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Relaxation and Nonconvex Problems in the Calculus of Variations

Fabián Flores-Bazán

Thesis submitted for the degree of "Doctor Philosophiae"
Academic Year 1991 - 92

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RELAXATION AND NONCONVEX PROBLEMS IN THE CALCULUS OF VARIATIONS

Contents

Introduction

In the study of minimization problems within the framework of the Calculus of Variations, the lower semicontinuity property, or a weak version, plays a prominent role. The so-called Direct Method is the most classical way to obtain the existence of minima for functionals that are lower semicontinuous and coercive with respect to a suitable topology. In many problems deriving from applications, the coerciveness property is already guaranteed. However, the lower semicontinuity is not an acceptable hypothesis as it is showed in [Er, Gur-T]. Precisely, the main point to be discussed throughout this thesis is the lack of lower semicontinuity or convexity in the so-called scalar problems. Recently, these kind of functionals are being active areas of investigations.

In Chapter 1 we give some notations and recall some preliminary facts in Convex Analysis and we describe the well known Direct Method in the Calculus of Variations. The Relaxation Method is also introduced as well as the classical Liapunov theorem used in Chapter 2. In this chapter, we deal with non-convex integrals on a symmetric domain of $\mathbf{R}^n$ involving the Laplacian. We present several classes of integrals for which the minimization problem admits at least one radially symmetric solution in $W_0^{2,p}$ or $W^{2,p} \cap W_0^{1,p}$. In contrast with previous papers in this direction [A-T1, A-T2, A-T3, R3, R4, T1], these integrands give rise to functionals that are l.s.c. only along some special minimizing sequences converging to a mimimun and non-differentiable ([C-F, F1, F2, F3]). Problems with obstacle are also discussed. Chapter 3 is devoted to the study of the asymptotic behaviour of the minimizing sequences for general integrals depending of an elliptic differential operator of order $2k$ defined in a closed affine subspace of $W^{2k,p} \cap W_0^{k,p}$ containing $W_0^{2k,p}$. The asymptotic behaviour of the minimizing sequences is studied with respect to weak convergence. The relevance of this kind of convergence ([Tar]) relies on the fact that in many physical applications, only averages of physical quantities are actually measured. The results ([F5]) established here improve and generalize those by Ekeland-teman [E-T, Thm.3.1 and Thm.4.1 in Chapter IX]. Some consequences of the lack of lower semicontinuity in the abstract setting and the lack of convexity for integrals of the gradient or of the Laplacian are discussed in Chapter 4 ([F4]). For a given non-lower l.s.c. function we show the existence of non-negative continuous functions (perturbed of the first), in such a way that the perturbed function does not attain its minimum. Afterwards, we deal with the simplest integrals involving either the gradient or the Laplacian. For both cases, we are able to construct continuous perturbations of integral-type whose integrand involves the state function and possibly the space variable (compare with Theorem 5.1 of [B-Mu]). Finally, in Chapter 5 we present some applications of Liapunov's theorem revisited recently established in [Br]. Given a multifunction $F : \mathbf{R}^n \to 2^{\mathbf{R}^m}$, we consider the so-called marginal distributions, obtained by integrating F along a family of parallel lines. We prove analogous results as in the case of Aumann's integral [Au]: the closure and convexity of these marginal distributions as elements of $L^1(\mathbf{R}^{n-1}, \mathbf{R}^m)$. A Bang-Bang theorem for the controlled wave equation and a non-convex optimization problem for the wave equation are presented. In contrast with papers concerning the latter point [Pul, Sur2, Sur3], we allow constrains to be placed on the entire boundary of the domain ([Br-F]).

Trieste, Summer 1992

Fabián Flores-Bazán.

CHAPTER 1

Some Preliminary Facts

Throughout this thesis, $m, n \in \mathbf{N}$, $n \geq 2$, $p \in \mathbf{R}$ $p > 1$. For any bounded and open set $\Omega \subset \mathbf{R}^n$; $W^{m,p}(\Omega)$ denotes the usual Sobolev space. If $p = 2$ we set $H^m(\Omega) = W^{m,2}\Omega$. In addition, we consider $W_0^{m,p}(\Omega)$ the closure of $C_0^m(\Omega)$ in $W^{m,p}(\Omega)$. For their properties as for the definition of fractional order Sobolev spaces $W^{s,p}(\Omega)$ $s > 0$, we refer to [Ad]. In particular, we recall just a proposition that will be used implicitly in the formulation of the minimization problems dealt in next chapter.

Proposition 1.1. ([Ad, Thm. 7.53]) *If Ω has a bounded smooth boundary, the operator $\Gamma : W^{2,p}(\Omega) \to W^{2-\frac{1}{p},p}(\partial\Omega) \times W^{1-\frac{1}{p},p}(\partial\Omega)$ defined as follows*

$$\Gamma(u) = \left\{ u \mid_{\partial\Omega}, \frac{\partial u}{\partial n} \mid_{\partial\Omega} \right\}$$

is linear, continuous and surjective. Moreover, $W_0^{2,p}(\Omega) = Ker\Gamma$, i.e.

$$W_0^{2,p}(\Omega) = \left\{ u \in W^{2,p}(\Omega) : u = \frac{\partial u}{\partial n} = 0 \quad on \quad \partial\Omega \right\}.$$

1.1. Some Facts in Convex Analysis.

We now collect some basic definitions and preliminary results in Convex Analysis.

Definitions. Let f be a function from $\mathbf{R}^n$ into $]-\infty, +\infty]$ such that $f(x_0) < +\infty$ at some x_0.

(i) The function $f^* : \mathbf{R}^n \to]-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup_{x \in \mathbf{R}^n} \left\{ \langle x, x^* \rangle - f(x) \right\}$$

($\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbf{R}^n$), is called the *conjugate*, or *polar* function of f. It is a lower semicontinuous (l.s.c.) convex function.

(ii) The function $f^{**} : \mathbf{R}^n \to]-\infty, +\infty]$ defined by

$$f^{**}(x) = \sup_{x^* \in \mathbf{R}^n} \left\{ \langle x, x^* \rangle - f^*(x^*) \right\}$$

is called the *biconjugate*, or *bipolar* function of f. Notice that $f^{**}(x) \leq f(x)$.
A relationship between f and f^{**} is given by

Proposition 1.2. ([Bré, Prop.I.8], [E-T, Prop.I.4.1; Coroll. I.2.3])
(a) *A function f is l.s.c. and convex if and only if $f \equiv f^{**}$.*
(b) *f^{**} is the largest convex l.s.c. function not larger than f.*
(c) *Assume f be convex, then f is continuous in the interior of* dom$f$*(the set where f is finite).*

We say that the function f has as an *exact minorant* at $x \in \mathbf{R}^n$ the affine function l if $l(y) \leq f(y) \ \ \forall y$ and $l(x) = f(x)$. Thus $f(x) < +\infty$ and $l(y) = \langle x^*, y - x\rangle + f(x)$.
The function f is said to be *subdifferentiable* at the point $x \in \mathbf{R}^n$ provided there exists an affine function which is an exact minorant of f at x. The slope x^* of such affine function is said to be a *subgradient* of f at x, and the set of all subgradients at x is denoted by $\partial f(x)$. The set valued map $\partial f : \mathbf{R}^n \to 2^{\mathbf{R}^n}$ is called the *subdifferential* of f and f is subdifferentiable at x provided $\partial f(x) \neq \emptyset$. Clearly, $x^* \in \partial f(x)$ if and only if $f(x)$ is finite and the subgradient inequality

$$f(y) - f(x) \geq \langle x^*, y - x\rangle \ \ \forall y \in \mathbf{R}^n,$$

holds.

Proposition 1.3. ([E-T, Prop.I.5.2])
Let f be a convex function of $\mathbf{R}^n$ into $]-\infty, +\infty]$. Then $\partial f(y) \neq \emptyset \ \ \forall y \in$ Int(dom$f$).

Let Ω be an open subset of $\mathbf{R}^n$ provided with the Lebesgue measure and denote by $\mathcal{L}$ the σ-algebra of (Lebesgue) measurable subsets of Ω and, by $\mathcal{B}(\mathbf{R}^m)$ the Borel σ-algebra of $\mathbf{R}^m$. We denote by $\mathcal{L} \otimes \mathcal{B}(\mathbf{R}^m)$ the product σ-algebra on $\Omega \times \mathbf{R}^m$ generated by all the sets of the form $A \times B$ with $A \in \mathcal{L}$ and $B \in \mathcal{B}(\mathbf{R}^m)$.
We recall that a function $f : \Omega \times \mathbf{R}^m \to]-\infty, +\infty]$ is called $\mathcal{L} \otimes \mathcal{B}(\mathbf{R}^m)$-measurable or simply measurable if the inverse image under f of every closed subset of $]-\infty, +\infty]$ is measurable, in other words, $f^{-1}(C) \in \mathcal{L} \otimes \mathcal{B}(\mathbf{R}^m)$ for every closed subset of $]-\infty, +\infty]$.

Let $\xi \mapsto h^{**}(x,\xi)$ be the bipolar of the function $\xi \mapsto h(x,\xi)$. We have the following

Proposition 1.4. ([E-T, Lemma IX.3.3; Prop. IX.3.1])

(a) *Let $h : \Omega \times \mathbf{R}^m \to]-\infty, +\infty]$ be such that:*

(h_1) *h is $\mathcal{L} \otimes \mathcal{B}(\mathbf{R}^m)$-measurable;*

(h_2) *$\xi \mapsto h(x,\xi)$ is lower semicontinuous for almost all x in Ω;*

(h_3) *there exists a positive constant α such that*

$$h(x,\xi) \geq \alpha|\xi|^p - \beta(x), \text{ where the function } \beta \text{ is in } L^1(\Omega).$$

Then

$$h^{**}(x,\xi) = \min\Big\{ \sum_{i=1}^{m+1} \lambda_i h(x,\xi_i) : \ \xi = \sum_{i=1}^{m+1} \lambda_i \xi_i; \ \lambda_i \geq 0; \ \sum_{i=1}^{m+1} \lambda_i = 1 \Big\}.$$

(b) *Let z be measurable. Then there exist $m+1$ measurable $p_i : \Omega \to [0,1]$ and $m+1$ measurable $v_i : \Omega \to \mathbf{R}$, such that:*

$$\sum_{i=1}^{m+1} p_i(x) = 1; \ z(x) = \sum_{i=1}^{m+1} p_i(x) v_i(x); \ h^{**}(x, z(x)) = \sum_{i=1}^{m+1} p_i(x) h(x, v_i(x)).$$

1.2. The Direct Method

In the sequel X denotes a topological space. For any function F from X into $]-\infty,+\infty]$,

$$\text{epi}F = \Big\{ (x,t) \in X \times \mathbf{R} : \ F(x) \leq t \Big\}$$

is the *epigraph* of F and

$$\text{dom}F = \Big\{ x \in X : \ F(x) < +\infty \Big\}$$

is the *effective domain* of F.

Definitions. A function $F : X \to]-\infty, +\infty]$ is said to be:

(i) *lower semicontinuous (l.s.c.)* on X if

$$F(x) \leq \liminf_{y \to x} F(y) \quad \text{for every } x \in X.$$

(ii) *sequential l.s.c. (s.l.s.c.)* on X if

$$F(x) \leq \liminf_{h\to+\infty} F(x_h) \quad \text{for every } x \text{ and for every sequence } (x_h) \text{ converging to } x.$$

(iii) *coercive* (resp. *sequentially coercive*) on X if $\forall t \in \mathbf{R}$ the set $\{x : F(x) \leq t\}$ is compact (resp. seq. compact).

We now state the so-called Direct Method in the Calculus of Variations.

Theorem 1.5. (for instance [DM, Thm.1.15]) *Let X be a topological space, $F : X \to]-\infty, +\infty]$ be a function such that*
(i) F is l.s.c. (resp. s.l.s.c.) on X and
(ii) coercive (resp. seq. coercive).
Then F attains its infimum.

To prove the sequential version of this theorem one may proceed as follows: we start by taking any minimizing sequence. The seq. coercivity implies that such sequence admits a convergent subsequence, thus F being s.l.s.c. we have that the limit is actually a minimum for F.

In concrete problems, X is usually a Banach space and the topology to be considered is the weak one. In this context, the classical approach to the problem of minimizing F over X consists in assuming X to be reflexive and (ii') below instead of (ii). Then, the sequential version of Theorem 1.5 is used.
(ii') There are constants c_1, c_2 with $c_1 > 0$ such that $F(x) \geq c_1 ||x||_X + c_2 \quad \forall x \in X$.
In this situation (ii') implies that every minimizing sequence is bounded in X, the reflexivity of X asserts that any bounded sequence has a weakly convergent subsequence, and from the (weakly) s.l.s.c. of F, one concludes that the weak limit is, in fact, a minimum for F. This is summarized in the following Theorem:

Theorem 1.6. ([E-T, Prop. II.1.2]) *Let X be a reflexive Banach space, $F : X \to]-\infty, +\infty]$ be a sequentially weakly l.s.c. function satisfying (ii'). Then F attains its imfimum.*

For weak l.s.c. functionals F of integral type depending of the gradient, the following characterization has been given.

Theorem 1.7. ([M-Sb, Thm.2.4; Da3, Thm. III.3.1 & Thm. 3.4]) *Let Ω be a bounded open set in $\mathbf{R}^n$, $p \in [1,+\infty]$, $u_0 \in W^{1,p}(\Omega)$. Let $f : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ be a function satisfying:*

(i) $0 \le f(x,u,\xi) \le g(x,|u|,|\xi|)$ *with g increasing with respect to $|u|$ and $|\xi|$, and locally integrable in x;*

(ii) $f(\cdot,u,\xi)$ *is $\mathcal{L}$-measurable for every $u \in \mathbf{R}$, $\xi \in \mathbf{R}^n$;*

(iii) *for almost every $x \in \Omega$, $f(x,\cdot,\cdot)$ is continuous.*

Then the functional

$$F(u) = \int_\Omega f(x,u(x),\nabla u(x))dx$$

is $W^{1,p}$-weakly (weakly*, if $p = +\infty$) lower semicontinuous if and only if $f(x,u,\cdot)$ is convex.

1.3. Relaxation

We return now to the general case where F no longer is l.s.c.. Thus we no longer have Theorem 1.5 at our disposal. It is natural to introduce the function $sc^- F$: the l.s.c. envelope (or relaxed function) of F, which is the greatest l.s.c. function majorized by F (see [Bu2, DM]). The relationships between the minimum problem $\min_{x\in X} F(x)$ and the relaxed problem $\min_{x\in X} sc^- F(x)$ is given by Theorem 1.8 below. In particular this theorem describes the behaviour of the minimizing sequences for F in terms of the minimum points for $sc^- F$.

Theorem 1.8. ([DM, Thms. 3.5 & 3.8; Bu2, Prop. 1.3.1]) *Let $F : X \to]-\infty,+\infty]$ be a given function. Then the following properties hold:*

(a) *for every $x \in X$,*

$$sc^- F(x) = \liminf_{y\to x} F(y) = \min\left\{ \liminf_{j\in J} F(x_j) : \quad (x_j) \text{ is a net converging to } x \right\};$$

(b) *if F is coercive then $sc^- F$ is coercive and l.s.c.. Thus attains its infimum by Theorem 1.5.;*

(c) *the epigraph of $sc^- F$ is the closure in $X \times \mathbf{R}$ of the epigraph of F;*

(d) $\min_{x\in X} sc^- F(x) = \inf_{x\in X} F(x)$;

(e) *every cluster point of a minimizing sequence for* F *is a minimum point for* sc^-F;
(f) *if* X *satisfies the first axiom of countability, then every minimum point for* sc^-F *is the limit of a minimizing sequence for* F *in* X.

Remark 1.9. If x is a minimum point for sc^-F such that $sc^-F(x) = F(x)$, then (c) of the previous theorem implies that x is also a minimum point for F.

Definition. By the *sequential version* of a topology τ on a space X, we mean the topology on X whose closed sets are the sequentially closed sets of X for the topology τ. This seq. version of a topology τ will be denoted by τ_{seq}.

The following proposition shows that the seq. lower semicontinuity is a topological concept.

Proposition 1.10. ([Bu2, Prop. 1.15])
(a) τ_{seq} *is the strongest topology on* X *for which the converging sequences so does (in* τ*);*
(b) F *is seq.l.s.c. if and only if* F *is l.s.c. in* τ_{seq};
(c) $\tau_{seq} = \tau$ *for every space* X *satisfying the first axiom of countability.*

We denote by $sc^-_{seq}F$ the relaxed function of F when the original topology is substituted by its sequential version.

Proposition 1.11. ([Bu2, Prop. 1.3.5])
Let $F : X \to]-\infty, +\infty]$ *be a function. Assume that*
(i) *every compact subset of* X *is metrizable;*
(ii) F *is coercive.*
Then

$$sc^-F(x) = sc^-_{seq}F(x) = \inf\left\{ \liminf_{h\to\infty} F(x_h) : \quad (x_h) \textit{ converges to } x \right\}.$$

Remark 1.12. (See [Du-Sch, page 434]) Hypothesis (i) of the previous proposition verifies either if X is a separable Banach and the topology to be considered is the weak one on X, or $X = V'$ with V a separable Banach space and X is equipped with the weak* topology.

As we mentioned before, concrete variational problems regards functions of integral type defined on Sobolev spaces. Thus, in order to apply the previous results we need to know the explicit form of the relaxed function, in particular, whether it is of integral type too. The latter gives rise to the problem of integral representation that can be studied in a more general setting (see for instance [DM, Bu2]).

In Chapter 3 we shall present some relaxation theorems for integrals depending of an elliptic differential operator of order $2k$ (for instance the Laplacian) defined on $W_0^{2k,p}(\Omega)$. The proof involves a technique widely used in the Theory of Γ-convergence. Indeed, relaxation may be considered as a particular case of Γ-convergence.

1.4. The Classical Liapunov Theorem

We start by recalling the classical version of the famous Liapunov's theorem extensively used in non-convex optimization problems.

Proposition 1.15.
Let Ω be a measurable subset of $\mathbf{R}^n$ (not necessarily with finite Lebesgue measure), let $f_1,...,f_k$ be measurable functions from Ω to $\mathbf{R}^m$ and let $p_1,...,p_k$ be measurable functions from Ω to $[0,1]$ such that:

$$\sum_{i=1}^{k} p_i(x) = 1 \text{ a.e. on } \Omega \text{ and for } j = 1,...,m \ \sum_{i=1}^{k} p_i|f_i^j| \text{ is in } L^1(\Omega),\ f_i = (f_i^1,...,f_i^m).$$

Then, there exists a measurable partition of Ω, $(\Omega_i)_i$, $i = 1,...,k$ such that

$$\int_\Omega \sum_{i=1}^{k} p_i(x) f_i(x)dx = \sum_{i=1}^{k} \int_{\Omega_i} f_i(x)dx.$$

Moreover, for all $i = 1,...,k$, the function f_i is in $L^1(\Omega_i, \mathbf{R}^m)$.

Remark 1.16. Actually the classical version assumes the integrability of the functions f_i and $meas(\Omega) < +\infty$. In this case we refer to [Ce, Thm.16.1.V]. The author in [R2, Prop.4.1] (see also [Ma1]), observed that it suffices to require the function $\sum_{i=1}^k p_i|f_i^j|$ be in $L^1(\Omega)$ instead of each f_i be in L^1. This weaker assumption is suggested in [C-C], and is verified in all the situations presented in optimization as a consequence of the growth condition usually imposed on the cost function. In fact, we actually know that

$\sum_{i=1}^{k} p_i f_i \in [L^1(\Omega)]^m$ and from the growth condition one gets $f_i^j \geq \delta_i^j$ for some δ_i^j in $L^1(\Omega)$. Both assertions imply that $\sum_{i=1}^{k} p_i |f_i^j|$ is in $L^1(\Omega)$. In case $meas(\Omega) = +\infty$, the result follows from the fact that $\mathbf{R}^n = \cup_{i=1}^{\infty} K_i$ for some measurable partition $(K_i)_i$, with K_i having finite Lebesgue measure. So that, $\Omega = \cup_{i=1}^{\infty}(K_i \cap \Omega)$, where $K_i \cap \Omega$ has finite Lebesgue measure, and one can repeat the argument used in [R2, Prop. 4.1]. The latter allows us to formulate the Liapunov theorem (in the form as above) in a measure space with a σ-finite measure, or possible in a measure space, where the measure (Borel) is regular and the space is σ-compact. Of course, such a space must fulfill the another standard requirements.

The relevance of previous proposition in non-convex minimization problems was pointed out for the first time by L.W.Neustadt [Ne]. He dealt with the so-called Bang-Bang principle in Optimal Control. This method then, was adapted for solving several problems in Optimal Control, mainly by Cesari [Ce]. Recently, Cellina-Colombo [C-C] applied such a theorem to the classical problem in the Calculus of Variations, giving, as a sufficient condition to get existence of minima, the concavity of the functional with respect to the state function, condition already formulated in [R1] in a very different setting. Further developments of this method can be found in [Ma1, Ma2, R2, R3, R4].

CHAPTER 2

Integral Functionals of the Laplacian

This Chapter is devoted to the problem of the existence of solutions to

$$\left.\begin{array}{l}\min \displaystyle\int_\Omega g(|x|, u(x)dx + \int_\Omega h(|x|, \Delta u(x) - \lambda u(x))dx \\ u \in X,\end{array}\right\} \tag{P}$$

where X is either $W_0^{2,p}$ or $W^{2,p} \cap W_0^{1,p}$ and Ω is the unit ball or an annulus in $\mathbf{R}^n$. The function $h(|x|, \cdot)$ being non convex, it is natural to consider the problem (P^{**}), i.e. problem (P) with h^{**} instead of h. The classical approach ([A-T2, A-T3, Ra, R3, R4, T1]) to get the existence of solutions is by imposing conditions such that every solution to problem (P^{**}) is, in fact, a solution to problem (P). In this situation one deals with functionals that are weakly lower semicontinuous (w.l.s.c.) along every minimizing sequence. Thus, one excludes problems (P), where the corresponding functional is w.l.s.c. only along special minimizing sequences. In particular, the method proposed in any of the papers mentioned above, cannnot be applied, for instance, if $n = 2$, $\lambda = 0$, $h(r, s) = (1 - s^2)^2$ and $g \equiv 0$. Of course, the corresponding problem (P) admits solution in X, $p = 4$. Indeed, the function u_1 defined by

$$u_1(x) = \begin{cases} \dfrac{1}{2}(-\dfrac{|x|^2}{2} + \log\sqrt{2}), & \text{if } \ 0 \le |x| \le \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{2}(\dfrac{|x|^2}{2} - \log|x|) - \dfrac{1}{4}, & \text{if } \ \dfrac{1}{\sqrt{2}} < |x| \le 1, \end{cases}$$

in case Ω is the unit ball, or that defined by

$$u_1(x) = \begin{cases} \dfrac{1}{2}(\dfrac{|x|^2}{2} - \log|x|) - \dfrac{1}{4}, & \text{if } \ 1 \le |x| \le a, \\ \dfrac{1}{2}(-\dfrac{|x|^2}{2} + (2a^2 - 1)\log|x| + \dfrac{2a^2 - 1}{2}) - a^2 \log(a), & \text{if } \ a \le |x| \le b, \\ \dfrac{1}{2}(\dfrac{|x|^2}{2} - 9\log|x| - \dfrac{9}{2}) + \dfrac{9}{2}\log(3), & \text{if } \ b \le |x| \le 3, \end{cases}$$

for some suitable constants a, b such that $b^2 - a^2 = 4$ in case Ω is the annulus $\{x : 1 < |x| < 3\}$, satisfies the boundary conditions: $u_1 = 0$ and $\frac{\partial u_1}{\partial n} = 0$ on $\partial\Omega$, and has a Laplacian taking values either $+1$ or -1, i.e. u_1 is a solution to the original problem. However, the convexified problem, where $h^{**}(r,s) = (1-s^2)_+^2$ has, among others, the solution u_2 identically zero, i.e., in this simple case, there are solutions to the convexified problem that are not solutions to the original problem. Another drawback presented in the papers mentioned above is that some regularity conditions on g and h had to be imposed, since the Euler-Lagrange equation associated to (P) is extensively used. So that, the simple case: $h(r,s) = (1-|s|)^2$ cannot be dealt with. Our results, to be stated presently, apply to functionals that are w.l.s.c. only along special minimizing sequences and not differentiable.

2.1. Notations and Auxiliary Results

Throughout this Chapter, n is an integer $2 \le n$, p is a real number such that $1 < p$. For any open bounded set $\Omega \subset \mathbf{R}^n$ with smooth boundary and for fixed $\lambda \ge 0$ we equip the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with the norm $||\Delta u - \lambda u||_{L^p(\Omega)}$ (or shortly $||\Delta u - \lambda u||_p$) which is equivalent to the usual one: $||u||_{W^{2,p}(\Omega)}$ (or shortly $||u||_{2,p}$). On the other hand, $W^{2,p}(\mathbf{R}^n)$ is equipped with the norm $||\Delta u - \lambda u||_{L^p(\mathbf{R}^n)}$, again equivalent to the usual one (see Proposition 2.1 below), here $\lambda > 0$ is fixed.

As we shall see, in case $\Omega = \mathbf{R}^n$, we need to solve equations of the form:

$$\begin{aligned} &\Delta w - \lambda w = f \\ &w \in W^{2,p}(\mathbf{R}^n), \end{aligned} \tag{$*$}$$

with $f \in L^p(\mathbf{R}^n)$, $0 < \lambda < +\infty$, $1 < p < +\infty$. Obviously, if $p = 2$ it suffices to apply the Lax-Milgram theorem in $H^2(\mathbf{R}^n)$ to the corresponding bilinear form to get the existence and uniqueness. The regularity can also be easily obtained (see the proof of Thm. IX.25 of [Bré]).

What follows will be devoted to solve problem $(*)$, in particular for $\lambda = 1$.

The Bessel kernel (see [S], [Z]); g_α, $\alpha > 0$ is defined as the function whose Fourier transform $\widehat{g}_\alpha$ is

$$\widehat{g}_\alpha(y) = (2\pi)^{-\frac{n}{2}}(1+|y|^2)^{-\frac{\alpha}{2}}.$$

where $\widehat{f}$ means the Fourier transform of the function f defined by

$$\widehat{f}(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-ixy} f(x) dx.$$

This definition can be extended to the space of Tempered distributions: $\mathcal{S}'(\mathbf{R}^n)$, being a continuous, linear, one-to-one mapping of $\mathcal{S}'(\mathbf{R}^n)$ onto $\mathcal{S}'(\mathbf{R}^n)$. For an analysis of the Fourier transform we refer to the books of Rudin [Ru], Stein-Weiss [S-W].

$\mathcal{L}^{\alpha,p}(\mathbf{R}^n)$, $\alpha > 0$, $1 \le p \le \infty$ denotes the space of those functions u such that $u = g_\alpha * f$, (the convolution of g_α and f) for some f in $L^p(\mathbf{R}^n)$. We have the following

Proposition 2.1. ([Z, Thm. 2.6.1], [S, Thm. V.3])
If k is a positive integer and $1 < p < \infty$, then

$$\mathcal{L}^{k,p}(\mathbf{R}^n) = W^{k,p}(\mathbf{R}^n).$$

Moreover, if $u \in \mathcal{L}^{k,p}(\mathbf{R}^n)$ with $u = g_k * f$, then

$$C^{-1}||f||_p \le ||u||_{k,p} \le C||f||_p$$

where $C = C(k,p,n)$.

As a consequence we have

Proposition 2.2. *Given f in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, the problem*

$$\Delta u - u = f, \quad u \in W^{2,p}(\mathbf{R}^n)$$

has a solution uniquely determined.

Proof. Define $u = g_2 * (-f)$. Then $u \in W^{2,p}(\mathbf{R}^n)$, basic properties of the Fourier transform yields $\widehat{(g_2 * (-f))} = \widehat{g_2}(-\widehat{f})$ in $\mathcal{S}'(\mathbf{R}^n)$, so that the following equality makes sense in $\mathcal{S}'(\mathbf{R}^n)$

$$\widehat{u} = -\frac{1}{1+|y|^2}\widehat{f} \ \text{ in } \ \mathcal{S}'(\mathbf{R}^n),$$

the right-hand side makes sense, because $\frac{1}{1+|y|^2}$ is in $C^\infty(\mathbf{R}^n)$ and it goes to zero as $|y|$ goes to $+\infty$. Therefore, the multiplication of this function by an element in $\mathcal{S}'(\mathbf{R}^n)$ is still in $\mathcal{S}'(\mathbf{R}^n)$. On the other hand, the multiplication of a Tempered distribution by a polynomial is still a Tempered distribution. Consequently,

$$(1+|y|^2)\widehat{u} = -\widehat{f} \quad \text{in} \quad \mathcal{S}'(\mathbf{R}^n).$$

But, $\widehat{\Delta u}(y) = -|y|^2\widehat{u}(y)$ in $\mathcal{S}'(\mathbf{R}^n)$. Therefore, $(\Delta u - u)\widehat{} = \widehat{\Delta u} - \widehat{u} = \widehat{f}$ in $\mathcal{S}'(\mathbf{R}^n)$. Hence

$$\Delta u - u = f \quad \text{in} \quad \mathcal{S}'(\mathbf{R}^n),$$

but, since $u \in W^{2,p}(\mathbf{R}^n)$, $f \in L^p(\mathbf{R}^n)$, we conclude

$$\Delta u - u = f \quad \text{a.e.} \quad \mathbf{R}^n.$$

A similar reasoning proves the uniqueness. ∎

$\mathbf{SO}(n)$ denotes the Rotation Group in $\mathbf{R}^n$ which has as elements the orthogonal matrices $A \in M(n)$ such that $det(A) = 1$: it is a compact and connected topological group [DNF]. Therefore, given $u \in W^{2,p}(\Omega)$, the integral

$$\int_{\mathbf{SO}(n)} u(Ax)d\mu(A)$$

is well defined, where μ is a left (or right) Haar measure on $\mathbf{SO}(n)$ with $\mu(\mathbf{SO}(n)) = 1$ [Co]. By the definition of $\mathbf{SO}(n)$, we have that its elements preserve the inner product, i.e.

$$\langle Ax, Ay\rangle = \langle x, y\rangle \qquad \forall A \in \mathbf{SO}(n).$$

Hence $|Ax| = |x|$. Furthermore, for fixed $x \in \mathbf{R}^n$, it is not difficult to show that:

$$\Big\{ Ax \in \mathbf{R}^n : \ A \in \mathbf{SO}(n) \Big\} = \Big\{ y \in \mathbf{R}^n : \ |y| = |x| \Big\}.$$

Henceforth $B = \{x \in \mathbf{R}^n : \ |x| < 1 \}$ and for fixed $b > a > 0$, $\Omega_a^b = \{x \in \mathbf{R}^n : \ a < |x| < b \}$ with boundary ∂B and $\partial\Omega_a^b = \Gamma_a \cup \Gamma_b$ respectively, where $\Gamma_a = \{x \in \mathbf{R}^n : \ |x| = a \}$ and Γ_b is defined similarly.

We have the following

Proposition 2.3. *Here Ω denotes either B or Ω_a^b. Let $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Define $\bar{u} : \Omega \to \overline{\mathbf{R}}$ by*

$$\bar{u}(x) = \int_{\mathbf{SO}(n)} u(Ax)d\mu(A).$$

Then

(a) $\bar{u}$ *is a radially symmetric function;*

(b) $\bar{u} \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$;

(c) $\frac{\partial \bar{u}}{\partial n} = 0$ *on* $\partial\Omega$;

(d) $\Delta\bar{u}(x) = \int_{\mathbf{SO}(n)} \Delta u(Ax)d\mu(A)$.

Proof. Clearly $\bar{u}$ is radially symmetric. We only prove when $\Omega = B$, the other case being entirely similar. Assume first $u \in C_0^1(\bar{\Omega})$, then $\bar{u}$ does so. Since $|Ax| = |x|$ we have that $\bar{u}$ vanishes on $\partial\Omega$ since u does so. We use Tonelli-Fubini's theorem (see [Co]) to prove that $\bar{u} \in W^{2,p}(\Omega)$. Let $A \in M(n)$, $A = (a_{ij})$, $|x| = 1$

$$\frac{\partial \bar{u}}{\partial n}(x) = \langle \nabla \bar{u}(x), \frac{x}{|x|}\rangle = \sum_{j=1}^{n} \frac{\partial \bar{u}}{\partial x_j}(x)\frac{x_j}{|x|} = \sum_{j=1}^{n} \int_{SO(n)} \sum_{i=1}^{n} \frac{\partial u}{\partial \xi_i}(Ax)a_{ij}\frac{x_j}{|x|}d\mu(A)$$
$$= \int_{\mathbf{SO}(n)} \sum_{i=1}^{n} \frac{\partial u}{\partial \xi_i}(\xi)\frac{\xi_i}{|x|}d\mu(A) = \int_{\mathbf{SO}(n)} \langle \nabla u(\xi), \frac{\xi}{|x|}\rangle d\mu(A)$$

where $\xi = Ax$ and $\xi_i = \sum_{j=1}^{n} a_{ij}x_j$. Since $|\xi| = |x|$: claim (c) is proved, and (d) is a consequence of the definition of $\mathbf{SO}(n)$. For general u, proceed by approximation. ■

Remark 2.4. In case $\Omega = \mathbf{R}^n$ we obtain exactly the same conclusion without the usual boundary conditions.

Next we establish a Liapunov's type theorem (see also [A-C]). We recall that h^{**} is the bipolar of the function h, i.e. h^{**} is the largest convex function not larger than h (see [E-T]).

Lemma 2.5. *Set* $I = [0,1]$ *and let* μ *be a non-atomic positive Radon measure on* I. *Let* z *be a measurable function and let* h *be a function such that* $t \mapsto h^{**}(z(t))$ *is in* $L^1(]0,1[)$. *Assume* $\{\xi \in \mathbf{R} : h^{**}(\xi) < h(\xi)\} = \cup_i]a_i, b_i[$, *where such intervals are supposed to be disjoint,* $-\infty < a_i < b_i < +\infty$, *where* i *runs over an at most numerable set. Let* $]a,b[$ *be such an interval and put* $E = \{t \in I : a < z(t) < b\}$. *Then there exists a measurable function* w *taking values in* $\{a, b\}$ *such that*

$$\text{(a)} \int_E w(t)d\mu = \int_I z(t)\chi_E(t)d\mu, \quad \text{(b)} \int_E h(w(t))d\mu = \int_I h^{**}(z(t))\chi_E(t)d\mu,$$

and for all $t \in I$

$$\text{(c)} \int_0^t w(s)\chi_E(s)d\mu \le \int_0^t z(s)\chi_E(s)d\mu.$$

Proof. By the definition of E and h^{**}, there exist measurable functions $p_i : I \to [0,1]$, $i = 1,2$ satisfying: $p_1(t) + p_2(t) = 1$, and such that

$$z(t) = p_1(t)a + p_2(t)b, \quad h^{**}(z(t)) = p_1(t)h(a) + p_2(t)h(b), \quad t \in E.$$

Define $\psi(\tau) = \int_0^\tau a\chi_E(t)d\mu + \int_\tau^1 b\chi_E(t)d\mu - \int_0^1 z(t)\chi_E(t)d\mu$. Clearly ψ is continuous from $[0,1]$ into $\mathbf{R}$, and $\psi(0) \ge 0$, $\psi(1) \le 0$. Therefore, there exists $\delta \in I$ such that $\psi(\delta) = 0$. We now consider the function $w(t) = a\chi_{E\cap[0,\delta[}(t) + b\chi_{E\cap[\delta,1]}(t)$, it is measurable and from the definition of δ, (a) follows. On the other hand, one can write for $t \in E$; $h^{**}(z(t)) = c(z(t)-a)+h(a)$ for some constant c, and since $w(t) \in \{a,b\}$ and $h(a) = h^{**}(a)$, $h(b) = h^{**}(b)$, we have

$$\begin{aligned}\int_I h^{**}(z(t))\chi_E(t)d\mu &= \int_E c(z(t)-a)d\mu + \int_E h(a)d\mu = \int_E c(w(t)-a)d\mu + \int_E h(a)d\mu \\ &= \int_E h(w(t))d\mu = \int_I h(w(t))\chi_E(t)d\mu,\end{aligned}$$

and so (b) holds. Fix t in I; if $t < \delta$

$$\int_0^t (w(s) - z(s))\chi_E(s)d\mu = \int_0^t (a - z(s))\chi_E(s)d\mu \le 0,$$

if $t \ge \delta$ then

$$\begin{aligned}\int_0^t (w(s) - z(s))\chi_E(s)d\mu &= \int_0^t w(s)\chi_E(s)d\mu - \int_0^1 z(s)\chi_E(s)d\mu + \int_t^1 z(s)\chi_E(s)d\mu \\ &= \int_t^1 (z(s) - b)\chi_E(s)d\mu \le 0,\end{aligned}$$

where we have used (a) and thus the proof is complete. ∎

Lemma 2.6. *In addition to the hypotheses of Lemma 2.5, assume that the function h satisfies the following growth condition: $h(\xi) \geq \gamma|\xi|^p - \beta$, for some constants γ, β, $\gamma > 0$ and $1 < p < +\infty$. Then if the function z is in $L^p(]0,1[)$, there exists a function w in $L^p(]0,1[)$ taking values in $\{a,b\}$ such that*

$$\text{(a)} \quad \int_I w(t)d\mu = \int_I z(t)d\mu, \quad \text{(b)} \quad \int_I h(w(t))d\mu = \int_I h^{**}(z(t))d\mu,$$

and for all $t \in I$

$$\text{(c)} \quad \int_0^t w(s)d\mu \leq \int_0^t z(s)d\mu.$$

Proof. Setting $E_i = \{t \in I : a_i < z(t) < b_i\}$, we apply Lemma 2.5 to obtain w_i a measurable function taking values either a_i or b_i, such that for every i

$$\text{(i)} \quad \int_{E_i} w_i(t)d\mu = \int_I z(t)\chi_{E_i}(t)d\mu, \quad \text{(ii)} \quad \int_{E_i} h(w_i(t))d\mu = \int_I h^{**}(z(t))\chi_{E_i}(t)d\mu,$$

and for all $t \in I$

$$\text{(iii)} \quad \int_0^t w_i(s)\chi_{E_i}(s)d\mu \leq \int_0^t z(s)\chi_{E_i}(s)d\mu.$$

Put $E_0 = I \setminus \cup_i E_i$ and define $w : I \to \overline{\mathbf{R}}$ by

$$w(t) = z(t)\chi_{E_0}(t) + \sum_i w_i(t)\chi_{E_i}(t).$$

We claim that w satisfies the requirements of the lemma. First of all, we show that w is in $L^p(]0,1[)$;

$$|w(t)|^p = |z(t)|^p\chi_{E_0}(t) + \sum_i |w_i(t)|^p\chi_{E_i}(t).$$

On one hand, since the integral $\int_I h^{**}(z(t))d\mu$ is finite, by standard arguments, one can prove by using (ii) and the growth condition on h, that, the integral $\int_I \sum_i \chi_{E_i} h(w_i(t))d\mu$ is finite. On the other, again, by taking into account the growth condition, we obtain

$$\sum_i |w_i(t)|^p\chi_{E_i}(t) \leq \frac{1}{\gamma}\Big[\sum_i h(w_i(t))\chi_{E_i}(t) + \sum_i \beta\chi_{E_i}(t)\Big],$$

and thus the first part of our lemma is proved. In particular w is in $L^1(]0,1[)$.

$$\int_I w(t)d\mu = \int_{E_0} z(t)d\mu + \int_I \sum_i w_i(t)\chi_{E_i}(t)d\mu = \int_{E_0} z(t)d\mu + \sum_i \int_I w_i(t)\chi_{E_i}(t)d\mu$$
$$= \int_E z(t)d\mu + \sum_i \int_I z(t)\chi_{E_i}(t)d\mu = \int_I z(t)d\mu,$$

so that (i) is verified. We have also

$$\int_I h^{**}(z(t))d\mu = \int_I h^{**}(z(t))\chi_{E_0}(t)d\mu + \sum_i \int_I h^{**}(z(t))\chi_{E_i}(t)d\mu$$
$$= \int_I h(z(t))\chi_{E_0}(t)d\mu + \sum_i \int_I h(w(t))\chi_{E_i}(t)d\mu = \int_I h(w(t))d\mu,$$

which proves (ii). Now, for any fixed t in I we have

$$\int_0^t w(s)d\mu = \int_0^t z(s)\chi_{E_0}(s)d\mu + \sum_i \int_0^t w_i(s)\chi_{E_i}(s)d\mu$$
$$\leq \int_0^t z(s)\chi_{E_0}(s)d\mu + \sum_i \int_0^t z(s)\chi_{E_i}(s)d\mu = \int_0^t z(s)d\mu.$$

This completes the proof of Lemma 2.6. ■

Remark 2.7. If we change "a" by "b" in the definiton of functions φ and w in Lemma 2.5, we obtain the same conclusion up to changing the sense of the inequality in (c).

We now recall the definition of the so-called spherical symmetric rearrangement, which will be used in the proof of the next Theorem. For any $f \in L^p(\Omega)$, the spherical symmetric rearrangement of f, denoted by $f^\star$, is the positive, radial, decreasing function, having the same distribution function than $|f|$, i.e.

$$measure\Big\{x \in \Omega : \ |f(x)| > t \Big\} = measure\Big\{x \in \Omega : \ f^\star(x) > t \Big\}, \quad \forall t > 0$$

where ω_n denotes the volume of the unit ball in $\mathbf{R}^n$. For an exhaustive statement of the properties of rearrangements we refer to [K] and to the appendix of [Ta]. We just recall the well-known Cavalieri principle.

Proposition 2.8. *For every continuous function* $f : \mathbf{R}_+ \to \mathbf{R}$ *and every function* $v : B \to \mathbf{R}$*, we have*

$$\int_B f(|v(x)|)dx = \int_B f(v^*(x))dx.$$

2.2. Functionals with Linear Dependence on the State Variable

In this Section, our main concerns is the existence of solutions to problems of the form:

$$\left.\begin{aligned} &\min \int_\Omega c(|x|)u(x)dx + \int_\Omega h(|x|, \Delta u(x) - \lambda u(x))dx \\ &u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \\ &\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}\right\} \tag{P}$$

when Ω is either a ball or an annulus in $\mathbf{R}^n$ and λ is a non negative number or Ω is the whole space $\mathbf{R}^n$ and λ is positive, and we seek radially symmetric solutions.
There are many papers (e.g. [A-T2, A-T3, Ra, R3, R4, T1]) devoted to the existence of solutions to problem (P) that avoid the convexity assumption on the function h, but most of them seek the minimum in the space $W^{2,p} \cap W_0^{1,p}$. In this case, for instance if $c \equiv 0$, the problem reduces to solving the following Dirichlet problem

$$\begin{aligned} \Delta u - \lambda u &= \sigma(x) \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{DP}$$

where σ is any L^p-selection from the map $x \mapsto argmin\{h(x, \cdot)\}$. Then the function u solution to (DP) will be a solution to the given minimization problem. Hence, in general, this problem admits several solutions, obtained simply as solutions to Dirichlet problems. However, this procedure cannot be used for the same minimization problem under the additional condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, since the corresponding Dirichlet-Neumann problem, in general, does not admit any solution, so that even for this case a more complex approach is needed. On the other hand, existence results for our problem have been given in [A-T2, Ra, R4], but they actually proved the existence of solutions by imposing conditions such that every solution to problem (P^{**}), i.e. problem (P) where h is replaced by h^{**}

is, in fact a solution to problem (P). The method of the proof goes by showing that, along any solution to (P^{**}), the functions h and h^{**} have to coincide almost everywhere, otherwise the Euler-Lagrange equation would be violated. Aubert-Tahraoui in [A-T2] used a method based on the Duality theory as presented in [E-T], by generalizing their earlier idea in dimension one [A-T1], whereas Raymond [R4] and Rabier [Ra] gave a direct proof derived from the Euler-Lagrange equation. Moreover, in all of these papers some regularity conditions on h had to be imposed. The method proposed in the papers quoted above cannot possibly be applied to cases, where there are solutions to problem (P^{**}) that are not solutions to (P).

We first present an existence result when Ω is a ball. The procedure employed in this case does not seem to be applicable in case Ω is an annulus. As a matter of fact, the boundary is a disconnected set, thus we do not consider Dirichlet-type auxiliary problems anymore but Mixed boundary value problems. In case Ω is the whole space $\mathbf{R}^n$, Proposition 2.2 above plays a crucial role. Our main tools are the notion of Rotation Group in $\mathbf{R}^n$ in order to obtain a radially symmetric solution to the convexified problem (P^{**}) from one which is not so, and a modified version of Liapunov's theorem presented in [Ce], [R2, Prop.4.1], that allows to construct a radially symmetric solution to problem (P) from a radially symmetric solution to (P^{**}). Liapunov's theorem has been used as a tool to prove existence of solution for a different minimum problem in [C-C]. In this chapter we had, in particular, to extend the applicability of this theorem to a more complex operator and boundary conditions.

HYPOTHESIS (H). Set I to be $]0,1[$ or $]a,b[$. The map $c : \bar{I} \to \mathbf{R}$ is such that $r \mapsto r^{n-1}c(r)$ is in $L^{p'}(I)$ with p' the exponent conjugate to p. The map $h : \bar{I} \times \mathbf{R} \to \overline{\mathbf{R}}$ is such that

(h_1) h is $\mathcal{L} \otimes \mathcal{B}(\mathbf{R})$-measurable;

(h_2) $\xi \mapsto h(r,\xi)$ is lower semicontinuous for almost all r in I;

(h_3) there exists a positive constant γ, such that

$$h(r,\xi) \geq \gamma|\xi|^p - \beta(r) \text{ where the function } r \mapsto r^{n-1}\beta(r) \text{ is in } L^1(I).$$

Theorem 2.9. ([**C-F, F1**]) *Let h and c satisfy hypothesis* (H) *and λ be non-negative. Assume that the functional $\int_\Omega h(|x|, \Delta u(x) - \lambda u(x))dx$ has a finite value for some u in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Then the problem*

$$\left.\begin{array}{l}\min \displaystyle\int_\Omega c(|x|)u(x)dx + \int_\Omega h(|x|, \Delta u(x) - \lambda u(x))dx \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega,\end{array}\right\} \qquad (P)$$

admits at least one radially symmetric solution.

Proof. The proof is divided into three steps: in (a) below we show that the relaxed problem admits at least one radially symmetric solution; in (b) we write several functions as convex combinations and apply Liapunov's theorem to start defining a candidate for a solution to the original problem and in (c) we complete the construction of the solution.

(a) We consider the relaxed problem

$$\left.\begin{array}{l}\min \displaystyle\int_\Omega c(|x|)u(x)dx + \int_\Omega h^{**}(|x|, \Delta u(x) - \lambda u(x))dx \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega,\end{array}\right\} \qquad (P^{**})$$

Clearly, h^{**} satisfies the growth condition (h_3), therefore a well-known result (see [E-T] or Thm. 1.6 of this thesis) assures that problem (P_0^{**}) has a solution $\tilde{u}$. We claim that we can assume the function $\tilde{u}$ to be radially symmetric. If it is not so, we can consider the function $\bar{u} : \Omega \to \overline{\mathbf{R}}$ defined by

$$\bar{u}(x) = \int_{\mathbf{SO}(n)} \tilde{u}(Ax)d\mu(A) \tag{2.1}$$

instead of $\tilde{u}$, which by Proposition 2.3 belongs to $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, is such that $\frac{\partial \bar{u}}{\partial n} = 0$ on $\partial\Omega$ and is radially symmetric. Let us show that $\bar{u}$ is another solution to problem (P^{**}). On one hand, Jensen inequality and (d) of Proposition 2.3 imply

$$h^{**}(|x|, \Delta\bar{u}(x) - \lambda\bar{u}(x)) \leq \int_{\mathbf{SO}(n)} h^{**}(|x|, \Delta\tilde{u}(Ax) - \lambda\tilde{u}(Ax))d\mu(A) < +\infty. \tag{2.2}$$

On the other hand, using Tonelli-Fubini's theorem and the fact that

$$\int_\Omega h^{**}(|x|, \Delta\tilde{u}(Ax) - \lambda\tilde{u}(Ax))dx = \int_\Omega h^{**}(|y|, \Delta\tilde{u}(y) - \lambda\tilde{u}(y))dy \quad \forall A \in \mathbf{SO}(n),$$

we have

$$\int_\Omega \int_{SO(n)} h^{**}(|x|, \Delta\tilde{u}(Ax) - \lambda\tilde{u}(Ax))d\mu(A)dx =$$

$$= \int_{SO(n)} \int_\Omega h^{**}(|x|, \Delta\tilde{u}(Ax) - \lambda\tilde{u}(Ax))dx d\mu(A) = \int_\Omega h^{**}(|y|, \Delta\tilde{u}(y) - \lambda\tilde{u}(y))dy.$$

Hence, from (2.2) it follows that

$$\int_\Omega h^{**}(|x|, \Delta\bar{u}(x) - \lambda\bar{u}(x))dx \le \int_\Omega h^{**}(|y|, \Delta\tilde{u}(y) - \lambda\tilde{u}(y))dy.$$

Similarly one can show that

$$\int_\Omega c(|x|)\bar{u}(x)dx = \int_\Omega c(|y|)\tilde{u}(y)dy,$$

i.e. $\bar{u}$ is a radially symmetric solution to problem (P^{**}).

(b) Using spherical coordinates we obtain

$$\int_\Omega h^{**}(|x|, \Delta\tilde{u}(x) - \lambda\tilde{u}(x))dx = n\omega_n \int_I r^{n-1} h^{**}(r, \tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r) - \lambda\tilde{u}(r))dr \quad (2.3)$$

where ω_n denotes the volume of the unit ball in $\mathbf{R}^n$ and we have used the same letter for a radial function to consider it as a function of x or $|x| = r$, so that, " $'$ " means differentiation with respect to $|x| = r$. By (b) of Proposition 1.4 there exist measurable functions p_i and v_i, $i = 1, 2$; such that

$$\sum_{i=1}^{2} p_i(r) = 1; \quad p_i(r) \ge 0, \quad i = 1, 2; \quad (2.4)$$

$$\sum_{i=1}^{2} p_i(r)v_i(r) = \tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r) - \lambda\tilde{u}(r); \quad (2.5)$$

$$\sum_{i=1}^{2} p_i(r)h(r, v_i(r)) = h^{**}(r, \tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r) - \lambda\tilde{u}(r)). \quad (2.6)$$

At this point of the proof we must distinguish the two cases: (i) $\Omega = B$ and (ii) $\Omega = \Omega_a^b$.

(i) $\underline{\Omega = B}$.

Consider the (radially symmetric, see for instance [G-T]) functions φ and ϕ, belonging to $W^{2,p'}(B)$, solutions to the Dirichlet problem

$$\begin{aligned} \Delta w - \lambda w &= f(|x|) \\ w &= 0 \quad \text{on} \quad \partial B, \end{aligned} \quad (2.7)$$

with right-hand side 1 and $c(|x|)$ respectively. We now apply Liapunov's theorem to construct from $\tilde{u}$ a new function u, which will be a solution to the original problem.
By Proposition 1.15 (the verification of the hypothesis is a simple exercise) there exists a measurable partition of $]0,1[$, $(E_i)_i, i = 1,2$, such that:

$$\int_0^1 \sum_{i=1}^2 p_i(r) r^{n-1} h(r, v_i(r)) dr = \sum_{i=1}^2 \int_0^1 \chi_{E_i}(r) r^{n-1} h(r, v_i(r)) dr; \tag{2.8}$$

$$\int_0^1 \sum_{i=1}^2 p_i(r) r^{n-1} v_i(r) dr = \sum_{i=1}^2 \int_0^1 \chi_{E_i}(r) r^{n-1} v_i(r) dr; \tag{2.9}$$

$$\int_0^1 \sum_{i=1}^2 p_i(r) r^{n-1} v_i(r) \varphi(r)\, dr = \sum_{i=1}^2 \int_0^1 \chi_{E_i}(r) r^{n-1} v_i(r) \varphi(r)\, dr; \tag{2.10}$$

$$\int_0^1 \sum_{i=1}^2 p_i(r) r^{n-1} v_i(r) \phi(r)\, dr = \sum_{i=1}^2 \int_0^1 \chi_{E_i}(r) r^{n-1} v_i(r) \phi(r)\, dr. \tag{2.11}$$

In particular, the map

$$r \mapsto \sum_{i=1}^2 \chi_{E_i}(r) r^{n-1} h(r, v_i(r))$$

belongs to $L^1(]0,1[)$, that together with (h_3) of Hyphotesis (H) imply that the map

$$r \mapsto \sum_{i=1}^2 \chi_{E_i}(r) r^{\frac{n-1}{p}} v_i(r)$$

belongs to $L^p(]0,1[)$ or, equivalently, the map

$$x \mapsto \sum_i \chi_{E_i}(|x|) v_i(|x|) \tag{2.12}$$

belongs to $L^p(B)$. On the other hand, (2.8) and (2.6) yield

$$\int_0^1 \sum_i \chi_{E_i}(r) r^{n-1} h(r, v_i(r)) dr = \int_0^1 r^{n-1} h^{**}(r, \tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r) - \lambda \tilde{u}(r)) dr. \tag{2.13}$$

Since E_i, $i = 1,2$, is a partition of $]0,1[$, we have

$$h(r, \sum_i \chi_{E_i}(r) v_i(r)) = \sum_i \chi_{E_i}(r) h(r, v_i(r)) \quad \text{for} \quad r \in]0,1[.$$

Therefore from (2.3) and (2.13), it follows that

$$\int_B h^{**}(|x|, \Delta\tilde{u}(x) - \lambda\tilde{u}(x))dx = \int_B h(|x|, \sum_i \chi_{E_i}(|x|)v_i(|x|))dx. \qquad (2.14)$$

(c) Now, let u be the (radially symmetric) solution to the Dirichlet problem

$$\Delta u - \lambda u = \sum_i \chi_{E_i}(|x|)v_i(|x|) \qquad (2.15)$$

$$u = 0 \quad \text{on} \quad \partial B. \qquad (2.16)$$

We actually know that $u \in W^{2,p}(B)$, i.e. $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$. Notice that, from (2.12), the right-hand side of (2.15) is in $L^p(B)$. In addition, $u, \tilde{u}$ are in $C^1(]0,1])$ since they are in $W^{2,p}(]\varepsilon, 1[)$ for every $\varepsilon > 0$ (we recall that $p > 1$).
We claim that the function u is a solution to problem (P). To infer it, we shall prove that:

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial B \quad \text{or, equivalently, in spherical coordinates, that} \quad u'(1) = 0; \qquad (2.17)$$

$$\int_B h^{**}(|x|, \Delta\tilde{u}(x) - \lambda\tilde{u}(x))dx = \int_B h(|x|, \Delta u(x) - \lambda u(x))dx; \qquad (2.18)$$

$$\int_B c(|x|)\tilde{u}(x)dx = \int_B c(|x|)u(x)dx. \qquad (2.19)$$

Ad (2.17). Taking into account (2.15) in spherical coordinates, (2.5) and (2.9) we obtain

$$\int_B (\Delta u(x) - \lambda u(x))dx = \int_B (\Delta\tilde{u}(x) - \lambda\tilde{u}(x))dx,$$

or, equivalently

$$\lambda \int_B (u(x) - \tilde{u}(x))dx = \int_B (\Delta u(x) - \Delta\tilde{u}(x))dx. \qquad (2.20)$$

Now, by applying Green's formula, the last integral equals

$$\int_{\partial B} (\frac{\partial u}{\partial n} - \frac{\partial \tilde{u}}{\partial n})dH^{n-1}(x) = \int_{\partial B} (\frac{\partial u}{\partial n} - \frac{\partial \tilde{u}}{\partial n})dH^{n-1}(x) = n\omega_n u'(1), \qquad (2.21)$$

where H^{n-1} denotes the $(n-1)$-dimensional Hausdorff measure. On the other hand, from (2.7) corresponding to φ, it follows that

$$\int_B (u(x) - \tilde{u}(x))dx = \int_B \Big(\Delta\varphi(x) - \lambda\varphi(x)\Big)(u(x) - \tilde{u}(x))dx.$$

By means of Green's formula again, the last integral can be written as

$$\int_B \varphi(x)\Big(\Delta u(x) - \lambda u(x) - \Delta\tilde{u}(x) + \lambda\tilde{u}(x)\Big)dx + \int_{\partial B} \frac{\partial\varphi}{\partial n}(u(x) - \tilde{u}(x))dH^{n-1}(x) +$$

$$+ \int_{\partial B} (\frac{\partial\tilde{u}}{\partial n} - \frac{\partial u}{\partial n})\varphi(x)dH^{n-1}(x).$$

The first integral equals zero because of (2.10) by noticing (2.5) and (2.15), and the last two integrals reduce zero too, because of (2.16) and (2.7) corresponding to φ. Consequently

$$\int_B (u(x) - \tilde{u}(x))dx = 0. \tag{2.22}$$

Therefore, (2.20), (2.21) and the last assertion imply that $u'(1) = 0$, i.e. $\frac{\partial u}{\partial n} = 0$ on ∂B, if λ is either positive or zero.

Ad (2.18). This is a straightforward consequence from (2.14) and (2.15).

Ad (2.19). By the definition of ϕ, (2.7):

$$\int_B c(|x|)u(x)dx = \int_B \Big(\Delta\phi(x) - \lambda\phi(x)\Big)u(x)dx.$$

Since the function u is in $W_0^{2,p}(B)$, by Green's Formula the right-hand side can be written as

$$\int_B \Big(\Delta u(x) - \lambda u(x)\Big)\phi(x)dx.$$

Taking into account (2.15) in spherical coordinates, the last integral equals

$$n\omega_n \int_0^1 r^{n-1} \sum_i \chi_{E_i}(r)v_i(r)\phi(r)dr = n\omega_n \int_0^1 r^{n-1} \sum_i p_i(r)v_i(r)\phi(r)dr$$
$$= \int_B \Big(\Delta\tilde{u}(x) - \lambda\tilde{u}(x)\Big)\phi(x)dx = \int_B \Big(\Delta\phi(x) - \lambda\phi(x)\Big)\tilde{u}(x)dx$$
$$= \int_B c(|x|)\tilde{u}(x)dx,$$

where we have used (2.11), (2.5) and (2.7) again. This proves that u is a radially symmetric solution to problem (P) in case Ω is the unit ball B.

(ii) $\underline{\Omega = \Omega_a^b}$

Instead of the Dirichlet problem (2.7), we consider the following Mixed boundary value problem:

$$\Delta w - \lambda w = f(|x|) \tag{2.23}$$

$$w = 0 \quad \text{on} \quad \Gamma_b, \tag{2.24.a}$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \Gamma_a. \tag{2.24.b}$$

Let φ, ϕ in $W^{2,p'}(\Omega)$, the radially symmetric solutions for $f(|x|) = 1, c(|x|)$, respectively. Moreover, let ψ in $W^{2,p'}(\Omega)$, the radially symmetric solution to problem

$$\Delta \psi - \lambda \psi = 1 \tag{2.25}$$

$$\psi = 0 \quad \text{on} \quad \Gamma_a, \tag{2.26.a}$$

$$\frac{\partial \psi}{\partial n} = 0 \quad \text{on} \quad \Gamma_b. \tag{2.26.b}$$

We now proceed as before. Apply Proposition 1.15 (the verification of the hypothesis is still a simple exercise) to obtain the existence of a measurable partition of $]a,b[$, $(E_i)_i, i = 1,2$ such that besides (2.8)-(2.11) hold over $]a,b[$, also the following equality is satisfied

$$\int_a^b \sum_{i=1}^{2} p_i(r) r^{n-1} v_i(r) \psi(r)\, dr = \sum_{i=1}^{2} \int_a^b \chi_{E_i}(r) r^{n-1} v_i(r) \psi(r)\, dr. \tag{2.27}$$

Here we also get (2.14) but over Ω.

(c) Consider the radially symmetric function $u \in W^{2,p}(\Omega)$ solution to the problem

$$\Delta u - \lambda u = \sum_i \chi_{E_i}(|x|) v_i(|x|) \tag{2.28}$$

$$u = 0 \quad \text{on} \quad \Gamma_b, \tag{2.29.a}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_a. \tag{2.29.b}$$

Certainly, as in the previous case, one can show that; the right-hand side of (2.28) is in $L^p(\Omega)$ and $u, \tilde{u} \in C^1([a,b])$ since $a > 0$. We claim that the function u is a solution to problem (P). For this purpose, we shall prove that:

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_b \text{ or, equivalently, in spherical coordinates, that } u'(b) = 0; \tag{2.30}$$

$$u = 0 \text{ on } \Gamma_a \text{ or, equivalently, in spherical coordinates, that } u(a) = 0; \tag{2.31}$$

$$\int_\Omega h^{**}(|x|, \Delta\tilde{u}(x) - \lambda\tilde{u}(x))dx = \int_\Omega h(|x|, \Delta u(x) - \lambda u(x))dx; \tag{2.32}$$

$$\int_\Omega c(|x|)\tilde{u}(x)dx = \int_\Omega c(|x|)u(x)dx. \tag{2.33}$$

Ad (2.30). First, remark that by taking into account (2.28) in spherical coordinates and (2.9), (2.5) we obtain

$$\int_\Omega (\Delta u(x) - \lambda u(x))dx = \int_\Omega (\Delta\tilde{u}(x) - \lambda\tilde{u}(x))dx,$$

or, equivalently

$$\lambda \int_\Omega (u(x) - \tilde{u}(x))dx = \int_\Omega (\Delta u(x) - \Delta\tilde{u}(x))dx. \tag{2.34}$$

Now, by applying Green's formula and then (2.29), the last integral equals

$$\int_{\partial\Omega} (\frac{\partial u}{\partial n} - \frac{\partial \tilde{u}}{\partial n})dH^{n-1}(x) = \int_{\Gamma_b} (\frac{\partial u}{\partial n} - \frac{\partial \tilde{u}}{\partial n})dH^{n-1}(x) = n\omega_n b^{n-1}u'(b), \tag{2.35}$$

where H^{n-1} denotes the $(n-1)$-dimensional Hausdorff measure. On the other hand, from (2.23) corresponding to φ, it follows that

$$\int_\Omega (u(x) - \tilde{u}(x))dx = \int_\Omega \Big(\Delta\varphi(x) - \lambda\varphi(x)\Big)(u(x) - \tilde{u}(x))dx.$$

By means of Green's formula again, the last integral can be written as

$$\int_\Omega \varphi(x)\Big(\Delta u(x) - \lambda u(x) - \Delta\tilde{u}(x) + \lambda\tilde{u}(x)\Big)dx + \int_{\partial\Omega} \frac{\partial\varphi}{\partial n}(u(x) - \tilde{u}(x))dH^{n-1}(x) +$$

$$+ \int_{\partial\Omega} (\frac{\partial\tilde{u}}{\partial n} - \frac{\partial u}{\partial n})\varphi(x)dH^{n-1}(x).$$

The first integral equals zero because of (2.10) by noticing (2.5) and (2.28), and the last two integrals reduce to zero too, because of (2.29) and (2.24) corresponding to φ. Consequently

$$\int_\Omega (u(x) - \tilde{u}(x))dx = 0. \tag{2.36}$$

On combining (2.34)-(2.36), we conclude that $u'(b) = 0$, i.e. $\frac{\partial u}{\partial n} = 0$ on Γ_b, if λ is either positive or zero.

Ad (2.31). Using (2.23), we obtain as before

$$\int_\Omega (u(x) - \tilde{u}(x))dx = \int_\Omega \psi(x)\Big(\Delta u(x) - \lambda u(x) - \Delta\tilde{u}(x) + \lambda\tilde{u}(x)\Big)dx +$$
$$+\int_{\partial\Omega} \frac{\partial\psi}{\partial n}(u(x) - \tilde{u}(x))dH^{n-1}(x) + \int_{\partial\Omega} (\frac{\partial\tilde{u}}{\partial n} - \frac{\partial u}{\partial n})\psi(x)dH^{n-1}(x). \tag{2.37}$$

The first integral is equal to zero because of (2.27) and by noticing (2.5) and (2.28), and by taking into account $u'(b) = 0$, (2.29) and (2.26) the last two integrals reduce

$$\int_{\Gamma_a} \frac{\partial\psi}{\partial n}(x)(u(x) - \tilde{u}(x))dH^{n-1}(x) = -n\omega_n a^{n-1}\psi'(a)u(a). \tag{2.38}$$

At this point, we claim actually that $\psi'(a) \neq 0$. In fact, this is a consequence of the uniqueness of the solutions to Neumann problems in case $\lambda > 0$, and if $\lambda = 0$ such a (Neumann) problem does not admit any solutions. Hence, (2.36)-(2.38) yield $u(a) = 0$ i.e. $u = 0$ on Γ_a.

Ad (2.32). This is a straightforward consequence from (2.14) and (2.28).

Ad (2.33). By the definition of ϕ, (see (2.23))

$$\int_\Omega c(|x|)u(x)dx = \int_\Omega \Big(\Delta\phi(x) - \lambda\phi(x)\Big)u(x)dx.$$

Since the function u is in $W_0^{2,p}(\Omega)$, by Green's Formula the right-hand side can be written as

$$\int_\Omega \Big(\Delta u(x) - \lambda u(x)\Big)\phi(x)dx.$$

Taking into account (2.28) in spherical coordinates, the last integral equals

$$n\omega_n \int_a^b r^{n-1}\sum_i \chi_{E_i}(r)v_i(r)\phi(r)dr = n\omega_n \int_a^b r^{n-1}\sum_i p_i(r)v_i(r)\phi(r)dr$$
$$= \int_\Omega \Big(\Delta\tilde{u}(x) - \lambda\tilde{u}(x)\Big)\phi(x)dx = \int_\Omega \Big(\Delta\phi(x) - \lambda\phi(x)\Big)\tilde{u}(x)dx$$
$$= \int_\Omega c(|x|)\tilde{u}(x)dx,$$

where we have used (2.11), (2.5) and (2.23) again. This concludes the proof when Ω is the annulus Ω_a^b, thus the proof of the Theorem is complete. ■

Theorem 2.10. ([**F1**]) *Let h and c satisfy hypothesis* (H) *with I being $]0,+\infty[$ and λ be positive. Assume that the functional $\int_{\mathbf{R}^n} h(|x|, \Delta u(x) - \lambda u(x))dx$ has a finite value for some u in $W^{2,p}(\mathbf{R}^n)$. Then the problem*

$$\left.\begin{aligned} \min \int_{\mathbf{R}^n} c(|x|)u(x)dx + \int_{\mathbf{R}^n} h(|x|, \Delta u(x) - \lambda u(x))dx \\ u \in W^{2,p}(\mathbf{R}^n), \end{aligned}\right\} \qquad (P_0)$$

admits at least one radially symmetric solution.

Proof. We argue as in the proof of Theorem 2.9, that is, we start by considering the problem

$$\min \int_{\mathbf{R}^n} c(|x|)u(x)dx + \int_{\mathbf{R}^n} h^{**}(|x|, \Delta u(x) - \lambda u(x))dx \qquad (P_0^{**})$$

in the space $W^{2,p}(\mathbf{R}^n)$, which has at least one radially symmetric solution $\tilde{u}$. Without loss of generality, we only deal with the case $\lambda = 1$. Using spherical coordinates one gets

$$\int_{\mathbf{R}^n} h^{**}(|x|, \Delta\tilde{u}(x) - \tilde{u}(x))dx = n\omega_n \int_0^{\infty} r^{n-1} h^{**}(r, \tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r) - \tilde{u}(r))dr. \quad (2.39)$$

Moreover, we also obtain (2.4)-(2.6) for some measurable functions p_i and v_i, $i = 1,2$. Consider the function φ in $W^{2,p'}(\mathbf{R}^n)$ (given by Proposition 2.2), the radially symmetric solution to the problem

$$\Delta\varphi - \varphi = c(|x|). \qquad (2.40)$$

Since $I =]0,\infty[$ is a σ-compact set, we can apply Proposition 1.15 to obtain a partition of I, $(E_i)_i$, $i = 1,2$ such that:

$$\int_0^{\infty} \sum_{i=1}^{2} p_i(r) r^{n-1} h(r, v_i(r))dr = \sum_{i=1}^{2} \int_0^{\infty} \chi_{E_i}(r) r^{n-1} h(r, v_i(r))dr; \qquad (2.41)$$

$$\int_0^{\infty} \sum_{i=1}^{2} p_i(r) r^{n-1} v_i(r)\varphi(r)dr = \sum_{i=1}^{2} \int_0^{\infty} \chi_{E_i}(r) r^{n-1} v_i(r)\varphi(r)dr. \qquad (2.42)$$

As a consequence, we have that the map

$$x \mapsto \sum_i \chi_{E_i}(|x|)v_i(|x|) \tag{2.43}$$

belongs to $L^p(\mathbf{R}^n)$. On the other hand, by taking into account that $(E_i)_i$ is a partition, (2.6) and (2.4) imply

$$\int_{\mathbf{R}^n} h^{**}(|x|, \Delta\tilde{u}(x) - \tilde{u}(x))dx = \int_{\mathbf{R}^n} h(|x|, \sum_i \chi_{E_i}(|x|)v_i(|x|))dx. \tag{2.44}$$

Now, let u be the (radially symmetric) solution of the following problem

$$\Delta u - u = \sum_i \chi_{E_i}(|x|)v_i(|x|), \tag{2.45}$$

which, by Proposition 2.2, belongs to $W^{2,p}(\mathbf{R}^n)$. Notice that the right-hand side of (2.45) is in $L^p(\mathbf{R}^n)$ because of (2.43). We claim that the function u is a solution to problem (P_0). To infer it, we shall prove that:

$$\int_{\mathbf{R}^n} h^{**}(|x|, \Delta\tilde{u}(x) - \tilde{u}(x))dx = \int_{\mathbf{R}^n} h(|x|, \Delta u(x) - u(x))dx; \tag{2.46}$$

$$\int_{\mathbf{R}^n} c(|x|)\tilde{u}(x)dx = \int_{\mathbf{R}^n} c(|x|)u(x)dx. \tag{2.47}$$

(2.46) follows directly from (2.44) and (2.45). Let us remark that, by definition of the distributional derivative and from the fact $C_0^\infty(\mathbf{R}^n)$ is dense in $W^{1,p}(\mathbf{R}^n)$, it follows that

$$\int_{\mathbf{R}^n} \Delta\varphi(x)\tilde{u}(x)dx = \int_{\mathbf{R}^n} \varphi(x)\Delta\tilde{u}(x)dx.$$

Therefore, from (2.40) it follows that

$$\begin{aligned}\int_{\mathbf{R}^n} c(|x|)\tilde{u}(x)dx &= \int_{\mathbf{R}^n} \Big(\Delta\varphi(x) - \varphi(x)\Big)\tilde{u}(x)dx = \int_{\mathbf{R}^n} \varphi(x)\Big(\Delta\tilde{u}(x) - \tilde{u}(x)\Big)dx \\ &= \int_{\mathbf{R}^n} \varphi(x)\Big(\Delta u(x) - u(x)\Big)dx,\end{aligned}$$

where the last equality has been obtained by using (2.5), (2.45) and (2.42). Again, by the above remark, the last integral equals

$$\int_{\mathbf{R}^n} \Big(\Delta\varphi(x) - \varphi(x)\Big)u(x)dx = \int_{\mathbf{R}^n} c(|x|)u(x)dx,$$

and thus (2.47) is proved, and thus the proof of theorem is complete. ■

Remark 2.11. Part (a) of the proof of Theorem 2.9 or Theorem 2.10 can also be applied if we have a convex dependence in u, because of Jense's inequality.

Remark 2.12. In the scalar case: $n = 1$, the proof remains as before (setting $n = 1$), except only that we take

$$\bar{u}(x) = \frac{1}{2}\hat{u}(x) + \frac{1}{2}\hat{u}(-x)$$

instead of that defined in (2.1).

2.3. Functionals with Non-linear Dependence on the State variable

We deal here with functionals that are not linear in u and, more precisely, we consider the two following non convex problems

$$\left.\begin{array}{l} \min \int_B \bar{g}(|u(x)|)dx + \int_B \bar{h}(|\Delta u(x)|)dx \\ u \in W^{2,p}(B) \cap W_0^{1,p}(B) \end{array}\right\} \qquad (P_1)$$

$$\left.\begin{array}{l} \min \int_B g(|x|, u(x))dx + \int_B h(\Delta u(x))dx \\ u \in W^{2,p}(B) \cap W_0^{1,p}(B) \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial B, \end{array}\right\} \qquad (P_2)$$

where B, as before, is the unit ball in $\mathbf{R}^n$ centered at the origin and we prove, for either case, the existence of radially symmetric solutions. In case the domain is not necessarily radial we refer to [A-T3, Ra, R3, R4, T1] for problems of the type (P_1) and to [A-T2, Ra, R4] for those of the form (P_2). In the same spirit as in the previous sections, the results presented here can be applied to cases where problem (P_i^{**}) admits solutions that are not solutions to problem (P_i) in constrast to that exposed in the papers quoted above.
In case of problem (P_1), under the assumption of monotonicity (actually decreasing, otherwise the result is not true) on the function $\bar{g}$ defined in $\mathbf{R}_+$, we prove the existence of at least one non-negative radially symmetric solution. Moreover, assuming that problem

(P_1^{**}) has a non-null solution or $h^{**}(0) < h(0)$, we show that problem (P_1) admits one positive radial solution. This result applies, for instance, to the functional

$$I(u) = \int_B e^{-|u(x)|^2} dx + \int_B (1 - |\Delta u(x)|)^2 dx.$$

Compare it with what Ekeland-Teman assert in [E-T, example 1 and Remark 5.1 of Chapter IX]. A second existence theorem, concerning problem (P_2), is given: in this case, we assume the convexity and (not necessarily strict) monotonicity on $g(|x|, \cdot)$.
In order to obtain a radially symmetric solution to (P_i^{**}), $i = 1, 2$, from one which is not so, we use the so-called spherical symmetric rearrangement in case of problem (P_1) and, the notion of Rotation Group in $\mathbf{R}^n$, already used previously, in case of problem (P_2). Then, we use Lemma 2.6 which allows to construct a radially symmetric solution to problem (P_i) from a radial solution to (P_i^{**}).

We shall assume the following hypothesis.

HYPOTHESIS (H1).-

The map $\bar{g} : \mathbf{R}_+ \to \mathbf{R}$ is such that

(g_1) $\bar{g}$ is continuous;

(g_2) $\bar{g}$ is decreasing on $\mathbf{R}_+$.

Moreover:

(g_3) there exist two constants; γ_1, β_1, with $\gamma_1 \geq 0$ such that
$\bar{g}(u) \geq -\gamma_1 |u|^p - \beta_1$ for every $u \geq 0$.

The map $\bar{h} : \mathbf{R}_+ \to \mathbf{R}$ is such that

(h_1) $\bar{h}$ is continuous;

(h_2) there exists two constants; γ_2, β_2 with $\gamma_2 > 0$ such that
$\bar{h}(\xi) \geq \gamma_2 |\xi|^p - \beta_2$ for every $\xi \geq 0$.

In addition, setting

$$K = K(p, n, \Omega) = \sup \left\{ \frac{||u||_p}{||\Delta u||_p} \; : \; u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \;\; u \neq 0 \right\},$$

so that, $K \in \mathbf{R}$ from $||\Delta \cdot ||_p$ being equivalent to $|| \cdot ||_{2,p}$ on $W^{2,p} \cap W_0^{1,p}$, we shall assume $1 - \frac{\gamma_1}{\gamma_2} K^p > 0$.

We are now in the position to state the first theorem of this section.

Theorem 2.13. ([F3]) *Let $\bar{g}$ and $\bar{h}$ satisfy hypothesis* (H1). *Assume that the functional $\int_B \bar{g}(|u(x)|)dx + \int_B \bar{h}(|\Delta u(x)|)dx$ has a finite value for some u in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Then problem (P_1) admits at least one non-negative radially symmetric solution, and thus also one non-positive. Furthermore, if either the relaxed problem (P_1^{**}) below has a non-null solution or $h^{**}(0) < h(0)$, then problem (P_1) admits one positive radially symmetric solution.*

Remark 2.14. If $\bar{g}$ is increasing on $\mathbf{R}_+$, there is no hope that the result holds as the following example shows

$$I(u) = \int_B |u(x)|^2 dx + \int_B (1 - |\Delta u(x)|)^2 dx$$

and we seek for the minima of I on the space $W^{2,2}(B) \cap W_0^{1,2}(B)$.

Proof of Theorem 2.13. Setting $h(\xi) := \bar{h}(|\xi|)$, clearly $h^{**}(\xi) = \phi(|\xi|)$ for some even and convex function ϕ.

(a) We consider the relaxed problem

$$\left.\begin{array}{l} \min \displaystyle\int_B \bar{g}(|u(x)|)dx + \int_B h^{**}(\Delta u(x))dx \\ u \in W^{2,p}(B) \cap W_0^{1,p}(B). \end{array}\right\} \qquad (P_1^{**})$$

which admits at least one solution, namely $\hat{u}$. From this solution, we shall construct another solution that will be non-negative and radially symmetric. To this end, we consider the radially symmetric function $\tilde{u}$, solution to the Dirichlet problem

$$\begin{aligned} -\Delta \tilde{u} &= (-\Delta \hat{u})^\star(x) = |\Delta \hat{u}|^\star(x) \\ \tilde{u} &= 0 \text{ on } \partial B, \end{aligned} \tag{2.48}$$

where $f^\star$ is the spherical symmetric rearrangement of f. For its properties we refer to [K] and to the appendix of [Ta]. Since $\Delta\hat{u}$ is in L^p, its rearrangement is also in L^p. Hence, $\tilde{u} \in W^{2,p}(B) \cap W_0^{1,p}(B)$. Moreover, from Theorem 1 of [Ta], it follows that

$$0 \leq \hat{u}^\star(x) \leq \tilde{u}(x) \quad \text{a.e. } x \text{ in } B. \tag{2.49}$$

We claim that $\tilde{u}$ is the function searched. In fact,

$$\int_B h^{**}(\Delta\tilde{u}(x))dx = \int_B \phi(-\Delta\tilde{u}(x))dx = \int_B \phi((-\Delta\hat{u})^\star(x))dx$$
$$= \int_B \phi(|-\Delta\hat{u}(x)|)dx = \int_B h^{**}(\Delta\hat{u}(x))dx, \tag{2.50}$$

where we have applied Cavalieri's principle (see Prop. 2.8 of this thesis or [K]). On the other hand, $\bar{g}$ being a decreasing function, (2.49) implies $\bar{g}(\hat{u}^\star(x)) \geq \bar{g}(\tilde{u}(x))$, a.e. x in B. Consequently, by using again, Cavalieri's principle

$$\int_B \bar{g}(|\tilde{u}(x)|)dx = \int_B \bar{g}(\tilde{u}(x))dx \leq \int_B \bar{g}(\hat{u}^\star(x))dx = \int_B \bar{g}(|\hat{u}(x)|)dx. \tag{2.51}$$

On combining (2.50) and (2.51), we conclude that $\tilde{u}$ is a solution to problem (P_1^{**}) and it is non-negative and radially symmetric, so that the claim is proved.

(b) Using spherical coordinates, we obtain

$$\int_B h^{**}(\Delta\tilde{u}(x))dx = n\omega_n \int_0^1 r^{n-1} h^{**}(\tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r))dr, \tag{2.52}$$

where ω_n denotes the volume of the unit ball in $\mathbf{R}^n$ and $\tilde{u}'$ means the derivative of the function $\tilde{u}$ with respect to $|x| = r$. It is known that under the assumptions (h_1) and (h_2) of Hyphotesis (H1), we can apply Lemma 2.6. Thus by putting $d\mu = r^{n-1}dr$, there exists a function w such that $r \mapsto r^{\frac{n-1}{p}} w(r)$ is in $L^p(]0,1[)$ satisfying

$$\int_0^1 r^{n-1} w(r)dr = \int_0^1 r^{n-1}(\tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r))dr; \tag{2.53}$$

$$\int_0^1 r^{n-1} h(w(r))dr = \int_0^1 r^{n-1} h^{**}(\tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r))dr; \tag{2.54}$$

and for all r in $]0,1]$

$$\int_0^r s^{n-1} w(s)ds \leq \int_0^r s^{n-1}(\tilde{u}''(s) + \frac{n-1}{s}\tilde{u}'(s))ds. \tag{2.55}$$

(c) At this point, we consider the function u in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, solution to the problem

$$\begin{aligned} \Delta u &= w(|x|) \\ u &= 0 \text{ on } \partial B. \end{aligned} \tag{2.9}$$

Certainly, u is a radially symmetric function and u and $\tilde{u}$ are in $C^1(]0,1])$ since they are in $W^{2,p}(]\varepsilon,1[)$ for all $\varepsilon > 0$. We claim that the function u is a solution to problem (P_1). To infer it, we shall prove that

$$\int_B h^{**}(\Delta\tilde{u}(x))dx = \int_B h(\Delta u(x))dx \tag{2.57}$$

and

$$\int_B \bar{g}(|\tilde{u}(x)|)dx = \int_B \bar{g}(|u(x)|)dx. \tag{2.58}$$

Ad (2.57). This is automatically verified because of (2.54).
Ad (2.58). By taking into account (2.56) in spherical coordinates, (2.55) becomes

$$\int_0^r \{(s^{n-1}\tilde{u}'(s))' - (s^{n-1}u'(s))'\}ds \geq 0. \tag{2.59}$$

Since $\nabla\tilde{u}$ is in $L^p(\Omega)$, we have $\int_0^1 r^{n-1}|\tilde{u}'(r)|^p dr < +\infty$. This implies that, $\lim r^{n-1}\tilde{u}'(r) = 0$ as $r \to 0^+$. Therefore, on integrating by parts (2.59) and by using the above remarks, we obtain

$$r^{n-1}\tilde{u}'(r) - r^{n-1}u'(r) \geq 0 \quad r \text{ in }]0,1]. \tag{2.60}$$

Consequently, by integrating again, and by noticing that $\lim u(r) = 0$ as $r \to 1^-$, we have $u(r) \geq \tilde{u}(r) \geq 0$ r in $]0,1]$, i.e. $u(x) \geq \tilde{u}(x) \geq 0$ a.e. x in Ω. So that, if $\bar{g}$ is a decreasing function, the latter gives $\bar{g}(u(x)) \leq \bar{g}(\tilde{u}(x))$ a.e. x in B. This implies that

$$\int_B \bar{g}(u(x))dx \leq \int_B \bar{g}(\tilde{u}(x))dx. \tag{2.61}$$

On the other hand, since $\tilde{u}$ is a solution to (P_1^{**}), by definition of h^{**},

$$\int_B \bar{g}(u(x))dx + \int_B h(\Delta u(x))dx \geq \int_B \bar{g}(\tilde{u}(x))dx + \int_B h^{**}(\Delta\tilde{u}(x))dx.$$

From (2.57) and (2.61), it follows that (2.58) holds. This proves that u is a non-negative radially symmetric solution to problem (P_1).

Let us prove the second part of our theorem. First assume $\hat{u} \not\equiv 0$. By (2.48) in spherical coordinates and by one of the remarks above, we have $-r^{n-1}\tilde{u}'(r) = \int_0^r s^{n-1}|\Delta\hat{u}|^\star(s)ds \geq 0$ for $r \in]0,1]$, i.e. $\tilde{u}$ is a decreasing function in $|x| = r$ and so u is too (see (2.60)). Since

$u \geq \tilde{u} \geq 0$, it is enough to prove that $\tilde{u} > 0$ in $]0,1[$. Suppose the contrary. We have $\tilde{u}(r) = 0$ for $r \in]r_0, 1]$, $1 > r_0 > 0$. So that, $\tilde{u}'(1) = 0$. Hence

$$\int_B |\Delta \hat{u}|(x)dx = \int_B |\Delta \hat{u}|^\star(x)dx = n\omega_n \int_0^1 r^{n-1}|\Delta \hat{u}|^\star(r)dr = -\tilde{u}'(1) = 0.$$

Then, $\hat{u} \equiv 0$. This yields a contradiction proving the claim if $\hat{u} \not\equiv 0$. It only remains to consider the case $h^{**}(0) < h(0)$ with $\hat{u} \equiv 0$. We also get $\tilde{u} \equiv 0$. We shall prove that u defined at point (c) is positive. Otherwise, by proceeding as before we obtain $u(r) = 0$ for $r \in [r_0, 1]$, $1 > r_0 > 0$. Then by a Stampacchia's theorem (for instance [G-T, Lemma 7.7]), $\Delta u = 0$ a.e. $x \in B \setminus \Omega_0$, $\Omega_0 = \{x \in \Omega : |x| < r_0\}$. Since u is a solution to problem (P_1), it has not to be identically zero because of $h^{**}(0) < h(0)$ and $\min P_1^{**} = \min P_1$ (see [E-T, Thm.41 of Chapter IX]). Therefore, $\Delta u \neq 0$ in a subset of positive measure of Ω_0. Consequently,

$$\begin{aligned}\int_B h(\Delta u)dx &= \int_{\Omega_0} h(\Delta u)dx + \int_{B\setminus\Omega_0} h(0)dx > \int_{\Omega_0} h^{**}(\Delta u)dx + \int_{B\setminus\Omega_0} h^{**}(0)dx \\ &= \int_B h^{**}(\Delta u)dx.\end{aligned}$$

On the other hand, u is also a solution to problem (P_1^{**});

$$\int_B h^{**}(\Delta u)dx + \int_B \bar{g}(u)dx = \int_B h^{**}(0)dx + \int_B \bar{g}(0)dx.$$

On combining the last two equalities and by taking into account (2.58), we have

$$\int_B h(\Delta u)dx > \int_B h^{**}(0)dx,$$

a contradiction with (2.57). ■

Corollary 2.14. *If, in addition to the hypothesis of the previous theorem, $\bar{g}$ is strictly decreasing in $\mathbf{R}_+$. Then every solution to problem (P_1^{**}) is a solution to the original (non convex) problem (P_1).*

Proof. Let $\hat{u}$ be any solution to problem (P_1^{**}). Then, in virtue of (2.50), (2.51), (2.57) and (2.58), the assertion will be proved if we show that $\tilde{u} \equiv u$. In fact, if it is so, because of (2.48) we have

$$\int_B h(\Delta\tilde{u})dx = \int_B \bar{h}(|\Delta\tilde{u}|)dx = \int_B \bar{h}((\Delta\hat{u})^*)dx = \int_B \bar{h}(|\Delta\hat{u}|)dx = \int_B h(\Delta\hat{u})dx,$$

which combined with (2.50) and (2.57) yield

$$\int_B h^{**}(\Delta\hat{u})dx = \int_B h(\Delta\hat{u})dx.$$

This shows that $\hat{u}$ is a solution to problem (P_1). Let us prove that $\tilde{u} \equiv u$ in B. Going back to part (c) of the proof of Theorem 2.13, we have $\bar{g}(u(x)) = \bar{g}(\tilde{u}(x))$ a.e. x in B because of (2.58). So that if $\bar{g}$ is strictly monotone we obtain $u \equiv \tilde{u}$ a.e. x in B. ■

Remark 2.15. Part (a) of the Proof of Theorem 2.13 can be applied directly to any solution to problem (P_1) to get one radially symmetric. Therefore, if the function g defined by $g(u) = \bar{g}(|u|)$ is concave, by Theorem 6.2 of [R3] (see also Annexe 1 of [R4]), Theorem 2.13 follows, i.e. in this case it is not necessary to show, a priori, that problem (P_1^{**}) admits a radially symmetric solution.

Remark 2.16. Sufficient conditions for $\bar{g}$ to be decreasing on $\mathbf{R}_+$ is that either $\bar{g}$ be concave or $\bar{g}$ be convex and $\bar{g}(u) < \bar{g}(0)$ for every $u > 0$. The first assertion follows directly from the assumptions. To prove the second one, we proceed by contradiction. Suppose there are $t_2 > t_1 > 0$ such that $\bar{g}(0) > \bar{g}(t_2) > \bar{g}(t_1)$, then the convexity of $\bar{g}$ implies $\bar{g}(t) > \bar{g}(t_2)\quad \forall t > t_2$. Hence $\bar{g}$ is a strictly increasing function in $[t_2, +\infty[$, being bounded from above by $\bar{g}(0)$; $\lim_{t\to+\infty} \bar{g}(t) = L \leq \bar{g}(0)$ does exist. Let us take $t_0 > t_2$ large enough and write for any $n \in \mathbf{N}$, $t_0 = \frac{(n-1)t_0}{nt_0-t_2}t_2 + \frac{t_0-t_2}{nt_0-t_2}nt_0$. Then $\bar{g}(t_0) \leq \frac{(n-1)t_0}{nt_0-t_2}\bar{g}(t_2) + \frac{t_0-t_2}{nt_0-t_2}\bar{g}(nt_0)$. Letting $n \to +\infty$ we obtain $\bar{g}(t_0) \leq \bar{g}(t_2)$, which is a contradiction since $\bar{g}$ is strictly increasing in $[t_2, +\infty[$.
Examples of such functions are: $\bar{g}(|u|) = -|u|^2$, $\bar{g}(|u|) = e^{-|u|}$. The last example shows that, in general, the function g defined by $g(u) = \bar{g}(|u|)$ is not convex even if $\bar{g}$ is so.

Remark 2.17. The use of rearrangements to obtain a radial solution from one which is not so, cannot be applied in the case when we look for the minima in $W_0^{2,p}(B)$ because of the lack of any possible relationship between the normal derivative of $\tilde{u}$ and that of $\hat{u}$ on ∂B. However, we are able to construct a new radial function u such that $\frac{\partial u}{\partial n} = \frac{\partial \tilde{u}}{\partial n} \leq 0$ on ∂B (see (2.53) and (2.56)). Therefore, we have also solved the problem of minimizing in the convex subset

$$\left\{u \in W^{2,p}(B) \cap W_0^{1,p}(B) : \frac{\partial u}{\partial n} \le 0 \quad on \quad \partial B\right\}.$$

Example. Let us consider the same functional mentioned in the introduction

$$I(u) = \int_B e^{-|u(x)|^2} dx + \int_B (1 - |\Delta u(x)|)^2 dx,$$

and we look for the minima of I on the space $W^{2,p}(B) \cap W_0^{1,p}(B)$. Since the function g defined by $g(u) = e^{-|u|^2}$ is neither concave nor convex, Theorem 6.2 of [R3] (see also Annexe 1 of [R4]) and Theorem 2.1 of [A-T3] cannot be applied, either Theorem 2.1 of [Ra] being $h^{**}(0) < h(0)$. However, according to the previous theorem, the associate problem (P_1) has solutions (compare with the assertion given in [E-T, Remark 5.1 of Chapter IX]).

Hypothesis (H2).-

The map $g : [0,1] \times \mathbf{R} \to \mathbf{R}$ is such that

(g_1) g is $\mathcal{L} \otimes \mathcal{B}(\mathbf{R})$-measurable;

(g_2) $u \mapsto g(r,u)$ is lower semicontinuous for almost all r in $[0,1]$;

(g_3) $u \mapsto g(r,u)$ is monotone and convex for almost all r in $[0,1]$;

Moreover:

(g_4) there exists a constant γ_1 such that

$g(r,u) \ge -\gamma_1 |u|^p - \beta_1(r)$ where the function $r \mapsto r^{n-1}\beta_1(r)$ is in $L^1(]0,1[)$.

The map $h : \mathbf{R} \to \overline{\mathbf{R}}$ is such that

(h_1) h is lower semicontinuous.

(h_2) there exist two constants; γ_2, β_2 with $\gamma_2 > 0$ such that

$h(\xi) \ge \gamma_2 |\xi|^p - \beta_2$.

In addition, we assume $1 - \frac{\gamma_1}{\gamma_2} K_0^p > 0$, where K_0 is given by

$$K_0 = K_0(p,n,\Omega) = \sup\left\{ \frac{||u||_p}{||\Delta u||_p} \quad : \quad u \in W_0^{2,p}(\Omega), \quad u \ne 0 \right\}.$$

We now establish the second Theorem

Theorem 2.18. **([F3])** *Let g and h satisfy hypothesis* (H2). *Assume that the functional $\int_B g(|x|, u(x))dx + \int_B h(\Delta u(x))dx$ has a finite value for some u in $W^{2,p}(B) \cap W_0^{1,p}(B)$ such that $\frac{\partial u}{\partial n} = 0$ on ∂B. Then problem (P_2) admits at least one radially symmetric solution.*

Remark 2.19. The condition of monotonicity on the function $g(|x|,\cdot)$ cannot be dropped in general. To see it, consider the same functional as in Remark 2.14 in the space $W_0^{2,2}(B)$.

Proof of Theorem 2.18. As usual, we first consider the relaxed problem

$$\left.\begin{array}{l}\min \displaystyle\int_B g(|x|,u(x))dx + \int_B h^{**}(\Delta u(x))dx \\ u \in W^{2,p}(B)\cap W_0^{1,p}(B) \\ \dfrac{\partial u}{\partial n} = 0 \quad \partial B.\end{array}\right\} \qquad (P_2^{**})$$

Let $\tilde{u}$ be a solution to problem (P_2^{**}), which, as in Theorem 2.9 and Remark 2.11, can be assumed radially symmetric.

From now on, we shall use the same argument applied in the Proof of Theorem 2.13. We can write

$$\int_B h^{**}(\Delta\tilde{u}(x))dx = n\omega_n \int_0^1 r^{n-1}h^{**}(\tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r))dr, \tag{2.62}$$

where, as before, ω_n denotes the volume of the unit ball in $\mathbf{R}^n$ and " $'$ " means the derivative with respect to $|x| = r$. By Lemma 2.6, there exists a function w such that $r \mapsto r^{\frac{n-1}{p}}w(r)$ is in $L^p(]0,1[)$ satisfying

$$\int_0^1 r^{n-1}w(r)dr = \int_0^1 r^{n-1}(\tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r))dr; \tag{2.63}$$

$$\int_0^1 r^{n-1}h(w(r))dr = \int_0^1 r^{n-1}h^{**}(\tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r))dr; \tag{2.64}$$

and for all r in $]0,1]$

$$\int_0^r s^{n-1}w(s)ds \leq \int_0^r s^{n-1}(\tilde{u}''(s) + \frac{n-1}{s}\tilde{u}'(s))ds. \tag{2.65}$$

Now, we consider the function u in $W^{2,p}(B)\cap W_0^{1,p}(B)$, solution to the problem

$$\begin{aligned}\Delta u &= w(|x|)\\ u &= 0 \text{ on } \partial B.\end{aligned} \tag{2.66}$$

Certainly u is a radially symmetric function. We claim that the function u is a solution to problem (P_2). To infer it, we shall prove that

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial B \text{ or equivalently } u'(1) = 0; \tag{2.67}$$

$$\int_B h^{**}(\Delta\tilde{u}(x))dx = \int_B h(\Delta u(x))dx \tag{2.68}$$

and

$$\int_B g(|x|, \tilde{u}(x))dx = \int_B g(|x|, u(x))dx. \tag{2.69}$$

Ad (2.67). Taking into account $\lim r^{n-1}u'(r) = 0$ as $r \to 0^+$, (2.66) in spherical coordinates implies

$$u'(r) = \frac{1}{r^{n-1}} \int_0^r s^{n-1}w(s)ds \quad \text{for } r \in]0,1]. \tag{2.70}$$

Then, from (2.63) it follows that

$$\begin{aligned} u'(1) &= \int_0^1 s^{n-1}w(s)ds = \int_0^1 s^{n-1}(\tilde{u}''(s) + \frac{n-1}{s}\tilde{u}'(s))ds \\ &= \int_0^1 (s^{n-1}\tilde{u}'(s))'ds = \tilde{u}'(1) = 0. \end{aligned}$$

Ad (2.68). This a direct consequence of (2.64) and (2.66).
Ad (2.69). By (2.66) in spherical coordinates, (2.65) becomes

$$\int_0^r \{(s^{n-1}\tilde{u}'(s))' - (s^{n-1}u'(s))'\}ds \geq 0. \tag{2.71}$$

We now integrate by parts to obtain $r^{n-1}\tilde{u}'(r) - r^{n-1}u'(r) \geq 0 \;\; r$ in $]0,1]$, because of the continuity of $\tilde{u}'$, u' as functions of $|x| = r$. Consequently, by integrating again, and by noticing that $\lim u(r) = 0$ as $r \to 1^-$, we have $u(r) - \tilde{u}(r) \geq 0 \;\; r$ in $]0,1]$, i.e. $u(x) \geq \tilde{u}(x)$ a.e. x in B. So that, if $g(|x|,\cdot)$ is a decreasing function for almost all x in B, the latter implies

$$g(|x|, u(x)) \leq g(|x|, \tilde{u}(x)) \;\text{ a.e. } x \text{ in } B. \tag{2.72}$$

Hence

$$\int_B g(|x|, u(x))dx \leq \int_B g(|x|, \tilde{u}(x))dx. \tag{2.73}$$

On the other hand, since $\tilde{u}$ is a solution to (P_2^{**}), by definition of h^{**}, we have

$$\int_B g(|x|, u(x))dx + \int_B h(\Delta u(x))dx \geq \int_B g(|x|, \tilde{u}(x))dx + \int_B h^{**}(\Delta \tilde{u}(x))dx.$$

From (2.68) and (2.73), it follows that (2.69) holds. This proves that u is a radially symmetric solution to problem (P_2). If $g(|x|, \cdot)$ is an increasing function for almost all x in B, we apply Remark 2.7 to obtain $u(x) \leq \tilde{u}(x)$ a.e. x in B. Then, (2.72) holds. We now repeat the argument previously applied. ■

Remark 2.20. From the proof of Theorem 2.18, it follows that we have also minimized such functional under the additional condition $u \geq \psi$ in B (problem with obstacle), if $g(|x|, \cdot)$ is decreasing and ψ is any fix radially symmetric function in L^1.

Remark 2.21. The convexity assumption has only been required to obtain a radially symmetric solution to problem (P_2^{**}). Consequently, if we deal with problems involving the first derivative instead of the Laplacian and, where the space variable x is in $\mathbf{R}$, as that in [M1], Lemma 2.6, combined with Remark 2.7, yields a new proof of theorem 2 of [M1].

Remark 2.22. In case $g(r, \cdot)$ is strictly convex, $\tilde{u}$ and $\bar{u}$ defined in (2.1), have to coincide, and thus $\tilde{u}$ is already a radially symmetric solution to problem (P_2^{**}), i.e. every solution to problem (P^{**}) is radially symmetric.

2.4. A further Existence Result

The previous sections were devoted to integrals defined on symmetric domains of $\mathbf{R}^n$, $n \geq 2$. Nevertheless, all of these results hold when $n = 1$, with obvious changes, the case $n = 1$ allows us to deal with a different class of functions g. In particular, the symmetry of the interval is not needed. More specifically, we shall consider the following minimization problem

$$\min_{x \in H_0^2(]0,T[)} \int_0^T g(t, x(t))dt + \int_0^T h(t, x''(t) - \lambda(t)x(t))dt, \qquad (P_3)$$

where λ is a non-negative real function, g and h are Caratheodory type functions with h satisfying no convexity assumption. Therefore, the classical Direct Method cannot be applied because of the lack of the lower semicontinuity of the functional. In spite of this fact we show that the problem (P_3) admits solutions, such a solution x will satisfy, letting $L = \frac{d^2}{dt^2} - \lambda(t)$

$$LD_2h(t, x''(t) - \lambda(t)x(t)) = -g_x(t, x(t)) \quad \text{in } \mathcal{D}'(]0, T[)$$

under some regularity assumptions on h and g, other than the usual growth ones. So that, we have solved, formally, a broader class of equations than those considered in [G2], [U] namely

$$\begin{aligned} &x^{(4)} + a(t)x = b(t) \\ &x(0) = x(T) = 0, \;\; x'(0) = x'(T) = 0. \end{aligned} \qquad (P')$$

Problems like (P') describe the deformation of a elastic beam with end-points cantilevered or fixed. For another type of boundary conditions which arise, according to the controls at the ends of the beam, we refer to [G1], [G2], [U].

We prove an existence theorem under the concavity assumption on the function $g(t, \cdot)$. It is done in a different setting from those presented in [A-T1], [A-T2] and [R1], but follows that of [C-F], [F1] in the vector case, i.e. where $]0, T[$ is replaced for a subset of $\mathbf{R}^n$.

Henceforth $T > 0$, $I =]0, T[$. Let $H_0^2(I)$ be the closure of $C_0^\infty(I)$ in $H^2(I)$. In this situation the immersion $H_0^2(I) \subset C^1(\bar{I})$ is compact (See [Ad] for instance). For fixed $\lambda \in L^\infty(I), \lambda \geq 0$ we equip the space $H_0^2(I)$ with the norm $||x'' - \lambda x||_{L^2(I)}$.

We here assume the following hypothesis:

HYPOTHESIS (H3).- The map $\lambda : I \to \mathbf{R}$ is in $L^\infty(I)$ such that $\lambda \geq 0$.

The map $g : I \times \mathbf{R} \to \overline{\mathbf{R}}$ is such that

(g_1) g is $\mathcal{L} \otimes \mathcal{B}(\mathbf{R})$-measurable;

(g_2) $x \mapsto g(t, x)$ is lower semicontinuous and concave for almost all t in I.

Moreover:

(g_3) there exists a positive constant γ_1, such that

$$g(t, x) \geq -\gamma_1 |x|^2 - \beta_1(t) \text{ where the function } \beta_1 \text{ is in } L^2(I).$$

The map $h : I \times \mathbf{R} \to \overline{\mathbf{R}}$ is such that

(h_1) h is $\mathcal{L} \otimes \mathcal{B}(\mathbf{R})$-measurable;

(h_2) $\xi \mapsto h(t, \xi)$ is lower semicontinuous for almost all t in I.

Moreover:

(h_3) there exists a positive constant γ_2, such that

$h(t,\xi) \geq \gamma_2|\xi|^2 - \beta_2(t)$ where the function β_2 is in $L^1(I)$.

Here, we also assume that $1 - \frac{\gamma_1}{\gamma_2}K_0^p > 0$, where K_0 as in Hypothesis (H2).

Theorem 2.23. ([F2]) *Under Hypothesis* (H) *and by assuming the functional be finite for some x in $H_0^2(]0,T[)$, problem (P_3) admits at least one solution.*

Proof. As usual, we start by considering the problem

$$\min_{x \in H_0^2(]0,T[)} \int_0^T g(t,x(t))dt + \int_0^T h^{**}(t, x''(t) - \lambda(t)x(t))dt. \qquad (P_3^{**})$$

Obviously h^{**} satifies the growth condition (h_3), then the well known Direct Method in the Calculus of Variations (see [E-T] for instance) assures that problem (P^{**}) has a solution. We denote it by $\tilde{x}$.

By (b) of Proposition 1.4 there exist measurable functions p_i and v_i, $i = 1, 2$, such that:

$$\sum_{i=1}^{2} p_i(t) = 1; \quad p_i(t) \geq 0 \ \ i = 1,2; \tag{2.74}$$

$$\sum_{i=1}^{2} p_i(t)v_i(t) = \tilde{x}''(t) - \lambda(t)\tilde{x}(t); \tag{2.75}$$

$$\sum_{i=1}^{2} p_i(t)h(t,v_i(t)) = h^{**}(t, \tilde{x}''(t) - \lambda(t)\tilde{x}(t)). \tag{2.76}$$

On the other hand, the subdifferential of $g(t,\tilde{x}(t))$ with respect to the second argument, i.e. the map $t \mapsto -\partial_x(-g(t,\tilde{x}(t)))$, admits a selection $\sigma(\cdot)$ in $L^2(I)$, for the existence of such a selection see [R3, Lemma 5.2].

Now, consider the function ϕ in $H^2(]0,T[)$, solution to the Cauchy problem

$$\phi'' - \lambda(t)\phi = \sigma(t) \tag{2.77}$$

$$\phi(T) = \phi'(T) = 0, \tag{2.78}$$

as we shall see, for our purpose it is enough to take any ϕ satisfying (2.77) without the boundary conditions (2.78).

Let X be the solution of the matrix differential equation

$$X' = A(t)X$$
$$X(0) = \mathrm{I},$$

here, I denotes the Identity matrix. Then the solution to

$$z' = A(t)z + \hat{f}(t)$$
$$z(0) = 0$$

can be written as

$$z(t) = X(t) \int_0^t X^{-1}(s)\hat{f}(s)ds. \tag{2.79}$$

Setting

$$A(t) = \begin{pmatrix} 0 & 1 \\ \lambda(t) & 0 \end{pmatrix},$$

we apply Proposition 1.15 to obtain a measurable partition $(E_i)_i$; $i = 1, 2$ of I, such that:

$$\int_0^T \sum_{i=1}^{2} p_i(t)h(t, v_i(t))dt = \int_0^T \sum_{i=1}^{2} \chi_{E_i}(t)h(t, v_i(t))dt; \tag{2.80}$$

$$\int_0^T \sum_{i=1}^{2} p_i(t)X^{-1}(t)\begin{pmatrix} 0 \\ v_i(t) \end{pmatrix} dt = \int_0^T \sum_{i=1}^{2} \chi_{E_i}(t)X^{-1}(t)\begin{pmatrix} 0 \\ v_i(t) \end{pmatrix} dt; \tag{2.81}$$

$$\int_0^T \sum_{i=1}^{2} p_i(t)\phi(t)v_i(t)dt = \int_0^T \sum_{i=1}^{2} \chi_{E_i}(t)\phi(t)v_i(t)dt. \tag{2.82}$$

In particular, the integral $\int_0^T \sum_{i=1}^2 \chi_{E_i}(t)h(t, v_i(t))dt$ is finite. Therefore, by using the growth condition (h_3), one can prove that the map

$$t \mapsto \sum_{i=1}^{2} \chi_{E_i}(t)v_i(t) \tag{2.83}$$

belongs to $L^2(]0, T[)$. From (2.80) and (2.76) it follows that

$$\int_0^T \sum_{i}^{2} \chi_{E_i}(t)h(t, v_i(t))dt = \int_0^T h^{**}(t, \tilde{x}''(t) - \lambda(t)\tilde{x}(t))dt. \tag{2.84}$$

Since E_i, $i = 1, 2$, is a partition of I, we have

$$h(t, \sum_{i=1}^{2} \chi_{E_i}(t) v_i(t)) = \sum_{i=1}^{2} \chi_{E_i}(t) h(t, v_i(t)) \quad \text{a. e. } t \in]0, T[.$$

Therefore (2.84) can be written as

$$\int_0^T h^{**}(t, \tilde{x}''(t) - \lambda(t)\tilde{x}(t))dt = \int_0^T h(t, \sum_{i=1}^{2} \chi_{E_i}(t) v_i(t))dt. \tag{2.85}$$

Now, let x be the solution to the Cauchy problem

$$x'' - \lambda(t)x = \sum_{i=1}^{2} \chi_{E_i}(t) v_i(t) \tag{2.86}$$

$$x(0) = x'(0) = 0. \tag{2.87}$$

Certainly $x \in H^2(]0, T[)$ since the right-hand side of (2.86) is in $L^2(]0, T[)$. We shall prove that the function x is actually a solution to problem (P_3). First of all we show that $x(T) = x'(T) = 0$. To this end, we use the first order differential equation associated to (2.86)-(2.87) and the representation formula (2.79) for its solution $z = (x, x')$ and we denote by $\tilde{z}$ the corresponding to $\tilde{x}$ (see (2.75)). Therefore, it suffices to prove $z(T) = 0$;

$$\begin{aligned} z(T) &= X(T) \int_0^T \sum_{i=1}^{2} \chi_{E_i}(t) X^{-1}(t) \begin{pmatrix} 0 \\ v_i(t) \end{pmatrix} dt \\ &= X(T) \int_0^T \sum_{i=1}^{2} p_i(t) X^{-1}(t) \begin{pmatrix} 0 \\ v_i(t) \end{pmatrix} dt = \tilde{z}(T) = 0, \end{aligned}$$

where we have used (2.81) and the representation formula for $\tilde{x}$, i.e. $\tilde{z}$, besides its initial conditions. This proves our claim, i.e. $x \in H_0^2(]0, T[)$. In order to show that x is a solution to problem (P_3), it only remains to prove

$$\int_0^T h^{**}(t, \tilde{x}''(t) - \lambda(t)\tilde{x}(t))dt = \int_0^T h(t, x''(t) - \lambda(t)x(t))dt \tag{2.88}$$

and

$$\int_0^T g(t, \tilde{x}(t))dt = \int_0^T g(t, x(t))dt. \tag{2.89}$$

Ad (2.88). This is a direct consequence of (2.85) and (2.86).

Ad (2.89). Notice that for any selection of $x \mapsto -\partial_x(-g(t, \tilde{x}(t)))$, we have

$$g(t, x(t)) \leq g(t, \tilde{x}(t)) + \sigma(t)(x(t) - \tilde{x}(t)), \quad \text{a. e.} \quad t \in I. \tag{2.90}$$

Assume we have showed

$$\int_0^T \sigma(t)(x(t) - \tilde{x}(t))dt = 0. \tag{2.91}$$

Then (2.90) implies

$$\int_0^T g(t, x(t))dt \leq \int_0^T g(t, \tilde{x}(t))dt. \tag{2.92}$$

Therefore, since $\tilde{x}$ is a solution to problem (P_3^{**}), by (b) of Proposition 1.2

$$\int_0^T g(t, x(t))dt + \int_0^T h(t, x''(t) - \lambda(t)x(t))dt \geq$$

$$\geq \int_0^T g(t, \tilde{x}(t))dt + \int_0^T h^{**}(t, \tilde{x}''(t) - \lambda(t)\tilde{x}(t))dt.$$

Finally from (2.88) and (2.92), (2.89) follows. We now prove assertion (2.91);

$$\int_0^T \sigma(t)(x(t) - \tilde{x}(t))dt = \int_0^T (\phi''(t) - \lambda(t)\phi(t))(x(t) - \tilde{x}(t))dt =$$

$$= \int_0^T \phi(t)(x''(t) - \lambda(t)x(t) - \tilde{x}''(t) + \lambda(t)\tilde{x}(t))dt,$$

where the last integral has been obtained by applying Green formula and taking into account only the boundary conditions of x and $\tilde{x}$, not that of ϕ. Then by using (2.75), (2.86) and (2.82), assertion (2.91) is proved, and thus the proof of theorem is complete. ■

Remark 2.24. The result holds even in the case when h and g are defined on $I \times \mathbf{R}^n$, $n > 1$. The proof is essentially the same up to adding components before applying Proposition 1.15.

CHAPTER 3

Relaxation Problems in the Calculus of Variations

In this chapter our main concern is the study of problems of the form

$$\inf_{u\in u_0+V}\int_\Omega f(x,u(x),Lu(x))dx, \tag{P}$$

where u_0 is any fixed function in $W^{2,p}(\Omega)$, V is a closed subspace of $W^{2,p}(\Omega)$ containing $W_0^{2,p}(\Omega)$, $p>1$ and L is an elliptic operator of second order.
The well-known Direct Method in the Calculus of Variations yields the existence of solutions to problem (P) under the following conditions on the functional to be minimized
(i) the functional is lower semicontinuous (l.s.c.);
(ii) the functional is coercive with respect to the same topology in which is l.s.c..
In this context, usually, the topology to be considered is the weak one, and condition (ii) is guaranteed by some growth assumptions on f.

Our aim is to study the existence of solutions for (P) precisely when the functional is not l.s.c., i.e. when $f(x,u,\cdot)$ is not convex, but maintaining condition (ii); then, in general, problem (P) will not have any solution. However, from the point of view of applications, the study of the asymptotic behaviour of the minimizing sequences for (P) is extremely important. More precisely, we will show that if $f^{**}(x,u,\xi)$ is the bipolar function of $f(x,u,\cdot)$, then

$$\inf_{u\in u_0+V}\int_\Omega f(x,u(x),Lu(x))dx = \inf_{u\in u_0+V}\int_\Omega f^{**}(x,u(x),Lu(x))dx, \tag{*}$$

and for every $\bar{u}$, solution to the right-hand side of $(*)$, there exists a minimizing sequence (u_h) for (P), $u_h\in u_0+V$ such that $u_h \rightharpoonup \bar{u}$ in $W^{2,p}$, and

$$\lim_{h\to+\infty}\int_\Omega f(x,u_h(x),Lu_h(x))dx = \int_\Omega f^{**}(x,\bar{u}(x),L\bar{u}(x))dx.$$

Thus, the asymptotic behaviour of the minimizing sequences is with respect to the weak convergence. The importance of this kind of convergence ([Tar]) relies on the fact that, in

many physical applications, only averages of physical quantities are actually measured. In this direction, the Relaxation theory provides a useful approach.

Some relaxation problems in the Calculus of Variations have been studied, among others, in [Ac-Fu, M-Sb] for functionals of the gradient, and in [E-T] for the present case. The results we shall present improve and generalize those of [E-T] (Thm 3.1.IX and Thm.4.1.IX). A rather similar problem, but in the framework of Optimal Control, was considered in [Bu1]. It is to be noted, however, that the analysis done there cannot be applied here. Other related results can be found in [E].

3.1. Notations and Preliminaries

We start by proving an elementary result on functionals defined on reflexive Banach spaces.

Proposition 3.1. *Let X be a reflexive Banach space, Y be a closed subspace of X, $u_0 \in X$ and $H : X \to]-\infty, +\infty]$ be a functional such that:*
(i) for every $t \in \mathbf{R}$ $\Big\{u \in X : \; H(u) \le t \Big\} \subset u_0 + Y$;
(ii) H restricted to $u_0 + Y$ (endowed with the topology of X) is sequentially weakly l.s.c..
Then H is l.s.c. (for the strong topology of X).

Proof. Let (u_h) be a sequence converging to a function u in X. We shall prove

$$H(u) \le \liminf_{h \to +\infty} H(u_h).$$

If the right-hand side equals $+\infty$, there is nothing to prove. Thus we may, up to extracting a subsequence, assume that $\lim_{h\to+\infty} H(u_h)$ exists and is less than $+\infty$. By (i), $u_h \in u_0 + Y$, since $(u_h - u_0)$ is bounded in Y, it being reflexive too, a subsequence of $(u_h - u_0)$ converges weakly to a function v in Y. But $u_h \to u$ in X, therefore $u - u_0 = v$, hence $u \in u_0 + Y$ and $u_h \rightharpoonup u$ in $u_0 + Y$. The conclusion now follows from assumption (ii). ■

We shall make the following assumptions:

Here we are given a bounded open set $\Omega \subset \mathbf{R}^n$ with boundary $\partial\Omega$ of class $C^{1,1}$, and $f : \Omega \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a function such that
(f_1) for every $u, \xi \in \mathbf{R}$ $f(\cdot, u, \xi)$ is $\mathcal{L}$-measurable;

(f_2) for almost every $x \in \Omega$ $f(x,\cdot,\cdot)$ is continuous;

(f_3) there are positive constants $\alpha_1 \leq \alpha_2$ such that $\alpha_1|\xi|^p \leq f(x,u,\xi) \leq \alpha_2(|u|^p + |\xi|^p) + \beta(x)$ for some function β in $L^1(\Omega)$ with $p > 1$.

Let L be the differential operator of second order:

$$Lu = \sum_{ij} a_{ij}(x)D_{ij}u + \sum_i b_i(x)D_iu + c(x)u$$

where $a_{ij} \in C^0(\bar{\Omega})$, $b_i, c \in L^\infty(\Omega)$, $i,j = 1,...n$ and $c \leq 0$. Further, for a given closed subspace V of $W^{2,p}$, let us consider the function $F : W^{2,p}(\Omega) \to [0,+\infty]$ defined by

$$F(u) = \begin{cases} \int_\Omega f(x,u(x),Lu(x))dx & \text{if } u \in u_0 + V \\ +\infty & \text{otherwise,} \end{cases}$$

here u_0 is any fixed function in $W^{2,p}(\Omega)$. We also consider the function H defined as follows

$$H(u) = \begin{cases} \int_\Omega f^{**}(x,u(x),Lu(x))dx & \text{if } u \in u_0 + V \\ +\infty & \text{otherwise,} \end{cases}$$

where $f^{**}(x,u,\xi)$ is the bipolar of the function $f(x,u,\cdot)$. By Proposition 3.1 H is l.s.c. for the strong topology of $W^{2,p}(\Omega)$. From Proposition 1.3.5 and Remark 1.3.6 of [Bu2], it follows that

$$\begin{aligned} sc^-(\omega - W^{2,p})F(u) &= sc^-_{seq}(\omega - W^{2,p})F(u) \\ &= \inf\left\{ \liminf_{h\to+\infty} F(u_h) : u_h \rightharpoonup u \text{ in } W^{2,p} \right\}, \end{aligned}$$

where $sc^-(\omega - W^{2,p})F$, $sc^-_{seq}(\omega - W^{2,p})F$ denote the lower semicontinuous envelope of F for the weak topology of $W^{2,p}$ and for its sequential version respectively (see [Bu2]), i.e. $sc^-(\omega - W^{2,p})F$ is the greatest l.s.c. function in the weak topology of $W^{2,p}$, not greater than F and $sc^-_{seq}(\omega - W^{2,p})F$ is the greatest sequentially l.s.c. function, in the weak topology of $W^{2,p}$ not greater than F.

Remark 3.2. The fact that, the infimum in the last equality is a minimum, can be deduced from a standard diagonalization argument since the weak topology of $W^{2,p}(\Omega)$, $p > 1$ is metrizable on bounded sets in $W^{2,p}(\Omega)$.

3.2. Main Results

We are now in position to state the first main result that will be used later, although is of importance by itself.

Theorem 3.3. ([F5]) *Assume* $V = W^{2,p} \cap W_0^{1,p}$ *and* L*, defined above, be a strictly elliptic operator in* Ω*. Then* $sc^-_{seq}(\omega - W^{2,p})F = sc^-(\omega - W^{2,p})F = H$*. More precisely, for every* $u \in u_0 + W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$*, there exists a sequence* (u_h) *in* $u_0 + W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *such that* $u_h \rightharpoonup u$ in $W^{2,p}$ *and*

$$\lim_{h \to +\infty} \int_\Omega f(x, u_h(x), Lu_h(x))dx = \int_\Omega f^{**}(x, u(x), Lu(x))dx.$$

In particular,

$$sc^-(\omega - W^{2,p})F(u) = \int_\Omega f^{**}(x, u(x), Lu(x))dx \quad \forall u \in u_0 + W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Proof. It is easy to see that H is sequentially weakly l.s.c. in $W^{2,p}$ majorized by F. Therefore

$$H(u) \leq sc^-_{seq}(\omega - W^{2,p})F(u) = \min\left\{ \liminf_{h \to +\infty} F(u_h) : u_h \rightharpoonup u \text{ in } W^{2,p} \right\} \quad \forall u \in W^{2,p}(\Omega).$$

in order to prove the equality in the preceding inequality, it suffices to show it for u in $u_0 + W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, otherwise the equality holds trivially. For $u \in u_0 + W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ we shall prove the existence of a sequence (u_h) in $u_0 + W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $u \rightharpoonup u$ in $W^{2,p}$ and

$$\int_\Omega f^{**}(x, u(x), Lu(x))dx = \lim_{h \to +\infty} \int_\Omega f(x, u_h(x), Lu_h(x))dx.$$

We apply Prop.1.2.IX and Prop.2.3.IX of [E-T] (or Theorem 2.6.4. of [Bu2]) to the function $\tilde{f}(x, v) = f(x, u(x), v)$, F being coercive, we obtain a sequence (v_h), $v_h \rightharpoonup Lu$ in $L^p(\Omega)$ such that

$$\int_\Omega f^{**}(x, u(x), Lu(x))dx = \lim_{h \to +\infty} \int_\Omega f(x, u(x), v_h(x))dx. \tag{3.1}$$

Setting (for instance [G-T, Thm.9.15]) $Lu_h = v_h$, $u_h \in u_0 + W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, we get $Lu_h \rightharpoonup Lu$ in $L^p(\Omega)$, and because of Lemma 9.17 of [G-T], up to extracting a subsequence, we have $u_h \rightharpoonup u$ in $W^{2,p}(\Omega)$.

We claim that for a subsequence (still indexed by h) the conclusion holds. To prove the claim, we first notice, by the Convergence lemma of [Ei], that

$$q_h(x) \doteq f(x, u_h(x), Lu_h(x)) - f(x, u(x), Lu_h(x))$$

converges to zero in measure. Extracting a subsequence we may suppose that $q_h(x) \to 0$ a.e. in Ω. Using the growth condition on f we deduce

$$|q_h(x)| \leq \alpha_2(|u(x)|^p + |u_h(x)|^p + 2|Lu_h(x)|^p) + 2\beta(x). \tag{3.2}$$

Since the family of $|u_h|^p$, $|Lu_h|^p$, $h \in \mathbf{N}$, is equi-integrable, the right-hand side of (3.2) is equi-integrable too. By Corollary 1.3.VIII of [E-T] we conclude that $q_h \to 0$ in $L^1(\Omega)$. In particular:

$$\lim_{h\to+\infty} \left[\int_\Omega f(x, u_h(x), Lu_h(x))dx - \int_\Omega f(x, u(x), Lu_h(x))dx \right] = 0,$$

which together with (3.1) we finally obtain

$$\lim_{h\to+\infty} \int_\Omega f(x, u_h(x), Lu_h(x))dx = \int_\Omega f^{**}(x, u(x), Lu(x))dx. \blacksquare$$

Remark 3.4. The previous theorem improves the results established in Theorems 3.1.IX and 4.1.IX of [E-T]; in Theorem 3.1, the authors consider integrands of the form $h(x,\xi) + g(x,u)$ with $g(x,\cdot)$ convex, whereas in Theorem 4.1 they do not prove that $sc^-(\omega - W^{2,p})F = H$ (see Remark 4.1. IX of [E-T]).

We next present a refinement of the preceding theorem that admits the functional be defined in an affine closed subspace of $W^{2,p} \cap W_0^{1,p}$. Thus, we obtain a generalization to that established in [E-T]. The proof involves a technique widely used in the theory of Γ-convergence ([DM, Bu2, [Bu-T]] and references therein).

Theorem 3.5. ([**F5**]) *Assume V to be a closed subspace of $W^{2,p} \cap W_0^{1,p}$ containing $W_0^{2,p}$ and L, defined as before, be a strictly elliptic operator in Ω. Then $sc^-_{seq}(\omega - W^{2,p})F = H$.*

More precisely, for every $u \in u_0 + V$*, there exists a sequence* (u_h) *in* $u_0 + V$ *such that* $u_h \rightharpoonup u$ in $W^{2,p}$ *and*

$$\lim_{h\to+\infty}\int_\Omega f(x,u_h(x),Lu_h(x))dx = \int_\Omega f^{**}(x,u(x),Lu(x))dx.$$

In particular,

$$sc^-(\omega - W^{2,p})F(u) = \int_\Omega f^{**}(x,u(x),Lu(x))dx \quad \forall u \in u_0 + V.$$

Proof. Clearly, as before H is sequentially weakly l.s.c. in $W^{2,p}$. So that,

$$H(u) \le sc^-_{seq}(\omega - W^{2,p})F(u) = \min\left\{\liminf_{h\to+\infty} F(u_h) : u_h \rightharpoonup u \text{ in } W^{2,p}\right\} \quad \forall u \in W^{2,p}(\Omega).$$

In order to prove that the last inequality is, in fact, an equality, we shall prove the following: given any $u \in W^{2,p}$ and $\varepsilon > 0$ there exists a sequence (u_h) in $W^{2,p}$ such that $u_h \rightharpoonup u$ in $W^{2,p}(\Omega)$ and

$$\liminf_{h\to+\infty} F(u_h) \le H(u) + \varepsilon. \tag{3.3}$$

If $u \notin u_0 + V$ the inequality trivially holds, so that we may assume $u \in u_0 + V \subset u_0 + W^{2,p} \cap W_0^{1,p}$, applying the previous theorem, we obtain the existence of a sequence (u_h) in $u_0 + W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $u_h \rightharpoonup u$ in $W^{2,p}(\Omega)$ and

$$\int_\Omega f^{**}(x,u(x),Lu(x))dx = \lim_{h\to+\infty}\int_\Omega f(x,u_h(x),Lu_h(x))dx \tag{3.4}$$

Let us fix a compact subset K of Ω, setting $\delta = dist(K,\partial\Omega) > 0$, we consider the sets A_l, $l = 1,...,\nu$ defined as follows: $A_0 = \text{Int}K$, $A_l = \left\{x \in \mathbf{R}^n : dist(x,K) < \frac{l\delta}{\nu}\right\}$ and functions $\varphi_l \in C_0^\infty(A_l)$ such that $0 \le \varphi_l \le 1$ in $\mathbf{R}^n$, $\varphi_l = 1$ in A_{l-1} and $|D^\alpha\varphi_l| \le M\left(\frac{\nu}{\delta}\right)^{|\alpha|}$ $|\alpha| \le 2$ where M is a fix positive constant. Define $w_{l,h} = \varphi_l u_h + (1-\varphi_l)u$, clearly $w_{l,h} \in u_0 + V$ for every $l = 1,...,\nu$ and $\forall h \in \mathbf{N}$ and $w_{l,h} \rightharpoonup u$ in $W^{2,p}$ for every l. On the other hand, since $Lw_{l,h} = Lu + \varphi_l(Lu_h - Lu) + (u_h - u)L\varphi_l - c(x)\varphi_l(u_h - u) + \sum_{ij} a_{ij}(x)\Big[(D_j\varphi_l)D_i(u_h - u) + (D_i\varphi_l)D_j(u_h - u)\Big]$, we have

$$\int_\Omega f(x,w_{l,h}(x),Lw_{l,h}(x))dx = \int_{\bar{A}_{l-1}} f(x,u_h(x),Lu_h(x))dx + \int_{\Omega\setminus A_l} f(x,u(x),Lu(x))dx +$$
$$+\int_{A_l\setminus\bar{A}_{l-1}} f(x,w_{l,h}(x),Lw_{l,h}(x))dx \le$$

$$\leq \int_\Omega f(x,u_h(x),Lu_h(x))dx + \int_{\Omega\setminus K} f(x,u(x),Lu(x))dx+$$
$$+\int_{A_l\setminus \bar{A}_{l-1}} f(x,w_{l,h}(x),Lw_{l,h}(x))dx. \tag{3.5}$$

We set $S_l = A_l \setminus \bar{A}_{l-1}$ and let us estimate the last integral, by the properties of φ_l and assumption (f_3):

$$I_{l,h} \doteq \int_{S_l} f(x,w_{l,h},Lw_{l,h})dx \leq$$
$$\leq 2^{p-1}\alpha_2 \int_{S_l}|u|^p dx + c_1\int_{S_l}|Lu|^p dx + c_2\int_{S_l}|Lu_h - Lu|^p dx + c_3\int_{S_l}|u_h-u|^p dx+$$
$$+c_4\int_{S_l}\sum_i |D_iu_h - D_iu|^p dx + \int_{S_l}\beta(x)dx,$$

where c_i; $i=1,...,4$ are positive constants with c_1, c_2 independently of δ,ν. Then,

$$\min_{1\leq l\leq \nu} I_{l,h} \leq \frac{1}{\nu}\sum_{l=1}^{\nu} I_{l,h} \leq$$
$$\leq 2^{p-1}\frac{\alpha_2}{\nu}\int_\Omega |u|^p dx + \frac{c_1}{\nu}\int_\Omega |Lu|^p dx + \frac{c_2}{\nu}\int_\Omega |Lu_h - Lu|^p dx+$$
$$+\frac{c_3}{\nu}\int_{\Omega\setminus K}|u_h-u|^p dx + \frac{c_4}{\nu}\int_{\Omega\setminus K}\sum_i |D_iu_h - D_iu|^p dx + \frac{1}{\nu}\int_{\Omega\setminus K}\beta(x)dx.$$

Since $Lu_h \rightharpoonup Lu$ in L^p, $||Lu_h - Lu||$ is bounded in L^p, say by c_0. Returning to (3.5), we obtain for some $l_h \in \{1,...,\nu\}$:

$$\int_\Omega f(x,w_{l_h,h},Lw_{l_h,h})dx = \min_{1\leq l\leq\nu}\int_\Omega f(x,w_{l,h},Lw_{l,h})dx \leq$$
$$\leq \int_\Omega f(x,u_h,Lu_h)dx + \int_{\Omega\setminus K} f(x,u,Lu)dx + 2^{p-1}\frac{\alpha_2}{\nu}||u||^p+$$
$$+\frac{c_1}{\nu}||Lu||^p + \frac{c_0c_2}{\nu} + \frac{c_3}{\nu}\int_\Omega |u_h-u|^p dx+$$
$$+\frac{c_4}{\nu}\int_{\Omega\setminus K}\sum_i |D^iu_h - D^iu|^p dx + \frac{1}{\nu}||\beta||_1.$$

By a standard diagonalization procedure, we have the existence of a sequence (w_h) converging weakly to u in $W^{2,p}(\Omega)$. By (3.4) and recalling that $u_h \rightharpoonup u$ in $W^{2,p}$, we get

$$\liminf_{h\to+\infty}\int_\Omega f(x,w_h,Lw_h)dx \le \int_\Omega f^{**}(x,u,Lu)dx + \int_{\Omega\setminus K} f(x,u,Lu)dx + 2^{p-1}\frac{\alpha_2}{\nu}||u||^p +$$
$$+\frac{c_1}{\nu}||Lu||^p + \frac{c_0c_2}{\nu} + \frac{1}{\nu}||\beta||_1.$$

Choosing K and ν such that

$$\int_{\Omega\setminus K} f(x,u,Lu)dx < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1}{\nu}(2^{p-1}\alpha_2||u||^p + c_1||Lu||^p + c_0c_2 + ||\beta||_1) < \frac{\varepsilon}{2},$$

we conclude that

$$\liminf_{h\to+\infty} F(w_h) \le H(u) + \varepsilon,$$

which proves $sc^-_{seq}(\omega - W^{2,p})F(u) = H(u)$. ■

As a consequence of the very definition of lower semicontinuous envelope, we have

$$\inf_{u\in W^{2,p}(\Omega)} sc^-_{seq}(\omega - W^{2,p})F(u) = \inf_{u\in W^{2,p}(\Omega)} F(u).$$

Therefore,

$$\inf_{u\in u_0+V}\int_\Omega f(x,u(x),Lu(x))dx = \inf_{u\in u_0+V}\int_\Omega f^{**}(x,u(x),Lu(x))dx. \tag{3.6}$$

So that, if $||L(\cdot)||_p$ is a norm on V equivalent to the usual one $||\cdot||_{2,p}$ in $W^{2,p}$, then the right-hand side of (3.6), because of (f_3), admits at least one solution. Consequently, for every solution $\tilde{u}$ of $(\bar{P})$:

$$\min_{u\in u_0+V}\int_\Omega f^{**}(x,u(x),Lu(x))dx, \tag{$\bar{P}$}$$

there exists a minimizing sequence (u_h) of (P) $u_h \in u_0 + V$ such that $u_h \rightharpoonup \tilde{u}$ in $W^{2,p}$ and

$$\lim_{h\to+\infty}\int_\Omega f(x,u_h(x),Lu_h(x))dx = \int_\Omega f^{**}(x,\tilde{u}(x),L\tilde{u}(x))dx.$$

The above theorem is still valid if f is supposed to satisfy, instead of (f_3), condition (f_3'):

(f_3') there is a positive constant α such that $0 \le f(x,u,\xi) \le \alpha(|u|^p + |\xi|^p) + \beta(x)$ for some β in L^1.

Thus, in particular, (3.6) still holds. The proof of the next theorem is similar to that given in [A-DC].

Theorem 3.6. *Assume f satisfies $(f_1),(f_2)$ and (f_3'), L,V,Ω as in the preceding theorem. Then, $sc^-_{seq}(\omega - W^{2,p})F = H$. In particular (3.6) holds.*

Proof. Let us apply Theorem 3.5 to the function F_ε corresponding to $f_\varepsilon(x,u,\xi) = f(x,u,\xi) + \varepsilon|\xi|^p$, $\varepsilon > 0$. Thus we obtain

$$sc^-_{seq}(\omega - W^{2,p})F_\varepsilon = sc^-(\omega - W^{2,p})F_\varepsilon = H_\varepsilon,$$

where $H_\varepsilon : W^{2,p} \longrightarrow [0,+\infty]$ is defined by

$$H_\varepsilon(u) = \begin{cases} \int_\Omega f_\varepsilon^{**}(x,u(x),Lu(x))dx & \text{if } u \in u_0 + V \\ +\infty & \text{otherwise.} \end{cases}$$

Since $f_\varepsilon^{**}(x,u,\xi) = f^{**}(x,u,\xi) + \varepsilon|\xi|^p$, we have

$$\lim_{\varepsilon\to 0+} sc^-_{seq}(\omega - W^{2,p})F_\varepsilon(u) = \lim_{\varepsilon\to 0+} H_\varepsilon(u) = H(u).$$

On the other hand, $F(u) \leq F_\varepsilon(u)$ $\forall\varepsilon > 0$, so that, $sc^-_{seq}(\omega - W^{2,p})F(u) \leq sc^-_{seq}(\omega - W^{2,p})F_\varepsilon(u) = H_\varepsilon(u)$. Therefore, $sc^-_{seq}(\omega - W^{2,p})F(u) \leq H(u)$. H being sequentially weakly l.s.c. not greater than F we deduce $H(u) \leq sc^-_{seq}(\omega - W^{2,p})F(u)$. Consequently $H(u) = sc^-_{seq}(\omega - W^{2,p})F(u)$. ■

Remark 3.7. Since f is not coercive in Thm. 3.6, we cannot conclude that $sc^-(\omega - W^{2,p})F(u) = sc^-_{seq}(\omega - W^{2,p})F(u)$.

Remark 3.8. We point out that Theorems 3.3, 3.5 and 3.6 as well, can be extended to the case of differential operator of order $2k$ with the usual assumptions on the coefficients. Here, the relaxation takes place in the weak topology of $W^{2k,p}$, and the closed subspace V is such that $W_0^{2k,p} \subset V \subset W^{2k,p} \cap W_0^{k,p}$ and the functions u_h arised in the proof of Thm. 3.3 will be the solutions to the equations (see [Ag-D-N]):

$$\begin{aligned} Lu_h &= v_h \\ \frac{\partial^m u_h}{\partial n^m} &= 0 \quad \text{on } \partial\Omega \ m = 0,1,...,k-1, \end{aligned}$$

which belongs to $W^{2k,p} \cap W_0^{k,p}$.

We finally present a result concerning the existence of minima for non-convex functionals depending only on the Laplacian. The proof outlines the different scope of the Relaxation method in the study of non-convex minimization problems in the Calculus of Variations. In particular, we do not require a boundedness from above for the integrand.

Theorem 3.9. *Assume $p > 1$, h be a Borel l.s.c. (not necessarily convex) function satisfying $\alpha|\xi|^p \leq h(\xi)$, $\alpha > 0$. Then*

$$\min_{u \in W_0^{2,p}(\Omega)} \int_\Omega h(\Delta u(x))dx = \min_{u \in W_0^{2,p}(\Omega)} \int_\Omega h^{**}(\Delta u(x))dx = meas(\Omega)h^{**}(0). \tag{3.7}$$

Moreover, the minimum on the left-hand side of (3.7) *is achieved in a non-negative $W_0^{2,q}(\Omega)$-function for all $q \in]1, +\infty[$. In particular, such a minimum belongs to $C^{1,r}(\bar{\Omega})$ for all $r \in]0,1[$ by a Theorem of Morrey (see [Bré]).*

Proof. Indeed, clearly for every $u \in W_0^{2,p}(\Omega)$, one has

$$\int_\Omega h^{**}(\Delta u(x))dx \geq meas(\Omega)h^{**}(0).$$

This gives the equality on the right-hand side of (3.7) and both equalities if $h^{**}(0) = h(0)$. In case $h^{**}(0) < h(0)$, by the growth assumption on h, we have that $0 = \lambda\xi_1 + (1-\lambda)\xi_2$, $h^{**}(0) = \lambda h(\xi_1) + (1-\lambda)h(\xi_2)$, for some $\lambda \in [0,1]$, $\xi_1 < 0 < \xi_2$ such that $h^{**}(\xi_i) = h(\xi_i)$ $i = 1,2$. On the other hand, setting $B_1 = B(0,1)$: the unit open ball in $\mathbf{R}^n$, consider the family of closed sets of the form $a + \varepsilon\bar{B}_1$ contained in Ω, where $a \in \mathbf{R}^n$, $\varepsilon > 0$. This family covers Ω in the sense of Vitali, and hence by the Vitali covering Theorem, there exists a finite or countable disjoint sequence $a_i + \varepsilon_i\bar{B}_1$ of subsets of Ω such that $meas(\Omega \setminus \cup_i(a_i + \varepsilon_i\bar{B}_1)) = 0$, so that $meas(\Omega) = \sum_i \varepsilon_i^n \omega_n$. Let $u_0 \in W_0^{2,p}(B_1)$ be the radial function as that considered in [C-F, F3] with obvious modifications, that is

$$\Delta u_0(x) = \begin{cases} \xi_1 & \text{if } x \in B_\lambda \\ \xi_2 & \text{if } x \in B_1 \setminus B_\lambda, \end{cases}$$

where $B_\lambda = B(0,\lambda)$ and $meas(B_\lambda) = \lambda meas(B_1)$. Obviously $u_0 \in W_0^{2,q}(B_1)$ for all $q \in]1, +\infty[$ and $u_0 > 0$ in B_1. Let us define

$$\tilde{u}(x) = \begin{cases} \varepsilon_i^2 u_0(\dfrac{x-a_i}{\varepsilon_i}) & \text{if } \ x \in a_i + \varepsilon_i \bar{B}_1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\tilde{u} \in W_0^{2,q}(\Omega)$ for all $q \in]1, +\infty[$ and

$$\begin{aligned} \int_\Omega h(\Delta \tilde{u}(x))dx &= \sum_i \int_{a_i+\varepsilon_i \bar{B}_1} h(\Delta u_0(\frac{x-a_i}{\varepsilon_i}))dx = \sum_i \varepsilon_i^n \int_{B_1} h(\Delta u_0(z))dz \\ &= \sum_i \varepsilon_i^n \Big(\int_{B_\lambda} h(\xi_1)dz + \int_{B_1 \setminus B_\lambda} h(\xi_2)dz \Big) = \sum_i \varepsilon_i^n h^{**}(0)\omega_n \\ &= meas(\Omega) h^{**}(0) = \min_{u \in W_0^{2,p}(\Omega)} \int_\Omega h^{**}(\Delta u(x))dx, \end{aligned}$$

which completes the proof of the theorem since $h^{**} \leq h$, the function $\tilde{u}$ is a minimum for the left-hand side of (3.7) and satisfies our requirements. ■

Problems like (3.7), in a more general context, have been dealt in [A-T, Ra, R4] and recently in [C-F, F1, F3] for functionals with radial symmetry.

CHAPTER 4

The Lack of Lower Semicontinuity in the Calculus of Variations

As we have mentioned in the introduction of this thesis, in the study of minimization problems in the Calculus of Variations, the lower semicontinuity (l.s.c.) property, or a weak version of it, plays a prominent role in the Direct Method. Recently, non-lower semicontinuous functions or non-convex integrals (for the so-called scalar problems) are being active areas of investigation (see [C1, C2, M2, Ra, R3, T2] and references therein). We are not aware of any general theory in this direction. However, some existence results of minima for non convex integrals have been given in particular cases, namely, when the integral can be split into the sum of two integrals; one that depends on the derivative of higher order (e.g. gradient, Laplacian), and the other depending on the state function. Thus, people actually, consider sequentially weakly continuous (s.w.c.) perturbations of integrals that are not sequentially weakly lower semicontinuous (s.w.l.s.c.), i.e. the non convex part. Then, under ad hoc assumptions on the continuous term, one gets the desired. Purpose of this chapter is to contribute for a better understanding on the phenomenon that arises when the l.s.c. or the convexity property fails. In the next section (Thm. 4.2), given a non-l.s.c. function, we show the existence of non-negative continuous functions (perturbations of the first), such that the perturbed function does not attain its minimum. Moreover, such a perturbation can be chosen as small as one wants under some additional assumptions (Thm. 4.4 and Remark 4.6). The argument used here, is similar to that in [M2], where the author establishes a relationship between the non existence of minima for non-convex integrals and the uniqueness for the convexified (relaxed) integral, which is not strictly convex.

In Section 2, we deal with the simplest integrals depending either of the gradient or of the Laplacian. For either case, we are able to construct continuous perturbations in an integral form (Thms. 4.11 and 4.13). However, we have to point out the difficulty presented here since we do not allow the boundary datum to vary as the one which occurs in [B-Mu, Theorem 5.1].

4.1. The Abstract Setting

In this section X is a metric space. For $r > 0$, $B(u,r)$ denotes the open ball in X centered at u with radius r, $\bar{B}(u,r)$ the closed ball.

Proposition 4.1. *Let $\Phi : X \to [0,+\infty]$ be a lower semicontinuous function, let u_0 be in X such that $\Phi(u_0) < +\infty$. Then there exists a non-negative continuous function Ψ defined in X such that $\Psi(u) \leq \Phi(u)$ for $u \in X$ and $\Psi(u_0) = \Phi(u_0) \geq \Psi(u)$.*

Proof. Set for every $n \in \mathbf{N}$ $a_n \doteq \inf \left\{ \Phi(u) : u \in B(u_0, \frac{1}{n}) \right\}$. Clearly $a_n \geq 0$ and (a_n) is a non decreasing sequence bounded from above by $\Phi(u_0)$. The lower semicontinuity of Φ implies that $a \doteq \Phi(u_0) = \sup_n a_n = \lim_n a_n$. Let us consider for every $n \in \mathbf{N}$ a continuous function φ_n defined in X satisfying $0 \leq \varphi_n(u) \leq 1$ for u in X, $\varphi_n \equiv 1$ in $\bar{B}(u_0, \frac{1}{n+1})$ and $\varphi_n \equiv 0$ in $X \setminus B(u_0, \frac{1}{n})$. Define the function Ψ as follows

$$\Psi(u) = \begin{cases} \displaystyle\sum_{n=1}^{+\infty} (a_{n+1} - a_n)\varphi_{n+1}(u) + a_1, & \text{if } u \in B(u_0, 1/2) \\ a_1\varphi_1(u) & \text{otherwise.} \end{cases}$$

This function will satisfy our requirements. First of all, notice that Ψ is well defined: it becomes a finite sum for every u in $B(u_0, \frac{1}{2})$, $\Psi(u) = 0$ in $X \setminus B(u_0, 1)$, $\Psi(u_0) = a = \Phi(u_0)$ since $\varphi_{n+1}(u_0) = 1$ for all n, and $0 \leq \Psi(u) \leq a = \Psi(u_0) \quad \forall u \in X$.

(a) We shall prove $\Psi(u) \leq \Phi(u) \quad \forall u \in X$. If $\tilde{u} \notin B(u_0, \frac{1}{2})$ then either $\Psi(\tilde{u}) = 0$ or $\Psi(\tilde{u}) \leq a_1$, therefore $\Psi(\tilde{u}) \leq \Phi(\tilde{u})$ whenever $\tilde{u} \notin B(u_0, 1)$. If $\tilde{u} \in B(u_0, \frac{1}{n_0}) \setminus \bar{B}(u_0, \frac{1}{n_0+1})$ for some $n_0 \in \mathbf{N}$ $n_0 \geq 2$, then $\varphi_{n+1}(\tilde{u}) = 0 \ \forall n \geq n_0$. So that, $\Psi(\tilde{u}) = \sum_{n=1}^{n_0-1} (a_{n+1} - a_n)\varphi_{n+1}(\tilde{u}) + a_1 \leq a_{n_0} \leq \Phi(u)$ for all $u \in B(u_0, \frac{1}{n_0})$, in particular we obtain $\Psi(\tilde{u}) \leq a_{n_0} \leq \Phi(\tilde{u})$.

(b) Let us now prove the continuity of Ψ. The continuity is obviously verified at $\tilde{u}$ if $0 < d(\tilde{u}, u_0) < \frac{1}{2}$ or $d(\tilde{u}, u_0) > \frac{1}{2}$. On the other hand, it is not difficult to prove it at $\tilde{u}$ with $d(\tilde{u}, u_0) = \frac{1}{2}$. We only check it at u_0. Let $\varepsilon > 0$, then by the convergence of (a_n) there is $\bar{n} \in \mathbf{N}$ such that $0 < a - a_n < \varepsilon \quad \forall n \geq \bar{n}$. The definition of continuity at u_0 is verified by taking $\delta \doteq \frac{1}{\bar{n}+1}$. Indeed, if u is such that $d(u, u_0) < \delta$, then $\varphi_n(u) = 1$ for $n = 1, ..., \bar{n}$. Hence,

$$
\begin{aligned}
\left|\Psi(u)-\Psi(u_0)\right| &= \left|\sum_n (a_{n+1}-a_n)\varphi_{n+1}(u)+a_1-[\sum_n (a_{n+1}-a_n)+a_1]\right| \\
&= \sum_n (a_{n+1}-a_n)(1-\varphi_{n+1}(u)) = \sum_{n\geq \bar n} (a_{n+1}-a_n)(1-\varphi_{n+1}(u)) \\
&\leq \sum_{n\geq \bar n} (a_{n+1}-a_n) = a-a_{\bar n} < \varepsilon,
\end{aligned}
$$

which proves the continuity at u_0, and the proof of the Theorem is complete. ■

We denote by $sc^- F$ the lower semicontinuous envelope (or relaxed function) of F, which is the greatest lower semicontinuous function majorized by F (see Chapter 1 of this thesis or [Bu2], [DM]).

Proposition 4.2. *Assume the function $F : X \to [0,+\infty]$ is not lower semicontinuous at some point u_0 such that $F(u_0) < +\infty$. Then, there exists a continuous function $G : X \to \mathbf{R}_+$ such that the minimization problem:*

$$\min_{u\in X}(F(u)+G(u)) \tag{P}$$

has no solution.

Proof. Since F is not lower semicontinuous at u_0, we have $F(u_0) > sc^- F(u_0) \geq 0$. Applying Proposition 4.1 to $sc^- F$, we obtain a continuous function Ψ such that $0 \leq \Psi(u) \leq sc^- F(u)$ in X and $\Psi(u_0) = sc^- F(u_0) \geq \Psi(u)$ for all $u \in X$. Let us define the function G by $G(u) = -\Psi(u) + sc^- F(u_0) + d(u,u_0)$. Clearly G is continuous, $G(u) \geq d(u,u_0) \geq 0$ and $G(u_0) = 0$. On the other hand, for any u in X one has

$$sc^- F(u)+G(u) = sc^- F(u)-\Psi(u)+sc^- F(u_0)+d(u,u_0) \geq sc^- F(u_0) = sc^- F(u_0)+G(u_0).$$

This implies that u_0 is a solution to the problem

$$\min_{u\in X}(sc^- F(u)+G(u)). \tag{$\bar P$}$$

Actually, one can show that u_0 is the unique solution. Let us prove that for the function G as above, the conclusion of the proposition holds. Suppose the contrary: let u be a solution to problem (P), then a well-known result (for instance [DM, Theorem 3.8]), asserts that $\min(F(u) + G(u)) = \min sc^{-}(F + G)(u) = \min(sc^{-}F(u) + G(u))$. Therefore, u is also a solution to problem $(\bar{P})$. Hence $u \equiv u_0$, thus $F(u_0) = sc^{-}F(u_0)$ reaching a contradiction. ■

Remark 4.3. The previous proposition has an equivalent formulation:
Let $F : X \to [0, +\infty]$ be a function such that problem (P) has solution for every continuous function $G : X \to \mathbf{R}$. Then F is lower semicontinuous.

Theorem 4.4. *Let $F : X \to [0, +\infty]$ be non lower semicontinuous at some point u_0 such that $F(u_0) < +\infty$. Then , given any $\eta > sc^{-}F(u_0) - \inf F \geq 0$, there exists a continuous function G_η such that $0 \leq G_\eta \leq \eta$ in X and problem (P_η), i.e. problem (P) where G is replaced by G_η, admits no solution.*

Proof. The function G_η defined by $G_\eta = G \wedge \eta$ with G given by Proposition 4.2, satisfy the requirements of the theorem. In fact, if on the contrary there is a function $\bar{u}$ solution to problem (P_η), then $\bar{u}$ also is a solution to problem $(\bar{P}_\eta)$, i.e. problem $(\bar{P})$ with $G = G_\eta$. Therefore since $0 \leq G_\eta \leq G$ and $G(u_0) = 0$ we obtain $sc^{-}F(\bar{u}) + G_\eta(\bar{u}) \leq sc^{-}F(u_0) + G(u_0) = sc^{-}F(u_0) < \eta + \inf F = \eta + \inf sc^{-}F \leq \eta + sc^{-}F(\bar{u})$. Then $G_\eta(\bar{u}) < \eta$, i.e. $G_\eta(\bar{u}) = G(\bar{u})$, by recalling that u_0 is the unique solution to problem $(\bar{P})$, we conclude that $\bar{u} \equiv u_0$. Consequently $sc^{-}F(u_0) = F(u_0)$, which is a contradiction because u_0 is a point where F is not lower semicontinuous. ■

Remark 4.5. That problem $(\bar{P})$ admits the unique solution u_0: the point just where F is not l.s.c., was the main fact used in the proof of Proposition 4.2 (a similar argument has been used in [M2] in a very particular situation, namely for functions of integral type). It turns out that problem $(\bar{P}_\eta)$ has at most one solution: u_0. This property fails if $0 < \eta < sc^{-}F(u_0) - \inf F$, by assuming u_0 is not a minimum for $sc^{-}F$, otherwise see Remark 4.6 below. In fact, suppose there is a function G_η such that $0 \leq G_\eta \leq \eta$ in X. Then, $\inf(sc^{-}F + G_\eta) \leq \inf sc^{-}F + \eta = \inf F + \eta < sc^{-}F(u_0)$. Thus, there exists $\bar{u}$ in X such that $sc^{-}F(\bar{u}) + G_\eta(\bar{u}) < sc^{-}F(u_0) \leq sc^{-}F(u_0) + G_\eta(u_0)$, i.e. u_0 is not a solution to problem $(\bar{P}_\eta)$.

Remark 4.6.- Assume in the previous theorem, that u_0 is a minimum for $sc^{-}F$ in X. Then, the set

$$\mathcal{G} = \left\{ G \in C^0(X, [0, +\infty[) : G(u_0) = 0 \text{ problem } (P) \text{ admits no solution} \right\}$$

is dense in the set of those functions in $C^0(X, [0, +\infty[)$ vanishing at u_0. Indeed, let $G \geq 0$, $G(u_0) = 0$ be in $C^0(X, [0, +\infty[)$ and let $\eta > 0$. Then $F + G$ is not a lower semicontinuous function at u_0. Applying Theorem 4.4 to the function $F + G$, we get a continuous function G_0, $G_0(u_0) = 0$ satisfying $0 \leq G_0 \leq \eta$ in X such that the minimization problem

$$\min_{u \in X} (F(u) + G(u) + G_0(u))$$

admits no solution, thus $G + G_0 \in \mathcal{G}$ and $0 \leq (G(u) + G_0(u)) - G(u) \leq \eta$. This proves our assertion.

4.2. Some Concrete Cases

In this Section we are concerned with the existence of functions g such that, for a given non convex function h, the following problem

$$\min_{u \in X} \int_\Omega h(\Lambda u(x))dx + \int_\Omega g(x, u(x))dx, \tag{P}$$

admits no solution, where X is, depending if Λ is either the gradient or the Laplacian operator, $W_0^{1,p}(\Omega)$ or $W_0^{2,p}(\Omega)$.

We start by studying the case $h(\Delta u) = (1 - |\Delta u|)^2$, $g(x, u) = |u|^2$. It is well-known that the corresponding problem (P) has no solution. One may prove it in the following way: suppose there is a function u_0, solution to (P), being $\min P^{**} = \min P$ (by (3.6) of previous chapter), u_0 also is a solution to (P^{**}). But, $u \equiv 0$ is the only solution to (P^{**}), thus $\int_\Omega h(0) = \int_\Omega h^{**}(0)$, which gives a contradiction. The main fact used here is the uniqueness of solutions for the problem (P^{**}). A similar argument has already been employed in [M2], although in a different context. The previous example offers no difficulty because 0 besides being in H_0^2, $\int h^{**}(0) < \int h(0)$ and $0 \in \partial h^{**}(0)$. If we consider a general non-convex function h (eventually $h^{**}(0) = h(0)$): are we able to construct a function $u_0 \in W_0^{2,p}$ such that $\int_\Omega h^{**}(\Delta u_0) < \int_\Omega h(\Delta u_0)$ and the set valued map $x \mapsto \partial h^{**}(\Delta u_0(x))$ admits a $W^{2,p}$-selection?. Suppose the answer be affirmative, then, for some $\rho \in W^{2,p}$ with $\rho(x) \in \partial h^{**}(\Delta u_0(x))$, by definition of subgradient and integrating, we have

$$\int_\Omega h^{**}(\Delta u(x))dx - \int_\Omega \Delta\rho(x)(u(x) - u_0(x))dx \geq \int_\Omega h^{**}(\Delta u_0(x))dx, \tag{4.1}$$

i.e. u_0 is the only solution for the problem (P^{**}) with

$$g(x,u) = -\Delta\rho(x)(u - u_0(x)) + |u - u_0(x)|^p. \tag{4.2}$$

Notice that, if h is, as in the example, g reduces $|u|^2$ for $u_0 \equiv 0$, $p = 2$. Of course, in order to get $\int h^{**}(\Delta u_0) < \int h(\Delta u_0)$, Δu_0 must belong, in a subset of Ω having positive measure, to the region where $h^{**} < h$.

We cannot expect that g be simpler than (4.2) for any given $u_0 \in W_0^{2,p}$ satisfying only $\int h^{**}(\Delta u_0) < \int h(\Delta u_0)$ for which u_0 continues to be a solution to the corresponding problem (P^{**}), as the following proposition and the discussion presently show.

Proposition 4.7. *Let $v \in W_0^{2,p}$ be given, setting $g(x,u) = |u - v(x)|^p$. Then v is not a solution to (P^{**}) iff*

$$\sup_{u \in W_0^{2,p}\setminus\{v\}} \frac{\int h^{**}(\Delta v) - \int h^{**}(\Delta u)}{\int |u - v|^p} = +\infty.$$

The proof is direct.

Without loss of generality we may assume $0 \in \Omega$. Thus, $B_\varepsilon = B(0,\varepsilon) \subset\subset \Omega$ for some $\varepsilon > 0$, let $t_0 \geq 0$ such that $h^{**}(t_0) < h(t_0)$ and let us consider $u_0 \in W_0^{2,p}(B_\varepsilon)$ with $\Delta u_0 \in \{-t_0, t_0\}$ (as in the introduction of Chapter 2 with obvious changes), we extend u_0 by zero outside B_ε. In the same way, we consider $u_n \in W_0^{2,p}(B_\varepsilon)$ with $\Delta u_n \in \{-t_0 - \frac{1}{n}, t_0 + \frac{1}{n}\}$, and extend it by zero outside B_ε. Then,

$$\frac{\int_\Omega h^{**}(\Delta u_0) - \int_\Omega h^{**}(\Delta u_n)}{||u_n - u_0||^p} \geq Cn^{p-1}\left[\frac{h^{**}(-t_0) - h^{**}(-t_0 - \frac{1}{n})}{\frac{1}{n}} + \frac{h^{**}(-t_0) - h^{**}(t_0 + \frac{1}{n})}{\frac{1}{n}}\right], \tag{4.3}$$

where we have used $||u||_{W^{2,p}} \leq C||\Delta u||_{L^p}$ for some constant $C > 0$ if $u \in W^{2,p} \cap W_0^{1,p}$ (for instance [G-T]). Letting $n \to +\infty$ in (4.3), the expression in brackets goes to $h_-^{**\prime}(-t_0) - h^{**\prime}(t_0) \in \mathbf{R}$. As a consequence, the right-hand side tends to $+\infty$. By the previous proposition, with $g(x,u) = |u - u_0(x)|^p$, u_0 is not a solution to problem (P^{**}), proving our assertion.

In this paragraph, as in those which follow, $|\cdot|$, $\langle\cdot,\cdot\rangle$ stand for the Euclidean norm and the inner product in $\mathbf{R}^n$, respectively. For a given function f, we recall that $\partial f(\xi)$ is the subdifferential at ξ: it is the set of subgradients of f at ξ (see [E-T, Chapter I]). To prove our main theorems, we shall need the following auxiliary lemmas.

Lemma 4.8. *Let $\phi : \mathbf{R} \to \mathbf{R}$ be a convex function such that $\phi(-t) = \phi(t)$ for $t \in \mathbf{R}$. Then*
(a) $\phi(t) \geq \phi(0)$ $\forall t \in \mathbf{R}$, and $\phi(t_1) \leq \phi(t_2)$ whenever $0 \leq t_1 \leq t_2$.
(b) $\phi'_-(0) \leq 0 \leq \phi'_+(0)$, i.e. $0 \in \partial\phi(0)$.
(c) $\phi'_+(t_1) \leq \phi'_-(t_2) \leq \phi'_+(t_2)$ whenever $t_1 < t_2$. Therefore, if $\phi'(t_0) = 0$ for some $t_0 > 0$, then $\phi(t) = \phi(0)$ for $t \in [-t_0, t_0]$.

Proof. Since ϕ is an even and convex function, we have $\phi(0) \leq \frac{1}{2}\phi(t) + \frac{1}{2}\phi(-t) = \phi(t)$. The second part of (a) follows from the assertion above, since one has $\phi(t_1) \leq \alpha\phi(0) + (1-\alpha)\phi(t_2)$, where we have written $t_1 = \alpha 0 + (1-\alpha)t_2$. It is well-known that the right and left-hand derivatives of any convex function exist everywhere. For $t > 0$ $(t < 0)$, using (a) we obtain

$$0 \leq \frac{\phi(t) - \phi(0)}{t} \quad (\geq),$$

letting $t \to +0$ $(t \to -0)$, part (b) is proved. Part (c) is left to the reader (or see Chapter I of [P-S]). ∎

Lemma 4.9. *Let ϕ as in Lemma 4.8 and let $f : \mathbf{R}^n \to \mathbf{R}$ be the function defined by $f(x) = \phi(|x|)$. Then (a) $\partial f(\xi) \subset \left\{\xi^* \in \mathbf{R}^n : |\xi^*| \in \partial\phi(|\xi|)\right\}$. (b) If $\langle\xi^*, \xi\rangle = |\xi||\xi^*|$ and $|\xi^*| \in \partial\phi(|\xi|)$ we have $\xi^* \in \partial f(\xi)$.*

Proof. (a) Let ξ^* be in $\partial f(\xi)$. Then $f(\eta) \geq f(\xi) + \langle\xi^*, \eta - \xi\rangle$ $\forall \eta \in \mathbf{R}^n$. Thus, $\phi(|t|) \geq \phi(|\xi|) + \langle\xi^*, \frac{t\eta}{|\eta|}\rangle - \langle\xi^*, \xi\rangle$ $\forall \eta \in \mathbf{R}^n \setminus \{0\}, \forall t \in \mathbf{R}$. Recalling that $|\xi^*| = \sup\left\{\langle\xi^*, \frac{x}{|x|}\rangle : x \in \mathbf{R}^n \setminus \{0\}\right\}$, the last inequality implies $\phi(t) \geq \phi(|\xi|) + |\xi^*|(|t| - |\xi|)$ $\forall t \in \mathbf{R}$. Therefore, $\phi'_-(|\xi|) \leq |\xi^*| \leq \phi'_+(|\xi|)$, which proves

$$\partial f(\xi) \subset \left\{\xi^* \in \mathbf{R}^n : |\xi^*| \in \partial\phi(|\xi|)\right\},$$

since $\partial\phi(r) = [\phi'_-(r), \phi'_+(r)]$.
(b) Let ξ, ξ^* be in $\mathbf{R}^n$ such that $\langle\xi^*, \xi\rangle = |\xi||\xi^*|$ and $|\xi^*| \in \partial\phi(|\xi|)$. We have

$$\phi(s) \geq \phi(|\xi|) + |\xi^*|(s - |\xi|) \;\; \forall s \in \mathbf{R}. \tag{4.4}$$

But $|\xi^*| = \sup\left\{\langle \xi^*, \frac{x}{|x|}\rangle : x \in \mathbf{R}^n \setminus \{0\}\right\}$, then the last inequality, because of $\langle \xi^*, \xi\rangle = |\xi||\xi^*|$, can be written as $\phi(s) \geq \phi(|\xi|) + \langle \xi^*, \frac{x}{|x|}\rangle s - \langle \xi^*, \xi\rangle \;\forall s \geq 0 \;\forall x \in \mathbf{R}^n \setminus \{0\}$. Thus, by setting $s = |x| > 0$, we obtain

$$\phi(|x|) \geq \phi(|\xi|) + \langle \xi^*, x - \xi\rangle \;\; \forall x \in \mathbf{R}^n \setminus \{0\}. \tag{4.5}$$

In virtue of (4.4), (4.5) holds for all $x \in \mathbf{R}^n$, so that, $\xi^* \in \partial f(\xi)$. ■

Lemma 4.10. *Let $\sigma : \mathbf{R} \to \mathbf{R}$ be a measurable, locally bounded and odd function. Then, given $\varepsilon > 0$ and $C > 0$, there exists a function φ in $C^\infty([0,\varepsilon])$ such that*

$$-C \leq \varphi(t) \leq C \;\; \textit{if } t \in [0,\varepsilon]$$

$$\varphi(t) = 0 \;\; \textit{if } 0 \leq t \leq \delta_1, \; \delta_6 \leq t \leq \varepsilon$$

$$\varphi(t) = C \; \textit{if } \; \delta_2 \leq t \leq \delta_3 \; \textit{and} \; \varphi(t) = -C \; \textit{if } \; \delta_4 \leq t \leq \delta_5$$

for some $0 < \delta_1 < \delta_2 < \delta_3 < \delta_4 < \delta_5 < \delta_6 < \varepsilon$, and

$$\int_0^\varepsilon \sigma(\varphi(r)) r^{n-1}\, dr = 0. \tag{4.6}$$

Proof. Let ψ_1 be an odd and C^∞-function in $[-\frac{\varepsilon^n}{2}, \frac{\varepsilon^n}{2}]$ satisfying the following properties (see figure 1)

$$-C \leq \psi_1(t) \leq C \;\; \text{if } t \in [-\frac{\varepsilon^n}{2}, \frac{\varepsilon^n}{2}]$$

$$\psi_1(t) = 0 \;\; \text{if } \; -\frac{\varepsilon^n}{2} \leq t \leq -\bar\delta_1, \; \bar\delta_3 \leq t \leq \frac{\varepsilon^n}{2}$$

$$\psi_1(t) = C \; \text{if } \; -\bar\delta_2 \leq t \leq -\bar\delta_1 \; \text{and} \; \psi_1(t) = -C \; \text{if } \; \bar\delta_1 \leq t \leq \bar\delta_2$$

for fixed $0 < \bar\delta_1 < \bar\delta_2 < \bar\delta_3 < \frac{\varepsilon^n}{2}$. Then, by putting $\psi_2(t) = \psi_1(t - \frac{\varepsilon^n}{2})$ we have

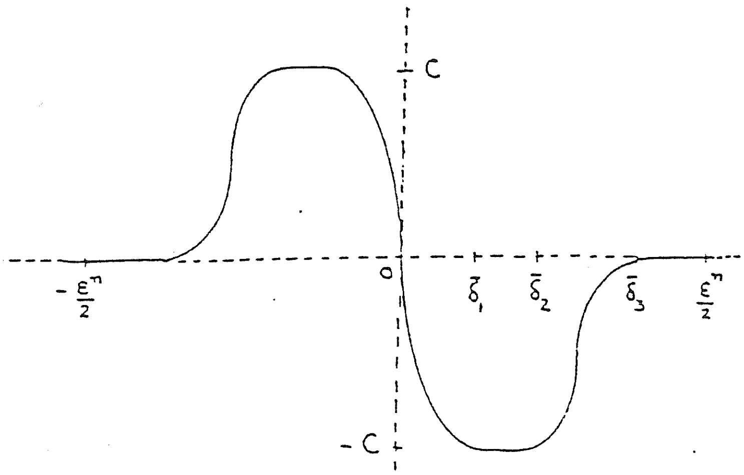
$$\int_0^{\varepsilon^n} \sigma(\psi_2(t))dt = \int_0^{\varepsilon^n} \sigma(\psi_1(t - \frac{\varepsilon^n}{2}))dt = \int_{-\frac{\varepsilon^n}{2}}^{\frac{\varepsilon^n}{2}} \sigma(\psi_1(t))dt = 0$$

since $\sigma \circ \psi_1$ is an odd function. Similarly, for the function $\varphi(r) = \psi_2(r^n)$, an easy computation shows, that

$$\int_0^{\varepsilon} \sigma(\varphi(r)) r^{n-1}\, dr = 0,$$

which proves our lemma with $\delta_i = \sqrt[n]{\frac{\varepsilon^n}{2} - \bar{\delta}_{4-i}}$, $\delta_{i+3} = \sqrt[n]{\frac{\varepsilon^n}{2} + \bar{\delta}_i}$ for $i = 1, 2, 3$. ■

figure 1



In the remainder of this Chapter, $1 < p < +\infty$, $h : \mathbf{R}^n \to \mathbf{R}$ is a continuous function such that

(h_1) $h(\xi) = \tilde{h}(|\xi|)$ for some even continuous function $\tilde{h}$ defined in $\mathbf{R}$;

(h_2) there are positive constants $\alpha_1 \leq \alpha_2$ such that $\alpha_1 |\xi|^p \leq h(\xi) \leq \alpha_2(|\xi|^p + 1)$.

In what follows, we denote by h^* and h^{**} the polar (or conjugate) and bipolar function of h, respectively. For theirs properties we refer to [E-T]. We just recall one: h^{**} is the largest convex function not larger than h. As a consequence of (h_2) h^{**} is also continuous. The set Ω will be bounded and open set of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$.

We notice that (h_1) implies the existence of an even and convex function ϕ such that $h^{**}(\xi) = \phi(|\xi|)$ and $h^*(\xi) = \phi^*(|\xi|)$ for all $\xi \in \mathbf{R}^n$.

The theorems in the preceding sections have been established in terms of the lower semi-continuity of a function, here by dealing with functions of integrals type, our theorems will be established by means of the convexity of the integrands.

We have the first main result of this section:

Theorem 4.11. [F4] *Assume the function h be as above. If h is not convex, in the sense that $h^{**}(\xi_0) \neq h(\xi_0)$ for some ξ_0, then, there exists a function $g : \Omega \times \mathbf{R} \to \mathbf{R}$, continuous in its second argument, such that the following problem*

$$\min_{u \in W_0^{1,p}(\Omega)} \int_\Omega h(\nabla u(x))dx + \int_\Omega g(x, u(x))dx, \qquad (P_1)$$

has no solution.

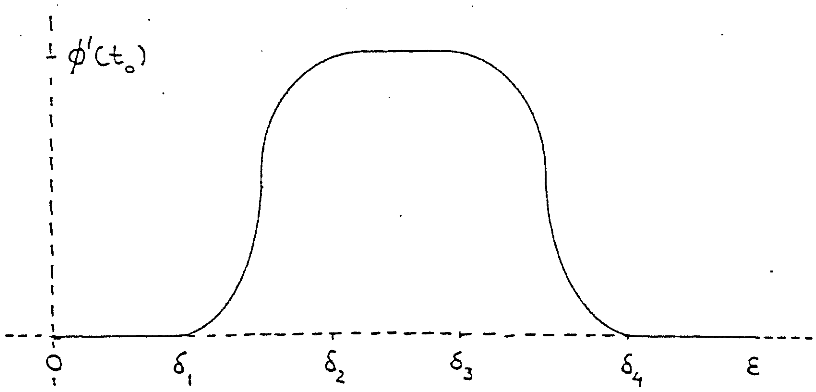
Proof. Set $t_0 = |\xi_0|$ and let us fix $x_0 \in \Omega$ and $\varepsilon > 0$ such that $B_\varepsilon \doteq B(x_0, \varepsilon) \subset\subset \Omega$. Let $0 < \delta_1 < \delta_2 < \delta_3 < \delta_4 < \varepsilon$ and let us consider a function φ in $C^\infty([0,\varepsilon])$ as in figure 2 satisfying

$$0 \leq \varphi(t) \leq \phi'(t_0) \text{ if } t \in [0, \varepsilon]$$

$$\varphi(t) = \phi'(t_0), \text{if } \delta_2 \leq t \leq \delta_3$$

$$\varphi(t) = 0, \text{ if } 0 \leq t \leq \delta_1, \delta_4 \leq t \leq \varepsilon.$$

figure 2



Since $\phi'(t_0) \in \partial\phi(t_0)$, that is, $t_0 \in (\partial\phi)^{-1}(\phi'(t_0))$, by recalling that $(\partial\phi)^{-1} \equiv \partial\phi^*$, $t_0 \in \partial\phi^*(\phi'(t_0))$, so that it is possible to choose a function $\sigma \in L^\infty_{loc}(\mathbf{R})$ selection from the map $s \mapsto \partial\phi^*(s)$ such that $\sigma(\phi'(t_0)) = t_0$, for the existence of such a measurable selection we refer to [Ku-RN], and by assumption (h_2) it is in L^∞_{loc}, after a modification (if necessary) we obtain the selection desired. Thus $\sigma(\varphi(t)) \in \partial\phi^*(\varphi(t))$ or, equivalently, $\varphi(t) \in \partial\phi(\sigma(\varphi(t)))$ for $t \in [0, \varepsilon]$. We now consider the function

$$u_0(x) = u_1(|x - x_0|) \text{ for } x \in B(x_0, \varepsilon)$$

where u_1 is the solution to the equation

$$u_1'(r) = \sigma(\varphi(r)) \quad r \in [0, \varepsilon]$$
$$u_1(\varepsilon) = 0.$$

From the definition of u_0 and σ, it follows that $u_0 \in W_0^{1,p}(B_\varepsilon)$ and $|\nabla u_0(x)| = |\sigma(\varphi(|x - x_0|))| = \sigma(\varphi(|x - x_0|))$ for $x \in B_\varepsilon$. We still denote by u_0 the function extended by zero outside B_ε. Clearly $u_0 \in W_0^{1,p}(\Omega)$, and by the choice of ξ_0 we have

$$\int_\Omega h^{**}(\nabla u_0(x))dx < \int_\Omega h(\nabla u_0(x))dx. \tag{4.7}$$

On the other hand, we know that $\varphi(|x-x_0|) \in \partial\phi(\sigma(\varphi(|x-x_0|))$ for all $x \in B_\varepsilon$. Then using Part (b) of Lemma 4.9 we conclude that $\varphi(|x - x_0|)\frac{x-x_0}{|x-x_0|} \in \partial h^{**}(\nabla u_0(x))$ a.e. $x \in B_\varepsilon$. Calling ρ_0 the C^∞-function such that $\rho_0'(r) = \varphi(r)$ $\rho_0(\varepsilon) = 0$, and if $\tilde{\rho}_0(x) = \rho_0(|x - x_0|)$ then $\tilde{\rho}_0 \in C_0^\infty(B_\varepsilon)$. We extend this function by zero outside $B(x_0, \varepsilon)$, to obtain a function, still denoted by $\tilde{\rho}_0$ defined in Ω. Taking into account that $0 \in \partial h^{**}(0)$ (see Lemma 4.8) and the properties of φ, we obtain $\nabla\tilde{\rho}_0(x) \in \partial h^{**}(\nabla u_0(x))$ a.e. $x \in \Omega$. Therefore,

$$h^{**}(\nabla u(x)) \geq h^{**}(\nabla u_0(x)) + \langle \nabla\tilde{\rho}_0(x), \nabla u(x) - \nabla u_0(x)\rangle \quad \text{for a.e. } x \in \Omega \tag{4.8}$$

for any function $u \in W_0^{1,p}(\Omega)$. On integrating (4.8) over Ω, and by using Green's Formula, we obtain

$$\int_\Omega h^{**}(\nabla u(x)) \geq \int_\Omega h^{**}(\nabla u_0(x))dx - \int_\Omega \Delta\tilde{\rho}_0(x)(u(x) - u_0(x))dx$$

for every function $u \in W_0^{1,p}(\Omega)$. Setting $\rho_1(x) = \Delta\tilde{\rho}_0(x)$, the last inequality implies that u_0 is a solution to the relaxed problem

$$\min_{u \in W_0^{1,p}(\Omega)} \int_\Omega h^{**}(\nabla u(x))dx + \int_\Omega g(x, u(x))dx, \tag{P_1^{**}}$$

where g is defined by

$$g(x, u) = \rho_1(x)(u - u_0(x)) + |u - u_0(x)|^p, \tag{4.9}$$

which is continuous and strictly convex in u, so that u_0 is the unique solution to problem (P_1^{**}). Let us prove that problem (P_1), with g as above, does not admit any solution.

Suppose the contrary, i.e. let u be a solution to problem (P_1), since $\min P_1^{**} = \inf P_1$ (see [E-T, Th. 3.7 of Chapter X]), u is also a solution to problem (P_1^{**}). Therefore, $u \equiv u_0$ because of the uniqueness of solutions to problem (P_1^{**}). It follows that $\int_\Omega h^{**}(\nabla u_0(x))dx = \int_\Omega h(\nabla u_0(x))dx$ reaching a contradiction with (4.7). ∎

Remark 4.12. In Thm. 5.1 of [B-Mu] a similar result was established, but here we had some difficulties to construct continuous perturbations of integral type since we do not allow the boundary datum to vary as the one which occurs in [B-Mu], in other words, the space where we seek the minimum is fixed from the beginning.

Theorem 4.13. [F4] *Let $h : \mathbf{R} \to \mathbf{R}$ satisfies hypothesis (h_1) and (h_2). If $h^{**}(t_0) \neq h(t_0)$ for some t_0, then there exists a function $g : \Omega \times \mathbf{R} \to \mathbf{R}$, continuous in its second argument such that the minimization problem*

$$\min_{u \in W_0^{2,p}(\Omega)} \int_\Omega h(\Delta u(x))dx + \int_\Omega g(x, u(x))dx, \tag{P_2}$$

has no solution.

Proof. We assume $t_0 \geq 0$, otherwise we consider $-t_0$. We proceed as in the proof of Theorem 4.11. First, fix $x_0 \in \Omega$ and $\varepsilon > 0$ such that $B_\varepsilon \doteq B(x_0, \varepsilon) \subset\subset \Omega$.
(a) Let us assume $h^{**\prime}(t_0) > 0$. Since $t_0 \in (\partial h^{**})^{-1}(h^{**\prime}(t_0)) = \partial h^*(h^{**\prime}(t_0))$, and $0 \in (\partial h^{**})^{-1}(0)$, we can take any measurable, locally bounded selection σ from the map $s \mapsto \partial h^*(s) = (\partial h^{**})^{-1}(s)$ verifying $\sigma(h^{**\prime}(t_0)) = t_0$ and $\sigma(0) = 0$. After a modification (if necessary) we can assume, because of the properties of h^{**}, that σ is an odd function. Consider the function φ given by Lemma 4.10, with $C = h^{**\prime}(t_0)$, and let u_0 be the solution to the Dirichlet problem

$$\begin{aligned} \Delta u_0 &= \sigma(\varphi(|x - x_0|)) \\ u_0 &= 0 \ \text{ on } \partial B_\varepsilon. \end{aligned} \tag{4.10}$$

Then $u_0 \in W^{2,p}(B_\varepsilon) \cap W_0^{1,p}(B_\varepsilon)$ and in virtue of (4.6), we obtain $\frac{\partial u_0}{\partial n} = 0$ on ∂B_ε, thus $u_0 \in W_0^{2,p}(B_\varepsilon)$. By the defintion of σ, we have $\varphi(r) \in \partial h^{**}(\sigma(\varphi(r)))$ $r \in [0, \varepsilon]$, or equivalently $\varphi(|x - x_0|) \in \partial h^{**}(\sigma(\varphi(|x - x_0|)))$ $x \in B_\varepsilon$, i.e. $\varphi(|x - x_0|) \in \partial h^{**}(\Delta u_0(x))$ $x \in B_\varepsilon$. Setting $\tilde{\varphi}(x) = \varphi(|x - x_0|)$ for $x \in B_\varepsilon$ and $\tilde{\varphi}(x) = 0$ for $x \in \Omega \setminus B_\varepsilon$, we have $\tilde{\varphi} \in C_0^\infty(\Omega)$. We also extend u_0 by zero outside B_ε, then $u_0 \in W_0^{2,p}(\Omega)$ and by the choice of t_0 we have

$$\int_\Omega h^{**}(\Delta u_0(x))dx < \int_\Omega h(\Delta u_0(x))dx. \tag{4.11}$$

Recalling the properties of h^{**} and σ we conclude that $\tilde{\varphi}(x) \in \partial h^{**}(\Delta u_0(x))$ a.e. $x \in \Omega$. Hence, for every $u \in W_0^{2,p}(\Omega)$:

$$h^{**}(\Delta u(x)) \geq h^{**}(\Delta u_0(x)) + \tilde{\varphi}(x)(\Delta u(x) - \Delta u_0(x)) \text{ for a.e. } x \in \Omega$$

On integrating the last inequality over Ω and using Green's Formula twice, we obtain

$$\int_\Omega h^{**}(\Delta u(x)) \geq \int_\Omega h^{**}(\Delta u_0(x))dx + \int_\Omega \Delta\tilde{\varphi}(x)(u(x) - u_0(x))dx$$

for every function $u \in W_0^{2,p}(\Omega)$. This shows that u_0 is a solution to the problem

$$\min_{u \in W_0^{2,p}(\Omega)} \int_\Omega h^{**}(\Delta u(x))dx + \int_\Omega g(x, u(x))dx, \tag{P_2^{**}}$$

where g is defined by

$$g(x, u) = \rho_1(x)(u - u_0(x)) + |u - u_0(x)|^p \tag{4.12}$$

and $\rho_1(x) = -\Delta\tilde{\varphi}(x)$. Obviously g is a continuous and strictly convex function in u. Therefore u_0 is the unique solution to problem (P_2^{**}). It remains to prove that problem (P_2) has no solution. If it is not so, any function u, solution to problem (P_2), is also a solution to problem (P_2^{**}) (see (3.6) of Chapter 3). Then $u \equiv u_0$, thus we obtain $\int_\Omega h^{**}(\Delta u_0(x))dx = \int_\Omega h(\Delta u_0(x))dx$, which is a contradiction with (4.11).
(b) If $h^{**\prime}(t_0) = 0$ we take any (for instance, radial) function $u_1 \in W_0^{2,p}(B(0, \varepsilon))$ whose Laplacian takes values in $\{-t_0, t_0\}$. Then, we define the function u_0 as follows: $u_0(x) = u_1(x - x_0)$ for $x \in B_\varepsilon = B(x_0, \varepsilon)$ and zero otherwise. Because of (a) and (c) of Lemma 4.8, we have for every function $u \in W_0^{2,p}(\Omega)$, $h^{**}(\Delta u(x)) \geq h^{**}(\Delta u_0(x)) = h^{**}(0)$ for a.e. $x \in \Omega$. This implies that problem (P_2^{**}) with $\rho_1 \equiv 0$ in (4.12), has the unique solution u_0. A similar reasoning as before allows us to conclude that the corresponding problem (P_2) has no solution. ■

Remark 4.14. The function g in the two previous theorems can be chosen non-negative. Indeed, it suffices to consider $g(x, u) = \tilde{g}(x, u) \vee 0$ with $\tilde{g}$ as above. On the other hand, Theorem 4.11 admits the following equivalent formulation:

If the minimization problem

$$\min_{u \in W_0^{1,p}(\Omega)} \int_\Omega h(\nabla u(x))dx + \int_\Omega g(x, u(x))dx$$

has solution for every function $g : \Omega \times \mathbf{R} \to \mathbf{R}$, *then* h *is convex.*

CHAPTER 5

Some Applications of the Liapunov theorem Revisited

Let $F : [a, b] \to 2^{\mathbf{R}^n}$ be a measurable multifunction with nonempty values. Its multi-valued integral is then defined as

$$\int_a^b F(x)dx \doteq \left\{ \int_a^b f(x)dx \ ; \qquad f \in L^1, \ \ f(x) \in F(x) \ a.e. \right\}. \tag{5.1}$$

The convexity of this integral, together with other properties, was proved in Aumann's classical paper [Au].

When F is defined on a multidimensional space $\mathbf{R}^n$, in addition to the Aumann integral $\int_{\mathbf{R}^n} F(x)dx$ over the entire space, one can consider the marginal distributions, obtained by integrating F along a family of parallel lines. More precisely, fix any unit vector $v \in \mathbf{R}^n$ and call $v^\perp$ its perpendicular hyperplane through the origin. For every $f \in L^1$, the line integrals

$$g(x') \doteq \int_{-\infty}^{+\infty} f(x' + \lambda v)d\lambda, \qquad\qquad x' \in v^\perp,$$

define an integrable map from $v^\perp$ into $\mathbf{R}^m$. Given any finite family of unit vectors $v_1, ..., v_k$ in $\mathbf{R}^n$, we define the *joint marginal integrals* of f as

$$\int_{(v_1,...,v_k)} f \ d\lambda \doteq (g_1, ..., g_k), \qquad g_i(x'_i) = \int_{-\infty}^{+\infty} f(x'_i + \lambda v_i)d\lambda \qquad \forall x'_i \in v_i^\perp. \qquad 5.2)$$

When $F : \mathbf{R}^n \to 2^{\mathbf{R}^m}$ is a multifunction, the set of joint marginal integrals of F in the directions of $v_1, \ldots, v_k$ is defined as

$$\int_{(v_1,...,v_k)} F \ d\lambda = \left\{ \int_{(v_1,...,v_k)} f \ d\lambda \ ; \quad f \in L^1, \quad f(x) \in F(x) \quad \text{a.e.} \right\}. \tag{5.3}$$

Since in (5.2) each function g_i is defined on the $n-1$ dimensional hyperplane $v_i^\perp$, making the obvious identifications one has

$$\int_{(v_1,...,v_k)} F \ d\lambda \ \subseteq \ \left[L^1(\mathbf{R}^{n-1};\ \mathbf{R}^m)\right]^k. \tag{5.4}$$

The aim of this chapter is to investigate the basic properties of the "multivariable Aumann integrals" (5.3), pointing out some possible applications, in connection with the control of the wave equation.

Our first theorems establish the closure and convexity of the set of joint marginal integrals, and some properties of their extreme points. In turn, they imply a new version of the classical theorem of Liapunov on the range of a vector measure, valid on a product space. A key technical tool used in most of our proofs is the multidimensional Liapunov-type theorem, recently established in [Br]:

Proposition 5.1. *Assume* $f_1, \ldots, f_\nu \in L^1(\mathbf{R}^n;\ \mathbf{R}^m)$, *let* $p_1, \ldots, p_\nu : \mathbf{R}^n \to [0,1]$ *be measurable weights with* $\sum p_i(x) \equiv 1$, *and let* $v_1, \ldots, v_k$ *be unit vectors in* $\mathbf{R}^n$. *Then there exists a measurable partition* $\{A_1, \ldots, A_\nu\}$ *of* $\mathbf{R}^n$ *such that, for every* $j = 1, \ldots, k$, *one has*

$$\int_{-\infty}^{+\infty} \sum_{i=1}^{\nu} p_i(x' + \lambda v_j) f_i(x' + \lambda v_j)\ d\lambda = \sum_{i=1}^{\nu} \int_{-\infty}^{+\infty} \chi_{A_i}(x' + \lambda v_j) f_i(x' + \lambda v_j)\ d\lambda \tag{5.5}$$

for almost every x' *in the perpendicular hyperplane* $v_j^\perp$ *to* v_j, *here, a.e. refers to the (n-1) dimensional Lebesgue measure on* $v_j^\perp$. *As a consequence, we have*

$$\int_{\mathbf{R}^n} \sum_{i=1}^{k} p_i(x) f_i(x) dx = \int_{\mathbf{R}^n} \sum_{i=1}^{k} \chi_{A_i}(x) f_i(x) dx.$$

For control systems governed by linear ordinary differential equations, without convexity assumptions, the closure of the reachable sets and the existence of optimal controls can be usually deduced as consequences of the closure and convexity of a classical Aumann integral. On the other hand, for certain control systems governed by partial differential equations of hyperbolic type, the validity of analogous results follows from [Br] and the multidimensional extension of Aumann's theorem proved in the present paper.

We shall not try to establish here the most general results in this direction. Instead, in order to illustrate the basic method, in the last sections we consider the simplest linear wave equation, with distributed control. In contrast with previous results in this direction [Pul, Sur2, Sur3], we allow constraints to be placed on the entire boundary of the domain. Exploiting the special structure of the Riemann-Green kernel for this particular system, we obtain a bang-bang theorem and an existence theorem for optimal controls without convexity assumptions.

5.1. Multivariable Aumann Integrals

Theorem 5.2. *Let $F : \mathbf{R}^n \to 2^{\mathbf{R}^m}$ be an arbitrary multifunction. Then, for any family of unit vectors $v_1, \ldots, v_k \in \mathbf{R}^n$, the joint marginal integral $\int_{(v_1,\ldots,v_k)} F\, d\lambda$ defined at (5.3) is a convex subset of $\left[L^1(\mathbf{R}^{n-1};\ \mathbf{R}^m)\right]^k$.*

Proof. Assume that f_1 and f_2 are two integrable selections of F. Given any $\alpha \in [0,1]$, we have to construct an integrable function f, with $f(x) \in F(x)$ almost everywhere, such that, for every $j = 1, ..., k$,

$$\int_{-\infty}^{+\infty} \alpha f_1(x' + \lambda v_j) d\lambda + \int_{-\infty}^{+\infty} (1-\alpha) f_2(x' + \lambda v_j) d\lambda = \int_{-\infty}^{+\infty} f(x' + \lambda v_j) d\lambda \tag{5.6}$$

for almost every $x' \in v_j^{\perp}$.
Applying Proposition 5.1 to the functions f_1, f_2 and the weights $p_1 \equiv \alpha,\ p_2 \equiv 1 - \alpha$, we obtain a measurable partition $\{A_1,\ A_2\}$ of $\mathbf{R}^n$ such that, for every $j = 1, ..., k$,

$$\begin{aligned}&\int_{-\infty}^{+\infty} \alpha f_1(x' + \lambda v_j) d\lambda + \int_{-\infty}^{+\infty} (1-\alpha) f_2(x' + \lambda v_j) d\lambda \\ &\quad = \int_{-\infty}^{+\infty} \chi_{A_1}(x' + \lambda v_i) f_1(x' + \lambda v_j) d\lambda + \chi_{A_2}(x' + \lambda v_j) f_1(x' + \lambda v_j) d\lambda,\end{aligned}$$

for almost every $x' \in v_j^{\perp}$. The requirements (5.6) are thus satisfied by defining f as

$$f(x) = \chi_{A_1}(x) f_1(x) + \chi_{A_2}(x) f_2(x). \quad \blacksquare$$

In the following, $B[0, \rho]$ denotes the closed ball centered at the origin with radius ρ, while $\overline{co}F(x)$ and $extF(x)$ denote the closed convex hull and the set of extreme points of $F(x)$, respectively. We say that the multifunction F is *integrally bounded* if there exists $\rho \in L^1(\mathbf{R}^n)$ such that

$$F(x) \subseteq B[0, \rho(x)] \qquad \text{for a.e.} \quad x \in \mathbf{R}^n. \tag{5.7}$$

Theorem 5.3. *Let $F : \mathbf{R}^n \to 2^{\mathbf{R}^m}$ be an integrably bounded, measurable multifunction with nonempty closed values. Then, for any family $v_1, \ldots, v_k$ of unit vectors in $\mathbf{R}^n$,* *the joint marginal integral of F defined at (5.3) is a closed, bounded, convex subset of* $\left[L^1(\mathbf{R}^{n-1};\ \mathbf{R}^m)\right]^k$. *Moreover,*

$$\int_{(v_1,\ldots,v_k)} \overline{co}F\ d\lambda = \int_{(v_1,\ldots,v_k)} F\ d\lambda = \int_{(v_1,\ldots,v_k)} extF\ d\lambda. \tag{5.8}$$

Proof. The convexity of $\int_{(v_1,\ldots,v_k)} F\ d\lambda$ was proved in Theorem 5.2. Its boundedness follows from (5.7) and Fubini's theorem, which implies $\|g_j\|_{L^1(v_j^\perp)} \leq \|\rho\|_{L^1(\mathbf{R}^n)}$, for every $j = 1, \ldots, k$.

In the next step, we establish the closure of $\int_{(v_1,\ldots,v_k)} \overline{co}F\ d\lambda$. Let $(g^\alpha)_{\alpha\geq 1}$ be a sequence contained in $\int_{(v_1,\ldots,v_k)} \overline{co}F\ d\lambda$, converging to $g = (g_1, \ldots, g_k)$ in the norm topology of $[L^1(\mathbf{R}^{n-1};\ \mathbf{R}^m)]^k$. For some integrable selections f^α of F we thus have

$$g^\alpha = (g_1^\alpha, \ldots, g_k^\alpha) = \int_{(v_1,\ldots,v_k)} f^\alpha\ d\lambda, \qquad \forall\alpha.$$

By assumption, there exists $\rho \in L^1$ such that $|f^\alpha(x)| \leq \rho(x)$. Hence, by possibly taking a subsequence, we can assume that $f^\alpha \rightharpoonup f^*$ weakly in $L^1(\mathbf{R}^n;\ \mathbf{R}^m)$, for some function f^*. By Mazur's theorem, there exists a sequence of convex combinations of f^α, say

$$\varphi^\alpha = \sum_{i=\alpha}^{N(\alpha)} \theta_{\alpha,i} f^i, \qquad \text{with} \qquad \theta_{\alpha,i} \in [0,1], \quad \sum_{i=\alpha}^{N(\alpha)} \theta_{\alpha,i} = 1 \qquad \forall\alpha,$$

which converges to f^* strongly in L^1. By taking again a subsequence, we can assume that (φ^α) converges to f^* pointwise almost everywhere. Therefore, $f^*(x) \in \overline{co}F(x)$ for a.e. $x \in \mathbf{R}^n$. Observing that the map $f \mapsto \int_{(v_1,\ldots,v_k)} f\ d\lambda$ is linear and continuous on $L^1(\mathbf{R}^n;\ \mathbf{R}^m)$, we have

$$\int_{(v_1,\ldots,v_k)} f^*\ d\lambda = \lim_{\alpha\to\infty} \int_{(v_1,\ldots,v_k)} \varphi^\alpha\ d\lambda = \lim_{\alpha\to\infty} \sum_{i=\alpha}^{N(\alpha)} \theta_{\alpha,i}(g_1^i, \ldots, g_k^i) = (g_1, \ldots, g_k).$$

Therefore, $g \in \int_{(v_1,\ldots,v_k)} \overline{co}F\ d\lambda$, proving that this set is closed.

To complete the proof of the theorem, it remains to establish the equalities (5.8). Since $extF(x) \subseteq F(x) \subseteq \overline{co}F(x)$, we only need to prove

$$\int_{(v_1,\ldots,v_k)} \overline{co}F\ d\lambda \subseteq \int_{(v_1,\ldots,v_k)} extF\ d\lambda. \tag{5.9}$$

Assume $f \in L^1$, with $f(x) \in \overline{co}F(x)$ for a.e. x. Since F is integrally bounded and has closed values, for almost every x the set $\overline{co}F(x)$ is the convex hull of $extF(x)$. Using a measurable selection theorem [Ku-RN], we find measurable functions $p_i : \mathbf{R}^n \to [0,1]$, $f_i \in \mathcal{L}^1$, such that

$$f(x) = \sum_{i=0}^{m} p_i(x) f_i(x), \qquad \sum_{i=0}^{m} p_i(x) = 1, \qquad f_i(x) \in extF(x) \quad \text{for a.e. } x \in \mathbf{R}^n. \tag{5.10}$$

By Proposition 5.1, there exists a measurable partition $\{A_0, \ldots, A_m\}$ of $\mathbf{R}^n$ such that, for every $j = 1, ..., k$,

$$\int_{-\infty}^{+\infty} \sum_{i=0}^{m} p_i(x' + \lambda v_j) f_i(x' + \lambda v_j)\, d\lambda = \int_{-\infty}^{+\infty} \sum_{i=0}^{m} \chi_{A_i}(x' + \lambda v_j) f_i(x' + \lambda v_j)\, d\lambda \tag{5.11}$$

for almost every $x' \in v_j^{\perp}$. Defining the function

$$f^*(x) \doteq \sum_{i=0}^{m} \chi_{A_i}(x) f_i(x),$$

from (5.10), (5.11) it follows

$$\int_{(v_1,\ldots,v_k)} f\, d\lambda = \int_{(v_1,\ldots,v_k)} f^*\, d\lambda \in \int_{(v_1,\ldots,v_k)} extF\, d\lambda,$$

proving (5.9). ∎

The last result of this section is concerned with selections of F whose corresponding marginal integrals are extreme points of $\int_{(v_1,\ldots,v_k)} F\, d\lambda$.

Theorem 5.4. *Let $F : \mathbf{R}^n \to 2^{\mathbf{R}^m}$ be an arbitrary multifunction and let $v_1, \ldots, v_k \in \mathbf{R}^n$ be unit vectors. If $g = (g_1, \ldots, g_k)$ is an extreme point of the set $\int_{(v_1,\ldots,v_k)} F\, d\lambda$, then there exists a unique $f \in L^1$ such that*

$$g = \int_{(v_1,\ldots,v_k)} f\, d\lambda, \qquad f(x) \in F(x) \quad \textit{for a.e. } x.$$

If F is measurable with closed values, then $f(x) \in extF(x)$ almost everywhere.

Proof. Assume that the functions f_1, f_2 satisfy $f_1(x), f_2(x) \in F(x)$ almost everywhere, and

$$\int_{(v_1,\ldots,v_k)} f_1\, d\lambda = \int_{(v_1,\ldots,v_k)} f_2\, d\lambda \in ext\left(\int_{(v_1,\ldots,v_k)} F\, d\lambda\right).$$

If $f_1 \neq f_2$, there exists a measurable set A such that

$$\int_A \big(f_1(x) - f_2(x)\big)\; dx = \int_{A^c} \big(f_2(x) - f_1(x)\big)\; dx \neq 0, \tag{5.12}$$

where $A^c = \mathbf{R}^n \setminus A$. Define

$$f' = \chi_{_A} f_1 + \chi_{_{A^c}} f_2, \qquad f'' = \chi_{_{A^c}} f_1 + \chi_{_A} f_2,$$

$$g' = \int_{(v_1,\ldots,v_k)} f'\; d\lambda, \qquad g'' = \int_{(v_1,\ldots,v_k)} f''\; d\lambda.$$

Since $\frac{1}{2}(f_1 + f_2) = \frac{1}{2}(f' + f'')$, we have

$$\frac{1}{2}\big(g' + g''\big) = g, \qquad g', g'' \in \int_{(v_1,\ldots,v_k)} F\; d\lambda. \tag{5.13}$$

The extremality of g is contradicted by (5.13), provided $g' \neq g''$. This is indeed the case because, for every $i = 1, \ldots, k$, Fubini's theorem and (5.12) imply

$$\begin{aligned} \int_{v_i^\perp} g_i'(z)\; dz &= \int_{\mathbf{R}^n} f'(x)\; dx = \int_{A \cup A^c} f_1(x)\; dx + \int_{A^c} \big(f_2(x) - f_1(x)\big)\; dx \\ &\neq \int_{A \cup A^c} f_1(x)\; dx + \int_A \big(f_2(x) - f_1(x)\big)\; dx = \int_{\mathbf{R}^n} f''(x)\; dx = \int_{v_i^\perp} g_i''(z)\; dz. \end{aligned}$$

This establishes the uniqueness of f. Next, assume that F is measurable with closed values. For every $m \geq 1$, the multifunction

$$H_m(x) \doteq \left\{ (y_1, y_2, \theta);\quad \theta \in \left[\frac{1}{m}, 1 - \frac{1}{m}\right], \quad y_1, y_2 \in F(x) \bigcap B[0, m], \right.$$
$$\left. |y_1 - y_2| \geq \frac{1}{m}, \qquad f(x) = \theta y_1 + (1 - \theta) y_2 \right\}$$

is measurable with closed, possibly empty values. Assuming that

$$meas\{x;\;\; f(x) \notin extF(x)\;\} > 0, \tag{5.14}$$

there exists some integer m and a set $S \subset \mathbf{R}^n$ such that

$$0 < meas(S) < \infty, \qquad H_m(x) \neq \emptyset \qquad \forall x \in S.$$

Let $x \mapsto \big(y_1(x), y_2(x), \theta(x)\big)$ be a measurable selection of H_m defined on S. Extend this function to the entire space $I\!R^n$ by setting $y_1(x) = y_2(x) = f(x)$, $\theta(x) = 0$ if $x \notin S$. By Proposition 5.1, there exists a measurable set A such that

$$\int_{(v_1,\ldots,v_k)} f\; d\lambda = \int_{(v_1,\ldots,v_k)} \theta y_1 + (1 - \theta) y_2\; d\lambda = \int_{(v_1,\ldots,v_k)} \chi_{_A} y_1 + \chi_{_{A^c}} y_2\; d\lambda.$$

The two distinct functions f and $\chi_A y_1 + \chi_{A^c} y_2$ therefore yield the same marginal integral. The assumption (5.14) is thus in contradiction with the uniqueness property obtained in the first part of the proof. ■

5.2. Product Measure Spaces

A classical theorem of Liapunov [H1, O] states the convexity of the range of a nonatomic vector measure. We now prove a similar result, valid for measures on product spaces.

Theorem 5.5. *For $i = 1, 2$, let X_i be a separable measure space with a finite, nonatomic vector measure μ_i. For each measurable subset A in the product space $X_1 \times X_2$, consider the functions*

$$\psi_1^A(x_1) \doteq \mu_2\Big(\Big\{x_2 \in X_2;\ (x_1, x_2) \in A\Big\}\Big),$$

$$\psi_2^A(x_2) = \mu_1\Big(\Big\{x_1 \in X_1;\ (x_1, x_2) \in A\Big\}\Big).$$

Then the set $\mathcal{S} \doteq \big\{(\psi_1^A, \psi_2^A);\ A \subseteq X_1 \times X_2\big\}$ is bounded, closed and convex in $L^1(X_1; \mathbf{R}^{m_1}) \times \mathcal{L}^1(X_2; \mathbf{R}^{m_2})$.

By the next lemma, the proof of Theorem 5.5 will be reduced to the case where each X_i is an interval on the real line. A similar isomorphism theorem can be found in [H2, p.173.]

Lemma 5.6. *Let (X, μ) be a separable measure space with a positive, finite nonatomic measure μ. Then there exists $\tau \geq 0$ and a measurable map $\Phi : X \to [0, \tau]$ such that the composition*

$$f \mapsto \Phi \circ f \tag{5.15}$$

determines an isometric isomorphism between $L^1([0, \tau], m)$ and $L^1(X, \mu)$, m being the usual Lebesgue measure on $[0, \tau]$.

Proof of the Lemma 5.6. Set $\tau = \mu(X)$. Because of the separability assumption, there exists a sequence of partitions $\mathcal{P}_\nu = \big\{A_{\nu,1}, \ldots, A_{\nu,N_\nu}\big\}$ of X such that the linear span of all characteristic functions $\chi_{A_{\nu,j}}$ is dense in $L^1(X, \mu)$. It is not restrictive to assume that, for $\nu > 1$, each set $A_{\nu,j}$ is entirely contained in some $A_{\nu-1,i(j)}$. By possibly relabelling, we can also assume that the map $j \mapsto i(j)$ is nondecreasing. For each $\nu \geq 1$, $x \in X$, define

$$\Phi_\nu(x) = \sum_{j=1}^{i(x)} \mu(A_{\nu,j}),$$

where $A_{\nu,i(x)}$ is the set in the partition $\mathcal{P}_\nu$ which contains x. Our previous assumption on the map $j \mapsto i(j)$ implies that the sequence $\Phi_\nu(x)$ is nonincreasing, hence the function

$$\Phi(x) = \lim_{\nu \to \infty} \Phi_\nu(x)$$

is well-defined. For each ν, j, set

$$t_{\nu,j} \doteq \sum_{i=1}^{j} \mu(A_{\nu,j}) \in [0,\tau].$$

Call μ' the measure on $[0,\tau]$ defined by $\mu'(S) = \mu\big(\Phi^{-1}(S)\big)$. The above definitions now yield

$$\mu'\big([t_{\nu,j-1},\ t_{\nu,j}]\big) = \mu\big(A_{\nu,j}\big) = t_{\nu,j} - t_{\nu,j-1}. \tag{5.16}$$

Since μ is nonatomic, the points t_ν, j are dense on $[0,\tau]$ and from (5.16) we conclude that μ' coincides with the Lebesgue measure on $[0,\tau]$. The assignment (5.15) is thus an isometric isomorphism from $span\{\chi_{[t_{\nu,j-1},\ t_{\nu,j}]};\ \nu, j \geq 1\}$ onto $span\{\chi_{A_{\nu,j}};\ \nu, j \geq 1\}$. Because of the separability assumption, this isomorphism can be continuously extended to a map, still defined by (5.15), from $L^1\big([0,\tau], m\big)$ onto $L^1\big(X,\mu\big)$. ■

Proof of Theorem 5.5. For $i = 1, 2$, set

$$\tau_i = |\mu_i|\big(X\big), \qquad f_i = \frac{d\mu_i}{d|\mu_i|},$$

where $|\mu_i|$ denotes the total variation of the measure μ_i. By the isomorphism between $\big(X_i, |\mu_i|\big)$ and $\big([0,\tau_i], m\big)$ constructed in Lemma 5.6, it suffices to prove the theorem in the case where

$$X_i = [0,\tau_i], \qquad d\mu_i = f_i\ dm,$$

for some functions $f_i : [0,\tau_i] \to \mathbf{R}^{m_i}$ with $|f_i(t)| = 1$ almost everywhere. On the product space $Q \doteq [0,\tau_1] \times [0,\tau_2]$, consider the bounded, measurable multifunction

$$F(t_1,t_2) \doteq \big\{f_1(t_1) \otimes f_2(t_2),\ 0\big\},$$

with values in the tensor space $\mathbf{R}^{m_1} \otimes \mathbf{R}^{m_2}$. Applying Theorem 5.3 with $v_1 = (0,1)$, $v_2 = (1,0)$, we obtain the closure and convexity of the set of joint marginal integrals

$$\mathcal{S}^{\otimes} \doteq \{(\phi_1^A, \phi_2^A);\ A \subseteq [0,\tau_1] \times [0,\tau_2]\},$$

with

$$\phi_1^A(t_1) = \int_{\{t_2;\ (t_1,t_2)\in A\}} f_1(t_1) \otimes f_2(t_2)\ dt_2 = f(t_1) \otimes \psi_1^A(t_1),$$

$$\phi_2^A(t_2) = \int_{\{t_1;\ (t_1,t_2)\in A\}} f(t_1) \otimes f_2(t_2)\ dt_1 = \psi_2^A(t_2) \otimes f_2(t_2).$$

Recalling that $|f_i(t)| \equiv 1$, one checks that the map

$$(\psi_1, \psi_2) \mapsto (f_1 \otimes \psi_1,\ \psi_2 \otimes f_2)$$

is a linear bicontinuous isomorphism between $L^1([0,\tau_1];\ \mathbf{R}^{m_1}) \times L^1([0,\tau_2];\ \mathbf{R}^{m_2})$ and its image. Hence, from the closure and convexity of $\mathcal{S}^{\otimes}$, it follows that the the set $\mathcal{S}$ is closed and convex as well. ■

5.3. A Bang-Bang Theorem for the Controlled Wave Equation

Consider the rectangle $Q = [0,a] \times [0,b]$, and let $\partial Q = \partial^+ Q \cup \partial^- Q$ be a decomposition of its boundary, with

$$\partial^+ Q = [0,a] \times \{b\} \cup \{a\} \times [0,b], \qquad \partial^- Q = [0,a] \times \{0\} \cup \{0\} \times [0,b]. \tag{5.17}$$

Given a continuous boundary condition $\psi : \partial^- Q \to \mathbf{R}^m$, we write $z(x,y,u)$ for the value at the point (x,y) of the solution to the linear wave equation

$$z_{xy} = u(x,y) \quad \text{on} \quad Q, \qquad z = \psi \quad \text{on} \quad \partial^- Q. \tag{5.18}$$

Assuming that u is integrable, we have the representation

$$z(x,y,u) = \psi(x,0) + \psi(0,y) - \psi(0,0) + \int_0^x \int_0^y u(r,s)\ dsdr. \tag{5.19}$$

If $F : Q \to 2^{\mathbf{R}^n}$ is a multifunction, we wish to compare solutions of $z_{xy} \in F(x,y)$ with solutions of $z_{xy} \in extF(x,y)$. In the following, we say that a curve γ is *monotone* if it can be parametrized by absolutely continuous functions $t \mapsto (x(t), y(t))$, such that either $\dot{x} \geq 0, \dot{y} \geq 0$ for a.e. t, or $\dot{x} \geq 0, \dot{y} \leq 0$ for a.e. t.

Theorem 5.7. *Let $F : Q \to \mathbf{R}^m$ be an integrably bounded, measurable multifunction with closed values. Then, for every u satisfying $u(x,y) \in \overline{co}F(x,y)$ a.e. and every monotone curve γ in Q, there exists u^* with $u^*(x,y) \in extF(x,y)$ a.e. such that the corresponding solutions of (5.18) satisfy*

$$z(x,y,u^*) = z(x,y,u) \qquad \text{for all } (x,y) \in \gamma \cup \partial Q. \tag{5.20}$$

Proof. To fix the ideas, assume that $\gamma = \{(x(t), y(t));\ \ t \in [t_0, t_1]\}$, with $\dot{x}(t) \geq 0,\ \dot{y}(t) \leq 0$ for almost every t, the other case being entirely similar. On Q, consider the order relation

$$(x,y) \prec (x',y') \qquad \text{iff} \qquad x < x' \quad \text{and} \quad y < y'$$

and define the two regions

$$Q^- \doteq \{(x,y);\ \ (x,y) \prec (x(t),y(t)) \ \text{ for some } t\}, \qquad\qquad Q^+ \doteq Q \setminus Q^-.$$

Construct measurable functions $u_i,\ \theta_i,\ i = 0, \ldots, m$, such that

$$\theta_i \in [0,1], \qquad \sum_{i=0}^{n} \theta_i = 1 \qquad \text{for a.e. } (x,y) \in Q,$$

$$u_i(x,y) \in extF(x,y), \qquad \sum_{i=0}^{n} \theta_i(x,y) u_i(x,y) = u(x,y) \qquad \text{for a.e. } (x,y) \in Q.$$

Applying Proposition 5.1 separately to the functions $\sum \theta_i u_i \chi_{Q^+}$ and $\theta_i u_i \chi_{Q^-}$, we obtain measurable partitions $\{A_0^+, \ldots, A_n^+\}$ of Q^+ and $\{A_0^-, \ldots, A_n^-\}$ of Q^- such that

$$\int_0^a \chi_{Q^\pm}(x,y) \sum_{i=0}^{m} \theta_i(x,y) u_i(x,y)\ dx = \sum_{i=0}^{m} \int_0^a \chi_{A_i^\pm}(x,y) u_i(x,y)\ dx \qquad \text{for a.e. } y,$$

$$\int_0^b \chi_{Q^\pm}(x,y) \sum_{i=0}^{m} \theta_i(x,y) u_i(x,y)\ dy = \sum_{i=0}^{m} \int_0^b \chi_{A_i^\pm}(x,y) u_i(x,y)\ dy \qquad \text{for a.e. } x.$$

Defining $u^* : Q \to \mathbf{R}^m$ as

$$u^*(x,y) = u_i(x,y) \qquad \text{iff} \qquad (x,y) \in A_i^+ \cup A_i^-,$$

we claim that (5.20) holds. Indeed, fix any $(\bar{x},\bar{y}) \in \gamma$. We then have

$$\begin{aligned}
&\int_0^{\bar{x}} \int_0^{\bar{y}} u(x,y)\ dydx \\
&= \int_0^{\bar{x}} \int_0^{b} \chi_{Q^-}(x,y)u(x,y)\ dydx - \int_{\bar{y}}^{b} \int_0^{a} \chi_{Q^-}(x,y)u(x,y)\ dxdy \\
&= \int_0^{\bar{x}} \left[\int_0^{b} \chi_{Q^-} \sum_{i=0}^{m} \theta_i u_i\ dy \right] dx - \int_{\bar{y}}^{b} \left[\int_0^{a} \chi_{Q^-} \sum_{i=0}^{m} \theta_i u_i\ dx \right] dy \\
&= \int_0^{\bar{x}} \sum_{i=0}^{m} \left[\int_0^{b} \chi_{A_i^-} u_i\ dy \right] dx - \int_{\bar{y}}^{b} \sum_{i=0}^{m} \left[\int_0^{a} \chi_{A_i^-} u_i\ dx \right] dy \\
&= \int_0^{\bar{x}} \int_0^{b} \chi_{Q^-}(x,y)u^*(x,y)\ dydx - \int_{\bar{y}}^{b} \int_0^{a} \chi_{Q^-}(x,y)u^*(x,y)\ dxdy \\
&= \int_0^{\bar{x}} \int_0^{\bar{y}} u^*(x,y)\ dydx.
\end{aligned}$$

By (5.19), this implies $w(\bar{x},\bar{y},u) = w(\bar{x},\bar{y},u^*)$. Next, consider a point P on the boundary of Q. If $P \in \partial^- Q$ there is nothing to prove. If, say, $P \equiv (a,\bar{y})$, then

$$\begin{aligned}
&\int_0^{\bar{y}} \int_0^{a} u(x,y)\ dxdy = \\
&\int_0^{\bar{y}} \left[\int_0^{a} \chi_{Q^-} \sum_{i=0}^{m} \theta_i u_i\ dx \right] dy + \int_0^{\bar{y}} \left[\int_0^{a} \chi_{Q^+} \sum_{i=0}^{m} \theta_i u_i\ dx \right] dy \\
&= \int_0^{\bar{y}} \sum_{i=0}^{n} \left[\int_0^{a} \chi_{A_i^-} u_i\ dx \right] dy + \int_0^{\bar{y}} \sum_{i=0}^{n} \left[\int_0^{a} \chi_{A_i^+} u_i\ dx \right] dy \\
&= \int_0^{\bar{y}} \int_0^{a} u^*(x,y)\ dxdy.
\end{aligned}$$

By (5.19), this again implies $z(a,\bar{y},u) = z(a,\bar{y},u^*)$. The computations in the case $P \equiv (\bar{x},b)$ are entirely similar. This completes the proof of the theorem. ■

5.4. A Non-convex Optimization Problem for the Wave Equation

On $Q = [0,a] \times [0,b]$, consider the controlled wave equation

$$z_{xy} = f(x,y,u) \qquad u(x,y) \in U, \tag{5.21}$$

with boundary conditions

$$z(x,y) = \psi(x,y) \qquad (x,y) \in \partial Q. \tag{5.22}$$

We assume that $U \subset \mathbf{R}^m$ is compact and contained in the ball $B(0,r)$, ψ is continuous on the boundary ∂Q and the restriction of ψ to each one of the four sides of the rectangle Q is absolutely continuous, with derivative in L^p, for some $p \in [1,\infty[$. The function $f : Q \times U \to \mathbf{R}^n$ is continuous in u, measurable w.r.t. x, y, and satisfies

$$|f(x,y,u)| \leq \alpha_1(x,y) \qquad \forall (x,y) \in Q, \quad u \in U, \tag{5.23}$$

for some function $\alpha_1 \in L^p$. We consider the space $W^{*,p}$ of all functions $z : Q \to \mathbf{R}^n$ whose distributional derivatives z_x, z_y, z_{xy} are in L^p. This space becomes a Banach space with the norm $\|z\|_* = \|z\|_p + \|z_x\|_p + \|z_y\|_p + \|z_{xy}\|_p$. For its basic properties we refer to [Sur1].

Every solution of (5.21) has the representation

$$z(x,y) = z(x,0) + z(0,y) - z(0,0) + \int_0^x \int_0^y f(r,s,u(r,s))dsdr. \tag{5.24}$$

Therefore, recalling (5.23), one concludes that any solution of (5.21), (5.22) satisfies an a priori bound in $W^{*,p}(Q)$ say $\|z\|_* \leq K$ for some positive constant K independent of u.

In connection with the system (5.21), (5.22), consider the optimal control problem:

$$\min_{(z,u)\in\mathcal{A}} J(z,u) \doteq \int_0^b \int_0^a h(x,y,u(x,y))dxdy + \int_0^b \int_0^a g(x,y,z(x,y))dxdy, \tag{P}$$

where

$$\mathcal{A} = \Big\{(z,u) \in W^{*,p}(Q) \times \mathcal{L}^1(Q) \qquad u(x,y) \in U, \quad z \text{ satisfies } (5.21),(5.22)\Big\}.$$

In the following, $\mathcal{L} \otimes \mathcal{B}$ denotes the product of the Lebesgue σ-algebra on Q with the Borel σ-algebra on U. On the cost functional J we assume:

(h_1) $h : Q \times U \to \overline{\mathbf{R}}$ is $\mathcal{L} \otimes \mathcal{B}$-measurable.

(h_2) $h(x,y,\cdot)$ is lower semicontinuous on U for almost all $(x,y) \in Q$.

(h_3) There exists a function $\alpha_2 \in L^1$ such that $h(x,y,u) \geq \alpha_2(x,y)$, for all x,y,u.

(g_1) $g : Q \times \mathbf{R}^n \to \overline{\mathbf{R}}$ is $\mathcal{L} \otimes \mathcal{B}(\mathbf{R}^n)$ measurable.

(g_2) $g(x,y,\cdot)$ is concave in $\mathbf{R}^n$ for almost all $(x,y) \in Q$.

(g_3) There exists a positive constant γ and a function $\alpha_3 \in L^{p'}(Q)$ such that $g(x,y,z) \geq \alpha_3(x,y) - \gamma|z|^p$. Here p' is the conjugate exponent of p.

Theorem 5.8. *Under the assumptions given above, if $J(z,u)$ has a finite value for some $(z,u) \in \mathcal{A}$, then the problem (P) admits an optimal solution.*

Proof. We first prove the existence of an optimal solution $(\tilde{z}, \tilde{u})$ to the convexified problem

$$\min_{(z,u)\in\widetilde{\mathcal{A}}} \bar{J}(z,u) = \int_0^b \int_0^a h^{**}(x,y,u(x,y),z_{xy}(x,y))dxdy + \int_0^b \int_0^a g(x,y,z(x,y))dxdy, \tag{P^{**}}$$

where

$$\widetilde{\mathcal{A}} = \Big\{(z,u) \in W^{*,p}(Q) \times L^\infty(Q); \quad z \ \text{ satifies } \ (5.22)\Big\},$$

and the function $(u,w) \mapsto h^{**}(x,y,u,w)$ is the bipolar function of $\tilde{h}(x,y,\cdot,\cdot)$, with

$$\tilde{h}(x,y,u,w) = \begin{cases} h(x,y,u) \ \text{ if } \ w = f(x,y,u) \ \text{ and } \ u \in U, \\ \qquad +\infty \ \text{ otherwise.} \end{cases}$$

The assumptions on h and g imply that there exists a constant C such that $\bar{J}(z,u) \geq C > -\infty$ for all $(z,u) \in \mathcal{A}$. Since $\mathcal{A}$ in nonempty, the infimum of $\bar{J}$ is finite. Let (z^k, u^k) be a minimizing sequence in $\widetilde{\mathcal{A}}$, so that $\lim_k \bar{J}(z^k,u^k) = \inf \bar{J} \in \mathbf{R}$. It follows that $u^k(x,y) \in coU \subseteq B(0,r)$ and $z^k_{xy}(x,y) \in co\{f(x,y,u);\ u \in U\}$. Thus, $||z^k||_* \leq K$. In particular, $||z^k_{xy}||_{\mathcal{L}^p} \leq K$ for all k. Therefore, there exists a subsequence, still indexed by k, and a limit function ξ, such that $z^k_{xy} \rightharpoonup \xi$ weakly in L^p. This follows from the weak compactness of balls in L^p if $1 < p < +\infty$, while for $p = 1$ we use the equi-integrability of the sequence z^k_{xy}: since for all k we have $u^k(x,y) \in coU \subseteq B(0,r)$ and $z^k_{xy}(x,y) \in co\{f(x,y,u);\ u \in U\}$. This, together with (5.23), implies

$$\int_E |z^k_{xy}(x,y)| \ dxdy \leq \int_E \alpha_1(x,y) \ dxdy \ \ \forall E \subset \Omega.$$

Similar arguments can be used for the sequences (z^k_x), (z^k_y). Moreover, one can easily check that the functions z^k are equicontinuous. Therefore, there exist a continuous function $\tilde{z}$ and a subsequence, still indexed by k, such that $z^k_{xy} \rightharpoonup \tilde{z}_{xy}$, $z^k_x \rightharpoonup \tilde{z}_x$, $z^k_y \rightharpoonup \tilde{z}_y$ weakly in L^p, while $z^k \to \tilde{z}$ uniformly on Q. Since $|u^k(x,y)| \leq r$ we can assume (up to a subsequence) $u^k \rightharpoonup \tilde{u}$ in the weak* topology of L^∞. Applying Theorem 2.1 in Chapter VIII of [E-T], we obtain $\bar{J}(\tilde{z},\tilde{u}) \leq \lim_k \bar{J}(z^k,u^k) = \inf \bar{J}$. Clearly $(\tilde{z},\tilde{u}) \in \widetilde{\mathcal{A}}$. Therefore, $(\tilde{z},\tilde{u})$ is a solution to problem $\bar{P}$.

In the second part of the proof, we construct $(z,u) \in \mathcal{A}$ such that $J(z,u) = \bar{J}(\tilde{z},\tilde{u})$. Observe that $h^{**}(x,y,\tilde{u}(x,y),\tilde{z}_{xy}(x,y)) < +\infty$ for almost all $(x,y) \in Q$. From Proposition

3.1 in Chapter IX of [E-T] it follows that there exist measurable functions p_i, u_i, w_i defined in Q, $i = 0, ..., n+m$, such that

$$h^{**}(x,y,\tilde{u}(x,y),\tilde{z}_{xy}(x,y)) = \sum_{i=0}^{n+m} p_i(x,y)\tilde{h}(x,y,u_i(x,y),w_i(x,y)) \tag{5.25}$$

$$(\tilde{u}(x,y),\tilde{z}_{xy}(x,y)) = \sum_{i=0}^{n+m} p_i(x,y)(u_i(x,y),w_i(x,y)) \tag{5.26}$$

$$\sum_{i=0}^{n+m} p_i(x,y) = 1, \qquad p_i(x,y) \in [0,1]. \tag{5.27}$$

Observe that $w_i(x,y)$ can be different from $f(x,y,u_i(x,y))$ or $u_i(x,y) \notin U$ on a set E_i of positive measure only if $p_i(x,y) = 0$ on E_i. We can then modify w_i on E_i so that (5.25) still holds, together with $w_i(x,y) = f(x,y,u_i(x,y))$ and $u_i(x,y) \in U$. Thus (5.25) becomes

$$h^{**}\big(x,y,\tilde{u}(x,y),\tilde{z}_{xy}(x,y)\big) = \sum_{i=0}^{n+m} p_i(x,y)h(x,y,u_i(x,y)) \tag{5.28}$$

with

$$\sum_{i=0}^{n+m} p_i(x,y)u_i(x,y) = \tilde{u}(x,y), \sum_{i=0}^{n+m} p_i(x,y)f(x,y,u_i(x,y)) = \tilde{z}_{xy}(x,y). \tag{5.29}$$

By Lemma 5.2 in [R3], there exists a selection $\delta \in L^{p'}(Q)$ from the map $(x,y) \mapsto \partial_z\big(-g(x,y,\tilde{z}(x,y))\big)$. The right-hand side denotes here the subdifferential of the convex function $-g(x,y,\cdot)$ at the point $\tilde{z}(x,y)$. Let $\phi \in W^{*,p'}(Q)$ be a function such that $\phi_{xy} = \delta(x,y)$. We now remark that Proposition 5.1 remains valid if the assumption $f_1, \dots, f_\nu \in L^1$ is replaced with $\sum p_i|f_i| \in L^1$. Indeed, in this latter case, one can construct a sequence of disjoint compact sets $K_j \subset \mathbf{R}^n$ such that $meas\big(\mathbf{R}^n \setminus \bigcup K_j\big) = 0$, and the restriction of each p_i, f_i to K_j is continuous. Applying Proposition 5.1 to the integrable functions $\chi_{K_j} f_i$, $\chi_{K_j}|f_i|$, we obtain a partition $\{A_{j,1}, \dots, A_{j,\nu}\}$ of each K_j such that

$$\int_{(v_1,\dots,v_k)} \sum_{i=1}^{\nu} \chi_{K_j} p_i f_i \; d\lambda = \int_{(v_1,\dots,v_k)} \sum_{i=1}^{\nu} \chi_{A_{j,i}} f_i \; d\lambda,$$

$$\int_{(v_1,\dots,v_k)} \sum_{i=1}^{\nu} \chi_{K_j} p_i |f_i| \; d\lambda = \int_{(v_1,\dots,v_k)} \sum_{i=1}^{\nu} \chi_{A_{j,i}} |f_i| \; d\lambda.$$

Setting $A_i \doteq \bigcup_{j\geq 1} A_{j,i}$, the previous equalities imply $\sum \chi_{A_i} f_i \in L^1$, together with (5.5). Using this generalized version of Proposition 5.1, we now obtain a partition $\{A_1, \ldots, A_{n+m+1}\}$ of Q such that

$$\int_0^a \sum_i p_i(x,y)h(x,y,u_i(x,y))\, dx = \sum_i \int_0^a \chi_{A_i}(x,y)h(x,y,u_i(x,y))dx \tag{5.0$_a$}$$

$$\int_0^a \sum_i p_i(x,y)f(x,y,u_i(x,y))\, dx = \sum_i \int_0^a \chi_{A_i}(x,y)f(x,y,u_i(x,y))dx \tag{5.31$_a$}$$

$$\int_0^a \sum_i p_i(x,y)\langle\phi(x,y), f(x,y,u_i(x,y))\rangle\, dx = \sum_i \int_0^a \chi_{A_i}(x,y)\langle\phi(x,y), f(x,y,u_i(x,y))\rangle dx, \tag{5.32$_a$}$$

for almost every $y \in [0,b]$. Here $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbf{R}^n$. Moreover, we can assume that similar equalities, say $(5.30)_b$ - $(5.32)_b$ hold for a.e. x, when the above expressions are integrated on $[0,b]$ w.r.t. y. In view of (5.28), $(5.30)_{a,b}$ one has

$$\begin{aligned}\int_0^b \int_0^a h^{**}(x,y,\tilde{u}(x,y),\tilde{z}_{xy}(x,y))dxdy &= \int_0^b \int_0^a \sum_i p_i(x,y)h(x,y,u_i(x,y))dxdy \\ &= \int_0^b \int_0^a \sum_i \chi_{A_i}(x,y)h(x,y,u_i(x,y))dxdy \\ &= \int_0^b \int_0^a h(x,y,\sum_i \chi_{A_i}(x,y)u_i(x,y))dxdy.\end{aligned} \tag{5.33}$$

We now define the measurable function $u : Q \mapsto U$ by setting

$$u(x,y) = \sum_i \chi_{A_i}(x,y)u_i(x,y),$$

and consider the solution z to the boundary value problem

$$z_{xy} = f(x,y,u(x,y)), \qquad z = \psi \text{ on } \partial^- Q, \tag{5.34}$$

where the lower part $\partial^- Q$ of the boundary is defined as in (5.17). Using Fubini's theorem and $(5.31)_a$ we obtain

$$\begin{aligned}z(a,y) &= \psi(a,0) + \psi(0,y) - \psi(0,0) + \int_0^a \int_0^y \sum_i \chi_{A_i}(x,y)f(r,s,u_i(r,s))dsdr \\ &= \psi(a,0) + \psi(0,y) - \psi(0,0) + \int_0^y \int_0^a \sum_i p_i(x,y)f(r,s,u_i(r,s))dsdr = \tilde{z}(a,y) = \psi(a,y).\end{aligned}$$

Similarly, by using $(5.31)_b$, we obtain $z(x,b) = \tilde{z}(x,b) = \psi(x,b)$ for every $x \in [0,b]$. This proves that $(z,u) \in \mathcal{A}$.

It remains to show that the pair (z,u) defined above is a solution to the original problem. Indeed, (5.33) and the definition of u imply

$$\int_0^b \int_0^a h^{**}(x,y,\tilde{u}(x,y),\tilde{z}_{x,y}(x,y))dxdy = \int_0^b \int_0^a h(x,y,u(x,y))dxdy. \tag{5.13}$$

Since $\delta(x,y) \in \partial_z\big(-g(x,y,\tilde{z}(x,y))\big)$, we have

$$g(x,y,z(x,y)) \leq g(x,y,\tilde{z}(x,y)) + \langle \delta(x,y),\ z(x,y) - \tilde{z}(x,y)\rangle. \tag{5.36}$$

We claim that

$$\int_0^b \int_0^a \langle \delta(x,y), z(x,y) - \tilde{z}(x,y)\rangle dxdy = 0.$$

Indeed, recalling the definition of ϕ, taking into account that $z = \tilde{z} = \psi$ on ∂Q, and denoting by v_j the $j-th$ component of a vector v, we compute

$$\int_0^b \int_0^a \langle \delta(x,y),\ z(x,y) - \tilde{z}(x,y)\rangle\ dxdy = \sum_{j=1}^n \int_0^b \int_0^a \phi_{j,xy}(x,y)(z(x,y) - \tilde{z}(x,y))dxdy$$

$$= -\sum_{j=1}^n \int_0^b \int_0^a \phi_{j,x}(x,y)(z_{j,y}(x,y) - \tilde{z}_{j,y}(x,y))dxdy.$$

Using Fubini's Theorem and the equality of the boundary values of z and $\tilde{z}$, the last integral can be written as

$$\sum_{j=1}^n \int_0^a \int_0^b \phi_j(x,y)(z_{j,xy}(x,y) - \tilde{z}_{j,xy}(x,y))dydx$$

$$= \int_0^b \int_0^a \langle \phi(x,y),\ z_{xy}(x,y) - \tilde{z}_{xy}(x,y)\rangle dxdy = 0.$$

The last equality holds because of $(5.32)_{a,b}$. Therefore, (5.36) implies

$$\int_0^b \int_0^a g(x,y,z(x,y))dxdy \leq \int_0^b \int_0^a g(x,y,\tilde{z}(x,y))dxdy. \tag{5.37}$$

Since $(\tilde{z},\tilde{u})$ is a solution of the problem $(\bar{P})$ and because of the definition of h^{**}, we have

$$\int_0^b \int_0^a h^{**}(x,y,\tilde{u}(x,y),\tilde{z}_{xy}(x,y))dxdy + \int_0^b \int_0^a g(x,y,\tilde{z}(x,y))dxdy \leq$$

$$\leq \int_0^b \int_0^a \tilde{h}(x, y, u(x, y), z_{xy}(x, y))dxdy + \int_0^b \int_0^a g(x, y, z(x, y))dxdy =$$

$$= \int_0^b \int_0^a h(x, y, u(x, y))dxdy + \int_0^b \int_0^a g(x, y, z(x, y))dxdy,$$

where the equality was obtained simply by the definition of $\tilde{h}$. The latter and (5.15)

$$\int_0^b \int_0^a g(x, y, \tilde{z}(x, y))dxdy \leq \int_0^b \int_0^a g(x, y, z(x, y))dxdy. \tag{5.18}$$

Together, (5.35), (5.37) and (5.38) prove that (z, u) is in fact a solution to the original problem (P). ■

Final Remarks

As a consequence of the existence results proved in Chapter 2, we have $\min P = \min P^{**}$. This fact is analogous to that provided by the Relaxation theory (part (d) of Theorem 1.8). However, from this point of view, as we have seen in Chapter 3, an estimate from above for the integrand is needed, besides the coercivity one. This drawback does not allow us to deal with integrands taking the value $+\infty$. In the present setting, the problem (P^{**}) is defined by means of an auxiliary functional which we have called "convexified functional" defined in a proper way. More precisely, the functional F^{**} associated to problem (P^{**}) is such that

$$F^{**}(u) \leq F(u) \quad \forall u \in X \text{ with } \min_{u \in X} F^{**}(u) \in \mathbf{R}$$

and we prove the existence of a function $\tilde{u} \in X$ satisfying

$$F(\tilde{u}) = \min_{u \in X} F^{**}(u)$$

showing that $\tilde{u}$ is also a solution to the original problem (P).
The method proposed by the relaxation Theory (Remark 1.9) to find solutions for non-convex problems relies on the fact $\inf P = \min P^{**}$ (see chapter 3). So that, if $\tilde{u}$ is a solution to problem (P), it has to satisfy

$$f(x, u(x), L\tilde{u}(x)) = f^{**}(x, u(x), L\tilde{u}(x)). \tag{1}$$

Consequently, if u is a solution to the (proper) relaxed problem (P^{**}) such that (1) holds then u is also a solution to problem (P). Such a reasoning has been used in most of the works concerning non-convex variational problems [A-T2, Ra, R4, T1], proving in particular that every solution to problem (P^{**}) is a solution to problem (P). Unfortunately, the approach used in Chapter 2, as far as I know, has not been applied for integrals that are not of the form $f(x, u, \xi) = g(x, u) + h(x, \xi)$. Of course, a direct approach for solving non-convex minimization problems without passing through of convexification is necessary. A first attempt in this direction would be to investigate under what conditions on the integrand, weak convergence of the minimizing sequences implies strong convergence.

For the study of integrals depending of Du with u being a vector-value function we refer to [Mo, Me, Fu] and for some relaxation results concernig this kind of functionals we refer to [Da1, Da2, B-C-O] and references therein.

Existence Theorems with no convexity assumption, for problems governed by hyperbolic partial differential equations have also been given by Suryanarayana in [Sur2]. I am not aware of other works in this direction.

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