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Thesis submitted for the degree of "Doctor Philosopihiæ"

Academic Year 1999/2000

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Il presente lavoro costituisce la tesi presentata da Duccio Papini, sotto la direzione del Prof. Fabio Zanolin, al fine di ottenere il diploma di *"Doctor Philosopiæ"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni.

Ai sensi del Decreto del Ministero della Pubblica Istruzione numero 419 del 24/04/1987, tale diploma di ricerca post-universitaria è equipollente al titolo di *"Dottore di Ricerca in Matematica"*.

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Special thanks to a great friend, Aldo, for all the assistance, for all the meals, for all the ever–good pieces of advice. It is a peculiar thing: he is very skilful in obtaining thanks in other people's theses!

Finally (even if this sentence should have been placed above the others) I think that I should thank three persons: Paolo, Maria and God, since I owe to their collaboration if I am here.

[1] a slang word, used mostly in Tuscany, which can be translated here by "things which are not correct"

Contents

Introduction

The study of boundary value problems for second order ordinary differential equations like:

$$\ddot{x} + f(t, x, \dot{x}) = 0 \tag{1}$$

when f grows at infinity faster than any linear function with respect to the x variable, is a topic which has been widely investigated in the literature, using various different approaches.

Just to cite some works, the case, in which f in (1) does not depend on $\dot{x}$ and the limit:

$$\lim_{x \to \pm\infty} \frac{f(t, x)}{x} = +\infty \tag{2}$$

holds uniformly with respect to t, was studied since the fifties by Morris [88], [89], Ehrmann [40], [41] and Nehari [93] for some particular forms of f (see also the references in [28]); on the other hand a controlled dependence on $\dot{x}$ is allowed in the works by Hartman [54] and by Struwe [116]. The main feature of such kind of problems is the absence of a priori bounds due to the strong oscillatory behaviour of the large solutions [80].

Two are the main classes of equations which we will deal with in this thesis: the forced Liénard equation:

$$\ddot{x} + f(x)\dot{x} + g(x) = e(t) \tag{3}$$

and the Duffing equation with time–dependent weight:

$$\ddot{x} + q(t)g(x) = 0, \tag{4}$$

assuming in both cases that $g : \mathbb{R} \to \mathbb{R}$ satisfies:

$$\lim_{x \to \pm\infty} g(x)\,\mathrm{sign}(x) = +\infty.$$

The first equation is considered in Chapter 1 mainly from the point of view of the existence of T-periodic solutions, when e is a T-perodic forcing term, while various boundary value problems associated to the second one are considered in the other five chapters, paying much attention to the effects that the changes of sign of q have on the dynamic of the solutions.

The periodic problem for (3) has been widely treated in literature; classical results are obtained either with dissipativity hypotheses, which are translated into some kind of sign-definiteness of f (for instance $f(x) \geq c > 0$ for large values of $|x|$, or $F(x)\,\mathrm{sign}(x) \to +\infty$ as $|x|$ tends to infinity, where $F(x) = \int_0^x f(s)ds$; see for instance [22], [51] and the references contained therein) or assuming some sublinear growth condition on g ([79], [85]).

Only in more recent years the case, in which f and g have polynomial growth and $f(x)\,\mathrm{sign}(x) \to +\infty$ as $|x|$ goes to infinity (and so it does not enter into the dissipativity setting), has been attacked. In the unforced situation, i.e. when $e \equiv 0$ in (3), an analysis of the plane system associated to equation (3):

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \tag{5}$$

shows that the structure of the orbits drastically changes according to a relationship between the rates of growth at infinity of F and g ([53], [117] and the references contained therein).

In [122] G. Villari and F. Zanolin considered the case in which g is odd and the function f is the sum of two terms, $f = f_o + f_e$, where f_o is such that the plane system associated to the equation:

$$\ddot{x} + f_o(x)\dot{x} + g(x) = 0$$

describes a global center in a neighbourhood of infinity (the subscript "o" is used here since we typically have this situation if f_o is odd), while f_e introduces some dissipativity effect (for instance it is even and sign-defined for large $|x|$). They proved the existence of at least one T-periodic solution by exploiting such dissipativity, if the even part f_e is not too small. That result can be applied to an equation like:

$$\ddot{x} + (|x|^{\alpha-1}x + p(x))\dot{x} + |x|^{\beta-1}x = e(t)$$

where $p(x) \geq p_0 > 0$ and the condition on the odd part reads:

$$\beta > 2\alpha + 1$$

as it follows from the works by Filippov [45], Opial [95] and Hara and Yoneyama [53], but the problem of what happens when $p \equiv 0$ is left open in [122]and [123]. In this case system (5) defines a center in a neighbourhood of infinity and hence the problem concerns the study of a center subjected to periodic perturbations which are not necessarily small.

Some previous results have been obtained by interpreting equation (3) as a perturbation of the Duffing equation:

$$\ddot{x} + g(x) = e(t)$$

when g has a superlinear growth at infinity in the sense that:

$$\lim_{x \to \infty} \frac{g(x)}{x} = +\infty.$$

In this situation, thanks to the hamiltonian structure of the problem, different methods (critical point theory in [11], Poincaré-Birkhoff theorem in [37] and [38], Moser's twist theorem in [90] and [36]) allow to obtain the existence of infinitely many periodic (harmonic and subharmonic) solutions and even boundedness of all the solutions, if g is a polynomial-like function. Partial extensions of these results have been obtained by Liu in [71] and [72] and by Liu and You in [74]), when f and g are odd polynomial and e is odd, too, by the use of KAM theory for reversible systems (for recent works see [75] and [128]). However the case in which f and g are odd, but e is not, remains still open (see [124]). A typical example is given by the equation:

$$\ddot{x} + A|x|^{\alpha-1}x\dot{x} + B|x|^{\beta-1}x = e(t), \tag{6}$$

where $\beta > 2\alpha + 1$ and $e : \mathbb{R} \to \mathbb{R}$ is a continuous T-periodic function such that $e(t + \tau) \neq -e(-t + \tau)$, for all $t, \tau \in \mathbb{R}$ (this last condition just ensures that e is not odd, still after a time shift). This model can be considered as a generalization of the example considered by M. Struwe in [116].

In Chapter 1 we prove the existence of at least one T-periodic solution for a class of equations which includes (6) and for an arbitrary forcing term e. Actually we study the plane system:

$$\begin{cases} \dot{x} = y - F(x) + E(t,x,y) \\ \dot{y} = -g(x) + p(t,x,y) \end{cases} \tag{7}$$

which includes equation (3) as a particular case. Our result shows that, if E and p are bounded, $F(x) \sim |x|^{\alpha+1}$ and $g(x) \sim |x|^{\beta-1}x$, with $\beta > 2\alpha + 1$, then (7) has at least one T-periodic solution. To obtain our main result we will apply a continuation theorem proved in [28, Theorem 2]: we embed system (7) into a one-parameter family of differential equations:

$$\dot{\xi} = \mathcal{F}(t,\xi;\lambda), \tag{8}$$

where $\lambda \in [0,1]$ is the parameter and $\mathcal{F} : \mathbb{R} \times \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ is T-periodic with respect to t, gives the original system (7) for $\lambda = 1$ and is chosen in such a way that for $\lambda = 0$ (8) has a "good" set of T-periodic solutions (namely, $\mathcal{F}(t,x;0)$ is indipendent of t and its Brower degree is not zero in a suitable ball); then one hopes that the solution set for $\lambda = 0$ is "continuable" for all $\lambda > 0$ and, in particular, up to $\lambda = 1$. This hope can be usually fulfilled by continuations theorems only if a fundamental ingredient is supplied: some a priori estimate on the periodic solutions of (8). We have already remarked that in general this is not possible in superlinear problems: what is possible to do (and what is actually needed in order to apply Theorem 2 in [28]) is to find a priori bounds for periodic solutions with *fixed number of zeros* (see also [27] and [82]). Therefore the main efforts in Chapter 1 are spent in the study of (7).

Such a study is done in Section 1.1: by a recursive procedure, which consists in using the smoothness of f and g to successively reduce the rate of growth of undesirable terms, it is proved that the solutions with sufficiently large initial values are defined in a fixed interval and satisfy the "elastic property" (see Corollaries 1.1.3 and 1.1.4 and Lemma 2.2.2). This technique, which need enough regularity as in the papers where KAM theory is used, may recall similar procedures exploited, for instance, in [36], but it concerns only one step and thus is much more elementary than it is in those works. Moreover, we show that larger solutions oscillates more in a fixed interval $[a,b]$, in the sense that a solution $(x(t),y(t))$ of (7) makes as many rotations as we want around the origin, while t spans $[a,b]$, if the length of the vector $(x(t_0),y(t_0))$ is large enough for some $t_0 \in [a,b]$. Therefore, the solutions with a fixed number of zeros are necessarily bounded and we obtain the required a priori bound in order to apply Theorem 2 in [28].

However, the results obtained, combined with the ideas of the proof of Theorem 1 in [116], allow to deduce also the existence of infinitely many solutions of Sturm–Liouville boundary value problems associated to equations like (7).

The other five chapters of the thesis deal with equation (4). In this case, the condition of superlinearity (2) reads:

$$\lim_{x\to\pm\infty} \frac{g(x)}{x} = +\infty \tag{9}$$

and the uniformity with respect to t of the limit in (2) corresponds to the condition of definite sign:

$$\inf_t q(t) > 0.$$

In this situation, according to Coffman and Ullrich [31], some weak regularity assumptions on the weight q (like q continuous and locally of bounded variation) are enough to guarantee the global continuability of the solutions for the associated Cauchy problems and this allows to apply shooting type techniques. A particular type of equation (4) is given by:

$$\ddot{x} + q(t)x^{2n-1} = 0, \tag{10}$$

which has deserved much attention for the rich dynamics exhibited by its solutions. After the work of Laederich and Levi [63] (see also [68], [69], for further progress in that direction) it is known that for $q(\cdot)$ positive, periodic and sufficiently smooth (e.g., at least of class C^5, according to [68]), all the solutions of (10) are bounded, there are infinitely many periodic solutions (harmonics and subharmonics of each order) and most of the solutions with large amplitude are quasiperiodic. The same kind of result holds for suitable perturbations of (10), as well as for:

$$\ddot{x} + f(t,x) = 0, \tag{11}$$

with f smooth enough in both variables and having a polynomial growth in x (see, e.g., [68], [69]). On the other hand, it was proved in [31], that the sole continuity of q is not sufficient even for the global continuability of all the solutions.

It seems that Waltman [126] was the first who considered a changing sign weight for the study of the oscillatory solutions for a superlinear equation of the form (10). Observe that here the global continuability of the solutions is no more guaranteed (independently of the degree of regularity of q). In fact, one can see, according to Burton and Grimmer [21], that some solutions will blow up in the intervals of negativity for $q(t)$. Actually Burton and Grimmer showed that a necessary and sufficient condition for the global continuability to fail (when $q < 0$) is that:

$$\text{either} \quad \int_1^{+\infty} [G(x)]^{-1/2} dx < +\infty \qquad \text{or} \qquad \int_{-\infty}^{-1} [G(x)]^{-1/2} dx < +\infty,$$

where $G(x) := \int_0^x g(s)ds$. Hence, in this situation, the problem of absence of a priori bounds is accompanied by the technical difficulty due to the noncontinuability of some solutions. This makes the phase-plane analysis somehow more delicate.

A study of the topological properties of the set of initial points from which depart globally defined solutions of (4) was initiated by Butler in [23], [24]. In [23] G. J. Butler assumed that both the integrals above were finite, that is:

$$\int_{-\infty}^{+\infty} [1 + G(x)]^{-1/2} dx < +\infty, \tag{12}$$

and this condition allowed him to determine some topological property of the sets of good initial conditions, although he remarked that in general one cannot expect that they may have any particular structure. Moreover he proved the existence of infinitely many periodic solutions for equation (2.1) assuming (12) and (9), in a situation for which the weight function $q(t)$ may change sign a finite number of times in a fixed compact interval and has some weak smoothness in order to avoid problems of continuability where $q \geq 0$, according to Coffman and Ullrich [31]. Butler's argument, which is an ingenious blend of the rapid oscillatory properties of the large solutions when $q > 0$ with the properties of the set of initial points of the continuable solutions when $q < 0$, seems to be flexible enough to be adapted to other boundary value problems.

Indeed in [99] the result by Butler has been extended to cover a more general class of boundary condition: namely, under Butler's same hypotheses, infinitely many solutions of (4) are found which satisfy the Floquet–type boundary condition:

$$(x(\omega), \dot{x}(\omega)) = \Lambda(x(0), \dot{x}(0)), \tag{13}$$

where $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous, nondegenerate and positively σ-homogeneous map, that is:

(i) $\Lambda(p) = (0,0) \iff p = (0,0)$,

(ii) $\Lambda(rp) = r^\sigma \Lambda(p) \quad \forall r > 0 \, \forall p \in \mathbb{R}^2$.

Such kind of boundary value problems have been studied for instance in [42, 43, 54, 56, 61]. Recently Mawhin in [81, 83] considered again the Floquet boundary value problem for a first order differential equation in the complex plane, since by a suitable change of variable he was able to transform a periodic problem into a Floquet problem which turned out to be easier to solve via a continuation approach. The same kind of problems have been considered in [120], again through a continuation approach, and in [114], using the fixed point index. Henrard [57] also uses a continuation theorem for a superlinear second order real ordinary differential equation of the form

$$\ddot{x} + f(x) = p(t, x, \dot{x}),$$

with p bounded, and a Floquet boundary condition, where Λ is the rotation of an angle $\theta \neq 0, \pi$, that is his result does not cover the periodic and the antiperiodic case. However he is able to find at least four different solutions with a prescribed number of zeros. In this case, that is when Λ is the rotation map, also in [99] the multiplicity of solutions is obtained by finding at least a solution having an arbitrary and sufficiently large number of zeros in the interval $[0, \omega]$. Moreover, the periodic problem for the damped equation:

$$\ddot{x} + c\dot{x} + q(t)g(x) = 0 \tag{14}$$

is transformed into a Floquet–type problem for (4) by employing a change of variables used in [21]; therefore, also periodic solutions, with an arbitrary and sufficiently large number of zero, as well as infinitely many subharmonic solutions of every order are found in [99] for equation (14).

In 1991, Lassoued [64] using a variational approach, obtained the existence of one non-constant T-periodic solution for the system of differential equations:

$$\ddot{x} + q(t)G'(x) = 0, \tag{15}$$

assuming $G : \mathbb{R}^N \to \mathbb{R}$ a superquadratic convex homogeneous function of class C^2 and $q \in L^1([0,T], \mathbb{R})$ a T-periodic function which changes sign. In the case of G even, also the existence of infinitely many solutions was proved. Also Le and Schmitt apply a variational method in [70] to find a periodic solution of an equation of the form:

$$\ddot{x} + q(t)e^x = p(t),$$

where p and q are ω–periodic, q changes sign and satisfies a suitable integral condition. Various investigations along this or related directions were then performed, in particular, with respect to the existence and multiplicity of solutions for Dirichlet, Neumann or mixed boundary value problems associated to elliptic equations [2], [3], [4], [5], [9], [10], [16], [17],

[18], [60], [67], [77], [107] or to the existence and multiplicity of periodic solutions for Hamiltonian systems [6], [7], [16], [25], [39], [44], [47], [48]; recently, a condition for the stability of the origin for a perturbed form of (10) with q continuous, periodic and changing sign, has been obtained by Liu in [73].

Most of the above quoted results apply to situations (like PDEs or systems) which are widely more general than (4); on the other hand, the assumptions involved therein on the nonlinearity require either symmetry conditions or a growth at infinity which is quite close to a power. For instance it is usually assumed that G is superquadratic in the sense that:

$$g(x) \cdot x \geq \beta G(x) \qquad \forall\, |x| \geq R$$

for some $\beta > 2$ (an assumption which is more restrictive than (12), since it does not allow potentials G that behave at infinity like $|x|^2 \log^\gamma |x|$ for $\gamma > 2$) and moreover homogeneous or satisfying a condition like:

$$|g(x) \cdot x - \beta G(x)| \leq c|x|^2 \qquad \forall, x \in \mathbb{R}^n,$$

for some $c > 0$ and $\beta > 2$. Under these hypotheses at least one nontrivial periodic solution is usually found in the works where the periodic problem for a systems like (15) is considered; if G is assumed to be also even, it is shown that there are infinitely many periodic solutions and a sequence of subharmonic solutions with orders tending to $+\infty$. Moreover, except for the case of the existence of positive solutions, the results obtained up to now, seem to be not very complete with respect to the nodal properties of the solutions.

In a recent article [121], S. Terracini and G. Verzini, dealing with the scalar equation:

$$\ddot{x} + q(t)x^3 + mx + h(t) = 0,$$

obtain a very sharp result for the two-point boundary value problem, proving the existence of solutions with precise nodal properties in the intervals of positivity of $q(t)$ and with at least one zero in the intervals of negativity of q. Similar conclusions are derived for the periodic problem and for the existence of bounded non-periodic oscillatory solutions defined on the whole real line and also when a more general superlinear function g is considered in place of the cubic x^3 : however it seems that in [121] some homogeneity condition on g has to be assumed in order to apply the Nehari method.

In this thesis we adopt the following plan for the last five chapters: in Chapters 2 and 3 we mainly study the cases of, respectively, $q \geq 0$ and $q \leq 0$ in equation (4); in Chapters 4 and 5 we use the results obtained in the preceding ones in order to prove the existence and multiplicity of solutions of various boundary value problems associated to (4) when g is superlinear (in some sense to be specified) and q changes sign; in Chapter 6 the existence of globally defined solutions with prescribed nodal properties is considered for (4) and applied to the case in which q is periodic and, in this case, some chaotic features of the dynamic are outlined.

About the conditions of superlinear growth that we consider throughout Chapters 2–6, they are related to the dynamical properties of the two autonomous equations:

$$(+) \quad \ddot{x} + g(x) = 0 \qquad \text{and} \qquad (-) \quad \ddot{x} - g(x) = 0. \tag{16}$$

When $g : \mathbb{R} \to \mathbb{R}$ satisfies the sign condition:

$$g(x)x > 0 \qquad \forall\, x \neq 0,$$

(or at least for all $|x| \geq x_0$) the primitive $G(x) = \int_0^x g(s)\,ds$ has a right inverse G_+^{-1} : $[0,+\infty[\to[0,+\infty[$ and a left one $G_-^{-1} : [0,+\infty[\to]-\infty,0]$; we take the two energy functions:

$$E^+(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + G(x) \qquad \text{and} \qquad E^-(x,\dot{x}) = \frac{1}{2}\dot{x}^2 - G(x)$$

respectively associated to the equations in (16). The first equation (+) describes a global center around the origin in the phase plane because all the level lines of E^+ are closed cicles around $(0,0)$; therefore, for each value $e > 0$ of the energy E^+ we have a unique (up to a time shift) solution of the equation (+) which is periodic and whose period is given by:

$$\tau_g^+(e) = \sqrt{2}\int_{G_-^{-1}(e)}^{G_+^{-1}(e)} \frac{1}{\sqrt{e - G(s)}}\,ds.$$

On the other hand the set of trajectories of equation (−) has the structure of a saddle around the origin. For every value $e \neq 0$ of the energy E^- there are two unbounded trajectories and the sum of the length of the two maximal intervals of continuability of the corresponding solutions can be written as:

$$\tau_g^-(e) = \begin{cases} \sqrt{2}\displaystyle\int_{-\infty}^{+\infty} \frac{1}{\sqrt{e+G(s)}}\,ds & \text{if } e > 0 \\ \sqrt{2}\displaystyle\int_{-\infty}^{G_-^{-1}(-e)} \frac{1}{\sqrt{G(s)+e}}\,ds + \sqrt{2}\displaystyle\int_{G_+^{-1}(-e)}^{+\infty} \frac{1}{\sqrt{G(s)+e}}\,ds & \text{if } e < 0. \end{cases}$$

When $g(x)/x \to +\infty$ for $x \to \pm\infty$, then it can be proved that $\tau_g^+(e) \to 0$ as $e \to +\infty$, so that the solutions of (+) with greater C^1-norm oscillate more and have more zeros in a fixed interval; when g is a power, like $x|x|^{\gamma-1}$, $\gamma > 1$, also $\tau_g^-(e) \to 0$, as $e \to \pm\infty$, all the solutions of (−) have a blow-up in finite time and the length of their maximal interval of existence decreases to zero as their energy tends to infinity. Hence, our assumptions of superlinearity will be the following ones:

$$(g2+) \qquad \lim_{e\to+\infty} \tau_g^+(e) = 0$$

and

$$(g2+) \qquad \lim_{e\to\pm\infty} \tau_g^-(e) = 0$$

and will be needed repectively in the intervals where $q \geq 0$ and $q \leq 0$. Actually $(g2+)$ and $(g2-)$ are satisfied by any function $g : \mathbb{R} \to \mathbb{R}$ such that:

$$\exists\, k > 0, \alpha > 2 : \quad |g(s)| \geq k|s|\log^\alpha(1+|s|) \qquad \forall\,|s| \gg 0,$$

therefore we can treat non-monotone functions, so that the potential G does not need to be convex. More general assumptions which ensure the validity of both $(g2+)$ and $(g2-)$, when $g(x)x > 0$ for large $|x|$, are:

$$(17) \qquad \lim_{x\to\pm\infty} \frac{g(x)}{x} = +\infty, \qquad \left|\int^{\pm\infty} \frac{1}{\sqrt{G(s)}}\,ds\right| < +\infty \quad \text{and} \quad \liminf_{x\to\pm\infty} \frac{G(\sigma x)}{G(x)} > 1,$$

for some $\sigma > 1$. The third condition in (17) is always satisfied when g is monotone nondecreasing in a neighbourhood of infinity. Indeed, if g has such a monotonicity property, then just the first two assumptions are enough for $(g2+)$ and $(g2-)$.

The results contained in Chapters 2 and 3 are mostly instrumental for the successive chapters. In Chapter 2 we study the continuability and the oscillatoric behaviour of the solutions of superlinear equations of the form (11), and we give an existence and multiplicity theorem with respect to general boundary conditions, which can be written as:

$$\begin{cases}(x(a),\dot{x}(a)) \in \Gamma_a \\ (x(b),\dot{x}(b)) \in \Gamma_b,\end{cases} \tag{18}$$

where Γ_a and Γ_b are two suitable unbounded continua, in the spirit of the class $\mathbb{B}$ of boundary conditions introduced in [116].

We formulate and prove a version of the main Theorem of [31] which gives sufficient conditions for the continuability of the solutions of:

$$\ddot{x} + q(t)G'(x) = p(t, x, \dot{x}),$$

when the weight function q is nonnegative and the growth of p with respect to x and $\dot{x}$ is no more than linear. In particular it turns out that for the continuability of the solutions of (4) it is sufficient that q has locally bounded variation and that is monotone in each left neighbourhood and in each right neighbourhood of its zeros. In the Theorem in [31] just the case of (10) was considered, but its approach is easily generalized to obtain Theorem 2.1.1, which is a result known in literature (see for instance [23] and [24]) even if we did not find an exact reference for its proof. In particular the global continuability of the solutions of (4) holds in an interval where $q \geq 0$, if q is locally of bounded variation and it is monotone in a left neighbourhood and in a right neighbourhood of each of its zeros.

Then we deal with the oscillatoric properties of the solutions of (11), when $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that:

$(f1)$ every solution of $\ddot{x} + f(t,x) = 0$ is defined on $[a, b]$

and:

$(f2)$ there are a nontrivial subinterval $I \subset [a, b]$, a number $x^* > 0$ and a continuous function $g : \mathbb{R} \to \mathbb{R}$, satisfying $(g1)$ and $(g2+)$, such that:

$$f(t,x)\,\mathrm{sign}(x) \geq |g(x)| \qquad \text{for a.e. } t \in I \text{ and } \forall\, |x| \geq x^*;$$

in particular we do not require the uniqueness for the Cauchy problems associated to (11). We show that under this assumptions, the number of turns, that the phase vector $(x(t), \dot{x}(t))$ makes around the origin as t varies in $[a, b]$, tends to infinity as the length of $(x(t_0), \dot{x}(t_0))$ goes to infinity for some $t_0 \in [a, b]$: this is a typical feature of superlinear problems (see [29], [54], [79]). As a consequence, we obtain by a shooting technique the existence of solutions of (11) satisfying boundary conditions like (18) and having an arbitrary, but sufficiently large, number of zeros.

In Chapter 3 the situation in which q is nonpositive and g is superlinear in the equation (4) is considered. When $(g2-)$ holds and $q \leq 0$, $q \not\equiv 0$ in an interval $[a, b]$, there are solutions of (4) which are not continuable over the whole $[a, b]$ and blow up. Therefore we consider the same kind of sets Ω_a^b introduced in [23] and made by all the points $p \in \mathbb{R}^2$ such that the solution starting at $t = a$ from p is defined on $[a, b]$, and we show that, under the assumption $(g2-)$, the intersection of Ω_a^b with every straight line from the origin is bounded.

Moreover we consider also the existence of continua of initial points that give raise to solutions which are defined on $]a, b[$ but present the blow up at a or at b or both. This kind of singular boundary value problems, arising from questions of differential geometry and mathematical physics, dates back to Bieberback [20] and Rademacher [106], who initiated the study of the solutions of:

$$\Delta u = f(u), \quad \text{in } \Omega,$$

such that $u(\mathrm{x}) \to +\infty$ as $\mathrm{dist}(\mathrm{x}, \partial\Omega) \to 0$. Further results were then obtained by Keller [59], Osserman [97], Walter [125], Loewner and Nirenberg [76], Rhee [110] and others (see [15]). More recent contributions and extensions can be found in [12], [13], [30], [35], [65], [66], [87], [130] and the references therein. The study of radially symmetric solutions of:

$$\Delta u = w(|\mathrm{x}|)g(u), \quad \text{in } \Omega,$$

which present the blow-up phenomenon at the boundary of Ω leads to the problem:

$$\begin{cases} \ddot{x} + q(t)g(x) = 0 & \text{in }]a, b[\\ x(a) = x(b) = +\infty \end{cases} \tag{19}$$

in the case of an annular domain, with sign conditions on q corresponding to appropriate sign conditions on $w(r)$ with $r = |\mathrm{x}|$. In [12], [13], [65], [130], the authors considered the situation in which $w(r) > 0$ for all r and this turns out to be equivalent to the sign condition $q(t) < 0$ for all $t \in]0, 1[$. Recently, under the assumption of monotonicity for g, the case of a weight function of constant sign but possibly vanishing on some subset of its domain, was considered too (see [30] and the references therein).

In Section 3.3, we take a superlinear and continuous g, which satifies the sign condition $(g1)$ only in a neighbourhood of infinity and of 0 and an integral condition in 0 in order to have the uniqueness of the constant zero solution (we do not assume any Lipschitz continuity), and a continuous and nonpositive q such that every neighbourhood of b (respectively, of a) contains points in which q is strictly negative: the we find unbounded continua of points p of the plane such that there is a solution of (4) starting at $t = a$ (respectively, at $t = b$) from p, which is defined in $]a, b[$, has either constant sign or exactly one zero in $]a, b[$, and presents the blow-up as $t \to b$ (respectively, as $t \to a$). The technique is based on a limit procedure over sequences of continua which are provided by the Leray–Schauder Continuation Theorem. We remark that, if we assume that $q(t) \leq 0$ for all t in a neighbourhood of a and b, then it turns out that our condition on the sign of q is also necessary for the existence of blow-up solutions. As a first consequence we have that it can be found a positive solution of (19), since we can show that two continua like those mentioned above intersects somewhere, so that we can "glue" a solutions blowing up as $t \to a$ to a solution blowing up as $t \to b$. Since, by standard rescaling procedures, equation (4) can be obtained from ODEs of the form:

$$u''(r) + c(r)u'(r) + h(r)f(u(r)) = 0,$$

our result, in principle, could be applied to the search of radially symmetric solutions of different classes of PDEs (like, e.g., the self-similar solutions for semilinear heat equations [102]).

In a similar manner, in Section 3.2 we suppose that $q \leq 0$ in $]-\infty, a]$ and in $[b, +\infty[$ and we study the existence of unbounded continua of points p in the plane such that the solution starting from p at $t = a$ or at $t = b$ tends to the origin as $t \to -\infty$ or $t \to +\infty$, respectively. Such continua are found using a Ważewski type argument considered by Conley in [32] and assuming that g is a locally Lipschitz continuous function satysfying the sign condition $(g1)$ and q has a primitive $Q(t)$ which is strictly decreasing and such that $Q(\pm\infty) = \mp\infty$.

At this point, in the remaining chapters we consider a changing sign weight function and we prove, following some lines inspired by Butler's approach [23] and like in [121], that various boundary value problems for equation (4) have solutions with precise number of zeros in each interval of positivity of q and, moreover, for each interval of negativity, we can fix a priori if the solution will have exactly one zero, being also strictly monotone, or will have no zeros and exactly one zero of the derivative. As in [23], we combine the oscillatory properties of the solutions, where $q \geq 0$, with the noncontinuability, where $q \leq 0$. In the proof, the main problem that we have to face is to "connect" a solution of (4) having a certain number n_1 of zeros in an interval $]a, b[$, where $q > 0$, to a solution with a possibly different number n_2 of zeros in another interval of positivity $]c, d[$, by passing through the intermediate interval $]b, c[$ where $q < 0$. This program is achieved by finding a continuum Γ in the phase-plane such that all the solutions departing from Γ at the time $t = a$ will be continuable till to $t = d$ and have n_1 zeros in $]a, b[$ and n_2 zeros in $]c, d[$. We can also prescribe that either the solution or its derivative, will vanish exactly once in $]b, c[$.

More precisely, in Chapter 4 we consider "separated" boundary value problems, in the sense that the boundary conditions can be written as two conditions such that each one is to be fulfilled at one of the two endpoints of the interval under consideration. We state and prove a general theorem for a problem of the form (4)–(18): solutions are found with an arbitrarily assigned (but sufficiently large) number of zeros in each interval of positivity of q and with none or one zero (we can choose it) in each interval of negativity. We remark that the minimum number of zeros in each interval of positivity depends only on the behaviour of q in that interval and in the two adjacent intervals of negativity: in this way the number of zeros in every interval of positivity can be chosen indipendently of the others. Considering in place of Γ_a and Γ_b in (18) the continua found in Chapter 3 for the solutions which blow up or go to zero, as well as the straight line $x = 0$ in the phase plane, we deduce the existence of solutions of (4) which have nodal behaviour precisely described as above and, moreover, are homoclinic to zero, or blow up at the endpoints of the interval, or satisfy the two-point boundary condition $x(a) = x(b) = 0$ or any mixture of these conditions.

On the other hand, in Chapter 5 the Floquet–type boundary condition (13) is considered and, again, solutions with precise distribution of zeros are found improving the result in [99]. In particular this applies to the existence of periodic solutions for (4) and for (14).

Finally, in Chapter 6 the case is considered in which the weight function $q :]a, b[\to \mathbb{R}$ possesses a doubly infinite sequence of intervals $[t_{2k}, t_{2k+1}]$, $k \in \mathbb{Z}$, where it is negative, while it is positive in the remaining intervals $[t_{2k+1}, t_{2k+2}]$, $k \in \mathbb{Z}$. Clearly the set of initial conditions, that produce solutions which are continuable in a certain interval I, becomes smaller and smaller as I is increased up to contain a greater number of consecutive intervals of negativity of q. Thus we are interested in the existence of solutions which are defined in the whole $]a, b[$, have a chosen number of zeros in the intervals of positivity and have either one or none zeros in the intervals of negativity. Therefore we choose a suitable sequence $\{n_k\}_{k\in\mathbb{Z}}$ of positive integers to represent the number of zeros in the intervals of positivity $[t_{2k+1}, t_{2k+2}]$ and another sequence $\{\delta_k\}_{k\in\mathbb{Z}}$, with $\delta_k \in \{0, 1\}$, to specify the number of zeros in the intervals of negativity $[t_{2k}, t_{2k+1}]$. Under the usual assumptions of superlinearity $(g2+)$ and $(g2-)$ on g and the sign condition $(g1)$, we prove that there are at least two solutions of (4) which are defined in $]a, b[$ and have nodal behaviour described by the two fixed sequences. This goal is reached considering for every $j \in \mathbb{N}$ the set Ω_j of initial conditions that give raise to solutions continuable on, say, $[t_{-2j}, t_{2j}]$ and, there, have the desired distribution of zeros: we show that the intersection of all these sets is nonempty and this provides the existence of solutions globally defined and with prescribed nodal properties.

Moreover, this result, when applied to the case of a ω-periodic weight function q, helps in showing the chaotic feature of equation (4), since it allows to define a topological semiconjugation between the Poincaré map $\Pi : (x(0), \dot{x}(0)) \mapsto (x(\omega), \dot{x}(\omega))$ and the Bernoulli shift on sequences of a suitable set of symbols. The literature in the this field is very reach. For the classical theory we referred to the books [34], [58], [91] and [105] and to the paper by Séré [111], and we just mention some recent works in this field, like [14], [113] and [115]: in [14] the existence of a transversal homoclinic orbit for the Poincaré map associated to a Duffing equation is studied and in [115] some geometric conditions for detection of chaos are given. There are also many works dealing with chaotic dynamics in the framework of variational methods: for more comments and a comprehensive bibliography, we refer to the thesis of Berti [19].

About our equation, we show the existence of an uncountable set of bounded and non-periodic solutions of (4) and, as a consequence of the topological semiconjugation with the Bernoulli shift, we obtain the positivity of the topological entropy of the Poincaré map, a fact whose importance in the detection of chaos has been pointed out in [50] and [113] and which is not deducible from the other calssical axioms of chaos in [34]. In particular, in the introduction of [113] a ω-periodic equation is called *chaotic* if the Poicaré map Π, restricted to a compact and invariant set, is semiconjugated to a suitable shift map and, moreover, "the counterimage (by the semiconjugacy) of any periodic point of the shift contains a periodic point of the Poincaré map". Actually this is what happens, when q is ω-periodic in (4), as a consequence of the theorem on the existence of periodic solutions with prescribed nodal properties which we obtain in Chapter 5. We underline also that, while various papers treat problems of perturbative type and the chaos is detected for particular values of some parameter, our result is of global nature and involves a very simple equation as (4).

We discuss here briefly some possible extensions of our results. At first we observe that every theorem we obtain for equation (4) in Chapters 4, 5 and 6, could be given for an equation of the form (11) under suitable assumptions on f like those in [23]. As an example, we could deal with:

$$f(t,x) = q^+(t)f_1(x) - q^-(t)f_2(x),$$

with $f_i(s)s > 0$ for $s \neq 0$ and f_1 and f_2 satisfying $(g2+)$ and $(g2-)$, respectively. Similar kind of nonlinearities, with $f_1(s) = s^\alpha$ and $f_2(s) = s^\beta$, have been recently considered in [4] for the study of positive solutions of an elliptic problem.

Concerning $g(x)$, we note that, by the use of mollifiers and proving the fact that the solutions of (4) with fixed nodal properties will be subjected to a priori bounds which are uniform with respect to perturbations of g which are small in the compact-open topology, it is possible to check that the condition of local lipschitzianity for g can be dropped and the continuity of g (paired by an upper bound for $g(x)/x$ in a neighbourhood of zero) is enough to prove all our results.

Moreover, all the results we proved in Chapters 2–6 for (4) can be obtained for the p-Laplacian scalar ODE:

$$(\phi_p(u'))' + q(t)g(u) = 0,$$

with $\phi_p(s) = |s|^{p-2}s$, for $p > 1$ or even for a more general ϕ-Laplacian scalar ODE of the form:

$$(\phi(u'))' + q(t)g(u) = 0,$$

with $\phi : \mathbb{R} \to \mathbb{R}$ an odd increasing homeomorphism satisfying suitable upper and lower σ-conditions (see [78]), provided we modify accordingly the growth assumptions on g and its primitive G. For instance, in the case of the p-Laplace operator, the condition $(g0)$ used in Section 3.3, in order to have the uniqueness of the zero solution, has to be replaced by:

$$(g_0^p) \qquad \int_{-\alpha_0}^{\alpha_0} G(s)^{-1/p}\, ds = +\infty.$$

In the same manner, the time-maps used in conditions $(g2+)$ and $(g2-)$ have to be replaced by:

$$\tau^+_{g,p}(e) = k_p \int_{G_-^{-1}(e)}^{G_+^{-1}(e)} \frac{1}{[e - G(s)]^{1/p}}\, ds$$

and:

$$\tau^-_{g,p}(e) = \begin{cases} k_p \displaystyle\int_{-\infty}^{+\infty} \frac{1}{[e + G(s)]^{1/p}}\, ds & \text{if } e > 0 \\ k_p \displaystyle\int_{-\infty}^{G_-^{-1}(-e)} \frac{1}{[G(s) + e]^{1/p}}\, ds + k_p \int_{G_+^{-1}(-e)}^{+\infty} \frac{1}{[G(s) + e]^{1/p}}\, ds & \text{if } e < 0, \end{cases}$$

for a suitable constant k_p, so that now the conditions of superlinear growth at infinity become:

$$(gp+) \qquad \lim_{e \to +\infty} \tau^+_{g,p}(e) = 0$$

and:

$$(gp-) \qquad \lim_{e \to \pm\infty} \tau^-_{g,p}(e) = 0.$$

Note that in this case, sufficient conditions ensuring the fact that the time-maps are infinitesimal at infinity are:

$$(20) \qquad \lim_{x \to \pm\infty} \frac{g(x)}{|x|^{p-2}x} = +\infty, \qquad \left| \int^{\pm\infty} G(s)^{-1/p}\, ds \right| < +\infty, \quad \text{and} \quad \liminf_{x \to \pm\infty} \frac{G(\sigma x)}{G(x)} > 1,$$

for some $\sigma > 1$ (cf. [87] and also Remark 2.0.5 and Section 3.4). Like for the case $p = 2$, also here, the condition (20) may be reduced just to the convergence of the integral of $G(s)^{-1/p}$ at infinity when g is monotone in a neighbourhood of infinity. As a consequence we can obtain existence and multiplicity of radial solutions of boundary value problems associated to the equation:

$$\Delta_p u + q(|\mathrm{x}|)g(u) = 0$$

in an annular domain.

Chapter 1 appears in [98], while Chapter 5 will be contained in [100]. The paper [101] in collaboration with F. Zanolin is divided into Chapter 4 and Sections 3.1, 3.2 and 3.4. The remaining Section 3.3 of Chapter 3 is part of the paper [84] in collaboration with J. Mawhin and F. Zanolin and the work [26] with A. Capietto and W. Dambrosio forms Chapter 6.

Chapter 1

A class of Liénard equations

Here, we are mainly concerned with the existence of periodic solutions for a second order nonlinear equation of the form:

$$\ddot{x} + f(x)\dot{x} + g(x) = e(t), \tag{1.1}$$

where e is T-periodic and:

$$f \sim x|x|^{\alpha-1} \qquad \text{and} \qquad g \sim x|x|^{\beta-1}.$$

A typical example is given by the equation:

$$\ddot{x} + A|x|^{\alpha-1}x\dot{x} + B|x|^{\beta-1}x = e(t). \tag{1.2}$$

This model can be considered as a generalization of the example considered by M. Struwe in [116].

We prove the existence of at least one T-periodic solution for a class of equations which includes (1.2) and for an arbitrary forcing term e. Actually we study the plane system:

$$\begin{cases} \dot{x} = y - F(x) + E(t,x,y) \\ \dot{y} = -g(x) + p(t,x,y) \end{cases} \tag{1.3}$$

which includes equation (1.1) as a particular case. Our result shows that, if E and p are bounded, $F(x) \sim |x|^{\alpha+1}$ and $g(x) \sim |x|^{\beta-1}x$, with $\beta > 2\alpha + 1$, then (1.3) has at least one T-periodic solution. The condition:

$$\beta > 2\alpha + 1$$

implies that the autonomous system:

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \tag{1.4}$$

defines a center in a neighbourhood of infinity, as it follows from the works by Filippov [45], Opial [95] and Hara and Yoneyama [53]; therefore the problem concerns with the study of a center subjected to periodic perturbations which are not necessarily small.

We cannot expect a multiplicity result, like in [72] and [74], since we are dealing with a general term E. To show this, let us consider again the autonomous system (1.4), where we now assume that F and g are C^1 functions, $g(x)x > 0$ for all $x \neq 0$ and $F(0) = 0$; moreover we suppose that the origin is a global center for system (1.4) (sufficient conditions to ensure that such a situation happens are given, for instance, in [53]).

By theorem 1.1 in [86] there exists a continuous first integral $I : \mathbb{R}^2 \to [0, +\infty)$ for (1.4) which has an isolated minimum at $(0,0)$, is of class C^1 out of the origin and $|\nabla I(x,y)| \neq 0$ if $(x,y) \neq (0,0)$. Hence every orbit of (1.4) is a level set of I and vice versa. This in particular means that:

$$I_x(x,y)(y - F(x)) + I_y(x,y)(-g(x)) = 0 \qquad \forall (x,y) \neq (0,0). \tag{1.5}$$

By this identity it is easy to show that:

$$I_x(x,y) = 0 \iff x = 0 \text{ and } y \neq 0;$$

then $I_x(x,y)x > 0$ for every $x \neq 0$. Let us consider a smooth and bounded function $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that:

- $E(t,0) = 0, \ \forall t \in \mathbb{R}$;
- $E(t+T,x) = E(t,x), \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}$;
- $E(t,x)x > 0, \ \forall x \neq 0, \forall t \in \mathbb{R}$;

for instance $E(t,x) = (2 + \sin t) \arctan x$. We claim that the perturbed system:

$$\begin{cases} \dot{x} = y - F(x) - E(t,x) \\ \dot{y} = -g(x) \end{cases} \tag{1.6}$$

has the only trivial T-periodic solution $(x,y) \equiv (0,0)$ (this has been already proved in [118] with more general assumptions, if E does not depend on t). In fact, let us suppose that (x,y) is another T-periodic solution of (1.6); then $(x(t),y(t)) \neq 0$ for every $t \in [0,T]$ and $I(x(\cdot),y(\cdot))$ is differentiable on $[0,T]$; thus we are allowed to write:

$$\begin{aligned} 0 &= I(x(T),y(T)) - I(x(0),y(0)) \\ &= \int_0^T \frac{d}{dt} I(x(t),y(t)) dt \\ &= \int_0^T \{I_x(x(t),y(t))[y(t) - F(x(t)) - E(t,x(t))]\} \, dt + \int_0^T I_y(x(t),y(t))[-g(x(t))] dt \\ &= -\int_0^T I_x(x(t),y(t)) E(t,x(t)) dt; \end{aligned}$$

on the other hand we have:

$$I_x(x(t),y(t))E(t,x(t)) \geq 0 \qquad \forall t \in [0,T].$$

Hence:

$$I_x(x(t),y(t))E(t,x(t)) = 0 \qquad \forall t \in [0,T].$$

This is possible only if $x \equiv 0$; from the first equation in (1.6) we get that also $y \equiv 0$, but we have already excluded this situation. Hence the origin is the unique T-periodic solution of (1.6). Using Liapunov's stability theory (see for instance [52, section X.3, theorem 3.2]) it is possible to prove that actually the origin is uniformly asymptotically stable for the system (1.6).

The main tool to obtain the existence of a periodic solution of (1.1) is a continuation theorem by Capietto, Mawhin and Zanolin [28, Theorem 2] which we state in Section 1.2 for imediate reference. In order to apply this theorem, in the first section we develop some estimates which need enough regularity and give some useful a priori bounds for periodic solutions of (1.1) which have a fixed number of zeros. All the proofs are developed for the model nonlinearities used in (1.2); however a wider class of functions, to which the results of Section 1.1 can be easily extended, is discussed in Remark 1.1.8: roughly speaking, it consists of regular functions such that they and their derivatives grows at infinity like powers.

We finally remark that the analysis carried on in the first section allows us to find also infinitely many solutions for Sturm-Liouville type boundary value problems for equation (1.1) (see Remark 1.2.3).

1.1 Study of the system

We want to determine some properties of the solutions of the system:

$$\begin{cases} \dot{x} = y - \dfrac{\lambda A}{\alpha+1}|x|^{\alpha+1} + E(t,x,y;\lambda) \\ \dot{y} = -B|x|^{\beta-1}x + p(t,x,y;\lambda) \end{cases} \tag{1.7}$$

where $\lambda \in [0,1]$, $A \geq 0$, $B > 0$ and $p, E : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ are bounded functions. It will be useful in the sequel to perform the following change of variable due to R. Conti [33] (see also [8] for another application):

$$z = \Psi(x) = \sqrt{2G(x)}\,\mathrm{sign}(x) = \sqrt{\frac{2B}{\beta+1}}|x|^{\frac{\beta+1}{2}}\,\mathrm{sign}(x), \tag{1.8}$$

where:

$$G(x) = \int_0^x g(s)ds$$

and in our case $g(x) = B|x|^{\beta-1}x$. We note that Ψ is strictly monotone increasing as $x \neq 0$, while $\Psi'(0) = 0$, so we will have to be careful in handling this case; let Φ be the inverse function of Ψ, that is:

$$\Phi(z) = c_1|z|^{\frac{2}{\beta+1}}\,\mathrm{sign}(z),$$

where we set $c_1 = \left(\frac{\beta+1}{2B}\right)^{\frac{1}{\beta+1}}$. Then the new system is :

$$\begin{cases} \dot{z} = c_0|z|^{\frac{\beta-1}{\beta+1}}\left[y - c_2|z|^{2\frac{\alpha+1}{\beta+1}} + \widehat{E}(t,z,y;\lambda)\right] \\ \dot{y} = -c_0|z|^{\frac{\beta-1}{\beta+1}}z + \widehat{p}(t,z,y;\lambda) \end{cases} \tag{1.9}$$

where we have defined:

$$\begin{aligned} \widehat{E}(t,z,y;\lambda) &= E(t,\Phi(z),y;\lambda) \\ \widehat{p}(t,z,y;\lambda) &= p(t,\Phi(z),y;\lambda) \\ c_0 &= B\left(\frac{\beta+1}{2B}\right)^{\frac{\beta}{\beta+1}} \\ c_2 &= \left(\frac{\beta+1}{2B}\right)^{\frac{\alpha+1}{\beta+1}}\frac{\lambda A}{\alpha+1}. \end{aligned}$$

We will carry on the analysis of the solutions of the system (1.9) and then we will state the corresponding results for the original system (1.7). If $(z(\cdot), y(\cdot))$ is a solution of (1.9), we introduce the polar coordinates:

$$\left\{ \begin{array}{l} y(t) = \rho(t)\cos\theta(t) \\ z(t) = \rho(t)\sin\theta(t) \end{array} \right. \tag{1.10}$$

for all the t in which $(z(t), y(t)) \neq (0,0)$; then the following formulas hold:

$$\dot{\theta} = \frac{y\dot{z} - \dot{y}z}{\rho^2} \qquad \text{and} \qquad \dot{\rho} = \frac{y\dot{y} + z\dot{z}}{\rho}. \tag{1.11}$$

Lemma 1.1.1 *Assume that:*

$$\alpha \geq 0 \qquad \textit{and} \qquad \beta > 2\alpha + 1; \tag{1.12}$$

then for every $a, b \in \mathbb{R}$ with $a < b$ there exist positive real numbers $\widehat{\rho}, \mu$ and ν such that every solution of (1.9) with $\rho(t_0) > \widehat{\rho}$ and $t_0 \in [a, b]$ is defined in the whole interval $[a, b]$ and satisfies:

$$\mu\rho(t_0) \leq \rho(t) \leq \nu\rho(t_0) \qquad \forall t \in [a, b]. \tag{1.13}$$

Proof. Let us consider a solution (z, y) of the system (1.9) with $(z(t_0), y(t_0)) = (z_0, y_0)$ for some $t_0 \in [a, b]$; let I_M be its maximal interval of prolungability and $\rho(\cdot)$ and $\theta(\cdot)$ its radial and angular functions with respect to the polar coordinates (1.10); if t is such that $\rho(t) \neq 0$ then $\theta(t)$ is well defined and the first formula in (1.11) leads to:

$$\dot{\theta}(t) = c_0|z(t)|^{1-\frac{2}{\beta+1}}\left[1 - \frac{c_2|z(t)|^{2\frac{\alpha+1}{\beta+1}}y(t)}{\rho(t)^2} + \frac{y(t)\widehat{E}}{\rho(t)^2} - \frac{z(t)\widehat{p}}{c_0\rho(t)^2|z(t)|^{1-\frac{2}{\beta+1}}}\right], \tag{1.14}$$

where $\widehat{E}$ and $\widehat{p}$ are evaluated at $(t, z(t), y(t); \lambda)$.

Since $\widehat{E}$ and $\widehat{p}$ are uniformly bounded and we are assuming (1.12), there exists a ρ^*, which depends only on α, β, A, B and the bounds of $\widehat{E}$ and $\widehat{p}$, such that the term in the square brackets is greater than a small fixed positive number—say 1/2—if $\rho(t) > \rho^*$; hence for all those t ranging in the set $I^+ = \{t \in I_M : \rho(t) > \rho^*\}$ we have that $\dot{\theta}(t) \geq 0$ and $\dot{\theta}(t) = 0$ only if also $z(t) = 0$. This means that the orbits of (1.9), which lie sufficiently far from the origin, turn around it in clockwise sense of rotation in the (z, y)-plane until they remain at a greater distance from $(0,0)$ than ρ^*. Moreover, if we define $D = \{t \in I^+ : \dot{\theta}(t) = 0\}$, we claim that we can choose ρ^* in such a way that D is made of isolated points in I^+. In fact let us pick a $t_1 \in D$; this means that $z(t_1) = x(t_1) = 0$, $\theta(t_1) = k\pi$, for some $k \in \mathbb{Z}$, and $|y(t_1)| = \rho(t_1) > \rho^*$; thus, if ρ^* is taken sufficiently large, $\dot{x}(t_1) \neq 0$ by the first equation of the system (1.7) and, by continuity, x—and then z, too— will be different from 0 in a neighbourhood of t_1; so $\dot{\theta}$ is different from zero in a neighbourhood of t_1.By these considerations we infer that ρ can be seen as a well defined function of θ at least along the arcs of the trajectory which correspond to the set I^+, and the formula:

$$\left.\frac{d\rho}{d\theta}(\theta)\right|_{\theta=\theta(t)} = \frac{\dfrac{d\rho}{dt}(t)}{\dfrac{d\theta}{dt}(t)}$$

holds for every $t \in I^+ \setminus D$. With an abuse of notation we will write:

$$\frac{d\rho}{d\theta}(t) \qquad \text{instead of} \qquad \frac{d\rho}{d\theta}(\theta)\bigg|_{\theta=\theta(t)}.$$

We now perform an analysis of these derivatives the aim of which is to show that ρ^* can be chosen in such a way that $\frac{d\rho}{d\theta}$ is actually bounded on $I^+ \setminus D$.

From (1.11) we can easily get:

$$\begin{aligned}\frac{d\rho}{dt} &= \frac{-c_3|z|^{1+\frac{2\alpha}{\beta+1}}z + c_0|z|^{1-\frac{2}{\beta+1}}z\widehat{E} + y\widehat{p}}{\rho} \\ &= \frac{-c_3\rho^{2+\frac{2\alpha}{\beta+1}}|\sin\theta|^{1+\frac{2\alpha}{\beta+1}}\sin\theta}{\rho} + \frac{c_0\rho^{2-\frac{2}{\beta+1}}\widehat{E}|\sin\theta|^{1-\frac{2}{\beta+1}}\sin\theta}{\rho} + \frac{\widehat{p}\rho\cos\theta}{\rho}\end{aligned}$$

with $c_3 = c_0c_2 = (\frac{\beta+1}{2B})^{1+\frac{\alpha}{\beta+1}}\frac{\lambda AB}{\alpha+1}$ and then:

$$\frac{d\rho}{d\theta} = \frac{\dfrac{-c_2\rho^{1+2\frac{\alpha+1}{\beta+1}}|\sin\theta|^{2\frac{\alpha+1}{\beta+1}}\sin\theta}{\rho} + \widehat{E}\sin\theta + \dfrac{\widehat{p}\cos\theta}{c_0\rho^{1-\frac{2}{\beta+1}}|\sin\theta|^{1-\frac{2}{\beta+1}}}}{1 - \dfrac{c_2\rho^{1+2\frac{\alpha+1}{\beta+1}}|\sin\theta|^{2\frac{\alpha+1}{\beta+1}}\cos\theta}{\rho^2} + \dfrac{\widehat{E}\cos\theta}{\rho} - \dfrac{\widehat{p}\sin\theta}{c_0\rho^{2-\frac{2}{\beta+1}}|\sin\theta|^{1-\frac{2}{\beta+1}}}}$$

on the set $I^+ \setminus D$. From now on the fact that those formulas *hold on* $I^+ \setminus D$ means that both ρ and θ are evaluated at $t \in I^+ \setminus D$, while $\widehat{E}$ and $\widehat{p}$ at $(t, \rho(t)\sin\theta(t), \rho(t)\cos\theta(t); \lambda)$. A direct computation, using the expression above and assumption (1.12), shows that, after a possible rearrangement of the constant ρ^*, we can write:

$$\begin{aligned}\frac{d\rho}{d\theta} &= \frac{\dfrac{-c_2\rho^{1+2\frac{\alpha+1}{\beta+1}}|\sin\theta|^{2\frac{\alpha+1}{\beta+1}}\sin\theta}{\rho}}{1 - \dfrac{c_2\rho^{1+2\frac{\alpha+1}{\beta+1}}|\sin\theta|^{2\frac{\alpha+1}{\beta+1}}\cos\theta}{\rho^2}} + K_0(\rho,\theta,t) \\ &= S_0(\rho,\theta) + K_0(\rho,\theta,t),\end{aligned} \tag{1.15}$$

with $K_0(\rho(t),\theta(t),t)$ uniformly bounded for $t \in I^+ \setminus D$ and:

$$S_0(\rho,\theta) = -\frac{c_2\rho^{1+2\frac{\alpha+1}{\beta+1}}|\sin\theta|^{2\frac{\alpha+1}{\beta+1}}\sin\theta}{\rho - c_2\rho^{2\frac{\alpha+1}{\beta+1}}|\sin\theta|^{2\frac{\alpha+1}{\beta+1}}\cos\theta}.$$

Now let $R_1(\rho,\theta)$ be any primitive of S_0 with respect to θ, say:

$$R_1(\rho,\theta) = \int_0^\theta S_0(\rho,\vartheta)d\vartheta, \tag{1.16}$$

so that R_1 is even and 2π-periodic with respect to θ. By (1.15) and (1.16) we obtain:

$$\begin{aligned}\frac{d}{d\theta}[\rho - R_1(\rho,\theta)] &= \frac{d\rho}{d\theta} - \frac{\partial R_1}{\partial\rho}\frac{d\rho}{d\theta} - \frac{\partial R_1}{\partial\theta} \\ &= -\frac{\partial R_1}{\partial\rho}S_0 - \frac{\partial R_1}{\partial\rho}K_0 + K_0 \qquad \text{on } I^+ \setminus D.\end{aligned}$$

Since S_0 is a rational function of real powers of ρ for fixed θ, we can say that the *degree* of S_0 is $2\frac{\alpha+1}{\beta+1}$, in the sense that S_0 behaves like $\rho^{2\frac{\alpha+1}{\beta+1}}$ as ρ tends to ∞; for short we will write $\deg S_0 = 2\frac{\alpha+1}{\beta+1}$. Then $\deg R_1 = \deg S_0$ by (1.16) and the degree will decrease by 1 after differentiating once with respect to ρ, while it remains unchanged if an integration with respect to θ is performed; so $\deg \frac{\partial R_1}{\partial \rho} = \deg S_0 - 1 < 0$ by assumption (1.12) and, therefore, $\frac{\partial R_1}{\partial \rho}$ is bounded for $\rho > \rho^*$ (after a suitable modification of ρ^*); this suggests to let:

$$K_1(\rho,\theta,t) = -\frac{\partial R_1}{\partial \rho}(\rho,\theta)K_0(\rho,\theta,t) + K_0(\rho,\theta,t),$$

so that $K_1(\rho(t),\theta(t),t)$ is also bounded as t ranges in $I^+\setminus D$. Moreover we have $\deg(\frac{\partial R_1}{\partial \rho}S_0) = 2\deg(S_0) - 1$, hence:

$$\deg\left[\frac{d}{d\theta}(\rho - R_1)\right] = \deg\left(\frac{d\rho}{d\theta}\right) - \tau,$$

where we set $\tau = \tau(\alpha,\beta) = \frac{\beta-2\alpha-1}{\beta+1} > 0$.

If we define:

$$S_1(\rho,\theta) = -\frac{\partial R_1}{\partial \rho}(\rho,\theta)S_0(\rho,\theta) \qquad \text{and} \qquad R_2(\rho,\theta) = \int_0^\theta S_1(\rho,\vartheta)d\vartheta,$$

we obtain:

$$\frac{d}{d\theta}(\rho - R_1) = S_1 + K_1 \qquad \text{on } I^+\setminus D$$

and:

$$\frac{d}{d\theta}(\rho - R_1 - R_2) = -\frac{\partial R_2}{\partial \rho}S_0 + \frac{\partial R_2}{\partial \rho}K_0 + K_1 \qquad \text{on } I^+\setminus D.$$

Arguing as above we find that:

$$K_2(\rho,\theta,t) = \frac{\partial R_2}{\partial \rho}(\rho,\theta)K_0(\rho,\theta,t) + K_1(\rho,\theta,t)$$

is bounded when ρ and θ are evaluated on $I^+\setminus D$ and $\deg(-\frac{\partial R_2}{\partial \rho}S_0) = 3\deg S_0 - 2$; thus:

$$\deg\left[\frac{d}{d\theta}(\rho - R_1 - R_2)\right] = 1 - 2\tau.$$

We repeat this procedure n times, until we have:

$$\deg\left[\frac{d}{d\theta}\left(\rho - \sum_{i=1}^n R_i\right)\right] = 1 - n\tau < 0 \tag{1.17}$$

and $\deg R_i < 1$ for all $i = 1,\ldots,n$; thus, if we define:

$$\mathcal{T}(\rho,\theta) = \sum_{i=1}^n R_i(\rho,\theta),$$

we obtain that:

$$\left|\frac{d}{d\theta}\left[\rho - \mathcal{T}(\rho,\theta)\right]_{\theta=\theta(t)}\right| \leq L \qquad \forall t \in I^+\setminus D, \tag{1.18}$$

where L depends on ρ^*. Let I be the open connected component of I^+ such that $t_0 \in I$; inequality (1.18) holds almost everywhere on I, since D is a discrete subset of I^+, then we can integrate it between $\theta(t)$, with $t \in I$, and $\theta(t_0)$ with the following result:

$$|\rho(t) - \rho(t_0)| \leq |\mathcal{T}(\rho(t), \theta(t))| + |\mathcal{T}(\rho(t_0), \theta(t_0))| + L|\theta(t) - \theta(t_0)|$$

for every $t \in I$. Since $\deg(R_i) < 1$ for every $i = 1, \dots, n$, we have that $|\mathcal{T}(\rho, \theta)| \leq \frac{1}{2}\rho$ for every $\rho > \rho^*$ and uniformly with respect to θ, up to a modification of ρ^*; then we obtain:

$$\frac{1}{3}\rho(t_0) - \frac{2L}{3}|\theta(t) - \theta(t_0)| \leq \rho(t) \leq 3\rho(t_0) + 2L|\theta(t) - \theta(t_0)| \qquad \forall t \in I. \tag{1.19}$$

From (1.14) we can easily deduce that $|\dot{\theta}| \leq c\rho$ when ρ is sufficiently large, say $\rho > \rho^*$; then we can integrate also this last equation between t and t_0 and get that:

$$|\theta(t) - \theta(t_0)| \leq c\left|\int_{t_0}^{t} \rho(s)ds\right| \qquad \forall t \in I.$$

This and (1.19) lead to:

$$\frac{1}{3}\rho(t_0) - \frac{2L'}{3}\left|\int_{t_0}^{t} \rho(s)ds\right| \leq \rho(t) \leq 3\rho(t_0) + 2L'\left|\int_{t_0}^{t} \rho(s)ds\right| \qquad \forall t \in I, \tag{1.20}$$

with $L' = Lc$. Now by Gronwall's lemma we deduce that:

$$\frac{1}{3}\rho(t_0)e^{-2L'|t-t_0|} \leq \rho(t) \leq 3\rho(t_0)e^{\frac{2}{3}L'|t-t_0|}, \tag{1.21}$$

for every $t \in I$.

If we define:

$$\mu = \frac{1}{3}e^{-2L'(b-a)} \qquad \text{and} \qquad \nu = 3e^{\frac{2}{3}L'(b-a)},$$

the inequalities in (1.21) imply that:

$$\mu\rho(t_0) \leq \rho(t) \leq \nu\rho(t_o) \tag{1.22}$$

for every $t \in I \cap [a, b]$.

We have only to prove that actually $[a, b] \subset I$ if $\rho(t_0)$ is chosen large enough. In particular let us take a $\widehat{\rho} > \rho^*/\mu$ and let $t_1 < t_2$ be the endpoints of $I \cap [a, b]$. By (1.22) we have that:

$$\rho(t_1) \geq \mu\rho(t_0) \geq \mu\widehat{\rho} > \rho^* \quad \text{and} \quad \rho(t_2) \geq \mu\rho(t_0) \geq \mu\widehat{\rho} > \rho^*;$$

then t_1 and t_2 are contained in I; thus the only possibility is that $t_1 = a$ and $t_2 = b$. ■

Remark 1.1.2 As it happens with the KAM theory, also here the regularity of the functions involved in the differential equations helps in proving the prolungability of the solutions of the initial value problems. In fact in the previous proof each function S_i is obtained by differentiating with respect to ρ the function R_i, which is built starting from S_{i-1}; hence we performed n successive differentiation to arrive to our goal, where n is estimated by the inequality (1.17) and depends on α and β.

Thanks to the continuability of sufficiently large solution we can prove that the solutions of our system satisfy the so–called "elastic property" (see also Lemma 2.2.2).

Corollary 1.1.3 *Let us suppose that $\beta > 2\alpha + 1$ and $a < b$; then the following two facts hold:*

(*i*) *for every $\rho_1 > 0$ there exists $\rho_2 \geq \rho_1$ such that, if $t_0 \in [a,b]$ and $\rho(t_0) \geq \rho_2$, then $\rho(t) \geq \rho_1$ for all $t \in [a,b]$;*

(*ii*) *for every $\rho_1 > 0$ there exists $\rho_2 \geq \rho_1$ such that, if $t_0 \in [a,b]$ and $\rho(t_0) \leq \rho_1$, then $\rho(t) \leq \rho_2$ for all $t \in [a,b]$.*

Proof. If we choose

$$\rho_2 > \max\left\{\widehat{\rho}, \frac{\rho_1}{\mu}\right\},$$

assertion (*i*) follows from the first inequality in (1.13), while assertion (*ii*) is easily deduced from (*i*) by a contradiction argument. ■

The next corollary is simply a rephrasing for the system (1.7) of what we have just proved and is a straightforward consequence of the properties of the change of variable we performed at the beginning of this section; then it is convenient to introduce here the polar coordinates:

$$\begin{cases} x(t) = r(t)\sin\omega(t) \\ y(t) = r(t)\cos\omega(t) \end{cases}$$

for each solution (x,y) of (1.7).

Corollary 1.1.4 *Let us suppose that $\beta > 2\alpha + 1$ and $a < b$; then there exist positive real numbers $\widehat{r}, M$ and N such that every solution of (1.7) with $r(t_0) > \widehat{r}$ and $t_0 \in [a,b]$ is defined in the whole interval $[a,b]$ and satisfies:*

$$Mr(t_0) \leq r(t) \leq Nr(t_0) \qquad \forall t \in [a,b].$$

Moreover the following two facts hold:

(*i*) *for every $r_1 > 0$ there exists $r_2 \geq r_1$ such that, if $t_0 \in [a,b]$ and $r(t_0) \geq r_2$, then $r(t) \geq r_1$ for all $t \in [a,b]$;*

(*ii*) *for every $r_1 > 0$ there exists $r_2 \geq r_1$ such that, if $t_0 \in [a,b]$ and $r(t_0) \leq r_1$, then $r(t) \leq r_2$ for all $t \in [a,b]$.*

The next results are about the number of rotations made by the orbits around the origin. It is shown that the orbits turn faster and faster as their distance from the origin increases; then they must be bounded if we fix their number of rotations.

Lemma 1.1.5 *Let us assume that (1.12) holds and $a < b$. If $\{(z_k, y_k)\}$ is a sequence of solutions of (1.9) such that:*

$$\lim_{k\to+\infty} \rho_k(t_k) = +\infty,$$

with $t_k \in [a,b]$, then:

$$\lim_{k\to+\infty} [\theta_k(b) - \theta_k(a)] = +\infty.$$

Proof. By corollary (1.1.3) for every $N > 0$ there exists $R_N \geq N$ such that, if we have a solution with $\rho(t_0) \geq R_N$ for some $t_0 \in [a, b]$, then $\rho(t) \geq N$ for all $t \in [a, b]$. In particular there exists k_N such that $\rho_k(t_k) \geq R_N$ for all $k \geq k_N$; then $\rho_k(t) \geq N$ for all $t \in [a, b]$ and for all $k \geq k_N$. This implies that actually:

$$\lim_{k \to +\infty} \left[\inf_{t \in [a,b]} \rho_k(t) \right] = +\infty. \tag{1.23}$$

Let us now give a look at the behaviour of the angular function θ along the orbits which lie far from the origin. By formula (1.14) and our assumptions we can write:

$$\dot{\theta}(t) = c_0[1 + \eta(\rho(t), \theta(t), t)][\rho(t)]^{1-\delta} |\sin \theta(t)|^{1-\delta},$$

where $\delta = \frac{2}{\beta+1} \in (0, 1)$ and $\eta(\rho, \theta, t)$ is a function which tends to zero uniformly with respect to θ and t as ρ tends to infinity. Hence there is a positive number ρ^* such that:

$$\eta(\rho, \theta, t) \geq -\frac{1}{2} \qquad \forall \rho \geq \rho^*, \quad \forall t \in [a, b], \quad \forall \theta.$$

By (1.23) and the previous two relations we have that:

$$\frac{\dot{\theta}_k(t)}{|\sin \theta_k(t)|^{1-\delta}} \geq \frac{c_0}{2} [\rho_k(t)]^{1-\delta} \qquad \forall t \in [a, b],$$

for all k which are large enough. Now we integrate on the interval $[a, b]$ and, after a change of variables, we obtain:

$$\int_{\theta_k(a)}^{\theta_k(b)} \frac{d\theta}{|\sin \theta|^{1-\delta}} \geq \frac{c_0}{2} \int_a^b [\rho_k(t)]^{1-\delta} dt, \tag{1.24}$$

if k is sufficiently large. Letting k go to infinity, we have:

$$\lim_{k \to +\infty} \int_{\theta_k(a)}^{\theta_k(b)} \frac{d\theta}{|\sin \theta|^{1-\delta}} = +\infty.$$

Since $1 - \delta \in (0, 1)$, the integrand is integrable on bounded intervals; then the thesis must follow. ∎

An easy consequence of this last lemma is the following corollary.

Corollary 1.1.6 *If* (1.12) *holds, then for all $a < b$, for all $\sigma \in [0, 2\pi)$ and for all $n \in \mathbb{N}$ there exists a constant K such that every solution of* (1.9) *with $\theta(b) - \theta(a) = \sigma + 2n\pi$ satisfies $\rho(t) \leq K$ for all $t \in [a, b]$.*

Now we translate this last property for the system (1.7).

Corollary 1.1.7 *If* (1.12) *holds, then for all $a < b$, for all $\sigma \in [0, 2\pi)$ and for all $n \in \mathbb{N}$ there exists a constant K such that every solution of* (1.7) *with $\omega(b) - \omega(a) = \sigma + 2n\pi$ satisfies $r(t) \leq K$ for all $t \in [a, b]$.*

Proof. Let us argue by contradiction: then we can find a sequence $\{(x_k, y_k)\}$ of solutions of (1.7) such that:

(*i*) $\lim_{k\to+\infty} \sup_{t\in[a,b]} r_k(t) = +\infty$;

(*ii*) $\omega_k(b) - \omega_k(a) = \sigma + 2n\pi$ for all $k \in \mathbb{N}$.

Fact (*i*) implies that, if we consider the corresponding solutions (z_k, y_k) of (1.9), they also satisfy:

$$\lim_{k\to+\infty} \sup_{t\in[a,b]} \rho_k(t) = +\infty.$$

On the other hand, a theorem in [108, theorem 4] proves that the angular coordinates in the xy-system and in the zy-system may differ at most by $\pi/4$; then from (*ii*) we deduce that:

$$\sigma + \left(2n - \frac{1}{4}\right)\pi \le \theta_k(b) - \theta_k(a) \le \sigma + \left(2n + \frac{1}{4}\right)\pi.$$

But this contradicts lemma 1.1.5. ∎

Remark 1.1.8 We proved the preceding results in detail starting with a system of the form:

$$\begin{cases} \dot{x} = y - \lambda F(x) + E(t,x,y;\lambda) \\ \dot{y} = -g(x) + p(t,x,y;\lambda) \end{cases}$$

where:

$$f(x) = A|x|^{\alpha-1}x, \qquad F(x) = \int_0^x f(s)ds, \qquad g(x) = B|x|^{\beta-1}x$$

and $\beta > 2\alpha + 1$; but actually our proofs can work also with more general f and g. To be more explicit and following a similar notation in [36], given $\alpha \in \mathbb{R}$, let us consider the set $\mathcal{P}(\alpha)$ of all the functions f such that:

(I) $f \in C^0(\mathbb{R})$ and $f \in C^\infty(\mathbb{R} \setminus [-r,r])$ for some positive r;

(II) for every $n \in \mathbb{N}$ there is a number $r_n > 0$ such that:

$$\sup_{|x|\ge r_n} \frac{|f^{(n)}(x)|}{|x|^{\alpha-n}} < +\infty.$$

Of course power functions, like $|x|^\alpha$ and $|x|^{\alpha-1}x$, are the most characteristic elements of that set and the number α plays here the role played by the degree with polynomials. $\mathcal{P}(\alpha)$ shares with spaces of polynomials some simple properties:

(*i*) if $\alpha < \beta$, then $\mathcal{P}(\alpha) \subset \mathcal{P}(\beta)$;

(*ii*) if $f \in \mathcal{P}(\alpha)$, then $f^{(n)} \in \mathcal{P}(\alpha - n)$;

(*iii*) if $f \in \mathcal{P}(\alpha)$ and $g \in \mathcal{P}(\beta)$, then $fg \in \mathcal{P}(\alpha+\beta)$ and $f + g \in \mathcal{P}(\max\{\alpha,\beta\})$;

(*iv*) if $f \in \mathcal{P}(\alpha)$ and there is $R, c > 0$ such that $|f(x)| \ge c|x|^\alpha$ for all $|x| \ge R$, then $1/f \in \mathcal{P}(-\alpha)$;

(*v*) if $f \in \mathcal{P}(\alpha)$ and $g \in \mathcal{P}(\beta)$, with
$\beta > 0$ and $|g(x)| \ge c|x|^\beta$ for all $|x| \ge r$, then $f \circ g \in \mathcal{P}(\alpha\beta)$.

Now let us consider $\alpha \geq 0$ and $\beta > 2\alpha + 1$, as in our results, and let us take $f \in \mathcal{P}(\alpha)$ and $g \in \mathcal{P}(\beta)$, with $g(x)x > 0$ for all $x \neq 0$ and $|g(x)| \geq c|x|^\beta$ for $|x| \geq R$. First we observe that we can split f and g in the following way:

$$f = f_0 + f_1 \qquad \text{and} \qquad g = g_0 + g_1,$$

where f_1 and g_1 are continuous with compact support in $\mathbb{R}$, while f_0 and g_0 maintain the same properties of f and g, respectively, but are also C^∞-functions in $\mathbb{R}$. Then we make the following positions:

$$F(x) = \int_0^x f_0(s)ds, \qquad G(x) = \int_0^x g_0(s)ds,$$

$$E_0(t,x,y;\lambda) = E(t,x,y;\lambda) - \lambda \int_0^x f_1(s)ds,$$

$$p_0(t,x,y;\lambda) = p(t,x,y;\lambda) - g_1(x),$$

in such a way that F belongs to $\mathcal{P}(\alpha+1) \cap C^\infty(\mathbb{R})$, G to $\mathcal{P}(\beta+1) \cap C^\infty(\mathbb{R})$, G is nonnegative, E_0 and p_0 are still bounded. So we apply Conti's change of variables (1.8) to the system:

$$\begin{cases} \dot{x} = y - \lambda F(x) + E_0(t,x,y;\lambda) \\ \dot{y} = -g(x) + p_0(t,x,y;\lambda) \end{cases}$$

obtaining the new system:

$$\begin{cases} \dot{z} = \gamma(z)[y - \lambda \widehat{F}(z) + \widehat{E}_0(t,z,y;\lambda)] \\ \dot{y} = -\gamma(z)z + \widehat{p}_0(t,z,y;\lambda) \end{cases}$$

with:

$$\gamma(z) = \frac{|g(\Phi(z))|}{\sqrt{2G(\Phi(z))}}, \qquad \widehat{F}(z) = F(\Phi(z)),$$

$$\widehat{E}_0(t,z,y;\lambda) = E_0(t,\Phi(z),y;\lambda) \qquad \text{and} \qquad \widehat{p}_0(t,z,y;\lambda) = p_0(t,\Phi(z),y;\lambda).$$

As we can expect, it is easy to show that $\widehat{F}$ belongs to $\mathcal{P}(2\frac{\alpha+1}{\beta+1})$ and γ to $\mathcal{P}(\frac{\beta-1}{\beta+1})$ and that $\gamma(z) \geq |z|^{\frac{\beta-1}{\beta+1}}$. Consequently, after an outlook of the proofs of the previous lemmas, one can be persuaded that they still work if we assume also that $\widehat{F}$ is an even function. Such an hypothesis implies that the orbits of the unperturbed autonomous system:

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \tag{1.25}$$

have "deformed mirror symmetry" with respect to the y-axis (in the sense that the symmetry comes out only after Conti's change of variable on the x-axis), is strictly related to the fact that the origin is a center for (1.25) (see [117], [118], [53] and [119]) and is satisfied when f and g are odd functions.

1.2 Existence of a periodic solution

To obtain our main result we will apply a continuation theorem proved in [28, theorem 2]. We consider a one-parameter family of differential equations:

$$x' = \mathcal{F}(t, x; \lambda),$$

where $\lambda \in [0,1]$ is the parameter and $\mathcal{F} : \mathbb{R} \times \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ is T-periodic with respect to t. The aim is to prove the existence of at least one T-periodic solution when $\lambda = 1$, knowing that the problem is solvable if $\lambda = 0$. Then one defines the functional:

$$\varphi(x, \lambda) = \frac{1}{2\pi} \left| \int_0^T (\mathcal{F}(t, x(t); \lambda) | \mathbf{J}x(t))\delta(x(t))dt \right|,$$

where:

$$\delta(a_1, a_2) = \begin{cases} 1 & \text{if } a_1^2 + a_2^2 < 1 \\ \frac{1}{a_1^2+a_2^2} & \text{if } a_1^2 + a_2^2 \geq 1 \end{cases}$$

and:

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If it is evaluated along a T-periodic solution x of $x' = \mathcal{F}(t, x; \lambda)$ which lays far enough from the origin, φ is a positive integer and gives the number of rotations that such solution makes around the origin. Finally $d_B(h, B(0,r), 0)$ denotes the usual Brouwer degree for the function h in 0 with respect the ball $B(0,r)$. Now we can write the statement of the continuation theorem we will apply.

Theorem 1.2.1 *[28, theorem 2] Let $\mathcal{F} : \mathbb{R} \times \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ be a Carathéodory function such that $\mathcal{F}(t, x; 0) = h(x)$ and $\mathcal{F}(t + T, x; \lambda) = \mathcal{F}(t, x; \lambda)$, for all $(t, x, \lambda) \in \mathbb{R} \times \mathbb{R}^2 \times [0,1]$. Let us assume:*

(i) $\exists r_0 > 0$ such that $\|x\|_\infty < r_0$ for every T-periodic solution of $x' = h(x)$;

(ii) $d_B(h, B(0,r), 0) \neq 0$, for any $r \geq r_0$;

(iii) $\forall r_1 \geq 0, \exists r_2 \geq r_1$ such that:

$$\min_{[0,T]} (x_1(t)^2 + x_2(t)^2)^{1/2} \leq r_1 \Rightarrow \|x\|_\infty \leq r_2,$$

for each T-periodic solution x of $x' = \mathcal{F}(t, x; \lambda)$ (with $\lambda \in [0,1]$);

(iv) $\forall n \in \mathbb{N}, \exists K_n \geq 0$ such that:

$$\varphi(x, \lambda) = n \Rightarrow \min_{[0,T]} (x_1(t)^2 + x_2(t)^2)^{1/2} \leq K_n,$$

for each solution x of $x' = \mathcal{F}(t, x; \lambda)$ (with $\lambda \in [0,1]$).

Then the system $x' = \mathcal{F}(t, x; 1)$ has at least one T-periodic solution.

In [28] itself theorem (1.2.1) has been applied to obtain the existence of at least one T-periodic solution for a Liénard equation like (1.1), but the assumptions on f, that have been made in that paper, exclude the case we are going to study.

Theorem 1.2.2 *Let us suppose that* $\alpha \geq 0$, $\beta > 2\alpha + 1$, $A \geq 0$, $B > 0$ *and* $e : \mathbb{R} \to \mathbb{R}$ *is a* T*-periodic continuous function; then the forced Liénard equation:*

$$\ddot{x} + A|x|^{\alpha-1}x\dot{x} + B|x|^{\beta-1}x = e(t) \tag{1.26}$$

has at least one T*-periodic solution.*

Proof. We start remarking that finding a T-periodic solution of (1.26) is equivalent to doing it for the following system:

$$\begin{cases} \dot{x} = y - \dfrac{A}{\alpha+1}|x|^{\alpha+1} + E_1(t) \\ \dot{y} = -B|x|^{\beta-1}x + C \end{cases} \tag{1.27}$$

with:

$$C = \frac{1}{T}\int_0^T e(s)ds \qquad \text{and} \qquad E_1(t) = \int_0^t [e(s) - C]ds,$$

where E_1 is T-periodic by construction.

In order to apply theorem 1.2.1 let us define the one parameter family of vector fields $\mathcal{F} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}^2$ in the following way:

$$\mathcal{F}(t, x, y; \lambda) = \begin{pmatrix} y - \dfrac{\lambda A}{\alpha+1}|x|^{\alpha+1} + E(t, x; \lambda) \\ -B|x|^{\beta-1}x + \lambda C \end{pmatrix},$$

where $E(t, x; \lambda) = \lambda E_1(t) - (1 - \lambda)\arctan(x)$. Hence we obtain our original system as $\lambda = 1$, while we have the autonomous system:

$$\begin{cases} \dot{x} = y - \arctan(x) \\ \dot{y} = -B|x|^{\beta-1}x \end{cases} \tag{1.28}$$

for $\lambda = 0$. Let us check that the hypotheses of theorem 1.2.1 are fulfilled. Of course $\mathcal{F}$ is a Carathéodory function and is T-periodic with respect to t.

(*i*) We have to show that all T-periodic solutions of (1.28) are uniformly bounded. Actually the origin is its unique T-periodic solution. In fact let (x, y) be a T-periodic solution of(1.28); let us multiply the first equation of (1.28) by $B|x|^{\beta-1}x$ and the second by y and sum; integrating the resulting equation between 0 and T and using the T-periodicity we obtain:

$$\int_0^T \dot{y}(t)\arctan x(t)dt = 0.$$

Integrating by parts this equality we get:

$$\begin{aligned} 0 &= \int_0^T \frac{y(t)}{1 + x(t)^2}\dot{x}(t)dt \\ &= \int_0^T \frac{\dot{x}(t) + \arctan x(t)}{1 + x(t)^2}\dot{x}(t)dt \\ &= \int_0^T \frac{\dot{x}(t)^2}{1 + x(t)^2}dt. \end{aligned}$$

Then $\dot{x} \equiv 0$ and x is a constant function; thus also y is constant and $x \equiv 0$ from the second equation of (1.28).

(*ii*) By construction, $\mathcal{F}$ does not depend on t for $\lambda = 0$; hence we can set $h(x,y) = \mathcal{F}(t,x,y;0)$ and standard calculations show that:

$$|d_B(h, B(0,r), (0,0))| = |d_B(g, (-r,r), 0)| = 1$$

(see for instance at the beginning of the proof of the first lemma in [94]).

(*iii*) This is the elastic property and it is a direct consequence of assertion (*ii*) in Corollary 1.1.4 with $a = 0$ and $b = T$.

(*iv*) Since we have already remarked that, if we evaluate the functional φ on a T-periodic solution (x,y), it gives the number of rotations made by (x,y) in the Liénard phase plane, we can conclude that this assumption is fulfilled by corollary 1.1.7, when we choose $a = 0$, $b = T$ and $\sigma = 0$. ■

Remark 1.2.3 By the results we proved in the previous section it is easy to deduce the the existence of infinitely many solutions of Sturm-Liouville type boundary value problem for our equation. As a matter of notation, given $\sigma \in [0, 2\pi)$, we denote by $\ell(\sigma)$ the half line which starts from the origin and forms an angle of σ radiants with the positive x-axis, where the angle is measured in clockwise sense from the positive x-axis. If we fix $\sigma_a, \sigma_b \in [0, 2\pi)$, the boundary conditions read:

$$\begin{cases} (x(a), \dot{x}(a)) \in \ell(\sigma_a) \\ (x(b), \dot{x}(b)) \in \ell(\sigma_b), \end{cases} \tag{1.29}$$

where $\ell(\sigma_a)$ and $\ell(\sigma_b)$ are interpreted as straight half lines in the usual phase plane for our equation (see for instance [56]). But the analysis we performed in the previous section was carried on in the Liénard plane, where the two straight half lines in (1.29) become more general curves. Hence we are led to consider a wider class of boundary conditions, namely the ones studied by M. Struwe in [116].

Explicitly we call $\mathbb{B}$ the class of all continua γ of $\mathbb{R}^2$ for which there exist a radius $r > 0$ and a straight line $\ell = \{(x,y) \in \mathbb{R}^2 : c_1 x + c_2 y = c_3\}$ (with $c_1^2 + c_2^2 \neq 0$, of course) such that:

(*i*) $\gamma \cap \ell = \emptyset$;

(*ii*) every continuum, which has a complement with a bounded component containing the ball $B(O,r)$, intersects γ.

The first condition essentially prevents the possibility that γ goes to infinity spinning around the origin and avoiding the orbits of our system, while the second one says that γ tends to infinity crossing every sufficiently far continuum surrounding the origin. Since $\ell(\sigma_a) = \{(s\cos\sigma_a, s\sin\sigma_a) : s \geq 0\}$, we have that the half straight line $\ell(\sigma_a)$ in the usual phase plane becomes in the Liénard plane the curve:

$$\gamma(\sigma_a) = \left\{ \left(s\cos\sigma_a, s\sin\sigma_a - \frac{A}{\alpha+1}|s\cos\sigma_a|^{\alpha+1} + E(a) \right) : s \geq 0 \right\}$$

and similarly $\ell(\sigma_b)$ is transformed into $\gamma(\sigma_b)$. It is not difficult to show that $\gamma(\sigma_a), \gamma(\sigma_b) \in \mathbb{B}$. Now the same proof made by M. Struwe in [116, theorem 1] works and the existence of infinitely many solutions of Sturm-Liouville boundary value problem follows for our equation.

Remark 1.2.4 In view of remark 1.1.8 we can prove the existence of at least one T-periodic solution and of infinite solutions of Sturm-Liouville type boundary value problems for Liénard equations in which the nonlinearities f and g have non symmetric behaviour at infinity, but satisfies the requirement of $\widehat{F}$ to be an even function. For instance this is the case of equations like the following one:

$$\ddot{x} + [2(x_+) - (x_-)^3]\dot{x} + [16(x_+)^4 - (x_-)^9] = e(t),$$

where x_+ and x_- respectively stand for the positive and the negative part of x.

Chapter 2

Superlinear equations with positive weight

Let us consider the equation

$$\ddot{x} + q(t)g(x) = 0 \tag{2.1}$$

where $q : [a, b] \to [0, +\infty[$ is a continuous nonnegative function and $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and satisfies the sign condition $g(x)x > 0$ for every $x \neq 0$. We are interested in the qualitative behaviour of the solutions of (2.1) when some condition of superlinear growth at infinity is assumed on g. To this aim let us consider at first the case in which $q \equiv 1$, so that we have to handle the autonomous equation $\ddot{x} + g(x) = 0$ or, equivalently, the planar first order system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x) \end{cases} \tag{2.2}$$

of which each trajectory $(x(t), y(t))$ lies in a level line $\{(x, y) \in \mathbb{R}^2 : E(x, y) = \text{const}\}$ of the energy function $E(x, y) = \frac{1}{2}y^2 + G(x)$, with $G(x) = \int_0^x g(s)\,ds$, as a straightforward computation shows. The sign condition on g leads to the fact that $G(x)$ is positive for all $x \neq 0$, is strictly monotone increasing on $[0, +\infty[$ and is strictly monotone decreasing on $]-\infty, 0]$; moreover, since we want to deal with an in–some–sense–superlinear function g, it is reasonable to assume that actually G is coercive, in the sense that $G(\pm\infty) = +\infty$. Thus a right inverse function $G_+^{-1} : [0, +\infty[\to [0, +\infty[$ and a left inverse function $G_-^{-1} : [0, +\infty[\to]-\infty, 0]$ are defined for G and all the trajectories of (2.2) are closed cycles around the trivial one $x \equiv y \equiv 0$, which are run by the solution in clockwise sense of rotation as time t increases. To be more explicit, a solution with energy $e > 0$ starts from $(0, \sqrt{2e})$ in the phase plane (x, y), when, say, $t = 0$ (the system is autonomous, so the initial time does not matter); then it enters the first quadrant and reaches the point $(G_+^{-1}(e), 0)$ at $t = t_1$; it continues through the fourth quadrant, touching $(0, -\sqrt{2e})$ at $t = 2t_1$, through the third quadrant, touching $(G_-^{-1}(e), 0)$ at $2t_1 + t_2$, and through the second quadrant, coming back to the initial point $(0, \sqrt{2e})$ at $t = 2t_1 + 2t_2$. As a consequence, all the solutions of (2.2) are periodic (in other words, the origin is a global center) and, therefore, there are no problems of continuability (they are defined over the whole real line) even if $g(x)$ grows at infinity much faster than any affine function. In particular, there is a unique solution $(x(t), y(t))$ of (2.2) for each value $e \geq 0$ of the energy function E : therefore we can associate to each positive real number e the period $\tau_g^+(e)$ of the corresponding solution $(x(t), y(t))$ of (2.2) satisfying $E(x(t), y(t)) = e$ for all t. The function $\tau_g^+ :]0, +\infty[\to]0, +\infty[$, which is usually called the *time–map* associated to the autonomous equation $\ddot{x} + g(x) = 0$ and to the system (2.2), can be explicitly evaluated in terms of G starting from the relation $\frac{1}{2}\dot{x}^2(t) + G(x(t)) \equiv e$ and straightforward computations show that:

$$\tau_g^+(e) = \sqrt{2} \int_{G_-^{-1}(e)}^{G_+^{-1}(e)} \frac{1}{\sqrt{e - G(s)}}\, ds. \tag{2.3}$$

The superscript "+" will be useful in the sequel, since in the next chapter we introduce an analogous of the time–map for the autonomous equation $\ddot{x} - g(x) = 0$ and we will call it τ_g^-.

When the usual assumption of superlinear growth is imposed on g, that is:

$$\lim_{x \to \pm\infty} \frac{g(x)}{x} = +\infty, \tag{2.4}$$

it can be proved that $\tau_g^+(e)$ tends to 0 as we let e go to infinity, that is, solutions with larger C_1-norm have smaller periods and, hence, in a fixed interval $[a, b]$ they oscillate more and have more zeros. At this stage it can be interesting to observe how unbounded curves of initial points are transformed by the flow of the system (2.2), when g satisfies (2.4) or, more generally, when we assume that:

$$\lim_{e \to +\infty} \tau_g^+(e) = 0, \tag{$g2+$}$$

which will be our weaker condition of superlinear growth at infinity for g (see also hypothesis $(f2)$, pag 38). Let us consider a fixed time interval $[a, b]$ and the positive y-axis $Y^+ = \{(0, s) : s \geq 0\}$ in the phase plane: we follow the solutions of (2.2) starting at $t = a$ from Y^+ and we determine their value at $t = b$, that is we evaluate on Y^+ the Poincaré map $\Pi : \mathbb{R}^2 \to \mathbb{R}^2$ which associates to every point (x_0, y_0) the value $(x(b), y(b))$ that the solution of (2.2) will have at time $t = b$, if $(x(a), y(a)) = (x_0, y_0)$. We can imagine taking a point $(0, s)$, moving it along the positive y-axis Y^+ towards the infinity and looking at which curve the final point $\Pi(0, s)$ will describe in the phase plane: as s increases, the correspondent solution runs along an orbit of (2.2) which is larger and farther from the origin, with a greater speed. Therefore it is not surprising that the curve $\Pi(0, s)$, $s > 0$, is an unbounded spiral starting from the origin and turning infinitely many times around the origin itself in clockwise sense of rotation. In particular each time the curve $\Pi(0, s)$, $s > 0$, crosses the y-axis, the first component x of the solution (x, y) of (2.2), with $(x(a), y(a)) = (0, s)$, solves the two–point boundary value problem

$$\begin{cases} \ddot{x} + g(x) = 0 \\ x(a) = x(b) = 0 \end{cases}.$$

In other words, this problem has infinitely many solutions which can be found by looking for the intersections between the spiral–like curve $\Pi(0, s)$, $s > 0$, and the y-axis. Of course, we can consider also Sturm–Liouville boundary conditions, that is we could ask that $(x(a), y(a)) \in r_a$ and $(x(b), y(b)) \in r_b$, where r_a and r_b are straight lines: in this case r_a is again transformed by the map Π in an unbounded spiral around the origin and we should look for its intersections with r_b. More generally, any boundary condition which can be written as $(x(a), y(a)) \in \Gamma_a$ and $(x(b), y(b)) \in \Gamma_b$ is suitable for the superlinear equation $\ddot{x} + g(x) = 0$ or for the equivalent system (2.2), when it is possible to show that $\Pi(\Gamma_a)$ is an unbounded spiral–like set around the origin and Γ_b, is unbounded, too, but does not spin too much around the origin, in such a way that it is possible to prove that $\Pi(\Gamma_a) \cap \Gamma_b \neq \emptyset$. In this sense, a suitable class of boundary condition, which generalizes the Sturm–Liouville ones, is the class $\mathbb{B}$ defined in [116] by Struwe and considered in Remark 1.2.3.

Remark 2.0.5 If $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that:

$$\lim_{s \to \pm\infty} g(s)\mathrm{sgn}(s) = +\infty,$$

then, clearly, the primitive G of g is strictly monotone in a neighbourhood of $\pm\infty$ and thus the inverses $G_{\pm}^{-1}$ are defined. If we look for some sufficient conditions ensuring the validity of $(g2+)$, we can use various results in the literature, starting with the classical work of Opial [96] (see also [112]). As already noted, a typical assumption which implies $(g2+)$ is the condition of superlinear growth at infinity:

$$\lim_{x \to \pm\infty} \frac{g(x)}{x} = +\infty.$$

Other possibilities for $(g2+)$ when the usual superlinear condition does not hold, are discussed in [37]. For instance, we have that $(g2+)$ follows from:

$$\exists\, A > 0 : \quad \lim_{x \to \pm\infty} \frac{G(x+A) - G(x)}{x^2} = +\infty.$$

2.1 Continuability of solutions

The first problem that one has to face, when in equation:

$$\ddot{x} + q(t)g(x) = 0 \tag{2.5}$$

a general weight function q is considered, is the possible lack of continuability of the solutions, if g is not dominated by an affine function: Coffman and Ullrich in [31] gave an example of a continuous $q : [0, +\infty[\to]0, +\infty[$ such that the equation:

$$\ddot{x} + q(t)x^3 = 0$$

has a solution x with $[0, \pi^2/6[$ as maximal interval of continuability. In particular the function q satisfies $\lim_{t \to \pi^2/6} q(t) = c > 0$ and $q(t) = c$ for all $t > \pi^2/6$, but the total variation of q on $[0, t]$ tends to infinity as t approaches $\pi^2/6$ from the left. Therefore the fact that q is close to a positive constant c around $\pi^2/6$ does not matter for what concerns the continuability of solutions and also the fact that c can be chosen arbitrarily small is of no influence: what really makes the difference is the lack of regularity of q nearby $\pi^2/6$. In fact Coffman and Ullrich proved in [31] that, if $q : [0, +\infty[\to]0, +\infty[$ is continuous and locally of bounded variation and if $n \in \mathbb{N}$, then the initial value problem:

$$\begin{cases} \ddot{x} + q(t)x^{2n+1} = 0 \\ (x(t_0), \dot{x}(t_0)) = (x_0, y_0) \end{cases}$$

has, for every $t_0 > 0$ and $(x_0, y_0) \in \mathbb{R}$, a unique solution which exists on $[0, +\infty[$. Their result can be easily modified to cover more general nonlinearities g and also the case in which q vanishes somewhere and this is the subject of the following theorem.

Theorem 2.1.1 *Let $q : [a,b] \to [0,+\infty[$ be a continuous function with bounded variation such that, for each $t_0 \in [a,b]$ with $q(t_0) = 0$, there are a left neighbourhood of t_0, where q is monotone decreasing, and a right neighbourhood of t_0, where q is monotone increasing; let $G \in C^1(\mathbb{R})$ be such that $\inf_{\mathbb{R}} G > -\infty$, and let $p : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and such that $|p(t,x,y)| \leq l(t)(1+|x|+|y|)$, for all $(t,x,y) \in [a,b] \times \mathbb{R} \times \mathbb{R}$ and for some nonnegative function $l \in L^1([a,b])$. Moreover we assume that the local uniqueness holds for the initial value problem:*

$$\begin{cases} \ddot{x} + q(t)G'(x) = p(t,x,\dot{x}) \\ (x(t_0),\dot{x}(t_0)) = (x_0,y_0) \end{cases} \tag{2.6}$$

for all $t_0 \in [a,b]$ and for all $(x_0,y_0) \in \mathbb{R}^2$. Then the solution of (2.6) is continuable on $[a,b]$ for all $t_0 \in [a,b]$ and for all $(x_0,y_0) \in \mathbb{R}^2$.

We did not find an exact reference for this theorem, even if it is clear that is known from the works by Butler [23, 24]. Therefore we provide here a proof of it.

Let us state and prove, at first, a preliminary result which is a simple generalization of Lemma 2 in [31], following the ideas in the proofs of Proposition 1 in [129] and Theorem 6 in [108]. We remark that in the next lemma we do not need the local uniqueness for the initial value problems associated to our equation: we will need it later in the proof of the theorem, when we use an approximation procedure, in order to be sure that the approximating solutions are converging to *the* solution we are considering.

Lemma 2.1.2 *Let $G \in C^1(\mathbb{R})$ be a function such that $\inf_{\mathbb{R}} G > -\infty$, and let $p : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and such that $|p(t,x,y)| \leq l(t)(1+|x|+|y|)$, for all $(t,x,y) \in [a,b] \times \mathbb{R} \times \mathbb{R}$ and for some nonnegative function $l \in L^1([a,b])$. Assume also that $q \in C^1([a,b])$ is a nonnegative function satisfying:*

$$|q'(t)| \leq h(t)q(t) \qquad \forall\, t \in [a,b], \tag{2.7}$$

for some nonnegative function $h \in L^1([a,b])$. Then every solution x of the initial value problem

$$\begin{cases} \ddot{x} + q(t)G'(x) = p(t,x,\dot{x}) \\ (x(t_0),\dot{x}(t_0)) = (x_0,y_0) \end{cases} \tag{2.8}$$

is continuable on $[a,b]$ for all $t_0 \in [a,b]$ and for all $(x_0,y_0) \in \mathbb{R}^2$. Moreover, if we set:

$$\Phi(t) = \frac{1}{2} + \frac{1}{2}x^2(t) + \frac{1}{2}\dot{x}^2(t) + q(t)[G(x(t)) + L],$$

where $L \geq -\inf_{\mathbb{R}} G$, then Φ satisfies:

$$\Phi(t) \leq \Phi(t_0) \exp\left(\left|\int_{t_0}^{t} [4l(s) + h(s) + 1]\, ds\right|\right) \qquad \forall\, t \in [a,b]. \tag{2.9}$$

Proof. Let x be a solution of the problem (2.8) and suppose that it is defined on $]t_1,t_2[\subset [a,b]$, with $t_1 < t_0 < t_2$. Clearly Φ is a C^1 function on $]t_1,t_2[$ and satisfies:

$$\begin{aligned} \Phi'(t) &= x(t)\dot{x}(t) + \ddot{x}(t)\dot{x}(t) + q(t)G'(x(t))\dot{x}(t) + q'(t)[G(x(t)) + L] \\ &= \dot{x}(t)[x(t) + p(t,x(t),\dot{x}(t))] + q'(t)[G(x(t)) + L]. \end{aligned}$$

Moreover, by (2.7) and the assumptions on p and G we obtain:

$$\begin{aligned}|\Phi'(t)| &\leq l(t)|\dot{x}(t)| + (l(t)+1)|x(t)\dot{x}(t)| + l(t)\dot{x}^2(t) + h(t)q(t)[G(x(t)) + L] \\ &\leq l(t)\left[\frac{1}{2} + \frac{1}{2}\dot{x}^2(t)\right] + (l(t)+1)\left[\frac{1}{2}x^2(t) + \frac{1}{2}\dot{x}^2(t)\right] 2l(t)\frac{\dot{x}^2(t)}{2} \\ &\quad + h(t)q(t)[G(x(t)) + L] \\ &\leq [4l(t) + h(t) + 1]\Phi(t),\end{aligned}$$

for every $t \in]t_1, t_2[$. By Gronwall's inequality we deduce:

$$\Phi(t) \leq \Phi(t_0) \exp\left(\left|\int_{t_0}^{t} [4l(s) + h(s) + 1]\, ds\right|\right) \qquad \forall\, t \in]t_1, t_2[\,.$$

Hence x is continuable on $[a, b]$ (it is bounded) and inequality (2.9) holds. ■

We can now prove the main theorem of the section.

Proof of Theorem 2.1.1. By the local existence theorem the solution x of (2.6) exists in an interval $[t_1, t_2]$, with $t_1 < t_0 < t_2$, such that $q(t_i) > 0$ for $i = 1, 2$: this is obvious if also $q(t_0) > 0$, while if $q(t_0) = 0$ the solution surely reaches a zone in which $q > 0$, since there are no problems of continuability wherever $q \equiv 0$. Therefore we can suppose that $q(t_0) > 0$, up to redefine $t_0 = t_1$ or $t_0 = t_2$. As a first step we will show that x is continuable in every interval $[c, d]$ where q is strictly positive. We use an approximation procedure: let $\{\eta_k\}$ be the classical C^∞ mollifiers given by $\eta_k(t) = k\eta(kt)$, where:

$$\eta(s) = \begin{cases} C\exp\left(\dfrac{1}{s^2-1}\right) & \text{if } |s| \leq 1 \\ 0 & \text{if } |s| > 1 \end{cases} \quad \text{and} \quad C = \left(2\int_0^1 \exp\left(\frac{1}{s^2-1}\right) ds\right)^{-1},$$

so that $\int_{-\infty}^{+\infty} \eta_k(t)\, dt = \int_{-\infty}^{+\infty} \eta(t)\, dt = 1$, for all $k \in \mathbb{N}$. For convenience we extend $q(t) = q(c)$ for $t < c$ and $q(t) = q(d)$ for $t > d$ and we set:

$$q_k(t) = \int_{-\infty}^{+\infty} q(\tau)\eta_k(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} q(t-\tau)\eta_k(\tau)\, d\tau.$$

It is well known that $q_k \to q$ uniformly in $[c, d]$ and that $\int_c^d |q_k'(t)|\, dt \to T(c, d)$, as $k \to +\infty$, if $T(c, d)$ is the total variation of q on $[c, d]$ (see [49, Proposition 1.15, Remark 1.16]). Let $m = \inf_{[c,d]} q > 0$: it is easy to see that $\inf_{\mathbb{R}} q_k(t) \geq m$. Therefore q_k satisfies

$$|q_k'(t)| \leq \frac{|q_k'(t)|}{m} q_k(t) \qquad \forall\, t \in [c, d]$$

and Lemma 2.1.2 implies that a fixed solution x_k of the problem

$$\begin{cases} \ddot{u} + q_k(t)G'(u) = p(t, u, \dot{u}) \\ (u(t_0), \dot{u}(t_0)) = (x_0, y_0) \end{cases} \tag{2.10}$$

is continuable on $[c, d]$ and satisfies

$$\Phi_k(t) \leq \Phi_k(t_0) \exp\left(\frac{1}{m}\int_c^d |q_k'(t)|\, dt\right) \qquad \forall\, t \in [c, d],\ \forall\, k, \tag{2.11}$$

where we have set:

$$\Phi_k(t) = \frac{1}{2} + \frac{1}{2}x_k^2(t) + \frac{1}{2}\dot{x}_k^2(t) + q_k(t)[G(x_k(t)) + L].$$

If I is the maximal interval of continuability of x with respect to the interval $[c, d]$, then $(x_k(t), \dot{x}_k(t)) \to (x(t), \dot{x}(t))$ for every $t \in I$ by the theorem on the continuous dependence; in particular, we can let k go to infinity in (2.11) for each $t \in I$, obtaining that:

$$\Phi(t) \leq \Phi(t_0)e^{T(b,c)/m} \qquad \forall t \in I,$$

that is x and $\dot{x}$ are bounded in I; therefore $I = [c, d]$ and the solution is defined on $[c, d]$.

Now, let us show that the solution x is continuable on an interval $[c, d]$ such that $q > 0$ in $]c, d[$ and $q(c) = q(d) = 0$. By our hypotheses on q, there is a sufficiently small $\varepsilon > 0$ such that $t_0 \in [c+\varepsilon, d-\varepsilon]$ and q is monotone increasing on $[c, c+\varepsilon]$ and monotone decreasing on $[d-\varepsilon, d]$; moreover q is strictly positive in $[c+\varepsilon, d-\varepsilon]$, so that x is defined on $[c+\varepsilon, d-\varepsilon]$ by what we have just proved. Therefore we have to prove that the solution can be continuated from $c + \varepsilon$ to c and from $d - \varepsilon$ to d. Let us show only that x is continuable up to d : in an analogous way it can be proved that it is continuable up to c. We have that q is monotone decreasing in $[d - \varepsilon, d]$ and positive in $[d - \varepsilon, d[$ and let us extend q to $\mathbb{R}$ in such a way that $q(t) = q(d - \varepsilon)$ for all $t < d - \varepsilon$ and $q(t) = 0$ for all $t > d$. It is easy to see that, now, the mollified weights $\{q_k\}$ are also monotone decreasing, so that Lemma 2.1.2 applies to the initial value problem:

$$\begin{cases} \ddot{u} + q_k(t)G'(u) = p(t, u, \dot{u}) \\ u(d - \varepsilon) = x(d - \varepsilon) \\ \dot{u}(d - \varepsilon) = \dot{x}(d - \varepsilon) \end{cases}$$

with the choice $h(t) \equiv 0$ in (2.7). Hence its solution x_k is defined in $[d - \varepsilon, d]$ and satisfies $\Phi_k(t) \leq \Phi_k(d - \varepsilon)$ for every $t \in [d - \varepsilon, d]$. As above, this ensures that x, too, is continuable on $[d - \varepsilon, d]$.

Finally we remark that the hypotheses about q imply that the set $\{t \in [a, b] : q(t) > 0\}$ can be decomposed into the union of a finite number of open intervals (relatively to $[a, b]$), while the set $\{t \in [a, b] : q(t) = 0\}$ is made by a finite number of isolated points and disjoint closed intervals. Therefore it can be shown that x is continuable on $[a, b]$ by applying a finite number of times the steps proved above. ∎

A direct consequence of Theorem 2.1.1 that we will use in the next chapters is stated here:

Corollary 2.1.3 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function such that $g(x)x > 0$ for every $x \neq 0$. Let $q : [a, b] \to [0, +\infty[$ be a nonnegative continuous function with bounded variation such that:*

1. *the set $\{t \in [a, b] : q(t) = 0\}$ has a finite number of connected components;*
2. *if $q > 0$ on $]b, c[\subset [a, b]$ and $q(c) = 0$ (or, respectively, $q(d) = 0$), then q is monotone nondecreasing in a right neighbourhood of c (or, respectively, monotone nonincreasing in a left neighbourhood of d).*

Then the solution of the initial value problem:

$$\begin{cases} \ddot{x} + q(t)g(x) = 0 \\ (x(t_0), \dot{x}(t_0)) = (x_0, y_0) \end{cases}$$

is defined in $[a, b]$ for every $t_0 \in [a, b]$ and for every $(x_0, y_0) \in \mathbb{R}^2$.

Proof. It's enough to apply Theorem 2.1.1 with:

$$G(x) = \int_0^x g(s)\, ds \qquad \text{and} \qquad p(t,x,y) \equiv 0$$

and the corollary is proved. ∎

2.2 Rapidly oscillating solutions

Now, we are going to generalize some of the consequences due to the superlinear growth of g in the autonomous equation $\ddot{x} + g(x) = 0$, to a wider class of equations. Namely we will consider an equation of the form:

$$\ddot{x} + f(t,x) = 0 \tag{2.12}$$

and we will give conditions on f in order to have that the solutions passing through points in the phase plane which are farther from the origin, oscillate more in a fixed time interval I, i.e. have more zeros and their orbits $(x(t), \dot{x}(t))$, $t \in I$, turn around the origin of the phase plane more times. The result we prove, will apply to the case of $f(t,x) = q(t)g(x)$ and $q(t) \geq 0$, but it is stated in the general framework of Carathéodory hypotheses.

Definition 2.2.1 A function $F : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$ is called a *Carathéodory function* if it satisfies the following conditions:

- for almost every $t \in [a,b]$ the function $x \mapsto F(t,x)$ is continuous;
- for every $x \in \mathbb{R}^N$ the function $t \mapsto F(t,x)$ is Lebesgue measurable;
- for every compact $K \subset \mathbb{R}^N$ there is a function $h_K \in L^1([a,b])$ such that $|F(t,x)| \leq h_K(t)$ for almost all $t \in [a,b]$ and for all $x \in K$.

In this case a function $x : [a,b] \supset I \to \mathbb{R}^N$ is a solution of:

$$\dot{x} = F(t,x)$$

if x is an *absolutely continuous* function (i.e. $x \in AC(I, \mathbb{R}^N)$) and satisfies $x(t) = F(t,(x(t))$ for *almost every* $t \in I$.

In our case, if $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $x : [a,b] \supset I \to \mathbb{R}$ is a solution of (2.12), then x actually belongs to the space $AC^1(I) = \{x : I \to \mathbb{R} : x \in C^1(I),\, \dot{x} \in AC(I)\}$, as can be seen writing the equation as the equivalent first order plane system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(t,x). \end{cases}$$

Besides the fact of f being a Carathéodory function, we will use the following hypotheses:

$(f1)$ the global continuability holds for every initial value problem associated to the equation (2.12);

($f2$) there are a nontrivial subinterval $I \subset [a,b]$, a number $x^* > 0$ and a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x)x > 0$ for all $x \neq 0$, $\tau_g^+(e) \to 0$ as $t \to +\infty$, where $\tau_g^+(e)$ is defined by (2.3), and

$$f(t,x)\,\mathrm{sign}(x) \geq |g(x)| \qquad \text{for a.e. } t \in I \text{ and } \forall\, |x| \geq x^*. \tag{2.13}$$

In fact, we will exploit a well known consequence of ($f1$), that is the so–called *elastic property*, which is stated and proved in the next lemma.

Lemma 2.2.2 *Let $F : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$, be a Carathéodory function such that all the solutions of the differential equation:*

$$\dot{x} = F(t,x) \tag{2.14}$$

are continuable on the whole $[a,b]$. Then the following properties hold:

1. *for all $R_1 > 0$ there exists $R_2 \geq R_1$ such that, if $x : [a,b] \to \mathbb{R}^N$ is a solution of* (2.14) *satisfying $|x(t_0)| \geq R_2$ for some $t_0 \in [a,b]$, then $|x(t)| \geq R_1$ for every $t \in [a,b]$;*

2. *for all $R_1 > 0$ there exists $R_2 \geq R_1$ such that, if $x : [a,b] \to \mathbb{R}^N$ is a solution of* (2.14) *satisfying $|x(t_0)| \leq R_1$ for some $t_0 \in [a,b]$, then $|x(t)| \leq R_2$ for every $t \in [a,b]$.*

Proof. Suppose by contradiction that statement 1 does not hold: then there is $R_1 > 0$ such that for all $R \geq R_1$ there exist some $t_R, s_R \in [a,b]$ and some solution x_R of equation (2.14) which satisfy $|x_R(t_R)| \geq R$ and $|x_r(s_R)| < R_1$. By a compactness argument it is possible to find two sequences $\{t_k\}$ and $\{s_k\}$ of points of $[a,b]$ and a sequence $\{x_k\}$ of solutions of (2.14) such that:

(i) $s_k \to s_0 \in [a,b]$ and $x_k(s_k) \to x_0 \in B[R_1]$;

(ii) $t_k \to t_0 \in [a,b]$ and $|x_k(t_k)| \to +\infty$;

(we denote by $B[R]$ the closed ball in $\mathbb{R}^N$ with center in the origin). The continuous dependence theorem and point (i) imply that x_k uniformly converge to a solution x_0 of the initial value problem:

$$\begin{cases} \dot{x} = F(t,x) \\ x(t_0) = x_0 \end{cases}$$

which is defined on $[a,b]$ by our hypothesis on F. However this is clearly in contradiction with (ii).

The proof of the second statement can be done again by contradiction in a completely similar way. ■

We start the work of estimating how much time a trajectory $(x(t), \dot{x}(t))$ of (2.12) takes to make a complete turn around the origin: fixed $M > 0$, we divide the phase plane into the strip $\{(x,y) \in \mathbb{R}^2 : |x| \leq M\}$ and the two remaining half planes $\{(x,y) \in \mathbb{R}^2 : x \leq -M\} \cup \{(x,y) \in \mathbb{R}^2 : x \geq M\}$ and we compute separately the time needed for the trajectory to cross each of these portions of the phase plane. The following lemma regards the vertical strip: for every $M > 0$ the time spent in it by the curve $(x(t), \dot{x}(t))$ can be made arbitrarily small by considering solutions x with sufficiently large first derivative. Here the superlinearity of f does not matter: the result is true just assuming that f is a Carathéodory function.

Lemma 2.2.3 *Assume that $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and let $M > 0$ and $\delta > 0$ be two arbitrary constants. Then there exists $N = N(M,\delta) > 0$ such that, for every interval $[t_1,t_2] \subset [a,b]$, for every $t_0 \in [t_1,t_2]$ and for every solution x of (2.12), with $x(t_0) = 0$, $|\dot{x}(t_0)| \geq N$ and $|x(t)| \leq M$ for every $t \in [t_1,t_2]$, then it turns out that $t_2 - t_1 \leq \delta$.*

Proof. Let us fix $M > 0$ and $\delta > 0$ and $h_M \in L^1([a,b])$ be the function given by the Carathéodory conditions on f relatively to the compact interval $K = [-M,M]$, so that $|f(t,x)| \leq h_M(t)$ for almost every $t \in [a,b]$ and every $x \in [-M,M]$. Then we define:

$$N = \max\left\{4\frac{M}{\delta}\,,\,2\int_a^b h_M(s)\,ds\right\}.$$

Let x, t_0, t_1 and t_2 be as in the statement of the lemma, that is, $t_0 \in [t_1,t_2] \subset [a,b]$ and x is a solution of (2.12) with $x(t_0) = 0$, $|\dot{x}(t_0)| \geq 0$ and $|x(t)| \geq M$ for every $t \in [t_1,t_2]$; we have:

$$|\ddot{x}(t)| = |f(t,x(t))| \leq h_M(t) \qquad \text{for a.e. } t \in [t_1,t_2],$$

since $|x(t)| \leq M$ for every $t \in [t_1,t_2]$. Thus, for every $t \in [t_1,t_2]$ we can estimate:

$$|\dot{x}(t) - \dot{x}(t_0)| \leq \int_a^b h_M(s)\,ds \leq \frac{N}{2},$$

by the choice of N, and we can conclude that:

$$\min_{t\in[t_1,t_2]} |\dot{x}(t)| \geq \frac{N}{2},$$

by the fact that $|\dot{x}(t_0)| \geq N$. In particular we have that $\dot{x}$ has a fixed sign in $[t_1,t_2]$ and x is monotone, so that:

$$2M \geq |x(t_2) - x(t_1)| = \int_{t_1}^{t_2} |\dot{x}(t)|\,dt \geq \min_{t\in[t_1,t_2]} |\dot{x}(t)|\,(t_2 - t_1) \geq \frac{N}{2}(t_2 - t_1).$$

In conclusion, this last inequality and the choice of N imply:

$$t_2 - t_1 \leq 4\frac{M}{N} \leq \delta$$

and the lemma is proved. ■

The next result says that the lenght of a time interval in which a solution x of (2.12) satisfies $|x(t)| \geq M$, tends to zero as M tends to infinity. Here the assumption of f being superlinear (that is condition $(f2)$) plays the main role.

Lemma 2.2.4 *Assume that $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $(f2)$ and let $I \subset [a,b]$ be the interval where (2.13) is satisfied. Then for every $\delta > 0$ there exists $M = M(\delta) > 0$ such that for every $[t_1,t_2] \subset I$ and for every solution x of (2.12) with $|x(t)| \geq M$ for all $t \in [t_1,t_2]$, it turns out that $t_2 - t_1 \leq \delta$.*

Proof. Let us fix $\delta > 0$ and let $g : \mathbb{R} \to \mathbb{R}$ be the function provided in $(f2)$; since $\tau_g^+(e) \to 0$ as $e \to +\infty$, there exists $E_\delta > 0$ such that:

$$\tau_g^+(e) \leq \delta \qquad \forall\, e \geq E_\delta.$$

We prove that the choice:

$$M = \max\{x^*, E_\delta\}$$

works, with x^* given in $(f2)$: suppose that x is a solution of (2.12) such that $x(t) \geq M$ for all $t \in [t_1, t_2] \subset I$. We will show that $t_2 - t_1 \leq \delta$. The other case, i.e. $x(t) \leq -M$ for all $t \in [t_1, t_2] \subset I$, is completely symmetric and its proof is omitted.

By the choice of M and assumption $(f2)$ we have that

$$\ddot{x}(t) = -f(t, x(t)) \leq -g(x(t)) \leq 0 \qquad \text{for a.e. } t \in [t_1, t_2],$$

thus $\dot{x}$ is a decreasing function in $[t_1, t_2]$. Then we set $t^* = t_1$, if $\dot{x}(t) < 0$ for every $t \in [t_1, t_2]$, and $t^* = \sup\{t \in [t_1, t_2] : \dot{x}(t) \geq 0\}$, otherwise. However the interval $[t_1, t_2]$ can be splitted into two disjoint intervals $[t_1, t^*[$ and $[t^*, t_2]$ in such a way that $\dot{x}(t) \geq 0$ for all $t \in [t_1, t^*[$ and $\dot{x}(t) \leq 0$ for all $t \in [t^*, t_2]$. The lemma will be proved if we show that both $t^* - t_1$ and $t_2 - t^*$ are bounded by $\delta/2$.

We consider the first interval and, calling as usual $G(s) = \int_0^s g(\xi)\, d\xi$, we evaluate the following derivative on $[t_1, t^*[$:

$$\frac{d}{dt}\left[G(x(t)) + \frac{1}{2}\dot{x}^2(t)\right] = [g(x(t)) - f(t, x(t))]\dot{x}(t) \leq 0 \quad \text{for a.e. } t \in [t_1, t^*[\,,$$

since $f(t, x(t)) \geq g(x(t))$, by hypothesis $(f2)$, and $\dot{x}(t) \geq 0$. Therefore the function:

$$t \longmapsto G(x(t)) + \frac{1}{2}\dot{x}^2(t) \tag{2.15}$$

is monotone decreasing in $[t_1, t^*[$ and:

$$G(x(t)) + \frac{1}{2}\dot{x}^2(t) \geq G(x(t^*)) + \frac{1}{2}\dot{x}^2(t^*) \geq G(x(t^*)) \qquad \forall\, t \in [t_1, t^*[\,,$$

that is:

$$1 \leq \frac{\dot{x}(t)}{\sqrt{2[G(x(t^*)) - G(x(t))]}} \qquad \forall\, t \in [t_1, t^*[\,.$$

We note that the square root on the right hand side is well defined, since $\dot{x}(t)$ is nonnegative and $x(t)$ in monotone increasing in $[t_1, t^*[$. If we integrate the last inequality between t_1 and t^*, after the change of variables $s = x(t)$, we get:

$$t^* - t_1 \leq \int_{x(t_1)}^{x(t^*)} \frac{ds}{\sqrt{2[G(x(t^*)) - G(s)]}} \leq \frac{1}{2}\tau_g^+(G(x(t^*))) \leq \frac{\delta}{2},$$

because $x(t^*) \geq M$.

In the second interval the situation is very similar, but reversed, in some sense; indeed we have:

$$\frac{d}{dt}\left[G(x(t)) + \frac{1}{2}\dot{x}^2(t)\right] \geq 0 \qquad \text{for a.e. } t \in [t^*, t_2],$$

since now $\dot{x}(t) \leq 0$. Therefore the function (2.15) is now monotone increasing in $[t^*, t_2]$ and:

$$G(x(t)) + \frac{1}{2}\dot{x}^2(t) \geq G(x(t^*)) + \frac{1}{2}\dot{x}^2(t^*) \geq G(x(t^*)) \qquad \forall\, t \in [t^*, t_2].$$

Thus we obtain that:

$$1 \leq \frac{-\dot{x}(t)}{\sqrt{2[G(x(t^*)) - G(x(t))]}} \qquad \forall\, t \in [t^*, t_2]$$

and the square root is well defined because x is monotone decreasing in $[t^*, t_2]$. As above, we integrate the last inequality between t^* and t_2 and we perform the change of variables $s = x(t)$ to get:

$$t_2 - t^* \le \int_{x(t_2)}^{x(t^*)} \frac{ds}{\sqrt{2[G(x(t^*)) - G(s)]}} \le \frac{1}{2}\tau_g^+(G(x(t^*))) \le \frac{\delta}{2},$$

concluding the proof of the lemma. ■

We introduce, now, the polar coordinates to evaluate the number of turns that a trajectory $(x(t), \dot{x}(t))$ makes around the origin of the plane as t ranges in an interval $[t_1, t_2]$. Let x be a solution of (2.12) such that $(x(t), \dot{x}(t)) \ne (0,0)$ for all $t \in [t_1, t_2]$. Then there are two absolutely continuous functions $\theta_x, \rho_x : [t_1, t_2] \to \mathbb{R}$ such that:

$$\begin{cases} x(t) = \rho_x(t)\cos\theta_x(t) \\ \dot{x}(t) = \rho_x(t)\sin\theta_x(t) \end{cases} \tag{2.16}$$

and $\rho_x(t) = \sqrt{x^2(t) + \dot{x}^2(t)}$, while θ_x is uniquely determined up to multiples of 2π. From (2.16) and equation (2.12) it is easy to deduce that ρ_x and θ_x satisfy:

$$\begin{cases} \dot{\rho}_x(t) = \dfrac{x(t)\dot{x}(t) + \dot{x}(t)\ddot{x}(t)}{\rho_x(t)} = \dfrac{x(t)\dot{x}(t) - f(t,x(t))\dot{x}(t)}{\rho_x(t)} & \text{for a.e. } t, \\ \dot{\theta}_x(t) = \dfrac{\ddot{x}(t)x(t) - \dot{x}^2(t)}{\rho_x^2(t)} = -\dfrac{f(t,x(t))x(t) + \dot{x}^2(t)}{\rho_x^2(t)} & \text{for a.e. } t. \end{cases}$$

In particular, the angle $\theta_x(t_1) - \theta_x(t_2)$, which is spanned by the vector $(x(t), \dot{x}(t))$ in the phase plane as t varies in the interval $[t_1, t_2]$ and is independent of the choice of θ_x, can be evaluated by the following integral:

$$\theta_x(t_1) - \theta_x(t_2) = \int_{t_2}^{t_1} \frac{f(t,x(t))x(t) + \dot{x}^2(t)}{x^2(t) + \dot{x}^2(t)}\, dt. \tag{2.17}$$

Note that in this formula the angle is measured in clockwise sense: this is just a matter of conventions and we choose this one since, when $|x(t)|$ is large and f satisfies $(f2)$, it turns out that

$$f(t,x(t))x(t) \ge g(x(t))x(t) \ge 0,$$

so that $\dot{\theta}_x(t) \le 0$ and the vector $(x(t), \dot{x}(t))$ rotates around the origin in clockwise sense.

We wish to prove that, when $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is superlinear in the sense that it satisfies $(f2)$, the solutions with large C^1 norm rotate many times around the origin. This goal is achieved for a particular solution x once it is shown that $\theta_x(a) - \theta_x(b)$ is "large". However in $(f2)$ the superlinear growth condition on f holds only on a subinterval $I = [c,d] \subset [a,b]$, so that, in fact, we will show that $\theta_x(c) - \theta_x(d)$ is large. Thus a result is needed to bound the rotation of the trajectories in $[a,c]$ and in $[d,b]$. This is the aim of the following lemma.

Lemma 2.2.5 *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and let x be a solution of $\ddot{x} + f(t,x) = 0$ such that $(x(t), \dot{x}(t)) \ne (0,0)$ for every $t \in [a,b]$. Then:*

$$\theta_x(a) - \theta_x(b) > -\pi. \tag{2.18}$$

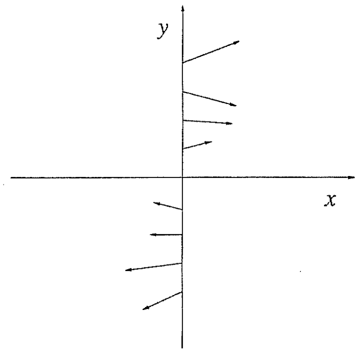

Figure 2.1: The vector field of (2.19) on the y-axis

This result can be heuristically justified by considering the plane system which is equivalent to the equation (2.12):

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(t,x) \end{cases} \tag{2.19}$$

and by studying the associated vector field $(x, y) \mapsto (y, -f(t, x))$ on the y-axis for any time t. Indeed, such a vector field restricted on $Y^+ = \{(0, y) : y > 0\}$ always points inward the first quadrant, while, if restricted on $Y^- = \{(0, y) : y < 0\}$, it always points inward the third quadrant, no matter what t is, as shown in figure 2.1. Therefore, as time increases, a trajectory $(x(t), \dot{x}(t))$ of (2.19) can cross Y^+ only from the left to the right and Y^- only from the right to the left. With this in mind, it is clear that, if x is such that $(x(a), \dot{x}(a))$ belongs to the right half plane and if $\theta_x(b) \geq, \theta_x(a) + \pi$, then the trajectory of x crosses Y^+ at some time t^* coming from the first quadrant and going to the second one, which is forbidden by the vector field. The same is when $(x(a), \dot{x}(a))$ lies in the left half plane.

However, we show how the inequality (2.18) can be proved analytically.

Proof. Let us argue by contradiction and suppose that (2.18) does not hold: then we can find $[t_1, t_2] \subset [a, b]$ such that $\theta_x(t_2) = \theta_x(t_1) + \pi$ and $\theta_x(t_1) < \theta_x(t) < \theta_x(t_1) + \pi$ for all $t \in]t_1, t_2[$. We have that $\theta_x(t_2) \in [k\pi + \pi/2, (k+1)\pi + \pi/2[$, for a unique $k \in \mathbb{Z}$, therefore the set $\{t \in [t_1, t_2] : \theta_x(t) = k\pi + \pi/2\}$ is nonempty and we can consider its supremum t^*. The equations (2.16) imply that $x(t^*) = 0$ and that $\dot{x}(t^*)$ is strictly positive, if k is even, or strictly negative if k is odd. The following alternative holds true:

1. either $t^* = t_2$ and $\theta_x(t) < \theta_x(t^*)$ for all t in a left neighbourhood of t^*;

2. or $t^* < t_2$ and $\theta_x(t) > \theta_x(t^*)$ in a right neighbourhood of t^*.

For the sake of definiteness let us assume that k is even (that is $\dot{x}(t^*) > 0$) and that alternative 2 holds (that is $t^* < t_2$): the other cases can be treated similarly. The cosine function is decreasing around $k\pi + \pi/2$ if k is even, therefore $\cos\theta_x(t) < 0$ and $x(t) < 0$ for every t in a right neighbourhood of t^*. This implies that $\dot{x}(t^*) \leq 0$, which is a contradiction. ∎

After these auxiliary lemmas, we can state the following theorem, which deals with the oscillatoric behaviour of large solutions.

Theorem 2.2.6 *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying $(f1)$ and $(f2)$. Then for every $H > 0$ there is $R = R(H) > 0$ such that, for every $t_0 \in [a,b]$ and for every solution x of (2.12) satisfying $x^2(t_0) + \dot{x}^2(t_0) \geq R$, the following properties hold:*

(*i*) $(x(t), \dot{x}(t)) \neq (0,0)$ *for all* $t \in [a,b]$;

(*ii*) $\theta_x(a) - \theta_x(b) \geq 2\pi H$.

Proof. By $(f1)$ and Lemma 2.2.2, there is $R_0 > 0$ such that every solution x of (2.12), with $x^2(t_0) + \dot{x}^2(t_0) \geq R_0^2$ for some $t_0 \in [a,b]$, will satisfies $(x(t), \dot{x}(t)) \neq (0,0)$ for every $t \in [a,b]$, so that the polar coordinates ρ_x and θ_x in (2.16) are defined. Suppose that the interval I in hypothesis $(f2)$ can be written as $[c,d]$. As anticipated before Lemma 2.2.5, we divide the total angle, which is spanned by the phase vector $(x(t), \dot{x}(t))$ as t ranges in $[a,b]$, into three parts:

$$\theta_x(a) - \theta_x(b) = (\theta_x(a) - \theta_x(c)) + (\theta_x(c) - \theta_x(d)) + (\theta_x(d) - \theta_x(b)).$$

By Lemma 2.2.5 we know that:

$$(\theta_x(a) - \theta_x(c)) + (\theta_x(d) - \theta_x(b)) > -2\pi. \tag{2.20}$$

In order to estimate $\theta_x(c) - \theta_x(d)$, we will consider the first time t^* after c, when $x(t^*) = 0$, so that we will be sure that the solution is on the y-axis, and then we will count the number of turns of the trajectory as t varies in $[t^*, d]$ by considering how many times the solution comes back on the y-axis. Moreover Lemmas 2.2.3 and 2.2.4 will help in evaluating the time needed to make a complete turn around the origin. Indeed, suppose that we have fixed a small $\delta > 0$: then, by lemma 2.2.4 there is $M = M(\delta)$ such that every solution of (2.12) will remain in $\{(x,y) \in \mathbb{R}^2 : |x| \geq M\}$ for a time interval smaller than δ; on the other hand, by Lemma 2.2.3 there is $N = N(M(\delta), \delta)$ such that every solution hitting the y-axis in a point farther from the origin than N will cross the strip $\{(x,y) \in \mathbb{R}^2 : |x| \leq M\}$ in a time interval smaller again than δ. Now, in order to make a complete turn around the origin, starting from a point on the y-axis, a "large" solution has to cross the vertical strip $\{(x,y) \in \mathbb{R}^2 : |x| \leq M\}$ three times and the region $\{(x,y) \in \mathbb{R}^2 : |x| \geq M\}$ twice: therefore the time to make a complete turn around the origin can be bounded by 5δ. On the other hand, the time $t^* - c$, which is needed to reach the y-axis the first time, can be bounded by 3δ, since the solution will have to cross the vertical strip twice and the other region once, in the worst case. The inequality in (*ii*) is satisfied if the trajectory makes at least $H + 1$ turns around the origin as t spans $[c,d]$, since in $[a,b] \setminus [c,d]$ it can loose no more than one turn by (2.20). Therefore, the solution has enough time to make $H + 1$ rotation around the origin in $[c,d]$ if we choose δ in such a way that $3\delta + 5\delta(H+1) \leq d - c$, that is:

$$\delta \leq \frac{d-c}{5H+8}.$$

So we rigorously prove the theorem with this choice for δ and the corresponding constants M and N, which are given by Lemmas 2.2.3 and 2.2.4.

Let $K = \sqrt{M^2 + N^2}$ and let $R_\delta > 0$ be such that every solution x of (2.12), with $x^2(t_0) + \dot{x}^2(t_0) \geq R_\delta$ for some $t_0 \in [a,b]$, also satisfies

$$x^2(t) + \dot{x}^2(t) \geq K^2 \qquad \forall\, t \in [a,b]$$

(this is possible by Lemma 2.2.2) and let us consider such a solution x. As in figure 2.2 we divide $\mathbb{R}^2 \setminus B(K)$ into four closed regions which can overlap only in correspondence of

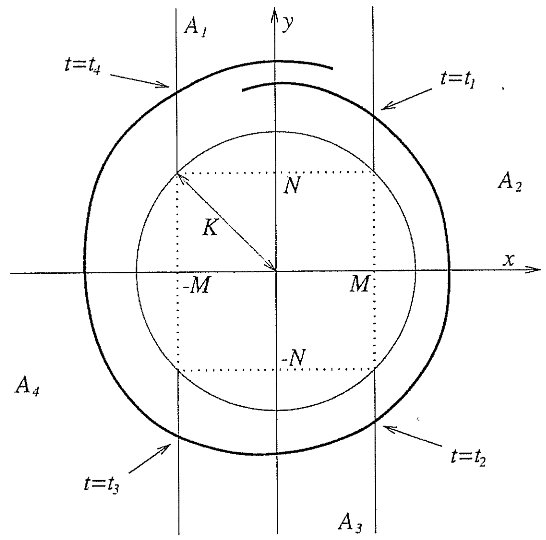

Figure 2.2: Following a complete turn of a solution

their boundaries: $A_1 = \{(x,y) \in \mathbb{R}^2 : y \geq N, |x| \leq M, x^2 + y^2 \geq K^2\}$, $A_2 = \{(x,y) \in \mathbb{R}^2 : x \geq M, x^2 + y^2 \geq K^2\}$, $A_3 = \{(x,y) \in \mathbb{R}^2 : y \leq -N, |x| \leq M, x^2 + y^2 \geq K^2\}$ and $A_4 = \{(x,y) \in \mathbb{R}^2 : x \leq -M, x^2 + y^2 \geq K^2\}$.

If $(x(c), \dot{x}(c)) \in A_1$, let $t_1 = \inf\{t \geq c : (x(t), \dot{x}(t)) \notin A_1\}$: then $\dot{x}(t) \geq N$ for all $t \in [c, t_1]$ and x is monotone increasing in $[c, t_1]$. In particular $t_1 - c \leq \delta$, by Lemma 2.2.3 and $x(t_1) = M$, that is $(x(t_1), \dot{x}(t_1)) \in A_2$. Now let $t_2 = \inf\{t \geq t_1 : (x(t), \dot{x}(t)) \notin A_2\}$: $x(t) \geq M \geq x^*$ for all $t \in [t_1, t_2]$, so that

$$\ddot{x} = -f(t,x) \leq -g(x(t)) < 0 \qquad \text{for a.e. } t \in [t_1, t_2]$$

and $\dot{x}$ is monotone decreasing in $[t_1, t_2]$. By Lemma 2.2.4, $t_2 - t_1 \leq \delta$ and t_2 belongs to $]c, d[$: therefore, as t varies from t_1 to t_2, $x(t)$ increases up to a maximum value greater than K and then decreases to $x(t_2) = M$. Thus we have that $(x(t_2), \dot{x}(t_2)) \in A_3$. The behaviour of the trajectory in A_3 is similar to that in A_1 : in particular, if $t_3 = \inf\{t \geq t_2 : (x(t), \dot{x}(t)) \notin A_3\}$, then $t_3 - t_2 \leq \delta$, $x(t_3) = -M$ and $(x(t_3), \dot{x}(t_3)) \in A_4$. Finally, letting $t_4 = \inf\{t \geq t_3 : (x(t), \dot{x}(t)) \notin A_4\}$ and arguing similarly as in A_2, we obtain that $t_4 - t_3 \leq \delta$, $x(t_4) = -M$ and $(x(t_4), \dot{x}(t_4)) \in A_1$, again.

It is clear, now, that $t^* - c \leq 3\delta$, if $t^* = \inf\{t \geq c : x(t) = 0\}$, and that the time needed to $(x(t), \dot{x}(t))$ to make a complete turn around the origin starting from the y-axis, is bounded by 5δ. By the choice of δ, there is $t_H \in]t^*, d[$ such that $\theta_x(t^*) - \theta_x(t_H) = 2\pi(H+1)$. Moreover we have that $\theta_x(c) - \theta_x(t^*) \geq 0$ and $\theta_x(t_H) - \theta_x(d) \geq 0$ (see (2.17)), hence we can conclude that:

$$\theta_x(c) - \theta_x(d) \geq 2\pi(H+1)$$

and this inequality, together with (2.20), ends the proof of the theorem. ∎

Remark 2.2.7 Another way of stating Theorem 2.2.6 is the following: if f satisfies $(f1)$ and $(f2)$ then, for every sequence $\{x_n\}$ of solutions of (2.12) such that:

$$\lim_{n \to +\infty} [x_n^2(t_n) + \dot{x}_n^2(t_n)] = +\infty,$$

for some $\{t_n\} \subset [a,b]$, we have that the angular functions θ_{x_n} are defined in $[a,b]$ for every large n and satisfy:

$$\lim_{n\to+\infty} [\theta_{x_n}(a) - \theta_{x_n}(b)] = +\infty.$$

Moreover we note that, if $\theta_x(t_1) - \theta_x(t_2) \geq \pi$ for some $t_1 < t_2$ and some solution x of (2.12), then there exists $t^* \in [t_1, t_2]$ such that $\theta_x(t^*) = k\pi$, for some $k \in \mathbb{Z}$, and, therefore, $x(t^*) = 0$. Similarly, if $x(t_1) = x(t_2) = 0$ and $x(t) \neq 0$ for every $t \in]t_1, t_2[$ then $\theta_x(t_1) - \theta_x(t_2) = \pi$. Hence Theorem 2.2.6 states that, if f satisfies $(f1)$ and $(f2)$, then larger solution of (2.12) will have more zeros in $[a,b]$ and will oscillate more.

We consider here an application of the previous result to the particular case which we will deal with in the next chapters: suppose that we have two functions $g : \mathbb{R} \to \mathbb{R}$ and $q : [a,b] \to [0,+\infty[$ as in the hypotheses of Corollary 2.1.3, in such a way that the uniqueness and the global continuability hold for all the initial value problems associated to the equation:

$$\ddot{x} + q(t)g(x) = 0. \tag{2.21}$$

Given $p \in \mathbb{R}^2$ with $p \neq (0,0)$, there is a unique solution x of the equation (2.21) such that $(x(a), \dot{x}(a)) = p$: we surely have $(x(t), \dot{x}(t)) \neq (0,0)$ for every $t \in [a,b]$, since $x \equiv 0$ is solution of (2.21); therefore we can use the formula (2.17) to define:

$$\operatorname*{rot}_{[a,b]}(p) = \int_a^b \frac{q(t)g(x(t))x(t) + \dot{x}^2(t)}{x^2(t) + \dot{x}^2(t)}\, dt, \tag{2.22}$$

that is, $\operatorname*{rot}_{[a,b]}(p)$ measure the angle spanned in clockwise sense by the vector $(x(t), \dot{x}(t))$ as t goes from a to b, p being the position of the vector at $t = a$. With this notation we have the following result.

Corollary 2.2.8 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function such that:*

$$g(x)x > 0 \qquad \forall x \neq 0 \tag{$g1$}$$

and:

$$\lim_{e\to+\infty} \tau_g^+(e) = 0. \tag{$g2+$}$$

Let $q : [a,b] \to [0,+\infty[$ be a nonnegative continuous function with bounded variation such that:

1. *the set $\{t \in [a,b] : q(t) = 0\}$ has a finite number of connected components;*
2. *if $q > 0$ on $]b,c[\subset [a,b]$ and $q(c) = 0$ (or, respectively, $q(d) = 0$), then q is monotone nondecreasing in a right neighbourhood of c (or, respectively, monotone nonincreasing in a left neighbourhood of d).*

Then:

$$\lim_{|p|\to+\infty} \operatorname*{rot}_{[a,b]}(p) = +\infty.$$

Proof. Simply apply Theorem 2.2.6 to the function $f(t,x) = q(t)g(x)$, which clearly satisfies $(f1)$ (by Corollary 2.1.3) and $(f2)$. ∎

2.3 An existence and multiplicity theorem for generalized Sturm–Liouville boundary value problems

Definition 2.3.1 We say that a set $C \subset \mathbb{R}^N$ is a *continuum* if it is nonempty, closed and connected.

Let us define $H^+ = (]0, +\infty[\times\mathbb{R}) \cup (\{0\}\times]0, +\infty[)$ and $H^- = -H^+ = (]-\infty, 0[\times\mathbb{R}) \cup (\{0\}\times]-\infty, 0[)$; thus $\overline{H^+}$ is the closed right half plane and $\overline{H^-}$ is the closed left half plane. Moreover we respectively denote by $B(R)$ and $B[R]$ the open and the closed ball in $\mathbb{R}^N$ which are centered at the origin and have radius R.

Our aim is to prove a result on boundary value problems for the equation:

$$\ddot{x} + f(t, x) = 0, \tag{2.23}$$

in which the boundary condition is expressed by imposing that $(x(t), \dot{x}(t))$ belongs to two fixed unbounded continua Γ_a and Γ_b at two fixed instants $t = a$ and $t = b$, respectively. We use a shooting technique: roughly speaking, we let $(x(a), \dot{x}(a))$ move along the first continuum Γ_a and we consider the set $\Pi(\Gamma_a)$ of all the corresponding final points $(x(b), \dot{x}(b))$, which turns out to be a continuum, too. A solution of the boundary value problem is found whenever we detect an intersection between $\Pi(\Gamma_a)$ and Γ_b. Then the following very simple topological lemma will be useful in the sequel.

Lemma 2.3.2 *Let us assume that $P, Q \subset \mathbb{R}^2$ are two continua which are both contained either in the closed right half plane $\overline{H^+}$ or in the closed left half plane $\overline{H^-}$. Moreover we suppose that there are $R_2 > R_1 > 0$ such that:*

(*i*) $P \subset B[R_2] \setminus B(R_1)$ *and* $P \cap (\{0\} \times [R_1, R_2]) \neq \emptyset \neq P \cap (\{0\} \times [-R_2, -R_1])$;

(*ii*) $Q \cap B(R_1) \neq \emptyset \neq Q \setminus B[R_2]$.

Then $P \cap Q \neq \emptyset$.

Proof. Without loss of generality we suppose that P and Q are both contained in $\overline{H^+}$. There exist $y_1 \in [R_1, R_2]$ and $y_2 \in [-R_2, -R_1]$ such that $(0, y_1), (0, y_2) \in P$, by (*i*). For every $\varepsilon > 0$ let $P_\varepsilon = \{(x, y) \in \overline{H^+} : |(x - x_0, y - y_0)| < \varepsilon$ for some $(x_0, y_0) \in P\}$ be the ε-neighbourhood of P in $\overline{H^+}$: clearly $P_\varepsilon \subset B(R_2 + \varepsilon) \setminus B[R_1 - \varepsilon]$ and P_ε is still a connected set. However, since it is also open relatively to $\overline{H^+}$, it is connected by arcs; therefore there exists a continuous curve $\gamma_\varepsilon : [0, 1] \to P_\varepsilon$ which links $(0, y_1)$ to $(0, y_2)$, that is $\gamma_\varepsilon(0) = (0, y_1)$ and $\gamma_\varepsilon(1) = (0, y_2)$. It is not restrictive to assume also that actually $\gamma_\varepsilon(s) \in]0, +\infty[\times\mathbb{R}$ for every $s \in]0, 1[$. Hence the union of the image of γ_ε and of the vertical segment $\{0\} \times [y_2, y_1]$ is the image of a Jordan curve in $\mathbb{R}^2$, which is the boundary (relatively to $\mathbb{R}^2$) of a bounded open region that we denote by U_ε. Let $V_\varepsilon = U_\varepsilon \cup (\{0\}\times]y_2, y_1[)$: with respect to the topology in $\overline{H^+}$, V_ε is open and its boundary is exactly the image of γ_ε.

If ε is sufficently small, Q intersects both $B(R_1 - \varepsilon)$ and the complement of $B[R_2 + \varepsilon]$: this implies that Q has some points in V_ε and in $\overline{H^+} \setminus \overline{V_\varepsilon}$. Since Q is connected, then Q must intersect the boundary of V_ε relatively to $\overline{H^+}$, that is there exists $p_\varepsilon \in Q \cap \gamma_\varepsilon([0, 1])$. Up to choose a suitable sequence, we have that $p_\varepsilon \to p_0$ as $\varepsilon \to 0$ and $p_0 \in P \cap Q$. ■

When a function $F : [a, b] \times \mathbb{R}^N \to \mathbb{R}^N$ is regular enough in order to ensure that all the solutions of the differential equation $\dot{x} = F(t, x)$ enjoy the local uniqueness and the global continuability, the Poincaré map $\Pi : \mathbb{R}^N \to \mathbb{R}^N$, which sends each $p \in \mathbb{R}^N$ to the value

$x(b)$ of the unique solution of the equation starting with $x(a) = p$, is well defined and is a homeomorphism of $\mathbb{R}^N$ onto itself. In particular, if $\Gamma \subset \mathbb{R}^N$ is a continuum, then $\Pi(\Gamma)$ is a continuum, too. When the local uniqueness holds no more, then, given a point $p \in \mathbb{R}^N$, one has to consider the set:

$$\Xi(p) = \{x \in AC([a,b],\mathbb{R}^N) : x \text{ solves } \dot{x} = F(t,x) \text{ and } x(a) = p\},$$

where $AC([a,b],\mathbb{R}^N)$ denotes the set of absolutely continuous functions $x : [a,b] \to \mathbb{R}^N$, and Π has no meaning as an ordinary map, but can be considered as a set–valued map setting:

$$\Pi(p) = \{x(b) : x \in \Xi(p)\}.$$

If the global continuability holds, the Theorem on the Peano Phenomenon (see [103, III, §2.3, p. 165]) and Kneser's Theorem (see [55, Theorem 4.1, II, §4, pp. 15–18] ensure that, for every $p \in \mathbb{R}^N$, both $\Xi(p)$ and $\Pi(p)$ are compact continua as subsets of $C([a,b],\mathbb{R}^N)$ (endowed of the norm $\|\cdot\|_\infty$) and $\mathbb{R}^N$, respectively. Moreover the following can be proved:

Lemma 2.3.3 *Let $F : [a,b]\times\mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that every solution of $\dot{x} = F(t,x)$ is continuable on $[a,b]$. If $\Gamma \subset \mathbb{R}^N$ is a continuum, then the set:*

$$\mathcal{X}(\Gamma) = \bigcup_{p\in\Gamma} [\{p\} \times \Xi(p)] = \{(p,x) \in \mathbb{R}^N \times C([a,b],\mathbb{R}^N) : p \in \Gamma \text{ and } x \in \Xi(p)\},$$

is a continuum with respect to the product topology of $\mathbb{R}^N \times C([a,b],\mathbb{R}^N)$; moreover, if Γ is compact, $\mathcal{X}(\Gamma)$ is compact, too. In particular the set:

$$\Pi(\Gamma) = \bigcup_{p\in\Gamma} \Pi(p)$$

is a continuum in $\mathbb{R}^N$.

Proof. The closedness of both $\mathcal{X}(\Gamma)$ and $\Pi(\Gamma)$ is a consequence of the continuous dependence on initial data as well as the compactness of $\mathcal{X}(\Gamma)$, when Γ is compact: so we confine ourselves to their connectedness.

Let $C_1, C_2 \subset \mathbb{R}^N \times C([a,b],\mathbb{R}^N)$ be two closed and disjoint subsets such that $C_1 \cup C_2 = \mathcal{X}(\Gamma)$: we want to show that one of them must be empty. Let $\pi_1 : \mathbb{R}^N \times C([a,b],\mathbb{R}^N) \to \mathbb{R}^N$ and $\pi_2 : \mathbb{R}^N \times C([a,b],\mathbb{R}^N) \to C([a,b],\mathbb{R}^N)$ be the two continuous projections. First, we observe that, if $(p,x_0) \in C_i$ for some $x_0 \in \Xi(p)$ and some $i \in \{1,2\}$, then $\{p\} \times \Xi(p) \subset C_i$: indeed, if $(p,x_1) \in C_1$ and $(p,x_2) \in C_2$ for two different solutions $x_1, x_2 \in \Xi(p)$, then we should have:

$$\{p\} \times \Xi(p) \cap C_1 \neq \emptyset \neq \{p\} \times \Xi(p) \cap C_2$$

and, therefore, $\{p\} \times \Xi(p)$ should not be connected in contradiction with the Theorem on Peano Phenomenon. Hence the following alternative holds for every $p \in \Gamma$:

1. either $\{p\} \times \Xi(p) \subset C_1$
2. or $\{p\} \times \Xi(p) \subset C_2$.

This implies that the two sets $\pi_1(C_1)$ and $\pi_1(C_2)$ are disjoint subsets of Γ. It is also clear that $\Gamma = \pi_1(C_1) \cup \pi_1(C_2)$. Moreover it is easy to see that $\pi_1(C_i)$ is closed for $i = 1,2$. Indeed, if $\{p_n\} \subset \pi_1(C_i)$ is a sequence such that $p_n \to p_0$, then it is possible to select some $x_n \in \Xi(p_n)$

and $x_0 \in \Xi(p_0)$ by the continuous dependence on initial data in such a way that $x_n \to x_0$, up to a subsequence; since $(p_n, x_n) \in C_i$ and C_i is closed, we have that $(p_0, x_0) \in C_i$ and, therefore, $p_0 \in \pi_1(C_i)$. We can conclude that either $\pi_1(C_1) = \emptyset$ or $\pi_1(C_2) = \emptyset$, because Γ is connected. Thus either $C_1 = \emptyset$ or $C_2 = \emptyset$.

We finally note that:

$$\Pi(\Gamma) = \mathcal{E}_b(\pi_2(\mathcal{X}(\Gamma))),$$

where $\mathcal{E}_b : C([a,b], \mathbb{R}^N) \to \mathbb{R}^N$ is the evaluation functional at b, which is defined by $\mathcal{E}_b(x) = x(b)$. Both $\mathcal{E}_b$ and π_2 are continuous and $\mathcal{X}(\Gamma)$ is connected: therefore $\Pi(\Gamma)$ is connected, too. ∎

With the previous auxiliary results we can now try to solve boundary value problems of the following kind:

$$\begin{cases} \ddot{x} + f(t,x) = 0 \\ (x(a), \dot{x}(a)) \in \Gamma_a \\ (x(b), \dot{x}(b)) \in \Gamma_b \end{cases} \tag{2.24}$$

which have the same form of those studied in [116].

Theorem 2.3.4 *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying $(f1)$ and $(f2)$. Let Γ_a and Γ_b be two unbounded continua in $\mathbb{R}^2$ such that one of the following alternatives holds:*

(i) $\Gamma_a \subset H^+$ and $\Gamma_b \subset H^-$ or, vice versa, $\Gamma_a \subset H^-$ and $\Gamma_b \subset H^+$;

(ii) Γ_a and Γ_b are both contained either in H^+ or in H^-.

Finally we suppose that $\Gamma_a \cap B[R_a] \neq \emptyset \neq \Gamma_b \cap B[R_b]$, for some $R_a, R_b > 0$. Then there exists a positive integer $n^* = n^*(R_a, R_b)$ such that:

1. *if (i) holds, problem (2.24) has at least a solution x with exactly n zeros in $]a,b]$, for every* odd *integer $n \geq n^*$;*
2. *if (ii) holds, problem (2.24) has at least a solution x with exactly n zeros in $]a,b]$, for every* even *integer $n \geq n^*$.*

Proof. By $(f1)$ and Lemma 2.2.2 there is $R_0 = R_0(R_b, R_a) \geq R_a$ such that each solution x of (2.23) satisfies $x^2(t) + \dot{x}^2(t) \geq R_b + 1$ for all $t \in [a,b]$, if $x^2(a) + \dot{x}^2(a) \geq R_0$. We select an unbounded connected component of $\Gamma_a \setminus B(R_0)$ such that $\Gamma_a \cap \partial B(R_0) \neq \emptyset$ and we continue to call it Γ_a with a little abuse of notation.

Let us consider only the case $\Gamma_a \subset H^+$; the other one, that is $\Gamma_a \subset H^-$, can be handled in a symmetric way. We take the angular coordinate function $\vartheta : \overline{H^+} \setminus B(R_b) \to [-\pi/2, \pi/2]$ which is continuous and satisfies:

$$p = (|p| \cos \vartheta(p), |p| \sin \vartheta(p)) \qquad \forall\, p \in H^+.$$

Moreover, we define the functions $\rho, \theta : [a,b] \times \mathcal{X}(\overline{H^+} \setminus B(R_b))$ (see Lemma 2.3.3 for the meaning of $\mathcal{X}$) in the following way:

$$\rho(t; p, x) = \sqrt{x^2(t) + \dot{x}^2(t)} \qquad \forall\, t \in [a,b],\ \forall\, (p,x) \in \mathcal{X}(\overline{H^+} \setminus B(R_b))$$

and:

$$\theta(t;p,x) = \vartheta(p) - \int_a^t \frac{f(s,x(s))x(s) + \dot{x}^2(s)}{\rho(s;p,x)}\,ds,$$

which turn out to be well defined and continuous in the topology inherited from $[a,b] \times \mathbb{R}^2 \times C^1([a,b])$ and satisfy:

$$\begin{cases} x(t) = \rho(t;p,x)\cos\theta(t;p,x) \\ \dot{x}(t) = \rho(t;p,x)\sin\theta(t;p,x) \end{cases} \qquad \forall\, t \in [a,b],\ \forall\,(p,x) \in \mathcal{X}(\overline{H^+} \setminus B(R_b)) \tag{2.25}$$

(see (2.16) and (2.17) for details). In particular, for every $(p,x) \in \mathcal{X}(\overline{H^+} \setminus B(R_b))$ the quantity:

$$\vartheta(p) - \theta(t;p,x) = \theta(a;p,x) - \theta(t;p,x)$$

measures the angle spanned in clockwise sense by the vector $(x(s),\dot{x}(s))$ as s increases from a to t. We observe that the solution x will have a zero in t_0 if and only if $\theta(t_0;p,x) = k\pi + \pi/2$ for some integer k, that is if and only if the point $(x(t_0),\dot{x}(t_0))$ lies on the y-axis in the phase plane; therefore, the solution x will have exactly n zeros in $]a,b]$ if and only if:

$$-\frac{\pi}{2} - n\pi < \theta(b;p,x) \le -\frac{\pi}{2} - (n-1)\pi, \tag{2.26}$$

as can be seen by arguments similar to those employed to justify and prove Lemma 2.2.5.

Let us come to the definition of n^*. Since θ is continuous and $\overline{H^+} \cap \partial B(R_0)$ is nonempty and compact, we can define:

$$\theta^* = \min\{\theta(b;p,x) : (p,x) \in \mathcal{X}(\overline{H^+} \cap \partial B(R_0))\}$$

which turns out to be finite by Lemma 2.3.3. As a consequence of this definition and of the fact that Γ_a is connected and intersects $\partial B(R_0)$, we have that the image of the function:

$$\mathcal{X}(\Gamma_a)(p,x) \longmapsto \theta(b;p,x)$$

contains the unbounded interval $]-\infty,\theta^*]$. Hence a good choice is to take n^* to be the least positive integer such that:

$$-\frac{\pi}{2} - (n^* - 1)\pi \le \theta^*,$$

as can be seen by considering the condition (2.26).

It is now clear from (2.26) and (2.25) that, if a solution x starts from $p \in H^+$ at $t = a$ and has n zeros in $]a,b]$, then the final point $(x(b),\dot{x}(b))$ belongs to H^+, if n is even, and to H^-, if n is odd. Without loss of generality we suppose that Γ_b lies in H^+, we consider an even number $n \ge n^*$ and we look for a couple $(p_0,x_0) \in \mathcal{X}(\Gamma_a)$ such that:

1. $-\frac{\pi}{2} - n\pi < \theta(b;p_0,x_0) \le -\frac{\pi}{2} - (n-1)\pi$;
2. $(x_0(b),\dot{x}_0(b)) \in \Gamma_b$.

The function x_0 will be the desired solution of (2.24).

Thanks to the choice of n^* there is a closed and connected subset Σ_n of $\mathcal{X}(\Gamma_a)$ such that:

- $-\frac{\pi}{2} - n\pi \le \theta(b;p,x) \le -\frac{\pi}{2} - (n-1)\pi$ for every $(p,x) \in \Sigma_n$;
- $\theta(b;p_1,x_1) = -\frac{\pi}{2} - (n-1)\pi$ and $\theta(b;p_2,x_2) = -\frac{\pi}{2} - n\pi$ for some couples (p_1,x_1), $(p_2,x_2) \in \Sigma_n$.

The set $P = \{(x(b), \dot{x}(b)) : (p, x) \in \Sigma_n\}$ is closed and connected (see the proof of Lemma 2.3.3 for similar arguments), is contained in $\overline{H^+} \setminus B(R_b + 1)$ and is surely bounded by Theorem 2.2.6, thus it is contained in a closed annulus $B[R_2] \setminus B(R_b + 1)$, for some $R_2 > R_b + 1$. On the other hand we know that the set $Q = \Gamma_b$ is a continuum contained in $\overline{H^+}$ which intersects $B[R_b]$ and, therefore, $B(R_b + 1)$; moreover intersects also the complement of $B[R_2]$ since it is unbounded. Hence $P \cap Q \neq \emptyset$ by Lemma 2.3.2: this implies that there exists a couple $(p_0, x_0) \in \mathcal{X}(\Gamma_a)$ such that statements 1 and 2 above hold and the theorem is proved. ∎

In the following chapters we will need to apply this result to the equation (2.1), that is we will consider boundary value problems like:

$$\begin{cases} \ddot{x} + q(t)g(x) = 0 \\ (x(a)\dot{x}(a)) \in \Gamma_a \\ (x(b)\dot{x}(b)) \in \Gamma_b. \end{cases} \tag{2.27}$$

What will be enough is the following corollary.

Corollary 2.3.5 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function such that $g(x)x > 0$ for every $x \neq 0$ and:*

$$\lim_{e \to +\infty} \tau_g^+(e) = 0.$$

Let $q : [a, b] \to [0, +\infty[$ be a nonnegative continuous function with bounded variation such that:

1. *the set $\{t \in [a, b] : q(t) = 0\}$ has a finite number of connected components;*
2. *if $q > 0$ on $]b, c[\subset [a, b]$ and $q(c) = 0$ (or, respectively, $q(d) = 0$), then q is monotone nondecreasing in a right neighbourhood of c (or, respectively, monotone nonincreasing in a left neighbourhood of d).*

Let Γ_a and Γ_b be two unbounded continua in $\mathbb{R}^2$ such that one of the following alternatives holds:

(i) $\Gamma_a \subset H^+$ and $\Gamma_b \subset H^-$ or, vice versa, $\Gamma_a \subset H^-$ and $\Gamma_b \subset H^+$;

(ii) Γ_a and Γ_b are both contained either in H^+ or in H^-.

Finally we suppose that $\Gamma_a \cap B[R_a] \neq \emptyset \neq \Gamma_b \cap B[R_b]$, for some $R_a, R_b > 0$. Then there exists a positive integer $n^* = n^*(R_a, R_b)$ such that:

1. *if (i) holds, problem (2.27) has at least a solution x with exactly n zeros in $]a, b]$, for every* odd *integer $n \geq n^*$;*
2. *if (ii) holds, problem (2.27) has at least a solution x with exactly n zeros in $]a, b]$, for every* even *integer $n \geq n^*$.*

Proof. Consider $f(t, x) = q(t)g(x)$ and apply Theorem 2.3.4: $(f1)$ holds by Corollary 2.1.3 and $(f2)$ holds by the assumptions we made on q and g. ∎

Chapter 3

Superlinear equations with negative weight

Our purpose here is to describe the qualitative behaviour of the solutions of the equation:

$$\ddot{x} + q(t)g(x) = 0,$$

when g is again superlinear in some sense, but the weight function q is nonpositive. Like in the preceding chapter, at first let us give a look to what happens in the case $q \equiv -1$, that is we initially deal with the autonomous equation $\ddot{x} - g(x) = 0$ or with the equivalent planar system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = g(x). \end{cases} \tag{3.1}$$

We suppose that the continuous function g satisfies the usual sign condition $g(x)x > 0$ for all $x \neq 0$ and we call $G(x) = \int_0^x g(s)\,ds$: G is a nonnegative function, is strictly monotone increasing on $[0, +\infty[$ and strictly monotone decreasing on $]-\infty, 0]$ and we assume also that $G(\pm\infty) = +\infty$, so that a continuous right inverse $G_+^{-1} : [0, +\infty[\to [0, +\infty[$ and a continuous left inverse $G_-^{-1} : [0, +\infty[\to]-\infty, 0]$ are defined. A direct computation shows that each trajectory $(x(t), \dot{x}(t))$ lies in a level line of the energy function $E(x, y) = \frac{1}{2}y^2 - G(x)$, that is $E(x(t), y(t)) =$ constant, so that the phase plane picture is the one represented in figure 3.1. Each positive value e of the energy E gives rise to the two unbounded orbits which cross the y-axis at $(0, \sqrt{e})$ and at $(0, -\sqrt{e})$ and which correspond respectively to a strictly monotone increasing solution and to a strictly monotone decreasing one, both with exactly one zero in which x changes its concavity. However each of the two branches of the level set $E = e > 0$ is run by the corresponding solution in a time that is given by the following integral:

$$\frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{e + G(s)}}\, ds.$$

On the other hand the level set $E = -e$, with $e > 0$, is filled by the two unbounded orbits which cross the x-axis at $(G_-^{-1}(e), 0)$ and at $(G_+^{-1}(e), 0)$ and which correspond respectively to a negative and concave solution x and to a positive and convex one, both with exactly one zero of the derivative $\dot{x}$. Moreover the time that the point $(x(t), \dot{x}(t))$ takes to run the

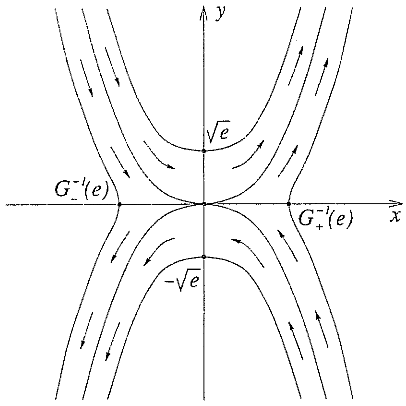

Figure 3.1: The phase portrait for (3.1)

left orbit (that is the one passing by $(G_-^{-1}(e), 0)$) is given by:

$$\sqrt{2}\int_{-\infty}^{G_-^{-1}(e)} \frac{1}{\sqrt{G(s)-e}}\, ds,$$

while the time needed to run the right orbit (the one passing by $(G_+^{-1}(e), 0)$) is given by:

$$\sqrt{2}\int_{G_+^{-1}(e)}^{+\infty} \frac{1}{\sqrt{G(s)-e}}\, ds.$$

Finally the zero level $E = 0$ is formed by five different orbits: the constant one $(x(t), \dot{x}(t)) \equiv 0$ and four unbounded branches which represents solutions that are asymptotic to $(0, 0)$ as the time tends to $+\infty$ or to $-\infty$. In other words, the set $E = 0$ is the union of the origin and of the stable and unstable manifolds relatively to $(0, 0)$.

If we define:

$$\tau_g^-(e) = \begin{cases} \sqrt{2}\displaystyle\int_{-\infty}^{+\infty} \frac{1}{\sqrt{e+G(s)}}\, ds & \text{if } e > 0 \\ \sqrt{2}\displaystyle\int_{-\infty}^{G_-^{-1}(-e)} \frac{1}{\sqrt{G(s)+e}}\, ds + \sqrt{2}\int_{G_+^{-1}(-e)}^{+\infty} \frac{1}{\sqrt{G(s)+e}}\, ds & \text{if } e < 0, \end{cases} \tag{3.2}$$

then $\tau_g^-(e)$ represents the whole time of moving along the level line $E = e$.

When superlinear functions g, like the powers $|x|^{\alpha-1}x$ with $\alpha > 1$, are considered, all the integrals above are finite and tend to zero as the value of the energy e tends to infinity, that is *all* the solutions of (3.1) with nonzero energy have a bounded maximal interval of continuability, whose length tends to zero as the absolute value of their energy goes to infinity. In particular every solution x presents a blow-up in finite time, in the sense that $|x(t)|$ and $|\dot{x}(t)|$ tend to infinity as t approaches some finite value t^*. However, since there is the trivial solution $x \equiv 0$, there are also solutions of (3.1) with nonzero energy and arbitrarily large maximal interval of continuability by the continuous dependence on initial data. If we fix a closed interval, say $[0, 1]$, and we consider the set Ω_0^1 of the points $p \in \mathbb{R}^2$ such that the solution of (3.1) with $(x(0), \dot{x}(0)) = p$ is defined on $[0, 1]$, then we have that Ω_0^1 is open by the continuous dependence on initial data, it contains the origin and its stable manifold (that is the two unbounded branches of the set $E = 0$ which are contained in the second and in the fourth quadrants) and, therefore, it is unbounded. Its boundary $\partial\Omega_0^1$ contains those points p such that the solution with $(x(0), \dot{x}(0)) = p$ has exactly $[0, 1[$ as right maximal interval of continuability and presents the blow-up at time 1. Since the orbits which are farther from the stable manifold correspond to solutions with smaller maximal interval of continuability, it turns out that the set Ω_0^1 develops around the stable manifold.

Finally we can consider the Poincaré map relatively to the system (3.1) on the interval $[0, 1]$, that is the map Π which associates to a point $p \in \mathbb{R}^2$ the point $(x(1), \dot{x}(1))$ reached at $t = 1$ by the solution of (3.1) satisfying $(x(0), \dot{x}(0)) = p$: the domain of Π is no more the whole $\mathbb{R}^2$, like it happens for the system (2.2), but it has to be reduced to Ω_0^1. Let us consider a straight line r through the origin in the phase plane and suppose that its equation can be written as $y = mx$ with positive angular coefficient $m > 0$. The points of r which lie far from the origin, belongs to level sets of the energy function that correspond to large values e : therefore Π is defined only on a bounded portion of r. If we take the points $p_s = (s, ms)$ of r letting s assume increasing positive values from $s = 0$, then the solutions starting from p_s at $t = 0$ are positive and monotone increasing in $[0, 1]$, so that $\Pi(p_s)$ belongs to the first quadrant until s reaches a value s^* such that $p_{s^*} \in \partial\Omega_0^1$ and the solutions starting from p_{s^*} has blow-up as t tends to 1 from the left. Hence, the segment $\{p_s : s \in [0, s^*[\,\}$ is transormed by Π into an unbounded curve in the first quadrant. Similarly there is $s_* < 0$ such that $\Pi(p_s)$ describes an unbounded curve in the third quadrant as s varies in $]s_*, 0]$.

In the same way, one can consider the "backward version" of Ω_0^1, that is the set Ω_1^0 of points $p \in \mathbb{R}^2$ such that the solution x with $(x(1), \dot{x}(1)) = p$ is defined on $[0, 1]$, and find that Ω_1^0 possesses the same properties of Ω_0^1 except for the fact that now Ω_1^0 develops around the unstable manifold (that is the two unbounded branches of the set $E = 0$ which are contained in the first and in the third quadrants) and the blow-up of solutions happens at $t = 0$ when they start from $p \in \partial\Omega_1^0$ at $t = 1$.

3.1 The sets of continuability

Let us come back to consider a general function $q \in C^0([a, b])$, with $q(t) \leq 0$ and $q \not\equiv 0$. About $g : \mathbb{R} \to \mathbb{R}$ we assume the usual sign condition:

$$(g1) \qquad g(x)x > 0 \qquad \forall x \neq 0,$$

and that g is locally Lipschitz continuous in order to have the local uniqueness for every initial value problem associated to the equation

$$(3.3) \qquad \ddot{x} + q(t)g(x) = 0.$$

For each $t_0 \in [a, b]$ and for each $p \in \mathbb{R}^2$ let $x(\cdot; t_0, p)$ be the unique solution of:

$$(3.4) \qquad \begin{cases} \ddot{x} + q(t)g(x) = 0 \\ (x(t_0), \dot{x}(t_0)) = p \end{cases}$$

defined in its maximal interval of existence, which is a neighbourhood of t_0, and let:

$$z(\cdot; t_0, p) = (x(\cdot; t_0, p), y(\cdot; t_0, p)) = (x(\cdot; t_0, p), \dot{x}(\cdot; t_0, p))$$

be the solution of the equivalent first order plane system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -q(t)g(x) \\ (x(t_0), y(t_0)) = p. \end{cases} \tag{3.5}$$

Following the notation introduced in [23], for every $t_1, t_2 \in [a, b]$ we define:

$$\Omega_{t_1}^{t_2} = \{p \in \mathbb{R}^2 : x(t; t_1, p) \text{ is defined for all } t \in [\min\{t_1, t_2\}, \max\{t_1, t_2\}]\}, \tag{3.6}$$

that is $\Omega_{t_1}^{t_2}$ is the set of all the points p of the phase plane such that the solution which starts at $t = t_1$ from p, is defined up to t_2.

Burton and Grimmer proved in [21] that the integrability of $1/\sqrt{G(x)}$ in a neighbourhood of either $-\infty$ or $+\infty$ is a necessary and sufficient condition in order to have at least one solution of (3.3) which is not defined on the whole $[a, b]$ and, therefore, explodes somewhere in $[a, b]$. In [23] and [24] Butler assumed that $1/\sqrt{1 + G(x)}$ is integrable on the real line, which is equivalent to require that:

$$\lim_{e \to +\infty} \tau_g^-(e) = 0$$

(see (3.2)), and proved that, if $q < 0$ in $]a, b[$, then the intersection of Ω_a^b with every vertical strip of the form $K \times \mathbb{R}$ is bounded, K being a compact subset of $\mathbb{R}$. In the next lemma we are going to prove that, actually, the intersection of Ω_a^b with every straight line passing by the origin is bounded, assuming that:

$$\lim_{e \to \pm\infty} \tau_g^-(e) = 0. \tag{$g2-$}$$

Lemma 3.1.1 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying $(g1)$ and $(g2-)$, let $q : [a, b] \to]-\infty, 0]$ be a continuous function with $q \not\equiv 0$ and let $r = \{(r_1 u, r_2 u) \in \mathbb{R}^2 : u \geq 0\}$, with $r_1^2 + r_2^2 = 1$, be an arbitrary half line starting from the origin. Then there exists $L = L(r, q) > 0$ such that $(r_1 u, r_2 u) \notin \Omega_a^b \cup \Omega_b^a$ for all $|u| \geq L$.*

Proof. We can assume without loss of generality that actually $q(t) \not\equiv 0$ in every right neighborhood of a and in every left neighborhood of b. In fact, if $q(t) \equiv 0$ on $[a, a']$ and on $[b', b]$, the Poincaré maps $p \mapsto z(a'; a, p)$ and $p \mapsto z(b'; b, p)$ homeomorphically transforms our half line r in another half line, leaving the origin fixed. This reduction implies that both a and b are now accumulation points of the set $\{t \in [a, b] : q(t) < 0\}$.

Moreover it is sufficient to show that $r \cap \Omega_a^b$ is bounded, since a simple inversion of the direction of time in the equation transforms the other case $r \cap \Omega_b^a$ in the previous one.

We will show the proof of the fact that $r \cap \Omega_a^b$ is bounded when r is given by $\{(u, -hu) : u > 0\}$ for some $h > 0$, since this turns out to be the most difficult case and the other ones can be proved with analogous (or even simpler) calculations. We set $x(t; u) := x(t; a, (u, -hu))$ to simplify the notation. Since $x(\cdot; u)$ starts positive in a, then it is convex and its graph lies over the line which is tangent in $t = a$, until it remains positive; hence we have that:

$$x(t; u) \geq (-hu)(t - a) + u, \qquad \forall t \in \left[a, a + \frac{1}{h}\right]$$

and:

$$x(t; u) \geq \frac{u}{2}, \qquad \forall t \in \left[a, a + \frac{1}{2h}\right], \tag{3.7}$$

whenever $x(t,u)$ is defined. By our initial remark there exist an interval $[a',b'] \subset]a, a+\frac{1}{2h}[$ and a constant $m > 0$ such that:

$$-q(t) \geq m, \qquad \forall t \in [a',b'].$$

We will actually show that the right maximal interval of continuability of $x(\cdot;u)$ is contained in $[a,a']$ for every u large enough: this will obviously implies the existence of L.

As a first step, we are going to find that, for all u sufficiently large, $x(\cdot;u)$ has an internal minimum whose abscissa lies in $]a,a']$. Let us argue by contradiction and suppose that there are a sufficiently small number $\eta > 0$ (for instance $\eta < b' - a'$) and two sequences $0 < u_n \to +\infty$ and $t_n \in]a'+\eta, b']$ such that $\dot{x}_n(t) \leq 0$ for every $t \in [a',t_n]$, where $x_n(t) := x(t;u_n)$.

Therefore we have:

$$\ddot{x}_n(t)\dot{x}_n(t) = -q(t)g(x_n(t))\dot{x}_n(t) \leq mg(x_n(t))\dot{x}_n(t), \quad \forall\, t \in [a',t_n].$$

Integrating this inequality between $t \in [a',t_n]$ and t_n we obtain:

$$\begin{aligned} \dot{x}_n^2(t) &\geq 2m[G(x_n(t)) - G(x_n(t_n))] + \dot{x}_n^2(t_n) \\ &\geq 2m[G(x_n(t)) - G(x_n(t_n))] \qquad \forall\, t \in [a',t_n]. \end{aligned}$$

Recalling that $\dot{x}(t) \leq 0$ on $[a',t_n]$, we have:

$$-\dot{x}_n(t) \geq \sqrt{2m}\sqrt{[G(x_n(t)) - G(x_n(t_n))]} \qquad \forall\, t \in [a',t_n]$$

and then:

$$1 \leq -\frac{1}{\sqrt{2m}}\frac{\dot{x}_n(t)}{\sqrt{G(x_n(t)) - G(x_n(t_n))}} \qquad \forall\, t \in [a',t_n].$$

A second integration between a' and t_n leads to:

$$\begin{aligned} \eta &< t_n - a' \\ &\leq \frac{1}{\sqrt{2m}}\int_{x_n(t_n)}^{x_n(a')} \frac{dx}{\sqrt{G(x) - G(x_n(t_n))}} \\ &\leq \frac{1}{\sqrt{2m}}\int_{x_n(t_n)}^{+\infty} \frac{dx}{\sqrt{G(x) - G(x_n(t_n))}} \\ &\leq \frac{1}{2\sqrt{m}}\tau^-(-G(x_n(t_n))). \end{aligned}$$

Thanks to our choices we know that $x_n(t_n) \geq \frac{u_n}{2}$ by (3.7), so that $x_n(t_n) \to +\infty$ as $n \to +\infty$. Hence we obtain a contradiction with hypotheses $(g2-)$.

Now, let $t_0 = t_0(u) \in]a,b[$, be the minimum point of $x(\cdot;u)$ for u sufficiently large. Then, for such large values of u, we have:

1. $\dot{x}(t_0;u) = 0$ and $x(\cdot;u)$ is a convex function, wherever it is defined in $[a,b]$;

2. $t_0(u) \leq a'$;

3. $x(t;u) \geq x(t_0;u) \to +\infty$ as $u \to +\infty$, by (3.7).

Now we can show that the right interval of continuability of $x(\cdot;u)$ is contained in $[a,a']$ for every sufficiently large u. Let us argue again by contradiction: let us suppose that there are a number $0 < \eta < b' - a'$ and two sequences $u_n \to +\infty$ and $t_n \in]a'+\eta, b'[$

such that $x_n(\cdot) := x(\cdot; u_n)$ is continuable up to t_n and $t_0(u_n) \in]a, a']$ for every n (this last condition can be fulfilled thanks to point 2 above). Then we have that $\dot{x}_n(t) \geq 0$ and $x_n(t) \geq x_n(a') \geq x_n(t_0(u_n))$, as $t \in [a', t_n]$. Estimates, which are completely analogous to those carried on above, lead to:

$$\begin{aligned} \eta &< t_n - a' \\ &\leq \frac{1}{\sqrt{2m}} \int_{x_n(a')}^{+\infty} \frac{dx}{\sqrt{G(x) - G(x_n(a'))}} \\ &\leq \frac{1}{2\sqrt{m}} \tau^-(-G(x_n(a'))) \end{aligned}$$

and this again leads to a contradiction with the assumption $(g2-)$, since we have that $x_n(a') \to +\infty$ as $n \to +\infty$. ■

3.2 Solutions asymptotic to zero

When $q \equiv -1$ we have that the solutions with zero energy of $\ddot{x} - g(x) = 0$ are asymptotic to the constant solution $x \equiv 0$. In the following lemmas we consider continuous and nonpositive weights q which are defined on unbounded intervals like $]-\infty, a]$ and $[b, +\infty[$ and we give conditions on q and g in order to have a similar situation fir the equation:

$$\ddot{x} + q(t)g(x) = 0.$$

We will look for unbounded continua of points $p \in \mathbb{R}^2$ such that:

$$\text{either} \quad \lim_{t \to -\infty} z(t; a, p) = (0, 0) \quad \text{or} \quad \lim_{t \to +\infty} z(t; b, p) = (0, 0)$$

(see (3.4) and (3.5) for the definitions of the Poincaré map $z(t; t_0, p)$). An essential tool we need is a topological lemma which is proved in [109] and we state it here for imediate reference.

Lemma 3.2.1 *Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, -1 < x < 1\}$, $P_0 = (0, -1)$, $P_1 = (0, 1)$, $Q_0 = (-1, 0)$ and $Q_1 = (1, 0)$ and assume that there is a set $C \subset D^\circ = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ such that $\overline{C} \cap D = C$ and every continuous curve in D from P_0 to P_1 meets C. Then there is a connected set $\Gamma \subseteq C$ such that $\overline{\Gamma} \cap D = \Gamma$ and $Q_0, Q_1 \in \overline{\Gamma}$.*

Here are the two lemmas about solutions which are asymptotic to the origin in the phase plane. Let $A_1 = [0, +\infty[\times [0, +\infty[$, $A_2 =]-\infty, 0] \times [0, +\infty[$, $A_3 =]-\infty, 0] \times]-\infty, 0]$, and $A_4 = [0, +\infty[\times]-\infty, 0]$ denote respectively the four closed quadrants in the plane.

Lemma 3.2.2 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying $(g1)$ and let $q :]-\infty, a] \to]-\infty, 0]$ be a continuous function such that $Q(t) = \int_a^t q(s)\, ds$ is strictly decreasing on $]-\infty, a]$ and $Q(-\infty) = +\infty$. Then there are two unbounded continua $\Gamma^+ \subset A_1$ and $\Gamma^- \subset A_3$ such that:*

(i) $(0, 0) \in \Gamma^+ \cap \Gamma^-$;

(ii) for all $p \in \Gamma^+ \cup \Gamma^-$, $x(\cdot; a, p)$ is continuable on $]-\infty, a]$ and:

$$\lim_{t \to -\infty} x(t; a, p) = \lim_{t \to -\infty} \dot{x}(t; a, p) = 0;$$

(iii) $x(t; a, p) > 0$ *for every* $t \in]-\infty, a]$, *and* $p \in \Gamma^+$, *while* $x(t; a, p) < 0$ *for every* $t \in]-\infty, a]$, *and* $p \in \Gamma^-$.

Lemma 3.2.3 *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a locally Lipschitz continuous function satisfying* $(g1)$ *and let* $q : [b, +\infty[\to]-\infty, 0]$ *be a continuous function such that* $Q(t) = \int_b^t q(s)\,ds$ *is strictly decreasing on* $[b, +\infty[$, *with* $Q(+\infty) = -\infty$. *Then there are two unbounded continua* $\Gamma^+ \subset A_4$ *and* $\Gamma^- \subset A_2$ *such that:*

(i) $(0,0) \in \Gamma^+ \cap \Gamma^-$;

(ii) *for all* $p \in \Gamma^+ \cup \Gamma^-$, $x(\cdot; b, p)$ *is continuable on* $[b, +\infty[$ *and:*

$$\lim_{t \to +\infty} x(t; b, p) = \lim_{t \to +\infty} \dot{x}(t; b, p) = 0;$$

(iii) $x(t; b, p) > 0$ *for every* $t \in [b, +\infty[$ *and* $p \in \Gamma^+$, *while* $x(t; b, p) < 0$ *for every* $t \in [b, +\infty[$ *and* $p \in \Gamma^-$.

It is sufficient to prove the second Lemma 3.2.3, since Lemma 3.2.2 can be deduced from the first by an inversion of the direction of time in the equation.

Proof of Lemma 3.2.3. We will show only the existence of Γ^+, since Γ^- can be found in a completely symmetric way.

We claim at first that if $\gamma : [0,1] \to A_4$ is any continuous curve with $\gamma(0) \in \mathbb{R}_0^+ \times \{0\}$ and $\gamma(1) \in \{0\} \times \mathbb{R}_0^-$ then there is $\bar{s} \in]0,1[$ such that:

(a) $x(\cdot; b, \gamma(\bar{s}))$ is continuable on $[b, +\infty[$;

(b) $x(t; b, \gamma(\bar{s})) > 0$ for all $t \in [b, +\infty[$;

(c) $\lim_{t \to +\infty} x(t; b, \gamma(\bar{s})) = \lim_{t \to +\infty} \dot{x}(t; b, \gamma(\bar{s})) = 0$.

To this aim let $W := \{(x, y, t) : x \geq 0,\ y \leq 0 \text{ and } t \geq b\}$ as subset of the extended phase space and let us remark that every solution x, whose trajectory $t \mapsto (x(t), \dot{x}(t), t)$ remains in W for $t \geq b$ (as long as it is defined), must satisfy (a), (b) and (c). In fact, such an x must be nonnegative and decreasing and, therefore, it satisfies (a) and (b) and:

$$\lim_{t \to +\infty} x(t) = \ell \geq 0 \qquad \text{and} \qquad \lim_{t \to +\infty} \dot{x}(t) = 0.$$

If $\ell > 0$, there is a constant $m > 0$ such that $g(x(t)) \geq m$ for all $t \in [b, +\infty[$. Then we can evaluate:

$$\begin{aligned} -\dot{x}(b) &= \int_b^{+\infty} \ddot{x}(t)dt \\ &= \int_b^{+\infty} [-q(t)]g(x(t))dt \\ &\geq m \int_b^{+\infty} [-q(t)]dt, \end{aligned}$$

that is q is integrable on $[b, +\infty[$, which is a contradiction. Hence x satisfies condition (c), too, and the problem is now finding $\bar{s} \in]0,1[$ such that the trajectory of $x(\cdot; b, \gamma(\bar{s}))$ remains in W.

Consider the semiflow $\pi : (w, s) \mapsto w \cdot s$ induced on the phase space $X := \{(x, y, t) : t \geq b\}$ by the system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -q(t)g(x) \\ \dot{t} = 1 \end{cases}$$

and define the sets $U := \{(x, 0, t) : x > 0 \text{ and } t \geq b\}$ and $V := \{(0, y, t) : y < 0 \text{ and } t \geq b\}$ which are contained in the boundary ∂W of W relatively to X.

If $w_0 = (x_0, 0, t_0) \in U$, we have that $w_0 \cdot s \notin W$ for $0 < s \leq \varepsilon$. In fact, if $\varepsilon > 0$ is small enough, we have that:

$$g(x(t_0 + s)) \geq \eta > 0 \qquad \text{for all } 0 < s \leq \varepsilon$$

and a suitable η. Hence:

$$\begin{aligned} y(t_0 + s) - y(t_0) &= -\int_0^s q(t_0 + \xi) g(x(t_0 + \xi)\, d\xi \\ &\geq \eta(Q(t_0) - Q(t_0 + s)) \\ &> 0, \end{aligned}$$

as $Q(\cdot)$ is strictly decreasing. Similarly, one can check that $w_0 \cdot s \notin W$ for $0 < s \leq \varepsilon$, when $w_0 \in V$. On the other hand, if $w_0 \in \partial W \setminus (U \cup V)$, then $w_0 \cdot s \in W$ for all s, since $x(t_0 + s) = y(t_0 + s) = 0$ for all $s \geq b - t_0$ and therefore, for each point w_0 of W, either $w_0 \cdot s \in W$, for all $s \geq 0$, or there is a first s, such that $w_0 \cdot s \in U \cup V$. In this manner, we have proved that $U \cup V$ is the set of exit points for W.

The Ważewski lemma (see [32] for a similar argument) now implies that the map ϕ, which sends a point $(x_0, y_0, t_0) \in W$ to the first exit point from W of the positive semitrajectory starting from (x_0, y_0, t_0), is continuous whenever defined.

We come back now to our curve γ, recalling that $\gamma(0) = (x_0, 0)$, with $x_0 > 0$ and $\gamma(1) = (0, y_0)$, with $y_0 < 0$, and we evaluate ϕ along $(\gamma(s), b)$. Clearly, $\phi(\gamma(0), b) = (\gamma(0), b) \in U$ and $\phi(\gamma(1), b) = (\gamma(1), b) \in V$. Assume, by contradiction, that we cannot find our $\bar{s} \in]0, 1[$: then $(\gamma(s), b)$ belongs to the domain of ϕ for every $s \in [0, 1]$ and therefore $\phi(\gamma(s), b) \in U \cup V$, for all $s \in [0, 1]$. Observe also that $\{\phi(\gamma(s), b) : s \in [0, 1]\}$ is a connected set, since ϕ and γ are continuous. But this is not possible since such connected set should be contained in $U \cup V$ and intersect both U and V, which are disjoint and open relatively to ∂W. Thus, our claim is proved.

Let now:

$$\Omega^+ := \{p \in A_4 : x(\cdot; b, p) \text{ satisfies (a), (b) and (c)}\}.$$

We wish to find a connected and unbounded component of Ω^+, which contains the origin, by Lemma 3.2.1. To this aim let us define:

$$\psi(x, y) := \left(2\frac{x + y}{1 + (x - y)^2}, 1 - \frac{2}{1 + (x - y)^2}\right).$$

It is possible to check that, using the notation of Lemma 3.2.1, ψ is a homeomorphism of A_4 onto $\overline{D} \setminus \{Q_1\}$, such that:

1. $\psi((0, 0)) = Q_0$;
2. $\psi(\mathbb{R}_0^+ \times \{0\}) = \{(x, y) : x^2 + y^2 = 1 \text{ and } x > 0\}$ and $\psi(\{0\} \times \mathbb{R}_0^-) = \{(x, y) : x^2 + y^2 = 1 \text{ and } x < 0\}$;

3. if Z is any neighbourhood of Q_1 then $\psi^{-1}(Z \cap D)$ is unbounded.

What we have proved above ensures that the set $C := \psi(\Omega^+)$ satisfies the hypotheses of Lemma 3.2.1 and, hence, there is a connected $\Gamma \subseteq C$ such that $Q_0, Q_1 \in \overline{\Gamma}$. Our search is concluded letting $\Gamma^+ := \psi^{-1}(\Gamma)$; in fact, with our choices, Γ^+ is connected, contains the origin by the connectedness and the fact that $(0,0) \in \overline{\Gamma} \cap C$ and it is unbounded by property 3 of ψ. ∎

3.3 Blow-up of solutions

In the autonomous case $\ddot{x} - g(x) = 0$, with g superlinear and such that $g(x)x > 0$ for every $x \neq 0$, we have seen at the beginning of this chapter that the solutions which start from points $p \in \partial\Omega_a^b$ at time $t = a$, have $[a, b[$ as right maximal interval of continuability and blow up as t tends to b; on the other hand the blow-up is at $t = a$, if a solution starts from $p \in \partial\Omega_b^a$ at $t = b$. Hence we are going to give sufficient conditions in order to obtain similar situations when a continuous weight $q : [a, b] \to]-\infty, 0]$ is considered in the equation:

$$\ddot{x} + q(t)g(x) = 0. \tag{3.8}$$

At first we observe that, when $q \leq 0$, a *necessary* assumption for the blow-up at a certain $t = t_0$ is that t_0 is an accumulation point of the set of t's where q is strictly negative. Indeed, if $q(t) = 0$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, then every solution of (3.8) satisfies

$$\ddot{x}(t) \equiv 0 \qquad \forall t \in [t_0 - \varepsilon, t_0 + \varepsilon],$$

so that x is an affine function in $[t_0 - \varepsilon, t_0 + \varepsilon]$ and the blow-up clearly cannot happen around t_0. Actually we will see that such a condition is also sufficient for suitable superlinear functions g.

However in this section we consider more general conditions about g then in the previous ones. Namely we are not going to consider locally Lipschitz continuous functions and we will not require the sign condition $g(x)x > 0$ for every $x \neq 0$. In fact the sign conditions we will employ are localized around 0 and ∞ in the following way:

($g1+$) there are two positive constants $\alpha_0 \leq \beta_0$ such that:

$$g(x) > 0 \qquad \forall x \in]0, \alpha_0] \cup [\beta_0, +\infty[;$$

($g1-$) there are two positive constants $\alpha_0 \leq \beta_0$ such that:

$$g(x) < 0 \qquad \forall x \in]-\infty, -\beta_0] \cup [-\alpha_0, 0[.$$

Since our g is just continuous, we will have no local uniqueness for the intial value problems associated to the equation (3.8). Anyhow we will need at least the uniqueness of the zero solution (our functions will satisfy $g(0) = 0$) and this is supplied by the two conditions:

$$\int_0^{\alpha_0} \frac{1}{\sqrt{G(s)}}\, ds = +\infty \tag{$g0+$}$$

and

$$\int_{-\alpha_0}^{0} \frac{1}{\sqrt{G(s)}}\, ds = +\infty, \tag{$g0-$}$$

with $G(x) = \int_0^x g(s)\,ds$, as shown by Propositions 3.3.1 and 3.3.2 below. The conditions $(g0-)$ and $(g0+)$ are weaker than assuming that the ratio $g(x)/x$ is bounded in a neighbourhood of the origin, since they clearly hold if $|g(x)| \leq L|x|$ for all $|x| \leq \alpha$, for some positive α.

Proposition 3.3.1 *Assume that $g : [0, +\infty[\to \mathbb{R}$ is a continuous function satisfying $g(0) = 0$, $(g0+)$ and $(g1+)$ and let $q : [a, b] \to]-\infty, 0]$ be continuous. Let x be a nonnegative solution of* (3.8) *defined in an interval $[t_1, t_2] \subset [a, b]$. If $t_0 \in [t_1, t_2]$ is such that $x(t_0) = \min_{[t_1,t_2]} x$ and $\dot{x}(t_0) = 0$, then $x(t_0) > 0$, unless $x \equiv 0$.*

Proof. If $x_{\min} := x(t_0) \geq \alpha_0$ the proposition is already proved, therefore we consider the case $x_{\min} < \alpha_0$. Without loss of generality we can also suppose that either $\dot{x}(t) > 0$ in $]t_0, s_2[$ with $x(s_2) \leq \alpha_0$ or $\dot{x}(t) < 0$ in $]s_1, t_0[$ with $x(s_1) \leq \alpha_0$ (for some $s_1 \leq t_0 \leq s_2$) since, if $x \not\equiv 0$, there are s_1 and s_2 such that $t_1 \leq s_1 \leq t_0 \leq s_2 \leq t_2$ and $\alpha := x_{\min} < \max\{x(s_1), x(s_2)\} \leq \alpha_0$ and we could substitute t_0 either by $\max\{t \in [t_1, t_2] : x(t) = x_{\min}\}$ or by $\min\{t \in [t_1, t_2] : x(t) = x_{\min}\}$. We develop the proof when the first situation occurs (namely, $\dot{x}(t) > 0$ in $]t_0, s_2[$ with $x(s_2) \leq \alpha_0$), the other one being analogous.

We have that:

$$\ddot{x}(t) \leq \|q\|_\infty g(x(t)),$$

for all $t \in [t_0, s_2]$. Multiplying by $\dot{x}(t)$ and integrating on $[t_0, t] \subset [t_0, s_2]$ yields that:

$$\dot{x}(t)^2 \leq 2\|q\|_\infty (G(x(t)) - G(x_{\min}))$$

for all $t \in [t_0, s_2]$. Hence,

$$\int_{x_{\min}}^{\alpha} \frac{1}{\sqrt{G(x)}}\,dx < \int_{x_{\min}}^{\alpha} \frac{1}{\sqrt{G(x) - G(x_{\min})}}\,dx \leq \sqrt{2\|q\|_\infty}(s_2 - t_0) < \sqrt{2\|q\|_\infty}.$$

From $(g0+)$ we have immediately that $x_{\min} > 0$. ∎

In a completely symmetric way it can be proved the following analogous result.

Proposition 3.3.2 *Assume that $g :]-\infty, 0] \to \mathbb{R}$ is a continuous function satisfying $g(0) = 0$, $(g0-)$ and $(g1-)$ and let $q : [a, b] \to]-\infty, 0]$ be continuous. Let x be a nonpositive solution of* (3.8) *defined in an interval $[t_1, t_2] \subset [a, b]$. If $t_0 \in [t_1, t_2]$ is such that $x(t_0) = \max_{[t_1,t_2]} x$ and $\dot{x}(t_0) = 0$, then $x(t_0) < 0$, unless $x \equiv 0$.*

Now we are in position to state and prove the main results about blow-up solutions: we have divided them into three lemmas about solutions which explode at $t = b$ and into the relative three lemmas about those exploding at $t = a$, in order to exhibit which set of hypotheses is sufficient in each situation. As a matter of notation, in this section $\pi_x, \pi_y : \mathbb{R}^2 \to \mathbb{R}$ will denote the projections of $\mathbb{R}^2$ onto the x-axis and the y-axis, respectively.

Lemma 3.3.3 *Let $g : [0, +\infty[\to \mathbb{R}$ be a continuous function satisfying $g(0) = 0$, $(g1+)$, $(g0+)$ and:*

$$\lim_{c \to +\infty} \sqrt{2} \int_c^{+\infty} \frac{1}{\sqrt{G(s) - G(c)}}\,ds = 0, \tag{3.9}$$

and let $q : [a,b[\to]-\infty, 0]$ *be a continuous and bounded function such that:*

$$b \in \overline{\{t \in]a,b[: q(t) < 0\}}.$$

Then, there is an unbounded continuum $\Gamma_a^+ \subset [0,+\infty[\times\mathbb{R}$*, with* $\pi_x(\Gamma_a^+) = [0,+\infty[$*, such that for each* $(x_0,y_0) \in \Gamma_a^+$ *there is a nonnegative solution* $x(\cdot)$ *of* (3.8) *with* $x(a) = x_0$*,* $\dot{x}(a) = y_0$*,* $x(t) > 0$ *for all* $t \in]a,b[$ *and* $x(t) \to +\infty$ *as* $t \to b^-$*. Moreover, the localization of the branch* Γ_a^+ *in the phase plane can be described as follows: there is* $\delta > 0$ *and:*

(i) there is $\varepsilon > 0$ *such that* $\pi_y(\Gamma_a^+ \cap ([0,\varepsilon[\times\mathbb{R})) \subset]\delta,+\infty[$*,*

(ii) there is $K > 0$ *such that* $\pi_y(\Gamma_a^+ \cap (]K,+\infty[\times\mathbb{R})) \subset]-\infty,-\delta[$*.*

We note that the substitution $e = G(c)$ shows that the integral in (3.9) represents the time needed to run the trajectory of the planar autonomous system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x) \end{cases}$$

which passes through $(G_+^{-1}(e), 0)$ and is contained in the right half plane (see figure 3.1). Therefore (3.9) is implied by the stronger condition:

$$\lim_{e\to-\infty} \tau_g^-(e) = 0$$

(cf. the definition of τ_g^- in (3.2)).

We remark also that, even if q is not defined in $t = b$, it has sense to consider solutions of (3.8) in the entire closed interval $[a,b]$: such solutions might not be twice differentiable at $t = b$, but will surely have bounded C^2-norm in $]a,b[$.

The proof of this lemma will be carried out via the following plan. First of all, we fix a number $\beta > \beta_0$ and take $n \in \mathbb{N}$ with $n > \beta$. Then, we consider the two-point boundary value problem:

$$\text{(3.10)} \qquad \begin{cases} \ddot{x} + q(t)g(x) = 0 \\ x(a) = r \\ x(b) = n \end{cases}$$

with $r \in [0,\beta]$ considered as a parameter. Using the Leray–Schauder Continuation Theorem [70] and a connectivity argument ([80]), we can find a compact connected set $\mathcal{S}_n \subset [0,\beta] \times C^1([a,b])$ of positive solution pairs (r,x) of (3.10) such that for each $r \in [0,\beta]$ there is $(r,x) \in \mathcal{S}_n$, with $x(a) = r$. From the assumptions, it is also possible to see that there is $N = N(\beta) > 0$, with N independent of n, such that $|\dot{x}(a)| \le N$, for all $x \in \mathcal{S}_n$. Thus, if we denote by Σ_n the image of $\mathcal{S}_n$ under the continuous map $[0,\beta]\times C^1([a,b])(r,x) \mapsto (r,\dot{x}(a)) \in \mathbb{R}_0^+\times\mathbb{R}$, we have that $\Sigma_n \subset [0,\beta] \times [-N,N]$ is a compact connected set, with $\pi_x(\Sigma_n) = [0,\beta]$ and for each $(x_0,y_0) \in \Sigma_n$ there is a solution of (3.8) with $x(a) = x_0$, $\dot{x}(a) = y_0$ and $x(b) = n$. As a next step, we let $n \to +\infty$, and after some computations, we prove that there is a compact connected set $\Sigma = \Sigma(\beta)$, $\Sigma \subset [0,\beta] \times [-N,N]$, with $\pi_x(\Sigma) = [0,\beta]$ and, for each $(x_0,y_0) \in \Sigma$ there is a solution $x(\cdot)$ of (3.8) on the interval $[a,b[$, with $x(a) = x_0$, $\dot{x}(a) = y_0$ and $x(b^-) = +\infty$. Finally, we make this construction for $\beta = k$, letting $k \to +\infty$. Having denoted by Γ_k the corresponding continua $\Sigma(k)$, we prove that the Γ_k "converge" to an unbounded continuum Γ_a^+ with the desired properties. The convergence of the continua in the first and the second steps of the proof is based on a topological lemma [62, p.171] and on some locally uniform estimates for the solutions (see [46] for a similar argument). In the course of the proof of these intermediate steps, we obtain some additional properties of the solutions that will be used to make more precise the localization of the continuum Γ_a^+.

For the reader's convenience we recall here the definitions of lim inf and lim sup of a sequence of sets and the topological lemma on the connectedness of the lim sup of sequence of continua in comact metric spaces.

Definition 3.3.4 Let $\{C_n\}$ be a sequence of subsets of a topological space X. We say that a point $p \in X$ belongs to $\liminf_{n\to+\infty} C_n$ if for every neighbourhood V of p there exists n_0 such that:

$$V \cap C_n \neq \emptyset \qquad \forall n \geq n_0.$$

On the other hand, we say that p belongs to $\limsup_{n\to+\infty} C_n$ if for every neighbourhood V of p we have that:

$$V \cap C_n \neq \emptyset \qquad \text{for infinitely many } n \in \mathbb{N}.$$

Roughly speaking, $\liminf_{n\to+\infty} C_n$ is made of all the points which are limit of converging *sequences* $\{p_n\}$ such that $p_n \in C_n$ for every n, while $\limsup_{n\to+\infty} C_n$ contains the limit points of converging *subsequences* of the sequences of the form $\{p_n\}$ with $p_n \in C_n$ for every n.

Lemma 3.3.5 [62, Theorem 6, §47, II, p.171]. *If $\{C_n\}$ is a sequence of continua of a compact metric space X then:*

$$\liminf_{n\to+\infty} C_n \neq \emptyset \qquad \Longrightarrow \qquad \limsup_{n\to+\infty} C_n \text{ is a continuum.}$$

Clearly, if X is a compact metric space and $\{C_n\}$ is a sequence of continua of X, then we can select a sequence $\{p_n\}$ with $p_n \in C_n$ for every n; since X is compact, we can find a subsequence $\{p_{k_n}\}$ such that $p_{k_n} \to p_0 \in X$. Therefore we have that $\liminf_{n\to+\infty} C_{k_n} \neq \emptyset$ and, by Lemma 3.3.5, $\limsup_{n\to+\infty} C_{k_n}$ is a continuum. Hence every sequence of continua of a compact metric space admits a subsequence whose lim sup is a continuum, too.

The following proposition provides some estimates which will be needed in the proof.

Proposition 3.3.6 *Assume that $g : [0, +\infty[\to \mathbb{R}$ is a continuous function satisfying $g(0) = 0$ and $(g0+)$ and $(g1+)$ and let $q : [a, b[\to]-\infty, 0]$ be continuous and bounded. For any $\gamma > \beta$ and any nonnegative solution x of:*

$$\begin{cases} \ddot{x} + \lambda q(t) g(x) = 0 \\ x(a) = r, \end{cases}$$

for some $\lambda \in [0, 1]$ and $r \in [0, \beta]$, and x defined in in a right maximal interval of continuability contained in $[a, b]$ there is $t^* = t^*_{(x,\gamma)} \in]a, b]$ such that:

$$[a, t^*_{(x,\gamma)}] = \{t \in [a, b] : x(t) \leq \gamma\}.$$

Moreover, for each $\varepsilon > 0$ there is $M_\varepsilon = M_\varepsilon(\gamma)$, with M_ε independent of x and of λ such that if $t^*_{(x,\gamma)} \geq a + \varepsilon$, then:

$$|\dot{x}(t)| < M_\varepsilon, \quad \forall t \in [a, t^*].$$

Proof. Consider $\gamma > \beta$. If $x(t) \leq \gamma$ for all $t \in [a,b]$, we have $t_{(x,\gamma)} = b$. Otherwise, if there is some $\hat{t} \in]a,b]$ such that $x(\hat{t}) > \gamma$, by the fact that $\dot{x}(t) \geq 0$ for all t where $x(t) \geq \beta_0$, and $x(a) \leq \beta < \gamma$, it is easy to prove that there is a unique $t = t^* \in]a, \hat{t}[$ such that $x(t) < \gamma$ for all $t \in [a, t^*[$ and $x(t) > \gamma$ for all $t > t^*$ where $x(t)$ is defined and this ends the proof of the first part of the statement.

We have that $|\ddot{x}(t)| < \|q\|_\infty K$, for all $t \in [a, t^*]$, where $K = K(\gamma) > \max\{|g(s)| : 0 \leq s \leq \gamma\}$. Since $x(t^*) > x(a)$, we have at least one point where $\dot{x} > 0$. Hence, if $\dot{x} \leq 0$ somewhere, then also $\dot{x} = 0$ at some point. If we assume that $\dot{x} > 0$ on $[a, t^*]$ and also $t^* \geq a + \varepsilon$, we immediately see that $\min_{[a,t^*]} \dot{x} \leq \gamma/\varepsilon$. Thus, in any case, there is some $\tilde{t} \in [a, t^*]$ where $|\dot{x}(\tilde{t})| \leq \gamma/\varepsilon$. Finally, using the bound on $\ddot{x}$, we have that:

$$|\dot{x}(t)| < M_\varepsilon(\gamma) := \frac{\gamma}{\varepsilon} + (b-a)\|q\|_\infty K, \quad \forall t \in [a, t^*]$$

and also the second part of the statement is proved. ∎

Proof of Lemma 3.3.3. For convenience we extend g on the whole real line just by requiring that $g(x) < 0$ for all $x < 0$. Let us fix $\beta > \beta_0$ and consider the boundary value problem:

$$\begin{cases} \ddot{x} + \lambda q(t) g(x) = 0 \\ x(a) = r \\ x(b) = n \end{cases} \tag{3.11}$$

with $\lambda \in [0,1]$, $r \in [0, \beta]$ and $n > \beta$. Observe that every solution of (3.11) is such that $x(t) < x(b)$ for each $t \in [a, b[$ by $(g1+)$ and the fact that $x > \beta_0$, and must be nonnegative since we extended g to be negative on $]-\infty, 0[$ (a solution which starts nonnegative and becomes negative a first time at some $t = t_0$, should remain negative for all $t > t_0$); hence we have that $x(t) > 0$ for all $t \in]a, b[$ by Proposition 3.3.1.

It is known that we can write (3.11) as an operator equation of the form:

$$x(t) = \varphi_\lambda(r, x)(t) = h_r(t) - \lambda \int_a^b \mathcal{G}(t,s) q(s) g(x(s))\, ds,$$

where $\mathcal{G}$ is the Green function for the homogeneous two-point boundary value problem:

$$\begin{cases} \ddot{x} = w \\ x(a) = x(b) = 0 \end{cases}$$

and $h_r(t) = r + \frac{n-r}{b-a}(t-a)$. The operator $\varphi : [0,1] \times [0,\beta] \times C^1([a,b]) \to C^1([a,b])$ is completely continuous (with respect to the C^1-norm) and any solution of the operator equation:

$$x = \varphi_\lambda(r, x),$$

with $\lambda \in [0,1]$ and $r \in [0, \beta]$ satisfies:

$$\begin{cases} 0 \leq x(t) < n+1 \\ |\dot{x}(t)| < M_n \end{cases} \qquad \forall t \in [a,b],$$

with $M_n = M_{(b-a)}(n+1)$ and $M_{(b-a)}(n+1)$ defined according to Proposition 3.3.6, (note that $\gamma = n+1$ and therefore $t^* = b$). Now, with respect to the open and bounded set $\Omega \subset C^1([a,b])$, defined by:

$$\Omega = \{x \in C^1([a,b]) : -1 < x(t) < n+1 \text{ and } |\dot{x}(t)| < M_n \ \ \forall t \in [a,b]\},$$

we have that:

$$\deg(I - \varphi_1(r,\cdot), \Omega, 0) = \deg(I - \varphi_0(r,\cdot), \Omega, 0) = \deg(I, \Omega, h_r) = 1.$$

By the Leray–Schauder Continuation Theorem applied to the equation $x = \phi_1(r,x)$ parameterized with respect to $r \in [0,\beta]$, we have that there is a compact connected set $\mathcal{S}_n \subset [0,\beta] \times C^1([a,b])$ such that for each $r \in [0,\beta]$ there is $x \in C^1([a,b])$, with $(r,x) \in \mathcal{S}_n$ and $x = \varphi_1(r,x)$, that is, x is a solution of (3.10). The projection of $\mathcal{S}_n$ via the operator $(r,x) \mapsto (r, \dot{x}(a))$ is a compact connected set $\Sigma_n \subset [0,\beta] \times [-M_n, M_n]$.

CLAIM 1. For each integer $k > 1/(b-a)$, there are $L_k > \beta$ and j_k such that every solution x of (3.10), with $r \in [0,\beta]$, satisfies $x(t) \le L_k$ for every $t \in [a, b-1/k]$ and each $n \ge j_k$.

Assume, by contradiction, that for some k there is a sequence of solutions x_{ℓ_n} of equation (3.8) such that $x_{\ell_n}(a) \in [0,\beta]$, $x_{\ell_n}(b) = \ell_n$ and $x_{\ell_n}(b-1/k) = \max_{[a,b-1/k]} x_{\ell_n} \to +\infty$ as $n \to +\infty$. By the assumption on q, there is a non trivial interval $[t_1,t_2] \subset [b-1/k, b]$ and a real number $m > 0$ such that $q(t) \le -m$ for each $t \in [t_1,t_2]$ so that we have $\ddot{x}_{\ell_n} \ge m g(x_{\ell_n})$ and $\dot{x} > 0$ on $[t_1,t_2]$. Hence we have that:

$$\frac{d}{dt}\left[\frac{1}{2}\dot{x}_{\ell_n}^2(t) - mG(x_{\ell_n}(t))\right] \ge 0 \qquad \forall\, t \in [t_1,t_2]$$

and therefore:

$$\begin{aligned}\frac{1}{2}\dot{x}_{\ell_n}^2(t) - mG(x_{\ell_n}(t)) &\ge \frac{1}{2}\dot{x}_{\ell_n}^2(t_1) - mG(x_{\ell_n}(t_1)) \\ &> -mG(x_{\ell_n}(t_1)) \qquad\qquad \forall\, t \in [t_1,t_2].\end{aligned}$$

From this, we have that:

$$\dot{x}_{\ell_n}(t) > \sqrt{2m(G(x_{\ell_n}(t)) - G(x_{\ell_n}(t_1)))} \qquad \forall\, t \in [t_1,t_2]$$

and then, integrating on $[t_1,t_2]$ and using (3.9), we find that:

$$\sqrt{m}(t_2 - t_1) \le \frac{1}{\sqrt{2}} \int_{x_{\ell_n}(t_1)}^{\ell_n} \frac{1}{\sqrt{G(x) - G(x_{\ell_n}(t_1))}}\, dx \to 0, \qquad \text{as } n \to +\infty,$$

since $x_{l_n}(t_1) \ge x_{\ell_n}(b-1/k)$. Hence, a contradiction is found and the claim is proved.

CLAIM 2. For each $\gamma > \beta$, there are $t_\gamma \in]a,b[$ and an index i_γ such that $x(t) > \gamma$ for all $t \in [t_\gamma, b]$ and each $n \ge i_\gamma$, if x is any solution of (3.10) with $r \in [0,\beta]$.

Assume, by contradiction, that there is $\gamma > \beta$ and a sequence x_{ℓ_n} of solutions of equation (3.8) such that $x_{\ell_n}(a) \in [0,\beta]$, $x_{\ell_n}(b) = \ell_n \to +\infty$ as $n \to +\infty$ and a sequence $t_n \to b$ with $x_{\ell_n}(t_n) = \gamma$. Without loss of generality, we can assume that $t_n \ge d$ for some fixed $d \in]a,b[$ and for all n. By Proposition 3.3.6, there is a constant $M = M_{d-a}(\gamma)$ such that $|x'_{\ell_n}(t)| < M$, for all $t \in [a, t_n]$. Using $\ddot{x} \le \|q\|_\infty g(x)$ and $\dot{x} > 0$ on $[t_n, b]$, we have:

$$\frac{d}{dt}\left[\frac{1}{2}\dot{x}_{\ell_n}^2(t) - \|q\|_\infty G(x_{\ell_n}(t))\right] \le 0 \qquad \forall\, t \in [t_n, b]$$

and therefore:

$$\begin{aligned}\frac{1}{2}\dot{x}_{\ell_n}^2(t) - \|q\|_\infty G(x_{\ell_n}(t)) &\leq \frac{1}{2}\dot{x}_{\ell_n}^2(t_n) - \|q\|_\infty G(x_{\ell_n}(t_n)) \\ &< \frac{1}{2}M^2 - \|q\|_\infty G(\gamma) \qquad \forall\, t \in [t_n, b].\end{aligned}$$

Hence:

$$\dot{x}_{\ell_n}(t) < \sqrt{M^2 + 2\|q\|_\infty (G(x_{\ell_n}(t)) - G(\gamma))} \qquad \forall\, t \in [t_n, b]$$

and then, integrating on $[t_n, b]$, we find that:

$$\int_\gamma^{\ell_n} \frac{1}{\sqrt{M^2 + 2\|q\|_\infty (G(x) - G(\gamma))}}\, dx < b - t_n \to 0.$$

Hence, a contradiction is found and the claim is proved.

From Claim 1 we know that if we fix an arbitrary constant $b_1 \in]a, b[$, there is a constant $R = R(b_1) > \beta$ such that if n is sufficiently large, then $x(t) \leq R$ for all $t \in [a, b_1]$ and any solution of problem (3.10) with $r \in [0, \beta]$. Now, we apply Proposition 3.3.6 for $\gamma = R$ and taking n sufficiently large ($x(b) = n > R$) in order to have $t^* > b_1$. Hence there is $N = M_{(b_1 - a)}(\gamma)$ such that $|\dot{x}(t)| < N$, for all $t \in [a, b_1]$. This proves that:

$$\Sigma_n \subset [0, \beta] \times [-N, N]$$

for each n sufficiently large. Being the Σ_n all contained in a compact set, we can take a subsequence (that we still indicate with $(\Sigma_n)_n$), such that $\liminf_{n\to\infty} \Sigma_n \neq \emptyset$ and, therefore, Lemma 3.3.5 ensures that:

$$\Sigma(\beta) = \limsup_{n\to\infty} \Sigma_n \subset [0, \beta] \times [-N, N]$$

is a continuum. Note that $\pi_x(\Sigma(\beta)) = [0, \beta]$.

Next, we describe how are the solutions of (3.8) with initial point in $\Sigma(\beta)$. Let us take $(r, s) \in \Sigma(\beta)$. By assumption, there is a sequence $(r_{j_n}, s_{j_n}) \in \Sigma_{j_n}$ which converges to (r, s). For each n, there is a solution x_{j_n} of (3.8) with $x_{j_n}(a) = r_n$, $\dot{x}_{j_n}(a) = s_n$ and $x_{j_n}(b) = j_n$. From Claim 1 and Claim 2, via Ascoli–Arzelà's theorem and a standard diagonal argument, we have that a subsequence of the x_{j_n} converges (C^2-uniformly on compact subsets of $[a, b[$) to a solution $\bar{x}$ of (3.8) defined on $[a, b[$, with $\bar{x}(a) = r$, $\dot{\bar{x}}(a) = s$ and $\bar{x}(b^-) = +\infty$. Therefore, we have proved that:

- *for each $r \in [0, \beta]$, there is $s \in [-N, N]$ such that $(r, s) \in \Sigma(\beta)$ and for each $(r, s) \in \Sigma(\beta)$, there is a solution x of equation* (3.8) *such that $x(a) = r$, $\dot{x}(a) = s$ and $x(b^-) = +\infty$.*

Our last goal is to analyze what happens when $\beta \to +\infty$. First of all, we note that the arguments in the proofs of Claim 1 and Claim 2, are valid for the solutions of (3.8) with $r \in [0, \beta]$ and $x(b^-) = +\infty$. Hence we have that:

- *For each $\beta > \beta_0$ there is $N = N(\beta)$ such that for each $\beta_1 \geq \beta$ it holds that:*

$$\Sigma(\beta_1) \cap ([0, \beta] \times \mathbb{R}) \subset [0, \beta] \times [-N, N].$$

In fact, from Proposition 3.3.6 and Claim 2 (applied with $\gamma = \beta + 1$) we see that any solution of (3.8) with initial point in $\Sigma(\beta_1)$, but with $x(a) \in [0, \beta]$ satisfies $|\dot{x}(a)| \le N(\beta) = M_{(d-a)}(\beta+1)$, for a fixed $d \in]a, b[$. We take now $\beta = k$ and define $\Gamma_k := \Sigma(k)$ and $N_k = N(k)$. Moreover, by a standard argument from real analysis, we can find a continuous function $f : [0, +\infty[\to [0, +\infty[$ such that any solution x of (3.8) satisying $x(a) = r$ and $(x(a), \dot{x}(a)) \in \Gamma_k$ for some $k > 0$, is such that $|\dot{x}(a)| \le f(r)$.

Consider now $k \to +\infty$. Using again Lemma 3.3.5, we can prove that there is a closed unbounded connected set $\Gamma_a^+ \subset [0, +\infty[\times \mathbb{R}$ "joining" $x = 0$ with $x = +\infty$ in the phase plane. Actually, to enter rigorously in the setting of the lemma, which works for compact metric spaces, one should first compactify the set $\mathcal{A} = \{(r, s) \in [0, +\infty[\times \mathbb{R} : |s| \le f(r)\}$, adding a point at infinity: $\mathcal{A}_0 := \mathcal{A} \cup \{p_\infty\}$. Then $\limsup_{k\to\infty} \Gamma_k \subset \mathcal{A}_0$ contains a compact connected set $\Xi \subset B[1]$ joining $x = 0$ and p_∞. If we consider now $\Xi \setminus \{p_\infty\}$, we see that it must contain a connected component Γ_a^+ which is closed relatively to $\mathcal{A}$ and satisfies $\overline{\Gamma_a^+} = \Gamma_a^+ \cup \{p_\infty\}$, so that $\pi_x(\Gamma_a^+) = [0, +\infty[$.

By the way in which Γ_a^+ is obtained and from the properties of the solutions with initial value in Γ_k for $t = a$, we have that for each $r \ge 0$, there is $s \in \mathbb{R}$, with $(r, s) \in \Gamma_a^+$ and also for each $(r, s) \in \Gamma_a^+$ there is at least one solution x of (3.8) with $x(a) = r, \dot{x}(a) = s \in [-f(r), f(r)]$ with $x(t)$ defined in $[a, b[$ and such that $\lim_{t\to b^-} x(t) = +\infty$.

As a last step, we observe that if $x(a) = r$ tends to zero, then $\dot{x}(a)$ cannot be too small, otherwise, $x(t)$ remains small for all $t \in [a, b]$. To check this, it is sufficient to repeat a computation similar to that at the end of Proposition 3.3.1. Analogously, we can prove (via a similar computation) that if $x(a) = r$ tends to infinity, then $\dot{x}(a)$ must be at least a little negative, otherwise, we would have a blow up of the solution before $t = b$. These remarks permit to check (i) and (ii) of Lemma 3.3.3 and thus we conclude the proof. ■

In a completely simmetric manner a "negative" version of Lemma 3.3.3 can be proved (and we omit the proof) assuming that the time to run only the trajectories of:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x), \end{cases}$$

which pass through the points like $(c, 0)$, with $c < 0$ (see figure 3.1), tends to zero as c goes to $-\infty$. Of course, the condition:

$$\lim_{e\to-\infty} \tau_g^-(e) = 0$$

is enough for (3.12), below.

Lemma 3.3.7 *Let $g :]-\infty, 0] \to \mathbb{R}$ be a continuous function satisfying $g(0) = 0$, $(g1-)$, $(g0-)$ and:*

$$\lim_{c\to-\infty} \sqrt{2} \int_{-\infty}^{c} \frac{1}{\sqrt{G(s) - G(c)}}\, ds = 0, \tag{3.12}$$

and let $q : [a, b[\to]-\infty, 0]$ be a continuous and bounded function such that:

$$b \in \overline{\{t \in]a, b[: q(t) < 0\}}.$$

Then, there is an unbounded continuum $\Gamma_a^- \subset]-\infty, 0] \times \mathbb{R}$, with $\pi_x(\Gamma_a^-) =]-\infty, 0]$, such that for each $(x_0, y_0) \in \Gamma_a^-$ there is a nonpositive solution $x(\cdot)$ of (3.8) with $x(a) = x_0$, $\dot{x}(a) = y_0$, $x(t) < 0$ for all $t \in]a, b[$ and $x(t) \to -\infty$ as $t \to b^-$. Moreover, the localization of the branch Γ_a^- in the phase plane can be described as follows: there is $\delta > 0$ and:

(i) *there is* $\varepsilon > 0$ *such that* $\pi_y(\Gamma_a^- \cap (]-\varepsilon, 0] \times \mathbb{R})) \subset]-\infty, -\delta[$,

(ii) *there is* $K > 0$ *such that* $\pi_y(\Gamma_a^- \cap (]-\infty, -K[\times\mathbb{R})) \subset]\delta, +\infty[$.

The next result concerns the existence of continua of blow-up solutions which have exactly one zero in $[a, b[$.

Lemma 3.3.8 *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $g(0) = 0$ *and satisfying* $(g1+)$, $(g1-)$, $(g0+)$ *and* $(g0-)$ *and let* $q : [a, b[\to]-\infty, 0]$ *be a continuous and bounded function such that:*

$$b \in \overline{\{t \in]a, b[: q(t) < 0\}}.$$

Then, the following facts hold:

(i) *if* (3.9) *holds, then there is an unbounded continuum* $\Gamma_a^- \subset [0, +\infty[\times\mathbb{R}$ *such that for each* $(x_0, y_0) \in \Gamma_a^-$ *there is a solution* $x(\cdot)$ *of* (3.8) *having exactly one zero in* $[a, b[$ *and satisfying* $x(a) = x_0$, $\dot{x}(a) = y_0$, *and* $x(t) \to +\infty$ *as* $t \to b^-$; *moreover* $\Gamma_a^- \cap (]-\infty, -\beta_0] \times \mathbb{R}) \subset]-\infty, -\beta_0] \times [0, +\infty[$ *and* $\pi_x(\Gamma_a^-) =]-\infty, 0]$;

(ii) *if* (3.12) *holds then there is an unbounded continuum* $\Gamma_a^+ \subset [0, +\infty[\times\mathbb{R}$ *such that for each* $(x_0, y_0) \in \Gamma_a^+$ *there is a solution* $x(\cdot)$ *of* (3.8) *having exactly one zero in* $[a, b[$ *and satisfying* $x(a) = x_0$, $\dot{x}(a) = y_0$ *and* $x(t) \to -\infty$ *as* $t \to b^-$; *moreover* $\Gamma_a^+ \cap ([\beta_0, +\infty[\times\mathbb{R}) \subset [\beta_0, +\infty[\times]-\infty, 0]$ *and* $\pi_x(\Gamma_a^+) = [0, +\infty[$.

Proof. The proof is quite similar to the one of Lemma 3.3.3 and we are just going to outline the main steps to be followed in order to prove the parts of the statement concerning Γ_a^-. For Γ_a^+ everything is quite symmetric.

At first we want to find a continuum of solutions of the boundary value problem:

$$\begin{cases} \ddot{x} + q(t)g(x) = 0 \\ x(a) = r \\ x(b) = n \end{cases} \tag{3.13}$$

with $r \in [-\beta, 0]$ and with fixed $\beta > \beta_0$ and fixed $n > \beta$. To this aim we consider again the family of boundary value problems:

$$\begin{cases} \ddot{x} + \lambda q(t)g(x) = 0 \\ x(a) = r \\ x(b) = n \end{cases} \tag{3.14}$$

with $\lambda \in [0, 1]$ and $r \in [-\beta, 0]$. Observe that every solution of (3.14) is such that $-\beta \leq x(t) \leq n$ for each $t \in [a, b]$. Indeed, if $x(t_0) < -\beta$, then $g(x(t)) < 0$ around t_0 by $(g1-)$: hence $\ddot{x}(t) \leq 0$ and $\dot{x}$ is monotone decreasing around t_0; in particular, if $\dot{x}(t_0) \geq 0$, x is increasing on $[a, t_0]$, while, if $\dot{x}(t_0) < 0$, x turns out to be decreasing on $[t_0, b]$. Therefore we have either $x(a) < -\beta$ or $x(b) < -\beta$, that is a contradiction. In a similar manner it is possible to show that $x(t_0) > n$ cannot hold by $(g1+)$.

We write (3.14) as an operator equation of the form:

$$x(t) = \varphi_\lambda(r, x)(t) = h_r(t) - \lambda \int_a^b \mathcal{G}(t, s)q(s)g(x(s))\, ds,$$

as in the proof of Lemma 3.3.3.

Any solution of $x = \varphi_\lambda(r, x)$, with $\lambda \in [0,1]$, $r \in [-\beta, 0]$ and $n > \beta$, satisfies $\|x\|_\infty \leq n$. Moreover, for every solution x of (3.14) there is $t_0 \in]a, b[$ such that:

$$0 < \dot{x}(t_0) = \frac{n-r}{b-a} \leq \frac{2n}{b-a},$$

by Lagrange's theorem, and we can estimate $|\dot{x}|$ as follows:

$$\begin{aligned} |\dot{x}(t)| &\leq \dot{x}(t_0) + \left| \int_{t_0}^{t} |\ddot{x}(s)|\, ds \right| \\ &\leq \frac{2n}{b-a} + \|q\|_\infty \int_a^b |g(x(s))|\, ds \\ &\leq M_n \qquad\qquad \forall\, t \in [a,b], \end{aligned}$$

where we have set:

$$M_n = \frac{2n}{b-a} + \|q\|_\infty (b-a) \max_{s\in[-n,n]} |g(s)|.$$

Now, with respect to the open and bounded set $\Omega \subset C^1([a,b])$, defined by:

$$\Omega = \{x \in C^1([a,b]) : |x(t)| < n+1 \text{ and } |\dot{x}(t)| < M_n + 1 \ \ \forall\, t \in [a,b]\},$$

we have that:

$$\deg(I - \varphi_\lambda(r,\cdot), \Omega, 0) = \deg(I - \varphi_0(r,\cdot), \Omega, 0) = \deg(I, \Omega, h_r) = 1,$$

for each $\lambda \in [0,1]$. If we consider r fixed and we apply the Leray–Schauder Continuation Theorem to $x = \phi_\lambda(r,x)$ with $\lambda \in [0,1]$ as a parameter, we find a continuous family $\{x^\lambda\}_{\lambda\in[0,1]} \subset C^1([a,b])$ of solutions with $x^0(t) = h_r(t)$; therefore x^0 has a unique simple zero in $[a,b]$. If x^1 has more than one zero, then by a continuity argument there should be a $\hat{\lambda} \in]0,1]$ and $t_0 \in [a,b]$ such that $x^{\hat{\lambda}}(t_0) = \dot{x}^{\hat{\lambda}}(t_0) = 0$ with $x^{\hat{\lambda}} \not\equiv 0$: this would be in contradiction with Propositions 3.3.1 and 3.3.2 by $(g0+)$ and $(g0-)$. We can conclude that x^1 has exactly one zero in $[a,b]$.

On the other hand, by the Leray–Schauder Continuation Theorem applied to the equation $x = \phi_1(r,x)$ parametrized with respect to $r \in [-\beta, 0]$, we have that there is a compact connected set $\mathcal{S}_n \subset [-\beta,0] \times C^1([a,b])$ such that for each $r \in [-\beta,0]$ there is $x \in C^1([a,b])$, with $(r,x) \in \mathcal{S}_n$ and $x = \varphi_1(r,x)$, that is, x is a solution of (3.13) with exactly one zero in $[a,b]$. The projection of $\mathcal{S}_n$ via the operator $(r,x) \mapsto (r, \dot{x}(a))$ is a compact connected set $\Sigma_n \subset [-\beta,0] \times [-M_n, M_n]$.

The same arguments used in the Claims 1 and 2 in the proof of Lemma 3.3.3 imply the following analogous claims, respectively.

Claim 1. For each integer $k > 1/(b-a)$, there are $L_k > \beta$ and j_k such that every solution x of (3.13), with $r \in [-\beta,0]$, satisfies $|x(t)| \leq L_k$ for every $t \in [a, b - 1/k]$ and each $n \geq j_k$.

Claim 2. For each $\gamma > \beta$, there are $t_\gamma \in]a,b[$ and an index i_γ such that $x(t) > \gamma$ for all $t \in [t_\gamma, b]$ and each $n \geq i_\gamma$, if x is any solution of (3.13) with $r \in [-\beta, 0]$.

From now on the proof proceeds almost in the same way. It is possible to show that there is $N > 0$ such that

$$\Sigma_n \subset [-\beta, 0] \times [N, N] \qquad \forall\, n,$$

hence, up to pass to a suitable subsequence, we have that $\liminf_{n\to+\infty} \Sigma_n \neq \emptyset$ and that:

$$\Sigma(\beta) = \limsup_{n\to+\infty} \Sigma_n \subset [-\beta, 0] \times [N, N]$$

is a compact continuum with $\pi_x(\Sigma(\beta)) = [-\beta, 0]$.

Then it is possible to show that for each $r \in [-\beta, 0]$, there is $s \in [-N, N]$ such that $(r, s) \in \Sigma(\beta)$ and for each $(r, s) \in \Sigma(\beta)$, there is a solution x of equation (3.8) with exactly one zero in $[a, b]$ and such that $x(a) = r$, $\dot{x}(a) = s$ and $x(b^-) = +\infty$.

Finally we let β go to infinity: it can be proved that for each $\beta > \beta_0$ there is $N = N(\beta)$ such that for each $\beta_1 \geq \beta$ it holds that:

$$\Sigma(\beta_1) \cap ([-\beta, 0] \times \mathbb{R}) \subset [-\beta, 0] \times [-N, N].$$

We take now $\beta = k \in \mathbb{N}$ and define $\Gamma_k := \Sigma(k)$ and $N_k = N(k)$. By a standard argument from real analysis, we can find a continuous function $f :]-\infty, 0] \to [0, +\infty[$ such that any solution x of (3.8) satisying $x(a) = r < 0$ and $(x(a), \dot{x}(a)) \in \Gamma_k$ for some $k > 0$, is such that $|\dot{x}(a)| \leq f(r)$. Using again Lemma 3.3.5, we consider the set $\mathcal{A} = \{(r, s) \in]-\infty, 0] \times \mathbb{R} : |s| \leq f(r)\}$ and its compactification $\mathcal{A}_0 = \mathcal{A} \cup \{p_\infty\}$ obtained by adding a point p_∞ at infinity, so that Γ_k is now a sequence of compact continua in $\mathcal{A}_0$. Then we find a continuum Γ_a^-, which is closed relatively to $\mathcal{A}$ and satisfies $\overline{\Gamma_a^-} = \Gamma_a^- \cup \{p_\infty\}$ (the closure here is with respect to $\mathcal{A}_0$) and $\pi_x(\Gamma_a^-) = [0, +\infty[$.

By the way in which Γ_a^- is obtained and from the properties of the solutions with initial value in Γ_k for $t = a$, we have that for each $r \leq 0$, there is $s \in \mathbb{R}$, with $(r, s) \in \Gamma_a^-$ and also for each $(r, s) \in \Gamma_a^-$ there is at least one solution x of (3.8) with $x(a) = r$, $\dot{x}(a) = s \in [-f(r), f(r)]$ with $x(t)$ defined in $[a, b[$ and such that $\lim_{t\to b^-} x(t) = +\infty$.

If x is a solution of (3.8) such that $x(a) \leq -\beta_0$ and $\dot{x}(a) \leq 0$, then it can be shown that actually x is decreasing wherever defined by $(g1-)$ and cannot satisfy $x(b^-) = +\infty$. Therefore, if $(r, s) \in \Gamma_a^-$ and $r \leq \beta_0$, then s must be positive, proving that $\Gamma_a^- \cap (]-\infty, -\beta_0] \times \mathbb{R}) \subset]-\infty, -\beta_0] \times [0, +\infty[$. ■

It is now possible to state analogous results about continua of solutions which present the blow-up at $t = a$: they can be obtained simply by the corresponding Lemmas 3.3.3, 3.3.7 and 3.3.8 and by an inversion of the time direction.

Lemma 3.3.9 *Let $g : [0, +\infty[\to \mathbb{R}$ be a continuous function such that $g(0) = 0$ and satisfying $(g1+)$, $(g0+)$ and (3.9) and let $q :]a, b] \to]-\infty, 0]$ be a continuous and bounded function such that:*

$$a \in \overline{\{t \in]a, b[: q(t) < 0\}}.$$

Then, there is an unbounded continuum $\Gamma_b^+ \subset [0, +\infty[\times \mathbb{R}$, with $\pi_x(\Gamma_b^+) = [0, +\infty[$, such that for each $(x_0, y_0) \in \Gamma_b^+$ there is a solution $x(\cdot)$ of (3.8) with $x(b) = x_0$, $\dot{x}(b) = y_0$, $x(t) > 0$ for all $t \in]a, b[$ and $x(t) \to +\infty$ as $t \to a^+$. Moreover, the localization of the branch Γ_b^+ in the phase plane can be described as follows: there is $\delta > 0$ and

(i) there is $\varepsilon > 0$ such that $\pi_y(\Gamma_b^+ \cap ([0, \varepsilon[\times \mathbb{R})) \subset]-\infty, -\delta[$,

(ii) there is $K > 0$ such that $\pi_y(\Gamma_b^+ \cap (]K, +\infty[\times \mathbb{R})) \subset]\delta, +\infty[$.

Lemma 3.3.10 *Let $g :]-\infty, 0] \to \mathbb{R}$ be a continuous function satisfying $g(0) = 0$ and satisfying $(g1-)$, $(g0-)$ and (3.12) and let $q :]a, b] \to]-\infty, 0]$ be a continuous and bounded function such that:*

$$a \in \overline{\{t \in]a, b[: q(t) < 0\}}.$$

Then, there is an unbounded continuum $\Gamma_b^- \subset]-\infty,0]\times\mathbb{R}$, *with* $\pi_x(\Gamma_b^-) =]-\infty,0]$, *such that for each* $(x_0,y_0)\in\Gamma_b^-$ *there is a nonpositive solution* $x(\cdot)$ *of* (3.8) *with* $x(b)=x_0$, $\dot{x}(b)=y_0$, $x(t)<0$ *for all* $t\in]a,b[$ *and* $x(t)\to-\infty$ *as* $t\to a^+$. *Moreover, the localization of the branch* Γ_b^- *in the phase plane can be described as follows: there is* $\delta>0$ *and:*

(i) *there is* $\varepsilon>0$ *such that* $\pi_y(\Gamma_b^-\cap(]-\varepsilon,0]\times\mathbb{R}))\subset]\delta,+\infty[$,

(ii) *there is* $K>0$ *such that* $\pi_y(\Gamma_b^-\cap(]-\infty,-K[\times\mathbb{R}))\subset]-\infty,-\delta[$.

Lemma 3.3.11 *Let* $g:\mathbb{R}\to\mathbb{R}$ *be a continuous function such that* $g(0)=0$ *and satisfying* $(g1+)$, $(g1-)$, $(g0+)$ *and* $(g0-)$ *and let* $q:]a,b]\to]-\infty,0]$ *be a continuous and bounded function such that:*

$$a\in\overline{\{t\in]a,b[:q(t)<0\}}.$$

Then the following facts hold:

(i) *if* (3.9) *holds, then there is an unbounded continuum* $\Gamma_b^-\subset]-\infty,0]\times\mathbb{R}$ *such that for each* $(x_0,y_0)\in\Gamma_b^-$ *there is a solution* $x(\cdot)$ *of* (3.8) *having exactly one zero in* $]a,b]$ *and satisfying* $x(b)=x_0$, $\dot{x}(b)=y_0$ *and* $x(t)\to+\infty$ *as* $t\to a^+$; *moreover* $\Gamma_b^-\cap(]-\infty,-\beta_0]\times\mathbb{R})\subset]-\infty,-\beta_0]\times[0,+\infty[$ *and* $\pi_x(\Gamma_b^-)=]-\infty,0]$;

(ii) *if* (3.12) *holds, then there is an unbounded continuum* $\Gamma_b^+\subset[0,+\infty[\times\mathbb{R}$ *such that for each* $(x_0,y_0)\in\Gamma_b^+$ *there is a solution* $x(\cdot)$ *of* (3.8) *having exactly one zero in* $]a,b]$ *and satisfying* $x(b)=x_0$, $\dot{x}(b)=y_0$ *and* $x(t)\to-\infty$ *as* $t\to a^+$; *moreover* $\Gamma_b^+\cap([\beta_0,+\infty[\times\mathbb{R})\subset[\beta_0,+\infty[\times]-\infty,0]$ *and* $\pi_x(\Gamma_b^+)=[0,+\infty[$.

An immediate consequence of Lemma 3.3.3 and Lemma 3.3.9 can be given in the case when g satisfies $g(0)=0$, $(g0+)$, $(g1+)$ and (3.9) and q is a bounded continuous function defined in $]0,1[$ and such that:

$$(q1)\qquad q(t)\le 0 \text{ for all } t\in]0,1[\text{ and } 0,1\in\overline{\{t\in]0,1[:q(t)<0\}}.$$

In fact, one can apply Lemma 3.3.3 on $[a,b[=[1/2,1[$, obtaining the unbounded continuum Γ_1^+ for the solutions blowing up at $t=1$, and 3.3.9 on $]a,b]=]0,1/2]$, obtaining the unbounded continuum Γ_0^+ for the solutions blowing up at $t=0$. Using a result like [92, Lemma 3], one can prove that the continua Γ_0^+ and Γ_1^+ intersect at some point $(\hat{r},\hat{s})\in]0,+\infty[\times\mathbb{R}$. Consequently we have the following:

Theorem 3.3.12 *Let* $g:[0,+\infty[\to\mathbb{R}$ *be a continuous function satisfying* $g(0)=0$, $(g0+)$, $(g1+)$ *and* (3.9) *and let* $q:]0,1[to\mathbb{R}$ *be a bounded and continuous function satisfying* $(q1)$. *Then, the problem:*

$$(3.15)\qquad \begin{cases}\ddot{x}+q(t)g(x)=0\\ x(0^+)=x(1^-)=+\infty\end{cases}$$

has at least one positive solution.

Remark 3.3.13 With respect to preceding works dealing with problem (3.15), we don't assume that $g(s)>0$ for all $s>0$, but only for the s in a neighborhood of zero and infinity. Moreover, our assumption (3.9) is more general than other growth conditions at infinity previously considered in the literature. In particular it is always satisfied if $g(s)$ is monotone nondecreasing for all s sufficiently large and $\int^{+\infty}G(s)^{-1/2}\,ds<+\infty$ (see Section

3.4). As to the sign condition on the weight $q(t)$, we observe that $(q1)$ holds true when $q(\cdot)$ is continuously defined on $[0,1]$, with $q(t) \leq 0$ for all $t \in [0,1]$ and $q(0), q(1) < 0$, but it may be satisfied also when $q(0) = 0$ or $q(1) = 0$, provided that in any neighbourhood of 0 and 1 there are points where q is negative. Such a weak form of sign conditions was recently considered by Cîrstea and Radulescu in [30] for PDEs, in the case of a monotone nonlinearity.

Remark 3.3.14 The same kind of results may be obtained for the p-Laplacian scalar ODE:

$$(\phi_p(u'))' + q(t)g(u) = 0, \tag{3.16}$$

with $\phi_p(s) = |s|^{p-2}s$, for $p > 1$ or even for a more general ϕ-Laplacian scalar ODEs of the form:

$$(\phi(u'))' + q(t)g(u) = 0,$$

with $\phi : \mathbb{R} \to \mathbb{R}$ an odd increasing homeomorphism satisfying suitable upper and lower σ-conditions (see [78]). The assumptions on g will have to be modified accordingly. For instance, in the case of the p-Laplace operator, $(g0+)$ has to be replaced by:

$$\int_0^{\alpha_0} G(s)^{-1/p}\, ds = +\infty. \tag{g_p0+}$$

In the same manner, the time-map $\tau(\cdot) = \tau_p(\cdot)$ is now expressed by means of an integral like $k_p \int_c^{+\infty} (G(s) - G(c))^{-1/p}\, ds$ (with k_p a suitable positive constant) and then (3.9) has to be changed to:

$$\lim_{c \to +\infty} \tau_p(c) = 0. \tag{3.17}$$

Remark 3.3.15 Denote by $B(R)$ and $B[R]$ the open and the closed ball in $\mathbb{R}^N$, with center in the origin and radius $R > 0$. Let $\Omega = B(R_2) \setminus B[R_1]$, with $0 < R_1 < R_2$, be an annular domain in $\mathbb{R}^N$, and let $w :]R_1, R_2[\to \mathbb{R}^+$ be a bounded and continuous function such that:

$$R_1, R_2 \in \overline{\{r \in]R_1, R_2[: w(r) > 0\}}.$$

We observe that our assumptions for w are of the same kind like those considered by Lazer and McKenna in [65, p.334] where for an arbitrary open domain they have a bounded and continuous weight. On the other hand, we allow a more general condition on the sign. Then, as a consequence of Theorem 3.3.12 and Remark 3.3.14, we have the following result.

Corollary 3.3.16 *If $g : [0, +\infty[\to \mathbb{R}$ is any continuous function satisfying $g(0) = 0$, (g_p0+), $(g1+)$ and (3.17) and $w(r)$ satisfies the above sign condition, then the boundary value problem:*

$$\begin{cases} \Delta_p u = w(|\mathrm{x}|)g(u) & \mathrm{x} \in \Omega \\ u(\mathrm{x}) \to +\infty & as\ \mathrm{x} \to \partial\Omega \end{cases} \tag{3.18}$$

(with $p > 1$) has at least one radially symmetric positive solution.

Proof. The search of the radially symmetric solutions of (3.18) yields to the study of the boundary value problem:

$$\begin{cases} (\phi_p(u'(r)))' + \dfrac{N-1}{r}\phi_p(u'(r)) - w(r)g(u(r)) = 0 & r \in]R_1, R_2[\\ u(r) \to +\infty & \text{as } r \to R_1,\ r \to R_2, \end{cases} \tag{3.19}$$

where $' = \frac{d}{dr}$ denotes the differentiation with respect to $r = |x|$ and $\phi_p(s) = |s|^{p-2}s$, $p > 1$. If we consider now the change of variable $t \mapsto r(t)$, $r \mapsto t(r)$, where:

$$t(r) = \frac{\displaystyle\int_{R_1}^{r} \xi^{-\frac{N-1}{p-1}}\, d\xi}{\displaystyle\int_{R_1}^{R_2} \xi^{-\frac{N-1}{p-1}}\, d\xi},$$

we transform problem (3.19) to:

$$\begin{cases} (\phi_p(x'(t)))' + q(t)g(x(t)) = 0, & t \in]0,1[\\ u(t) \to +\infty & \text{as } t \to 0,\ t \to 1, \end{cases}$$

where:

$$q(t) = -\frac{w(r(t))}{\left.\frac{dt}{dr}\right|_{r=r(t)}},$$

and for this problem we can apply Theorem 3.3.12 with Remark 3.3.14. ∎

Note that no other restriction on the growth of g is needed here. This result answers a question raised in [87, §6]. Moreover, with respect to [87], we allow more general conditions on g than those considered in [87, Theorem 2] and, as remarked above, the assumption $g(s) > 0$ for all $s > 0$ is not required as well. Furthermore, a general weight function is also permitted.

3.4 Remarks about the conditions of superlinear growth at infinity

We have expressed our assumption of superlinear growth at infinity for g by means of an asymptotic condition on the time map associated to the autonomous system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = g(x) \end{cases}$$

(cf. τ_g^- in (3.2) with conditions ($g2-$), (3.9) and (3.12)). When we deal with the map τ_g^- in (3.2), the following kind of integrals have to be evaluated:

$$f_1(x) := \int_{-\infty}^{+\infty} \frac{1}{\sqrt{x + G(s)}}\, ds, \tag{3.20}$$

and:

$$f_2(x) := \int_{G_+^{-1}(x)}^{+\infty} \frac{1}{\sqrt{G(s) - x}}\, ds \tag{3.21}$$

(there is a similar integral depending on $G_-^{-1}(x)$ which can be handled in the same way like f_2). Our goal is to study under which conditions $f_1(x), f_2(x) \to 0$ as $x \to +\infty$. We assume throughout this section that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that:

$$\lim_{x \to \pm\infty} g(x)\,\mathrm{sign}(x) = +\infty;$$

therefore, the primitive $G(x) = \int_0^x g(s)\,ds$ of g is strictly monotone in a neighbourhood of $\pm\infty$ and the inverses $G_{\pm}^{-1}$, which have been considered at the beginning of this chapter, are defined.

For the first integral we easily see that $f_1(x) \to 0$ as $x \to +\infty$ if and only if:

$$\left| \int^{\pm\infty} \frac{1}{\sqrt{G(s)}}\,ds \right| < +\infty. \tag{3.22}$$

For the second one we have the following lemma.

Lemma 3.4.1 *Assume that:*

$$\lim_{x\to\pm\infty} \frac{g(x)}{x} = +\infty, \tag{3.23}$$

that there is a constant $k > 1$ such that:

$$\liminf_{x\to\pm\infty} \frac{G(kx)}{G(x)} > 1 \tag{3.24}$$

and that (3.22) *holds. Then* $(g2-)$ *holds, that is:*

$$\lim_{e\to\pm\infty} \tau_g^-(e) = 0.$$

Proof. By the above remark we can confine ourselves to the proof of $f_2(x) \to 0$ as $x \to +\infty$. With a simple change of variables, we consider the function:

$$f_3(x) = \int_x^{+\infty} \frac{1}{\sqrt{G(s) - G(x)}}\,ds.$$

Clearly, $f_2(x) \to 0$ if and only if $f_3(x) \to 0$. Now we split the integral as:

$$\int_x^{kx} \frac{1}{\sqrt{G(s) - G(x)}}\,ds + \int_{kx}^{+\infty} \frac{1}{\sqrt{G(s) - G(x)}}\,ds.$$

For the first one, we have that:

$$\begin{aligned}
\int_x^{kx} \frac{1}{\sqrt{G(s) - G(x)}}\,ds &= \int_x^{kx} \frac{1}{\sqrt{\int_x^s g(\xi)\,d\xi}}\,ds \\
&\leq \int_x^{kx} \frac{1}{\sqrt{s-x}\sqrt{g_{\min}(x)}}\,ds \\
&= \frac{2\sqrt{(k-1)x}}{\sqrt{g_{\min}(x)}} \\
&= 2\sqrt{(k-1)}\sqrt{\frac{x}{\xi_x}}\sqrt{\frac{\xi_x}{g(\xi_x)}},
\end{aligned}$$

where we have set $g_{\min}(x) = \min_{[x,kx]} g(\xi)$ and $\xi_x \in [x, kx]$ is such that $g(\xi_x) = g_{\min}(x)$. From (3.23) easily follows that the first integral tends to zero as $x \to +\infty$.

By (3.24) there exist $c > 1$ and $\bar{s} > 0$ such that:

$$G(ks) \geq cG(s) \qquad \forall, s \geq \bar{s};$$

moreover we can assume without loss of generality that G is monotone increasing on $[\bar{s}, +\infty[$. Hence, for the second integral we can estimate:

$$\begin{aligned}
\int_{kx}^{+\infty} \frac{1}{\sqrt{G(s)-G(x)}}\, ds &= \int_{kx}^{+\infty} \frac{1}{\sqrt{G(s)}\sqrt{1-\frac{G(x)}{G(s)}}}\, ds \\
&\leq \int_{kx}^{+\infty} \frac{1}{\sqrt{G(s)}\sqrt{1-\frac{G(x)}{G(kx)}}}\, ds \\
&\leq \sqrt{\frac{c}{c-1}} \int_{kx}^{+\infty} \frac{1}{\sqrt{G(s)}}\, ds \qquad \forall x \geq \bar{s}.
\end{aligned}$$

Thus also the second integral tends to zero as $x \to +\infty$ since (3.22) holds. ∎

We remark that condition (3.24) is related to a well-known assumption which appears in the theory of Orlicz–Sobolev spaces. Indeed, it is easy to prove that (3.24) holds when G_+^{-1} and $|G_-^{-1}|$ satisfy a *Δ_2-condition at infinity/* (see [1, p.232]).

We notice that a sufficient condition for the validity of (3.24) is that g is *monotone increasing* in a neighbourhood of infinity. In fact, if $x > 0$ is large enough, by the monotonicity, we have that $G(2x) = \int_0^{2x} g(s)\, ds \geq 2\int_0^x g(s)\, ds = 2G(x)$ (the case for $x << 0$ is treated in the same manner) and therefore (3.24) is proved. In this case, however, we can obtain much more, namely we have the following result.

Lemma 3.4.2 *Assume that g is a monotone function in a neighbourhood of infinity satifying (3.22). Then (g_2^-) holds.*

Proof. Via the change of variables $\xi = G(s) - G(x)$ and $s = G_+^{-1}(\xi + G(x))$, the function $f_3(x)$ defined in the proof of Lemma 3.4.1 can be written in the following way:

$$f_3(x) = \int_0^{\infty} \frac{d\xi}{g(G_+^{-1}(\xi + G(x)))\sqrt{\xi}}.$$

By definition, the function $\phi_x : \xi \mapsto \frac{1}{g(G_+^{-1}(\xi+G(x)))\sqrt{\xi}}$ is positive and summable on $]0, +\infty[$ for large $x > 0$. Moreover, by the monotonicity of g, we see that $\phi_x(\xi)$ is monotonically pointwise decreasing to zero as $x \to +\infty$. As a consequence, the monotone convergence theorem implies that $f_3(x) \to 0$ as $x \to +\infty$. ∎

Remark 3.4.3 Suppose that $h : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying:

$$\lim_{s \to \pm\infty} h(s)\mathrm{sgn}(s) = +\infty$$

and assume also that:

$$|g(x)| \geq |h(x)|, \quad \text{for } |x| >> 1.$$

Then, it is straightforward to check that the validity of the condition (g_2) with respect to the function h implies the same for the function g. In particular, if, for $|x|$ large we have that g "dominates" a monotone increasing function h at infinity, with $h(x)/x \to +\infty$ as $x \to \pm\infty$ and $|\int^{\pm\infty} \frac{ds}{\sqrt{H(s)}}| < \infty$, where $H' = h$, then (g_2) holds for g.

From this remark, we see that a (non-necessarily monotone) function g which is larger than $kx\log^{\alpha}(|x|+1)$ for $|x|$ large and for some $k>0$ and $\alpha>2$, satisfies (g_2).

Chapter 4

Separated boundary conditions for superlinear equations with indefinite weight

We consider now the equation:

$$\ddot{x} + q(t)g(x) = 0 \tag{4.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and superlinear, in the sense specified in the previous chapters, and the weight function $q : I \to \mathbb{R}$ is allowed to change sign in I. In particular, suppose that I can be written as the union of $2k+1$ adjacent and nontrivial closed intervals:

$$I_1^+, I_1^-, \ldots, I_k^+, I_k^-, I_{k+1}^+$$

such that:

$$q \geq 0,\ q \not\equiv 0 \text{ on } I_i^+ \quad \text{and} \quad q \leq 0,\ q \not\equiv 0 \text{ on } I_i^-.$$

From what we have seen in the preceding two chapters, we expect to find solutions of (4.1) with a large number of zeros in I_i^+ and with at most one zero in I_i^-. Actually in the theorems we are going to prove we will find solutions x whose nodal behaviour in I (that is, the distribution of their zeros in I) is described by a couple of vectors $\mathbf{n} = (n_1, \ldots, n_{k+1}) \in \mathbb{N}^{k+1}$ and $\delta = (\delta_1, \ldots, \delta_k) \in \{0,1\}^k$ in the following way:

- n_i is the number of zeros of x in the interval I_i^+;
- δ_i is the number of zeros of x in the interval I_i^-;

Moreover we recall that in I_i^- a solution of (4.1) is strictly monotone, if it has one zero, while it is convex or concave (depending on the sign of x) otherwise; therefore our solutions will also have exactly $1 - \delta_i$ *changes of sign* of the first derivative $\dot{x}$ in I_i^-. If q does not vanish identically in any subinterval of I_i^-, then we can say that $\dot{x}$ will have exactly $1 - \delta_i$ zeros, otherwise it might happen that $\dot{x} \equiv 0$ in a subinterval of I_i^- where $q \equiv 0$.

We will prove at first a result about existence and multiplicity of solutions of a generalized Sturm–Liouville boundary value problem. Then we will apply this result to obtain solutions of usual Sturm–Liouville boundary value problems as well as homoclinic solutions and solutions presenting blow-up at a given instant t^*.

Let us fix now some notation broadly used in this chapter. The function $g : \mathbb{R} \to \mathbb{R}$ in (4.1) will be here a a locally Lipschitz continuous function and $q : I \to \mathbb{R}$ will be continuous, so that the local uniqueness holds for every intial value problem associated to (4.1). In particular, for every $p \in \mathbb{R}^2$ and every $t_0 \in I$ we can define $x(\cdot; t_0, p)$ to be the solution of (4.1) which satisfies $(x(t_0), \dot{x}(t_0)) = p$ and is defined in its maximal interval of continuability, and $z(\cdot; t_0, p) = (x(\cdot; t_0, p), y(\cdot; t_0, p)) = (x(\cdot; t_0, p), \dot{x}(\cdot; t_0, p))$ to be the corresponding solution of the equivalent plane system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -q(t)g(x). \end{cases}$$

Throughout this chapter we will frequently use the following assumptions:

$$(g1) \qquad g(x)x > 0 \qquad \forall x \neq 0,$$

$$(g2+) \qquad \lim_{e \to +\infty} \tau_g^+(e) = 0,$$

$$(g2-) \qquad \lim_{e \to \pm\infty} \tau_g^-(e) = 0$$

(see (2.3) and (3.2)) and:

($q0$) if $q(t) \geq 0$ for all t in an interval I and $q|_I \not\equiv 0$, then q is of bounded variation in I and the set $\{t \in I : q(t) > 0\}$ is the union of a finite number of open intervals; moreover if $q(t) > 0$ for every $t \in]c, d[$ and $q(c) = 0$ (or $q(d) = 0$), then q is monotone in a right neighbourhood of c (or, respectively, in a left neighbourhood of d).

Condition ($g1$) is the usual sign condition on g, while ($g2+$) and ($g2-$) are the hypotheses of superlinearity needed, respectively, where $q \geq 0$ and where $q \leq 0$, in order to apply the results discussed in the previous two chapters. We finally recall that ($q0$) ensures that there are no problems of continuability for equation (4.1) in the intervals where $q \geq 0$, as shown by Corollary 2.1.3.

4.1 Generalized Sturm–Liouville boundary value problems

In the next two lemmas the situation in which q is nonnegative in $[a, b]$ and nonpositive in $[b, c]$ is considered: taken an unbounded curve γ, we describe how it is transformed by the composition of the two Poincaré maps:

$$p \longmapsto z(b; a, p) \qquad \text{and} \qquad p \longmapsto z(c; b, p).$$

Lemma 4.1.1 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying ($g1$), ($g2+$) and ($g2-$) and let $q : [a, c] \to \mathbb{R}$ be a continuous function satisfying ($q0$). Assume that there is $b \in]a, c[$ such that:*

$$q \geq 0 \text{ and } q \not\equiv 0 \text{ on } [a, b], \quad q \leq 0 \text{ and } q \not\equiv 0 \text{ on } [b, c].$$

Then there is a constant R^* (depending only on g and $q|_{[b,c]}$) such that the following holds:

- *for each $R > 0$, there is $n^* = n^*_R > 0$ such that, for each $n \geq n^*$, and for each continuous curve $\gamma : [\alpha, \beta[\to H^+$ (respectively, $\gamma : [\alpha, \beta[\to H^-$), with:*

$$|\gamma(\alpha)| \leq R, \quad \text{and } |\gamma(s)| \to +\infty, \text{ as } s \to \beta^-,$$

there is an interval $]\alpha_n, \beta_n] \subset]\alpha, \beta[$ such that, for each $s \in]\alpha_n, \beta_n]$, $z(t; a, \gamma(s))$ is defined for all $t \in [a, c]$ and $x(\cdot; a, \gamma(s))$ has exactly n zeros in $]a, b[$, no zeros in $[b, c]$ and exactly one change of sign of the derivative in $]b, c[$;

- *if we set set:*

$$\gamma_n(s) = z(c; a, \gamma(s)), \quad \forall\, s \in]\alpha_n, \beta_n],$$

we have that:

$$|\gamma_n(\beta_n)| \leq R^*, \quad \text{and } |\gamma_n(s)| \to +\infty, \text{ as } s \to \alpha_n^+$$

and γ_n lies in A_1 or in A_3, according to the fact that n is even or odd (respectively, γ_n lies in A_3 or in A_1, according to the fact that n is even or odd).

Proof. By applying Lemma 3.1.1 to $q|_{[b,c]}$ with $(r_1, r_2) = (0, \pm 1)$ and $(r_1, r_2) = (\pm 1, 0)$, we find that there is $R^* > 0$ such that for every $p \in (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$, with $|p| \geq R^*$, the solutions $z(\cdot; b, p)$ and $z(\cdot; c, p)$ are not continuable on the whole $[b, c]$; moreover let:

$$R' = \max\{|\xi| : |z(b; a, \xi)| \leq R^*\},$$

which is finite by Corollary 2.1.3 and Lemma 2.2.2. For every $R > 0$ we set

$$n^* = n^*_R = \left\lceil \frac{\sup\left\{ \operatorname*{rot}_{[a,b]}(p) : 0 < |p| \leq \max\{R, R'\} \right\}}{\pi} \right\rceil + 1.$$

In other words, n^*_R is the minimum integer n such that:

$$\operatorname*{rot}_{[a,b]}(p) \leq (n-1)\pi \qquad \forall\, p \in \mathbb{R}^2 : 0 < |p| \leq \max\{R, R'\}$$

and, roughly speaking, represents an upper bound for the number of zeros that a solution x of our equation may have in $]a, b]$, if the starting point $(x(a), \dot{x}(a))$ is inside the closed ball of radius $\max\{R, R'\}$. Note that, since $g(s)/s$ is bounded in a neighbourhood of zero, n^*_R is finite.

Let us consider a curve $\gamma : [\alpha, \beta[\to H^+$ such that

$$|\gamma(\alpha)| \leq R \quad \text{and} \quad \lim_{s \to \beta^-} |\gamma(s)| = +\infty;$$

without loss of generality we may assume also that $\gamma(s) \neq (0,0)$ for every $s \in [\alpha, \beta[$: thus $z(t; a, \gamma(s)) \neq (0,0)$ for every t and for every s by the uniqueness of the zero solution. By Corollary 2.2.8 we can find two continuous functions $\rho : [\alpha, \beta[\to]0, +\infty[$ and $\theta : [\alpha, \beta[\to \mathbb{R}$ such that

1) $z(b; a, \gamma(s)) = (\rho(s)\cos\theta(s), \rho(s)\sin\theta(s)), \qquad \forall s \in [\alpha, \beta[$;

2) $\lim_{s \to \beta^-} \rho(s) = +\infty$ and $\lim_{s \to \beta^-} \theta(s) = -\infty$;

3) $\theta(s) + \operatorname*{rot}_{[a,b]}(\gamma(s)) \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\quad \forall s \in [\alpha, \beta[$

(condition 3 can be achieved since $\gamma(s) \in H^+$ and θ is uniquely determined up to multiples of 2π). Then we have that:

$$\theta(\alpha) \geq -(n^* - 1)\pi,$$

by condition 3) above and by the definition of n^*. Therefore, for every $n \geq n^*$, we can determine $\alpha', \beta' \in [\alpha, \beta[$ such that $\theta(\alpha') = -(n - \frac{1}{2})\pi$, $\theta(\beta') = -(n + 1/2)\pi$ and $-(n + 1/2)\pi < \theta(s) < -(n - \frac{1}{2})\pi$ for every $s \in]\alpha', \beta'[$. We have that $x(\cdot; a, \gamma(s))$ has exactly n zeros in $]a, b]$ for all $s \in [\alpha', \beta'[$. We suppose for definiteness that n is even: then $z(b; a, \gamma(s)) \in]0, +\infty[\times \mathbb{R}$ as s ranges in $]\alpha', \beta'[$; the other case can be treated in a completely symmetric way. We remark that $z(\cdot; a, \gamma(\alpha'))$ and $z(\cdot; a, \gamma(\beta'))$ are not continuable on the whole $[a, c]$ since $z(b; a, \gamma(\alpha'))$ and $z(b; a, \gamma(\beta'))$ belong to $\{0\} \times \mathbb{R}$, $|z(b; a, \gamma(\alpha'))| \geq R^*$ and $|z(b; a, \gamma(\beta'))| \geq R^*$, by the definition of n^*; therefore, it can be deduced, by a simple analysis of equation (4.1), that there are $t_1, t_2 \in]b, c]$ such that:

$$\lim_{t \to t_1^-} x(t; a, \gamma(\alpha')) = \lim_{t \to t_1^-} \dot{x}(t; a, \gamma(\alpha')) = +\infty$$

and:

$$\lim_{t \to t_2^-} x(t; a, \gamma(\beta')) = \lim_{t \to t_2^-} \dot{x}(t; a, \gamma(\beta')) = -\infty.$$

Let:

$$\beta'' = \inf\{s \in]\alpha', \beta'[: x(t; a, \gamma(s)) < 0 \text{ for some } t \in]b, c]\}.$$

In the definition of β'' we do not care if $x(\cdot; a, \gamma(s))$ is not continuable up to c: we simply ask that it is negative somewhere in the part of its interval of continuability which lies to the right of b. Of course $\alpha' < \beta'' < \beta'$.

Now we show that $x(\cdot; a, \gamma(\beta''))$ is actually continuable up to c and is nonnegative and decreasing on $[b, c]$, with $x(c; a, \gamma(\beta'')) = 0$ and $\dot{x}(c; a, \gamma(\beta'')) < 0$. By the continuous dependence on initial data we deduce that $x(t; a, \gamma(\beta'')) \geq 0$ for every $t \in [b, c]$, wherever it is defined. Moreover, if $x(t; a, \gamma(s)) < 0$ for some $t \in]b, c]$ and $s \in]\alpha', \beta'[$, then $x(\cdot; a, \gamma(s))$ has a zero in $]b, c[$, since $x(b; a, \gamma(s)) > 0$ for every $s \in]\alpha', \beta'[$. On the other hand, every solution of (4.1) on $[b, c]$ is convex, wherever it is positive, and concave elsewhere; this implies that every solution x of (4.1) may have at most one zero and, if this is the case, x is strictly monotone decreasing. Thus $x(\cdot; a, \gamma(\beta''))$ is decreasing since it is limit of decreasing functions by the continuous dependence on the initial data. Together with the nonnegativity, the decreasing monotonicity implies that $x(\cdot; a, \gamma(\beta''))$ is continuable on $[a, c]$. Finally, if $x(c; a, \gamma(\beta''))$ were positive, we should have, again by the continuous dependence on the initial data, that $x(t; a, \gamma(s)) > 0$ for every $t \in [b, c]$ and every s in a neighbourhood of β'', and the definition of β'' should be violated. Moreover we obtain also that $\dot{x}(c; a, \gamma(\beta'')) < 0$ by the uniqueness of the trivial solution. In particular we can deduce that all the solutions $x(\cdot; a, \gamma(s))$ are positive (and then convex) on $[b, c[$ (where defined) for all $s \in [\alpha', \beta'']$.

Now let:

$$\alpha_n = \inf\{\sigma \in]\alpha', \beta''] : z(\cdot; a, \gamma(s)) \text{ is continuable on } [a, c], \quad \forall s \in [\sigma, \beta'']\},$$

thus $\alpha' \leq \alpha_n < \beta''$,

$$\lim_{s \to \alpha_n^+} x(c; a, \gamma(s)) = \lim_{s \to \alpha_n^+} \dot{x}(c; a, \gamma(s)) = +\infty$$

and $x(\cdot; a, \gamma(s))$ is continuable on $[a, c]$ for all $s \in]\alpha_n, \beta'']$. Since $\dot{x}(c; a, \gamma(\beta'')) < 0$, there is $\beta_n \in]\alpha_n, \beta''[$ such that $\dot{x}(c; a, \gamma(\beta_n)) = 0$ and $\dot{x}(c; a, \gamma(s)) > 0$ for all $s \in]\alpha_n, \beta_n[$. As a consequence of this construction, $\dot{x}(\cdot; a, \gamma(s))$ must change sign exactly once in $]b, c[$ and, by the convexity, the zero set of $\dot{x}(\cdot)$ is either a point or a closed subinterval of $]b, c[$.

If we set $\gamma_n(s) = z(c; a, \gamma(s))$ as s ranges in $]\alpha_n, \beta_n]$, then $|\gamma_n(s)| \to +\infty$ as $s \to \alpha_n^+$ and $|\gamma_n(\beta_n)| < R^*$, since, by construction, $\gamma_n(\beta_n) \in \mathbb{R} \times \{0\}$ and $z(\cdot; c, \gamma_n(\beta_n))$ is continuable on $[b, c]$.

An analogous argument can be used in the case $\gamma : [\alpha, \beta[\to H^-$. ∎

A completely symmetric result is the following.

Lemma 4.1.2 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying (g1), (g2+) and (g2−) and let $q : [a, c] \to \mathbb{R}$ be a continuous function satisfying (q0). Assume that there is $b \in]a, c[$ such that:*

$$q \geq 0 \ \text{ and } \ q \not\equiv 0 \ \text{ on } \ [a, b], \quad q \leq 0 \ \text{ and } \ q \not\equiv 0 \ \text{ on } \ [b, c].$$

Then there is a constant R^* (depending only on g and $q|_{[b,c]}$) such that the following holds:

- *For each $R > 0$, there is $n^* = n_R^* > 0$ such that, for each $n \geq n^*$, and for each path $\gamma : [\alpha, \beta[\to H^+$ (respectively, $\gamma : [\alpha, \beta[\to H^-$), with:*

$$|\gamma(\alpha)| \leq R, \quad \text{and } |\gamma(s)| \to \infty, \text{ as } s \to \beta^-,$$

  there is an interval $[\alpha_n, \beta_n[\subset]\alpha, \beta[$ such that, for each $s \in]\alpha_n, \beta_n[$, $z(t; a, \gamma(s))$ is defined for all $t \in [a, c]$ and $x(\cdot; a, \gamma(s))$ has exactly n zeros in $]a, b[$, exactly one zero in $]b, c[$ and no zeros of the derivative in $[b, c]$;

- *if we set:*

$$\gamma_n(s) = z(c; a, \gamma(s)) \quad \forall\, s \in [\alpha_n, \beta_n[,$$

  we have that:

$$|\gamma_n(\alpha_n)| \leq R^*, \quad \text{and } |\gamma_n(s)| \to \infty, \text{ as } s \to \beta_n^-$$

  and γ_n lies in $\bar{A}_3$ or in $\bar{A}_1$ according to the fact that n is even or odd (respectively, γ_n lies in $\bar{A}_1$ or in $\bar{A}_3$ according to the fact that n is even or odd).

Proof. We describe only the main changes to be performed on the proof of Lemma 4.1.1 in order to obtain Lemma 4.1.2. The first part of the proof can be left unchanged until it is remarked that there are $t_1, t_2 \in]b, c]$ such that:

$$\lim_{t \to t_1^-} x(t; a, \gamma(\alpha')) = \lim_{t \to t_1^-} \dot{x}(t; a, \gamma(\alpha')) = +\infty$$

and

$$\lim_{t \to t_2^-} x(t; a, \gamma(\beta')) = \lim_{t \to t_2^-} \dot{x}(t; a, \gamma(\beta')) = -\infty.$$

At this point we set:

$$\alpha_n = \sup\{s \in]\alpha', \beta'[: x(t; a, \gamma(s)) > 0 \text{ for all } t \in]b, c] \text{ where it is defined}\},$$

hence $\alpha' < \alpha_n < \beta'$.

Now it is possible to show by the same kind of argument that $x(\cdot; a, \gamma(\alpha_n))$ is actually continuable up to c and on $[b, c]$ is nonnegative and decreasing, with $x(c; a, \gamma(\alpha_n)) = 0$ and $\dot{x}(c; a, \gamma(\alpha_n)) < 0$. In particular we deduce that all the solutions $x(\cdot; a, \gamma(s))$ are strictly decreasing (hence their derivatives have no zeroes) and with exactly one zeroes on $]b,c]$ for all $s \in [\alpha_n, \beta']$.

Now let:

$$\beta_n = \sup\{\sigma \in]\alpha_n, \beta'] : z(\cdot; a, \gamma(s)) \text{ is continuable on } [a,c]\ \forall s \in [\alpha_n, \sigma]\},$$

thus $\alpha_n < \beta_n \leq \beta'$,

$$\lim_{s \to \beta_n^-} x(c; a, \gamma(s)) = \lim_{s \to \beta_n^-} \dot{x}(c; a, \gamma(s)) = -\infty$$

and $x(\cdot; a, \gamma(s))$ is continuable on $[a,c]$ for all $s \in [\alpha_n, \beta_n[$.

If we set $\gamma_n(s) = z(c; a, \gamma(s))$ as s ranges in $[\alpha_n, \beta_n[$, then $|\gamma_n(s)| \to +\infty$ as $s \to \beta_n^-$ and $|\gamma_n(\alpha_n)| < R^*$, since, by construction, $\gamma_n(\alpha_n) \in \{0\} \times \mathbb{R}$ and $z(\cdot; c, \gamma_n(\alpha_n))$ is continuable on $[b,c]$.

An analogous argument can be used in the case $\gamma : [\alpha, \beta[\to H^-$. ■

Remark 4.1.3 Note that the same choices of R^* and n_R^* work for both Lemma 4.1.1 and Lemma 4.1.2.

Here is the main result of this section.

Theorem 4.1.4 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locallly Lipschitz continuous function such that:*

$(g1)$
$$g(x)x > 0 \qquad \forall x \neq 0,$$

$(g2+)$
$$\lim_{e \to +\infty} \tau_g^+(e) = 0$$

and:

$(g2-)$
$$\lim_{e \to \pm\infty} \tau_g^-(e) = 0$$

(see (2.3) and (3.2) for the definitions of τ_g^+ and of τ_g^-); let $q : [a,b] \to \mathbb{R}$ be a continuous function such that:

- $(q0)$ *if $q(t) \geq 0$ for all t in an interval $I \subset [a,b]$ and $q|_I \not\equiv 0$, then q is of bounded variation in I and the set $\{t \in I : q(t) > 0\}$ is the union of a finite number of open intervals; moreover if $q(t) > 0$ for every $t \in]c,d[$ and $q(c) = 0$ (or $q(d) = 0$), then q is monotone in a right neighbourhood of c (or, respectively, in a left neighbourhood of d).*

We assume that $[a,b]$ can be written as the union of $2k+1$ ($k > 0$) consecutive, adjacent, nondegenerate and closed intervals:

$$I_1^+, I_1^-, \ldots, I_k^+, I_k^-, I_{k+1}^+,$$

such that $q \geq 0$, $q \not\equiv 0$ on I_i^+ and $q \leq 0$, $q \not\equiv 0$ on I_i^-. Let $\Gamma_a, \Gamma_b \subset \mathbb{R}^2$ be two unbounded continua such that one of the following two alternatives holds:

- (i) *$\Gamma_a \subset H^+$ and $\Gamma_b \subset H^-$ or, vice versa, $\Gamma_a \subset H^-$ and $\Gamma_b \subset H^+$,*
- (ii) *$\Gamma_a \cup \Gamma_b \subset H^+$ or $\Gamma_a \cup \Gamma_b \subset H^-$.*

Then, there are $k+1$ positive integers $n_1^*, \dots, n_{k+1}^*$ such that for each $(k+1)$-tuple $\mathbf{n} = (n_1, \dots, n_{k+1})$, with $n_i \geq n_i^*$, and each k-tuple $\delta = (\delta_1, \dots, \delta_k)$, with $\delta_i \in \{0,1\}$, such that:

$$\sum_{i=1}^{k+1} n_i + \sum_{i=1}^{k} \delta_i \text{ is odd, if (i) holds, and even, if (ii) holds,}$$

there is at least one solution $x = x_{\mathbf{n},\delta}(\cdot)$ of (4.1) such that:

$$(x(a), \dot{x}(a)) \in \Gamma_a, \quad (x(b), \dot{x}(b)) \in \Gamma_b \tag{4.2}$$

and:

1. *$x(\cdot)$ has exactly n_i zeros in I_i^+, exactly δ_i zeros in I_i^- and exactly $1-\delta_i$ changes of sign of the derivative in I_i^-;*
2. *for each i, $|x_{\mathbf{n},\delta}(t)| + |\dot{x}_{\mathbf{n},\delta}(t)| \to +\infty$, as $n_i \to +\infty$, uniformly in $t \in I_i^+$.*

We remark that the case $k = 0$ is contained in Corollary 2.3.5.

Proof. We write the intervals $I_i^{\pm}$ in the following way:

$$I_i^+ = [a_i, b_i] \qquad \text{and} \qquad I_i^- = [b_i, c_i],$$

so, in particular, $a = a_1$, $c_i = a_{i+1}$ and $b_{k+1} = b$.

We will deal only with the case $\Gamma_a \subset H^+$, since the other one can be treated analogously. We first develop the proof assuming that Γ_a is the image of a continuous curve in $\mathbb{R}^2$ and then we will solve the general case by approximating a suitable portion of Γ_a by images of continuous curves.

Let $\gamma : [0, +\infty[\to H^+$ be an unbounded continuous curve and let:

$$R_a = \min_{s \in [0,1[} |\gamma(s)|.$$

STEP 1. Let $R_1^* > 0$ and n_1^* be respectively the numbers R^* and n^* given by the two Lemmas 4.1.1 and 4.1.2 applied to our equation on $[a, c] = [a_1, c_1]$ with $b = b_1$ and $R = R_a$ (see Remark 4.1.3). Fix any $n_1 \geq n_1^*$ and suppose that $\delta_1 = 0$ in $\boldsymbol{\delta}$. In this case there is an interval $]\alpha_{(n_1,\delta_1)}, \beta_{(n_1,\delta_1)}] \subset]0, +\infty[$ such that for each $s \in]\alpha_{(n_1,\delta_1)}, \beta_{(n_1,\delta_1)}]$, we have:

- $z(t; a_1, \gamma(s))$ is defined for all $t \in [a_1, c_1]$,
- $x(\cdot; a_1, \gamma(s))$ has exactly n_1 zeros in $]a_1, b_1[$, no zeros in $[b_1, c_1]$ and exactly one change of sign of the derivative in $]b_1, c_1[$.

Moreover, setting:

$$\gamma_{(n_1,\delta_1)}(s) = z(c_1; a_1, \gamma(s)) \quad \forall\, s \in]\alpha_{(n_1,\delta_1)}, \beta_{(n_1,\delta_1)}],$$

we have that:

$$|\gamma_{(n_1,\delta_1)}(\beta_{(n_1,\delta_1)})| \leq R_1^*, \quad \text{and} \quad |\gamma_{(n_1,\delta_1)}(s)| \to \infty, \text{ as } s \to \alpha^+_{(n_1,\delta_1)}$$

and $\gamma_{(n_1,\delta_1)}$ lies in A_1 or in A_3 according to the fact that n_1 is even or odd.

On the other hand, if $\delta_1 = 1$ in $\boldsymbol{\delta}$, we obtain a completely similar result, but with:

- $x(\cdot; a_1, \gamma(s))$ has exactly n_1 zeros in $]a_1, b_1[$, exactly one zero in $]b_1, c_1[$ and no zeros of the derivative in $[b_1, c_1]$

and $\gamma_{(n_1,\delta_1)}$ lies in A_3 or in A_1 according to the fact that n_1 is even or odd.

If $k = 1$ we have to skip the next two steps, to set $\phi = \gamma_{(n_1,\delta_1)}$ and to go directly to STEP 4.

STEP 2. We repeat now inductively this kind of argument. Without loss of generality, we suppose that γ_{n_1} lies in A_1. The proof is completely symmetric if γ_{n_1} lies in A_3.

Let $R_2^* > 0$ and n_2^* be respectively the numbers R^* and n^* given by the two Lemmas 4.1.1 and 4.1.2 applied to our equation on $[a, c] = [a_2, c_2]$, with $b = b_2$ and $R = R_1^*$. Observe that n_2^* depends on R_1^*, but not on n_1^*. Fix any $n_2 \geq n_2^*$ and suppose that $\delta_2 = 0$ in $\boldsymbol{\delta}$. There is an interval:

$$[\alpha_{((n_1,n_2),(\delta_1,\delta_2))}, \beta_{((n_1,n_2),(\delta_1,\delta_2))}[\subset]\alpha_{(n_1,\delta_1)}, \beta_{(n_1,\delta_1)}],$$

such that for each $s \in [\alpha_{((n_1,n_2),(\delta_1,\delta_2))}, \beta_{((n_1,n_2),(\delta_1,\delta_2))}[$, we have:

- $z(t; a_1, \gamma(s))$ is defined for all $t \in [a_1, c_2]$,
- $x(\cdot; a_1, \gamma(s))$ has exactly n_2 zeros in $]a_2, b_2[$, no zeros in $[b_2, c_2]$ and exactly one change of sign of the derivative in $]b_2, c_2[$.

Moreover, setting:

$$\gamma_{((n_1,n_2),(\delta_1,\delta_2))}(s) = z(c_2; a_1, \gamma(s)) \quad \forall\, s \in [\alpha_{((n_1,n_2),(\delta_1,\delta_2))}, \beta_{((n_1,n_2),(\delta_1,\delta_2))}[,$$

we have that:

$$|\gamma_{((n_1,n_2),(\delta_1,\delta_2))}(\alpha_{((n_1,n_2),(\delta_1,\delta_2))})| \leq R_2^*,$$

and:

$$|\gamma_{((n_1,n_2),(\delta_1,\delta_2))}(s)| \to \infty, \quad \text{as } s \to \beta^-_{((n_1,n_2),(\delta_1,\delta_2))}$$

and $\gamma_{((n_1,n_2),(\delta_1,\delta_2))}$ lies in A_1 or in A_3 according to the fact that n_2 is even or odd.

On the other hand, if $\delta_2 = 1$ in $\boldsymbol{\delta}$, we obtain a completely similar result, but with:

- $x(\cdot; a_1, \gamma(s))$ has exactly n_2 zeros in $]a_2, b_2[$, exactly one zero in $]b_2, c_2[$ and no zeros of the derivative in $[b_2, c_2]$

and $\gamma_{((n_1,n_2),(\delta_1,\delta_2))}$ lies in A_3 or in A_1 according to the fact that n_2 is even or odd.

As remarked above, it is clear that the same argument works if $\gamma_{(n_1,\delta_1)}$ lies in $\bar{A}_3$. Actually, we could now summarize all the different possibilities as follows: $\gamma_{((n_1,n_2),(\delta_1,\delta_2))}$ lies in A_1 or in A_3 according to the fact that $n_1 + n_2 + \delta_1 + \delta_2$, that is the total number of zeros in $]a_1, c_2]$, is even or odd.

STEP 3. Repeating this argument k-times, we find $R_1^*, \ldots, R_k^*$ and $n_1^*, \ldots, n_k^*$ and an unbounded continuum Γ^* which is the image of a continuous curve:

$$\phi = \gamma_{((n_1,\ldots,n_k),\delta)},$$

defined on a suitable half-open bounded interval $I \subset]\alpha_{(n_1,\delta_1)}, \beta_{(n_1,\delta_1)}]$, such that $\Gamma^* \cap B[R_k^*] \neq \emptyset$ and:

$$\Gamma^* \subset A_1 \quad \text{or} \quad \Gamma^* \subset A_3$$

according to the fact that $\sum_{i=1}^{k}(n_i + \delta_i)$ is even or odd.

We have that for each $s \in I$ the solution $x(\cdot; a_1, \gamma(s))$ is defined on $[a_1, c_k]$ and, for each $i = 1, \dots, k$, it possesses exactly n_i zeros on $]a_i, b_i[$, exactly δ_i zeros in $[b_i, c_i]$ and exactly $1 - \delta_i$ changes of sign of the derivative in $[b_i, c_i]$. By construction, we notice that the curve $\phi : I \to \Gamma^*$ is a homeomorphism and the numbers $n_i \geq n_i^*$ (for $i = 1, \dots, k$) may be chosen independently one from each other, depending only on the choice of n_i^* which, in turn, depends on R_{i-1}^* and R_i^*.

STEP 4. At this point, we can use Corollary 2.3.5 on the interval $[a_{k+1}, b_{k+1}] = [a_{k+1}, b]$ to find that there exists n_{k+1}^*, which depends on R_k^* and on $R_b = \min\{|p| : p \in \Gamma_b\}$, such that, for every fixed $n_{k+1} \geq n_{k+1}^*$ (even or odd, depending on the relative positions of Γ^* and Γ_b), there is at least one solution the following boundary value problem:

$$\begin{cases} \ddot{x} + q(t)g(x) = 0 \\ (x(a_{k+1}), \dot{x}(a_{k+1})) \in \Gamma^* \\ (x(b), \dot{x}(b)) \in \Gamma_b. \end{cases}$$

In other words, there is at least one point $p \in \Gamma^*$ such that the solution $x(\cdot; a_{k+1}, p)$ of (4.1) satisfies $z(b; a_{k+1}, p) \in \Gamma_b$ and has exactly n_{k+1} zeros in $]a_{k+1}, b_{k+1}]$. Moreover, we have that $z(b_{k+1}; a_{k+1}, p)$ and $p \in \Gamma^*$ belong to the same vertical half plane or to opposite ones, according to the fact that n_{k+1} is even or is odd. More precisely, if $\Gamma^* \subset A_1$ (respectively, if $\Gamma^* \subset A_3$), then $\dot{x}(b_{k+1}; a_{k+1}, p) > 0$ if and only if n_{k+1} is odd (respectively, if and only if n_{k+1} is even).

In conclusion, we can take:

$$s^* = \phi^{-1}(p)$$

and have that the solution $x(\cdot; a_1, \gamma(s^*))$ is defined on $[a_1, b_{k+1}] = [a, b]$, it has precisely n_i zeros in the interior of the I_i^+ intervals, for each $i = 1, \dots, k+1$ and also it has exactly δ_i zeros and $1 - \delta_i$ changes of sign of the derivative in the interior of the I_i^- intervals. Thus, the main part of the proof is concluded.

We make here two remarks that will be useful when we deal with the case in which Γ_a is a general unbounded continuum. The first one concerns the behaviour on ∂I_i^- of the solutions we find; more precisely, we want to show that the choice of n_i^* implies that any solution x with nodal behaviour described, as in the statement of the theorem, by the vectors $\boldsymbol{\delta}$ and $\mathbf{n}$ with $n_i \geq n_i^*$, is such that neither x nor $\dot{x}$ vanish on ∂I_i^-, that is on the points b_i and c_i for all $i = 1, \dots, k$. Indeed, if $x(b_i) = 0$ or $\dot{x}(b_i) = 0$ for instance, then the point $(x(b_i), \dot{x}(b_i))$ belongs to $(\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$. Moreover, if we look to the definition of n_i^*, that is to the definition of n^* in the Lemmas 4.1.1 and 4.1.2 applied on $[a_i, c_i]$, it is clear that $|(x(b_i), \dot{x}(b_i))| \geq R^*$ and, therefore, x should not be continuable on $[b_i, c_i]$ by the definition of R^* in those lemmas, that is a contradiction since x is a solution on $[a, b]$. In the same way it can be seen that $x(b_i) \neq 0$ and $\dot{x}(b_i) \neq 0$. We note also that this argument is independent of the fact that Γ_a is or is not the image of a continuous curve: it's just a matter of how we have defined n_i^* for $i = 1, \dots, k+1$.

The second remark regards a priori bounds for solutions which have a fixed amount of zeros. In fact, if we fix a $\boldsymbol{\delta} \in \{0, 1\}^k$ and $\mathbf{n} \in \mathbb{N}^{k+1}$, with $n_i \geq n_i^*$, Corollary 2.2.8 implies that there is an apriori bound for the modulus $|p|$, if $x(\cdot; a, p)$ has exactly n_1 zeros in $[a, a_1]$. More precisely, there exists $R_a^* > R_a$ such that every solution of (4.1)–(4.2) which has the nodal behaviour in $[a, b]$ prescribed by the vectors $\boldsymbol{\delta}$ and $\mathbf{n}$, as in the statement of the theorem, must start from a point $p \in \Gamma_a$ at $t = a$ such that $|p| \in B[R_a^*] \setminus B(R_a)$. Therefore, when dealing with an arbitrary unbounded continuum Γ_a, we need to approximate only a compact and connected portion of Γ_a which lie in $B[R_a^*] \setminus B(R_a)$.

To this aim we fix $\varepsilon > 0$ and we denote by $\gamma_\varepsilon : [0, 1] \to H^+$ a continuous curve such that:

- $|\gamma_\varepsilon(0)| \leq R_a$ and $|\gamma_\varepsilon(1)| \geq R_a^*$;
- for every $s \in [0,1]$ the point $\gamma_\varepsilon(s)$ belongs to the ε-neighbourhood of Γ_a.

Moreover we extend γ_ε to an unbounded and continuous curve $[0,+\infty[\to H^+$ by defining $\gamma_\varepsilon(s) = s\gamma_\varepsilon(1)$ for every $s > 1$. By what we have already proved in the steps above and by the previous remark, there exists $s_\varepsilon \in [0,1]$ such that $x(\cdot; a, \gamma_v e(s_\varepsilon))$ satisfies:

$$z(b; a, \gamma_\varepsilon(s_\varepsilon)) \in \Gamma_b$$

and has nodal behaviour prescribed by tha vectors $\boldsymbol{\delta}$ and $\mathbf{n}$. We have that $\gamma_\varepsilon(s_\varepsilon)$ belongs to the compact annulus $B[R_a^*] \setminus B(R_a)$, therefore $\gamma_{\varepsilon_m}(s_{\varepsilon_m}) \to p_0 \in \Gamma_a$, up to choose a suitable sequence $\varepsilon_m \to 0$. Clearly $x(\cdot; a, p_0)$ is a solution of (4.1)-(4.2) by the continuous dependence on initial data.

What remains to be checked is the behaviour of the zeros of the functions $x_m(t) = x(t; a, \gamma_{\varepsilon_m}(s_{\varepsilon_m}))$ as $m \to +\infty$. By the first of the two remarks above, none of the zeros that x_m and $\dot{x}_m$ have in the open intervals $]a_i, b_i[$ and $]b_i, c_i[$, can approach the boundary of those intervals as m goes to $+\infty$, otherwise the limit solution $x(\cdot; a, p_0)$ or its derivative would have a zero on some ∂I_i^- and this situation has been excluded. Moreover, the uniqueness of the zero solution implies that all the zeros of x_m and $x(\cdot; a, p_0)$ are simple and this actually shows that the nodal behaviour of $x(\cdot; a, p_0)$ is the one prescribed by the vectors $\boldsymbol{\delta}$ and $\mathbf{n}$. ■

4.2 Applications

We apply the general result of the previous section to some particular boundary value problems associated to our equation:

$$\ddot{x}(t) + q(t)g(x) = 0. \tag{4.3}$$

The first one deals with the two-point boundary condition:

$$x(0) = x(\omega) = 0. \tag{4.4}$$

Theorem 4.2.1 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying (g1), (g2+) and (g2−); let $q : [0,\omega] \to \mathbb{R}$ be a continuous function satisfying (q0). Let $k \geq 0$, be an integer. Suppose that there are $2k+1$ consecutive, adjacent, nondegenerate and closed intervals:*

$$I_1^+\,,\ I_1^-\,,\ \ldots\ I_k^+\,,\ I_k^-\,,\ I_{k+1}^+\,,$$

such that $q \geq 0$, $q \not\equiv 0$ on I_i^+ and $q \leq 0$, $q \not\equiv 0$ on I_i^-. We assume also that $q \leq 0$, $q \not\equiv 0$ on:

$$J = [0,\omega] \setminus \left(\left(\cup_{i=1}^{k+1} I_i^+ \right) \cup \left(\cup_{i=1}^{k} I_i^- \right) \right),$$

if $J \neq \emptyset$. Then, there are $k+1$ positive integers $n_1^*, \ldots, n_{k+1}^*$ such that for each $(k+1)$-tuple $\mathbf{n} = (n_1, \ldots, n_{k+1}) \in \mathbb{N}^{k+1}$, with $n_i \geq n_i^*$, and each k-tuple $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k) \in \{0,1\}^k$, there are at least two solutions $x_{\mathbf{n},\boldsymbol{\delta}}^+(\cdot)$ and $x_{\mathbf{n},\boldsymbol{\delta}}^-(\cdot)$ of (4.3)-(4.4) such that:

1. *$\dot{x}_{\mathbf{n},\boldsymbol{\delta}}^-(0) < 0 < \dot{x}_{\mathbf{n},\boldsymbol{\delta}}^+(0)$;*
2. *$x_{\mathbf{n},\boldsymbol{\delta}}^\pm(\cdot)$ has exactly n_i zeros in I_i^+, exactly δ_i zeros in I_i^- and exactly $1-\delta_i$ changes of sign of the derivative in I_i^-;*

3. *neither* $x^{\pm}_{\mathbf{n},\delta}(\cdot)$, *nor* $\dot{x}^{\pm}_{\mathbf{n},\delta}(\cdot)$, *may vanish in* $J \setminus \{0, \omega\}$;

4. *for each* $i = 1, \dots, k+1$, $|x^{\pm}_{\mathbf{n},\delta}(t)| + |\dot{x}^{\pm}_{\mathbf{n},\delta}(t)| \to +\infty$, *as* $n_i \to +\infty$, *uniformly in* $t \in I_i^+$.

Proof. Let $Y^+ = \{0\} \times [0, +\infty[$ and $Y^- = \{0\} \times]-\infty, 0]$ be respectively the positive and the negative vertical semiaxes. If $J = \emptyset$, the existence of $x^+_{\mathbf{n},\delta}(\cdot)$ follows immediately by applying Theorem 4.1.4 on $[a, b] = [0, \omega]$ with $\Gamma_a = Y^+$ and Γ_b being the positive or the negative y-axis (it depends as usual on the total number of zeros that is required); on the other hand we choose $\Gamma_a = Y^-$ to find $x^-_{\mathbf{n},\delta}(\cdot)$.

If $J \supset [0, a]$, we have that $q \le 0$ and $q \not\equiv 0$ in $[0, a]$ and the sets $Y^+ \cap \Omega_0^a$ and $Y^- \cap \Omega_0^a$ are bounded by Lemma 3.1.1 (see (3.6) for the definition of the set Ω_0^a) and contain the origin. As a consequence there are $\alpha^- < 0 < \alpha^+$ such that the solution $z(\cdot; 0, (0, s))$ is defined on $[0, a]$ for every $s \in]\alpha^-, \alpha^+[$, but not for $s = \alpha^{\pm}$. In particular the curve $\gamma_a :]\alpha^-, \alpha^+[\to \mathbb{R}^2$, defined by:

$$\gamma_a(s) = z(a; 0, (0, s)),$$

satisfies:

- $\gamma_a(0) = (0, 0)$;
- $\lim_{s \to \alpha^{\pm}} |\gamma_a(s)| = +\infty$;
- $\gamma_a(]0, \alpha^+[) \subset H^+$ and $\gamma_a(]\alpha^-, 0[) \subset H^-$.

Therefore we can set $\Gamma_a = \gamma_a(]0, \alpha^+[)$, to find solutions of (4.3)–(4.4) with $\dot{x}(0) > 0$, or $\Gamma_a = \gamma_a(]\alpha^-, 0[)$ to find those with $\dot{x}(0) < 0$. Similarly, if $J \supset [b, \omega]$, we can apply again Lemma 3.1.1 to $Y^+ \cap \Omega_\omega^b$ and to $Y^- \cap \Omega_\omega^b$ to find that there are $\beta^- < 0 < \beta^+$ such that the curve $\gamma_b :]-beta^-, \beta^+[\to \mathbb{R}^2$, given by $\gamma_b(s) = z(b; \omega, (0, s)$, satisfies:

- $\gamma_b(0) = (0, 0)$;
- $\lim_{s \to \beta^{\pm}} |\gamma_b(s)| = +\infty$;
- $\gamma_b(]0, \beta^+[) \subset H^-$ and $\gamma_b(]\beta^-, 0[) \subset H^+$.

We can set $\Gamma_b = \gamma_b(]0, \beta^+[)$ or $\Gamma_b = \gamma_b(]\beta^-, 0[)$ and apply Theorem 4.1.4 on $[a, b]$ with the choices made above for Γ_a and Γ_b. ∎

Let us consider now the Sturm–Liouville boundary conditions

$$\begin{cases} u_1 x(0) + u_2 \dot{x}(0) = 0 \\ v_1 x(\omega) + v_2 \dot{x}(\omega) = 0 \end{cases} \tag{4.5}$$

where $|u_1| + |u_2| > 0$ and $|v_1| + |v_2| > 0$. With the same technique of Theorem 4.2.1, the following result can be proved.

Theorem 4.2.2 *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a locally Lipschitz continuous function satisfying* ($g1$), ($g2+$) *and* ($g2-$); *let* $q : [0, \omega] \to \mathbb{R}$ *be a continuous function satisfying* ($q0$). *Let* $k \ge 0$, *be an integer. Suppose that there are* $2k + 1$ *consecutive, adjacent, nondegenerate and closed intervals:*

$$I_1^+, I_1^-, \dots I_k^+, I_k^-, I_{k+1}^+,$$

such that $q \geq 0$, $q \not\equiv 0$ on I_i^+ and $q \leq 0$, $q \not\equiv 0$ on I_i^-. We assume also that $q \leq 0$, $q \not\equiv 0$ on:

$$J = [0,\omega] \setminus \left((\cup_{i=1}^{k+1} I_i^+) \cup (\cup_{i=1}^{k} I_i^-) \right),$$

if $J \neq \emptyset$. Then there are $k+1$ positive integers $n_1^*, \dots, n_{k+1}^*$ such that for each $(k+1)$-tuple $\mathbf{n} = (n_1, \dots, n_{k+1}) \in \mathbb{N}^{k+1}$, with $n_i \geq n_i^*$, and each k-tuple $\delta = (\delta_1, \dots, \delta_k) \in \{0,1\}^k$, there are at least two solutions $x_{\mathbf{n},\delta}^+(\cdot)$ and $x_{\mathbf{n},\delta}^-(\cdot)$ of (4.3))-(4.5), such that:

1. $u_1 \dot{x}_{\mathbf{n},\delta}^-(0) - u_2 x_{\mathbf{n},\delta}^-(0) < 0 < u_1 \dot{x}_{\mathbf{n},\delta}^+(0) - u_2 x_{\mathbf{n},\delta}^+(0)$;
2. *$x_{\mathbf{n},\delta}^\pm(\cdot)$ has exactly n_i zeros in I_i^+, exactly δ_i zeros in I_i^- and exactly $1-\delta_i$ changes of sign of the derivative in I_i^-;*
3. *for each i, $|x_{\mathbf{n},\delta}^\pm(t)| + |\dot{x}_{\mathbf{n},\delta}^\pm(t)| \to +\infty$, as $n_i \to +\infty$, uniformly in $t \in I_i^+$.*

Combining Lemmas 3.2.3 and 3.2.2 with Theorem 4.1.4 we imediately obtain the following theorem about solutions which are homoclinic to zero.

Theorem 4.2.3 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying (g1), (g2+) and (g2−); let $q : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (q0). Assume that there are $a, b \in \mathbb{R}$ with $a < b$ such that $Q(t) = \int_0^t q(s)\,ds$ is strictly decreasing on $]-\infty, a] \cup [b, +\infty[$, with $Q(\mp\infty) = \pm\infty$. Suppose that there are $2k+1$ consecutive, adjacent, nondegenerate and closed intervals:*

$$I_1^+,\ I_1^-,\ \dots\ I_k^+,\ I_k^-,\ I_{k+1}^+,$$

with $q \geq 0$, $q \not\equiv 0$ on I_i^+ and $q \leq 0$, $q \not\equiv 0$ on I_i^-, such that:

$$[a,b] = \left((\cup_{i=1}^{k+1} I_i^+) \cup (\cup_{i=1}^{k} I_i^-) \right).$$

Then, there are $k+1$ positive integers $n_1^*, \dots, n_{k+1}^*$ such that for each $(k+1)$-tuple $\mathbf{n} = (n_1, \dots, n_{k+1}) \in \mathbb{N}^{k+1}$, with $n_i > n_i^*$, and each k-tuple $\delta = (\delta_1, \dots, \delta_k) \in \{0,1\}^k$, there are at least two solutions $x_{\mathbf{n},\delta}^+(\cdot)$ and $x_{\mathbf{n},\delta}^-(\cdot)$ of (4.3) such that:

1. *$x_{\mathbf{n},\delta}^-(t) < 0$ and $\dot{x}_{\mathbf{n},\delta}^-(t) < 0$ for all $t \in]-\infty, a]$, while $x_{\mathbf{n},\delta}^+(t) > 0$ and $\dot{x}_{\mathbf{n},\delta}^+(t) > 0$ for all $t \in]-\infty, a]$;*
2. *$x_{\mathbf{n},\delta}^\pm(\cdot)$ has exactly n_i zeros in I_i^+, exactly δ_i zeros in I_i^- and exactly $1-\delta_i$ changes of sign of the derivative in I_i^-;*
3. *$x_{\mathbf{n},\delta}^\pm(t) \times \dot{x}_{\mathbf{n},\delta}^\pm(t) \neq 0$ for all $t \in [b, +\infty[$;*
4. *for each i, $|x_{\mathbf{n},\delta}^\pm(t)| + |\dot{x}_{\mathbf{n},\delta}^\pm(t)| \to +\infty$, as $n_i \to +\infty$, uniformly in $t \in I_i^+$;*
5. *$|x_{\mathbf{n},\delta}^\pm(t)| + |\dot{x}_{\mathbf{n},\delta}^\pm(t)| \to 0$ as $t \to \pm\infty$.*

Using Theorem 4.1.4 with Γ_a and Γ_b provided by Lemmas 3.3.3, 3.3.7, 3.3.8, 3.3.9, 3.3.10 and 3.3.11, it is easy to prove the following theorem about solutions with prescribed nodal behaviour and blow-up at the boundary of the domain.

Theorem 4.2.4 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying (g1), (g2+) and (g2−); let $q :]0, \omega[\to \mathbb{R}$ be a continuous and bounded function satisfying (q0). Assume that:*

$$0, \omega \in \overline{\{t \in]0,\omega[: q(t) < 0\}}$$

and that there are an integer $k \geq 0$, two points $a, b \in]0, \omega[$, with $a < b$, and $2k+1$ consecutive, adjacent, nondegenerate and closed intervals:

$$I_1^+ \,,\; I_1^- \,,\; \dots\; I_k^+ \,,\; I_k^- \,,\; I_{k+1}^+ \,,$$

such that $q \leq 0$, $q \not\equiv 0$ on $]0, a] \cup [b, \omega[$, $q \geq 0$, $q \not\equiv 0$ on I_i^+, $q \leq 0$, $q \not\equiv 0$ on I_i^- and:

$$[a, b] = \left(\left(\cup_{i=1}^{k+1} I_i^+ \right) \cup \left(\cup_{i=1}^{k} I_i^- \right) \right).$$

Then, there are $k+1$ positive integers $n_1^*, \dots, n_{k+1}^*$ such that for each $(k+1)$-tuple $\mathbf{n} = (n_1, \dots, n_{k+1}) \in \mathbb{N}^{k+1}$, with $n_i > n_i^*$, and each $(k+2)$-tuple $\delta = (\delta_0, \dots, \delta_{k+1}) \in \{0,1\}^{k+2}$, there are at least two solutions $x^+_{\mathbf{n},\delta}(\cdot)$ and $x^-_{\mathbf{n},\delta}(\cdot)$ of (4.3) defined on $]0, \omega[$ and such that:

1. $\lim_{t\to 0} x^-_{\mathbf{n},\delta}(t) = -\infty$, $\lim_{t\to 0} \dot{x}^-_{\mathbf{n},\delta}(t) = +\infty$, $\lim_{t\to 0} x^+_{\mathbf{n},\delta}(t) = +\infty$ *and* $\lim_{t\to 0} \dot{x}^+_{\mathbf{n},\delta}(t) = -\infty$;
2. $\lim_{t\to\omega} |x^-_{\mathbf{n},\delta}(t)| = \lim_{t\to\omega} |\dot{x}^-_{\mathbf{n},\delta}(t)| = \lim_{t\to\omega} |x^+_{\mathbf{n},\delta}(t)| = \lim_{t\to\omega} |\dot{x}^+_{\mathbf{n},\delta}(t)| = +\infty$;
3. *$x^\pm_{\mathbf{n},\delta}(\cdot)$ has exactly n_i zeros in I_i^+, exactly δ_i zeros in I_i^- and exactly $1-\delta_i$ changes of sign of the derivative in I_i^-;*
4. *$x^\pm_{\mathbf{n},\delta}(\cdot)$ has exactly δ_0 zeros and exactly $1-\delta_0$ changes of sign of the derivative in $]0, a[$ and has exactly δ_{k+1} zeros and exactly $1 - \delta k + 1$ changes of sign of the derivative in $]b, \omega[$;*
5. *for each i, $|x^\pm_{\mathbf{n},\delta}(t)| + |\dot{x}^\pm_{\mathbf{n},\delta}(t)| \to +\infty$, as $n_i \to +\infty$, uniformly in $t \in I_i^+$.*

Finally we note that in the same way we can obtain existence and multiplicity results for boundary value problems associated to the equation (4.3) and different boundary condition at the two endpoints of the domain of the weight function q. For instance theorems like those stated above could be proved looking for solutions tending to zero as $t \to -\infty$ and having blow-up at some $t = t_0$, as well as solutions satisfying a Sturm–Liouville condition at one endpoint and the blow-up condition at the other one, and so on.

Chapter 5

Floquet–type boundary value problems for superlinear equations with indefinite weight

We give here some existence and multiplicity results for solutions of the equation:

$$\ddot{x} + q(t)g(x) = 0 \tag{5.1}$$

on an interval $[0, \omega]$ and satisfying a boundary condition which can be expressed in the form:

$$(x(\omega), \dot{x}(\omega)) = \Lambda(x(0), \dot{x}(0)), \tag{5.2}$$

$\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ being a suitable continuous map. Such a boundary condition is called of Floquet–type since generalizes the usual Floquet boundary condition which corresponds to the map Λ given by the rotation of the plane by a fixed angle. Our map Λ will be *nondegenerate*, in the sense that $\Lambda(p) = (0,0)$ only for $p = (0,0)$, and *positively σ-homogeneous*, that is, there is a real number $\sigma > 0$ such that:

$$\Lambda(rp) = r^{\sigma}\Lambda(p) \qquad \forall\, p \in \mathbb{R}^2\ \forall\, r > 0.$$

Thus $\Lambda(p) = Mp$ is allowed for every 2×2 real and invertible matrix M. In particular the periodic and the antiperiodic boundary conditions are included, since they are realized by choosing M equal to the identity matrix I and to its opposite $-I$, respectively.

The hypotheses on q and g are the same we used in Chapter 4, that is the sign condition on g :

$$g(x)x > 0 \qquad \forall\, x \neq 0, \tag{$g1$}$$

the two superlinearity conditions on g :

$$\lim_{e \to +\infty} \tau_g^{+}(e) = 0 \tag{$g2+$}$$

and

$$\lim_{e \to \pm\infty} \tau_g^{-}(e) = 0. \tag{$g2-$}$$

(see also (2.3) and (3.2)) and some regularity on q in order to have no problems of continuability for the solutions of (5.1) in the intervals where $q \geq 0$:

(q0) if $q(t) \geq 0$ for all t in an interval I and $q|_I \not\equiv 0$, then q is of bounded variation in I and the set $\{t \in I : q(t) > 0\}$ is the union of a finite number of open intervals; moreover if $q(t) > 0$ for every $t \in]c, d[$ and $q(c) = 0$ (or $q(d) = 0$), then q is monotone in a right neighbourhood of c (or, respectively, in a left neighbourhood of d)

(see Corollary 2.1.3). Moreover we will always deal with a locally Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$ so that the local uniqueness holds for every intial value problem associated to (5.1). In particular, for every $p \in \mathbb{R}^2$ and every $t_0 \in I$ we can define $x(\cdot; t_0, p)$ to be the solution of (5.1) which satisfies $(x(t_0), \dot{x}(t_0)) = p$ and is defined in its maximal interval of continuability, and $z(\cdot; t_0, p) = (x(\cdot; t_0, p), y(\cdot; t_0, p)) = (x(\cdot; t_0, p), \dot{x}(\cdot; t_0, p))$ to be the corresponding solution of the equivalent plane system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -q(t)g(x). \end{cases}$$

Therefore, from now on, finding a solution of (5.1)–(5.2) is equivalent to determine some $p \in \mathbb{R}^2$ such that the solution $x(t; 0, p)$ is defined for all $t \in [0, \omega]$ and satisfies:

$$z(\omega; 0, p) = \Lambda(p).$$

We recall also that, if $p \neq (0, 0)$ and $x(t; a, p)$ is defined in $[a, b]$, then $z(t; a, p) \neq (0, 0)$ for every $t \in [a, b]$ and we can measure the angle spanned by the vector $z(t; a, p)$, as t varies in $[a, b]$, by the quantity:

$$\operatorname*{rot}_{[a,b]}(p) = \int_a^b \frac{q(t)g(x(t; a, p))x(t; a, p) + \dot{x}^2(t; a, p)}{x^2(t; a, p) + \dot{x}^2(t; a, p)}\, dt.$$

To fix some notation, let $A_1 = [0, +\infty[\times[0, +\infty[$, $A_2 =]-\infty, 0] \times [0, +\infty[$, $A_3 =]-\infty, 0]\times]-\infty, 0]$, and $A_4 = [0, +\infty[\times]-\infty, 0]$ denote respectively the four closed quadrants in the plane.

5.1 A general theorem

We prove at first a theorem for a general nondegenerate and positively σ-homogeneous map $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$. We postpone to the next sections the discussion of what can be said when more precise choices are considered for Λ.

Theorem 5.1.1 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying (g1), (g2+) and (g2−), let $q : [0, \omega] \to \mathbb{R}$ be a continuous function satisfying (q0) and let $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map such that:*

- *it is* positively σ-homogeneous *for some $\sigma > 0$, that is $\Lambda(rp) = r^\sigma \Lambda(p)$ for all $p \in \mathbb{R}^2$ and for all $r > 0$;*
- *it is* nondegenerate*, that is $\Lambda(p) = (0, 0)$ if and only if $p = (0, 0)$.*

Suppose that there are $2k + 1$ points $t_i \in [0, \omega]$, $i = 0, \ldots, 2k$ such that:

$$0 = t_0 < t_1 < t_2 < \cdots < t_{2k-1} < t_{2k} = \omega$$

and:

$$q \leq 0,\ q \not\equiv 0 \text{ in } I_i^- = [t_{2i-2}, t_{2i-1}] \quad \text{and} \quad q \geq 0,\ q \not\equiv 0 \text{ in } I_i^+ = [t_{2i-1}, t_{2i}],$$

for all $i = 1, \ldots, k$.
Then there exist $k-1$ natural numbers $n_1^*, \ldots, n_{k-1}^*$ such that, for all $\mathbf{n} = (n_1, \ldots, n_{k-1}) \in \mathbb{N}^{k-1}$, with $n_i \geq n_i^*$, and for all $\delta = (\delta_1, \ldots, \delta_k) \in \{0,1\}^k$, there exist two unbounded continua $\Gamma^+ \subset A_4$ and $\Gamma^- \subset A_2$ such that the following facts hold for every $p \in \Gamma^+ \cup \Gamma^-$:

1. *$x(t;0,p)$ is defined on $[0,\omega]$ and satisfies:*
$$|z(\omega;0,p)| = |\Lambda(p)|;$$
2. *$x(\cdot;0,p)$ has exactly n_i zeros in $]t_{2i-1}, t_{2i}[$ for all $i = 1, \ldots, k-1$;*
3. *$x(\cdot;0,p)$ has exactly δ_i zeros in $]t_{2i-2}, t_{2i-1}[$ for all $i = 1, \ldots, k$;*
4. *$\dot{x}(\cdot;0,p)$ has exactly $1-\delta_i$ changes of sign in $]t_{2i-2}, t_{2i-1}[$ for all $i = 1, \ldots, k$;*
5. $\displaystyle\lim_{|p|\to+\infty} \left[\inf_{t\in[t_{2k-1},\omega]} |z(t;0,p)| \right] = +\infty.$

Moreover, if we set:
$$\Gamma_{2k-1}^+ = \{z(t_{2k-1};0,p) : p \in \Gamma^+\} \qquad \text{and} \qquad \Gamma_{2k-1}^- = \{z(t_{2k-1};0,p) : p \in \Gamma^-\},$$
then Γ_{2k-1}^+ is an unbounded continuum contained in the first quadrant, if the number:
$$\delta_1 + n_1 + \cdots + \delta_{k-1} + n_{k-1} + \delta_k \tag{5.3}$$
(that is the number of zeros that the solution $x(t;0,p)$ has in $[0, t_{2k-1}]$ for every $p \in \Gamma^+ \cup \Gamma^-$) is even, or in the third one, if (5.3) is odd, while Γ_{2k+1}^- is contained in the first quadrant, if (5.3) is odd, or in the third one, if (5.3) is even.
Finally there exist two unbounded sequences $\{p_l^+\} \subset \Gamma^+$ and $\{p_l^-\} \subset \Gamma^-$ such that:
$$z(\omega;0,p_l^+) = \Lambda(p_l^+) \qquad \text{and} \qquad z(\omega;0,p_l^-) = \Lambda(p_l^-) \qquad \forall\, l \in \mathbb{N}.$$

Proof. For every $i = 1, \ldots, k$ let:
$$R_i = \sup\{|p| : p \in [\Omega_{t_{2i-1}}^{t_{2i-2}} \cup \Omega_{t_{2i-2}}^{t_{2i-1}}] \cap [(\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})]\}$$
be the supremum of the moduli of the initial conditions p that lie on the coordinate axes and produce a solution $x(t;t_{2i-2},p)$ or $x(t;t_{2i-1},p)$ which is continuable on the whole interval $[t_{2i-2}, t_{2i-1}]$: each R_i is finite by Lemma 3.1.1 applied to our equation on $I_i^- = [t_{2i-2}, t_{2i-1}]$ with the choices $(r_1, r_2) = (\pm 1, 0)$ and $(r_1, r_2) = (0, \pm 1)$ and depends only on $q|_{[t_{2i-2}, t_{2i-1}]}$. We note that, if we fix $i = 1, \ldots, k-1$ and we apply Lemmas 4.1.1 and 4.1.2 to our equation on the interval $[a,c] = [t_{2i-1}, t_{2i+1}]$ with $b = t_{2i}$, we obtain a constant R^* which coincides with R_{i+1}, as can be seen in the proof of those lemmas, and, for every $R > 0$ we obtain a positive integer n_R^* : let n_i^* be that positive integer obtained with the choice $R = R_i$; therefore each n_i^* depends on the values of q restricted only on $[t_{2i-2}, t_{2i+1}]$, that is on the three consecutive intervals I_i^-, I_i^+ and I_{i+1}^-.

If we set:
$$\mu = \min_{|p|=1} |\Lambda(p)| \qquad \text{and} \qquad \nu = \max_{|p|=1} |\Lambda(p)|,$$
then $0 < \mu \leq \nu$ by the nondegeneracy of Λ and:
$$\mu|p|^\sigma \leq |\Lambda(p)| \leq \nu|p|^\sigma \qquad \forall\, p \in \mathbb{R}^2 \tag{5.4}$$

by the positive σ-homogeneity of Λ. Since $q \geq 0$ on $[t_{2k-1}, t_{2k}] = [t_{2k-1}, \omega]$, Lemma 2.1.2 and Lemma 2.2.2 apply to our equation (5.1) on the interval $[t_{2k-1}, t_{2k}]$ and we can define:

$$R_{k+1} = \max_{|p| \leq R_k} |z(\omega; t_{2k-1}, p)| < +\infty \tag{5.5}$$

and:

$$R_0 = \max\left\{ R_1, \left(\frac{R_{k+1}}{\mu}\right)^{1/\sigma} \right\}. \tag{5.6}$$

Let us fix $\delta = (\delta_1, \ldots, \delta_k) \in \{0,1\}^k$ and $\mathbf{n} = (n_1, \ldots, n_{k-1}) \in \mathbb{N}^{k-1}$ such that $n_i \geq n_i^*$ for all $i = 1, \ldots, k-1$. To find the unbounded continuum $\Gamma^+ \subset A_4$ we use an argument similar to that employed in the proof of Lemma 3.2.3, therefore we first need to prove the following claim.

CLAIM: For every continuous curve $\gamma : [0,1] \to A_4 \setminus B(R_0)$ with $\gamma(0) \in [R_0, +\infty[\times \{0\}$ and $\gamma(1) \in \{0\} \times]-\infty, -R_0]$ there exists $s_0 \in [0,1]$ such that the following facts hold:

1. $x(t; 0, \gamma(s_0))$ is defined for all $t \in [0, \omega]$;
2. $x(\cdot; 0, \gamma(s_0))$ has exactly n_i zeros in $]t_{2i-1}, t_{2i}[$, for all $i = 1, \ldots, k-1$, and δ_i zeros in $]t_{2i-2}, t_{2i-1}[$, for all $i = 1, \ldots, k$;
3. $\dot{x}(\cdot; 0, \gamma(s_0))$ has exactly $1 - \delta_i$ changes of sign in $]t_{2i-2}, t_{2i-1}[$, for all $i = 1, \ldots, k$;
4. $|z(\omega; 0, \gamma(s_0))| = |\Lambda(\gamma(s_0))|$;
5. $z(t_{2k-1}; 0, \gamma(s_0))$ belongs to the first or to the third quadrant, depending on the fact that $\delta_1 + n_1 + \cdots + \delta_{k-1} + n_{k-1} + \delta_k$ is even or odd, respectively.

Suppose that $\delta_1 = 0$. We follow the idea of the proof of Lemma 4.1.1: by the definition of R_0, $x(t; 0, \gamma(0))$ and $x(t; 0, \gamma(1))$ are not defined up to t_1, $x(t; 0, \gamma(0))$ is strictly positive and monotone increasing, $x(t; 0, \gamma(1))$ is nonpositive and monotone decreasing and they satisfy:

$$\lim_{t \to b_0^-} x(t; 0, \gamma(0)) = \lim_{t \to b_0^-} \dot{x}(t; 0, \gamma(0)) = +\infty$$

and

$$\lim_{t \to b_1^-} x(t; 0, \gamma(1)) = \lim_{t \to b_1^-} \dot{x}(t; 0, \gamma(1)) = -\infty$$

for some $0 < b_0, b_1 \leq t_1$. Letting:

$$\beta = \inf\{s \in [0,1] : x(t; 0, \gamma(s)) < 0 \text{ for some } t \in [0, t_1]\}$$

and using the continuous dependence on the initial data together with the monotonicity properties of the solutions of (5.1) where $q \leq 0$, it is possible to show that $x(\cdot; 0, \gamma(\beta))$ is defined, nonnegative and monotone decreasing on $[0, t_1]$ and $x(t_1; 0, \gamma(\beta)) = 0$, while $\dot{x}(t_1; 0, \gamma(\beta)) < 0$. Then we can set:

$$\alpha_1 = \inf\{s \in [0, \beta] : x(\cdot; 0, \gamma(u)) \text{ is defined on } [0, t_1]\ \forall\, u \in [s, \beta]\}$$

and:

$$\beta_1 = \inf\{s \in [\alpha_1, \beta] : \dot{x}(t_1; 0, \gamma(s)) < 0\}.$$

Arguing like in the proof of Lemma 4.1.1 one can see that, for every $s \in]\alpha_1, \beta_1]$, the solution $x(\cdot; 0, \gamma(s))$ is defined up to t_1, is positive and convex and has exactly one change of sign of the derivative in $[0, t_1]$ and moreover we have that:

$$\lim_{s\to\alpha_1^+} x(t_1; 0, \gamma(s)) = \lim_{s\to\alpha_1^+} \dot{x}(t_1; 0, \gamma(s)) = +\infty$$

and:

$$\dot{x}(t_1; 0, \gamma(\beta_1)) = 0.$$

Hence, the curve $\gamma_1 :]\alpha_1, \beta_1] \to A_1$ given by:

$$\gamma_1(s) = z(t_1; 0, \gamma(s)),$$

is an unbounded curve in the first closed quadrant with $\gamma_1(\beta_1) \in [0, +\infty[\times\{0\}$: we have that $\gamma_1(\beta_1) \in B[R_1]$ by the definition of R_1.

In a similar manner, but following the proof of Lemma 4.1.2, if $\delta_1 = 1$, we can find α_1 and β_1 with $0 < \beta_1 < \alpha_1 < 1$ such that, for every $s \in [\beta_1, \alpha_1[$, the solution $x(\cdot; 0, \gamma(s))$ is defined up to t_1, is strictly monotone decreasing in $[0, t_1]$, has exactly one zero in $]0, t_1]$ and moreover we have that:

$$\lim_{s\to\alpha_1^-} x(t_1; 0, \gamma(s)) = \lim_{s\to\alpha_1^-} \dot{x}(t_1; 0, \gamma(s)) = -\infty$$

and:

$$x(t_1; 0, \gamma(\beta_1)) = 0,$$

so that again $\gamma_1(\beta_1) \in B[R_1]$ by the definition of R_1.

We define $J_1 =]\alpha_1, \beta_1]$, if $\delta_1 = 0$, or $J_1 = [\beta_1, \alpha_1[$, if $\delta_1 = 1$, and $\gamma_1(s) = z(t_1; 0\gamma(s))$, for $s \in J_1$, so that γ_1 is a continuous and unbounded curve whose image is contained either in the first or in the third quadrant and which intersects the closed ball $B[R_1]$ and corresponds to solutions having the desired nodal behaviour in the first interval $[0, t_1]$.

Now, for $i = 1, \ldots, k-1$, we inductively apply either Lemma 4.1.1, if $\delta_{i+1} = 0$, or Lemma 4.1.2, if $\delta_{i+1} = 1$, to the interval $[a, c] = [t_{2i-1}, t_{2i+1}]$ and the curve $\gamma = \gamma_i$, with $b = t_{2i}$, $R = R_i$ and $n = n_i$. We find, therefore, a subinterval $J_{i+1} \subset J_i$, which is of the form $]\alpha_{i+1}, \beta_{i+1}]$, if $\delta_{i+1} = 0$, and $[\beta_{i+1}, \alpha_{i+1}[$, if $\delta_{i+1} = 1$, such that the following facts hold:

- $x(t; t_{2j-1}, \gamma_i(s))$ is defined for every $t \in [t_{2i-1}, t_{2i+1}]$ and for every $s \in J_{i+1}$;
- $x(\cdot; t_{2j-1}, \gamma_i(s))$ has exactly n_i zeros in $]t_{2i-1}, t_{2i}[$, δ_{i+1} zeros in $]t_{2i}, t_{2i+1}[$ and exactly $1 - \delta_{i+1}$ changes of sign of the derivative in $]t_{2i}, t_{2i+1}[$ for every $s \in J_{i+1}$;
- if we consider the curve $\gamma_{i+1} : J_{i+1} \to \mathbb{R}^2$ given by:

$$\gamma_{i+1}(s) = z(t_{2i+1}; t_{2j-1}, \gamma_i(s)) = z(t_{2i+1}; 0, \gamma(s)),$$

  then $\gamma_{i+1}(\beta_{i+1}) \in B[R_{i+1}]$, $\lim_{s\to\alpha_{i+1}} |\gamma_{i+1}(\alpha_{i+1})| = +\infty$ and $\gamma_{i+1}(s)$ belongs to the first or to the third quadrant for every $s \in J_{i+1}$, depending on the fact that $\delta_1 + n_1 + \cdots + \delta_i + n_i + \delta_{i+1}$ is even or odd, respectively.

When we arrive at $i = k-1$ we obtain the final interval $J_k \subset [0, 1]$ such that the solutions $x(t; 0, \gamma(s))$, $s \in J_k$, are defined in $[0, t_{2k-1}]$ (and, therefore, in $[0, \omega]$ since $q \geq 0$ in $[t_{2k-1}, \omega]$) and have nodal behaviour in $[0, t_{2k-1}]$ prescribed by the vectors $\boldsymbol{\delta}$ and $\mathbf{n}$ (see points 1, 2 and 3 of the claim). Moreover we have that:

(*i*) $|z(\omega;0,\gamma(\beta_k))| \leq R_{k+1}$, by the fact that $|z(t_{2k-1};0,\gamma(\beta_k))| \leq R_k$ and by (5.5);

(*ii*) $\lim_{s\to\alpha_k} |z(\omega;0,\gamma(s))| = \lim_{s\to\alpha_k} |z(t_{2k-1};0,\gamma(s))| = +\infty$;

(*iii*) $z(t_{2k-1};0,\gamma(s))$ belongs to the first or to the third quadrant for every $s \in J_k$ depending on the fact that $\delta_1 + n_1 + \cdots + \delta_{k-1} + n_{k-1} + \delta_k$ is even or odd, respectively.

From (5.4), (5.6) and (*i*) we deduce that:

$$|\Lambda(\gamma(\beta_k))| \geq \mu|\gamma(\beta_k)|^\sigma \geq \mu R_0^\sigma \geq R_{k+1} \geq |z(\omega;0,\gamma(\beta_k))|.$$

On the other hand, from (*ii*) and the fact that $\gamma([0,1])$ is compact and Λ is continuous, we have that:

$$|\Lambda(\gamma(s))| \leq |z(\omega;0,\gamma(s))|,$$

for every s in a (right or left) neighbourhood of α_k. Therefore there is $s_0 \in J_k$ such that $|\Lambda(\gamma(s_0))| = |z(\omega;0,\gamma(s_0))|$. This s_0 satisfies the statements 1, 2, 3, 4, and 5 of the claim, which is now proved.

The existence of Γ^+ follows from this Claim by an application of Lemma 3.2.1 like in the proof of Lemma 3.2.3. We note that, by statement 5 of the Claim, $z(t_{2k-1};0,p)$ belongs to the same quadrant (the first or the third one, depending as usual on the total number of zeros on $[0,t_{2k-1}]$) for every $p \in \Gamma^+$; therefore the set:

$$\Gamma^+_{2k-1} = \{z(t_{2k-1};0,p) : p \in \Gamma^+\}$$

is an unbounded continuum contained either in the first or in the third quadrant.

In a completely analogous way we can find the other unbounded continuum $\Gamma^- \subset A_2 \setminus B(R_0)$ announced in the statement of the theorem.

What remains to do is to select two unbounded sequences $\{p_l^+\} \subset \Gamma^+$ and $\{p_l^-\} \subset \Gamma^-$ such that $z(\omega;0,p_l^\pm) = \Lambda(p_l^\pm)$. Actually the two vectors $z(\omega;0,p_l^\pm)$ and $\Lambda(p_l^\pm)$ have already the same modulus, hence we have just to impose that they have the same angular coordinate, up to multiples of 2π. Let us show how to do only for the "+" case, the "−" one being of course analogous at all.

Since trivially $z(0;0,p) = p$, it is possible to define a continuous angular function $\theta(0;p)$: $\{0\} \times [A_4 \setminus B(R_0)] \to \mathbb{R}$ such that for all $p \in A_4 \setminus B(R_0)$:

- $\theta(0;p) \in \left[-\frac{\pi}{2}, 0\right]$;
- $p = (|p|\cos\theta(0;p); |p|\sin\theta(0;p))$.

Then we extend continuously θ to $[0,\omega] \times [A_4 \setminus B(R_0)]$ by setting:

$$\theta(t;p) = \theta(0;p) - \int_0^t \frac{q(t)g(x(s;0,p))x(s;0,p) + \dot{x}^2(s;0,p)}{x^2(s;0,p) + \dot{x}^2(s;0,p)}\,ds.$$

We have that:

$$\begin{cases} x(t;0,p) = |z(t;0,p)|\cos\theta(t;p) \\ \dot{x}(t;0,p) = |z(t;0,p)|\sin\theta(t;p) \end{cases} \qquad \forall\, t \in [0,\omega]\ \forall\, p \in A_4 \setminus B(R_0)$$

(see (2.16) and (2.17) for more details). Moreover we can also define a continuous angular function $\vartheta : A_4 \setminus \{(0,0)\} \to \mathbb{R}$ for Λ, in the sense that for every $p \in A_4 \setminus \{(0,0)\}$ we can write:

$$\Lambda(p) = (|\Lambda(p)|\cos\vartheta(p), |\Lambda(p)|\sin\vartheta(p))$$

and we have that $\vartheta(rp) = \vartheta(p)$ for every $r > 0$ since Λ is positively homogeneous.

Let us define $\psi : \Gamma^+ \to \mathbb{R}$ as follows:

$$\psi(p) = \theta(\omega; p) - \vartheta(p).$$

The theorem is proved if we find an unbounded sequence $\{p_l^+\} \subset \Gamma^+$ such that:

$$\psi(p_l^+) \in 2\pi\mathbb{Z}$$

and this will be possible if we show that $\psi(\Gamma^+)$, which is an interval by the connectedness of Γ^+, is also unbounded. We can write $\psi(p)$ in the following way:

$$\psi(p) = [\theta(\omega; p) - \theta(t_{2k-1}; p)] + [\theta(t_{2k-1}; p) - \vartheta(p)].$$

Let $N = \delta_1 + n_1 + \cdots + \delta_{k-1} + n_{k-1} + \delta_k$ be the constant number of zeros that the solutions starting from Γ^+ have on $[0, t_{2k-1}]$. Therefore we have that:

$$-\frac{\pi}{2} + N\pi < \theta(t_{2k-1}; p) \leq -\frac{\pi}{2} + (N-1)\pi \qquad \forall\, p \in \Gamma^+.$$

Since ϑ is continuous and constant along the straight half lines starting from the origin, we can set:

$$M = \sup\{|\vartheta(p)| : p \in A_4 \setminus \{(0,0)\}\} = \sup\{|\vartheta(p)| : p \in A_4,\ |p| = 1\} < +\infty.$$

Hence the term $[\theta(t_{2k-1}; p) - \vartheta(p)]$ is bounded for all $p \in \Gamma^+$. On the other hand, applying Corollary 2.2.8 on $[t_{2k-1}, \omega]$, where q is nonnegative, we obtain that:

$$\lim_{|p| \to +\infty} [\theta(t_{2k-1}; p) - \theta(\omega; p)] = +\infty,$$

thus $\psi(\Gamma^+)$ is unbounded. ■

Remark 5.1.2 What we can further obtain from the Claim in the preceding proof and Lemma 3.2.1 is that the two unbounded continua Γ^+ and Γ^- also satisfy:

$$\Gamma^\pm \cap \partial B[R_0] \neq \emptyset,$$

where R_0 is defined in (5.6) and, therefore, depends on g, Λ and q restricted to $[0, t_1] \cup [t_{2k-2}, \omega] = I_1^- \cup I_k^- \cup I_k^+$, but it is independent of the choice of $\mathbf{n}$ and δ. If we take, now, $p_0 \in \Gamma^\pm \cap \partial B[R_0] \neq \emptyset$, we have that:

$$|z(\omega; 0, p_0)| = |\Lambda(p_0)| =\leq \nu R_0^\sigma,$$

by (5.4). Hence we deduce that:

$$\Gamma_{2k-1}^\pm \cap B[K] \neq \emptyset,$$

if we set:

$$K = \max\{|z(t_{2k-1}; \omega, p)| : |p| \leq \nu R_0^\sigma\},$$

so that again K is a constant independent of $\mathbf{n}$ and δ.

5.2 Applications

When particular choices are considered for the map Λ, one can hope to be able to improve the statement of Theorem 5.1.1 and to prescribe the nodal behaviour of the solutions of:

$$(5.7)\qquad \begin{cases} \ddot{x} + q(t)g(x) = 0 \\ (x(\omega), \dot{x}(\omega)) = \Lambda(x(0), \dot{x}(0)) \end{cases}$$

also in the final interval $[t_{2k-1}, \omega]$. This happens for instance if we consider the classical Floquet boundary conditions, i.e.:

$$(5.8)\qquad \begin{pmatrix} x(\omega) \\ \dot{x}(\omega) \end{pmatrix} = Q_\lambda \begin{pmatrix} x(\omega) \\ \dot{x}(\omega) \end{pmatrix} = \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix},$$

where Q_λ is the 2×2 matrix which is associated to a rotation of λ radiants in $\mathbb{R}^2$. To determine the nodal behaviour we will use again the quantity:

$$\operatorname*{rot}_{[a,b]}(p) = \int_a^b \frac{q(t)g(x(t;a,p))x(t;a,p) + \dot{x}^2(t;a,p)}{x^2(t;a,p) + \dot{x}^2(t;a,p)}\,dt,$$

which is considered in (2.22) and which measures the angle spanned in clockwise sense by the vector $z(t; a, p)$ as t goes from a to b, if $p \neq (0,0)$. Indeed we have the following theorem.

Theorem 5.2.1 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying $(g1)$, $(g2+)$ and $(g2-)$, let $q : [0,\omega] \to \mathbb{R}$ be a continuous function satisfying $(q0)$ and let $\lambda \in]-\pi,\pi]$. Suppose that there are $2k+1$ points $t_i \in [0,\omega]$, $i = 0,\dots,2k$ such that:*

$$0 = t_0 < t_1 < t_2 < \cdots < t_{2k-1} < t_{2k} = \omega$$

and:

$$q \le 0,\ q \not\equiv 0 \text{ in } I_i^- = [t_{2i-2}, t_{2i-1}] \quad \text{and} \quad q \ge 0,\ q \not\equiv 0 \text{ in } I_i^+ = [t_{2i-1}, t_{2i}],$$

for all $i = 1,\dots,k$.
Then there exist k natural numbers $n_1^*,\dots,n_k^*$ such that, for all $\mathbf{n} = (n_1,\dots,n_k) \in \mathbb{N}^k$, with $n_i \ge n_i^*$, and for all $\boldsymbol{\delta} = (\delta_1,\dots,\delta_k) \in \{0,1\}^k$, with:

$$|\mathbf{n}| + |\delta| = n_1 + \cdots + n_k + \delta_1 + \cdots + \delta_k = \text{even number},$$

there exist two points $p^+_{\mathbf{n},\delta} \in A_4$ and $p^-_{\mathbf{n},\delta} \in A_2$ such that the following facts hold:

1. *$x(t; 0, p^\pm_{\mathbf{n},\delta})$ is defined on $[0,\omega]$ and satisfies:*
$$z(\omega; 0, p^\pm_{\mathbf{n},\delta}) = Q_\lambda(p^\pm_{\mathbf{n},\delta});$$
2. *$x(\cdot; 0, p^\pm_{\mathbf{n},\delta})$ has exactly n_i zeros in $]t_{2i-1}, t_{2i}[$ for all $i = 1,\dots,k-1$;*
3. *$x(\cdot; 0, p^\pm_{\mathbf{n},\delta})$ has exactly δ_i zeros in $]t_{2i-2}, t_{2i-1}[$ for all $i = 1,\dots,k$;*
4. *$\dot{x}(\cdot; 0, p^\pm_{\mathbf{n},\delta})$ has exactly $1-\delta_i$ changes of sign in $]t_{2i-2}, t_{2i-1}[$ for all $i = 1,\dots,k$;*
5. *$\operatorname*{rot}_{[0,\omega]}(p^\pm_{\mathbf{n},\delta}) = (|\mathbf{n}| + |\delta|)\pi - \lambda$.*

Proof. Let us apply Theorem 5.1.1 with Λ given by the rotation Q_λ in (5.8): of course, this Λ is nondegenerate and positively 1-homogeneous and satisfies (5.4) with $\mu = \nu = \sigma = 1$. Theorem 5.1.1 provides the first $k-1$ integer $n_1^*, \ldots, n_{k-1}^*$. We define the last one n_k^* using the uniform constant K determined in Remark 5.1.2:

$$n_k^* = \left\lceil \frac{\lambda + \sup\left\{ \operatorname*{rot}_{[t_{2k-1},\omega]}(p) : 0 < |p| \leq K \right\}}{\pi} \right\rceil, \tag{5.9}$$

that is, n_k^* is the minimum integer n such that:

$$\operatorname*{rot}_{[t_{2k-1},\omega]}(p) \leq n\pi - \lambda \qquad \forall\, p \in \mathbb{R}^2 \, : \, 0 < |p| \leq K.$$

Roughly speaking, n_k^* is an upper bound for the number of zeros that a solution x of our equation may have if the starting point $(x(t_{2k-1}), \dot{x}(t_{2k-1}))$ is inside the closed ball of radius K. Note that n_k^* is finite since $g(s)/s$ is bounded in a neighbourhood of zero.

Let now fix $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k) \in \{0,1\}^k$ and $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ such that:

- $n_i \geq n_i^*$, for every $i = 1, \ldots, k$;
- $|\mathbf{n}| + |\boldsymbol{\delta}| = n_1 + \cdots + n_k + \delta_1 + \cdots + \delta_k$ is an even integer.

Then, if we consider the $(k-1)$-tuple $(n_1, \ldots, n_{k-1})$ and the k-tuple $\boldsymbol{\delta}$ and we apply Theorem 5.1.1, we find the two unbounded continua $\Gamma^+ \subset A_4$ and $\Gamma^- \subset A_2$ such that the solutions starting at $t = 0$ from a point p in $\Gamma^\pm$ are defined in $[0, \omega]$ and satisfy:

$$|z(\omega; 0, p)| = |Q_\lambda(p)| = |p|$$

and statements 2, 3 and 4 in Theorem 5.2.1. It remains to find $p^\pm_{\mathbf{n},\boldsymbol{\delta}} \in \Gamma^\pm$ such that statement 5 holds and:

$$z(\omega; 0, p^\pm_{\mathbf{n},\boldsymbol{\delta}}) = Q_\lambda(p^\pm_{\mathbf{n},\boldsymbol{\delta}}). \tag{5.10}$$

We show just the existence of $p^+_{\mathbf{n},\boldsymbol{\delta}} \in \Gamma^+$, since the other point can be found by similar considerations.

As in the proof of Theorem 5.1.1, we define the angular coordinate $\theta : [0,\omega] \times [A_4 \setminus \{(0,0)\}]$ of the vector $z(t; 0, p)$ in such a way that:

- $\theta(0; p) \in \left[-\frac{\pi}{2}, 0\right]$ for every $p \in A_4 \setminus \{(0,0)\}$;
- $p = (|p| \cos\theta(0;p); |p| \sin\theta(0;p))$ for every $p \in A_4 \setminus \{(0,0)\}$;
- $\theta(t;p) = \theta(0;p) - \displaystyle\int_0^t \frac{q(t)g(x(s;0,p))x(s;0,p) + \dot{x}^2(s;0,p)}{x^2(s;0,p) + \dot{x}^2(s;0,p)} = \theta(0;p) - \operatorname*{rot}_{[0,t]}(p);$
- $\begin{cases} x(t;0,p) = |z(t;0,p)| \cos\theta(t;p) \\ \dot{x}(t;0,p) = |z(t;0,p)| \sin\theta(t;p) \end{cases} \qquad \forall\, t \in [0,\omega] \;\forall\, p \in A_4 \setminus B(R_0)$

(see (2.16) and (2.17) for more details).

Moreover we consider the continuous angular function $\vartheta : A_4 \setminus \{(0,0)\} \to \mathbb{R}$ for Λ, which satisfies:

$$\Lambda(p) = (|\Lambda(p)| \cos\vartheta(p), |\Lambda(p)| \sin\vartheta(p))$$

for every $p \in A_4 \setminus \{(0,0)\}$. Actually, since now Λ is the rotation of λ radiants in the plane, we have that:

$$|\Lambda(p)| = 1 \qquad \text{and} \qquad \vartheta(p) = \theta(0;p) + \lambda.$$

Hence the function ψ we considered in the proof of Theorem 5.1.1 can be written as:

$$\psi(p) = \theta(\omega;p) - \vartheta(p) = \theta(\omega;p) - \theta(0;p) - \lambda = -\operatorname*{rot}_{[0,\omega]}(p) - \lambda.$$

Therefore, if we show that statement 5 holds, then also (5.10) is already proved, since $\psi(p)$ turns out to be an integer multiple of 2π by the fact that $|\mathbf{n}| + |\delta|$ is even.

We split $\operatorname*{rot}_{[0,\omega]}(p)$ in the following way:

$$\operatorname*{rot}_{[0,\omega]}(p) = [\theta(0;p) - \theta(t_{2k-1};p)] + [\theta(t_{2k-1};p) - \theta(\omega;p)].$$

Since $x(\cdot;0,p)$ has exactly $N_{2k-1} = n_1 + \cdots + n_{k-1} + |\delta|$ in $[0, t_{2k-1}]$ for every $p \in \Gamma^+$ and since we know from Theorem 5.1.1 that Γ^+_{2k-1} (that is the image of Γ^+ via the Poincaré map $p \mapsto z(t_{2k-1};0,p)$) is contained in the first or in the third quadrant, we have the following estimate:

$$-N_{2k-1}\pi \le \theta(t_{2k-1};p) \le -N_{2k+1}\pi + \frac{\pi}{2} \qquad \forall\, p \in \Gamma^+$$

and also:

$$(N_{2k-1} - 1)\pi \le \theta(0;p) - \theta(t_{2k-1};p) \le N_{2k-1}\pi \qquad \forall\, p \in \Gamma^+.$$

Hence:

$$\operatorname*{rot}_{[0,\omega]}(p) \le N_{2k-1}\pi + [\theta(t_{2k-1};p) - \theta(\omega;p)] \qquad \forall\, p \in \Gamma^+.$$

If we take $p_0 \in \Gamma^+ \cap \partial B[R_0]$ and we let $p_{2k-1} = z(t_{2k-1};0,p_0) \in \Gamma^+_{2k-1}$ and $p_{2k} = z(\omega;0,p_0) = z(\omega;t_{2k-1},p_{2k-1})$, then we have $|p_{2k}| = |\Lambda(p_0)| = |p_0| = R_0$ and, thus, $|p_{2k-1}| \le K$ (see Remark 5.1.2, where K is defined for a general Λ). By (5.9) we obtain that:

$$\theta(t_{2k-1};p_0) - \theta(\omega;p_0) = \operatorname*{rot}_{[t_{2k-1},\omega]}(p_{2k-1}) \le n_k^*\pi - \lambda$$

and:

$$\operatorname*{rot}_{[0,\omega]}(p_0) \le (N_{2k-1} + n_k^*)\pi - \lambda. \tag{5.11}$$

On the other hand, we have that, if $p \in \Gamma_0$ and $|p| \to +\infty$, then $|z(\omega;0,p)| = |\Lambda(p)| = |p| \to +\infty$ and, therefore, $|z(t_{2k-1};0,p)| \to +\infty$, too, since $q \ge 0$ on $[t_{2k-1},\omega]$ and we can apply there Corollary 2.1.3 and Lemma 2.2.2. A consequence of Corollary 2.2.8 on $[t_{2k-1},\omega]$ is that:

$$\lim_{|p|\to+\infty} \operatorname*{rot}_{[t_{2k-1},\omega]}(p) = +\infty,$$

hence, if $p \in \Gamma_0$ and $|p| \to +\infty$, then:

$$\theta(t_{2k-1};p) - \theta(\omega;p) = \operatorname*{rot}_{[t_{2k-1},\omega]}(z(t_{2k-1};0,p)) \to +\infty.$$

This relation and (5.11) imply that for every $n_k \ge n_k^*$ there is $p^+_{\mathbf{n},\delta} \in \Gamma^+$ such that:

$$\operatorname*{rot}_{[0,\omega]}(p^+_{\mathbf{n},\delta}) = (N_{2k-1} + n_k)\pi - \lambda = (|\mathbf{n} + |\delta||)\pi - \lambda,$$

which is statement 5. ∎

Remark 5.2.2 In [57] Henrard proves a similar theorem for an equation of the form:

$$\ddot{x} + f(x) = p(t, x, \dot{x}) \qquad t \in [a, b],$$

with a bounded function p and a superlinear f. He proves that there is a n^* such that for every $n \geq n^*$ there exist at least *four* solutions such that satisfy:

$$(x(b), \dot{x}(b)) = Q_\lambda(x(a), \dot{x}(a))$$

and:

$$\operatorname*{rot}_{[a,b]}(x(a), \dot{x}(a)) = 2n\pi - \lambda.$$

However, its proof does not work in the cases $\lambda = 0$ and $\lambda = \pm\pi$, that is in the cases of periodic and antiperiodic boundary conditions, cases that we are able to handle. In fact, an immediate consequence of this theorem is the existence of ω-periodic solutions of:

$$\ddot{x} + q(t)g(x) = 0$$

with nodal behaviour precisely assigned. Indeed it is sufficient to apply Theorem 5.2.1 with $\lambda = 0$.

Corollary 5.2.3 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying $(g1)$, $(g2+)$ and $(g2-)$, let $q : [0, \omega] \to \mathbb{R}$ be a continuous function satisfying $(q0)$. Suppose that there are $2k + 1$ points $t_i \in [0, \omega]$, $i = 0, \dots, 2k$ such that:*

$$0 = t_0 < t_1 < t_2 < \dots < t_{2k-1} < t_{2k} = \omega$$

and:

$$q \leq 0,\ q \not\equiv 0 \ \text{in}\ I_i^- = [t_{2i-2}, t_{2i-1}] \qquad \text{and} \qquad q \geq 0,\ q \not\equiv 0 \ \text{in}\ I_i^+ = [t_{2i-1}, t_{2i}],$$

for all $i = 1, \dots, k$.
Then there exist k natural numbers $n_1^*, \dots, n_k^*$ such that, for all $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$, with $n_i \geq n_i^*$, and for all $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k) \in \{0, 1\}^k$, with:

$$|\mathbf{n}| + |\boldsymbol{\delta}| = n_1 + \dots + n_k + \delta_1 + \dots + \delta_k = \text{even number},$$

there exist two points $p^+_{\mathbf{n},\delta} \in A_4$ and $p^-_{\mathbf{n},\delta} \in A_2$ such that the following facts hold:

1. *$x(t; 0, p^\pm_{\mathbf{n},\delta})$ is defined on $[0, \omega]$ and is ω-periodic, that is, it satisfies:*

$$z(\omega; 0, p^\pm_{\mathbf{n},\delta}) = p^\pm_{\mathbf{n},\delta};$$

2. *$x(\cdot; 0, p^\pm_{\mathbf{n},\delta})$ has exactly n_i zeros in $]t_{2i-1}, t_{2i}[$ for all $i = 1, \dots, k$;*
3. *$x(\cdot; 0, p^\pm_{\mathbf{n},\delta})$ has exactly δ_i zeros in $]t_{2i-2}, t_{2i-1}[$ for all $i = 1, \dots, k$;*
4. *$\dot{x}(\cdot; 0, p^\pm_{\mathbf{n},\delta})$ has exactly $1 - \delta_i$ changes of sign in $]t_{2i-2}, t_{2i-1}[$ for all $i = 1, \dots, k$;*

Using a change of variables from [21], it is possible to find periodic solutions with prescribed nodal properties and infinitely many subharmonic solutions also for the damped equation:

$$\xi'' + c\xi' + \overline{q}(s)g(\xi) = 0, \tag{5.12}$$

where c is a real constant and the apices denote derivation with respect to s.

Theorem 5.2.4 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying (g1), (g2+) and (g2−), let $\overline{q} : [0, \varpi] \to \mathbb{R}$ be a continuous function satisfying (q0) and let $c \in \mathbb{R}$. Suppose that there are $2k+1$ points $s_i \in [0, \varpi]$, $i = 0, \dots, 2k$ such that:*

$$0 = s_0 < s_1 < s_2 < \cdots < s_{2k-1} < s_{2k} = \varpi$$

and:

$$\overline{q} \le 0, \overline{q} \not\equiv 0 \text{ in } I_i^- = [s_{2i-2}, s_{2i-1}] \quad \text{and} \quad \overline{q} \ge 0, \overline{q} \not\equiv 0 \text{ in } I_i^+ = [s_{2i-1}, s_{2i}],$$

for all $i = 1, \dots, k$.
Then there exist k natural numbers $n_1^*, \dots, n_k^*$ such that, for all $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$, with $n_i \ge n_i^*$, and for all $\delta = (\delta_1, \dots, \delta_k) \in \{0,1\}^k$, with:

$$|\mathbf{n}| + |\delta| = n_1 + \cdots + n_k + \delta_1 + \cdots + \delta_k = \text{even number},$$

there exist two ω-periodic solutions $\xi^+_{\mathbf{n},\delta}$ and $\xi^-_{\mathbf{n},\delta}$ of (5.12) such that:

1. *$\xi^-_{\mathbf{n},\delta}(0) < 0 < \xi^+_{\mathbf{n},\delta}(0)$;*
2. *$\xi^\pm_{\mathbf{n},\delta}$ has exactly n_i zeros in $]s_{2i-1}, s_{2i}[$ for all $i = 1, \dots, k$;*
3. *$\xi^\pm_{\mathbf{n},\delta}$ has exactly δ_i zeros in $]s_{2i-2}, s_{2i-1}[$ for all $i = 1, \dots, k$;*
4. *$(\xi^\pm_{\mathbf{n},\delta})'$ has exactly $1 - \delta_i$ changes of sign in $]s_{2i-2}, s_{2i-1}[$ for all $i = 1, \dots, k$;*

Moreover for every $m \ge 2$ there are infinitely many subharmonic solutions of (5.12) of order m.

Proof. We carry out the proof in the case $c > 0$, since the other case can be managed in the same way.

Let us make the following change of variables (we found it used in [21]):

$$t = \tau(s) = \frac{1}{c}\left(1 - e^{-cs}\right),$$

that is:

$$s = \varsigma(t) = \frac{1}{c} \log \frac{1}{1 - ct},$$

and define:

$$x(t) = \xi(\varsigma(t));$$

then straightforward computations show that ξ is a solution of the equation (5.12) if and only if x satisfies the equation:

$$\ddot{x} + q(t)g(x) = 0, \tag{5.13}$$

with q defined by:

$$q(t) = \left(\frac{1}{1 - ct}\right)^2 \overline{q}(\varsigma(t)).$$

Looking for periodic solutions of (5.12), that is for solutions of (5.12) which are defined on $[0, \varpi]$, satisfy the boundary conditions:

$$\begin{cases} \xi(\varpi) = \xi(0) \\ \xi'(\varpi) = \xi'(0) \end{cases}$$

and have the nodal behaviour prescribed by two vectors $\mathbf{n}$ and δ, turns out to be equivalent to find solutions of (5.13) which are defined on $[0,\omega]$, with $\omega = \tau(\overline{\omega})$, satisfy:

$$\begin{cases} x(\omega) = x(0) \\ \dot{x}(\omega) = e^{c\,\overline{\omega}}\dot{x}(0) \end{cases}$$

and also have nodal behaviour prescribed by the same $\mathbf{n}$ and δ.

In fact, if we define:

$$\Lambda(p_1,p_2) = \begin{pmatrix} 1 & 0 \\ 0 & e^{c\,\omega} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

our problem is now of the form (5.7) and Λ and q verify the assumptions of Theorem 5.1.1, with the positions:

$$t_i = \tau(s_i) \qquad \forall\, i = 0,\ldots,2k,$$

so that in the remainder of the proof we will forget the equation (5.12) and we will concentrate to the usual equation (5.13).

Theorem 5.1.1 provides as usual the first $k-1$ positive integers $n_1^*,\ldots,n_{k-1}^*$ and, as in the proof of Theorem 5.2.1, we define the last one by means of the uniform constant K defined in Remark 5.1.2: let n_k^* be the minimum integer such that:

$$\operatorname*{rot}_{[t_{2k-1},\omega]}(p) \le (n-1)\pi \qquad \forall\, p \in \mathbb{R}^2 \,:\, 0 < |p| \le K;$$

again, it exists and is finite since $g(x)/x$ is bounded in a neighbourhood of the origin.

We fix now $\delta = (\delta_1,\ldots,\delta_k) \in \{0,1\}^k$ and $\mathbf{n} = (n_1,\ldots,n_k) \in \mathbb{N}^k$ such that:

- $n_i \ge n_i^*$, for every $i = 1,\ldots,k$;
- $|\mathbf{n}| + |\delta| = n_1 + \cdots + n_k + \delta_1 + \cdots + \delta_k$ is an even integer,

and we apply Theorem 5.1.1 with the $(k-1)$-tuple $(n_1,\ldots,n_{k-1})$ and the k-tuple δ, finding the two unbounded continua $\Gamma^+ \subset A_4$ and $\Gamma^- \subset A_2$ such that the solutions starting at $t=0$ from a point p in $\Gamma^\pm$ are defined in $[0,\omega]$, satisfy:

$$|z(\omega;0,p)| = |\Lambda(p)|$$

and have the prescribed nodal behaviour in $[0,t_{2k-1}]$. It remains to find $p^\pm_{\mathbf{n},\delta} \in \Gamma^\pm$ such that:

(a) $x(\cdot;0,p^\pm_{\mathbf{n},\delta})$ has exactly n_k zeros in $]t_{2k-1},\omega]$;

(b) $z(\omega;0,p^\pm_{\mathbf{n},\delta}) = \begin{pmatrix} 1 & 0 \\ 0 & e^{c\,\omega} \end{pmatrix}$.

As usual, we show only the existence of $p^+_{\mathbf{n},\delta} \in \Gamma^+$, since the other point can be similarly found. As in the proof of Theorems 5.1.1 and 5.2.1, we define the angular coordinate θ : $[0,\omega] \times [A_4 \setminus \{(0,0)\}]$ of the vector $z(t;0,p)$ in such a way that:

- $\theta(0;p) \in \left[-\frac{\pi}{2}, 0\right]$ for every $p \in A_4 \setminus \{(0,0)\}$;
- $p = (|p|\cos\theta(0;p); |p|\sin\theta(0;p))$ for every $p \in A_4 \setminus \{(0,0)\}$;
- $\theta(t;p) = \theta(0;p) - \displaystyle\int_0^t \frac{q(t)g(x(s;0,p))x(s;0,p) + \dot{x}^2(s;0,p)}{x^2(s;0,p) + \dot{x}^2(s;0,p)} = \theta(0;p) - \operatorname*{rot}_{[0,t]}(p);$

- $\begin{cases} x(t;0,p) = |z(t;0,p)|\cos\theta(t;p) \\ \dot{x}(t;0,p) = |z(t;0,p)|\sin\theta(t;p) \end{cases} \qquad \forall t \in [0,\omega]\ \forall p \in A_4 \setminus B(R_0)$

(see (2.16) and (2.17) for more details). In particular, a point $p \in A_4$ is such that $x(\cdot;0,p)$ has exactly n zeros in $[0,t]$ if and only if:

$$-\frac{\pi}{2} - n\pi < \theta(t;p) \leq -\frac{\pi}{2} - (n-1)\pi.$$

Now, all the solutions starting at $t=0$ from $p \in \Gamma^+$ have exactly $N_{2k-1} = n_1 + \cdots + n_{k-1} + |\delta|$ zeros in $[0, t_{2k-1}]$; therefore, in order to satisy statement (a), we have to consider those $p \in \Gamma^+$ such that:

$$-\frac{\pi}{2} - (N_{2k-1} + n_k)\pi < \theta(\omega;p) \leq -\frac{\pi}{2} - (N_{2k-1} + n_k - 1)\pi,$$

that is:

$$-\frac{\pi}{2} - (|\mathbf{n}| + |\delta|)\pi < \theta(\omega;p) \leq -\frac{\pi}{2} - (|\mathbf{n}| + |\delta| - 1)\pi.$$

Arguing like in the proof of Theorem 5.2.1 and using the definition of n_k^* it can be shown that:

$$\theta(\omega;\Gamma^+) = \{\theta(\omega;p) : p \in \Gamma^+\} \supset \left]-\infty, -(N_{2k-1} + n_k^*)\pi + \frac{\pi}{2}\right].$$

Since θ is continuous, Γ^+ is connected and $n_k \geq n_k^*$, we can find a connected and closed $\Gamma_{n_k}^+ \subset \Gamma^+$ such that:

$$\theta(\omega;\Gamma_{n_k}^+) = \left[-\frac{\pi}{2} - (|\mathbf{n}| + |\delta|)\pi, -\frac{\pi}{2} - (|\mathbf{n}| + |\delta| - 1)\pi\right].$$

We consider also the continuous angular function $\vartheta : A_4 \setminus \{(0,0)\} \to \mathbb{R}$ for Λ, which satisfies:

$$\Lambda(p) = (|\Lambda(p)|\cos\vartheta(p), |\Lambda(p)|\sin\vartheta(p))$$

for every $p \in A_4 \setminus \{(0,0)\}$. Since Λ now is a dilatation in the vertical direction it can be easily shown that for every point p in the fourth quadrant A_4 there holds that:

$$-\frac{\pi}{2} \leq \theta(0;0,p) \leq 0 \qquad \forall p \in A_4 \setminus \{(0,0)\}.$$

Finally we define again the function $\psi : \Gamma \to \mathbb{R}$ by:

$$\psi(p) = \theta(\omega;p) - \vartheta(p),$$

so that we have to find $p_{\mathbf{n},\delta}^+ \in \Gamma_{n_k}^+$ such that $\psi(p_{\mathbf{n},\delta}^+) \in 2\pi\mathbb{Z}$, in order to reach goals (a) and (b).

Let $p_1 \in \Gamma_{n_k}^+$ be such that:

$$\theta(\omega;p_1) = -\frac{\pi}{2} - (|\mathbf{n}| + |\delta|)\pi;$$

then we can evaluate:

$$\begin{aligned} \psi(p_1) &\leq -\frac{\pi}{2} - (|\mathbf{n}| + |\delta|)\pi + \frac{\pi}{2} \\ &\leq -(|\mathbf{n}| + |\delta|)\pi. \end{aligned}$$

On the other hand, if $p_2 \in \Gamma_{n_k}^+$ is such that:

$$\theta(\omega;p_1) = -\frac{\pi}{2} - (|\mathbf{n}| + |\delta| - 1)\pi,$$

then:

$$\begin{aligned}\psi(p_1) &\geq -\frac{\pi}{2} - (|\mathbf{n}| + |\delta| - 1)\pi \\ &\geq -(|\mathbf{n}| + |\delta|)\pi + \frac{\pi}{2}.\end{aligned}$$

Therefore there exists $p^+_{\mathbf{n},\delta} \in \Gamma^+_{n_k}$ such that $\psi(p^+_{\mathbf{n},\delta}) = -(|\mathbf{n}| + |\delta|)\pi$.

Now the trick to find infinitely many subharmonic solutions is standard and without loss of generality we can assume that ω is the minimal period of q (otherwise such minimal period should be of the form ω/l, with $l \in \mathbb{N}$, and we could repeat the argument below with ω/l in place of ω); we recall that a solution x of (5.12) is subharmonic if it is $m\omega$-periodic with $m \in \mathbb{N}$, $m > 1$, and minimal period $m\omega$. Then we apply what the part of the theorem that we have just proved to the interval $[0, m\omega]$ (q is obviously $m\omega$–periodic) and with $\mathbf{n}$ and δ such that $2N = |\mathbf{n}| + |\delta|$ is an even number and N and m are relatively prime numbers: then we find a $m\omega$–periodic solution x of (5.12) with exactly $2N$ zeros in $]0, m\omega]$; let us verify that $m\omega$ is the minimal period of x, since it could happen that x has a minimal period which is incommensurable with ω, as remarked in [104, pp. 10–11]. Let T be the minimal period of x; a direct inspection shows that this implies that T is also a period of q, then $T = k\omega$ for some $k \in \mathbb{N}$, $k \leq m$. Now let $2M$ be the number of zeroes of x in $[0, k\omega)$; if we look to $[0, mk\omega)$ as the union of either k intervals of length $m\omega$ or m intervals of length $k\omega$, we find $2Nk$ zeroes of x in $[0, mk\omega)$, in the first case, and $2Mm$ zeroes, in the second one, that is $2Nk = 2Mm$; but this implies that:

$$\frac{m}{N} = \frac{k}{N},$$

which is not possible for $k < m$ since m and N are relatively prime numbers, so $k = m$. ∎

Chapter 6

Globally defined solutions for superlinear indefinite equations and chaotic dynamics

We have seen (and actually this is the main subject of the paper by Burton and Grimmer [21]) that there are problems of continuability for the solutions of the equation:

$$(6.1) \qquad \ddot{x} + q(t)g(x) = 0,$$

when g is a superlinear function and q is negative somewhere. In this chapter we are consider a function q with infinitely many zeros and changes of sign and we look for solutions of (6.1) which are "globally defined", that is defined in the entire domain of q, and have prescribed nodal behaviour in the sense of the preceding chapters: we divide the domain of q into a double sequence of alternating intervals in which q is either positive or negative and we wish to assign in every interval of positivity an arbitrary (but sufficiently large) number of zeros and to decide in each interval of negativity if the solutions will be monotone with exactly one zero or concave/convex with the first derivative that changes sign exactly once.

In Chapter 3 we have associated to each interval $[t_0, t_1]$, where $q \leq 0$ and $q \not\equiv 0$, the sets $\Omega_{t_0}^{t_1}$ and $\Omega_{t_1}^{t_0}$ of all the points p of the plane such that the solution of (6.1) starting from p at $t = t_0$ and $t = t_1$, respectively, is defined in the interval $[t_0, t_1]$. If g satisfies:

$$(g2-) \qquad \lim_{e \to \pm\infty} \tau_g^-(e) = 0,$$

where τ_g^- is defined in (3.2), then the sets like $\Omega_{t_0}^{t_1}$ are such that their intersections with every straight line from the origin are bounded (cf Lemma 3.1.1). Suppose now that the domain of q is an open interval $]a, b[$ and that there is an increasing sequence $\{t_k\}_{k \in \mathbb{Z}} \subset]a, b[$ such that:

$$\lim_{k \to -\infty} t_k = a \qquad \text{and} \qquad \lim_{k \to +\infty} t_k = b$$

and:

$$q \leq 0, \ q \not\equiv 0 \text{ in }]t_{2k}, t_{2k+1}[\qquad \forall k \in \mathbb{Z},$$

$$q \geq 0, \ q \not\equiv 0 \text{ in }]t_{2k+1}, t_{2k+2}[\quad \forall k \in \mathbb{Z}.$$

If we consider the set of initial points which produce solutions from $t = t_0$ that are defined on $[t_{-j}, t_j]$, then this set becomes smaller and smaller as j increases. However we are able to prove that, in the limit as $j \to \infty$, some point "survives" and gives raise to a solution globally defined.

The method we use is based on the construction of solutions of (6.1) defined in intervals of ever-increasing width; to be precise, let $x(\cdot; t_0, p)$ be the solution of the initial value problem:

$$\begin{cases} \ddot{x} + q(t)g(x) = 0 \\ (x(t_0), \dot{x}(t_0)) = p, \end{cases}$$

for some $p \in \mathbb{R}^2$, and let $z(t; t_0, p) = (x(t; t_0, p), \dot{x}(t; t_0, p))$ be the corresponding phase vector.

Let $\mathbf{n} = \{n_k\}_{k\in\mathbb{Z}} \subset \mathbb{N}^{\mathbb{Z}}$ and $\delta = \{\delta_k\}_{k\in\mathbb{Z}} \subset \{0,1\}^{\mathbb{Z}}$ be two double sequences; for every $j \in \mathbb{N}$ we define the open set:

$$\begin{aligned} \Omega_j^+ = \{p \in A_4 : {} & x(\cdot; t_0, p) \text{ is defined in } [t_{-2j}, t_{2j+2}], \text{ has exactly } n_k \text{ zeros} \\ & \text{in }]t_{2k+1}, t_{2k+2}[\text{ and } \delta_k \text{ zeros in }]t_{2k}, t_{2k+1}[\text{ and } \dot{x}(\cdot; t_0, p) \text{ has} \\ & \text{exactly } 1 - \delta_k \text{ changes of sign in }]t_{2k}, t_{2k+1}[\,,\ \forall |k| \le j\} \end{aligned} \tag{6.2}$$

(an analogous definition is given in order to consider solutions such that $x(t_0) < 0$ and $\dot{x}(t_0) > 0$). Then we show that the intersection $\cap_{j\in\mathbb{N}}\Omega_j^+$ is non-empty. To this aim, it is sufficient to prove that every set Ω_j^+ is non-empty and bounded and that the family $\{\Omega_j^+\}_{j\in\mathbb{N}}$ is "compactly" nasted, i.e.:

$$\overline{\Omega_{j+1}^+} \subset \Omega_j^+, \quad \forall j \in \mathbb{N}. \tag{6.3}$$

Indeed, the fact that, for every $j \in \mathbb{N}$, the set Ω_j^+ is non-empty and bounded directly follows from the existence theorems of the preceding two chapters. Finally, in order to prove (6.3) we need a detailed study of the boundary of Ω_j^+ (see Lemma 6.1.7) together with the application of the so-called "elastic property" (see Lemma 6.1.5), which gives an a priori estimate for the solutions of (6.1) having a prescribed number of zeros.

In the last section we restrict ourselves to the study of (6.1) when q is ω-periodic. In this situation we provide, by the use of the Poincaré map, some consequences of the results of the first section, which point out the chaotic features of our problem. First, we show (see Proposition 6.2.3) the existence of an uncountable number of bounded non-periodic solutions of (6.1). Then, we obtain that the Poincaré map associated to (6.1) has positive topological entropy.

6.1 An existence and multiplicity result

We consider the differential equation:

$$\ddot{x} + q(t)g(x) = 0, \tag{6.4}$$

where $q :]a,b[\to \mathbb{R}$ is a continuous function, with $-\infty \le a < b \le +\infty$, and satsfies the following assumption:

$(q0)$ there exists an increasing sequence $\{t_k\} \subset]a,b[\,,\ k \in \mathbb{Z}$, such that:

$$\lim_{k\to-\infty} t_k = a, \qquad \lim_{k\to+\infty} t_k = b$$

and:

$$q \le 0,\ q \not\equiv 0 \text{ in }]t_{2k}, t_{2k+1}[\qquad \forall k \in \mathbb{Z},$$

$$q \ge 0,\ q \not\equiv 0 \text{ in }]t_{2k+1}, t_{2k+2}[\qquad \forall k \in \mathbb{Z};$$

moreover, the function q is locally of bounded variation in every interval of the form $]t_{2k+1}, t_{2k+2}[$, with $k \in \mathbb{Z}$ and, for every $c \in [t_{2k+1}, t_{2k+2}]$ such that $q(c) = 0$, there exist a left neighbourhood and a right neighbourhood of c where q is monotone.

About $g : \mathbb{R} \to \mathbb{R}$ we assume that it satisfies the usual sign condition:

$$(g1) \qquad g(x)x > 0, \quad \forall x \neq 0$$

and the two hypotheses of superlinear growth at infinity:

$$(g2+) \qquad \lim_{e \to +\infty} \tau_g^+(e) = 0$$

and:

$$(g2-), \qquad \lim_{e \to \pm\infty} \tau_g^-(e) = 0$$

where τ^+ and τ^- are defined by (2.3) and (3.2), respectively.

Under these assumptions, we are able to prove the following result:

Theorem 6.1.1 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying $(g1)$, $(g2+)$ and $(g2-)$ and let $q :]a, b[\to \mathbb{R}$ be a continuous function satisfying $(q0)$, with $-\infty \leq a < b \leq +\infty$. Then, there exists a sequence $\mathbf{n}^* = \{n_k^*\}_{k \in \mathbb{Z}}$ of natural numbers such that, for every sequence $\mathbf{n} = \{n_k\}_{k \in \mathbb{Z}}$, with $n_k \in \mathbb{N}$ and $n_k \geq n_k^*$, and for every sequence $\delta = \{\delta_k\}_{k \in \mathbb{Z}}$, with $\delta_k \in \{0, 1\}$, there exist two points $p^+_{\mathbf{n},\delta} \in A_4$ and $p^-_{\mathbf{n},\delta} \in A_2$ such that:*

1. *$x(t; t_0, p^{\pm}_{\mathbf{n},\delta})$ is defined for all $t \in]a, b[$;*
2. *for every $k \in \mathbb{Z}$, $x(\cdot; t_0, p^{\pm}_{\mathbf{n},\delta})$ has exactly n_k zeros in $]t_{2k+1}, t_{2k+2}[$ and exactly δ_k zeros in $]t_{2k}, t_{2k+1}[$;*
3. *for every $k \in \mathbb{Z}$, $\dot{x}(\cdot; t_0, p^{\pm}_{\mathbf{n},\delta})$ has exactly $1 - \delta_k$ changes of sign in $]t_{2k}, t_{2k+1}[$.*

We underline the fact that, if the measure of the set where q vanishes is zero, then Statement 3 in Theorem 6.1.1 can be improved by stating that the functions $\dot{x}(\cdot; t_0, p^{\pm}_{\mathbf{n},\delta})$ has exactly $1 - \delta_k$ *zeros* in $]t_{2k}, t_{2k+1}[$, for every $k \in \mathbb{Z}$.

Remark 6.1.2 It is possible to show that, for every $k \in \mathbb{Z}$, the integer n_k^* only depends on the behaviour of the function q on the interval $[t_{2k}, t_{2k+3}]$, as can be seen from Lemma 6.1.5 and Lemma 6.1.4.

Remark 6.1.3 We first observe that, in contrast with various situations which can be found in the literature, we do not assume any periodicity on q. A recent contribution where oscillatory functions (not necessarily periodic) are considered, is due to S. Terracini and G. Verzini [121]; however, in [121] an appropriate boundedness condition on q is required. We also remark that the regularity assumptions on q (see condition $(q0)$) guarantee the continuability of the solutions of the equation in (6.4) in every interval where q is positive by Corollary 2.1.3.

Assumptions $(g2+)$ and $(g2-)$ represent a superlinear behaviour of the function g at infinity. They are clearly satisfied by functions g like $g(x) = x|x|^{p-1} \log^q(|x| + 2)$, with either $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q > 2$. Condition $(g2+)$ is needed to deduce the properties of the number of zeros of any solution to (6.4) in the intervals where q is positive (see Lemma 6.1.5); furthermore, we have to assume $(g2-)$ in order to describe the sets of continuability for the solutions of equation (6.4) in the intervals where q is negative (see Lemma 6.1.4 and Section 3.1.

For every $k \in \mathbb{Z}$, let us define the set:

$$\Omega_i^j = \{p \in \mathbb{R}^2 : z(\cdot; t_i, p) \text{ is defined in } [\min\{t_i, t_j\}, \max\{t_i, t_j\}]\}, \tag{6.5}$$

which is the set of all the points p in the plane such that the solution of (6.4) starting from p at $t = t_i$, is defined in the closed interval spanned by t_i and t_j. Lemma 3.1.1 implies that the intersections of these set with the coordinate axes (actually, with every straight line passing by the origin) are bounded whenever q is negative somewhere between t_i and t_j. Indeed, the following result holds true.

Lemma 6.1.4 *For every $k \in \mathbb{Z}$ there exists $l_k > 0$ such that*

$$\left.\begin{array}{c} p \in \Omega_{2k}^{2k+1} \cap [(\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})] \\ or \\ p \in \Omega_{2k+1}^{2k} \cap [(\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})] \end{array}\right\} \quad \Rightarrow \quad |p| < l_k.$$

Proof. Apply Lemma 3.1.1 on $[t_{2k}, t_{2k+1}]$. ■

Now, it is easy to see, from the sign condition $(g1)$ and the local Lipschitz continuity of g, that every solution x of (6.4), $x \not\equiv 0$, has only a finite number of zeros in every bounded interval. We denote this number by $\mathbf{n}_{[t_j, t_{j+1}]}(x)$. Moreover, for every $k \in \mathbb{Z}$, let us set:

$$L_k = \max(l_k, l_{k+1}).$$

We state our next preliminary result:

Lemma 6.1.5 *The following statements hold:*

1. *for every $k \in \mathbb{Z}$ there exists $n_k^* \in \mathbb{N}$ such that for every solution $x \not\equiv 0$ of (6.4) we have:*

$$|(x(t^*), \dot{x}(t^*))| < L_k, \quad \text{for some } t^* \in [t_{2k+1}, t_{2k+2}] \quad \Rightarrow \quad \mathbf{n}_{[t_{2k+1}, t_{2k+2}]}(x) < n_k^*.$$

2. *for every $k \in \mathbb{Z}$ and for every $N \in \mathbb{N}$ there exists $C = C(k, N)$ such that for every solution x of (6.4) we have:*

$$\mathbf{n}_{[t_{2k+1}, t_{2k+2}]}(x) \le N \quad \Rightarrow \quad |(x(t), \dot{x}(t))| \le C, \quad \forall\, t \in [t_{2k+1}, t_{2k+2}].$$

Proof. Statement 1 in Lemma 6.1.5 is a direct consequence of the uniqueness of the zero solution $x \equiv 0$, due to the local Lipschitz continuity of g : in fact, if Statement 1 is false, it is possible to find a sequence of solutions which are bounded in C^1 and whose number of zeros increase to infinity; passing to a suitable subsequence and using Rolle's Theorem, it is possible to show that they converge to a nonzero solution x such that $x(t^*) = \dot{x}(t^*) = 0$, a contradiction with the uniqueness of the zero solution.

Statement 2 follows from Corollary 2.2.8 applied on $[t_{2k+1}, t_{2k+2}]$, where we have that $q \ge 0$ and $q \not\equiv 0$. ■

We also observe that Statement 2 implies that the number of zeros of a solution of (6.4), in any interval where q is positive, is arbitrarily large provided that the norm of this solution is large.

Remark 6.1.6 From the statements of Lemma 6.1.4 and Lemma 6.1.5, it is clear that, for every $k \in \mathbb{Z}$, the number n_k^* only depends on the restriction of the function q to the interval $[t_{2k}, t_{2k+3}]$.

Now, we pass to the first step for the proof of Theorem 6.1.1. To this aim let us fix two sequences $\mathbf{n} = \{n_k\}_{k\in\mathbb{Z}}$, with $n_k \in \mathbb{N}$ and $n_k > n_k^*$, and $\delta = \{\delta_k\}_{k\in\mathbb{Z}}$, with $\delta_k \in \{0,1\}$, as in the statement of Theorem 6.1.1.

For every $j \in \mathbb{N}$, let us define:

$$\begin{aligned}\Omega_j^+ = \{p \in A_4 : x(\cdot;t_0,p) &\text{ is defined in } [t_{-2j}, t_{2j+2}], \text{ has exactly } n_k \text{ zeros in} \\ &]t_{2k+1}, t_{2k+2}[\text{ and exactly } \delta_k \text{ zeros in }]t_{2k}, t_{2k+1}[\text{ and } \dot{x}(\cdot;t_0,p) \\ &\text{has exactly } 1-\delta_k \text{ changes of sign in }]t_{2k}, t_{2k+1}[\text{ for every } |k| \le j\}\end{aligned} \tag{6.6}$$

and:

$$\begin{aligned}\Omega_j^- = \{p \in A_2 : x(\cdot;t_0,p) &\text{ is defined in } [t_{-2j}, t_{2j+2}], \text{ has exactly } n_k \text{ zeros in} \\ &]t_{2k+1}, t_{2k+2}[\text{ and exactly } \delta_k \text{ zeros in }]t_{2k}, t_{2k+1}[\text{ and } \dot{x}(\cdot;t_0,p) \\ &\text{has exactly } 1-\delta_k \text{ changes of sign in }]t_{2k}, t_{2k+1}[\text{ for every } |k| \le j\}.\end{aligned} \tag{6.7}$$

By the definition of Ω_j^+, it is easy to see that $x(t;t_0,p)$ is a solution of (6.4) defined on $]a,b[$, with $p \in A_4$ and the nodal properties prescribed by the vectors $\mathbf{n}$ and δ as stated in Theorem 6.1.1, if and only if:

$$p \in \bigcap_{j=0}^{+\infty} \Omega_j^+;$$

analogously, $x(t;t_0,p)$ is a solution of (6.4) defined on $]a,b[$, with $p \in A_4$ and the nodal properties prescribed by the vectors $\mathbf{n}$ and δ, if and only if:

$$p \in \bigcap_{j=0}^{+\infty} \Omega_j^-.$$

Therefore Theorem 6.1.1 holds true if we prove that:

$$\bigcap_{j=0}^{+\infty} \Omega_j^\pm \neq \emptyset. \tag{6.8}$$

Actually we will show only that:

$$\bigcap_{j=0}^{+\infty} \Omega_j^+ \neq \emptyset, \tag{6.9}$$

since the proof of the same relation for the sets Ω_j^- is similar. The crucial step of our method is the following lemma, which describes the boundary of every set Ω_j^+.

Lemma 6.1.7 *For every $j \in \mathbb{N}$ and for every $p \in \partial\Omega_j^+$ the solution $z(\cdot;t_0,p)$ is defined in $[t_{-2j+1}, t_{2j+2}]$, but it is not defined in $[t_{-2j}, t_{2j+2}]$.*

Proof. We first show that, for $j \in \mathbb{N}$ and $p \in \partial\Omega_j^+$, the solution $z(\cdot;t_0,p)$ is not defined in $[t_{-2j}, t_{2j+2}]$. We argue by contradiction: let $j \in \mathbb{N}$ and $p \in \partial\Omega_j^+$ be such that $z(\cdot;t_0,p)$ is defined in $[t_{-2j}, t_{2j+2}]$. Therefore, at least one of the following conditions holds true:

(a) $x(t_0) = 0$;

(b) $\dot{x}(t_0) = 0$;

(c) there exists k, with $|k| \leq j$, such that $x(t_{2k+1}; t_0, p) \cdot x(t_{2k+2}; t_0, p) = 0$ (i.e. the solution $x(t; t_0, p)$ has a zero on the boundary of $[t_{2k+1}, t_{2k+2}]$);

(d) there exists k, with $|k| \leq j$ and $\delta_k = 1$, such that $x(t_{2k}; t_0, p) \cdot x(t_{2k+1}; t_0, p) = 0$ (i.e. $x(t; t_0, p)$ has a zero on the boundary of $[t_{2k}, t_{2k+1}]$);

(d') there exists k, with $|k| \leq j$ and $\delta_k = 0$, such that $\dot{x}(t_{2k}; t_0, p) \cdot \dot{x}(t_{2k+1}; t_0, p) = 0$ (i.e. $\dot{x}(t; t_0, p)$ has a zero on the boundary of $[t_{2k}, t_{2k+1}]$).

We show that in any case we get a contradiction. Indeed, assume that (a) holds; then, we have:

$$p \in \Omega_0^1 \cap (\{0\} \times \mathbb{R})$$

and thus, by Lemma 6.1.4, $|p| < l_0 \leq L_{-1}$. Therefore, by Lemma 6.1.5, we should have:

$$n_{-1} \leq \mathbf{n}_{[t_{-1}, t_0]} < n^*_{-1} \leq n_{-1},$$

which is absurd.

When (b) holds, we have $p \in \Omega_0^1 \cap (\mathbb{R} \times \{0\})$ and we can conclude as in the previous case.

Assume now that (c) is fulfilled; if $x(t_{2k+1}; t_0, p) = 0$, then:

$$z(t_{2k+1}; t_0, p) \in \Omega_{2k+1}^{2k} \cap (\{0\} \times \mathbb{R})$$

and $|z(t_{2k+1}; t_0, p)| < l_k \leq L_k$ by Lemma 6.1.4. Therefore, by Lemma 6.1.5, we should have:

$$n_k \leq \mathbf{n}_{[t_{2k+1}, t_{2k+2}]} < n^*_k \leq n_k,$$

which is absurd. If $x(t_{2k+2}; t_0, p) = 0$, then we can similarly show that $|z(t_{2k+2})| < L_k$ and this leads to the same contradiction.

In the same manner one can see that also (d) and (d') cannot hold and this concludes the first part of the proof.

Now we prove that, for $j \in \mathbb{N}$ and $p \in \partial\Omega_j^+$, the solution $z(\cdot; t_0, p)$ is defined in $[t_{-2j+1}, t_{2j+2}]$. We argue again by contradiction: assume then that there exist $j \in \mathbb{N}$ and $p^* \in \partial\Omega_j^+$ such that $z(\cdot; t_0, p^*)$ is not defined in $[t_{-2j+1}, t_{2j+2}]$. Then, there exists $t^* \in [t_{-2j+1}, t_{2j+2}]$ such that:

$$\lim_{t \to t^*} |z(t; t_0, p^*)| = +\infty$$

(actually $t* \in [t_{-2j+2}, t_{2j+1}]$, since there are no problems of continuability in the intervals $[t_{-2j+1}, t_{-2j+2}]$ and $[t_{2j+1}, t_{2j+2}]$, in which q is nonnegative). Therefore, for every $M > 0$ there exists a (right or left) neighbourhood $V(t^*, M)$ of t^* such that:

$$|z(t; t_0, p^*)| \geq M, \quad \forall\, t \in V(t^*, M),\ t \neq t^*. \tag{6.10}$$

On the other hand, since the number of zeros of x on every interval $[t_{2k+1}, t_{2k+2}]$, for $|k| \leq j$, is bounded, from Lemma 6.1.5 (Statement 2) we deduce that there exists $C_j > 0$ such that:

$$|z(t; t_0, p)| \leq C_j, \quad \forall\, t \in [t_{2k+1}, t_{2k+2}]\ \forall\, |k| \leq j,\ \forall\, p \in \Omega_j^+. \tag{6.11}$$

Now, we observe that in every interval of the form $[t_{2k}, t_{2k+1}]$, that is in every interval where q is nonpositive, a kind of maximum principle holds for a solution x of (6.4), in the sense that the modulus $|(x(t), \dot{x}(t))|$ attains its maximum value on the boundary of the interval, that is in t_{2k} or in t_{2k+1}; this can be easily seen by checkig the sign of the derivative of $x^2(t) + \dot{x}^2(t)$ and by making use of the fact that, where $q \leq 0$, the solutions of (6.4) are monotone or convex and positive or concave and negative. However, this implies that the estimate (6.11) also holds in those intervals of negativity of q which lie between t_{-2j+1} and t_{2j+2}, i.e. in $[t_{2k}, t_{2k+1}]$ for every $k = -j+1, \ldots, j$. Putting all these things together we have:

$$(6.12) \qquad |z(t;t_0,p)| \leq C_j \quad \forall\, t \in [t_{-2j+1}, t_{2j+2}]\, \forall\, p \in \Omega_j^+.$$

Since $p^* \in \partial\Omega_j^+$, there exists $p_n \in \Omega_j^+$ such that $p_n \to p^*$; hence, from (6.12) we have:

$$(6.13) \qquad |z(t;t_0,p_n)| \leq C_j, \quad \forall\, t \in [t_{-2j+1}, t_{2j+2}]\, \forall\, n \in \mathbb{N}.$$

If we take $M = C_j + 1$ in (6.10), an elementary continuous dependence argument shows that (6.13) contradicts (6.10) and the result is proved. ■

We are now able to prove the following result.

Proposition 6.1.8 *For every $j \in \mathbb{N}$ we have:*

$$(6.14) \qquad \Omega_j^+ \neq \emptyset,$$

$$(6.15) \qquad \Omega_j^+ \quad \textit{is bounded}$$

and:

$$(6.16) \qquad \overline{\Omega_{j+1}^+} \subset \Omega_j^+.$$

Proof. The fact that, for every $j \in \mathbb{N}$, the set Ω_j^+ is not empty, directly follows by applying one of the existence theorems discussed in the previous two chapters: for instance we could apply Theorem 4.2.1 on the interval $[t_{-2j}, t_{2j+2}]$, in order to get a solution of (6.4) wich is defined in this interval and has the prescribed nodal behaviour.

Moreover, from Lemma 6.1.5, Statement 2, we deduce that there exists $C = C(-1, n_{-1}) > 0$ such that for every solution x of (6.4) with exactly n_{-1} zeros in $[t_{-1}, t_0[$ we have:

$$|(x(t), \dot{x}(t)| \leq C, \quad \forall\, t \in [t_{-1}, t_0].$$

In particular, taking $t = t_0$, this shows that $|p| \leq C$, for every $p \in \Omega_j^+$, for every $j \in \mathbb{N}$, i.e. Ω_j^+ is bounded.

Finally, from the definition of Ω_j^+ we plainly deduce that:

$$\Omega_{j+1}^+ \subset \Omega_j^+$$

and:

$$\overline{\Omega_{j+1}^+} \subset \overline{\Omega_j^+},$$

for every $j \in \mathbb{N}$. Now, suppose that $p \in \partial\Omega_j^+$ for some $j \in \mathbb{N}$; then, by Lemma 6.1.7, we necessarily have that $z(\cdot; t_0, p)$ is defined in $[t_{-2j+1}, t_{2j+2}]$, but not in $[t_{-2j}, t_{2j+2}]$. On the other hand, if $p \in \overline{\Omega_{j+1}^+}$, we should have either $p \in \Omega_{j+1}^+$ or $p \in \partial\Omega_{j+1}^+$ and, in both cases,

$z(\cdot; t_0, p)$ should be defined at least in $[t_{-2j-1}, t_{2j+4}]$ by the definition of Ω_{j+1} and by Lemma 6.1.7 applied to $\partial\Omega_{j+1}^+$. Henceforth, $p \notin \overline{\Omega_{j+1}^+}$ and this shows that (6.16) holds. ■

We are ready to prove our main theorem.

Proof of Theorem 6.1.1. Let us consider the sets Ω_j^+ defined in (6.6); from Proposition 6.1.8 and an easy topological argument, it is immediate to see that (6.9) holds. By the previous discussion, this concludes the proof. ■

Remark 6.1.9 Theorem 6.1.1 provides solutions of (6.4) with prescribed nodal behaviour in the *open* intervals $]t_i, t_{i+1}[$ for every $i \in \mathbb{Z}$, but, may be, up to now it is not clear what happens on the *boundary* of these intervals. Actually it can be proved that if x is a solution of (6.4) such that $x(t_i) \cdot \dot{x}(t_i) = 0$ for some $i \in \mathbb{Z}$, then the number of zeros of x either in $[t_{i-1}, t_i]$, if i is even, or in $[t_i, t_{i+1}]$, if i is odd, is less than the correspondent minimum number n_k^* given by the Theorem (6.1.1) (combine Lemma 6.1.4 and Statement 1 of Lemma 6.1.5 as in the proof of Lemma 6.1.7). Therefore the solutions we find via Theorem 6.1.1 (and their derivatives, too) cannot vanish in t_i for every $i \in \mathbb{Z}$.

Remark 6.1.10 Let us consider the damped equation:

$$\ddot{x} + c\dot{x} + q(t)g(x) = 0, \tag{6.17}$$

where c is a real constant, and suppose for instance that $c > 0$ (the case $c < 0$ can be treated in a similar way). According to [21] (see also the proof of [99, Th. 3.2]), we set:

$$t(\tau) = \frac{1}{c}\log\frac{1}{1-c\tau}, \quad \forall \tau \in \left]\frac{1-e^{-ac}}{c}, \frac{1-e^{-bc}}{c}\right[,$$

and:

$$q^*(\tau) = \frac{1}{(1-c\tau)^2}q(t(\tau)), \quad \forall \tau \in \left]\frac{1-e^{-ac}}{c}, \frac{1-e^{-bc}}{c}\right[\tag{6.18}$$

(if $a = -\infty$ or $b = +\infty$, we let $e^{-ac} = +\infty$ and $e^{-bc} = 0$, respectively). Then, $x(t)$ is a solution of equation (6.17) if and only if $\xi(\tau) = x(t(\tau))$ is a solution of:

$$\ddot{\xi} + q^*(\tau)g(\xi) = 0, \quad \tau \in \left]\frac{1-e^{-ac}}{c}, \frac{1-e^{-bc}}{c}\right[, \tag{6.19}$$

where q^* is defined in (6.18). Now, it is easy to see that the function q^* satisfies $(q0)$ if q does; then, we are allowed to apply Theorem 6.1.1 to equation (6.19) and this gives that the same Theorem 6.1.1 holds for the damped equation (6.17).

6.2 The periodic case: chaotic dynamics

Let us now consider the equation:

$$\ddot{x} + q(t)g(x) = 0, \tag{6.20}$$

where $q : \mathbb{R} \to \mathbb{R}$ satisfies $(q0)$ and is ω-periodic and $g : \mathbb{R} \to \mathbb{R}$ satisfies $(g1)$, $(g2+)$ and $(g2-)$. Without loss of generality we can assume that there exist:

$$0 = t_0 < t_1 < \cdots < t_{2l-1} < t_{2l} = \omega$$

such that:

$$q(t) \leq 0,\ q \not\equiv 0, \quad \forall\, t \in [t_{2k-2}, t_{2k-1}] \text{ and } k = 1, \ldots, l,$$
$$q(t) \geq 0,\ q \not\equiv 0, \quad \forall\, t \in [t_{2k-1}, t_{2k}] \text{ and } k = 1, \ldots, l.$$

According to the notation of Section 6.1, we are taking:

$$t_{2li+k} = t_k + 2li\omega, \qquad \forall\, i \in \mathbb{Z} \text{ and } \forall\, k = 0, \ldots, 2l-1, \tag{6.21}$$

so that $\{t_j\}_{j\in\mathbb{Z}}$ is an increasing sequence of real numbers such that $t_i \to \pm\infty$ as $i \to \pm\infty$. The aim of this section is to prove some results which show that the equation (6.20) exhibits a chaotic-like behaviour. Our results will be consequences of Theorem 6.1.1 proved in Section 2; in the particular case of equation (6.20), it can be stated in the following way:

Corollary 6.2.1 *Under the above assumptions there exist l positive integers $n_1^*, \ldots, n_l^*$ such that for every pair of sequences of l-tuples $\mathbf{n} = \{(n_1^i, \ldots, n_l^i) \in \mathbb{N}^l\}_{i\in\mathbb{Z}}$ and $\delta = \{(\delta_1^i, \ldots, \delta_l^i) \in \{0,1\}^l\}_{i\in\mathbb{Z}}$, there exist two points $p_{\mathbf{n},\delta}^+ \in A_4$ and $p_{\mathbf{n},\delta}^- \in A_2$ such that:*

1. *$x(t; 0, p_{\mathbf{n},\delta}^\pm)$ is defined on the whole real line $\mathbb{R}$;*
2. *$x(\cdot; 0, p_{\mathbf{n},\delta}^\pm)$ has exactly $n_k^* + n_k^i$ zeros in $]t_{2k-1} + i\omega, t_{2k} + i\omega[$ for every $i \in \mathbb{Z}$ and for every $k = 1, \ldots, l$;*
3. *$x(\cdot; 0, p_{\mathbf{n},\delta}^\pm)$ has exactly δ_k^i zeros in $]t_{2k-2} + i\omega, t_{2k-1} + i\omega[$ for every $i \in \mathbb{Z}$ and for every $k = 1, \ldots, l$;*
4. *$\dot{x}(\cdot; 0, p_{\mathbf{n},\delta}^\pm)$ has exactly $1 - \delta_k^i$ changes of sign in $]t_{2k-2} + i\omega, t_{2k-1} + i\omega[$ for every $i \in \mathbb{Z}$ and for every $k = 1, \ldots, l$.*

The difference between Corollary 6.2.1 and Theorem 6.1.1 is the fact that, in the present case, the sequence (n_k^*), $i \in \mathbb{Z}$, whose existence is ensured in the statement of Theorem 6.1.1, is such that $n_k^* = n_{k+2li}^*$, for every $i \in \mathbb{Z}$ and for every $k = 1, \ldots, l$. Indeed, recalling (6.21) and Remark 6.1.2, this is a direct consequence of the ω-periodicity of q.

Now, we pass to the description of the chaotic behaviour of the solutions of (6.20). We first show that the boundedness of the solutions of (6.20) can be characterized by means of the boundedness of the sequence $\mathbf{n}$ in Corollary 6.2.1. As a consequence, we are able to prove that (6.20) has an uncountable set of bounded non-periodic solutions; as it can be seen in [105], this is one of the features of a chaotic dynamics.

Proposition 6.2.2 *Let us consider $\mathbf{n}$ and δ as in the statement of Corollary 6.2.1. Then:*

$$x(\cdot; 0, p_{\mathbf{n},\delta}^\pm) \text{ is bounded in } C^1\text{-norm} \iff \mathbf{n} \text{ is bounded.}$$

Proof. Assume that $\mathbf{n}$ is bounded; since q is periodic, from Lemma 6.1.5, Statement 2, we know that there exists $C = C(q, \mathbf{n})$ such that:

$$|x(t; 0, p_{\mathbf{n},\delta}^\pm)| \leq C, \qquad \forall\, t \in [t_{2k-1} + i\omega, t_{2k} + i\omega],\ \forall\, k = 1, \ldots, l,\ \forall\, i \in \mathbb{Z}. \tag{6.22}$$

This means that $x(\cdot;0,p^{\pm}_{\mathbf{n},\delta})$ is bounded on every interval where q is nonnegative. Now, as observed in the proof of Lemma 6.1.7, in every interval $[t_{2k}+i\omega, t_{2k+1}+i\omega]$, in which q is nonpositive, the maximum of $|z(t;0,p^{\pm}_{\mathbf{n},\delta})|$ is attained on the boundary, which is in common with some interval where q is nonnegative. This implies that the estimate (6.22) also holds in the intervals of negativity of q, i.e. $x(t;0,p^{\pm}_{\mathbf{n},\delta})$ is globally bounded on $\mathbb{R}$.

In order to show that when $x(\cdot;0,p^{\pm}_{\mathbf{n},\delta})$ is bounded also $\mathbf{n}$ is bounded, it is sufficient to apply Corollary 2.2.8 on every interval of positivity of q, recalling also that q is a periodic function. ∎

Proposition 6.2.3 *There exists an uncountable set of bounded nonperiodic solutions of* (6.20) *with an arbitrarily large number of zeros in every interval of positivity of q and at most one zero in any interval of negativity of q.*

Proof. Let us denote by S the set of the pairs $(\mathbf{n},\boldsymbol{\delta})$ of sequences of l-tuples $\mathbf{n} = \{(n^i_1,\ldots,n^i_l) \in \mathbb{N}^l\}_{i\in\mathbb{Z}}$ and $\boldsymbol{\delta} = \{(\delta^i_1,\ldots,\delta^i_l) \in \{0,1\}^l\}_{i\in\mathbb{Z}}$ such that $\mathbf{n}$ is bounded and at least one between $\mathbf{n}$ and $\boldsymbol{\delta}$ is not a periodic sequence (that is, for every $m \in \mathbb{N}$ there are $i \in \mathbb{Z}$ and $k \in \{1,\ldots,l\}$ such that either $n^i_k \neq n^{i+m}_k$ or $\delta^i_k \neq \delta^{i+m}_k$). Clearly S is an uncountable set (since the set of periodic sequences of integers is countable, while the set of all bounded sequences of integers is not).

Now, let $(\mathbf{n},\boldsymbol{\delta}) \in S$: $x(\cdot;0,p^{\pm}_{\mathbf{n},\delta})$, whose existence is guaranteed by Corollary 6.2.1, is bounded by Proposition 6.2.2 and obviously cannot be periodic, since the sequece of its zeros is not periodic. ∎

We observe that a result on the existence of bounded non-periodic solutions to (6.20) has been proved also in [121].

In order to present our next results, we recall some basic facts on chaos theory and symbolic dynamics. For every $M \in \mathbb{N}$, let $S_M = \{1,2,\ldots,M\}$ and let Σ^M be the set $[S_M]^{\mathbb{Z}}$ of the bi-infinite sequences $\mathbf{s} = (s_i)_{i\in\mathbb{Z}}$, with $s_i \in S_M$; Σ^M is a compact metric space equipped with the distance:

$$d(\mathbf{s},\overline{\mathbf{s}}) = \sum_{i=-\infty}^{+\infty} \frac{1}{2^{|i|}} \frac{|s_i - \overline{s}_i|}{1+|s_i-\overline{s}_i|}$$

and let $\sigma : \Sigma^M \to \Sigma^M$ the Bernoulli shift map defined by:

$$(\sigma(\mathbf{s}))_i = s_{i+1}, \qquad \text{for every } \mathbf{s} \in \Sigma^M$$

(see [127, p. 94–107] for details about σ and Σ_M).
Moreover, we recall the following definition of the topological semiconjugation of maps.

Definition 6.2.4 A map $\Psi : \Omega \to \Omega \subset \mathbb{R}^d$ is *topologically semiconjugated* to the Bernoulli shift if there exist a compact set $A \subset \Omega$ and a continuous surjection $\tau : A \to \Sigma^M$ such that the diagram:

$$\begin{array}{ccc} A & \xrightarrow{\ \tau\ } & \Sigma^M \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\sigma} \\ A & \xrightarrow{\ \tau\ } & \Sigma^M \end{array} \tag{6.23}$$

commutes.

Remark 6.2.5 In the case when the function τ in Definition 6.2.4 is also injective, we say that the maps Ψ and σ are *topologically conjugated.* In this situation Ψ inherits all the chaotic features of the Bernoulli shift and we can actually call it a chaotic map.

Another feature of chaotic maps is the fact that they possess positive *topological entropy.*

Definition 6.2.6 Let $X \subset \mathbb{R}^d$ is a compact set and $f : X \to X$ is a continuous map, then the *topological entropy* of f is given by :

$$\lim_{\varepsilon \to 0^+} \limsup_{m \to +\infty} \frac{\log s(\varepsilon, m)}{m},$$

with:

$$s(\varepsilon, m) = \max\{\text{card}(E) : E \subset X \text{ such that } \forall\, x, y \in E, \text{ with } x \neq y,$$
$$|f^r(x) - f^r(y)| \geq \varepsilon \text{ for some } 0 \leq r \leq m\};$$

on the other hand, if X is not compact, the topological entropy can be defined by:

$$\sup_{R>0} \lim_{\varepsilon \to 0^+} \limsup_{m \to +\infty} \frac{\log s(\varepsilon, m, R)}{m},$$

with:

$$s(\varepsilon, m, R) = \max\{\text{card}(E) : E \subset X \text{ such that } f^r(E) \subset B(O, R)\ \forall\, r = 0, \ldots, m, \text{ and}$$
$$|f^r(x) - f^r(y)| \geq \varepsilon\ \forall\, x, y \in E\, x \neq y, \text{ for some } 0 \leq r \leq m\ \}.$$

In both cases the topological entropy measures the exponential growth of the number of bounded orbits which diverge as successive iterations of f are considered.

In particular, the entropy of the Bernoulli shift on Σ^M is equal to $\log M$. Moreover, any map semiconjugated to the Bernoulli shift has positive entropy; in fact, we have the following result.

Proposition 6.2.7 [105, Prop. 3.11] *If* $\Psi : \Omega \to \Omega$ *is topologically semiconjugated to the Bernoulli shift on* Σ^M, *then the entropy of* Ψ *is at least* $\log M$.

Our aim is to show that the Poincaré map associated to (6.20) is topologically semiconjugated to the Bernoulli shift and so it has positive entropy. To do this, we recall that, for $p \in \mathbb{R}^2$, we denoted by $z(\cdot; 0, p)$ the pair $(x(\cdot; 0, p), \dot{x}(\cdot; 0, p))$, where $x(\cdot; 0, p)$ is the solution of (6.20) satisfying the initial condition $(x(0), \dot{x}(0)) = p$; then, we introduce the Poincaré map Π by setting:

$$\Pi(p) = z(\omega; 0, p)$$

for all those $p \in \mathbb{R}^2$ for which the solution $x(\cdot; 0, p)$ is defined on $[0, \omega]$. Moreover, with n_k^* given in Corollary 6.2.1, let us define:

$$\Omega = \{p \in \mathbb{R}^2 : z(\cdot; 0, p) \text{ is defined on } \mathbb{R} \text{ and } x \text{ has at least } n_k^* \text{ zeros in}$$
$$]t_{2k-1} + i\omega, t_{2k} + i\omega[\,, \text{ for every } k = 1, \ldots, l \text{ and for every } i \in \mathbb{Z}\}. \tag{6.24}$$

Now, for any given natural number N, we can consider the subset $\Omega_N \subset \Omega$ defined by:

$$\Omega_N = \{p \in \mathbb{R}^2 : z(\cdot; 0, p) \text{ is defined on } \mathbb{R} \text{ and } x \text{ has at least } n_k^* \text{ and}$$
$$\text{at most } n_k^* + N \text{ zeros in }]t_{2k-1} + i\omega, t_{2k} + i\omega[\tag{6.25}$$
$$\text{for every } k = 1, \ldots, l \text{ and for every } i \in \mathbb{Z}\}.$$

The following lemma summarizes the properties of the sets Ω and Ω_N.

Lemma 6.2.8 *The following results hold true:*

1. $\cup_{N=0}^{\infty}\Omega_N \subset \Omega$;
2. $\Pi(\Omega) = \Omega$;
3. *for every* $N \in \mathbb{N}$ *we have* $\Pi(\Omega_N) = \Omega_N$;
4. *for every* $N \in \mathbb{N}$ *the set* Ω_N *is nonempty and compact.*

Proof. Statement 1 is trivial. Statements 2 and 3 are easy consequences of the fact that q is ω-periodic. As far as Statement 4 is concerned, the fact that every set Ω_N is nonempty follows from Corollary 6.2.1. Moreover, from an application of Lemma 6.1.5, Statement 2, we immediately deduce that Ω_N is bounded.

Finally, in order to show that it is closed, let us consider a sequence $p_n \in \Omega_N$ such that $p_n \to p_0$, for some $p_0 \in \mathbb{R}^2$. For every $n \in \mathbb{N}$, we set $x_n(\cdot) = x(\cdot; 0, p_n)$ and $x_0 = x(\cdot; 0, p_0)$. According to proof of Proposition 6.2.2, the sequence (x_n) is bounded in $C^2(\mathbb{R})$; then, an application of Ascoli-Arzelà's Theorem, together with a continuous dependence argument, shows that, possibly passing to a subsequence, $x_n \to x_0$ in $C^2_{\text{loc}}(\mathbb{R})$. This means that x_n and $\dot{x}_n$ uniformly converge to x_0 and $\dot{x}_0$, respectively, on every interval of the form $[-j\omega, (j+1)\omega]$, for every $j \in \mathbb{Z}$; this fact guarantees that x_0 is globally defined in $\mathbb{R}$. Now, we need to prove that the number of zeros of x_0 in every interval of the form $]t_{2k-1} + i\omega, t_{2k} + i\omega[$ is greater than n_k^* and less or equal to $n_k^* + N$; to this aim, let us fix an interval $]t_{2k-1} + i\omega, t_{2k} + i\omega[$ and let us denote it by $I_{k,i}$. Recalling Remark 6.1.9, we know that neither x_n nor x_0 may have a zero on the boundary of $I_{k,i}$; moreover, let us denote by $\tau_1 < \tau_2 < \ldots < \tau_K$ the zeros (they are all simple) of x_0 in $I_{k,i}$. Using the continuous dependence of the solutions to (6.20) on initial data, we deduce that, for every neighbourhood V_j of τ_j, $j = 1, \ldots, K$, and for n sufficiently large, x_n has exactly one zero in V_j; this shows that x_n, for n sufficiently large, has K zeros in $I_{k,i}$. Therefore, x_n has the same number of zeros of x_0 in every interval $]t_{2k-1} + i\omega, t_{2k} + i\omega[$, for n sufficiently large; since $p_n \in \Omega_N$, the number of zeros of x_n is greater than n_k^* and less or equal to $n_k^* + N$ and then the same holds true for x_0, that is $p_0 \in \Omega_N$. ∎

The next result proves that the Poincaré map Π is topologically semiconjugated, on the sets Ω_N, to the Bernoulli shift.

Theorem 6.2.9 *For every* $N \in \mathbb{N}$, *the Poincaré map* $\Pi : \Omega_N \to \Omega_N$ *is topologically semiconjugated to the shift map* $\sigma : \Sigma^M \to \Sigma^M$, *with* $M = (2N + 2)^l$.

Proof. If $p \in \Omega$ we can define:

$$\delta_k^i(p) = \text{number of zeros of } x(\cdot; 0, p) \text{ in }]t_{2k-2} + i\omega, t_{2k-1} + i\omega[$$

$$n_k^i(p) = \{\text{number of zeros of } x(\cdot; 0, p) \text{ in }]t_{2k-1} + i\omega, t_{2k} + i\omega[\} - n_k^*,$$

for every $i \in \mathbb{Z}$ and every $k = 1, \ldots, l$. In particular we have associated to every $p \in \Omega$ a bi-infinite sequence of $2l$-tuples of the form $\{(\delta_1^i, n_1^i, \delta_2^i, n_2^i, \ldots, \delta_l^i, n_l^i)\}_{i \in \mathbb{Z}}$, with $\delta_k^i \in \{0, 1\}$ and $n_k^i \in \mathbb{N}$, which describes the precise nodal behaviour of the solution of (6.20) starting from p at time 0. We stress the fact that such an association is surjective thanks to Corollary 6.2.1.

Now, let ξ_N be the map defined by:

$$\xi_N(\delta_1, n_1, \delta_2, n_2, \ldots, \delta_l, n_l) = \sum_{k=1}^{l}(2N+2)^{k-1}[(N+1)\delta_k + n_k] + 1;$$

ξ_N is inspired by the map which converts a number of l figures, expressed in a numeration system with an alphabet of $2N+2$ symbols, into the correspondent number in the decimal system and therefore it is a bijection between the set of all the $2l$-tuples of the form $(\delta_1, n_1, \delta_2, n_2, \ldots, \delta_l, n_l)$, with $\delta_k \in \{0,1\}$ and $n_k \in \{0, \ldots, N\}$, and the set S_M, with $M = (2N+2)^l$. The map $\tau_N : \Omega_N \to \Sigma^M$ defined by:

$$(\tau_N(p))_i = \xi_N(\delta_1^i(p), n_1^i(p), \delta_2^i(p), n_2^i(p), \ldots, \delta_l^i(p), n_l^i(p)), \quad \forall i \in \mathbb{Z},$$

is a topological semiconjugation of $\Pi : \Omega_N \to \Omega_N$ with $\sigma : \Sigma^M \to \Sigma^M$. Indeed, it is easy to see, using Corollary 6.2.1, that τ_N is surjective; moreover, the fact that $\sigma \circ \tau_N = \tau_N \circ \Pi$ on Ω_N is a direct consequence of the definitions of τ_N, Π and σ.

Finally, to show that τ_N is continuous on Ω_N, let us consider $p_n \in \Omega_N$ such that $p_n \to p_0$, for some $p_0 \in \Omega_N$. Arguing as in the proof of Lemma 6.2.8, we deduce that, for n large, the functions $x_n(\cdot) = x(\cdot; 0, p_n)$ and $x_0(\cdot) = x(\cdot; 0, p_0)$ have the same number of zeros on every interval of positivity or negativity of q. This implies that $(\tau_N(p_n))_i = (\tau_N(p_0))_i$, for n large and for every $i \in \mathbb{Z}$, i.e. $\tau_N(p_n) \to \tau_N(p_0)$. ■

From Theorem 6.2.9 and Proposition 6.2.7, we immediately deduce the following:

Corollary 6.2.10 *For every $N \in \mathbb{N}$ the map $\Pi : \Omega_N \to \Omega_N$ has topological entropy greater or equal to $l \log(2N+2)$ and $\Pi : \Omega \to \Omega$ has infinite topological entropy.*

Remark 6.2.11 In the literature several approaches to chaotic dynamics are considered; in particular, in the introduction of [113] a ω-periodic equation is called *chaotic* if the Poincaré map Π, restricted to a compact and invariant set, is semiconjugated to a suitable shift map and, moreover, "the counterimage (by the semiconjugacy) of any periodic point of the shift contains a periodic point of the Poincaré map". If we translate the last condition into our setting, it would mean that, if $\mathrm{s} \in \Sigma^M$ is such that $\sigma^m(\mathrm{s}) = \mathrm{s}$ for some $m \in \mathbb{N}$, then there should be a point $p \in \tau_N^{-1}(\mathrm{s})$ such that the solution $x(\cdot; 0, p)$ is $m\omega$-periodic: in our case this is actually *impossible* for some choices of s. Indeed, let us consider a $2l$-tuple $(\delta_1, n_1, \ldots, \delta_l, n_l)$ such that $\sum_{k=1}^{l}(\delta_k + n_k + n_k^*)$ is an odd integer and let us take the constant sequence $\mathrm{s} = (s_i)_{i\in\mathbb{Z}}$ with $s_i = \xi_N(\delta_1, n_1, \ldots, \delta_l, n_l)$ for all $i \in \mathbb{Z}$, so that $\sigma(\mathrm{s}) = \mathrm{s}$. Then, $\tau_N^{-1}(\mathrm{s})$ cannot contain the initial value of a ω-periodic solution of our equation, otherwise we would have a periodic solution with an odd number of zeros in a period (recall that the zeros of nontrivial solutions of (6.20) are all simple by the uniqueness of the constant solution $x \equiv 0$). However, if we call *admissible* those $2l$-tuples whose entries give an even sum we have proved in Corollary 5.2.3 that there are at least two ω-periodic solutions with nodal behaviour prescribed by every fixed admissible $2l$-tuple: in other words, if we restrict our maps Π, σ and τ_N to the "admissible" sets:

$$\Omega_N^{\mathrm{adm}} = \{p \in \Omega_N : x(\cdot; 0, p) \text{ has an even number of zeros in } [0, \omega)\}$$

and:

$$\Sigma_{\mathrm{adm}}^M = \{\xi_N(\delta_1, n_1, \ldots, \delta_l, n_l) : (\delta_1, n_1, \ldots, \delta_l, n_l) \text{ is an admissible } 2l\text{-tuple}\},$$

then our equation (6.20) turns out to be chaotic in the sense of [113].

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