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Thesis submitted for the degree of "Doctor Philosophiae"

THE MIXED BOUNDARY VALUE PROBLEM FOR ELLIPTIC EQUATIONS WITH CRITICAL NONLINEARITY

Candidate:

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Supervisor:

Prof. Filomena Pacella

Academic Year 1990-91

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Introduction.

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, whose boundary is Lipschitz continuous and is made of two manifolds Γ_0 and Γ_1, with Γ_0 having positive (N-1)-dimensional Hausdorff measure.

We consider in Ω the following equation

$$(0.1) \qquad -\Delta u = u^p + \lambda u \qquad \text{in } \Omega$$

with the mixed boundary conditions

$$(0.2) \qquad \begin{cases} u = 0 & \text{on } \Gamma_0 \\ \partial u/\partial \nu = 0 & \text{on } \Gamma_1 \end{cases}$$

where ν denotes the outer normal to $\partial\Omega$, $p+1 = \frac{2N}{N-2}$ and λ is a real parameter.

It turns out that, despite of its simple form, problem (0.1) has a very rich structure and provides open problems and new ideas. We start recalling some well known facts in the case we have $\partial\Omega = \Gamma_0$ (i.e the Dirichlet problem). The first contribute to this problem was given in a pioneering paper of Brezis and Nirenberg (see [BN]).

In their work they look for solutions to (0.1) minimizing the functional

$$F_\lambda(u) = \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx$$

constrained on the manifold $M = \{u \in H_0^1(\Omega)$ such that $\int_\Omega |u|^{p+1} dx = 1\}$.

The major difficulty in proving that the infimum is achieved comes from the fact that the function $u \rightarrow \|u\|_{p+1}$ is not continuous under weak convergence in $H_0^1(\Omega)$.

This is because p+1 is the critical exponent for the Sobolev imbedding $H_0^1(\Omega) \rightarrow L^{p+1}(\Omega)$ that is continuous but not compact.

We remark that in the subcritical case $p < \frac{N+2}{N-2}$ it is not difficult to see that there is always a solution of (0.1) if $\lambda < \lambda_1(\Omega)$. Here $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary condition. In the critical case the situation is different. In fact, using the Pohozaev identity (see [Po]), it is possible to prove that the infimum

$$S_\lambda(\Omega) = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{p+1}=1}} F_\lambda(u)$$

is never achieved if $\lambda \leq 0$, for every Ω in $\mathbb{R}^N$ (see [BN]). Moreover, if $\lambda \leq 0$ and Ω is starshaped there is no solutions to (0.1). To overcome these difficulties Brezis and Nirenberg consider the case $0 < \lambda < \lambda_1(\Omega)$ and they obtain the following results :

(a) when $N \geq 4$, problem (0.1) has a solution for every $0 < \lambda < \lambda_1(\Omega)$.

(b) when $N = 3$, problem (0.1) has a solution if $\lambda \in (\lambda^*, \lambda_1(\Omega))$, $\lambda^* > 0$ and $\lambda^* = \frac{\lambda_1}{4}$ if Ω is a ball.

As we will show the situation is quite different if we deal with functions which do not belong to $H_0^1(\Omega)$. In fact, the space related to problem (0.1), (0.2), is $V(\Omega) = \{u \in H^1(\Omega) : u \equiv 0 \text{ on } \Gamma_0\}$.

First of all, if we consider the best Sobolev constant for the imbedding $V(\Omega) \to L^{p+1}(\Omega)$, it is easy to see that the infimum

$$S(\Omega) = S_0(\Omega) = \inf_{\substack{u \in V(\Omega) \\ \|u\|_{p+1}=1}} \left[\int_\Omega |\nabla u|^2 dx \right] \tag{0.3}$$

does depend on Ω and that it is not even possible to find a positive constant which bounds from below all the numbers $S(\Omega)$. In fact if we consider, for example, the sector $S(\alpha,R)$ defined in chapter 1 and take the functions $u_\varepsilon(x)$ which appear in the proof of Lemma 1.3 , we see that : $S(\alpha,R)) \to 0$ as the amplitude α tends to zero.

Hence there is a sharp contrast with the Dirichlet case, where

$$S_0(\Omega) = S = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{p+1}=1}} \left[\int_\Omega |\nabla u|^2 dx \right]$$

(best constant for the Sobolev imbedding $H_0^1(\Omega) \to L^{p+1}(\Omega)$)

is not depending on Ω (see [BN]).

Then, a natural question is to obtain a new "Sobolev inequality" , taking Ω in some fixed class of open sets, defined according to some geometric properties of Γ_1. The class that we consider is the class of open sets Ω which exhibit the same isoperimetric constant relative to Γ_1 , $Q(\Gamma_1,\Omega)$ (see Chapter 1) .

Then we give a sufficient condition for the infimum

$$\text{(0.4)} \qquad \inf_{\substack{u \in V(\Omega) \\ \|u\|_{p+1}=1}} \left[\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 dx \right]$$

to be achieved and we present some examples where this condition holds. The main result of this chapter is the following (see [Gr1]).

Theorem 1.1

Let Ω belong to $\mathcal{E}_{\alpha_N}$. Then there exists a constant $\lambda^* < \lambda_1$, $\lambda^* = \lambda^*(\Omega, N)$ such that for all $\lambda \in]\lambda^*, \lambda_1[$ problem (0.1) admits at least one solution . Moreover ,

(i) *if Γ_1 is regular and $N \geq 4$, then $\lambda^* \leq 0$.*

(ii) *for any $k < 0$ and $N \geq 3$ there exists a domain Ω_k such that $\lambda^*(\Omega_k) < k$.*

(here $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ with mixed boundary condition (0.2)).

By ii) we note the difference between our case and the Dirichlet case. In fact, with boundary zero data, we have that the infimum $S_\lambda(\Omega)$ is never achieved for every $\lambda \leq 0$.

In Chapter 2 we consider the case $\lambda = 0$. Since in general it is not possible to establish if the infimum (0.3) is achieved (in Chapter 1 some case are considered where (0.3) is not achieved) we consider domains with some geometrical or topological assumptions. Let us point out that recently a sufficient condition for the minimum (0.4) to be achieved has been derived by Adimurthi - Mancini (see[AM]). Their theorem is the following

Theorem (Adimurthi - Mancini)

Let $\alpha(x)\in L^{\infty}(\Omega)$ *such that* $-\Delta+\alpha$ *is a positive operator on* $H(\Gamma_0)$ *and assume that the following geometrical condition on* Γ_1 *holds :*

there exists an x_0 *in the interior of* Γ_1 *such that in a neighbourhood of* x_0 , Ω *lies on one side of the tangent plane at* x_0 *and the mean curvature with respect to the unit outward at* x_0 *is positive, then the problem*

$$\begin{cases} -\Delta u + \alpha(x)u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \partial u/\partial \nu = 0 & \text{on } \Gamma_1 \end{cases}$$

admits a solution u_0 *such that* $J(u_0) < \frac{S^{N/2}}{2N}$.

(for $N = 2$, *results similar to this theorem appear also in* Adimurthi - Yadava (*see* [AY])).

Therefore nothing can be deduced by the previous theorem about the case of Γ_1 having negative or zero mean curvature. The result of chapter 2 answers also this open problem in the sence that among the domains for which we get an existence theorem (in the case $\lambda = 0$) there are some where the mean curvature on Γ_1 is always negative. A case with zero curvature on Γ_1 is also treated in Chapter 3.

We look for solutions to (0.1) as critical points of the functional

$$J(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 dx - \frac{1}{p+1}\int_{\Omega} (u^+)^{p+1} dx \tag{0.5}$$

It is easy to see that there is a one - to - one correspondence between the critical points of J on $V(\Omega)$ and the critical points of F on the manifold $M=\{u\in V(\Omega)$ such that $\int_{\Omega} |u|^{p+1} dx = 1\}$. More precisely if u is a critical point of F in $V(\Omega)$ then $v = \frac{ku}{\|ku\|_{p+1}}$,

$k = \left(\int_{\Omega} |\nabla u|^2 dx\right)^{(N-2)/4}$ is a critical point of J on M.

In the homogeneous Dirichlet problem Bahri and Coron (see [BC1], [BC2], [Co]) exploit the topology of the domain Ω in order to prove the existence of a positive solution to the equation (0.1).

Their results rely on the fundamental remark that the Palais-Smale condition holds, for the functional J(u) defined in the space $H_0^1(\Omega)$, in certain ranges of energy. This result is a consequence of a careful analysis of how the compactness of the Palais-Smale sequences can be lost in the space $H_0^1(\Omega)$. To this analysis many authors have contributed such as Brezis, Nirenberg, P.L.Lions, Struwe, Bahri, Coron and others. However we refer the reader to the papers [Br2] and [Br4] for more details and for an extensive bibliography.

In this chapter we carry out the same analysis for the Palais-Smale sequences of the functional (0.4) in the space $V(\Omega)$ in order to prove the existence of a positive solution of (0.1) - (0.2) (recall that we are considering the case $\lambda = 0$). Once this has been done then one should look for geometrical assumptions on Ω (and Γ_1 in the mixed boundary problem) which allow to construct Palais-Smale sequences which lie in a range of energy where compactness is restored, as it has been done in the Dirichlet case by Bahri-Coron ([BC1],[BC2], Coron ([Co]), Ding([Di]), Passaseo [Pas]). At the end we emphasize that while in the Dirichlet problem there are geometrical or topological conditions only on the domain Ω, in the mixed boundary problem we need to make geometrical and topological assumptions also on Γ_1.

In Chapter 3 we consider some existence and multiplicity results for domains with symmetries (see [Gr2]). These results are inspired by the fundamental concentration - compactness principle of P.L. Lions. As in [L2] - [L3] we consider open domains in $\mathbb{R}^N$ which are invariant by suitable subgroup G of the orthogonal group *O(N)*. Then when we look for solutions of (0.1) as functions that achieve $S_\lambda(\Omega)$, we can restrict $F_\lambda(u)$ to the subspace consisting of all symmetric functions with respect to G. Precisely we consider

(0.5) $$\tilde{S}_\lambda(\Omega) = \inf_{\substack{u \in V^*(\Omega) \\ \|u\|_{p+1}=1}} F_\lambda(u)$$

where $V^*(\Omega) = \{u \in V(\Omega) \text{ such that } u(x) = u(gx) \quad x \in \bar{\Omega} \text{ and } g \in G\}$.

Using the concentration - compactness principle it is possible to observe that the minimizing sequences for $S_\lambda(\Omega)$ which are not relatively compact, " concentrate" at a point y belonging to $\bar{\Gamma}_1$ (see [LPT] for a precise statement). If instead we consider the infimum (0.5) then we have that if a minimizing sequences u_n concentrates at a point y then u_n concentrates also at every point $z = g \cdot y$ for every $g \in G$ (see Theorem [3.1]).

By this remark we obtain a sufficient condition for the minimizing sequences for (0.5) to be relatively compact. A first application of this theorem is the existence of a solution of (0.1) in the case $\lambda = 0$ on a contractible domain (with suitable boundary condition) having also zero corvature at any point of Γ_1.

A second application is a multiplicity result for a Lipschitz continuous domain Ω_r defined by $\Omega_r = \{x \in \mathbb{R}^2$ such that $r < |x| < r+d\} x\ \omega$, ω bounded domain in $\mathbb{R}^{N-2}$.

More precisely, when $\lambda^*(\Omega_r) < \lambda < \lambda_1(\Omega_r)$, $\lambda^*(\Omega_r) > 0$, we prove that the number of solutions of (0.1) increases as $r \to \infty$. The proof of this last result is inspired by a paper of Yan Yan Li (see [Ly]), where a different group action is used. We conclude pointing out that the results of Chapters 1,2,3, are contained respectively in the papers [Gr1], [GP], and [Gr2].

Chapter 1

EXISTENCE AND NONEXISTENCE THEOREMS IN DEPENDENCE OF THE PARAMETER λ.

1.1 – PRELIMINARIES.

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$. Here we consider the following problem:

$$(1.1) \qquad \begin{cases} -\Delta u = u^p + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \end{cases}$$

where $p = \frac{N+2}{N-2}$, $\lambda \in \mathbb{R}$, $\partial\Omega = \Gamma_0 \cup \Gamma_1$ is Lipschitz continuous and Γ_0 has positive (N-1)-dimensional Hausdorff measure. By ν we denote the outer normal to $\partial\Omega$. The aim of this chapter is to prove Theorem 1.1; for this, we start with some preliminaries.

Let Σ_α be an open cone in $\mathbb{R}^N$, $N \geq 3$, with vertex at the origin and solid angle $\alpha \in]0,\omega_{N-1}[$, where ω_{N-1} is the (N–1)-dimensional Hausdorff measure of the unit sphere S^{N-1}. To be more precise, if A_α is a subset of S^{N-1} with $H_{N-1}(A_\alpha) = \alpha$ then $\Sigma_\alpha = \{\lambda x, x \in A_\alpha \text{ and } \lambda \in]0,+\infty[\}$. We assume that $\partial\Sigma_\alpha$ is Lipschitz continuous and denote by $\Sigma(\alpha,R)$ the open sector in $\mathbb{R}^N$ with solid angle α and radius $R > 0$, that is $\Sigma(\alpha,R) = \Sigma_\alpha \cap B_R$, where B_R is the ball in $\mathbb{R}^N$ with center at the origin and radius R.

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, whose boundary is Lipschitz continuous and is given by the union of the manifolds Γ_0 and Γ_1, Γ_0 having positive (N–1)-dimensional Hausdorff measure. We recall that the "isoperimetric constant of Ω relative to Γ_1", denoted by $Q(\Gamma_1, \Omega)$, is defined by (see [PT])

$$Q(\Gamma_1, \Omega) = \sup_E \frac{|E|^{(N-1)/N}}{P_\Omega(E)} \quad ,$$

where $|E|$ denotes the Lebesgue measure of a set E and the supremum is taken over all measurable subsets E of Ω such that $\partial E \cap \Gamma_0$ does not contain any set of positive (N–1)-dimensional Hausdorff measure. Moreover $P_\Omega(E)$ represents the De Giorgi perimeter of E relative to Ω, that is

$$P_\Omega(E) = \sup \{ | \int_\Omega \operatorname{div} \psi \, dx |,\ \psi \in [C_0^\infty(\Omega)]^N,\ |\psi| \le 1 \}.$$

Some properties of $Q(\Gamma_1, \Omega)$ are shown in [PT], [LPT] and [Gr3]; we just recall here that if $\Sigma(\alpha,R)$ is a convex sector and $\tilde{\Gamma}_0 = \{ x \in \partial\Sigma(\alpha,R),\ |x|=R \}$, $\tilde{\Gamma}_1 = \partial\Sigma(\alpha,R) \setminus \Gamma_0$ then $Q(\tilde{\Gamma}_1, \Sigma(\alpha,R)) = (N\alpha_N^{1/N})^{-1}$, with α_N the measure of the unit sector $\Sigma(\alpha,1)$ (see [PT] for further details). Now let us consider the class $\mathcal{E}_{\alpha_N}$ of domains Ω of the above type such that $Q(\Gamma_1,\Omega) = (N\alpha_N^{1/N})^{-1}$. Of course any convex sector $\Sigma(\alpha,R)$ belongs to the class $\mathcal{E}_{\alpha_N}$ determinate by $\alpha_N = |\Sigma(\alpha,1)|$.

Then we consider the Hilbert space

$$V(\Omega) = \{ u \in H^1(\Omega) : u \equiv 0 \text{ on } \Gamma_0 \}.$$

As a consequence of Theorem 2.1 of [LPT] we have that if $\Omega \in \mathcal{E}_{\alpha_N}$, then a positive constant Σ exists such that

$$(1.2) \qquad \int_\Omega |\nabla u|^2 dx \ge S(\alpha_N) \left(\int_\Omega |u|^{p+1} dx \right)^{2/(p+1)}, \quad \forall u \in V(\Omega),\ p = \frac{N+2}{N-2}.$$

where $S(\alpha_N) = \left[\dfrac{N(\alpha_N)^{1/N}}{B^{\frac{N-2}{2N}}} \right]^2$ and $B = \dfrac{\Gamma(N)}{\Gamma(N/2)\,\Gamma(N/2+1)} (1/2)^{\frac{3N-2}{2}}$ (see [Au], [T]).

So that it makes sense to define

$$(1.3) \qquad S_\lambda(\Omega) = \inf_{\substack{u \in V(\Omega) \\ \|u\|_{p+1}=1}} \left[\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega u^2 \, dx \right].$$

Let us remark that in the case $V(\Omega) \equiv H_0^1(\Omega)$ (i.e. $H_{N-1}(\Gamma_1) = 0$) the constant

$$S(\Omega) = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{p+1}=1}} \int_\Omega |\nabla u|^2 \, dx$$

is precisely $\left(\frac{N C_N^{1/N}}{B^{(N-2)/2N}}\right)$ for any Ω belonging to $\mathbb{R}^N$ (here C_N is the measure of the unit ball in $\mathbb{R}^N$.

Since $S(\Omega)$ does not depend on Ω for the space $H_0^1(\Omega)$ it will be denoted simply by S. This is not the case when $H_{N-1}(\Gamma_1) > 0$, as it is shown in [LPT]. Moreover, again from [LPT], it follows that if $H_{N-1}(\Gamma_1) > 0$, then

$$S_0(\Omega) \leq S_0(A) = \left(\frac{N(C_N/2)^{1/N}}{B^{(N-2)/2N}}\right)^2$$

where A is a half ball in $\mathbb{R}^N$ with radius $R > 0$ and its boundary is made by $\partial A = \tilde{\Gamma}_0 \cup \tilde{\Gamma}_1$, $\tilde{\Gamma}_0 = \{x \in \partial A \, , \, |x| = R\}$, $\tilde{\Gamma}_1 = \partial A \setminus \tilde{\Gamma}_0$.

Lemma 1.1.
There is no solution of problem (1.1) *for* $\lambda \geq \lambda_1$.

Proof.
Let ϕ_1 be the eigenfunction of $-\Delta$ corresponding to λ_1 with $\phi_1 > 0$ on Ω. For a solution u of (1.1) we have :

$$\lambda \int_\Omega u\phi_1 = \int_\Omega (-\Delta u - u^p)\phi_1 < -\int_\Omega \Delta u \phi_1 = \int_\Omega \nabla u \nabla \phi_1 - \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\phi_1 =$$

$$= \int_\Omega \nabla u \cdot \nabla \phi_1 = -\int_\Omega u\Delta\phi_1 + \int_{\partial\Omega} u\frac{\partial \phi_1}{\partial \nu} = -\int_\Omega u\Delta\phi_1 = \lambda_1 \int_\Omega u\phi_1$$

having denoted by ν the outer normal to $\partial\Omega$. Therefore $\lambda < \lambda_1$. ♦

Now we introduce a" concentration constant " $C_{x_0}^\lambda$ " defined by

$$C^{\lambda}_{x_0} = \lim_{\rho\to 0^+} S_\lambda(\Omega_\rho) = \lim_{\rho\to 0^+} [\inf_{\substack{u\in V(\Omega_\rho)\\ \|u\|_{p+1}=1}} \int_{\Omega_\rho} |\nabla u|^2\,dx - \lambda \int_{\Omega_\rho} u^2\,dx] \quad ,$$

where $\Omega_\rho = \Omega \cap B(x_0,\rho)$, $B(x_0,\rho) = \{x\in \mathbb{R}^N : |x - x_0| < \rho\}$ and $V(\Omega_\rho) = \{u\in H^1(\Omega_\rho) : u = 0$ on $\Gamma^\rho_0\}$ with $\Gamma^\rho_0 = \partial\Omega_\rho \cap \Omega$. We prove some properties of $C^{\lambda}_{x_0}$. First of all we consider the number

$$S_\lambda(\Gamma_1) = \inf_{x_0\in\bar{\Gamma}_1} C^{\lambda}_{x_0}$$

and we will proof the following

Proposition 1.1.
$C^{\lambda}_{x_0}$ does not depend on λ and

$$C^{\lambda}_{x_0} = C_{x_0} = \lim_{\rho\to 0} (\inf_{\substack{u\in V(\Omega_\rho)\\ \|u\|_{p+1}=1}} \int_{\Omega_\rho} |\nabla u|^2\,dx) \, .$$

Proof.

If $\lambda \geq 0$, then $\int_{\Omega_\rho} |\nabla u|^2 - \lambda \int_{\Omega_\rho} u^2 \leq \int_{\Omega_\rho} |\nabla u|^2$, so that

$$C^{\lambda}_{x_0} \leq \lim_{\rho\to 0} (\inf_{\substack{u\in V(\Omega_\rho)\\ \|u\|_{p+1}=1}} \int_{\Omega_\rho} |\nabla u|^2) \, .$$

On the other hand we have (see [LPT]) that

$$\int_{\Omega_\rho} |\nabla u|^2 \geq C(\int_{\Omega_\rho} |u|^{p+1})^{2/(p+1)} \geq C|\Omega_\rho|^{2/N} \int_{\Omega_\rho} |u|^2.$$

if $u\in V(\Omega_\rho)$.
Therefore,

$$\int_{\Omega_\rho} |\nabla u|^2 - \lambda \int_{\Omega_\rho} u^2 \geq (1-\lambda C^{-1} |\Omega_\rho|^{2/N}) \int_{\Omega_\rho} |\nabla u|^2 \quad ,$$

which implies

$$C^{\lambda}_{x_0} \geq \lim_{\rho\to 0}\Big(\inf_{\substack{u\in V(\Omega_\rho)\\ \|u\|_{p+1}=1}} \int_{\Omega_\rho} |\nabla u|^2\Big)$$

and our claim is proved in the case $\lambda \geq 0$. If $\lambda < 0$ we can repeat the same proof just reversing the inequalities. ♦

By the previous proposition we may always write C_{x_0} and $S(\Gamma_1)$ instead of $C^{\lambda}_{x_0}$ and $S_\lambda(\Gamma_1)$. Therefore $S(\Gamma_1) = \inf_{x_0\in\bar{\Gamma}_1} C^{\lambda}_{x_0}$. In the next proposition we compute C_{x_0} in the case that x_0 is a regular point of Γ_1.

Proposition 1.2.

Let x_0 *be a regular point of* Γ_1. *Then* $C_{x_0}=\left(\frac{N(C_N/2)^{1/N}}{B^{(N-2)/2N}}\right)^2$.

Proof.

Since x_0 is a regular point we have:

(1.4) $$\lim_{\rho\to 0^+} \inf_{\substack{u\in V(\Omega_\rho)\\ \|u\|_{p+1}=1}} \Big(\int_{\Omega_\rho} |\nabla u|^2\Big) \geq \inf_{\substack{u\in V(A)\\ \|u\|_{p+1}=1}} \Big(\int_{\Omega_\rho} |\nabla u|^2\Big) = \left[\frac{N(C_N/2)^{1/N}}{B^{(N-2)/2N}}\right]^2 = S_0(A)$$

where A is a half ball in $\mathbb{R}^N$.

In fact if the previous inequality is not true, there exist $\varepsilon > 0$, $r_0 > 0$ such that for any $0 <\rho< r_0$ there exists a function $u_\rho \in V(\Omega_\rho)\cap C^\infty(\Omega_\rho)$, with $\|u\|_{p+1}=1$, satisfying

(1.5) $$\int_{\Omega} |\nabla u_\rho(x)|^2 dx < S_0(A) - \varepsilon.$$

Because x_0 is a regular point and ε does not depend on $\rho\in\,]0,r_0[$, (1.5) is not possible. In fact by the definition of regular point it is evidently possible to construct a diffeomorfism H_ρ of class C^1, between Ω_ρ and A_ρ = {the half ball centered in x_0 with radius ρ such that the outer normal vector ν to Γ_1 at x_0 is not contained in $A_\rho\}$ and such that $H(\Gamma_0^\rho) = \partial A_\rho \cap \partial B(x_0,\rho)$ if ρ is sufficiently small.

Then the functions $v_\rho = u_\rho \circ H_\rho$ belong to $V(A_\rho)$ and we have:

(1.6)
$$\frac{\int_{A_\rho} |\nabla v_\rho|^2 dx}{\left(\int_{A_\rho} |v_\rho|^{p+1} dx\right)^{2/(p+1)}} \leq C_\rho \int_{\Omega_\rho} |\nabla u_\rho|^2$$

where the numbers C_ρ depend on the diffeomorfism H_ρ that by the definition of regular point, can be chosen in such a way that $C_\rho \to 1$ as $\rho \to 0$.
Then, choosing ρ sufficiently small, by (1.5) we have:

$$\inf_{\substack{u \in V(A_\rho) \\ \|u\|_{p+1}=1}} \left(\int_{A_\rho} |\nabla u|^2 \right) < S_0(A)$$

which is impossible because, for the invariance by scaling (see [LPT]),

$$\inf_{\substack{u \in V(A_\rho) \\ \|u\|_{p+1}=1}} \left(\int_{A_\rho} |\nabla u|^2 \right) = \inf_{\substack{u \in V(A) \\ \|u\|_{p+1}=1}} \left(\int_{A} |\nabla u|^2 \right. = S_0(A)$$

Now, if we consider the functions $\phi_r(t) \in C_0^\infty([0,+\infty[)$, $\phi_r(t) \geq 0$ such that $\phi_r(t) = 1$ if $0 \leq t \leq \frac{r}{4}$ and $\phi_r(t) = 0$ if $t \geq \frac{r}{2}$ by a standard argument (see, for example, [BN] or [LPT]) it can be proved that for $r, \varepsilon \to 0$

$$\frac{\|\nabla u_\varepsilon^r\|_2^2}{\|u_\varepsilon^r\|_{p+1}^2} \to \left(\frac{N(C_N/2)^{1/N}}{B^{(N-2)/2N}} \right)^2$$

where

$$U_\varepsilon^r(x) = \frac{\varphi_r(|x - x_0|)}{(\varepsilon + |x - x_0|^2)^{(N-2)/2}}$$

Then, $C_{x_0} \leq \left(\frac{N(C_N/2)^{1/N}}{B^{(N-2)/2N}} \right)^2$. The proof is complete. ♦

Remark 1.1

By the definition of $S_\lambda(\Omega)$ and the proofs of the previous propositions we have that for every $\lambda \in \mathbb{R}$ the following holds

$$S_\lambda(\Omega) \leq \inf_{x_0 \in \bar{\Gamma}_1} C_{x_0} = S(\Gamma_1) \leq \left(\frac{N(C_N/2)^{1/N}}{B^{(N-2)/2N}} \right)^2 .$$

1.2. PROOF OF THEOREM 1.1

We start with a lemma which is analogous to Theorem 2.2. of [LPT].

Lemma 1.2

Let u_m *be a minimizing sequence for* $S_\lambda(\Omega)$ *. Then either* u_m *is relatively compact or the weak limit* $u = 0$*. In the latter case there exist a subsequence* u_{m_k} *and a point* $x_0 \in \bar{\Gamma}_1$ *such that*

(1.7) $$|u_{m_k}|^{2N/(N-2)} \rightarrow \delta_{x_0} \quad , \quad |\nabla u_{m_k}|^2 \rightarrow S_\lambda(\Omega)\, \delta_{x_0}$$

weakly in the sense of measures .

Proof.

We distinguish the case $\lambda \geq 0$ and $\lambda < 0$.

Case $\lambda \geq 0$.

Then $S_\lambda(\Omega) \leq S_0(\Omega)$. If $S_\lambda(\Omega) < S_0(\Omega)$ then repeating the same proof as in Lemma 1.2 of [BN] we first deduce that $u \neq 0$ and then that $S_\lambda(\Omega)$ achieves a minimum at u. On the other side, if $S_\lambda(\Omega) = S_0(\Omega)$, then u_m is minimizing for $S_0(\Omega)$. Hence we can repeat the same proof as in [LPT], Theorem (2.2), to obtain (1.7) for a point $x_0 \in \bar{\Gamma}_1$.

Case $\lambda \leq 0$.

In this case the statement follows from an analogous theorem of P.L. Lions for minimizing sequences in $H^1(\Omega)$ (see [L2], Section 4 and [L1], [L3] for further details). ♦

Now we are the following proposition

Proposition 1.3
If $S_\lambda(\Omega) < S_\lambda(\Gamma_1)$, *then the infimum* (1.3) *is achieved.*

Proof

Let u_m be a minimizing sequence for $S_\lambda(\Omega)$ and u its weak limit in $V(\Omega)$. Thus, either u_m is relatively compact and consequently the infimum is achieved, or there exists $x_0 \in \overline{\Gamma}_1$ for which (1.7) holds (up to a subsequence that we still denote by u_m). Then, we consider the functions

$$\tilde{u}^r_m(x) = u_m(x)\phi_r(\,|x-x_0|\,) \ ,$$

where ϕ_r are the same functions considered in Proposition 1.2.

We have that $\tilde{u}^r_m(x)$ satisfy

$$(1.8) \qquad \int|\tilde{u}^r_m(x)|^{p+1} \to 1 \quad \text{and} \quad \int|\nabla\tilde{u}^r_m(x)|^2 - \lambda \int|\,\tilde{u}^r_m(x)|^2 \to S_\lambda(\Omega)\ .$$

for $m \to \infty$ and for every $r > 0$. On the other hand by the definition of $S(\Gamma_1)$ we have

$$\lim_{r\to 0}\ (\ \int|\nabla\tilde{u}^r_m(x)|^2 - \lambda \int|\,\tilde{u}^r_m(x)|^2\,) \geq S(\Gamma_1)$$

which contradicts (1.8) since $S_\lambda(\Omega) < S(\Gamma_1)$. Hence $u_m(x)$ is relatively compact. ♦

Remark 1.2.

From Proposition 1.3 it follows immediately the existence of a solution to problem (1.1) for those domains Ω for which $S_\lambda(\Omega) < S(\Gamma_1)$. In fact, let u be a function which realizes $S_\lambda(\Omega)$. We have

$$\|u\|_{p+1} = 1 \ \text{ and } \int_\Omega |\nabla u|^2\, dx - \lambda \int_\Omega u^2\, dx = S_\lambda(\Omega)\ .$$

Of course we can assume that $u \geq 0$ on Ω (otherwise we replace u by $|u|$). Since u is a minimum on the constraint $\{\|u\|_{p+1}=1\ \}$, a Lagrange multiplier $\mu \in \mathbb{R}^N$ exists such that

$$-\Delta u = \lambda u + \mu u^p \quad \text{on } \Omega\ .$$

In fact, $\mu = S_\lambda(\Omega)$ and $S_\lambda(\Omega) > 0$ since $\lambda < \lambda_1$. Then the function $S_\lambda^{1/(p+1)}u$ satisfies (1.1) since $u > 0$ in Ω by the strong maximum principle. In fact since u solves the differential inequality $-\Delta u - \lambda u \geq 0$ then by the classical maximum principle, $u > 0$ in Ω (even if $\lambda < 0$).

Our next purpose is to give a bound on the numbers λ for which a solution to (1.1) exists by using Proposition 1.3. We start with the following lemma .

Lemma 1.3

Let Γ_1 *be regular and* $N \geq 4$. *Then* $S_\lambda(\Omega) < S(\Gamma_1)$ *for any* $\lambda \in]0,\lambda_1[$.

Proof.

As in the proof of Lemma 1.1 of [BN], we use some test functions to prove that $S_\lambda(\Omega) < S(\Gamma_1)$ for any $\lambda \in]0,\lambda_1[$. Since Γ_1 is regular we have $S(\Gamma_1) = \left(\frac{N(C_N/2)^{1/N}}{B^{(N-2)/2N}}\right)^2$.

For $x_0 \in \Gamma_1$ let

$$U_\varepsilon(x) = \frac{\varphi(|x - x_0|)}{(\varepsilon + |x - x_0|^2)^{(N-2)/2}}$$

where $\phi(t) \in C_0^\infty([0,+\infty[\,)$, $\phi(t) \geq 0$, with $\phi(t) = 1$, if $0 \leq t \leq \frac{r}{4}$ and $\phi(t) = 0$, if $t \geq \frac{r}{2}$ where r is chosen in such that $B(x_0, \frac{r}{2}) \subset \Omega$. We claim that the following estimates hold

(i) $$\|\nabla u_\varepsilon\|_2^2 = \frac{k_1}{\varepsilon^{(N-2)/2}} + O(1)$$

(ii) $$\|u_\varepsilon\|_{p+1}^2 = \frac{k_2}{\varepsilon^{(N-2)/2}} + O(1)$$

(iii) $$\|u_\varepsilon\|_2^2 = \begin{cases} \dfrac{k_3}{\varepsilon^{(N-2)/2}} + O(1) & \text{if } N \geq 5 \\ k_4|\log\varepsilon| + O(1) & \text{if } N = 4 \end{cases}$$

where $k_1,\ldots,k_4$ denote positive constants depending only on N and such that $\frac{k_1}{k_2} = S(\Gamma_1)$. Let us prove (i) :

$$\|\nabla u_\varepsilon\|_2^2 = \int_\Omega \frac{|\nabla\varphi(|x-x_0|)|^2}{(\varepsilon+|x-x_0|^2)^{(N-2)}}dx - 2(N-2)\int_\Omega \frac{\nabla\varphi(|x-x_0|)\cdot(x-x_0)}{(\varepsilon+|x-x_0|^2)^{(N-1)}}dx +$$

$$+ (N-2)^2\int_\Omega \frac{\varphi^2(|x-x_0|^2)\cdot|x-x_0|^2}{(\varepsilon+|x-x_0|^2)^N}dx = (N-2)^2\int_\Omega \frac{\varphi^2(|x-x_0|)\cdot|x-x_0|^2}{(\varepsilon+|x-x_0|^2)^N}dx + O(1) =$$

$$(N-2)^2\int_\Omega \frac{|x-x_0|^2}{(\varepsilon+|x-x_0|^2)^N}dx + O(1) = (N-2)^2\frac{1}{\varepsilon^{(N-2)/2}}\int_{(\Omega-x_0)/\sqrt{\varepsilon}} \frac{|y|^2}{(1+|y|^2)^N}dx + O(1)$$

It follows that, being Γ_1 regular

$$\|\nabla u_\varepsilon\|_2^2 = (N-2)^2\frac{1}{\varepsilon^{(N-2)/2}}\int_{A(\infty)} \frac{|x-x_0|^2}{(1+|x-x_0|^2)^N}dx + O(1) = \frac{k_1}{\varepsilon^{(N-2)/2}} + O(1)$$

where $A(\infty)$ is the half-space tangent to Ω in x_0.

In a similar way we can prove (ii) and (iii). Then, we have for $N \geq 5$

$$S_\lambda(\Omega) \leq \frac{\|\nabla u_\varepsilon\|_2^2 - \|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{p+1}^2} = \frac{k_1 - \lambda k_3\varepsilon + O(\varepsilon^{(N-2)/2})}{k_2 + O(\varepsilon^{(N-2)/2})} = \frac{k_1}{k_2} - \lambda\frac{k_3}{k_2}\varepsilon + O(\varepsilon^{(N-2)/2}) < S(\Gamma_1)$$

for $\varepsilon > 0$ small enough . In the same way we get the assertion for the case $N = 4$. ♦

Lemma 1.4

If $\Omega \in \mathcal{E}_{\alpha_N}$ *and* $S_0(\Omega) < S(\Gamma_1)$, *then* $\forall\lambda < 0$ *there exists* $\mu > 0$ *such that* $S_\lambda(\mu\Omega) < S(\Gamma_1)$.

Proof.

Let $\mu\Omega = \{y \in \mathbb{R}^N,\ y=\mu x,\ x \in \Omega\}$, $\mu > 0$. By the very definition $S(\Gamma_1)$ is the same for Ω and $\mu\Omega$. Since $S_0(\Omega) < S(\Gamma_1)$, by Proposition 1.3 the infimum $S_0(\Omega)$ is achieved at a function $v \in V(\Omega)$ such that

$$\frac{\int_\Omega |\nabla v(x)|^2 dx}{\left(\int_\Omega |v(x)|^{p+1} dx\right)^{2/(p+1)}} = S_0(\Omega) < S(\Gamma_1).$$

Thus, for the function $u(x) = v(\frac{x}{\mu}) \in V(\mu\Omega)$ we have

$$\frac{\int_{\mu\Omega} |\nabla u(x)|^2 dx - \lambda \int_{\mu\Omega} u(x)^2}{\left(\int_{\mu\Omega} |u(x)|^{p+1} dx \right)^{2/(p+1)}} \leq S_0(\Omega) - \lambda\mu^2 |\Omega|^{2/n}$$

using Hölder's inequality. Then the assertion follows by choosing $\mu^2 < \frac{S(\Gamma_1)-S_0(\Omega)}{-\lambda|\Omega|^{2/N}}$. ♦

Proof of Theorem 1.1.

Since $\Omega \in \mathcal{E}_{\alpha_N}$, $S(\Gamma_1) \geq S_0(\Omega)$ in virtue of definition of $S(\Gamma_1)$. Moreover $S_{\lambda_1}(\Omega) = 0$ and $S_\lambda(\Omega) > S_{\lambda_1}(\Omega)$ for $\lambda < \lambda_1$. Therefore there exists a constant $\lambda^* < \lambda_1$ such that $S_{\lambda^*}(\Omega) < S(\Gamma_1)$ for $\lambda \in]\, \lambda^*, \lambda_1[$ (note that $S_\lambda(\Omega)$ is a continuous function with respect to λ). Thus,by Proposition 1.3 and Remark 1.2 the first assertion of Theorem 1.1 follows. Obviously (i) is a consequence of Lemma 1.3 and Proposition 1.3.

Let us prove (ii). We consider the domain $\Omega = \Omega(h) = \{x \in \mathbb{R}^N,\ 1 < |x| < h\}$, where $N \geq 3$, $h > 1$. We set $\Gamma_0^h = \{x \in \partial\Omega(h) : |x| = 1\}$, $\Gamma_1^h = \partial\Omega(h) \setminus \Gamma_0$. We consider the functions: $U(x) = |x|^q - 1$ where $q > 0$ will be fixed later. It is clear that $u \in V(\Omega)$ for any $q > 0$. Then

$$S_0(\Omega) \leq \frac{\int_\Omega |\nabla U(x)|^2 dx}{\left(\int_\Omega |U(x)|^{p+1} dx \right)^{2/(p+1)}} = \frac{\int_\Omega q^2 |x|^{2q-2} dx}{\left(\int_\Omega (|x|^q - 1)^{p+1} dx \right)^{2/(p+1)}} = K \frac{q^2 (h^{2q+n-2} - 1)}{(2q+N-2)\left(\int_1^h (t^q-1)^{p+1} t^{N-1} \right)^{2/(p+1)}},$$

where K is a positive constant. From this we have

$$(1.9) \qquad \lim_{h\to\infty} \frac{\int_\Omega |\nabla U(x)|^2 dx}{\left(\int_\Omega |U(x)|^{p+1} dx \right)^{2/(p+1)}} = K \frac{q^2}{2q+N-2} \left[\frac{N}{N-2} (2q+N-2) \right]^{(N-2)/N} < \left(\frac{N\, (C_N/2)^{1/N}}{B^{(N-2)/2N}} \right)^2$$

if we take q sufficiently small. Note that $S(\Gamma_1^h) = \left(\frac{N\,(C_N/2)^{1/N}}{B^{(N-2)/2N}}\right)^2$. Hence $S(\Gamma_1^h)$ does not depend on h, as Γ_1 is regular. Therefore, from (1.9) we get that $S_0(\Omega_h) < S(\Gamma_1)$ for h sufficiently large. Applying Lemma 1.4 and Proposition 1.3 we get (ii). ♦

Remark 1.3

If Ω is a sector $\Sigma(\alpha,R)\subset\mathbb{R}^N$, $N\geq 4$ with $\Gamma_0 = \{x\in\partial\Sigma(\alpha,\ R),\ |x| = R\}$ and $\Gamma_1 = \partial\Sigma(\alpha,R)\setminus\Gamma_0$ we have again (by repeating the proof of Lemma 1.3) that there exists a solution to (1.1) for $0<\lambda<\lambda_1$ and $N\geq 4$. Furthermore in this case there is no solution to (1.1) when $\lambda\leq 0$. In fact, by Pohozaev's identity (see [Po]) for a solution u to (1.1) we would have

$$(1.10)\qquad \lambda\int_\Omega u^2-\lambda\int_{\Gamma_1}(x\cdot\nu)u^2+\frac{N-2}{2N}\int_{\Gamma_1}(x\cdot\nu)u^{2N/N-2}=-\frac{1}{2}\int_{\Gamma_0}(x\cdot\nu)(\partial u/\partial\nu)^2 d\sigma+\frac{1}{2}\int_{\Gamma_1}(x\cdot\nu)|\nabla u|^2$$

Since $x\cdot\nu = 0$ on Γ_1 and $x\cdot\nu > 0$ on Γ_0, (1.10) is not possible. ♦

Remark 1.4

The domains $\Omega(h)$ considered in the proof of Theorem 1.1 are not the only ones for which it is possible to prove that $S_0(\Omega) < S(\Gamma_1)$. Another example can be given by the domains

$$C_h = \{x\equiv(x',y)\in\mathbb{R}^{N-1}\times\mathbb{R}\ ,\ |x'|<1 \ \text{and}\ 0<y<h\},$$

with $\Gamma_0 = \{(x',y)\in\partial C_h\ ,\ y = h\}$.

Then, by using the function $u(x',y) = y$ as before it is possible to prove that

$$S_0(\Omega_h)\to 0 \ \text{as}\ h\to+\infty .$$

Remark 1.5

The case $N = 3$ presents the same difficulties appearing in the study of the Dirichlet problem, even for $\lambda > 0$, (see [BN]). Moreover, if Ω is a particular convex sector $\Sigma(\alpha,R)$, we can use a symmetry theorem of Berestycki and Pacella ([BP]) to claim that all positive solutions to (1.1) are radial functions. Then, by repeating the previous arguments for the case of the ball it is possible to prove that $\lambda^*(\Sigma(\alpha,R)) = \frac{\lambda_1}{4}$ and the result is optimal.

Proposition 1.4

Let us suppose that $\partial\Omega$ *is smooth. Then there exists* $\varepsilon > 0$ *such that* $S_0(\Omega)$ *is achieved whenever* $\mathrm{diam}(\Gamma_0) < \varepsilon$.

Proof.

Suppose that the diameter of Γ_0 is smaller than a positive number ε. Thus Γ_0 is contained in a ball B_ε with radius ε. We recall the definition of capacity of E respect to a set Ω

$$\mathrm{cap}(E,\Omega) = \inf_{\substack{u \in H_0^1(\Omega) \\ u \geq 1 \text{ on } E}} \int_\Omega |\nabla u|^2 \, dx$$

Then the capacity of B_ε with respect to a ball B_R with fixed radius $R > \varepsilon$, $\mathrm{cap}\,(B_\varepsilon, B_R)$, tends to zero as $\varepsilon \to 0$ (see [Ma], section 2.2). Therefore, if ε is sufficiently small, by the definition of capacity it is possible to construct a function $u \in H_0^1(B_R)$ which is identically 1 in a neighborhood of B_ε and such that $\int_{B_R} |Du| \, dx$ is very small. By considering the function $v = 1 - u$ and extending it to 1 in $\Omega \setminus B_R$ we can define a new function $\phi \in V(\Omega)$ which coincides with v in $B_R \cap \Omega$. Consequently $\int_\Omega |D\phi| \, dx$ is very small, which implies that $S_0(\Omega) < S_0(C_n/2) = S(\Gamma_1)$ and hence by Proposition 1.3 the infimum is achieved. ♦

The last proposition allows to construct many domains Ω for which $S_0(\Omega)$ is achieved. It also shows that if the mixed boundary problem (1.1) is "close" to a homogeneous Neumann problem then a positive solution of (1.1) exists with energy, obviously, very small.

Let us observe that for the domains Ω satisfying the hypotheses of the previous proposition the isoperimetric constant $Q(\Gamma_1,\Omega)$ is very large. Therefore, even if not mentioned explicitly, the sufficient condition given in Proposition 1.4 is again a condition on the isoperimetric properties of Ω with respect to Γ_1.

Finally we would like to point out that the previous propositions have been stated for smooth domains Ω only for simplicity. Similar results apply in the case of open sets with Lipschitz continuous boundary after suitable modifications.

Remark 1.6

Another result which empathizes the deep connection between the "relative" isoperimetric properties of Ω and the solutions of mixed boundary problems has been obtained in [BP]. The result obtained there is of different nature since it does not deal with an existence problem but gives a description of the geometrical structure of positive solutions of semilinear elliptic equations in the general from $\Delta u + f(u) = 0$ with mixed boundary conditions of (0.2) type. In particular it is shown that all positive solutions of such equations in some convex sectors are spherically symmetric.

Remark 1.7

The question whether there are domains Ω such that $S_\lambda(\Omega)$ is achieved for all $\lambda < \lambda_1$ is different from showing the existence of domains Ω for which a solution to (1.1) exists for all $\lambda < \lambda_1$. In fact the last question has a positive answer as it can be easily shown by taking $\Omega = \{x \in \mathbb{R}^N : z < |x| < R\} \cap \Sigma(\alpha,R)$, $\Gamma_0 = \{x \in \partial\Omega : |x| = z \text{ or } |x| = R\}$, $0 < z < R$ and by using the result of Brezis and Nirenberg on the existence of positive radial solutions for the analogous Dirichlet problem in the annulus (see [BN]).

We end this chapter exhibiting some sufficient conditions for which the infimum (1.3) is not achieved for $\lambda = 0$. This results are contained in [EPT].

Theorem 1.2

Let $\Omega \in \mathcal{E}_{\alpha N}$. If $S_0(\Omega) = S_0(\alpha_N)$, *then* $S_0(\Omega)$ *is not achieved.*

Proof

In [EPT] have been proved the inequality

$$\|\nabla u\|_2^2 \geq S_0(\alpha_N)\|u\|_{p+1} + \lambda\|u\|_s^2 \tag{1.11}$$

which holds for a positive constant $\lambda = \lambda(\alpha_N,s)$ depending on the class $\mathcal{E}_{\alpha N}$. Because of the hypothesis we can replace $S_0(\alpha_N)$ by $S_0(\Omega)$ in (1.11). Thus, if $S_0(\Omega)$ were achieved by a function $v \in V(\Omega)$, we would have

$$S_0(\Omega)\Big(\int_\Omega |v|^{p+1}dx\Big)^{2/(p+1)} = \int_\Omega |\nabla v|^2 dx$$

Combining this equality with (1.11) we would get $\|v\|_s = 0$. But this is a contradiction since v gives the minimum $S_0(\Omega)$. ♦

One way of checking the condition $S_0(\Omega) = S_0(\alpha_N)$, $\Omega \in \mathcal{E}_{\alpha N}$, is to investigate the isoperimetric constant $Q(\Gamma_1,\Omega)$. As an example we have the following

Corollary 1.1
If $Q(\Gamma_1,\Omega) = (N(C_{N/2})^{1/N})^{-1}$ *(i.e.* Ω *belongs to the class* $\mathcal{E}C_{N/2}$ *) then* $S_0(\Omega)$ *is not achieved.*

Proof
As a consequence of Theorem 2.1 of [LPT] we get $S_0(\Omega) \geq S_0(C_{N/2}) = S_0(\Sigma(\pi,R))$. On the other hand, repeating the Lemma 1.3 for the case $\lambda = 0$ we have that $S_0(\Omega) \leq S_0(C_N/2)$. Thus $S_0(\Omega) = S_0(C_{N/2})$ and Theorem 1.2 applies. ♦

Remark 1.8
By the Corollary 1.1 the final part of the proof of the Theorem 1.1 would not work if the role of Γ_1 and Γ_0 would be reversed for the domains $\Omega(h)$ considered. Actually, if $\Gamma_1 = \{x \in \mathbb{R}^n , |x| = 1\}$ it is shown in [PT] that $Q(\Gamma_1,\Omega) = (N(C_{N/2})^{1/N})^{-1}$ and then by Corollary 1.1 we have that the infimum S_0 is not achieved. The aim of the next chapter is proving the existence of a solution also for this class of domains.

Chapter 2

EXISTENCE THEOREMS UNDER GEOMETRICAL ASSUMPTIONS.

2.1 - PRELIMINARIES.

For an open set $G \subseteq \mathbb{R}^N$ we denote by $D^1(G)$ the Sobolev space $D^1(G) = \{u \in L^{p+1}(G)$ such that $|\nabla u| \in L^2(G)\}$, where $p+1 = \frac{2N}{N-2}$. If G has finite measure then $D^1(G) = H^1(G)$, otherwise $H^1(G) \subseteq D^1(G)$.

Let us denote by Σ_α an open cone in $\mathbb{R}^N$, $N \geq 3$, with vertex at the origin and solid angle α.

Theorem 2.1

If Σ_α *is a convex cone then for any* $\mu > 0$ *the functions* $u(x) = \frac{\mu^{(N-2)/2}}{(|x|^2+\mu^2)^{(N-2)/2}}$ *are the only solutions of the problem*

$$(2.1) \qquad \begin{cases} -\Delta u = u^{(N+2)/(N-2)} & \text{in } \Sigma_\alpha \\ u > 0 & \text{in } \Sigma_\alpha \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Sigma_\alpha \\ u \in D^1(\Sigma_\alpha) & \end{cases}$$

where ν *denotes the outer normal to* $\partial\Sigma_\alpha$.

Proof

It is contained in the proof of Theorem 2.4. of [LPT]. ♦

Now we recall some nonexistence results which will be used later.

Theorem 2.2

i) *Let* A *be the half space*: $A=\{x=(x_1,...,x_N)\in \mathbb{R}^N, x_1>0\}$. *If* u *is a solution of*

$$(2.1)' \qquad \begin{cases} -\Delta u = u^{(N+2)/(N-2)} & \text{in } A \\ u = 0 & \text{on } \partial A \\ u\in D^1(A) \end{cases}$$

then u *is identically zero in* A.

ii) *Let* $\Sigma_{\pi/2}$ *be the cone*: $\Sigma_{\pi/2} = \{x\in \mathbb{R}^N,\ x=(\rho,\Theta_1,..,\Theta_{N-1})\ \Theta_i\in\,]0,\pi[,\ \ i=1,...,N-2, \Theta_{N-1}\in\,]0,\pi/2[,\ |x|=\rho>0\}$ *where* $(\rho,\Theta_1,...,\Theta_{N-1})$ *are the polar coordinates in* $\mathbb{R}^N$. *If* u *is a solution of*

$$(2.1)'' \qquad \begin{cases} -\Delta u = u^{(N+2)/(N-2)} & \text{in } \Sigma_{\pi/2} \\ u = 0 & \text{on } \Gamma_0=\{x\in\partial\Sigma_{\pi/2} \text{ such that } \Theta_{N-1}=0\} \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1=\{x\in\partial\Sigma_{\pi/2} \text{ such that } \Theta_{N-1}=\frac{\pi}{2}\} \\ u\in D^1(\Sigma_{\pi/2}) \end{cases}$$

where ν *denotes the outer normal to* $\partial\Sigma_{\pi/2}$, *then u is identically zero in* $\Sigma_{\pi/2}$.

Proof

The proof of i) is given in [Gi] if $u > 0$ and in [EL] in the general case (actually in [EL] much more general non existence results are proved).

The assertion ii) is easily reduced to i) by observing that the Neumann condition on Γ_1 allows to extend the solution u to the whole half space A, by reflection with respect to Γ_1. ♦

2.2 - ANALYSIS OF PALAIS-SMALE SEQUENCES

Let Ω be a open set of the type considered in the previous section, belonging to a certain class $\mathcal{E}_{\alpha_N}$. From now on we will make the assumption that either $\partial\Omega$ is regular (of class C^1) and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ or Γ_0 and Γ_1 are regular and if $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 \neq \emptyset$ then Γ_0 and Γ_1 intersect orthogonally.

We start with a lemma

Lemma 2.1

For any $\alpha > 0$ *let us consider the set* $\Omega_\alpha = \alpha\Omega$. *Then there exists a family of operators* $\{P_\alpha\}$ *such that*

$$P_\alpha : V(\Omega_\alpha) \longrightarrow D^1(\mathbb{R}^N)$$

where $V(\Omega_\alpha) = \{u \in H^1(\Omega_\alpha)$ *such that* $u = 0$ on $\alpha\Gamma_0\}$ *and*

i) $$P_\alpha(u(x)) = u(x) \qquad \forall x \in \Omega_\alpha$$

ii) $$\int_{\mathbb{R}^N} |\nabla P_\alpha u|^2 dx \leq C \int_{\Omega_\alpha} |\nabla u|^2 dx$$

iii) $$\int_{\mathbb{R}^N} |P_\alpha u|^q dx \leq C \int_{\Omega_\alpha} |u|^q dx$$

where C *is a constant which does not depend on* α.

Proof

The existence of the operators P_α for each $\alpha > 0$, is a classical statement for which we refer, for example, to [Br1], Theorem IX.7 when $\partial\Omega$ is of class C^1 or to [Ma], Section 1.1 if $\partial\Omega$ is only Lipschitz continuous. Here we want to point out that it is possible to choose the operators P_α in such a way that, in the inequalities ii) and iii) the constant C is independent of α.

To do this let us fix $\alpha=1$ and consider the operator $P_1 : V(\Omega) \longrightarrow D^1(\mathbb{R}^N)$ with the properties

i) $P_1(u(x)) = u(x) \qquad \forall x \in \Omega$

ii) $$\int_{\mathbb{R}^N} |\nabla P_1 u|^2 dx \le K \int_{\Omega} |\nabla u|^2 dx$$

iii) $$\int_{\mathbb{R}^N} |P_1 u|^q dx \le K \int_{\Omega} |u|^q dx$$

where K is a positive constant dependent on Ω. Now, if $v(x) \in V(\Omega_\alpha)$, the function $v_\alpha(x)=v(\alpha x)$ belongs to $V(\Omega)$ and, for any $\alpha > 0$, we can define the operator

$$P_\alpha(v)(x)= P_1(v_\alpha)(\frac{x}{\alpha}).$$

Consequently if $x \in \Omega_\alpha$, $\frac{x}{\alpha} \in \Omega$ and $P_1(v_\alpha)(\frac{x}{\alpha}) = v_\alpha(\frac{x}{\alpha}) = v(x) = P_\alpha(v)(x)$.

Moreover $$\int_{\mathbb{R}^N} |\nabla P_\alpha v|^2 dx = \frac{1}{\alpha^2} \int_{\mathbb{R}^N} |\nabla P_1(v_\alpha)(\frac{x}{\alpha})|^2 dx = \alpha^{N-2} \int_{\mathbb{R}^N} |\nabla P_1(v_\alpha)(y)|^2 dy \le$$

$$\le K \alpha^{N-2} \int_{\Omega} |\nabla v_\alpha(y)|^2 dy = K \int_{\Omega_\alpha} |\nabla v(x)|^2 dx$$

In this way we have obtained ii) with C=K and similarly one gets iii). ♦

Now let us consider the functional

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\Omega} (u^+)^{p+1} dx \qquad \forall u \in V(\Omega)$$

where $u^+=\max(u,0)$.

We say that J(u) satisfies the Palais-Smale condition at the level $c \in \mathbb{R}$ if

(2.2) $\begin{cases} \text{every sequence } \{u_n\} \text{ in } V(\Omega) \text{ such that } F(u_n) \to c \text{ and } F'(u_n) \to 0 \\ \text{in the dual space } [V(\Omega)]^* \text{ is relativaly compact in } V(\Omega). \end{cases}$

As in the previous chapter we consider the constant Σ and S defined as

$$(2.3) \qquad \Sigma = \inf_{\substack{u \in D^1(A) \\ u \neq 0}} \frac{\int_A |\nabla u|^2 dx}{\left(\int_A |u|^{p+1} dx\right)^{2/(p+1)}} \qquad A=\{x=(x_1,\dots,x_N)\in \mathbb{R}^N, x_1 > 0\}$$

$$(2.4) \qquad S = \inf_{\substack{u \in D^1(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{p+1} dx\right)^{2/(p+1)}}$$

Lemma 2.2.

The Palais-Smale condition for the functional F fails at the levels

$$c = J(u_0) + \frac{h}{N}\Sigma^{N/2}$$

where h is a positive integer and u_0 is a solution of

$$(2.5) \qquad \begin{cases} -\Delta u = u^{(N+2)/(N-2)} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \end{cases}$$

Proof

Let us start by proving that if u_0 is a solution of (2.5) then the Palais-Smale condition fails at the level $c = J(u_0) + \frac{1}{N}\Sigma^{N/2}$.

To this aim let us consider the sequence:

$$(2.6) \qquad u_n(x) = u_0 + \Psi(x)U_{\varepsilon_n}(x), \quad x \in \Omega$$

with

(2.7) $$U_{\varepsilon_n}(x) = \frac{\varepsilon_n^{(N-2)/2}}{(\varepsilon_n + |x-x_0|^2)^{(N-2)/2}}, \quad x \in \Omega$$

where $x_0 \in \Gamma_1$, $\varepsilon_n \to 0$ and $\Psi(x) \in C_0^\infty(B(x_0,\rho))$, $\Psi(x) \geq 0$, $\Psi(x) \equiv 1$ in $B(x_0,\rho/2)$, and $\rho \geq 0$ is sufficiently small in such a way that $u_n(x) = 0$ for $x \in \Gamma_0$.

By making the same computations as in [BC1] (see also [LPT] or [Gr1]) it is possible to prove that

(2.8) $$J(\Psi(x)U_{\varepsilon_n}(x)) \to \frac{1}{N}\Sigma^{N/2}$$

because $x_0 \in \Gamma_1$ and Γ_1 is smooth. It is also easy to see that ([BN])

(2.9) $$\begin{cases} \Psi(x)U_{\varepsilon_n}(x) \to 0 & \text{weakly in } V(\Omega) \\ \Psi(x)U_{\varepsilon_n}(x) \to 0 & \text{strongly in } L^S(\Omega) \text{ for any } s < p+1 \end{cases}$$

Moreover we have that

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla u_0|^2 dx + \int_\Omega |\nabla(\Psi U_{\varepsilon_n})|^2 dx + o(1)$$

because of (2.9), and

(2.10) $$\int_\Omega |u_n|^{p+1} dx = \int_\Omega |u_0|^{p+1} dx + \int_\Omega |\Psi U_{\varepsilon_n}|^{p+1} dx + o(1)$$

because of a result of [BL] (see also [BN]).

Therefore $$J(u_n) \to J(u_0) + \frac{1}{N}\Sigma^{N/2}.$$

Let us prove that $J'(u_n) \to 0$ in $[V(\Omega)]^*$.

We have :

(2.11) $$J'(u_n) = -\Delta u_n - (u_n^+)^p = -\Delta u_0 - \Delta(\Psi U_{\varepsilon_n}) - (u_0+\Psi U_{\varepsilon_n})^p = u_0^p + + (\Psi U_{\varepsilon_n})^p - (u_0+\Psi U_{\varepsilon_n})^p + \phi_n$$

where $\phi_n \to 0$ in $[V(\Omega)]^*$, because u_0 is a solution of (2.5) and $J'(\Psi u_n) \to 0$ in $[V(\Omega)]^*$ by standard computations ([BC1]). Now

(2.12) $$(u_0+\Psi U_{\varepsilon_n})^p = u_0^p + \zeta_n$$

where $\zeta_n \to 0$ in $L^{(2N)/(N-2)}$ and hence in $[V(\Omega)]^*$, since $L^{(2N)/(N-2)} \subset [V(\Omega)]^*$; (2.12) follows from the inequality

$$|(u_0+\Psi U_{\varepsilon_n})^p - u_0^p - (\Psi U_{\varepsilon_n})^p| \le K\left(u_0^{p-1} |\Psi U_{\varepsilon_n}| + |u_0|\,|\Psi U_{\varepsilon_n}|^{p-1}\right).$$

From (2.11) and (2.12) we deduce that $J'(u_n) \to 0$ in $[V(\Omega)]^*$.

Obviously the sequence u_n is not relatively compact in $V(\Omega)$ since $\Psi U_{\varepsilon_n} \to 0$ weakly in $V(\Omega)$ but not strongly because of (2.8).

If, in the definition (2.6) - (2.7) of the sequence u_n, we take the point x_0 in the interior of Ω, arguing as in [BC1] we have that:

(2.13) $$J(\Psi U_{\varepsilon_n}) \to \frac{1}{N} S^{N/2} = \frac{2}{N} \Sigma^{N/2}$$

because $S = 2^{2/N}\Sigma$.

Therefore $J(u_n) \to J(u_0) + \frac{2}{N}\Sigma^{N/2}$ and $J'(u_n) \to 0$ as before, while u_n is not relatively compact in $V(\Omega)$. Finally, considering the sequences

$$u_n = u_0 + \sum_{i=1}^{m} \Psi_i(x) \frac{\varepsilon_n^{(N-2)/2}}{(\varepsilon_n + |x-x_0^i|^2)^{(N-2)/2}}$$

with the points x_0^i belonging either to Γ_1 or Ω, and taking the function $\Psi_i(x)$ with disjoint supports we obtain the assertion. ♦

Now we are ready to characterize the levels of J at which the Palais-Smale condition fails.

Theorem 2.3

Let u_n *be a Palais-Smale sequence for the functional* J, *that is*

(2.14) $$J(u_n) \to c \ \textit{and}\ J'(u_n) \to 0 \ \textit{in}\ [V(\Omega)]^*.$$

Then

(2.15) $$c = J(u_0) + \frac{h}{N}\Sigma^{N/2}$$

where h is a nonnegative integer and u_0 *is a solution of* (2.5).

Proof

We adapt to our case a blow-up method based on the concentration-compactness principle of P.L.Lions ([L2]) and used already in [BC]. From (2.14) we have

(2.16) $$\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \frac{1}{p+1}\int_\Omega (u_n^+)^{p+1} dx = c + o(1)$$

and

(2.17) $$\phi_n = -\Delta u_n - (u_n^+)^p \to 0 \qquad \text{in } [V(\Omega)]^*.$$

Hence, multiplying (2.17) by $\frac{1}{2}\, u_n$ and integrating

(2.18) $$\frac{1}{2}\int_\Omega \left[|\nabla u_n|^2 - (u_n^+)^{p+1}\right] dx = \frac{1}{2} < \phi_n , u_n >$$

Subtracting (2.18) to (2.16) we get

(2.19) $$\frac{1}{N}\int_\Omega (u_n^+)^{p+1} dx \le c + o(1) + \|\phi_n\|_{[V(\Omega)]^*}\|u_n\|_{V(\Omega)}$$

which, together with (2.16) implies that u_n is bounded in $V(\Omega)$. Therefore, up to a subsequence that we still denote by u_n, we have

$$(2.20)\qquad \begin{cases} u_n \rightarrow u_0 & \text{weakly in } V(\Omega) \\ u_n \rightarrow u_0 & \text{strongly in } L^s(\Omega) \text{ for any } s < p+1 \end{cases}$$

for some functions $u_0 \in V(\Omega)$.

Now if $u_n \rightarrow u_0$ strongly in $V(\Omega)$, the theorem is proved with $h = 0$. Therefore we assume that u_0 is not relatively compact in $V(\Omega)$, so that u_n does not converge strongly to u_0 in $V(\Omega)$. Consequently denoting by v_n the sequence $v_n = u_n - u_0$ we have

$$(2.21)\qquad \begin{cases} v_n \rightarrow 0 & \text{weakly in } V(\Omega) \\ v_n \rightarrow 0 & \text{strongly in } L^s(\Omega) \text{ for any } s < p+1 \\ v_n \text{ does not converge strongly in } V(\Omega) \end{cases}$$

Moreover

$$J(v_n) = \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx - \frac{1}{p+1}\int_\Omega (v_n^+)^q dx = \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \int_\Omega \nabla u_n \nabla u_0\, dx + \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx -$$

$$- \frac{1}{p+1}\int_\Omega ((u_n - u_0)^+)^{p+1} dx = \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{1}{p+1}\int_\Omega (u_n^+)^{p+1} dx + \frac{1}{p+1}\int_\Omega (u_0^+)^{p+1} dx$$

$$+ o(1)$$

Hence

$$(2.22)\qquad J(v_n) = J(u_n) - J(u_0) + o(1)$$

To obtain (2.22) we have used the weak convergence of u_n to u_0 in $V(\Omega)$ and a result of [BL] as in (2.11). Moreover $J'(v_n) \rightarrow 0$ in $[V(\Omega)]^*$. In fact, since $J'(u_n) \rightarrow 0$ and $u_n \rightarrow u_0$ weakly, we have that u_0 is a solution of (2.5). Therefore

$$-\Delta u_n = (u_n^+)^p + \phi_n \qquad \text{with } \phi_n \rightarrow 0 \text{ in } [V(\Omega)]^*.$$

and hence

$-\Delta v_n - \Delta u_0 = u_0^p + v_n^p + \phi_n +$ other terms which go to zero strongly in $[V(\Omega)]^*$ because they go to zero in $[L^{p+1}(\Omega)]^* = L^{(2N)/(N-2)}(\Omega)$ and $[L^{p+1}(\Omega)]^* \subset [V(\Omega)]^*$ as in the case of (2.12).

Since u_0 is a solution of (2.5) we have $-\Delta u_0 = u_0^{q-1}$ and consequently $J'(v_n) \to 0$ in $[V(\Omega)]^*$ so that we have

(2.23)
$$\begin{cases} -\Delta v_n = v_n^p + \phi_n & \text{in } \Omega \\ v_n = 0 & \text{on } \Gamma_0 \end{cases}$$

with $\phi_n \to 0$ in $[V(\Omega)]^*$.

Let us introduce the concentration function of v_n: (see[L2])

$$Q_n(t) = \max_{x \in \mathbb{R}^N} \int_{\Omega \cap (x+tB)} (v_n^+)^{p+1}$$

where B is the ball centered in the origin with radius 1. Obviously we have $\lim_{t \to \infty} Q_n(t) = \int_\Omega (v_n^+)^{p+1}$ and there exists $R > 0$ such that $Q_n(R) = \int_\Omega (v_n^+)^{p+1}$. Since v_n is not relatively compact in $V(\Omega)$, there exists $\alpha > 0$ such that $\int_\Omega (v_n^+)^{p+1} \geq \alpha > 0$. Therefore if $\mu \in]0,\alpha[$ there exist a sequence of positive number ε_n and a sequence of points $a_n \in \Omega$ such that:

$$Q_n(\varepsilon_n) = \mu \Leftrightarrow \int_{\Omega \cap (a_n + \varepsilon_n B)} (v_n^+)^{p+1} = \mu$$

Now we set $\tilde{u}_n = \varepsilon_n^{(N-2)/2} v_n(\varepsilon_n x + a_n)$; evidently $\tilde{u}_n$ is defined in $\Omega_n = \dfrac{\Omega - a_n}{\varepsilon_n}$ and we have

(2.24)
$$\int_B (\tilde{u}_n^+)^{p+1} = \mu$$

(2.25)
$$\int_{z+B} (\tilde{u}_n^+)^{p+1} \leq \mu \qquad \forall z \in \mathbb{R}^N$$

Let us prove now that $\varepsilon_n \to 0$, as $n \to \infty$. Arguing by contradiction we suppose that $\varepsilon_n \to \lambda \neq 0$. Then

(2.26) $$\int_{\Omega_n} (\tilde{u}_n)^2 = \frac{1}{\varepsilon_n^2} \int_{\Omega} |v_n|^2 \to 0$$

Let $\zeta \in C^\infty(\mathbb{R}^N)$ a cut-off function whose support is contained in z+B. Since $J'(v_n) \to 0$ in $[V(\Omega)]^*$ we have

(2.27) $$\int_{\Omega_n} \nabla\tilde{u}_n \nabla(\tilde{u}_n\zeta^2) = \int_{\Omega_n} (\tilde{u}_n^+)^{p-1}(\tilde{u}_n^+)^2\zeta^2 + o(1)$$

On the other hand

$$\int_{\Omega_n} \nabla\tilde{u}_n \nabla(\tilde{u}_n\zeta^2) = \int_{\Omega_n} |\nabla\tilde{u}_n|^2\zeta^2 + 2\int_{\Omega_n} \zeta\tilde{u}_n\nabla\tilde{u}_n\nabla\zeta = \int_{\Omega_n} |\nabla(\tilde{u}_n\zeta)|^2 - \int_{\Omega_n} |\nabla\zeta|^2\tilde{u}_n^2$$

and hence using (2.27)

$$\int_{\Omega_n\cap(z+B)} |\nabla(\tilde{u}_n\zeta)|^2 - \int_{\Omega_n\cap(z+B)} |\nabla\zeta|^2\tilde{u}_n^2 = \int_{\Omega_n\cap(z+B)} (\tilde{u}_n^+)^{p-1}(\tilde{u}_n^+)^2\zeta^2 + o(1) \leq \Big(\int_{\Omega_n\cap(z+B)} (\tilde{u}_n^+)^{p+1}\Big)^{(p-1)/p+1} \cdot$$

$$\cdot \Big(\int_{\Omega_n\cap(z+B)} |\tilde{u}_n\zeta|^{p+1}\Big)^{2/(p+1)} + o(1).$$

From (2.25) and (2.26) we get

$$\int_{\Omega_n\cap(z+B)} |\nabla(\tilde{u}_n\zeta)|^2 \leq \frac{\mu^{2/N}}{S_0(\Omega)}\Big(\int_{\Omega_n\cap(z+B)} |\nabla(\tilde{u}_n\zeta)|^2\Big) + o(1) \quad (1)$$

---

(1) since the ratio $\dfrac{\int |\nabla u|^2 dx}{(\int |u|^q dx)^{2/q}}$ is invariant by dilatation we have the Sobolev inequality

$\int_{\Omega_n} |\nabla u|^2 dx \geq S_0(\Omega)(\int_{\Omega_n} |u|^q dx)^{2/q}$ for any function u in $H^1(\Omega_n)$ such that $u = 0$ on $\dfrac{\Gamma_0 - a_n}{\varepsilon_n}$.

Therefore, choosing μ sufficiently small we have $\int_{\Omega_n\cap(z+B)} |\nabla(\tilde{u}_n\zeta)|^2 \leq o(1)$ for any $z\in \mathbb{R}^N$ which implies that $\tilde{u}_n \to 0$ strongly in $H^1_{loc}(\mathbb{R}^N)$, contradicting (2.24). In this way we have proved that $\varepsilon_n \to 0$. Now let us extend the functions $\tilde{u}_n$ to the whole $\mathbb{R}^N$ by using the operators $P_n = P_{1/\varepsilon_n}$. We set $w_n = P_n\tilde{u}_n$, $w_n \in D^1(\mathbb{R}^N)$ and, from ii) and iii) of Lemma 2.1

(2.28) $$\int_{\mathbb{R}^N} |\nabla w_n|^2 \leq C \int_{\Omega_n} |\nabla \tilde{u}_n|^2 = C \int_{\Omega} |\nabla \tilde{v}_n|^2 \leq C'$$

(2.29) $$\int_{\mathbb{R}^N} |w_n|^{p+1} dx \leq C \int_{\Omega_n} |\tilde{u}_n|^{p+1} dx) = C \int_{\Omega_n} |\tilde{v}_n|^{p+1} dx \leq C''$$

where C' and C'' do not depend on n. Therefore the sequence w_n is bounded in $D^1(\mathbb{R}^N)$ and hence

(2.30) $$w_n \to w \qquad \text{weakly in } D^1(\mathbb{R}^N)$$

(2.31) $$w_n \to w \qquad \text{weakly in } L^{p+1}(\mathbb{R}^N)$$

for some $w \in D^1(\mathbb{R}^N)$. The function w is not identically zero on $\mathbb{R}^N$, otherwise (2.26) would hold and repeating the same argument we have just used to prove that $\varepsilon_n \to 0$, we would get that $w_n \to 0$ strongly in $H^1_{loc}(\mathbb{R}^N)$ which contradicts (2.24).

Now let us denote by I the limit of the domains Ω_n; obviously $w_n \to w$ weakly in $D^1(I)$ so that

$$\int_I \nabla w_n \nabla g = \int_I \nabla w \nabla g + o(1) \qquad \text{for any } g \in D^1(I)$$

But

$$\int_I \nabla w_n \nabla g = \int_{I\cap\Omega_n} \nabla \tilde{u}_n \nabla g + \int_{I\setminus\Omega_n} \nabla w_n \nabla g$$

and

$$\left|\int_{I\setminus\Omega_n}\nabla w_n\nabla g\right| \leq \left(\int_{I\setminus\Omega_n}|\nabla w_n|^2\right)^{1/2}\left(\int_{I\setminus\Omega_n}|\nabla g|^2\right)^{1/2} \leq C\left(\int_{I\setminus\Omega_n}|\nabla g|^2\right)^{1/2} = o(1)$$

because of (2.28) and $\Omega_n \to I$. Therefore

$$\int_I \nabla w \nabla g = \int_{I\cap\Omega_n}\nabla\tilde{u}_n\nabla g + o(1) \qquad \text{for any } g\in D^1(I)$$

Repeating the same thing for $\int_I (w_n^+)^p g$ we finally get

(2.32) $$\int_{I\cap\Omega_n}\nabla\tilde{u}_n\nabla g - \int_{I\cap\Omega_n}(\tilde{u}_n^+)^p g = \int_I \nabla w\nabla g - \int_I (w^+)^p g + o(1)$$

for every $g\in D^1(I)$

Since $J'(\tilde{u}_n) \to 0$ in $[V(\Omega_n)]^*$, $V(\Omega_n) = \{u\in H^1(\Omega_n)$, such that $u = 0$ on $\frac{\Gamma_0 - a_n}{\varepsilon_n}\}$ and $\Omega_n \to I$, from (2.32) we deduce

(2.33) $$-\Delta w = (w^+)^p \qquad \text{in } I.$$

Now, denoting by $d(\cdot,\cdot)$ the euclidean distance in $\mathbb{R}^N$, we have the following alternatives

a) $\lim_{n\to\infty}\frac{d(a_n,\partial\Omega)}{\varepsilon_n} = +\infty$, so that $a_n \to x_0 \in \Omega$, $I=\mathbb{R}^N$, and w is a positive solution of (0.1) in $D^1(\mathbb{R}^N)$.

b) $\lim_{n\to\infty}\frac{d(a_n,\partial\Omega)}{\varepsilon_n} = m\in\mathbb{R}$, and $a_n \to x_0\in\Gamma_1$, so that $I=A$ the half space and w is a solution of (2.1) in $A=\Sigma_\alpha$.

c) $\lim_{n\to\infty}\frac{d(a_n,\partial\Omega)}{\varepsilon_n} = m\in\mathbb{R}$, and $a_n \to x_0\in\Gamma_0$, so that $I=A$ and w is a solution of (2.1)'.

d) $\lim_{n\to\infty}\frac{d(a_n,\partial\Omega)}{\varepsilon_n} = m\in\mathbb{R}$, and $a_n \to x_0\in\bar{\Gamma}_0\cap\bar{\Gamma}_1\neq\emptyset$ so that $I = \Sigma_{\pi/2}$ and w is a solution of (2.1)", since we have assumed that Γ_0 and Γ_1 intersect orthogonally.

Case c) and d) are easily excluded since Theorem 2.2 ensures that any solution of (2.1)' or (2.1)" is identically zero, while w is not.

If case a) occurs then $w = \dfrac{\mu^{(N-2)/2}}{(|x|^2+\mu^2)^{(N-2)/2}}$ for some $\mu > 0$ since by known results (see [Gi], [Au], [T], [Br3], [Br2]), these are the only positive solutions of (0.1) in $D^1(\mathbb{R}^N)$. If case b) occurs then $w = \dfrac{\mu^{(N-2)/2}}{(|x|^2+\mu^2)^{(N-2)/2}}$ for some $\mu > 0$ because of Theorem 2.1. Let us first treat case b). We set

$$\phi_n^1(x) = v_n(x) - \frac{1}{\varepsilon_n^{(N-2)/2}} w\left(\frac{x-a_n}{\varepsilon_n}\right), \qquad x\in\Omega$$

We have

$$\frac{1}{2}\int_\Omega |\nabla\phi_n^1|^2 = \frac{1}{2}\int_\Omega |\nabla v_n|^2 - \frac{1}{\varepsilon_n^{(N-2)/2}}\int_\Omega \nabla v_n \nabla w\left(\frac{x-a_n}{\varepsilon_n}\right) + \frac{1}{\varepsilon_n^{(N-2)/2}}\int_\Omega \left|\nabla w\left(\frac{x-a_n}{\varepsilon_n}\right)\right|^2 =$$

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2 - \int_{\Omega_n} \nabla\tilde{u}_n \nabla w + \frac{1}{2}\int_{\Omega_n} |\nabla w|^2 = \frac{1}{2}\int_\Omega |\nabla v_n|^2 - \frac{1}{2}\int_\Gamma |\nabla w|^2 + o(1).$$

Arguing in the same way to compute $\int_\Omega |\phi_n|^{p+1}$ we finally get

(2.34) $$J(v_n) = J(\phi_n^1) + \frac{1}{2}\int_\Gamma |\nabla w|^2 - \frac{1}{p+1}\int_\Gamma |w|^{p+1} + o(1)$$

Since

$$\frac{1}{2}\int_\Gamma |\nabla w|^2 - \frac{1}{p+1}\int_\Gamma |w|^{p+1} = \frac{1}{N}\Sigma^{N/2} \qquad \text{(see(2.8))}$$

by (2.32) and (2.34) we get

(2.35) $$J(u_n) = J(u_0) + \frac{1}{N}\Sigma^{N/2} + J(\phi_n^1) + o(1).$$

If case a) occurs, then repeating the same argument we obtain

(2.36) $$J(u_n) = J(u_0) + \frac{2}{N}\Sigma^{N/2} + J(\phi_n^1) + o(1).$$

since $$\frac{1}{2}\int_I |\nabla w|^2 - \frac{1}{p+1}\int_I |w|^{p+1} = S = 2^{2/N}\Sigma \qquad \text{when } I = \mathbb{R}^N$$

Now, proceeding as we did for the sequence v_n, it is possible to prove that

(2.37) $$J'(\phi_n^1) \to 0, \quad \phi_n^1 \to 0 \text{ weakly in } V(\Omega)$$

and obviously

(2.38) $$J(\phi_n^1) \to b^1 = c - J(u_0) - \frac{k}{N}\Sigma^{N/2} \qquad (k = 1 \text{ or } 2)$$

From (2.38) we obtain

(2.39) $$\frac{1}{2}\int_I |\nabla \phi_n^1|^2 - \frac{1}{p+1}\int_I \left((\phi_n^1)^+\right)^{p+1} = b^1 + o(1)$$

and, from (2.37)

(2.40) $$-\Delta\phi_n^1 = \left((\phi_n^1)^+\right)^p + f_n \qquad \text{with } f_n \to 0 \text{ in } [V(\Omega)]^*.$$

Then we get

(2.41) $$\int_\Omega |\nabla \phi_n^1|^2 = \int_\Omega \left((\phi_n^1)^+\right)^{p+1} + o(1)$$

Combining (2.39) and (2.41) we deduce

$$\int_\Omega |\nabla \phi_n^1|^2 = Nb^1 + o(1) \qquad \text{and} \qquad \int_\Omega \left((\phi_n^1)^+\right)^{p+1} = Nb^1 + o(1)$$

Using the Poincaré-Sobolev inequality we obtain

(2.42) $$Nb^1 \geq S_0(\Omega)(Nb^1)^{2/(p+1)}$$

from which it follows that either $b^1 = 0$ or

(2.43) $$b^1 \geq \frac{S_0(\Omega)^{N/2}}{N}$$

If $b^1 = 0$ (or equivalently $\phi_n^1 \to 0$ in $V(\Omega)$) the assertion is proved with h = 1 (or 2). Otherwise we could repeat for the sequence ϕ_n^1 the same procedure we used for the sequence v_n obtaining the analogous of (2.34)

(2.44) $$J(\phi_n^1) = J(\phi_n^2) + \frac{k}{N}\Sigma^{N/2} + o(1). \qquad (k = 1 \text{ or } 2)$$

for some sequence ϕ_n^2 which behaves exactly as the sequence ϕ_n^1.

In particular, by (2.43)

(2.45) $$b^2 = \lim_{n\to\infty} J(\phi_n^2) \geq \frac{S_0(\Omega)^{N/2}}{N}$$

and hence by (2.44)

(2.46) $$b^1 \geq \frac{1}{N}\Sigma^{N/2}.$$

After a finite number of steps we reach a sequence ϕ_n^j such that

(2.47) $$\begin{cases} J(\phi_n^j) \to c - J(u_0) - \frac{k}{N}\Sigma^{N/2} = b^j < \frac{1}{N}\Sigma^{N/2} \\ J'(\phi_n^j) \to 0 \qquad \text{and} \quad \phi_n^j \to 0 \text{ weakly in } V(\Omega). \end{cases}$$

Then, either $b^j = 0$ or, by the analogous of (2.46), b^j must be greater than $\frac{1}{N}\Sigma^{N/2}$. Since this contradicts (2.47) b^j is zero and the theorem is proved. ♦

Corollary 2.1

If (2.5) *has no nontrivial solutions in* $V(\Omega)$ *then the Palais-Smale condition for the functional* J *fails only at the levels*

(2.48) $$c = \frac{k}{N}\Sigma^{N/2}$$

where k is any positive integer.

Remark 2.1

A crucial point in the proof of Theorem 2.1 is to know exactly all positive solutions of problems a) - d). This is the reason why we made the assumption that Γ_0 and Γ_1 are regular and intersect orthogonally. In particular in the case d) we have used the fact that the mixed boundary problem (2.1)" does not have nontrivial solution. If $\partial\Omega$ was regular and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 \neq \emptyset$, the situation d) would become: $I = A$ and w is a nonnegative solution of

$$(2.49) \qquad \begin{cases} -\Delta u = u^{(N+2)/(N-2)} & \text{in } A \\ u = 0 & \text{on } \Gamma_0 = \partial A \cap \{x_N > 0\} \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 = \partial A \cap \{x_N < 0\} \\ u \in D^1(A) \end{cases}$$

Since, as far as we know, the positive solutions of (2.49) are not known (perhaps there are not any) we could not conclude the proof of the theorem. However more general hypotheses on the way $\bar{\Gamma}_0$ and $\bar{\Gamma}_1$ intersect could be allowed by using the nonexistence results of [EL]. The same remark applies to Γ_0, in case it is only Lipschitz continuous.

Regarding the situation b), we could use Theorem 2.1 in case Γ_1 is not of class C^1 but has only some "convex corners". To be more precise if $a_n \to x_0 \in \Gamma_1$ and x_0 is the vertex of a convex sector, then w would be a positive solution of (2.1) in a certain convex cone Σ_α. Thus by Theorem 2.1 we know that $w = \dfrac{\mu^{(N-2)/2}}{(|x|^2+\mu^2)^{(N-2)/2}}$ for some $\mu > 0$ and, by standard computations (see [LPT]), we have

$$(2.50) \qquad \frac{1}{2}\int_I |\nabla w|^2 - \frac{1}{p+1}\int_I |w|^{p+1} = \frac{1}{N}[S(\Sigma(\alpha_n,R))]^{N/2} = \frac{1}{N}\frac{2\alpha_n}{C_N}\Sigma^{N/2}$$

Thus if, for example, Γ_1 has only a convex corner, the critical value c would be

$$(2.51) \qquad c = J(u_0) + \frac{k}{N}\Sigma^{N/2} + \frac{h}{N}\frac{2\alpha_n}{C_N}\Sigma^{N/2}$$

where h and k are nonnegative integer.

2.3 EXISTENCE RESULTS FOR DOMAINS WITH HOLES

Let D and G be bounded domains in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary such that $\bar{G} \subset D$ and the origin $O \in \partial G$. For a point $x_0 \in D \setminus \bar{G}$ we denote by $B(x_0,r)$ the ball with center in x_0 and radius $r > 0$ and consider the bounded domain $\Omega = \Omega_\lambda = D \setminus (\bar{G} \cup \lambda^2 B(x_0,r))$ where x_0 and $r > 0$ have been chosen in such a way that $\lambda^2 B(x_0,r) \subset D \setminus \bar{G}$, for any $\lambda \in]0,1[$. In other words Ω is a domain with two holes, one of which is going to be chosen very small.

Finally we set

$$\Gamma_0 = \partial D \qquad\qquad \Gamma_1 = \partial G \cup \partial(\lambda^2 B(x_0,r))$$

and obviously $\Gamma_0 \cap \Gamma_1 = \varnothing$ and $\partial\Omega = \Gamma_0 \cup \Gamma_1$.

In this section we prove the following existence result.

Theorem 2.4

Let us consider the problem

$$(2.52) \qquad \begin{cases} -\Delta u = u^{(N+2)/(N-2)} & \text{in } \Omega_\lambda \\ u > 0 & \text{in } \Omega_\lambda \\ u = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \end{cases}$$

where ν *is the outer normal to* Γ_1.

Then there exists λ_0 *such that for any* $\lambda \in]0,\lambda_0[$, *problem* (2.52) *has at least one solution.*

To prove Theorem (2.52) we need some preliminary lemmata. We recall that $F_0(u)$ is the functional

$$F_0(u) = F(u) = \int_\Omega |\nabla u|^2$$

Moreover we define

(2.53) $$X(\Omega)=\{u\in V(\Omega),\ u\geq 0\},\qquad M^+=X(\Omega)\cap M$$

Lemma 2.3

Let u_n *be a sequence in* M^+ *such that*

(2.54) $$F(u_n)\to c\in\,]\Sigma,\ 2^{2/N}\Sigma[\qquad \text{and}\qquad F'_M\to 0 \text{ in } [V(\Omega)]^*$$

where F'_M *is the derivative of the functional* F *defined on* M. *Then there exists at least one solution of* (2.52).

Proof

It is easy to see that there exists a sequence λ_n of Lagrange multipliers such that the functions $v_n=\lambda_n^{(N-2)/4}u_n$ make up a Palais-Smale sequence for the functional J on $V(\Omega)$. Moreover, by (2.54)

$$J(v_n)\to c\in\,]\frac{1}{N}\Sigma^{N/2},\ \frac{2}{N}\Sigma^{N/2}[$$

If there are no solutions of (2.52), by the strong maximum principle, the problem (2.5) does not have any nontrivial solution. Thus Corollary 2.1 applies and the Palais -Smale conditions holds at the level c. This implies that v_n converges (up to a subsequence to a critical point of J which is a solution of (2.52)). ♦

Lemma 2.4 (Deformation Lemma)

Let us suppose that there are no solutions of (2.52). *If* $c\in\,]\Sigma,\ 2^{2/N}\Sigma[$ *is not a critical value for* F *on* M *then there exists a map*

$$\eta : [0,1] \times M\to M$$

and a positive number ε_0 *such that for any* $\varepsilon\leq\varepsilon_0$

i) $\eta(0,u) = u \qquad \forall u\in M$

ii) $\eta(t,u) = u \qquad \forall u\in \{u\in M,\ F(u)\leq c-\varepsilon\}\cup M\setminus\{u\in M,\ F(u)\leq c+\varepsilon\}\qquad \forall t\in[0,1]$

iii) $\eta(t,u)\in M^+ \qquad \forall u\in M^+,\ \forall t\in[0,1]$

iv) $\eta(1,u)\in M^+\cap\{u\in M,\ F(u)\le c-\varepsilon/2\}$, $\qquad \forall u\in M^+\cap\{u\in M,\ F(u)\le c+\varepsilon/2\}$.

Proof

This lemma is a variant of the classical Deformation Lemma and it holds because if (2.52) has no solutions then F satisfies the Palais-Smale condition in the interval $]\Sigma,\ 2^{2/N}\Sigma[$ (see the previous lemma). Moreover M^+ is invariant for the gradient flow associated to F'_M and this gives iii). However we refer the reader to [Ho] for more details. ♦

Now let us consider the functions

(2.55)
$$\phi_\mu^a(x)=\frac{\mu^{(N-2)/2}}{(\,|x-a|^2+\mu^2)^{(N-2)/2}}\,,\qquad \mu>0,\quad a\in\mathbb{R}^N,\quad x\in\mathbb{R}^N$$

and the "cut-off" function $\sigma\in C^\infty(\bar\Omega)$, $\sigma(x)\in[0,1]$ $\forall x\in\bar\Omega$, such that

$$\sigma(x)=\begin{cases}1 & \text{in } K_0\\ 0 & \text{outside of } K_1\end{cases}$$

where K_0 is a compact set in D containing $\overline{B(x_0,r)\cup G}$ in its interior and K_1 is a compact set such that $\bar K_0\subset \overset{\circ}{K}_1$ and $\bar K_1\subset D$.

Then the functions

(2.56)
$$v_\mu^a=\sigma(x)\cdot\phi_\mu^a(x)\Big|_\Omega \quad\text{and}\quad w_\mu^a=\frac{v_\mu^a(x)}{\left(\int_\Omega |v_\mu^a|^{p+1}dx\right)^{1/(p+1)}}$$

belong to $X(\Omega)$ and M^+ respectively, for any $\mu>0$, $a\in\mathbb{R}^N$.

As already shown in Chapter 1 that we have

Lemma 2.5

(2.57)
$$\lim_{\mu\to 0} F(w_\mu^a)=\Sigma \qquad \textit{for any } a\in\Gamma_1$$

and

(2.58) $$\lim_{\mu\to 0} F(w^a_\mu) = \Sigma \quad \textit{uniformly for } a\in \partial(\lambda^2 B(x_0,r))\subset \Gamma_1$$

Proof of Theorem 2.4

We will prove the theorem arguing by contradiction. More precisely we assume that (2.52) has no solutions and will show that F has at least one critical point v on M^+ such that $F(v)\in\,]\Sigma, 2^{2/N}\Sigma[$. Then the function $u = [F(v)]^{(N-2)/4} v$ will be a critical point of J in $X(\Omega)$ and hence will give a solution of (2.52) contradicting the assumption. From now on let us denote by B the ball $\lambda^2 B(x_0,r)$ and by y_0 the point $\lambda^2 x_0$. We define a map $h : \bar{B} \to M^+$ in the following way

(2.59) $$h(a) = \frac{z(a)}{\left(\int_\Omega |z(a)|^{p+1} dx\right)^{1/(p+1)}}$$

where

$$z(a) = \begin{cases} (1-2t)\, v^{O}_{\bar\mu} + 2t\, v^{g(a)}_{\bar\mu} & \text{if } t = \dfrac{|a-y_0|}{\lambda^2 r} \le \dfrac{1}{2} \\ v^{g(a)}_{\mu(t)} & \text{if } t > \dfrac{1}{2} \end{cases}$$

O is the origin in $\mathbb{R}^N$ $\mu(t) = (t-\frac{1}{2})\mu_* + (2-2t)\bar\mu$ and g(a) is the point on ∂B which lays on the straight line through y_0 and a on the same side of a with respect to y_0. The numbers μ^* and $\bar\mu$ will be fixed later.

Let us choose a number $\delta > 0$ such that $\Sigma + \delta < 2^{2/N}\Sigma$. By (2.57) and the hypothesis that $O\in\Gamma_1$ we deduce

(2.60) there exists $\bar\mu > 0$ such that $F(w^{O}_{\bar\mu}) = F(h(O)) < \Sigma + \delta$

Then we consider the function

(2.61) $$H : M^+ \to \mathbb{R}^N \quad , \quad H(u) = \int_\Omega x|u|^{p+1} dx$$

Since we are assuming that (2.52) has no solutions, Lemma 2.6 implies that $S_0(\Omega) = \Sigma$. In addition, by Lemma 2.5 and the hypothesis $y_0 \notin \bar{\Gamma}_1$ we get that there exists a number $\alpha_\lambda(\Omega) > 0$ such that

(2.62) $$H(u) = x_0 \Rightarrow F(u) \geq \Sigma + \alpha_\lambda(\Omega) \quad \text{and} \quad \Sigma + \alpha_\lambda(\Omega) < 2^{2/N}\Sigma$$

On the other hand by (2.58) we can choose $\mu^* > 0$ so small that

(2.63) $$F(h(a)) = F(w^a_{\mu^*/2}) < \Sigma + \frac{\alpha_\lambda(\Omega)}{2} \qquad \text{for any } a \in \partial B$$

because $\partial B = \partial(\lambda^2 B(x_0, r)) \subset \Gamma_1$. Note that if $a \in \partial B$, $\mu(t) = \mu_*/2$ in the definition (2.59). Moreover we have that

(2.64) there exists $\lambda_0 > 0$ such that if $\bar{\mu} = \lambda < \lambda_0$, then $F(h(a)) < 2^{2/N}\Sigma$ for any $a \in \bar{B}$.

For convenience we postpone the proof of (2.64) and assume from now on that $\Omega = \Omega_\lambda$ is such that $\lambda < \lambda_0$. Let us define the class of maps

$$\mathcal{F} = \{ f \in C(\bar{B}, M^+) , f|_{\partial B} = h|_{\partial B} \}$$

and set

$$c = \inf_{f \in \mathcal{F}} \max_{a \in \bar{B}} F(f(a))$$

Since $h \in \mathcal{F}$ and (2.64) holds, $c < 2^{2/N}\Sigma$. Using some argument of topological degree theory that we will detail below it is possible to show that

(2.65) for any $f \in \mathcal{F}$ there exists $a \in B$ such that $F(f(a)) \geq \Sigma + \alpha_\lambda(\Omega)$

Therefore

(2.66) $$c \geq \Sigma + \alpha_\lambda(\Omega)$$

Then c must be a critical value for J on M^+. In fact if it was not we could apply the Deformation Lemma 2.4 obtaining a map η and a number $\varepsilon_0 > 0$ satisfying i) - iv). By definition of c, there exists $\tilde{f} \in \mathcal{F}$ such that

(2.67) $$c \le \max_{a \in \bar{B}} F((\tilde{f}(a)) < c + \frac{\varepsilon}{2} \quad , \qquad \varepsilon \le \varepsilon_0$$

We claim that the function $\eta_1 \circ \tilde{f} = \eta\ (1,\cdot\ ) \circ \tilde{f}$ belongs to $\mathcal{F}$. In fact, by (2.63) and (2.66) we have

(2.68) $$F(h(a)) < c - \varepsilon \qquad \text{for any } a \in \partial B$$

choosing $\varepsilon \le \varepsilon_0$ sufficiently small. Thus by ii) of Lemma 2.4

(2.69) $$(\eta_1 \circ \tilde{f}\,)(a) = (\eta_1 \circ h\,)(a) = h(a) \qquad \text{for any } a \in \partial B$$

which implies that $\eta_1 \circ \tilde{f} \in \mathcal{F}$.
On the other side, by iv) of Lemma 2.4

$$\max_{a \in \bar{B}} F(\eta_1 \circ \tilde{f}\,)(a) < c - \frac{\varepsilon}{2}$$

and this contradicts the definition of c. Therefore c must be a critical value of J on M^+ and the assertion is proved.

Let us explain (2.65)

Proof of 2.65

By (2.62) it is enough to prove that for each $f \in \mathcal{F}$ there exists $a \in B$ such that

$$H(f(a)) = \int_\Omega x|f(a)|^{p+1} dx = y_0 .$$

Fixed $f \in \mathcal{F}$ we consider the map $G = H \circ f$ from $\bar{B}$ to $\mathbb{R}^N$ which is homotopic to the identity map $I : \bar{B} \to \bar{B}$ by the homotopy

$$G_s = sG + (1 - s)I \quad , \qquad s \in \,]0,1[$$

We claim that $G_s(a) \neq y_0$ for every $a\in \partial B$ and $s\in]0,1[$. In fact if $a\in \partial B$ $G(a) = H(f(a)) = H(h(a)) \neq y_0$ because of (2.62) and (2.63). Moreover by P.L. Lions principle and Lemma 2.5 the function $|h(a)|^{p+1} = |w^{a}_{\mu^*}|^{p+1} \to \delta_a$ weakly in the sense of measure if $a\in \partial B \subset \Gamma_1$, as $\mu^* \to 0$. Therefore $H(h(a))$ is very close to $a\in \partial\lambda^2 B(x_0,r)$ (if μ^* is sufficiently small) and hence the segment joining $G(a) = H(h(a))$ to a does not pass through the point y_0 . This proves that $G_s(a) \neq y_0$ for every $a\in \partial B$ and $s\in]0,1[$. Thus, by the property of invariance under homotopy of the topological degree, $\deg(G; \bar{B}; x_0) = \deg(I; \bar{B}; x_0) = 1$ since $x_0\in \bar{B}$, so that there exists a solution of the equation $G(a) = H(f(a)) = y_0$.

The last thing to be proved is (2.64)

Proof of 2.64

Let us first prove (2.64) in the case $0 < t < \dfrac{|a-y_0|}{\lambda^2 r} < \dfrac{1}{2}$.

Since the ratio $\dfrac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{2/(p+1)}}$ is homogeneous of degree zero, to prove that $F(h(a)) < 2^{2/N}\Sigma$ is equivalent to show

$$\frac{\int_\Omega |\nabla z(a)|^2}{\left(\int_\Omega |z(a)|^{p+1}\right)^{2/(p+1)}} < 2^{2/N}\Sigma \tag{2.70}$$

where $z(a) =(1-2t)\, \overset{o}{v}_{\bar{\mu}} + 2t\, v^{g(a)}_{\bar{\mu}}$, $t\in]0, \frac{1}{2}[$ and $g(a)$ being defined in (2.59). We now take $\bar{\mu} = \lambda$ and, for simplicity, set

$$u_1 = \overset{o}{v}_{\bar{\mu}} = \overset{o}{v}_{\lambda} = \sigma(x)\left(\frac{\lambda}{\lambda^2+ |x|^2}\right)^{(N-2)/2}\Big|_{\Omega_\lambda}$$

and

$$u_2 = v^{g(a)}_{\bar{\mu}} = v^{g(a)}_{\lambda} = \sigma(x)\left(\frac{\lambda}{\lambda^2+ |x - \lambda^2 b|^2}\right)^{(N-2)/2}\Big|_{\Omega_\lambda}$$

where we have written $\lambda^2 b$, $b \in \partial B(x_0,r)$, instead of $g(a) \in \partial(\lambda^2 B(x_0,r))$.

The following estimates hold

(i) $$\int_{\Omega_\lambda} |\nabla u_1|^2 = K_1 + o(1)$$

(ii) $$\int_{\Omega_\lambda} |\nabla u_2|^2 = K_1 + o(1)$$

(iii) $$\int_{\Omega_\lambda} \nabla u_1 \nabla u_2 = K_1 + o(1)$$

(iv) $$\left(\int_{\Omega} |(1-2t)u_1 + 2tu_2|^{p+1}\right)^{2/(p+1)} = K_1 + o(1) \qquad \forall t \in]0, \tfrac{1}{2}[$$

where $K_1 = \int_A |\nabla U|^2$, $K_2 = \left(\int_A |U|^{p+1}\right)^{2/(p+1)}$ with $U(x) = \left(\frac{1}{1+|x|^2}\right)^{(N-2)/2}$ and A is the half space defined in Theorem 2.2. As it is well known (see [Au],[T] or [LPT]) $\frac{K_1}{K_2} = \Sigma$ since U is one of the functions which minimize the ratio $\dfrac{\int_A |\nabla u|^2}{\left(\int_A |u|^{p+1}\right)^{2/(p+1)}}$ in the space $D^1(A)$. The estimates (i) and (ii) follow by standard computations for which we refer to [BC1] or [Gr1]. Let us prove (iii).

$$\int_{\Omega_\lambda} \nabla u_1 \nabla u_2 = \int_{\Omega_\lambda} \left(\frac{\nabla\sigma}{(\lambda^2 + |x|^2)^{(N-2)}} - (2-N) \frac{\sigma \cdot x}{(\lambda^2 + |x|^2)^{N/2}} \right) \cdot$$

$$\cdot \left(\frac{\nabla\sigma}{(\lambda^2 + |x-\lambda^2 b|^2)^{(N-2)}} - (2-N) \frac{\sigma(x-\lambda^2 b)}{(\lambda^2 + |x-\lambda^2 b|^2)^{N/2}} \right) \lambda^{N-2}$$

and

$$\int_{\Omega_\lambda} \frac{|\nabla\sigma|^2}{(\lambda^2+|x|^2)^{(N-2)}(\lambda^2+|x-\lambda^2 b|^2)^{(N-2)}} = \int_{\Omega_\lambda\setminus K_0} \frac{|\nabla\sigma|^2}{(\lambda^2+|x|^2)^{(N-2)}(\lambda^2+|x-\lambda^2 b|^2)^{(N-2)}}$$

where K_0 is the compact set where $\sigma = 1$. Since O and $\lambda^2 b$ belong to K_0 we have

$$\int_{\Omega_\lambda} \frac{|\nabla\sigma|^2}{(\lambda^2+|x|^2)^{(N-2)}(\lambda^2+|x-\lambda^2 b|^2)^{(N-2)}} = o(1) \qquad \text{as } \lambda\to 0$$

The same estimate holds for each integral where the term $\nabla\sigma$ appear. Therefore

$$\int_{\Omega_\lambda} \nabla u_1 \nabla u_2 = (N-2)^2 \int_{\Omega_\lambda} \frac{x\cdot(x-\lambda^2 b)\lambda^{(N-2)}}{(\lambda^2+|x|^2)^{N/2}(\lambda^2+|x-\lambda^2 b|^2)^{N/2}} + O(\lambda^{(N-2)})$$

because $\sigma = 1$ in K_0

By the change of variable $\lambda y = x-\lambda^2 b$ we get

$$\int_{\Omega_\lambda} \nabla u_1 \nabla u_2 = (N-2)^2 \int_{(\Omega_\lambda-\lambda^2 b)/\lambda} \frac{y\cdot(\lambda b + y)}{(1+|y|^2)^{N/2}(1+|y+\lambda b|^2)^{N/2}} + O(\lambda^{(N-2)}) =$$

$$= (N-2)^2 \int_A \frac{|y|^2}{(1+|y|^2)^N} + o(1) = K_1 + o(1)$$

Let us prove (iv). By making the same change of variable as before we obtain

$$\int_{\Omega_\lambda} |(1-2t)u_1+2tu_2|^{p+1} = \int_{\Omega_\lambda} \sigma^{p+1}|(1-2t)(\frac{\lambda}{\lambda^2+|\lambda y+\lambda^2 b|^2})^{(N-2)/2} + 2t\,(\frac{\lambda}{\lambda^2+|y|^2})^{(N-2)/2}|^{p+1}\lambda^N =$$

$$\int_{(\Omega_\lambda-\lambda^2 b)/\lambda} \left| \frac{2t}{(1+|y+\lambda^2 b|^2)^{(N-2)/2}} + \frac{1-2t}{(1+|y|^2|)^{(N-2)/2}} \right|^{p+1} + O(\lambda^N) = (N-2)^2 \int_A \frac{1}{(1+|y|^2)^N} + o(1)$$

by standard computation (see[BN]) and because $\sigma = 1$ in K_0. Hence

$$\int_\Omega |(1-2t)u_1+2tu_2|^{p+1})^{2/(p+1)} = K_2 + o(1) \qquad \forall t \in]0,\frac{1}{2}[$$

From (i) - (iv) we finally get

$$\frac{\int_\Omega |\nabla z(a)|^2}{(\int_\Omega |z(a)|^{p+1})^{2/(p+1)}} = \frac{4t^2K_1+(1-2t)^2K_1+4t(1-t)K_1+o(1)}{K_2+o(1)} \to \frac{K_1}{K_2} = \Sigma < 2^{2/N}\Sigma$$

as $\lambda \to 0$.

Hence (2.70) holds if $\bar{\mu} = \lambda$ is chosen sufficiently small and $0 < \frac{|a-y_0|}{\lambda^2 r} < \frac{1}{2}$. Obviously by (2.57) and (2.60),(2.64) also holds if $\bar{\mu} = \lambda$ and μ^* are taken sufficiently small.

Therefore, we first choose λ_0 and fix $\lambda < \lambda_0$ in such a way that (2.64) holds for any $a \in \bar{B}$ and μ^* sufficiently small. Finally (once λ is fixed) we choose a value of μ^* which also satisfies (2.63). ♦

Remark 2.2

The topological explanation of the previous existence result is the following. If there are no solutions to (2.52) then, by Corollary 2.1 and the P.L. Lions principle the level set

$$F_{\Sigma+\eta} = \{u \in M^+ , F(u) \le \Sigma + \eta\}$$

has the homotopy type of $\bar{\Gamma}_1$, i.e. two holes. While moving from $J_{\Sigma+\eta}$ to $J_{2^{2/N}\Sigma} = J_S$ this homotopy type changes because we can also consider functions obtained connecting two functions v_μ^a , $a \in \Gamma_1$ by a segment. Therefore, since the Palais - Smale condition holds in the interval $]\Sigma, 2^{2/N}\Sigma[$ there must be a critical value of J between Σ and $2^{2/N}\Sigma$. The assumption on the hole $\lambda^2 B(x_0,r)$ (i.e. it is small and close to G) ensures that the critical value c belongs to $]\Sigma, 2^{2/N}\Sigma[$. Of course other topological hypotheses could be made in order to get the same existence result. For example we can assume that Ω is the following domain

$$\Omega = D \setminus \lambda^2 B(x_0,r)$$

where D is a bounded domain whose boundary is made by two smooth manifolds Γ_0 and Γ_2 such either $\bar{\Gamma}_0 \cap \bar{\Gamma}_2 = \emptyset$ or they intersect orthogonally and $O \in \Gamma_2$. Then if $\Gamma_1 = \Gamma_2 \cup \partial(\lambda^2 B(x_0,r))$ and λ is very small it is possible to prove that there exists a solution to (2.52) by repeating essentially the same proof. The assumption that Γ_0 and Γ_2 intersect orthogonally is obviously needed to apply Corollary 2.1.

Chapter 3

MULTIPLICITY RESULTS IN THE PRESENCE OF SYMMETRY.

1. THE GENERAL THEOREM.

In this chapter we consider some existence and multiplicity result to (1.1) in symmetric domains. Both results follow from a general theorem regarding minimizing sequences for some suitable functional, consequence of concentration-compactness principle of P.L Lions (see [L2]), which we recall.

Theorem 3.1.

Let u_n converge weakly in $V(\Omega)$ to some function u and assume, as it is always the case for some subsequences, that there exist two bounded measures $\mu, \tilde{\mu}$ on $\bar{\Omega}$ such that $|u_n|^{p+1}$ and $|\nabla u|^2$ converge weakly in the sense of measures to μ, $\tilde{\mu}$ respectively.

Then, there exists a countable set J, distinct points $(x_j)_{j\in J} \in \bar{\Omega}$ and real numbers $\mu_j > 0$ such that

$$(3.0)\qquad \begin{cases} \mu = |u|^{p+1} + \sum_{j\in J} \mu_j \delta_{x_j} \\ \tilde{\mu} \geq |\nabla u|^2 + S(\Omega) \sum_{j\in J} \mu_j^{2/(p+1)} \delta_{x_j} \end{cases}$$

where δ_{x_j} is the Dirac mass at the point x_j.

We start with some definitions

Definition 3.1 :

We say that Ω belongs to the class $\mathcal{E}_{\alpha_N}^T$ if the following conditions hold

(3.1) $\Omega \in \mathcal{E}_{\alpha_N}$ *for some* $\alpha_N \in (0, C_N/2\,]$

(3.2) *there exists a map* $T \in O(N)$ *such that* $T(\bar{\Omega}) = \bar{\Omega}$, $T(\Gamma_0) = \Gamma_0$ *and* $T(\Gamma_1) = \Gamma_1$

(3.3) $T(x) \neq x$, $\forall x \in \bar{\Gamma}_1$ ($\bar{\Gamma}_1$ *is the closure of* Γ_1 *relatively to* $\partial\Omega$).

Here $O(N)$ denotes the group of orthogonal matrices in $\mathbb{R}^N$.

Definition 3.2 :

For $\Omega \in \mathcal{E}^{T}_{\alpha_N}$ *we call* $V^T(\Omega)$ *the space of all functions in* $V(\Omega)$ *which are invariant by the action of* T: $V^T(\Omega) = \{u \in V(\Omega) : u \circ T = u$ *a.e. in* $\Omega\}$.

Let

$$(3.4) \qquad \tilde{S}_\lambda(\Omega) = \inf_{\substack{u \in V^T(\Omega) \\ \|u\|_{p+1}=1}} \left(\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2 \right) = \inf_{\substack{u \in V^T(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2}{\left(\int_\Omega |u|^{p+1} \right)^{2/(p+1)}} .$$

Remark 3.1

By the properties of T one can easily verify that $\Delta^{-1}(|u|^{p-1} u)$ lies in $V^T(\Omega)$ for every $u \in V^T(\Omega)$. Consequently every minimizer u for $\tilde{S}(\Omega)$ is a critical point of $F(u)$ on M.

Definition 3.3 :

For $T \in O(N)$ *we denote by* G *the cyclic group (finite or infinite) generated by* T . *Let* $O_G(x) = \{gx, g \in G\}$ *be the orbit of a point* $x \in \Omega$. *If there exists some* $x \in \bar{\Omega}$ *such that* $O_G(x)$ *is finite (which is always the case if* G *is finite) we define* $\zeta = \zeta(T,\Omega) = \underset{x \in \bar{\Omega} \setminus F}{\inf} d(O_G(x))$

where $d(O_G(x))$ *is the number of points in* $O_G(x)$ *and* $F = \{x \in \bar{\Omega}$ *such that* $x = T(x)\}$. *If* $O_G(x)$ *is never finite we set* $\zeta = +\infty$.

We recall that, we cannot have any positive solutions to (1.1) if $\lambda \geq \lambda_1(\Omega)$ (see Lemma 1.1). Hence we consider the case $\lambda < \lambda_1(\Omega)$.
We have the following theorem:

Theorem 3.2 :

Let us suppose $\Omega \in \mathcal{E}^{T}_{\alpha_N}$, *and* Γ_1 *is smooth. If* T *has no fixed point in* $\bar{\Omega}$, $\lambda < \lambda_1(\Omega)$ *and*

$$(3.5) \qquad \tilde{S}_\lambda(\Omega) < \zeta^{2/N} S(\Omega)$$

then there is a solution of (1.1) *in* $V^T(\Omega)$*(as a consequence of it, if* $\zeta = +\infty$ *there is always a solution of* (1.1) *for any* $\lambda < \lambda_1(\Omega)$*). On the other side if* T *has at least one fixed point in* $\bar{\Omega}\setminus\bar{\Gamma}_1$ *and*

$$(3.6) \qquad \tilde{S}_\lambda(\Omega) < 2^{2/N}\, S(\Sigma(\pi, R)) = S$$

again there is a solution of (1.1) *in* $V^T(\Omega)$.
(See (2.3) *and* (2.4) *for the definition of* Σ *and* S *).*

Proof.

We argue as in [L2], Part 2, Corollary 4.3.

Let u_n be a minimizing sequence for $\tilde{S}_\lambda(\Omega)$. Hence u_n satisfies

$$(3.7) \qquad \begin{cases} \|u_n\|_{p+1} = 1 & u_n \in V^T(\Omega) \\ \int_\Omega |\nabla u_n|^2 - \lambda \int_\Omega u_n^2 = \tilde{S}_\lambda(\Omega) + o(1) \end{cases}$$

where $o(1)$ goes to zero when n goes to infinity.

It is easy to verify that u_n is bounded in $V(\Omega)$. Then there exists $u \in V(\Omega)$ such that

$$(3.8) \qquad \begin{cases} u_n \to u & \text{weakly in } V(\Omega) \\ u_n \to u & \text{a. e. in } \Omega \end{cases} .$$

We remark that from (3.8) it follows that $u \in V^T(\Omega)$. Then, by Theorem 3.1 we have the following alternatives:

i) u_n is relatively compact

ii) there exists a countable set J, distinct points $(x_j)_{j\in J} \in \bar{\Omega}$ and real numbers $\mu_j > 0$ such that (3.0) holds.

In the case i) we can pass to the limit in (3.7) and then we obtain that the infimum is achieved by the function u. Of course, such an infimum provides a solution to (1.1) (see Remark 1.2).

Let us suppose that case ii) occurs. Our first aim is to prove that the weak limit u is identically zero. Our proof is contained in [Br3], Lemma 2, but we repeat it for the reader's convenience. By the definition of $\tilde{S}_\lambda(\Omega)$ we have

$$(3.9) \qquad \int_\Omega |\nabla(u_n+\phi)|^2 - \lambda \int_\Omega |(u_n+\phi)|^2 \geq \tilde{S}_\lambda(\Omega)\, \|u_n+\phi\|_{p+1} \qquad \forall \phi \in V^T(\Omega)$$

On the other hand, we have by convexity

$$(3.10) \qquad \|u_n+\phi\|_{p+1}^2 \geq \left[\left(1 + (p+1)\int_\Omega |u_n|^{p-1}u_n\phi\right)^+\right]^{2/(p+1)}$$

Combining (3.9) and (3.10) and passing to the limit, we obtain

$$\tilde{S}_\lambda(\Omega) + 2\int_\Omega \nabla u \nabla\phi + \int_\Omega |\nabla\phi|^2 - 2\lambda \int_\Omega u\phi - \lambda \int_\Omega |\phi|^2 \geq$$

$$\geq \tilde{S}_\lambda(\Omega)\left[\left(1 + (p+1)\int_\Omega |u|^{p-1}u\phi\right)^+\right]^{2/(p+1)}$$

Replacing ϕ by $t\phi$, we find, as $t \to 0$

$$\int_\Omega \nabla u \nabla \phi = \tilde{S}_\lambda(\Omega) \int_\Omega |u|^{p-1}u\phi + \lambda \int_\Omega u\phi \qquad \forall \phi \in V^T(\Omega)$$

and, in particular

$$(3.11) \qquad \int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 = \tilde{S}_\lambda(\Omega) \int_\Omega |u|^{p+1}$$

However we have

$$(3.12) \qquad \int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 \geq \tilde{S}_\lambda(\Omega) \left(\int_\Omega |u|^{p+1}\right)^{2/(p+1)}$$

Combining (3.11) and (3.12), we obtain that either $\|u\|_{p+1} \geq 1$ or $\|u\|_{p+1} = 0$. In the first case we must have $\|u\|_{p+1} = 1$ (since $\|u\|_{p+1} \leq 1$). It follows that, by standard arguments, we deduce that u_n is relatively compact in $V^T(\Omega)$(and we return in the case i)). Hence $u = 0$ if u_n is not relatively compact. By Theorem 3.1 we have that there exists a countable set J , distinct points $(x_j)_{j\in J} \in \bar{\Omega}$ and real numbers $\mu_j > 0$ such that

$$(3.13)\qquad \begin{cases} \mu = \sum_{j\in J} \mu_j \delta_{x_j} \\ \tilde{\mu} \geq S(\Omega) \sum_{j\in J} \mu_j^{2/(p+1)} \delta_{x_j} \end{cases}$$

where μ and $\tilde{\mu}$ are the weak limits (in the sense of measure) of $|u_n|^{p+1}$ and $|\nabla u_n|^2$ respectively.

Now we prove that the set J is finite. In fact we have (by the variational principle of I. Ekeland)

$$-\Delta u_n = \tilde{S}_\lambda(\Omega)\, |u_n|^{p-1} u_n + \lambda u_n + f_n \quad \text{with } f_n \to 0 \text{ in } [V^T(\Omega)]^*$$

and we observe that for all $\psi \in C_0^1(\bar{\Omega})$

$$< -\Delta u_n , u_n \psi > = \tilde{S}_\lambda(\Omega) \int_\Omega |u_n|^{p+1} \psi + \lambda \int_\Omega |u_n|^2 \psi + < f_n , u_n \psi >$$

i.e.

$$\int_\Omega |\nabla u_n|^2 \psi + \int_\Omega u_n \nabla\psi \nabla u_n = \tilde{S}_\lambda(\Omega) \int_\Omega |u_n|^{p+1} \psi + \lambda \int_\Omega |u_n|^2 \psi + < f_n , u_n \psi >$$

and because $u_n \to 0$ weakly in $V^T(\Omega)$ and strongly in $L^2(\Omega)$, we have passing to the limit for $n \to \infty$

$$(3.13)' \qquad \int_\Omega \psi d\tilde{\mu} = \tilde{S}_\lambda(\Omega) \int_\Omega \psi d\mu \quad \Rightarrow \tilde{\mu} = \tilde{S}_\lambda(\Omega)\mu$$

and (3.13) implies that J is finite.

Now we consider the case when T has no fixed point in $\bar{\Omega}$ and $\zeta < \infty$. Our aim is to prove that if "u is concentrated at y " then u is also concentrated at T(y). In fact, since $|u_n|^{p+1} \to \mu$ weakly in the sense of measures we have as $n \to \infty$

$$\int_\Omega |u_n(x)|^{p+1} \phi(x) dx \to \sum_{j\in J} \mu_j\, \phi(x_j) \qquad \forall \phi \in C^0(\bar{\Omega})$$

and

$$\int_{\Omega} |u_n(x)|^{p+1} \phi(T(x))dx \to \sum_{i\in J} \mu_i \, \phi(T(x_i)) \qquad \forall \phi \in C^0(\bar{\Omega})$$

Now

$$\int_{\Omega} |u_n(x)|^{p+1} \phi(x)dx = \int_{T^{-1}(\Omega)} |u_n(T(x))|^{p+1} \phi(T(x))|\det(\mathrm{Jac}T)|dx = \int_{\Omega} |u_n(x)|^{p+1} \phi(T(x))dx$$

for $|\det T| = 1$ being T an orthogonal matrix (see Definition 2.1). Hence

$$\sum_{j\in J} \mu_j \, \phi(x_j) = \sum_{i\in J} \mu_i \, \phi(T(x_i)) \qquad \forall \phi \in C^0(\bar{\Omega}) \tag{3.14}$$

Since the set of the "points of concentration" $(x_j)_{j\in J}$ is finite, we will denote it by $C = \{x_1,..,x_s\}$, $s\in \mathbb{N}$. We claim that if x_i is a point which belongs to C then also $T(x_i)$ belongs to C.

Arguing by contradiction, let us suppose that there exists $x_1 \in C$ such that $T(x_1) \notin C$, that is $T(x_1) \neq x_j \;\; \forall j = 1,..,s$.. We can choose a function $\eta \in C^0(\bar{\Omega})$ such that

$$\eta(x) = \begin{cases} 0 & \text{if } x \in \{x_1,.,x_s,T(x_2),..,T(x_s)\} \\ 1 & \text{if } x = T(x_1) \end{cases} \tag{3.15}$$

Note that by the invertibility of T we have $T(x_1) = T(x_j)$ if $j \neq 1$ and therefore η is well defined.

Then (3.14) becomes $0 = \mu_1$, a contradiction since $\mu_1 > 0$. Then we have that $T(x_1) = x_m$ for some $m \in \{2,..,s\}$ and hence from (3.14)

$$\mu_m \phi(T(x_1)) + \sum_{i\neq m} \mu_i \phi(x_i) = \mu_1 \phi(T(x_1)) + \sum_{i=2}^{s} \mu_i \, \phi(T(x_i)) \tag{3.16}$$

and choosing the same function η of (3.15) we have $\mu_m = \mu_1$

Now, if we repeat such a procedure considering the points $T(x_1)$, $T^2(x_1)$ we have that $O_G(x_1) \subset C$. Since C is finite, we also have that $O_G(x_1)$ is finite and hence $O_G(x_1) = \{x_1, T(x_1), .., T^{k_1}(x_1)\}$ for some $k_1 \in \mathbb{N}$. Hence (3.13) becomes

$$\mu = \mu_1(\delta_{x_1} + \delta_{T(x_1)} + .. + \delta_{T^{k_1}(x_1)}) + \sum_{j=2}^{s'} \mu_j \, \delta_{x_j} \tag{3.17}$$

Then considering the other points $x_i \in C$ we have

$$\begin{cases} \mu = \sum_{i \in J'} \mu_i(\delta_{x_i} + \delta_{T(x_i)} + .. + \delta_{T^{k_i}(x_i)}) \\ \tilde{\mu} \geq S(\Omega) \sum_{i \in J'} \mu_i^{2/(p+1)}(\delta_{x_i} + \delta_{T(x_i)} + .. + \delta_{T^{k_i}(x_i)}) \end{cases} \tag{3.18}$$

k_i having the same meaning as k_1 and $J' \subset J$.
Recalling the definition of u_n we have

$$1 = \int_\Omega d\mu = \sum_{i \in J'} k_i \mu_i \tag{3.19}$$

and

$$\tilde{S}_\lambda(\Omega) = \int_\Omega d\tilde{\mu} \geq S(\Omega) \sum_{i \in J'} k_i \mu_i^{2/(p+1)} \tag{3.20}$$

Now we prove that J' is a singleton. Let us set $z_i = k_i \mu_i$ and we choose a function $\phi_i(x) \in C^\infty(\bar{\Omega}) \cap V^T(\Omega)$ such that

$$\phi_i(x) = \begin{cases} 1 & \text{on } x_i, T(x_i), .., T^{k_i}(x_i) \\ 0 & \text{on } x_j, T(x_j), .., T^{k_j}(x_j) \quad j \neq i \end{cases}$$

Hence the function $v_n(x) = u_n(x)\phi_i(x) \in V^T(\Omega)$ and from (3.13)' and (3.18) and the definition of $\phi_i(x)$ we get

$$\lim_{n\to\infty}\frac{\int_\Omega|\nabla v_n|^2-\lambda\int_\Omega v_n^2}{\left(\int_\Omega|v_n|^{p+1}\right)^{2/(p+1)}}=\lim_{n\to\infty}\frac{\int_\Omega|\nabla u_n|^2\phi_i^2}{\left(\int_\Omega|u_n|^{p+1}\phi_i^{p+1}\right)^{2/(p+1)}}=\tilde{S}_\lambda(\Omega)\frac{k_i\mu_i}{(k_i\mu_i)^{2/(p+1)}}=$$

$$\tilde{S}_\lambda(\Omega)(k_i\mu_i)^{2/N}\le\tilde{S}_\lambda(\Omega)\quad\text{since } u_n\to 0 \text{ weakly in } V^T(\Omega).$$

But u_n is a minimizing sequence for $\tilde{S}_\lambda(\Omega)$ and then we must have $k_i\mu_i=1\ \forall i\in J'$. This last fact, together with the relation $\sum_{i\in J'} k_i\mu_i=1$ implies that J' is a singleton. Therefore the set C is $C=\{\bar{x}, T(\bar{x}),..,T^{\bar{k}}(\bar{x})\}$ for some $\bar{x}\in\bar{\Omega}$; hence, from (3.18) and (3.19) we get

$$\mu=1/\bar{k}\ (\delta_{\bar{x}}+\delta_{T(\bar{x})}+..+\delta_{T^{\bar{k}}(\bar{x})})$$

Now we consider the sequence of functions

$$w_n^\varepsilon(x)=u_n(x)\psi_\varepsilon(|x-\bar{x}|)\qquad x\in\Omega,\ r>0$$

where $\psi_\varepsilon\in C^\infty([0,+\infty[)$ is a function satisfying $\psi_\varepsilon(t)=1$ for $t\in[0,\frac{\varepsilon}{4}]$ and $\psi_\varepsilon(t)=0$ for $t>\frac{\varepsilon}{2}$.

We have that, for every $\varepsilon\le\varepsilon_0$

$$\lim_{n\to\infty}\frac{\int_\Omega|\nabla w_n^\varepsilon(x)|^2-\lambda\int_\Omega(w_n^\varepsilon)^2(x)}{\left(\int_\Omega|w_n^\varepsilon(x)|^{p+1}\right)^{2/(p+1)}}=\tilde{S}_\lambda(\Omega)(\bar{k})^{-2/N}$$

On the other side, arguing as in [LPT] (Corollary 2.1) we have that

$$\lim_{\varepsilon\to 0} \frac{\int_\Omega |\nabla w_n^\varepsilon(x)|^2 - \lambda \int_\Omega (w_n^\varepsilon)^2(x)}{\left(\int_\Omega |w_n^\varepsilon(x)|^{p+1}\right)^{2/(p+1)}} \geq \Sigma$$

Then we must have

$$\tilde{S}_\lambda(\Omega)(\bar{k})^{-2/N} \geq \Sigma$$

which implies

$$\tilde{S}_\lambda(\Omega) \geq \Sigma\,(\bar{k})^{2/N} \geq \Sigma\,\zeta^{2/N}$$

and we have a contradiction with the hypothesis. Then u_n is relatively compact.

Now we consider the case when $\zeta = +\infty$; here we cannot have a "concentration Phenomenon" of u_n. In fact utilizing the invariance of the set C for the action of the group G, we have that if $x \in C$ then $y \in C$ $\forall y \in O_G(x)$. But then we have that C is an infinite set while we showed that C is finite.

Finally we consider the case when T has some fixed points in Ω. Obviously, if C does not contain any fixed point of T, we can repeat just the same proof of the case i) and the claim follows by the assumption $\tilde{S}_\lambda(\Omega) < 2^{2/N}\,\Sigma \leq \zeta^{2/N}\,\Sigma$. Otherwise let us suppose that there exists $x_\alpha \in C$ such that $x_\alpha = T(x_\alpha)$, with $x_\alpha \in \bar{\Omega} \setminus \bar{\Gamma}_1$. Let $\phi(x) \in C^\infty(\bar{\Omega})$ be a positive function such that

$$\phi(x_\alpha) = 1,\ \phi(x) = 0 \ \text{on } \Gamma_1 \text{ and } \phi(x) = 0 \text{ on the set } C \setminus \{x_\alpha\}.$$

Let $v_n(x) = u_n(x)\phi(x) \in H_0^1(\Omega)$.

We have that

$$\int_\Omega |v_n(x)|^{p+1} dx = \int_\Omega |u_n(x)|^{p+1} \phi(x)^{p+1} dx \to \mu_\alpha \phi(x_\alpha)^{p+1} = \mu_\alpha \leq 1\,.$$

Moreover

$$\int_\Omega |\nabla v_n(x)|^2 dx = \int_\Omega |\nabla u_n(x)|^2 \phi^2(x) dx + o(1)$$

and since from (3.13)' $\tilde{\mu} = \tilde{S}_\lambda(\Omega)\mu$ we have that

$$\int_\Omega |\nabla v_n(x)|^2 dx \to \int_\Omega \phi^2 d\tilde{\mu} = \tilde{S}_\lambda(\Omega) \int_\Omega \phi^2 d\mu = \mu_\alpha \tilde{S}_\lambda(\Omega).$$

Hence

$$\frac{\int |\nabla v_n|^2}{\left(\int |v_n|^{p+1}\right)^{2/(p+1)}} \to \frac{\mu_\alpha \tilde{S}_\lambda(\Omega)}{\mu_\alpha^{2/(p+1)}} = \mu_\alpha^{2/N}(\Omega)\, \tilde{S}_\lambda(\Omega) \le \tilde{S}_\lambda(\Omega) < 2^{2/N}\, \Sigma$$

by hypothesis, and it is not possible since

$$2^{2/N}\, \Sigma = S = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int |\nabla u|^2}{\left(\int_\Omega |u(x)|^{p+1}\right)^{2/(p+1)}}$$

and $v_n(x) \in H_0^1(\Omega)$. ♦

Remark 3.2

If Γ_1 is not smooth, but has only "convex corners", the critical constant in (3.5) and (3.6) will be $S(\Sigma(\alpha,R))$ where $\Sigma(\alpha,R)$ is a convex sector with vertex in some point of Γ_1 (See also Remark 2.2) .

3.2. AN EXISTENCE RESULT

Let us consider $\Omega^a = (]-a, a[)^{N-1}\times]0,a[\quad a>1$, where $\Gamma_1^a = \left[(]-a,a[)^{N-1}\times\{0\}\setminus (]-\frac{1}{a},\frac{1}{a}[^{N-1}\times\{0\})\right]\cup\left[(]-a, a[)^{N-1}\times\{a\}\right]\setminus(]-\frac{1}{a},\frac{1}{a}[^{N-1}\times\{a\})$ and $\Gamma_0^a = \partial\Omega\setminus\Gamma_1^a$.

Let T be the "rotation of the angle $\frac{\pi}{2}$ " around the axis x_N

$$T(x_1,x_2,x_3,.., x_N) = (-x_2,x_1,x_3,.., x_N) .$$

Obviously T verifies all the hypothesis of Definition 3.1 and the fixed points of T with respect to $\bar{\Omega}$ do not belong to Γ_1^a for every a. Moreover $T^4 = Id$.

Then by (3.6) of Theorem 3.2, we have a solution of (1.1) in $V^T(\Omega^a)$ if we proof that

$$\tilde{S}_0(\Omega) < 2^{2/N}\,\Sigma = S .$$

We denote by $\mathbb{R}_+^N = \{x\in\mathbb{R}^N : x_N \geq 0\}$

Let us consider $\phi = \phi(|x|)\in C^\infty(\mathbb{R}_+^N, [0,1])$ such that

$$\phi(|x|) = \begin{cases} 0 & \text{if } |x| \leq \frac{1}{4} \text{ or } |x| \geq 2 \\ 1 & \text{if } \frac{1}{2} \leq |x| \leq \frac{3}{2} \end{cases}$$

and for any integer $k \geq 1$ let $\varphi_k(|x|)\in C^\infty(\mathbb{R}_+^N, [0,1])$ be

$$\varphi_k(|x|) = \begin{cases} \phi(kx) & \text{if } |x| \leq \frac{1}{k} \\ \phi(\frac{x}{k}) & \text{if } |x| \geq k \\ 1 & \text{elsewhere} \end{cases}$$

Let

$$u_k(x) = \frac{\varphi_k(|x|)}{(1+|x|^2)^{(N-2)/2}} \quad , x\in\mathbb{R}^N$$

Arguing as in [Co] we have that

$$\lim_{k\to\infty} \frac{\int_{\mathbb{R}^N_+} |\nabla u_k|^2}{\left(\int_{\mathbb{R}^N_+} |u_k|^{p+1}\right)^{2/p+1}} = \lim_{k\to\infty} 2^{-2/N} \frac{\int_{\mathbb{R}^N} |\nabla u_k|^2}{\left(\int_{\mathbb{R}^N} |u_k|^{p+1}\right)^{2/p+1}} = 2^{-2/N} S = \Sigma$$

Then there exists a k_0 such that $\frac{\int_{\mathbb{R}^N_+} |\nabla u_k|^2}{\left(\int_{\mathbb{R}^N_+} |u_k|^{p+1}\right)^{2/(p+1)}} < 2^{2/N}\,\Sigma$ for any $k \geq k_0$. We remark that, for $a \geq 4k_0$ the functions $u_{k_0}(x) \in V^T(\Omega^a)$ Hence for all a large enough we have $\tilde{S}_0(\Omega^{a_0}) < 2^{2/N}\, S(\Sigma(\pi, R))$ and then we have a solution of (1.1) in $V^T(\Omega^a)$.

Let us remark that in this way we have proved the existence of a symmetric solution of (1.1) in Ω^a (i.e. invariant by the action of T). Note that if the infimum $S(\Omega^a)$ is also achieved by a function $v \in V(\Omega)$ we cannot say that there are two solutions of (1.1) in Ω^a since it could happen that $v \in V^T(\Omega^a)$. We end by observing that the domain Ω^a has zero curvature at any point of his boundary (see Introduction).

3.3. A MULTIPLICITY RESULT

Let us consider a Lipschitz continuous domain Ω_r defined by $\Omega_r = \{x \in \mathbb{R}^2$ such that $r < |x| < r + d\} \times \omega$, where $r, d > 0$, $N \geq 3$ and ω is a bounded domain in $\mathbb{R}^{N-2}$. Since Ω_r is supposed to be Lipschitz continuous it belongs to $\mathcal{E}_{\alpha_N}$ for some $\alpha_N \in]0, C_N/2[$ as we pointed out in chapter 1. Let $\Gamma_1 = \{x \in \mathbb{R}^2 / |x| = r\} \times \omega$ and $\Gamma_0 = \partial\Omega \setminus \Gamma_1$.

We consider the map $T_\alpha : \mathbb{R}^N \to \mathbb{R}^N$, $\alpha \in \mathbb{R}$

$$T_\alpha(\rho, \theta, x') = (\rho, \theta + \frac{2\pi}{\alpha}, x')$$

where (ρ, θ, x') is the system of coordinates

$$(3.21) \qquad \begin{cases} x_1 = \rho \cos \theta \\ x_2 = \rho \sin \theta \\ x_3' = x_3 \\ \quad \vdots \\ x_N' = x_N \end{cases}$$

In the new system of coordinates we have $\Omega_r = \{(\rho, \theta, x')$ such that $r < \rho < r + d$ $0 \leq \theta \leq 2\pi$, $x' \in \omega\}$.

Let

$$J^\lambda_{r,k} = \inf_{\substack{u \in V^{T_k}(\Omega_r) \\ \|u\|_{p+1} = 1}} \left(\int |\nabla u|^2 - \lambda \int u^2 \right) \quad \text{and} \quad J^\lambda_{r,\infty} = \inf_{\substack{u \in V^\infty(\Omega_r) \\ \|u\|_{p+1} = 1}} \left(\int |\nabla u|^2 - \lambda \int u^2 \right)$$

where k is a positive integer and $V^{T_k}(\Omega_r) = \{u \in V(\Omega_r)$ such that $u(T_k(x)) = u(x)\}$, $V^\infty(\Omega_r) = \{u \in V(\Omega_r)$ u such that $u(T_\alpha(x)) = u(x)$, $\forall \alpha \in \mathbb{R}\}$.

In other words u belongs to $V^\infty(\Omega_r)$ if u does not depend on θ. Using the same notation of the previous section, we have $\zeta = \zeta(T_k, \Omega_r) = k$ in the case $T = T_k$.

Obviously

$$J^{\lambda}_{r,1} \le J^{\lambda}_{r,2} \le \dots \le J^{\lambda}_{r,2^s} \le J^{\lambda}_{r,2^{s+1}} \le \dots \le J^{\lambda}_{r,\infty} .$$

Proposition 3.1.

There exists a constant $\lambda^*(r) > 0$ *such that* $J^{\lambda}_{r,k}$ *is achieved for every* $\lambda^*(r) < \lambda < \lambda_1(\Omega_r)$. *If* $N \ge 4$ *then* $\lambda^*(r) = 0$. *Moreover* $J^{\lambda}_{r,\infty}$ *is always achieved for every* $\lambda < \lambda_1(\Omega_r)$, $N \ge 3$.

Proof.

First we consider the case $J^{\lambda}_{r,\infty}$. Let u_n be a minimizing sequence for $J^{\lambda}_{r,\infty}$. As in the case of Theorem 3.1 u_n satisfies

$$-\Delta u_n = J^{\lambda}_{r,\infty} |u_n|^{p-1} u_n + \lambda u_n + f_n \qquad \text{with } f_n \to 0 \text{ in } [V^{\infty}(\Omega_r)]^*$$

If u_n is not relatively compact, arguing as in Theorem 3.2, we have that u_n "concentrates at a finite number of points of Ω_r". Let us denote by C this set. Because the sequence u_n belongs to $V^{T_k}(\Omega_r)$ for every $k \ge 1$, repeating the same proof of Theorem 3.2, we have that the cardinality of C is greater than k for every $k \ge 1$. Then C is an infinite set and we have a contradiction. Hence u_n is relatively compact.

Now let us consider $J^{\lambda}_{r,k}$, with $k < +\infty$. Let us define

$$\Omega^m_{r,k} = \{x \in \Omega_r \text{ such that } \frac{2\pi}{k}(m-1) \le \theta \le \frac{2\pi}{k} m , \ m \text{ integer } 1 \le m \le k\}$$

We have $\bigcup_{m=1}^{k} \Omega^m_{r,k} = \Omega_r$ and $T_k(\Omega^m_{r,k}) = \Omega^{m+1}_{r,k}$

We consider the functions

$$U^1_{\varepsilon,r}(x) = \frac{\phi(x)}{(\varepsilon + |x-x_0|^2)^{(N-2)/2}} \ , \ x \in \Omega^1_{r,k} \ , \ x_0 \in \partial\Omega^1_{r,k} \cap \Gamma_1 \ , \ x_0 \notin (\partial\Omega^k_{r,k} \cup \partial\Omega^2_{r,k})$$

where ϕ is a C^{∞}-function such that $\phi(x) = 1$ in $B_{\delta/2}(x_0)$, $\phi(|x|) = 0$ outside of $B_{\delta}(x_0)$ and $\delta > 0$ such that $B_{\delta}(x_0) \cap \Omega^1_{r,k} \subset \overset{o}{\Omega}{}^1_{r,k}$.

Then we set

$$U_{\varepsilon,r}(x) = \begin{cases} U^1_{\varepsilon,r}(x) & \text{for } x \in \Omega^1_{r,k} \\ U^1_{\varepsilon,r}(T^{-m+1}(x)) & \text{for } x \in \Omega^m_{r,k} \end{cases}$$

Then as in Brezis-Nirenberg([BN] (Lemma 1.1)) or also [Gr1] we have, for ε sufficiently small

$$\frac{\displaystyle\int_{\Omega^m_{r,k}} |\nabla U^m_{\varepsilon,r}|^2 - \lambda \int_{\Omega^m_{r,k}} U^m_{\varepsilon,r}}{\left(\displaystyle\int_{\Omega^m_{r,k}} |U^m_{\varepsilon,r}|^{p+1}\right)^{2/(p+1)}} < \Sigma \tag{3.22}$$

for $\lambda > 0$ if $N \geq 4$, for $\lambda > \lambda^*(r)$ if $N = 3$ where $\lambda^*(r)$ is a positive constant depending on Ω_r (see again [BN]).

Finally

$$\frac{\displaystyle\int_{\Omega_r} |\nabla U_{\varepsilon,r}|^2 - \lambda \int_{\Omega_r} U_{\varepsilon,r}}{\left(\displaystyle\int_{\Omega_r} |U_{\varepsilon,r}|^{p+1}\right)^{2/(p+1)}} < k^{2/N}\Sigma$$

and by Theorem 3.2 we have that $J^\lambda_{r,k}$ is achieved for any $k \geq 1$.

Lemma 3.1.

$$\lim_{r\to\infty} J^\lambda_{r,\infty} = +\infty \qquad \forall\lambda < \lambda_1(\Omega_r).$$

Proof.

By the previous lemma we have that for every $r > 0$ there exists $u_r = u_r(\rho,x') > 0$ such that

$$J^{\lambda}_{r,\infty} = \frac{\int_{\Omega_r} |\nabla u_r|^2 - \lambda \int_{\Omega_r} u_r^2}{\left(\int_{\Omega_r} |u_r|^{p+1} \right)^{2/(p+1)}}$$

If we consider the coordinates (ρ, θ, x') we have that

$$\int_{\Omega_r} |\nabla u_r|^2 \, dx_1 \ldots dx_N = 2\pi \int_{\omega} \int_{r}^{r+d} \left[\left(\frac{\partial u_r}{\partial \rho}\right)^2 + |\nabla_{x'} u_r|^2 \right] \rho d\rho dx'$$

Where

$$|\nabla_{x'} u_r|^2 = \sum_{j=3}^{N} \left(\frac{\partial u_r}{\partial x_i}\right)^2$$

and

$$\int_{\Omega_r} u_r^s = 2\pi \int_{r}^{r+d} \int_{\omega} u_r^s \rho d\rho dx' \qquad s > 0$$

Hence

$$(3.23) \qquad J^{\lambda}_{r,\infty} = \frac{2\pi \int_{r}^{r+d} \int_{\omega} \left[\left(\frac{\partial u_r}{\partial \rho}\right)^2 + |\nabla_{x'} u_r|^2 \right] \rho d\rho dx' - 2\pi\lambda \int_{r}^{r+d} \int_{\omega} u_r^2 \rho d\rho dx'}{(2\pi)^{2/(p+1)} \left(\int_{\omega} \int_{r}^{r+d} |u_r^{p+1}| \rho d\rho dx' \right)^{2/(p+1)}}$$

Let $D_r = \omega \times]r, r+d[$. Now

$$(3.24) \qquad \int_{\omega} \int_{r}^{r+d} |u_r^{p+1}| \rho d\rho dx' = \int_{D_r} |u_r^{p+1}| \rho^{p+1} \rho^{-p} d\rho dx' \leq r^{-p} \int_{D_r} |u_r \rho|^{p+1} d\rho dx'$$

First we suppose $N \geq 4$. If we consider the function $\rho u_r(\rho,x') : D_r \subset \mathbb{R}^{N-1} \to \mathbb{R}$ by the Sobolev Theorem we have $\rho u_r(\rho,x') \in L^{(2N-2)/(N-3)}$. Hence

$$r^{-p} \int_{D_r} |u_r \rho|^{p+1} d\rho dx' \leq C r^{-p} \Big(\int_{D_r} |u_r \rho|^{(2N-2)/(N-3)} d\rho dx' \Big)^{N(N-3)/(N-2)(N-1)} \leq \leq C$$

$$\frac{r^{-p}}{I} \Big(\int_{D_r} |\nabla(\rho u_r)|^2 d\rho dx' \Big)^{N/(N-2)}$$

where C does not depend on r and

$$I = \inf_{\substack{u \in \tilde{V}(D_r) \\ u \neq 0}} \frac{\int_{D_r} |\nabla u|^2}{\Big(\int_{D_r} |u|^{(2N-2)/(N-3)} d\rho dx' \Big)^{(N-3)/(N-1)}} = \inf_{\substack{u \in \tilde{V}(D_0) \\ u \neq 0}} \frac{\int_{D_0} |\nabla u|^2}{\Big(\int_{D_0} |u|^{(2N-2)/(N-3)} d\rho dx' \Big)^{(N-3)/(N-1)}} > 0$$

$\tilde{V}(D_r) = \{u \in H^1(D_r)$ such that $u = 0$ on $\partial D_r \setminus (\{r\} \times \omega)\}$.
(Note that I does not depend on r).

If $N = 3$ we have the same conclusion, because $\rho u_r(\rho,x') \in L^q(D_r)$ $\forall q \geq 1$.
Hence, from (3.24) and the previous computations we get

$$(3.25) \qquad \int_{D_r} u_r^{p+1} \rho d\rho dx' \leq C_1 r^{-p} \Big(\int_{D_r} |\nabla(\rho u_r)|^2 d\rho dx' \Big)^{N/(N-2)}$$

Now

$$|\nabla(\rho u_r)|^2 = \Big(|\nabla_{x'}(\rho u_r)|^2 + \Big(\frac{\partial(\rho u_r)}{\partial \rho}\Big)^2 \Big) = \rho^2 \Big[\Big(|\nabla_{x'} u_r|^2 + \Big(\frac{\partial u_r}{\partial \rho}\Big)^2 \Big) \Big] + 2\rho u_r \frac{\partial u_r}{\partial \rho} + u_r^2 \leq$$

$$\leq 2\rho^2 \Big[\Big(|\nabla_{x'} u_r|^2 + \Big(\frac{\partial u_r}{\partial \rho}\Big)^2 \Big) \Big] + 2u_r^2$$

Then

$$\Big(\int_{D_r} |\nabla(\rho u_r)|^2 d\rho dx' \Big)^{N/(N-2)} \leq C_2 \Big[\int_{D_r} \rho^2 \Big[\Big(|\nabla_{x'} u_r|^2 + \Big(\frac{\partial u_r}{\partial \rho}\Big)^2 \Big) \Big] d\rho dx' \Big]^{N/(N-2)} +$$

$$+ C_2[\int_{D_r} u_r^2]^{N/(N-2)} = K_1 + K_2$$

$$K_1 \le C_3(r+d)^{N/(N-2)}\Big(\int_{D_r} \rho\Big[\Big(|\nabla_{x'} u_r|^2 + \Big(\frac{\partial u_r}{\partial \rho}\Big)^2\Big)\Big]d\rho dx'\Big)^{N/(N-2)}$$

and

$$K_2 \le C_4 \int_{D_r} u_r^{p+1}\Big(\int_{D_r} 1\Big)^{(p-1)/(p+1)} = C_5 \int_{D_r} u_r^{p+1} = C_5 \int_{D_r} u_r^{p+1}\rho\rho^{-1} \le \frac{C_6}{r} \int_{D_r} u_r^{p+1}\rho$$

Then if r is sufficiently large (3.25) becomes

$$\int_{D_r} u_r^{p+1}\rho d\rho dx' \le C_7 r^{-p}(r+d)^{(p+1)/2}\Big(\int_{D_r} \rho\Big[\Big(|\nabla_{x'} u_r|^2 + \Big(\frac{\partial u_r}{\partial \rho}\Big)^2\Big)\Big]d\rho dx'\Big)^{N/(N-2)} +$$

$$+ \frac{C_7}{r^{p+1}} \int_{D_r} u_r^{p+1}\rho \le C_8 r^{(1-p)/2}\Big(\int_{D_r} \rho\Big[\Big(|\nabla_{x'} u_r|^2 + \Big(\frac{\partial u_r}{\partial \rho}\Big)^2\Big)\Big]d\rho dx'\Big)^{N/(N-2)} + \frac{1}{2} \int_{D_r} u_r^{p+1}\rho$$

Consequently

$$\Big(\int_{D_r} |u_r|^{p+1}\Big)^{2/(p+1)} \le C_9 r^{(1-p)/(1+p)}\Big(\int_{D_r} \rho\Big[\Big(|\nabla_{x'} u_r|^2 + \Big(\frac{\partial u_r}{\partial \rho}\Big)^2\Big)\Big]d\rho dx'\Big)$$

Then (by Hölder inequality)

$$J^{\lambda}_{r,\infty} = \frac{\int_{\Omega_r} |\nabla u_r|^2 - \lambda \int_{\Omega_r} u_r^2}{\left(\int_{\Omega_r} |u_r|^{p+1}\right)^{2/(p+1)}} = \tag{3.26}$$

$$= \frac{2\pi\int_r^{r+d}\int_\omega[(\frac{\partial u_r}{\partial\rho})^2+|\nabla_{x'}u_r|^2]\rho d\rho dx' - 2\pi\lambda\int_r^{r+d}\int_\omega u_r^2\rho d\rho dx'}{(2\pi)^{2/(p+1)}\left(\int_\omega\int_r^{r+d}|u_r^{p+1}|\,\rho d\rho dx'\right)^{2/(p+1)}} \geq$$

$$\geq \frac{r^{(p-1)/(1+p)}}{C_{10}} - (2\pi)^{2/N}\lambda\frac{\int_r^{r+d}\int_\omega u_r^2\rho^{2/(p+1)}\rho^{2/N}d\rho dx'}{(2\pi)^{2/N}\left(\int_\omega\int_r^{r+d}|u_r^{p+1}|\,\rho d\rho dx'\right)^{2/(p+1)}} \geq$$

$$\geq \frac{r^{(p-1)/(1+p)}}{C_{10}} - \lambda C_{11} \geq \frac{r^{(p-1)/(1+p)}}{C_{10}} - \lambda_1(\Omega_r)C_{11}$$

On the other side, for $r \geq 1$

$$\lambda_1(\Omega_r) = \inf_{\substack{u\in V(\Omega_r)\\ u\neq 0}} \frac{\int_{\Omega_r}|\nabla u|^2}{\int_{\Omega_r}|u(x)|^2} = \inf_{\substack{u\in V(\Omega_r)\\ u\neq 0}} \frac{\int_\omega\int_0^{2\pi}\int_r^{r+d}\left[(\frac{\partial u}{\partial\rho})^2 + \frac{1}{\rho^2}(\frac{\partial u}{\partial\theta})^2 + |\nabla_{x'}u_r|^2\right]\rho d\rho d\theta dx'}{\int_\omega\int_0^{2\pi}\int_r^{r+d}|u(\rho,\theta,x')|^2\rho d\rho d\theta dx'} \leq$$

$$\leq \frac{r+d}{r}\inf_{\substack{u\in V(\Omega_r)\\ u\neq 0}} \frac{\int_\omega\int_0^{2\pi}\int_r^{r+d}\left[(\frac{\partial u}{\partial\rho})^2 + (\frac{\partial u}{\partial\theta})^2 + |\nabla_{x'}u_r|^2\right]d\rho d\theta dx'}{\int_\omega\int_0^{2\pi}\int_r^{r+d}|u(\rho,\theta,x')|^2 d\rho d\theta dx'}.$$

Therefore if we set $\tilde{\Omega}_r = [0,2\pi[\times]r,r+d[\times\omega$ we have for r large enough

$$(3.27) \qquad \lambda_1(\Omega_r) \leq \frac{r+d}{r}\inf_{\substack{u\in V(\tilde{\Omega}_r)\\ u\neq 0}}\frac{\int_{\tilde{\Omega}_r}|\nabla u|^2}{\int_{\tilde{\Omega}_r}|u|^2} \leq 2\inf_{\substack{u\in V(\tilde{\Omega}_0)\\ u\neq 0}}\frac{\int_{\tilde{\Omega}_0}|\nabla u|^2}{\int_{\tilde{\Omega}_0}|u|^2} = 2\lambda_1(\tilde{\Omega}_0)$$

Hence, by (3.26) and (3.27) we have

$$\lim_{r\to\infty} J^{\lambda}_{r,\infty} = +\infty \qquad \forall\lambda < \lambda_1(\Omega_r).$$

Theorem 3.3.
Let $\Omega_r = \{x\in \mathbb{R}^2 / r < |x| < r+d\}\times\omega$ *and* $\lambda^*(r) < \lambda < \lambda_1(\Omega_r)$, $\lambda^*(r) = 0$ *for* $N \geq 4$, $\lambda^*(r) > 0$ *for* $N = 3$. *Then the number of nonrotationally equivalent solutions of* (1.1) *tends to* $+\infty$ *as* r *tends to infinity.*

Proof.
According to Proposition 3.1 we have that, for every $r > 0$, $\lambda^*(r) < \lambda < \lambda_1(\Omega_r)$

$$J^{\lambda}_{r,1}, J^{\lambda}_{r,2}, \dots, J^{\lambda}_{r,2^s}, J^{\lambda}_{r,2^{s+1}}, \dots, J^{\lambda}_{r,\infty}$$

are achieved by some nonnegative function. By a regularity result (consequence of a Theorem of Brezis and Kato, see[BK]) we have that the minimizers are functions of class C^2 in Ω_r.

Our aim is to prove that, for r large, $\exists k_0 = k_0(r)$ such that

$$J^{\lambda}_{r,1} < J^{\lambda}_{r,2} < \dots < J^{\lambda}_{r,2^{k_0}} .$$

Let us prove that, for r large $J^{\lambda}_{r,1} < J^{\lambda}_{r,2}$

Let u_r be a function such that $\|u_r\|_{p+1} = 1$ and $J^{\lambda}_{r,2} = \int_{\Omega_r} |\nabla u_r|^2 - \lambda \int_{\Omega_r} |u_r|^2$. If we consider the change of coordinates (3.21) we have

$$\int_{\Omega_r} |\nabla u_r|^2 - \lambda \int_{\Omega_r} |u_r|^2 = \int_r^{r+d}\int_0^{2\pi}\int_{\omega} \Big[\Big(\frac{\partial u_r}{\partial\rho}\Big)^2 + \frac{1}{\rho^2}\Big(\frac{\partial u_r}{\partial\theta}\Big)^2 + \Big(\frac{\partial u_r}{\partial x'}\Big)^2\Big]\rho\, d\rho\, d\theta\, dx' -$$
$$- \lambda \int_0^{2\pi}\int_r^{r+d}\int_{\omega} |u_r|^2\, \rho\, d\rho\, d\theta\, dx'$$

and

$$\int_{\Omega_r} |u_r|^{p+1} = \int_r^{r+d}\int_0^{2\pi}\int_{\omega} |u_r|^{p+1}\, \rho\, d\rho\, d\theta\, dx' .$$

Define $v \in V^{T_1}(\Omega)$ by

$$v_r(\rho, \theta, x') = u_r(\rho, \frac{\theta}{2}, x') \qquad 0 \leq \theta \leq 2\pi .$$

Therefore

$$\int_0^{2\pi} \int_r^{r+d} \int_\omega |\nabla v_r|^2 \rho d\rho d\theta dx' = \int_0^{2\pi} \int_r^{r+d} \int_\omega \left[\left(\frac{\partial u_r}{\partial \rho}\right)^2 + \frac{1}{4\rho^2} \cdot \left(\frac{\partial u_r}{\partial \theta}\right)^2 + \left(\frac{\partial u_r}{\partial x'}\right)^2 \right] \rho d\rho d\theta dx' .$$

Similarly we have

$$\int_{\Omega_r} |u_r|^{p+1} = \int_{\Omega_r} |v_r|^{p+1} \text{ and } \int_{\Omega_r} |u_r|^2 = \int_{\Omega_r} |v_r|^2 .$$

In the proof of Proposition 3.1 we showed that $J^\lambda_{r,2} < 2^{2/N} \Sigma$ for any $r > 0$. Therefore by Lemma 3.1 we have, for r large, $J^\lambda_{r,2} < J^\lambda_{r,\infty}$. Then $\frac{\partial u_r}{\partial \theta}$ is not identically zero. Hence

$$J^\lambda_{r,1} \leq \int_{\Omega_r} |\nabla v_r|^2 - \lambda \int_{\Omega_r} |v_r|^2 < \int_{\Omega_r} |\nabla u_r|^2 - \lambda \int_{\Omega_r} |u_r|^2 = J^\lambda_{r,2} .$$

Similarly, we can prove that $J^\lambda_{r,2} < J^\lambda_{r,4}$ if r is so large that $J^\lambda_{r,4} < J^\lambda_{r,\infty}$ and so on. Hence, for r large we obtain the solutions $u_1, u_2, \dots, u_{2k_0}$ (k_0 depending on r) that, clearly, are nonrotationally equivalent and the number $k_0 \to \infty$ as $r \to \infty$. ♦

References

[AM] ADIMURTHI - MANCINI G. : The Neumann problem for elliptic equations with critical nonlinearity *Nonlinear analysis. A tribute in honour of G. Prodi.* 1991. S.N.S. Pisa.

[Au] AUBIN T. : Problèmes isopérimétriques et espaces de Sobolev, *J. Diff. Geom* **11** (1976), 573-598.

[AY] ADIMURTHI - YADAVA S.L. : Critical exponent problem in $\mathbb{R}^2$ with Neumann boundary condition , *Comm. Part. Diff. Eq.*(to appear).

[BC1] BAHRI A. , CORON J. M. : On a nonlinear elliptic equation involving the critical Sobolev exponent : the effect of the topology of the domain, *Comm. Pure Appl. Math.* **XLI** (1988), 253 - 294.

[BC2] BAHRI A. , CORON J. M. : Sur une équation elliptique non linéaire avec l'exposant critique de Sobolev, *C. R. Acad. Sc.Paris.* **301** série I (1985), 345- 348.

[BP] BERESTYCKI H. , PACELLA F. : Symmetry properties for positive solutions of elliptic equations with mixed boundary conditions, *J. Funct. Anal.*, (to appear).

[Br1] BREZIS H. : Analyse fonctionelle, Masson - Paris (1983).

[Br2] BREZIS H. : Elliptic equations with limiting Sobolev exponents. The impact of topology, *Comm. Pure Appl. Mat.* **XXXIX** (1986), 17 - 39.

[Br3] BREZIS H. : Nonlinear elliptic equations involving the critical Sobolev exponent - survey and perspectives, *Directions in Part. Diff. Eq. - Acad. Press* (1987), 17 - 36.

[Br4] BREZIS H. : Some variational problem with lack of compactness. *Proceedings of Symposia in Pure Mathematics.* **45** (1986), 165-201.

[BC] BREZIS H. , CORON J. M. : Convergence of solutions of H-systems or how to blow bubbles, *Arch. Rat. Mech. Anal.* **89** (1985), 21 - 56.

[BL] BREZIS H. , LIEB E. : A relation between pointwise convergence of functions and convergence of functions, *Proc. A.M.S.* **88** (1983), 486 - 490.

[BK] BREZIS H. , KATO T : Remarks on the Schrödinger operator with singular complex potential, *J. Math. Pures Appl.* **58** (1979), 137-151.

[BN] BREZIS H. , NIRENBERG L. : Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **XXXVI** (1983), 437-477.

[Co] CORON J. M. : Topologie et cas limite des injections de Sobolev, *C.R.Acad. Sc. Paris.* **299** série I (1984), 209 - 212.

[Da]DANCER E. N. : A note on an equation with critical exponent, *Bull. London Math. Soc.* **20** (1988), 600-602.

[Di] DING W. : Positive solutions of $\Delta u + u^{(N+2)/(N-2)} = 0$ on contractible domains, (to appear).

[Eg] EGNELL H. : Elliptic boundary value problems with singular coefficients and critical nonlinearities, *Uppsala Univ. U.U.D.M. Report* **21** (1987).

[EPT] EGNELL H. , PACELLA F. , TRICARICO M. : Some remarks on Sobolev inequalities, *Nonlinear Analysis T.M.A.***9** (1989), 763-773.

[EL] ESTEBAN M. J. , LIONS P. L. : Existence and non-existence results for semilinear elliptic problems in unbounded domains, *Proceed. Royal Soc. Edinburgh,* **93A** (1982), 1 - 14.

[Gi] GIDAS B. : Symmetry properties and isolated singularities of positive solutions of nonlinear elliptic problems, *Nonlinear Partial Differential Equations in Engineering and Applied Science*, ed.R. Sternberg, A. Kalinowski and J. Papadakis. (New York : Dekker 1980).

[GNN] GIDAS B., NI W. M., NIRENBERG L. : Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209-243.

[Gr1] GROSSI M. : On some semilinear elliptic equations with critical nonlinearities and mixed boundary conditions, Rend Mat, Serie VII , Vol. **10**, 1990, 287 - 302.

[Gr2] GROSSI M. : Existence and multiplicity result in the presence of symmetry for elliptic equations with critical sobolev exponent , *Nonlinear Analysis T.M.A.*(to appear)

[Gr3] GROSSI M. : Elliptic equations with critical growth: the mixed boundary value problem, Master thesis (1989), S.I.S.S.A. Trieste.

[GP] GROSSI M. , PACELLA F. : Positive solution of nonlinear elliptic equation with critical Sobolev exponent and mixed boundary condition, *Proceed.Royal Soc.Edinb*, **116A**, 1990, 23 -43.

[Ho] HOFER H. : Variational and topological methods in partially ordered Hilbert spaces, *Math. Ann.* **261** (1982), 493 - 514.

[Ly] LI Y. Y. : Existence of many positive solutions of semilinear elliptic equations on Annulus, (Ph.D. Thesis - Courant Institute N.Y.U.(1988)).

[L1] LIONS P. L. : Applications de la méthode de concentration-compacité à l'existence de fonctions extrémales. *C.R. Acad. Sci. Paris* **296** (1983), 645-648.

[L2] LIONS P. L. : The concentration compactness principle in the calculus of variations. The limit case (part 1 and part 2), *Riv. Mat. Iberoamericana* **1** (1985), 145 -201 and 45-121.

[L3] LIONS P. L. : The concentration compactness principle in the calculus of variations . The locally compact case (part 1 and part 2). *Ann. I.H.P. Anal. Non Lin.* , (1984), 109-145 and 223-283.

[LP] LIONS P. L. , PACELLA F. : An isoperimetric inequality for convex cones, *Proceed. Amer. Math. Society* (to appear).

[LPT] LIONS P. L. , PACELLA F. , TRICARICO M. : Best constant in Sobolev inequalities for functions vanishing on some part of the boundary and related questions, *Indiana Univ. Math. J.* **2** (1988), 301 - 324 .

[Ma] MAZ'JA V. G. : Sobolev spaces, Springer - Verlag (1985).

[Pa] PACELLA F. : Some relative isoperimetric inequalities and applications to nonlinear problems, *Proceedings of the Meeting "Variational Problems"*, Paris, June 1988.

[PT] PACELLA F. , TRICARICO M. : Symmetrizazion for a class of elliptic equations with mixed boundary conditions, *Atti Sem. Mat. e Fis. Univ. Modena* **XXXIV**(1986), 75-94.

[Pas] PASSASEO D. : Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains, *Manuscripta Mathematica* **2**(1989), 147-165.

[Po] POHOZAEV S. : Eigenfunctions of the equations $\Delta u + f(u) = 0$, *Soviet Math. Dokl.* **6** (1965), 1408-1411.

[St] STRUWE M. : A global compactness result for elliptic boundary value problems involving limiting nonlinearities, *Math. Z.* **187** (1984), 511 - 517.

[T] TALENTI G. : Best constant in Sobolev inequality, *Annali di Mat.* **110** (1976), 353-372.