'abelianizing' the problem. Notwithstanding the differences with that case, it is possible to reproduce the same trick on eq.(5.36), thanks to (D.23). We set

$$C'\mathcal{V}^{rs} = \mathcal{X}^{rs} \quad \text{and} \quad C'\mathcal{S} = \mathcal{T} \ ,$$

and assume that

$$[\mathcal{X}^{rs}, \mathcal{T}] = 0$$

(of course this has to be checked *a posteriori*). Notice however that we cannot assume that C' commutes with $\mathcal{S}$, but we assume that

$$C'\mathcal{S}C' = \tilde{\mathcal{S}} \ .$$

By multiplying (5.36) from the left by C' we get:

$$\mathcal{T} = \mathcal{X}^{11} + (\mathcal{X}^{12}, \mathcal{X}^{21})(\mathbf{I} - \Sigma\mathcal{V})^{-1} \begin{pmatrix} \mathcal{T}\mathcal{X}^{21} \\ \mathcal{T}\mathcal{X}^{12} \end{pmatrix} \tag{5.37}$$

For instance $\tilde{\mathcal{S}}\mathcal{V}^{12} = \tilde{\mathcal{S}}C'C'\mathcal{V}^{12} = \mathcal{T}\mathcal{X}^{12}$, etc. In the same way,

$$(\mathbf{I} - \Sigma\mathcal{V})^{-1} = \begin{pmatrix} \mathbb{I} - \mathcal{T}\mathcal{X}^{11} & -\mathcal{T}\mathcal{X}^{12} \\ -\mathcal{T}\mathcal{X}^{21} & \mathbb{I} - \mathcal{T}\mathcal{X}^{11} \end{pmatrix}^{-1}$$

where $\mathbb{I}^{\alpha}_{\beta,MN} = \delta^{\alpha}_{\beta}\,\delta_{MN}$. Now all the entries are commuting matrices, so the inverse can be calculated straight away.

From now on everything is the same as in [34, 37], therefore we limit ourselves to a quick exposition. Using (D.24) and (D.25), one arrives at an equation only in terms of $\mathcal{T}$ and $\mathcal{X} \equiv \mathcal{X}^{11}$:

$$(\mathcal{T} - \mathbb{I})(\mathcal{X}\mathcal{T}^2 - (\mathbb{I} + \mathcal{X})\mathcal{T} + \mathcal{X}) = 0 \tag{5.38}$$

This gives two solutions:

$$\mathcal{T} = \mathbb{I} \tag{5.39}$$

$$\mathcal{T} = \frac{1}{2\mathcal{X}}\left(\mathbb{I} + \mathcal{X} - \sqrt{(\mathbb{I} + 3\mathcal{X})(\mathbb{I} - \mathcal{X})}\right) \tag{5.40}$$

The third solution, with a + sign in front of the square root, is not acceptable. In both cases we see that the solution commutes with $\mathcal{X}^{rs}$. Naturally we are talking about solutions of the abelianized eq.(5.37). The true solution we are looking for is, in both cases, $\mathcal{S} = C'\mathcal{T}$.

As for (5.39), it is easy to see that it leads to the identity state. Therefore, from now on we will consider (5.40) alone.

Now, let us deal with the normalization of $|S_\perp\rangle$. Imposing $|S_\perp\rangle * |S_\perp\rangle = |S_\perp\rangle$ we find

$$\mathcal{N}^2 = K_2^{-1}\,\mathrm{DET}\,(\mathbf{I} - \Sigma\mathcal{V})^{1/2}$$

Replacing in it the solution one finds

$$\mathrm{DET}(\mathbf{I} - \Sigma\mathcal{V}) = \mathrm{Det}\,((\mathbb{I} - \mathcal{X})(\mathbb{I} + \mathcal{T})) \tag{5.41}$$

Det denotes the determinant with respect to the indices α, β, M, N. Using this equation and (5.21), and borrowing from Chapter 1 the expression for $|S_{||}\rangle$, one finally gets for the 23-dimensional tachyonic lump:

$$\begin{aligned} |S\rangle \;=\; & \left\{\det(1-X)^{1/2}\det(1+T)^{1/2}\right\}^{24} \exp\left(-\frac{1}{2}\eta_{\bar\mu\bar\nu}\sum_{m,n\geq 1} a_m^{\bar\mu\dagger} S_{mn} a_n^{\bar\nu\dagger}\right)|0\rangle \otimes \qquad (5.42)\\ & \frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\mathrm{Det}G)^{1/4}}\left(\mathrm{Det}(\mathbb{I}-\mathcal{X})^{1/2}\mathrm{Det}(\mathbb{I}+\mathcal{T})^{1/2}\right)\exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha\dagger}\mathcal{S}_{\alpha\beta,MN}a_N^{\beta\dagger}\right)|\tilde{0}\rangle, \end{aligned}$$

where $\mathcal{S} = C'\mathcal{T}$ and $\mathcal{T}$ is given by (5.40). The quantities in the first line are defined in ref.[37] with $\bar\mu, \bar\nu = 0, \ldots 23$ denoting the parallel directions to the lump.

The value of the action corresponding to (5.42) is easily calculated

$$\begin{aligned} \mathcal{S}_{\mathcal{S}} \;=\; & \mathcal{K}\frac{V^{(24)}}{(2\pi)^{24}}\left\{\det(1-X)^{3/4}\det(1+3X)^{1/4}\right\}^{24} \\ & \cdot\frac{A^4(3+4a^2)^2}{2\pi b^3(\mathrm{Det}G)^{1/2}}\,\mathrm{Det}(\mathbb{I}-\mathcal{X})^{3/4}\mathrm{Det}(\mathbb{I}+3\mathcal{X})^{1/4} \end{aligned} \tag{5.43}$$

where $V^{(24)}$ is the volume along the parallel directions and $\mathcal{K}$ is the constant of eq.(E.44).

Finally, let $\mathfrak{e}$ denote the energy per unit volume, which coincides with the brane tension when $B = 0$. Then one can compute the ratio of the D23–brane energy density $\mathfrak{e}_{23}$ to the D25–brane energy density $\mathfrak{e}_{25}$;

$$\frac{\mathfrak{e}_{23}}{\mathfrak{e}_{25}} = \frac{(2\pi)^2}{(\mathrm{Det}G)^{1/4}}\cdot\mathcal{R} \tag{5.44}$$

$$\mathcal{R} = \frac{A^4(3+4a^2)^2}{2\pi b^3(\mathrm{Det}G)^{1/4}}\,\frac{\mathrm{Det}(\mathbb{I}-\mathcal{X})^{3/4}\mathrm{Det}(\mathbb{I}+3\mathcal{X})^{1/4}}{\det(1-X)^{3/2}\det(1+3X)^{1/2}} \tag{5.45}$$

Since $\mathcal{R}$ equals 1 (see Appendix D), this equation is exactly what is expected for the ratio of a flat static D25–brane action and a D23–brane action per unit volume in the presence of the B field (5.17). In fact the DBI Lagrangian for a flat static Dp–brane is,

$$\mathcal{L}_{DBI} = \frac{1}{g_s(2\pi)^p}\sqrt{\mathrm{Det}(1+2\pi B)} \tag{5.46}$$

where g_s is the closed string coupling. Substituting (5.17) and taking the ratio the claim follows.

Let us briefly discuss the generalization of the above results to lower dimensional lumps. As remarked at the end of section 2, every couple of transverse directions corresponding to an eigenvalue B_i of the field B can be treated in the same way as the 24–th and 25–th directions. One has simply to replace in the above formulas B with B_i. The derivation of the above formulas for the case of $25-2i$ dimensional lumps is straightforward.

Switching on a constant B field on VSFT does not obstruct the possibility to find exact results. On the contrary, we have found that (matter) squeezed states representing tachyonic lumps are still solutions of the equations of motion, and that we can give compact explicit formulas for these solutions, much like in the $B=0$ case. Indeed these are still interpretable as (lower dimensional) D–branes.

5.4 Wedge state with B field and deconfinement of string midpoint

In this section we present a couple of results which are natural extensions of analogous results with $B=0$, namely the possibility of defining wedge-like states and orthogonal projectors. But we investigate also a particular phenomenon, the confinement or not of the midpoint of the string, where the presence of the B field determines makes a strong difference with the $B=0$ case.

5.4.1 Wedge state with B

We saw that wedge states are geometrical states in that they can be defined simply by means of a conformal map of the unit disk to a portion of it. They are spanned by an integer n: the limit for $n \to \infty$ is the sliver $|\Xi\rangle$, which is interpreted as the D25–brane. Wedge states also admit a representation in terms of oscillators $a_n^\dagger$ with

$n > 0$,

$$|W_n\rangle = \mathcal{N}_n^{26} e^{-\frac{1}{2}a^\dagger C T_n a^\dagger}|0\rangle \tag{5.47}$$

which is specified by the matrix T_n, $n > 1$. It can be shown that, see [55], T_n satisfy a recursive relation which can be solved in terms of the matrix T characterizing the sliver state ($T = CS$, S being the sliver matrix). The normalization $\mathcal{N}$ can also be derived from a recursion relation. Since all these results are essentially based on equations which are generalized to the case when a B–field is present and are in fact reported in Appendix D, it is easy to deduce that analogous results hold also when a B field is turned on.

The generalized wedge states will be the tensor product of a factor like (5.47) for the the 24 directions in which the components of the B field are zero and

$$|\mathcal{W}_n\rangle = \mathcal{N}_n^2 e^{-\frac{1}{2}a^\dagger C' \mathcal{T}_n a^\dagger}|\tilde{0}\rangle \tag{5.48}$$

for the other two directions. From now on we will be concerned with the determination of $\mathcal{T}_n$ and $\mathcal{N}_n$. We start from the hypothesis that

$$[\mathcal{X}^{rs}, \mathcal{T}_n] = 0, \qquad C'\mathcal{T}_n = \tilde{\mathcal{T}}_n C' \tag{5.49}$$

whose consistency we will verify a posteriori.

Now we define $\mathcal{T}_2 = 0$ and the sequence of states

$$|\mathcal{W}_{n+1}\rangle = |\mathcal{W}_n\rangle * |\mathcal{W}_2\rangle \tag{5.50}$$

Using eq.(5.32) and (5.36), with $\Sigma = \begin{pmatrix} C'\tilde{\mathcal{T}}_n & 0 \\ 0 & 0 \end{pmatrix}$, we find the recursion relation

$$\begin{aligned} \mathcal{T}_{n+1} &= \mathcal{X}^{11} + (\mathcal{X}^{12}, \mathcal{X}^{21}) \left(1 - \begin{pmatrix} \mathcal{T}_n\mathcal{X}^{11} & \mathcal{T}_n\mathcal{X}^{12} \\ 0 & 0 \end{pmatrix}\right)^{-1} \begin{pmatrix} \mathcal{T}_n\mathcal{X}^{21} \\ 0 \end{pmatrix} \\ &= \mathcal{X}\frac{1-\mathcal{T}_n}{1-\mathcal{T}_n\mathcal{X}} \end{aligned} \tag{5.51}$$

where use has been made of the second equation in (D.27). Solving this recursion relation, [55], we can write

$$\mathcal{T}_n = \frac{\mathcal{T} + (-\mathcal{T})^{n-1}}{1-(-\mathcal{T})^n} \tag{5.52}$$

Notice that this sequence of states can be extended to $|\mathcal{W}_1\rangle$ defined by $\mathcal{T}_1 = 1$. An analogous recursion relation applies also to the normalization factors. Once solved, it gives

$$\mathcal{N}_n = K_2^{-1}\det\left(\frac{1-\mathcal{T}^2}{1-(-\mathcal{T})^{n+1}}\right)^{1/2} \tag{5.53}$$

The constant K_2 is defined in eq.(2.19) of [75]. The relations (5.49) are now easy to verify.

The limit of $\mathcal{T}_n$ as $n \to \infty$ is $\mathcal{T}$ (i.e. the deformation of the lump), provided $\lim \mathcal{T}^n = 0$. In turn, the latter holds if the eigenvalues of $\mathcal{T}$ are in absolute value less then 1, as those of T are.

5.4.2 Orthogonal projectors

In the presence of a background B field it is also possible to construct other projectors than the one shown in (5.42). To show this we follow Chapter 2. The treatment is very close to what can be found there, and the main purpose of this subsection is to stress some differences with it. As usual we will be concerned only with the transverse part of the projectors, the parallel being exactly the same as in (2.62), and will denote the transverse part of the solution (5.42) by $|\mathcal{S}_\perp\rangle$.

We start by introducing the projection operators parallel to that ones of eq.(2.62)

$$\rho_1 = \frac{1}{(\mathbb{I}+\mathcal{T})(\mathbb{I}-\mathcal{X})}\left[\mathcal{X}^{12}(\mathbb{I}-\mathcal{T}\mathcal{X})+\mathcal{T}(\mathcal{X}^{21})^2\right] \tag{5.54}$$

$$\rho_2 = \frac{1}{(\mathbb{I}+\mathcal{T})(\mathbb{I}-\mathcal{X})}\left[\mathcal{X}^{21}(\mathbb{I}-\mathcal{T}\mathcal{X})+\mathcal{T}(\mathcal{X}^{12})^2\right] \tag{5.55}$$

They satisfy

$$\rho_1^2 = \rho_1, \qquad \rho_2^2 = \rho_2, \qquad \rho_1 + \rho_2 = \mathbb{I} \tag{5.56}$$

i.e. they project onto orthogonal subspaces. Moreover, if we use the superscript ^{T} to denote transposition with respect to the indices N, M and α, β, we have

$$\rho_1^T = \tilde{\rho}_1 = C'\rho_2 C', \qquad \rho_2^T = \tilde{\rho}_2 = C'\rho_1 C'. \tag{5.57}$$

Now, in order to find another solution of the equation $|\Psi\rangle * |\Psi\rangle = |\Psi\rangle$, distinct from $|\mathcal{S}_\perp\rangle$, we make the following ansatz:

$$|\mathcal{P}_\perp\rangle = \left(-\xi\tau a^\dagger\ \zeta\cdot a^\dagger + \kappa\right)|\mathcal{S}_\perp\rangle \tag{5.58}$$

where $\xi = \{\xi_N^\alpha\}$, $\zeta = C'\xi$ and τ is the matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ acting on the indices α and β. κ is a constant to be determined and ξ is required to satisfy the constraints:

$$\rho_1\xi = 0, \qquad \rho_2\xi = \xi, \qquad \text{i.e.} \quad \tilde{\rho}_1\zeta = \zeta, \qquad \tilde{\rho}_2\zeta = 0 \tag{5.59}$$

Using (5.56,5.59) it is simple to prove that

$$\zeta^T f(\mathcal{X}^{rs},\mathcal{T})\,\xi = 0, \qquad \xi^T f(\tilde{\mathcal{X}}^{rs},\tilde{\mathcal{T}})\zeta = 0$$

for any function f. Now, imposing $|\mathcal{P}_\perp\rangle * |\mathcal{S}_\perp\rangle = 0$ we determine κ:

$$\kappa = -\frac{1}{2}\zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{11}\xi - \frac{1}{2}\xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{11}\tau\zeta \tag{5.60}$$

where

$$\mathcal{K} = \mathbb{I} - \mathcal{T}\mathcal{X}, \qquad \mathcal{V} = \begin{pmatrix} \mathcal{V}^{11} & \mathcal{V}^{12} \\ \mathcal{V}^{21} & \mathcal{V}^{22} \end{pmatrix} \tag{5.61}$$

Next we compute $|\mathcal{P}_\perp\rangle * |\mathcal{P}_\perp\rangle$. This gives

$$|\mathcal{P}_\perp\rangle * |\mathcal{P}_\perp\rangle = \frac{1}{2}\left(\xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{12}\tau\zeta + \zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{21}\xi\right)\left(-a^\dagger\tau\xi\, a^\dagger\cdot\zeta + \kappa\right)|\mathcal{S}_\perp\rangle \tag{5.62}$$

where use has been made of the identities

$$\begin{aligned}
\zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{11}\xi &= \zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{22}\xi = -\zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{12}\xi = \xi^T\tau\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\xi \\
\xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{11}\tau\zeta &= \xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{22}\tau\zeta = -\xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{21}\tau\zeta = \zeta^T\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\tau\zeta \\
\xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{12}\tau\zeta &= \zeta^T\frac{1}{\mathbb{I}-\mathcal{T}^2}\tau\zeta, \qquad \zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{21}\xi = \xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\xi
\end{aligned} \tag{5.63}$$

Similarly one can prove that

$$\zeta^T\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\tau\zeta = \xi^T\tau\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\xi, \qquad \zeta^T\frac{1}{\mathbb{I}-\mathcal{T}^2}\tau\zeta = \xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\xi \tag{5.64}$$

So, in order for $|\mathcal{P}_\perp\rangle$ to be a projector, we have to impose

$$\left(\xi^T\left(\mathcal{V}\mathcal{K}^{-1}\right)_{12}\tau\zeta + \zeta^T\tau\left(\mathcal{V}\mathcal{K}^{-1}\right)_{21}\xi\right) = 2\,\xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\xi = 2 \tag{5.65}$$

Using this and following [38], it is simple to prove that

$$\langle\mathcal{P}_\perp|\mathcal{P}_\perp\rangle = \left(\zeta^T\frac{1}{\mathbb{I}-\mathcal{T}^2}\tau\zeta\;\xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\xi\right)\langle\mathcal{S}_\perp|\mathcal{S}_\perp\rangle = \langle\mathcal{S}_\perp|\mathcal{S}_\perp\rangle \tag{5.66}$$

thanks to (5.64,5.65).

Therefore, under the condition

$$\xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\xi = 1 \tag{5.67}$$

the BPZ norm of $|\mathcal{P}_\perp\rangle + |\mathcal{S}_\perp\rangle$ is twice the norm of $|\mathcal{S}_\perp\rangle$. As a consequence the sum of these two states, once they are tensored by the corresponding 24–dimensional complements defined in Chapter 2, represent a couple of parallel D23–branes.

Similarly one can construct the more complicated brane configurations as we saw at the end of the Chapter 2.

5.4.3 The string midpoint

It was shown in [43] that, in the absence of a B field, the string midpoint in the lower dimensional lumps is confined to the hyperplane (D–brane) of vanishing transverse coordinates. Evaluating the exact string midpoint position in the full VSFT is in fact a nontrivial and interesting problem.

The oscillator expansion for the transverse string coordinates is, (4.20), setting $\alpha' = \frac{1}{2}$,

$$x^\alpha(\sigma) = x_0^\alpha + \frac{\theta^{\alpha\beta}}{\pi} p_{0,\beta}\Big(\sigma - \frac{\pi}{2}\Big) + \sqrt{2}\sum_{n=1}^{\infty}\left[x_n^\alpha \cos(n\sigma) + \frac{\theta^{\alpha\beta}}{\pi}\frac{1}{n} p_{n,\beta}\sin(n\sigma)\right] \quad (5.68)$$

Therefore the string midpoint is specified by

$$x^\alpha\Big(\frac{\pi}{2}\Big) = x_0^\alpha + \sqrt{2}\sum_{n=1}^{\infty}(-1)^n\left[x_{2n}^\alpha - \frac{\theta^{\alpha\beta}}{\pi}\frac{1}{2n-1}p_{2n-1,\beta}\right] \quad (5.69)$$

It is more convenient to pass to the operator basis $a_N^\alpha, a_N^{\alpha\dagger}$, which satisfies the algebra

$$[a_M^{(r)\alpha}, a_N^{(s)\beta\dagger}] = G^{\alpha\beta}\delta_{MN}\delta^{rs}$$

and are related to x_n, p_n by

$$x_n^\alpha = \frac{i}{\sqrt{2n}}\left(a_n^\alpha - a_n^{\alpha\dagger}\right), \qquad p_{n,\alpha} = \sqrt{\frac{n}{2}}G_{\alpha\beta}\left(a_n^\beta + a_n^{\beta\dagger}\right), \quad (5.70)$$

while the analogous relation for x_0, p_0 is given by eq.(5.9) with the specification that throughout this section, for simplicity, we fix $b = 2$.

Now, confinement of the string midpoint means

$$x^\alpha\Big(\frac{\pi}{2}\Big)|\mathcal{S}_\perp\rangle = 0 \quad (5.71)$$

Evaluating the LHS we get

$$\begin{aligned} x^\alpha\Big(\frac{\pi}{2}\Big)|\mathcal{S}_\perp\rangle &= -\frac{i}{\sqrt{2}}(a^\dagger + a^\dagger\mathcal{S})_0^\alpha|\mathcal{S}_\perp\rangle - i\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{2n}}(a^\dagger + a^\dagger\mathcal{S})_{2n}^\alpha|\mathcal{S}_\perp\rangle \\ &\quad -\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{2n-1}}\frac{\theta^{\alpha\beta}}{\pi}G_{\beta\gamma}(a^\dagger - a^\dagger\mathcal{S})_{2n-1}^\gamma|\mathcal{S}_\perp\rangle \end{aligned} \quad (5.72)$$

Confinement requires that this vanish. In order to write this condition in compact form, we introduce the 2×2–matrix–valued vector

$$\Theta = |\nu\rangle\,\mathbf{1} + |\mu\rangle\,\mathbf{e} \quad (5.73)$$

where

$$|\nu\rangle = \{\nu_0, \nu_{2n}\}, \qquad \nu_0 = \frac{1}{\sqrt{2}}, \qquad \nu_{2n} = \frac{(-1)^n}{\sqrt{2n}}$$
$$|\mu\rangle = \{\mu_{2n-1}\}, \qquad \mu_{2n-1} = i\pi\, B\, \frac{(-1)^n}{\sqrt{2n-1}} \tag{5.74}$$

Now the confinement condition for the string midpoint can be written as

$$\mathcal{S}C'\,\Theta = -\Theta, \qquad \text{or, equivalently,} \qquad \tilde{\mathcal{T}}\,\Theta = -\Theta, \quad \text{i.e.} \quad \mathcal{T}\,\tilde{\Theta} = -\tilde{\Theta}. \tag{5.75}$$

Due to (5.40) an eigenvalue -1 of $\mathcal{T}$ corresponds to an eigenvalue $-\frac{1}{3}$ of $\mathcal{X}$ with the same eigenvector. Let us rewrite $\mathbb{I} + 3\mathcal{X}$ as

$$\mathbb{I} + 3\mathcal{X} = \mathcal{Y}\,\mathbf{1} + \mathcal{Z}\,\mathbf{e} \tag{5.76}$$

Then eq.(5.75) becomes $(\mathbb{I} + 3\mathcal{X})\,\tilde{\Theta} = 0$, which in turn corresponds to the two equations

$$\mathcal{Y}\,|\nu\rangle + \mathcal{Z}\,|\mu\rangle = 0 \tag{5.77}$$
$$\mathcal{Z}\,|\nu\rangle - \mathcal{Y}\,|\mu\rangle = 0 \tag{5.78}$$

It is useful to further split $\mathcal{Y}$ as $\mathcal{Y} = \mathcal{Y}_0 + \mathcal{Y}_1$, where $\mathcal{Y}_0 = \mathcal{Y}(B = 0)$. Using (E.7) one obtains

$$\mathcal{Y}_0 = \begin{pmatrix} 4(1 - A^{-1}) & -4A^{-1}\langle v_e| \\ -4A^{-1}|v_e\rangle & 1 + 3X - 4\,A^{-1}(|v_e\rangle\langle v_e| - |v_o\rangle\langle v_o|) \end{pmatrix} \tag{5.79}$$

$$\mathcal{Y}_1 = 12\,H \begin{pmatrix} 1 & \langle v_e| \\ |v_e\rangle & |v_e\rangle\langle v_e| - |v_o\rangle\langle v_o| \end{pmatrix} \tag{5.80}$$

$$\mathcal{Z} = 8\sqrt{3}\,i\,a\,K \begin{pmatrix} 0 & \langle v_o| \\ |v_o\rangle & |v_e\rangle\langle v_o| + |v_o\rangle\langle v_e| \end{pmatrix} \tag{5.81}$$

where $H = \frac{4}{3}\frac{a^2 A^{-1}}{4a^2+3}$.

Now let us express the previous equations in a more explicit form.

$$|\nu\rangle = \nu_0 \oplus |\nu_e\rangle,$$
$$|\mu\rangle = -i\pi B|\lambda_o\rangle,$$

where

$$|\nu_e\rangle_n = \frac{1+(-1)^n}{2}\nu_n, \qquad \nu_n = \frac{(-1)^{n/2}}{\sqrt{n}} \tag{5.82}$$

$$|\lambda_o\rangle_n = \frac{1-(-1)^n}{2}\lambda_n, \qquad \lambda_n = \frac{(-1)^{(n+1)/2}}{\sqrt{n}} \tag{5.83}$$

We remark that $|\nu\rangle$ is the eigenvector corresponding to the eigenvalue $-\frac{1}{3}$ of $\mathcal{X}(B=0)$, introduced in [43]; and that $|\lambda_o\rangle$ is the eigenvector with eigenvalue $-\frac{1}{3}$ of X, introduced in [47]. As a consequence one has

$$\mathcal{Y}_0\,|\nu\rangle = 0, \qquad (1+3X)|\lambda_o\rangle = 0 \tag{5.84}$$

The first equation can be rewritten as

$$\langle v_e|\nu_e\rangle = V_{00}\nu_0 \tag{5.85}$$

$$(1+3X)|\nu_e\rangle = 4\nu_0|v_e\rangle \tag{5.86}$$

Remarkably enough, all the other equations from (5.77, 5.78), after using (5.85) and the second equation in (5.84), reduce to a single one

$$\langle v_o|\lambda_o\rangle = \sqrt{\frac{2}{3}}\pi \tag{5.87}$$

Therefore, since eqs.(5.84) have been proved independently, confinement of the string midpoint holds or not according to whether eq.(5.87) is true or not. Now, the LHS of this equation is

$$\langle v_o|\lambda_o\rangle = \sum_{n\ \mathrm{odd}} (-1)^{(n+1)/2}\frac{A_n}{n} \tag{5.88}$$

The latter series can be summed with standard methods and gives

$$\langle v_o|\lambda_o\rangle = \frac{9-2\sqrt{3}\pi}{6}$$

Therefore (5.87) is definitely not satisfied. So we can conclude that the string midpoint in the presence of a B field is *not confined* on the hyperplane that identifies the D23–brane.

In this section we have shown that the introduction of a B field in VSFT does not prevent us from obtaining parallel results to those obtained when $B=0$.

On the other hand a nonvanishing background B field may have advantageous aspects. The smoothing out effects of B on the UV divergences of noncommutative field theories are well–known.

We have verified that the singular geometry of the lump solutions, pointed out in [43], disappears in the presence of a B field, in particular the string midpoint is not confined any longer to stay on the D–brane.

We remark that this *deconfinement* might mean also that the left–right factorization characteristic of the sliver solution, [38, 50, 51], is not possible for lump solutions with B field. However it looks like there are other aspects of VSFT which may be fruitfully extended to VSFT with B field. For instance, the series of wedge–like states introduced before seem to suggest that the geometric nature of the wedge states, [35], persists also in the presence of a B field. This is confirmed by the results obtained in [62], where the presence of a B field has been dealt with entirely geometrically.

5.5 VSFT Ancestors of the GMS solitons

In this section, starting from the squeezed state, we construct an infinite sequence of solutions of eq.(5.56), denoted $|\Lambda_n\rangle$ for any natural number n. $|\Lambda_n\rangle$ is generated by acting on a tachyonic lump solution $|\Lambda_0\rangle$ with $(-\kappa)^n L_n(\mathbf{x}/\kappa)$, where L_n is the n-th Laguerre polynomial, $\mathbf{x}$ is a quadratic expression in the string creation operators, see below eqs.(6.71, 6.70), and κ is an arbitrary constant. These states satisfy the remarkable properties

$$|\Lambda_n\rangle * |\Lambda_m\rangle = \delta_{n,m}|\Lambda_n\rangle \tag{5.89}$$

$$\langle\Lambda_n|\Lambda_m\rangle = \delta_{n,m}\langle\Lambda_0|\Lambda_0\rangle \ . \tag{5.90}$$

Each $|\Lambda_n\rangle$ represents a D23–brane, parallel to all the others. The field theory limit of $|\Lambda_n\rangle$ factors into the sliver state (D25–brane) and the n-th GMS soliton. The algebra (6.75) and the property (6.76) exactly reflect isomorphic properties of the GMS solitons (in terms of Moyal product). In other words, the GMS solitons are nothing but the relics of the $|\Lambda_n\rangle$ D23–branes in the low energy limit.

To define the states $|\Lambda_n\rangle$ we start from the lump solution (5.42). I.e. we take $|\Lambda_0\rangle = |\mathcal{S}\rangle$. However, in the following, we will limit ourselves only to the transverse part of it, the parallel one being universal and irrelevant for our construction. We will denote the transverse part by $|\mathcal{S}_\perp\rangle$.

First we introduce two 'vectors' $\xi = \{\xi_{N\alpha}\}$ and $\zeta = \{\zeta_{N\alpha}\}$, which are chosen to satisfy the conditions

$$\rho_1\xi = 0, \qquad \rho_2\xi = \xi, \qquad \text{and} \quad \rho_1\zeta = 0, \qquad \rho_2\zeta = \zeta, \tag{5.91}$$

Next we define

$$\mathrm{x} = (a^\dagger \tau \xi)\,(a^\dagger C' \zeta) = (a_N^{\alpha\dagger} \tau_\alpha{}^\beta \xi_{N\beta})(a_N^{\alpha\dagger} C'_{NM} \zeta_{M\alpha}) \tag{5.92}$$

where τ is the matrix $\tau = \{\tau_\alpha{}^\beta\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and introduce the Laguerre polynomials $L_n(\mathrm{x}/\kappa)$. The definition of $|\Lambda_n\rangle$ is as follows

$$|\Lambda_n\rangle = (-\kappa)^n L_n\Big(\frac{\mathrm{x}}{\kappa}\Big)|\mathcal{S}_\perp\rangle \tag{5.93}$$

where κ is an arbitrary constant. Hermiticity requires that

$$(a\tau\xi^*)(aC'\zeta^*) = (a\tau C'\xi)(a\zeta) \tag{5.94}$$

Finally we impose that the two following conditions be satisfied

$$\xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2}\zeta = -1, \qquad \xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2}\zeta = -\kappa \tag{5.95}$$

Let us spend a few words to motivate the definition of the states $|\Lambda_n\rangle$. The definition (6.70) is not, as one might suspect, dictated in the first place by the similarity with the form of the GMS solitons. Rather it has been selected due to its apparently unique role in the framework of Witten's star algebra.

In [76], on the wake of [38], starting from the (transverse) lump solution $|\mathcal{S}_\perp\rangle$ we introduced a new lump solution $|\mathcal{P}_\perp\rangle = (\mathrm{x}-\kappa)|\mathcal{S}_\perp\rangle$. Imposing that $|\mathcal{P}_\perp\rangle * |\mathcal{P}_\perp\rangle = |\mathcal{P}_\perp\rangle$ and $|\mathcal{P}_\perp\rangle * |\mathcal{S}_\perp\rangle = 0$ and, moreover, that $\langle\mathcal{P}_\perp|\mathcal{P}_\perp\rangle = \langle\mathcal{S}_\perp|\mathcal{S}_\perp\rangle$, we found the conditions (5.95).

The next most complicated state one is lead to try is of the form

$$|\mathcal{P}'\rangle = (\alpha + \beta\mathrm{x} + \gamma\mathrm{x}^2)|\mathcal{S}_\perp\rangle \tag{5.96}$$

The conditions this state has to satisfy turn out to be more restrictive than for $|\mathcal{P}\rangle$, but, nevertheless, are satisfied if, besides conditions (5.95), the following relations hold

$$-2(\alpha)^{1/2} = \beta, \qquad \gamma = \frac{1}{2} \tag{5.97}$$

and then, putting $\alpha = \kappa$

$$|\mathcal{P}'\rangle = \Big(\kappa^2 - 2\kappa\mathrm{x} + \frac{1}{2}\mathrm{x}^2\Big)|\mathcal{S}_\perp\rangle \tag{5.98}$$

The polynomial in the RHS is nothing but the second Laguerre polynomial of x/κ multiplied by κ^2. We deduce from this that the Laguerre polynomials must play a fundamental role in this problem and, as a consequence, put forward the general ansatz (6.70).

Proving the necessity of the conditions (5.95) for general n is very cumbersome, so we will limit ourselves to showing that these conditions are sufficient. However it is instructive and rather easy to see, at least, that the second condition (5.95) is necessary in general. In fact, by requiring that the state $|\Lambda_n\rangle$ be orthogonal to the 'ground state' $|\mathcal{S}_\perp\rangle$, we get:

$$
\begin{aligned}
|\Lambda_n\rangle * |\mathcal{S}_\perp\rangle &= (-\kappa)^n \sum_{j=0}^{\infty} \binom{n}{j} \frac{(-\mathrm{x}/\kappa)^j}{j!} |\mathcal{S}_\perp\rangle * |\mathcal{S}_\perp\rangle \\
&= (-\kappa)^n \sum_{j=0}^{\infty} \binom{n}{j} (\kappa)^{-j} \\
&\cdot \quad (\xi\tau C')_{l_1}^{\alpha_1} \dots (\xi\tau C')_{l_j}^{\alpha_j} \zeta_{k_1}^{\beta_1} \dots \zeta_{k_j}^{\beta_j} \frac{\partial}{\partial \mu_{l_1}^{\alpha_1}} \dots \frac{\partial}{\partial \mu_{l_j}^{\alpha_j}} \frac{\partial}{\partial \mu_{k_1}^{\beta_1}} \dots \frac{\partial}{\partial \mu_{k_j}^{\beta_j}} \\
&\cdot \quad \exp\Big(-(\chi^T \mathcal{K}_1)^{-1}\mu - \frac{1}{2}\mu^T (\mathcal{V}\mathcal{K}^{-1})_{11}\mu \Big) |\mathcal{S}_\perp\rangle \Big|_{\mu=0} \\
&= (-\kappa)^n \sum_{j=0}^{\infty} \binom{n}{j} (\kappa)^{-j} \left(\xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \zeta \right)^j |\mathcal{S}\perp\rangle \\
&= (-\kappa)^n \left(1 + \frac{1}{\kappa} \xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \zeta \right)^n |\mathcal{S}_\perp\rangle = 0
\end{aligned} \tag{5.99}
$$

which is true for the choice κ given by the second eq.(5.95).

The complete proof of eqs.(6.75) and (6.76) is presented in appendix D.

5.6 GMS as low energy limit of "Ancestors"

We saw in Chapter 3 that soliton solutions of field theories defined on a noncommutative space, GMS solitons, describe Dp–branes. It is then interesting to see if we can recover such solutions using the Seiberg-Witten limit, that gives a noncommutative field theory from a string theory with a B field turned on.

To discuss this limit we first reintroduce the closed string metric $g_{\alpha\beta}$ as $g\,\delta_{\alpha\beta}$. Now we take $\alpha' B \gg g$, in such a way that G, θ and B are kept fixed. The limit is described by means of a parameter ϵ going to 0. ($\alpha' \sim \epsilon$). We could also choose to parametrize the $\alpha' B \gg g$ condition by sending B to infinity, keeping g and α' fixed and operating a rescaling of the string modes as in [31], of course at the end we get identical results. By looking at the exponential of the 3-string field theory vertex in the presence of a B field

$$\sum_{r,s=1}^{3}\left(\frac{1}{2}\sum_{m,n\geq 1} G_{\alpha\beta}a_m^{(r)\alpha\dagger}V_{mn}^{rs}a_n^{(s)\beta\dagger}+\sqrt{\alpha'}\sum_{n\geq 1}G_{\alpha\beta}p_{(r)}^{\alpha}V_{0n}^{rs}a_n^{(s)\beta\dagger}\right.$$
$$\left.+\,\alpha'\frac{1}{2}G_{\alpha\beta}p_{(r)}^{\alpha}V_{00}^{rs}p_{(s)}^{\beta}+\frac{i}{2}\sum_{r<s}p_{\alpha}^{(r)}\theta^{\alpha\beta}p_{\beta}^{(s)}\right) \tag{5.100}$$

we see that the limit is characterized by the rescalings

$$\begin{aligned} V_{mn} &\to V_{mn}\\ V_{m0} &\to \sqrt{\epsilon}V_{m0}\\ V_{00} &\to \epsilon V_{00}\end{aligned} \tag{5.101}$$

$G_{\alpha\beta}$ and $\theta^{\alpha\beta}$ are kept fixed. Their explicit dependence on g, α' and B will be reintroduced at the end of our calculations in the form

$$G_{\alpha\beta}=\frac{(2\pi\alpha' B)^2}{g}\delta_{\alpha\beta},\qquad \theta=\frac{1}{B} \tag{5.102}$$

Substituting the leading behaviors of V_{MN} in eqs.(5.25), and keeping in mind that $A=V_{00}+\frac{b}{2}$, the coefficients $\mathcal{V}_{MN}^{\alpha\beta,rs}$ become

$$\mathcal{V}_{00}^{\alpha\beta,rs} \to G^{\alpha\beta}\delta^{rs}-\frac{4}{4a^2+3}\left(G^{\alpha\beta}\phi^{rs}-ia\hat{\epsilon}^{\alpha\beta}\chi^{rs}\right) \tag{5.103}$$
$$\mathcal{V}_{0n}^{\alpha\beta,rs} \to 0 \tag{5.104}$$
$$\mathcal{V}_{mn}^{\alpha\beta,rs} \to G^{\alpha\beta}V_{mn}^{rs} \tag{5.105}$$

We see that the squeezed state (5.42) factorizes in two parts: the coefficients $\mathcal{V}_{mn}^{\alpha\beta,11}$ reconstruct the full 25 dimensional sliver, while the coefficients $\mathcal{V}_{00}^{\alpha\beta,11}$ take a very simple form

$$\mathcal{S}_{00}^{\alpha\beta}=\frac{2|a|-1}{2|a|+1}G^{\alpha\beta}\equiv s\,G^{\alpha\beta} \tag{5.106}$$

The soliton lump with this choice of the coefficients $\mathcal{V}_{MN}^{\alpha\beta,rs}$ will be called $|\hat{\mathrm{S}}\rangle$

$$|\hat{\mathrm{S}}\rangle = \left\{\det(1-X)^{1/2}\det(1+T)^{1/2}\right\}^{24}\exp\left(-\frac{1}{2}\eta_{\bar{\mu}\bar{\nu}}\sum_{m,n\geq 1}a_m^{\bar{\mu}\dagger}S_{mn}a_n^{\bar{\nu}\dagger}\right)|0\rangle\otimes \tag{5.107}$$

$$\exp\left(-\frac{1}{2}G_{\alpha\beta}\sum_{m,n\geq 1}a_m^{\alpha\dagger}S_{mn}a_n^{\beta\dagger}\right)|0\rangle\otimes$$
$$\frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\mathrm{Det}G)^{1/4}}\left(\mathrm{Det}(\mathbb{I}-\mathcal{X})^{1/2}\mathrm{Det}(\mathbb{I}+\mathcal{T})^{1/2}\right)\exp\left(-\frac{1}{2}sa_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle,$$

where $\bar{\mu},\bar{\nu}=0,\ldots 23$ and $\alpha,\beta=24,25$. In the low energy limit we have also

$$\mathrm{Det}(\mathbb{I}-\mathcal{X})^{1/2}\mathrm{Det}(\mathbb{I}+\mathcal{T})^{1/2}=\frac{4}{4a^2+3}\det(1-X)\,\frac{4a}{2a+1}\det(1+T)\qquad(5.108)$$

So the complete lump state becomes

$$|\hat{\mathbb{S}}\rangle = \left\{\det(1-X)^{1/2}\det(1+T)^{1/2}\right\}^{26}\exp\left(-\frac{1}{2}G_{\mu\nu}\sum_{m,n\geq 1}a_m^{\mu\dagger}S_{mn}a_n^{\nu\dagger}\right)|0\rangle\otimes$$
$$\frac{4a}{2a+1}\,\frac{b^2}{\sqrt{2\pi b^3}(\det G)^{1/4}}\exp\left(-\frac{1}{2}sa_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle,\qquad(5.109)$$

where $\mu,\nu=0,\ldots 25$ and $G_{\mu\nu}=\eta_{\bar{\mu}\bar{\nu}}\oplus G_{\alpha\beta}$. The first line of (5.109) is the usual 25-dimensional sliver up to a simple rescaling of $a_n^{\alpha\dagger}$. The norm of the lump is now regularized by the presence of a which is directly proportional to B: $a=-\frac{\pi^2}{A}B$. Using

$$|x\rangle=\sqrt{\frac{2\sqrt{\det G}}{b\pi}}\exp\left[-\frac{1}{b}x^\alpha G_{\alpha\beta}x^\beta-\frac{2}{\sqrt{b}}ia_0^{\alpha\dagger}G_{\alpha\beta}x^\beta+\frac{1}{2}a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}\right]|\Omega_{b,\theta}\rangle$$

we can calculate the projection onto the basis of position eigenstates of the transverse part of the lump state

$$\langle x|e^{-\frac{s}{2}a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle = \sqrt{\frac{2\sqrt{\det G}}{b\pi}}\frac{1}{1+s}\,e^{-\frac{1-s}{1+s}\frac{1}{b}x^\alpha x^\beta G_{\alpha\beta}}\qquad(5.110)$$

The transverse part of the lump state in the x representation is then

$$\langle x|\hat{\mathbb{S}}_\perp\rangle=\frac{1}{\pi}e^{-\frac{1}{2|a|b}x^\alpha x^\beta G_{\alpha\beta}}\,.\qquad(5.111)$$

Finally, the lump state projected into the x representation is

$$\langle x|\hat{\mathbb{S}}\rangle=\frac{1}{\pi}\exp\left[-\frac{1}{2|a|b}x^\alpha x^\beta G_{\alpha\beta}\right]|\Xi\rangle=\frac{1}{\pi}\exp\left[-\frac{x^\alpha x^\beta\delta_{\alpha\beta}}{\theta}\right]|\Xi\rangle\,.\qquad(5.112)$$

$|\Xi\rangle$ is the sliver state (RHS of first line in eq.(5.109)) and $\theta = \frac{1}{B}$. We recall that B has been chosen nonnegative. The coefficient in front of the sliver $|\Xi\rangle$ is nothing but the simplest GMS soliton solution (4.70):

$$\psi(r) = 2e^{-r^2} \tag{5.113}$$

which corresponds to the $|0\rangle\langle 0|$ projector in the harmonic oscillator Hilbert space. Strictly speaking there is a discrepancy between these coefficients and the corresponding GMS soliton, given by the normalizations which differ by a factor of 2π. This can be traced back to the traditional normalizations used for the eigenstates $|x\rangle$ and $|p\rangle$ in the SFT theory context and in the Moyal context, respectively. This discrepancy can be easily dealt with with a simple redefinition.

We notice that the profile and the normalization of $\langle x|\hat{\mathcal{S}}_\perp\rangle$ do not depend on b.

As compared to [43], the B field provides a natural realization of the regulator for the tachyonic soliton introduced ad hoc there. This beneficial effect of the B field is confirmed by the fact that the projector (5.109) is no longer annihilated by x_0

$$\begin{aligned} x_0 \exp\left(-\frac{1}{2} s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle &= i\,\frac{\sqrt{b}}{2}\,(a_0 - a_0^\dagger)\exp\left(-\frac{1}{2} s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle \\ &= -i\,\frac{\sqrt{b}}{2}\left[\frac{4a}{2a+1}\right] a_0^\dagger \exp\left(-\frac{1}{2} s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right)|\Omega_{b,\theta}\rangle \end{aligned}$$

Therefore, also in the low energy limit, the singular structure found in [43] has disappeared.

In order to analyze the same limit for any $|\Lambda_n\rangle$, first of all we have to find the low energy limit of the projectors ρ_1, ρ_2. Also these two projectors factorize into the zero mode and non–zero mode part. The former is given by

$$(\rho_1)_{00}^{\alpha\beta} \to \frac{1}{2}\Big[G^{\alpha\beta} + i\epsilon^{\alpha\beta}\Big], \qquad (\rho_2)_{00}^{\alpha\beta} \to \frac{1}{2}\Big[G^{\alpha\beta} - i\epsilon^{\alpha\beta}\Big], \tag{5.114}$$

Now, in order to single out the appropriate limit of $|\Lambda_n\rangle$, we take, in the definition (6.71), $\xi = \hat{\xi} + \eta$ and $\zeta = \hat{\zeta} + \vartheta$, where η, ϑ vanish in the limit $\alpha' \to 0$. Then we make the choice $\hat{\xi}_n = \hat{\zeta}_n = 0\ \forall n > 0$. We will see that the two zero components $\hat{\xi}_0$ and $\hat{\zeta}_0$ are enough to define a consistent low energy limit. In the field theory limit the defining conditions (6.72) become

$$\hat{\xi}_{0,24} + i\hat{\xi}_{0,25} = 0, \qquad \hat{\zeta}_{0,24} + i\hat{\zeta}_{0,25} = 0, \tag{5.115}$$

From now on we set $\hat{\xi}_0 = \hat{\xi}_{0,25} = -i\hat{\xi}_{0,24}$ and, similarly, $\hat{\zeta}_0 = \hat{\zeta}_{0,25} = -i\hat{\zeta}_{0,24}$. The conditions (5.95) become

$$\xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2} \zeta \to -\frac{1}{1-s^2} \frac{2}{\sqrt{\det G}} \hat{\xi}_0 \hat{\zeta}_0 = -1 \tag{5.116}$$

$$\xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \zeta \to -\frac{s}{1-s^2} \frac{2}{\sqrt{\det G}} \hat{\xi}_0 \hat{\zeta}_0 = -\kappa \tag{5.117}$$

Compatibility requires

$$\frac{2\hat{\xi}_0 \hat{\zeta}_0}{\sqrt{\det G}} = 1 - s^2, \qquad \kappa = s \tag{5.118}$$

At the same time

$$(\xi \tau a^\dagger)(\zeta C' a^\dagger) \to -\hat{\xi}_0 \hat{\zeta}_0 ((a_0^{24\dagger})^2 + (a_0^{25\dagger})^2) = -\frac{\hat{\xi}_0 \hat{\zeta}_0}{\sqrt{\det G}} a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger} \tag{5.119}$$

Hermiticity (6.74) requires that the product $\hat{\xi}_0 \hat{\zeta}_0$ be real. In order to be able to compute $\langle x | \Lambda_n \rangle$ in the field theory limit, we have to evaluate first

$$\begin{aligned} \langle x | \left(a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger} \right)^k e^{-\frac{s}{2} a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}} | \Omega_{b,\theta} \rangle &= (-2)^k \frac{d^k}{ds^k} \left(\langle x | e^{-\frac{s}{2} a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}} | \Omega_{b,\theta} \rangle \right) \\ &= (-2)^k \frac{d^k}{ds^k} \left(\sqrt{\frac{2\sqrt{\det G}}{b\pi}} \frac{1}{1+s} e^{-\frac{1-s}{1+s} \frac{1}{b} x^\alpha G_{\alpha\beta} x^\beta} \right) \end{aligned} \tag{5.120}$$

An explicit calculation gives

$$\begin{aligned} &\frac{d^k}{ds^k} \left(\frac{1}{1+s} e^{-\frac{1-s}{1+s} \frac{1}{b} x^\alpha x^\beta G_{\alpha\beta}} \right) = \\ &= \sum_{l=0}^{k} \sum_{j=0}^{k-l} \frac{(-1)^{k+j}}{(1-s)^j (1+s)^{k+1}} \frac{k!}{j!} \binom{k-l-1}{j-1} \langle x, x \rangle^j e^{-\frac{1}{2}\langle x, x \rangle} \end{aligned} \tag{5.121}$$

where we have set

$$\langle x, x \rangle = \frac{1}{ab} x^\alpha G_{\alpha\beta} x^\beta = \frac{2r^2}{\theta} \tag{5.122}$$

with $r^2 = x^\alpha x^\beta \delta_{\alpha\beta}$. In this equation it must be understood that, by definition, the binomial coefficient $\binom{-1}{-1}$ equals 1.

Now, inserting (5.121) in the definition of $|\Lambda_n\rangle$, we obtain after suitably reshuffling the indices:

$$
\begin{aligned}
&\langle x|(-\kappa)^n L_n\Big(\frac{\mathrm{X}}{\kappa}\Big) e^{-\frac{1}{2}s a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle\\
&\to \langle x|(-s)^n L_n\Big(-\frac{1-s^2}{2s}\; a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}\Big) e^{-\frac{1}{2}s a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle\\
&= \frac{(-s)^n}{(1+s)}\sum_{j=0}^{n}\sum_{k=j}^{n}\sum_{l=j}^{k}\binom{n}{k}\binom{l-1}{j-1}\frac{1}{j!}\frac{(1-s)^k}{(1+s)^j s^k}\\
&\cdot (-1)^j\langle x,x\rangle^j\, e^{-\frac{1}{2}\langle x,x\rangle}\sqrt{\frac{2\sqrt{\det G}}{b\pi}}
\end{aligned}
\tag{5.123}
$$

The expression can be evaluated as follows. First one uses the result

$$
\sum_{l=j}^{k}\binom{l-1}{j-1}=\binom{k}{j}
\tag{5.124}
$$

Inserting this into (5.123) one is left with the following summation, which contains an evident binomial expansion,

$$
\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j}\Big(\frac{1-s}{s}\Big)^k=\binom{n}{j}\frac{(1-s)^j}{s^n}
\tag{5.125}
$$

Replacing this result into (5.123) we obtain

$$
\begin{aligned}
&\langle x|(-\kappa)^n L_n\Big(\frac{\mathrm{X}}{\kappa}\Big)\, e^{-\frac{1}{2}s a_0^{\alpha\dagger}G_{\alpha\beta}a_0^{\beta\dagger}}|\Omega_{b,\theta}\rangle\\
&\to \frac{2|a|+1}{4|a|}\sqrt{\frac{2\sqrt{\det G}}{b\pi}}(-1)^n\sum_{j=0}^{n}\binom{n}{j}\frac{1}{j!}\Big(-\frac{2r^2}{\theta}\Big)^j e^{-\frac{r^2}{\theta}}
\end{aligned}
$$

Recalling now that the definition of $|\hat{\mathrm{S}}\rangle$ includes an additional numerical factor (see eq.(5.109)), we finally obtain

$$
\begin{aligned}
\langle x|\Lambda_n\rangle &\to \frac{1}{\pi}(-1)^n\sum_{j=0}^{n}\binom{n}{j}\frac{1}{j!}\Big(-\frac{2r^2}{\theta}\Big)^j e^{-\frac{r^2}{\theta}}|\Xi\rangle\\
&= \frac{1}{\pi}(-1)^n L_n\Big(\frac{2r^2}{\theta}\Big)\, e^{-\frac{r^2}{\theta}}|\Xi\rangle
\end{aligned}
\tag{5.126}
$$

The coefficient in front of the sliver $|\Xi\rangle$ is the $n-th$ GMS solution.

In Chapter 3 it was shown that a generic noncommutative scalar field theory with polynomial interaction allows for solitonic solutions in any space dimension. The solutions are very elegantly constructed in terms of harmonic oscillators eigenstates $|n\rangle$. In particular, solitonic solutions correspond to projectors $P_n = |n\rangle\langle n|$. Via the Weyl transform these projectors can be mapped to classical functions $\psi_n(x,y)$ of two variables x, y, in such a way that the operator product in the Hilbert space correspond to the Moyal product in (x, y) space.

This construction is rather universal and does not depend in any essential way on the form of the potential. Now, as we have noticed in the introduction, the low energy effective tachyonic field theory derived from SFT in the presence of a background B field is a noncommutative scalar field theory of the type described above. Therefore it is endowed with the GMS noncommutative solitons. It is reasonable to expect that these solitons may emerge from soliton–type solutions of the SFT, which has the noncommutative scalar tachyonic field theory as its low energy effective action. Therefore the low energy GMS solitons we found in the previous sections are no surprise. What is surprising however is the isomorphism we find between the lump solutions $|\Lambda_n\rangle$ in VSFT and the corresponding GMS solitons. Setting $r^2 = x^2 + y^2$ and $\psi_n(x,y) = 2(-1)^n L_n(\frac{2r^2}{\theta})\, e^{-\frac{r^2}{\theta}}$, we have in fact the following correspondences

$$\begin{array}{ccccc} |\Lambda_n\rangle & \longleftrightarrow & P_n & \longleftrightarrow & \psi_n(x,y) \\ |\Lambda_n\rangle * |\Lambda_{n'}\rangle & \longleftrightarrow & P_n P_{n'} & \longleftrightarrow & \psi_n \star \psi_{n'} \end{array} \tag{5.127}$$

where $\star$ denotes the Moyal product. Moreover

$$\langle\Lambda_n|\Lambda_{n'}\rangle \longleftrightarrow \mathrm{Tr}(P_n P_{n'}) \longleftrightarrow \int dxdy\, \psi_n(x,y)\psi_{n'}(x,y) \tag{5.128}$$

up to normalization (see (6.76)). This correspondence seems to indicate that the Laguerre polynomials hide a universal structure of these noncommutative algebras.

It is evident from the above that the GMS solitons are the low energy remnants of corresponding D–branes in SFT. This explains many features of the former: why, for instance, the energy of the soliton given by $\sum_{k=0}^{n-1} |k\rangle\langle k|$ is n time the energy of the soliton $|0\rangle\langle 0|$; this is nothing but a low energy relic of the same property for the tensions of the corresponding D–branes.

This parallelism can actually be pushed still further. In fact we can easily construct the correspondents of the operators $|n\rangle\langle m|$. Let us first define

$$X = a^\dagger \tau \xi \qquad Y = a^\dagger C' \zeta \tag{5.129}$$

so that $\mathrm{x} = XY$. The definitions we are looking for are as follows

$$|\Lambda_{n,m}\rangle = \sqrt{\frac{n!}{m!}}(-\kappa)^n Y^{m-n} L_n^{m-n}\left(\frac{\mathrm{x}}{\kappa}\right)|\mathbb{S}_\perp\rangle, \qquad n \le m \tag{5.130}$$

$$|\Lambda_{n,m}\rangle = \sqrt{\frac{m!}{n!}}(-\kappa)^m X^{n-m} L_m^{n-m}\left(\frac{\mathrm{x}}{\kappa}\right)|\mathbb{S}_\perp\rangle, \qquad n \ge m \tag{5.131}$$

where $L_n^{m-n}(z) = \sum_{k=0}^{m}\begin{pmatrix} m \\ n-k \end{pmatrix}(-z)^k/k!$. With the same techniques as in the previous sections one can prove that

$$|\Lambda_{n,m}\rangle * |\Lambda_{r,s}\rangle = \delta_{m,r}|\Lambda_{n,s}\rangle \tag{5.132}$$

for all natural numbers n, m, r, s. It is clear that the previous states $|\Lambda_n\rangle$ coincide with $|\Lambda_{n,n}\rangle$. In view of (5.132), we can extend the correspondence (5.127) to $|n\rangle\langle m| \leftrightarrow |\Lambda_{n,m}\rangle$. Therefore, following [56], [61], we can apply to the construction of projectors in the VSFT star algebra the solution generating technique, in the same way as in the harmonic oscillator Hilbert space $\mathcal{H}$. Recalling from chapter 3 the solution generating tecnique and the partial isometry structure, we write the analog of the shift operator in VSFT as

$$S = \sum_{n=0}^{\infty} |\Lambda_{n+1,n}\rangle \tag{5.133}$$

$$S^\dagger = \sum_{m=0}^{\infty} |\Lambda_{m,m+1}\rangle \tag{5.134}$$

Then follows

$$S * S^\dagger = \sum_{n,m=0}^{\infty} |\Lambda_{n+1,n}\rangle * |\Lambda_{m,m+1}\rangle = \sum_{n=0}^{\infty}|\Lambda_n\rangle - |\Lambda_0\rangle \tag{5.135}$$

$$S^\dagger * S = \sum_{n,m=0}^{\infty} |\Lambda_{n,n+1}\rangle * |\Lambda_{m+1,m}\rangle = \sum_{n=0}^{\infty}|\Lambda_n\rangle \tag{5.136}$$

which looks like the partial isometry relations we saw in chapter 4

$$SS^\dagger = 1 - P = 1 - |0\rangle\langle 0| \tag{5.137}$$

$$S^\dagger S = 1 \tag{5.138}$$

related to the solution generating technique [?]

$$\phi = S^n \lambda_i I S^{\dagger n} = \lambda_i (1 - P_n) \tag{5.139}$$

At first sight, one is tempted to identify $\sum_{n=0}^{\infty} |\Lambda_n\rangle$ with a sort of identity of string fields and consider the Ancestors as a "complete basis" beeing also orthogonal. We will see in the last chapter other motivations to suspect this but we must say that this problem is more subtle and we will not treat it for the moment [81, 80]. To conclude, we notice that the parallel between Ancestors and GMS works also at the solution generating technique level because we have

$$\Lambda_{n+m} = S^m \Lambda_n S^{\dagger m} \tag{5.140}$$

Chapter 6

Playing with star projectors

6.1 The General squeezed state form

In this section we will show that all the string fields and star-algebra projectors we described in chapter 2, can be written in a general squeezed state form involving a matrix U:[1]

$$|G_U\rangle = \frac{\mathcal{N}_\Xi}{det\,(1+TU)}\, e^{-\frac{1}{2}\,CG\,a^\dagger a^\dagger}|0\rangle \tag{6.1}$$

where

$$G = \frac{U+T}{1+TU} \tag{6.2}$$

Depending which matrix we choose to be U among a very simple group of possibilities, the null matrix or the identity or even and odd powers of the matrix S (sliver matrix), we recover the projectors. (For details about indices and diagonal basis, see chapter 2 and [70, 73]). In particular we have:

- $U = 0$ corresponds to the Sliver state
- $U = 1$ corresponds to the Identity state

[1]Kawano and Okuyama wrote down the same state with u beeing a number in []: in particular they showed that this state interpolates between the Sliver ($u = 0$) and the Identity ($u = 1$). In a certain sense we generalize their idea but it will be more clear when we will show how we derived such state

- $U = C$ corresponds to the Nothing state
- $U = -S$ corresponds to the Butterfly state
- $U = (-T)^{N-1}$ corresponds to the Nth Wedge state
- $U = (-S)^{N-1}$ (N even) corresponds to the $\alpha = \frac{2}{N}$ Generalized Butterfly state

For our calculation, we will use the diagonal basis used in [73], which is the basis of eigenvectors of the operator K_1^2 instead of the usual K_1 [66, 70].[2] This allows us, for instance, to write the sliver matrix in a simple two times two matrix form

$$S = -s(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.3}$$

where

$$s(k) = -e^{-\frac{k\pi}{2}} \tag{6.4}$$

is the Sliver eigenvalue. In the same basis the twist matrix C is

$$C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{6.5}$$

and thus $T = CS$

$$T = s(k) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.6}$$

The "General" state then takes the form

$$G = \frac{U + s(k)\mathbf{1}}{\mathbf{1} + s(k)U} \tag{6.7}$$

We will refer to [73] for other equivalent representation of star projectors we will use in the following steps.

[2]This diagonal basis in just a change of oscillator basis, defining two type of oscillators, twist even and twist odd, using the eigenvectors of K_1 which are k dependent, where k is the continous eigenvalue of K_1. It turns out that on this basis the Witten's star can be rewritten as a continous Moyal star product in suitable noncommutative k-dependent cordinate

6.1.1 $U = 0$ or the Sliver

Choosing $U = 0$ is immediately to see

$$G = s(k)\mathbf{1} = T \tag{6.8}$$

Since there is a C matrix multiplication between the diagonal oscillator basis and the basis of the a oscillators, we will write

$$|G_{U=0}\rangle = \mathcal{N}_{\Xi}\, e^{-\frac{1}{2} S\, a^{\dagger} a^{\dagger}} |0\rangle = |\Xi\rangle \tag{6.9}$$

which is the sliver with the correct constant in front of the exponential factor.

6.1.2 $U = 1$ or the Identity

Putting $U = 1$, we have

$$G = \mathbf{1} \tag{6.10}$$

then

$$|G_{U=1}\rangle = \frac{\mathcal{N}_{\Xi}}{det(1+T)}\, e^{-\frac{1}{2} C\, a^{\dagger} a^{\dagger}} |0\rangle = |I\rangle \tag{6.11}$$

which has also the correct constant factor. From now on, we will skip such constant factor for space reason.

6.1.3 $U = C$ or the Nothing

If we take for the matrix $U = C$,

$$G = \frac{C+T}{1+S} = C \tag{6.12}$$

and recalling the oscillator representation of the Nothing state

$$|G_{U=C}\rangle = e^{-\frac{1}{2} \mathbf{1}\, a^{\dagger} a^{\dagger}} |0\rangle = |\mathbb{N}\rangle \tag{6.13}$$

6.1.4 $U = -S$ or the Butterfly

Choosing $U = -S$ and using for S the form

$$S = -s(k)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.14}$$

we have

$$G = \frac{s(k)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + s(k)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + s(k)s(k)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \tag{6.15}$$

Then

$$G = \frac{s(k)}{1+s^2(k)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = -\frac{1}{2ch(k\pi/2)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{6.16}$$

which is the diagonal K_1^2 representation of the butterfly, as showed in [73]. Thus

$$|G_{U=-S}\rangle = e^{-\frac{1}{2}V_{\mathcal{B}}\,a^\dagger a^\dagger}|0\rangle = |\mathcal{B}\rangle \tag{6.17}$$

It is interesting to notice that from such "diagonal" form of the butterfly, using the form of matrices S and T, is it possible to rewrite the butterfly matrix in the form

$$V_B = \frac{S-T}{1-ST} \tag{6.18}$$

thus

$$|\mathcal{B}\rangle = \mathcal{N}_{\mathcal{B}}\, e^{-\frac{1}{2}\frac{S-T}{1-ST}a^{\dagger 2}}|0\rangle \tag{6.19}$$

which is immediately understood to be twist odd: the matrix elements V_{nm} with $n+m$ even are zero while the odd ones are not, as it must be. It will be useful also the diagonal form of such state

$$|\mathcal{B}_g(n=1)\rangle = \mathcal{N}_{\mathcal{B}}\, e^{-\frac{1}{2}\frac{2s}{1+s^2}o^{\dagger 2}}|0\rangle \tag{6.20}$$

6.1.5 $U = (-T)^{N-1}$ or the Wedge state

Putting $U = (-T)^{N-1}$ and recalling that T is simply proportional to identity matrix in two dimensions

$$G = \frac{(-s(k))^{N-1}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}^{N-1} + s(k)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + (-s(k))^N\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}^N} \tag{6.21}$$

which gives immediately

$$G = \frac{(-s(k))^{N-1} + s(k)}{1 + (-s(k))^N}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{6.22}$$

which is exactly the definition of the Nth wedge state matrix T_N, recalling that in our basis

$$T = s(k)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{6.23}$$

So, we get

$$|G_{U=(-T)^{N-1}}\rangle = e^{-\frac{1}{2}CT_N\, a^\dagger a^\dagger}|0\rangle = |T_N\rangle \tag{6.24}$$

6.1.6 $U = (-S)^{N-1}$ or the Generalized Butterfly

It is important to notice that in this formalism, apart the constant s(k), the matrix T behaves as an even power of the matrix S. In this sense, even powers of matrix S lead to wedge states, odd ones to generalized butterflies as we will see now. Indeed, taking $U = (-S)^{N-1}$ with N even, we get

$$G = \frac{(-s(k))^{N-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{N-1} + s(k)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + (s(k))^N\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{N-1}} \tag{6.25}$$

and after a bit of algebra

$$G = \frac{\begin{pmatrix} s & s^{N-1} \\ s^{N-1} & s \end{pmatrix}}{\begin{pmatrix} 1 & s^N \\ s^N & 1 \end{pmatrix}} \tag{6.26}$$

$$G = \frac{s}{1-s^{2N}} \begin{pmatrix} 1-s^{2N-2} & s^{N-2}(1-s^2) \\ s^{N-2}(1-s^2) & 1-s^{2N-2} \end{pmatrix} \tag{6.27}$$

which we rewrite as

$$G = \frac{s}{1-s^{2N}} \left[(1-s^{2N-2})\mathbb{I} - s^{N-2}(1-s^2)C\right] \tag{6.28}$$

Let us separate the twist even and twist odd sectors

$$G_{even} = \frac{s}{1-s^{2N}} \left[(1-s^{2N-2}) - s^{N-2}(1-s^2)\right] \tag{6.29}$$

$$G_{odd} = \frac{s}{1-s^{2N}} \left[(1-s^{2N-2}) + s^{N-2}(1-s^2)\right] \tag{6.30}$$

thus

$$G_{even} = \frac{s - s^{N-1}}{1-s^N} \tag{6.31}$$

$$G_{odd} = \frac{s + s^{N-1}}{1+s^N} \tag{6.32}$$

and finally

$$|G\rangle = e^{-\frac{1}{2}\left(G_{even}\, e^{\dagger 2} + G_{odd}\, o^{\dagger 2}\right)}|0\rangle \tag{6.33}$$

In order to compare this with the expected state, we need a diagonal oscillator representation of generalized butterflies. We will derive it from the Moyal coordinate wave-function representation [73]. Let us make first of all, an example with the sliver state. It is very easy to obtain the gaussian form of the Sliver in the $x(k), y(k)$ Moyal coordinate

$$\Psi_{\Xi}(x(k), y(k)) = \langle x(k), y(k)|\Xi\rangle = 2e^{-\frac{x^2(k)+y^2(k)}{\theta}} \tag{6.34}$$

we must take the braket between the sliver state in diagonal form

$$|\Xi\rangle = \mathcal{N}_{\Xi} \quad e^{-\frac{1}{2}\int_0^\infty dk s(k)\left(e^{\dagger 2}+o^{\dagger 2}\right)}|0\rangle \tag{6.35}$$

and the Moyal coordinate squeezed state (the same for $y(k)$)

$$\langle x(k)| = \langle 0|exp\left(e^2 + i\sqrt{2}ex(k) - 1/2x^2(k)\right) \tag{6.36}$$

In order to do this, we use the same formula we saw in chapter 2

$$\begin{aligned} &\langle 0|\exp\left(-\frac{1}{2}a\cdot Pa + \lambda\cdot a\right)\exp\left(-\frac{1}{2}a\cdot Qa^{\dagger} + \mu\cdot a\right)|0\rangle \\ = \; & \det(1-PQ)^{-1}\exp\left(\mu^T\cdot(1-PQ)^{-1}\cdot\lambda - \frac{1}{2}\mu^T\cdot Q(1-PQ)^{-1}\cdot\mu \right. \\ & \left. -\frac{1}{2}\lambda^T\cdot P(1-PQ)^{-1}\cdot\lambda\right), \end{aligned} \tag{6.37}$$

with

$$\lambda = i\sqrt{2}x(k) \quad , P = 1, \quad \mu = 0\, Q = s(k) \tag{6.38}$$

Now, if we would like to do the opposite, starting from the Moyal function representation and get the oscillator one, we should start from the coefficient p of $x(k)$ (or $y(k)$, for the sliver is the same) and, after some algebra, obtain the coefficient Q from

$$\frac{Q}{1+Q} - \frac{1}{2} = \; p \tag{6.39}$$

In our case

$$p = -\frac{1}{\theta} \tag{6.40}$$

$$\theta(\kappa) = 2\tanh\left(\frac{\kappa\pi}{4}\right) , \tag{6.41}$$

then

$$Q = \frac{\theta - 2}{\theta + 2} = s(k) \tag{6.42}$$

which is the well-known coefficient of the diagonal form of the sliver [69].

Starting from the Moyal function form of the butterfly, [73], the same game leads to

$$\frac{Q_x}{1+Q_x} - \frac{1}{2} = p_x \tag{6.43}$$

$$\frac{Q_y}{1+Q_y} - \frac{1}{2} = p_y \tag{6.44}$$

with

$$p_x = -\frac{1}{2} \quad , \quad p_y = -\frac{2}{\theta^2} \tag{6.45}$$

thus

$$Q_x = 0 \quad , \quad Q_y = \frac{\theta^2 - 4}{\theta^2 + 4} = \frac{2s}{1+s^2} \tag{6.46}$$

and we obtain

$$|\mathcal{B}\rangle = \mathcal{N}_{\mathcal{B}}\, e^{-\frac{1}{2}\frac{2s}{1+s^2}o^{\dagger 2}}|0\rangle \tag{6.47}$$

which coincides with the form we saw before.

Wave functions in $\vec{X}_\kappa \equiv (x_\kappa, y_\kappa)$ are proportional to the Gaussian

$$\exp\left(-\frac{1}{2}\int_0^\infty \vec{X}_\kappa \tilde{L}_\kappa \vec{X}_\kappa d\kappa\right) . \tag{6.48}$$

The generalized butterfly $|\mathcal{B}_g\rangle$ in this representation is [73]

$$\tilde{L}_\kappa = \coth\left(\frac{\kappa\pi}{4}\right)\begin{pmatrix} \tanh(\frac{\kappa\pi(2-\alpha)}{4\alpha}) & 0 \\ 0 & \coth(\frac{\kappa\pi(2-\alpha)}{4\alpha}) \end{pmatrix} \tag{6.49}$$

The sliver is limit $\alpha \to 0$

$$\tilde{L}_\kappa = \coth\left(\frac{\kappa\pi}{4}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{2}{\theta}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.50}$$

We want to derive the diagonal oscillator form for the generalized butterflies. We put

$$n = \frac{2-\alpha}{\alpha} \tag{6.51}$$

$$N = n + 1 = \frac{2}{\alpha} \tag{6.52}$$

and we write down

$$\frac{Q_x}{1+Q_x} - \frac{1}{2} = \ -\tfrac{1}{2}coth\left(\tfrac{k\pi}{4}\right) th\left(\tfrac{k\pi n}{4}\right) \tag{6.53}$$

$$\frac{Q_y}{1+Q_y} - \frac{1}{2} = \ -\tfrac{1}{2}coth\left(\tfrac{k\pi}{4}\right) coth\left(\tfrac{k\pi n}{4}\right) \tag{6.54}$$

It is useful to define

$$t = e^{-\frac{k\pi}{2}} \tag{6.55}$$

in order to simplify calculations:

$$coth\left(\frac{k\pi}{4}\right) = \frac{1+t}{1-t} \tag{6.56}$$

$$coth\left(\frac{k\pi n}{4}\right) = \frac{1+t^n}{1-t^n} \tag{6.57}$$

$$th\left(\frac{k\pi n}{4}\right) = \frac{1-t^n}{1+t^n} \tag{6.58}$$

Easily we get

$$Q_x = \frac{-t+t^n}{1-t^{n+1}} \tag{6.59}$$

$$Q_y = \frac{-t-t^n}{1+t^{n+1}} \tag{6.60}$$

so

$$|\mathcal{B}_g\rangle = \mathcal{N}_{\mathcal{B}_g}\, e^{-\frac{1}{2}\left(\frac{-t+t^n}{1-t^{n+1}}\, e^{\dagger 2}+\frac{-t-t^n}{1+t^{n+1}}\, o^{\dagger 2}\right)}|0\rangle \tag{6.61}$$

Note that for $n=1$ or alternatevely, $\alpha = N = 2$, we find the simple Butterfly

$$|\mathcal{B}_g(n=1)\rangle = \mathcal{N}_{\mathcal{B}}\, e^{-\frac{1}{2}\frac{2s}{1+s^2}\, o^{\dagger 2}}|0\rangle \tag{6.62}$$

Now, we compare the form for the Generalized Butterfly we found with the General state with $U = \left(-S\right)^{N-1}$, N even

$$|G_{U=(-S)^{N-1}}\rangle = e^{-\frac{1}{2}\left(\frac{s-s^{N-1}}{1-s^N}\, e^{\dagger 2}+\frac{s+s^{N-1}}{1+s^N}\, o^{\dagger 2}\right)}|0\rangle \tag{6.63}$$

Recalling $-t=s$, $N=n+1$, it is straightforward to see that the even sector of $|\mathcal{B}_g\rangle$

$$\frac{-t+t^n}{1-t^{n+1}} = \frac{s+(-s)^{N-1}}{1-(-s)^N} = \frac{s-s^{N-1}}{1-s^N} \tag{6.64}$$

coincides with G_{even} and the odd sectors of $|\mathcal{B}_g\rangle$

$$\frac{-t-t^n}{1+t^{n+1}} = \frac{s+s^{N-1}}{1+s^N} \tag{6.65}$$

coincides with G_{odd}.

6.2 General state from "Ancestors"

Let us define the function $P(u)$ with u a real constant starting from generic Λ_n star projectors:

$$P(u) = \sum_{n=0}^{\infty} u^n |\Lambda_n\rangle \tag{6.66}$$

Kawano and Okuyama showed [74] that if Λ_n are a complete basis this function has the form

$$|P(u)\rangle = \frac{1}{det\,(1+Tu)} e^{-\frac{1}{2}\frac{u+CS}{1+Tu} a^\dagger a^\dagger} |0\rangle \tag{6.67}$$

and behaves as

$$P(u) * P(v) = P(uv) \tag{6.68}$$

under star product, so interpolates between the Sliver ($u = 0$) and the Identity state ($u = 1$) if such projectors Λ_n are a complete basis. Indeed,

$$P(u = 1) = \sum_{n=0}^{\infty} |\Lambda_n\rangle = |I\rangle \tag{6.69}$$

We have an infinite set of projectors and, making some assumptions about some arbitrary terms, their Laguerre form leads to the same structure of $P(u)$, with a generalization respect to the constant u: we will use a matrix U. We already said that the structure of Ancestors solutions suggests the possibility they are a sort of complete basis of star-algebra but this problem, the completness problem with string fields, is very subtle, so we will not jump to definetely conclusions but make simply an interesting exercise [80, 81].

Let us recall the definition of the "Ancestors" $|\Lambda_n\rangle$

$$|\Lambda_n\rangle = (-\kappa)^n L_n\left(\frac{\mathrm{X}}{\kappa}\right)|\Xi\rangle \tag{6.70}$$

where

$$\mathrm{x} = (a^\dagger \xi)\,(a^\dagger C' \zeta) \tag{6.71}$$

and the two 'vectors' inside $\xi = \{\xi_n^\mu\}$ and $\zeta = \{\zeta_n^\mu\}$ are chosen to satisfy the conditions

$$\rho_1 \xi = 0, \qquad \rho_2 \xi = \xi, \qquad \text{and} \quad \rho_1 \zeta = 0, \qquad \rho_2 \zeta = \zeta, \tag{6.72}$$

and

$$\xi^T \frac{1}{1-T^2} \zeta = -1, \qquad \xi^T \frac{T}{1-T^2} \zeta = -\kappa \tag{6.73}$$

and κ is an arbitrary real constant. $|\Xi\rangle$ is the sliver.
Hermiticity requires that

$$(a\xi^*)(aC\zeta^*) = (aC\xi)(a\zeta) \tag{6.74}$$

We know that

$$|\Lambda_n\rangle * |\Lambda_m\rangle = \delta_{n,m} |\Lambda_n\rangle \tag{6.75}$$

$$\langle \Lambda_n | \Lambda_m \rangle = \delta_{n,m} \langle \Lambda_0 | \Lambda_0 \rangle \tag{6.76}$$

Therefore, if we consider the sum $|\Lambda_\infty\rangle = \sum_{n=0}^{\infty} |\Lambda_n\rangle$, this is still a projector, but it has infinity action

$$\langle \Lambda_\infty | \Lambda_\infty \rangle = \langle \Lambda_0 | \Lambda_0 \rangle \sum_{n=0}^{\infty} 1 \tag{6.77}$$

If we define

$$|G_U\rangle = \sum_{n=0}^{\infty} U^n |\Lambda_n\rangle \tag{6.78}$$

and if make two assumptions about the value of the constant k and the two, related infinite but, up their constraints, arbitrary vectors

1.

$$\kappa = \frac{1}{2}Tr(T) = s(k) \tag{6.79}$$

2.

$$(\tau\xi)_n \zeta_m = -\frac{1}{2}\left(1 - S^2\right)_{nm} = -\frac{1}{2}\left(1 - s^2(k)\right)\left(e^{\dagger 2} + o^{\dagger 2}\right) \tag{6.80}$$

where we write the assumptions also in the diagonal formalism skipping some integral over k in order not to complicate the expression. It is easy to see that these assumptions are coherent with the constraints about the vectors and k. Using a well–known resummation formula for Laguerre polynomials

$$\sum_{n=0}^{\infty} z^n L_n(x) = \frac{1}{1-z} e^{-\frac{zx}{1-z}} \tag{6.81}$$

and putting

$$z = -s(k)U \quad , \quad \mathrm{x} = -\frac{1}{2}\left(1 - s^2(k)\right)\left(e^{\dagger 2} + o^{\dagger 2}\right) \tag{6.82}$$

we obtain

$$|G_U\rangle = \frac{1}{det(1 + s(k)U)} e^{\frac{s(k)U}{1+s(k)U}\left[-\frac{1}{2}\left(1-s^2(k)\right)\left(e^{\dagger 2}+o^{\dagger 2}\right)\right]-\frac{1}{2}s(k)\left(e^{\dagger 2}+o^{\dagger 2}\right)}|0\rangle \tag{6.83}$$

and finally

$$|G_U\rangle = \frac{\mathcal{N}_\Xi}{det\,(1 + TU)} e^{-\frac{1}{2}\,\frac{U+CS}{1+TU}\left(e^{\dagger 2}+o^{\dagger 2}\right)}|0\rangle \tag{6.84}$$

Appendix A

Star products:rules and definitions

A.1 computations of $*$ products

As seen earlier, the matter part of the sliver state is given by

$$|\Xi\rangle = \mathcal{N}^{26} \exp\Big(-\frac{1}{2} a^\dagger \cdot S \cdot a^\dagger\Big)|0\rangle\,. \tag{A.1}$$

Coherent states are defined by letting exponentials of the creation operator act on the vacuum. Treating the sliver as the vacuum we introduce coherent like states of the form

$$|\Xi_\beta\rangle = \exp\Big(\sum_{n=1}^{\infty}(-)^{n+1}\beta_{\mu n} a_n^{\mu\dagger}\Big)|\Xi\rangle = \exp(-a^\dagger \cdot C\beta)|\Xi\rangle\,. \tag{A.2}$$

As built, the states satisfy a simple BPZ conjugation property:

$$\langle \Xi_\beta| = \langle\Xi|\exp\Big(\sum_{n=1}^{\infty}\beta_{n\mu} a_n^\mu\Big) = \langle\Xi|\exp(\beta\cdot a)\,. \tag{A.3}$$

We compute the $*$ product of two such states using the procedure discussed in refs.[34, 37]. We begin by writing out the product using two by two matrices encoding the oscillators of strings one and two:

$$\begin{aligned}
\Big(|\Xi_{\beta_1}\rangle * |\Xi_{\beta_2}\rangle\Big)_{(3)} &= \Big(\exp(-a^\dagger\cdot C\beta_1)|\Xi\rangle * \exp(-a^\dagger\cdot C\beta_2)|\Xi\rangle\Big)_{(3)} \\
&= {}_{(1)}\langle\Xi|\exp(\beta_1\cdot a_{(1)})\ {}_{(2)}\langle\Xi|\exp(\beta_2\cdot a_{(2)})|V_{123}\rangle \\
&= \langle 0_{12}|\exp\Big(\beta\cdot a - \frac{1}{2}a\cdot\Sigma\cdot a\Big)\exp\Big(-\frac{1}{2}a^\dagger\cdot\mathcal{V}\cdot a^\dagger - \chi^T\cdot a^\dagger\Big)|0_{12}\rangle \\
&\qquad \cdot\exp\Big(-\frac{1}{2}a^\dagger_{(3)}\cdot V^{11}\cdot a^\dagger_{(3)}\Big)|0_3\rangle\,,
\end{aligned} \tag{A.4}$$

where $a = (a_{(1)}, a_{(2)})$, and

$$\Sigma = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{pmatrix},$$

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \qquad \chi^T = (a^\dagger_{(3)} V^{12}\,,\, a^\dagger_{(3)} V^{21})\,. \tag{A.5}$$

Explicit evaluation continues by using the equation

$$\begin{aligned} &\langle 0| \exp\Big(\beta_i a_i - \frac{1}{2} P_{ij} a_i a_j\Big) \exp\Big(- \chi_i a_i^\dagger - \frac{1}{2} Q_{ij} a_i^\dagger a_j^\dagger\Big) |0\rangle \qquad \text{(A.6)} \\ &= \det(K)^{-1/2} \exp\Big(- \chi^T K^{-1} \beta - \frac{1}{2}\beta^T Q K^{-1} \beta - \frac{1}{2}\chi^T K^{-1} P \chi\Big)\,, \; K \equiv 1 - PQ\,. \end{aligned}$$

At this time we realize that since $|\Xi\rangle * |\Xi\rangle = |\Xi\rangle$ the result of the product is a sliver with exponentials acting on it; the exponentials that contain β. This gives

$$|\Xi_{\beta_1}\rangle * |\Xi_{\beta_2}\rangle = \exp\Big(-\chi^T \mathcal{K}^{-1} \beta - \frac{1}{2}\beta^T \mathcal{V}\mathcal{K}^{-1} \beta\Big) |\Xi\rangle\,, \quad \mathcal{K} = (1 - \Sigma\mathcal{V})\,. \tag{A.7}$$

The expression for $\mathcal{K}^{-1}$, needed above is simple to obtain given that all the relevant submatrices commute. One finds that

$$\mathcal{K}^{-1} = (1 - \Sigma\mathcal{V})^{-1} = \frac{1}{(1+T)(1-X)} \begin{pmatrix} 1 - TX & TM^{12} \\ TM^{21} & 1 - TX \end{pmatrix}\,. \tag{A.8}$$

We now recognize that the projectors ρ_1 and ρ_2 defined in (2.62) make an appearance in the oscillator term of (A.7)

$$\begin{aligned} -\chi^T \mathcal{K}^{-1} \beta &= -a^\dagger \cdot C\,(M^{12}, M^{21}) \mathcal{K}^{-1} \beta = -a^\dagger \cdot C\,(\rho_1, \rho_2) \beta \qquad \text{(A.9)} \\ &= -a^\dagger C \cdot (\rho_1 \beta_1 + \rho_2 \beta_2)\,. \end{aligned}$$

One can verify that

$$\begin{aligned} \mathcal{C}(\beta_1, \beta_2) &\equiv \frac{1}{2} (\beta_1, \beta_2)\, \mathcal{V}\mathcal{K}^{-1} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \qquad \text{(A.10)} \\ &= \frac{1}{2} (\beta_1, \beta_2) \frac{1}{(1+T)(1-X)} \begin{pmatrix} V^{11}(1-T) & V^{12} \\ V^{21} & V^{11}(1-T) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}\,. \end{aligned}$$

Since the matrix in between is symmetric we have

$$\mathcal{C}(\beta_1, \beta_2) = \mathcal{C}(\beta_2, \beta_1)\,. \tag{A.11}$$

Using (A.9) and (A.10) we finally have:

$$|\Xi_{\beta_1}\rangle * |\Xi_{\beta_2}\rangle = \exp\Big(-\mathcal{C}(\beta_1,\beta_2)\Big)|\Xi_{\rho_1\beta_1+\rho_2\beta_2}\rangle\,. \tag{A.12}$$

This is a useful relation that allows one to compute $*$-products of slivers acted by oscillators by simple differentiation. In particular, using eq.(A.2) we get

$$\begin{aligned}(a^{\mu_1\dagger}_{m_1}\cdots a^{\mu_k\dagger}_{m_k}|\Xi\rangle) * (a^{\nu_1\dagger}_{n_1}\cdots a^{\nu_l\dagger}_{n_l}|\Xi\rangle) \;&=\; (-1)^{\sum_{i=1}^{k}(m_i+1)+\sum_{j=1}^{l}(n_j+1)} \\ &\Big(\frac{\partial}{\partial\beta_{1m_1\mu_1}}\cdots\frac{\partial}{\partial\beta_{1m_k\mu_k}}\frac{\partial}{\partial\beta_{2n_1\nu_1}}\cdots\frac{\partial}{\partial\beta_{2n_l\nu_l}}(|\Xi_{\beta_1}\rangle * |\Xi_{\beta_2}\rangle)\Big)_{\beta_1=\beta_2=0}\,.\end{aligned} \tag{A.13}$$

Since $\rho_1+\rho_2=1$, for $\beta_1=\beta_2$ eq.(A.12) reduces to

$$|\Xi_{\beta}\rangle * |\Xi_{\beta}\rangle = \exp\Big(-\mathcal{C}(\beta,\beta)\Big)|\Xi_{\beta}\rangle\,. \tag{A.14}$$

Using the definition of $\mathcal{C}$ in (A.10) one can show that $\mathcal{C}(\beta,\beta)$ simplifies down to

$$\mathcal{C}(\beta,\beta) = \frac{1}{2}\,\beta C(1-T)^{-1}\beta\,. \tag{A.15}$$

It follows from (A.14) that by adjusting the normalization of the Ξ_β state

$$P_\beta \equiv \exp\Big(\mathcal{C}(\beta,\beta)\Big)|\Xi_\beta\rangle\,, \tag{A.16}$$

we obtain projectors

$$P_\beta * P_\beta = P_\beta\,. \tag{A.17}$$

Using eq.(A.6) one can also check that

$$\langle P_\beta|P_\beta\rangle = \langle\Xi|\Xi\rangle\,. \tag{A.18}$$

Appendix B

The coefficients V^{rs}_{mn}

In this Appendix we give the coefficients V^{rs}_{mn} introduced in Chapter 5. These results are taken from refs.[32, 33, 37]. First we define the coefficients A_n and B_n for $n \geq 0$ through the relations:

$$
\begin{aligned}
\left(\frac{1+ix}{1-ix}\right)^{1/3} &= \sum_{n \text{ even}} A_n x^n + i \sum_{n \text{ odd}} A_n x^n \,, \\
\left(\frac{1+ix}{1-ix}\right)^{2/3} &= \sum_{n \text{ even}} B_n x^n + i \sum_{n \text{ odd}} B_n x^n \,.
\end{aligned}
\tag{B.1}
$$

In terms of A_n and B_n we define the coefficients $N^{r,\pm s}_{mn}$ as follows:

$$
\begin{aligned}
N^{r,\pm r}_{nm} &= \frac{1}{3(n \pm m)} (-1)^n (A_n B_m \pm B_n A_m) \quad \text{for} \quad m+n \text{ even}, \; m \neq n \,, \\
&= 0 \quad \text{for} \quad m+n \text{ odd} \,, \\
N^{r,\pm(r+1)}_{nm} &= \frac{1}{6(n \pm m)} (-1)^{n+1} (A_n B_m \pm B_n A_m) \quad \text{for} \quad m+n \text{ even}, \; m \neq n \,, \\
&= \frac{1}{6(n \pm m)} \sqrt{3}\, (A_n B_m \mp B_n A_m) \quad \text{for} \quad m+n \text{ odd} \,, \\
N^{r,\pm(r-1)}_{nm} &= \frac{1}{6(n \mp m)} (-1)^{n+1} (A_n B_m \mp B_n A_m) \quad \text{for} \quad m+n \text{ even}, \; m \neq n \,, \\
&= -\frac{1}{6(n \mp m)} \sqrt{3}\, (A_n B_m \pm B_n A_m) \quad \text{for} \quad m+n \text{ odd} \,.
\end{aligned}
\tag{B.2}
$$

The coefficients V^{rs}_{mn} are then given by

$$
\begin{aligned}
V^{rs}_{nm} &= -\sqrt{mn}\, (N^{r,s}_{nm} + N^{r,-s}_{nm}) \quad \text{for} \quad m \neq n, \; m,n \neq 0 \,, \\
V^{rr}_{nn} &= -\frac{1}{3}[2 \sum_{k=0}^{n} (-1)^{n-k} A_k^2 - (-1)^n - A_n^2], \quad \text{for} \quad n \neq 0 \,,
\end{aligned}
$$

$$
\begin{aligned}
V_{nn}^{r(r+1)} &= V_{nn}^{r(r+2)} = \frac{1}{2}[(-1)^n - V_{nn}^{rr}] \quad \text{for} \quad n \neq 0\,, \\
V_{0n}^{rs} &= -\sqrt{2n}\,(N_{0n}^{r,s} + N_{0n}^{r,-s}) \quad \text{for} \quad n \neq 0\,, \\
V_{00}^{rr} &= \ln(27/16)\,. \qquad \text{(B.3)}
\end{aligned}
$$

The value of V^{rr}_{nn} quoted above corrects the result for $N^{rr}_{nn}(\equiv -V^{rr}_{nn}/n)$ quoted in eqn.(1.18) of [33]. In writing down the expressions for V^{rs}_{0n} and V^{rr}_{00} has been taken into account the fact that we are using $\alpha' = 1$ convention, as opposed to the $\alpha' = 1/2$ convention used in refs.[32, 33].

Appendix C

Conversion from momentum to oscillator basis

We start with the three string vertex in the matter sector as given in Chapter 2:

$$|V_3\rangle = \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\delta^{(26)}(p_{(1)} + p_{(2)} + p_{(3)}) \exp(-E)|0,p\rangle_{123} \qquad \text{(C.1)}$$

where

$$E = \frac{1}{2}\sum_{\substack{r,s\\ m,n\geq 1}} \eta_{\mu\nu}a_m^{(r)\mu\dagger}V_{mn}^{rs}a_n^{(s)\nu\dagger} + \sum_{\substack{r,s\\ n\geq 1}}\eta_{\mu\nu}p_{(r)}^{\mu}V_{0n}^{rs}a_n^{(s)\nu\dagger} + \frac{1}{2}\sum_r \eta_{\mu\nu}p_{(r)}^{\mu}V_{00}^{rr}p_{(r)}^{\nu}\,. \qquad \text{(C.2)}$$

Note that using the freedom of redefining V_{00}^{rs} using momentum conservation, we have chosen V_{00}^{rs} to be zero for $r \neq s$. Due to the same reason, a redefinition $V_{0n}^{rs} \to V_{0n}^{rs} + A_n^s$ by some r independent constant A_n^s leaves the vertex unchanged. We shall use this freedom to choose:

$$\sum_r V_{0n}^{rs} = 0\,. \qquad \text{(C.3)}$$

It can be easily verified that V_{0n}^{rs} given in eq.(B.3) satisfy these conditions.

We now pass to the oscillator basis for a subset of the space-time coordinates x^α ($(26-k) \leq \alpha \leq 25$), by relating the zero mode operators $\hat{x}^\alpha$ and $\hat{p}^\alpha$ to oscillators a_0^α and $a_0^{\alpha\dagger}$. For this one writes:

$$a_0^\alpha = \frac{1}{2}\sqrt{b}\,\hat{p}^\alpha - \frac{1}{\sqrt{b}}i\hat{x}^\alpha, \qquad a_0^{\alpha\dagger} = \frac{1}{2}\sqrt{b}\,\hat{p}^\alpha + \frac{1}{\sqrt{b}}i\hat{x}^\alpha\,, \qquad \text{(C.4)}$$

where b is an arbitrary constant. Then a_0^α, $a_0^{\alpha\dagger}$ satisfy the usual commutation rule $[a_0^\alpha, a_0^{\beta\dagger}] = \delta^{\alpha\beta}$ (we are assuming that the directions x^α are space-like; otherwise we shall need $\eta^{\alpha\beta}$), and we can define a new vacuum state $|\Omega_b\rangle$ such that $a_0^\alpha|\Omega_b\rangle = 0$. The relation between the momentum basis and the new oscillator basis is given by (for each string)

$$|\{p^\alpha\}\rangle = (2\pi/b)^{-k/4}\exp\Big[-\frac{b}{4}p^\alpha p^\alpha + \sqrt{b}a_0^{\alpha\dagger}p^\alpha - \frac{1}{2}a_0^{\alpha\dagger}a_0^{\alpha\dagger}\Big]|\Omega_b\rangle\,. \tag{C.5}$$

In the above equation $\{p^\alpha\}$ label momentum eigenvalues. Substituting eq.(C.5) into eq.(C.1), and integrating over $p^\alpha_{(i)}$, we can express the three string vertex as

$$\begin{aligned}|V_3\rangle \;=\; &\int d^{26-k}p_{(1)}d^{26-k}p_{(2)}d^{26-k}p_{(3)}\delta^{(26-k)}(p_{(1)}+p_{(2)}+p_{(3)})\\ &\exp\Big(-\frac{1}{2}\sum_{\substack{r,s\\ m,n\geq 1}}\eta_{\bar\mu\bar\nu}a_m^{(r)\bar\mu\dagger}V^{rs}_{mn}a_n^{(s)\bar\nu\dagger} - \sum_{\substack{r,s\\ n\geq 1}}\eta_{\bar\mu\bar\nu}p^{\bar\mu}_{(r)}V^{rs}_{0n}a_n^{(s)\bar\nu\dagger} - \frac{1}{2}\sum_r \eta_{\bar\mu\bar\nu}p^{\bar\mu}_{(r)}V^{rr}_{00}p^{\bar\nu}_{(r)}\Big)|0,p\rangle_{123}\\ &\otimes\Big(\frac{\sqrt{3}}{(2\pi b^3)^{1/4}}(V^{rr}_{00}+\frac{b}{2})\Big)^{-k}\exp\Big(-\frac{1}{2}\sum_{\substack{r,s\\ M,N\geq 0}}a_M^{(r)\alpha\dagger}V'^{rs}_{MN}a_N^{(s)\alpha\dagger}\Big)|\Omega_b\rangle_{123}\,.\end{aligned} \tag{C.6}$$

In this expression the sums over $\bar\mu, \bar\nu$ run from 0 to $(25-k)$, and the sum over α runs from $(26-k)$ to 25. Note that in the last line the sums over M, N run over 0, 1, 2 .... The new b-dependent V' coefficients are given in terms of the V coefficients by

$$\begin{aligned}V'^{rs}_{mn}(b) &= V^{rs}_{mn} - \frac{1}{V^{rr}_{00}+\frac{b}{2}}\sum_{t=1}^{3}V^{tr}_{0m}V^{ts}_{0n}\,, && m,n\geq 1\,,\\ V'^{rs}_{0n}(b) &= V'^{sr}_{n0} = \frac{1}{V^{rr}_{00}+\frac{b}{2}}\sqrt{b}\,V^{rs}_{0n}\,, && n\geq 1\,,\\ V'^{rs}_{00}(b) &= \frac{1}{3}\frac{b}{V^{rr}_{00}+\frac{b}{2}}\,, && r\neq s\,,\\ V'^{rr}_{00}(b) &= 1-\frac{2}{3}\frac{b}{V^{rr}_{00}+\frac{b}{2}}\,.\end{aligned} \tag{C.7}$$

In deriving the above relations we have used eq.(C.3). These relations can be readily inverted to find

$$\begin{aligned}V^{rs}_{mn} &= V'^{rs}_{mn}(b) + \frac{2}{3}\frac{1}{1-V'^{rr}_{00}(b)}\sum_{t=1}^{3}V'^{rt}_{m0}(b)V'^{ts}_{0n}(b)\,, && m,n\geq 1\,,\\ V^{rs}_{0n} &= \frac{2}{3}\frac{1}{1-V'^{rr}_{00}(b)}\sqrt{b}\,V'^{rs}_{0n}(b)\,, && n\geq 1\,,\\ V^{rr}_{00} &= \frac{b}{6}\frac{1+3V'^{rr}_{00}(b)}{1-V'^{rr}_{00}(b)}\,.\end{aligned} \tag{C.8}$$

We shall now describe how our variables V^{rs}_{mn} and V'^{rs}_{mn} are related to the variables introduced in ref.[32]. For this we begin by comparing the variables in the oscillator representation. Since ref.[32] uses the $\alpha' = 1/2$ convention rather than the $\alpha' = 1$ convention used here, every factor of p (x) in [32] should be multiplied (divided) by $\sqrt{2\alpha'}$, and then α' should be set equal to one in order to compare with our equations. With this prescription eqs.(2.5b) of [32] giving $a_0 = \frac{1}{2}\hat{p} - i\hat{x}$ becomes $a_0 = \frac{1}{\sqrt{2}}\hat{p} - \frac{i}{\sqrt{2}}\hat{x}$, which corresponds to (C.4) for $b = 2$. Thus, we can directly compare our variables with those of [32] for the case $b = 2$.

Ref.[32] introduced a matrix U which appears, for example, in their eq.(2.47). We shall denote this matrix by U^{gj}. This matrix appears in the construction of the vertex in the oscillator basis ([32], eqn.(2.52) and (2.53)). This implies that the V' coefficients for $b = 2$ can be expressed in terms of U^{gj} using their results. In particular, defining V'^{rs} to be the matrices V'^{rs}_{mn} with m, n now running from 0 to ∞, we have (see [32], eqn.(2.53)):[1]

$$V'^{rs}(2) = \frac{1}{3}(C' + \omega^{s-r}U^{gj} + \omega^{r-s}\bar{U}^{gj}), \tag{C.9}$$

where $\omega = \exp(2\pi i/3)$, $C'_{mn} = (-1)^m \delta_{mn}$ with $m, n \geq 0$, and the matrix U^{gj} satisfies the relations (eq.(2.51) of [32]):

$$U^{gj\,\dagger} = U^{gj}, \quad \bar{U}^{gj} \equiv (U^{gj})^* = C'U^{gj}C', \quad U^{gj}U^{gj} = 1. \tag{C.10}$$

Eq.(C.9) gives us, $V'^{rr}_{00}(2) = \frac{1}{3}(1 + 2U^{gj}_{00})$. With this result, the last equation in (C.8) can be used with $b = 2$ to find

$$V^{rr}_{00} = \frac{1 + U^{gj}_{00}}{1 - U^{gj}_{00}}. \tag{C.11}$$

Similarly, the second equation in (C.8) gives:

$$V^{rs}_{0n} = \frac{1}{1 - U^{gj}_{00}} \sqrt{2}\, V'^{rs}_{0n}(2), \qquad \text{for} \quad n \geq 1. \tag{C.12}$$

Making use of (C.9) and $\bar{U}^{gj}_{0n} = (U^{gj}_{0n})^*$ we find that we can write, for $n \geq 1$:

$$V^{rs}_{0n} = \frac{1}{3}(\omega^{s-r}W_n + \omega^{r-s}W_n^*), \tag{C.13}$$

where

$$W_n = \frac{\sqrt{2}U^{gj}_{0n}}{1 - U^{gj}_{00}}. \tag{C.14}$$

[1] As explained at the end of appendix E.36, U^{gj} should really be identified with $\bar{U}$ of ref.[32].

The first equation in (C.8) together with (C.9) gives us [34]

$$V^{r\,s} = \frac{1}{3}(C + \omega^{s-r}U + \omega^{r-s}\bar{U})\,, \tag{C.15}$$

where V^{rs}, U and C are regarded as matrices with indices running over $m, n \geq 1$, $C_{mn} = (-1)^m \delta_{mn}$ and U is given as

$$U_{mn} = U^{gj}_{mn} + \frac{U^{gj}_{m0} U^{gj}_{0n}}{1 - U^{gj}_{00}}\,. \tag{C.16}$$

By virtue of this relation, and the identities in (C.10) we have that the matrix U satisfies

$$\bar{U} \equiv U^* = CUC, \qquad U^2 = \bar{U}^2 = 1, \qquad U^\dagger = U\,, \quad \bar{U}^\dagger = \bar{U}\,. \tag{C.17}$$

It follows from (C.10) and (C.14) that W_n satisfies the relations:

$$W^*_n = (-1)^n W_n, \qquad \sum_{n\geq 1} W_n U_{np} = W_p, \qquad \sum_{m\geq 1} W^*_m W_m = 2V^{rr}_{00}\,. \tag{C.18}$$

Appendix D

The coefficients $\mathcal{V}_{MN}^{\alpha\beta,rs}$

In this Appendix we derive the properties of the coefficients $\mathcal{V}_{MN}^{\alpha\beta,rs}$ which has been essential for the definition of lump solutions with B field. These properties are parallel to those enjoyed by the ordinary coefficients, reported in Appendix B, and first found in [32, 33, 34, 37].

Let us quote first two straightforward properties of $\mathcal{V}_{MN}^{\alpha\beta,rs}$:

- (i) they are symmetric under the simultaneous exchange of all the three couples of indices;
- (ii) they are endowed with the property of cyclicity in the r,s indices, i.e. $\mathcal{V}^{rs} = \mathcal{V}^{r+1,s+1}$, where $r,s=4$ is identified with $r,s=1$ and we have dropped the other indices.

The first property is immediate. The second can also be proven directly from eqs.(5.25). However, since it will be an easy consequence of eq.(D.11) below, we pass immediately to the derivation of the latter.

To this end we need the following representation of the coefficients V_{0n}^{rs}, derived from [32]:

$$V_{0n}^{rs} = \begin{cases} Z_n \chi^{rs}, & n \quad \text{odd} \\ -\frac{2}{\sqrt{3}} Z_n \phi^{rs}, & n \quad \text{even} \end{cases} \tag{D.1}$$

where

$$Z_n = \sqrt{\frac{2}{3n}} B_0 A_n \tag{D.2}$$

The numbers B_0 and A_n were defined in ref.[32]. Notice that, since we have assumed $Z_n^{rs} = Z_n^{sr}$, we must have, by definition, $V_{0n}^{rs} = V_{n0}^{rs}$ for n even and $V_{0n}^{rs} = -V_{n0}^{rs}$ for n odd. Finally, for convenience, we introduce $Z_0 = \sqrt{\frac{b}{3}}$.

Substituting (D.1) into eqs.(5.25) and using (5.27), we obtain

$$\mathcal{V}_{NM}^{\alpha\beta,rs} = \begin{cases} \mathcal{V}_{NM}^{\alpha\beta,rs}(\infty) - \frac{6A^{-1}}{4a^2+3} K_\infty^{\alpha\beta,rs} Z_N Z_M, & N+M \quad \text{even} \\ \mathcal{V}_{NM}^{\alpha\beta,rs}(\infty) + \frac{\sqrt{3}A^{-1}}{4a^2+3} H_\infty^{\alpha\beta,rs} (-1)^N Z_N Z_M, & N+M \quad \text{odd} \end{cases} \tag{D.3}$$

In these equations

$$K_\infty^{\alpha\beta,rs} = G^{\alpha\beta}\phi^{rs} - ia\hat{\epsilon}^{\alpha\beta}\chi^{rs} \tag{D.4}$$

$$H_\infty^{\alpha\beta,rs} = 3G^{\alpha\beta}\chi^{rs} + 4ia\hat{\epsilon}^{\alpha\beta}\phi^{rs} \tag{D.5}$$

and $\mathcal{V}_{NM}^{\alpha\beta,rs}(\infty)$ is

$$\begin{aligned} \mathcal{V}_{00}^{\alpha\beta,rs}(\infty) &= G^{\alpha\beta}\delta^{rs} \\ \mathcal{V}_{0m}^{\alpha\beta,rs}(\infty) &= 0 \\ \mathcal{V}_{nm}^{\alpha\beta,rs}(\infty) &= G^{\alpha\beta} V_{nm}^{rs} \end{aligned} \tag{D.6}$$

The coefficients V_{nm}^{rs} are the same as in ref.[37] for $n, m \geq 1$.

We can also express the $\mathcal{V}_{NM}^{\alpha\beta,rs}$ in the following way

$$\mathcal{V}_{NM}^{\alpha\beta,rs} = \begin{cases} \mathcal{V}_{NM}^{\alpha\beta,rs}(0) + \frac{6A^{-1}}{4a^2+3} K_0^{\alpha\beta,rs} Z_N Z_M, & N+M \quad \text{even} \\ \mathcal{V}_{NM}^{\alpha\beta,rs}(0) + \frac{\sqrt{3}A^{-1}}{4a^2+3} H_0^{\alpha\beta,rs} (-1)^N Z_N Z_M, & N+M \quad \text{odd} \end{cases} \tag{D.7}$$

where

$$K_0^{\alpha\beta,rs} = \frac{4}{3}a^2 G^{\alpha\beta}\phi^{rs} + ia\hat{\epsilon}^{\alpha\beta}\chi^{rs} \tag{D.8}$$

$$H_0^{\alpha\beta,rs} = -4a^2 G^{\alpha\beta}\chi^{rs} + 4ia\hat{\epsilon}^{\alpha\beta}\phi^{rs} \tag{D.9}$$

and $\mathcal{V}_{NM}^{\alpha\beta,rs}(0) = G^{\alpha\beta} V_{NM}^{\prime rs}$ are the values taken by $\mathcal{V}_{NM}^{\alpha\beta,rs}$ for $B=0$. As expected, the symbols $V_{NM}^{\prime rs}$ are the same as the coefficients $V_{nm}^{\prime rs}(b)$ with $n, m \geq 0$, used in [37].

Next we introduce the third root of unity $\omega = e^{i\frac{2\pi}{3}}$ and notice that

$$\phi^{rs} = \frac{1}{2}(\omega^{r-s} + \omega^{s-r}), \qquad \chi^{rs} = \frac{i}{\sqrt{3}}(\omega^{r-s} - \omega^{s-r}), \tag{D.10}$$

Inserting these relations into (D.3,D.7) and rearranging the terms we find the basic relation

$$\mathcal{V}_{NM}^{\alpha\beta,rs} = \frac{1}{3}\left(C'_{NM} G^{\alpha\beta} + \omega^{s-r}\mathcal{U}_{NM}^{\alpha\beta} + \omega^{r-s}\bar{\mathcal{U}}_{NM}^{\alpha\beta}\right) \tag{D.11}$$

where

$$\mathcal{U}_{NM}^{\alpha\beta} = \begin{cases} G^{\alpha\beta}\mathcal{U}_{NM}(\infty) + R^{\alpha\beta} Z_N Z_M, & N+M \quad \text{even} \\ G^{\alpha\beta}\mathcal{U}_{NM}(\infty) + iR^{\alpha\beta}(-1)^N Z_N Z_M, & N+M \quad \text{odd} \end{cases} \tag{D.12}$$

Moreover

$$\bar{\mathcal{U}}^{\alpha\beta} = (\mathcal{U}^{\beta\alpha})^* \tag{D.13}$$

where * denotes complex conjugation. In (D.11) $C'_{NM} = (-1)^N \delta_{NM}$ and

$$R^{\alpha\beta} = \frac{6A^{-1}}{4a^2+3}\left(-\frac{3}{2}G^{\alpha\beta} + \sqrt{3}a\hat{\epsilon}^{\alpha\beta}\right) \tag{D.14}$$

Moreover

$$\begin{aligned} &\mathcal{U}^{\alpha\beta}_{00}(\infty) = G^{\alpha\beta}, \qquad \mathcal{U}^{\alpha\beta}_{0n} = 0 \\ &\mathcal{U}^{\alpha\beta}_{nm}(\infty) = G^{\alpha\beta} U_{nm} \end{aligned} \tag{D.15}$$

In the last equation U_{nm} coincides with the same symbol used in [37] (see eq.(B.15) in that reference).

Alternatively one can split $\mathcal{U}$ into the $B = 0$ part and the rest. Then

$$\mathcal{U}^{\alpha\beta}_{NM} = \begin{cases} G^{\alpha\beta}\mathcal{U}_{NM}(0) + T^{\alpha\beta} Z_N Z_M, & N+M \quad \text{even} \\ G^{\alpha\beta}\mathcal{U}_{NM}(0) + iT^{\alpha\beta}(-1)^N Z_N Z_M, & N+M \quad \text{odd} \end{cases} \tag{D.16}$$

where

$$T^{\alpha\beta} = \frac{12A^{-1}}{4a^2+3}\left(a^2 G^{\alpha\beta} + \frac{\sqrt{3}}{2}a\hat{\epsilon}^{\alpha\beta}\right) \tag{D.17}$$

and $\mathcal{U}^{\alpha\beta}_{NM} = G^{\alpha\beta} U'_{NM}$. The coefficients $U'_{nm}, U'_{0n}, U'_{00}$ are the same as in ref.[37] (see eq.(B.19) therein).

Let us discuss the properties of $\mathcal{U}$. Since

$$(\mathcal{U}^{\alpha\beta}_{NM})^* = \begin{cases} \mathcal{U}^{\alpha\beta}_{NM}, & N+M \quad \text{even} \\ -\mathcal{U}^{\alpha\beta}_{NM}, & N+M \quad \text{odd} \end{cases}$$

it is easy to prove the following properties (where we use the matrix notation for the indices N, M)

$$(\mathcal{U}^{\alpha\beta})^* = C'\mathcal{U}^{\alpha\beta}C' \tag{D.18}$$

and

$$(\mathcal{U}^{\alpha\beta})^\dagger = (\mathcal{U}^{\alpha\beta})^{*T} = (C'\mathcal{U}^{\alpha\beta}C')^T = \mathcal{U}^{\alpha\beta} \tag{D.19}$$

Finally, if tilde denotes transposition in the indices α, β, it is possible to prove that (the proof is rather technical and deferred to the end of this Appendix)

$$(\mathcal{U}\tilde{\mathcal{U}})^{\alpha\beta}_{NM} = (\tilde{\mathcal{U}}\mathcal{U})^{\alpha\beta}_{NM} = G^{\alpha\beta}\delta_{NM} + \left(RG + G\tilde{R} + \frac{2}{3}AR\tilde{R}\right) Z_N Z_M \tag{D.20}$$

Now, remembering that $\hat{\epsilon}^{\alpha\gamma}\hat{\epsilon}_\gamma{}^\beta = -G^{\alpha\beta}$, it is elementary to prove that

$$RG + G\tilde{R} + \frac{2}{3}AR\tilde{R} = 0 \tag{D.21}$$

Therefore, finally,

$$(\mathcal{U}\tilde{\mathcal{U}})^{\alpha\beta}_{NM} = (\tilde{\mathcal{U}}\mathcal{U})^{\alpha\beta}_{NM} = G^{\alpha\beta}\delta_{NM} \tag{D.22}$$

Eqs.(D.18, D.19, D.22) are the generalization of the analogous ones in [32, 33, 34, 37]. Using in particular (D.22), it is easy to prove that

$$[C'\mathcal{V}^{rs}, C'\mathcal{V}^{r's'}] = 0. \tag{D.23}$$

This follows from

$$9[C'\mathcal{V}^{rs}, C'\mathcal{V}^{r's'}] = \omega^{s-r+r'-s'}(C'\mathcal{U}\tilde{\mathcal{U}}C' - \tilde{\mathcal{U}}\mathcal{U}) + \omega^{s-r+s'-r'}(\tilde{\mathcal{U}}\mathcal{U} - C'\mathcal{U}\tilde{\mathcal{U}}C')$$

and from eq.(D.22). In the two previous equations matrix multiplication is understood both in the indices M, N and α, β. In the same sense, on the wake of [34, 37], we can also write down the following identities

$$C'\mathcal{V}^{12}C'\mathcal{V}^{21} = C'\mathcal{V}^{21}C'\mathcal{V}^{12} = (C'\mathcal{V}^{11})^2 - C'\mathcal{V}^{11} \tag{D.24}$$

$$(C'\mathcal{V}^{12})^3 + (C'\mathcal{V}^{21})^3 = 2(C'\mathcal{V}^{11})^3 - 3(C'\mathcal{V}^{11})^2 + G \tag{D.25}$$

which will be needed in the next section.

Notice however that, unlike refs.[32, 33, 34, 37], we have

$$C'\mathcal{V}^{rs} = \tilde{\mathcal{V}}^{sr}C' \qquad C'\mathcal{X}^{rs} = \tilde{\mathcal{X}}^{sr}C' \tag{D.26}$$

where tilde denotes transposition with respect to the α, β indices. Finally one can prove that

$$\begin{aligned}
&\mathcal{X} + \mathcal{X}^{12} + \mathcal{X}^{21} = \mathbb{I}\\
&\mathcal{X}^{12}\mathcal{X}^{21} = \mathcal{X}^2 - \mathcal{X}\\
&(\mathcal{X}^{12})^2 + (\mathcal{X}^{21})^2 = \mathbb{I} - \mathcal{X}^2\\
&(\mathcal{X}^{12})^3 + (\mathcal{X}^{21})^3 = 2\mathcal{X}^3 - 3\mathcal{X}^2 + \mathbb{I}
\end{aligned} \tag{D.27}$$

In the matrix products of these identities, as well as throughout the paper, the indices α, β must be understood in alternating up/down position: $\mathcal{X}^{\alpha}{}_{\beta}$. For instance, in (D.27) $\mathbb{I}$ stands for $\delta^{\alpha}{}_{\beta}\,\delta_{MN}$.

Derivation of $(\mathcal{U}\tilde{\mathcal{U}})^{\alpha\beta}_{NM}$

We derive now eq.(D.20). This can be done starting both from the representation (D.12) and from (D.16). In the first case we need the following identities taken from the Appendix B of [37].

$$\sum_{n\geq 1} W_n U_{nm} = W_m, \qquad \sum_{n\geq 1} W_n^* W_n = 2V_{00} \tag{D.28}$$

The numbers W_n are defined via the equation

$$V_{0n}^{rs} = \frac{1}{3}(\omega^{s-r}W_n + \omega^{r-s}W_n^*) \tag{D.29}$$

On the other hand we have

$$\begin{aligned} V_{0n}^{rs} &= \frac{i}{\sqrt{3}}(\omega^{r-s} - \omega^{s-r})Z_n, \qquad n \quad \text{odd} \\ V_{0n}^{rs} &= -\frac{1}{\sqrt{3}}(\omega^{r-s} + \omega^{s-r})Z_n, \qquad n \quad \text{even} \end{aligned} \tag{D.30}$$

This allows us to identify W_n and Z_n as follows:

$$\begin{aligned} W_n &= -i\sqrt{3}Z_n, \qquad n \quad \text{odd} \\ W_n &= -\sqrt{3}Z_n, \qquad n \quad \text{even} \end{aligned} \tag{D.31}$$

In particular, from the second equation in (D.28), we get

$$\sum_{n\geq 1} Z_n^2 = \frac{2}{3}V_{00} \tag{D.32}$$

Next one has to consider $(\mathcal{U}\tilde{\mathcal{U}})_{NM}$ case by case according to the various possibilities for N, M. As a sample, let us consider $N = n$ odd and $M = m$ odd. Then

$$(\mathcal{U}\tilde{\mathcal{U}})_{nm} = \mathcal{U}_{n0}\tilde{\mathcal{U}}_{0m} + \sum_{k\,\text{odd}} \mathcal{U}_{nk}\tilde{\mathcal{U}}_{km} + \sum_{k\,\text{even}} \mathcal{U}_{nk}\tilde{\mathcal{U}}_{km}$$

Now we replace on the RHS the values extracted from eq.(D.12). After rearranging the terms we get

$$(\mathcal{U}\tilde{\mathcal{U}})_{nm} \;=\; G\delta_{nm} + \frac{b}{3}R\tilde{R}Z_n Z_m + R\tilde{R}Z_n Z_m \sum_{k\geq 1} Z_k^2$$

$$
\begin{aligned}
&-\frac{i}{\sqrt{3}}G\tilde{R}\sum_{k\geq 1}U_{nk}W_k^* Z_m + \frac{i}{\sqrt{3}}RGZ_n\sum_{k\geq 1}W_k U_{km} \\
= \quad & G\delta_{nm}\left(RG + G\tilde{R} + \frac{2}{3}(V_{00} + \frac{b}{2})R\tilde{R}\right) Z_n Z_m \qquad \text{(D.33)}
\end{aligned}
$$

where use has been made of (D.28) and (D.32). In the same way all other cases of the identity (D.20) can be proved.

Alternatively one can prove (D.20) by means of the representation (D.16). The procedure is the same, but the matrix involved is U' instead of U. For this reason we need, instead of the second eq.(D.28), the identity

$$
\sum_{n\geq 1} W_n U'_{nm} = \frac{\frac{b}{2} - V_{00}}{\frac{b}{2} + V_{00}} W_m \qquad \text{(D.34)}
$$

Appendix E

Some proofs

In this Appendix we collect some proofs that otherwise would have uselessy made heavy the treatment of VSFT with B field. First, we explicitly show that the ratio $\mathcal{R}$ defined in (5.45)

$$\mathcal{R} = \frac{A^4(3+4a^2)^2}{2\pi b^3(\mathrm{Det}G)^{1/4}} \frac{\mathrm{Det}(\mathbb{I}-\mathcal{X})^{3/4}\mathrm{Det}(\mathbb{I}+3\mathcal{X})^{1/4}}{\det(1-X)^{3/2}\det(1+3X)^{1/2}},$$

is indeed equal to 1. Second, we prove the fundamental properties (6.75) and (6.76) of the states $|\Lambda_n\rangle$, i.e.

$$|\Lambda_n\rangle * |\Lambda_m\rangle = \delta_{n,m}|\Lambda_n\rangle$$
$$\langle\Lambda_n|\Lambda_m\rangle = \delta_{n,m}\langle\Lambda_0|\Lambda_0\rangle$$

E.1 Proof that $\mathcal{R} = 1$

This section is devoted to the proof of

$$\mathcal{R} = 1 \tag{E.1}$$

What we need is compute the ratio of $\mathrm{Det}(\mathbb{I}-\mathcal{X})$ and $\mathrm{Det}(\mathbb{I}+3\mathcal{X})$ with respect to the squares of $\mathrm{Det}(1-X)$ and $\mathrm{Det}(1+3X)$, respectively. To this end we follow the lines of ref.[52]. To start with we rewrite $\mathcal{V}^{11} \equiv \mathcal{V}$ in a more convenient form. Following [52], we introduce the vector notation $|v_e\rangle$ and $|v_0\rangle$ by means of

$$|v_e\rangle_n = \frac{1+(-1)^n}{2}\frac{A_n}{\sqrt{n}}, \qquad |v_o\rangle_n = \frac{1-(-1)^n}{2}\frac{A_n}{\sqrt{n}},$$

The constants A_n are as in [32]. Now we can write

$$
\begin{aligned}
\mathcal{V}_{00} &= \left(1 - \frac{2A^{-1}b}{4a^2+3}\right)\mathbf{1} \\
\mathcal{V}_{0n} &= -\frac{2A^{-1}\sqrt{2b}}{4a^2+3}\mathbf{1}\,\langle v_e|_n + i\sqrt{\frac{2b}{3}}\frac{4aA^{-1}}{4a^2+3}\,\mathsf{e}\,\langle v_o|_n, \qquad \mathcal{V}_{0n} = (-1)^n\mathcal{V}_{n0} \\
\mathcal{V}_{nm} &= \left(V_{nm} - \frac{4A^{-1}}{4a^2+3}(|v_e\rangle\langle v_e| + |v_o\rangle\langle v_o|)_{nm}\right)\mathbf{1} + i\frac{8}{\sqrt{3}}\frac{aA^{-1}}{4a^2+3}(|v_e\rangle\langle v_o| - |v_o\rangle\langle v_e|)_{nm}\,\mathsf{e}
\end{aligned}
\tag{E.2}
$$

where we have understood the indices α, β. They can be reinserted using

$$
\mathbf{1}^\alpha{}_\beta = \delta^\alpha{}_\beta, \qquad \mathsf{e}^\alpha{}_\beta = \epsilon^\alpha{}_\beta
$$

Now $\mathcal{X} = C'\mathcal{V}$ can be written in the following block matrix form

$$
\mathcal{X} = \begin{pmatrix} (1-2Kb)\mathbf{1} & -2K\sqrt{2b}\,\mathbf{1}\,\langle v_e| + 4iaK\sqrt{\frac{2b}{3}}\,\mathsf{e}\,\langle v_o| \\ -2K\sqrt{2b}|v_e\rangle\,\mathbf{1} & X\mathbf{1} - 4K\,\mathbf{1}\,(|v_e\rangle\langle v_e| - |v_o\rangle\langle v_o|) \\ +4iaK\sqrt{\frac{2b}{3}}|v_o\rangle\,\mathsf{e} & +\frac{8}{\sqrt{3}}iaK\,\mathsf{e}\,(|v_e\rangle\langle v_o| + |v_o\rangle\langle v_e|) \end{pmatrix}
\tag{E.3}
$$

where all m, n as well as all α, β indices are understood, $K = \frac{A^{-1}}{4a^2+3}$.

The first determinant we have to compute is the one of the matrix $\mathbb{I} - \mathcal{X}$. Using (E.3) we extract from $\mathbb{I} - \mathcal{X}$ the factor $2bK$ and represent the rest in the block form

$$
\frac{1}{2bK}(\mathbb{I} - \mathcal{X}) = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}
$$

By a standard formula, the determinant of the RHS is given by the determinant of $\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}$. After some algebra and using the obvious identity $\langle v_o|v_e\rangle = 0$, one gets

$$
\begin{aligned}
\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B} &= \begin{pmatrix} 1 - X - \frac{4}{3}A^{-1}|v_o\rangle\langle v_o| & 0 \\ 0 & 1 - X - \frac{4}{3}A^{-1}|v_o\rangle\langle v_o| \end{pmatrix} \\
&= \left(1 - X - \frac{4}{3}A^{-1}|v_o\rangle\langle v_o|\right)\mathbb{I}
\end{aligned}
$$

The rest of the computation is straightforward,

$$
\begin{aligned}
\mathrm{Det}(\mathbb{I} - \mathcal{X}) &= (2bK)^2\left(\mathrm{Det}\Big(1 - X - \frac{4}{3}A^{-1}|v_o\rangle\langle v_o|\Big)\right)^2 \\
&= (2bK)^2\,(\mathrm{Det}(1-X))^2\left(\mathrm{Det}\Big(1 - \frac{4}{3}A^{-1}\frac{1}{1-X}|v_o\rangle\langle v_o|\Big)\right)^2 \\
&= \left(\frac{b}{A}\right)^4\left(\frac{1}{4a^2+3}\right)^2(\mathrm{Det}(1-X))^2
\end{aligned}
\tag{E.4}
$$

In the last step we have used the identities, see [52],

$$\mathrm{Det}\left(1 - \frac{4}{3}A^{-1}\frac{1}{1-X}|v_o\rangle\langle v_o|\right) = 1 - \frac{4}{3}A^{-1}\langle v_o|\frac{1}{1-X}|v_o\rangle \tag{E.5}$$

and

$$\langle v_o|\frac{1}{1-X}|v_o\rangle = \frac{3}{4}V_{00} \tag{E.6}$$

The treatment of $\mathrm{Det}(\mathbb{I} + 3\mathcal{X})$ is less trivial. We start again by writing $(\mathbb{I} + 3\mathcal{X})$ in block matrix form

$$\mathbb{I} + 3\mathcal{X} = \begin{pmatrix} (4 - 6Kb)\mathbf{1} & -6K\sqrt{2b}\,\mathbf{1}\,\langle v_e| + 4iaK\sqrt{6b}\,\mathbf{e}\,\langle v_o| \\ -6K\sqrt{2b}\,|v_e\rangle\,\mathbf{1} & (1+3X)\mathbf{1} - 12K\,\mathbf{1}\,(|v_e\rangle\langle v_e| - |v_o\rangle\langle v_o|) \\ +4iaK\sqrt{6b}\,|v_o\rangle\,\mathbf{e} & +8\sqrt{3}iaK\,\mathbf{e}\,(|v_e\rangle\langle v_o| + |v_o\rangle\langle v_e|) \end{pmatrix} \tag{E.7}$$

and set

$$\mathbb{I} + 3\mathcal{X} \equiv (4 - 6bK)\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \tag{E.8}$$

Therefore

$$\begin{aligned} \mathrm{Det}(\mathbb{I} + 3\mathcal{X}) &= (4 - 6bK)^2 \det\left(\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right) \\ &= (4 - 6bK)^2\,(\det(1+3X))^2\,\det\left(\frac{1}{1+3X}\left(\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right)\right) \end{aligned} \tag{E.9}$$

The last expression is formal. In fact X has an eigenvalue $-\frac{1}{3}$ which renders the RHS of (E.9) ill–defined. To avoid this we follow [52] and introduce the regularized inverse

$$Y_\varepsilon = \frac{1}{1 + 3X - \varepsilon^2 X} \tag{E.10}$$

where ε is a small parameter, and replace it into (E.9). After some algebra we find

$$Y_\varepsilon\left(\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right) = \mathcal{A} \cdot \mathfrak{B} \tag{E.11}$$

The matrices in the RHS are given by

$$\mathcal{A} = \begin{pmatrix} 1 + \alpha Y_\varepsilon|v_e\rangle\langle v_e| + \beta Y_\varepsilon|v_o\rangle\langle v_o| & 0 \\ 0 & 1 + \alpha Y_\varepsilon|v_e\rangle\langle v_e| + \beta Y_\varepsilon|v_o\rangle\langle v_o| \end{pmatrix} \tag{E.12}$$

where

$$\alpha = -\frac{24K}{2 - 3bK}, \qquad \beta = 12K\frac{2 - A^{-1}}{2 - 3bK}, \tag{E.13}$$

and

$$\mathcal{B} = \begin{pmatrix} 1 & \lambda Y_\varepsilon |v_2\rangle\langle v_o| + \mu Y_\varepsilon |v_o\rangle\langle v_e| \\ -\lambda Y_\varepsilon |v_e\rangle\langle v_o| - \mu Y_\varepsilon |v_o\rangle\langle v_e| & 1 \end{pmatrix} \tag{E.14}$$

where,

$$\lambda = \frac{\gamma}{1+\alpha\langle v_e|Y_\varepsilon|v_e\rangle}, \qquad \mu = \frac{\gamma}{1+\beta\langle v_0|Y_\varepsilon|v_0\rangle}, \qquad \gamma^2+\alpha\beta = -\frac{4}{V_{00}}\beta \tag{E.15}$$

Now, after some computation,

$$\det\mathcal{A} = (1+\alpha\langle v_e|Y_\varepsilon|v_e\rangle)^2\ (1+\beta\langle v_o|Y_\varepsilon|v_o\rangle)^2 \tag{E.16}$$

and

$$\det\mathcal{B} = \left(1 + \frac{\gamma^2\langle v_e|Y_\varepsilon|v_e\rangle\langle v_o|Y_\varepsilon|v_0\rangle}{(1+\alpha\langle v_e|Y_\varepsilon|v_e\rangle)\ (1+\beta\langle v_o|Y_\varepsilon|v_o\rangle)}\right)^2 \tag{E.17}$$

As a consequence

$$\det\mathcal{A}\,\det\mathcal{B} = \left(1+\alpha\langle v_e|Y_\varepsilon|v_e\rangle + \beta\langle v_o|Y_\varepsilon|v_0\rangle\left(1-\frac{4}{V_{00}}\langle v_e|Y_\varepsilon|v_e\rangle\right)\right)^2 \tag{E.18}$$

Now we can remove the regulator ε by using the basic result of [52]:

$$\lim_{\varepsilon\to 0}\left(1-\frac{4}{V_{00}}\langle v_e|Y_\varepsilon|v_e\rangle\right)\ \langle v_o|Y_\varepsilon|v_o\rangle = \frac{\pi^2}{12V_{00}} \tag{E.19}$$

and

$$\langle v_e|\frac{1}{1+3X}|v_e\rangle = \frac{V_{00}}{4}.$$

Inserting this result in (E.18) we find

$$\det\mathcal{A}\,\det\mathcal{B} = \frac{A^2}{(8a^2A+6V_{00})^2}\left(8a^2+\frac{2\pi^2}{A^2}\right)^2 \tag{E.20}$$

As a consequence of eqs.(E.9,E.11,E.18,E.20) we find

$$\frac{\mathrm{Det}(\mathbb{I}+3\mathcal{X})}{(\mathrm{Det}(1+3X))^2} = \frac{4}{(4a^2+3)^2}\left(8a^2+\frac{2\pi^2}{A^2}\right)^2 \tag{E.21}$$

Finally, substituting this and (E.4) into $\mathcal{R}$, we get

$$\mathcal{R} \;=\; \frac{A^4(3+4a^2)^2}{2\pi b^3(\mathrm{Det}G)^{1/4}}\frac{\mathrm{Det}(\mathbb{I}-\mathcal{X})^{3/4}\mathrm{Det}(\mathbb{I}+3\mathcal{X})^{1/4}}{\det(1-X)^{3/2}\mathrm{Det}(1+3X)^{1/2}} = 1 \tag{E.22}$$

This is what we wanted to show. It implies

$$\frac{\mathfrak{e}_{23}}{\mathfrak{e}_{25}} = \frac{(2\pi)^2}{(\mathrm{Det}G)^{1/4}} \tag{E.23}$$

which corresponds to the expected result for this ratio, as explained in [75]. We remark that (E.21) implies that the eigenvalue $-\frac{1}{3}$ is also contained in the spectrum of $\mathcal{X}$ with double multiplicity with respect to X.

E.2 Proofs of eqs.(6.75) and (6.76)

The star product $|\Lambda_n\rangle * |\Lambda_{n'}\rangle$ can be evaluated by using the explicit expression of the Laguerre polynomials

$$|\Lambda_n\rangle * |\Lambda_{n'}\rangle = \Big((-\kappa)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-\mathrm{x}/\kappa)^k}{k!}|\mathcal{S}_\perp\rangle\Big) * \Big((-\kappa)^{n'} \sum_{p=0}^{n'} \binom{n'}{p} \frac{(-\mathrm{x}/\kappa)^p}{p!}|\mathcal{S}_\perp\rangle\Big) \tag{E.24}$$

Therefore we need to compute $(\mathrm{x}^k|\mathcal{S}_\perp\rangle) * (\mathrm{x}^p|\mathcal{S}_\perp\rangle)$. According to [38], this is given by

$$\begin{aligned}(\mathrm{x}^k|\mathcal{S}_\perp\rangle) * (\mathrm{x}^p|\mathcal{S}_\perp\rangle) &= (\xi\tau C')^{\alpha_1}_{l_1}\ldots(\xi\tau C')^{\alpha_k}_{l_k}\zeta^{\beta_1}_{j_1}\ldots\zeta^{\beta_k}_{j_k}\frac{\partial}{\partial\mu^{\alpha_1}_{l_1}}\cdots\frac{\partial}{\partial\mu^{\alpha_k}_{l_k}}\frac{\partial}{\partial\mu^{\beta_1}_{j_1}}\cdots\frac{\partial}{\partial\mu^{\beta_k}_{j_k}}\\
&\cdot\ (\xi\tau C')^{\bar\alpha_1}_{\bar l_1}\ldots(\xi\tau C')^{\bar\alpha_p}_{\bar l_p}\zeta^{\bar\beta_1}_{\bar j_1}\ldots\zeta^{\bar\beta_p}_{\bar j_p}\frac{\partial}{\partial\bar\mu^{\bar\alpha_1}_{\bar l_1}}\cdots\frac{\partial}{\partial\bar\mu^{\bar\alpha_p}_{\bar l_p}}\frac{\partial}{\partial\bar\mu^{\bar\beta_1}_{\bar j_1}}\cdots\frac{\partial}{\partial\bar\mu^{\bar\beta_p}_{\bar j_p}}\\
&\cdot\ \exp\Big(-\chi^T\mathcal{K}^{-1}M - \frac{1}{2}M^T\mathcal{V}\mathcal{K}^{-1}M\Big)|\mathcal{S}_\perp\rangle\Big|_{\mu=\bar\mu=0}\end{aligned} \tag{E.25}$$

where

$$\mathcal{K} = \mathbb{I} - \mathcal{T}\mathcal{X}, \qquad \mathcal{V} = \begin{pmatrix}\mathcal{V}^{11} & \mathcal{V}^{12}\\ \mathcal{V}^{21} & \mathcal{V}^{22}\end{pmatrix} \tag{E.26}$$

and

$$M = \begin{pmatrix}\mu\\ \bar\mu\end{pmatrix}, \qquad \chi^T = (a^\dagger\mathcal{V}^{12}, a^\dagger\mathcal{V}^{21}), \qquad \chi^T\mathcal{K}^{-1}M = a^\dagger C'(\rho_1\mu + \rho_2\bar\mu) \tag{E.27}$$

The explicit computation, at first sight, looks daunting. However, we may avail ourselves of the following identities

$$\begin{aligned}\xi^T(\mathcal{V}\mathcal{K}^{-1})_{\alpha\alpha}\zeta &= \xi^T\tau C'(\mathcal{V}\mathcal{K}^{-1})_{\alpha\alpha}\tau C'\zeta = \xi^T C'\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta = 0\\
\xi^T\tau C'(\mathcal{V}\mathcal{K}^{-1})_{\alpha\alpha}\zeta &= \xi^T(\mathcal{V}\mathcal{K}^{-1})_{\alpha\alpha}\tau C'\zeta = \xi^T\tau\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta = -\kappa\end{aligned} \tag{E.28}$$

for $\alpha = 1, 2$, and

$$
\begin{aligned}
\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\zeta &= \xi^T\tau C'(\mathcal{V}\mathcal{K}^{-1})_{21}\tau C'\zeta = -\xi^T C'\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta = 0\\
\xi^T(\mathcal{V}\mathcal{K}^{-1})_{21}\zeta &= \xi^T\tau C'(\mathcal{V}\mathcal{K}^{-1})_{12}\tau C'\zeta = \xi^T C'\frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta = 0\\
\xi^T(\mathcal{V}\mathcal{K}^{-1})_{12}\tau C'\zeta &= \xi^T\tau C'(\mathcal{V}\mathcal{K}^{-1})_{21}\zeta = \xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta = -1\\
\xi^T\tau C'(\mathcal{V}\mathcal{K}^{-1})_{12}\zeta &= \xi^T(\mathcal{V}\mathcal{K}^{-1})_{21}\tau C'\zeta = -\xi^T\tau\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta = \kappa
\end{aligned}
\tag{E.29}
$$

Moreover

$$
\begin{aligned}
&(\chi^T\mathcal{K}^{-1})_1\xi = 0, \qquad (\chi^T\mathcal{K}^{-1})_1\tau C'\xi = a^\dagger\tau\xi\\
&(\chi^T\mathcal{K}^{-1})_2\xi = a^\dagger C'\xi, \qquad (\chi^T\mathcal{K}^{-1})_1\tau C'\xi = 0
\end{aligned}
\tag{E.30}
$$

with analogous equations for ζ.

In evaluating (E.28, E.29, E.30) we have used the methods of ref.[38] (see also [76]), together with eqs.(6.72, 5.95). These results are all we need to explicitly compute (E.25). In fact it is easy to verify that the latter can be mapped to a rather simple combinatorial problem. To show this we introduce generic variables $x, y, \bar{x}, \bar{y}$, and make the following formal replacements:

$$
\begin{aligned}
A &\equiv \chi^T\mathcal{K}^{-1}M \longrightarrow x(a^\dagger\tau\xi) + \bar{y}(a^\dagger C'\zeta),\\
B &\equiv M^T\mathcal{V}\mathcal{K}^{-1}M \longrightarrow (-\kappa xy + \kappa x\bar{y} - \bar{x}y - \kappa\bar{x}\bar{y})
\end{aligned}
\tag{E.31}
$$

and

$$
(\tau C'\xi)_l^\alpha\frac{\partial}{\partial\mu_l^\alpha} = \partial_x, \quad \zeta_j^\beta\frac{\partial}{\partial\mu_j^\beta} = \partial_y, \quad (\tau C'\xi)_{\bar{l}}^{\bar{\alpha}}\frac{\partial}{\partial\bar{\mu}_{\bar{l}}^{\bar{\alpha}}} = \partial_{\bar{x}}, \quad \zeta_{\bar{j}}^{\bar{\beta}}\frac{\partial}{\partial\bar{\mu}_{\bar{j}}^{\bar{\beta}}} = \partial_{\bar{y}}, \tag{E.32}
$$

Then (E.25) is equivalent to

$$
\partial_x^k\partial_y^k\partial_{\bar{x}}^p\partial_{\bar{y}}^p\, e^{-A-\frac{1}{2}B}\Big|_{x=\bar{x}=y=\bar{y}=0} \tag{E.33}
$$

This in turn can be easily calculated and gives

$$
\sum_{m=0}^{[p,k]}\mathrm{x}^m\frac{k!p!}{m!}\sum_{l=m}^{[p,k]}(-1)^{l+m}\binom{k}{l}\binom{p}{l}\binom{l}{m}\kappa^{p+k-l-2m} \tag{E.34}
$$

where $[n, m]$ stands for the minimum between n and m. Now we insert this back into the original equation (E.24), we find

$$|\Lambda_n\rangle * |\Lambda_{n'}\rangle = \sum_{k=0}^{n}\sum_{p=0}^{n'}\sum_{m=0}^{[p,k]}\sum_{l=0}^{[p-m,k-m]} \frac{(-1)^{p+k+l}}{m!} \tag{E.35}$$
$$\cdot \kappa^{n+n'-l-2m} \binom{n}{k}\binom{n'}{p}\binom{k}{m}\binom{k-m}{l}\binom{p}{l+m} \mathrm{x}^m|\mathcal{S}_\perp\rangle$$

In order to evaluate these summations we split them as follows

$$\sum_{k=0}^{n}\sum_{p=0}^{n'}\sum_{m=0}^{[p,k]}\sum_{l=0}^{[p-m,k-m]}(\ldots) = \sum_{k=0}^{n}\left(\sum_{p=k+1}^{n'}\sum_{m=0}^{k}\sum_{l=0}^{k-m} + \sum_{p=0}^{k}\sum_{m=0}^{p}\sum_{l=0}^{p-m}\right)(\ldots) \tag{E.36}$$

Next we replace $l \to l+m$ and (E.36) becomes

$$\sum_{k=0}^{n}\left(\sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=k+1}^{n'} + \sum_{m=0}^{k}\sum_{p=m}^{k}\sum_{l=m}^{p}\right)(\ldots) =$$
$$= \sum_{k=0}^{n}\left(\sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=k+1}^{n'} + \sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=l}^{k}\right)(\ldots) = \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=l}^{n'}(\ldots) \tag{E.37}$$

Summarizing, we have now to calculate

$$|\Lambda_n\rangle * |\Lambda_{n'}\rangle = \sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{p=l}^{n'} \frac{(-1)^{p+k+l+m}}{m!} \tag{E.38}$$
$$\cdot \kappa^{n+n'-l-m} \binom{n}{k}\binom{n'}{p}\binom{k}{m}\binom{k-m}{l-m}\binom{p}{l} \mathrm{x}^m|\mathcal{S}_\perp\rangle$$

Now

$$\sum_{p=l}^{n'}(-1)^{p+l}\binom{n'}{p}\binom{p}{l} = \binom{n'}{l}\sum_{p=0}^{n'-l}(-1)^p\binom{n'-l}{p} = \binom{n'}{l}(1-1)^{n'-l} \tag{E.39}$$

This vanishes unless $l = n'$. In the case $n' > n$, $l < n'$. Inserting this into (E.38), for $n' > n$ we get 0.

In the case $n = n'$, l can take the value n'. This corresponds to the case $k = p = l = n = n'$ in eq.(E.38). The result is easily derived

$$|\Lambda_n\rangle * |\Lambda_n\rangle = \sum_{m=0}^{n} \frac{(-1)^{n+m}}{m!}\binom{n}{m}\kappa^{n-m}\mathrm{x}^m|\mathcal{S}_\perp\rangle = (-\kappa)^n L_n\left(\frac{\mathrm{x}}{\kappa}\right)|\mathcal{S}_\perp\rangle = |\Lambda_n\rangle \tag{E.40}$$

This proves eq.(6.75).

One could as well derive these results numerically. For instance, in order to obtain (E.40) one could proceed, alternatively, as follows. After setting $n = n'$ in (E.35), one realizes that $|\Lambda_n\rangle * |\Lambda_n\rangle$ has the form

$$|\Lambda_n\rangle * |\Lambda_n\rangle = \sum_{m=0}^{n} F_m^{(n)} \left(\frac{\mathrm{x}}{\kappa}\right)^m |\mathcal{S}_\perp\rangle \tag{E.41}$$

where

$$\begin{aligned} F_m^{(n)} &= 2\sum_{p=0}^{n-m}\sum_{k=0}^{p}\sum_{l=0}^{k} \frac{(-1)^{p+k+l}\kappa^{2n-l-m}(n!)^2}{(m!)^2(n-k-m)!(n-p-m)!l!(l+m)!(k-l)!(p-l)!} \\ &- \sum_{p=0}^{n-m}\sum_{l=0}^{p} \frac{(-1)^l\kappa^{2n-l-m}(n!)^2}{[m!(n-p-m)!(p-l)!]^2 l!l+m)!} \end{aligned} \tag{E.42}$$

This corresponds to the desired result if

$$F_m^{(n)} = \frac{(-1)^{n+m}}{m!}\kappa^n \binom{n}{m} \tag{E.43}$$

Using *Mathematica* one can prove (numerically) that this is true for any value of n and m a computer is able to calculate in a reasonable time.

The value of the SFT action for any solution $|\Lambda_n\rangle$ is given by

$$\mathcal{S}(\Lambda_n) = \mathcal{K}\langle\Lambda_n|\Lambda_n\rangle \tag{E.44}$$

where $\mathcal{K}$ contains the ghost contribution. As shown in [39], $\mathcal{K}$ is infinite unless it is suitably regularized. Nevertheless, as argued there, $|\Lambda_n\rangle$, together with the corresponding ghost solution, can be taken as a representative of a corresponding class of smooth solutions.

Our task now is to calculate $\langle\Lambda_n|\Lambda_n\rangle$. However it may be important to consider states which are linear combinations of $|\Lambda_n\rangle$. In order to evaluate their action we have to be able to compute $\langle\Lambda_n|\Lambda_{n'}\rangle$. Without loss of generality we can assume $n' > n$. By defining $\tilde{\mathrm{x}} = (a^\dagger\tau C'\xi)\,(a^\dagger\zeta)$ we get

$$\begin{aligned} \langle\Lambda_n|\Lambda_{n'}\rangle &= (-\kappa)^{n+n'}\langle\tilde{0}|L_n(\tilde{\mathrm{x}}/\kappa)e^{-\frac{1}{2}a\tilde{\mathcal{S}}a}L_{n'}(\mathrm{x}/\kappa)e^{\frac{1}{2}a^\dagger\mathcal{S}a^\dagger}|\tilde{0}\rangle \\ &= L_n\left(\frac{1}{\kappa}(\tau C'\xi)_l^\alpha\zeta_j^\beta\frac{\partial}{\partial\lambda_l^\alpha}\frac{\partial}{\partial\lambda_j^\beta}\right)L_{n'}\left(\frac{1}{\kappa}(\tau\xi)_l^\alpha(C'\zeta)_j^\beta\frac{\partial}{\partial\mu_l^\alpha}\frac{\partial}{\partial\mu_j^\beta}\right) \\ &\cdot \frac{1}{\sqrt{\det(\mathbb{I}-\mathcal{T}^2)}}\, e^{\lambda C'\frac{1}{\mathbb{I}-\mathcal{T}^2}C'\mu-\frac{1}{2}\lambda C'\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\lambda-\frac{1}{2}\mu\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}C'\mu}\Big|_{\lambda=\mu=0} \end{aligned} \tag{E.45}$$

For the derivation of this equation, see [37, 34, 38]. Now, let us set

$$A = \lambda C' \frac{1}{\mathbb{I} - \mathcal{T}^2} C' \mu, \qquad B = \lambda C' \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \lambda, \qquad C = \mu \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} C' \mu$$

and introduce the symbolic notation

$$(\tau C'\xi)_l^\alpha \frac{\partial}{\partial \lambda_l^\alpha} = \partial_x, \quad \zeta_j^\beta \frac{\partial}{\partial \lambda_j^\beta} = \partial_y, \qquad (\tau\xi)_l^\alpha \frac{\partial}{\partial \mu_l^\alpha} = \partial_{\bar{x}}, \qquad (C'\zeta)_j^\beta \frac{\partial}{\partial \mu_j^\beta} = \partial_{\bar{y}}, \tag{E.46}$$

Then, using (5.95) and (E.28, E.29), we find

$$\begin{aligned} &\partial_x \partial_{\bar{x}} A = 0, \qquad \partial_x \partial_{\bar{y}} A = -1, \qquad \partial_y \partial_{\bar{x}} A = -1, \qquad \partial_y \partial_{\bar{y}} A = 0 \\ &\partial_x \partial_x B = 0, \qquad \partial_x \partial_y B = -2\kappa, \qquad \partial_y \partial_y B = 0, \\ &\partial_{\bar{x}} \partial_{\bar{x}} C = 0, \qquad \partial_{\bar{x}} \partial_{\bar{y}} C = -2\kappa, \qquad \partial_{\bar{y}} \partial_{\bar{y}} C = 0 \end{aligned} \tag{E.47}$$

We can therefore make the replacement

$$A - \frac{1}{2} B - \frac{1}{2} C \to \kappa x y + \kappa \bar{x} \bar{y} - x \bar{y} - \bar{x} y \tag{E.48}$$

In (E.45) we have to evaluate such terms as

$$\partial_x^k \partial_y^k \partial_{\bar{x}}^p \partial_{\bar{y}}^p (\kappa x y + \kappa \bar{x} \bar{y} - x \bar{y} - \bar{x} y)^{k+p}$$

for any two natural numbers k and p. It is easy to obtain

$$\frac{1}{(p+k)!} \partial_x^k \partial_y^k \partial_{\bar{x}}^p \partial_{\bar{y}}^p (\kappa x y + \kappa \bar{x} \bar{y} - x \bar{y} - \bar{x} y)^{k+p} = \sum_{s=0}^{[p,k]} \binom{k}{s} \binom{p}{s} k!\, p!\, \kappa^{p+k-2s} \tag{E.49}$$

Therefore we have

$$\begin{aligned} &\langle \Lambda_n | \Lambda_{n'} \rangle \\ &= \sum_{k=0}^{n} \sum_{p=0}^{n'} \frac{(-1)^{k+p} \kappa^{n+n'-p-k}}{k! p!} \binom{n}{k} \binom{n'}{p} \partial_x^k \partial_y^k \partial_{\bar{x}}^p \partial_{\bar{y}}^p \, e^{A - \frac{1}{2} B - \frac{1}{2} C} \Big|_{x=y=\bar{x}=\bar{y}=0} \langle \mathcal{S}_\perp | \mathcal{S}_\perp \rangle \\ &= \sum_{k=0}^{n} \sum_{p=0}^{n'} \frac{(-1)^{k+p}}{k! p!} \binom{n}{k} \binom{n'}{p} \sum_{s=0}^{[p,k]} \binom{k}{s} \binom{p}{s} k! p! \kappa^{n+n'-2s} \langle \mathcal{S}_\perp | \mathcal{S}_\perp \rangle \end{aligned} \tag{E.50}$$

As in the previous subsection, we can rearrange the summations as follows,

$$\begin{aligned} \sum_{k=0}^{n} \sum_{p=0}^{n'} \sum_{p=0}^{[p,k]} (\ldots) &= \sum_{k=0}^{n} \left(\sum_{p=0}^{k} \sum_{s=0}^{p} + \sum_{p=k+1}^{n'} \sum_{s=0}^{k} \right) (\ldots) \\ &= \sum_{k=0}^{n} \left(\sum_{s=0}^{k} \sum_{p=s}^{k} + \sum_{s=0}^{k} \sum_{p=k+1}^{n'} \right) (\ldots) = \sum_{k=0}^{n} \sum_{s=0}^{k} \sum_{p=s}^{n'} (\ldots) \end{aligned} \tag{E.51}$$

In conclusion we have to compute

$$\langle\Lambda_n|\Lambda_{n'}\rangle = \sum_{k=0}^{n}\sum_{s=0}^{k}\sum_{p=s}^{n'}(-1)^{p+k}\frac{n!n'!}{(n-k)!(n'-p)!(k-s)!(p-s)!(s!)^2}\kappa^{n+n'-2s} \tag{E.52}$$

Now,

$$\sum_{p=s}^{n'}(-1)^p\frac{1}{(n'-p)!(p-s)!} = \sum_{p=0}^{n'-s}\frac{(-1)^{p+s}}{(n'-s)!}\begin{pmatrix} n'-s \\ p \end{pmatrix} = \frac{(-1)^s}{(n'-s)!}(1-1)^{n'-s} \tag{E.53}$$

The right end side vanishes if $n' \neq s$, which is certainly true if $n' > n$. Therefore in such a case, inserting (E.53) into (E.52) we get $\langle\Lambda_n|\Lambda_{n'}\rangle = 0$. When $s = n'$, eq.(E.53) is ambiguous. But this corresponds to $p = k = s = n = n'$ in (E.52). The relevant contribution is elementary to compute, and one gets

$$\langle\Lambda_n|\Lambda_n\rangle = \langle\Lambda_0|\Lambda_0\rangle \tag{E.54}$$

This completes the proof of (6.76).

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