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A Fourier transform for sheaves on Lagrangian families of real tori

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A FOURIER TRANSFORM FOR SHEAVES ON LAGRANGIAN FAMILIES OF REAL TORI

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INTRODUCTION

Mirror symmetry has developed during the last decade as a result of the work of many physicists and mathematicians. It takes its origin in Physics, as a duality between string theories compactified on different Calabi-Yau manifolds, however it translates into deep geometrical problems when properly formalized. From this point of view, mirror symmetry may be regarded, in a very simplified way, as a correspondence between symplectic and complex manifolds identifying suitable structures on these manifolds. This is what is known as the homological mirror symmetry conjecture, formulated by M. Kontsevich (see [18]).

From the physical viewpoint, mirror symmetry is an example of duality. A given physical theory may admit more than one theoretical formulation and the respective levels of difficulty in analysing these distinct formulations can be widely different. This happens in mirror symmetry where two topologically distinct Calabi-Yau compactifications of string theory give rise to identical physical models. The consistency of the quantized theory requires the target manifold (i.e., the manifold where the strings moves) to be 10-dimensional (this in the case of superstrings, string models which incorporate fermion fields and exhibit supersymmetry). A part of M will play the role of the four Minkowski space-time. The most intuitive representation of this idea is that in which six of the ten spatial dimensions in the flat space approach are compactified, that is, are replaced by a compact six dimensional manifold X, so that $M = M_4 \times X$. It can be prooved that the symmetries of a theory are encoded in the geometry of X. In particular, $N = 2$ supersymmetry and conformal invariance imply that X is a Calabi-Yau manifold.

String theories on X are parametrized by a "moduli space" which contains the moduli space $\mathcal{M}_X^C$ of complex structures on X and the moduli space $\mathcal{M}_X^K$ of complexified Kähler structures on X. When string theories compactified on different Calabi-Yau manifolds are mirror-symmetric (have the same quantum spectra), then $\mathcal{M}^C_X \cong \mathcal{M}^K_{\hat{X}}$ and $\mathcal{M}^C_{\hat{X}} \cong \mathcal{M}^K_X$, which implies, considering the tangent spaces, that $H^{n-1,n}(X,\mathbb{C}) \cong H^{1,1}(\hat{X},\mathbb{C})$ and $H^{n-1,n}(\hat{X},\mathbb{C}) \cong H^{1,1}(X,\mathbb{C})$. This is what is known as topological mirror symmetry. Quantum spectra of string theories corresponding to type IIB and IIA also include extended objects, called A-cycles and B-cycles respectively. An A-cycle is a special Lagrangian submanifold U of X together with a line bundle L on U. A B-cycles is a complex submanifold of $\hat{X}$ supporting a holomorphic vector bundle with a compatible connection satisfying a deformed Einstein-Yang-Mills condition. The form of mirror symmetry suggested by physicists is just a correspondence between A-cycles on X and B-cycles on $\hat{X}$.

From the mathematical viewpoint, to state precisely the homological mirror conjecture seems to be still very difficult, however we will try to give a simple idea, inevitably with few details, according to what Fukaya explains in [13].

Given a symplectic manifold (M,ω), the conjecture first of all asserts the existence of a complex manifold $(M,\omega)^\wedge$, called a mirror for M. Which conditions on M would ensure the existence of a mirror are still unknown. A sufficient condition might be for example the existence of a fibration $M \to B$, possibly singular, such that the general fibre is a Lagarangian torus. This is just the case this thesis will be concerned with, under the further assumption that the fibres are smooth. Moreover, when a mirror exists, it is not even known if it is unique. The first part of the conjecture can be stated as follows:

For each pair $(L,\mathcal{L})$ of unobstructed Lagrangian submanifolds and flat line bundles $\mathcal{L}$, we can associate an object $\mathcal{E}(L,\mathcal{L})$ in the derived category of coherent sheaves on $(M,\omega)^\wedge$.

Here unobstructed is referred to the existence of Lagrangian intersection Floer homology. For example L is unobstructed when $H_*(L, \mathbb{Q}) \to H_*(M, \mathbb{Q})$ is injective. If $(L_1, \mathcal{L}_1)$ and $(L_2, \mathcal{L}_2)$ are unobstructed, we can define Floer homology $HF((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2))$ between them. The second part of the conjecture can be formulated as follows:

There exists a canonical isomorphism

$$HF((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2)) \cong Ext(\mathcal{E}(L_1, \mathcal{L}_1), \mathcal{E}(L_2, \mathcal{L}_2)) \tag{1}$$

between Floer homology and extension.

On Floer homology it can be defined a product structure

$$m_2 : HF((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2)) \otimes HF((L_2, \mathcal{L}_2), (L_3, \mathcal{L}_3)) \to HF((L_1, \mathcal{L}_1), (L_3, \mathcal{L}_3)) \tag{2}$$

on the other hand, given objects $\mathcal{E}_i$ in the derived category of coherent sheaves on $M^\vee$, we have a map

$$Ext(\mathcal{E}(L_1, \mathcal{L}_1), \mathcal{E}(L_2, \mathcal{L}_2)) \otimes Ext(\mathcal{E}(L_2, \mathcal{L}_2), \mathcal{E}(L_3, \mathcal{L}_3)) \to Ext(\mathcal{E}(L_1, \mathcal{L}_1), \mathcal{E}(L_3, \mathcal{L}_3)) \tag{3}$$

The third part of the conjecture can be expressed as follows:

If $\mathcal{E}_i = \mathcal{E}(L_i, \mathcal{L}_i)$, then the map (3) coincides with the map (2) through the isomorphism (1).

Before stating the last part of the conjecture, we review the notion of $A_\infty-$ category, which extends that of category by not requiring the associativity of the composition of morphisms. This a collection $\mathcal{C}$ of a set $Ob\mathcal{C}$ (the set of objects), of chain complexes $\mathcal{C}(c, d)$ for each $c, d \in Ob\mathcal{C}$ (the set of morphisms), and of maps

$$m_k : \mathcal{C}(c_0, c_1) \otimes ... \otimes \mathcal{C}(c_{k-1}, c_k) \to \mathcal{C}(c_0, c_k)$$

such that $m_1 : \mathcal{C}(c_0, c_1) \to \mathcal{C}(c_0, c_1)$ is the boundary operator and

$$\sum_{0<i\leq j\leq k} \pm m_{k-j+i}(x_1 \otimes ... \otimes x_{l-1} \otimes m_{j-i+1}(x_i \otimes ... \otimes x_j) \otimes x_{j+1} \otimes ... \otimes x_k) = 0$$

We can rewrite this equation in a different way introducing the bar complex $\mathcal{BC}(c, d)$ for each $c, d \in Ob\mathcal{C}$, defined as

$$\mathcal{BC}(c,d) = \bigoplus_{k=1}^{\infty} \bigoplus_{c=c_0,c_1,\dots,c_k=d} \mathcal{C}(c_0, c_1) \otimes ... \otimes \mathcal{C}(c_{k-1}, c_k)$$

m_k then induces a homomorphism $\hat{m}_k : \mathcal{BC}(c,d) \to \mathcal{BC}(c,d)$

$$\hat{m}_k(x_1 \otimes ... \otimes x_m) = \sum_i \pm x_1 \otimes ... \otimes m_k(x_i, ..., x_{i+k-1}) \otimes ... \otimes x_m$$

satisfying

$$\sum_{k_1+k_2=k+1} \pm \hat{m}_{k_1} \circ \hat{m}_{k_2} = 0$$

To each symplectic manifold (M, ω) we can now associate an $A_\infty-$ category $\mathcal{LAG}(M, \omega)$ such that:

-the objects are pairs $(L, \mathcal{L})$, where L is an unobstructed Lagrangian submanifold and $\mathcal{L}$ is a flat line bundle on it;

-the homology (with respect to the boundary operator m_1) of $\mathcal{LAG}((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2))$ is canonically isomorphic to the Floer homology $HF((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2))$;

-the map m_2 of the $A\infty$-structure induces the map (2) in homology.

In this thesis, though we do not work on a generic manifold M, we will be able to define a Fourier-Mukai transform only on a subcategory of $\mathcal{LAG}(M, \omega)$. It would be interesting to extend the transform to the whole Fukaya category and compare our construction with that proposed by Fukaya in [13]. Consider now $M^\wedge$ and let $\mathcal{O}$ be the set of chain homotopy equivalence classes of chain complexes of sheaves of $\mathcal{O}_{M^\wedge}$-modules on $M^\wedge$ with coherent homology sheaves. For each $\mathcal{F} \in \mathcal{O}$ fix a representative $C(\mathcal{F})$ of it such that $C^k(\mathcal{F})$ is locally free and flabby. We define an $A_\infty-$category $\mathcal{SH} = \mathcal{SH}(M^\wedge)$ as

follows:
-the set of objects of $\mathcal{SH}$ is $\mathcal{O}$;
-if $\mathcal{F}_1$,$\mathcal{F}_2 \in \mathcal{O}$, then $\mathcal{SH}^k(\mathcal{F}_1, \mathcal{F}_2) = \oplus_l Hom(C^l(\mathcal{F}_1), C^{l+k}(\mathcal{F}_2))$;
-m_2 is the usual composition of homomorphisms, $m_i = 0$ for $i > 2$.
The definition of $\mathcal{SH}(M^\wedge)$ is independent from the representatives up to homotopy equivalence. The last part of the conjecture can now be stated as follows:

There exists an A_∞-functor $\mathcal{F} : \mathcal{LAG}(M, \omega) \to \mathcal{SH}(M, \omega)^\wedge)$ such that:
-the quasi isomorphism class of $\mathcal{F}(L, \mathcal{L})$ is $\mathcal{E}(L, \mathcal{L})$;
-the homomorphism

$$\mathcal{F}_{((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2))} : \mathcal{LAG}((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2)) \to \mathcal{SH}(\mathcal{E}(L_1, \mathcal{L}_1), \mathcal{E}(L_2, \mathcal{L}_2))$$

induces the canonical isomorphism (1).

It seems that the A_∞-functor $\mathcal{F}$ is not a homotopy equivalence. It is not clear yet which singular Lagrangian submanifolds must be included in order to make $\mathcal{F}$ a homotopy equivalence.

The idea of studying mirror symmetry through Lagrangian fibrations originates from a conjecture proposed by Strominger, Yau and Zaslow in [25], according to which all mirror dual pairs of Calabi-Yau manifolds are equipped with dual special Lagrangian tori fibrations. The geometry of such fibrations and their compactification were studied by Gross in [14] and [15]. On the other hand, as already explained, in his contruction Fukaya is concerned with the A_∞-category $\mathcal{LAG}(M, \omega)$ where Lagrangian submanifolds are considered up to Hamiltonian diffeomorphisms of M. Presumably these two constructions are compatible, since one can expect that for some class of Lagrangian submanifolds the speciality condition picks up a unique representative in each orbit of Hamiltonian diffeomorphisms group. From this point of view, the construction performed in this thesis is close to that developed by Fukaya.

The idea that, in accordance with the Strominger-Yau-Zaslow conjecture, a kind of Fourier-Mukai transform should describe mirror symmetry dates back to 1996 (see [10]). The original Fourier-Mukai transform, mapping coherent sheaves on an abelian variety X to coherent sheaves on the dual abelian variety $\hat{X}$, was introduced by Mukai in [23], as a functor $D(X) \to D(\hat{X})$, where $D(X)$ is the derived category of coherent sheaves of $\mathcal{O}_X$-modules. A relative Fourier-Mukai transform for elliptic surface was developed by Bartocci, Bruzzo, Hernández Ruipérez and Muñoz Porras in [4] and [5], by Bridgeland in [7] and again by Hernández Ruipérez and Muñoz Porras in [16], and was shown to describe, as conjectured in [25], D-branes transformation pattern under mirror symmetry in [4], in the paper [1] of Andreas, Curio, Hernández Ruipérez and Yau, and by Aspinwall and Donagi in [3]. In the case of Calabi-Yau threefolds which are fibred in (special Lagrangian) real 3-tori, the construction of such a transform should be provided by a real Fourier-Mukai transform. That is the aim of this thesis. A big problem is represented by the presence of singular fibres, and, since it is not clear how to handle them, we consider the simplified case when the fibration has only smooth fibres. Before outlining the content of the thesis, we mention the most relevant works on this problem.

The mirror symmetry equivalence was proved for elliptic curves by Polishchuk and Zaslow in [24], by Arinkin and Polishchuk in [2] and by Kreußler in [19]. The construction given in [2] is similar to the one presented in this thesis. However the transform is defined only for skyscraper sheaves or, in the relative setting, only for sections of the fibration, and the exposition lacks many details. A more geometric approach is exhibited in the work [21] by Leung, Yau and Zaslow. There the transform is defined for local systems supported on Lagrangian sections. They consider, when the fibration has real dimension 6, a special Lagrangian submanifolds such that its projection on the base of the fibration has real dimension 1, computing the transformed object on the mirror manifolds, showing that the transformed support is a complex submanifold of $\hat{X}$ and that the transformed vector bundle over it has a holomorphic structure. In both works the problem of singular fibres is not considered and moreover Lagrangian sections are supposed to be not ramified over the base of the family.

What is presented in this thesis is the construction of a relative Fourier-Mukai transform on a symplectic family of smooth Lagrangian tori X, yielding a correspondence between local systems supported on Lagrangian submanifolds and holomorphic vector bundles supported on complex subvarieties of the mirror $\hat{X}$ which are flat along the fibre directions of $\hat{X}$. Besides considering only fibrations with smooth fibres, we too assume that Lagrangian submanifolds of X are not ramified covers over the base of the family. Moreover to obtain a complex structure on the transformed objects we have to require further conditions on the Lagrangian submanifolds. We also prove that, with some hypotheses, the transform is invertible. This correspondence closely resembles Fukaya's homological mirror symmetry in [13] and it would be interesting, as already remarked, to compare these two constructions, besides including singular fibres.

In the first chapter we extend certain results given for line bundles over complex tori by Lange and Birkenhake in their book [20]. We will describe the group $Pic(X)$ of isomorphism classes of $U(1)$ complex bundles on a real torus in terms of factors of automorphy. More precisely a factor of automorphy is a smooth map $f : \Lambda \times V \to U(1)$ satisfying the relation

$$f(\lambda + \mu, v) = f(\lambda, v + \mu) f(\mu, v)$$

for all λ,$\mu \in \Lambda$ and $v \in V$. These factors form a group $Z(\Lambda)$ under multiplication. The factors of the form $(\lambda, v) \to h(\lambda + v)h(v)^{-1}$, for some non vanishing smooth function on V, are called boundaries and form a subgroup

$B(\Lambda)$ of $Z(\Lambda)$. The quotient $Z(\Lambda)/B(\Lambda)$, denoted with $H(\Lambda)$, turns out to be canonically isomorphic to $Pic(T)$. Now, if $P(\Lambda)$ is the set of all pairs (A, χ) where $A \in Alt^2(\Lambda, \mathbb{Z}) \cong H^2(T, \mathbb{Z})$ and χ is a semicharacter for A, then the group homomorphism $P(\Lambda) \to Pic(T)$, which associates to $(A, \chi) \in P(\Lambda)$ the line bundle generated by the factor of automorphy $a_{(A,\chi)} : \Lambda \times V \to U(1)$ given by

$$a(\lambda, v) = \chi(\lambda) e^{\pi i A(x,\lambda)}$$

is an isomorphism. This is a real version of the Appel-Humbert theorem, which says that $Pic(T)$ is an extension of $H^2(T, \mathbb{Z})$ by the group $Hom(\Lambda, U(1)) \cong Pic^0(T)$. As an example, we will show the construction of the Poincarè bundle for a real torus T. The generalization of the results in [20] ends at this point, since it is hard to extend theta functions to real tori and also it is not necessary to our purposes.

The second chapter is devoted to the construction of the Fourier-Mukai transform for a real torus T, the absolute case. This will be useful to define the Fourier-Mukai transform in the relative setting, that is for a fibration in Lagrangian tori, since local computations will be quite similar, but for the presence of the coordinates of the base of the fibrations. We outline now some ideas of this construction. The Levi-Civita connection $\nabla_{\mathcal{P}}$ of the Poincaré sheaf $\mathcal{P}$ on $T \times \hat{T}$ has a Künneth splitting into two operators

$$\nabla_1 \colon \mathcal{P} \to \mathcal{P} \otimes \Omega^{1,0}, \qquad \nabla_2 \colon \mathcal{P} \to \mathcal{P} \otimes \Omega^{0,1}$$

both squaring to zero, but with anticommutator equal to curvature of $\nabla_{\mathcal{P}}$. Given a pair $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a $\mathcal{C}^\infty_T$-module and ∇ is its flat connection, we pull it back to $T \times \hat{T}$ and couple it with the pair $(\mathcal{P}, \nabla_1)$, obtaining a complex

$$\begin{aligned} 0 \quad &\to \quad \ker \nabla_1^{\mathcal{E}} \to p^*\mathcal{E} \otimes \mathcal{P} \xrightarrow{\nabla_1^{\mathcal{E}}} p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{1,0} \\ &\xrightarrow{\nabla_1^{\mathcal{E}}} \quad p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{2,0} \to \dots \end{aligned}$$

Since locally the operator $\nabla_1^{\mathcal{E}}$ coincides with the exterior differential, this sheaf complex is exact, and is a fine resolution of the sheaf $\ker \nabla_1^{\mathcal{E}}$. This gives an isomorphism between the cohomology of the sheaf $\ker \nabla_1^{\mathcal{E}}$ and the cohomology of the complex $(\Gamma(p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{\bullet,0}), \nabla_1^{\mathcal{E}})$. When the support of $\mathcal{E}$ is a subtorus of T of dimension k, we find out that $R^j p_* \ker \nabla_1^{\mathcal{E}} = 0$ for every $j \neq k$ ($\mathcal{E}$ is said to be WIT_k), so as a sheaf, the transform of the pair $(\mathcal{E}, \nabla)$, is defined as

$$\hat{\mathcal{E}} = R^k p_* \ker \nabla_1^{\mathcal{E}}$$

If $\nabla_2^{\mathcal{E}}$ anticommutes with $\nabla_1^{\mathcal{E}}$, then a flat connection will be induced on each higher direct images. The problem is that the anticommutator of $\nabla_1^{\mathcal{E}}$ and $\nabla_2^{\mathcal{E}}$ is the curvature of the Poincaré bundle, however when $\mathcal{E}$ is supported on a subtorus the curvature of the Poincaré sheaf $\mathcal{P}$ vanishes on the product of the support of $\mathcal{E}$ and $\hat{\mathcal{E}}$. As a first step, following this procedure, we will introduce funtors $\mathbf{Sky}(T) \to \mathbf{Loc}(\hat{T})$ and $\mathbf{Loc}(\hat{T}) \to \mathbf{Sky}(T)$, where $\mathbf{Sky}(T)$ is category of skyscraper sheaves on T while $\mathbf{Loc}(\hat{T})$ is the category of flat vector bundles on $\hat{T}$, and we will show that their composition is the identity. The second step will be the extension of the transform to a local system supported on a generic affine subtorus of T. The main features of this transform can be resumed as follows: if the support of $\mathcal{E}$ is a subtorus of dimension k then $\hat{\mathcal{E}}$ is supported on a subtorus of $\hat{T}$ of dimension $g-k$ and "orthogonal" to the support of E; the support of $\hat{\mathcal{E}}$ is partly determined by ∇, while $\hat{\nabla}$ is determined by the support of $\mathcal{E}$.

The third chapter contains the construction of the Fourier-Mukai transform for a symplectic family of smooth Lagrangian subtori $X \to B$. This is similar to the construction developed in Chapter 2. Consider the split exact sequence

$$0 \to \hat{p}^*\Omega^1_{\hat{X}} \to \Omega^1_Z \xrightarrow{r} \Omega^1_{Z/\hat{X}} \to 0 \tag{4}$$

which defines the sheaf $\Omega^1_{Z/\hat{X}}$ of $\hat{p}$-relative differentials. For every sheaf $\mathcal{E}$ of $\mathcal{C}^\infty_X$-modules endowed with a flat connection ∇, one defines two differential operators:

$$\nabla^{\mathcal{E}} \colon p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^\bullet_Z \to p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{\bullet+1}_Z ,$$

obtained by coupling the pullback of the connection ∇ with the connection of the Poincaré sheaf $\nabla_{\mathcal{P}}$, and $\nabla^{\mathcal{E}}_r$, obtained by composing $\nabla^{\mathcal{E}}$ with the projection r onto the relative differentials. One has $(\nabla^{\mathcal{E}}_r)^2 = 0$, and so, as for tori, we consider the higher direct images $R^i\hat{p}_* \ker \nabla^{\mathcal{E}}_r$, which are the cohomology sheaves of the complex

$$\hat{p}_*(p^*\mathcal{E} \otimes \mathcal{P}) \xrightarrow{\nabla^{\mathcal{E}}_r} \hat{p}_*(p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^1_{Z/\hat{X}}) \xrightarrow{\nabla^{\mathcal{E}}_r} \hat{p}_*(p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^2_{Z/\hat{X}}) \to \dots \quad (5)$$

If the sheaf $\mathcal{E}$ is supported on a closed submanifold $S \subset X$ and $R^j\hat{p}_* \ker \nabla^{\mathcal{E}}_r$ is supported on a closed submanifold $\hat{S} \subset \hat{X}$, then a connection is induced on the sheaf $R^j\hat{p}_* \ker \nabla^{\mathcal{E}}_r$ when the curvature of the Poincaré bundle vanishes on $S \times_B \hat{S} \subset Z$. We first analyze the two extreme cases, that is when the submanifold is a fibre of X and when it is a Lagrangian section of X. In the first case, a local sheaf $\mathcal{L}$ on a fibre turns out to be WIT_g and it is transformed into a skyscraper sheaf, supported on the point of $\hat{X}$ parametrizing the flat line bundle corresponding to $\mathcal{L}^*$. In the second case one finds that the transform is a holomorphic line vector bundle with unitary connection on $\hat{X}$ which is flat along any fibre of $\hat{X}$, or in other words we get what may be called a holomorphic family of flat vector bundles. This sets up a bijective correspondence between local systems supported on Lagrangian sections of X and holomorphic line bundles with unitary connection, flat along the fibre directions of $\hat{X}$ (and satisfying some further conditions). The intermediate cases (when one considers a Lagrangian submanifold $S \subset X$ whose projection onto B has a dimension strictly between 0 and $\dim B$) are more involved. To get a well-behaved transform one needs to assume that S intersects the fibres X_b of X (here $b \in B$) along subtori S_b of X_b, and that the vertical tangent spaces to S undergo parallel displacement under the natural Gauss-Manin connection defined in TX. Under this condition the transform of a local system on S is a holomorphic vector bundle supported on a complex submanifold $\hat{S}$ of $\hat{X}$, which intersects the fibres of $\hat{X}$ along affine subtori of complex dimension k and such that the connection $\hat{\nabla}$ is invariant under the action of T^g on $\hat{X}$ while its horizontal part is flat. These results hold true whatever is the dimension of X, and do not require X to be Calabi-Yau (and not even complex). One should note that, when X is a Calabi-Yau manifold, the additional condition on the support S we have previously described is in general quite unrelated to the condition of S being special (in addition to being Lagrangian), and coincides with the latter only when X is complex 3-dimensional, and the projection of the Lagrangian submanifold onto the base is (real) 1-dimensional (this corresponds to a transformed sheaf which is a line bundle supported on a curve in $\hat{X}$).

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Chapter 1

LINE BUNDLES ON REAL TORI

The purpose of this chapter is the description of $U(1)$ bundles over a real torus T in terms of factors of automorphy. This theory is well known for holomorphic line bundles over a complex torus (see [20]). What we are going to do is a simple extension of this theory to real tori. As a consequence we will give an explicit construction of the Poincaré bundle over $T \times \hat{T}$, where $\hat{T}$ is the dual torus. Along all this chapter, for simplicity by line bundle we will mean a $U(1)$ line bundle

If V is g-dimensional real vector space and Λ a lattice in V, that is a discrete subgroup of rank g of V, we call a g-dimensional real torus T the quotient V/Λ.

Definition 1.0.1. *A factor of automorphy is a smooth map $f : \Lambda \times V \to U(1)$ satisfying the relation*

$$f(\lambda + \mu, v) = f(\lambda, v + \mu) f(\mu, v)$$

for all λ,$\mu \in \Lambda$ and $v \in V$

These factors form a group $Z(\Lambda)$ under multiplication.

Any factor f defines a line bundle L on T in the following way: on the trivial line bundle $V \times \mathbb{C} \to V$ consider the action of Λ

$$\lambda(v, s) = (v + \lambda, f(\lambda, v)s)$$

for $\lambda \in \Lambda$. Set

$$L = (V \times \mathbb{C})/\Lambda$$

then it can be prooved that L is a complex manifold and that the projection $L \to T$ is a complex line bundle on T. We call f the factor of automorphy of L.

The factors of the form $(\lambda, v) \to h(\lambda + v)h(v)^{-1}$, for some non vanishing smooth function h on V, are called boundaries and form a subgroup $B(\Lambda)$ of $Z(\Lambda)$. The quotient $Z(\Lambda)/B(\Lambda)$ is denoted with $H(\Lambda)$. Now let $Pic(T)$ be the group of isomorphism classes of $U(1)$ bundles on T. The following is a transposition of a result proved in [20]

Proposition 1.0.2. *There is a canonical functorial isomorphism* $Pic(T) \cong H(\Lambda)$.

The following lemmas are consequences of the previous proposition:

Lemma 1.0.3. *Let* L_1, $L_2 \in Pic(T)$ *whose associated factors of automorphy are respectively* f_1 *and* f_2, *then the factor of automorphy of* $L_1 \otimes L_2$ *is the product* $f_1 f_2$.

Proof. It follows from the fact that the isomorphism of Proposition 1.0.2 is a group homomorphism. □

Lemma 1.0.4. *Let* $\varphi : T_1 \to T_2$ *be a smooth map of real tori and* L *a smooth line bundle over* T_2 *with factor of automorphy* f, *let* $\tilde{\varphi} : V_1 \to V_2$ *be the induced map on the universal covers* V_i *of* T_i *and* $\varphi_* : \Lambda_1 \to \Lambda_2$ *be the induced map on lattices, then the factor of automorphy of* $\varphi^* L$ *is given by* $(\varphi_* \times \tilde{\varphi})^* f$.

Proof. This again is a transposition of the same result prooved for complex tori in [20]. $\square$

The factor of automorphy f associated to a line bundle L on T allows to describe the group of global sections of L. These turn out to be the smooth functions $t : V \to \mathbb{C}$ satisfying

$$t(v + \lambda) = f(\lambda, v)t(v)$$

for all $v \in V$ and $\lambda \in \Lambda$.

Remark 1.0.5. *To change the factor of automorphy of a line bundle L by a boundary is equivalent to act on L by an automorphism (or to change gauge). Indeed, an automorphism of L is induced by a map $\phi: V \to U(1)$ which changes the factor of automorphy as follows*

$$a'_L(x, \lambda) = \phi(x + \lambda)\, a_L(x, \lambda)\, \phi(x)^{-1}.$$

The isomorphism $Alt^2(\Lambda, \mathbb{Z}) \cong H^2(T, \mathbb{Z})$, where $Alt^2(\Lambda, \mathbb{Z})$ denotes the space of alternating 2-forms on Λ, allows to identify the first Chern class of a line bundle L on T with an alternating 2-form on the lattice Λ. If $A \in Alt^2(\Lambda, \mathbb{Z})$, a semicharacter for A is a map $\chi : \Lambda \to U(1)$ such that

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{\pi i A(\lambda, \mu)}$$

for all $\lambda, \mu \in \Lambda$. Call $P(\Lambda)$ the set of all pairs (A, χ) where $A \in Alt^2(\Lambda, \mathbb{Z})$ and χ is a semicharacter for A. With respect to the composition $(A_1, \chi_1) \circ (A_2, \chi_2) = (A_1 + A_2, \chi_1\chi_2)$, it is easy to see that $P(\Lambda)$ has a group structure. Consider the group homomorphism $P(\Lambda) \to Pic(T)$ defined as follows: to each $(A, \chi) \in P(\Lambda)$ we associate the line bundle generated by the factor of automorphy $a_{(A,\chi)} : \Lambda \times V \to U(1)$ given by

$$a(\lambda, v) = \chi(\lambda)e^{\pi i A(x, \lambda)}$$

As for complex tori, provided that we define $Pic^0(T)$ as the dual torus $\hat{T}$, it is possible to prove the Appell-Humbert theorem, which says that $Pic(T)$ is an extension of $H^2(T,\mathbb{Z})$ by the group $Hom(\Lambda, U(1)) \cong Pic^0(T)$. This means that we do not consider only the $\mathcal{C}^\infty$ structure over line bundles on T, but also a flat connection, otherwise $Pic^0(T)$ would be trivial.

Theorem 1.0.6. *There is a canonical isomorphism of exact sequences*

$$\begin{tikzcd} 1 \arrow[r] & Hom(\Lambda, U(1)) \arrow[r] \arrow[d, "\cong"'] & P(\Lambda) \arrow[r] \arrow[d, "\cong"'] & H^2(T,\mathbb{Z}) \arrow[r] \arrow[d, "="'] & 0 \\ 1 \arrow[r] & Pic^0(T) \arrow[r] & Pic(T) \arrow[r] & H^2(T,\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

Proof. It is enough to show the isomorphism $Hom(\Lambda, U(1)) \cong Pic^0(T)$. We have defined $Pic^0(T)$ as the dual torus $\hat{T}$ which is the quotient $\Omega/\hat{\Lambda}$, where $\Omega = Hom(V,\mathbb{R})$ and $\hat{\Lambda} = \{\psi \in \Omega | \psi(\Lambda) \subseteq \mathbb{Z}\}$. The canonical morphism $\Omega \to Hom(\Lambda, U(1))$, given by $\psi \to exp(2\pi i \psi)$, since it is surjective and has $\hat{\Lambda}$ as its kernel, induces an isomorphism $\hat{T} \to Pic^0(X)$. □

So, given a line bundle L with $c_1(L) = A$, there is a semicharacter χ for A, such that $a_{(A,\chi)}$ is a factor of automorphy of L called the canonical factor of automorphy. For flat line bundles the canonical factor of automorphy reduces to a character, that is a semicharacter corresponding to $A = 0$. If y is a point of $\hat{T}$, then the flat line bundle L_y on T parametrized by y has factor of automorphy given by

$$a_y = e^{2\pi i y(\lambda)}$$

Even in the real case, it is possible to prove the existence of a Poincaré bundle for T, that is a line bundle $\mathcal{P}$ on the product $T \times \hat{T}$ satisfying

$$\mathcal{P}\mid_{X\times\{y\}} \cong L_y \quad \forall y \in \hat{T} \qquad \text{and}$$

$$\mathcal{P}\mid_{\{0\}\times\hat{T}} \quad \text{is trivial}$$

The next theorem gives an explicit construction.

Theorem 1.0.7. *There exists a Poincaré bundle $\mathcal{P}$ on $T \times \hat{T}$, uniquely determined up to isomorphisms.*

Proof. Consider the pair $(A, \chi) \in P(\Lambda \times \hat{\Lambda})$ where

$$A((\lambda_1, \mu_1), (\lambda_2, \mu_2) = \mu_1(\lambda_2) - \mu_2(\lambda_1)$$
$$\chi(\lambda, \mu) = e^{i\pi\mu(\lambda)}$$

According to the Appell-Humbert theorem this pair defines a line bundle $\mathcal{P}$ on $T \times \hat{T}$ with factor of automorphy given by

$$a_{\mathcal{P}}(x, y, \lambda, \mu) = e^{\pi i[y(\lambda) - \mu(x) - \mu(\lambda)]}$$

We are showing that $\mathcal{P}$ is a Poincaré bundle for $T \times \hat{T}$. To this purpose it is convenient to apply the automorphism induced by the map

$$\phi \colon V \times \hat{V} \to U(1), \qquad \phi(x, y) = e^{\pi i\, y(x)}$$

thus obtaining a new factor of automorphy

$$a'_{\mathcal{P}}(x, y, \lambda, \mu) = e^{2\pi i\, y(\lambda)}. \tag{1.1}$$

This description of the Poincaré bundle shows explicitly that $\mathcal{P}_{|T\times\{y\}} \cong L_y$. If instead we act on $a_{\mathcal{P}}$ with the automorphism $\phi(x, y) = e^{-i\pi\, y(x)}$ we obtain the factor of automorphy $a''_{\mathcal{P}}(x, y, \lambda, \mu) = e^{-2i\pi\, \mu(x)}$ which shows that the second property defining the Poincaré bundle holds for $\mathcal{P}$. □

Remark 1.0.8. *Note that since the factor of automorphy $a_{\mathcal{P}}$ and $a''_{\mathcal{P}}$ are equivalent, it follows that after the identification $\hat{\hat{T}} \simeq T$, the dual bundle $\mathcal{P}^\vee$ is a Poincaré bundle for $\hat{T} \times T$.*

Let $\nabla_{\mathcal{P}}$ be the connection of $\mathcal{P}$. The universality property of this connection implies that its connection form $\mathbb{A}$, in the gauge where the factor of automorphy of $\mathcal{P}$ has the form $a''_{\mathcal{P}}$, is written as

$$\mathbb{A} = 2i\pi \sum_{j=1}^{g} x^j\, dy_j \tag{1.2}$$

while its curvature is

$$\mathbb{F} = 2i\,\pi \sum_{j=1}^{g} dx^j \wedge dy_j \tag{1.3}$$

where $(x^1, \ldots, x^g)$ are flat coordinates on T and $(y_1, \ldots, y_g)$ are dual flat coordinates on $\hat{T}$. The restriction $\nabla_{\mathcal{P}|\{x\}\times\hat{T}}$ is the flat connection of L_x on $\hat{T}$.

Chapter 2

FOURIER-MUKAI TRANSFORM FOR REAL TORI

The second chapter is devoted to the study of a Fourier-Mukai transform for a real g-dimensional torus T. The motivation lies in the fact that this is the first step towards the construction of a Fourier-Mukai transform for a fibration whose fibre are smooth real tori. The objects that we want to transform are subtori of T together with a smooth vector bundle endowed with a flat unitary connection, including the limit case when the subtorus is the whole torus T or it reduces to a point supporting a skyscraper sheaf.

Consider the Poincaré sheaf $\mathcal{P}$ on $T\times\hat{T}$ and denote by p, $\hat{p}$ the projections onto the two factors of $T\times\hat{T}$. To simplify the notation we shall set

$$\Omega^{m,n} = \Omega^m_T \boxtimes \Omega^n_{\hat{T}} \equiv p^*\Omega^m_T \otimes_{\mathcal{C}^\infty_{T\times\hat{T}}} \hat{p}^*\Omega^n_{\hat{T}}$$

where p^* denotes the pullback of $\mathcal{C}^\infty$-modules, i.e.

$$p^*\mathcal{E} = p^{-1}\mathcal{E} \otimes_{p^{-1}\mathcal{C}^\infty_T} \mathcal{C}^\infty_{T\times\hat{T}},$$

and similarly for $\hat{p}^*$. Remember that in general the box product $\boxtimes$, for sheaves of modules over the structure sheaf of a differentiable manifold, is defined as follows: if X, Y are differentiable manifolds, and $\mathcal{F}$, $\mathcal{G}$ are sheaves of $\mathcal{C}^\infty_X$- and $\mathcal{C}^\infty_Y$-modules, respectively, then

$$\mathcal{F} \boxtimes \mathcal{G} = p^*\mathcal{F} \otimes_{\mathcal{C}^\infty_{X\times Y}} q^*\mathcal{G}\,,$$

where p and q are the projections onto the two factors of $X \times Y$. The connection $\nabla_\mathcal{P}$ of $\mathcal{P}$ has a Künneth splitting into two operators

$$\nabla_1 \colon \mathcal{P} \to \mathcal{P} \otimes \Omega^{1,0}, \qquad \nabla_2 \colon \mathcal{P} \to \mathcal{P} \otimes \Omega^{0,1}$$

both squaring to zero (but their anticommutator is the curvature of $\nabla_\mathcal{P}$). The action of ∇_1, ∇_2 on sections is locally written in the form

$$\nabla_1 f = \sum_{j=1}^{g} \frac{\partial f}{\partial x^j}\, dx^j, \qquad \nabla_2 f = \sum_{j=1}^{g} \left(\frac{\partial f}{\partial y_j} - 2i\,\pi\, x^j\, f \right) dy_j,$$

or

$$\nabla_1 f = \sum_{j=1}^{g} \left(\frac{\partial f}{\partial x^j} + 2i\,\pi\, y_j\, f \right) dx^j, \qquad \nabla_2 f = \sum_{j=1}^{g} \frac{\partial f}{\partial y_j}\, dy_j,$$

and we switch from the former to the latter simply acting on $\mathcal{P}$ with an automorphism as already done in the proof of theorem 1.0.7.

Let $\mathcal{E}$ be a $\mathcal{C}^\infty_T$-module with a flat connection ∇, where by a connection on a $\mathcal{C}^\infty_T$-module $\mathcal{E}$ (not necessarily locally free) we mean a map $\nabla \colon \mathcal{E} \to \mathcal{E} \otimes \Omega^1_T$ satisfying the Leibniz rule

$$\nabla(fs) = f\nabla(s) + s \otimes df.$$

The purpose is to develop a procedure to construct a $\mathcal{C}^\infty_{\hat{T}}$-module $\hat{\mathcal{E}}$ with a flat unitary connection $\hat{\nabla}$ on $\hat{T}$. Of course, this procedure does not work for every $\mathcal{C}^\infty_T$-module. In other words, we will have to determine on which categories to define the functor. By pulling the pair $(\mathcal{E}, \nabla)$ back to $T \times \hat{T}$ and coupling it with the pair $(\mathcal{P}, \nabla_1)$ we obtain a complex

$$\begin{aligned} 0 \quad \to \quad & \ker \nabla_1^{\mathcal{E}} \to p^*\mathcal{E} \otimes \mathcal{P} \xrightarrow{\nabla_1^{\mathcal{E}}} p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{1,0} \\ \xrightarrow{\nabla_1^{\mathcal{E}}} \quad & p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{2,0} \to \dots \end{aligned}$$

Since locally the operator $\nabla_1^{\mathcal{E}}$ coincides with the exterior differential, this sheaf complex is exact, and is a fine resolution of the sheaf $\ker \nabla_1^{\mathcal{E}}$. Thus we obtain an isomorphism

$$H^i(T \times \hat{T}, \ker \nabla_1^{\mathcal{E}}) \simeq H^i(\Gamma(p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{\bullet,0}), \nabla_1^{\mathcal{E}}), \qquad i \geq 0$$

between the cohomology of the sheaf $\ker \nabla_1^{\mathcal{E}}$ and the cohomology of the complex $(\Gamma(p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{\bullet,0}), \nabla_1^{\mathcal{E}})$, where Γ is the global sections functor. Similar results hold for the operator $\nabla_2^{\mathcal{E}}$. Consider the higher direct images $R^j p_* \ker \nabla_1^{\mathcal{E}}$. To compute these sheaves, by definition of direct image, we have to study the presheaves

$$U \rightsquigarrow H^i(T \times \hat{U}, \ker \nabla_1^{\mathcal{E}}) \simeq H^i(\Gamma(p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{\bullet,0})(T \times \hat{U}), \nabla_1^{\mathcal{E}})$$

where $\hat{U}$ is an open subset of $\hat{T}$.

Definition 2.0.9. *The pair $(\mathcal{E}, \nabla)$ is said to be WIT_k if $R^i \hat{p}_* \ker \nabla_1^{\mathcal{E}} = 0$ for $i \neq k$.*

The operator $\nabla_2^{\mathcal{E}}$ defines a connection on each higher direct images provided that it anticommutes with $\nabla_1^{\mathcal{E}}$. The anticommutator $\nabla_1^{\mathcal{E}} \circ \nabla_2^{\mathcal{E}} + \nabla_2^{\mathcal{E}} \circ \nabla_1^{\mathcal{E}}$, as an operator $p^*\mathcal{E} \otimes \mathcal{P} \to p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^2_{T \times \hat{T}}$, coincides with $1 \otimes \mathbb{F}$, where $\mathbb{F}$ is the curvature of the Poincaré bundle. This is clear from local computations (cf. equation (1.3))

$$\nabla_1^{\mathcal{E}} \circ \nabla_2^{\mathcal{E}} f + \nabla_2^{\mathcal{E}} \circ \nabla_1^{\mathcal{E}} f =$$

$$= \sum_{j,k=1}^{g} \left(\frac{\partial f}{\partial x^j \partial y_k} + 2i\,\pi\, f + 2i\,\pi\, y_j\, \frac{\partial f}{\partial x^j} - \frac{\partial f}{\partial x^j \partial y_k} - 2i\,\pi\, y_j\, \frac{\partial f}{\partial x^j} \right) dx^j \wedge dy_k =$$

$$= 2i\pi\, f dx^j \wedge dy_k$$

As a consequence of this, we can state the following proposition

Proposition 2.0.10. *Assume that the sheaf $\mathcal{E}$ is supported on a closed submanifold $S \subset T$, the sheaf $R^j\hat{p}_* \ker \nabla_1^{\mathcal{E}}$ is supported on a closed submanifold $\hat{S} \subset \hat{T}$, and the curvature operator $\mathbb{F}$ vanishes on $S \times \hat{S} \subset Z$. Then the operator $\hat{\nabla}_2^{\mathcal{E}}$ induces a connection on the sheaf $R^j\hat{p}_* \ker \nabla_1^{\mathcal{E}}$.*

Definition 2.0.11. *If the pair $(\mathcal{E}, \nabla)$ is WIT_k and satisfies the condition in Proposition 2.0.10, the pair $(\hat{\mathcal{E}}, \hat{\nabla})$, where $\hat{\mathcal{E}} = R^k p_* \ker \nabla_1^{\mathcal{E}}$ and $\hat{\nabla}$ is the connection induced as in Proposition 2.0.10, is called the Fourier-Mukai transform of $(\mathcal{E}, \nabla)$. We denote the transform by $\mathcal{F}$.*

There is a class of submanifolds of T which satisfies theorem 2.0.10 and which we are going to define.

Definition 2.0.12. *Let $T = V/\Lambda$ be a real torus, where V is a real vector space and Λ a lattice in V. An affine subtorus S of T is a subset of the form $U/(U \cap \Lambda) + x$ for some $x \in T$ and some vector subspace U of V such that $U \cap \Lambda$ is a maximal sublattice of U. For simplicity, we will call subtorus an affine subtorus.*

We introduce now the categories we will be concerned with.

Definition 2.0.13. *(1) the category $\mathbf{Mod}(\mathbb{C}_T)$ of $\mathbb{C}_T$-modules, where $\mathbb{C}_T$ is the constant sheaf on T*
(1a) the subcategory $\mathbf{Sky}(T)$ of skyscrapers of total finite length (i.e., $\dim H^0(T, M) < \infty$ for all $M \in \mathrm{Ob}(\mathbf{Sky}(T))$)
(1b) the category $\mathbf{Loc}(T)$ of unitary local systems on T (see [9]). Objects of this category are pairs (E, ∇), where E is a smooth complex vector bundle on T, and ∇ is a flat unitary connection. Morphisms in this category are vector bundle morphisms compatible with the connections. The objects in $\mathbf{Loc}(T)$

can also be regarded as locally free $\mathbb{C}_T$-modules of finite rank equipped with a hermitian metric defined up to homothety (cf. [8])

More generally, we can consider unitary local systems supported on affine subtori and define the categories $\mathbf{Loc}_k(T)$, for $k = 0, ..., g$, in such a way that $\mathbf{Sky}(T) \cong \mathbf{Loc}_0(T)$ and $\mathbf{Loc}(T) \cong \mathbf{Loc}_g(T)$.

Definition 2.0.14. *We call $\mathbf{Loc}_k(T)$ the category of $U(1)$ local systems supported on affine subtori of T of dimension k. Objects of this category are triples (S, E, ∇) (where S is an affine subtorus in T, E a line bundle on S, and ∇ a flat unitary connection on E) modulo isomorphisms, i.e., modulo vector bundle isomorphisms which commute with the actions of the connections (the two line bundles having the same support). The space of morphisms between two objects (S_1, E_1, ∇_1) and (S_2, E_2, ∇_2) of $\mathbf{Loc}_k(T)$ is defined by taking into account that the intersection $S = S_1 \cap S_2$ is a (possibly empty) finite collection of (possibly zero-dimensional) affine tori R_i, and one sets*

$$\mathrm{Mor}((S_1, E_1, \nabla_1), (S_2, E_2, \nabla_2)) = \bigoplus_i \mathrm{Mor}_\nabla((R_i, E_1, \nabla_1), (R_i, E_2, \nabla_2)),$$

where $\mathrm{Mor}_\nabla((R_i, E_1, \nabla_1), (R_i, E_2, \nabla_2))$ is the set of morphisms between $E_{1|R_i}$ and $E_{2|R_i}$ compatible with the connections ∇_1 and ∇_2.

To build a real Fourier-Mukai transform $\mathcal{F}$ for real tori we start from the easiest case, that is we apply the construction just outlined to the category $\mathbf{Sky}(T)$. As a result we will have defined a functor $\mathcal{F}\colon \mathbf{Sky}(T) \to \mathbf{Loc}(\hat{T})$. Let M be a an object of $\mathbf{Sky}(T)$, and denote by $\mathcal{M}$ the corresponding $\mathcal{C}_T^\infty$-module, with multiplication given by evaluation of functions. In this case we have $\ker \nabla_1 = p^*\mathcal{M} \otimes \mathcal{P}$, and the higher direct images vanish apart from R^0, in other words, M is WIT_0. Moreover

$$\hat{p}_*(p^*\mathcal{M} \otimes \mathcal{P})$$

is locally free of finite rank, so it is the sheaf of sections $\hat{\mathcal{E}}$ of a vector bundle $\hat{E}$ on $\hat{T}$, with $\operatorname{rk}\hat{E} = \operatorname{length}(M)$. The operator ∇_2 naturally extends to $p^*\mathcal{M} \otimes \mathcal{P}$, and since the latter sheaf is supported on $\{x\} \times \hat{T}$, it induces an operator $\nabla : \hat{E} \to \hat{E} \otimes \Omega^1_{\hat{T}}$ which is a flat unitary connection.

Example 2.0.15. *Let $\mathbb{C}(0)$ denote the one-dimensional skyscraper at $0 \in T$. One has $\mathcal{F}(\mathbb{C}(0)) \simeq \mathbb{C}_{\hat{T}}$. Indeed, in this case we have $p^*\mathcal{M} \otimes \mathcal{P} \simeq \mathcal{C}^\infty_{\{0\}\times\hat{T}}$, while the operator ∇_2 reduces on this sheaf to the exterior differential along the $\hat{T}$ direction. As a consequence, $(\hat{\mathcal{E}}, \hat{\nabla}) = (\mathcal{C}^\infty_{\hat{T}}, d)$, and $\mathcal{F}(\mathbb{C}(0)) = \ker d \simeq \mathbb{C}_{\hat{T}}$.*

For every $x \in T$ let t_x be the associated translation, $t_x(x') = x + x'$. Moreover, identify $\hat{\hat{T}}$ with T.

Proposition 2.0.16. *For every $x \in T$ and $M \in \mathrm{Ob}(\mathbf{Sky}(T))$ there is an isomorphism $\mathcal{F}(t_x^{-1}M) \simeq L_{-x} \otimes \mathcal{F}(M)$.*

Proof. It is a consequence of the following property of the Poincaré bundle (see [23]): $t_{(x,0)}^{-1}\mathcal{P} \simeq \mathcal{P} \otimes \hat{\pi}^{-1}L_x$ as follows from lemma 1.0.3 and 1.0.4. □

As a consequence, in view of Example 2.0.15, we have

Corollary 2.0.17. *For every $x \in T$ one has $\mathcal{F}(\mathbb{C}(x)) \simeq L_{-x}$.*

This defines the action of $\mathcal{F}$ on the whole category $\mathbf{Sky}(T)$.

The morphisms transform as morphisms of sheaves. Observe that the situation is not so involved since $\mathrm{Hom}(\mathbb{C}(x), \mathbb{C}(y))$ and $\mathrm{Hom}(\mathfrak{L}_x, \mathfrak{L}_y)$ are empty when $x \neq y$. We have only to check that the transformed morphism commutes with the transformed connection This is clear because if φ is a morphism of a skyscraper sheaf, then $p^*\varphi$, being supported on $\{x\} \times \hat{T}$ in $T \times \hat{T}$, commutes with ∇_2.

Therefore this procedure does define a functor.

It is not clear how to define an inverse for the functor $\mathcal{F}$ by means of the adjunction theory for $\mathbb{C}$-modules. The next step is a direct construction of a functor $\hat{\mathcal{F}}: \mathbf{Loc}(\hat{T}) \to \mathbf{Sky}(T)$ which inverts the functor $\mathcal{F}$. Let (E, ∇) be an object in $\mathbf{Loc}(\hat{T})$, let $\mathcal{E}$ be the sheaf of sections of E. Reverting the roles of T and $\hat{T}$, we consider on the sheaf $\hat{p}^*\mathcal{E} \otimes \mathcal{P}^\vee$ an operator ∇_2^E obtained by coupling the pullback of ∇ with the operator ∇_2. The following result characterizes the functor $\hat{\mathcal{F}}$.

Proposition 2.0.18. *1. $R^j p_* \ker \nabla_2^{\mathcal{E}} = 0$ for $j = 0, \ldots, g-1$ ($\mathcal{E}$ is WIT_g)*
2. The sheaf $R^g p_* \ker \nabla_2^{\mathcal{E}}$ is a skyscraper of finite length.

Hence the functor $\hat{\mathcal{F}}$ is defined as $\hat{\mathcal{F}}((E, \nabla)) = R^g p_* \ker \nabla_2^E$.

Proof. As a first step we compute the action of $\hat{\mathcal{F}}$ on the trivial line bundle, i.e, we take $\mathcal{E} = \mathcal{C}^\infty_{\hat{T}}$ and $\nabla = d$. Thus we want to compute the sheaves $R^j p_* \ker \nabla_2$, and, as explained, we have to study the presheaves

$$U \rightsquigarrow H^j(U \times \hat{T}, \ker \nabla_2) \simeq H^j\left((\mathcal{P}^\vee \otimes \Omega^{0,\bullet})(U \times \hat{T}), \nabla_2\right)$$

Proposition 2.0.19. *$H^0(U \times \hat{T}, \ker \nabla_2) = 0$ for all open sets $U \subset T$, so that $p_* \ker \nabla_2 = 0$.*

Proof. An element of $H^0(U \times \hat{T}, \ker \nabla_2)$ restricted to $\{x\} \times \hat{T}$, with $x \in U$, yields a global section of the flat line bundle L_x, which is zero unless $x = 0$. By a density argument we get the result. □

To compute the higher-order direct images we first consider the case $g = 1$.

Proposition 2.0.20. *If $g = 1$, then $R^1 p_* \ker \nabla_2 \simeq k(0)$.*

Proof. We compute the cohomology groups $H^1(U \times \hat{T}, \ker \nabla_2) \simeq H^1((\mathcal{P}^\vee \otimes \Omega^{0,\bullet})(U \times \hat{T}), \nabla_2)$. We represent T as $\mathbb{R}/\mathbb{Z}\lambda$, with $\lambda \in \mathbb{R}\backslash\{0\}$, and $\hat{T}$ as $\mathbb{R}/\mathbb{Z}\mu$, with $\mu = 1/\lambda$. Let W be the inverse image of U in $\mathbb{R}$. We work now in a gauge where the factor of automorphy of $\mathcal{P}^\vee$ is $e^{2i\pi\,\mu(x)}$, and the operator ∇_2 is the $\hat{T}$-part of the exterior differential. An element in $((\mathcal{P}^\vee \otimes \Omega^{0,1})(U \times \hat{T}), \ker \nabla_2)$ may be written as $\tau = t(x,y)\,dy$, where t is a function on $W \times V^\vee$ satisfying the automorphy condition

$$t(x, y+\mu) = t(x,y)\,e^{2i\pi\,\mu(x)}.$$

Since $g = 1$ clearly τ is closed. If τ is a coboundary, $\tau = \nabla_2 s$, one has

$$s(x,y) = \int_0^y t(x,u)\,du + c(x).$$

The function s must satisfy the automorphy condition

$$s(x, y+\mu) = \int_0^{y+\mu} t(x,u)\,du + c(x) = \int_0^{\mu} t(x,u)\,du + \int_\mu^{y+\mu} t(x,u)\,du + c(x) =$$

changing varaibles $u = v + \mu$

$$= \int_0^{\mu} t(x,u)\,du + \int_0^{y} t(x,v+\mu)\,dv + c(x) =$$

and using the automorphy condition for t we obtain

$$= e^{2i\pi\,\mu(x)} \int_0^{y} t(x,v)\,dv + \int_0^{\mu} t(x,u)\,du + c(x) =$$

$$= e^{2i\pi\,\mu(x)}(s(x,y) - c(x,y)) + \int_0^{\mu} t(x,u)\,du + c(x) =$$

$$= e^{2i\pi\,\mu(x)} s(x,y) + c(x)(1 - e^{2i\pi\,\mu(x)}) + \int_0^{\mu} t(x,u)\,du$$

hence the automorphy condition for s amounts to the following condition on c:

$$c(x)(1 - e^{2i\pi\,\mu(x)}) = -\int_0^{\mu} t(x,u)\,du. \tag{2.1}$$

If $0 \notin U$ this condition may be solved for c, which means that τ is a coboundary, so that $H^1(U \times \hat{T}, \ker \nabla_2) = 0$. Thus $R^1 p_* \ker \nabla_2$ is a skyscraper supported at $0 \in T$. If $0 \in U$, the condition (2.1) may be solved if and only if

$$\int_0^\mu t(0,u)\,du = 0$$

this gives a surjective map $((\mathcal{P}^\vee \otimes \Omega^{0,1})(U \times \hat{T}), \ker \nabla_2) \to \mathbb{C}$ whose kernel are the coboudaries, so that $H^1(U \times \hat{T}, \ker \nabla_2) \simeq \mathbb{C}$. This proves the claim. $\square$

We move to the higher-dimensional case by means of a Künneth-type argument.

Proposition 2.0.21. *If* $\dim T = g$ *we have*

1. $R^j p_* \ker \nabla_2 = 0$ *for* $j = 0, \ldots, g-1$*;*

2. $R^g p_* \ker \nabla_2 \simeq k(0)$.

Proof. A choice of flat coordinates $(x^1, \ldots, x^g)$ on T fixes an isomorphism $T \simeq S^1 \times \ldots S^1$. The Poincaré sheaf $\mathcal{P}$ on $T \times \hat{T}$ is the box product of the Poincaré sheaves $\mathcal{P}_i$ on the i factors of $T \times \hat{T}$, as one can check for instance by describing the Poincaré bundles by their factors of automorphy. Let $U \subset T$ be of the form $U = U_1 \times \ldots \times U_g$ where each U_i lies in a factor of $\hat{T}$. If $g = 2$, a word-by-word translation of the Künneth theorem for de Rham cohomology (cf. e.g. [6]) gives a decomposition

$$H^j(U \times \hat{T}, \ker \nabla_2) = \bigoplus_{m+n=j} H^m(U_1 \times S^1, \ker \nabla_2^1) \otimes H^n(U_2 \times S^1, \ker \nabla_2^2)$$

whence we have, by Proposition 2.0.19,

$$H^j(U \times \hat{T}, \ker \nabla_2) = 0 \quad \text{for} \quad j = 0, 1, \quad H^2(U \times \hat{T}, \ker \nabla_2) \simeq \mathbb{C}.$$

Induction on g then yields, for every g,

$$H^j(U \times \hat{T}, \ker \nabla_2) = 0 \quad \text{for} \quad j = 0, \ldots, g-1, \quad H^g(U \times \hat{T}, \ker \nabla_2) \simeq \mathbb{C}.$$

This proves both claims. $\square$

We have also obtained

$$H^j(T \times \hat{T}, \ker \nabla_2) = \begin{cases} 0 & \text{for} \quad j = 0, \dots, g-1, \\ \mathbb{C} & \text{for} \quad j = g. \end{cases}$$

The $\mathcal{C}^\infty(T)$-module structure of the g-th cohomology group is given by $f \cdot \alpha = f(0)\,\alpha$. This concludes the proof of proposition 2.0.18. $\square$

Let $\mathfrak{L}_x$ be the local system corresponding to the line bundle L_x with its flat connection. In analogy with Proposition 2.0.16, we have

Proposition 2.0.22. *$\hat{\mathcal{F}}(\mathfrak{L}_{-x} \otimes_{\mathbb{C}_{\hat{T}}} \mathfrak{S}) \simeq t_x^{-1}\hat{\mathcal{F}}(\mathfrak{S})$ for every $x \in T$ and every local system $\mathfrak{S}$ on $\hat{T}$.*

Corollary 2.0.23. *$\hat{\mathcal{F}}(\mathfrak{L}_{-x}) \simeq \mathbb{C}(x)$ for every $x \in T$.*

Remark 2.0.24. *Since any flat vector bundle on a torus is a direct sum of flat line bundles (i.e., every local system on $\hat{T}$ is a direct sum of local systems of the type $\mathfrak{L}_x$), Corollary 2.0.23 completely describes the action of the functor $\hat{\mathcal{F}}$.*

Nothing to prove as to the behaviour of morphisms under the functor $\hat{\mathcal{F}}$: in fact these transform as morphisms of sheaves and moreover both $\operatorname{Hom}(\mathbb{C}(x), \mathbb{C}(y))$ and $\operatorname{Hom}(\mathfrak{L}_x, \mathfrak{L}_y)$ are empty when $x \neq y$. Corollaries 2.0.17 and 2.0.23 and Remark 2.0.24 eventually prove

Theorem 2.0.25. *The functors $\mathcal{F}$, $\hat{\mathcal{F}}$ are inverse to each other, and establish an equivalence between the categories $\mathbf{Sky}(T)$ and $\mathbf{Loc}(\hat{T})$.*

After the easiest case, we want to extend the transform to $U(1)$ local systems supported on subtori of T of general dimension, that is to define the transform on the categories $\mathbf{Loc}_k(T)$. So let $\mathfrak{L}$ be a $U(1)$ local system on a k-dimensional affine subtorus S of T, and let $\mathcal{L} = \mathfrak{L} \otimes_{\mathbb{C}} \mathcal{C}_S^\infty$ be the corresponding flat line bundle on S. By restricting the sheaves $\mathcal{P} \otimes \Omega^{m,0}$ to the closed submanifold $S \times \hat{T} \subset T \times \hat{T}$ one obtains, as already explained, a complex

$$\begin{aligned} 0 \quad &\rightarrow \quad \ker \nabla_1^{\mathcal{L}} \rightarrow p^* j_* \mathcal{L} \otimes \mathcal{P}_{|S \times \hat{T}} \xrightarrow{\nabla_1^{\mathcal{L}}} p^* j_* \mathcal{L} \otimes (\mathcal{P} \otimes \Omega^{1,0})_{|S \times \hat{T}} \\ &\xrightarrow{\nabla_1^{\mathcal{L}}} \quad p^* j_* \mathcal{L} \otimes (\mathcal{P} \otimes \Omega^{2,0})_{|S \times \hat{T}} \rightarrow \dots \end{aligned}$$

where $j : S \hookrightarrow T$ is the natural embedding.

Proposition 2.0.26. *1. $R^j \hat{p}_* \ker \nabla_1^{\mathcal{L}} = 0$ for all $j \neq k$ (so $\mathfrak{L}$ is WIT_k);*

2. $R^k \hat{p}_* \ker \nabla_1^{\mathcal{L}}$ is supported by a $(g-k)$-dimensional affine subtorus $\hat{S}$ of $\hat{T}$, which is normal to S;

3. if $\mathcal{L}$ is trivial, then $\hat{S}$ goes through the origin of $\hat{T}$, otherwise it is an affine subtorus translated by the element of $\hat{T}$ corresponding to $\mathcal{L}^*$.

4. The sheaf $R^k \hat{p}_* \ker \nabla_1^{\mathcal{L}}$ is a line bundle, and has a flat connection which makes it into a local system $\hat{\mathfrak{L}}$.

Proof. We first consider the case when S is a 1 dimensional subtorus of T. The direct images $R^i \hat{p}_* (\ker \nabla_1^{\mathcal{L}})$ we want to compute are by definition the sheaf associated to the presheaf

$$\hat{U} \rightsquigarrow H^i(S \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) \simeq H^i \left(\Omega^{\bullet,0}(p^* L \otimes \mathcal{P})(S \times \hat{U}), \nabla_1^{\mathcal{L}} \right)$$

where $\hat{U}$ is an open subset of the dual torus $\hat{T}$.
For $i = 0$, take an element s in $H^0(S \times \hat{U}, \ker \nabla_1^{\mathcal{L}})$ and consider, for $y \in \hat{U}$, its restriction to $S \times \{y\}$, which is a global section of $\mathcal{L} \otimes \mathcal{P}_{S \times \{y\}}$. A line bundle over T is a representation of $\pi_1(T)$ in $U(1)$, that is, an element of $Hom(\pi_1(T), U(1)) \cong \hat{T}$. The embedding $S \hookrightarrow T$ gives a map $\pi_1(S) \rightarrow \pi_1(T)$, which, considering the long exact sequence in homotopy associated to (S,T) and dualizing, induces a map $\psi : \hat{T} \rightarrow Hom(\pi_1(S), U(1))$ such

that $ker\psi = Hom(\Lambda/\Lambda_S, U(1))$, where Λ_S is the lattice associated to S. Its decomposition in a free part and a torsion part is

$$Hom(\Lambda_f, U(1)) \times Hom(\Lambda_t, U(1))$$

so, since $rk\Lambda_f < rk\Lambda$ and the torsion is finite, it follows that $ker\psi$ is a non dense subgroup of $\hat{T}$. This means that $\mathcal{P}_{S\times\{y\}}$ is non trivial for every y in the complement of $ker\psi$, which is a dense subset of $\hat{T}$. The same holds for $\mathcal{L}\otimes\mathcal{P}_{S\times\{y\}}$. Since $\nabla_1^{\mathcal{L}} |_{S\times\{y\}} s |_{S\times\{y\}}= 0$ for every $y \in \hat{T}$, the restriction of s to $S \times \{y\}$ vanishes in a dense subset of $\hat{T}$. The continuity of s implies that s vanishes everywhere.

When $i = 1$ we need to write the equation of S explicitly. In order to simplify the computations involved we will restrict to the case when $\dim T = 2$. Let x^i be flat coordinates on T and y_i the dual coordinates on $\hat{T}$. We pick a gauge where the Poincaré bundle has factor of automorphy

$$a_{\mathcal{P}}(x^1, x^2, y_1, y_2, \lambda_1, \lambda_2, \mu_1, \mu_2) = e^{2\pi i(\lambda_1 y_1+\lambda_2 y_2)}.$$

The equation of S in the universal cover of T is given by the affine line $x^2 = ax^1 + \chi$. Let $A = \xi_1 dx^1$ be the 1-form connection of the local system $(\mathcal{L}, \nabla^{\mathcal{L}})$ on S. We need to compute $H^1(S \times \hat{U}, \ker \nabla_1^{\mathcal{L}})$. So take an element $\tau \in (\Omega^{1,0}(p^*\mathcal{L}\otimes\mathcal{P})(S\times\hat{U}), \nabla_1^{\mathcal{L}})$ which can be written as $t_1(x^1, x^2, y_1, y_2)dx^1 + t_2(x^1, x^2, y_1, y_2)dx^2$. Observe that τ is exact with respect to $\nabla_1^{\mathcal{L}}$ because $\dim S = 1$. The automorphy condition satisfied by τ can be expressed with respect to the natural coordinate ζ on S in the following way: having set $a = q/p$ with p, q coprime, the relations between ζ and p, q are

$$\zeta = \frac{x^1\sqrt{p^2+q^2}}{p} = \frac{x^2\sqrt{p^2+q^2}}{q}$$

and substituting in τ we obtain

$$\tau(x^1, x^2, y_1, y_2) = \frac{p}{\sqrt{p^2+q^2}} t_1\Big(\frac{p\zeta}{\sqrt{p^2+q^2}}, \frac{q\zeta}{\sqrt{p^2+q^2}}, y_1, y_2\Big)d\zeta +$$

$$\frac{q}{\sqrt{p^2+q^2}}t_2\Big(\frac{p\zeta}{\sqrt{p^2+q^2}},\frac{q\zeta}{\sqrt{p^2+q^2}},y_1,y_2\Big)d\zeta \equiv \phi(\zeta,y_1,y_2)d\zeta$$

so the automorphy condition becomes

$$\phi(\zeta+\sqrt{p^2+q^2},y_1,y_2)=$$

$$=\frac{p}{\sqrt{p^2+q^2}}t_1\Big(\frac{p\zeta}{\sqrt{p^2+q^2}}+p,\frac{q\zeta}{\sqrt{p^2+q^2}}+q,y_1,y_2\Big)d\zeta+$$

$$\frac{q}{\sqrt{p^2+q^2}}t_2\Big(\frac{p\zeta}{\sqrt{p^2+q^2}}+p,\frac{q\zeta}{\sqrt{p^2+q^2}}+q,y_1,y_2\Big)d\zeta=$$

$$=\frac{p}{\sqrt{p^2+q^2}}t_1(x^1+p,x^2+q,y_1,y_2)+\frac{q}{\sqrt{p^2+q^2}}t_2(x^1+p,x^2+q,y_1,y_2)=$$

$$=e^{p(y_1+\xi_1)+qy_2}\Big(\frac{p}{\sqrt{p^2+q^2}}t_1(x^1,x^2,y_1,y_2)+\frac{q}{\sqrt{p^2+q^2}}t_2(x^1,x^2,y_1,y_2)\Big)=$$

$$=e^{p(y_1+\xi_1)+qy_2}\phi(\zeta,y_1,y_2)$$

Suppose that τ is exact so that we can write $\tau=\nabla_1^{\mathcal{L}}s$ with $s\in C^\infty(S\times\hat{U},\ker\nabla_1^{\mathcal{L}})$. Then s can be written in the form

$$s(\zeta,y_1,y_2)=\int_0^\zeta \phi(u,y_1,y_2)du+c(y_1,y_2)$$

which, to be well defined, must satisfy the automorphy condition. This happens when

$$c(y_1,y_2)(1-e^{2\pi i(p(y_1+\xi_1)+qy_2)})=-\int_0^{\sqrt{p^2+q^2}}\phi(u,y_1,y_2)\,du. \tag{2.2}$$

This equation may be solved for c in the complement of the set $\hat{S}$ defined by

$$p(y_1+\xi_1)+qy_2=0$$

or in term of a by

$$y_2=-\frac{1}{a}y_1-\frac{\xi_1}{a}$$

this means that τ is exact when U does not intersect $\hat{S}$ and that the cohomology vanishes on such U, thus the support of $R^1\hat{p}_* ker \nabla_1^{\mathcal{L}}$ is exactly $\hat{S}$. In the case $\dim T = n > 2$ the computations are similar: indeed, let

$$x^i = a^i x^1 + \chi^i \tag{2.3}$$

be the equations defining S and $A = \xi_1 dx_1$ the 1-form connection of the local system $(\mathcal{L}, \nabla^{\mathcal{L}})$ on S, then the equation of the support is given by

$$y_1 + a^2 y_2 + \ldots + a^n y_n + \xi_1 = 0 \tag{2.4}$$

To compute the sheaf $R^1\hat{p}_* ker \nabla_1^{\mathcal{L}}$, note that the map

$$\psi : \Omega^{1,0}(p^*\mathcal{L} \otimes \mathcal{P})(S \times \hat{U}) \to \mathcal{C}^\infty(\hat{S} \cap \hat{U})$$

$$\tau \to -\int_0^{\sqrt{p^2+q^2}} \phi(u, y_1, y_2)\, du$$

is surjective.
Indeed suppose that the local system is trivial and $a = 0$, so that $\hat{S}$ is represented by $\{y_1 = 0)\}$, and let $\hat{S}' = \{y_2 = 0\}$ be the dual torus of S: if $f \in \mathcal{C}^\infty(\hat{S} \cap \hat{U})$ and s is a section of the Poincaré bundle over $S \times \hat{S}'$, or, in other words, a function on the universal cover V satisfying the automorphy condition

$$s(x^1 + p, y_1) = e^{2\pi i p y_1} s(x^1, y_1)$$

then the 1-form $\tau = \phi d\zeta$ defined by

$$\phi(\zeta, y_1, y_2) = \beta s(\zeta, y_1) f(y_2)$$

with $1/\beta = -\int_0^{\sqrt{p^2+q^2}} s(u, 0) du$, satisfies $\psi(\tau) = f$. Note that we can choose s such that its restriction $s(u, 0)$, which is a periodic function, satisfies $\int_0^{\sqrt{p^2+q^2}} s(u, 0) du \neq 0$. When $a \neq 0$, we call again $\hat{S}'$ the dual torus of S, which is a line orthogonal to $\hat{S}$ and s a function on $S \times \hat{S}'$ satisfying the following automorphy condition

$$s(\zeta + \sqrt{p^2 + q^2}, y_1, y_2) = e^{2\pi i (p y_1 + q y_2)} s(\zeta, y_1, y_2),$$

With a coordinate η on $\hat{S}'$

$$\eta = \frac{y_1\sqrt{p^2+q^2}}{p} = \frac{y_2\sqrt{p^2+q^2}}{q}$$

the automorphy condition for s can be rewritten as

$$s(\zeta + \sqrt{p^2+q^2}, \eta) = e^{2\pi i\sqrt{p^2+q^2}} s(\zeta, \eta)$$

Given now a couple (y_1, y_2) in $\hat{T} \cap \hat{U}$, we define $s(\zeta, y_1, y_2)$ as $s(\zeta, pr(y_1, y_2))$, where $pr(y_1, y_2)$ is the orthogonal projection of (y_1, y_2) on S'. The 1-form $\tau = \phi d\zeta$ defined by

$$\phi(\zeta, y_1, y_2) = \beta s(\zeta, y_1, y_2) f(y_2)$$

with $1/\beta = -\int_0^{\sqrt{p^2+q^2}} s(u, y_1, y_2) du$, where $y_1,\ y_2 \in S$, satisfies $\psi(\tau) = f$. Moreover ϕ is a section of $p^*\mathcal{L}\otimes\mathcal{P}$, since it satisfies the automorphy condition. As before we can choose s such that β is a nowhere zero function.

This shows that ψ is surjective, thus that $H^1(S\times\hat{U}, \ker\nabla_1^{\mathcal{L}}) = \mathbb{C}^\infty(\hat{S}\cap\hat{U})$.

The transformed sheaf is endowed with a flat connection induced by $\nabla_2^{\mathcal{L}}$, the $(0,1)$ part of the connection $p^*\nabla^{\mathcal{L}} \otimes \nabla_{\mathcal{P}}$. The hypotheses of proposition 2.0.10 are satisfied. Consider, indeed, for simplicity the case when $n = 2$. Since the equation of S is $x^2 = ax^1 + \chi$ and the equation of $\hat{S}$ is $y_2 = -\frac{1}{a}y_1 - \frac{\xi_1}{a}$, the curvature of the Poincaré bundle, given by $2\pi i(dx^1 \wedge dy_1 + dx^2 \wedge dy_2)$, vanishes on $S \times \hat{S}$. This happens also in general because S and $\hat{S}$ are "orthogonal". In fact, substituting in the curvature of the Poincaré bundle $\mathbb{F} = 2i\,\pi\sum_{j=1}^{g} dx^j \wedge dy_j$ the expressions of S and S' given by equations (2.3) and (2.4) respectively, we realize that

$$\mathbb{F} = dx^1 \wedge dy_1 + a^2 dx^1 \wedge dy_2 + \ldots + a^{n-1} dx^1 \wedge dy_{n-1} +$$

$$+ a^n dx^1 \wedge \left(-\frac{1}{a^n} dy_1 - \frac{a^2}{a^n} dy_2 - \ldots - \frac{a^{n-1}}{a^n} dy_{n-1} \right) = 0$$

So it follows that the $(0,1)$-part of the connection $p^*\nabla^{\mathcal{L}} \otimes \nabla_{\mathcal{P}}$ induces a connection on $R^1\hat{p}_* ker \nabla_1^{\mathcal{L}}$, which has the following expression in coordinates

$$A = -x^1 dy_1 - x^2 dy_2 =$$

$$= -x^1 dy_1 - (ax^1 + \chi)(-\frac{1}{a} dy_1) = -(-\frac{\chi}{a}) dy_1$$

Note that the expression of A using the coordinate y_2 is

$$A = -\chi dy_2$$

. In a similar way when $n > 2$

$$A = -x^1 dy_1 - \dots - x^n dy_n =$$

$$= -x^1(-a^2 dy_2 - \dots - a^n dy_n) - (a^2 x^1 + \chi^2) dy_2 + \dots =$$

$$= -\chi^2 dy_2 - \dots - \chi^n dy_n$$

In order to show that $R^i\hat{p}_*(\ker \nabla_1^{\mathcal{L}}) = 0$ for $i > 1$, it is enough to observe that the groups

$$H^i(S \times \{y\}, \ker \nabla_1^{\mathcal{L}} \mid_{S\times\{y\}}) \cong (R^i\hat{p}_*(\ker \nabla_1^{\mathcal{L}}))_y$$

vanish because S is has dimension 1.

We can now extend this result to the case when $\dim S = k > 1$ using a Künneth formula. To this purpose we write S as a product $S = S_1^1 \times \dots \times S_k^1$ where S_i^1 are 1-dimensional subtori of T. Since by "universality" the Poincaré bundle $(\mathcal{P}, \nabla_{\mathcal{P}})$ on $S \times \hat{T}$ is the box product of the Poincaré bundles $(\mathcal{P}_i, \nabla_i)$ on $S_i \times \hat{T}$, we can apply the Künneth formula getting:

$$R^i\hat{p}_* ker \nabla_1^{\mathcal{L}} = 0 \quad \text{when } i \neq k$$

$R^k\hat{p}_* ker \nabla_1^{\mathcal{L}}$ is a local system with support on an $(n-k)$-subtorus of $\hat{T}$.

For example, if S^2 is a 2-subtorus, then we can write $S^2 = S^1_1 \times S^1_2$ for suitable 1-subtori S^1_1 and S^1_2, so by the Künneth formula

$$H^0(S^2 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) = H^0(S^1_1 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) \otimes H^0(S^1_2 \times \hat{U}, \ker \nabla_1^{\mathcal{L}})$$

$$H^1(S^2 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) = H^0(S^1_1 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) \otimes H^1(S^1_2 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) +$$
$$+ H^1(S^1_1 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) \otimes H^0(S^1_2 \times \hat{U}, \ker \nabla_1^{\mathcal{L}})$$

$$H^2(S^2 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) = H^2(S^1_1 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) \otimes H^0(S^1_2 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) +$$
$$+ H^1(S^1_1 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) \otimes H^1(S^1_2 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) + H^0(S^1_1 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) \otimes H^2(S^1_2 \times \hat{U}, \ker \nabla_1^{\mathcal{L}})$$

and so on. Since, as seen, only $H^1(S^1_1 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) \neq 0$ and $H^1(S^1_2 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) \neq 0$, it follows that only $H^1(S^1_1 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) \otimes H^1(S^1_2 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) \neq 0$, so only $H^2(S^2 \times \hat{U}, \ker \nabla_1^{\mathcal{L}}) \neq 0$. Its support is the intersection of the supports of $\hat{S}^1_1$ and $\hat{S}^1_2$, which is non empty since, on the universal cover, it is given by the intersection of two $(n-1)$-hyperplanes orthogonal to two incident lines. So this intersection is a $(n-2)$-subtorus. The transformed connection on S^2 is the transformed connection of S^1_1 or S^1_2 (these in fact coincides on the transformed support $\hat{S}_1 \cap \hat{S}_2$). For a generic k-subtorus $S^k = S^1_1 \times \ldots \times S^1_k$ there is only one non vanishing term, and it is the one which contains the term $\otimes_{j=1}^k H^1(S^1_j \times \hat{U}, \ker \nabla_1^{\mathcal{L}})$, so it is $H^k(S^k \times \hat{U}, \ker \nabla_1^{\mathcal{L}})$. The support is given by the intersection of the supports of $\hat{S}^1_j$, which are $(n-k)$-subtori, and the transformed connection is the transformed connections of S^1_j. □

Let us describe the content of this Proposition in local coordinates. Let $(x^1, \ldots, x^g)$ be flat coordinates in T, $(y_1, \ldots, y_g)$ the corresponding dual flat coordinates in the dural torus $\hat{T}$, and write the equation for the affine subtorus S in the form

$$\sum_{j=1}^{g} a^i_j x^j + \chi^i = 0, \qquad i = 1, \ldots, g-k\,.$$

The equations $\sum_{j=1}^{g} a_j^i\, x^j = 0$ describe a corresponding "linear subtorus" S_0; the equations of the dual torus $\hat{S}_0$ may be written implicitly as

$$\sum_{j,\ell=1}^{g} a_j^i\, g^{j\ell}\, y_\ell = 0\,, \qquad i = 1, \ldots, g-k\,,$$

where the constant functions $g^{j\ell}$ are the components of the natural flat metric on $\hat{T}$, or explicitly as

$$y_\ell = \sum_{m=1}^{k} \tilde{a}_\ell^m\, \xi_m\,, \qquad \ell = 1, \ldots, g \tag{2.5}$$

for a suitable $k \times (g-k)$ matrix $\tilde{a}$. The specification of the local system $\mathfrak{L}$ corresponds to a choice of the parameters $(\xi_1, \ldots, \xi_k)$ in equation (2.5). The support $\hat{S}$ of the transformed local system is given by equations

$$\sum_{j=1}^{g} \gamma_m^j\, y_j + \xi_m = 0\,, \qquad m = 1, \ldots, k\,,$$

where γ_ℓ^j is a matrix satisfying $\sum_{j=1}^{g} \gamma_\ell^j\, a_j^i = 0$. The local system $\hat{\mathfrak{L}}$ is given by the point in S_0 (regarded as the dual to the linear subtorus $\hat{S}_0$ corresponding to $\hat{S}$) whose coordinates are the numbers χ^i.

Definition 2.0.27. *The pair* $\hat{\mathcal{F}}(S, \mathfrak{L}) = (\hat{S}, \hat{\mathfrak{L}})$ *is the* Fourier-Mukai transform *of the pair* $(S, \mathfrak{L})$.

Of course we may perform the same transformation from $\hat{T}$ to T, and we have:

Proposition 2.0.28. *The Fourier-Mukai transform of* $(\hat{S}, \hat{\mathfrak{L}})$ *is naturally isomorphic to the pair* $(S, \mathfrak{L})$.

Proof. We start cheking the invertibility when the local system is supported on a subtorus S of dimension 1, since, when the support is an higher dimensional subtorus, through the Künneth formula, we reduce to transform local systems supported on subtori of dimension 1.

If S has equation $x^2 = ax^1 + \chi$ and the local system on it is defined by the flat connection $A = \xi_1 dx^1$, then the transformed local system has support and connection given respectively by $y_2 = -\frac{1}{a}y_1 - \frac{\xi_1}{a}$ and $\hat{A} = \frac{\chi}{a}dy_1$. We apply now to this local system the inverse transform, which is performed by means of the dual Poincaré bundle: the support is given by solving the equation

$$-a(-x^1 - \frac{\chi}{a}) - x^2 = 0$$
$$\Leftrightarrow x^2 = ax^1 + \chi$$

while, as to connection, we take the (1,0)-part of the connection of the dual Poincaré bundle

$$A = -y_1 dx^1 - y_2 dx^2 =$$
$$= -y_1 dx^1 - (-\frac{1}{a}y_1 - \frac{\xi_1}{a})adx^1 = \xi_1 dx^1$$

This shows the invertibility of the transform.

When $\dim T$ is bigger than 2, and $\dim S = 1$, written S as $x^i = a^i x^1 + \chi^i$, with $i = 2, ..., n$, and the connection of the local system on it as $A = \xi_1 dx^1$, the transformed local system is defined by a support $\hat{S}$, whose equation is $y_1 + a^2 y_2 + ... + a^n y_n + \xi_1 = 0$, and a connection $\hat{A} = -\chi^2 dy_2 - ... - \chi^n dy_n$. To transform back this local system, we have to find $n-1$ independent lines lying in the hyperplane just found, transform them and take the intersection, according to the Künneth formula, of the corresponding transformed hyperplanes. A natural choice of these lines is the following: the line r_i, with $i = 2, ..., n$, is defined by the equations

$$y_1 + a^i y_i + \xi_1 = 0$$

$$y_j = 0$$

with $j = 2, ..., n$, $j \neq i$. But transforming these lines with their connections, we get the hyperplanes whose intersection is the line we started with. The same for the connection, though with longer computations:

$$A = -y_1dx^1 - ... - y_ndx^n =$$

$$= -y_1dx^1 - \frac{1}{a^2}(-y_1 - \xi_1 - a^3y_3 - ... - a^ny_n)a^2dx^1 +$$

$$-\frac{1}{a^3}(-y_1 - \xi_1 - a^2y_2 - a^4y_4 - ... - a^ny_n)a^3dx^1 - ... +$$

$$-\frac{1}{a^n}(-y_1 - \xi_1 - a^2y_2 - ... - a^{n-1}y_{n-1})a^ndx^1 =$$

$$= \xi_1dx^1.$$

In the general case we proceed as in the previous one. If $\dim S = k$, S is defined by the equations

$$x^{k+i} = a_1^{k+1}x_j + ... + a_k^{k+1}x_k + \chi^{k+i}$$

with $i = 1, ..., n-k$, and connection

$$\sum_{j=1}^{k} \xi_j dy_j$$

To use the Künneth formula, we choose k independent lines lying in S, for example the family of lines $\{S_1, ..., S_k\}$ where S_j has equations

$$x^t = 0 \qquad \text{for } t = 1, ..., k, \quad t \neq j$$

$$x^{k+i} = a_j^{k+i}x^j + \chi^{k+i} \qquad \text{for } i = 1, ..., n-k$$

transforming the S_j's with their connection, and applying the Künneth formula, we obtain the transformed local system $\hat{S}$ with support

$$y_j + a_j^{k+1} + ... + a_j^n y_n + \xi_j = 0$$

for $j = 1, ..., k$, and connection

$$\hat{A} = \sum_{i=1}^{n-k} \chi^{k+i} dy_{k+i}$$

To apply now the inverse transform we choose $n-k$ lines $\hat{S}_1, ..., \hat{S}_{n-k}$ lying in $\hat{S}$, for example we can take $\hat{S}_i$ to be the line whose equations are

$$y_j = -a_j^{k+i} y_{k+i} - \xi_j \qquad \text{for } j = 1, ..., k$$

$$y_{k+s} = 0 \qquad \text{for } s \neq i$$

It is quite easy to check now that the inverse transform of this family of lines with their connections gives, after having applied the Künneth formula, the initial local system S. □

So the Fourier-Mukai transforms $\mathcal{F}$ and $\hat{\mathcal{F}}$ yield an equivalence of categories

$$\mathbf{Loc}_k(T) \cong \mathbf{Loc}_{g-k}(T)$$

When a local system $\mathcal{E}$, supported on a subtorus of dimension k, has rank n bigger than 1, it splits as direct sum of local sytems $\mathcal{E}_i$ of rank 1. So it follows that its transform is a collection of n local systems of rank 1 supported on parallel subtori of dimension $g-k$. This happens because the support of $\mathcal{E}$ (which coincides with those of $\mathcal{E}_i$) is "orthogonal" to the supports of $\hat{\mathcal{E}}_i$ (this means that they are parallel), moreover these are displaced according to the connections ∇_i of $\mathcal{E}_i$. The transformed support of $\hat{\mathcal{E}}_i$ and $\hat{\mathcal{E}}_j$ concide when the parameters $(\xi_1^i, \ldots, \xi_k^i)$ and $(\xi_1^j, \ldots, \xi_k^j)$, which appear in equation (2.5) and represent the connections ∇_i and ∇_j respectively, differ by an element of the lattice. What just explained can be generalized to a family of local systems of generic rank supported on a family of parallel subtori of T.

Chapter 3

THE TRANSFORM FOR A LAGRANGIAN FAMILY OF TORI

In this chapter we extend the construction described in the second one for a real torus to a family of real tori, providing a real relative Fourier-Mukai transform. A big problem is represented by singular fibres because it is not clear how to handle them. So as a first step, one may consider the simplified case when there are no singular fibres.
Let (X, ω) be a connected symplectic manifold admitting a map $f\colon X \to B$ whose fibres are g-dimensional smooth Lagrangian tori. We assume that f admits a Lagrangian section $\sigma\colon B \to X$. The following theorem comes from a work of Duistermaat [11]

Theorem 3.0.29. *Let (X, ω) be a connected symplectic manifold of dimension $2n$ which supports a fibration $\pi : X \to B$ such that for every $b \in B$ the fibre $F_b = \pi^{-1}(b)$ is a compact connected Lagrangian submanifold of X. Then the following statements are equivalent:*
1. $X \simeq T^*B/\Lambda$ as symplectic manifolds fibred over B in Lagrangian sub-

manifolds, where Λ is a covering of B such that $\Lambda_b = \Lambda \cap T^*_b B$ is a lattice for every $b \in B$.
2. the bundle $\pi : X \to B$ admits a global Lagrangian section

X clearly satisfies the hypothesis of this theorem, so the existence of the section σ gives the isomorphism $X \simeq T^*B/\Lambda$. The symplectic form ω provides an isomorphism between TX and T^*X and this, in turn, an isomorphism $\operatorname{Vert} TX \simeq f^*T^*B$. We also have an identification $TB \simeq R^1 f_*\mathbb{R} \otimes \mathcal{C}^\infty_B$, which endows TB with a connection ∇_{GM}, given by $1 \otimes d$, which is flat and torsion-free, and called the Gauss-Manin connection of the local system $R^1 f_*\mathbb{R}$. The holonomy of this connection coincides with the monodromy of the covering Λ (indeed, the horizontal tangent spaces may be identified with the first homology groups of the fibres with real coefficients). Let $\hat{X} = R^1 f_*\mathbb{R}/R^1 f_*\mathbb{Z}$ be the dual family, with projection $\hat{f}\colon \hat{X} \to B$. Dualizing the isomorphism $\operatorname{Vert} TX \simeq f^*T^*B$ we get a new isomorphism $\operatorname{Vert} T\hat{X} \simeq \hat{f}^*TB$; combining this with the splitting of the Atiyah sequence

$$0 \to \operatorname{Vert} T\hat{X} \to T\hat{X} \to \hat{f}^*TB \to 0$$

provided by the Gauss-Manin connection (which can be regarded as a connection on $T\hat{X}$), one has a splitting

$$T\hat{X} \simeq \hat{f}^*TB \oplus \hat{f}^*TB\,.$$

By letting $J(\alpha, \beta) = (-\beta, \alpha)$ this induces a complex structure on $\hat{X}$, such that the holomorphic tangent bundle to $\hat{X}$ is isomorphic, as a smooth bundle, to $\hat{f}^*TB \otimes \mathbb{C}$. On X we may consider local symplectic coordinates $(x^1, \ldots, x^g, y_1, \ldots, y_g)$ such that the x's are local coordinates on B, and for fixed values of the x's, the y's are flat coordinates on the corresponding torus (local action-angle coordinates). Analogously, we may consider on $\hat{X}$ local coordinates $(x^1, \ldots, x^g, w^1, \ldots, w^g)$ such that the w's are dual coordinates to the y's. Local holomorphic coordinates on $\hat{X}$ are given by $z^j = x^j + iw^j$.

In this relative context it is natural to consider the fibre product $Z = X \times_B \hat{X}$ of the fibrations X and $\hat{X}$. We shall denote by p, $\hat{p}$ the projections of Z onto its factors. On Z there is a Poincaré bundle $\mathcal{P}$ which may be described in an intrinsic way, however, it is enough to say that $\mathcal{P}$ a line bundle on $X \times_B \hat{X}$ equipped with a $U(1)$ connection $\nabla_{\mathcal{P}}$ whose connection form may be written in a suitable gauge as

$$\mathbb{A} = 2i\,\pi \sum_{j=1}^{g} w^j\, dy_j\,.$$

Moreover, $\mathcal{P}$ has the property that for every $\xi \in \hat{X}$, $\mathcal{P}_{|\hat{p}^{-1}(\xi)}$ is isomorphic to $\mathcal{L}_\xi$ (the line bundle on $X_{f^{-1}(\xi)}$ parametrized by ξ) as a $U(1)$ bundle. Let us consider the exact sequence

$$0 \to \hat{p}^*\Omega^1_{\hat{X}} \to \Omega^1_Z \xrightarrow{r} \Omega^1_{Z/\hat{X}} \to 0 \tag{3.1}$$

which defines the sheaf $\Omega^1_{Z/\hat{X}}$ of $\hat{p}$-relative differentials. Sections of this sheaf are locally written as $\eta = \sum_{j=1}^{g} \eta^j(x,y,w)\, dy_j$. The Gauss-Manin connection ∇_{GM} provides a splitting of this exact sequence. Analogously, we have the exact sequence

$$0 \to p^*\Omega^1_X \to \Omega^1_Z \xrightarrow{\hat{r}} \Omega^1_{Z/X} \to 0 \tag{3.2}$$

which defines the sheaf $\Omega^1_{Z/X}$ of p-relative differentials. For every sheaf $\mathcal{E}$ of $\mathcal{C}^\infty_X$-modules endowed with a flat connection ∇, one defines the following differential operators:

(i) the operator

$$\nabla^{\mathcal{E}} \colon p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^\bullet_Z \to p^*\mathcal{E} \otimes \mathcal{P} \otimes \Omega^{\bullet+1}_Z\,,$$

obtained by coupling the pullback of the connection ∇ with the connection of the Poincaré sheaf $\nabla_{\mathcal{P}}$;

(ii) the operators $\nabla^{\mathcal{E}}_r$, $\nabla^{\mathcal{E}}_{\hat{r}}$ obtained by composing $\nabla^{\mathcal{E}}$ with respectively the projections r, $\hat{r}$ onto the relative differentials.

One has $(\nabla^{\mathcal{E}}_r)^2 = (\nabla^{\mathcal{E}}_{\hat{r}})^2 = 0$.

We shall consider the higher direct images $R^i\hat{p}_* \ker \nabla^{\mathcal{E}}_r$, which are the cohomology sheaves of the complex

$$\hat{p}_*(p^*\mathcal{E}\otimes\mathcal{P}) \xrightarrow{\nabla^{\mathcal{E}}_r} \hat{p}_*(p^*\mathcal{E}\otimes\mathcal{P}\otimes\Omega^1_{Z/\hat{X}}) \xrightarrow{\nabla^{\mathcal{E}}_r} \hat{p}_*(p^*\mathcal{E}\otimes\mathcal{P}\otimes\Omega^2_{Z/\hat{X}}) \to \dots \quad (3.3)$$

Also in the relative context, we introduce the WIT notion, whose definition is the same as in Chapter 2 (cf. definition 2.0.11) and which we recall now

Definition 3.0.30. *The pair $(\mathcal{E},\nabla)$ is said to be WIT_k if $R^i\hat{p}_* \ker \nabla^{\mathcal{E}}_r = 0$ for $i \neq k$.*

Now we want to state a condition for the sheaves $R^jp_* \ker \nabla^{\mathcal{E}}_r$ to admit a connection induced, so to say, by the part of the operator $\nabla^{\mathcal{E}}$ complementary to $\nabla^{\mathcal{E}}_r$. The splitting of the exact sequence (3.1) provided by the Gauss-Manin connection ∇_{GM} allows one to make a splitting

$$\nabla^{\mathcal{E}} = \nabla^{\mathcal{E}}_r + \hat{\nabla}^{\mathcal{E}}.$$

where

$$\hat{\nabla}^{\mathcal{E}}\colon p^*\mathcal{E}\otimes\mathcal{P}\otimes\hat{p}^*\Omega^1_{\hat{X}}\otimes\Omega^\bullet_Z \to p^*\mathcal{E}\otimes\mathcal{P}\otimes\hat{p}^*\Omega^1_{\hat{X}}\otimes\Omega^{\bullet+1}_Z.$$

The $\hat{\nabla}^{\mathcal{E}}$ operator induces connections on the higher direct images $R^j\hat{p}_* \ker \nabla^{\mathcal{E}}_r$ provided it anticommutes with the operator $\nabla^{\mathcal{E}}_r$. The anticommutator $\nabla^{\mathcal{E}}_r\circ\hat{\nabla}^{\mathcal{E}}+\hat{\nabla}^{\mathcal{E}}\circ\nabla^{\mathcal{E}}_r$ may be regarded as an operator $p^*\mathcal{E}\otimes\mathcal{P} \to p^*\mathcal{E}\otimes\mathcal{P}\otimes\Omega^2_Z$ and as such it coincides with $1\otimes\mathbb{F}$, where $\mathbb{F}$ is the curvature of the connection $\nabla_{\mathcal{P}}$ of the Poincaré bundle. As a consequence, we have (cf. proposition 2.0.10):

Proposition 3.0.31. *Assume that the sheaf $\mathcal{E}$ is supported on a closed submanifold $S\subset X$, that the sheaf $R^j\hat{p}_* \ker \nabla^{\mathcal{E}}_r$ is supported on a closed submanifold $\hat{S}\subset\hat{X}$, and that the curvature operator $\mathbb{F}$ vanishes on $S\times_B\hat{S}\subset Z$. Then the operator $\hat{\nabla}^{\mathcal{E}}$ induces a connection on the sheaf $R^j\hat{p}_* \ker \nabla^{\mathcal{E}}_r$.*

Eventually, we may introduce the Fourier-Mukai transform

Definition 3.0.32. *If the pair* $(\mathcal{E}, \nabla)$ *is* WIT_k *and satisfies the condition in Proposition 3.0.31, the pair* $(\hat{\mathcal{E}}, \hat{\nabla})$*, where* $\hat{\mathcal{E}} = R^k p_* \ker \nabla_2^{\mathcal{E}}$ *and* $\hat{\nabla}$ *is the connection induced as in Proposition 3.0.31, is called the Fourier-Mukai transform of* $(\mathcal{E}, \nabla)$ *.*

The following lemma is useful when one wants to check if the WIT property holds for some sheaf and connection.

Lemma 3.0.33. *Let* $\mathcal{E}$ *be a sheaf of* $\mathcal{C}_X^\infty$*-modules supported on a closed submanifold of* X *which intersects every fibre* X_b *along a closed submanifold, and let* ∇ *be a flat connection on* $\mathcal{E}$*. For every* $j = 1, \dots, g$ *there is a canonical isomorphism:*

$$(R^j \hat{p}_* \ker \nabla_r^{\mathcal{E}})_{|\hat{X}_b} \xrightarrow{\sim} R^j \hat{p}_{b,*} \ker \nabla_1^{\mathcal{E}_b}\,, \tag{3.4}$$

where $b = \hat{p}(\xi)$*,* $\hat{p}_b \colon X_b \times \hat{X}_b \to \hat{X}_b$ *is the canonical projection onto* $\hat{X}_b$*,* $\mathcal{E}_b$ *is the restriction of* $\mathcal{E}$ *to* X_b *and* $\ker \nabla_1^{\mathcal{E}_b}$ *is the operator introduced in chapter 2.*

Proof. Standard homological arguments provide a morphism as in equation (3.4). Moreover, one has the canonical isomorphism

$$(R^j \hat{p}_* \ker \nabla_r^{\mathcal{E}})_\xi \simeq H^j(\hat{p}^{-1}(\xi), (\ker \nabla_r^{\mathcal{E}})_{|\hat{p}^{-1}(\xi)})\,.$$

Since the restriction of $\nabla_r^{\mathcal{E}}$ to X_b is $\nabla_1^{\mathcal{E}_b}$, the group in the r.h.s. is isomorphic to $H^j(X_b,\, \ker \nabla_1^{\mathcal{E}_b})$, so that the restriction of the morphism (3.4) to every stalk is an isomorphism. □

The case of a local system $\mathfrak{L}$ supported on a fibre X_b is the simplest to deal with. It is enough to consider the case $\operatorname{rank} \mathfrak{L} = 1$, since the higher rank case reduces immediately to this. Notice that the isomorphism class of the local sistem $\mathfrak{L}^*$ singles out a point in $\hat{X}_b$, which we denote by $[\mathfrak{L}^*]$. We obtain the usual "tautological" property of the Fourier-Mukai transform.

Proposition 3.0.34. *Let $\mathcal{L} = \mathfrak{L} \otimes \mathcal{C}^\infty_X$. Then $\mathcal{L}$ is WIT$_g$, and the sheaf $\hat{\mathcal{L}} = R^g\hat{p}_* \ker \nabla^{\mathcal{L}}_r$ is isomorphic to the skyscraper $\mathbb{C}_{[\mathfrak{L}^*]}$.*

Proof. The sheaf $R^j\hat{p}_* \ker \nabla^{\mathcal{L}}_r$ is obtained by "sheafifying" the presheaf

$$U \rightsquigarrow H^j(\Gamma(X \times U, p^*\mathcal{L} \otimes \mathcal{P} \otimes \Omega^\bullet_{Z/\hat{X}})) \tag{3.5}$$

where U is an open set in $\hat{X}$. The sheaf $p^*\mathcal{L} \otimes \mathcal{P} \otimes \Omega^\bullet_{Z/\hat{X}}$ vanishes outside the set $X_b \times U_b$, where $U_b = U \cap \hat{X}_b$, so that the group in the r.h.s. of (3.5) is actually $H^j(\Gamma(X_b \times U_b, p_b^*\mathcal{L} \otimes \mathcal{P}_b \otimes p_b^*\Omega^\bullet_{X_b}))$ where $\mathcal{P}_b = \mathcal{P}_{|X_b \times \hat{X}_b}$ and p_b is the projection onto the first factor of $X_b \times \hat{X}_b$. Comparison with the absolute case in chapter 2 shows that $\mathcal{L}$ is WIT$_g$, and that $\hat{\mathcal{L}} \simeq k_{[\mathfrak{L}^*]}$. □

We turn now to the costruction of a transform for local systems supported on sections of $X \to B$. This will generalize the tautological correspondence that in the absolute case holds between skycrapers of length one on a torus and $U(1)$ local systems on the dual torus. The transform will produce holomorphic line bundles on $\hat{X}$ with compatible $U(1)$ connections which satisfy some further conditions.

Let $S \subset X$ be the image of a Lagrangian section of $X \to B$, and $\mathfrak{L}$ a unitary local system on S. Set $\mathcal{L} = \mathfrak{L} \otimes \mathcal{C}^\infty_S$ and call p_S and $\hat{p}_S$ the canonical projections of $S \times_B \hat{X}$ onto the factors S and $\hat{X}$, and $\mathcal{P}_S$ the restriction of the Poincaré bundle $\mathcal{P}$ to $S \times_B \hat{X}$.

Proposition 3.0.35. *1. The pair $(\mathcal{L}, \nabla)$ is WIT$_0$;*

2. $\hat{\mathcal{L}} = \hat{p}_{S,*} \ker \nabla^{\mathcal{L}}_r$ is a rank-one locally free $\mathcal{C}^\infty_{\hat{X}}$-module.

Proof. Both claims follows from Lemma 3.0.33 and the absolute case. □

Since $\mathbb{F}_{|S \times_B \hat{X}} = 0$ the conditions of Proposition 3.0.31 are met, so that $\hat{\mathcal{L}}$ carries a $U(1)$ connection $\hat{\nabla}$.

Let us express this connection in coordinates. We write the local equations of S as $y_j = \epsilon_j(x)$. Moreover, the x's can be thought of as local coordinates on S. If the connection form associated with the local system $\mathfrak{L}$ is $A = i\sum_{j=1}^{g} A_j(x)\, dx^j$, with $\frac{\partial A_j}{\partial x^\ell} = \frac{\partial A_\ell}{\partial x^j}$, then $\hat{\nabla}$ may be represented by the connection form

$$\hat{A} = i\sum_{j=1}^{g} A_j(x)\, dx^j - 2i\pi \sum_{j=1}^{g} \epsilon_j(x)\, dw^j \,.$$

Remark 3.0.36. *In these coordinates the components of the connection form $\hat{A}$ do not depend on the w's. Moreover, both the horizontal and vertical part (with respect to the splitting given by the Gauss-Manin connection) are flat, and in particular, the restriction of $\hat{\nabla}$ to any fiber $\hat{X}_b$ of $\hat{X} \to B$ is flat. The independence of the components $\hat{A}$ on the w's can be stated invariantly in a variety of ways. For instance, one can use the fact that the zero-section of $\hat{X}$ makes the latter into a (trivial) principal T^g-bundle over B; then, $\hat{\nabla}$ commutes with the action of T^g on $\hat{X}$.*

The Hodge components of curvature form $\hat{F}$ of this connection may be written — recalling that in the complex structure we have given to $\hat{X}$ the coordinates $z^j = x^j + iw^j$ are complex holomorphic — as

$$\hat{F}^{2,0} = \frac{\pi}{2}\sum_{k,j} \frac{\partial \epsilon_j}{\partial x^k}\, dz^k \wedge dz^j$$

$$\hat{F}^{0,2} = -\frac{\pi}{2}\sum_{k,j} \frac{\partial \epsilon_j}{\partial x^k}\, d\bar{z}^k \wedge d\bar{z}^j$$

$$\hat{F}^{1,1} = \frac{\pi}{2}\sum_{k,j} (\frac{\partial \epsilon_k}{\partial x^j} + \frac{\partial \epsilon_j}{\partial x^k})\, dz^k \wedge d\bar{z}^j.$$

Since S is Lagrangian we have $\dfrac{\partial \epsilon_j}{\partial x^k} = \dfrac{\partial \epsilon_k}{\partial x^j}$, thus $\hat{F}^{0,2} = \hat{F}^{2,0} = 0$, so that $\hat{\mathcal{L}}$ may be given a holomorphic structure compatible with the connection $\hat{\nabla}$. Moreover, we have

$$\hat{F}^{1,1} = \pi \sum_{k,j} \frac{\partial \epsilon_j}{\partial x^k} dz^k \wedge d\bar{z}^j.$$

Definition 3.0.37. *The Fourier transform of* $(S, \mathcal{L})$ *is the triple* $(\hat{\mathcal{L}}, \hat{\nabla})$.

We now want to generalize this construction including local system supported on more general Lagrangian submanifolds of X. To this purpose we select the following class of submanifolds of X:

(C1) S is a Lagrangian subvariety of X;

(C2) the intersection $S_b = S \cap X_b$ of S with a fibre of X, when nonempty, is a (possibly affine) subtorus S_b of X_b whose dimension does not depend on b.

Let $\mathfrak{L}$ be a $U(1)$ local system on S and ∇ the corresponding flat connection on $\mathcal{L} = \mathfrak{L} \otimes_{\mathbb{C}} C^\infty_X$.

We define as before a Fourier-Mukai transform of the local system $(S, \mathcal{L}, \nabla)$ at the sheaf level as

$$\hat{\mathcal{L}} = R^m \hat{p}_{S,*} \ker \nabla^{\mathcal{L}}_r$$

where m is the dimension of the tori S_b. This definition has its motivation in the following result.

Proposition 3.0.38. *Let* $(S, \mathcal{L}, \nabla)$ *be a local system supported on a Lagrangian subvariety* S *which satisfies the conditions* $C1$, $C2$. *Then the sheaf* $\mathcal{L}$ *is* WIT_m.

Proof. It follows from Lemma 3.0.33 and Proposition 2.0.26. □

Lemma 3.0.33 and Proposition 2.0.26 also imply that after restriction to its support, $\hat{\mathcal{L}}$ is a line bundle. We shall now show that, under some suitable conditions on the support S, the transform $\hat{\mathcal{L}}$ is supported on a complex submanifold $\hat{S}$ of the dual family $\hat{X}$. More precisely, we assume:

(C3) the vertical tangent spaces of the family of subtori $\{S_b\}_{b \in f(S)}$ are parallelly transported by the Gauss-Manin connection ∇_{GM} regarded as a connection in TX.

This requirement can be translated into a more explicit form in terms of the action-angle coordinates (x, y) we have previously introduced, in that it amounts to the condition that the family of subtori $\{S_b\}$ can be written as

$$\sum_{j=1}^{g} a_i^j\, y_i + \chi_i = 0\,, \qquad i = 1, \ldots, g - m$$

with the matrix a_i^j constant and the χ_i's are local functions on B.

Lemma 3.0.39. *Conditions C1, C2 and C3 imply that $f(S)$ is a submanifold of B of dimension $k = g - m$, and that it can be parametrized by the first k action coordinates x^j.*

Proof. The first claim follows from the fact that the horizontal part of the tangent space to S has costant dimension; the second from the Lagrangian condition which implies that the local equations of $f(S)$ in B are linear in the action coordinates. $\square$

Proposition 3.0.40. *Let $(S, \mathcal{L}, \nabla)$ be a local system supported on a Lagrangian submanifold S fulfilling the conditions $C1$ and $C2$. The condition C3 is satisfied if and only if the support $\hat{S}$ of the transform $\hat{\mathcal{L}}$ is a complex submanifold of $\hat{X}$.*

Proof. For notational convenience we suppose that $k \leq g/2$; the complementary case $k > g/2$ can be treated similarly. In the usual coordinates x, y, w we can write the local equations for S as

$$\begin{cases} y_{g-k+j} = \eta_{g-k+j}(x^1, \ldots, x^k, y_1, \ldots, y_{g-k})\,, & j = 1, \ldots, k \\ x^{k+i} = \zeta^{k+i}(x^1, \ldots, x^k)\,, & i = 1, \ldots, g-k \end{cases} \tag{3.6}$$

Since S is Lagrangian the symplectic form vanishes on S. So one has

$$\begin{cases} \delta_j^m + \displaystyle\sum_{\ell=g-k+1}^{g} \frac{\partial \zeta^\ell}{\partial x^j} \frac{\partial \eta_\ell}{\partial y_m} = 0\,, & j, m = 1, \dots, k \\ \dfrac{\partial \zeta^{k+i}}{\partial x^m} + \displaystyle\sum_{\ell=g-k+1}^{g} \frac{\partial \zeta^\ell}{\partial x^m} \frac{\partial \eta_\ell}{\partial y_{k+i}} = 0\,, & i = 1, \dots, g-2k; m = 1, \dots, k; \\ \displaystyle\sum_{\ell=g-k+1}^{g} \left[\frac{\partial \zeta^\ell}{\partial x^j} \frac{\partial \eta_\ell}{\partial x^m} - \frac{\partial \zeta^\ell}{\partial x^m} \frac{\partial \eta_\ell}{\partial x^j} \right] = 0\,, & 1 \leq j < m \leq k. \end{cases} \tag{3.7}$$

The equations of the subtori S_b, with $b = (x_1, \dots, x_k)$, can be written in a linear form

$$y_{g-k+j} = \sum_{m=1}^{g-k} a_{g-k+j}^m (x^1, \dots, x^k)\, y_m + \chi_{g-k+j}(x^1, \dots, x^k)\,, \quad j = 1, \dots, k\,. \tag{3.8}$$

To find the equations of $\hat{S}$ we shall perform a fibrewise transform and use the Künneth formula. First we split every subtorus S_b as a product of 1-dimensional tori $r_i(b)$ which have linear equations given by

$$\begin{cases} y_\ell = 0\,, & \ell = 1, \dots, g-k,\ \ell \neq i; \\ y_{g-k+j} = a_{g-k+j}^i (x^1, \dots, x^k) y_i + \chi_{g-k+j}(x^1, \dots, x^k)\,, & j = 1, \dots, k. \end{cases}$$

Observe that we can also split the local system $\mathcal{L}$ on S_b as a box product of local systems $\mathcal{L}_i(b)$ on $r_i(b)$ where $i = 1, \dots, g-k$. Transforming the local system $\mathcal{L}_i(b)$ on $r_i(b)$ we get the following equations for the support of $\mathcal{L}_i(b)$:

$$w^i + \sum_{\ell=g-k+1}^{g} a_\ell^i (x^1, \dots, x^k)\, w^\ell + \xi^i \tag{3.9}$$

where the constant term ξ^i describes the automorphy of $\mathcal{L}_i$ (here i is fixed). Then $\hat{S}$ is the intersection of the supports $\hat{r}_i$, so that its equations are of the form

$$w^{k+i} = \sum_{j=1}^{k} \tilde{\gamma}_j^{k+i} (x^1, \dots, x^k)\, w^j + \varsigma^{k+i}(x^1, \dots, x^k)\,, \quad i = 1, \dots, g-k \tag{3.10}$$

together with the second set of equations (3.6). Here we have solved with respect to $w^1, \ldots, w^k$. These equations may be used to substitute the functions η in the system (3.6), thus getting

$$\begin{cases} \delta_j^m + \displaystyle\sum_{\ell=g-k+1}^{g} \frac{\partial \zeta^\ell}{\partial x^j} a_\ell^m = 0\,, & j, m = 1, \ldots, k \\ \dfrac{\partial \zeta^{k+i}}{\partial x^m} + \displaystyle\sum_{\ell=g-k+1}^{g} \frac{\partial \zeta^\ell}{\partial x^m} a_\ell^{k+i} = 0\,, & i = 1, \ldots, g-2k; m = 1, \ldots, k \end{cases} \tag{3.11}$$

$$\sum_{\ell=g-k+1}^{g} \left[\frac{\partial \zeta^\ell}{\partial x^j} \frac{\partial \chi_\ell}{\partial x^m} - \frac{\partial \zeta^\ell}{\partial x^m} \frac{\partial \chi_\ell}{\partial x^i} \right] = 0\,, \quad 1 \leq j < m \leq k. \tag{3.12}$$

The equations in (3.11) can be rewritten as the collection of the following sets of equations for $m = 1, \ldots, k$

$$\begin{cases} \delta_j^m + \displaystyle\sum_{\ell=g-k+1}^{g} a_\ell^m \frac{\partial \zeta^\ell}{\partial x^j} = 0\,, & j = 1, \ldots, k \\ \dfrac{\partial \zeta^{k+i}}{\partial x^m} + \displaystyle\sum_{\ell=g-k+1}^{g} a_\ell^{k+i} \frac{\partial \zeta^\ell}{\partial x^m} = 0\,, & i = 1, \ldots, g-2k. \end{cases} \tag{3.13}$$

To solve this system, observe that it is similar to (3.9) (where $i = 1, ..., g-k$), in the sense that the unknowns have the same coefficients. Therefore the solution of (3.13) is

$$\frac{\partial \zeta^{k+i}}{\partial x^m} = \tilde{\gamma}_m^{k+i}\,, \qquad m = 1, \ldots, k, \quad i = 1, \ldots, g-k\,. \tag{3.14}$$

If the submanifold S is Lagrangian, the conditions (3.14) admit solutions in ζ. We must check that the support $\hat{S}$ is holomorphic, i.e., the equations that define it fulfil the Cauchy-Riemann conditions. The latters are satisfied if and only if the coefficients $\tilde{\gamma}_m^{k+i}$ do not depend on the x's, but this is true if and only if the coefficients a_{g-k+j}^m are in turn independent of the x's. As a result, we have proved that when S is Lagrangian, the tangent spaces to the S_b's are parallelly transported by ∇_{GM} if and only if $\hat{S}$ is holomorphic. □

One may note that the coefficients χ^j play no role in the specification of the complex structure of $\hat{S}$.

Remark 3.0.41. *In our setting there is no constraint on the dimension of X, the latter space is assumed to be just symplectic, and we consider local systems supported on Lagrangian submanifolds of X. On the other hand, string-theoretic mirror symmetry assumes, on physical grounds, that X is a (usually 3-dimensional) Calabi-Yau manifold, and one considers special Lagrangian supports (let us recall that a special Lagrangian submanifold of a Calabi-Yau n-fold X is an oriented real n-dimensional submanifold Y which is Lagrangian w.r.t. the Kähler form of X, and such that the global trivialization Ω of the canonical bundle of X may be chosen so that its imaginary part vanishes on Y. For more details cf. [17]). In this case, the condition that S is special Lagrangian implies, for $k = 1$, that the coefficients a_i^j's are constant, so that this is a particular case within our treatment. On the contrary, for $k = 2$ the speciality property seems to be unrelated to the conditions that ensure the support $\hat{S}$ to be complex holomorphic.*

Proposition 3.0.42. *Under the conditions of Proposition 3.0.40, the operator $\hat{\nabla}^{\mathcal{L}}$ induces on $\hat{\mathcal{L}}$ a $U(1)$ connection.*

Proof. We know that $\hat{\nabla}^{\mathcal{L}}$ induces a connection on the Fourier-Mukai transform if the curvature $\mathbb{F}$ of the Poincaré bundle on $Z = X \times_B \hat{X}$ vanishes on $S \times_B \hat{S}$, where S and $\hat{S}$ are the supports of $\mathcal{L}$ and $\hat{\mathcal{L}}$, respectively. In view of the form of $\mathbb{F}$, this condition is met if the intersections of S and $\hat{S}$ with the fibres X_b, $\hat{X}_b$ yield for all $b \in B$ subtori of X_b, $\hat{X}_b$ that are normal to each other. But looking at the equations of the supports, (3.6) and (3.10), and comparing with the absolute case (Proposition 2.0.26), we see that this condition is fulfilled. $\square$

Proposition 3.0.43. *If the support $\hat{S}$ of the transformed sheaf $\hat{\mathcal{L}}$ is a complex submanifold of $\hat{X}$, then $\hat{\mathcal{L}}$ has a holomorphic structure.*

Proof. The connection 1-form of the connection ∇ can be written in an appropriate gauge as

$$A = i\sum_{j=1}^{k} \alpha_j(x^1,\ldots,x^k)\, dx^j + 2i\,\pi \sum_{\ell=1}^{g-k} \xi^\ell\, dy_\ell\,,$$

with the quantities ξ^ℓ constant. From the proof of Proposition 2.0.26 we know that the transformed connection $\hat{\nabla}$ is given in local coordinates by the 1-form

$$\hat{A} = -2i\,\pi \sum_{\ell=g-k+1}^{g} \chi_\ell(x^1,\ldots,x^k)\, dw^\ell + i\sum_{j=1}^{k} \alpha_j(x^1,\ldots,x^k)\, dx^j\,.$$

Rewriting this in terms of $w^1,\ldots,w^k$ we obtain

$$\hat{A} = -2i\,\pi \sum_{\ell=g-k+1}^{g} \sum_{j=1}^{k} \chi_\ell(x^1,\ldots,x^k)\, \tilde{\gamma}^\ell_j\, dw^j + i\sum_{j=1}^{k} \alpha_j(x^1,\ldots,x^k)\, dx^j$$

where the coefficients $\tilde{\gamma}^\ell_j$ are constant. Since $d(\sum_j \alpha_j\, dx^j) = 0$ because of the flatness of ∇, it follows that the curvature of $\hat{\nabla}$ is given by

$$\hat{F} = -2i\,\pi \sum_{\ell=g-k+1}^{g} \sum_{j,m=1}^{k} \frac{\partial \chi_\ell}{\partial x^j}\, \tilde{\gamma}^\ell_m\, dx^j \wedge dw^m\,.$$

Since $\tilde{\gamma}^{g+j}_m = \dfrac{\partial \zeta^{g+j}}{\partial x^m}$, where the functions ζ^{g+j} are those of the equations (3.6), the equation $\hat{F}^{0,2} = 0$ can be written as

$$\sum_{\ell=g-k+1}^{g} \left[\frac{\partial \zeta^\ell}{\partial x^j}\frac{\partial \chi_\ell}{\partial x^m} - \frac{\partial \zeta^\ell}{\partial x^m}\frac{\partial \chi_\ell}{\partial x^j}\right] = 0\,, \qquad 1 \le j < m \le k.$$

These are the equations which comes from the Lagrangianity of S, more precisely they are those which were not used in the proof of the complex structure of the transformed support, therefore when S is Lagrangian, this condition is automatically satisfied. $\square$

Remark 3.0.44. *(The higher rank case.) So far we have for simplicity considered only the transformation of local systems of rank one. However the higher rank case, under the same conditions, can be treated along the same lines, obtaining on the $\hat{X}$ side holomorphic vector bundles of the corresponding rank supported on complex submanifolds of $\hat{X}$.*

The aim is now to prove that the Fourier-Mukai transform inverts. However we shall only discuss the inverse transform of rank 1 sheaves, since the higher rank case requires to consider Lagrangian submanifolds of X which ramify over B.

We shall therefore consider a holomorphic line bundle $\hat{\mathcal{L}}$ supported on a k-dimensional complex submanifold $\hat{S}$ of $\hat{X}$, equipped with a compatible $U(1)$ connection $\hat{\nabla}$. Moreover, we shall assume that:

(D1) $\hat{S}$ intersects the fibres of $\hat{X}$ along affine subtori of complex dimension k;

(D2) the horizontal part of the connection $\hat{\nabla}$ is flat (horizontality is given by the Gauss-Manin connection);

(D3) the connection $\hat{\nabla}$ is invariant under the action of T^g on $\hat{X}$ (cf. Remark 3.0.36).

These conditions allow us to write the local connection form of $\hat{\nabla}$ as

$$\hat{A} = i\sum_{j=1}^{k} \alpha_j(x^1,\ldots,x^k)\,dx^j + 2i\pi\sum_{j=1}^{k}\beta_j(x^1,\ldots,x^k)\,dw^j\,,$$

where the functions α_j satisfy (as a consequence of D2) the closure condition $\frac{\partial\alpha_j}{\partial x^\ell} = \frac{\partial\alpha_\ell}{\partial x^j}$. This shows that the restriction of $\hat{\nabla}$ to any fiber $\hat{X}_b$ of $\hat{X}\to B$ yields a flat connection on $\hat{\mathcal{L}}_{|\hat{X}_b}$.

We consider the operator

$$\nabla_{\hat{r}}^{\hat{\mathcal{L}}} = \hat{r}\circ(\hat{p}^*\nabla^{\mathcal{L}}\otimes 1 + 1\otimes\nabla_{\mathcal{P}^\vee})$$

and in terms of it we define a Fourier-Mukai transform from sheaves on $\hat{X}$ to sheaves on X (notice that we twist with the dual Poincaré bundle $\mathcal{P}^\vee$).

Proposition 3.0.45. *$\mathcal{L}$ is WIT_k, and $\mathcal{L} = R^k p_* \ker \nabla_{\hat{f}}^{\hat{\mathcal{L}}}$ is supported on a Lagrangian submanifold S of X such that the intersection $S_b = S \cap X_b$ of S is an affine subtorus of dimension $g-k$ (when nonempty). Moreover the family of subtori S_b is parallelly transported by the Gauss-Manin connection ∇_{GM}. Finally, a flat connection ∇ is naturally induced on $\mathcal{L}$.*

Proof. The WIT condition follows immediatly from Lemma 3.0.33. To show the remaining part of the claim we write local equations for $\hat{S}$ as

$$\begin{cases} x^{k+j} = \zeta^{k+j}(x^1,\dots,x^k)\,, & j = 1,\dots,g-k \\ w^{k+j} = \sum_{i=1}^k P_i^{k+j}(x^1,\dots,x^k)\, w^i + Q^{k+j}(x^1,\dots,x^k)\,, & j = 1,\dots,g-k. \end{cases}$$

Performing a fibrewise transform we obtain the following equations for the support S of the transform $\mathcal{L}$:

$$y_l + \sum_{m=k+1}^{g} P_l^m(x^1,\dots,x^k)\, y_m + \beta_l(x^1,\dots,x^k) = 0$$

where $l = 1,\dots,k$. It remains to show that S is Lagrangian and that the family $\{S_b\}_{b\in \hat{f}(S)}$ is parallelly transported by the Gauss-Manin connection. The latter point follows from the complex structure of $\hat{S}$ (cf. Proposition 3.0.40): the Cauchy-Riemann equations for $\hat{S}$ imply that the coefficients P_l^{k+j} and Q^{k+j} are constant. As far as the Lagrangian property of S is concerned, the holomorphicity of $\hat{S}$ and $\hat{\mathcal{L}}$ imply equations (3.7) in the proof of Proposition 3.0.40 . Therefore S is Lagrangian. Observe that the transformed connection ∇ has a 1-form given by

$$A = i\sum_{j=1}^{k} \alpha_j(x^1,\dots,x^k)\, dx^j - 2i\,\pi \sum_{m=k+1}^{g} Q^m\, dy_m\,,$$

whence we can immediatly deduce its flatness. $\square$

So, as a consequence, we obtain

Theorem 3.0.46. *The Fourier-Mukai transform $\mathcal{F}$ we have defined is invertible.*

This can be checked in a more intrinsic way by resorting to the absolute case. From Proposition 2.0.26 it follows that the fibrewise transforms $\mathcal{F}_b$ and $\hat{\mathcal{F}}_b$ satisfy the condition $\hat{\mathcal{F}}_b \circ \mathcal{F}_b \simeq \mathrm{Id}_{\hat{X}_b}$ for every $b \in f(S)$ and $\mathcal{F}_b \circ \hat{\mathcal{F}}_b \simeq \mathrm{Id}_{\hat{X}_b}$ for every $b \in \hat{f}(\hat{S})$. Since $\mathcal{F}_b$ and $\hat{\mathcal{F}}_b$ are the restrictions of $\mathcal{F}$ and $\hat{\mathcal{F}}$ respectively (cf. Lemma 3.0.33), it follows that the compositions $\mathcal{F} \circ \hat{\mathcal{F}}$ and $\hat{\mathcal{F}} \circ \mathcal{F}$ differ from the identity functor by a term which comes from the base B. Since the transforms $\mathcal{F}$ and $\hat{\mathcal{F}}$ preserve such terms, it follows that $\mathcal{F} \circ \hat{\mathcal{F}} \simeq \mathrm{Id}_{\hat{X}}$, $\hat{\mathcal{F}} \circ \mathcal{F} \simeq \mathrm{Id}_X$. This parallels the classical result in [23].

These results generalize the one in [2], whose authors consider the case where X and $\hat{X}$ are S^1-fibrations over S^1 ($\hat{X}$ is actually an elliptic curve) and $\mathcal{L}$ is a local system on an affine line $S \subset X$. Observe that in this case the conditions C1, C2, C3 and D1, D2, D3 are trivially satisfied.

Finally, we would like to comment upon the relation of the construction we have described in this paper with Fukaya's homological mirror symmetry. First we notice that, in the absence of the B-field and with no singular fibres, our "mirror manifold" $\hat{X}$ coincides with Fukaya's, also taking into account its complex structure. Let S be a Lagrangian submanifold of X, and $\beta = (\mathcal{L}, \nabla)$ a local system on it. Fukaya proposes to construct on $\hat{X}$ a coherent sheaf whose fibre at a point $(b, \alpha) \in \hat{X}$ (where $\alpha = (L_\alpha, \nabla_\alpha)$ is a local system on the fibre X_b) is given by the Floer homology

$$HF^\bullet((X_b, \alpha), (S, \beta)).$$

This homology may be proved [12] to be isomorphic to

$$H^{\bullet - \eta(X_b, S)}(S \cap X_b, \mathcal{H}om_\nabla(\mathcal{L}_\alpha, \mathcal{L})),$$

where $\eta(X_b, S)$ is a Maslov index, and $\mathcal{H}om_\nabla(\mathcal{L}_\alpha, \mathcal{L})$ is the sheaf of ∇-compatible morphisms between $\mathcal{L}_\alpha$ and $\mathcal{L}$. It is not difficult to show that only one of these cohomology groups does not vanish (in the correct degree), and that it is isomorphic, up to a dual, to the fibre of our transform $\hat{\mathcal{L}}$. However, the concrete construction done in [13] is not in terms of Floer homology, but it is an *ad hoc* one, which may be compared with ours when $X = T^{2g}$, $B = T^g$ and S is a Lagrangian embedding of T^g. In this case the vector bundle constructed on $\hat{X}$ coincides with ours.

It should be noted that our construction provides on the "mirror side" $\hat{X}$ more data, in that we obtain on $\hat{\mathcal{L}}$ a connection, which is not present in Fukaya's approach. It is interesting to note that this connection is not invariant under Hamiltonian diffeomorphisms of X, while the remaining geometric data on $\hat{X}$ are.

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