Scuola Internazionale Superiore di Studi Avanzati - Trieste

Orbifold Cohomology of ADE-singularities

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Advisors: Prof. Barbara Fantechi and
Prof. Lothar Göttsche

Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy
in the International School for Advanced Studies
Via Beirut 2-4, 34014 Trieste, Italy

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Chapter 1

Introduction

Orbifolds arise in geometry in two different ways and as a consequence they can be given two different descriptions. On one hand, as a topological space Y (respectively an algebraic variety) which is the union of open subsets U of the form $U = \tilde{U}/G$, where $\tilde{U}$ is a smooth manifold (respectively a smooth variety) and G is a finite group acting on it by diffeomorphisms (respectively algebraically). The data $(U, \tilde{U}, G)$, as U varies, define an orbifold structure $[Y]$ on Y. This way of thinking about orbifolds is in some sense concrete and geometric, but, to define a good notion of maps between such objects requires some work. On the other hand, from moduli problems, orbifolds arise in a more abstract but stil natural way. Orbifolds arising from moduli problems are called smooth algebraic stacks in the sense of Deligne-Mumford [21]. If $[Y]$ denotes such a stack, the coarse moduli space Y associated to it is, locally in the étale topology, the quotient of a smooth variety by the action of a finite group. This variety parametrizes objects modulo isomorphisms.

Gromov-Witten invariants for proper orbifolds have been defined first by Chen and Ruan [17] for almost complex orbifolds. Their construction arises from orbifold string theory introduced by Dixon, Harvey, Vafa and Witten [22], [23], for orbifolds which are global quotients. In the algebraic category, orbifold Gromov-Witten theory has been developed by Abramovich, Corti, Graber and Vistoli [1], [2]. The definition of Gromov-Witten invariants for orbifolds is similar to that for almost complex manifolds and smooth varieties. The main change is that one replaces stable maps by twisted stable maps; these are orbifold morphisms from nodal curves with orbifold structure to a target orbifold. These morphisms have to satisfy some stability conditions that guarantee the existence of a compact moduli space. Similar to the smooth case, the genus zero orbifold Gromov-Witten invariants can be used to define an orbifold quantum cohomology. The degree zero part of the orbifold quantum cohomology, i.e. using only degree zero maps, is the orbifold cohomology ring of $[Y]$.

The orbifold cohomology of $[Y]$ has been defined by Chen and Ruan [16] for almost complex orbifolds. This has been extended to a noncommutative ring by Fantechi and Göttsche [28] in the case where the orbifold is a global quotient. Abramovich, Graber and Vistoli defined the orbifold cohomology in the algebraic case [2].

The underlying vector space of the orbifold cohomology is the direct sum of the cohomology of Y and the cohomology of the twisted sectors. These are orbifolds that parametrize points of $[Y]$ together with nontrivial automorphisms. Note that, if $U = \tilde{U}/G$ is an open subset of Y and $\tilde{y} \in \tilde{U}$, by an automorphism of $\tilde{y}$ we mean an element of the stabilizer of $\tilde{y}$ in G. Twisted stable maps, being orbifold morphisms, carry informations about automorphisms of points. So, twisted sectors come naturally in the picture of orbifold cohomology.

We can see here a difference between the smooth case and the orbifold case. In the smooth case, the degree zero part of the quantum cohomology is the ordinary cohomology of the target space. In the orbifold case, the degree zero part contains the ordinary cohomology of Y as a subring.

Let $[Y]$ be a global quotient orbifold, i.e. its underlying space Y is of the form $Y = X/G$. Originating in Physics [22], [23], [62], cohomological invariants of the orbifold $[Y]$ have been defined and studied [6]. These invariants can be recovered from the additive structure of $H^*_{orb}([Y])$. However, the orbifold cohomology ring contains much more information, notably the orbifold cup product.

Assume that $[Y]$ is a Gorenstein orbifold (see Definition 3.4.3) and that the coarse moduli space Y admits a crepant resolution $\rho : Z \to Y$: this means that Z is a smooth variety, ρ is an isomorphism outside the singular locus and $\rho^*(K_Y) = K_Z$ (see also Definition 3.4.4). Here K_Y and K_Z are the canonical bundle of Y and Z respectively.

Motivated from Physics, Ruan stated the following *cohomological crepant resolution conjecture* [52], see also Conjecture 3.4.16. Let $\rho : Z \to Y$ be a crepant resolution, assume further that we have chosen an integral basis $\beta_1, ..., \beta_n$ of the kernel Ker ρ_* of the group homomorphism $\rho_* : H_2(Z, \mathbb{Q}) \to H_2(Y, \mathbb{Q})$. Then, we assign a formal variable q_l to each class β_l. Using contributions from genus zero Gromov-Witten invariants of Z whose homology classes belong to Ker ρ_*, we can deform the ordinary cup product on $H^*(Z, \mathbb{C})$ in order to obtain new ring structures. This deformation will depend on the complex parameters $q_1, ..., q_n$. Giving these parameters the same value -1, we get the so called *quantum corrected cohomology ring* $H^*_\rho(Z, \mathbb{C})$. Then the conjecture states that, under suitable assumptions, $H^*_\rho(Z, \mathbb{C})$ is isomorphic to $H^*_{orb}([Y])$.

There are several motivations for studying this conjecture. It is related to the problem of understanding the behaviour of quantum cohomology under birational transformations. Another motivation comes from mirror symmetry as most of the known Calabi-Yau 3-folds are crepant resolution of Calabi-Yau orbifolds.

The conjecture has been proved in the following cases: for $Y = S^{(2)}$, the symmetric product of a compact complex surface S, and $Z = S^{[2]}$ the Hilbert scheme of two points in S, in [38]; for $Y = S^{(r)}$, the symmetric product of a complex projective surface S with trivial canonical bundle $K_S \cong \mathcal{O}_S$, and $Z = S^{[r]}$ the Hilbert scheme of r points in S, in [28], Theorem 3.10; for $Y = V/G$ the quotient of a symplectic vector space of finite dimension by a finite group of symplectit automorphisms, and Z a crepant resolution, when it does exists, in [32], Theorem 1.2. Notice that in the last two cases, the crepant resolution Z has a holomorphic symplectic form, so the Gromov-Witten invariants vanish and there are no quantum corrections. This means that the orbifold cohomology ring is isomorphic to the ordinary cohomology ring of Z.

In this thesis, we study Ruan's conjecture for orbifolds with transversal ADE singularities, which are in a sense the simplest ones to which the conjecture applies. Indeed, assume that in the orbifold $[Y]$ the open subset of points with trivial automorphisms group is dense. If $[Y]$ is Gorenstein, and even under the much more general assumptions that no chart $(U, \tilde{U}, G)$ contains a pseudoreflection, then the orbifold structure $[Y]$ can be recovered by the singular variety or analytic space Y. The simplest possible Gorenstein singularities are the ADE surface singularities (or rational double points also called Du Val singularities).

An orbifold $[Y]$ has transversal ADE singularities if, étale locally, the coarse moduli space Y is isomorphic to a product $R \times \mathbb{C}^k$, where R is a germ of an ADE singularity. In general for any Gorenstein orbifold $[Y]$, there exists a closed subset $W \subset Y$ of codimension ≥ 3 such that $Y \backslash W$ has transversal ADE singularities. So the case we study is in a natural sense the simplest, ignoring higher codimension phenomena. On the other hand, transversal ADE singularities occur natural in many contexts: transversal A_1 in $S^{(2)}$ for S a surface; transversal A_n in many complete intersections in weighted projective spaces; they also are, at least locally, examples of symplectic orbifolds.

In this thesis, we lay down the structure to deal with general ADE singularities. After that, we concentrate on the transversal A_n case, with a further mild technical assumption we address Ruan's conjecture by computing explicitly both the orbifold cohomology and the quantum corrections. The former is achieved in general, for the later we have an explicit conjecture (see Conjecture 6.1.6) which is only verified under additional, and somewhat unnatural, technical assumptions. We include a sketch of an argument which would prove the conjecture in general, assuming some technical results which we cannot so far prove. The conjecture is proven fully in the transversal A_1 case. Finally, we construct an explicit isomorphism between the orbifold cohomology ring $H^*_{orb}([Y])$ and the quantum corrected cohomology ring $H^*_\rho(Z)$ in the transversal A_1 case, verifying Ruan's conjecture. In the transversal A_2 case, the quantum corrected 3-point function (see Chapter 3.4.2) can not be evaluated in $q_1 = q_2 = -1$, so that $H^*_\rho(Z, \mathbb{C})$ is not defined. Thus Ruan's conjecture has to be slightly modified. However we have found that, if $q_1 = q_2$ is a third root of the unit different to 1, then the resulting ring $H^*(Z)(q_1, q_2)$ is isomorphic to the orbifold cohomology ring $H^*_{orb}([Y])$. Also in this case we give an explicit isomorphism between them (see Theorem 7.2.4). Thus proving a slightly modified version of Ruan's conjecture. We expect that in the A_n case, the same modification of Ruan's conjecture holds, i.e. if $q_1 = ... = q_n$ is an $(n+1)$-th root of unity such that $q_1 \cdots q_n \neq 1$, the resulting ring $H^*(Z)(q_1, ..., q_n)$ is isomorphic to the orbifold cohomology ring $H^*_{orb}([Y])$.

The structure of the thesis is the following.

In Chapter 1 we collect some basic definitions on orbifolds, morphisms of orbifolds and orbifold vector bundles. In Chapter 3 we first review the definition of orbifold cohomology ring for a complex orbifold, then we state the *cohomological crepant resolution conjecture* as given by Ruan in [52]. In Chapter 4 we define orbifolds with transversal ADE-singularities, see Definition 4.2.5. Then we give a description of the twisted sectors in general. Finally we specialize to orbifolds with transversal A_n-singularities and, under the technical assumption of trivial monodromy, we compute the orbifold cohomology ring. In Chapter 5 we study the crepant resolution. We first show that any variety with transversal ADE-singularities Y has a unique crepant resolution $\rho : Z \to Y$, Proposition 5.2.1. Then we restrict our attention to the case of transversal A_n-singularities and trivial monodromy and we give an explicit description of the cohomology ring of Z. Chapter 6 contains the computations of the Gromov-Witten invariants of Z in the A_n case. We also give a description of the quantum corrected cohomology ring of Z. In Chapter 7 we prove Ruan's conjecture in the A_1 case and, in the A_2 case with minor modifications.

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Chapter 2

Orbifolds

We collect here some basic definitions. First of all we define smooth (or C^∞) orbifolds and complex holomorphic orbifold using local charts, then we give some examples of orbifolds to explain the definition. We give the definition of a morphism between orbifolds and we show that the category of orbifolds is a 2-category, which means that given two orbifolds the set of morphisms between them has a structure of a groupoid. Finally we define orbifold vector bundles over a given orbifold. As an example of orbifold vector bundle, we will recall the definition of the tangent bundle to any given orbifold.

2.1 Definition of orbifold

The notion of orbifold was first introduced by Satake in [53] under the name of *V-manifold*. A characterization of orbifolds, as defined by Satake, in terms of their sheaves is given in [43]. A different definition was given by Chen and Ruan in [17]. The second definition is slightly general then the first, indeed it leads to the notion of non reduced orbifold while an orbifold as defined by Satake is called reduced. In [17], section 4.1, the authors prove that the two definitions are equivalent, once restricted to reduced orbifolds.

In this paper we will follow [53] and [43] with minor modifications in order to include non reduced orbifolds.

Let Y be a paracompact Hausdorff topological space. A *uniformizing system ! uniformizing system* for an open subset $U \subset Y$ is a collection of the following objects:

$\tilde{U}$ be a connected open subset of $\mathbb{R}^d$;

G be a finite group of C^∞-automorphisms of $\tilde{U}$ such that: the fixed-point set of each element of the group is either the whole space or of codimension at least 2, the multiplication in G is given by $g_1 \cdot g_2 = g_1 \circ g_2$ where $\circ$ is the composition;

χ be a continuous map $\tilde{U} \to U$ that induces an homeomorphism from $\tilde{U}/G$ to U, where $\tilde{U}/G$ is the quotient space with the quotient topology. Here G acts on $\tilde{U}$ on the left.

We will call the subgroup of G which consists of elements fixing the whole space the *kernel* of the action, and will be denoted by Ker(G).

Notation 2.1.1. Given an open subset U of Y, a uniformizing system for U will be denoted by $(\tilde{U}, G_U, \chi_U)$. If the dependence from U is clear from the context, it will also be denoted by $(\tilde{U}, G, \chi)$.

Definition 2.1.2. *The* ***dimension*** *of an uniformizing system* $(\tilde{U}, G, \chi)$ *is the dimension of* $\tilde{U}$ *as a real manifold.*

Let $(\tilde{U}, G, \chi)$ and $(\tilde{U}', G', \chi')$ be uniformizing systems for U and U' respectively, and let $U \subset U'$. An *embedding* between such uniformizing systems is a pair (φ, λ), where $\varphi : \tilde{U} \to \tilde{U}'$ is a smooth embedding such that $\chi' \circ \varphi = \chi$ and $\lambda : G \to G'$ is a group homomorphism such that $\varphi \circ g = \lambda(g) \circ \varphi$ for all $g \in G$. Furthermore, λ induces an isomorphism from Ker(G) to Ker(G').

Definition 2.1.3. *An* ***orbifold atlas*** *on Y is a family $\mathcal{U}$ of uniformizing systems for open sets in Y satisfying the following conditions:*

1. *The family $\chi(\tilde{U})$ is an open covering of Y, for $(\tilde{U}, G, \chi) \in \mathcal{U}$.*
2. *Let $(\tilde{U}, G, \chi), (\tilde{U}', G', \chi') \in \mathcal{U}$ be uniformizing systems for U and U' respectively, and let $U \subset U'$. Then there exists an embedding $(\varphi, \lambda) : (\tilde{U}, G, \chi) \to (\tilde{U}', G', \chi')$.*

3. *Let $(\tilde{U}, G, \chi), (\tilde{U}', G', \chi') \in \mathcal{U}$ be uniformizing systems for U and U' respectively. Then, for any point $y \in U \cap U'$, there exists an open neighbourhood $U'' \subset U \cap U'$ of y and a uniformizing system $(\tilde{U}'', G'', \chi'')$ for U'' which belong to the family $\mathcal{U}$.*

Two such atlases are said to be equivalent if they have a common refinement, where an atlas $\mathcal{U}$ is said to refine $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists an embedding into some chart in $\mathcal{V}$.

*A **smooth orbifold** structure on a paracompact Hausdorff topological space Y is an equivalence class of orbifold atlases on Y.*

Notation 2.1.4. We denote by $[Y]$ the smooth orbifold structure on the topological space Y. We will call this simply an orbifold.

Definition 2.1.5. *The orbifold $[Y]$ is said of **real dimension** d if all the uniformizing systems of an atlas have dimension d.*

Remark 2.1.6. Every orbifold atlas for Y is contained in a unique maximal one, and two orbifold atlases are equivalent if and only if they are contained in the same maximal one. Therefore we shall often tacitly work with a maximal atlas.

Proposition 2.1.7 ([43] Proposition A.1). Let (φ, λ) and (ψ, μ) be two embeddings from $(\tilde{U}, G, \chi)$ to $(\tilde{U}', G', \chi')$. Then, there exists a $g' \in G$ such that

$$\psi = g' \circ \varphi \quad \text{and} \quad \mu = \mathrm{g}' \cdot \lambda \cdot \mathrm{g}'^{-1}.$$

Moreover, this g' is unique up to composition by an element of $\mathrm{Ker(G')}$.

Remark 2.1.8. Notice that, for any uniformizing system $(\tilde{U}, G, \chi)$ and $g \in G$, there exists a $g' \in G'$ such that $\varphi \circ g = g' \circ \varphi$. Moreover this g' is unique up to an element in $\mathrm{Ker(G')}$. Thus, in the definition of an embedding (φ, λ), the existence of λ is required to guarantee a continuity for the kernels of the actions.

Remark 2.1.9. The previous remark implies that, for any embedding (φ, λ), the group homomorphism λ is injective.

Lemma 2.1.10 ([43], Lemma A.2). Let $(\varphi, \lambda) : (\tilde{U}, G, \chi) \to (\tilde{U}', G', \chi')$ be an embedding. If $g' \in G'$ is such that $\varphi(\tilde{U}) \cap (g' \circ \varphi)(\tilde{U}) \neq \emptyset$, then g' belongs to the image of λ.

Remark 2.1.11. Let $(\tilde{U}, G, \chi)$ be a uniformizing system for the open subset U of Y. Let $U' \subset U$ be an open subset. Then we have an *induced uniformizing system* $(\tilde{U}', G', \chi')$ for U', where $\tilde{U}'$ is a connected component of $\chi^{-1}(U')$ and G' is the maximal subgroup of G that acts on $\tilde{U}'$. Clearly there is an embedding of $(\tilde{U}', G', \chi')$ in $(\tilde{U}, G, \chi)$.

It follows that, for a given orbifold $[Y]$, we can choose an orbifold atlas $\mathcal{U}$ arbitrarily fine.

Remark 2.1.12. Let $[Y]$ be an orbifold and let $(\tilde{U}_1, G_1, \chi_1)$, $(\tilde{U}_2, G_2, \chi_2)$ be uniformizing systems for open subsets U_1, U_2 of Y in the same orbifold structure $[Y]$. For any point $y \in U_1 \cap U_2$, there is an open neighbourhood U_{12} of y such that $U_{12} \subset U_1 \cap U_2$, a uniformizing system $(\tilde{U}_{12}, G_{12}, \chi_{12})$ for U_{12}, compatible with $[Y]$, and embeddings $(\varphi_i, \lambda_i) : (\tilde{U}_{12}, G_{12}, \chi_{12}) \to (\tilde{U}_i, G_i, \chi_i)$ for $i \in \{1, 2\}$. So, we ge the isomorphism:

$$\varphi_{12} := \varphi_2 \circ \varphi_1^{-1} : \varphi_1(\tilde{U}_{12}) \to \varphi_2(\tilde{U}_{12}).$$

Let now U_1, U_2 and U_3 be open subsets of Y such that $U_1 \cap U_2 \cap U_3 \neq \emptyset$, and assume that there are uniformizing systems $(\tilde{U}_1, G_1, \chi_1)$, $(\tilde{U}_2, G_2, \chi_2)$ and $(\tilde{U}_3, G_3, \chi_3)$ for U_1, U_2 and U_3 respectively. Then, from Proposition 2.1.7, there exists $g \in G_3$ such that

$$\varphi_{23} \circ \varphi_{12} = g \circ \varphi_{13},$$

where the equation holds if we restrict the functions to some open subsets of the domains.

Definition 2.1.13. *A* ***reduced*** *orbifold is an orbifold structure* $[Y]$ *on* Y *such that there exists an orbifold atlas* $\mathcal{U}$ *for* $[Y]$ *with the following propertiy: for any uniformizing system* $(\tilde{U}, G, \chi) \in \mathcal{U}$, Ker(G) *is the trivial group.*

Definition 2.1.14. *Let* $[Y]$ *be a smooth orbifold and* $y \in Y$ *be a point. A* ***uniformizing system for*** $[Y]$ ***at*** y *is given by an open neighbourhood* U_y *of* y *in* Y *and a uniformizing system* $(\tilde{U}, G, \chi)$ *for* U_y *in the orbifold structure* $[Y]$ *such that,* $\tilde{U} \subset \mathbb{R}^d$ *is a ball centered in the origin* $0 \in \mathbb{R}^n$, G *acts trivially on* 0 *and* $\chi^{-1}(y) = 0$.

Notation 2.1.15. For a given orbifold $[Y]$ and a point $y \in Y$, a uniformizing system at y will be denoted by $(\tilde{U}_y, G_y, \chi_y)$ and $\chi(\tilde{U}_y)$ by U_y. The group G_y will be also called the local group at y.

We now define a complex orbifold. We will use the same notation as in the smooth case.

Let Y be a paracompact Hausdorff topological space. A *complex uniformizing system* for an open subset U of Y is a triple $(\tilde{U}, G, \chi)$, where $\tilde{U} \subset \mathbb{C}^d$ is a connected open subset, G is a finite group of holomorphic automorphisms of $\tilde{U}$ and χ is a continuous map satisfying the same properties required in the smooth case.

Definition 2.1.16. *The* ***complex dimension*** *of a complex uniformizing system* $(\tilde{U}, G, \chi)$ *is the dimension of* $\tilde{U}$ *as a complex manifold.*

Let $(\tilde{U}, G, \chi)$ and $(\tilde{U}', G', \chi')$ be complex uniformizing systems for U and U' respectively, and let $U \subset U'$. A *complex embedding* between such uniformizing systems is a pair (φ, λ) satisfying the same properties stated in the smooth case but where $\varphi : \tilde{U} \to \tilde{U}'$ is holomorphic.

Definition 2.1.17. *A* ***complex orbifold atlas*** *on* Y *is a family* $\mathcal{U}$ *of complex uniformizing systems for open sets in* Y *satisfying the conditions 1., 2. and 3. of Definition 2.1.3 where we replace embeddings with complex embeddings.*

Two such atlases are said to be equivalent if they have a common refinement, where an atlas $\mathcal{U}$ *is said to refine* $\mathcal{V}$ *if for every chart in* $\mathcal{U}$ *there exists a complex embedding into some chart in* $\mathcal{V}$.

A ***complex orbifold*** *structure on a paracompact Hausdorff topological space* Y *is an equivalence class of complex orbifold atlases on* Y.

Definition 2.1.18. *The complex orbifold* $[Y]$ *is said of* ***complex dimension*** d *if all the uniformizing systems of a complex atlas have complex dimension* d.

Remark 2.1.19. Proposition 2.1.7, Lemma 2.1.10 and Remarks 2.1.6, 2.1.8, 2.1.9, 2.1.11 and 2.1.12, holds in the complex case. The notions of reduced complex orbifold and of uniformizing systems at a point are defined for complex orbifolds in the same way of the smooth case.

Lemma 2.1.20 (*Linearization lemma*,[15] Theorem 4.). Let $[Y]$ be a complex orbifold and let $y \in Y$ be a point. Then we can choose a local uniformizing system $(\tilde{U}_y, G_y, \chi_y)$ at y such that G_y acts linearly on $\tilde{U}_y$.

2.2 Examples of orbifolds

In this section we give some examples of orbifolds, principally of complex orbifolds.

Example 2.2.1 (*Smooth and complex manifolds*). Let Y be a smooth (resp. complex) manifold. Let (U, α) be a chart, where $U \subset Y$ is open and $\alpha : U \to \mathbb{R}^d$ (resp. $\mathbb{C}^d$) is an homeomorphism with the image. Then $(\tilde{U} = \alpha(U), G = \{1\}, \chi = \alpha^{-1})$ is a smooth (resp. complex) uniformizing system for U. Let (U', α') be another chart in the same smooth (resp. complex) atlas of (U, α). Then the previous construction gives smooth (resp. complex) uniformizing system for $U \cap U'$ and for U'. Moreover we have smooth (resp. complex) embeddings corresponding to $U \cap U' \subset U, U'$. It follows that a smooth (resp. complex) manifold has a natural orbifold structure.

Example 2.2.2 (*Global quotients*). Let X be a smooth (resp. complex) manifold with the action of a finite group of diffeomorphisms (resp. biholomorphic transformations) G. Let Y be the quotient space with the quotient topology, $Y = X/G$. Clearly $\mathcal{U} = \{(U = X, G, \chi)\}$ is a smooth (resp. complex) orbifold atlas for Y, where $\chi : X \to Y$ is the quotient map. So the class of $\mathcal{U}$ define a smooth (resp. complex) orbifold structure over Y. This orbifold structure will be denoted by $[X/G]$ and is called *global quotient*.

Example 2.2.3 (*Weighted projective space* $[\mathbb{P}(a_0, ..., a_d)]$). Let $a_0, ..., a_d$ be positive integers. Then $\mathbb{P}(a_0, ..., a_d)$ is, by definition, the quotient

$$\mathbb{P}(a_0, ..., a_d) = (\mathbb{C}^{d+1} - \{0\})/\mathbb{C}^*$$

of $\mathbb{C}^{d+1} - \{0\}$ under the equivalence relation

$$(x_0, ..., x_d) \sim (\lambda^{a_0} x_0, ..., \lambda^{a_a} x_d) \qquad \text{for} \quad \lambda \in \mathbb{C}^*.$$

For any $(x_0, ..., x_d) \in \mathbb{C}^{d+1} - \{0\}$ we will denote with $[x_0, ..., x_d] \in \mathbb{P}(a_0, ..., a_d)$ the equivalence class of $(x_0, ..., x_d)$.

Let $X_i = \{x_i \neq 0\} \subset \mathbb{C}^{d+1} - \{0\}$ and $U_i = X_i/\mathbb{C}^*$. For any $i = 0, ..., d$, let $\tilde{U}_i = \mathbb{C}^d$ with coordinates $(z_0, ..., \hat{z}_i, ..., z_d)$ and $G_i = \mu_{a_i}$, the group of a_i-th rooth of unity, acting on $\tilde{U}_i$ as follows

$$\epsilon \cdot (z_0, ..., \hat{z}_i, ..., z_d) = (\epsilon^{-a_0} z_0, ..., \epsilon^{-a_d} z_d) \qquad \text{for} \quad \epsilon \in \mu_{a_i}.$$

Then the map $\tilde{U}_i \to \mathbb{P}(a_0, ..., a_d)$ defined by

$$(z_0, ..., \hat{z}_i, ..., z_d) \mapsto [z_0, ..., z_{i-1}, 1, z_{i+1}, ..., z_d]$$

induces an isomorphism between $\tilde{U}_i/\mu_{a_i}$ and U_i.

In order to give an orbifold atlas on $\mathbb{P}(a_0, ..., a_d)$ it is enough to construct, for any $[x] \in \mathbb{P}(a_0, ..., a_d)$, a uniformizing system at $[x]$, say $(\tilde{U}_{[x]}, G_{[x]}, \chi_{[x]})$, and embeddings $(\tilde{U}_{[x]}, G_{[x]}, \chi_{[x]}) \to (\tilde{U}_i, \mu_{a_i}, \chi_i)$ whenever $U_{[x]} \subset U_i$.

Let $[x] \in U_i$, then take any $z \in \tilde{U}_i$ such that $[z] = [x]$. Define $G_{[x]}$ to be the stabilizer in μ_{a_i} of z. For $\tilde{U}_{[x]}$ take a small ball in $\tilde{U}_i$ such that $G_{[x]}$ acts on it and, for any $g \in \mu_{a_i} - G_{[x]}$, $g(\tilde{U}_{[x]}) \cap \tilde{U}_{[x]} = \emptyset$. Let $U_{[x]}$ be the quotient of $\tilde{U}_{[x]}$ by $G_{[x]}$ and $\chi_{[x]} : \tilde{U}_{[x]} \to U_{[x]}$ be the quotient map. By construction there is an embedding $(\tilde{U}_{[x]}, G_{[x]}, \chi_{[x]}) \to (\tilde{U}_i, \mu_{a_i}, \chi_i)$.

If $U_{[x]} \subset U_j$ for some $j \neq i$ we have to construct an embedding

$$(\tilde{U}_{[x]}, G_{[x]}, \chi_{[x]}) \to (\tilde{U}_j, \mu_{a_j}, \chi_j).$$

Notice that, if $U_i \cap U_j \neq \emptyset$, for any $z \in \tilde{U}_i$ such that $[z] \in U_i \cap U_j$ there is a biholomorphic map with domain a suitable neighbourhood of z in $\tilde{U}_i$ and values in $\tilde{U}_j$ defined as follow

$$\tilde{\phi}_{ij} : (z_0, ..., \hat{z}_i, ..., z_d) \mapsto \left(\frac{z_0}{z_j^{a_0/a_j}}, ..., \frac{1}{z_j^{a_i/a_j}}, ..., \hat{z}_j, ..., \frac{z_d}{z_j^{a_d/a_j}} \right)$$

where z_j^{1/a_j} is a choosed a_j-th rooth of z_j. Then using this we obtain the required embedding. Notice that, if $z' = \tilde{\phi}_{ij}(z)$, then $[z'] = [x]$ and the stabilizers of z' in μ_{a_j} and of z in μ_{a_i} are isomorphic.

Example 2.2.4 (*Hypersurfaces in weighted projective spaces*). We use the same notation of the previous example. A polynomial F in the variables $x_0, ..., x_d$ is *weighted homogeneous* of degree D, if the following holds

$$F(\lambda^{a_0} x_0, ..., \lambda^{a_d} x_d) = \lambda^D F(x_0, ..., x_d),$$

for any $\lambda \in \mathbb{C}^*$. In this case, the equation $F = 0$ defines a closed subvariety Y of $\mathbb{P}(a_0, ..., a_d)$.

Assume that the affine variety $\{F = 0\}$ is smooth in $\mathbb{C}^{d+1} - \{0\}$. Than the same construction of the previous example can be used to give an orbifold structure over Y.

Remark 2.2.5. There is a more algebraic construction of $\mathbb{P}(a_0, ..., a_d)$ which goes as follows ([24]). Define $S(a_0, ..., a_d)$ to be the polynomial ring $\mathbb{C}[x_0, ..., x_d]$ graded by the condition $deg(x_i) = a_i$, for $i \in \{0, ..., d\}$. Then

$$\mathbb{P}(a_0, ..., a_d) = \text{Proj}(S(a_0, ..., a_d)).$$

Here we use the definition given in [33] for the Proj of a graded $\mathbb{C}$-algebra.

The condition for F to be weighted homogeneous of degree D means that F is homogeneous of degree D in $S(a_0, ..., a_d)$.

Let Y to be a variety as in the last example. Then the canonical sheaf K_Y of Y is given as follows

$$K_Y = \mathcal{O}_Y(D - \sum_{i=0}^{d} a_i),$$

where, for any integer m, $\mathcal{O}_Y(m)$ is the sheaf associated to the graded module $(S(a_0, ..., a_d)/(F))\,(m)$, [24]. See Notation 3.4.2 for the definition of K_Y.

This construction has been extensively used to give examples of orbifold with trivial canonical bundle, i.e. Calabi-Yau orbifold.

Remark 2.2.6. In [11], the authors introduces the notion of *toric Deligne-Mumford stack*, this is an orbifold structure over a simplicial toric variety. Note that a simplicial toric variety is an algebraic variety with quotient singularities ([31], Section 2.2).

A toric Deligne-Mumford stack corresponds to a combinatorial object called a *stacky fan*. A stacky fan $\mathbf{\Sigma}$ is a triple consisting of a finitely generated abelian group N, a simplicial fan Σ in $\mathbb{Q} \otimes_{\mathbb{Z}} N$ with d rays (see [31], Section 2.2, for the definition of a simplicial fan), and a map $\beta : \mathbb{Z}^d \to N$ where the image of the standard basis of $\mathbb{Z}^d$ generates the rays in Σ.

A rational simplicial fan Σ in some lattice $N \cong \mathbb{Z}^d$ (where rational means that any simplex of the fan is generated by vectors in the lattice) gives a canonical stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$ where β is the map defined by the minimal lattice points on the rays. Hence, there is a natural toric Deligne-Mumford stack associated to every simplicial toric variety.

Notice that weighted projective spaces and hypersurfaces in weighted projective spaces are toric varieties and the orbifold structures constructed in Examples 2.2.3 and 2.2.4 are examples of toric Deligne-Mumford stacks.

2.3 Morphisms of orbifolds

In this section we review the notion of morphism between two given orbifolds and of *natural transformation* between two morphisms of the same orbifolds.

We will see that orbifolds and morphisms form a 2-category: 1-morphisms are morphisms and 2-morphisms are natural transformations.

Satake in the paper [53] defined C^∞*-map* between orbifolds. With this definition, smooth orbifolds and C^∞-maps form a category. It turns out that, given a C^∞-map from the orbifold $[X]$ to the orbifold $[Y]$, and an orbifold vector bundle $[E]$ over $[Y]$, it is not possible, in general, to pull-back $[E]$ using this map. This point is explained, for example, in [17] Section 4.4, [43] Section 2 and in [46] Section 2.

In order to be able to pull-back vector bundles, the correct definition of a morphism between two orbifolds is that of *strong map* defined by I. Moerdijk and D. A. Pronk in [43] (Section 5), or *good map* defined by W. Chen and Y. Ruan in [17], Definition 4.4.1. As proved in [39], Proposition 5.1.7, the two definitions are equivalent.

Definition 2.3.1. *Let $[X]$ and $[Y]$ be two orbifolds, $f : X \to Y$ be a continuous map. A* ***compatible system*** *for f is given by the following objects:*

1. *two atlases $\mathcal{V}$ and $\mathcal{U}$ for $[X]$ and $[Y]$ respectively;*
2. *a correspondence that associate to any uniformizing system $(\tilde{V}, H, \rho)$ in $\mathcal{V}$, an uniformizing system $(\tilde{U}, G, \chi)$ in $\mathcal{U}$ and a smooth function $f_{\tilde{V}} : \tilde{V} \to \tilde{U}$, such that*
$$\chi \circ f_{\tilde{V}} = f \circ \rho;$$
3. *a correspondence that associate to any embedding $(\psi, \mu) : (\tilde{V}, H, \rho) \to (\tilde{V}', H', \rho')$ an embedding $(\varphi, \lambda) : (\tilde{U}, G, \chi) \to (\tilde{U}', G', \chi')$ between the corresponding uniformizing systems such that*
$$f_{\tilde{V}'} \circ \psi = \varphi \circ f_{\tilde{V}}.$$

Moreover we require that the assignment of the above objects is functorial with respect to the composition of embeddings $(\psi, \mu) : (\tilde{V}, H, \rho) \to (\tilde{V}', H', \rho')$ and $(\psi', \mu') : (\tilde{V}', H', \rho') \to (\tilde{V}'', H'', \rho'')$.

Remark 2.3.2. From the definition we have that for any uniformizing system $(\tilde{V}, H, \rho)$ in $\mathcal{V}$, let $(\tilde{U}, G, \chi)$ be the corresponding uniformizing system of $\mathcal{U}$, then there is a group homomorphism $f_H : H \to G$ such that
$$f_{\tilde{V}} \circ h = f_H(h) \circ f_{\tilde{V}} \qquad \text{for any } h \in H.$$

Notation 2.3.3. A compatible system for f will be denoted by $\tilde{f} : \mathcal{V} \to \mathcal{U}$. Where $\mathcal{V}$ and $\mathcal{U}$ are atlases of $[X]$ and $[Y]$ respectively. For any uniformizing system $(\tilde{V}, H, \rho) \in \mathcal{V}$, we will denote by $\tilde{f}(\tilde{V}, H, \rho)$ the corresponding uniformizing system in $\mathcal{U}$. For any embedding (ψ, μ) in $\mathcal{V}$ we will denote by $\tilde{f}(\psi, \mu)$ the corresponding embedding in $\mathcal{U}$.

Lemma 2.3.4 ([17] Remark 4.4.7). *Let $\tilde{f}_i : \mathcal{V}_i \to \mathcal{U}_i$, for $i \in \{1, 2\}$, be two compatible systems of the same function $f : X \to Y$. Then there exists a common refinement $\mathcal{V}$ of both $\mathcal{V}_1$ and $\mathcal{V}_2$, a refinement $\mathcal{U}$ of both $\mathcal{U}_1$ and $\mathcal{U}_2$, and compatible systems $\tilde{f}_i^! : \mathcal{V} \to \mathcal{U}$, $i \in \{1, 2\}$, such that $\tilde{f}_i^!$ is induced by $\tilde{f}_i$ for any $i \in \{1, 2\}$.*

The previous Lemma means that given two compatible systems we can always assume that they are defined over the same atlases.

Definition 2.3.5. *Two compatible systems $\tilde{f}_i : \mathcal{V}_i \to \mathcal{U}_i$, for $i = 1, 2$, of the same map f, are said* ***equivalent*** *if they coincide when defined over the same atlases.*

A ***morphism*** *$[f]$ from the orbifold $[X]$ to the orbifol $[Y]$ is given by a continuous map $f : X \to Y$ and an equivalence class of compatible systems for f.*

We now review the notion of natural transformation between two morphisms.

Definition 2.3.6. *Let $\tilde{f}_1$ and $\tilde{f}_2$ be compatible systems for $f : X \to Y$. Assume that they are defined over the same atlases $\mathcal{V}$ and $\mathcal{U}$ of $[X]$ and $[Y]$ respectively. A* ***natural transformation*** *from $\tilde{f}_1$ to $\tilde{f}_2$ is a correspondence that associate to any uniformizing system $(\tilde{V}, H, \rho) \in \mathcal{V}$ an automorphism $\delta_{\tilde{V}}$ of $\tilde{f}_1(\tilde{V}, H, \rho) = \tilde{f}_2(\tilde{V}, H, \rho)$ such that*

$$(f_2)_{\tilde{V}} = \delta_{\tilde{V}} \circ (f_1)_{\tilde{V}},$$

and such that, for any embedding $(\psi, \mu) : (\tilde{V}, H, \rho) \to (\tilde{V}', H', \rho')$ in $\mathcal{V}$, the following diagram commutes

$$\begin{array}{ccc} \tilde{f}_1(\tilde{V}, H, \rho) & \xrightarrow{\tilde{f}_1(\psi,\mu)} & \tilde{f}_1(\tilde{V}', H', \rho') \\ \delta_{\tilde{V}} \downarrow & & \downarrow \delta_{\tilde{V}'} \\ \tilde{f}_2(\tilde{V}, H, \rho) & \xrightarrow{\tilde{f}_2(\psi,\mu)} & \tilde{f}_2(\tilde{V}', H', \rho'). \end{array}$$

Remark 2.3.7. Let $\tilde{f}_1, \tilde{f}_2 : \mathcal{V} \to \mathcal{U}$ be compatible systems for $f : X \to Y$, and let $\{\delta_{\tilde{V}} : (\tilde{V}, H, \rho) \in \mathcal{V}\}$ be a natural transformation between them. Let $\mathcal{V}'$ and $\mathcal{U}'$ be refinement of $\mathcal{V}$ and $\mathcal{U}$ respectively. Then there is a natural transformation $\{\delta'_{\tilde{V}} : (\tilde{V}', H', \rho') \in \mathcal{V}'\}$ between the compatible systems $\tilde{f}'_1, \tilde{f}'_2 : \mathcal{V}' \to \mathcal{U}'$ that are induced by $\tilde{f}_1$ and $\tilde{f}_2$. We will say that $\{\delta_{\tilde{V}}\}$ and $\{\delta'_{\tilde{V}}\}$ are equivalent.

Definition 2.3.8. *Let $[f]_1$ and $[f]_2$ be morphisms from $[X]$ to $[Y]$. A **natural transformation** between $[f]_1$ and $[f]_2$ is an equivalence class of natural transformations between two compatible systems that represents $[f]_1$ and $[f]_2$.*

Notation 2.3.9. We will denote a natural transformation between the morphisms $[f_1]$ and $[f_2]$ by $[f_1] \Rightarrow [f_2]$. We will use the same symbol for natural transformations between compatible systems.

Remark 2.3.10. *(Orbifolds as groupoids)* Let $[Y]$ be an orbifold. Any atlas $\mathcal{U}$ of $[Y]$ determines a groupoid which represents $[Y]$. This is shown in [44] Theorem 4.1.1; for a detailed study of the relations between orbifolds and groupoids see [43], [42].

By abuse of notation, we will denote also with $\mathcal{U}$ the groupoid associated to the covering $\mathcal{U}$.

Let $[X]$ and $[Y]$ be orbifolds and $f : X \to Y$ a continuous map. Then a compatible system $\tilde{f} : \mathcal{V} \to \mathcal{U}$ induces a morphism of the corresponding groupoids. Conversely, for any pair of groupoids $\mathcal{V}$ and $\mathcal{U}$ that represents $[X]$ and $[Y]$ respectively, and a groupoid morphism $\tilde{f} : \mathcal{V} \to \mathcal{U}$, there is an induced compatible system for f. This is the content of Proposition 5.1.7 in [39]. Moreover, natural transformations between compatible systems corresponds to natural transformations between the associated morphisms of groupoids (see [42] Section 2.2 for the definition of natural transformation of morphisms of groupoids). So, the notion of compatible system is the same of *strong map* introduced in [43], Section 5.

Remark 2.3.11. *(Orbifolds as stacks)* There is another way to define orbifolds, that is as *smooth Deligne-Mumford stacks* ([21], [27], [59]). The way to pass from our definition to stacks is through groupoids. Indeed, given an atlas $\mathcal{U}$, we can construct the corresponding groupoid $\mathcal{U}$, then there is a procedure that associate to the groupoid $\mathcal{U}$ a stack ([59], Appendix).

We now show that our definition of morphisms between orbifolds corresponds to the definition of morphisms between stacks.

Proposition 2.3.12. *The 2-category of orbifolds with orbifold morphisms is equivalent to the 2-category of smooth Deligne-Mumford stacks with morphisms of stacks.*

Proof. Let (Orb) denote the category of orbifolds and orbifold morphisms, and let (Stacks) denote the category of stacks and stack morphisms. From Remark 2.3.11 it follows that we have a correspondence that associate to any object $[Y]$ in (Orb) an object $\mathcal{Y}$ in (Stacks). We now construct a correspondence that associate to any morphism in (Orb) a morphism in (Stacks) in such a way that this gives a functor (Orb) $\to$ (Stacks) such that, for any pair of orbifolds $[X]$ and $[Y]$, the map

$$\mathrm{Mor}_{(\mathrm{Orb})}([X],[Y]) \to \mathrm{Mor}_{(\mathrm{Stacks})}(\mathcal{X},\mathcal{Y}) \tag{2.1}$$

is bijective. Note that this would be an equivalence, indeed any smooth Deligne-Mumford stack is represented by a groupoid ([59] Appendix), then apply Remark 2.3.10.

Let $[f] : [X] \to [Y]$ be a morphism of orbifolds. We construct a morphism $F : \mathcal{X} \to \mathcal{Y}$ of stacks in the following way:

1. we give open coverings $\{[\tilde{V}_i/H_i]\}_{i\in I}$ of $\mathcal{X}$ and $\{[\tilde{U}_j/G_j]\}_{j\in J}$ of $\mathcal{Y}$;
2. we define morphisms $F_i : [\tilde{V}_i/H_i] \to [\tilde{U}_{\nu(i)}/G_{\nu(i)}]$, where $\nu : I \to J$ is a function;
3. for any $i, i' \in I$, we give a natural transformation $\delta_{ii'} : F_{ii'} \Rightarrow F_{i'i}$, such that $\delta_{i'i''} \circ \delta_{ii'} = \delta_{ii''}$ on triple intersections, where $F_{ii'}$ denote the restriction of F_i to $[\tilde{V}_i/H_i] \times_{\mathcal{X}} [\tilde{V}_{i'}/H_{i'}]$.

Let $\tilde{f} : \mathcal{V} \to \mathcal{U}$ be a compatible system representing $[f]$. Then define

$$\{[\tilde{V}_i/H_i]\}_{i\in I} := \{[\tilde{V}/H] \;:\; (\tilde{V}, H, \rho) \in \mathcal{V}\}$$

and

$$\{[\tilde{U}_j/G_j]\}_{j\in J} := \{[\tilde{U}/G] \;:\; (\tilde{U}, U, \chi) \in \mathcal{U}\}.$$

The function $\nu : I \to J$ is the correspondence given in point 2 of Definition 2.3.1. We now define a morphism $F_i : [\tilde{V}_i/H_i] \to [\tilde{U}_{\nu(i)}/G_{\nu(i)}]$ for any $i \in I$. Let $f_{\tilde{V}_i} : \tilde{V}_i \to \tilde{U}_{\nu(i)}$ and $f_{H_i} : H_i \to G_{\nu(i)}$ given as in Definition 2.3.1 and Remark 2.3.2. An object in $[\tilde{V}_i/H_i]$ over the base B is represented by the following diagram:

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & \tilde{V}_i \\ \downarrow & & \\ B & & \end{array}$$

where $P \to B$ is a principal H_i-bundle, and $\alpha : P \to \tilde{V}_i$ is an H_i-equivariant smooth function. This object can be also given by an open covering $\{B_k\}_K$ of B and the following data

$$\begin{aligned} h_{lm} &\in H_i, \text{ for any } l, m \text{ such that} B_l \cap B_m \neq \emptyset, \text{ such that } h_{mn} \cdot h_{lm} = h_{ln} \\ \alpha_k &: B_k \to \tilde{V}_i, \text{ smooth functions such that } \alpha_l = h_{lm} \circ \alpha_m \end{aligned}$$

modulo the equivalence relation that identifies h_{lm} with $h_l \cdot h_{lm} \cdot h_m^{-1}$ and α_k with $h_k \circ \alpha_k$ for $h_k \in H_i$, for any $k \in K$. So, using this description of the objects, we define F_i on the objects to be given by the following correspondence

$$F_i : \{h_{lm},\ \alpha_k\}/\tilde{\mapsto}\{g_{lm} = f_{H_i}(h_{lm}),\ \beta_k = f_{\tilde{V}_i} \circ \alpha_k\}/\tilde{}$$

On morphisms, F_i is defined in an analogous way.

Let $i, i' \in I$. Then, the natural transformation $\delta_{ii'} : F_{ii'} \Rightarrow F_{i'i}$ is constructed as follows. We can cover $[\tilde{V}_i/H_i] \times_{\mathcal{X}} [\tilde{V}_{i'}/H_{i'}]$ using open embeddings that are induced by embeddings of uniformizing systems as follows

$$\Psi : [\tilde{V}_{lm}/H_{lm}] \to [\tilde{V}_i/H_i] \times_{\mathcal{X}} [\tilde{V}_{i'}/H_{i'}],$$

where $(\tilde{V}_{lm}, H_{lm}, \rho_{lm}) \in \mathcal{V}$. Point 3 of Definition 2.3.1 gives natural transformations from the restriction of $F_{ii'}$ on $[\tilde{V}_{lm}/H_{lm}]$ to the restriction of $F_{i'i}$ on $[\tilde{V}_{lm}/H_{lm}]$. The functoriality assumption in Definition 2.3.1 ensures that these natural transformations patch together to give $\delta_{ii'}$. Finally, the cocycle condition on natural transformations follows from the functoriality condition given in Definition 2.3.1.

To show that the map (2.1) is bijective, we will construct an inverse. Let

$$F : \mathcal{X} \to \mathcal{Y}$$

be a morphism of stacks. Let $\tilde{x} \in X$ be a point, and let $(\tilde{V}_x, H_x, \rho_x)$ be a uniformizing system at x. We assume that $\tilde{V}_x$ is simply connected. We can assume that the restriction of F to $[\tilde{V}_x/H_x]$ take values in a global quotient stack $[\tilde{U}/G]$. So, we have

$$F_x : [\tilde{V}_x/H_x] \to [\tilde{U}/G].$$

The following object

$$\begin{array}{ccc} \tilde{V}_x \times H_x & \longrightarrow & \tilde{V}_x \\ \downarrow & & \\ \tilde{V}_x & & \end{array} \tag{2.2}$$

has group of automorphisms H_x, where the horizontal map $\tilde{V}_x \times H_x \to \tilde{V}_x$ is given by $(\xi, k) \mapsto k(\xi)$ and the vertical map is the projection. Indeed, for any $h \in H_x$, the following diagram is cartesian

$$\begin{array}{ccc} \tilde{V}_x \times H_x & \longrightarrow & \tilde{V}_x \times H_x \\ \downarrow & & \downarrow \\ \tilde{V}_x & \longrightarrow & \tilde{V}_x \end{array}$$

where the vertical arrows are projections, the lower horizontal is $\xi \mapsto h(\xi)$ and the upper horizontal $(\xi, k) \mapsto (h(\xi), k \cdot h^{-1})$. So we get a group homomorphism $f_{H_x} : H_x \to G$. Now notice that we have a morphism $\tilde{V}_x \to [\tilde{U}/G]$ which is the composition of $\tilde{V}_x \to [\tilde{V}_x/H_x]$ with F_x. This correspond to the following diagram

$$\begin{array}{ccc} Q & \xrightarrow{\beta} & \tilde{U} \\ \downarrow & & \\ \tilde{V}_x & & \end{array} \tag{2.3}$$

where $Q \to \tilde{V}_x$ is a principal G-bundle and $Q \to \tilde{U}$ is G-equivariant. But since $\tilde{V}_x$ is simply connected, $Q \to \tilde{V}_x$ has a section s. So, if we define $f_{\tilde{V}} = \beta \circ s$, we get a map $f_{\tilde{V}} : \tilde{V}_x \to \tilde{U}$. Let $h \in H_x$. If we see h as an automorphism of (2.2), then $F_x(h)$ is an automorphism of (2.3). It follows that $f_{\tilde{V}} \circ h = f_H(h) \circ f_{\tilde{V}}$.

So, we have costructed a map $\mathrm{Mor}_{(\mathrm{Stacks})}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Mor}_{(\mathrm{Orb})}([X], [Y])$. We now show that it is an inverse of (2.1). Consider an object in $[\tilde{V}_x/H_x](B)$. Let it be described by the data $\{h_{lm},\ \alpha_k\}$ with respect to the covering $\{B_k\}_{k \in K}$. Let us denote the image of this object under F_x by $\{g_{lm},\ \beta_k\}$. Then we have the following commutative diagram

$$\begin{array}{ccccc} B_k & \xrightarrow{\alpha_k} & \tilde{V}_x & \xrightarrow{f_{\tilde{V}}} & \tilde{U} \\ {\scriptstyle id}\downarrow & & \downarrow & & \downarrow \\ B_k & \longrightarrow & [\tilde{V}_x/H_x] & \xrightarrow{F_x} & [\tilde{U}/G]. \end{array}$$

This shows that $\beta_k = f_{\tilde{V}} \circ \alpha_k$. Now, for any $l, m \in K$, the equation $\alpha_l = h_{lm} \circ \alpha_m$ means that h_{lm} is a morphism between two objects over the same base B_{lm}. Then, if we apply the functor F_x, we have that $F_x(h_{lm})$ is a morphism between the corresponding objects. This shows that $g_{lm} = f_{H_x}(h_{lm})$. $\square$

2.4 Orbifold vector bundles

We shall now review the definition of orbifold vector bundles. We will follow the definition given in [43]. Then, following [53], we will review the construction of the *orbifold tangent bundle*. The orbifold cotangent bundle and its exterior powers are constracted in the same way. Then we review the notion of *differential form* and also of the de Rham cohomology for orbifolds. Finally we recall a Theorem, due to Satake, which state that the de Rham cohomology of $[Y]$ is isomorphic to the singular cohomology of the underlying topological space.

Definition 2.4.1. *Let $[Y]$ be an orbifold and let $\mathcal{U}$ be the maximal atlas. A* ***smooth orbifold vector bundle*** *$[E]$ on $[Y]$ is given by:*

1. *for any uniformizing system $(\tilde{U}, G, \chi) \in \mathcal{U}$ a (ordinary) vector bundle $E_{\tilde{U}}$ on $\tilde{U}$;*

2. *for each embedding $(\varphi, \lambda) : (\tilde{U}, G, \chi) \to (\tilde{U}', G', \chi')$ an isomorphism $E(\varphi, \lambda) : E_{\tilde{U}} \to \varphi^* E_{\tilde{U}'}$, moreover we require that these isomorphisms are functorial in (φ, λ).*

Let $[E]$ be an orbifold vector bundle on $[Y]$. A ***section*** *$[s]$ of $[E]$ is given by a (ordinary) section $s_{\tilde{U}}$ of $E_{\tilde{U}}$, for any uniformizing system $(\tilde{U}, G, \chi) \in \mathcal{U}$, such that, for each embedding $(\varphi, \lambda) : (\tilde{U}, G, \chi) \to (\tilde{U}', G', \chi')$,*

$$E(\varphi, \lambda)(s_{\tilde{U}}) = s_{\tilde{U}'}.$$

Remark 2.4.2. To define an orbifold vector bundle $[E]$ on $[Y]$ (up to isomorphism), it is enough to specify the bundles $E_{\tilde{U}}$ and vector bundle maps $E(\varphi, \lambda)$ for all uniformizing systems $(\tilde{U}, G, \chi)$ in some atlas with the property that the images $U = \chi(\tilde{U}) \subset Y$ form a basis for the topology on Y. This is Remark 1, in Section 2, [43].

Remark 2.4.3. Let $[E]$ be an orbifold vector bundle on $[Y]$, and let $(\tilde{U}, G, \chi)$ be a uniformizing system. Then each $g \in G$ defines an embedding $(g, Ad_g) : (\tilde{U}, G, \chi) \to (\tilde{U}, G, \chi)$, where $Ad_g : G \to G$ is given by $g' \mapsto g \circ g' \circ g^{-1}$. So, we have an isomorphism

$$E(g, Ad_g) : E_{\tilde{U}} \to g^* E_{\tilde{U}}.$$

This defines an action of G on $E_{\tilde{U}}$. Thus we see that $E_{\tilde{U}}$ is a *G-equivariant vector bundle* on $\tilde{U}$.

Example 2.4.4 (*Orbifold tangent and cotangent bundle*). Let $[Y]$ be a smooth orbifold, and let $\mathcal{U}$ be the maximal compatible atlas. The *orbifold tangent bundle* of $[Y]$ is the orbifold vector bundle $T_{[Y]}$ defined as follows:

1. $(T_{[Y]})_{\tilde{U}} = T_{\tilde{U}}$ is the tangent bundle of $\tilde{U}$, for any uniformizing system $(\tilde{U}, G, \chi)$;

2. for any embedding $(\varphi, \lambda) : (\tilde{U}, G, \chi) \to (\tilde{U}', G', \chi')$, $T_{[Y]}(\varphi, \lambda) = T\varphi$ is the tangent morphism of φ.

In the same way we define the *orbifold cotangent bundle* $T^{\vee}_{[Y]}$. Then we can form the p-th exterior product $\wedge^p T^{\vee}_{[Y]}$.

Definition 2.4.5. *Let $[Y]$ be a smooth orbifold. A **differential p-form** on $[Y]$ is a section of $\wedge^p T^{\vee}_{[Y]}$.*
The space of differential p-forms over $[Y]$ will be denoted by $\Omega^p_{[Y]}$.

We can define the exterior differential $d : \Omega^p_{[Y]} \to \Omega^{p+1}_{[Y]}$ and the wedge product on differential forms (see [53], Section 3, for more details). So, we define the p-th de Rham cohomology group of $[Y]$ in the usual way:

$$H^p_{dR}([Y], \mathbb{R}) = \frac{\mathrm{Ker}(\mathrm{d} : \Omega^{\mathrm{p}}_{[\mathrm{Y}]} \to \Omega^{\mathrm{p+1}}_{[\mathrm{Y}]})}{\mathrm{Im}(\mathrm{d} : \Omega^{\mathrm{p-1}}_{[\mathrm{Y}]} \to \Omega^{\mathrm{p}}_{[\mathrm{Y}]})}.$$

The following result holds, see [53] Theorem 1, [37] pag. 78.

Theorem 2.4.6. *For any $p \geq 0$, there is a natural isomorphism between the p-th singular cohomology group $H^p(Y, \mathbb{R})$ of the topological space Y and the p-th de Rham cohomology group $H^p_{dR}([Y], \mathbb{R})$ of the orbifold $[Y]$. Moreover, under this isomorphism the exterior product in $\Omega^*_{[Y]}$ corresponds to the cup product in $H^p(Y, \mathbb{R})$.*

We can define connections on orbifold bundles. For an orbifold vector bundle with a linear connection we have characteristic forms by Weil homomorphism. The cohomology class of a characteristic form is independent of the choice of the connection. So we have *Euler classes* for oriented orbifold vector bundles, *Chern classes* for complex orbifold vector bundles, and *Pontrjagin classes* for real orbifold vector bundles. Moreover one can see that these characteristic classes are defined over the rational numbers.

Integration over compact orbifolds is defined as follows. First of all, assume that $[Y] = [V/G]$ is a global quotient orbifold. Let ω be a differential p-form on $[Y]$. By definition, ω is a G-equivariant p-form $\tilde{\omega}$ on V. Then the integration of ω on $[Y]$ is defined by

$$\int_{[Y]}^{orb} \omega := \frac{1}{|G|} \int_V \tilde{\omega},$$

where $|G|$ is the order of the group G. We use the convention such that, if the degree of the differential form is different from the dimension of the manifold, then the integral is zero.

Let $[Y]$ be a compact orbifold. Fix a C^∞ partition of unity $\{\rho_i\}$ subordinated to the covering $\{U_i\}$, where each U_i is an uniformized open set in Y. For any p-form ω on $[Y]$, the integration of ω over $[Y]$ is defined by

$$\int_{[Y]}^{orb} \omega := \sum_i \int_{[U_i]}^{orb} \rho_i \cdot \omega_{|U_i},$$

where $[U_i]$ has the global quotient structure. This definition is independent of the choice of the partition of unity ([53], Section 8).

The following Theorem holds, which is a form of Poincaré Theorem for orbifolds, see also [53], Theorem 3.

Theorem 2.4.7. *The bilinear form*

$$\begin{array}{ccc} H^p_{dR}([Y], \mathbb{R}) & \times & H^{n-p}_{dR}([Y], \mathbb{R}) \to \mathbb{R} \\ (\omega, \eta) & \mapsto & \int_{[Y]}^{orb} \omega \wedge \eta \end{array}$$

is non degenerate.

We now recall the definition of *complex orbifold vector bundle* and of *holomorphic section* over a complex orbifold.

Definition 2.4.8. *Let $[Y]$ be a complex orbifold and let $\mathcal{U}$ be the maximal atlas. A* ***complex orbifold vector bundle*** *$[E]$ on $[Y]$ is given by:*

1. *an ordinary complex vector bundle $E_{\tilde{U}}$ on $\tilde{U}$, for any uniformizing system $(\tilde{U}, G, \chi) \in \mathcal{U}$;*

2. *an isomorphism $E(\varphi, \lambda) : E_{\tilde{U}} \to \varphi^* E_{\tilde{U}'}$ of complex vector bundles, for any embedding $(\varphi, \lambda) : (\tilde{U}, G, \chi) \to (\tilde{U}', G', \chi')$ such that these isomorphisms are functorial in (φ, λ).*

Let $[E]$ *be an orbifold vector bundle on* $[Y]$*. A* ***holomorphic section*** $[s]$ *of* $[E]$ *is given by a ordinary holomorphic section* $s_{\tilde{U}}$ *of* $E_{\tilde{U}}$*, for any uniformizing system* $(\tilde{U}, G, \chi) \in \mathcal{U}$*, such that, for each embedding* $(\varphi, \lambda) : (\tilde{U}, G, \chi) \to (\tilde{U}', G', \chi')$*,*

$$E(\varphi, \lambda)(s_{\tilde{U}}) = s_{\tilde{U}'}.$$

Chapter 3

Orbifold cohomology

The notion of *orbifold cohomology ring* was introduced by Chen and Ruan for an almost complex orbifold [16]. This has been extended by Fantechi and Göttsche to a noncommutative ring [28], in the case where the orbifold is a global quotient. Abramovich, Graber and Vistoli gave the definition of orbifold cohomology ring in the algebraic case, that is for a smooth Deligne-Mumford stack [2].

This chapter can be divided in two parts. In the first part, Sections 1,2,3, we review the definition of orbifold cohomology ring for a complex orbifold. We follow closely the paper [16]. In the second part, Section 4 we state the *cohomological crepant resolution conjecture* given by Ruan [52]. The aim of this paper is to verify this conjecture for a certain class of orbifolds which will be defined in the next chapter.

Notation 3.0.9. In this chapter all orbifolds will be complex orbifolds. So, morphisms will be holomorphic and orbifold vector bundles will be complex. Furthermore, we assume that $[Y]$ has complex dimension $rmdim_{\mathbb{C}} = d$.

We will use the same notation 2.1.15, so, for any $y \in Y$, $(\tilde{U}_y, G_y, \chi_y)$ will denote a uniformizing system at y.

3.1 Inertia orbifold

Let $[Y]$ be an orbifold. Let us consider the set

$$Y_1 = \{(y, (g)_y) : \ y \in Y, \ (g)_y \subset G_y \text{ is a conjugacy class}\},$$

as usual we will denote by $(g)_y$ the conjugacy class of $g \in G_y$. We want to define an orbifold $[Y_1]$ in such a way that there is a morphism $[Y_1] \to [Y]$ that induces the continuous map $Y_1 \to Y$ given by $(y, (g)_y) \mapsto y$. This orbifold is the *inertia orbifold*.

We introduce an equivalence relation on the set Y_1. Let $y \in Y$ and let $(\tilde{U}_y, G_y, \chi_y)$ be an uniformizing system at y (2.1.15). For any $y' \in U_y = \chi_y(\tilde{U}_y)$, let $(\tilde{U}_{y'}, G_{y'}, \chi_{y'})$ be a uniformizing system at y' such that $U_{y'} \subset U_y$. Then we have an embedding $(\varphi, \lambda) : (\tilde{U}_{y'}, G_{y'}, \chi_{y'}) \to (\tilde{U}_y, G_y, \chi_y)$. By Remark 2.1.9, $\lambda : G_{y'} \to G_y$ is injective, so, for any conjugacy class $(g')_{y'} \subset G_{y'}$, we can associate the conjugacy class of $\lambda(g')$ in G_y, i.e. $(\lambda(g'))_y \subset G_y$. Notice that, the class $(\lambda(g'))_y \subset G_y$ does not depend on the chosen embedding (Proposition 2.1.7). In this situation, we say that $(y', (g')_{y'})$ is *equivalent* to $(y, (\lambda(g'))_y)$. This generate the claimed equivalence relation on Y_1.

Notation 3.1.1. We will denote by T the set of equivalence classes of Y_1 just described. Elements of T will be denoted by (g). For any $(g) \in T$, the equivalence class (g) will be denoted by $Y_{(g)}$. The class of $(y, (1)_y) \in Y_1$ will be denoted by (1), so $Y_{(1)} = Y$.

Lemma 3.1.2 ([16] Lemma 3.1.1). *For any $(g) \in T$, the set $Y_{(g)}$ has a topology and an orbifold structure $[Y_{(g)}]$ which is given as follows: for any point $(y, (g)_y) \in (Y_1)_{(g)}$, an uniformizing system at $(y, (g)_y)$ is given by*

$$(\tilde{U}_y^g, C(g), \chi_y \mid),$$

where g is a representative of $(g)_y$, $\tilde{U}_y^g$ is the fixed-point set of g in $\tilde{U}$, $C(g) \subset G_y$ is the centralizer of g in G_y and $\chi_y \mid$ denotes the restriction of χ_y to $\tilde{U}_y^g$.

There is an orbifold morphism

$$[\pi_{(g)}] : [Y_{(g)}] \to [Y]$$

which locally is given by the inclusion $\tilde{U}_y^g \to \tilde{U}_y$ and the group injection $C(g) \to G_y$.

The orbifold structure $[Y_1]$ is defined to be the disjoint union of $[Y_{(g)}]$, as (g) varies in T, so that

$$[Y_1] = \bigsqcup_{(g) \in T} [Y_{(g)}].$$

There is a morphism $[\pi] : [Y_1] \to [Y]$ defined by requiring that its restriction to $[Y_{(g)}]$ is $[\pi_{(g)}]$.

Moreover, if $[Y]$ is a complex orbifold, so is $[Y_1]$ and $[\pi] : [Y_1] \to [Y]$ is holomorphic.

Definition 3.1.3. *Let $[Y]$ be an orbifold. The* ***inertia orbifold*** *of $[Y]$ is the orbifold $[Y_1]$ descrbed in Proposition 3.1.2.*

For any $(g) \in T$, $(g) \neq (1)$, the orbifold $[Y_{(g)}]$ is called a ***twisted sector****. On the other hand the orbifold $[Y_{(1)}]$ is the* ***nontwisted sector****.*

Proposition 3.1.4. *Let $[Y]$ be an orbifold. Then, there is a morphism*

$$[I] : [Y_1] \to [Y_1]$$

which induces the map $I : Y_1 \to Y_1$ defined by: $(y, (g)_y) \mapsto (y, (g^{-1})_y)$. It is an involution, i.e. $[I] \circ [I] = [id]$.

Proof. For any $y \in Y$ and an uniformizing system at y, $(\tilde{U}_y, G_y, \chi_y)$, we get the following uniformizing system for $[Y_1]$:

$$\Big(\bigsqcup_{(g)_y \subset G_y} \tilde{U}_y^g, \quad \oplus_{(g)_y \subset G_y} C(g), \quad \bigsqcup_{(g)_y \subset G_y} (\chi_y) \mid \tilde{U}_y^g \Big)$$

with the property that, if g represent $(g)_y$, then g^{-1} represent $(g^{-1})_y$. Then, the restriction of $[I]$ to this uniformizing system is given by

$$\begin{array}{rcl} \tilde{U}_y^g & \to & \tilde{U}_y^{g^{-1}}, \quad \tilde{x} \mapsto \tilde{x} \\ C(g) & \to & G(g^{-1}), \quad h \mapsto h. \end{array}$$

□

3.2 Orbifold cohomology group

As vector space, $H^*_{orb}([Y])$ is the cohomology of the inertia orbifold $[Y_1]$. The grading takes the *degree shifting* of the elements of the local groups into account.

We review now the definition of degree shifting. Let $y \in Y$ be any point and let $(\tilde{U}_y, G_y, \chi_y)$ be a uniformizing system at y (Definition 2.1.14). The origin 0 of $\tilde{U}_y$ is fixed by the action of G_y, so we have an action of G_y on the tangent space of $\tilde{U}_y$ at 0. We represent this action by a group homomorphism $R_y : G_y \to GL(d, \mathbb{C})$, where $d = \dim_{\mathbb{C}} \mathrm{Y}$. For every $g \in G_y$, $R_y(g)$ can be written as a diagonal matrix ([55], Chapter 2, Proposition 3):

$$R_y(g) = \mathrm{diag}\left(\exp(2\pi \mathrm{i} \mathrm{m}_{1,\mathrm{g}}/\mathrm{m}_\mathrm{g}), ..., \exp(2\pi \mathrm{i} \mathrm{m}_{\mathrm{d},\mathrm{g}}/\mathrm{m}_\mathrm{g})\right),$$

where m_g is the order of $R_y(g)$, and $0 \leq m_{i,g} < m_g$ is an integer. Since this matrix depends only on the conjugacy class $(g)_y$ of $g \in G_y$, we define a function $\iota : Y_1 \to \mathbb{Q}$ by

$$\iota(y, (g)_y) = \sum_{i=1}^{d} \frac{m_{i,g}}{m_g}.$$

Lemma 3.2.1. *For any $(g) \in T$ (see Notation 3.1.1), the function $\iota : Y_{(g)} \to \mathbb{Q}$ is constant on each connected components.*

Proof. From the definition of $Y_{(g)}$ (see Lemma ??), it follows that it is enough to prove the following statement. Let $(y, (g)_y), (y', (g')_{y'}) \in Y_{(g)}$ such that there exists an embedding $(\varphi, \lambda) : (\tilde{U}_{y'}, G_{y'}, \chi_{y'}) \to (\tilde{U}_y, G_y, \chi_y)$ which sends the origin $0_{y'} \in \tilde{U}_{y'}$ to the origin $0_y \in \tilde{U}_y$, then $\iota(y, (g)_y) = \iota(y', (g')_{y'})$.

This follows from the fact that φ is λ-equivariant, so the tangent map $T\varphi : T_{0_{y'}}\tilde{U}_{y'} \to T_{0_y}\tilde{U}_y$ is a morphism of representations. □

Definition 3.2.2. *For any $(g) \in T$, the* ***degree shifting number of*** *(g) is the locally constant function*

$$\iota(y, (g)_y) : Y_{(g)} \to \mathbb{Q}.$$

If $Y_{(g)}$ is connected, we identify $\iota(y, (g)_y)$ with its value, and we will denote it also by $\iota_{(g)}$.

Remark 3.2.3. In the literature, the degree shifting number is also called the *age*.

Remark 3.2.4. Note that, for any $(y, (g)_y) \in Y_{(g)}$, $\exp(2\pi i \iota(y, (g)_y))$ is the eigenvalue of the linear map

$$\det R_y(g) : \wedge^d \mathbb{C}^d \to \wedge^d \mathbb{C}^d$$

Definition 3.2.5. *For any integer p, the degree p* ***orbifold cohomology group*** *of $[Y]$, $H^p_{orb}([Y])$, is defined as follows*

$$H^p_{orb}([Y]) = \oplus_{(g)\in T} H^{p-2\iota_{(g)}}(Y_{(g)}),$$

where $H^*(Y_{(g)})$ is the singular cohomology of $Y_{(g)}$ with complex coefficients. The ***total orbifold cohomology group*** *of $[Y]$ is*

$$H^*_{orb}([Y]) = \oplus_p H^p_{orb}([Y]).$$

Remark 3.2.6. Note that $H^*_{orb}([Y])$ is a priori rationally graded. It is integrally graded if and only if all the degree shifting numbers are half-integers.

Remark 3.2.7. From Remark 3.2.4 it follows that the degree shiftings are integers if and only if the induced action of G_y on $\wedge^d T_{0_y}\tilde{U}_{0_y}$ is trivial. This means that the canonical sheaf of the singular variety Y is locally free which imply (and indeed is equivalent to say) that Y is Gorenstein, see Definition 3.4.3.

Remark 3.2.8. The orbifold cohomology group $H^*_{orb}([Y]) = \oplus_p H^p_{orb}([Y])$ can be splitted into even and odd parts, [28] Definition 1.8. By definition

$$H^{\text{even}}_{orb}([Y]) = \oplus_{(g)\in T} H^{\text{even}}(Y_{(g)}),$$

and analogously for the odd part. In general this decomposition is not related with the even/odd decomposition given by the orbifold cohomology grading. On the other hand, for Gorenstein orbifolds, the two gradings coincides.

On the vector space $H^*_{orb}([Y])$ there is a complex valued pairing $\langle , \rangle_{orb}$ which we will call the *Poincaré duality.*

Definition 3.2.9. *Let $[I] : [Y_1] \to [Y_1]$ be the holomorphic morphism defined in Proposition 3.1.4. Then $[I]$ sends $[Y_{(g)}]$ to $[Y_{(g^{-1})}]$. The **Poincaré duality pairing** is the following pairing*

$$\langle , \rangle_{orb} : H^p_{orb}([Y]) \times H^{2n-p}_{orb}([Y]) \to \mathbb{C}, \quad \textit{for } 0 \le p \le 2n,$$

defined as the direct sum of

$$\langle , \rangle^{(g)}_{orb} : H^{p-2\iota_{(g)}}(Y_{(g)}) \times H^{2n-p-2\iota_{(g^{-1})}}(Y_{(g^{-1})}) \to \mathbb{C},$$

where

$$\langle \alpha, \beta \rangle^{(g)}_{orb} = \int^{orb}_{[Y_{(g)}]} \alpha \cdot I^*(\beta).$$

We recall here Proposition 3.3.1 of [16].

Proposition 3.2.10. *The Poincaré duality pairing is nondegenerate.*

3.3 Orbifold cup product

In this section we review the definition of *orbifold cup product.* It is an associative product on the total orbifold cohomology group, the resulting ring is the *orbifold cohomology ring.*

For any positive integer k, consider the following set

$$Y_k = \{(y, (\underline{g})_y) : \ y \in Y, \ \underline{g} = (g_1, ..., g_k), \ g_i \in G_y\},$$

where $(\underline{g})_y$ is the conjugacy class of $\underline{g}$. Here, two k-tuples $(g_1^{(1)}, ..., g_k^{(1)})$ and $(g_1^{(2)}, ..., g_k^{(2)})$ are conjugate if there exists $g \in G_y$ such that $g_i^{(2)} = g \cdot g_i^{(1)} \cdot g^{-1}$ for all $i = 1, ..., k$.

We can define an equivalence relation on the set Y_k as we did for Y_1, see Notation 3.1.1. The resulting set of equivalence classes will be denoted by T_k.

Lemma 3.3.1 ([16] Lemma 4.1.1). *For any $(\underline{g}) \in T_k$, let $(Y_k)_{(\underline{g})}$ be the corresponding equivalence class of Y_k. Then, there is a topology and an orbifold structure $[(Y_k)_{(\underline{g})}]$ over $(Y_k)_{(\underline{g})}$ such that, for any point $(y, (\underline{g})_y) \in (Y_k)_{(\underline{g})}$, an uniformizing system at $(y, (\underline{g})_y)$ is given by*

$$(\tilde{U}_y^{\underline{g}}, C(\underline{g}), \chi_y \mid),$$

where $\underline{g} = (g_1, ..., g_k)$ is a representative of $(\underline{g})_y$, $\tilde{U}_y^{\underline{g}} = \tilde{U}_y^{g_1} \cap ... \cap \tilde{U}_y^{g_k}$, and $C(\underline{g}) = C(g_1) \cap ... \cap C(g_k)$.

We obtain an orbifold structure $[Y_k]$ defined as follows:

$$[Y_k] = \bigsqcup_{(\underline{g}) \in T_k} [Y_{(\underline{g})}].$$

For any $i = 1, ..., k$, we have morphisms

$$[e_i] : [Y_k] \to [Y_1]$$

that locally are given by: the topological inclusions $\tilde{U}_y^{\underline{g}} \to \tilde{U}_y^{g_i}$ and the group injections $C(\underline{g}) \to C(g_i)$.

Moreover, if $[Y]$ is a complex orbifold, so is $[Y_k]$ and each $[e_i] : [Y_k] \to [Y_1]$ is holomorphic.

Definition 3.3.2. *Let $[Y]$ be an orbifold. For any positive integer k, the* ***k-multisector*** *of $[Y]$ is the orbifold $[Y_k]$ constructed in Lemma 3.3.1.*

Consider the map $\mathrm{o} : \mathrm{T_k} \to \mathrm{T}$ induced by $(g_1, ..., g_k) \mapsto g_1 \cdot ... \cdot g_k$. The set $T_k^{\mathrm{o}} = \mathrm{o}^{-1}(1)$ is the subset of T_k consisting of equivalence classes $(\underline{g})$ such that $\underline{g} = (g_1, ..., g_k)$ satisfies the condition $g_1 \cdot ... \cdot g_k = 1$.

Notation 3.3.3. We will denote by Y_k^{o} the set

$$Y_k^{\mathrm{o}} = \bigsqcup_{(\underline{g}) \in T_k^{\mathrm{o}} Y_{(\underline{g})}} ,$$

and similarly, by $[Y_k^{\mathrm{o}}]$ the orbifold

$$[Y_k^{\mathrm{o}}] = \bigsqcup_{(\underline{g}) \in T_k^{\mathrm{o}}} [Y_{(\underline{g})}].$$

The definition of the orbifold cup product requires the construction of an *obstruction bundle* $[E_{(\underline{g})}]$ over each component $[Y_{(\underline{g})}]$ of $[Y_3^{\mathrm{o}}]$. We now review the definition of $[E_{(\underline{g})}]$.

Let $Y_{(\underline{g})}$ be a component of Y_3^{o}, $(y, (\underline{g})_y) \in Y_{(\underline{g})}$ be a point and $(\tilde{U}_y^{\underline{g}}, C(\underline{g}), \chi_y \mid)$ be a uniformizing system of $[Y_{(\underline{g})}]$ at $(y, (\underline{g})_y)$.

If $\underline{g} = (g_1, g_2, g_3)$ is a representative of $(\underline{g})_y$, then $g_1 \cdot g_2 \cdot g_3 = 1$. So we have a morphism

$$\begin{aligned} \pi_1(S^2 - \{0, 1, \infty\}) &\to G_y \\ \gamma_i &\mapsto g_i. \end{aligned}$$

Here, S^2 is the unit sphere in $\mathbb{R}^3$, so $\pi_1(S^2 - \{0, 1, \infty\})$ is the free group generated by three elements $\gamma_1, \gamma_2, \gamma_3$ with the unique relation $\gamma_1 \cdot \gamma_2 \cdot \gamma_3 = 1$. Geometrically, we can represent γ_1, γ_2 and γ_3 as loops around $0, 1$ and ∞ respectively.

There is a compact Riemann surface Σ and a projection $\tilde{\pi} : \Sigma \to S^2$ which is a Galois covering, with Galois group the subgroup G of G_y generated by g_1, g_2, g_3, branched over $0, 1, \infty$, see [28] Appendix. In particular, the group G acts on the vector space $H^1(\Sigma, \mathcal{O}_\Sigma)$.

Then we define a vector bundle over $\tilde{U}_y^{\underline{g}}$ as follows:

$$(E_{(\underline{g})})_{\tilde{U}_y^{\underline{g}}} := \left(H^1(\Sigma, \mathcal{O}_\Sigma) \otimes (T_{\tilde{U}_y}) \mid \tilde{U}_y^{\underline{g}} \right)^G , \qquad (3.1)$$

where $()^G$ means the G-invariant part. Note that $H^1(\Sigma, \mathcal{O}_\Sigma) \otimes (T_{\tilde{U}_y}) \mid \tilde{U}_y^{\underline{g}}$ is a G-bundle, so $(E_{(\underline{g})})_{\tilde{U}_y^{\underline{g}}}$ is a vector bundle. See [28], page 201, for further details.

Let $(\psi, \mu) : (\tilde{U}_{y'}^{\underline{g}'}, C(\underline{g}'), \chi_{y'} \mid) \to (\tilde{U}_y^{\underline{g}}, C(\underline{g}), \chi_y \mid)$ be an embedding compatible with $[Y_{(\underline{g})}]$. Then we can suppose that it is induced by an embedding $(\varphi, \lambda) : (\tilde{U}_{y'}, G_{y'}, \chi_{y'}) \to (\tilde{U}_y, G_y, \chi_y)$ compatible with $[Y]$. Then, if G' denote the subgroup of $G_{y'}$ generated by $\underline{g}'$, λ induces an isomorphism $G' \to G$, where G is the subgroup of G_y generated by $\underline{g}$. This induces an isomorphism $\Sigma' \to \Sigma$ which is compatible with the group actions, where $\Sigma' \to S^2$ is the Galois covering associated to $\underline{g}'$. So, we obtain an isomorphism

$$(E_{(\underline{g}')})_{\tilde{U}_{y'}^{\underline{g}'}} \to \psi^*(E_{(\underline{g})})_{\tilde{U}_y^{\underline{g}}}. \tag{3.2}$$

We have the following result.

Proposition 3.3.4 ([16]). *The vector bundles given by (3.1), for any uniformizing system, and the isomorphisms (3.2) associated to any embedding of local uniformizing systems compatible with $[Y_{(\underline{g})}]$, satisfies the conditions of Definition 2.4.1. So these data defines an orbifold vector bundle on $[Y_3^\circ]$.*

Definition 3.3.5. *The* ***obstruction bundle*** *for the orbifold cohomology of $[Y]$ is the orbifold vector bundle defined in Proposition 3.3.4.*

Notation 3.3.6. We will denote the obstruction bundle for the orbifold cohomology of $[Y]$ by $[E]$. The restriction of $[E]$ to the component $[Y_{(\underline{g})}]$, for any $(\underline{g}) \in T_3^\circ$, will be denoted by $[E_{(\underline{g})}]$.

Definition 3.3.7. *Let $[Y]$ be a complex orbifold such that Y is compact. For $\alpha, \beta, \gamma \in H^*_{orb}([Y])$, the* ***3-point function*** *is defined as follows*

$$\langle \alpha, \beta, \gamma \rangle_{orb} = \sum_{(\underline{g}) \in T_3^\circ} \int_{[Y_{(\underline{g})}]}^{orb} [e_1]^*\alpha \wedge [e_2]^*\beta \wedge [e_3]^*\gamma \wedge c_{top}([E_{(\underline{g})}]),$$

where $[e_i] : [Y_{(\underline{g})}] \to [Y_1]$, for $i = 1, 2, 3$, is the morphism defined in Lemma 3.3.1.

The ***orbifold cup product*** *of $[Y]$ is the product on $H^*_{orb}([Y])$*

$$\begin{aligned} \cup_{orb} : H^*_{orb}([Y]) \times H^*_{orb}([Y]) &\to H^*_{orb}([Y]) \\ (\alpha, \beta) &\mapsto \alpha \cup_{orb} \beta \end{aligned}$$

where $\alpha \cup_{orb} \beta$ is defined by the following relation

$$\langle \alpha \cup_{orb} \beta, \gamma \rangle_{orb} = \langle \alpha, \beta, \gamma \rangle_{orb} \quad \textit{for any } \gamma \in H^*_{orb}([Y]).$$

Remark 3.3.8. The orbifold cup product can be defined also for an *almost complex orbifold* that is not compact. We will not recall the definition in this general case, it can be found in [16], Definition 4.1.2.

We will report in the following Theorem the most important properties of the orbifold cup product. This is the main result of the paper [16], see Theorem 4.1.5. Even if the Theorem holds for almost complex orbifolds which are not necessarily compact, we will present the result under stronger assumptions.

Theorem 3.3.9. *Let $[Y]$ be a complex orbifold such that the underlying topological space Y is compact. Assume that $[Y]$ has complex dimension* $\dim_{\mathbb{C}}[\mathrm{Y}] = \mathrm{d}$. *The orbifold cup product preserves the orbifold grading, i.e.*

$$\cup_{orb} : H^p_{orb}([Y]) \times H^q_{orb}([Y]) \to H^{p+q}_{orb}([Y]),$$

for any $0 \leq p, q \leq 2d$ such that $p + q \leq 2d$, and has the following properties.

Associativity. *The orbifold cup product is associative and has a unity $e_{[Y]}$. Moreover, $e_{[Y]} \in H^0_{orb}([Y]) = H^0(Y)$ and it coincides with the unity of the usual cup product of Y.*

Poincaré duality. *For any $(\alpha, \beta) \in H^p_{orb}([Y]) \times H^{2d-p}_{orb}([Y])$, with $0 \leq p \leq 2d$, we have*

$$\int^{orb}_{[Y]} \alpha \cup_{orb} \beta = \langle \alpha, \beta \rangle_{orb}.$$

Deformation invariance. *The orbifold cup product $\cup_{orb}$ is invariant under deformations of the complex structure of $[Y]$.*

Supercommutativity. *If $[Y]$ is Gorenstein, the total orbifold cohomology is integrally graded, and we have supercommutativity*

$$\alpha \cup_{orb} \beta = (-1)^{\deg \alpha \cdot \deg \beta} \beta \cup_{orb} \alpha.$$

Compatibility with the usual cup product. *The restriction of $\cup_{orb}$ to the cohomology of the nontwisted sector, i.e. $H^*(Y)$, is equal to the usual cup product of Y.*

3.4 Ruan's conjecture

In this section we recall the statement of the *cohomological crepant resolution conjecture*, given by Ruan in [52]. The conjecture give a precise relation between the orbifold cohomology ring of a complex orbifold $[Y]$ and the cohomology ring of a crepant resolution of singularities of Y, when such a resolution exists.

Notation 3.4.1. In this section Y will be a complex algebraic variety. For an orbifold $[Y]$, we mean a complex orbifold structure over the topological space Y, where Y has the strong (or complex) topology, see [45] Chapter I, Section 10.

3.4.1 Crepant resolutions

We first recall the definition of Gorenstein variety, Gorenstein orbifold, and crepant resolution. For more details see [49] and [50].

Notation 3.4.2. For any normal variety Y, we will denote by Y_0 the smooth locus of Y and by $l : Y_0 \to Y$ the inclusion. Then K_Y will denote the sheaf $l_* K_{Y_0}$, where K_{Y_0} is the canonical sheaf of Y_0.

Definition 3.4.3. *Y is **Gorenstein** if it is Cohen-Macaulay and K_Y is locally free.*

*A **Gorenstein orbifold** is a complex orbifold structure $[Y]$ over a Gorenstein variety Y.*

Definition 3.4.4. *Let Y be a Gorenstein variety. A resolution of singularities $\rho : Z \to Y$ is crepant if $\rho^*(K_Y) \cong K_Z$.*

Remark 3.4.5. Crepant resolution of Gorenstein varieties with quotient singularities are known to exists in dimension $d = 2, 3$.

In dimension $d = 2$ the following stronger result hold, ([5], Chapter III, Theorem 6.2).

Theorem 3.4.6. *Every normal surface Y admits a unique crepant resolution of singularities.*

In dimension $d = 3$ the existence of a crepant resolution is proven in [51], Main Theorem, pag. 493. However, in this case, the uniqueness result does not hold.

In dimension $d \geq 4$ crepant resolution exists only in rather special cases.

Example 3.4.7. (*Hilbert scheme of points on surfaces*). An important class of examples for which a crepant resolution always exists, is the symmetric product of a compact complex surface S.

Let S be a compact complex surface. The r-th symmetric product of S is the quotient of $S \times ... \times S$, r-times, by the symmetric group $\mathfrak{S}_r$ acting by permutation. We will denote this quotient by $S^{(r)}$. Note that $S^{(r)}$ is a variety of dimension $2r$ with quotient singularities. Now, let $S^{[r]}$ be the Hilbert scheme parametrizing r-points in S. Then, there is a morphism $\rho : S^{[r]} \to S^{(r)}$ which is a crepant resolution. See [7], Section 6, for a review.

Example 3.4.8. (*Symplectic quotient singularities*). Let V be a finite dimensional complex vector space equipped with a non degenerate symplectic form. Let $G \subset Sp(V)$ be a finite subgroup of the group $Sp(V)$ of symplectic automorphisms. The quotient V/G has a natural structure of irreducible affine algebraic variety with coordinate ring $\mathbb{C}[V/G] = \mathbb{C}[V]^G$, the subalgebra of G-invariants polynomials on V. Moreover the variety V/G is Gorenstein ([8], Proposition 2.4).

There are strong necessary conditions on G in order for V/G to have a crepant resolution ([58], Theorem 1.2). Moreover there are examples of G that does not match these conditions, [32] Theorem 1.1.

3.4.2 Quantum corrections

We review the definition of *quantum corrected cohomology ring* given by Ruan in [52].

Notation 3.4.9. In this section Y will be a Gorenstein projective algebraic variety. So, for any crepant resolution $\rho : Z \to Y$, Z will be a nonsingular projective algebraic variety.

Let $[Y]$ be a Gorenstein orbifold and let $\rho : Z \to Y$ be a crepant resolution. Consider the group homomorphism

$$\rho_* : H_2(Z, \mathbb{Q}) \to H_2(Y, \mathbb{Q}) \tag{3.3}$$

induced by ρ. Choose a base $\beta_1, ..., \beta_n$ of Ker $\rho_* \subset H_2(Z, \mathbb{Q})$ which consists of elements that belong to the image of the morphism $H_2(Z, \mathbb{Z})/\text{torsion} \to H_2(Z, \mathbb{Q})$ which is induced by the group morphism $\mathbb{Z} \to \mathbb{Q}$. We will call $\beta_1, ..., \beta_n$ an *integral base* of Ker ρ_*. Then, the homology class of any effective curve that is contracted by ρ can be written in a unique way as $\Gamma = \sum_{l=1}^{n} a_l \beta_l$, for $a_l \geq 0$.

For each β_l, we assign a formal variable q_l. Then Γ corresponds to $q_1^{a_1} \cdots q_n^{a_n}$. Define the *quantum corrected* 3*-point function* as

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{qc}(q_1, ..., q_n) := \sum_{a_1,...,a_n \geq 0} \Psi_\Gamma^Z(\gamma_1, \gamma_2, \gamma_3) q_1^{a_1} \cdots q_n^{a_n}, \tag{3.4}$$

where $\gamma_1, \gamma_2, \gamma_3 \in H^*(Z)$ are cohomology classes, $\Gamma = \sum_{l=1}^n a_l \beta_l$, and $\Psi_\Gamma^Z(\gamma_1, \gamma_2, \gamma_3)$ is the genus zero Gromov-Witten invariant.

Notation 3.4.10. We assume that the quantum corrected 3-point function is represented by an analytic function in the variables $q_1, ..., q_n$ in some region of the complex space $\mathbb{C}^n$. We will denote this function by $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{qc}(q_1, ..., q_n)$. In the following, when we valuate this function on particular values of the q_i's, we will implicitly understand that the analytic function is defined on such values.

We now define a family of rings $H_\rho^*(Z)(q_1, ..., q_n)$ depending on the parameters $q_1, ..., q_n$, where $q_1, ..., q_n$ belong to the domain of definition of the quantum corrected 3-point function.

Definition 3.4.11. *The quantum corrected triple intersection* $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{qc}(q_1, ..., q_n)$ *is defined as follows*

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\rho(q_1, ..., q_n) = \langle \gamma_1, \gamma_2, \gamma_3 \rangle + \langle \gamma_1, \gamma_2, \gamma_3 \rangle_{qc}(q_1, ..., q_n),$$

where $\langle \gamma_1, \gamma_2, \gamma_3 \rangle := \int_Z \gamma_1 \cup \gamma_2 \cup \gamma_3$. *The quantum corrected cup product* $\gamma_1 *_\rho \gamma_2$ *is defined by the equation*

$$\langle \gamma_1 *_\rho \gamma_2, \gamma \rangle = \langle \gamma_1, \gamma_2, \gamma \rangle_\rho(q_1, ..., q_n)$$

for arbitrary $\gamma \in H^*(Z)$*, where* $\langle \gamma_1, \gamma_2 \rangle := \int_Z \gamma_1 \cup \gamma_2$.

Remark 3.4.12. Note that the quantum corrected cup product is a family of products on $H^*(Z)$ depending on the parameters $q_1, ..., q_n$. These parameters belong to the domain of definition of the quantum corrected 3-point function $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{qc}(q_1, ..., q_n)$.

Remark 3.4.13. Our definition of quantum corrected triple intersection and of quantum corrected cup product is slightely general from that given by Ruan in [52]. We can recover the definition given by Ruan by giving to the parameters the value $q_1 = ... = q_n = -1$, provided that this point belongs to the domain of the quantum corrected 3-point function.

Theorem 3.4.14. *For any $(q_1, ..., q_n)$ belonging to the domain of the quantum corrected 3-point function, the quantum corrected cup product $*_\rho$ satisfies the following properties.*

Associativity. *It is an associative product on $H^*(Z)$, moreover it has an identity which coincides with the identity of the usual cup product of Z.*

Supercommutativity. *It is supercommutative, that is*

$$\gamma_1 *_\rho \gamma_2 = (-1)^{\deg \gamma_1 \cdot \deg \gamma_2} \gamma_2 *_\rho \gamma_1$$

for any $\gamma_1, \gamma_2 \in H^*(Z)$.

Homogeneity. *For any $\gamma_1, \gamma_2 \in H^*(Z)$, the following equality holds*

$$deg\ (\gamma_1 *_\rho \gamma_2) = \deg \gamma_1 + \deg \gamma_2.$$

For any $(q_1, ..., q_n)$ as before, the resulting ring structure on $H^*(Z)$ given by $*_\rho$, will be denoted by $H^*_\rho(Z)(q_1, ..., q_n)$.

Proof. Notice that the definition of the quantum corrected cup product, $*_\rho$, is analogous to the definition of the small quantum product for a smooth projective algebraic variety Z, as given for example in [19], Definition 8.1.1. (see also [19], Proposition 8.1.6.). The only difference is in the set where the effective curves belongs. Indeed, for the small quantum product, the quantum corrected 3-point function is defined as a sum over all the effective curves in Z. In our definition, we take into account only effective curves that are contracted by ρ.

Let $B \subset H_2(Z, \mathbb{Z})/\text{tor}$ be the set of homology classes β of effective curves in Z such that $\rho_*(\beta) = 0$, where ρ_* is defined in (3.3). Then, from Lemma 3.4.15 below it follows that the proof of associativity and supercommutativity is the same proof of Theorem 8.1.4. in [19] with $H_2(Z, \mathbb{Z})$ replaced by B.

To prove the homogeneity property, notice that, for any $\beta \in B$, the following equality holds

$$\int_\beta c_1(K_Z) = 0.$$

Then, apply Proposition 8.1.5., [19]. □

Lemma 3.4.15. *Let $B \subset H_2(Z, \mathbb{Z})/tor$ be the set of homology classes β of effective curves in Z such that $\rho_*(\beta) = 0$, where ρ_* is defined in (3.3). Then B satisfies the following properties:*

- *B is a semigroup under addition and contains the zero of $H_2(Z, \mathbb{Z})/tor$;*

- *for any $\beta \in B$, if $\beta = \alpha_1 + \alpha_2$ with $\alpha_1, \alpha_2 \in H_2(Z, \mathbb{Z})/tor$, then $\alpha_1, \alpha_2 \in B$.*

Proof. The first condition is clear. So, we prove the second.
Let $\beta = \alpha_1 + \alpha_2 \in B$, with $\alpha_1, \alpha_2 \in H_2(Z, \mathbb{Z})/\text{tor}$. Then

$$0 = \rho_*(\beta) = \rho_*(\alpha_1) + \rho_*(\alpha_2).$$

But, $\rho_*(\alpha_1)$ and $\rho_*(\alpha_2)$ are homology classes of effective curves in Y, and since Y is assumed to be projective (Notation 3.4.9), $\rho_*(\alpha_1) = \rho_*(\alpha_2) = 0$.

To see this, take any very ample line bundle L on Y. Then $\int_{\rho_*(\alpha_i)} c_1(L) \geq 0$ for any $i = 1, 2$. Since $\int_{\rho_*(\beta)} c_1(L) = 0$, it follows that $\int_{\rho_*(\alpha_1)} c_1(L) = \int_{\rho_*(\alpha_2)} c_1(L) = 0$. This implies that $\rho_*(\alpha_1) = \rho_*(\alpha_2) = 0$. □

3.4.3 The conjecture

We review the statement of the *cohomological crepant resolution conjecture.*
We use the same notations of the previous section.

The *quantum corrected cohomology ring* of Z is the ring obtained from $H^*(Z)(q_1, ..., q_n)$ by giving to the variables $q_1, ..., q_n$ the same value -1. It will be denoted by $H^*_\rho(Z)$, so

$$H^*_\rho(Z) = H^*(Z)(-1, ..., -1).$$

Conjecture 3.4.16 (*Ruan*, [52])**.** *The rings $H^*_\rho(Z)$ and $H^*_{orb}([Y])$ are isomorphic.*

Remark 3.4.17. The conjecture has been proved in the following cases:

1. for $Y = S^{(2)}$, be the symmetric product of a compact complex surface S, and $Z = S^{[2]}$ be the Hilbert scheme of two points in S, in [38];

2. for $Y = S^{(r)}$, be the symmetric product of a complex projective surface S with trivial canonical bundle $K_S \cong \mathcal{O}_S$, and $Z = S^{[r]}$ be the Hilbert scheme of r points in S, in [28], Theorem 3.10;

3. for $Y = V/G$ be the quotient of a symplectic vector space of finite dimension by a finite group of symplectic automorphisms, and Z is a crepant resolution, when it does exists, in [32], Theorem 1.2.

Note that in cases 2 and 3, the crepant resolution Z has a holomorphic symplectic form, so the Gromov-Witten invariants vanish and there are no quantum corrections. This means that the orbifold cohomology ring is isomorphic to the ordinary cohomology ring of Z.

Remark 3.4.18. The aim of this thesis is to verify this conjecture for orbifolds with transversal ADE-singularities, see Definition 4.2.5. We will prove the conjecture in the case of transversal A_1-singularities in Chapter 7.1 and for transversal A_2-singularities with minor modifications. Indeed, in this case, the quantum corrected 3-point function can not be evaluated in $q_1 = q_2 = -1$. However, we will show that by giving to q_1 and q_2 other values (which we have found), the resulting ring $H^*_\rho(Z)(q_1, q_2)$ is isomorphic to the orbifold cohomology ring $H^*_{orb}([Y])$. This is the content of Theorem 7.2.4.

Chapter 4

Orbifolds with transversal ADE-singularities

In this Chapter we define orbifolds with transversal ADE-singularities. We describe the inertia orbifold in terms of the *monodromy*, which we introduce in Section 4. Finally, we compute the orbifold cohomology ring of orbifolds with transversal A-singularities and trivial monodromy.

Orbifolds with transversal ADE-singularities are generalizations of orbifolds associated to quotient surface singularities which are Gorenstein, also called *rational double points*. So, in the first section we recall the definition of such surface singularities and collect some properties.

4.1 Rational double points

Let $G \subset SL(2, \mathbb{C})$ be a finite subgroup, $G \neq \{1\}$. The inclusion $G \subset SL(2, \mathbb{C})$ induces an action of G on $\mathbb{C}^2$. This action has $0 \in \mathbb{C}^2$ as the only fixed point, and is free on $\mathbb{C}^2 \backslash \{0\}$. The quotient $\mathbb{C}^2/G$ has a structure of algebraic variety whose ring of regular functions is $\mathbb{C}[u, v]^G$, the subring of the polynomial ring $\mathbb{C}[u, v]$ consisting of functions which are invariants under the action of G. Moreover, such quotient can be represented as an hypersurface R in $\mathbb{C}^3$ passing through the origin $0 \in \mathbb{C}^3$ and with $0 \in \mathbb{C}^3$ as the only singular point ([26], Chapter 5, Section 40).

Definition 4.1.1. *A **rational double point** is the germ of a surface singularity $R \subset \mathbb{C}^3$ which can be obtained as a quotient $\mathbb{C}^2/G$ of $\mathbb{C}^2$ by a finite subgroup G of $SL(2, \mathbb{C})$.*

Remark 4.1.2. Rational double points are Gorenstein, see Example 3.4.8.

The finite subgroups of $SL(2,\mathbb{C})$ can be classified in the following way. There is a group homomorphism $SL(2,\mathbb{C}) \to PGL(2,\mathbb{C})$ which is onto and two-to-one. Then, the finite subgroups of $SL(2,\mathbb{C})$ are inverse image of finite subgroups of $PGL(2,\mathbb{C})$.

By identifying the sphere S^2 with the complex projective line $\mathbb{P}^1$, we can see that the symmetry groups of the five regular polyhedra are finite subgroups of $PGL(2,\mathbb{C})$. These groups are called the *polyhedral groups*. Now, since the cube and the dodecahedron are duals respectively to the octahedron and icosahedron, their symmety groups are isomorphic ([26], Chapter 2, Section 8). So, the symmety groups of the regular polyhedra provides three finite subgroups of $PGL(2,\mathbb{C})$.

The classification of finite subgroups of $PGL(2,\mathbb{C})$ is given by the following theorem. The proof and other details can be found in [26], Chapter 2, Section 10.

Theorem 4.1.3. *Any finite subgroups of $PGL(2,\mathbb{C})$ is conjugate to one of the following subgroups: the symmetry group of the tetrahedron, E_6, of order 12; the symmetry group of the octahedron, E_7, of order 24; the symmetry group of the icosahedron, E_8, of order 60; the dihedral group, D_n for $n \geq 4$, of order $4(n-2)$; the cyclic group, A_n, of order $n+1$.*

To this classification of the finite subgroups of $SL(2,\mathbb{C})$, corresponds a classification of rational double points. We report now the classification of rational double points as hypersurfaces in $\mathbb{C}^3$ with coordinate (x, y, z). In the left column we report the group, while in the right the equation of the corresponding singularity.

$$\begin{array}{lll} \mathrm{A}_n: & xy - z^{n+1} & \text{for } n \geq 1 \\ \mathrm{D}_n: & x^2 + y^2 z + z^{n-1} & \text{for } n \geq 4 \\ \mathrm{E}_6: & x^2 + y^3 + z^4 & \\ \mathrm{E}_7: & x^2 + y^3 + yz^3 & \\ \mathrm{E}_8: & x^2 + y^3 + z^5 & . \end{array} \tag{4.1}$$

This is proved in [26], Chapter 5, Section 39.

Remark 4.1.4. Can be proved that rational double points are the only rational surface singulerities. For more details on this and on other characterizations of rational double points, see [25].

Remark 4.1.5. *(Resolution graph).* As pointed out in Remark 3.4.5, for any rational double point R, there exists a unique crepant resolution $\rho : \tilde{R} \to R$ ([5], Chapter III, Theorem 6.2). The exceptional locus of ρ is union of rational curves $C_1, ..., C_n$ with autointersection numbers equal to -2. Moreover, it is possible to associate a graph to the collection of these curves in the following way: there is a vertex for any irreducible component of the exceptional locus; two vertices are joined by an edge if and only if the corresponding components have non zero intersection. The list of the graphs obtained by resolving rational double points is given in [25], Table 1, and in [5], Chapter III, Proposition 3.6. Each of this graph is called *resolution graph* of the corresponding rational double point.

Remark 4.1.6. *(McKay correspondence).* McKay observed that the resolution graph of a rational double point can be recovered from the representation theory of the corresponding subgroup $G \subset SL(2, \mathbb{C})$ [41], see also [20].

Let $G \subset SL(2, \mathbb{C})$ be a finite subgroup and $Q = \mathbb{C}^2$ be the representation induced by the inclusion $G \subset SL(2, \mathbb{C})$. Let $\rho_0, ..., \rho_m$ be the irreducible representations of G, with ρ_0 being the trivial one. Then, for any $j = 1, ..., m$ we can decompose $Q \otimes \rho_j$ as follows

$$Q \otimes \rho_j = \oplus_{i=0}^{m} a_{ij} \rho_i, \quad a_{ij} = \mathrm{dim_{\mathbb{C}} Hom_G}(\rho_i, Q \otimes \rho_j).$$

The *McKay graph* of $G \subset SL(2, \mathbb{C})$ is the graph with one vertex for any irreducible representation, two vertices are joined by a_{ij} arrows. This graph is denoted by $\tilde{\Gamma}_G$.

The following theorem holds, [41], see also [20] Theorem 1.19.

Theorem 4.1.7. *The McKay graph $\tilde{\Gamma}_G$ is an extended Dynkin graph of $\tilde{A}\tilde{D}\tilde{E}$ type. Moreover the subgraph Γ_G consisting of nontrivial representations is the resolution graph of the corresponding rational double point.*

4.2 Definition

Convention 4.2.1. All the varieties are dfined over the field $\mathbb{C}$ of complex numbers. By an open subset of a variety we mean open in the strong topology, [45], Chapter I, Section 10. We will specify when we use a Zariski-open subset.

Notation 4.2.2. From now on, R will denote a surface in $\mathbb{C}^3$ defined by one of the equations (4.1), i.e. a surface with a rational double point at the origin $0 \in \mathbb{C}^3$. The crepant resolution of R will be denoted by $\tilde{R}$.

Let Y be a projective variety over $\mathbb{C}$. We say that Y has *transversal ADE-singularities* if the singular locus S of Y is connected, smooth, and the couple (S, Y) is locally isomorphic to $(\mathbb{C}^k \times \{0\}, \mathbb{C}^k \times R)$.

Remark 4.2.3. Let Y be a 3-fold with canonical singularities. Then, with the exception of at most a finite number of points, every point in Y has an open neighbourhood which is nonsingular or isomorphic to $\mathbb{C} \times R$, [49], Corollary 1.14.

The following Proposition is a particular case of the fact that every complex variety with quotient singularities has a unique reduced orbifold structure, [56], Theorem 1.3.

Proposition 4.2.4. *Let Y be a variety with transversal ADE-singularities. Then there is a unique reduced complex holomorphic orbifold structure $[Y]$ on Y.*

Proof. The surface R is isomorphic to the quotient of $\mathbb{C}^2$ by the action of a finite subgroup G of $SL(2, \mathbb{C})$.

For each point $y \in Y$, if $y \in Y - S$, we can take $U_y = \tilde{U}_y = C^{k+2}$ and $G_y = \{1\}$, otherwise y has a uniformized neighbourhood $U_y \cong \mathbb{C}^k \times R$ with uniformizing system $(\tilde{U}_y \cong \mathbb{C}^k \times \mathbb{C}^2, G_y \cong G, \chi_y)$, where G_y acts trivially on the first factor of $\mathbb{C}^k \times \mathbb{C}^2$, while the action on the second factor is induced by the inclusion $G_y \subset SL(2, \mathbb{C})$. Then one can see that these charts patch together to give an orbifold structure $[Y]$ over Y.

The uniqueness of the orbifold structure follows from [56], Theorem 1.3. □

Definition 4.2.5. *An **orbifold with transversal** ADE**-singularities** is the reduced orbifold $[Y]$ associated to a variety Y with transversal ADE-singularities.*

Remark 4.2.6. An orbifold with transversal ADE-singularities is Gorenstein. This follows from the fact that rational double points are Gorenstein.

4.3 Inertia orbifold and monodromy

We describe the inertia orbifold for orbifolds with transversal ADE-singularities. We will use the same notation introduced in Section 1 of Chapter 3, in partucular Notation 3.1.1.

Notation 4.3.1. By a *topological covering* we mean a covering space as defined in [41], Chapter 5, Section 2, with the difference that we don't require the connectedness.

Let $p : \tilde{X} \to X$ be a topological covering. For any point $x \in X$, the fundamental group $\pi_1(X, x)$ of X in x acts on the fiber $p^{-1}(x)$, as defined in [41], Chapter 5, Section 7. We will call this action the *monodromy* of the coevring.

Proposition 4.3.2. *Let $[Y]$ be an orbifold with transversal ADE-singularities, and let*

$$\tilde{Y}_1 := \bigsqcup_{(g)\in T, (g)\neq(1)} Y_{(g)}.$$

Then, the restriction of $\pi : Y_1 \to Y$ to $\tilde{Y}_1$ is a topological covering

$$\tilde{\pi} : \tilde{Y}_1 \to S,$$

moreover, the connected components of $\tilde{Y}_1$ are the sets $Y_{(g)}$.

For any $y \in S$, the fundamental group $\pi_1(S, y)$ of S at y acts on the fiber $\tilde{\pi}^{-1}(y)$, and the map from the quotient set $\tilde{\pi}^{-1}(y)/\pi_1(S, y)$ to T defined as:

$$\begin{array}{rcl} \tilde{\pi}^{-1}(y)/\pi_1(S, y) & \to & T \\ (g)_y & \mapsto & [(y, (g)_y)] \end{array}$$

is a bijection, where T is as in Notation 3.1.1. Where $(g)_y$ is a conjugacy class of G_y and $[(y, (g)_y)]$ is the equivalence class of $(y, (g)_y)$ in T.

Proof. The proposition follows easily from the definition of Y_1 and the fact that Y has transversal ADE-singularities. □

Notation 4.3.3. For any $y \in S$, the fiber $\tilde{\pi}^{-1}(y)$ is the set of conjugacy classes of the local group G_y. We will denote it by T_y.

For later use, we describe the monodromy of $\tilde{\pi} : \tilde{Y}_1 \to S$ explicitly.

Proposition 4.3.4. *Let $[Y]$ be an orbifold with transversal ADE-singualarities. Then the orbifold structure $[Y]$ determines a group homomorphism from the fundamental group of S in y, $\pi_1(S, y)$, to the automorphism group of T_y, $Aut(T_y)$. Let us denote such homomorphism by*

$$\mathfrak{m}_y : \pi_1(S, y) \to Aut(T_y).$$

Moreover, if $y_1 \in S$ is another point, for any continuous curve γ in S from y to y_1, we get isomorphisms $\pi_\gamma : \pi_1(S,y) \to \pi_1(S,y_1)$ and $A_\gamma : Aut(T_y) \to Aut(T_{y_1})$ such that the following diagram commutes

$$\begin{array}{ccc} \pi_1(S,y) & \xrightarrow{\mathfrak{m}_y} & Aut(T_y) \\ {\scriptstyle \pi_\gamma}\downarrow & & \downarrow{\scriptstyle A_\gamma} \\ \pi_1(S,y_1) & \xrightarrow{\mathfrak{m}_{y_1}} & Aut(T_{y_1}). \end{array} \tag{4.2}$$

Proof. Let $y \in S$ be a fixed point of S, let $(\tilde{U}_y, G_y, \chi_y)$ be a uniformizing system at y, and let $U_y = \chi_y(\tilde{U}_y)$.

For any $y' \in U_y \cap S$ such that $U_{y'} \subset U_y$, there is an embedding $(\varphi, \lambda) : (\tilde{U}_{y'}, G_{y'}, \chi_{y'}) \to (\tilde{U}_y, G_y, \chi_y)$. Moreover $\lambda : G_{y'} \to G_y$ is an isomorphism. So, using λ, we get a bijective correspondence $T_{y'} \to T_y$ given by $(g)'_y \mapsto (\lambda(g))_y$, where $(g)'_y \in T_{y'}$ and g is a representative of $(g)'_y$. But if $(\psi, \mu) : (\tilde{U}_{y'}, G_{y'}, \chi_{y'}) \to (\tilde{U}_y, G_y, \chi_y)$ is another embedding, then $\mu = g \cdot \lambda \cdot g^{-1}$, for some $g \in G_y$, Remark 2.1.7. So the correspondence $T_{y'} \to T_y$ does not depends on the chosen embedding.

Let $[\alpha] \in \pi_1(S,y)$ be a class of a loop α based on y. Choose a finite number of points $y = y_0, y_1, ..., y_k, y_{k+1} = y$ in α such that the sets U_{y_i} form a cover of α. The previous argument gives bijective maps $T_{y_{i+1}} \to T_{y_i}$, for any $i = 0, ..., k$. The composition of these maps is a bijection $T_y \to T_y$. We will define $\mathfrak{m}_y([\alpha])$ to be this bijection. Note that it depends only on the homotopy class of α. It is easy to see that this defines a group homomorphism $\mathfrak{m}_y$.

Let $y_1 \in S$ be another point. Let $\pi_\gamma : \pi_1(S,y) \to \pi_1(S,y_1)$ be the standard homomorphism, $[\alpha] \mapsto [\gamma^{-1} \cdot \alpha \cdot \gamma]$. In the same way we defined $\mathfrak{m}_y([\alpha])$, we can define a correspondence $T_\gamma : T_y \to T_{y_1}$. Then $A_\gamma : \mathrm{Aut}(T_y) \to \mathrm{Aut}(T_{y_1})$ is defined by $a \mapsto T_\gamma \circ a \circ T_\gamma^{-1}$. It is easy to see that 4.2 is commutative. □

Definition 4.3.5. *Let $[Y]$ be an orbifold with transversal ADE-singularities, and let $y \in S$. The* ***monodromy*** *of $[Y]$ in y is the morphism*

$$\mathfrak{m}_y : \pi_1(S,y) \to Aut(T_y)$$

constructed in Proposition 4.3.4.

Proposition 4.3.6. *Let $y \in S$ be a fixed point. Then the map*

$$\begin{array}{rcl} T_y/\mathfrak{m}_y(\pi_1(S,y)) & \to & T \\ (g)_y & \mapsto & [(y,(g)_y)] \end{array} \tag{4.3}$$

is bijective. Where T is the set of equivalence classes defined in Notation 3.1.1, and $[(y,(g)_y)] \in T$ is the equivalence class of $(y,(g)_y)$.

Let $y_1 \in S$ be another point, and let γ be a continuous curve from y to y_1. Then the application $T_\gamma : T_y \to T_{y_1}$ defined in the proof of Proposition 4.3.4 induces a bijection

$$T_y/\mathfrak{m}_y(\pi_1(S,y)) \to T_{y_1}/\mathfrak{m}_{y_1}(\pi_1(S,y_1)) \tag{4.4}$$

which does not depend on the curve γ and is such that the following diagram commutes

$$\begin{array}{ccc} T_y/\mathfrak{m}_y(\pi_1(S,y)) & \longrightarrow & T \\ \downarrow & & \downarrow = \\ T_{y_1}/\mathfrak{m}_{y_1}(\pi_1(S,y_1)) & \longrightarrow & T. \end{array}$$

Proof. The fact that the map 4.3 is bijective follows easily from the definition of the monodromy. The only thing to be proved is the second statement.

The map 4.4 is well defined. Indeed, let $\mathfrak{c}, \mathfrak{c}' \in T_y$ such that $\mathfrak{c}' = \mathfrak{m}_y(\alpha)\cdot\mathfrak{c}$, for some $\alpha \in \pi_1(S,y))$. Then

$$\begin{aligned} T_\gamma(\mathfrak{m}_y(\alpha)\cdot\mathfrak{c}) &= T_\gamma \circ \mathfrak{m}_y(\alpha) \circ T_\gamma^{-1}(T_\gamma(\mathfrak{c}) \\ &= \mathfrak{m}_{y_1}(\pi_\gamma(\alpha)) \cdot T_\gamma(\mathfrak{c}), \end{aligned}$$

in the last equality we have used the commutativity of 4.2. It doesn't depend on the curve γ. Indeed, if δ is another curve from y to y_1, then $T_\delta = \mathfrak{m}_{y_1}(\delta^{-1}\cdot\gamma)\circ T_\gamma$. □

Remark 4.3.7. For any finite group G, the set of equivalence clesses of irreducible representations of G modulo isomorphisms is in bijection with the set of characters of G. Moreover the characters form a basis for the vector space of class functions, where as usual a class function is a complex valued function on G that is constant on each conjugacy class. So, the set of characters of G is bijective with the set of cnjugacy classes of G.

Remark 4.3.8. Let $[Y]$ be an orbifold with transversal ADE-singularities, let $y \in S$ and G_y be the local group of $[Y]$ at y. We identify each conjugacy class $(g)_y$ of G_y with its characteristic function $\mathfrak{c}_{(g)_y}$, which is a class function. The monodromy $\mathfrak{m}_y$ extends to an automorphism of the set of class functions on G_y.

Proposition 4.3.9. *Under the same hypothesis of the previous Remark 4.3.8, the set of characters of irreducible representations of G_y is invariant under the action of the monodromy $\mathfrak{m}_y$ on the set of class functions on G_y.*

Moreover, under the identification of any irreducible representation with its character, we have that the image of $\mathfrak{m}_y$ is a subgroup of the group of automorphisms of the graph $\Gamma_{G_{y'}}$.

Proof. Let $(\tilde{U}_y, G_y, \chi_y)$ be a uniformizing system at y, and let $U_y = \chi_y(\tilde{U}_y)$. For any $y' \in U_y \cap S$ such that $U_{y'} \subset U_y \cap S$, there is an embedding $(\varphi, \lambda) : (\tilde{U}_{y'}, G_{y'}, \chi_{y'}) \to (\tilde{U}_y, G_y, \chi_y)$. Moreover $\lambda : G_{y'} \to G_y$ is an isomorphism. We obtain a map which sends a representation of G_y to a representation of $G_{y'}$ as follows. Let $\mathfrak{r} : G_y \to GL(V)$ be a linear representation of G_y, then $\mathfrak{r} \circ \lambda : G_{y'} \to GL(V)$ is a linear representation of $G_{y'}$.

This map has the following properties: 1. is bijective; 2. the character of $\mathfrak{r} \circ \lambda$ is the composition of the character of $\mathfrak{r}$ with λ; 3. sends isomorphic representations of G_y in isomorphic representations of $G_{y'}$; 4. sends irreducible representations of G_y in irreducible representations of $G_{y'}$; 5. is compatible with the tensor product of two representations, that is $(\mathfrak{r}_1 \otimes \mathfrak{r}_2) \circ \lambda = \mathfrak{r}_1 \circ \lambda \otimes \mathfrak{r}_2 \circ \lambda$; 6. sends the representation of G_y on $N_{\tilde{U}_y^{G_y}/\tilde{U}_y,0}$ in the representation of $G_{y'}$ in $N_{\tilde{U}_{y'}^{G_{y'}}/\tilde{U}_{y'},0}$. Where $N_{\tilde{U}_y^{G_y}/\tilde{U}_y,0}$ (resp. $N_{\tilde{U}_{y'}^{G_{y'}}/\tilde{U}_{y'},0}$) is the fiber at the origin $0 \in \tilde{U}_y$ (resp. $0 \in \tilde{U}_{y'}$) of the normal vector bundle of $\tilde{U}_y^{G_y}$ in $\tilde{U}_y$ (resp. of $\tilde{U}_{y'}^{G_{y'}}$ in $\tilde{U}_{y'}$. The last assertion is proved as follows. The tangent map

$$T_0\varphi : T_{\tilde{U}_{y'},0} \to T_{\tilde{U}_y,0}$$

induces a linear map

$$N_{\tilde{U}_{y'}^{G_{y'}}/\tilde{U}_{y'},0} \to N_{\tilde{U}_y^{G_y}/\tilde{U}_y,0}$$

such that the following diagram commutes for any $g' \in G_{y'}$

$$\begin{array}{ccc} N_{\tilde{U}_{y'}^{G_{y'}}/\tilde{U}_{y'},0} & \xrightarrow{g'} & N_{\tilde{U}_{y'}^{G_{y'}}/\tilde{U}_{y'},0} \\ \downarrow & & \downarrow \\ N_{\tilde{U}_y^{G_y}/\tilde{U}_y,0} & \xrightarrow{\lambda(g')} & N_{\tilde{U}_y^{G_y}/\tilde{U}_y,0}. \end{array} \tag{4.5}$$

If $(\psi, \mu) : (\tilde{U}_{y'}, G_{y'}, \chi_{y'}) \to (\tilde{U}_y, G_y, \chi_y)$ is anothe embedding, then $\mu = g \cdot \lambda \cdot g^{-1}$, for some $g \in G_y$, Remark 2.1.7. So, for any representation of G_y, the representations of $G_{y'}$ obtained using λ and μ are isomorphic.

The previous arguments shows that we have a map from the set of isomorphism classes of representations of G_y to the set of isomorphism classes of representations of $G_{y'}$ that satisfies properties 1., 2., 4., 5., 6. as before. We will denote with $\mathfrak{n}_y$ the class of $N_{\tilde{U}_y^{G_y}/\tilde{U}_y,0}$ and by $\mathfrak{n}_{y'}$ the class of $N_{\tilde{U}_{y'}^{G_{y'}}/\tilde{U}_{y'},0}$.

Property 2. gives the first assertion. To show the second claim, let $\mathfrak{r}_1, ..., \mathfrak{r}_m$ be the classes of irreducible representations of G_y, and let $\mathfrak{r}_1', ..., \mathfrak{r}_m'$ the classes of irreducible representations of $G_{y'}$. Suppose that the correspondence sends $\mathfrak{r}_i$ to $\mathfrak{r}_i'$. Let $\mathfrak{n}_y \otimes \mathfrak{r}_i = \sum_j a_{ji}\mathfrak{r}_j$, and $\mathfrak{n}_y \otimes \mathfrak{r}_i' = \sum_j a'_{ji}\mathfrak{r}_j'$. Then $a_{ji} = a'_{ji}$ for all i, j. □

Remark 4.3.10. For $G = A_n,\ n \geq 1,\ D_n\ n \geq 4, E_6, E_7, E_8$ (see Theorem 4.1.3), the automorphism group of Γ_G is given as follows:

G		$\mathrm{Aut}(\Gamma_G)$
A_1		$\{1\}$
A_n	$n \geq 2$	$\mathbb{Z}_2$
D_4		$\mathfrak{S}_3$
D_n	$n \geq 5$	$\mathbb{Z}_2$
E_6		$\mathbb{Z}_2$
E_7		$\{1\}$
E_8		$\{1\}$

where we have written on the left side the group G, and on the right $\mathrm{Aut}(\Gamma_G)$.

Proposition 4.3.11. *Let $[Y]$ be an orbifold with transversal singularities of type A_1, E_7 or E_8, then, for any $(g) \neq (1)$, the topological space $Y_{(g)}$ is isomorphic to S.*

Let $[Y]$ be an orbifold with transversal singularities of type A_n, for $n \geq 2$, or D_n, for $n \geq 5$, then, for any $(g) \neq (1)$, the topological space $Y_{(g)}$ is isomorphic to S if the monodromy is trivial, it is a double covering of S if the monodromy is not trivial.

As a consequence of the previous propositions we have the following description of the inertia orbifold in the case of transversal A_n-singularities with trivial monodromy.

Proposition 4.3.12. *Let $[Y]$ be an orbifold with transversal A_n-singularities and assume that the monodromy is trivial. Then the twisted sectors $[Y_{(g)}]$, for $(g) \neq (1)$, are isomorphic to $[S/\mu_{n+1}]$, where μ_{n+1} is the group of $(n+1)$-th roots of unity acting trivially on S.*

Moreover we can identify the local groups G_y with μ_{n+1} for all $y \in S$ in a consistent way and, under this identification, we have

$$[Y_1] \cong [Y] \bigsqcup_{g \in \mu_{n+1}, g \neq 1} [S/\mu_{n+1}].$$

4.4 Orbifold cohomology ring

We now describe the orbifold cohomology ring of an orbifold $[Y]$ with transversal A_n-singularities and trivial monodromy.

Convention 4.4.1. We identify in a consistent way G_y with μ_{n+1} for all $y \in S$. We also identify μ_{n+1} with the group $\mathbb{Z}_{n+1}$ of integers modulo $n+1$ via the morphism:

$$a \in \mathbb{Z}_{n+1} \mapsto exp(\frac{2\pi i}{n+1} a).$$

Then, g_1, g_2 and g_3 in μ_{n+1} correspond respectively to a_1, a_2 and a_3 in $\mathbb{Z}_{n+1}$. We denote by $\underline{a}$ the vector (a_1, a_2, a_3).

Lemma 4.4.2. *For any $y \in S$, let $(\tilde{U}_y, G_y, \chi_y)$ be a uniformizing system of $[Y]$ at y. Then, the normal vector bundles*

$$N_{\tilde{U}_y^{G_y}/\tilde{U}_y} \to S \cap \tilde{U}_y \qquad \textit{for } y \in S,$$

for any $y \in S$, defines an orbifold vector bundle $[N]$ over $[S/\mathbb{Z}_{n+1}]$, of rank 2. Where the action of $\mathbb{Z}_{n+1}$ on S is trivial.

If $n \geq 2$, then $[N]$ is isomorphic to the direct sum of two orbifold vector bundles $[N]^{\mathfrak{g}}$ and $[N]^{\mathfrak{g}^{-1}}$ of rank 1, that is

$$[N] \cong [N]^{\mathfrak{g}} \oplus [N]^{\mathfrak{g}^{-1}}.$$

Proof. Let $y \in S$, and let $(\tilde{U}_y, G_y, \chi_y)$ be a uniformizing system for $[Y]$ at y. Then $(\tilde{U}_y^{G_y}, G_y, \chi_y \mid)$ is a uniformizing system for $[S/\mathbb{Z}_{n+1}]$ at y, where $\chi_y \mid$ denotes the restriction of χ_y at $\tilde{U}_y^{G_y}$. Moreover, let $y' \in S$ and $(\tilde{U}_{y'}, G_{y'}, \chi_{y'})$ be a uniformizing system at y' such that $\chi_y(\tilde{U}_y) \subset \chi_{y'}(\tilde{U}_{y'})$. Then any embedding $(\varphi, \lambda) : (\tilde{U}_y, G_y, \chi_y) \to (\tilde{U}_{y'}, G_{y'}, \chi_{y'})$ compatible with $[Y]$ induces an embedding $(\varphi \mid, \lambda) : (\tilde{U}_y^{G_y}, G_y, \chi_y \mid) \to (\tilde{U}_{y'}^{G_{y'}}, G_{y'}, \chi_{y'} \mid)$ compatible with $[S/\mathbb{Z}_{n+1}]$. So, the orbifold $[Y]$ induces an orbifold structure on S equivalent to $[S/\mathbb{Z}_{n+1}]$.

For any uniformizing system $(\tilde{U}_y^{G_y}, G_y, \chi_y \mid)$, we define

$$N_{\tilde{U}_y^{G_y}} := N_{\tilde{U}_y^{G_y}/\tilde{U}_y}.$$

For any embedding $(\varphi \mid, \lambda) : (\tilde{U}_y^{G_y}, G_y, \chi_y \mid) \to (\tilde{U}_{y'}^{G_{y'}}, G_{y'}, \chi_{y'} \mid)$, we get an isomorphism $N_{\tilde{U}_y^{G_y}} \to \varphi \mid^* N_{\tilde{U}_{y'}^{G_{y'}}}$ which is induced by the tangent morphism

$$T\varphi : T_{\tilde{U}_y} \to T_{\tilde{U}_{y'}}.$$

This data defines $[N]$.

To define $[N]^{\mathfrak{g}}$ and $[N]^{\mathfrak{g}^{-1}}$, notice that $\mathbb{Z}_{n+1}$ acts on $N_{\tilde{U}_y^{G_y}}$. So,

$$N_{\tilde{U}_y^{G_y}} \cong \left(N_{\tilde{U}_y^{G_y}}\right)^{\mathfrak{g}} \oplus \left(N_{\tilde{U}_y^{G_y}}\right)^{\mathfrak{g}^{-1}}, \tag{4.6}$$

where $\mathfrak{g} : \mathbb{Z}_{n+1} \to \mathbb{C}^*$ is the character $a \mapsto \exp(\frac{2\pi i a}{n+1})$, $\mathfrak{g}^{-1}$ is the dual of $\mathfrak{g}$, and $\left(N_{\tilde{U}_y^{G_y}}\right)^{\mathfrak{g}}$ (resp. $\left(N_{\tilde{U}_y^{G_y}}\right)^{\mathfrak{g}^{-1}}$) is the subbundle of $N_{\tilde{U}_y^{G_y}}$ on which $\mathbb{Z}_{n+1}$ acts by the character $\mathfrak{g}$ (resp. $\mathfrak{g}^{-1}$). Since the monodromy is tivial, the line bundle $\left(N_{\tilde{U}_y^{G_y}}\right)^{\mathfrak{g}}$ (resp. $\left(N_{\tilde{U}_y^{G_y}}\right)^{\mathfrak{g}^{-1}}$) define the orbifold vector bundle $[N]^{\mathfrak{g}}$ (resp. $[N]^{\mathfrak{g}^{-1}}$). □

Lemma 4.4.3. Let $n \geq 2$, and consider the natural morphism of orbifolds

$$[S/\mathbb{Z}_{n+1}] \to S,$$

where the variety S is considered as an orbifold. Then, there are line bundles L, M and K on S whose pull-back under this morphism are $([N]^{\mathfrak{g}^{-1}})^{\otimes n+1}$, $([N]^{\mathfrak{g}})^{\otimes n+1}$ and $[N]^{\mathfrak{g}^{-1}} \otimes [N]^{\mathfrak{g}}$ respectively.

Proof. We prove the statement about $[N]^{\mathfrak{g}^{-1}}$ and L, the other are similar. Let $\{(\tilde{U}_i, G_i, \chi_i)\}_{i\in I}$ be a set of uniformizing systems for $[Y]$ such that $\{(\tilde{U}_i \cap S, G_i, \chi_i \mid)\}_{i\in I}$ is an orbifold atlas. We can assume that, for any $i, j \in I$, we have isomorphisms

$$g_{ij} : \left([N]^{\mathfrak{g}^{-1}}\right)_{\tilde{U}_{ij}\cap S} \to \left([N]^{\mathfrak{g}^{-1}}\right)_{\tilde{U}_{ji}\cap S}.$$

On triple intersections $\tilde{U}_{ijk} \cap S$, the isomorphisms $g_{jk} \circ g_{ij}$ differ from g_{ik} by the action of an element of the group. So,

$$g_{jk}^{n+1} \circ g_{ij}^{n+1} = g_{ik}^{n+1}.$$

This proves the claim. □

Theorem 4.4.4. *We have the following identification of vector spaces*

$$H^p_{orb}([Y]) = H^p(Y_{(0)}) \oplus_{a \in \mathbb{Z}_{n+1}, a \neq 0} H^{p-2}(Y_{(a)})$$

for all p. The orbifold cup product is given as follows for $\alpha \in H^*(Y_{(a_1)})$ *and* $\beta \in H^*(Y_{(a_2)})$*:*

1. $\alpha \cup_{orb} \beta = \alpha \cup \beta \in H^*(Y)$ *if* $a_1 = a_2 = 0$
2. $\alpha \cup_{orb} \beta = \alpha \cup i^*(\beta) \in H^*(Y_{(a_1)})$ *if* $a_1 \neq 0,\ a_2 = 0$
3. $\alpha \cup_{orb} \beta = \frac{1}{n+1} i_*(\alpha \cup \beta) \in H^*(Y)$ *if* $a_1 \neq 0,\ a_2 = a_1^{-1}$
4. $\alpha \cup_{orb} \beta = \frac{1}{n+1} \alpha \cup \beta \cup c_1(L) \in H^*(Y_{(a_1+a_2)})$ *if* $a_1 \neq 0,\ a_2 \neq 0,\ a_1 + a_2 < n+1$ *in* $\mathbb{Z}$
5. $\alpha \cup_{orb} \beta = \frac{1}{n+1} \alpha \cup \beta \cup c_1(M) \in H^*(Y_{(a_1+a_2-n+1)})$ *if* $a_1 \neq 0,\ a_2 \neq 0$, $a_1 + a_2 > n+1$ *in* $\mathbb{Z}$,

where L and M are the line bundles defined in Lemma 4.4.3, $i : S \to Y$ *is the inclusion of the singular locus in* Y*, and* $\cup$ *is the ordinary cup product of* Y*.*

Note that, since $[Y]$ *is Gorenstein,* $\cup_{orb}$ *is supercommutative, see Theorem 3.3.9.*

Proof. The orbifold vector bundles $[E_{(\underline{a})}]$ have rank 0 if $a_1 = 0,\ a_2 = 0$ or $a_3 = 0$. This follows e.g. from [28], Lemma 1.12, where a relation between the rank of $[E_{(\underline{a})}]$ and the degree shifting numbers of a_1, a_2 and a_3 is proved.

Let $y \in S$ and $\underline{a} = (a_1, a_2, a_3) \neq (0,0,0)$. By definition (see equation 3.1) we have

$$(E_{(\underline{a})})_{\tilde{U}_y^{\underline{a}}} := \left(H^1(\Sigma, \mathcal{O}_\Sigma) \otimes (T_{\tilde{U}_y}) \mid \tilde{U}_y^{\underline{a}} \right)^G,$$

where $G \subset \mathbb{Z}_{n+1}$ is the subgroup generated by a_1, a_2, a_3, $\Sigma \to \mathbb{P}^1$ is the Galois cover with Galois group G branched over $0, 1, \infty \in \mathbb{P}^1$ and monodromy respectively a_1, a_2, a_3.

We first notice that we can replace $(T_{\tilde{U}_y}) \mid \tilde{U}_y^{\underline{a}}$ with the normal bundle $N_{\tilde{U}_y^{\underline{a}}/\tilde{U}_y}$. Then, we can replace Σ with the Galois cover $C \to \mathbb{P}^1$, induced by the inclusion $G \subset \mathbb{Z}_{n+1}$, which has Galois group $\mathbb{Z}_{n+1}$. So we have:

$$(E_{(\underline{a})})_{\tilde{U}_y^{\underline{a}}} \cong \left(H^1(C, \mathcal{O}_C) \otimes N_{\tilde{U}_y^{\underline{a}}/\tilde{U}_y} \right)^{\mathbb{Z}_{n+1}}.$$

Note that $p : C \to \mathbb{P}^1$ is an abelian cover in the sense of [47], so

$$p_* \mathcal{O}_C = \oplus_{\mathfrak{c} \in \mathbb{Z}^*_{n+1}} (L^{-1})^{\mathfrak{c}}$$

where $\mathbb{Z}_{n+1}^*$ is the group of characters of $\mathbb{Z}_{n+1}$ and $\mathbb{Z}_{n+1}$ acts on $(L^{-1})^{\mathfrak{c}}$ via the character $\mathfrak{c}$. Note that the characters $\mathfrak{g}$ and $\mathfrak{g}^{-1}$ defined in Lemma 4.4.2 are generators of $\mathbb{Z}_{n+1}^*$.

Using the fact $H^1(C, \mathcal{O}_C) \cong H^1(\mathbb{P}^1, p_*\mathcal{O}_C)$ and the decomposition (4.6), we have

$$\left(H^1(C, \mathcal{O}_C) \otimes N_{\tilde{U}_y^{\underline{a}}/\tilde{U}_y}\right)^{\mathbb{Z}_{n+1}} \cong$$
$$H^1(\mathbb{P}^1, (L^{-1})^{\mathfrak{g}}) \otimes \left(N_{\tilde{U}_y^{\underline{a}}/\tilde{U}_y}\right)^{\mathfrak{g}^{-1}} \oplus H^1(\mathbb{P}^1, (L^{-1})^{\mathfrak{g}^{-1}}) \otimes \left(N_{\tilde{U}_y^{\underline{a}}/\tilde{U}_y}\right)^{\mathfrak{g}}.$$

By Proposition 2.1 and in particular by example 2.1 i) of [47] we have that

$$L^{\mathfrak{g}} = \begin{cases} \mathcal{O}(2) & \text{if } a_1 + a_2 < n+1, \\ \mathcal{O}(1) & \text{if } a_1 + a_2 \geq n+1 \end{cases}$$

and

$$L^{\mathfrak{g}^{-1}} = \begin{cases} \mathcal{O}(1) & \text{if } a_1 + a_2 \leq n+1, \\ \mathcal{O}(2) & \text{if } a_1 + a_2 > n+1. \end{cases}$$

It follows that

$$(E_{(\underline{a})})_{\tilde{U}_y^{\underline{a}}} = \begin{cases} \left(N_{\tilde{U}_y^{\underline{a}}/\tilde{U}_y}\right)^{\mathfrak{g}^{-1}} & \text{if } \; a_1 + a_2 < n+1 \\ \left(N_{\tilde{U}_y^{\underline{a}}/\tilde{U}_y}\right)^{\mathfrak{g}} & \text{if } \; a_1 + a_2 > n+1 \end{cases}$$

and the obstruction bundle is

$$[E_{(\underline{a})}] = \begin{cases} [N]^{\mathfrak{g}^{-1}} & \text{if } \; a_1 + a_2 < n+1 \\ [N]^{\mathfrak{g}} & \text{if } \; a_1 + a_2 > n+1 \end{cases}$$

□

4.5 Examples

We give here some special examples of orbifold cohomology rings.

Example 4.5.1. *(Surface case).* We give now a description of the orbifold cohomology of a surface with an A_n-singularity:

$$Y = \{(x, y, z) \in \mathbb{C}^3 : xy - z^{n+1} = 0\}.$$

Y is the quotient of $\mathbb{C}^2$ by the action of the group μ_{n+1} given by $\epsilon \cdot (u, v) = (\epsilon \cdot u, \epsilon^{-1} \cdot v)$, $\epsilon \in \mu_{n+1}$.
As a vector space

$$H^*_{orb}([Y]) = H^*(Y) \oplus H^{*-2}(S)\langle e_1 \rangle \oplus ... \oplus H^{*-2}(S)\langle e_n \rangle$$

where e_i is a generator of $H^*(Y_{(i)})$ as $H^*(S)$-module.
The product rule is given by

$$e_i \cup_{orb} e_j = \begin{cases} 0 & \text{if } i + j \neq 0(\text{mod } n+1), \\ \frac{1}{n+1} i_*[S] \in H^4(Y) & \text{if } i + j = 0(\text{mod } n+1). \end{cases}$$

Example 4.5.2. *(Transversal A_1-case).*

$$H^*_{orb}([Y]) = H^*(Y) \oplus H^{*-2}(S)$$

as vector space. Given $(\delta_1, \alpha_1), (\delta_2, \alpha_2) \in H^*_{orb}(Y)$, we have the following espression for the orbifold cup product:

$$(\delta_1, \alpha_1) \cup_{orb} (\delta_2, \alpha_2) = (\delta_1 \cup \delta_2 + \frac{1}{2} i_*(\alpha_1 \cup \alpha_2), i^*(\delta_1) \cup \alpha_2 + \alpha_1 \cup i^*(\delta_2))$$

Note that in this case the ostruction bundle $[E]$ has rank zero.

Example 4.5.3. *(Transversal A_2-case).*

$$H^*_{orb}([Y]) = H^*(Y) \oplus H^{*-2}(S) \oplus H^{*-2}(S)$$

as vector space. Given $(\delta_1, \alpha_1, \beta_1), (\delta_2, \alpha_2, \beta_2) \in H^*_{orb}(Y)$, we have the following espression for the orbifold cup product:

$$\begin{aligned} (\delta_1, \alpha_1, \beta_1) \quad \cup_{orb} \quad (\delta_2, \alpha_2, \beta_2) &= (\delta_1 \cup \delta_2 + \frac{1}{2} i_*(\alpha_1 \cup \beta_2 + \beta_1 \cup \alpha_2), \\ & i^*(\delta_1) \cup \alpha_2 + \alpha_1 \cup i^*(\delta_2) + \beta_1 \cup \beta_2 \cup c_1(L), \\ & i^*(\delta_1) \cup \beta_2 + \beta_1 \cup i^*(\delta_2) + \alpha_1 \cup \alpha_2 \cup c_1(M)). \end{aligned}$$

Chapter 5

Crepant resolutions

In this Chapter we show that any variety with transversal ADE-singularities Y (see Chapter 4.2) has a unique crepant resolution $\rho : Z \to Y$. Then we restrict our attention to the case of transversal A_n-singularities and trivial monodromy. In this case we describe the exceptional locus E in terms of the line bundles L and M defined in Lemma 4.4.3. We compute the cohomology ring $H^*(Z)$ of Z in terms of the cohomology of Y and of E.

In the first section we recall some facts about the resolution of rational double points.

5.1 Crepant resolutions of rational double points

Let $R \subset \mathbb{C}^3$ be a rational double point. Then, by Theorem 3.4.6, R has a unique crepant resolution $\rho : \tilde{R} \to R$. $\tilde{R}$ can be obtained by blowing-up successively the singular locus. The exceptional locus $C \subset \tilde{R}$ is union of rational curves C_l whose autointersection numbers are $C_l \cdot C_l = -2$. The shape of C inside $\tilde{R}$ is described by the resolution graph, see Remark 4.1.5. We explain with the next example the A_n case.

Example 5.1.1. *(Resolution of A_n-surface singulerities).* Let

$$R = \{(x, y, z) \in \mathbb{C}^3 : \; xy - z^{n+1} = 0\}$$

be a surface singularity of type A_n. Let $r : R_1 = Bl_0 R \to R$ be the blow-up of R at the origin. Then R_1 is covered by three open affine varieties U, V

and W, where

$$\begin{aligned} U &= \{\left(x, \frac{v}{u}, \frac{w}{u}\right) \in \mathbb{C}^3 : \left(\frac{v}{u}\right) - x^{n-1}\left(\frac{w}{u}\right)^{n+1} = 0\} \\ V &= \{\left(y, \frac{u}{v}, \frac{w}{v}\right) \in \mathbb{C}^3 : \left(\frac{u}{v}\right) - y^{n-1}\left(\frac{w}{v}\right)^{n+1} = 0\} \\ W &= \{\left(z, \frac{u}{w}, \frac{v}{w}\right) \in \mathbb{C}^3 : \frac{u}{w}\frac{v}{w} - z^{n-1} = 0\}. \end{aligned}$$

and the restruction of r to U, V, W is given as follows

$$\begin{aligned} r_{|U} : \left(x, \frac{v}{u}, \frac{w}{u}\right) &\mapsto (x,y,z) = \left(x, x\frac{v}{u}, x\frac{w}{u}\right) \\ r_{|V} : \left(y, \frac{u}{v}, \frac{w}{v}\right) &\mapsto (x,y,z) = \left(y\frac{u}{v}, y, y\frac{w}{v}\right) \\ r_{|W} : \left(z, \frac{u}{w}, \frac{v}{w}\right) &\mapsto (x,y,z) = \left(z\frac{u}{w}, z\frac{v}{w}, z\right). \end{aligned}$$

If $n = 1$, R_1 is smooth and the exceptional locus is given by one rational curve C. A direct computation shows that $C \cdot C = -2$. If $n \geq 2$, R_1 has a singularity of type A_{n-2} at the origin of W and the exceptional locus is given by the union of two rational curves meeting at the singular point. Then, after a finite number of blow-up, we get a smooth surface.

Let $\tilde{R}$ be the first smooth surface obtained in this way and $\rho : \tilde{R} \to R$ the composition of the blow-up morphisms. Let $C = C_1, ..., C_n$ be the components of the exceptional locus. A direct computation shows that $C_l \cdot C_l = -2$, for any $l = 1, ..., n$. From adjunction formula we get $K_{\tilde{R}} \cdot C_l = 0$, for any $l = 1, ..., n$. We want to prove that $\rho^* K_R \cong K_{\tilde{R}}$. But

$$\rho^* K_R \cong K_{\tilde{R}} + \sum_{l=1}^{n} a_l C_l,$$

for some integers $a_1, ..., a_n$. The intersection of the right and left side of the previous espression gives:

$$\sum_{l=1}^{n} a_l C_l \cdot C_k = 0 \quad \text{for any } k = 1, ..., n.$$

Since the matrix with entries $(C_l \cdot C_k)$ is negative definite ([5], Chapter III, Theorem 2.1), it follows $a_l = 0$ for all $l = 1, ..., n$.

From this description, it is clear that the exceptional locus C is a chain of rational curves whose dual graph is Γ_{A_n}, (see Remark 4.1.6).

5.2 Existence and unicity

Proposition 5.2.1. *Let Y be a variety with transversal ADE-singularities. Then, there exists a unique crepant resolution $\rho : Z \to Y$.*

Proof. To prove existence, one can proceed as follows. Let $r : Bl_S Y \to Y$ be the blow-up of $S \subset Y$. If $Bl_S Y$ is smooth, then define $Z := Bl_S Y$ and $\rho = r$. Otherwise, blow-up again. As in the surface case, after a finite number of blow-up, we will end with a smooth variety. Define Z to be the first smooth variety obtained in this way, and ρ be the composition of the blow-up morphisms. We will show that $\rho^* K_Y \cong K_Z$. In general we have

$$\rho^* K_Y \cong K_Z + \sum_{l=1}^{n} a_l E_l,$$

where E_l are the components of the exceptional divisor E of ρ and a_l are integers defined as follows. Let $z \in E_l$ be a generic point, and $g_l = 0$ be an equation for E_l in a neighbourhood of z. Let s be a (local) generator of K_Y in a neighbourhood of $\rho(z)$. Then a_l is defined by the following equation

$$\rho^*(s) = g^{a_l}(dz_1 \wedge ... \wedge dz_n),$$

where $z_1, ..., z_d$ are local coordinates for Z in z. For more details see [18], Lecture 6. In our case, Y is locally a product $R \times \mathbb{C}^k$, so Z is locally isomorphic to $\tilde{R} \times \mathbb{C}^k$. Then, since $\tilde{R} \to R$ is crepant, $a_l = 0$ for all $l = 1, ..., d$.

We now prove unicity. Assume that $\rho_1 : Z_1 \to Y$ is another resolution of Y. By [29], Lemma 2.10, the exceptional locus of ρ_1 is of pure codimension 1 in Z_1. Let $I_{S/Y}$ be the ideal sheaf of S in Y. The sheaf $J := \rho_1^{-1}(I_{S/Y}) \cdot \mathcal{O}_{Z_1}$ is a sheaf of ideals of $\mathcal{O}_{Z_1}$ and the associated subscheme of Z_1 is supported on the exceptional locus of ρ_1. Let J_{red} be the sheaf of ideals of $\mathcal{O}_{Z_1}$ such that $\mathcal{O}_{Z_1}/J_{red}$ is the sheaf of regular functions on the exceptional locus of ρ_1 with the reduced structure. By [33], Chapter II, Corollary 7.15, we get a morphism $Bl_J Z_1 \to Bl_S Y$ which lifts ρ_1. Since $J \subset J_{red}$, we have an inclusion $\oplus_{p \geq 0} J^p \subset \oplus_{\geq 0} J^p_{red}$ of algebras which gives a morphism $Bl_{J_{red}} Z_1 \to Bl_J Z_1$. Using the isomorphism of $Bl_{J_{red}} Z_1$ with Z_1 we get a morphism $Z_1 \to Bl_S Y$ which lifts ρ_1. Repeating this argument we get a morphism $f : Z_1 \to Z$, furthermore $f^* K_Z = K_{Z_1}$ becouse $\rho_1 : Z_1 \to Y$ is crepant. Then f is an isomorphism. Indeed, f induces a morphism $T_{Z_1} \to f^* T_Z$, taking the determinant we have a morphism

$$\wedge^d T_{Z_1} \to \wedge^d T_Z, \tag{5.1}$$

so we get a global section of $\mathcal{O}_{Z_1}(K_{Z_1} - f^* K_Z) \cong \mathcal{O}_{Z_1}$. Since Z_1 is projective and f is birational, this section is constant equal to 1. So it is a local isomorphism and one to one. □

5.3 Geometry of the exceptional divisor

In this section we restrict our attention to varieties with transversal A_n-singularities and whose associated orbifold $[Y]$ has trivial monodromy. In this case, any component E_l of the exceptional divisor has a structure of $\mathbb{P}^1$-bundle on S. We will describe E_l as the projectivization of vector bundles over S of rank 2. These vector bundles are defined in terms of the line bundles L, M and K, defined in Lemma 4.4.3.

Notation 5.3.1. We will denote by $E \subset Z$ the exceptional locus of ρ and by $E_1, ..., E_n$ the irreducible components of E. The restriction of ρ to E will be denoted $\pi : E \to S$ and the restriction of π to E_l with $\pi_l : E_l \to S$.

Proposition 5.3.2. *Let Y be a variety with transversal A_1-singularities. Then E is irredicible and there exists a vector bundle F on S, of rank two, such that*

$$E \cong \mathbb{P}(F), \tag{5.2}$$

where $\mathbb{P}(F)$ is the projective bundle of lines in F as defined in [30], Appendix B.5.5, where it is denoted by $P(F)$.

Moreover, the normal bundle $N_{E/Z}$ is given as follows

$$N_{E/Z} \cong \mathcal{O}_F(-2) \otimes \pi^* L \tag{5.3}$$

where L is defined by $\wedge^2 F \otimes L \cong R^1\pi_* N_{E/Z}$, $\mathcal{O}_F(-2)$ is defined in [30], Appendix B.5.1.

Proof. In this case, $Z = Bl_S Y$, and the normal cone $C_S Y$ of S in Y is a conic bundle with fiber isomorphic to $\{(x, y, z) \in \mathbb{C}^3 : xy - z^2 = 0\}$, so the projection $C_S Y \to S$ induces to $\pi : E = \mathbb{P}(C_S Y) \to S$ a structure of a $\mathbb{P}^1$ bundle over S. In particular it is irreducible.

Since S is smooth, there exists a rank two vector bundle on S, say F, such that $E \cong \mathbb{P}(F)$. This follows from [33], Chapter II, Exercise 7.10(c). Let us fix one of these bindles and denote it by F.

The normal bundle $N_{E/Z}$ is a line bundle whose restriction on each fiber $\pi^{-1}(s)$ is isomorphic to $\mathcal{O}_{\pi^{-1}(s)}(-2)$. So, equation 5.3 follows. The description of L is a consequence of the projection formula, [33], Chapter III, Exercise 8.3. □

Proposition 5.3.3. *Let Y be a variety with transversal A_n-singularities, $n \geq 2$. Then, for any component E_l of E, $\pi_l : E_l \to S$ is a $\mathbb{P}^1$-bundle and it can be written as follows,*

$$E_l \cong \mathbb{P}(L_l \oplus M_l) \quad \text{for } l = 1, ..., n,$$

where L_l and M_l are line bundles on S that satisfies the following equation

$$L_l \otimes M_l^{\vee} \cong M \otimes (K^{\vee})^{\otimes l} \quad \text{for } l = 1, ..., n. \tag{5.4}$$

Moreover the intersection $E_k \cap E_l$ has the following expression

$$E_k \cap E_l = \begin{cases} \emptyset & \text{if } |k-l| > 1, \\ \mathbb{P}(M_{l-1}) \subset E_{l-1} & \text{if } k = l-1, \\ \mathbb{P}(L_l) \subset E_l & \text{if } k = l-1. \end{cases}$$

Proof. In our case the monodromy is trivial, so we can identify the local groups G_y, $y \in S$, with $\mathbb{Z}_{n+1}$ in a natural way. From this it follows that the normal cone $C_S Y$ of S in Y is the union of two components, each of these being a vector bundle of rank 2 on S. So, we get two components of E with the structures of $\mathbb{P}^1$-bundles over S. If we blow-up again, we get other components of E. More precisely, if $n = 3$ we get one component, if $n \geq 4$, then we must get two other componets. Each of these components have clearly a structure of $\mathbb{P}^1$-bundles over S.

Since S is smooth, we can choose rank two vector bundles F_l on S such that $E_l \cong \mathbb{P}(F_l)$. But, for later use, we want F_l to be of the claimed form. Notice that, if $E_l \cong \mathbb{P}(F_l)$, then for any line bundle L on S, $E_l \cong \mathbb{P}(F_l \otimes L)$. So, if $F_l = L_l \oplus M_l$, E_l is dermined by $L_l \otimes M_l^{\vee}$.

We prove the Proposition in the following way: we think of Z as a finite number of blow-ups; at each blow-up we compute the transition functions of the vector bundles that forms the normal cone of the singular locus.

Let $\{U_i\}_{i \in I}$ be an open covering of a neighbourhood of S in Y. Assume that each U_i is given as follows

$$U_i = \{(\underline{w}_i, x_i, y_i, z_i) \in \mathbb{C}^k \times \mathbb{C}^3 : x_i y_i - z_i^{n+1} = 0\}.$$

For any $i \in I$, let $(\tilde{U}_i, \mathbb{Z}_{n+1}, \chi_{i|})$ be a uniformizing system for $[Y]$ such that $\chi_i(\tilde{U}_i) = \tilde{U}_i$. Let us assume that

$$\tilde{U}_i = \{(\underline{w}_i, u_i, v_i) \in \mathbb{C}^k \times \mathbb{C}^2\}.$$

We can suppose further that, for any $i, j \in I$, there are isomorphisms

$$\varphi_{ij} : \chi_i^{-1}(U_i \cap U_j) \to \chi_j^{-1}(U_i \cap U_j)$$

which are $\mathbb{Z}_{n+1}$-equivariant.

Let $\varphi = (\Phi, F, G)$, where Φ, F, G are the components of φ with respect to the coordinates $(\underline{w}_i, u_i, v_i)$. The fact that φ is $\mathbb{Z}_{n+1}$-equivariant imply that Φ depends only on $\underline{w}_i$ so it is an isomorphism between open subsets of S.

Using this coordinates we get bases for the normal bundles $N_{\tilde{U}_i^{\mathbb{Z}_{n+1}}/\tilde{U}_i} = \langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial v_i} \rangle$. Moreover we have

$$\begin{aligned} \frac{\partial}{\partial u_i} &\mapsto \frac{\partial F}{\partial u_i}\frac{\partial}{\partial u_j} + \frac{\partial G}{\partial u_i}\frac{\partial}{\partial v_j} \\ \frac{\partial}{\partial v_i} &\mapsto \frac{\partial F}{\partial v_i}\frac{\partial}{\partial u_j} + \frac{\partial G}{\partial v_i}\frac{\partial}{\partial v_i}. \end{aligned}$$

The condition for $T\varphi_{ij} : N_{\tilde{U}_i^{\mathbb{Z}_{n+1}}/\tilde{U}_i} \to N_{\tilde{U}_j^{\mathbb{Z}_{n+1}}/\tilde{U}_j}$ to be an isomorphism of $\mathbb{Z}_{n+1}$-representations implies that F is independent on v_i and that G does not depend on u_i.

The two conditions $F(\epsilon \cdot u_i) = \epsilon \cdot F(u_i)$, $G(\epsilon^{-1} \cdot v_i) = \epsilon^{-1} \cdot G(v_i)$ imply

$$\begin{aligned} F(u_i) &= \sum_{k=0}^{\infty} u_i^{k(n+1)+1} \frac{\partial^{k(n+1)+1} F}{\partial u_i^{k(n+1)+1}} \\ G(v_i) &= \sum_{k=0}^{\infty} v_i^{k(n+1)+1} \frac{\partial^{k(n+1)+1} G}{\partial v_i^{k(n+1)+1}}. \end{aligned}$$

From these expressions we see that F^{n+1} and G^{n+1} depends only on $u_i^{n+1} = x_i$ and $v_i^{n+1} = y_i$ respectively and $F \cdot G$ depends on (x_i, y_i, z_i). So $(F^{n+1}, G^{n+1}, F \cdot G)$ is an automorphism of $U_i \cap U_j$. Moreover we have the following change of variable expression

$$\begin{aligned} x_j &= x_i \left(\frac{\partial F}{\partial u_i}\right)^{n+1} + \text{h.o.t.'s} \\ y_j &= y_i \left(\frac{\partial G}{\partial v_i}\right)^{n+1} + \text{h.o.t.'s} \\ z_j &= z_i \frac{\partial F}{\partial u_i}\frac{\partial G}{\partial v_i} + \text{h.o.t.'s}. \end{aligned}$$

From the previous calculations it is clear that the normal cone of S in Y is the union of two irreducible components, C_1 and C_2. C_1 and C_2 have a structure of vector bundles of rank 2 over S and they are given as follows

$$\begin{aligned} C_1 &\cong M \oplus K \\ C_2 &\cong L \oplus K, \end{aligned}$$

where L, M and K are defined in Lemma 4.4.3. Moreover, the intersection $C_1 \cap C_2$ in $C_S Y$ is given by the line bundle K.

After the first blow-up, we get a variety over Y with transversal A_{n-2}-singularities, the exceptional divisor is $\mathbb{P}(C_S Y) = \mathbb{P}(C_1) \cup \mathbb{P}(C_2)$, the singular locus is $\mathbb{P}(C_1) \cap \mathbb{P}(C_2)$. So, if $n > 2$, we have to blow-up again. Let (a_i, b_i, z_i), (a_j, b_j, z_j) be coordinates in a neighbourhood of the singular locus. The blow-up morphism, in these coordinates, is given by: $x_i = a_i z_i$, $y_i = b_i z_i$, $z_i = z_i$. Then the two systems of coordinates, (a_i, b_i, z_i) and (a_j, b_j, z_j) are related ss follows:

$$a_j = \frac{x_j}{z_j} = \frac{F^{n+1}(a_i z_i)}{F \cdot G} \tag{5.5}$$

$$b_j = \frac{y_j}{z_j} = \frac{G^{n+1}(a_i z_i)}{F \cdot G} \tag{5.6}$$

$$z_j = F \cdot G. \tag{5.7}$$

Note that, in the first two equations, numerator and denominator on the right side, are both multiples of z_i. So, after dividing by z_i, 5.5 becomes

$$\begin{aligned} a_j &= a_i \frac{\left(\frac{\partial F}{\partial u_i}\right)^{n+1}}{\left(\frac{\partial F}{\partial u_i}\frac{\partial G}{\partial v_i}\right)} + \text{h.o.t.'s} \\ b_j &= b_i \frac{\left(\frac{\partial G}{\partial v_i}\right)^{n+1}}{\left(\frac{\partial F}{\partial u_i}\frac{\partial G}{\partial v_i}\right)} + \text{h.o.t.'s} \\ z_j &= F \cdot G. \end{aligned}$$

These calculations shows that the normal cone of the singular locus, after the first blow-up, is the union of the following irreducible components $\mathbb{P}((M \otimes K^\vee) \oplus K)$ and $\mathbb{P}(K \oplus (L \otimes K^\vee))$ intersecting along $\mathbb{P}(K)$.

Under the identification of the strict transform of $\mathbb{P}(C_S Y)$ with $\mathbb{P}(M \oplus K) \cup \mathbb{P}(K \oplus L)$, we claim that $\mathbb{P}((M \otimes K^\vee) \oplus K) \cap \mathbb{P}(M \oplus K) = \mathbb{P}(K) \subset \mathbb{P}(M \oplus K)$, $\mathbb{P}((M \otimes K^\vee) \oplus K) \cap \mathbb{P}(M \oplus K) = \mathbb{P}(M \otimes K^\vee) \subset \mathbb{P}(M \otimes K^\vee \oplus K)$ and $\mathbb{P}(M \oplus K) \cap \mathbb{P}(K \oplus (L \otimes K^\vee)) = \emptyset$. This follows from the surface case. In this case $Y = \{xy - z^{n+1} = 0\}$ and the blow-up is covered by three open sets with coordinates $(x, \frac{c}{a})$, $(y, \frac{c}{b})$ and $(\frac{a}{c}, \frac{b}{c}, z)$. The strict transform of the $x-$axis is contained in the open with coordinates $(x, \frac{c}{a})$, $\frac{c}{a} = \frac{z}{x}$. Under the identification of the exceptional locus with $\mathbb{P}(\{(x, y, z) : xy = 0\})$. □

5.4 Cohomology ring of the crepant resolution

Let Y be a variety with transversal A_n-singularities, suppose that the monodromy of $[Y]$ is trivial. In this section we describe the cohomology ring of the crepant resolution Z of Y in terms of the cohomology of Y and the geometry of the exceptional divisor E.

Notation 5.4.1. We will use the same notation of the preceding section. Moreover, we will denote by $j : E \to Z$ the embedding of E in Z, $i : S \to Y$ the embedding of S in Y. The restriction of j to the component E_l will be denoted by $j_l : E_l \to Z$. We will denote by $k_l : E_l \to E$ the morphism defined by the equality $j \circ k_l = j_l$.

Proposition 5.4.2. *Let Y be a variety with transversal A_1-singularities. Then the following map is an isomorphism of vector spaces*

$$\begin{aligned} H^*(Y) \oplus H^{*-2}(S)\langle E\rangle &\cong H^*(Z) \\ \delta + \alpha E &\mapsto \rho^*(\delta) + j_*\pi^*(\alpha). \end{aligned}$$

Under the identification of $H^*(Z)$ with $H^*(Y) \oplus H^{*-2}(S)\langle E\rangle$ by means of this map, the cup product of Z is given as follows

$$\begin{aligned} &(\delta_1 + \alpha_1 E) \cdot (\delta_2 + \alpha_2 E) = \delta_1 \cup \delta_2 - 2i_*(\alpha_1 \cup \alpha_2) \\ &+ \left(i^*(\delta_1) \cup \alpha_2 + \alpha_1 \cup i^*(\delta_2) + 2c_1(R^1\pi_* N_{E/Z}) \cup \alpha_1 \cup \alpha_2\right) E. \end{aligned}$$

Proof. From projection formula we get

$$j_*\pi^*(\alpha) \cdot \rho^*(\delta) = j_*\left(\pi^*(\alpha) \cdot j^*\rho^*(\delta)\right) = j_*\pi^*(\alpha \cdot i^*\delta).$$

So, $\alpha E \cdot \delta = (\alpha \cup i^*\delta)E$. To get the product rule between α_1 and α_2, for $\alpha_1, \alpha_2 \in H^*(S)$, we write

$$j_*\pi^*(\alpha_1) \cup j_*\pi^*(\alpha_2) = \rho^*(\delta) + j_*\pi^*(\alpha)$$

for some $\delta \in H^*(Y)$ and $\alpha \in H^*(S)$. But, from projection formula and the equality $\rho_* \circ j_* = (\rho \circ j)_* = (i \circ \pi)_* = i_* \circ \pi_*$, we have

$$\begin{aligned} \delta = \quad \rho_*(j_*\pi^*(\alpha_1) \cup j_*\pi^*(\alpha_2)) &= -2i_*(\alpha_1 \cup \alpha_2), \quad \text{and} \\ j_*\pi^*(\alpha_1) \cup j_*\pi^*(\alpha_2) &= j_*\left(c_1(N_{E/Z}) \cup \pi^*(\alpha_1 \cup \alpha_2)\right). \end{aligned}$$

On the other hand, $\pi^*(\alpha)$ is the coefficient of $c_1(\mathcal{O}_F(-2))$ in $j^*(j_*\pi^*(\alpha_1) \cup j_*\pi^*(\alpha_2))$, which is $2\alpha_1 \cup \alpha_2 \cup c_1\left(R^1\pi_* N_{E/Z}\right)$. ☐

Notation 5.4.3. For any variety X and line bundle L on X, we will denote by L the first Chern class $c_1(L) \in H^2(X)$. If $\alpha \in H^*(X)$, then we will denote by αL the cup product $\alpha \cup c_1(L) \in H^*(X)$.

Convention 5.4.4. The variety Y has complex dimension d and so also Z is of complex dimension d and S has complex dimension $k := d - 2$.

Proposition 5.4.5. *The following map is an isomorphism of vector spaces*

$$\begin{aligned} H^*(Y) \oplus_{l=1}^{n} H^{*-2}(S)\langle E_l\rangle &\rightarrow H^*(Z) \\ \delta + \alpha_1 E_1 + \ldots + \alpha_n E_n &\mapsto \rho^*(\delta) + \sum_{l=1}^{n} j_{l*}\pi_l^*(\alpha_l). \end{aligned}$$

Under this identification the cup product of Z is given as follows:

$$\begin{aligned} E_{i-1} \cup E_i &= [S] + \sum_{l=1}^{n}\{[(c_n^{-1})_{il} - (c_n^{-1})_{i-1,l}]M + [i(c_n^{-1})_{i-1,l} - (i-1)(c_n^{-1})_{il}]K\}E_l \\ E_i \cup E_i &= -2[S] + \sum_{l=1}^{n}\{[(c_n^{-1})_{i-1,l} - (c_n^{-1})_{i+1,l}]M \\ &\quad +[-(i-1)(c_n^{-1})_{i-1,l} - 4(c_n^{-1})_{il} + (i+1)(c_n^{-1})_{i+1,l}]K\}E_l \\ E_i \cup E_j &= 0 \quad \text{if} \quad |i-j| > 1 \end{aligned}$$

where $[S] \in H^4(X)$ denotes the fundamental class $i_*([S])$ of S in X, and $(c_n)_{ij}$ is the element in the i-th row and j-th column of the following $n \times n$ matrix

$$c_n = \begin{pmatrix} -2 & 1 & 0 & \ldots & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & -2 & 1 \\ 0 & \ldots & \ldots & 0 & 1 & -2 \end{pmatrix} \tag{5.8}$$

The proof will use the following lemmas.

Lemma 5.4.6. *The following sequence is exact for any q,*

$$0 \rightarrow H^q(Y) \xrightarrow{\rho^*} H^q(Z) \xrightarrow{[j^*]} H^q(E)/\pi^*(H^q(S)) \rightarrow 0,$$

where $[j^*]$ is the composition of j^* with the projection $H^q(E) \rightarrow H^q(E)/\pi^*(H^q(S))$. The sequence splits, so we get an isomorphism of vector spaces

$$H^*(Z) \cong H^*(Y) \oplus H^*(E)/\pi^*(H^*(S)).$$

Proof. The exactness follows by comparing the exact sequences of the pairs (E, Z) and (S, Y). The sequence is split since there is a push-forward morphism $\rho_* : H^*(Z) \to H^*(Y)$ which satisfies $\rho_* \circ \rho^* = id_{H^*(Y)}$. □

Lemma 5.4.7. *There is a canonical isomorphism of vector spaces*

$$H^*(E)/\pi^*(H^*(S)) \cong \oplus_{l=1}^{n} H^*(E_l)/\pi_l^*(H^*(S)).$$

Proof. Let $\hat{E}_1 = \overline{E - E_1}$ be the closure of the complement of E_1 in E. Then $E = E_1 \cup \hat{E}_1$ and $E_1 \cap \hat{E}_1 \cong S$. From the Mayer-Vietoris sequence with respect to the covering $\{E_1, \hat{E}_1\}$ of E it follows that the following morphism is an isomorphism

$$H^q(E)/\pi^*(H^q(S)) \xrightarrow{(k_1^*, -\hat{k}_1^*)} H^q(E_1)/\pi_1^*(H^q(S)) \oplus H^q(\hat{E}_1)/\hat{\pi}_1^*(H^q(S))$$

where $k_1 : E_1 \to E$ and $\hat{k}_1 : \hat{E}_1 \to E$ are inclusions, and $\hat{\pi}_1$ is the restriction of π to $\hat{\pi}$. The result follows by induction. □

Proof of Proposition 5.4.5. As a consequence of the above lemmas we have that the vector spaces $H^*(Y) \oplus_{l=1}^{n} H^{*-2}(S)$ and $H^*(Z)$ have the same dimension, so it is enough to show that the map is injective. Assume that $\rho^*(\delta) + \sum_{l=1}^{n} j_{l*}\pi_l^*(\alpha_l) = 0$. Then $\delta = \rho_*(\rho^*(\delta) + \sum_{l=1}^{n} j_{l*}\pi_l^*(\alpha_l)) = 0$. Next, applying j_k^* we get

$$j_k^*(\rho^*(\delta) + \sum_{l=1}^{n} j_{l*}\pi_l^*(\alpha_l)) = 0,$$

the left side of this equation, up to terms of the form $\pi_k^*(..)$ assumes the following expression:

$$\pi_k^*(-\frac{1}{2}\alpha_{k-1} + \alpha_k - \frac{1}{2}\alpha_{k+1}) \cup e_k,$$

where $e_k \in H^2(E_k)$ is the class $j_k^*(c_1(\mathcal{O}_Z(E_k)))$. We use the convenction $\alpha_{-1} = \alpha_{n+1} = 0$. It follows that

$$(c_n)_{kl}\alpha_l = 0 \quad \text{for any } k = 1, ..., n.$$

So, $\alpha_l = 0$ for all l, since c_n is nondegenerate.

Finally note that the following maps are isomorphisms

$$\begin{array}{rcl} H^{q-2}(S) & \to & H^q(E_l)/\pi_l^*(H^q(S)) \\ \alpha & \mapsto & [\pi_l^*(\alpha) \cup e_l]. \end{array}$$

To prove 5.8, write

$$E_i \cup E_j = \rho^*(\delta) + \sum_{l=1}^{n} j_{l*}\pi_l^*(\alpha_l), \tag{5.9}$$

where $\delta \in H^*(Y)$ and $\alpha_l \in H^*(S)$.

We immediately get

$$\begin{aligned} \delta &= \rho_*(E_i \cup E_j) = \rho_*(j_{i*}([E_i]) \cup j_{j*}([E_j])) \\ &= \rho_* j_{i*}([E_i]) \cup j_i^* j_{j*}([E_j])) = i_*\pi_{i*}([E_i]) \cup j_i^* j_{j*}([E_j])). \end{aligned}$$

So,

$$\delta = \begin{cases} 0 & \text{if } |i-j| > 1 \\ [S] & \text{if } |i-j| = 1 \\ -2[S] & \text{if } |i-j| = 0. \end{cases}$$

We now pull-back both sides of (5.9) with j_k. We get the following equation

$$\begin{aligned} j_k^*(E_i \cup E_j) &= j_k^*\rho^*(\delta) + \sum_{l=1}^{n} j_k^* j_{l*}\pi_l^*(\alpha_l) \\ &= \pi_k^* i^*(\delta) + j_k^*(\alpha_{k-1}E_{k-1} + \alpha_k E_k + \alpha_{k+1}E_{k+1} \\ &= \pi_k^* i^*(\delta) + \pi_k^*(\alpha_{k-1})[E_{k-1} \cap E_k \subset E_k] + \pi_k^*(\alpha_k) N_{E_k/Z} \\ &+ \pi_k^*(\alpha_{k+1})[E_{k+1} \cap E_k \subset E_k] \end{aligned}$$

where by $[E_{k-1} \cap E_k \subset E_k]$ (resp. $[E_{k+1} \cap E_k \subset E_k]$) we mean the cohomology class dual of the homology class of $E_{k-1} \cap E_k$ (resp. $E_{k+1} \cap E_k$) in E_k. Using our description of E_l (see Proposition 5.3.3) and [30], Appendix B.5.6, we have the following expression for $j_k^*(E_i \cup E_j)$ up to terms which are pulled-back from S with π_k:

$$j_k^*(E_i \cup E_j) = \pi_k^*(\alpha_{k-1} - 2\alpha_k + \alpha_{k+1})\mathcal{O}_{F_k}(1), \tag{5.10}$$

where, as usual, $\alpha_{-1} = \alpha_{n+1} = 0$.

On the other hand, the left side of (5.9) gives

$$\begin{aligned} j_k^*(E_i \cup E_j) &= j_k^*(j_{i*}([E_i]) \cup j_{j*}([E_j])) && (5.11) \\ &= [E_i \cap E_k \subset E_k] \cap [E_j \cap E_k \subset E_k]. && (5.12) \end{aligned}$$

We now distingiush three cases.

Case $|i-j| > 1$. Then clearly $E_i \cup E_j = 0$.

Case $|i-j|=1$. Then

$$j_k^*(E_{i-1}\cup E_i)=\begin{cases}0 & \text{for } k<i-1,\\ N_{E_{i-1}/Z}\cup[E_{i-1}\cap E_i\subset E_{i-1}] & \text{for } k=i-1,\\ N_{E_i/Z}\cup[E_{i-1}\cap E_i\subset E_i] & \text{for } k=i,\\ 0 & \text{for } k>i.\end{cases}$$

In order to compute $N_{E_l/Z}$, let us denote by $S_{l-1}=E_{l-1}\cap E_l$. Then notice that $N_{E_l/Z|S_{l-1}}\cong N_{S_{l-1}/E_{l-1}}$ and then, from [30] Appendix B.5.6, it follows that

$$N_{E_i/Z}\cong \mathcal{O}_{F_i}(-2)+\pi_i^*(K-L_i-M_i). \tag{5.13}$$

A direct application of [30] Appendix B.5.6 gives

$$[E_{l-1}\cap E_l\subset E_{l-1}]=\mathcal{O}_{F_{l-1}}(1)+\pi_{l-1}^*L_{l-1}. \tag{5.14}$$

From (5.13) and (5.14) it follows that, up to terms of the form $\pi_k^*(...)$, $j_k^*(E_{i-1}\cup E_i)$ is equal to

$$j_k^*(E_{i-1}\cup E_i)=\begin{cases}0 & \text{for } k<i-1,\\ \mathcal{O}_{F_{i-1}}(1)(iK-M) & \text{for } k=i-1,\\ \mathcal{O}_{F_i}(1)(M-(i-1)K) & \text{for } k=i,\\ 0 & \text{for } k>i.\end{cases}$$

Then we get the following system of equations for the α_l's.

$$\begin{pmatrix}0\\ \dots\\ 0\\ iK-M\\ M-(i-1)K\\ 0\\ \dots\\ 0\end{pmatrix}=\begin{pmatrix}-2 & 1 & 0 & \dots & \dots & \dots & \dots & 0\\ 1 & -2 & 1 & 0 & \dots & \dots & \dots & 0\\ 0 & 1 & -2 & 1 & 0 & \dots & \dots & 0\\ 0 & 0 & 1 & -2 & 1 & 0 & \dots & 0\\ 0 & 0 & 0 & 1 & -2 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & \dots & .. & .... & \dots & 1 & -2\end{pmatrix}\begin{pmatrix}\alpha_1\\ \dots\\ \dots\\ \alpha_{i-1}\\ \alpha_i\\ \dots\\ \dots\\ \alpha_n\end{pmatrix}$$

Case $|i-j|=0$. Then

$$j_k^*(E_i\cup E_i)=\begin{cases}0 & \text{for } k<i-1,\\ [E_{i-1}\cap E_i\subset E_{i-1}]^2 & \text{for } k=i-1,\\ N^2_{E_i/Z} & \text{for } k=i,\\ [E_{i+1}\cap E_i\subset E_{i+1}]^2 & \text{for } k=i+1,\\ 0 & \text{for } k>i+1.\end{cases}$$

Using again (5.13) and (5.14), we have that

$$j_k^*(E_i \cup E_i) = \begin{cases} 0 & \text{for } k < i-1, \\ \mathcal{O}_{F_{i-1}}(1)(M-(i-1)K) & \text{for } k = i-1, \\ \mathcal{O}_{F_i}(1)(-4K) & \text{for } k = i, \\ \mathcal{O}_{F_{i+1}}(1)((i+1)K-M) & \text{for } k = i+1, \\ 0 & \text{for } k > i+1. \end{cases}$$

Then we get the following system of equations for the α_l's.

$$\begin{pmatrix} 0 \\ \dots \\ 0 \\ M-(i-1)K \\ -4K \\ (i+1)K-M \\ 0 \\ \dots \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & .. & .... & \dots & \dots & 1 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \dots \\ \dots \\ \alpha_{i-1} \\ \alpha_i \\ \alpha_{i+1} \\ \dots \\ \dots \\ \alpha_n \end{pmatrix}$$

□

Chapter 6

Quantum corrections

We compute the *quantum corrected cup product* (see Definition 3.4.11) in the transversal A_n case and with trivial monodromy.

In the first section we give a description of the genus zero Gromov-Witten invariants of Z that are needed in order to compute the quantum corrected cup product. Some of these invariants are computed by a direct understanding of the *virtual fundamental class.* To compute the ramaining, we use the property that Gromov-Witten invariants are invariants under deformation of the complex structure of Z, so we will impose some technical hypothesis on Z which guarantee that some deformations of Z are unobstructed. In the last Section, using also the results of Chapter 5, we give a presentation of the ring $H^*_\rho(Z)(q_1, ..., q_n)$, as defined in Theorem 3.4.14. We review in Section 2, some basic facts about virtula fundamental classes that are used in the proof of the results.

6.1 Gromov-Witten invariants of the resolution of transversal A_n-singularities

We describe some of the genus zero Gromov-Witten invariants of Z in order to compute the quantum corrected cup product. Since we are able to compute some of the invariants in complete generality and some others under particular hypothesis, we decided to collect these results in three different Theorems.

Convention 6.1.1. Let X be a variety of dimension d, then we define to be zero the integral of any cohomology class $\alpha \in H^*(X)$ on X of degree different to $2d$.

Notation 6.1.2. Through this Chapter Y will be a variety with transversal A_n-singularities such that the corresponding orbifold $[Y]$ has trivial monodromy. We will denote by $\rho : Z \to Y$ the crepant resolution.

We have the following isomorphism of vector spaces (see Proposition 5.4.5):

$$\begin{aligned} H^*(Y) \oplus_{l=1}^{n} H^{*-2}(S)\langle E_l \rangle &\to H^*(Z) \\ \delta + \alpha_1 E_1 + ... + \alpha_n E_n &\mapsto \rho^*(\delta) + \sum_{l=1}^{n} j_{l*}\pi_l^*(\alpha_l). \end{aligned}$$

So, a cohomology class $\gamma \in H^*(Z)$ of Z will be denoted by

$$\gamma = \delta + \alpha_1 E_1 + ... + \alpha_n E_n, \quad \text{with } \delta \in H^*(Y),\ \alpha_l \in H^{*-2}(S).$$

The homology group $H_2(Z, \mathbb{Z})$ of Z can be described as follows,

$$H_2(Z, \mathbb{Z}) \cong H_2(Y\mathbb{Z}) \oplus H_0(S)\langle \beta_1 \rangle \oplus ... \oplus H_0(S)\langle \beta_n \rangle,$$

where $\beta_l \in H_2(Z, \mathbb{Z})$ is the class of a fiber of $\pi_l : E_l \to S$, see Notation 5.3.1. We have that $\beta_1, ..., \beta_n$ is an integral basis of Ker ρ_*, where $\rho_* : H_2(Z, \mathbb{Q}) \to H_2(Y, \mathbb{Q})$ is the group homomorphism induced by ρ, see Chapter 3.4. So, any class $\Gamma \in H_2(Z, \mathbb{Q})$ of a rational curve that is contracted by ρ, i.e. $\rho_*(\Gamma) = 0$, is written in a unique way as

$$\Gamma = a_1\beta_1 + ... + a_n\beta_n \quad \text{with } a_l \text{ positive integers.}$$

We now compute the 3-point, genus zero Gromov-Witten invariants

$$\Psi_\Gamma^Z(\gamma_1, \gamma_2, \gamma_3) = \int_{[\bar{\mathcal{M}}_{0,3}(Z,\Gamma)]^{vir}} ev_3^*(\gamma_1 \otimes \gamma_2 \otimes \gamma_3) \tag{6.1}$$

where $\gamma_i \in H^*(Z)$, $\Gamma = a_1 \cdot \beta_1 + ... + a_n \cdot \beta_n \in H_2(Z, \mathbb{Z})$ is an homology class such that $\rho_*(\Gamma) = 0$, $\bar{\mathcal{M}}_{0,3}(Z, \Gamma)$ is the moduli space of 3-pointed stable maps $[\mu : (C, p_1, p_2, p_3) \to Z]$ such that $\mu_*[C] = \Gamma$, the arithmetic genus of C is 0, and $ev_3 : \bar{\mathcal{M}}_{0,3}(Z, \Gamma) \to Z \times Z \times Z$ is the evaluation map.

Theorem 6.1.3. *Let Y be a variety with transversal A_1-singularities and let $\rho : Z \to Y$ be the crepant resolution. Then,*

$$\Psi_{a\beta}^Z(\gamma_1, \gamma_2, \gamma_3) = \begin{cases} 0 & \text{if } \gamma_1, \gamma_2 \text{ or } \gamma_3 \text{ are in } H^*(Y); \\ -8\int_S \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot c_1(R^1\pi_* N_{E/Z}) & \text{if } \gamma_i = \alpha_i E \text{ for } i = 1, 2, 3. \end{cases}$$

Where $a \geq 1$ is an integer, $\pi : E \to S$ is the restriction to E of ρ, and the dots denote the cup product of S.

Theorem 6.1.4. *Let Y be a variety with transversal A_n-singularities, $n \geq 2$, and let $\rho : Z \to Y$ be the crepant resolution. Then,*

$$\Psi_\Gamma^Z(\gamma_1, \gamma_2, \gamma_3) = \begin{cases} 0 & \text{if } \gamma_1, \gamma_2 \text{ or } \gamma_3 \text{ are in } H^*(Y); \\ (E_{l_1} \cdot \beta_{ij})(E_{l_2} \cdot \beta_{ij})(E_{l_3} \cdot \beta_{ij}) \int_S \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot c_1(K) \end{cases}$$

where the second possibility holds if $\Gamma = \beta_{ij} := \beta_i + ... + \beta_j$ for $1 \leq i \leq j \leq n$, and $\gamma_i = \alpha_i \cdot E_{l_i}$ for $i = 1, 2, 3$. Here K is the line bundle on S defined in Lemma 4.4.3.

Theorem 6.1.5. *Let Y be a variety with transversal A_n-singularities, $n \geq 2$, and let $\rho : Z \to Y$ be the crepant resolution. Assume further that the line bundle K (defined in Lemma 4.4.3) is sufficiently ample, that $H^2(Z, T_Z) = 0$ and $H^1(S, T_S) = 0$. Then, we have the following expression for the Gromov-Witten invariants:*

$$\Psi_\Gamma^Z(\gamma_1, \gamma_2, \gamma_3) = \begin{cases} 0 & \text{if } \gamma_1, \gamma_2 \text{ or } \gamma_3 \text{ are in } H^*(Y); \\ (E_{l_1} \cdot \beta_{ij})(E_{l_2} \cdot \beta_{ij})(E_{l_3} \cdot \beta_{ij}) \int_S \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot c_1(K) \\ 0 & \text{in the remaining cases.} \end{cases}$$

where the second possibility holds if $\Gamma = a \cdot \beta_{ij} := \beta_i + ... + \beta_j$ for a be a positive integer, $1 \leq i \leq j \leq n$, and $\gamma_i = \alpha_i \cdot E_{l_i}$ for $i = 1, 2, 3$.

Conjecture 6.1.6. Theorem 6.1.5 is true in complete generality, i.e. without the hypothesis on K, on $H^2(Z, T_Z)$ and $H^1(S, T_S)$. We give in Section 6.3 an outline of the proof of this conjecture.

Remark 6.1.7. Notice that, if $[Y]$ carries a global holomorphic symplectic 2-form ω, then we can identify L with $M^\vee$ by means of ω. So,

$$(n+1)K \cong M \otimes L \cong \mathcal{O}_S,$$

so, all the Gromov-Witten invariants vanish.

6.2 Virtual fundamental class for Gomov-Witten invariants

We review here some basic properties about the virtual fundamental class $[\bar{\mathcal{M}}_{g,n}(Z,\Gamma)]^{vir}$. We follow clesely [10] and [9].

We recall the definition of *obstruction theory* for a Deligne-Mumford stack $\mathcal{X}$. We will denote by $D(\mathcal{O}_{\mathcal{X}_{ét}})$ the derived category of $\mathcal{O}_{\mathcal{X}_{ét}}$-modules, where $\mathcal{X}_{ét}$ denote the small étale site of $\mathcal{X}$. The cotangent complex of $\mathcal{X}$ will be denoted by $L^{\bullet}_{\mathcal{X}_{ét}} \in D(\mathcal{O}_{\mathcal{X}_{ét}})$.

Definition 6.2.1. *Let $\mathcal{X}$ be a Deligne-Mumford stack. Let $E^{\bullet} \in obD(\mathcal{O}_{\mathcal{X}_{ét}})$ be an object that satisfies the following conditions:*

- $h^i(E^{\bullet}) = 0$, *for* $i > 1$;

- $h^i(E^{\bullet})$ *is coherent, for* $i = 0, -1$.

Then, a morphism $\phi : E^{\bullet} \to L^{\bullet}_{\mathcal{X}_{ét}}$ in $D(\mathcal{O}_{\mathcal{X}_{ét}})$ is called an ***obstruction theory*** *for $\mathcal{X}$, if $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. By abuse of language we also say that $E^{\bullet}$ is an obstruction theory for $\mathcal{X}$.*

The obstruction theory $\phi : E^{\bullet} \to L^{\bullet}_{\mathcal{X}_{ét}}$ is ***perfect,*** *if $E^{\bullet}$ is of perfect amplitude in $[-1,0]$.*

Let $E^{\bullet}$ be a perfect obstruction theory for $\mathcal{X}$. Assume that locally $E^{\bullet}$ is written as a complex of vector bundles $[E^{-1} \to E^0]$. Then, the rank of $E^{\bullet}$ is defined to be

$$\mathrm{rk}\ \mathrm{E}^{\bullet} = \mathrm{dimE}^0 - \mathrm{dimE}^{-1}.$$

Definition 6.2.2. *The* ***virtual dimension*** *of $\mathcal{X}$ with respect to the perfect obstruction theory $E^{\bullet}$ is defined to be the rank* rk E$^{\bullet}$ *of $E^{\bullet}$. We will denote the virtual dimension by ν.*

Remark 6.2.3. The virtual dimension is a locally constant function on $\mathcal{X}$. We shall assume that the virtual dimension of $\mathcal{X}$ with respect to $E^{\bullet}$ is constant, equal to ν.

Remark 6.2.4. Let $E^{\bullet}$ be a perfect obstruction theory for $\mathcal{X}$. Then $E^{\bullet}$ give rise to a vector bundle stack $\mathfrak{E}$, into which the intrinsic normal cone $\mathfrak{C}_{\mathcal{X}}$ of $\mathcal{X}$ can be embedded as a closed subcone stack ([10], page 72). Under the hypothesis that $E^\bullet$ has a global resolution ([10], Definition 5.2), the *virtual fundamental class* $[\mathcal{X}, E^\bullet]$ of $\mathcal{X}$, with respect to $E^\bullet$, is the class in the rational Chow group $A_\nu(\mathcal{X})$, obtained by intersecting $\mathfrak{C}_{\mathcal{X}}$ with the zero section of $\mathfrak{E}$, [10], Proposition 5.3.

Notation 6.2.5. We will denote by $\bar{\mathcal{M}}_{g,n}(Z,\Gamma)$ the Deligne-Mumford stack of stable maps of class $\Gamma \in H_2(Z)$ from an n-marked prestable curve of genus g to Z. The *universal curve* on $\bar{\mathcal{M}}_{g,n}(Z,\Gamma)$ will be denoted by $p : \mathcal{C} \to \bar{\mathcal{M}}_{g,n}(Z,\Gamma)$, and the *universal stable map* by $f : \mathcal{C} \to Z$.

The universal curve $p : \mathcal{C} \to \bar{\mathcal{M}}_{g,n}(Z,\Gamma)$ can also be seen in the following way. Consider the stack $\bar{\mathcal{M}}_{g,n+1}(Z,\Gamma)$, then, there is a morphism

$$f_{n+1,n} : \bar{\mathcal{M}}_{g,n+1}(Z,\Gamma) \to \bar{\mathcal{M}}_{g,n}(Z,\Gamma) \tag{6.2}$$

which forgets the last marked point and contracts the unstable components. Then $f_{n+1,n}$ can be identified with the universal curve, the universal stable map is the evaluation morphism of the $(n+1)$-th point.

The *cotangent complex* of $\bar{\mathcal{M}}_{g,n}(Z,\Gamma)$ will be denoted with $L^\bullet_{\bar{\mathcal{M}}_{g,n}(Z,\Gamma)}$. It is an element in the derived category $D(\mathcal{O}_{\bar{\mathcal{M}}_{g,n}(Z,\Gamma)_{ét}})$ of the category of $\mathcal{O}_{\bar{\mathcal{M}}_{g,n}(Z,\Gamma)_{ét}}$-modules, where $\bar{\mathcal{M}}_{g,n}(Z,\Gamma)_{ét}$ is the small étale site of $\bar{\mathcal{M}}_{g,n}(Z,\Gamma)$.

Remark 6.2.6. The moduli stack $\bar{\mathcal{M}}_{g,n}(Z,\Gamma)$ has a natural perfect obstruction theory given by

$$E^\bullet = R^\bullet p_* \{[f^*\Omega_Z \to \Omega_p] \otimes \omega_p\}, \tag{6.3}$$

where Ω_Z is the sheaf of relative differentials of Z over Spec($\mathbb{C}$), Ω_p is the sheaf of relative differentials of $\mathcal{C}$ over $\mathcal{M}$ ([33], page 175) and ω_p is the relative dualizing sheaf in degree -1, see [10], page 82. Moreover $E^\bullet$ has a global resolution. The resulting virtula fundamental class will be denoted by $[\bar{\mathcal{M}}_{g,n}(Z,\Gamma)]^{vir}$.

This is proved in [10] Proposition 6.3, [9] Proposition 5, for the relative case.

There are some basic properties that holds for $[\bar{\mathcal{M}}_{g,n}(Z,\Gamma)]^{vir}$, they are given and proved for the virtual fundamental class given by any perfect obstruction theory on a given Deligne-Mumford stack in [10], Propositions 5.5-5.10 and Propositions 7.2-7.5. Here we report some of these properties in the special case of $[\bar{\mathcal{M}}_{g,n}(Z,\Gamma)]^{vir}$.

Proposition 6.2.7. *The virtual dimension ν of $\bar{\mathcal{M}}_{g,n}(Z,\Gamma)$ is constant and equal to*

$$\nu = (1-g)(dimZ - 3) - (\Gamma \cdot K_Z) + n.$$

Proposition 6.2.8. *Let $\bar{\mathcal{M}}_{g,n}(Z,\Gamma)$ be smooth. Then $h^1(E^{\bullet\vee})$ is locally free and the virtual fundamental class is*

$$[\bar{\mathcal{M}}_{g,n}(Z,\Gamma)]^{vir} = c_r(h^1(E^{\bullet\vee})) \cdot [\bar{\mathcal{M}}_{g,n}(Z,\Gamma)],$$

where $r = \mathrm{rkh}^1(\mathrm{E}^{\bullet\vee})$.

Proposition 6.2.9. *Let*

$$f_{n+1,n} : \bar{\mathcal{M}}_{g,n+1}(Z,\Gamma) \to \bar{\mathcal{M}}_{g,n}(Z,\Gamma)$$

be the forgetful morphism. Then $f^*_{n+1,n}([\bar{\mathcal{M}}_{g,n}(Z,\Gamma)]^{vir}) = [\bar{\mathcal{M}}_{g,n+1}(Z,\Gamma)]^{vir}$.

Consider the following diagram of Deligne-Mumford stacks,

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{u} & \mathcal{X} \\ \downarrow & & \downarrow \\ B' & \xrightarrow{v} & B, \end{array} \tag{6.4}$$

where B and B' are smooth of constant dimension, v has finite unramified diagonal.

Proposition 6.2.10. *Let $E^\bullet \to L_{\mathcal{X}/B}$ be a perfect obstruction theory for $\mathcal{X}$ over B. If (6.4) is cartesian, then $u^*E^\bullet$ is a perfect obstruction theory for $\mathcal{X}'$ over B'. If $E^\bullet$ has a global resolution so does $u^*E^\bullet$ and for the induced virtual fundamental classes we have*

$$v^![\mathcal{M}]^{vir} = [\mathcal{M}']^{vir}$$

at least in the following cases.

1. *v is flat,*

2. *v is a regular local immersion.*

Proposition 6.2.11. *Let $E^\bullet$ be the obstruction theory defined in (6.3). Then we have the following exact sequence of coherent sheaves on $\mathcal{M}$,*

$$0 \to p_*\Omega_p^\vee \to p_*f^*T_Z \to h^0(E^{\bullet\vee}[1]) \to R^1p_*\Omega_p^\vee \to R^1p_*f^*T_Z \to h^1(E^{\bullet\vee}[1]) \to 0, \tag{6.5}$$

where $\Omega_p^\vee = R\mathcal{H}om_{\mathcal{O}_{\bar{\mathcal{M}}_{g,n}(Z,\Gamma)_{ét}}}(\Omega_p, \mathcal{O}_{\bar{\mathcal{M}}_{g,n}(Z,\Gamma)_{ét}})$.

Proof. First of all notice that $E^{\bullet\vee} \cong Rp_*\left([f^*\Omega_Z \to \Omega_p]^\vee\right)$. Indeed, by duality we have

$$\begin{aligned} E^{\bullet\vee} &:= R\mathcal{H}om_{\mathcal{O}_{\bar{\mathcal{M}}_{g,n}(Z,\Gamma)_{ét}}}\left(Rp_*([f^*\Omega_Z \to \Omega_p] \otimes \omega_p), \mathcal{O}_{\bar{\mathcal{M}}_{g,n}(Z,\Gamma)_{ét}}\right) \\ &\cong Rp_*R\mathcal{H}om_{\mathcal{O}_{\bar{\mathcal{M}}_{g,n}(Z,\Gamma)_{ét}}}\left([f^*\Omega_Z \to \Omega_p] \otimes \omega_p, \omega_p\right) \end{aligned}$$

(see [34], Sect. VII.4, page 393), then, since ω_p is locally free, we have

$$E^\bullet \cong Rp_*R\mathcal{H}om_{\mathcal{O}_{\bar{\mathcal{M}}_{g,n}(Z,\Gamma)_{ét}}}\left([f^*\Omega_Z \to \Omega_p] \otimes \omega_p \otimes \omega_p^\vee, \mathcal{O}_{\bar{\mathcal{M}}_{g,n}(Z,\Gamma)_{ét}}\right).$$

Consider the following distinguished triangle

$$f^*\Omega_Z \to \Omega_p \to [f^*\Omega_Z \to \Omega_p] \xrightarrow{+1},$$

where the sheaves $f^*\Omega_Z$ and Ω_p here means complexes centered in degree 0, the complex $[f^*\Omega_Z \to \Omega_p]$ is the mapping cone of the morphism $df : f^*\Omega_Z \to \Omega_p$. Taking duals in derived category, we get the following distinguished triangle

$$[f^*\Omega_Z \to \Omega_p]^\vee \to \Omega_p^\vee \to f^*T_Z \xrightarrow{+1}.$$

The axioms of triangulated categories implies that the following triangle is distinguished

$$\Omega_p^\vee \to f^*T_Z \to ([f^*\Omega_Z \to \Omega_p]^\vee)[1] \xrightarrow{+1}.$$

Now apply the derived functor Rp_* to get the following distinguished triangle

$$Rp_*(\Omega_p^\vee) \to Rp_*f^*T_Z \to Rp_*([f^*\Omega_Z \to \Omega_p]^\vee)[1] \xrightarrow{+1}.$$

Taking cohomology, we get the following long exact sequence

$$\begin{aligned} 0 &\to R^{-1}p_*([f^*\Omega_Z \to \Omega_p]^\vee)[1] \to p_*(\Omega_p^\vee) \to p_*f^*T_Z \to p_*([f^*\Omega_Z \to \Omega_p]^\vee)[1] \\ \to &\ R^1p_*(\Omega_p^\vee) \to R^1p_*f^*T_Z \to R^1p_*([f^*\Omega_Z \to \Omega_p]^\vee)[1] \to 0, \end{aligned}$$

notice that $[f^*\Omega_Z \to \Omega_p]$ is centered in $[-1,0]$ so $([f^*\Omega_Z \to \Omega_p]^\vee)[1]$ is also centered in $[-1,0]$.

Since f is a stable map, the morphism $p_*\Omega_p^\vee \to p_*f^*T_Z$ is injective. This conclude the proof. □

6.3 Proof of the Theorems

Notation 6.3.1. Through this Section, R will denote a rational double point of type A_n and $\tilde{R} \to R$ the crepant resolution. See Definition 4.1.1 and Remark 4.1.5.

6.3.1 Genaral considerations

The key point in the proof of the Theorems is to notice that S is canonically isomorphic to the moduli space $\bar{\mathcal{M}}_{0,0}(Z, \beta_{ij})$, for any $i \leq j$, and that $\bar{\mathcal{M}}_{0,0}(Z, \Gamma)$ has a fibered structure over S with fibers isomorphic to $\bar{\mathcal{M}}_{0,0}(\tilde{R}, \Gamma)$.

Lemma 6.3.2. *There is a morphism*

$$\phi : \bar{\mathcal{M}}_{0,0}(Z, \Gamma) \to S$$

such that, for any point $p \in S$, the fiber $\phi^{-1}(p)$ is isomorphic to $\bar{\mathcal{M}}_{0,0}(\tilde{R}, \Gamma)$. Moreover, there is an étale cover $U \to S$ and a cartesian diagram

$$\begin{array}{ccc} U \times \bar{\mathcal{M}}_{0,0}(\tilde{R}, \Gamma) & \longrightarrow & \bar{\mathcal{M}}_{0,0}(Z, \Gamma) \\ {\scriptstyle pr_1} \downarrow & & \downarrow {\scriptstyle \phi} \\ U & \longrightarrow & S. \end{array}$$

If $\Gamma = \beta_{ij} := \beta_i + \ldots + \beta_j$, for $i \leq j$, then ϕ is an isomorphism.

Proof. *Step 1.* We first prove that for any scheme B of finite type over $\mathbb{C}$ and any object

$$\begin{array}{ccc} C & \xrightarrow{f} & Z \\ {\scriptstyle p} \downarrow & & \\ B & & \end{array}$$

in $\bar{\mathcal{M}}_{0,0}(Z, \Gamma)(B)$, there is a morphism $g : C \to E$ such that $f = j \circ g$, where $j : E \to Z$ is the inclusion.

Let $\mathcal{O}_Z(E)$ be the line bundle over Z associated to the divisor E, and let s be the section of $\mathcal{O}_Z(E)$ defined by E, that is, $s = \{s_i\}$ where s_i are functions which defines the Cartier divisor E. Then f factors through E if and only if f^*s vanishes as section of $f^*\mathcal{O}_Z(E)$. We show that $p_* f^* \mathcal{O}_Z(E)$ is the zero sheaf.

First of all we assume that $B = \text{Spek}(\mathbb{C})$. Then

$$\rho_* f_*([C]) = 0,$$

where ρ_* and f_* are the morphisms of Chow groups induced by ρ and f respectively and $[C]$ is the fundamental class of C. It follows that the image of $\rho \circ f$ is a point $y \in Y$. This point must belong to S because, outside S, ρ is an isomorphism. So $f(C) \subset E$ and, since C is reduced, f factors through E ([33] exercise 3.11(d) chapter II). Note that $f(C)$ is contained in a fiber of $\pi : E \to S$.

Assume now that B is a scheme of finite type over $\mathbb{C}$ and $f : C \to Z$ is a stable map over B. Given a point $b \in B$, let X be the subvariety of B whose generic point is b, namely $X = \overline{\{b\}}$. For any closed point $x \in X$ we have that $H^0(C_x, f^*\mathcal{O}_Z(E)_x) = 0$ becouse $f \mid C_x$ factors through a fiber of E over S. By Cohomology and Base Change it follows that $(p_*f^*\mathcal{O}_Z(E))_x \otimes k(x) = 0$, since $p_*f^*\mathcal{O}_Z(E)$ is coherent it follows that it is zero on a neighbourhood of x, so it is zero on b.

Step 2. Let $\varphi := \pi \circ g : C \to S$, we prove that there exists a morphism $\phi : B \to S$ such that $\varphi = \phi \circ p$.

First of all we define a continuous map $\phi : B \to S$ such that $\varphi = \phi \circ p$. From *Step 1* it follows that, if $b \in B$ is a closed point, then we can define $\phi(b)$ by:

$$\phi(b) = \pi(f(C_b)).$$

Now, let $b \in B$ be any point, and let X be the subvariety whose generic point is b. Then we define $\phi(b)$ to be the generic point of $\overline{\phi(X)}$, the closure of $\phi(X)$ in S. The condition $\varphi = \phi \circ p$ implies that ϕ is continuous. Indeed, p is surjective and so, for any closed subset $T \subset S$, $\phi^{-1}(T) = \pi(\pi^{-1}(\phi^{-1}(T))) = \pi(\varphi^{-1}(T))$. Note that p is surjective since it is dominant and proper (the flatness of p implies that $p^\sharp : \mathcal{O}_B \to p_*\mathcal{O}_C$ is injective).

In order to give a morphism $\phi : B \to S$ it remains to give a morphism of sheaves

$$\phi^\sharp : \mathcal{O}_S \to \phi_*\mathcal{O}_B.$$

We have the following diagram

$$\begin{array}{ccc} \mathcal{O}_S & \xrightarrow{\varphi^\sharp} & \varphi_*\mathcal{O}_C \\ & & \uparrow {\scriptstyle \varphi_*p^{-1}(p^\sharp)} \\ & & \varphi_*p^{-1}\mathcal{O}_B \end{array} \tag{6.6}$$

where $p^{-1}(p^\sharp) : p^{-1}\mathcal{O}_B \to \mathcal{O}_C$ denote the adjoint of $p^\sharp : \mathcal{O}_B \to p_*\mathcal{O}_C$. Note that $\varphi_*p^{-1}(p^\sharp)$ is injective.

Since p is proper and surjective, a direct analysis show that the canonical morphism $\mathcal{O}_B \to p_*p^{-1}\mathcal{O}_B$ is an isomorphism. So the morphism $\phi_*\mathcal{O}_B \to$

$\varphi_* p^{-1}\mathcal{O}_B = \phi_*(p_* p^{-1}\mathcal{O}_B)$ is also an isomorphism. Then we can replace in the previous diagram (6.6), $\varphi_* p^{-1}\mathcal{O}_B$ with $\phi_*\mathcal{O}_B$, and get

$$\begin{array}{ccc} \mathcal{O}_S & \xrightarrow{\varphi^\sharp} & \varphi_*\mathcal{O}_C \\ & & \uparrow \\ & & \phi_*\mathcal{O}_B \end{array}$$

with the vertical arrow being an inclusion. It follows that we can consider $\phi_*\mathcal{O}_B$ as a subsheaf of $\varphi_*\mathcal{O}_C$ and so it is enough to show that the image of $\mathcal{O}_S$ under $\varphi^\sharp$ is contained in $\phi_*\mathcal{O}_B$. This is equivalent to say that the morphism $\mathcal{O}_S \to \varphi_*\mathcal{O}_C/\phi_*\mathcal{O}_B$ induced by $\varphi^\sharp$ is the zero morphism. But this is true on the geometric points and so it is true everywhere.

The last statement follows from the fact that, if $\Gamma = \beta_1 + ... + \beta_n$, then we have an inverse of ϕ. It is given by sending any morphism $B \to S$ to the following stable map

$$\begin{array}{ccc} B \times_S E & \xrightarrow{j \circ pr_2} & Z \\ {\scriptstyle pr_1}\downarrow & & \\ B & & \end{array}$$

□

Lemma 6.3.3. *Let γ_1, γ_2 or γ_3 be elements of $H^*(Y)$. Then*

$$\Psi^Z_\Gamma(\gamma_1, \gamma_2, \gamma_3) = 0$$

for any $\Gamma = a_1\beta_1 + ... + a_n\beta_n$.

Proof. By the Equivariance Axiom for Gromov-Witten invariants, see [19] Chapter 7.3, we can assume that $\gamma_3 = \rho^*(\delta_3)$. The virtual dimension of $\bar{\mathcal{M}}_{0,3}(Z,\Gamma)$ is equal to the dimension of Z, dimZ. So, let $\gamma_1, \gamma_2, \gamma_3$ be cohomology classes such that

$$\deg(\gamma_1) + \deg(\gamma_2) + \deg(\delta_3) = \text{dimZ}.$$

We have the following commutative diagram

$$\begin{array}{ccc} \bar{\mathcal{M}}_{0,3}(Z,\Gamma) & \xrightarrow{ev_3} & Z \times Z \times Z \\ {\scriptstyle f_{3,2}\times f_{3,0}}\downarrow & & \downarrow{\scriptstyle id\times id\times\rho} \\ \bar{\mathcal{M}}_{0,2}(Z,\Gamma) \times \bar{\mathcal{M}}_{0,0}(Z,\Gamma) & \xrightarrow{ev_2\times\varphi} & Z \times Z \times Y \end{array}$$

where $\varphi = i \circ \phi$. Then

$$\begin{aligned} ev_3^*(\gamma_1 \otimes \gamma_2 \otimes \rho^*(\delta_3)) &= (f_{3,2} \times f_{3,0})^*(ev_2 \times \varphi)^*(\gamma_1 \otimes \gamma_2 \otimes \delta_3) \\ &= \left[f_{3,2}^* ev_2^*(\gamma_1 \otimes \gamma_2)\right] \cdot \left[f_{3,0}^* \varphi^*(\delta_3)\right] \\ &= f_{3,2}^* \left(\left[ev_2^*(\gamma_1 \otimes \gamma_2)\right] \cdot \left[f_{2,0}^* \varphi^*(\delta_3)\right]\right) \end{aligned}$$

we have used the equality $f_{3,0} = f_{2,0} \circ f_{3,2}$. On the other hand, the following equalities hold

$$\begin{aligned} \left[\bar{\mathcal{M}}_{0,3}(Z,\Gamma)\right]^{vir} &= f_{3,0}^* \left[\bar{\mathcal{M}}_{0,0}(Z,\Gamma)\right]^{vir} \\ &= f_{3,2}^* f_{2,0}^* \left[\bar{\mathcal{M}}_{0,0}(Z,\Gamma)\right]^{vir} \\ &= f_{3,2}^* \left[\bar{\mathcal{M}}_{0,2}(Z,\Gamma)\right]^{vir}. \end{aligned}$$

So,

$$\begin{aligned} \Psi_\Gamma^Z(\gamma_1, \gamma_2, \rho^*(\delta_3)) &= \int_{f_{3,2}^*\left[\bar{\mathcal{M}}_{0,2}(Z,\Gamma)\right]^{vir}} f_{3,2}^* \left(\left[ev_2^*(\gamma_1 \otimes \gamma_2)\right] \cdot \left[f_{2,0}^* \varphi^*(\delta_3)\right]\right) \\ &= (\text{constant}) \cdot \int_{\left[\bar{\mathcal{M}}_{0,2}(Z,\Gamma)\right]^{vir}} \left[ev_2^*(\gamma_1 \otimes \gamma_2)\right] \cdot \left[f_{2,0}^* \varphi^*(\delta_3)\right] \end{aligned}$$

which is zero since the virtual dimension of $\mathcal{M}_{0,2}(Z,\Gamma)$ is the virtual dimension of $\mathcal{M}_{0,3}(Z,\Gamma)$ minus 1. □

It remains the computation of the invariants of the following form

$$\Psi_\Gamma^Z(j_{l_1 *}\pi_{l_1}^*(\alpha_1), j_{l_2 *}\pi_{l_2}^*(\alpha_2), j_{l_2 *}\pi_{l_2}^*(\alpha_2)),$$

where $\alpha_1, \alpha_2, \alpha_3 \in H^*(S)$ satisfy the following equation

$$\deg \alpha_1 + \deg \alpha_2 + \deg \alpha_3 = \dim \mathrm{S} - 1.$$

With the next Lemma, we reduce this computation to an integral over the following class:

$$\phi_*[\mathcal{M}_{0,0}(Z,\Gamma)]^{vir} \in A_*(S)$$

Lemma 6.3.4. *The following equality holds,*

$$\Psi_\Gamma^Z(j_{l_1 *}\pi_{l_1}^*(\alpha_1), j_{l_2 *}\pi_{l_2}^*(\alpha_2), j_{l_2 *}\pi_{l_2}^*(\alpha_2)) = \tag{6.7}$$

$$= (E_{l_1} \cdot \Gamma)(E_{l_2} \cdot \Gamma)(E_{l_3} \cdot \Gamma) \int_{\phi_*[\bar{\mathcal{M}}_{0,0}(Z,\Gamma)]^{vir}} (\alpha_1 \cdot \alpha_2 \cdot \alpha_3). \tag{6.8}$$

Proof. Consider the following cartesian diagram, which defines $E\times_S E\times_S E$,

$$\begin{array}{ccc} E\times_S E\times_S E & \longrightarrow & E\times E\times E \\ \downarrow & & \downarrow \\ S & \longrightarrow & S\times S\times S \end{array}$$

where the botom arrow is the diagonal embedding.

From the previous Lemma 6.3.2 it follows that the evaluation morphism $ev_3 : \bar{\mathcal{M}}_{0,3}(Z,\Gamma)\to Z\times Z\times Z$ factors through a morphism $\tilde{ev}_3 : \bar{\mathcal{M}}_{0,3}(Z,\Gamma)\to E\times E\times E$ and the inclusion $E\times E\times E\to Z\times Z\times Z$. Moreover, since the image of any stable map over a geometric point is contained in a fiber of $\pi : E\to S$, $\tilde{ev}_3$ factors through a morphism $\tilde{\tilde{ev}}_3 : \bar{\mathcal{M}}_{0,3}(Z,\Gamma)\to E\times_S E\times_S E$ and the inclusion $E\times_S E\times_S E\to E\times E\times E$. So, we have the following equalities

$$\begin{aligned} &\Psi^Z_\Gamma(j_{l_1*}\pi^*_{l_1}(\alpha_1), j_{l_2*}\pi^*_{l_2}(\alpha_2), j_{l_2*}\pi^*_{l_2}(\alpha_2)) = \\ = &\int_{[\bar{\mathcal{M}}_{0,3}(Z,\Gamma)]^{vir}} \tilde{ev}_3^*\left(j^*j_{l_1*}\pi^*_{l_1}(\alpha_1)\otimes j^*j_{l_2*}\pi^*_{l_2}(\alpha_2)\otimes j^*j_{l_3*}\pi^*_{l_3}(\alpha_3)\right) \\ = &\int_{[\bar{\mathcal{M}}_{0,3}(Z,\Gamma)]^{vir}} \tilde{ev}_3^*\left(\mathcal{O}_E(E_{l_1})\pi^*(\alpha_1)\otimes \mathcal{O}_E(E_{l_2})\pi^*(\alpha_2)\otimes \mathcal{O}_E(E_{l_3})\pi^*(\alpha_3)\right). \end{aligned}$$

Notice that $j^*j_{l*}\pi^*_l(\alpha) = \mathcal{O}_E(E_l)\pi^*(\alpha)$, for any $l=1,...,n$ and $\alpha\in H^*(S)$. Indeed, for any $m=1,...,n$,

$$k^*_m j^* j_{l*}\pi^*_l(\alpha) = j^*_m j_{l*}\pi^*_l(\alpha) = [E_m\cap E_l\subset E_m]\cdot\pi^*_m(\alpha)$$

and

$$k^*_m(\mathcal{O}_E(E_l)\pi^*(\alpha)) = j^*_m\mathcal{O}_Z(E_l)\pi^*_m(\alpha),$$

which are equal (see [30], Chapter 2.3). Using the fact that $\tilde{ev}_3$ factors through $\tilde{\tilde{ev}}_3$ we get

$$\begin{aligned} &\int_{[\bar{\mathcal{M}}_{0,3}(Z,\Gamma)]^{vir}} \tilde{ev}_3^*\left(\mathcal{O}_E(E_1)\pi^*(\alpha_1)\otimes \mathcal{O}_E(E_2)\pi^*(\alpha_2)\otimes \mathcal{O}_E(E_3)\pi^*(\alpha_3)\right) \\ = &\int_{[\bar{\mathcal{M}}_{0,3}(Z,\Gamma)]^{vir}} \tilde{\tilde{ev}}_3\left(\mathcal{O}_E(E_1)\otimes \mathcal{O}_E(E_2)\otimes \mathcal{O}_E(E_3)\pi^*(\alpha_1\cdot\alpha_2\cdot\alpha_3)\right). \end{aligned}$$

Now apply the divisor axion and get

$$\Psi^Z_\Gamma(j_{l_1*}\pi^*_{l_1}(\alpha_1), j_{l_2*}\pi^*_{l_2}(\alpha_2), j_{l_2*}\pi^*_{l_2}(\alpha_2)) = \tag{6.9}$$

$$= (E_{l_1}\cdot\Gamma)(E_{l_2}\cdot\Gamma)(E_{l_3}\cdot\Gamma)\int_{[\bar{\mathcal{M}}_{0,0}(Z,\Gamma)]^{vir}}\phi^*(\alpha_1\cdot\alpha_2\cdot\alpha_3). \tag{6.10}$$

Let pt denote the unique morphism from any scheme over $\mathbb{C}$ to Spec($\mathbb{C}$). Then the integral (6.1) is the degree of the homology class

$$pt_*\left(ev_3^*(\gamma_1 \otimes \gamma_2 \otimes \gamma_3) \cap \left[\bar{\mathcal{M}}_{0,3}(Z,\Gamma)\right]^{vir}\right).$$

Using the projection formula ([41] pag. 328) and the equality $pt = \phi \circ pt$, (6.9) becomes:

$$(E_{l_1}\cdot\Gamma)(E_{l_2}\cdot\Gamma)(E_{l_3}\cdot\Gamma)pt_*\left(\alpha_1\cdot\alpha_2\cdot\alpha_3 \cap \phi_*[\bar{\mathcal{M}}_{0,0}(Z,\Gamma)]^{vir}\right).$$

□

At this point, to conclude the proof, we proceed as follows. First we show that

$$\phi_*[\bar{\mathcal{M}}_{0,0}(Z,\Gamma)]^{vir} = \begin{cases} \frac{1}{a^3}[\bar{\mathcal{M}}_{0,0}(Z,\beta_{ij})]^{vir} & \text{if } \Gamma = a\beta_{ij}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.11}$$

Second, that

$$[\bar{\mathcal{M}}_{0,0}(Z,\beta_{ij})]^{vir} = c_1(K). \tag{6.12}$$

Let us explain our idea to prove (6.11), it will be motivated by our proof of Theorems 6.1.3, 6.1.4 and 6.1.5. Let us denote by $\mathcal{M}$ the moduli stack $\bar{\mathcal{M}}_{0,0}(Z,\Gamma)$ and by

$$\phi_{\mathcal{M}} : E^\bullet_{\mathcal{M}} \to L^\bullet_{\mathcal{M}}$$

the obstruction theory on $\mathcal{M}$ given as in (6.3). We will identify $\bar{\mathcal{M}}_{0,0}(Z,\beta_{ij})$ with S and the obstruction theory will be denoted by

$$\phi_S : E^\bullet_S \to L^\bullet_S.$$

Remark 6.3.5. Since S is smooth, $L^\bullet_S$ is given by the locally free sheaf Ω_S in degree zero.

Suppose that we have a morphism $\phi^*(E^\bullet_S) \to E^\bullet_{\mathcal{M}}$ such that the following diagram commutes

$$\begin{array}{ccc} \phi^*(E^\bullet_S) & \longrightarrow & E^\bullet_{\mathcal{M}} \\ {\scriptstyle \phi^*(\phi_S)}\downarrow & & \downarrow{\scriptstyle \phi_{\mathcal{M}}} \\ \phi^*(L^\bullet_S) & \longrightarrow & L^\bullet_{\mathcal{M}} \end{array} \tag{6.13}$$

where the botom row is the morphism induced by $\phi : \mathcal{M} \to S$. Then, we can complete (6.13) to a morphism of distinguished triangles as follows

$$\begin{array}{ccccccc} \phi^*(E_S^\bullet) & \longrightarrow & E_{\mathcal{M}}^\bullet & \longrightarrow & E_{\mathcal{M}/S}^\bullet & \xrightarrow{+1} & \\ \downarrow{\scriptstyle \phi^*(\phi_S)} & & \downarrow{\scriptstyle \phi_{\mathcal{M}}} & & \downarrow{\scriptstyle \phi_{\mathcal{M}/S}} & & \\ \phi^*(L_S^\bullet) & \longrightarrow & L_{\mathcal{M}}^\bullet & \longrightarrow & L_{\mathcal{M}/S}^\bullet & \xrightarrow{+1} & \end{array} \tag{6.14}$$

where $L_{\mathcal{M}/S}^\bullet$ is the relative cotangent complex of $\mathcal{M}$ over S and $E_{\mathcal{M}/S}^\bullet$ is, a priori, an object in $D(\mathcal{O}_{\mathcal{M}_{\acute{e}t}})$. We want to find conditions such that

$$\phi_{\mathcal{M}/S} : E_{\mathcal{M}/S}^\bullet \to L_{\mathcal{M}/S}^\bullet$$

is a relative obstruction theory for $\mathcal{M}$ over S, see [10] Section 7. From diagram (6.14) it follows that, if the map

$$h^{-1}(\phi^*(E_S^\bullet)) \to h^{-1}(E_{\mathcal{M}}^\bullet)$$

is injective, then $\phi_{\mathcal{M}/S} : E_{\mathcal{M}/S}^\bullet \to L_{\mathcal{M}/S}^\bullet$ is a relative obstruction theory.

Suppose that this is the case. Then, we have an induced perfect obstruction theory on each fiber of ϕ, [10] Proposition 7.2. An easy dimension count shows that this obstruction theory has virtual dimension zero.

The fact that $\phi : \mathcal{M} \to S$ is locally trivial implies that, under the identification of each fiber of ϕ with $\bar{\mathcal{M}}_{0,0}(\tilde{R}, \Gamma)$, the obstruction theory induced on the former stack does not depends on the point in S.

Notation 6.3.6. Let us denote by F the stack $\bar{\mathcal{M}}_{0,0}(\tilde{R}, \Gamma)$, by $E_F^\bullet$ the obstruction theory on F induced by $E_{\mathcal{M}/S}^\bullet$ and by $[F]^{vir}$ the virtual fundamental class.

Using the previous facts, we can prove the following formula

$$\phi_*[\mathcal{M}]^{vir} = \text{degree}([F]^{vir}) \cdot [S]^{vir} \tag{6.15}$$

So, to prove (6.11), it remains to show that

$$\text{degree}([F]^{vir}) = \begin{cases} \frac{1}{a^3} & \text{if } \Gamma = a\beta_{ij}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.16}$$

We have to understand $E_F^\bullet$. The obstruction theory $E_F^\bullet$ can be obtained as follows. Let $X \to \Delta$ be a generic deformation of $\tilde{R}$, where $\Delta \subset \mathbb{C}$ is a small open disc centered at the origin $0 \in \mathbb{C}$. We mean that $\tilde{R}$ is isomorphic to the fiber of $X \to \Delta$ in 0. Notice that X is a Calabi-Yau threefold. The embedding $\tilde{R} \to X$ gives a group homomorphism $H_2(\tilde{R}, \mathbb{Z}) \to H_2(X, \mathbb{Z})$ and, by abuse of notation, we will denote by $\Gamma \in H_2(X, \mathbb{Z})$ the image of $\Gamma \in H_2(\tilde{R}, \mathbb{Z})$. Since $X \to \Delta$ is generic, we will have an isomorphism as follows,

$$F \cong \bar{\mathcal{M}}_{0,0}(X, \Gamma).$$

This follows easily from the explicit description of the semiuniversal deformation space of $\tilde{R}$ given in [36] Theorem 1, see also [57] Section 3. Then $E_F^\bullet$ coincides with the obstruction theory (6.3) for to $\bar{\mathcal{M}}_{0,0}(X, \Gamma)$.

Now, (6.16) follows from the computation given in [13] Proposition 2.10.

6.3.2 The A_1 case

Here we prove Theorem 6.1.3. This is a straightforward generalization of [38] Theorem 3.5 (ii).

Notation 6.3.7. The exceptional divisor E is irreducible and moreover the following description holds: $E \cong \mathbb{P}(F)$, where F is a rank two vector bundle on S, $N_{E/Z} \cong \mathcal{O}_F(-2) \otimes \pi^* L$, where L is defined by $\wedge^2 F \otimes L \cong R^1\pi_* N_{E/Z}$. See Proposition 5.3.2.

In this section Γ will be $a\beta$, for a positive integer a.

The universal stable morphism $f : \mathcal{C} \to Z$ factors through a morphism $g : \mathcal{C} \to E$. See Lemma 6.3.2.

Lemma 6.3.8. *The moduli stack $\bar{\mathcal{M}}_{0,0}(Z, a\beta)$ is smooth of dimension dim $S+$ $2a-2$. The virtual fundamental class is given by*

$$[\bar{\mathcal{M}}_{0,0}(Z, a\beta)]^{vir} = c_r(h^1(E^{\bullet\vee})) \cdot [\bar{\mathcal{M}}_{0,0}(Z, a\beta)] \tag{6.17}$$

where

$$h^1(E^{\bullet\vee}) \cong R^1 p_*(g^* N_{E/Z}) \tag{6.18}$$

is a vector bundle of rank $r = 2a-1$.

Proof. The smoothness of $\bar{\mathcal{M}}_{0,0}(Z, a\beta)$ follows from the fact that the fibers of ϕ are smooth, where ϕ is defined in Lemma 6.3.2. Indeed they are all isomorphic to $\bar{\mathcal{M}}_{0,0}(\mathbb{P}^1, a\beta)$. Moreover from Proposition 6.2.7 it follows that they have dimension $2a-1$.

Equation (6.17) follows from Proposition 6.2.8, so, it remains to prove equation (6.18). We will show that

$$R^1 p_*(f^* T_Z) \text{ is a vector bundle of rank } 2a-1,$$

and that

$$R^1p_*(g^*N_{E/Z}) \cong R^1p_*(f^*T_Z). \tag{6.19}$$

Then the Lemma will follows from Proposition 6.2.11.

Since $H^2(p^{-1}(u), f^*T_Z|_{p^{-1}(u)}) = 0$, $R^2p_*(f^*T_Z)$ is locally free of rank 0, see [33] Theorem 12.11 (Cohomology and Base Change). So, it is enough to prove that $H^1(p^{-1}(u), f^*T_Z)|_{p^{-1}(u)})$ is independent from u.

Let $u = [\mu : D \to Z] \in \bar{\mathcal{M}}_{0,0}(Z, a\beta)$ be a stable map. From the following exact sequence of locally free sheaves on E

$$0 \to T_E \to T_Z|_E \to N_{E/Z} \cong \mathcal{O}_E(-2) \to 0,$$

we get the following

$$0 \to \mu^*T_E \to \mu^*T_Z|_E \to \mu^*\mathcal{O}_E(-2) \to 0.$$

Since $H^1(D, \mu^*T_E) = 0$, we get

$$H^1(D, \mu^*T_Z) \cong H^1(D, \mu^*\mathcal{O}_{\mu(D)}(-2))$$

which has dimension $2a - 1$.

To prove (6.19), consider the following exact sequence

$$0 \to T_E \to T_{Z|E} \to N_{E/Z} \to 0,$$

then apply $R^\bullet p_*$. □

Remark 6.3.9. From the previous Lemma, it follows that

$$[\bar{\mathcal{M}}_{0,0}(Z, \beta)]^{vir} = c_1(R^1\pi_*N_{E/Z}) \cdot [\bar{\mathcal{M}}_{0,0}(Z, \beta)].$$

Lemma 6.3.10. *We have the following exact sequence:*

$$0 \to \phi^*(R^1\pi_*N_{E/Z}) \to R^1p_*(g^*N_{E/Z}) \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is a vector bundle of rank $2a - 2$ whose restriction on each fiber of ϕ is given as follows. Let $p \in S$ and consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_{\phi^{-1}(p)} & \xrightarrow{g|} & E_p \\ {\scriptstyle p|}\downarrow & & \downarrow{\scriptstyle \pi_p} \\ \phi^{-1}(p) & \xrightarrow{\phi|} & \{p\} \end{array}$$

where $\mathcal{C}_{\phi^{-1}(p)}$ is $p^{-1}(\phi^{-1}(p))$ and $p \mid$ (resp.$g \mid$) is the restriction of p (resp. g) on it, E_p is the fiber $\pi^{-1}(p)$ and π_p is the restriction of π. Then, the restriction of $\mathcal{F}$ to $p^{-1}(\phi^{-1}(p))$ is:

$$R^1 p \mid_* \left(g \mid^* (\mathcal{O}_{E_p}(-1) \oplus \mathcal{O}_{E_p}(-1))\right).$$

Proof. Since $E \cong \mathbb{P}(F)$, we have the surjective morphism: $\pi^*(F^\vee) \to \mathcal{O}_F(1)$. Its kernel is $(\wedge^2 \pi^*(F^\vee)) \otimes \mathcal{O}_F(-1)$. So, we have the following exact sequence

$$0 \to (\wedge^2 \pi^*(F^\vee)) \otimes \mathcal{O}_F(-1) \to \pi^*(F^\vee) \to \mathcal{O}_F(1) \to 0,$$

which, tensorized with $(\pi^* \wedge^2 F \otimes L) \otimes \mathcal{O}_F(-1)$ gives the following one

$$0 \to N_{E/Z} \to \pi^*(F \otimes L) \otimes \mathcal{O}_F(-1) \to \pi^*(R^1\pi_* N_{E/Z}) \to 0, \tag{6.20}$$

see Proposition 5.3.2.

The pull back, under g, of (6.20) on $\mathcal{C}$ gives a short exact sequence of vector bundles, and then taking $R^\bullet p_*$ we have the following long exact sequence:

$$\begin{aligned} 0 &\to p_* g^* N_{E/Z} \to p_* \left(p^*\phi^*(F \otimes L) \otimes g^*\mathcal{O}_F(-1)\right) \to p_* p^* \phi^* R^1\pi_* N_{E/Z} \\ &\to R^1 p_*(g^* N_{E/Z}) \to R^1 p_* \left(p^*\phi^*(F \otimes L) \otimes g^*\mathcal{O}_F(-1)\right) \\ &\to R^1 p_*(p^*\phi^* R^1 \pi_* N_{E/Z}) \to 0. \end{aligned}$$

Notice that

$$p_* \left(p^*\phi^*(F \otimes L) \otimes g^*\mathcal{O}_F(-1)\right) \cong 0$$

by Cohomology and Base Change. Moreover, projection formula gives

$$\begin{aligned} p_* p^* \phi^* R^1 \pi_* N_{E/Z} &\cong \phi^*(R^1\pi_* N_{E/Z}) \quad \text{and} \\ R^1 p_*(p^*\phi^* R^1\pi_* N_{E/Z}) &\cong \phi^*(R^1\pi_* N_{E/Z}) \otimes R^1 p_* \mathcal{O}_{\mathcal{C}} \cong 0. \end{aligned}$$

So, the result follows by defining

$$\mathcal{F} := R^1 p_* \left(p^*\phi^*(F \otimes L) \otimes g^*\mathcal{O}_F(-1)\right).$$

□

Finally, to prove (6.11), and so to complete the proof of Theorem 6.1.3, we have to show that

$$\int_{\phi^{-1}(p)} c_{2a-2}\left(R^1 p_{|*}\left(g_|^*(\mathcal{O}_{E_p}(-1) \oplus \mathcal{O}_{E_p}(-1))\right)\right) = \frac{1}{a^3}.$$

This is proved in [19] Theorem 9.2.3.

6.3.3 The A_n case, $n \geq 2$

We prove Theorem 6.1.4.

Notation 6.3.11. For any $i, j \in \{1, ..., n\}$ with $1 \leq i \leq j \leq n$, we will denote by E_{ij} the union $E_i \cup ... \cup E_j$. The restriction of π to E_{ij} will be denoted by π_{ij}.

The moduli stack $\bar{\mathcal{M}}_{0,0}(Z, \beta_{ij})$ will be denoted by S, see Lemma 6.3.2.

Notice that we can assume that $i = 1$ and $j = n$, so that $E_{ij} = E$ and $\beta_{ij} = \beta_1, ..., \beta_n$. The moduli stack is smooth of virtual dimension dim $S - 1$, so the virtual fundamental class is given by

$$[S]^{vir} = c_1(h^1(E_S^{\bullet\vee})) \cdot S$$

where $E_S^\bullet$ is given as follows, see (6.3):

$$E_S^\bullet = R^\bullet \pi_*([\Omega_{Z|E} \to \Omega_\pi] \otimes \omega_\pi).$$

Lemma 6.3.12.

$$h^1(E_S^{\bullet\vee}) \cong R^1\pi_* N_{E/Z}.$$

proof. The complex of sheaves on E:

$$\Omega_{Z|E} \to \Omega_\pi, \tag{6.21}$$

is isomorphic, in the derived category $D(\mathcal{O}_E)$, to a locally free sheaf G in degree -1. Indeed, the morphism $\Omega_{Z|E} \to \Omega_\pi$ is surjective and, denoting by G its kernel, we have the following exact sequence

$$0 \to G \to \Omega_Z \mid E \to \Omega_\pi \to 0.$$

Since Ω_π is of projective dimension one, it follows that G is locally free.

So,

$$h^1(E_S^{\bullet\vee}) \cong R^1\pi_*(G^\vee).$$

We have the following exact sequence

$$0 \to \mathcal{O}_Z(-E) \mid E \to G \to \pi^*\Omega_S \to 0. \tag{6.22}$$

This follows from a diagram chasing in the next diagram,

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \pi^*\Omega_S & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_Z(-E) \mid E & \longrightarrow & \Omega_Z \mid E & \longrightarrow & \Omega_E & \longrightarrow & 0 \\
 & & & & \| & & \downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & \Omega_Z \mid E & \longrightarrow & \Omega_\pi & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Then, taking the dual of (6.22) and applying $R^\bullet\pi_*$ we get the following isomorphism

$$R^1\pi_*(G^\vee) \cong R^1\pi_* N_{E/Z},$$

which complete the proof. □

The proof of Theorem 6.1.4 will be completed by the following Lemma.

Lemma 6.3.13.

$$R^1\pi_{ij*}N_{E_{ij}/Z} \cong K.$$

Proof. Let us first assume that $i < j$. Let a be an integer which satisfies $i \leq a < j$. Let $\tilde{E}_a = E_i \cup ... \cup E_a$ and $\bar{E}_{a+1} = E_{i+1} \cup ... \cup E_j$. Let us denote by $S_a = \tilde{E}_a \cap \bar{E}_{a+1}$. Consider the following exact sequence

$$0 \to \mathcal{O}_{E_{ij}} \to \mathcal{O}_{\tilde{E}_a} \oplus \mathcal{O}_{\bar{E}_{a+1}} \to \mathcal{O}_{S_a} \to 0,$$

this gives the following,

$$0 \to N_{E_{ij}/Z} \to N_{\tilde{E}_a/Z}(S_a) \oplus N_{\bar{E}_{a+1}/Z}(S_a) \to N_{E_{ij}/Z_{|S_a}},$$

notice that $N_{E_{ij}/Z} \mid \tilde{E}_a = (\mathcal{O}_Z(\tilde{E}_a + \bar{E}_{a+1})) \mid \tilde{E}_a = N_{\tilde{E}_a/Z} \otimes \mathcal{O}_Z(\bar{E}_{a+1}) \mid \tilde{E}_a$ and $\mathcal{O}_Z(\bar{E}_{a+1}) \mid \tilde{E}_a$ is, by definition, the line bundle associated to the intersection of $\tilde{E}_a$ with the divisor $\bar{E}_{a+1}$, so it is $\mathcal{O}_{\tilde{E}_a}(S_a)$.

We now claim that

$$R^1\pi_{ij*}N_{E_{ij}/Z} \cong N_{E/Z} \mid S_a.$$

This follows from the long exact sequence associated to the functor $R^\bullet\pi_*$ once we notice that

$$R^p\pi_* N_{\tilde{E}_a/Z}(S_a) = R^p\pi_* N_{\bar{E}_{a+1}/Z}(S_a) = 0 \quad \text{for all } p \geq 0. \tag{6.23}$$

To see this, let $\tilde{C}_a$ be a fiber of $\tilde{E}_a$, $\bar{C}_{a+1}$ be a fiber of $\bar{E}_{a+1}$ and $p_a = \tilde{C}_a \cap \bar{C}_{a+1}$. We can assume that $\tilde{C}_a$, $\bar{C}_{a+1}$ and p_a are embedded in the surface $\tilde{R}$ as a part of the exceptional divisor of $\tilde{R} \to R$. Then, to prove (6.23) is equivalent to prove the following identities,

$$H^p(\tilde{C}_a, N_{\tilde{C}_a/\tilde{R}}(p_a)) = H^p(\bar{C}_{a+1}, N_{\bar{C}_{a+1}/\tilde{R}}(p_a)) = 0 \quad \text{for all } p \geq 0.$$

To prove the equality: $H^p(\tilde{C}_a, N_{\tilde{C}_a/\tilde{R}}(p_a) = 0$, we proceed by induction on a. As before we have the exact sequence

$$0 \to N_{\tilde{C}_a/\tilde{R}} \to N_{\tilde{C}_{a-1}/\tilde{R}}(p_{a-1}) \oplus N_{C_a/\tilde{R}}(p_{a-1}) \to N_{\tilde{C}_a/\tilde{R}} \mid p_{a-1} \to 0$$

which tensorized by $\mathcal{O}_{\tilde{C}_a}(p_a)$ gives

$$0 \to N_{\tilde{C}_a/\tilde{R}}(p_a) \to N_{\tilde{C}_{a-1}/\tilde{R}}(p_{a-1}) \oplus N_{C_a/\tilde{R}}(p_{a-1} + p_a) \to N_{\tilde{C}_a/\tilde{R}} \mid p_{a-1}.$$

To prove the second isomorphism notice that

$$\begin{aligned} N_{E/Z} \mid S_a &= (\mathcal{O}_Z(E_a + E_{a+1})) \mid S_a \\ &= (N_{E_a/Z} \otimes \mathcal{O}_Z(E_{a+1}) \mid E_a) \mid S_a \\ &= N_{E_a/Z} \mid S_a \otimes \mathcal{O}_{E_a}(S_a) \mid S_a. \end{aligned}$$

The result follows from the explicit description of the divisors E_i in terms of the line bundles L_i, M_i and K given in Proposition 5.3.3. □

We now prove Theorem 6.1.5. We will use the fact that Gromov-Witten invariants are invariants under deformation of the complex structure of Z. The hypothesis of the Theorem will allow us to deform Z in a convenient way.

Notation 6.3.14. For any variety X, we will denote by T_X the sheaf of $\mathbb{C}$-derivations, i.e.,

$$T_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X),$$

where Ω^1_X is the sheaf of differentials of X.

Proposition 6.3.15. *Under the hypothesis $H^2(Y, T_Y) = 0$, we have the following exact sequence of cohomology groups:*

$$0 \to H^1(Y, T_Y) \to H^1(Z, T_Z) \to \oplus_{l=1}^n H^0(S, R^1\pi_{l*}N_{E_l/Z}) \to 0. \qquad (6.24)$$

Proof. From the Leray spectral sequence, we have the following exact sequence of cohomology groups,

$$\begin{aligned} 0 &\to H^1(Y, T_Y) \to H^1(Z, T_Z) \to H^0(Y, R^1\rho_*T_Z) \\ &\to H^2(Y, T_Y) \to H^2(Z, T_Z), \end{aligned}$$

where we have used the fact that ρ_*T_Z is isomorphic to T_Y. This is true since Y has quotient singularities, see [56] Lemma 1.11 for the proof, see also [14] for a proof in dimension 2. The vanishing of $H^2(Y, T_Y)$ gives the following short exact sequence:

$$0 \to H^1(Y, T_Y) \to H^1(Z, T_Z) \to H^0(Y, R^1\rho_*T_Z) \to 0.$$

We now claim that $R^1\rho_*T_Z$ is isomorphic to $i_*\left(\oplus_{l=1}^n R^1\pi_{l*}N_{E_l/Z}\right)$. First of all notice that there is a morphism

$$R^1\rho_*T_Z \to i_*\left(\oplus_{l=1}^n R^1\pi_{l*}N_{E_l/Z}\right). \qquad (6.25)$$

It is the composition of the morphism $i^*\left(R^1\rho_*T_Z\right) \to R^1\pi_*T_Z \mid E$, as defined in [33] remark 9.3.1 Chapter III, and the morphism induced by $T_Z \mid E \to \oplus_{l=1}^n N_{E_l/Z}$, as the sum of $T_Z \mid E \to N_{E_l/Z}$. We will prove that (6.25) is an isomorphism.

Since this is a local problem, let us suppose that $Z = \mathbb{C}^k \times \tilde{R}$. Then, from Künneth formula and the fact that the surface singularity is rational, we have the following isomorphism

$$H^1(\mathbb{C}^k \times \tilde{R}, T_{\mathbb{C}^k \times \tilde{R}}) \cong H^0(\mathbb{C}^k, \mathcal{O}_{\mathbb{C}^k}) \otimes H^1(\tilde{R}, T_{\tilde{R}}).$$

Let $C = C_1 \cup ... \cup C_n \subset \tilde{R}$ be the exceptional locus with the C_l be the irreducible components. Then, we have the following isomorphism:

$$H^1(\tilde{R}, T_{\tilde{R}}) \cong \oplus_{l=1}^n H^1(C_l, N_{C_l/\tilde{R}}),$$

see [14] (1.8), and this shows that (6.25) is an isomorphism. □

Remark 6.3.16. It is known that $H^1(Y, T_Y)$ is in $1-1$ correspondence with the set of locally trivial first order deformations of Y modulo isomorphism. In the same way $H^1(Z, T_Z)$ is in $1 - 1$ correspondence with the set of first order deformations of Z modulo isomorphisms. So, the sequence (6.24) has the following meaning in deformation theory: to any first order locally trivial deformation of Y we can associate a first order deformation of Z, the remaining deformations of Z come from $H^0(Y, R^1\rho_* T_Z)$. We are interested in understanding the last deformations.

Lemma 6.3.17. *For any subset $I \subset \{1, ..., n\}$, let us denote by $E_I = \cup_{l\in I} E_l$. Let $\pi_I : E_I \to S$ be the restriction of π. Then $H^0(S, R^1\pi_{I*}N_{E_I/Z})$ is an obstruction space for deformations of E_I in Z. This means that, there is a morphism*

$$\mathrm{ob} : \mathrm{H}^0(\mathrm{S}, \oplus_{\mathrm{l}=1}^{\mathrm{n}} \mathrm{R}^1\pi_{\mathrm{l}*}\mathrm{N}_{\mathrm{E_l/Z}}) \to \mathrm{H}^0(\mathrm{S}, \mathrm{R}^1\pi_{\mathrm{I}*}\mathrm{N}_{\mathrm{E_I/Z}})$$

which associates to any $\sigma \in H^0(S, \oplus_{l=1}^n R^1\pi_{l*}N_{E_l/Z})$ the obstruction $\mathrm{ob}(\sigma)$ to extend E_I to the first order deformation associated to σ, see Remark 6.3.16.

Proof. There is a morphism

$$\mathrm{ob} : \mathrm{H}^1(\mathrm{Z}, \mathrm{T_Z}) \to \mathrm{H}^1(\mathrm{E_I}, \mathrm{N_{E_I/Z}})$$

which associate, to any first order deformation $\mathcal{Z}$ of Z, the obstruction to the existence of a first order deformation $\mathcal{E}_I$ of E_I in $\mathcal{Z}$, see [54] Proposition II.3.3. Since $H^1(Y, T_Y)$ corresponds to locally trivial deformations $\mathcal{Y}$ of Y, any element $\mathcal{Y}_t$ of the family has tranversal A_n singularities. The deformation $\mathcal{Z}$ of Z induced by $\mathcal{Y}$ is a simultaneous resolution, so it contains a deformation of E_I. It follows that ob factors through $H^0(S, \oplus_{l=1}^n R^1\pi_{l*}N_{E_l/Z})$. By abuse of notations we will denote by ob this induced morphism.

To conclude the proof, we will show that

$$H^1(E_I, N_{E_I/Z}) \cong H^0(S, R^1\pi_{I*}N_{E_I/Z}).$$

We use the Leray spectral sequence, $H^p(S, R^q\pi_{I*}N_{E_I/Z}) \Rightarrow H^{p+q}(E_I, N_{E_I/Z})$, to get the following exact sequence:

$$\begin{aligned} 0 \;\to\; & H^1(S, \pi_{I*}N_{E_I/Z}) \to H^1(E_I, N_{E_I/Z}) \to H^0(S, R^1\pi_{I*}N_{E_I/Z}) \\ \to\; & H^2(S, \pi_{I*}N_{E_I/Z}), \end{aligned}$$

where $\pi_I : E_I \to S$ is the restriction of π to E_I. Since $\pi_{I*}N_{E_I/Z} = 0$ and $H^2(S, \pi_{I*}N_{E_I/Z}) = 0$, we have an isomorphism between $H^1(E_I, N_{E_I/Z})$ and $H^0(S, R^1\pi_{I*}N_{E_I/Z})$. □

Let $\sigma \in H^0(S, \oplus_{l=1}^n R^1\pi_{l*}N_{E_l/Z})$ be a section, and let

$$\begin{array}{ccc} Z & \longrightarrow & \mathcal{Z}_1 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}\left(\frac{\mathbb{C}[\epsilon]}{(\epsilon^2)}\right) \end{array}$$

be the corresponding first order deformation of Z. Then, there exists a finite deformation of Z which, at the first order, coincides with the one given by σ, we will denote this deformation as follows

$$\begin{array}{ccc} Z & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \Delta \end{array} \tag{6.26}$$

where Δ is a small disc in $\mathbb{C}$ around the origin $0 \in \mathbb{C}$. Indeed, by the Kodaira-Nirenberg-Spencer Theorem (1958), there exists a complete family of deformations of Z, see [54] for a review.

Let V be a neighbourhood of E in Z and $r : V \to S$ be a morphism whose restriction to E coincides with π. Now, the deformation (6.26) induces a deformation $\mathcal{V}$ of V. We can choose $\mathcal{V}$ such that r extends to $\tilde{r} : \mathcal{V} \to S$, here we use the hypothesis $H^1(S, T_S) = 0$. Moreover, for K sufficiently ample, the deformation (6.26) can be chosen in such a way the locus of points $p \in S$ such that there exists a rational curve in $\tilde{r}^{-1}(p)$ whose homology class is Γ has codimension one, if $\Gamma = a\beta_{ij}$, and has codimension strictly greater then one if $\Gamma \neq a\beta_{ij}$.

The previous considerations shows that, if $\Gamma \neq a\beta_{ij}$, then all the Gromov-Witten invariants vanish. Indeed the virtual dimension of $\bar{\mathcal{M}}_{0,0}(Z, \Gamma)$ is dim $\mathrm{S} - 1$. If $\Gamma = a\beta_{ij}$, then the locus of points $p \in S$ such that there exists a rational curve in $\tilde{r}^{-1}(p)$ whose homology class is Γ is $\{p \in S : \mathrm{ob}(\sigma)(\mathrm{p}) = 0\}$, where $\sigma \in H^0(S, \oplus_{l=1}^n R^1\pi_{l*}N_{E_l/Z})$ and ob is the morphism defined in Lemma 6.3.17.

6.4 Quantum corrected cohomology ring for A_n-singularities

Let Y be a variety with transversal A_n-singularities and $\rho : Z \to Y$ be the crepant resolution. We compute the quantum corrected cup product $*_\rho$ Defined in Chapter 3.4.

Convention 6.4.1. We will tacitly assume that uor spaces S, Y, Z fulfil the required hypothesis in order to be able to apply Theorem 6.1.3, 6.1.4 or 6.1.5.

Notation 6.4.2. Since the crepant resolution $\rho : Z \to Y$ is unique, the quantum corrected cup product $*_\rho$ will be denoted by $*$.

Let $\gamma_1, \gamma_2 \in H^*(Z)$, then the product $\gamma_1 * \gamma_2$ will be represented by a family of cohomology classes in $H^*(Z)$ which depend on n complex parameters $q_1, ..., q_n$. More precisely, for any $\gamma_1, \gamma_2, \gamma_3 \in H^*(Z)$, one defines the quantum corrected triple intersection $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\rho(q_1, ..., q_n)$ as follows:

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\rho(q_1, ..., q_n) = \langle \gamma_1, \gamma_2, \gamma_3 \rangle + \langle \gamma_1, \gamma_2, \gamma_3 \rangle_{qc}(q_1, ..., q_n),$$

where $\langle \gamma_1, \gamma_2, \gamma_3 \rangle = \int_Z \gamma_1 \cup \gamma_2 \cup \gamma_3$ and $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{qc}(q_1, ..., q_n)$ is the quantum corrected 3-point function, see (3.4). Note that, $\langle \gamma_1, \gamma_2, \gamma_3 \rangle_\rho(q_1, ..., q_n)$, is a complex valued function defined on the domain of definition of the quantum corrected 3-point function. Then, the quantum corrected cup product, $\gamma_1 * \gamma_2$, is defined by the following equation:

$$\langle \gamma_1 * \gamma_2, \gamma \rangle = \langle \gamma_1, \gamma_2, \gamma \rangle_\rho(q_1, ..., q_n), \quad \text{for all } \gamma \in H^*(Z).$$

So, $\gamma_1 * \gamma_2$ is a cohomology class in $H^*(Z)$ which depends on the parameters $q_1, ..., q_n$. The vector space $H^*(Z)$ with the product $*$ forms a family of rings depending on the parameters $q_1, ..., q_n$, see Theorem 3.4.14. This family will be denoted by $H^*(Z)(q_1, ..., q_n)$. Clearly the parameters $q_1, ..., q_n$ belongs to the domain where the quantum corrected 3-point function is defined.

Notation 6.4.3. Let us define $\gamma_1 *_\epsilon \gamma_2$ by the following equation

$$\langle \gamma_1 *_\epsilon \gamma_2, \gamma \rangle = \langle \gamma_1, \gamma_2, \gamma \rangle_{qc}(q_1, ..., q_n), \quad \text{for all } \gamma \in H^*(Z). \tag{6.27}$$

Then, if $\gamma_1 \cup \gamma_2 = \delta + \alpha_1 E_1 + ... + \alpha_n E_n$ and $\gamma_1 *_\epsilon \gamma_2 = \delta_\epsilon + \epsilon_1 E_1 + ... + \epsilon_n E_n$,

$$\gamma_1 * \gamma_2 = \delta + \delta_\epsilon + (\alpha_1 + \epsilon_1) E_1 + ... + (\alpha_n + \epsilon_n) E_n.$$

Remark 6.4.4. If $\gamma_1 \in H^*(Y)$ or $\gamma_2 \in H^*(Y)$, then $\gamma_1 * \gamma_2 = 0$. Indeed, under these hypothesis, all the Gromov-Witten invariants $\Psi^Z_\Gamma(\gamma_1, \gamma_2, \gamma_3)$ are zero.

So, we can assume $\gamma_1 = E_i$ and $\gamma_2 = E_j$.

Remark 6.4.5. $E_i *_\epsilon E_j \in H^*(Y)^\perp$, where $H^*(Y)^\perp$ is the subspace of $H^*(Z)$ which is orthogonal to $H^*(Y)$ with respect to the ususal pairing $\langle , \rangle = \int_Z$.

Lemma 6.4.6. *The following expression holds:*

$$E_i *_\epsilon E_j = \sum_{l,m=1}^{n} (c_n^{-1})_{lm} R_{ijm}(\underline{q}) c_1(K) E_l, \tag{6.28}$$

where, c_n is the $n \times n$ matrix (5.8), K is the line bundle on S defined in Lemma 4.4.3 if $n \geq 2$ and is $R^1\pi_*(N_{E/Z})$ if $n = 1$, $(\underline{q})$ denote $(q_1, ..., q_n)$, and $R_{ijm}(\underline{q})$ is defined by the following expression:

$$R_{ijm}(\underline{q}) = \sum_{r \leq s} (E_i \cdot \beta_{rs})(E_j \cdot \beta_{rs})(E_m \cdot \beta_{rs}) \frac{q_r \cdots q_s}{1 - q_r \cdots q_s}.$$

As usual, $\beta_{rs} = \beta_r + ... + \beta_s$.

Proof. Let $E_i *_\epsilon E_j = \epsilon_1 E_1 + ... + \epsilon_n E_n$. Then, the left hand side of (6.27) becomes

$$\begin{aligned}
\langle E_i *_\epsilon E_j, \alpha E_k \rangle &= \int_Z \sum_{l=1}^{n} j_{l*}\pi_l^*(\epsilon_l) \cup \alpha E_k \\
&= \sum_{l=1}^{n} \int_Y \rho_*(j_{l*}\pi_l^*(\epsilon_l) \cup \alpha E_k) \\
&= \sum_{l=1}^{n} \int_Y \rho_* j_{l*}(\pi_l^*(\epsilon_l) \cup j_l{}^*(\alpha E_k)) \\
&= \sum_{l=1}^{n} \int_Y i_*\pi_{l*}(\pi_l^*(\epsilon_l \cup \alpha) \cup [E_l \cap E_k \subset E_l]) \\
&= \int_S (\epsilon_{k-1} - 2\epsilon_k + \epsilon_{k+1}) \cup \alpha.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\langle E_i, E_j, \alpha E_k \rangle_{qc}(\underline{q}) &= \sum_{a=1}^{\infty} \sum_{1 \leq r \leq s \leq n} (q_r \cdots q_s)^a (E_i \cdot \beta_{rs})(E_j \cdot \beta_{rs})(E_k \cdot \beta_{rs}) \int_S \alpha \cup c_1(K) \\
&= R_{ijk}(\underline{q}) \int_S \alpha \cup c_1(K).
\end{aligned}$$

So, we get the following equations for $\epsilon_1, ..., \epsilon_n$:

$$\int_S (\epsilon_{k-1} - 2\epsilon_k + \epsilon_{k+1}) \cup \alpha = R_{ijk}(\underline{q}) \int_S \alpha \cup c_1(K) \quad \text{for } \alpha \in H^*(S).$$

□

As a consequence, we have the following result.

Proposition 6.4.7. *The following expression holds for* $E_i * E_j$*:*

$$E_i * E_j = \sum_{l,m=1}^{n} (c_n^{-1})_{lm}\{R_{ijm}(\underline{q})c_1(K) + \alpha_{ijl}\}E_l, \tag{6.29}$$

where

$$(\alpha_{ij1}, ..., \alpha_{ijn}) = \begin{cases} 0 \\ \text{if } |i-j| > 1; \\ (0, ..., 0, iK - M, M - (i-1)K, 0, ..., 0) \\ \text{if } j = i-1; \\ (0, ..., 0, M - (i-1)K, -4K, (i+1)K - M, 0, ..., 0) \\ \text{if } j = i, \end{cases}$$

where, in the second row, $iK - M$ *is in the* $(i-1)$*-th place, and in the third* $-4K$ *is in the* i*-th place.*

Chapter 7

Comparisons for A_1, A_2 and conclusions

We put together the computations of the previous chapters in order to verify Ruan's conjecture for orbifolds with transversal A_n-singularities. Actually we have a complete picture only when $n = 1, 2$. These cases gives informations about how things should go in the general case.

7.1 The A_1 case

Form Lemma 6.4.6 we immediately get

$$E *_\epsilon E = 4\frac{q}{1-q}c_1(R^1\pi_* N_{E/Z})E.$$

So, from Proposition 5.4.2, we get the following expression for the quantum corrected cup product:

$$E * E = \left(2 + 4\frac{q}{1-q}\right) c_1(R^1\pi_* N_{E/Z}).$$

Notice that for $q = -1$, this is zero.

The orbifold cohomology ring $H^*_{orb}([Y])$ has been computed in Example 4.5.2. It is easy to see that the following morphism is a ring isomorphism

$$\begin{aligned} H^*_{orb}([Y]) &\to H^*(Z)(-1) \\ (\delta, \alpha) &\mapsto (\delta, \frac{\sqrt{-1}}{2}\alpha), \end{aligned}$$

where $H^*(Z)(-1)$ is the ring $H^*(Z)(q)$, defined in Theorem 3.4.14, for $q = -1$.

7.2 The A_2 case

We will tacitly assume that uor spaces S, Y, Z fulfil the required hypothesis in order to be able to apply Theorem 6.1.3, 6.1.4 or 6.1.5.

Notation 7.2.1. We will use the following notation: $\delta_1 = \frac{q_1}{1-q_1}$, $\delta_2 = \frac{q_2}{1-q_2}$ and $\delta_3 = \frac{q_1 q_2}{1-q_1 q_2}$.

The following expressions holds for $*_\epsilon$:

$$\begin{aligned} E_1 *_\epsilon E_1 &= -2[S] + ((2+4\delta_1+\delta_3)M + (3+4\delta_1+\delta_3)L)\, E_1 \\ &\quad + ((\delta_1+\delta_3)M + (2+\delta_2+\delta_3)L)\, E_2 \end{aligned}$$

$$\begin{aligned} E_1 *_\epsilon E_2 &= [S] + ((-1-2\delta_1+\delta_3)M + (-2\delta_1+\delta_3)L)\, E_1 \\ &\quad + ((-2\delta_2+\delta_3)M + (-1-2\delta_2+\delta_3)L)\, E_2 \end{aligned}$$

$$\begin{aligned} E_2 *_\epsilon E_2 &= -2[S] + ((2+\delta_1+\delta_3)M + (\delta_1+\delta_3)L)\, E_1 \\ &\quad + ((3+4\delta_2+\delta_3)M + (2+\delta_2+\delta_3)L)\, E_2. \end{aligned}$$

Remark 7.2.2. Notice that, for $q_1 = q_2 = -1$, $\delta_3 = \infty$, so, we have to modify slightly the cohomological crepant resolution conjecture (see Conjecture 3.4.16). In any case,

The orbifold cohomology ring for transversal A_2-singularities has been computed in Example 4.5.3. The resulting ring is described as follows:

$$\begin{aligned} e_1 \cup_{orb} e_1 &= Le_2 \\ e_1 \cup_{orb} e_2 &= \frac{1}{3}[S] \\ e_2 \cup_{orb} e_2 &= Me_1. \end{aligned}$$

Remark 7.2.3. Notice that the previous expressions, for the quantum corrected cup product $*$ and for the orbifold cup product, are symmetric if we exchange $E_1 \leftrightarrow E_2$, $L \leftrightarrow M$ and $\delta_1 \leftrightarrow \delta_2$.

Theorem 7.2.4. *The pairs (q_1, q_2) for which there exists a ring isomorphism $H^*(Z)(q_1, q_2) \to H^*_{orb}([Y])$ which respects the symmetry described in Remark 7.2.3 are: $q_1 = q_2 = exp(\frac{2}{3}\pi i)$ and $q_1 = q_2 = exp(\frac{4}{3}\pi i)$.*

Proof. We are looking for a linear isomorphism

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \tag{7.1}$$

In order for it to respects the symmetry, we must have $a = d$ and $b = c$. The condition to be a ring homomorphism gives a lot of equations, in particular we have $\delta_1 = \delta_2$. Moreover we get the following two possibilities: $a = \sqrt{3}\exp(\frac{\pi}{6}\mathrm{i})$ and $b = \sqrt{3}\exp(\frac{5}{6}\pi\mathrm{i})$, or $a = -\sqrt{3}\exp(\frac{5}{6}\pi\mathrm{i})$ and $b = \sqrt{3}\exp(\frac{\pi}{6}\mathrm{i})$. The first choice corresponds to $q_1 = q_2 = \exp(\frac{2}{3}\pi\mathrm{i})$, the second to $q_1 = q_2 = \exp(\frac{4}{3}\pi\mathrm{i})$. □

Comments 7.2.5. As pointed out in Remark 6.1.7, if the orbifold $[Y]$ carries a holomorphic symplectic 2-form, then our Gromov-Witten invariants are zero. It is easy to see that the isomorphisms $H^*(Z)(q_1, q_2) \to H^*_{orb}([Y])$ founded in Theorem 7.2.4 are still isomorphisms in this case, i.e. in the holomorphic symplectic case. So, to find an isomorphism in the A_n case, one can try to find an isomorphic under the additional hypothesis for $[Y]$ to carries an holomorphic symplectic 2-form, and then try to prove that this is still an isomorphism in the general case. Of course the natural candidates for the parameters $q_1, ..., q_n$ is $q_1 = ... = q_n$ equal to a nth root of unit, such that $q_1 \cdots q_n \neq 1$.

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