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Some Results on the Uniqueness in the Cauchy Problem for Partial Differential Operators

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Doctor Philosophiæ Thesis

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Contents

Introduction

Let Ω be an open set of $\mathbf{R}^n$; let φ be a smooth real function defined in Ω, such that $\varphi'(y) \neq 0$ for all $y \in \Omega$; let S be a level surface of φ, $S = \{y \in \Omega \,|\, \varphi(y) = 0\}$. We consider the following partial differential equation of order m:

$$F(y, (\partial_y^\alpha u(y))_{\alpha \in A}) = 0\,, \tag{0.1}$$

where A is a finite subset of $\mathbf{N}^n$ and $m = \max_{\alpha \in A} |\alpha|$.

If it happens that: *for two any solutions to* (0.1), u_1 *and* u_2:

$$\begin{array}{lcl} u_1 = u_2 \quad \textit{in} \quad \text{“}\textit{the past}\text{”} & & u_1 = u_2 \quad \textit{in} \quad \text{“}\textit{the future}\text{”} \\ \textit{i. e. in } \{y \in \Omega \,|\, \varphi(y) \leq 0\} & \Rightarrow & \textit{i. e. in } \{y \in \Omega \,|\, \varphi(y) \geq 0\}\,, \end{array} \tag{0.2}$$

then we say that the equation (0.1) has the *uniqueness in the Cauchy problem* with respect to S. The investigation on the relations between the equation (0.1), the surface S, and the property (0.2), is a classical subjects of the theory of partial differential equations and it is one of the pre–eminent parts of the more general topic known as the Cauchy problem.

There exist a large number of studies concerning the uniqueness in the Cauchy problem: the intent is to give sufficient conditions to obtain the property (0.2), conversely to deduce from (0.2) necessary conditions on F and S, and, as far as possible, to make necessary conditions and sufficient conditions close to one another, in order to have a criterion for the uniqueness.

In the following we will not face this problem in its maximal generality but we will bound ourself to some aspects only. First of all we will consider the problem locally, i. e. in a neighborhood of a point y_0 of S. Moreover the equation (0.1) will be taken linear. Let P be a linear partial differential operator:

$$P(y, D_y) = \sum_{|\alpha| \leq m} a_\alpha(y) D_y^\alpha\,,$$

(here and in the following D_y^α means $(1/i)^{|\alpha|}\partial_y^\alpha$). The problem we are interested in is: *to find conditions on P and S such that if there exists a neighborhood V of y_0 and a function u such that $Pu \equiv 0$ in V and $u \equiv 0$ in $V \cap \{y \in \Omega \,|\, \varphi(y) \leq 0\}$, then $u \equiv 0$ in a neighborhood of y_0.*

From the historical point of view, the first theorem on this subject goes back to a classical work of Holmgren. In [Hol] it was shown that if the coefficients of P and the function φ are analytic and S is non–characteristic, i.e. $p(y_0, \varphi'(y_0)) = \sum_{|\alpha|=m} a_\alpha(y_0)(\varphi'(y_0))^\alpha \neq 0$, then P has the uniqueness in the class of the distributions solutions. The proof is based on the Cauchy–Kowalewski theorem which makes impossible to adapt it to the case of $\mathcal{C}^\infty$ coefficients.

Many years after the result of Holmgren, Carleman proved in [Car] a uniqueness theorem for operators with real $\mathcal{C}^\infty$ coefficients and simple complex characteristics, in the two dimensional case. Apart from the very particular result obtained, the work of Carleman has a basic importance as it introduced the technique, today known under the name of Carleman estimates, which is still now the only useful tool in the proofs of the uniqueness in the Cauchy problem in the $\mathcal{C}^\infty$ framework. Let us explain it very briefly. Suppose that, given a real smooth function ψ defined on Ω, such that $\psi(y_0) = 0$, and given a neighborhood W of y_0, the following integral inequality holds:

$$\int_W e^{-2\gamma\psi}|u|^2 dy \le \frac{C}{\gamma}\int_W e^{-2\gamma\psi}|Pu|^2 dy\,, \tag{0.3}$$

for all $u \in \mathcal{C}_0^\infty(W)$ and for all $\gamma \ge \gamma_0$. ((0.3) is called Carleman estimate, where the function $e^{-\gamma\psi}$ is the corresponding weight function).

Suppose now that $w \in \mathcal{C}^\infty(W)$, $Pw \equiv 0$ in W, and $\operatorname{supp} w \subseteq \{y \in \Omega \,|\, \psi(y) \ge \delta|y-y_0|^2\}$, where $\delta > 0$. Then (0.3) implies that $w \equiv 0$ near y_0. In fact considering a fuction $\theta \in \mathcal{C}^\infty(\mathbf{R})$ such that $\theta(t) = 1$ for $t \le 1/2$ and $\theta(t) = 0$ for $t \ge 1$, we have that there exists $M > 0$ such that $w(y)\theta(M\psi(y)) \in \mathcal{C}_0^\infty(W)$. Applying (0.3) we have that:

$$\begin{aligned}
&\int_{0\le\psi(y)\le\frac{1}{3M}} e^{-2\gamma\psi}|w|^2 dy \\
&\le \int_{0\le\psi(y)\le\frac{1}{M}} e^{-2\gamma\psi}|\theta(M\psi)w|^2 dy \le \frac{C}{\gamma}\int_{0\le\psi(y)\le\frac{1}{M}} e^{-2\gamma\psi}|P(\theta(M\psi)w)|^2 dy \\
&\qquad\qquad\qquad\qquad \le \frac{C}{\gamma}\int_{\frac{1}{2M}\le\psi(y)\le\frac{1}{M}} e^{-2\gamma\psi}|P(\theta(M\psi)w)|^2 dy\,.
\end{aligned}$$

We get:

$$e^{-\frac{2\gamma}{3M}}\int_{0\le\psi(y)\le\frac{1}{3M}} |w|^2 dy \le \frac{C}{\gamma}e^{-\frac{\gamma}{M}}\int_{\frac{1}{2M}\le\psi(y)\le\frac{1}{M}} |P(\theta(M\psi)w)|^2 dy\,,$$

and finally:

$$\int_{0\le\psi(y)\le\frac{1}{3M}} |w|^2 dy \le \frac{C'}{\gamma}e^{-\frac{\gamma}{3M}}\,.$$

Letting γ go to $+\infty$ we deduce that $w \equiv 0$ for $\psi(y) \le \frac{1}{3M}$.

In the early 50's the study of the uniqueness received a new impulsion by the works of Pliś [P1] and De Giorgi [DG]: using functions defined in strips and glued together by cut off functions, they constructed operators with $\mathcal{C}^\infty$ coefficients, having $\mathcal{C}^\infty$ solutions which are identically zero for $t \le 0$ but having 0 in the support. These techniques improved by Cohen in [Co], were also used by Hörmander in [Hör1, Th.8.9.2] where he first gave a non–uniqueness theorem under simple geometrical hypotheses.

A fundamental step was made few years later by Calderón. In his famous paper of 1958 [Cal1], he showed that for operators with real principal part, if the surface of initial data is non–characteristic, the uniqueness is deduced by some properties of the roots of the characteristic polynomial. More precisely if the roots of the polynomial in τ, $p(y_0, \tau\varphi'(y_0) + \eta)$, are simple for all $\eta \in \mathbf{R}^n$ non–parallel to $\varphi'(y_0)$, then P has the uniqueness in the class of the classical solutions. The importance of this result lies not only on its conclusions about the Cauchy problem, but also on that it is one of the first and more interesting applications of the theory of pseudo–differential operators, which easily gives the way to obtain, for operators having the named properties, an inequality like (0.3).

Later on Hörmander, improving the original Calderón result, introduced, in [Hör1, Ch. 8], new general concepts like the set of principally normal operators and the strong pseudo–convexity of the surface of Cauchy data (see Chapter 2 for precise definitions and statements). Roughly speaking he related the uniqueness to some convexity properties of the hypersurface carrying the initial data with respect to the bicharacteristic curves of the operator.

Moreover, starting from the more and more precise non–uniqueness results given by Pliś [P2], [P3], [P4], Hörmander worked out in [Hör2] a general technique of construction of operators having C^∞ solutions not satisfying the uniqueness property. This technique is based on the asimptotic developments of the geometrical optic and it is now universally used for such kind of results.

These last years have seen the issue of many works on all these themes; referring to the monography of Zuily [Zu2] and to a survey paper of Alinhac [A2] for an almost complete bibliography, we want only to mention some of them.

The Calderón's theorem has been generalized in many directions, we remind the papers of Calderón [Cal2], Smith [S], Nirenberg [Ni1], Zeman [Ze1], [Ze2], Watanabe [Wa1], Watanabe and Zuily [WZ], Hörmander [Hör3, Ch. 28], and others [Rob], [Na1], [Na2], [CDS2].

The uniqueness result of Hörmander [Hör1, Th. 8.9.1] has been extended to larger classes of operators by Lerner in [L], where the definition of principally normal operator is weakened, and by Hörmander in [Hör3, Ch. 28]. Let us cite also the works of Strauss and Treves [ST], Zuily [Zu1], Alinhac [A3], Lerner and Robbiano [LR], Bahouri and Robbiano [BR], Saint Raymond [SR3], [SR4], and others [LZ], [D], [DSSR1].

Refined non–uniqueness results have been obtained by Alinhac, who essentially showed the necessity of the hypotheses of Hörmander uniqueness theorem in [A1]. Other precise discussions on the necessity of the hypotheses for the uniqueness in various different cases can be found in the papers by Alinhac and Zuily [AZ], Saint Raymond [SR1], [SR2], Robbiano [R1], [R2], Bahouri [Ba1] and others [D], [Na1], [Na2].

For the study of the degenerate operators and the questions connected with the lower order terms of the operators, let us recall the works of Watanabe [Wa2], Nirenberg [Ni2], Bahouri [Ba2], Colombini, Del Santo and Zuily [CDSZ1], [CDSZ2], for the degenerate elliptic case and the papers of Colombini, Jannelli and Spagnolo [CJS] and Colombini and Spagnolo [CS] for the weakly hyperbolic one.

Finally let us say that we have completly neglected in this introduction the so called unique continuation of solutions for the elliptic partial differential equations, which has many common aspects and similar techniques with the uniqueness of the Cauchy problem. Exhaustive references on this subject can be found in the paper of Kenig [K].

Let us come now to the contents of this Thesis. Before describing the results presented in each chapter, we want to say few words about the techniques we exploit in the proofs. A new fundamental idea from the point of view of the tools handled, is the use of para–differential operators instead of pseudo–differential ones. The theory of para–differential operators, as worked out by Bony in [Bo] and adapted to our pourposes by the introduction of symbols depending on a parameter or on a "large parameter"(essentially following the ideas of Hörmander [Hör3, Ch. 28] and Métivier [Mét2]) gives us the possibility of proving, for operators with a low regularity in the coefficients, results already known for operators with $\mathcal{C}^\infty$ leading part. In the Appendix we have collected all the definitions needed to outline this theory and all the results utilized in the proofs of the uniqueness theorems and reminded in the chapters of the Thesis. Such results on para–differential operators are well known (at least in the case of operators not depending on parameters), therefore in the Appendix we give only a sketch of some interesting proofs, providing, by the way, precise references for all the unproved statements. Apart from this the techniques are classical: we use Carleman estimates for the proofs of uniqueness, and the asimptotics of the geometrical optic in the construction of the non–uniqueness example of Paragraph 1.7.

In Chapter 1 we present some results, obtained in collaboration with F. Colombini, regarding Calderón's uniqueness theorem. In the first part of the chapter, by using para–differential operators, we prove for operators having $\mathcal{C}^3$ coefficients, the improved version of Calderón's theorem stated for operators with $\mathcal{C}^\infty$ principal part, by Nirenberg in [Ni1]. More precisely, if the principal symbol of the operator P admits the following factorization:

$$p(x,t,\xi,\tau) = \prod_{j=1}^{s}(\tau - \lambda_j(x,t,\xi))^2 \prod_{j=2s+1}^{m} (\tau - \lambda_j(x,t,\xi)) \,,$$

the uniqueness, with respect to $\{t=0\}$ at the origin, is proved under the following hypotheses: $\lambda_j \in {}^t\Gamma^1_\rho(\omega)$, with $\rho > 1$, and if $\lambda_j(x,t,\xi) = a_j(x,t,\xi) + ib_j(x,t,\xi)$ with a_j, $b_j \in \mathbf{R}$:

$$(0.4) \qquad \begin{aligned} &b_j \neq 0 \qquad \text{if } j = 1,\ldots,s\,, \\ &b_j \neq 0 \quad \text{or} \quad b_j \equiv 0 \qquad \text{if } j = 2s+1,\ldots,m\,, \end{aligned}$$

(see Paragraph A.5 for the definition of ${}^t\Gamma^m_\rho(\omega)$). In this way we recover the Calderón's result as stated in [Cal2]. Moreover if for simple roots, i. e. for $j = 2s+1,\ldots,m$, the following conditions are verified: $\lambda_j \in {}^t\Gamma^1_3(\omega)$ and:

$$(0.5) \qquad \begin{aligned} &b_j \geq 0\,, \\ \text{or}& \\ &Cb_j + \partial_t b_j + \sum_{k=1}^{n-1}(\partial_{x_k} a_j \partial_{\xi_k} b_j - \partial_{x_k} b_j \partial_{\xi_k} a_j) \leq 0\,, \end{aligned}$$

then we prove the compact uniqueness with respect to $\{t=0\}$ at the origin (i.e. uniqueness with respect to all the surfaces lying in $\{t \geq 0\}$, tangent to $\{t=0\}$ and having $\{0\}$ as intersection with $\{t=0\}$). Apart from the regularity of the coefficients, condition (0.5) is slightly weaker than Nirenberg's one and it can be found in [Hör3, Prop. 28.1.6]. All these claims are proved by showing a certain number of Carleman estimates of type (0.3), where the weight function is $e^{\gamma(t-T)^2}$. The estimates on P are obtained by a method similar to the original Calderón's one, starting from estimates on first order para–differential operators.

In the second part of the chapter we prove some other Carleman estimates for first order para–differential operators using singular weight functions like $t^{-\gamma}$ and $e^{\gamma/t}$. By these estimates we obtain some new results: e. g. for operators of order two we can weaken the condition (0.5) up to:

$$
b_j \geq 0\,,
$$

or

(0.6)
$$
b_j + t(\partial_t b_j + \sum_{k=1}^{n-1}(\partial_{x_k} a_j \partial_{\xi_k} b_j - \partial_{x_k} b_j \partial_{\xi_k} a_j)) \leq 0\,,
$$

proving under this condition the the $\mathcal{C}^\infty$–compact uniqueness (i.e. compact uniqueness for $\mathcal{C}^\infty$ solutions). Moreover, following the works of Roberts [Rob], and Nakane [Na1], [Na2], the last estimates permit us also to show new uniqueness theorems for second order singular operators of Fuchsian type and second order degenerate operators admitting a factorization like, e. g.:

$$
p_1(x,t,\xi,\tau) = (\tau - t^k\lambda_1(x,t,\xi))(\tau - t^k\lambda_2(x,t,\xi))\,,
$$

or

$$
p_2(x,t,\xi,\tau) = (\tau - e^{-\frac{1}{t}}\lambda_1(x,t,\xi))(\tau - e^{-\frac{1}{t}}\lambda_2(x,t,\xi))\,.
$$

Finally we give a non–uniqueness example, showing that the condition (0.6) is almost necessary at least with respect to a family of operators. Precisely let α be ≥ 0 and consider the operator:

$$
p_\alpha(x,t,D_x,D_t) = D_t^2 - D_{x_1}^2 - D_{x_2}^2 + i(D_{x_1}^2 + 2tD_{x_1}D_{x_2} + (1+\alpha)t^2D_{x_2}^2)\,.
$$

If $\alpha > 1/3$ then the roots of the characteristic polynomial of p_α satisfy (0.6) and therefore $p_\alpha +$ *lower order terms* has the $\mathcal{C}^\infty$–compact uniqueness. If $0 \leq \alpha < 1/3$ we show that there exist two $\mathcal{C}^\infty$ functions a, u such that $0 \in \operatorname{supp} u \subseteq \{t \geq 0\}$ and $p_\alpha u + au \equiv 0$.

Chapter 2 is devoted to the Hörmander's uniqueness theorem. We state a result obtained in collaboration with X. Saint Raymond, [DSSR1]. The operator P, having principal symbol p, is said to be principally normal in weak sense if:

$$
\mathrm{H}_p\bar{p}(y,\eta) = 2i\,\mathrm{Re}(\bar{q}(y,\eta)p(y,\eta))\,,
$$

where $\mathrm{H}_p = \sum_j \partial_{\eta_j} p(y,\eta)\partial_{y_j} - \partial_{y_j} p(y,\eta)\partial_{\eta_j}$ is the Hamiltonian vector field associated to p on the cotangent boundle $T^\star\Omega$, and q is a L^∞_{loc} function defined on $T^\star\Omega$, homogeneous of degree $m-1$ in η.

The Hörmander's uniqueness theorem states that if P has $\mathcal{C}^\infty$ principal part and it is principally normal in weak sense, and S is strongly pseudo–convex with respect to P at y_0 (see Definition 2.3), then P has the uniqueness in the Cauchy problem. The same in true if P has $\mathcal{C}^2$ principal part and q is $\mathcal{C}^1$ in the y's and $\mathcal{C}^\infty$ in the η's. The para–differential operators depending on a large parameter, introduced by Métivier in [Mét2], play a central rôle in the proof, which is different from Hörmander's one, as we do not dispose of a Fefferman–Phong inequality for para–differential operators.

Chapter 3 is the precise reproduction of a joint work with X. Saint Raymond. Dealing with second order operators with real principal part, the hypotheses of Hörmander's uniqueness theorem are very simple: the strong pseudo–convexity of the hypersurface S with respect to P at y_0, turn out to be equivalent to the following condition: for all $\eta \in \mathbf{R}^n \setminus \{0\}$:

$$p(y_0,\eta) = \mathrm{H}_p\varphi(y_0,\eta) = 0 \qquad \Rightarrow \qquad \mathrm{H}_p^2(y_0,\eta) < 0\,. \tag{0.7}$$

Under (0.7), P has the uniqueness with respect to S at y_0. If the operator P is of principal type, i.e.

$$\partial_\eta p(y_0,\eta) \neq 0\,,$$

for all $\eta \in \mathbf{R}^n \setminus \{0\}$, then Lerner and Robbiano showed in [LR], that P has the compact uniqueness under the following weaker form of (0.7): for all $\eta \in \mathbf{R}^n \setminus \{0\}$ and for all $y \in \Omega$:

$$\varphi(y) = p(y,\eta) = \mathrm{H}_p\varphi(y,\eta) = 0 \qquad \Rightarrow \qquad \mathrm{H}_p^2(y,\eta) \leq 0\,.$$

As usual this result is obtained for operators with $\mathcal{C}^\infty$ principal part. We give an alternative proof for this theorem, and we obtain the same conclusion for operators with coefficients in $\mathcal{C}^{7/3}$. Again our tool is the theory of para–differential operators. In some particular cases this result is proved for operators with $\mathcal{C}^{5/3}$ or $\mathcal{C}^1$ coefficients.

Chapter 1. Calderón's Uniqueness Theorem

1.1. Introduction

The aim of the first part of this chapter is to show how para–differential operators can be used to prove the classical Calderón's uniqueness theorem [Cal1, Th. 6]. In particular we will state and prove a uniqueness result for operators with non–smooth coefficients in the principal part and Calderón's theorem will follow. Our generalization is inspired to a work of Nirenberg [Ni1] on operators with $\mathcal{C}^\infty$ leading part, and our main idea is to substitute pseudo–differential operators with para–differential ones, avoiding in this way the smooth regularity assumption.

The proof of the main result is very similar to Nirenberg's one and it is divided into two fundamental steps: the Calderón's reduction of the equation to a first order system, and some Carleman estimates for first order para–differential operators. In establishing the estimates as well as in the reduction, the properties of para–differential operators will be essential.

As they may have a certain interest independently on the application in the proof of the Calderón's theorem, we first state and prove the Carleman estimates in Paragraph 1.3 and then we only give a sketch of the proof of the main result in Paragraph 1.4.

The second part of the chapter, which summarizes a joint work with F. Colombini [CDS2], is devoted to prove other Carleman estimates for first order para–differential operators and to show some applications of them.

In particular in Paragraph 1.5 we prove Carleman estimates with a singular weight function and, following the ideas of Roberts [Rob] and Nakane [Na1], [Na2], we deduce some uniqueness theorems for some classes of operators with non–smooth coefficients, namely Fuchsian type operators and degenerate operators.

In Paragraph 1.6 the same is done for more singular weight functions, and we obtain a result for operators which have an infinite order degeneration.

In the last paragraph we present an interesting example. We consider the following operator:

$$p_\alpha(x,t,D_x,D_t) = D_t^2 - D_{x_1}^2 - D_{x_2}^2 + i(D_{x_1}^2 + 2tD_{x_1}D_{x_2} + (1+\alpha)t^2D_{x_2}^2) \quad .$$

If $\alpha > 1/3$, then the hypotheses under which the Carleman estimates of the Paragraph 1.5 hold, are verified, therefore the operator $p_\alpha+$ *lower order terms* has the $\mathcal{C}^\infty$–compact uniqueness. If $0 \leq \alpha < 1/3$, the previous hypotheses are no more valid and the surprising fact is that there exist $u, a \in \mathcal{C}^\infty$ such that $0 \in \operatorname{supp} u \subseteq \{t \geq 0\}$ and $p_\alpha u + au \equiv 0$: this example gives a partial information about the necessity of the hypotheses of our version of Calderón's uniqueness theorem. Moreover the operator p_α satisfies , for all $\alpha \geq 0$, the condition (P) of Nirenberg and Treves [NT1], [NT2], [NT3], which consequently is not sufficient to guarantee the uniqueness in the Cauchy problem.

1.2. Calderón's Uniqueness Theorem

The celebrated uniqueness theorem of Calderón [Cal1, Th. 6] gives sufficient conditions for the uniqueness of the solutions to the Cauchy problem for a partial differential operator in terms of the roots of the principal part of the operator itself. To recall this result precisely let us introduce some notation. Let Ω be an open set of $\mathbf{R}^n$; let P be a linear partial differential operator of order m, with complex bounded coefficients in Ω,:

$$P(y, D_y) = \sum_{|\alpha| \leq m} a_\alpha(y) D_y^\alpha \quad , \tag{1.1}$$

with $a_\alpha \in L^\infty_{\mathrm{loc}}(\Omega, \mathbf{C})$. The operator:

$$p(y, D_y) = \sum_{|\alpha| = m} a_\alpha(y) D_y^\alpha \quad , \tag{1.2}$$

will be called principal part of the operator P, and the function:

$$p(y, \eta) = \sum_{|\alpha| = m} a_\alpha(y) \eta^\alpha \quad , \tag{1.3}$$

the principal symbol of P. Let S be a $\mathcal{C}^2$ hypersurface in Ω, $S = \{y \in \Omega \,|\, \varphi(y) = 0\}$, where φ is a $\mathcal{C}^2$ real valued function such that $\varphi' \neq 0$ on S. Let y_0 be a point of Ω such that $y_0 \in S$. We set $S^+ = \{y \in \Omega \,|\, \varphi(y) \geq 0\}$.

We require that S is non–characteristic at y_0, with respect to P, i.e.:

$$p(y_0, \varphi'(y_0)) \neq 0 \quad .$$

Theorem 1.1 [Cal1, Th. 6]. *Suppose that there exists $\rho > 1$ such that $a_\alpha \in \mathcal{C}^\rho(\Omega, \mathbf{R})$ for all $\alpha \in \mathbf{N}^n$ such that $|\alpha| = m$ (for the definition of $\mathcal{C}^\rho$ see Paragraph A.1). Suppose that the polynomial in τ:*

$$p(y_0, \tau\varphi'(y_0) + \eta) \tag{1.4}$$

has, for all $\eta \in \mathbf{R}^n \setminus \{0\}$, η non–parallel to $\varphi'(y_0)$, m distinct roots.

Then P has the uniqueness in the Cauchy problem with respect to S at y_0, i.e. if $u \in \mathcal{C}^m(\Omega)$, $Pu \equiv 0$ near y_0 and $\operatorname{supp} u \subseteq S^+$, then $u \equiv 0$ near y_0.

This result was obtained in the last 50's as an application of the theory of singular integral operators Calderón was working out in those years (let us remark that the original statement in [Cal1] has some more technical hypotheses, see also [Cal2]).

In his book of 1963, Hörmander gave a completly different proof of this result extending it to the case of $\mathcal{C}^1$ real or $\mathcal{C}^2$ complex principal part (see Theorem 2.4).

Later on, with the progress of the pseudo–differential operators theory, the result was revisited and refined by several authors (see [S],[WZ],[Zu2], [Ze1],[Ze2]), each of them giving weaker conditions on the roots to satisfy for the same conclusion, but, as the classical theory of pseudo–differential operators works with smooth symbols, requiring essentially that the coefficients of the principal part and the roots of (1.4) are C^∞ functions. Nirenberg's version [Ni1, Th. 5 and Th. 5'] can be considered the model of such results. In order to present it, let us observe that there exists a change of variables such that in the new coordinates S and y_0 become the hypersurface $\{(x_1, \ldots, x_{n-1}, t) \in \Omega \mid t = 0\}$ and the origin respectively. The non–characteristic condition on S implies that the operator P has, up to a non zero factor, the following Kowalewskian form:

$$P(x,t,D_x,D_t) = D_t^m + \sum_{\substack{|\alpha|+j \leq m \\ j<m}} a_{\alpha,j}(x,t) D_x^\alpha D_t^j \quad , \tag{1.5}$$

(in the following we call S the hypersurace $\{t = 0\}$).

Theorem 1.2 [Ni1, Th. 5 and Th. 5']. *Assume that for the operator P defined in (1.5), $a_{\alpha,j} \in C^\infty(\Omega, \mathbf{C})$ for all $\alpha \in \mathbf{N}^{n-1}$, $j \in \mathbf{N}$ such that $|\alpha| + j = m$. Assume that there exists a neighborhood of the origin, V, such that the multiplicity of the roots of the polynomial in τ, $p(x,t,\xi,\tau)$, is constant and ≤ 2, for all $(x,t,\xi) \in V \times S^{n-2}$ (here and in the following $S^{n-2} = \{\xi \in \mathbf{R}^{n-1} \mid |\xi| = 1\}$), and p admits the following factorization:*

$$p(x,t,\xi,\tau) = \prod_{j=1}^{s} (\tau - \lambda_j(x,t,\xi))^2 \prod_{j=2s+1}^{m} (\tau - \lambda_j(x,t,\xi)) \quad , \tag{1.6}$$

with $\lambda_j \neq \lambda_k$ if $j \neq k$, and $\lambda_j(x,t,\xi) = a_j(x,t,\xi) + i b_j(x,t,\xi)$, where a_j, b_j are continuous real valued functions defined on $V \times \mathbf{R}^{n-1} \setminus \{0\}$. Suppose that, for $1 \leq j \leq s$,

$$b_j(x,t,\xi) \neq 0 \qquad \text{for all} \quad (x,t,\xi) \in V \times S^{n-2} \quad , \tag{1.7}$$

and, for each $2s+1 \leq j \leq m$, one of the following conditions is true:

$$b_j(x,t,\xi) \geq 0 \qquad \text{for all} \quad (x,t,\xi) \in V \times S^{n-2} \quad ; \tag{1.8}$$

$$(Cb_j + \partial_t b_j + \sum_{k=1}^{n-1} (\partial_{x_k} a_j \partial_{\xi_k} b_j - \partial_{x_k} b_j \partial_{\xi_k} a_j))(x,t,\xi) \leq 0 \quad , \tag{1.9}$$

for some constant $C > 0$ and for all $(x,t,\xi) \in V \times S^{n-2}$.

Then P has the compact uniqueness in the Cauchy problem with respect to S at the origin, i.e. there exists W, open neighborhood of the origin, $W \subseteq V$, such that if $u \in C^m(V)$, $Pu \equiv 0$ near the origin, $\operatorname{supp} u \cap S \subset\subset W$ and $\operatorname{supp} u \subseteq S^+$, then $u \equiv 0$ near the origin.

Moreover, if for each $2s+1 \le j \le m$, (1.7) or:

$$b_j(x,t,\xi) \equiv 0 \qquad \text{for all} \quad (x,t,\xi) \in V \times S^{n-2} \quad , \tag{1.10}$$

holds, then P has the uniqueness in the Cauchy problem with respect to S at the origin.

Even if the hypotheses of Theorem 1.2 are less restrictive than those of Theorem 1.1, the latter is a corollary of the former only in the case of smooth coefficients in the principal part. This unpleasant situation can be avoided: by using the Bony's para–differential operators theory [Bo] and its variants for operators depending on a parameter, as presented in the Appendix, we will prove the following result.

Theorem 1.3. *In the hypotheses of Theorem 1.2, suppose that $a_{\alpha,j} \in \mathcal{C}^3(\Omega, \mathbf{C})$ for all $\alpha \in \mathbf{N}^{n-1}$, $j \in \mathbf{N}$ such that $|\alpha| + j = m$.*

Then the same conclusions of Theorem 1.2 hold.

Remark 1.4. The proof of Theorem 1.3 will show that if conditions (1.7) and (1.10) are satisfied, then $\mathcal{C}^\rho$–regularity, with $\rho > 1$, is sufficient to guarantee uniqueness, and consequently Theorem 1.1 is recovered. Moreover using the proof of Theorem 1.3 together with a localization argument it is possible to show that a result similar to Theorem 1.1 is valid for operators with $\mathcal{C}^\rho$ complex principal part admitting double complex roots (see [Cal2] and [Hör3, Th. 28.1.1]).

Remark 1.5. As it will be clear in the proof it would be sufficient to require the conditions (1.7), (1.8), (1.9) and (1.10) to hold only in $S^+ \times S^{n-2}$. Let us observe that hypotheses (1.8) and (1.9) are not symmetric with respect to t: this is not surprising as we are proving forward continuation only. Moreover, as already pointed out by Nirenberg, it would be sufficient that only one of the conditions (1.8) or (1.9) holds locally, i.e. for all $\xi_0 \in S^{n-2}$ there exists a neighborhood of $(0,0,\xi_0)$ in which one of the two named conditions holds.

Remark 1.6. There exist in literature many uniqueness results which don't require the constant multiplicity condition, substituting it with a smooth regularity condition on the roots of polynomial (1.5) (see [WZ],[Ze2],[Zu2]). It is not known if analogous results holds for less regular roots, i.e. in the hypotheses of Theorem 1.3, as the technique of the proof of these results is not directy applicable to the situation presented in the following paragraphs.

Example 1.7. The following simple example is inspired to an example of [Ni1]. Suppose that f is a $\mathcal{C}^3$ real function and there exists $C > 0$ such that $f(0) = 0$ and $Cf(t) + f'(t) \ge 0$ for $t \ge 0$. Then the operator:

$$P_1 = (D_t - ifD_x)(D_t^2 + D_x^2) + \textit{lower order terms}$$

satisfies the hypotheses of Theorem 1.3 and consequently has the compact uniqueness in the Cauchy problem with respect to $\{t = 0\}$ at the origin.

Analogously, consider the operator:

$$P_2 = D_t^2 - D_x^2 - D_y^2 + if(t)(D_x^2 + 2D_y^2) + \textit{lower order terms} .$$

P_2 satisfies the hypotheses of Theorem 1.3 and has the compact uniqueness.

Example 1.8. Let $(a_{j,k}(t))$ be a $(n-1)\times(n-1)$ matrix of C^3 real functions. Suppose that there exists $C > 0$ such that the matrix:

$$(Ca_{j,k}(t) + a'_{j,k}(t))$$

is positive semidefinite. Then the operator:

$$P_3 = D_t^2 - \sum_{j=1}^{n-1} D_{x_j}^2 + i \sum_{j,k=1}^{n-1} a_{j,k}(t) D_{x_j} D_{x_k} + \textit{lower order terms}$$

satisfies the hypotheses of Theorem 1.3 and consequently P_3 has the compact uniqueness.

1.3. Carleman Estimates for Para–Differential Operators of First Order

As said, we start proving Theorem 1.3 giving some Carleman estimates for para–differential operators of first order. We will use the notation and the results collected in the Appendix. ${}^t\Sigma_\rho^m(\omega)$ will indicate the set of para–differential symbols of order m, regularity ρ, depending on a parameter t, and defined on ω. ${}^t\Gamma_\rho^m(\omega) \subseteq {}^t\Sigma_\rho^m(\omega)$ will be the set of the homogeneous symbols. Let $l \in {}^t\Sigma_\rho^m(\omega)$ be a para–differential symbol; we will denote with T_l the para–differential operator of symbol l. Let now ω be a neighborhood of the origin in $\mathbf{R}^{n-1}$ and $T > 0$ a constant such that $\omega \times [0, T] \subseteq V$, the set in which conditions (1.7), (1.8), (1.9) and (1.10) hold; according to the Definition A.25, under the hypotheses of Theorem 1.3, the roots λ_j are in ${}^t\Gamma_3^1(\omega)$. The first lemma will provide two Carleman inequalities under conditions (1.7) and (1.10).

Lemma 1.9. *Let $\rho > 1$ and let $\lambda = a + ib$, where a and b are real symbols of ${}^t\Gamma_\rho^1(\omega)$. Assume that:*

$$b(x,t,\xi) \neq 0 \qquad \textit{for all} \quad (x,t,\xi) \in \omega \times [0,T] \times S^{n-2} \quad , \tag{1.11}$$

or

$$b(x,t,\xi) \equiv 0 \qquad \textit{for all} \quad (x,t,\xi) \in \omega \times [0,T] \times S^{n-2} \quad . \tag{1.12}$$

Then there exist T, r, γ_0 and C positive constants such that, for $\gamma > \gamma_0$,:

$$\int_0^T e^{2\gamma(t-T)^2} \|u\|^2 dt \le \frac{C}{\gamma} \int_0^T e^{2\gamma(t-T)^2} \|D_t u - T_\lambda u\|^2 dt \quad , \tag{1.13}$$

and, under condition (1.11),

$$(1.14)\quad \int_0^T e^{2\gamma(t-T)^2} \sum_{|\alpha|+j\le 1} \|D_x^\alpha D_t^j u\|^2 dt \le C(1+T^2\gamma) \int_0^T e^{2\gamma(t-T)^2} \|D_t u - T_\lambda u\|^2 dt \quad ,$$

for all $u \in C_0^\infty$ *such that* $\operatorname{supp} u \subseteq \bar{B}_r \times [0, T/2]$ ($\|\cdot\|$ *denotes the norm of* $L^2(\mathbf{R}_x^{n-1})$, $\|\cdot\|_s$ *denotes the norm of* $H^s(\mathbf{R}_x^{n-1})$, *and* $B_r = \{x \in \mathbf{R}^{n-1} \mid |x| < r\}$).

Proof. We set $v = e^{\gamma(t-T)^2} u$, where $u \in C_0^\infty$ with $\operatorname{supp} u \subseteq \bar{B}_r \times [0, T/2]$. We obtain:

$$e^{\gamma(t-T)^2}(D_t u - T_\lambda u) = D_t v - T_a v - iT_b v + 2i\gamma(t-T)v \quad .$$

So:

$$(1.15)\quad \begin{aligned} &\int_0^T e^{2\gamma(t-T)^2} \|D_t u - T_\lambda u\|^2 dt \\ &\qquad = \int_0^T \|D_t v - T_a v\|^2 dt + \int_0^T \|T_b v - 2\gamma(t-T)v\|^2 dt \\ &\qquad\qquad + 2\mathrm{Re} \int_0^T (D_t v - T_a v \,,\, -iT_b v + 2i\gamma(t-T)v) dt \\ &\qquad = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 \quad , \end{aligned}$$

where:

$$\begin{aligned} J_1 &= \int_0^T \|D_t v - T_a v\|^2 dt \quad ; \\ J_2 &= \int_0^T \|T_b v - 2\gamma(t-T)v\|^2 dt \quad ; \\ J_3 &= 2\mathrm{Re} \int_0^T (D_t v \,,\, -iT_b v) dt \quad ; \\ J_4 &= 2\mathrm{Re} \int_0^T (D_t v \,,\, 2i\gamma(t-T)v) dt \quad ; \\ J_5 &= 2\mathrm{Re} \int_0^T (-T_a v \,,\, -iT_b v) dt \quad ; \\ J_6 &= 2\mathrm{Re} \int_0^T (-T_a v \,,\, 2i\gamma(t-T)v) dt \quad . \end{aligned}$$

Easy computations, together with the Theorem A.20 (see also Paragraph A.5), give:

$$\begin{aligned} J_4 &= 2\gamma \int_0^T \|v\|^2 dt \quad , \\ J_6 &= \mathrm{Re} \int_0^T 2i\gamma(t-T)((T_a - T_a^*)v \,,\, v) \ge -CT\gamma \int_0^T \|v\|^2 dt \quad , \end{aligned}$$

and this is enough to prove (1.13) under condition (1.12), if T is sufficiently small. Let now (1.11) hold. From the inequality: $\|D_t v\| \leq \|D_t v - T_a v\| + \|T_a v\|$, we immediately deduce that: for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that:

$$\|D_t v\| \, \|v\| \leq \varepsilon \|D_t v - T_a v\|^2 + C_\varepsilon \|v\|^2 + C\|v\|_1 \|v\| \quad ,$$

Using this, the Theorem A.26 and the Theorem A.20, we obtain:

$$\begin{aligned} J_3 &= -\operatorname{Re}\int_0^T (T_{\partial_t b} v\, , \, v)dt + \operatorname{Re}\int_0^T (v\, , \, i(T_b^* - T_b)D_t v)dt \\ &\geq -\frac{1}{2}J_1 - C(\int_0^T \|v\|_1 \|v\| dt + \int_0^T \|v\|^2 dt) \quad . \end{aligned} \tag{1.16}$$

Moreover:

$$J_5 = \operatorname{Re}\int_0^T i((T_a^* T_b - T_b^* T_a)v\, , \, v)dt \geq -C\int_0^T \|v\|_1 \|v\| dt \quad , \tag{1.17}$$

where, as always, C represents different constants independent from v, γ, T. Now we use the ellipticity of b. By the Theorem A.22 (see also Paragraph A.5) we have that:

$$\|v\|_1 \leq C(\|T_b v\| + \|v\|) \quad .$$

Consequently:

$$\|v\|_1 \leq C(\|T_b v - 2\gamma(t-T)v\| + (1 + T\gamma)\|v\|) \quad , \tag{1.18}$$

and so we obtain that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that:

$$\int_0^T \|v\|_1 \|v\| dt \leq \varepsilon J_2 + C_\varepsilon (1 + T\gamma) \int_0^T \|v\|^2 dt \quad .$$

Therefore, by (1.15), (1.16) and (1.17), if T^{-1} and γ_0 are sufficiently great,:

$$\int_0^T e^{2\gamma(t-T)^2} \|D_t u - T_\lambda u\|^2 dt \geq \frac{1}{2}J_1 + \frac{1}{2}J_2 + \gamma \int_0^T \|v\|^2 dt \quad , \tag{1.19}$$

which implies (1.13). Moreover, from (1.18),:

$$\int_0^T \|v\|_1^2 dt \leq CJ_2 + C(1 + T^2\gamma^2) \int_0^T \|v\|^2 dt \quad ,$$

and from this and (1.19) we obtain that:

$$\int_0^T \|v\|_1^2 dt \leq C(1 + T^2\gamma) \int_0^T e^{2\gamma(t-T)^2} \|D_t u - T_\lambda u\|^2 dt \quad ,$$

from which (1.14) is easily obtained. This ends the proof.

The following lemma will furnish the estimate (1.13) under the hypothesis (1.8).

Lemma 1.10. *Let $\rho \geq 2$ and let $\lambda = a + ib$, where a and b are real symbols of ${}^t\Gamma^1_\rho(\omega)$. Assume that:*

$$(1.20) \qquad b(x,t,\xi) \geq 0 \qquad \text{for all} \quad (x,t,\xi) \in \omega \times [0,T] \times S^{n-2} \quad .$$

Then (1.13) follows.

Proof. With the notation in the proof of Lemma 1.9,:

$$\begin{aligned}
-\mathrm{Im} \int_0^T & e^{\gamma(t-T)^2}(D_t u - T_\lambda u\,,\, v)dt \\
&= -\mathrm{Im} \int_0^T (D_t v - T_\lambda v + 2i\gamma(t-T)v\,,\, v)dt \\
&= -\mathrm{Im} \int_0^T (-T_\lambda v\,,\, v)dt - 2\gamma \int_0^T (t-T)\|v\|^2 dt \\
&= \mathrm{Re} \int_0^T (T_{-i\lambda} v\,,\, v)dt + \frac{T}{2}\gamma \int_0^T \|v\|^2 dt \quad .
\end{aligned}$$

From this and the Schwarz inequality we deduce:

$$\int_0^T e^{2\gamma(t-T)^2}\|D_t u - T_\lambda u\|^2 dt \geq \mathrm{Re} \int_0^T (T_{-i\lambda} v\,,\, v)dt + (\frac{T}{2}\gamma - C) \int_0^T \|v\|^2 dt \quad ,$$

and we get (1.13) by bounding from below the first right–hand–side term, using (1.20) and the sharp Gårding inequality for para–differential operators as stated in the Theorem A.24 (see also Paragraph A.5). The proof is concluded.

Let us finally state the Carleman estimate regarding the last situation, i.e. condition (1.9).

Lemma 1.11. *Let $\rho \geq 3$ and let $\lambda = a + ib$, where a and b are real symbols of ${}^t\Gamma^1_\rho(\omega)$. Assume that there exists $C > 0$ such that:*

$$(1.21) \qquad (Cb + \partial_t b + \sum_{k=1}^{n-1}(\partial_{x_k} a \partial_{\xi_k} b - \partial_{x_k} b \partial_{\xi_k} a))(x,t,\xi) \leq 0 \quad ,$$

for all $(x,t,\xi) \in \omega \times [0,T] \times S^{n-2}$.

Then (1.13) follows.

Proof. Following again the notation used in the previous proofs, and defining:

$$
\begin{aligned}
\tilde a(t,x,\xi) &= a(t,x,\xi) - \frac{i}{2}\sum_{k=1}^{n-1}\partial_{x_k}\partial_{\xi_k}a(t,x,\xi) \quad ,\\
\tilde b(t,x,\xi) &= b(t,x,\xi) - \frac{i}{2}\sum_{k=1}^{n-1}\partial_{x_k}\partial_{\xi_k}b(t,x,\xi) \quad ,
\end{aligned} \tag{1.22}
$$

we have that $T_a - T_{\tilde a}$ and $T_b - T_{\tilde b}$ are bounded in L^2 (see Paragraph A.5); therefore:

$$
\begin{aligned}
2\int_0^T e^{2\gamma(t-T)^2}\|D_t u - T_\lambda u\|^2 dt \geq \int_0^T &\|D_t v - T_{\tilde a}v - iT_{\tilde b}v + 2i\gamma(t-T)v\|^2 dt\\
&-C\int_0^T \|v\|^2 dt \quad .
\end{aligned}
$$

Calling I the first right–hand–side term, we have that:

$$
\begin{aligned}
I = &\int_0^T \|D_t v - T_{\tilde a}v\|^2 dt + \int_0^T \|T_{\tilde b}v - 2\gamma(t-T)v\|^2 dt\\
&+ 2\mathrm{Re}\int_0^T (D_t v - T_{\tilde a}v\,,\, -iT_{\tilde b}v + 2i\gamma(t-T)v)dt\\
=&\tilde J_1 + \tilde J_2 + \tilde J_3 + \tilde J_4 + \tilde J_5 + \tilde J_6 \quad ,
\end{aligned} \tag{1.23}
$$

where, as in Lemma 1.9,:

$$
\begin{aligned}
\tilde J_4 &= 2\mathrm{Re}\int_0^T (D_t v\,,\, 2i\gamma(t-T)v)dt = 2\gamma\int_0^T \|v\|^2 dt \quad ,\\
\tilde J_6 &= 2\mathrm{Re}\int_0^T (-T_{\tilde a}v\,,\, 2\gamma(t-T)v)dt \geq -CT\gamma\int_0^T \|v\|^2 dt \quad .
\end{aligned}
$$

Moreover, by the choice of $\tilde a$ and $\tilde b$, and by the Theorem A.26 and the Theorem A.20, remarking that $T^*_{\tilde b} - T_{\tilde b}$ is 1–regularizing, we obtain:

$$
\begin{aligned}
\tilde J_3 &= 2\mathrm{Re}\int_0^T (D_t v\,,\, -iT_{\tilde b}v)dt = -\mathrm{Re}\int_0^T (v\,,\, T_{\partial_t \tilde b}v)dt + \mathrm{Re}\int_0^T (v\,,\, i(T^*_{\tilde b} - T_{\tilde b})D_t v)dt\\
&\geq -\mathrm{Re}\int_0^T (T_{\partial_t b}v\,,\, v)dt - C\int_0^T \|v\|^2 dt - \frac{1}{2}\tilde J_1 \quad ,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde J_5 &= 2\mathrm{Re}\int_0^T (-T_{\tilde a}v\,,\, -iT_{\tilde b}v)dt\\
&\geq \mathrm{Re}\int_0^T (T_{-\sum_{k=1}^{n}(\partial_{x_k}a\partial_{\xi_k}b - \partial_{x_k}b\partial_{\xi_k}a)}v\,,\, v)dt - C\int_0^T \|v\|^2 dt \quad ,
\end{aligned}
$$

By these estimates and (1.23), we get:

$$
(1.24)\qquad \begin{aligned} I \geq \frac{1}{2}\tilde{J}_1+\tilde{J}_2 + ((2-CT)\gamma - C)\int_0^T \|v\|^2 dt \\ + \operatorname{Re}\int_0^T (T_{-\partial_t b - \sum_{k=1}^{n-1}(\partial_{x_k} a \partial_{\xi_k} b - \partial_{x_k} b \partial_{\xi_k} a)} v\, ,\, v)dt \end{aligned} \quad .
$$

Let us now use the condition (1.21) and define the symbol:

$$
r(x,t,\xi) = -(Cb + \partial_t b + \sum_{k=1}^{n-1}(\partial_{x_k} a \partial_{\xi_k} b - \partial_{x_k} b \partial_{\xi_k} a))(x,t,\xi) \quad .
$$

Obviously $r \in {}^t\Gamma^1_2(\omega)$ and $r \geq 0$ in $\omega \times [0,T] \times S^{n-2}$. Using the sharp Gårding inequality, it follows that there exist $C > 0$ such that:

$$
\operatorname{Re}\int_0^T (T_r v\, ,\, v)dt \geq -C\int_0^T \|v\|^2 dt \quad ,
$$

for all $v \in C_0^\infty$ such that $\operatorname{supp} u \subseteq \bar{B}_r \times [0, T/2]$. Now it is easy to prove that:

$$
(1.25)\qquad \begin{aligned} \operatorname{Re}\int_0^T (T_{-\partial_t b - \sum_{k=1}^{n}(\partial_{x_k} a \partial_{\xi_k} b - \partial_{x_k} b \partial_{\xi_k} a)} v\, ,\, v)dt \\ \geq -C(1+T\gamma)\int_0^T \|v\|^2 dt - \frac{1}{2}\tilde{J}_2 \end{aligned} \quad .
$$

By choosing T^{-1} and γ_0 sufficiently great, (1.24) and (1.25) imply (1.13). The proof is complete.

Remark 1.12. If all the roots satisfy condition (1.7) or (1.10), C^ρ–regularity, with $\rho > 1$, is enough for the conclusions of the Theorem 1.3 to hold. In fact under the named conditions Lemma 1.9 furnishes all the estimates needed in the next proof. Analogously, if only (1.7), (1.8) or (1.10) are involved C^2–regularity is sufficient to the compact uniqueness.

Remark 1.13. The example of Paragraph 1.7 shows that the condition:

$$
(1.26)\qquad b(x,t,\xi) \leq 0 \qquad \textit{for all} \quad (x,t,\xi) \in [0,T] \times \omega \times S^{n-2}
$$

is not sufficient to guarantee the uniqueness for the operator $D_t - T_\lambda$. It is not clear how much the condition (1.26) must be strengthened to imply this property or the validity of a Carleman estimate like (1.13). This problem seems to be connected with open problems on the local solvability of pseudo–differential operators.

1.4. Proof of Theorem 1.3

We will follow very closely the proof of the Theorem 1.2 as given by Nirenberg [Ni1, Th. 5]. First of all we remark that, as conditions (1.7) and (1.10) are invariant under Holmgren's change of variable, it will be sufficient simply to prove compact uniqueness. This one will be classically deduced by the following result.

Lemma 1.14. *Under the hypotheses of Theorem 1.3, there exist T, r, γ_0 and C positive constants such that:*

$$\int_0^T e^{2\gamma(t-T)^2} \sum_{|\alpha|+j\leq m-1} \|D_x^\alpha D_t^j u\|^2 dt \leq C(\frac{1+T^2\gamma}{\gamma}) \int_0^T e^{2\gamma(t-T)^2} \|Pu\|^2 dt \quad , \tag{1.27}$$

for $\gamma > \gamma_0$ and for all $u \in C_0^\infty$ such that $\operatorname{supp} u \subseteq \bar{B}_r \times [0, T/2]$.

Proof. First of all we remark that, by Poincaré's formula, (1.27) is equivalent to the following condition: there exist T, r, γ_0 and C positive constants such that:

$$\int_0^T e^{2\gamma(t-T)^2} \sum_{|\alpha|+j=m-1} \|D_x^\alpha D_t^j u\|^2 dt \leq C(\frac{1+T^2\gamma}{\gamma}) \int_0^T e^{2\gamma(t-T)^2} \|pu\|^2 dt \quad , \tag{1.28}$$

for $\gamma > \gamma_0$ and for all $u \in C_0^\infty$ such that $\operatorname{supp} u \subseteq \bar{B}_r \times [0, T/2]$. We write:

$$p(x, t, D_x, D_t) = D_t^m + \sum_{j=1}^m l_j(x, t, D_x) D_t^{m-j} \quad ,$$

where:

$$l_j(x, t, \xi) = \sum_{|\alpha|=j} a_{\alpha, m-j}(x, t)\xi^\alpha \quad ,$$

and we consider the operator:

$$T_p = D_t^m + \sum_{j=1}^m T_{l_j} D_t^{m-j} \quad , \tag{1.29}$$

where, for $j = 1, \ldots, m$, T_{l_j} is the para-differential operator of symbol $l_j \in {}^t\Gamma_\rho^j(\omega)$, with $\rho \geq 3$. It is important to remark that $\rho > 1$ will be enough all over the proof up to the Carleman estimates for the first order para-differential operators. By the Theorem A.15 and Poincaré's formula we have that (1.28) is obtained by the following estimate: there exist T, r, γ_0 and C positive constants such that:

$$\int_0^T e^{2\gamma(t-T)^2} \sum_{|\alpha|+j=m-1} \|D_x^\alpha D_t^j u\|^2 dt \leq C(\frac{1+T^2\gamma}{\gamma}) \int_0^T e^{2\gamma(t-T)^2} \|T_p u\|^2 dt \quad , \tag{1.30}$$

for $\gamma > \gamma_0$ and for all $u \in C_0^\infty$ such that $\operatorname{supp} u \subseteq \bar{B}_r \times [0, T/2]$.

Denoting by Λ the para–differential operator of symbol $1 + |\xi|$, we set:

$$u_j = \Lambda^{m-j} D_t^{j-1} u \quad , \tag{1.31}$$

and

$$U = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} .$$

Therefore we obtain:

$$\begin{aligned} &D_t u_j = \Lambda u_{j+1} \qquad \text{for } j = 1, \dots, m-1 \\ &D_t u_m + \sum_{j=1}^{m} T_{l_j} \Lambda^{1-j} u_{m+1-j} + (RU)_m = T_p u \quad , \end{aligned} \tag{1.32}$$

where R is σ–regularizing operator for every $\sigma \in \mathbf{R}$. Let us now indicate with $h(x,t,\xi)$ the matrix symbol:

$$h(x,t,\xi) = \begin{pmatrix} 0 & -|\xi| & 0 & \dots & 0 \\ 0 & 0 & -|\xi| & \ddots & \vdots \\ \vdots & \dots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & -|\xi| \\ l_m|\xi|^{-m+1} & \dots & \dots & l_2|\xi|^{-1} & l_1 \end{pmatrix} .$$

We have $h \in {}^t\Sigma^1_\rho(\omega)$ and (1.32) can now be written:

$$D_t U + T_h U + T_r U = \begin{pmatrix} 0 \\ \vdots \\ T_p u \end{pmatrix} ,$$

where T_r is a matrix para–differential operator with symbol in ${}^t\Sigma^0_\rho(\omega)$. With this notation, again using Poincaré's formula, (1.30) can be easily deduced by the following (see [Ni1, p. 33]):

$$\begin{aligned} \int_0^T e^{2\gamma(t-T)^2} \|U\|^2 dt \leq & C(\frac{1+T^2\gamma}{\gamma}) \int_0^T e^{2\gamma(t-T)^2} \|D_t U + T_h U\|^2 dt \\ & + C \int_0^T e^{2\gamma(t-T)^2} (\sum_{|\alpha|+j<m-1} \|D_x^\alpha D_t^j u\|^2 + \|D_t^{m-1} u\|^2_{-1}) dt . \end{aligned} \tag{1.33}$$

The idea now is to reduce the matrix h to its Jordan canonical form. To this end we remark that the eigenvalues of h are the negative of the roots of the polynomial (1.6).

Using the hypothesis of constant multiplicity we have that it is possible to find a C^ρ non singular $m \times m$ matrix s such that:

$$(s\,h\,s^{-1})(x,t,\xi) = j(x,t,\xi) \quad ,$$

where j is the Jordan canonical form of h; this matrix s is defined locally in a neighborhood of each point $(0,0,\xi_0)$, with $\xi_0 \in S^{n-2}$. From this covering of S^{n-2} we extract a finite covering $\{\Omega_\nu\}$, and we consider $\sum_\nu \phi_\nu^2(\xi) \equiv 1$, a partition of unity subordinate to $\{\Omega_\nu\}$. Extending, for each ν, ϕ_ν as an homogeneous function of order 0 in ξ, we can consider the corresponding para–differential operator T_{ϕ_ν} and, defining $U_\nu = T_{\phi_\nu} U$, it is easy to see that (1.33) is now reduced to:

$$\begin{aligned} \int_0^T e^{2\gamma(t-T)^2} \|U_\nu\|^2 dt \le C(\frac{1+T^2\gamma}{\gamma}) \int_0^T e^{2\gamma(t-T)^2} \|D_t U_\nu + T_h U_\nu\|^2 dt \\ + C\int_0^T e^{2\gamma(t-T)^2} \|U_\nu\|_{-1}^2 dt \quad . \end{aligned} \tag{1.34}$$

Always following the proof of Nirenberg, let ψ_ν be a smooth map on S^{n-2} such that $\psi_\nu(S^{n-2}) \subseteq \Omega_\nu$ and $\psi_\nu \equiv 1$ on the support of ϕ_ν. We define $h_\nu(x,t,\xi) = h(x,t,\psi_\nu(\xi))$ and we extend it as an homogeneous function of degree 1 in ξ. Defining analogously s_ν and j_ν, we have that:

$$s_\nu\, h_\nu\, s_\nu^{-1} = j_\nu \quad ,$$

and $j_\nu = j$ on the support of ϕ_ν. By the Theorem A.23 we deduce that:

$$T_h U_\nu = T_{h_\nu} U_\nu + M_\nu U_\nu \quad ,$$

where M_ν is a $(\rho - 1)$–regularizing operator. Moreover, setting $T_{s_\nu} U_\nu = V_\nu$, we can write:

$$U_\nu = T_{s_\nu^{-1}} V_\nu + N_\nu U_\nu \quad ,$$

and as N_ν is a (-1)–regularizing operator, we obtain:

$$\|U_\nu\| \le C(\|V_\nu\| + \|U_\nu\|_{-1}) \quad . \tag{1.35}$$

Finally we have that:

$$\begin{aligned} T_{s_\nu}(D_t U_\nu - T_h U_\nu) =& D_t V_\nu + T_{\partial_t s_\nu} U_\nu - T_{s_\nu}(T_h - T_{h_\nu}) U_\nu \\ & - T_{s_\nu} T_{h_\nu} T_{s_\nu^{-1}} V_\nu - T_{s_\nu} T_{h_\nu} N_\nu U_\nu \\ =& D_t V_\nu - T_{j_\nu} V_\nu + L_\nu U_\nu \quad , \end{aligned}$$

where L_ν is bounded in L^2, and this implies that :

$$\|D_t V_\nu - T_{j_\nu} V_\nu\| \le C(\|D_t U_\nu + T_h U_\nu\| + \|U_\nu\|) \quad . \tag{1.36}$$

Using (1.35) and (1.36), is now not so hard a work to see that (1.34) can be deduced by the following:

$$(1.37) \qquad \int_0^T e^{2\gamma(t-T)^2}\|V_\nu\|^2 dt \le C(\frac{1+T^2\gamma}{\gamma}) \int_0^T e^{2\gamma(t-T)^2}\|D_t V_\nu + T_{j_\nu} V_\nu\|^2 dt \quad .$$

Recalling that j_ν is the Jordan form of the matrix h_ν, (1.37) is now obtained acting on each block of the Jordan form with Lemmas 1.9, 1.10, 1.11. This completes the proof of Lemma 1.12 and consequently the thesis of Theorem 1.3 is achieved.

1.5. Carleman Estimates for Degenerate Operators and Applications

The Calderón's uniqueness theorem has been extended to many particular classes of differential operators and its proof has been adapted and improved for each different situation. Nevertheless the pseudo–differential technique has always been used as a basic tool and consequently only operators with C^∞ coefficients, at least in the principal part, have been considered. We want to give here some other examples of how operators with non–smooth coefficients can be studied with respect to uniqueness, by using para–differential operators instead of pseudo–differential ones.

Again we start giving some Carleman estimates for first order para–differential operators. The main difference between the following lemmas and those of Paragraph 1.3 is the choice of the weight function: in the integral inequalities we substitute the function $e^{\gamma(t-T)^2}$ with the function $t^{-\gamma}$. This brings some considerable consequences: on one hand we will obtain estimates useful for the study of operators which degenerate on the surface of initial data, i.e. Fuchsian type operators and other degenerate operators, but on the other hand we will obtain the compact uniqueness or the uniqueness only for the C^∞ solutions (we will call these properties C^∞–compact uniqueness and C^∞–uniqueness).

The first lemma is the analogous of the Lemma 1.9.

Lemma 1.15. *Let $k \ge 0$ be an integer. In the hypotheses of Lemma* 1.9 *there exist T, r, γ_0 and C positive constants such that:*

$$(1.38) \qquad \int_0^T t^{-2\gamma-2}\|u\|^2 dt \le \frac{C}{\gamma}\int_0^T t^{-2\gamma}\|D_t u - T_{t^k\lambda}u\|^2 dt \quad ,$$

and, under condition (1.11),

$$(1.39) \qquad \int_0^T t^{-2\gamma} \sum_{|\alpha|+j\le 1} t^{2k|\alpha|}\|D_x^\alpha D_t^j u\|^2 dt \le C\gamma \int_0^T t^{-2\gamma}\|D_t u - T_{t^k\lambda}u\|^2 dt \quad ,$$

for $\gamma > \gamma_0$ and for all $u \in C_0^\infty$ such that $\operatorname{supp} u \subseteq \bar{B}_r \times [0, T/2]$.

Proof. The proof is similar to that of Lemma 1.9. Let τ, $\sigma \in \mathbf{R}$. We set $v = t^{-\tau}u$, where $u \in \mathcal{C}_0^\infty$ and $\operatorname{supp} u \subseteq \bar{B}_r \times [0, T/2]$. We have:

$$t^{\sigma-\tau}(D_t u - T_{t^k\lambda}u) = t^\sigma D_t v - T_{t^{k+\sigma}a}v - iT_{t^{k+\sigma}b}v - i\tau t^{\sigma-1}v \quad .$$

So that:

$$\begin{aligned}(1.40)\qquad &\int_0^T t^{2(\sigma-\tau)}\|D_t u - T_{t^k\lambda}u\|^2 dt \\ &= \int_0^T \|t^\sigma D_t v - T_{t^{k+\sigma}a}v\|^2 dt + \int_0^T \|T_{t^{k+\sigma}b}v + \tau t^{\sigma-1}v\|^2 dt \\ &+ 2\mathrm{Re}\int_0^T (t^\sigma D_t v - T_{t^{k+\sigma}a}v\,,\, -iT_{t^{k+\sigma}b}v - i\tau t^{\sigma-1}v)dt\,.\end{aligned}$$

Easily:

$$2\mathrm{Re}\int_0^T (t^\sigma D_t v\,,\, -i\tau t^{\sigma-1}v)dt = -(2\sigma-1)\tau\int_0^T \|t^{\sigma-1}v\|^2 dt \quad ,$$

$$2\mathrm{Re}\int_0^T (-T_{t^{k+\sigma}a}v\,,\, -i\tau t^{\sigma-1}v)dt \geq -CT^{k+1}\tau\int_0^T \|t^{\sigma-1}v\|^2 dt \quad .$$

If $b \equiv 0$, posing $\sigma = 0$ and $\tau = \gamma$, (1.38) follows from these inequalities and (1.40).

If (1.11) holds, let $\sigma = -k/2$. Then:

$$\begin{aligned}(1.41)\qquad &2\mathrm{Re}\int_0^T (t^\sigma D_t v\,,\, -iT_{t^{k+\sigma}b}v)dt \\ &= -\mathrm{Re}\int_0^T (T_{\partial_t b}v\,,\, v)dt + \mathrm{Re}\int_0^T (v\,,\, i(T_b^* - T_b)D_t v)dt \\ &\geq -C\int_0^T t^{-\sigma+1}(\|D_t v\| + \|v\|_1)\|t^{\sigma-1}v\|dt \\ &\geq -C\int_0^T t^{-2\sigma+1}\|t^\sigma D_t v - T_{t^{k+\sigma}a}v\|\,\|t^{\sigma-1}v\|dt - C\int_0^T t^{-\sigma+1}\|v\|_1\|t^{\sigma-1}v\|dt \\ &\geq -\frac{1}{2}\int_0^T \|t^\sigma D_t v - T_{t^{k+\sigma}a}v\|^2 dt - C\int_0^T t^{-2\sigma}\|v\|^2 dt \\ &\qquad - C\int_0^T t^{-\sigma+1}\|v\|_1\|t^{\sigma-1}v\|dt \quad .\end{aligned}$$

Moreover:

$$\begin{aligned}(1.42)\qquad &2\mathrm{Re}\int_0^T (-T_{t^{k+\sigma}a}v\,,\, -iT_{t^{k+\sigma}b}v)dt \\ &\geq \mathrm{Re}\int_0^T i((T^*_{t^{k+\sigma}a}T_{t^{k+\sigma}b} - T^*_{t^{k+\sigma}b}T_{t^{k+\sigma}a})v\,,\, v)dt \\ &\geq -C\int_0^T t^{-3\sigma+1}\|v\|_1\|t^{\sigma-1}v\|dt \quad .\end{aligned}$$

We use the ellipticity of T_b. We get:

$$t^{-\sigma}\|v\|_1 \le C(\|T_{t^k+\sigma b}v + \tau t^{\sigma-1}v\| + \tau\|t^{\sigma-1}v\|) \quad , \tag{1.43}$$

and this implies that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that:

$$\int_0^T t^{-\sigma+1}\|v\|_1\|t^{\sigma-1}v\|dt \le \varepsilon \int_0^T \|T_{t^k+\sigma b}v + \tau t^{\sigma-1}v\|^2 dt + C_\varepsilon(1+T\tau)\int_0^T \|t^{\sigma-1}v\|^2 dt\,.$$

From this, (1.40), (1.41) and (1,.42), if T and γ^{-1} are sufficiently small, we deduce:

$$\int_0^T t^{-k-2\tau}\|D_t u - T_{t^k\lambda}u\|^2 dt \ge C\tau \int_0^T t^{-k-2}\|v\|^2 dt \quad ,$$

and finally setting $\gamma = k/2 + \tau$, (1.38) follows.

From (1.43) we immediately obtain:

$$\int_0^T t^k\|v\|_1^2 dt \le C\int_0^T \|T_{t^k+\sigma b}v + \tau t^{\sigma-1}v\|^2 dt + C\tau^2 \int_0^T \|t^{\sigma-1}v\|^2 dt \quad ,$$

and arguing as in the Lemma 1.9 we obtain (1.39). The proof is complete.

The next lemma gives the Carleman estimate under condition (1.8).

Lemma 1.16. *Let $k \ge 0$ be an integer. In the hypotheses of Lemma* 1.10, (1.38) *follows.*

Proof. We set $v = t^{-\gamma}u$ and we have:

$$\begin{aligned} -\mathrm{Im}\int_0^T t^{-\gamma}(D_t u - T_{t^k\lambda}u\,,\,t^{-1}v)dt &= -\mathrm{Im}\int_0^T (-T_{t^k\lambda}v\,,\,t^{-1}v)dt + \gamma\int_0^T \|t^{-1}v\|^2 dt \\ &= \mathrm{Re}\int_0^T (T_{-it^{k-1}\lambda}v\,,\,v)dt + \gamma\int_0^T \|t^{-1}v\|^2 dt \quad . \end{aligned}$$

By the sharp Gårding inequality for para–differential operators we have that:

$$\mathrm{Re}\int_0^T (T_{-it^{k-1}\lambda}v\,,\,v)dt \ge -C\int_0^T t^{k-1}\|v\|^2 dt \quad .$$

(1.38) is easily reached by using the Schwarz inequality. This concludes the proof.

The last lemma will furnish the Carleman estimate in the remaining situation.

Lemma 1.17. *Let $k \geq 0$ be an integer. Let $\rho \geq 3$ and let $\lambda = a + ib$, where a and b are real symbols of ${}^t\Gamma^1_\rho(\omega)$. Assume that there exists $\varepsilon > 0$ such that:*

$$((1+k-\varepsilon)b + t\partial_t b + t^{k+1}\sum_{j=1}^{n-1}(\partial_{x_j}a\partial_{\xi_j}b - \partial_{x_j}b\partial_{\xi_j}a))(x,t,\xi) \leq 0 \quad , \tag{1.44}$$

for all $(x,t,\xi) \in \omega \times [0,T] \times S^{n-2}$.
Then (1.38) follows.

Proof. Let $\tilde{a}$ and $\tilde{b}$ be the symbols defined in (1.22). As usual we set $v = t^{-\gamma}u$ and we obtain:

$$2\int_0^T t^{-2\gamma}\|D_t u - T_{t^k\lambda}u\|^2 dt \geq \int_0^T \|D_t v - T_{t^k\tilde{a}}v - iT_{t^k\tilde{b}}v - i\gamma t^{-1}v\|^2 dt - CT^{2k}\int_0^T \|v\|^2 dt\,.$$

We call I the first right–hand–side term and we have:

$$\begin{aligned} I \geq \frac{1}{2}\int_0^T \|D_t v - T_{t^k\tilde{a}}v\|^2 dt + \int_0^T \|T_{t^k\tilde{b}}v + \gamma t^{-1}v\|^2 dt + \gamma\int_0^T \|t^{-1}v\|^2 dt \\ - CT^{2k}\int_0^T \|v\|^2 dt + \mathrm{Re}\int_0^T (T_{-(kt^{k-1}b + t^k\partial_t b)}v\,,\, v)dt \\ + \mathrm{Re}\int_0^T (T_{-t^{2k}(\sum_{j=1}^n(\partial_{x_j}a\partial_{\xi_j}b - \partial_{x_j}b\partial_{\xi_j}a))}v\,,\, v)dt - C\int_0^T \|v\|^2 dt\,. \end{aligned} \tag{1.45}$$

We define the symbol:

$$r(x,t,\xi) = -[(1+k-\varepsilon)b + t\partial_t b + t^{k+1}\sum_{j=1}^{n-1}(\partial_{x_j}a\partial_{\xi_j}b - \partial_{x_j}b\partial_{\xi_j}a)](x,t,\xi) \quad .$$

Using the sharp Gårding inequality we deduce from (1.44) that:

$$\mathrm{Re}\int_0^T (T_{t^{k-1}r}v\,,\, v)dt \geq -CT^{k-1}\int_0^T \|v\|^2 dt \quad ,$$

and this implies that:

$$\begin{aligned} \mathrm{Re}\int_0^T (T_{-(kt^{k-1}b + t^k\partial_t b + t^{2k}(\sum_{j=1}^n(\partial_{x_j}a\partial_{\xi_j}b - \partial_{x_j}b\partial_{\xi_j}a)))}v\,,\, v)dt \\ \geq \mathrm{Re}\int_0^T (T_{t^{k-1}r}v\,,\, v)dt + (1-\varepsilon)\mathrm{Re}\int_0^T (T_{t^k b}v\,,\, t^{-1}v)dt \\ \geq -((1-\varepsilon)\gamma + 1)\int_0^T \|t^{-1}v\|^2 dt - \frac{1}{2}\int_0^T \|T_{t^k\tilde{b}}v + \gamma t^{-1}v\|^2 dt\,. \end{aligned} \tag{1.46}$$

The proof is easily concluded by using (1.45) and (1.46).

Let us now come to some applications of these estimates. Let P be a second order differential operator of Fuchsian type, i.e.:

$$P(x,t,D_x,D_t) = t^2 p_2(x,t,D_x,D_t) + tp_1(x,t,D_x,D_t) + p_0(x,t) \quad ,$$

where, for $j = 0,1,2$, p_j is a j–th order homogeneous operator with bounded complex coefficients. Suppose that p_2 has the following form:

$$(1.47) \qquad p_2(x,t,D_x,D_t) = D_t^2 + \sum_{j=1}^{n-1} \alpha_j(x,t) D_{x_j} D_t + \sum_{j,k=1}^{n-1} \beta_{j,k}(x,t) D_{x_j} D_{x_k} \, .$$

Theorem 1.18. *Let the coefficients of p_2 be in $C^3(\Omega, \mathbf{C})$. For $j = 1,2$, let $\lambda_j(x,t,\xi) = a_j(x,t,\xi) + ib_j(x,t,\xi)$ be the roots of the polynomial $p_2(x,t,\xi,\tau)$. Let V be a neighborhood of the origin.*

If the roots are simple and for each of them one of the following conditions is verified:

$$(1.8) \qquad b_j(x,t,\xi) \geq 0 \qquad \text{for all} \quad (x,t,\xi) \in V \times S^{n-2} \quad ;$$

there exists $\varepsilon > 0$ such that:

$$(1.48) \qquad ((1-\varepsilon)b_j + t\partial_t b_j + t\sum_{k=1}^{n} (\partial_{x_k} a_j \partial_{\xi_k} b_j - \partial_{x_k} b_j \partial_{\xi_k} a_j))(x,t,\xi) \leq 0 \quad ,$$

for all $(x,t,\xi) \in V \times S^{n-2}$, then P has the C^∞–compact uniqueness with respect to $S = \{(x,t) \,|\, t = 0\}$ at the origin.

If the roots are simple and for each of them the condition (1.7) *or* (1.10) *is satisfied, then P has the C^∞–uniqueness with respect to S at the origin.*

If $\lambda_1 \equiv \lambda_2$ and (1.7) *holds, then P has the C^∞–uniqueness under the following Levi type condition: the coefficients of p_1 are in C^ρ with $\rho > 1$ and there exists $f \in C^\rho$ such that:*

$$(1.49) \qquad p_1(x,0,\xi,\tau) = f(x)(\tau - \lambda_1(x,0,\xi)) \quad ,$$

Remark 1.19. This problem has been studied, for operators with C^∞ coefficients, by Roberts in [Rob]. In that work operators of the form $P = t^m p_m + t^{m-1} p_{m-1} + \ldots + p_0$, where, for each j, p_j is a homogeneous operator of order j, are considered and the C^∞–uniqueness is obtained under two main assumptions: the roots of p_m are of constant multiplicity ≤ 2, where double roots satisfy (1.7) while simple roots satisfy (1.7) or (1.10), and a Levi type condition, analogous to condition (1.49), holds. Under the condition (1.8) and a condition a little stronger than (1.48) on the simple roots, C^∞–compact uniqueness is also shown. We were not able to prove this result for operator with C^3 coefficients, though it would be possible to use para–differential operators theory to show an analogous result for operators with C^m coefficients: in this case the proof of Roberts can be modified and the regularity of the coefficients is sufficient to avoid the troubles coming from the commutation of the para–differential operators with the operator D_t.

Remark 1.20. Also in this case the regularity of the coefficients can be weakened if only the conditions (1.7) and (1.10) are fulfilled. The same in the case of (1.7) (1.8) and (1.10).

Using the proof of the Theorem 1.18 it is easy to show the following result, which is also a particular case of the Theorem 1.23 but which we prefer to state here.

Theorem 1.21. *Let p_2 be the operator defined in* (1.47) *and let the coefficients of p_2 be in $C^3(\Omega, \mathbf{C})$. If the roots of the characteristic polynimial of p_2 are simple and for each of them* (1.8) *or* (1.48) *holds, then the operator:*

$$p_2(x, t, D_x, D_t) + \textit{ lower order terms}$$

has the C^∞–compact uniqueness.

Example 1.22. Let $(a_{j,k}(t))$ be a $(n-1) \times (n-1)$ matrix of C^3 real functions. Suppose that there exists $\varepsilon > 0$ such that the matrix:

$$((1-\varepsilon)a_{j,k}(t) + ta'_{j,k}(t))$$

is positive semidefinite. Consider the operator $P = t^2 p_2 + tp_1 + p_0$, where:

$$p_2(x, t, D_x, D_t) = D_t^2 - \sum_{j=1}^{n-1} D_{x_j}^2 + i \sum_{j,k=1}^{n} a_{j,k}(t) D_{x_j} D_{x_k} \quad ,$$

and p_1, p_0 are homogeneous operator of order 1 and 0 respectively. Then P satisfies the hypotheses of the Theorem 1.18 and consequently has the C^∞–compact uniqueness. By Theorem 1.21 also the operator:

$$p_2(x, t, D_x, D_t) + \textit{ lower order terms}$$

has the same property.

Proof of the Theorem 1.18. Using a singular change of variables (see [BZ]) under which the conditions (1.7), (1.10) and (1.49) are invariant , it is possible to reduce the proof to the establishing of the C^∞–compact uniqueness in all the cases. This property will be easily deduced by the following Carleman estimate: there exist T, r, γ_0 and C positive constants such that:

$$\int_0^T t^{-2\gamma-4} \|u\|^2 dt \leq C \int_0^T t^{-2\gamma} \|Pu\|^2 dt \quad , \tag{1.50}$$

for $\gamma > \gamma_0$ and for all $u \in C_0^\infty$ such that $\operatorname{supp} u \subseteq \bar{B}_r \times [0, T/2]$.

Let us first consider the case of simple roots. We write:

$$p_2(x,t,D_x,D_t) = D_t^2 + \sum_{j=1}^{n-1} \alpha_j(x,t) D_{x_j} D_t + \sum_{j,k=1}^{n-1} \beta_{j,k}(x,t) D_{x_j} D_{x_k} ,$$

and defining $\alpha(x,t,\xi) = \sum_{j=1}^{n-1} \alpha_j(x,t)\xi_j$ and $\beta(x,t,\xi) = \sum_{j,k=1}^{n-1} \beta_{j,k}(x,t)\xi_j\xi_k$, we consider the operator:

$$T_{p_2} = D_t^2 + T_\alpha D_t + T_\beta \quad .$$

By the Theorem A.15 we have that:

$$\int_0^T t^{-2\gamma} \|p_2 u - T_{p_2} u\|^2 dt \le C \int_0^T t^{-2\gamma} \sum_{|\alpha|+j\le 1} \|D_x^\alpha D_t^j u\|^2 dt \quad , \tag{1.51}$$

for all $u \in C_0^\infty$ such that $\operatorname{supp} u \subseteq \bar{B}_r \times [0, T/2]$.

From the Theorem A.20 and the Theorem A.26 we obtain:

$$\begin{aligned} T_{p_2} &= (D_t - T_{\lambda_1})(D_t - T_{\lambda_2}) + R_t \quad , \\ T_{p_2} &= (D_t - T_{\lambda_2})(D_t - T_{\lambda_1}) + R'_t \quad , \end{aligned}$$

where R_t and R'_t are bounded operator from H^1 to L^2 with norms bounded in t. Using (1.38) with $k = 0$, we obtain:

$$\begin{aligned} \int_0^T t^{-2\gamma} \|T_{p_2} u\|^2 dt \ge & C\gamma \int_0^T t^{-2\gamma-2} (\|D_t u - T_{\lambda_1} u\|^2 + \|D_t u - T_{\lambda_2} u\|^2) dt \\ & - C \int_0^T t^{-2\gamma} \|u\|_1^2 dt + C\gamma^2 \int_0^T t^{-2\gamma-4} \|u\|^2 dt . \end{aligned} \tag{1.52}$$

We remark that $T_{\lambda_1 - \lambda_2} u = (D_t - T_{\lambda_2}) - (D_t - T_{\lambda_1})$ and this implies that:

$$\|T_{\lambda_1-\lambda_2} u\|^2 \le 2(\|D_t u - T_{\lambda_1} u\|^2 + \|D_t u - T_{\lambda_2} u\|^2) \quad .$$

As the roots are simple, $T_{\lambda_1-\lambda_2}$ is elliptic, so that:

$$\|D_t u - T_{\lambda_1} u\|^2 + \|D_t u - T_{\lambda_2} u\|^2 \ge C \sum_{|\alpha|+j\le 1} \|D_x^\alpha D_t^j u\|^2 \quad . \tag{1.53}$$

From (1.52) and (1.53) we deduce:

$$\begin{aligned} \int_0^T t^{-2\gamma} \|T_{p_2} u\|^2 dt \ge C(\gamma - C') \int_0^T t^{-2\gamma-2} \sum_{|\alpha|+j\le 1} \|D_x^\alpha D_t^j u\|^2 dt \\ + C\gamma^2 \int_0^T t^{-2\gamma-4} \|u\|^2 dt . \end{aligned} \tag{1.54}$$

It is very easy to reach the conclusion using (1.51).

Suppose now that p_2 has a double root verifying condition (1.7). Hypothesis (1.49) implies that:

$$\|p_1(x,t,D_x,D_t)u\| \le C(\|D_t u - T_\lambda u\| + t \sum_{|\alpha|+j\le 1} \|D_x^\alpha D_t^j u\|) \quad . \tag{1.55}$$

Using (1.38), we have:

$$\begin{aligned}\int_0^T t^{-2\gamma}\|T_{p_2}u\|^2 dt \ge & C\gamma \int_0^T t^{-2\gamma-2}\|D_t u - T_\lambda u\|^2 dt \\ & - C\int_0^T t^{-2\gamma}\|u\|_1^2 dt + C\gamma^2 \int_0^T t^{-2\gamma-4}\|u\|^2 dt\,.\end{aligned}$$

Therefore, by (1.39), there exists $\varepsilon > 0$ such that:

$$\begin{aligned}\int_0^T t^{-2\gamma}\|T_{p_2}u\|^2 dt \ge & C\gamma \int_0^T t^{-2\gamma-2}\|D_t u - T_\lambda u\|^2 dt \\ & + \varepsilon \int_0^T t^{-2\gamma-2} \sum_{|\alpha|+j\le 1} \|D_x^\alpha D_t^j u\|^2 dt \\ & - C\int_0^T t^{-2\gamma}\|u\|_1^2 dt + C\gamma^2 \int_0^T t^{-2\gamma-4}\|u\|^2 dt\,.\end{aligned} \tag{1.56}$$

Putting together the information coming from (1.55) and (1.56), if T is sufficiently small,:

$$\int_0^T t^{-2\gamma}\|T_{p_2}u\|^2 dt \ge 4\int_0^T t^{-2\gamma-2}\|p_1 u\|^2 dt \quad ,$$

from which the inequality (1.50) is reached by using (1.51) and (1.56). This completes the proof.

Let us finally show another application of the estimates (1.38) and (1.39). Let P be a second order differential operator with bounded complex coefficients, $P = p_2 + p_1 + p_0$ where, for each j, p_j is a homogeneous operator of order j.

Theorem 1.23. *Let $k \ge 0$ be an integer. Assume that the coefficients of p_2 are in $C^3(\Omega, \mathbf{C})$ and that p_2 admits the following factorization:*

$$p_2(x,t,\xi,\tau) = (\tau - t^k\lambda_1(x,t,\xi))(\tau - t^k\lambda_2(x,t,\xi)) \quad ,$$

where, for $j = 1, 2$, $\lambda_j = a_j + ib_j$ with a_j and b_j real symbols in ${}^t\Gamma_3^1(\omega)$.

Suppose that for $j = 0, \dots, n$, there exists $\delta_j \in L^\infty_{\text{loc}}(\Omega, \mathbf{C})$, such that:

$$p_1(x,t,\xi,\tau) = \delta_0(x,t)\tau + t^{k-1}\sum_{j=1}^{n-1} \delta_j(x,t)\xi_j \quad . \tag{1.57}$$

If $\lambda_1 \neq \lambda_2$ and for each of them the condition (1.8) *or the condition* (1.44) *holds, then P has the $\mathcal{C}^\infty$–compact uniqueness with respect to $S = \{(x,t)\,|\,t=0\}$ at the origin.*

If $\lambda_1 \neq \lambda_2$ and for each of them the condition (1.7) *or the condition* (1.10) *holds, then P has the $\mathcal{C}^\infty$– uniqueness.*

If $\lambda_1 \equiv \lambda_2$ and (1.7) *holds, then P has the $\mathcal{C}^\infty$– uniqueness under the following Levi type condition: for $j = 0, \ldots, n$, the function δ_j are in $\mathcal{C}^\rho(\Omega, \mathbf{C})$ with $\rho > 1$ and:*

$$\sum_{j=1}^{n-1} \delta_j(x,0)\xi_j = \lambda(x,0,\xi) \quad .$$

Remark 1.24. The above $\mathcal{C}^\infty$– uniqueness result for operators with smooth coefficients in the principal part was given in [Na2]. In that work operators of order m are studied. Again we were not able to prove an analogous result for operator of order m and $\mathcal{C}^3$ coefficients, but it is possible to give a result for operators with $\mathcal{C}^m$ coefficients. Finally the Remark 1.20 can be done in this case also.

Example 1.25. Let $(a_{j,k}(t))$ be a $(n-1)\times(n-1)$ matrix of $\mathcal{C}^3$ real functions. Suppose that there exists $\varepsilon > 0$ such that the matrix:

$$((1+k-\varepsilon)a_{j,k}(t) + ta'_{j,k}(t))$$

is positive semidefinite. Consider the operator $P = p_2 + p_1 + p_0$, where:

$$p_2(x,t,D_x,D_t) = D_t^2 - t^{2k}\Big(\sum_{j=1}^{n} D_{x_j}^2 + i\sum_{j,k=1}^{n} a_{j,k}(t)D_{x_j}D_{x_k}\Big) \quad ,$$

$$p_1(x,t,D_x,D_t) = \delta_0(x,t)D_t + t^{k-1}\sum_{j=1}^{n}\delta_j(x,t)D_{x_j} \quad ,$$

and p_0 is a homogeneous operator of order 0, then P satisfies the hypotheses of the Theorem 1.23 and consequently has the $\mathcal{C}^\infty$–compact uniqueness.

Proof of the Theorem 1.23. The proof is similar to that of the Theorem 1.18. We write

$$p_2(x,t,D_x,D_t) = D_t^2 + t^k\sum_{j=1}^{n-1}\alpha_j(x,t)D_{x_j}D_t + t^{2k}\sum_{j,l=1}^{n-1}\beta_{j,l}(x,t)D_{x_j}D_{x_l}\,,$$

so that defining $\alpha(x,t,\xi) = \sum_{j=1}^{n-1}\alpha_j(x,t)\xi_j$ and $\beta(x,t,\xi) = \sum_{j,l=1}^{n-1}\beta_{j,l}(x,t)\xi_j\xi_l$ and considering the operator:

$$T_{p_2} = D_t^2 + t^k T_\alpha D_t + t^{2k}T_\beta \quad ,$$

we obtain:

$$\int_0^T t^{-2\gamma}\|p_2u - T_{p_2}u\|^2dt \le C\int_0^T t^{-2\gamma+2k}\sum_{|\alpha|+j\le 1} t^{2|\alpha|k}\|D_x^\alpha D_t^j u\|^2 dt \quad , \tag{1.58}$$

In the case of two simple roots we have:

$$T_{p_2} = (D_t - T_{t^k\lambda_1})(D_t - T_{t^k\lambda_2}) + R_t \quad ,$$
$$T_{p_2} = (D_t - T_{t^k\lambda_2})(D_t - T_{t^k\lambda_1}) + R'_t \quad ,$$

where:

$$\int_0^T t^{-2\gamma}\|R_t u\|^2 dt \le C \int_0^T t^{-2(\gamma-k+1)}\|u\|_1^2 dt \quad ,$$
$$\int_0^T t^{-2\gamma}\|R'_t u\|^2 dt \le C \int_0^T t^{-2(\gamma-k+1)}\|u\|_1^2 dt \quad .$$

Therefore, by (1.38), we deduce:

$$\begin{aligned}\int_0^T t^{-2\gamma}\|T_{p_2}u\|^2 dt \ge & C\gamma \int_0^T t^{-2\gamma-2}(\|D_t u - T_{t^k\lambda_1}u\|^2 + \|D_t u - T_{t^k\lambda_2}u\|^2)dt \\ & - C\int_0^T t^{-2(\gamma-k+1)}\|u\|_1^2 dt + C\gamma^2 \int_0^T t^{-2\gamma-4}\|u\|^2 dt .\end{aligned} \tag{1.59}$$

By the ellipicity of $T_{\lambda_1-\lambda_2}$ we get that:

$$\|D_t u - T_{t^k\lambda_1}u\|^2 + \|D_t u - T_{t^k\lambda_2}u\|^2 \ge C(t^{2k}\|u\|_1^2 + \|D_t u\|^2) \quad .$$

From this and (1.59) we deduce:

$$\begin{aligned}\int_0^T t^{-2\gamma}\|T_{p_2}u\|^2 dt \ge C(\gamma - C')\int_0^T t^{-2(\gamma-k+1)}\|u\|_1^2 dt + C\gamma \int_0^T t^{-2\gamma-2}\|D_t u\|^2 dt \\ + C\gamma^2 \int_0^T t^{-2\gamma-4}\|u\|^2 dt .\end{aligned} \tag{1.60}$$

Using (1.58), (1.60), and (1.57) we achieve the conclusion. The case of the double root is similar to that of Theorem 1.18. This concludes the proof.

1.6. Operators with Degeneration of Infinite Order

Inspired by a work of Nakane [Na1], we want to give here two new Carleman estimates for infinite order degenerate operators. As an application of them we will obtain a $\mathcal{C}^\infty$–compact uniqueness result for a class of second order differential operators with non-smooth coefficients having an infinite order degeneration. We exploit the main idea of Nakane. First of all we will consider a set of functions vanishing of infinite order in $t = 0$. The reciprocals of these functions will have the good properties to be the weight functions in some Carleman estimates. It will be possible to apply these estimates to a class of degenerate operators, if the degeneration of them will have a certain relation with the weight functions. The Carleman estimates will be valid only for a proper subspace of $\mathcal{C}_0^\infty$. Nevertheless these estimates will be sufficient to recover the $\mathcal{C}^\infty$–compact uniqueness in our application, by using a lemma on ordinary differential equations.

Let us now present precisely all the matter. Let μ be a $\mathcal{C}^3$ real function on $[0,T]$ such that:

$$\mu(0) = \mu'(0) = 0 \qquad \text{and} \qquad \mu(t) > 0 \quad \text{for} \quad t > 0 \ . \tag{1.61}$$

To the function μ we associate the following function defined on $[0,T]$:

$$\sigma(t) = \begin{cases} e^{-\int_t^T \mu^{-1}(s)ds} & \text{for} \ \ t > 0 \\ 0 & \text{for} \ \ t = 0 \ . \end{cases} \tag{1.62}$$

It is easy to see that σ is an increasing $\mathcal{C}^3$ real function with an infinite order zero in $t = 0$.

We denote by $\mathcal{B}_\sigma(\bar{B}_r \times [0,T/2])$ (or more simply $\mathcal{B}_\sigma$) the set of all the functions u of $\mathcal{C}_0^\infty$ such that $\operatorname{supp} u \subseteq \bar{B}_r \times [0,T/2]$ and for all $\gamma \in \mathbf{R}$, the functions $u(x,t)\sigma^{-\gamma}(t)$ and $D_t u(x,t)\sigma^{-\gamma}(t)$ are bounded.

The following two lemmas are the analogous ones of Lemma 1.10 and 1.11, where now the weight function is $\sigma^{-\gamma}$ and the estimates are valid only for the elements of $\mathcal{B}_\sigma$.

Lemma 1.26. *In the hypotheses of Lemma* 1.10, *there exist T, r, γ_0 and C, positive constants, such that:*

$$\int_0^T \sigma^{-2\gamma}\mu^{-2}\|u\|^2 dt \le \frac{C}{\gamma}\int_0^T \sigma^{-2\gamma}\|D_t u - T_{\sigma\lambda}u\|^2 dt \quad , \tag{1.63}$$

for $\gamma > \gamma_0$ and for all $u \in \mathcal{B}_\sigma$.

Proof. Let us set $v = \sigma^{-\gamma}u$, with $u \in \mathcal{B}_\sigma$. We have:

$$\begin{aligned} -\operatorname{Im}\int_0^T \sigma^{-\gamma}(D_t u - T_{\sigma\lambda}u\,,\,\frac{v}{\mu})dt & \\ &= -\operatorname{Im}\int_0^T(-T_{\sigma\lambda}v\,,\,\frac{v}{\mu})dt + \gamma\int_0^T \|\frac{v}{\mu}\|^2 dt \\ &= \operatorname{Re}\int_0^T(T_{-i\frac{\sigma}{\mu}\lambda}v\,,\,v)dt + \gamma\int_0^T \|\frac{v}{\mu}\|^2 dt \ . \end{aligned}$$

The sharp Gårding inequality for para–differential operators easily provides the conclusion.

Lemma 1.27. *Let $\rho \ge 3$ and let $\lambda = a + ib$, where a and b are real symbols of ${}^t\Sigma_\rho^1(\omega)$. Assume that there exists $\varepsilon > 0$ such that:*

$$((1-\varepsilon)b + \mu\partial_t b + \mu\sigma\sum_{j=1}^n(\partial_{x_j}a\partial_{\xi_j}b - \partial_{x_j}b\partial_{\xi_j}a))(x,t,\xi) \le 0 \quad , \tag{1.64}$$

for all $(x,t,\xi) \in \omega \times [0,T] \times S^{n-2}$.
Then (1.63) *follows.*

Proof. Let us set again $v = \sigma^{-\gamma}u$, with $u \in \mathcal{B}_\sigma$. We have:

$$\sigma^{-\gamma}(D_t u - T_{\sigma\lambda}u) = D_t v - T_{\sigma a}v - iT_{\sigma b}v - i\gamma\frac{v}{\mu} \quad .$$

Considering $\tilde{a}$ and $\tilde{b}$ as defined in (1.22) and arguing as in Lemma 1.17, we obtain:

$$2\int_0^T \sigma^{-2\gamma}\|D_t u - T_{\sigma\lambda}u\|^2 dt \geq \int_0^T \|D_t v - T_{\sigma\tilde{a}}v - iT_{\sigma\tilde{b}}v - i\gamma\frac{v}{\mu}\|^2 dt - C\int_0^T \|\frac{v}{\mu}\|^2 dt\,.$$

Calling I the first right–hand–side term it follows:

$$\begin{aligned} I \geq\ & \frac{1}{2}\int_0^T \|D_t v - T_{\sigma\tilde{a}}v\|^2 dt + \frac{1}{2}\int_0^T \|T_{\sigma\tilde{b}}v + \gamma\frac{v}{\mu}\|^2 dt \\ & + \gamma\int_0^T \mu'\|\frac{v}{\mu}\|^2 dt + \mathrm{Re}\int_0^T (T_{-(\frac{\varepsilon}{\mu}b + \sigma\partial_t b)}v\,,\, v)dt \\ & + \mathrm{Re}\int_0^T (T_{-\sigma^2(\sum_{j=1}^{n-1}(\partial_{x_j}a\partial_{\xi_j}b - \partial_{x_j}b\partial_{\xi_j}a))}v\,,\, v)dt - C\int_0^T \|\frac{v}{\mu}\|^2 dt\,. \end{aligned}$$

We define the symbol:

$$r(x,t,\xi) = -[(1-\varepsilon)b + \mu\partial_t b + \mu\sigma(\sum_{j=1}^{n}(\partial_{x_j}a\partial_{\xi_j}b - \partial_{x_j}b\partial_{\xi_j}a))](x,t,\xi) \quad .$$

Again from the sharp Gårding inequality we get:

$$\mathrm{Re}\int_0^T (T_{\frac{\sigma}{\mu}r}v\,,\, v)dt \geq -C\int_0^T \|\frac{v}{\mu}\|^2 dt \quad .$$

Consequently:

$$\begin{aligned} I \geq\ & \frac{1}{2}\int_0^T \|D_t v - T_{\sigma\tilde{a}}v\|^2 dt + \frac{1}{2}\int_0^T \|T_{\sigma\tilde{b}}v + \gamma\frac{v}{\mu}\|^2 dt \\ & + \gamma\int_0^T \mu'\|\frac{v}{\mu}\|^2 dt - C\int_0^T \|\frac{v}{\mu}\|^2 dt \\ & - \varepsilon\,\mathrm{Re}\int_0^T (T_{\sigma\tilde{b}}v\,,\, \frac{v}{\mu})dt \quad . \end{aligned}$$

Finally, by the Schwarz inequality:

$$-\varepsilon\,\mathrm{Re}\int_0^T (T_{\sigma\tilde{b}}v\,,\, \frac{v}{\mu})dt \quad \geq -\frac{1}{4}\int_0^T \|T_{\sigma\tilde{b}}v + \gamma\frac{v}{\mu}\|^2 dt + (\varepsilon\gamma - C)\int_0^T \|\frac{v}{\mu}\|^2 dt\,.$$

Putting all together we obtain:

$$I \geq \frac{1}{2}\int_0^T \|D_t v - T_{\sigma\tilde{a}} v\|^2 dt + \frac{1}{4}\int_0^T \|T_{\sigma\tilde{b}} v + \gamma\frac{v}{\mu}\|^2 dt$$
$$+\gamma\int_0^T (\varepsilon+\mu')\|\frac{v}{\mu}\|^2 dt - C\int_0^T \|\frac{v}{\mu}\|^2 dt \quad .$$

As $\lim_{t\to 0}\mu'(t) = 0$, choosing T and γ_0^{-1} sufficiently small we reach the thesis. The proof is complete.

It is easy to imagine how these estimates can be used. Let us only sketch the result paraphrasing Theorem 1.23. Let P be a second order differential operator with complex bounded coefficients, $P = p_2 + p_1 + p_0$ where, for each j, p_j is a homogeneous operator of order j. Let μ and σ two function as described in (1.62) and (1.63).

Theorem 1.28. *Assume that the coefficients of p_2 are in $C^3(\Omega, \mathbf{C})$ and that p_2 admits the following factorization:*

$$p_2(x,t,\xi,\tau) = (\tau - \sigma\lambda_1(x,t,\xi))(\tau - \sigma\lambda_2(x,t,\xi)) \quad ,$$

where, for $j = 1, 2$, $\lambda_j = a_j + ib_j$, with a_j and b_j real symbols in ${}^t\Gamma_3^1(\omega)$.
Suppose that for $j = 0, \dots, n$, there exists $\delta_j \in L^\infty_{\mathrm{loc}}(\Omega, \mathbf{C})$, such that:

$$p_1(x,t,\xi,\tau) = \delta_0(x,t)\tau + \frac{\sigma(t)}{\mu(t)}\sum_{j=1}^n \delta_j(x,t)\xi_j \quad .$$

If $\lambda_1 \neq \lambda_2$ and for each of them the condition (1.8) *or the condition* (1.64) *holds, then P has the C^∞–compact uniqueness with respect to $S = \{(x,t) \,|\, t = 0\}$ at the origin.*

Example 1.29. Let $\mu(t) = t^2$, so that $\sigma(t) = e^{-1/t}$. Let f be a real C^3 function such that there exists $\varepsilon > 0$ such that $(1-\varepsilon)f(t) + t^2 f'(t) \geq 0$. Then the operator:

$$P_1 = D_t^2 - e^{-\frac{2}{t}}(1 + if(t))^2 D_x^2 + \frac{e^{-\frac{1}{t}}}{t^2}\alpha D_x + \beta D_t + \gamma \quad ,$$

where α, β, γ are complex bounded functions, satisfies the hypotheses of the Theorem 1.28 and consequently has the C^∞–compact uniqueness.

Analogously let $(a_{j,k}(t))$ be a $(n-1)\times(n-1)$ matrix of C^3 real functions. Suppose that there exists $\varepsilon > 0$ such that the matrix:

$$((1-\varepsilon)a_{j,k}(t) + t^2 a'_{j,k}(t))$$

is positive semidefinite. Consider the operator $P = p_2 + p_1 + p_0$, where:

$$p_2(x,t,D_x,D_t) = D_t^2 - e^{-\frac{2}{t}}(\sum_{j=1}^n D_{x_j}^2 + i\sum_{j,k=1}^n a_{j,k}(t)D_{x_j}D_{x_k}) \quad ,$$

$$p_1(x,t,D_x,D_t) = \delta_0(x,t)D_t + \frac{e^{-\frac{1}{t}}}{t^2}\sum_{j=1}^n \delta_j(x,t)D_{x_j} \quad ,$$

and p_0 is a homogeneous operator of order 0, then P satisfies the hypotheses of the Theorem 1.28 and has the $\mathcal{C}^\infty$–compact uniqueness.

Proof of the Theorem 1.28. The only new point in the proof is that we have to deduce $\mathcal{C}^\infty$–compact uniqueness from an estimate in $\mathcal{B}_\sigma$. This can be done in the following way: let $u \in \mathcal{C}^\infty$ such that $Pu \equiv 0$, $\operatorname{supp} u \subseteq S^+$ and $\operatorname{supp} u \cap S \subset\subset B_r$. Then there exist $T > 0$ and a real smooth function θ, such that $\theta(t) = 1$ for $t < T/3$, $\theta(t) = 0$ for $t > T/2$ and $\theta u \in \mathcal{C}_0^\infty$. A consequence of Lemma 1 in [CDS1] show that $\theta u \in \mathcal{B}_\sigma$. Finally a Carleman estimate similar to (1.50) gives the conclusion.

1.7. A Non–Uniqueness Example

Let us consider the following operator:

$$p_\alpha(x,t,D_x,D_t) = D_t^2 - D_{x_1}^2 - D_{x_2}^2 + i(D_{x_1}^2 + 2tD_{x_1}D_{x_2} + (1+\alpha)t^2D_{x_2}^2)\,.$$

We ask if the operator $P_\alpha = p_\alpha +$ *lower order terms* has the uniqueness in the Cauchy problem with respect to the hypersurface $\{(x,t) \in \mathbf{R}^3 \,|\, t = 0\}$ at the origin. To this end we consider the roots of the characteristic polynomial. The roots are the following:

$$\begin{aligned}\lambda_1(x,t,\xi) = &- \sqrt{\frac{\sqrt{(\xi_1^2+\xi_2^2)^2 + (\xi_1^2 + 2t\xi_1\xi_2 + (1+\alpha)t^2\xi_2^2)^2} + (\xi_1^2+\xi_2^2)}{2}} \\ &+ i\frac{\xi_1^2 + 2t\xi_1\xi_2 + (1+\alpha)t^2\xi_2^2}{\sqrt{2\sqrt{(\xi_1^2+\xi_2^2)^2 + (\xi_1^2 + 2t\xi_1\xi_2 + (1+\alpha)t^2\xi_2^2)^2} + 2(\xi_1^2+\xi_2^2)}}\,, \\ \lambda_2(x,t,\xi) = &- \lambda_1(x,t,\xi)\,.\end{aligned}$$

Obviously $\lambda_1, \lambda_2 \in {}^t\Gamma^1_\rho(\omega)$ for all ρ. Let as usual $\lambda_j = a_j + ib_j$, with $a_j, b_j \in \mathbf{R}$, for $j = 1, 2$. We immediately have that:

$$b_1(x,t,\xi) \geq 0 \qquad \text{for all} \quad (x,t,\xi) \in \mathbf{R}^3 \times S^1\,. \tag{1.64}$$

and, if $\alpha > 1/3$, an easy computation gives that there exist $\varepsilon > 0$ and $T > 0$ such that:

$$(1-\varepsilon)b_2(x,t,\xi) + t\partial_t b_2(x,t,\xi) \leq 0 \qquad \text{for all} \quad (x,t,\xi) \in \mathbf{R}^2 \times [0,T] \times S^1\,. \tag{1.65}$$

Consequently the Theorem 1.21 ensures that P_α has the $\mathcal{C}^\infty$–compact uniqueness if $\alpha > 1/3$.

Theorem 1.30. *If $0 \leq \alpha < 1/3$, then there exist $a, u \in \mathcal{C}^\infty(\mathbf{R}^3)$ such that $0 \in \operatorname{supp} u \subseteq \{(x,t) \in \mathbf{R}^3 \,|\, t \geq 0\}$ and $p_\alpha u + au \equiv 0$ in a neighborhood of 0.*

Remark 1.31. From the point of view of Calderón's theorem, this example shows that requiring only the simplicity of the roots it is not enough to get the uniqueness, even if the roots are smooth and their imaginery part has a definite sign. This means that it is necessary to make some other hypothesis. At least in the case of operator p_α, the couple condition (1.64) and (1.65) seems to be near a necessary and sufficient condition for the uniqueness.

Remark 1.32. For every $\alpha \geq 0$, P_α satisfies the condition (P) of Nirenberg and Treves [NT1], [NT2], [NT3] (i.e. if we call p_1 and p_2 respectively the real and the imaginery part of p_α, p_2 does not change sign along any null bicharacteristic strip of p_1). Therefore the Theorem 1.30 shows that the condition (P) is not sufficient for the uniqueness with respect to a strongly pseudo–convex hypersurface of initial data (see Definition 2.3). The problem of knowing if the condition (P) is sufficient or how much it must be streghtened to guarantee the uniqueness was raised by Hörmander in [Hör3, Notes on Ch. 28].

Proof of the Theorem 1.30. The proof is rather long and technical; it is based on a refinement of some results of the geometrical optic. Examples of this technique can be found in [AZ], [Zu2] and [A1], but our proof will follow very closely the proof of the Theorem 3 in [CDSZ2].

Let us set:

$$\begin{aligned} \xi_1(\delta) &= \frac{1}{\delta}, \\ \xi_2(\delta) &= -\frac{2}{3(1+\alpha)\delta^2}. \end{aligned} \tag{1.66}$$

We consider the following functions:

$$f_1(s,\delta) = \frac{1 - \frac{4s}{3(1+\alpha)} + \frac{4s^2}{9(1+\alpha)}}{\sqrt{2\sqrt{(\delta^2 + \frac{4}{9(1+\alpha)^2})^2 + \delta^4(1 - \frac{4s}{3(1+\alpha)} + \frac{4s^2}{9(1+\alpha)})^2} + 2(\delta^2 + \frac{4}{9(1+\alpha)^2})}},$$

$$f_2(s,\delta) = \sqrt{\frac{\sqrt{(\delta^2 + \frac{4}{9(1+\alpha)^2})^2 + \delta^4(1 - \frac{4s}{3(1+\alpha)} + \frac{4s^2}{9(1+\alpha)})^2} + (\delta^2 + \frac{4}{9(1+\alpha)^2})}{2}}.$$

f_1 and f_2 are C^∞ functions, defined in $[1-\varepsilon_0, 1+\varepsilon_0] \times [0, \delta_0]$, while it is easy to see that:

$$(f_1(s,\delta) + i\frac{1}{\delta^2} f_2(s,\delta))^2 = -\frac{1}{\delta^2} - \frac{4}{9(1+\alpha)^2\delta^4} + i(\frac{1}{\delta^2} - \frac{4s}{3(1+\alpha)\delta^2} + \frac{4s^2}{9(1+\alpha)\delta^2}). \tag{1.67}$$

Let $\tilde{\phi}_1$, $\tilde{\phi}_2$ be the following functions:

$$\tilde{\phi}_1(s,\delta) = \int_1^s f_1(\sigma,\delta)d\sigma, \qquad \tilde{\phi}_2(s,\delta) = \int_1^s f_2(\sigma,\delta)d\sigma,$$

we define:

$$\phi(s,\delta) = \tilde{\phi}_1(s,\delta) + i\frac{\tilde{\phi}_2(s,\delta)}{\delta^2}. \tag{1.68}$$

Lemma 1.33. *There exists $\delta_0 > 0$ such that the function ϕ satisfies the following conditions:*

$$\operatorname{Re}\phi(1,\delta) = 0 \qquad \text{for all} \quad \delta \in [0,\delta_0] \quad ; \tag{1.69}$$

there exist $M_1, M_2 > 0$ such that:

$$M_1 \leq -\operatorname{Re}\partial_s\phi(1,\delta) - \operatorname{Re}\partial_s^2\phi(1,\delta) \leq M_2 \quad , \tag{1.70}$$

for all $\delta \in [0,\delta_0]$.

Proof. Let us verify (1.70). We know that $\operatorname{Re}\partial_s\phi(s,\delta) = f_1(s,\delta)$. Let us set:

$$g(s) = 1 - \frac{4s}{3(1+\alpha)} + \frac{4s^2}{9(1+\alpha)} \quad \text{and} \quad K(\delta) = \delta^2 + \frac{4}{9(1+\alpha)^2}.$$

Therefore:

$$\begin{aligned}
&\operatorname{Re}\partial_s^2\phi(s,\delta) \\
&= \frac{g'(s)}{\sqrt{2\sqrt{K^2(\delta)+\delta^4g^2(s)}+2K(\delta)}}\left(1 - \frac{\delta^4g^2(s)}{(2\sqrt{K^2(\delta)+\delta^4g^2(s)}+2K(\delta))\sqrt{K^2(\delta)+\delta^4g^2(s)}}\right) \\
&= \frac{g'(s)}{\sqrt{2\sqrt{K^2(\delta)+\delta^4g^2(s)}+2K(\delta)}} + \delta^4F(s,\delta) \quad ,
\end{aligned}$$

where $F(s,\delta)$ is bounded on $[1-\varepsilon_0, 1+\varepsilon_0]\times[0,\delta_0]$. It will be sufficient, taking a sufficiently small $\delta_0 > 0$, to verify that there exist $M_1', M_2' > 0$ such that:

$$M_1' \leq -g(s) - g'(s) \leq M_2' \quad ,$$

for all $s \in [1-\varepsilon_0, 1+\varepsilon_0]$, for a suitable $\varepsilon_0 > 0$. The proof is complete.

Before going on with the proof of the theorem, let us explain the structure of it. The function u will have the form:

$$u = \sum_{k\geq k_0} \tilde{u}_k \quad ,$$

where every $\tilde{u}_k$ has the support contained in $\mathbf{R}^2 \times [b_{k+1}, b_{k-1}]$, (b_k) being a suitable decreasing sequence going to zero when k goes to infinity. Each $\tilde{u}_k$ will be obtained from a function u_k, by multiplication with a cut off function. The functions u_k are defined as follows:

$$u_k(x,t) = e^{-\gamma(b_k)} e^{i\mu_k \xi(b_k)\cdot x} e^{\nu_k \phi(t/b_k, b_k)} w(\frac{t}{b_k}, b_k) \quad ,$$

where the functions $\xi(\delta)$ and $\phi(s,\delta)$ have been already defined in (1.66) and (1.68) respectively, $\gamma(\delta)$ and $w(s,\delta)$ will be constructed in the following pages and (μ_k), (ν_k) will be suitable sequences. Finally the function a will be obtained posing $a = -p_\alpha u/u$ and verifying the correctness of this definition. For a detailed description of this method we refer to [Zu2] and [SR2].

Let u be the following function:

$$u(x,t,\delta) = e^{-\gamma(\delta)} e^{i\mu\xi(\delta)\cdot x} e^{\nu\phi(t/\delta,\delta)} w(\frac{t}{\delta}, \delta) \quad ,$$

where $\xi(\delta), \phi(s,\delta)$ have already been defined. We set $\nu = \delta^{-\sigma/\rho}$ and $\mu = \nu/\delta = \delta^{-(\sigma+\rho)/\rho}$ (ρ and σ will be chosen later on), and however we will often continue to write ν and μ. Let finally be $b_k = k^{-\rho}$. We set:

$$u_k(x,t) = u(x,t,b_k) \quad ,$$

and u_k will be considered, as said, on $\mathbf{R}^2 \times [b_{k+1}, b_{k-1}]$.

Let us now choose the function γ. For $t \in [b_{k+1}, b_{k-1}]$, we define:

$$G_k(t) = \nu_k \mathrm{Re}\, \phi(\frac{t}{b_k}, b_k) - \nu_{k+1} \mathrm{Re}\, \phi(\frac{t}{b_{k+1}}, b_{k+1}) \quad . \tag{1.71}$$

Lemma 1.34. *Let $m_k = \frac{1}{2} b_k + \frac{1}{2} b_{k+1}$ and $I_k = G_k(m_k)$.*

Then there exist $\omega > 1$ such that:

$$I_k = -\mathrm{Re}\, \partial_s \phi(1,0) \rho k^{\sigma-1} + O(k^{\sigma-\omega}) \quad .$$

Proof. It is enough to use the Taylor formula in (1.71).

We shall take:

$$\gamma_k = -\sum_{j=k_0}^{k-1} I_j \quad ,$$

where $k \geq k_0 - 1$, with $k_0 >> 1$. We have:

$$\gamma_k = \mathrm{Re}\, \partial_s \phi(1,0) \frac{\rho}{\sigma} k^\sigma + O(k^{\sigma+1-\omega}) \quad .$$

It is now possible to choose $\tilde{\gamma} \in C^\infty[0, \delta_0]$ and define $\gamma(\delta) = \delta^{-\sigma/\rho} \tilde{\gamma}(\delta)$ in such a way that, for all $k \geq k_0$,:

$$\gamma(b_k) = \gamma_k \quad . \tag{1.72}$$

Let us now come to the construction of the function w. By (1.66), (1.67), and (1.68) we have:

$$p_\alpha u = \frac{\nu}{\delta^4}[-2\delta^2\partial_s\phi(\frac{t}{\delta},\delta)\partial_s w(\frac{t}{\delta},\delta) - \delta^2\partial_s^2\phi(\frac{t}{\delta},\delta)w(\frac{t}{\delta},\delta) - \frac{\delta^2}{\nu}\partial_s^2 w(\frac{t}{\delta},\delta)]e^{-\gamma(\delta)}e^{i\tau\xi(\delta)\cdot x}e^{\nu\phi(t/\delta,\delta)}$$

We consider the following operators:

$$\begin{aligned} L_0 &= -\,2\delta^2\partial_s\phi(s,\delta)\partial_s - \delta^2\partial_s^2\phi(s,\delta) \\ L_1 &= -\,\partial_s^2 \quad . \end{aligned}$$

Reminding (1.68) we have that $\delta^2\partial_s\phi(s,\delta) = \delta^2 f_1(s,\delta) + if_2(s,\delta)$ and $f_2(s,\delta) > 0$ in $[1-\varepsilon_0, 1+\varepsilon_0] \times [0,\delta_0]$. So that the following Cauchy problems for O.D.E.:

$$\begin{cases} L_0 g_0 = 0 \\ g_{0\,|s=1} = 1 \end{cases} ,$$

and

$$\begin{cases} L_0\, g_j = -L_1 g_{j-1} & \text{for} \quad j \geq 1 \\ g_{j\,|s=1} = 0 \end{cases} ,$$

have solutions g_j which are C^∞ functions on $[1-\varepsilon_0, 1+\varepsilon_0] \times [0,\delta_0]$. Let ψ be a real valued C^∞ function such that $\operatorname{supp}\psi \subseteq [-1,1]$, $0 \leq \psi \leq 1$ and $\psi(t) = 1$ for $|t| \leq 3/4$. We define:

$$\Phi(s,\delta,\theta) = \sum_{j=0}^{\infty} \psi(\lambda_j\theta)\theta^j g_j(s,\delta) \quad ,$$

where (λ_j) is an increasing sequence of real numbers going to infinity fast enough. Finally we set:

$$w(s,\delta) = \Phi(s,\delta,\frac{\delta^2}{\nu}) \quad .$$

Lemma 1.35. *For $t \in [b_{k+1}, b_{k-1}]$, we define:*

$$r_k(t) = \frac{p_\alpha u_k}{u_k} \quad .$$

Then for each $j \in \mathrm{N}$ there exists $\mathcal{H}_j > 0$ such that for each $N \in \mathrm{N}$ there exists $C_{j,N} \geq 0$ such that:

$$|D_t^j r_k(t)| \leq C_{j,N} k^{-\mathcal{H}_j N} \qquad \textit{for each} \quad k \geq k_0 \,. \tag{1.74}$$

Proof. First we prove (1.74) for $j=0$. We have:

$$\begin{aligned}
w_k(s) &= w(s,b_k)\\
&= \Phi(s,b_k,\frac{b_k^2}{\nu_k})\\
&= \sum_{j=0}^{N-1}\psi(\lambda_j\frac{b_k^2}{\nu_k})(\frac{b_k^2}{\nu_k})^j g_j(s,b_k) + (\frac{b_k^2}{\nu_k})^N \sum_{j=N}^{+\infty}\psi(\lambda_j\frac{b_k^2}{\nu_k})(\frac{b_k^2}{\nu_k})^{j-N} g_j(s,b_k)\\
&= \sum_{j=0}^{N-1}\psi(\lambda_j\frac{b_k^2}{\nu_k})(\frac{b_k^2}{\nu_k})^j g_j(s,b_k) + (\frac{b_k^2}{\nu_k})^N R_N(s,b_k)\,,
\end{aligned}$$

where R_N is a $\mathcal{C}^\infty$ function in s such that for all j there exist $C_j,\delta_j>0$ such that $|\partial_s^j R_N(s,\delta)|<C_j$ for $0<\delta<\delta_j$ and for all $s\in[1-\varepsilon_0,1+\varepsilon_0]$. Remarking that $b_k^2/\nu_k = k^{-2\rho-\sigma}$, we have that there exists k_N such that for $k\geq k_N$, $\psi(\lambda_j b_k^2/\nu_k)=1$ if $0\leq j<N$. Then:

$$L_0 w_k(s) + \frac{b_k^2}{\nu_k}L_1 w_k(s) = (\frac{b_k^2}{\nu_k})^N L_1 g_{N-1}(s,b_k) + (\frac{b_k^2}{\nu_k})^N L_0 R_N(s,b_k) + (\frac{b_k^2}{\nu_k})^{N+1} L_1 R_N(s,b_k).$$

So that:

$$\begin{aligned}
r_k(t) &= k^{\sigma+4\rho}\frac{1}{w_k(\frac{t}{b_k})}[L_0 w_k(\frac{t}{b_k}) + \frac{b_k^2}{\nu_k}L_1 w_k(\frac{t}{b_k})]\\
&= k^{(\sigma+4\rho)-(\sigma+2\rho)N}\frac{1}{w_k(\frac{t}{b_k})}[L_1 g_{N-1}(\frac{t}{b_k},b_k) + L_0 R_N(\frac{t}{b_k},b_k)\\
&\qquad + k^{-\sigma-2\rho}L_1 R_N(\frac{t}{b_k},b_k)]\,,
\end{aligned}$$

and this implies that:

$$|r_k(t)| \leq C_N k^{(\sigma+4\rho)-(\sigma+2\rho)N}\,,$$

for $k\geq k_N$. Eventually taking a bigger C_N we reach the conclusion. The case $j>0$ is analogous.

We study now the set where $|u_k|=|u_{k+1}|$. For $(x,t)\in\mathbf{R}^2\times[b_{k+1},b_k]$, we consider:

$$\begin{aligned}
F_k(t) &= \log|\frac{u_k(x,t)}{u_{k+1}(x,t)}|\\
&= -\gamma(b_k) + \nu_k \mathrm{Re}\,\phi(\frac{t}{b_k},b_k) + \gamma(b_{k+1}) - \nu_{k+1}\mathrm{Re}\,\phi(\frac{t}{b_{k+1}},b_{k+1}) + \log|\frac{w(\frac{t}{b_k},b_k)}{w(\frac{t}{b_{k+1}},b_{k+1})}|.
\end{aligned}$$

Lemma 1.36. *Suppose that $\sigma>1$ and that there exist $k_0'\in\mathbf{N}$ and $C>0$ such that, for all $k\geq k_0'$:*

$$-(\sigma+\rho)\mathrm{Re}\,\partial_s\phi(1,b_k) - \rho\mathrm{Re}\,\partial_s^2\phi(1,b_k) > C\,.$$

Then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and for all $t \in [b_{k+1}, b_k]$:

$$F_k'(t) > \frac{C}{2} k^{\sigma+\rho-1} . \tag{1.75}$$

Proof. We have:

$$\begin{aligned} F_k'(t) &= \frac{\nu_k}{b_k} \mathrm{Re}\, \partial_s \phi(\frac{t}{b_k}, b_k) - \frac{\nu_{k+1}}{b_{k+1}} \mathrm{Re}\, \partial_s \phi(\frac{t}{b_{k+1}}, b_{k+1}) + O(k^\rho) \\ &= (\frac{\nu_k}{b_k} - \frac{\nu_{k+1}}{b_{k+1}}) \mathrm{Re}\, \partial_s \phi(\frac{t}{b_k}, b_k) + \frac{\nu_{k+1}}{b_{k+1}} (\mathrm{Re}\, \partial_s \phi(\frac{t}{b_k}, b_k) - \mathrm{Re}\, \partial_s \phi(\frac{t}{b_{k+1}}, b_k)) \\ &\quad + \frac{\nu_{k+1}}{b_{k+1}} (\mathrm{Re}\, \partial_s \phi(\frac{t}{b_{k+1}}, b_k) - \mathrm{Re}\, \partial_s \phi(\frac{t}{b_{k+1}}, b_{k+1})) + O(k^\rho) \\ &= ① + ② + ③ + ④ . \end{aligned}$$

Consequently:

$$\begin{aligned} ① &= -(\sigma+\rho) k^{\sigma+\rho-1} \mathrm{Re}\, \partial_s \phi(1, b_k) + O(k^{\sigma+\rho-2}) , \\ ② &\geq -\rho k^{\sigma+\rho-1} \mathrm{Re}\, \partial_s^2 \phi(1, b_k) + O(k^{\sigma+\rho-2}) , \\ ③ &= O(k^{\sigma-1}) , \\ ④ &= O(k^\rho) , \end{aligned}$$

and this completes the proof.

Let us finally choose σ and ρ. We set $\sigma = 3$ and we fix ρ and k_0' such that:

$$-(\sigma+\rho) \mathrm{Re}\, \partial_s \phi(1, b_k) - \rho \mathrm{Re}\, \partial_s^2 \phi(1, b_k) > \frac{M_1}{2} > 0 ,$$

for all $k \geq k_0'$; this is possible, thank to condition (1.70). From (1.71) and (1.72) we have that:

$$F_k(m_k) = F_k(\frac{1}{2} b_k + \frac{1}{2} b_{k+1}) = \log |\frac{w(\frac{m_k}{b_k}, b_k)}{w(\frac{m_k}{b_{k+1}}, b_{k+1})}| = O(1) .$$

Then, by (1.75) we deduce that there exists a unique point $t_k^* \in [b_{k+1}, b_k]$ such that $|u_k(t_k^*)| = |u_{k+1}(t_k^*)|$, and $|t_k^* - m_k| \leq C k^{-\rho-\sigma+1}$, so that t_k^* is always "in the center" of the interval $[b_{k+1}, b_k]$. From now on the proof is exactly the same as that one of the Theorem 3, **g)**, **h)** of [CDSZ2]. We give only a sketch.

Using Whitney's theorem [Wh] it is possible to show that there exist functions z_k which are C^∞ on $[b_{k+1}, b_{k-1}]$ and such that, if we define:

$$\tilde{u}_k(x,t) = e^{-\gamma_k} e^{i\tau_k \xi(b_k)\cdot x} e^{\nu_k \phi(t/b_k, b_k)} [w(\frac{t}{b_k}, b_k) + z_k(t)] \quad ,$$

$$\tilde{F}_k(t) = \log \left| \frac{\tilde{u}_k(x,t)}{\tilde{u}_{k+1}(x,t)} \right| \quad ,$$

and :

$$\tilde{r}_k(t) = \frac{p_\alpha \tilde{u}_k(x,t)}{\tilde{u}_k(x,t)} \quad ,$$

then for $\tilde{F}_k$ the same statement of Lemma 1.36 holds, while for $\tilde{r}_k$ the same statement of Lemma 1.35 is valid and $\tilde{r}_k$ is flat on t_k^* and t_{k+1}^*.

Finally, letting χ be a real valued $\mathcal{C}^\infty$ function such that $\operatorname{supp}\chi \subseteq [-1,1]$, $0 \le \chi \le 1$, $\chi(t) = 1$ for $|t| \le 3/4$, and

$$\chi_k(t) = \chi\left(\frac{t - b_k}{b_k - b_{k+1}}\right) \quad ,$$

we define:

$$u(x,t) = \sum_{k \ge k_0} \chi_k(t)\tilde{u}_k(x,t)\,,$$

and

$$a(x,t) = -\frac{p_\alpha u(x,t)}{u(x,t)}\,.$$

It is a standard computation to verify the correct definition and smoothness of these two functions. The proof is complete.

Chapter 2. Hörmander's Uniqueness Theorem

2.1. Introduction

In this chapter we will collect some results on the uniqueness in the Cauchy problem which are essentially due to Hörmander [Hör1, Th. 8.9.1], [Hör3, Th. 28.3.4]. Although there are some common aspects with the subject of the previous chapter, as we will see, the approach to the problem of the uniqueness is here different, the new matter being the joint geometry of the operator and the oriented hypersurface of initial data. The geometrical notion of strong pseudo–convexity of a hypersurface with respect to an operator at a point, will be crucial in giving new sufficient conditions for the uniqueness for large classes of operators. The most important of these ones is the class of principally normal operators. The definitions of strongly pseudo–convex hypersurface and of principally normal operator, together with the main uniqueness results, are the content of the first paragraph.

Paragraph 2.2 is devoted to a theorem obtained in collaboration with X. Saint Raymond [DSSR1]. By using the para–differential calculus we extend to operators with $\mathcal{C}^2$ coefficients a result of Hörmander for principally normal operators with $\mathcal{C}^\infty$ coefficients. As the proof of Hörmander's result is based on the Fefferman–Phong inequality, and this one has not been proved for para–differential operators, we are forced to add a normality assumption and we give a proof which use the Gårding inequality for para–differential operators "with a large parameter", first introduced by Métivier in [Mét2].

2.2. Hörmander's Uniqueness Theorem

Let Ω be an open set of $\mathbf{R}^n$; let $P(x, D)$ be the following operator:

$$P(x, D_x) = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha \quad , \tag{2.1}$$

with $a_\alpha \in L^\infty_{\mathrm{loc}}(\Omega, \mathbf{C})$. Let S be a hypersurface in Ω, $S = \{x \in \Omega \mid \varphi(x) = 0\}$ where $\varphi \in \mathcal{C}^2(\Omega, \mathbf{R})$, and let $x_0 \in S$ such that $\varphi'(x_0) \neq 0$. As usual we will denote by $p(x, D_x) = \sum_{|\alpha|=m} a_\alpha(x) D_x^\alpha$ the principal part of P, and by $p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$, its principal symbol.

Definition 2.1. *The operator P defined in* (2.1) *will be called elliptic at x_0 if there exists a neighborhood V of x_0 such that:*

$$p(x,\xi) \neq 0 \qquad \textit{for all} \qquad (x,\xi) \in V \times \mathbf{R}^n \setminus \{0\} \quad .$$

Let now $T^\star\Omega$ be the cotangent boundle of Ω, and let f, g be two complex $\mathcal{C}^1$ functions on $T^\star\Omega$. The Poisson bracket of f, g is defined by:

$$\{f\,,\,g\}(x,\xi)=\sum_{j=1}^{n}(\partial_{x_j}f\partial_{\xi_j}g-\partial_{x_j}g\partial_{\xi_j}f)(x,\xi) \quad .$$

Definition 2.2 [Hörl, Def. 8.5.1]. *The operator P defined in* (2.1) *will be called principally normal (in strong sense) at x_0, if the coefficients of p are in $\mathcal{C}^1(\Omega,\mathbf{C})$, there exists a function $q(x,\xi)$ such that q is a polynomial in ξ of degree $m-1$ with coefficients in $\mathcal{C}^1(\Omega,\mathbf{C})$, and there exists a neighborhood V of x_0 such that:*

$$\{\bar{p}\,,\,p\}(x,\xi) \;=\; 2i\,\mathrm{Re}\,(\bar{q}(x,\xi)p(x,\xi)) \quad , \tag{2.2}$$

for all $(x,\xi)\in V\times\mathbf{R}^n\setminus\{0\}$.

Let us remark that the operators with real $\mathcal{C}^1$ principal part are principally normal (in strong sense).

Definition 2.3 [Hör3, Def. 28.3.1]. *Let P, defined in* (2.1), *be elliptic or principally normal at x_0. The oriented hypersurface $S=\{x\in\Omega\mid\varphi(x)=0\}$ is said to be strongly pseudo–convex at x_0, with respect to P, if, for each $\zeta=\xi-i\tau\varphi'(x_0)\neq0$, $\xi\in\mathbf{R}^n$, $\tau\in\mathbf{R}$, such that $p(x_0,\zeta)=\{p\,,\varphi\}(x_0,\zeta)=0$, we have:*

$$\frac{1}{2i\tau}\{\overline{p_{\tau\varphi}}\,,\,p_{\tau\varphi}\}(x_0,\xi)>0 \qquad \textit{if} \quad \tau\neq0 \quad , \tag{2.3}$$

$$\mathrm{Re}\,\{\bar{p}\,,\,\{p\,,\,\varphi\}\}(x_0,\xi)<0 \qquad \textit{if} \quad \tau=0 \quad , \tag{2.4}$$

where $p_{\tau\varphi}(x,\xi)=p(x,\xi-i\tau\varphi'(x))$.

We can now state the main result.

Theorem 2.4 [Hörl, Th. 8.9.1]. *Let P be the operator defined in* (2.1). *Assume either that P is elliptic at x_0, with $\mathcal{C}^1$ principal part, or else that P has a $\mathcal{C}^1$ real principal part, or that P is principally normal (in strong sense) at x_0, with $\mathcal{C}^2$ pincipal part. Assume also that S is a strongly pseudo–convex hypersurface at x_0 with respect to P.*

Then P has the uniqueness in the Cauchy problem with respect to S at x_0.

Remark 2.5. The Theorem 1.1 is a corollary of the Theorem 2.4. In fact it is easy to see that under the hypotheses of the Theorem 1.1 there are no $\zeta=\xi-i\tau\varphi'(x_0)\neq0$, such that $p(x_0,\zeta)=\{p\,,\varphi\}(x_0,\zeta)=0$. Better than this, as already pointed out by Hörmander, the condition: the polynomial in τ: $p(x_0,\tau N+\eta)$ has, for all η non parallel to N, m distinct roots, is a necessary and sufficient condition to the strong pseudo–convexity for every hypersurface through x_0, with normal N.

Remark 2.6. The Theorem 2.4 doesn't require that the hypersurface S is non–characteristic, so it can be applied also in the case of characteristic but strongly pseudo–convex surface of initial data.

Remark 2.7. There is an interesting geometrical interpretation of the strong pseudo–convexity for an operator P of order 2 with real principal part, in terms of the bicharacteristic curves associated to p (see [Hör1, Def. 1.8.6]): S is strongly pseudo–convex (an consequently P has the uniqueness in the Cauchy problem), if all the null bicharacteristic curves which are tangent to S at x_0, have a contact of order 2 in x_0 and lie "in the past", i.e. in $\{x \in \Omega \,|\, \varphi(x) \leq 0\}$.

Remark 2.8. There exists a wide number of works which investigate the necessity of the hypotheses of the Theorem 2.4. Let us only mention the well known paper of Alinhac [A1].

Let us now give a weaker version of the Definition 2.2 (see also [L] for another version of this definition).

Definition 2.9 [Hör3, Def. 28.2.4]. *The operator P defined in* (2.1) *will be called principally normal (in weak sense) at x_0, if the coefficients of p are in $C^1(\Omega, \mathbf{C})$, and there exist a neighborhood V of x_0 and a function $q \in L^\infty_{\mathrm{loc}}(V \times \mathbf{R}^n \setminus \{0\}, \mathbf{C})$ such that $q(x,\xi)$ is homogeneous in ξ of degree $m-1$, and* (2.2) *holds.*

The result for principally normal operators in weak sense is the following.

Theorem 2.10 [Hör3, Th. 28.3.4]. *Let P, defined in* (2.1), *be principally normal (in weak sense), with C^∞ principal symbol. Let S be strongly pseudo–convex at x_0, with respect to P.*

Then P has the uniqueness in the Cauchy problem with respect to S at x_0.

Remark 2.11. The proof of the theorem 2.10 is based on the Weyl calculus and on the highly non trivial result of Fefferman and Phong on lower bound for non–homogeneous pseudo–differential operators (see [FP] and [Hör3, Th. 18.6.8]). This result has not been proved for para–differential operators (even if it seems reasonable that a similar inequality must be valid for these operators too). Consequently the proof cannot be simply adapted to para–differential operators.

2.3. A Result for Operators with C^2 Coefficients

At the end of Chapter 28 of his book [Hör 3], Hörmander remarks that it is not clear how regular the coefficients of p must be for the Theorem 28.3.4, i.e. Theorem 2.10 above, to hold. Our main aim is to give a partial answer to this question. We think that the requirement on the regularity of the coefficients in the principal part essentially depends on the strength of the normality assumption. More clearly, if the normality condition is strong (e. g. the operator has a real principal part), then a low regularity of the coefficients (actually $\mathcal{C}^1$) is sufficient to the uniqueness. On the contrary if the normality condition is weaker (e. g. the operator is principally normal in weak sense), then the regularity must be higher (we conjecture $\mathcal{C}^4$).

We show here that under a certain normality assumption on the operator, $\mathcal{C}^2$ regularity will be sufficient to the uniqueness. Our condition will be stronger than the principal normality in weak sense , but weaker than principal normality in strong sense.

Theorem 2.12. *Assume that the differential operator P defined in* (2.1) *has $\mathcal{C}^2$ coefficients in the principal part. Assume that its principal symbol p satisfies condition* (2.2) *for a function $q(x,\xi)$ of class $\mathcal{C}^1$ in x and $\mathcal{C}^\infty$ in ξ (this means that $\partial_\xi^\alpha q$ is a $\mathcal{C}^1$ function of (x,ξ) for all $\alpha \in \mathbf{N}^n$), homogeneous in ξ of degree $m-1$. Assume also that S is strongly pseudo–convex with respect to P at x_0.*

Then P has the uniqueness in the Cauchy problem with respect to S at x_0.

Remark 2.13. The normality condition in Theorem 2.12 will be satisfied if it can be shown that an equality $\{\bar{p}, p\} = 2i\,Re\,(\bar{q}p)$ holds near every zero of p for some local function q; indeed a global q can then be constructed by using a partition of unity.

Remark 2.14. The part of the Theorem 2.4 regarding the principally normal operators in strong sense is a corollary of the Theorem 2.12.

Remark 2.15. We could also give a result under the weaker assumption that P is principally normal in weak sense (i.e. q is merely locally bounded, see Definition 2.9) if we were able to prove a Fefferman–Phong inequality for paradifferential operators, as it is easily seen in the proof below.

Proof of the Theorem 2.12. As in the model Theorem 28.3.4 of [Hör3], the uniqueness result follows from a Carleman estimate proved for a weight function admitting a given family of smooth strongly pseudo–convex level surfaces (see [Hör3, Prop. 28.3.3]). Thus, if this family is given as the level surfaces of a function ψ, we may assume, after a change of variables, that $\psi(x) = x_n$, then we set $\phi(x) = -e^{-A\psi(x)}$ (ϕ is then an increasing negative function of x_n); introducing the norms:

$$\|u\|_{s,\lambda}^2 = (2\pi)^{-n}\int(\lambda^2+|\xi|^2)^s|\hat{u}(\xi)|^2 d\xi \quad ,$$

and the neighborhoods of x_0 : $\Omega_\varepsilon = \{x \in \Omega \mid |x-x_0| < \varepsilon\}$, the result stated in Theorem 2.12 can be classically deduced from the following estimate.

Lemma 2.16. *Let P be as in the statement of the Theorem 2.12 and assume that the hypersurface $S = \{x \in \Omega \mid \psi(x) = 0\}$, $\psi(x) = x_n$, is strongly pseudo–convex at x_0.*

Then there exist constants A, C, and $\varepsilon > 0$ such that, with $\phi = -e^{-A\psi}$,

$$\lambda^{\frac{1}{2}}\|e^{-\lambda\phi}u\|_{m-1,\lambda} \le C\|e^{-\lambda\phi}p(x,D_x)u\|_0 \tag{2.5}$$

for all $u \in C_0^\infty(\Omega_\varepsilon)$ *and* $\lambda \geq \frac{1}{\varepsilon}$.

Proof. As usual we set $v = e^{-\lambda\phi}u$ and we consider the operator:

$$P_\lambda = e^{-\lambda\phi} \circ p(x, D_x) \circ e^{\lambda\phi} = p_\lambda(x, D_x) + r_\lambda(x, D_x) \quad ,$$

where $p_\lambda(x,\xi) = p_{\lambda\phi}(x,\xi) = p(x, \xi - i\lambda\phi'(x))$, and $r_\lambda(x,\xi)$ is a $(m-1)$–st degree polynomial in (λ, ξ); with this notation, the estimate (2.5) becomes:

$$\lambda^{\frac{1}{2}} \|v\|_{m-1,\lambda} \leq C\|P_\lambda v\|_0 \tag{2.6}$$

for all $v \in C_0^\infty(\Omega_\varepsilon)$ and $\lambda \geq \frac{1}{\varepsilon}$.

At this point we will use a para–differential calculus with a large parameter as given in [Mét2, App. B] (see Paragraph A.6), that is we'll use the spaces ${}^\lambda\Gamma_\rho^m$ and ${}^\lambda\Sigma_\rho^m$ introduced by the Paragraph A.6, and their properties; here, it is clear by homogeneity that $p_\lambda \in {}^\lambda\Gamma_2^m$. In the proof of estimate (2.6) we will use the following lemmas.

Lemma 2.17. *Under the assumptions of Lemma 2.16, there exist two constants* A *and* $\delta > 0$, *and a symbol* $e_\lambda \in {}^\lambda\Gamma_1^{2m-2}$ *such that for* x *sufficiently close to* x_0 *and* $\zeta = \xi - i\lambda\phi'(x)$,

$$\frac{1}{i}\{\overline{p_\lambda}, p_\lambda\} - 2\,\mathrm{Re}\,[\overline{(q + \lambda A^2\phi\frac{p_\lambda}{|\zeta|^2})}\,p_\lambda] = \lambda e_\lambda \qquad \textit{and} \qquad e_\lambda(x,\xi) \geq \delta(\lambda^2 + |\xi|^2)^{m-1} \quad .$$

Proof. Let us set $\tau = -\lambda A\phi(x)$, so that $\zeta = \xi - i\lambda\phi'(x) = \xi - i\tau\psi'(x)$; reminding that $\psi'' \equiv 0$, an easy computation gives:

$$\frac{1}{i}\{\overline{p_\lambda}, p_\lambda\} = 2\,\mathrm{Im}\,\langle\overline{p'_\xi(x,\zeta)}, p'_x(x,\zeta)\rangle + 2A\tau|\{p,\psi\}(x,\zeta)|^2 \quad ;$$

(here $\langle\cdot\,,\,\cdot\rangle$ denotes the scalar product in $\mathbf{R}^n$).

On the other hand, our normality assumption shows that the polynomial in τ:

$$\mathrm{Im}\,\langle\overline{p'_\xi(x,\zeta)}, p'_x(x,\zeta)\rangle - \mathrm{Re}\,(\bar{q}(x,\xi)p(x,\zeta))$$

vanishes for $\tau = 0$ and therefore it can be written $\tau\, l(x,\xi,\tau)$ where l is a C^1 function in x, C^∞ and homogeneous in (ξ,τ) of degree $2m-2$; furthermore, the convexity assumption implies that for x in a compact neighborhood of x_0,

$$\inf_{|\zeta|=1}[l(x,\xi,\tau) + A(|\{p,\psi\}(x,\zeta)|^2 + |p(x,\zeta)|^2)] = \delta > 0 \quad ,$$

for some large constant A. Thus we can use homogeneity to write:

$$\begin{aligned}
&\frac{1}{i}\{\overline{p_\lambda}, p_\lambda\} - 2\,\mathrm{Re}\,[\overline{(q + \lambda A^2\phi\frac{p_\lambda}{|\zeta|^2})}\,p_\lambda] \\
&= 2\tau l(x,\xi,\tau) + 2A\tau|\{p,\psi\}(x,\zeta)|^2 + 2A\tau\frac{|p(x,\zeta)|^2}{|\zeta|^2} \\
&= 2\tau|\zeta|^{2m-2}[l(x,\frac{\xi}{|\zeta|},\frac{\tau}{|\zeta|}) + A(|\{p,\psi\}(x,\frac{\zeta}{|\zeta|})|^2 + |p(x,\frac{\zeta}{|\zeta|})|^2)] \\
&= \lambda\, e_\lambda(x,\xi)
\end{aligned}$$

where e_λ has the wanted properties. The proof is complete.

Lemma 2.18. *With the notations of Lemma 2.17, the symbol:*

$$\tilde{p}_\lambda = p_\lambda + q + \lambda A^2 \phi \frac{p_\lambda}{|\zeta|^2}$$

satisfies $\tilde{p}_\lambda \in {}^\lambda\Sigma_2^m$. Moreover the operator:

$$R_\lambda = (T^\lambda_{p_\lambda})^* T^\lambda_{p_\lambda} - T^\lambda_{\tilde{p}_\lambda}(T^\lambda_{\tilde{p}_\lambda})^* - \lambda T^\lambda_{e_\lambda}$$

is bounded from H^{m-1} into H^{1-m} uniformly in λ (for the definition of $T^\lambda_{p_\lambda}$ see Definition A.28).

Proof. The property $\tilde{p}_\lambda \in {}^\lambda\Sigma_2^m$ simply follows from the fact that $\tilde{p}_\lambda$ is a sum of homogeneous functions of (ξ, λ). As for the boundedness of R_λ, we continue to use that all our symbols are sums of homogeneous terms, so that it is an easy consequence of the following result which refines Proposition B.1.3 of [Mét2], but the proof of which is a straightforward extension of the classical paradifferential machinery (see [Bo, Th. 3.9], [GR, Par. 3] and Paragraph A.6).

Lemma 2.19. *Let ρ, σ, m and l be positive integers and set, for homogeneous symbols $a \in {}^\lambda\Gamma^m_\rho$ and $b \in {}^\lambda\Gamma^l_\sigma$:*

$$a^* = \sum_{|\alpha|<\rho} \frac{1}{\alpha!} \partial^\alpha_\xi D^\alpha_x \bar{a} \qquad \text{and} \qquad a\#b = \sum_{|\alpha|<\sigma} \frac{1}{\alpha!} \partial^\alpha_\xi a D^\alpha_x b \quad .$$

Then $(T^\lambda_a)^* - T^\lambda_{a^*}$ (respectively $T^\lambda_a T^\lambda_b - T^\lambda_{a\#b}$) is bounded from H^s into $H^{s-m+\rho}$ (respectively into $H^{s-m-l+\sigma}$) uniformly in λ.

Let us now come to the proof of the estimate (2.6). First, we can write:

$$\begin{aligned} \|P_\lambda v\|_0 &\geq \|p_\lambda(x,D)v\|_0 - \|r_\lambda(x,D)v\|_0 \\ &\geq \|T^\lambda_{p_\lambda} v\|_0 - \|(p_\lambda(x,D) - T^\lambda_{p_\lambda})v\|_0 - \|r_\lambda(x,D)v\|_0 \\ &\geq \|T^\lambda_{p_\lambda} v\|_0 - C\,\|v\|_{m-1,\lambda} \quad , \end{aligned}$$

thanks to estimates similar to that of Theoerm A.15. On the other hand, we have:

$$\|T^\lambda_{p_\lambda} v\|_0^2 = (R_\lambda v\,,\, v) + \|(T^\lambda_{\tilde{p}_\lambda})^* v\|_0^2 + \lambda(T^\lambda_{e_\lambda} v\,,\, v) \geq \lambda(T^\lambda_{e_\lambda} v\,,\, v) - C\|v\|^2_{m-1,\lambda}$$

thanks to the uniform boundedness of R_λ given by Lemma 2.18. Finally the estimate we have for e_λ in Lemma 2.17 and a Gårding inequality such as in Proposition B.1.4 of [Mét2] (see also Theoerm A.22), imply:

$$\|T^\lambda_{p_\lambda} v\|_0^2 \geq \frac{\delta}{2}\lambda\|v\|^2_{m-1,\lambda} - C\lambda\|v\|^2_{m-2,\lambda} - C\|v\|^2_{m-1,\lambda} \quad ,$$

whence we get the estimate (2.6) if ε is chosen sufficiently small and λ sufficiently large to absorb all the error terms in the main term $\frac{\delta}{2}\lambda\|v\|^2_{m-1,\lambda}$.

ON THE UNIQUENESS THEOREM OF LERNER AND ROBBIANO

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Few years after the famous result of CALDERÓN [2], Hörmander proved in 1963 a uniqueness theorem for operators with C^1 coefficients (resp. C^2 coefficients) in the principal part simply by integrating by parts explicitly once (resp. twice) [4]. Then the use of pseudo-differential Gårding inequalities allowed many authors to extend these results in the case of operators with C^∞ coefficients in the principal part, see e.g. HÖRMANDER [5].

Recently [3], the authors of this paper revisited Hörmander's uniqueness theorem [5, th. 28.3.4] to weaken the smoothness assumption on the coefficients of the principal part. In the present paper, we now consider Lerner-Robbiano's uniqueness theorem [7], as improved by HÖRMANDER [5, th. 28.4.3], to examine again the smoothness assumption on the coefficients of the principal part. Easy arguments, parallel to those of [3] and consisting essentially in replacing pseudo-differential Gårding inequalities with para-differential Gårding inequalities, show that Lerner-Robbiano's result is still valid for second order operators of real principal type with C^4 coefficients in the principal part. But we can get much better, namely that this result is still valid when the coefficients belong to the Hölder class $C^{7/3}$.

This improvement follows from two different facts : first, that we do not need to change variables explicitly to work with the operator in the standard form $D_1^2 + q(x, D')$, and second that we can treat a not very good error term coming from our Gårding inequality using an interpolation estimate suggested by C. Zuily, whom we thank.

We also show in our last section that the result is still valid for $C^{5/3}$ coefficients (or even C^1) is some special situations; the trick is to replace the paradifferential Gårding inequality with a differential Gårding inequality :

indeed, the error terms in the latter situation are better than in the former. This modification of the proof is based on an algebraic result on real quadratic forms which is recalled in an appendix.

1. Statement of the main result

1.a. General assumptions. Let φ be a smooth real valued function defined near a point $x_0 \in \mathbf{R}^n$ and such that $\varphi(x_0) = 0 \neq \varphi'(x_0)$. Near this point, we are also given a second order linear partial differential operator

$$P = \sum_{|\alpha|\leq 2} a_\alpha(x) D^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ and $D_j = -i\partial/\partial x_j$. We assume that the complex-valued coefficients a_α satisfy

$$a_\alpha \in C^1 \quad \text{if} \quad |\alpha| = 2, \qquad a_\alpha \in L^\infty \quad \text{if} \quad |\alpha| \leq 1,$$

and that P is of real principal type at x_0 (in the strong sense), i.e. its principal symbol

$$p(x,\xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha$$

is real valued and has the property

$$p'_\xi(x_0,\xi) \neq 0 \quad \text{if} \quad \xi \in \mathbf{R}^n \setminus \{0\} .$$

Finally, we also assume that the initial surface $\varphi(x) = 0$ is noncharacteristic at x_0, a property which can be written

$$p(x_0, \varphi'(x_0)) \neq 0 .$$

1.b. The pseudo-convexity property. For the class of operators we have just described, HÖRMANDER [4] proved a uniqueness result under the assumption of strong pseudo-convexity at x_0; introducing the Hamiltonian vector field $H_p = \langle p'_\xi(x,\xi), \partial_x\rangle - \langle p'_x(x,\xi), \partial_\xi\rangle$ associated with the symbol p, this condition of strong pseudo-convexity at x_0 can be written

$$\forall \xi \in \mathbf{R}^n \setminus \{0\}, \quad p(x_0,\xi) = H_p\varphi(x_0,\xi) = 0 \Rightarrow H_p^2\varphi(x_0,\xi) < 0 .$$

Later, LERNER and ROBBIANO [7], then HÖRMANDER [5, th. 28.4.3] proved another uniqueness result under the weaker assumption of pseudo-convexity we give in theorem 1 below, but for operators with C^∞ coefficients in the principal part. The situation we want to examine here is the case of coefficients with a limited regularity as in [4], under the weak pseudo-convexity assumption of [5, th. 28.4.3].

1.c. Measuring the smoothness of the coefficients. For $k \in \mathbb{N}$, we denote by C^k the class of k times continuously differentiable functions defined in a neighborhood of x_0. Then we introduce the Hölder classes C^ρ : first if $\rho \in (0,1)$, we say that $a \in C^\rho$ if there is a neighborhood V of x_0 and a constant C such that

$$|a(x) - a(y)| \leq C\,|x-y|^\rho \quad \text{for all} \quad x, y \in V \ ;$$

next if $\rho \in \mathbb{R}_+ \setminus \mathbb{N}$, we denote by $[\rho]$ the largest integer smaller than ρ, and we say that $a \in C^\rho$ if $a \in C^{[\rho]}$ and $D^\alpha a \in C^{\rho-[\rho]}$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = [\rho]$.

We can now state our result, which is exactly theorem 28.4.3 of HÖRMANDER [5] with the exception of the smoothness assumption on the coefficients.

THEOREM 1. *Let P be, as above, a second order linear partial differential operator of real principal type at x_0, and $\varphi(x) = 0$ be a smooth, noncharacteristic hypersurface through x_0. Assume that $a_\alpha \in C^{7/3}$ for $|\alpha| = 2$, and that the following weak pseudo-convexity condition holds near x_0 :*

$$\forall x \in \mathbb{R}^n,\ \forall \xi \in \mathbb{R}^n,\ \varphi(x) = p(x,\xi) = H_p\varphi(x,\xi) = 0 \Rightarrow H_p^2\varphi(x,\xi) \leq 0 \ .$$

Then, we have the following uniqueness property: any solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ of $Pu = 0$ vanishing in a neighborhood of $\{x \neq x_0\,;\, \varphi(x) \leq 0\}$ must also vanish near x_0 in $\mathbb{R}^n$.

We give in section 4 below variants of this theorem where the assumption $a_\alpha \in C^{7/3}$ is replaced with $a_\alpha \in C^{5/3}$ (resp. $a_\alpha \in C^1$).

2. Standard reductions and estimates

2.a. General notation. First we fix a set of notation which will be used throughout the paper.

Let $\rho \geq 1$ be a real number; we are given $n^2 - n$ coefficients $(a_j)_{j>1}$ and $(a_{jk})_{j>1,k>1}$ with $a_{jk} = a_{kj}$; these coefficients are C^ρ real valued functions defined near $0 \in \mathbb{R}^n$ and we set

$$\ell(x,\xi) = \xi_1 + \sum_{j>1} a_j(x)\xi_j \ ,$$

$$\alpha(x) = \sum_{j>1} (\partial a_j/\partial x_j)(x) \ ,$$

$$q(x,\xi') = \sum_{j>1,k>1} a_{jk}(x)\xi_j\xi_k \ ,$$

$$r(x,\xi') = H_\ell q(x,\xi') + \alpha(x)q(x,\xi') = \sum_{j>1,k>1} b_{jk}(x)\xi_j\xi_k$$

where the real valued coefficients b_{jk} can be chosen so that $b_{jk} \in C^{\rho-1}$ and $b_{jk} = b_{kj}$. For any $a \in L^\infty$ defined near $0 \in \mathbf{R}^n$ and any $\varepsilon > 0$, we set $a^\varepsilon(x) = a(\varepsilon x)$. With the previous ℓ, q and r, we now associate the following operators

$$L^\varepsilon u = D_1 u + \sum_{j>1} a_j^\varepsilon D_j u \ ,$$
$$Q^\varepsilon u = \sum_{j>1,k>1} D_j(a_{jk}^\varepsilon D_k u) \ ,$$
$$R^\varepsilon u = \sum_{j>1,k>1} D_j(b_{jk}^\varepsilon D_k u) \ .$$

In our proof, which follows the standard method of Carleman estimates, we are going to use the L^2 scalar product and norm

$$(u,v) = \int u(x)\bar{v}(x)dx \ \text{ and } \ \|u\|^2 = \int |u(x)|^2 \, dx$$

for $u, v \in C_0^\infty(\mathbf{R}^n)$, the weight function $\phi(x_1) = (x_1^2/2) - x_1$ and the compact set $W = \{x \in \mathbf{R}^n \, ; 0 \le x_1 \le 1/2, \ |x'| \le 1\}$, and we recall that $u \in C_0^\infty(W)$ means $u \in C_0^\infty(\mathbf{R}^n)$ with $\operatorname{supp} u \subset W$. A little bit further, we will also use the more general Sobolev norms

$$\|u\|_s^2 = (2\pi)^{-n} \int (1 + |\xi|^2)^s \, |\hat{u}(\xi)|^2 \, d\xi$$

for $s \in \mathbf{R}$ and $u \in C_0^\infty(\mathbf{R}^n)$. Theorem 1 will be easily deduced from the Carleman estimate that can be stated as follows.

PROPOSITION 2. *With the notation given above, assume that $\rho = 7/3$, that the quadratic form $q(0,\xi')$ is non degenerate and that for some small $\delta > 0$,*

$$x \in \delta W \ \text{ and } \ q(x,\xi') = 0 \Rightarrow r(x,\xi') \ge 0 \ .$$

Then, for some large constant C and all sufficiently small $\varepsilon > 0$ we have

$$\tau^3 \left\|e^{\tau\phi} u\right\|^2 + \tau \left\|e^{\tau\phi} L^\varepsilon u\right\|^2 + \sum_{j\ge 1} \left\|e^{\tau\phi} D_j u\right\|^2 \le C \left\|e^{\tau\phi}((L^\varepsilon)^2 + Q^\varepsilon) u\right\|^2$$

for all $u \in C_0^\infty(W)$ and $\tau \ge 4$.

2.b. Proof of theorem 1. With the help of proposition 2, we can now prove theorem 1.

First, using a smooth change of variables, we can assume that $x_0 = 0$ and that $\varphi(x) = x_1$. Since $x_1 = 0$ is noncharacteristic at 0, we can divide P by the nonvanishing coefficient of D_1^2, then using the smooth change of variables of HÖRMANDER [5, lemma 28.4.2], we get that P can be written, with the notation of section 2.a.,

$$P = (L^1)^2 + Q^1 + \sum_{j\geq 1} b_j D_j + c$$

where q and r satisfy the assumptions of proposition 2 (including the $C^{7/3}$ smoothness of the coefficients), and $(b_j)_{j\geq 1}$ and c are bounded complex-valued coefficients.

For small $\varepsilon > 0$, a straightforward computation gives

$$(Pu)^\varepsilon = \varepsilon^{-2}((L^\varepsilon)^2 + Q^\varepsilon)(u^\varepsilon) + \varepsilon^{-1} \sum_{j\geq 1} b_j^\varepsilon D_j(u^\varepsilon) + c^\varepsilon(u^\varepsilon)$$

so that we can write

$$((L^\varepsilon)^2 + Q^\varepsilon)(u^\varepsilon) = \varepsilon^2 (Pu)^\varepsilon - \varepsilon \sum_{j\geq 1} b_j^\varepsilon D_j(u^\varepsilon) - \varepsilon^2 c^\varepsilon(u^\varepsilon) \ .$$

Thus it follows from proposition 2 that for small $\varepsilon > 0$,

$$\begin{aligned} &\left\|e^{\tau\phi}(u^\varepsilon)\right\|^2 + \sum_{j\geq 1} \left\|e^{\tau\phi} D_j(u^\varepsilon)\right\|^2 \\ &\qquad \leq C\left(\varepsilon^4 \left\|e^{\tau\phi}(Pu)^\varepsilon\right\|^2 + \varepsilon^2 \sum_{j\geq 1} \left\|e^{\tau\phi} D_j(u^\varepsilon)\right\|^2 + \varepsilon^4 \left\|e^{\tau\phi}(u^\varepsilon)\right\|^2\right), \end{aligned}$$

which proves that for a small fixed $\varepsilon > 0$ and all u such that $u^\varepsilon \in C_0^\infty(W)$,

$$\left\|e^{\tau\phi}(u^\varepsilon)\right\|^2 \leq \left\|e^{\tau\phi}(Pu)^\varepsilon\right\|^2 \ .$$

Finally, the uniqueness property of theorem 1 follows from this estimate thanks to standard arguments such as in [5]. □

2.c. Estimates of some commutators. In this end of section 2, we show that most of the estimates needed in the proof of proposition 2 can be obtained by integrating by parts just once, and this is why we need no more than the assumption $\rho \geq 1$ in corollary 4.

The commutator of two operators Q and R is defined as the operator $[Q, R]v = Q(Rv) - R(Qv)$. In the following statement, ix_j stands for the operator of multiplication by ix_j.

LEMMA 3. *For all* $v, w \in C_0^\infty(\mathbb{R}^n)$, *we have*

(i) $(v, L^\varepsilon w) = (L^\varepsilon v, w) - i\varepsilon(\alpha^\varepsilon v, w)$;

(ii) $(Q^\varepsilon v, L^\varepsilon w) = (L^\varepsilon v, Q^\varepsilon w) - i\varepsilon(R^\varepsilon v, w)$;

(iii) $[Q^\varepsilon, ix_j] = Q_j^\varepsilon - i\varepsilon\alpha_j^\varepsilon$ *where* $Q_j^\varepsilon = 2\sum_{k>1} a_{jk}^\varepsilon D_k$ *and* $\alpha_j = \sum_{k>1} \partial a_{jk}/\partial x_k$;

(iv) $[(L^\varepsilon)^2, i\,x_j] = 2\,a_j^\varepsilon L^\varepsilon + \varepsilon(L^1 a_j)^\varepsilon$.

PROOF : Formulas (i), (iii) and (iv) are easy to obtain, and are therefore left to the reader; as for (ii), we just have to sum over $j > 1$, $k > 1$ and $\ell \geq 1$ the identity

$$\begin{aligned}\Big(D_j(a_{jk}^\varepsilon D_k v), a_\ell^\varepsilon D_\ell w\Big) - \Big(a_\ell^\varepsilon D_\ell v, D_k(a_{jk}^\varepsilon D_j v)\Big)\\ = i\varepsilon\Big((\partial a_\ell/\partial x_j)^\varepsilon a_{jk}^\varepsilon D_k v, D_\ell w\Big)\\ - i\varepsilon\Big((\partial(a_\ell a_{jk})/\partial x_\ell)^\varepsilon D_k v, D_j w\Big)\\ + i\varepsilon\Big(a_{jk}^\varepsilon(\partial a_\ell/\partial x_k)^\varepsilon D_\ell v, D_j w\Big)\end{aligned}$$

which comes from the identities

$$\begin{aligned}\Big(a_\ell^\varepsilon D_j(a_{jk}^\varepsilon D_k v), D_\ell w\Big) &= \Big(D_j(a_\ell^\varepsilon a_{jk}^\varepsilon D_k v), D_\ell w\Big)\\ &\quad + i\varepsilon\Big((\partial a_\ell/\partial x_j)^\varepsilon a_{jk}^\varepsilon D_k v, D_\ell w\Big)\ ,\\ \Big(D_\ell(a_\ell^\varepsilon a_{jk}^\varepsilon D_k v), D_j w\Big) &= \Big(a_\ell^\varepsilon a_{jk}^\varepsilon D_\ell D_k v, D_j w\Big)\\ &\quad - i\varepsilon\Big((\partial(a_\ell a_{jk})/\partial x_\ell)^\varepsilon D_k v, D_j w\Big)\ ,\\ \Big(a_{jk}^\varepsilon a_\ell^\varepsilon D_k D_\ell v, D_j w\Big) &= \Big(a_{jk}^\varepsilon D_k(a_\ell^\varepsilon D_\ell v), D_j w\Big)\\ &\quad + i\varepsilon\Big(a_{jk}^\varepsilon(\partial a_\ell/\partial x_k)^\varepsilon D_\ell v, D_j w\Big)\ .\end{aligned}$$

□

COROLLARY 4. *With $\phi(x_1) = (x_1^2/2) - x_1$, we define the operator*

$$J_1^\varepsilon = (L^\varepsilon)^2 + \tau - (\tau\phi')^2 + Q^\varepsilon\ ;$$

then, for sufficiently small $\varepsilon > 0$, we have :

(i) *For all $\tau \geq 3$ and $v = e^{\tau\phi}u \in C_0^\infty(W)$,*

$$\begin{aligned}&\left\|e^{\tau\phi}((L^\varepsilon)^2 + Q^\varepsilon)\,u\right\|^2\\ &\qquad \geq 2\varepsilon\tau(-\phi' R^\varepsilon v, v) + \frac{1}{2}\|J_1^\varepsilon v\|^2 + \frac{\tau^3}{4}\|v\|^2 + \tau^2\|L^\varepsilon v\|^2\ ;\end{aligned}$$

(ii) *If the quadratic form $q(0,\xi')$ is nondegenerate, then there is a constant K such that for all $v \in C_0^\infty(W)$,*

$$\|v\|_1^2 \leq K(\|J_1^\varepsilon v\|^2 + \|v\|^2 + \|L^\varepsilon v\|^2)\ .$$

PROOF : We have $e^{\tau\phi}((L^\varepsilon)^2 + Q^\varepsilon)u = e^{\tau\phi}((L^\varepsilon)^2 + Q^\varepsilon)(e^{-\tau\phi}v)$, and

$$\begin{aligned}e^{\tau\phi}((L^\varepsilon)^2 + Q^\varepsilon)e^{-\tau\phi} &= e^{\tau\phi}L^\varepsilon e^{-\tau\phi}L^\varepsilon + i\tau\, e^{\tau\phi}L^\varepsilon\phi' e^{-\tau\phi} + Q^\varepsilon\\ &= (L^\varepsilon)^2 + 2i\tau\phi' L^\varepsilon + \tau - (\tau\phi')^2 + Q^\varepsilon = J_1^\varepsilon + J_2^\varepsilon\end{aligned}$$

with $J_2^\varepsilon = 2i\tau\phi' L^\varepsilon$. Thanks to lemma 3 (i) and (ii), we have

$$(J_1^\varepsilon v, L^\varepsilon w) = (L^\varepsilon v, J_1^\varepsilon w) + i\varepsilon(\alpha^\varepsilon L^\varepsilon v, L^\varepsilon w)$$
$$- i\varepsilon\Big(\alpha^\varepsilon v, (\tau - (\tau\phi')^2)w\Big) + \Big((2i\tau^2\phi' - i\varepsilon R^\varepsilon)v, w\Big) ,$$

and this gives

$$\begin{aligned}(J_1^\varepsilon v, J_2^\varepsilon v) &= \Big(J_1^\varepsilon v, [2i\tau\phi', L^\varepsilon]v\Big) + \Big(J_1^\varepsilon v, L^\varepsilon(2i\tau\phi' v)\Big)\\
&= (J_1^\varepsilon v, -2\tau v) + \Big(L^\varepsilon v, J_1^\varepsilon(2i\tau\phi' v)\Big) + i\varepsilon\Big(\alpha^\varepsilon L^\varepsilon v, L^\varepsilon(2i\tau\phi' v)\Big)\\
&\quad - i\varepsilon\Big(\alpha^\varepsilon v, (\tau - (\tau\phi')^2)2i\tau\phi' v\Big) + \Big((2i\tau^2\phi' - i\varepsilon R^\varepsilon)v, 2i\tau\phi' v\Big)\\
&= -2\tau(J_1^\varepsilon v, v) + \Big(L^\varepsilon v, [J_1^\varepsilon, 2i\tau\phi']v\Big) - (J_2^\varepsilon v, J_1^\varepsilon v)\\
&\quad + O\Big(\varepsilon\tau\,\|L^\varepsilon v\|^2 + \varepsilon\tau^3\,\|v\|^2\Big) + 4\tau^3\,\|\phi' v\|^2 + 2\varepsilon\tau(-\phi' R^\varepsilon v, v) ,\end{aligned}$$

and since $[J_1^\varepsilon, 2i\tau\phi'] = 4\tau L^\varepsilon$, we get for small $\varepsilon > 0$

$$2\Re e(J_1^\varepsilon v, J_2^\varepsilon v) \geq -\frac{1}{2}\,\|J_1^\varepsilon v\|^2 - 2\tau^2\,\|v\|^2 + \frac{11}{12}\tau^3\,\|v\|^2 + 2\varepsilon\tau(-\phi' R^\varepsilon v, v) .$$

The obvious estimate $\|J_2^\varepsilon v\|^2 \geq \tau^2\,\|L^\varepsilon v\|^2$ finally gives

$$\begin{aligned}\left\|e^{\tau\phi}((L^\varepsilon)^2 + Q^\varepsilon)u\right\|^2 &= \|J_1^\varepsilon v\|^2 + \|J_2^\varepsilon v\|^2 + 2\Re e(J_1^\varepsilon v, J_2^\varepsilon v)\\
&\geq 2\varepsilon\tau(-\phi' R^\varepsilon v, v) + \frac{1}{2}\,\|J_1^\varepsilon v\|^2 + \frac{\tau^3}{4}\,\|v\|^2 + \tau^2\,\|L^\varepsilon v\|^2\end{aligned}$$

for $\tau \geq 3$, and this is estimate (i).

Next, the assumption that $q(0, \xi')$ is nondegenerate can be written

$$\sum_{j>1,k>1,\ell>1} a_{jk}^\varepsilon a_{j\ell}^\varepsilon \xi_k \xi_\ell \geq \frac{1}{C_0}\,|\xi'|^2$$

for a fixed $C_0 > 0$ if $\varepsilon > 0$ is sufficiently small. Thus we have

$$\begin{aligned}\|v\|_1^2 &= \|v\|^2 + \sum_{j\geq 1}\|D_j v\|^2\\
&\leq \|v\|^2 + \|L^\varepsilon v\|^2 + C_1\sum_{j>1}\|D_j v\|^2\\
&\leq \|v\|^2 + \|L^\varepsilon v\|^2 + C_0 C_1 \sum_{j>1,k>1,\ell>1} \Re e(a_{jk}^\varepsilon D_k v, a_{j\ell}^\varepsilon D_\ell v)\\
&\leq \|v\|^2 + \|L^\varepsilon v\|^2 + \frac{C_0 C_1}{4}\sum_{j>1}\|Q_j^\varepsilon v\|^2\end{aligned}$$

for the operators Q_j^ε considered in lemma 3 (iii). Thanks to lemma 3 (iii) and (iv), we can write, for all $j > 1$,

$$Q_j^\varepsilon = [J_1^\varepsilon, ix_j] - 2a_j^\varepsilon L^\varepsilon + \varepsilon\beta_j^\varepsilon$$

where β_j is a continuous function. This gives

$$\left\|Q_j^\varepsilon v\right\|^2 = \left(J_1^\varepsilon(ix_jv), Q_j^\varepsilon v\right) - (ix_j J_1^\varepsilon v, Q_j^\varepsilon v) + \left((\varepsilon\beta_j^\varepsilon - 2a_j^\varepsilon L^\varepsilon)v, Q_j^\varepsilon v\right) .$$

Using tedious calculations similar to the proof of lemma 3, we can show that

$$\left|\left(J_1^\varepsilon(ix_jv), Q_j^\varepsilon v\right) - \left(Q_j^\varepsilon(ix_jv), J_1^\varepsilon v\right)\right| \le C\varepsilon(\|v\|_1^2 + \|L^\varepsilon v\|^2)$$

and since $[Q_j^\varepsilon, ix_j] = 2a_{jj}^\varepsilon$, we obtain an estimate

$$\begin{aligned}\left\|Q_j^\varepsilon v\right\|^2 &= 2\Re e(ix_jQ_j^\varepsilon v, J_1^\varepsilon v) + (2a_{jj}^\varepsilon v, J_1^\varepsilon v) + \left((\varepsilon\beta_j^\varepsilon - 2a_j^\varepsilon L^\varepsilon)v, Q_j^\varepsilon v\right)\\ &\qquad + O(\varepsilon\|v\|_1^2 + \varepsilon\|L^\varepsilon v\|^2)\\ &\le \frac{1}{2}\left\|Q_j^\varepsilon v\right\|^2 + 5\left\|J_1^\varepsilon v\right\|^2 + C\|v\|^2 + C\|L^\varepsilon v\|^2 + C\varepsilon\|v\|_1^2\end{aligned}$$

which yields

$$\left\|Q_j^\varepsilon v\right\|^2 \le 10\left\|J_1^\varepsilon v\right\|^2 + 2C\|v\|^2 + 2C\|L^\varepsilon v\|^2 + 2C\varepsilon\|v\|_1^2 .$$

Since we previously proved the estimate $\|v\|_1^2 \le \|v\|^2 + C\sum_{j>1}\left\|Q_j^\varepsilon v\right\|^2$, this completes the proof of our claim (ii) in corollary 4. □

It is easy to see that proposition 2 follows from the estimates in corollary 4 as soon as we have an estimate for the term $2\varepsilon\tau(-\phi' R^\varepsilon v, v)$; this term will be treated in the next section.

3. Estimates under the pseudo-convexity assumption

It is in this section, which will provide the estimate of the term $(\phi' R^\varepsilon v, v)$, that we need the additional smoothness of the coefficients we assumed in theorem 1. This smoothness assumption can be taken into account conveniently through Bony's paradifferential calculus that we review in section 3.a.

3.a. On the paradifferential calculus. We begin with the simplest paradifferential operator, the operator of paramultiplication : let $\sigma > 0$; then corresponding to any $a \in C^\sigma$, BONY [1] constructed an operator T_a with the following properties.

LEMMA 5. *Let $\sigma > 0$ and $a \in C^\sigma$; then there exist constants C_s (depending linearly on the C^σ norm of a) such that for all $v \in C_0^\infty(\mathbb{R}^n)$:*

(i) $\forall s \in \mathbf{R},\ \|T_a v\|_s \le C_s\|v\|_s$;

(ii) $\forall s \in \mathbf{R},\ \forall j \ge 1,\ \|[T_a, D_j]v\|_s \le C_s\|v\|_{s+\max(0,1-\sigma)}$;

(iii) $\forall s \in (0,\sigma),\ \|(a - T_a)v\|_s \le C_s\|v\|_{s-\sigma}$.

PROOF : Properties (i) and (ii) are given by BONY [1]. Using the same ideas, we can prove (iii) as follows.

Let us choose a Littelewood-Paley decomposition $a = \Sigma_p a_p$ and $v = \Sigma_q v_q$; we have $av = \Sigma_{p,q} a_p v_q$ while $T_a v = \Sigma_{p,q>p+N} a_p v_q$; thus

$$(a - T_a)v = \sum_{p,|r|\le N} a_p v_{p+r} + \sum_{p,q<p-N} a_p v_q \ .$$

In the first sum and for each fixed r, $\operatorname{supp} \widehat{a_p v_{p+r}}$ is contained in some ball of radius $C2^p$; moreover we have

$$\sum_p 2^{2ps} \|a_p v_{p+r}\|^2 \le C_a \sum_p 2^{2p(s-\sigma)} \|v_{p+r}\|^2 \le C_{a,r} \|v\|^2_{s-\sigma}$$

which gives an estimate $\left\|\Sigma_{p,|r|\le N} a_p v_{p+r}\right\|_s \le C \|v\|_{s-\sigma}$ for $s > 0$.

For the second sum, we set $w_p = \Sigma_{q<p-N} a_p v_q$ which satisfies, for $p \neq 0$, $\operatorname{supp} \widehat{w}_p \subset \{\xi \in \mathbb{R}^n ; C^{-1}2^p \le |\xi| \le C2^p\}$ for some constant C ; moreover, we have

$$2^{ps} \|w_p\| \le C_a \sum_{q\le p} \left(\|v_q\| \, 2^{q(s-\sigma)}\right) \left(2^{(p-q)(s-\sigma)}\right) = (u_2 * u_1)_p$$

where $u_{2,p} = \|v_p\| \, 2^{p(s-\sigma)}$ is an ℓ^2 sequence with norm smaller than $C \|v\|_{s-\sigma}$, and $u_{1,p} = C_a 2^{p(s-\sigma)}$ is, for $s < \sigma$, an ℓ^1 sequence with norm equal to $C_a \ (1 - 2^{s-\sigma})^{-1}$; it follows that the sequence $2^{ps} \|w_p\|$ is an ℓ^2 sequence with norm estimated by $\|v\|_{s-\sigma}$, and this gives the second estimate

$$\left\| \sum_{p,q<p-N} a_p v_q \right\|^2_s \le C_0 \sum_p 2^{2ps} \|w_p\|^2 \le C_1 \|v\|^2_{s-\sigma}$$

which proves the lemma. □

Next, BONY [1] defined more general paradifferential operators : for $\sigma > 0$ and $m \in \mathbb{R}$, he considers the class Σ^m_σ of symbols $a(x,\xi)$ satisfying estimates

$$\left|\partial^\alpha_x \partial^\beta_\xi a(x,\xi) - \partial^\alpha_x \partial^\beta_\xi a(y,\xi)\right| \le C_{\alpha\beta} |x-y|^{\sigma-[\sigma]} (1+|\xi|)^{m-|\beta|}$$

for all α and $\beta \in \mathbb{Z}^n_+$ with $|\alpha| = [\sigma]$, then he constructs operators T_a corresponding to these symbols.

However, we are going to deal here with symbols depending only on ξ', where $\xi = (\xi_1, \xi')$, and therefore, we need actually a variant of Bony's paradifferential calculus where the variable x_1 is merely a parameter. Thus we define the class ${}'\Sigma^m_\sigma$ of symbols $a(x,\xi')$ satisfying estimates

$$\left|\partial^\alpha_x \partial^\beta_{\xi'} a(x,\xi') - \partial^\alpha_x \partial^\beta_{\xi'} a(y,\xi')\right| \le C_{\alpha\beta} |x-y|^{\sigma-[\sigma]} (1+|\xi'|)^{m-|\beta|}$$

for all $\alpha \in \mathbb{Z}^n_+$ and $\beta \in \mathbb{Z}^{n-1}_+$ with $|\alpha| = [\sigma]$, then we construct operators T'_a corresponding to these symbols through Littlewood-Paley decompositions in x' only.

It is not difficult to modify Bony's proofs to get the following estimates.

LEMMA 6. *Let $\sigma > 1$, $m \geq 0$ and $\ell \geq 1 - m$. Then we have :*

(i) $\forall a \in {'\Sigma}^m_{\sigma-1}$, $\exists C : \forall v \in C_0^\infty(\mathbf{R}^n)$, $\|T'_a v\| \leq C \|v\|_m$;

(ii) $\forall a \in {'\Sigma}^m_\sigma$, $\forall j \geq 1$, $[D_j, T'_a] = T'_{D_j a}$;

(iii) $\forall a \in {'\Sigma}^m_\sigma$, $\forall b \in {'\Sigma}^\ell_\sigma$, $\exists C : \forall v \in C_0^\infty(\mathbf{R}^n)$,
$\|(T'_{ab} - T'_a T'_b) v\| \leq C \|v\|_{m+\ell-1}$;

(iv) *For any quadratic form* $s(x, \xi') = \Sigma_{j>1,k>1} c_{jk}(x) \xi_j \xi_k \in {'\Sigma}^2_\sigma$ *with* $c_{jk} = c_{kj}$, *there is a* C *such that for all* $v \in C_0^\infty(\mathbf{R}^n)$,

$$\|(S - T'_s) v\| \leq C \|v\|_1$$

where $Sv = \sum_{j>1,k>1} D_j(c_{jk} D_k v)$.

PROOF : As an example, we give the proof of (iv) : it is convenient to introduce the norms

$$\|v\|'_t = \left((2\pi)^{-n} \int (1 + |\xi'|^2)^t \, |\widehat{u}(\xi)|^2 \, d\xi\right)^{1/2}$$

which satisfy $\|v\|'_t \leq \|v\|_t$ when $t \geq 0$; then we have

$$\begin{aligned}
\|(S - T'_s) v\| &\leq \sum_{j>1,k>1} \left\{ \left\| D_j (c_{jk} - T'_{c_{jk}}) D_k v \right\| + \left\| [D_j, T'_{c_{jk}}] D_k v \right\| \right\} \\
&\leq \sum_{j>k,k>1} \left\{ \left\| (c_{jk} - T'_{c_{jk}}) D_k v \right\|'_1 + \left\| T'_{D_j c_{jk}} D_k v \right\| \right\} \\
&\leq C(\|v\|'_{2-\sigma} + \|v\|_1) \leq C \|v\|_1
\end{aligned}$$

by using lemma 5 (iii) and lemma 6 (i). □

Our last statement in this section can be easily deduced from the paradifferential Gårding inequality of HÖRMANDER [**6**].

LEMMA 7. *Let* $0 < \sigma \leq 1$ *and* $s \in {'\Sigma}^2_{2\sigma}$ *satisfying* $\Re e\, s \leq 0$ *on* $\delta W \times \{\xi' \in \mathbf{R}^{n-1} ; |\xi'| \geq 1\}$. *Then there is a constant* C *such that for any* $\varepsilon \leq \delta/2$ *and for all* $v \in C_0^\infty(W)$,

$$\Re e(T'_{s^\varepsilon} v, v) \leq C \|v\|^2_{1-(\sigma/2)}$$

where $s^\varepsilon(x, \xi') = s(\varepsilon x, \xi')$.

REMARK. Since the estimates of lemma 6 will be used for symbols of the form $a^\varepsilon(x, \xi') = a(\varepsilon x, \xi')$, it is important to point out that these estimates hold with constants C independent of ε. Indeed, these constants depend only on the constants $C_{\alpha\beta}$ occuring in the assumption $a \in {'\Sigma}^m_\sigma$, and it is obvious that $a \in {'\Sigma}^m_\sigma$ implies $a^\varepsilon \in {'\Sigma}^m_\sigma$ with the same constants $C_{\alpha\beta}$ if $\varepsilon \leq 1$.

3.b. Estimate of $(\phi' R^\varepsilon v, v)$. This estimate is obtained thanks to the following decomposition of the quadratic form r.

PROPOSITION 8. *Under the assumptions of proposition 2, there exist two symbols* $\lambda \in {}'\Sigma^0_{4/3}$ *and* $s \in {}'\Sigma^2_{4/3}$ *such that* $r = \lambda q + s$ *and* $s \geq 0$ *on* $\delta W \times \{\xi' \in \mathbb{R}^{n-1}\,;\ |\xi'| \geq 1\}$.

PROOF : Let us denote by a the set of variables $(a_{jk})_{j>1,k>1} \in \mathbb{R}^{(n-1)^2}$, then set $Q(a,\xi') = \sum_{j>1,k>1} a_{jk}\xi_j\xi_k$ which is an analytic function of the variables (a,ξ'). If $a_{jk}(x)$ are the coefficients of $q(x,\xi')$, we write $a^0 = (a_{jk}(0))_{j>1,k>1}$ and finally we set

$$K = \left\{(a,\xi') \in \mathbb{R}^{(n-1)^2} \times \mathbb{R}^{n-1}\,;\, a = a^0 \text{ and } |\xi'| = 1\right\}.$$

If Q does not vanish at a point $(a^0, \xi'_0) \in K$, we define the analytic functions $\Xi(a,\xi') \equiv 0$ and $\Lambda(a,b,\xi') = Q(b,\xi')/Q(a,\xi')$ for (a,ξ') close to (a^0,ξ'_0) and $b \in \mathbb{R}^{(n-1)^2}$. If Q does vanish at the point $(a^0,\xi'_0) \in K$, we can use the assumption "$q(0,\xi')$ is nondegenerate" and standard arguments (implicit function theorem, Taylor formula) to construct two analytic functions $\Xi(a,\xi')$ and $\Lambda(a,b,\xi')$, for (a,ξ') close to (a^0,ξ'_0) and $b \in \mathbb{R}^{(n-1)^2}$, such that

$$Q(a, \Xi(a,\xi')) \equiv 0$$

and

$$Q(b,\xi') = Q(b,\Xi(a,\xi')) + \Lambda(a,b,\xi')\,Q(a,\xi')\,.$$

In both situations, we set $S(a,b,\xi') = Q(b,\Xi(a,\xi'))$.

Now, using a nonnegative partition of unity $1 = \sum_{1\leq\nu\leq N}\varphi_\nu(\xi')$ and the homogeneity in ξ' of Q, we see that we can construct two C^∞ functions $\Lambda(a,b,\xi')$ and $S(a,b,\xi')$ globally defined on $A \times \mathbb{R}^{(n-1)^2} \times \mathbb{R}^{n-1}$, where A is a neighborhood of a^0 in $\mathbb{R}^{(n-1)^2}$, respectively homogeneous of degree 0 and 2 in ξ' for $|\xi'| \geq 1$ and such that

$$Q(b,\xi') = \Lambda(a,b,\xi')\,Q(a,\xi') + S(a,b,\xi')$$

where, for $|\xi'| \geq 1$,

$$S(a,b,\xi') = \sum_{1\leq\nu\leq N} \varphi_\nu(\xi'/\,|\xi'|)\,Q\Big(b, \Xi^\nu(a,\xi'/\,|\xi'|)\Big)\,|\xi'|^2$$

and the functions Ξ^ν all satisfy $Q(a,\Xi^\nu(a,\xi')) \equiv 0$.

Finally, if $a_{jk}(x)$ and $b_{jk}(x)$ are respectively the coefficients of the quadratic forms $q(x,\xi')$ and $r(x,\xi')$, we set $\lambda(x,\xi') = \Lambda(a(x),b(x),\xi')$ and $s(x,\xi') = S(a(x),b(x),\xi')$. Using the homogeneity for the derivatives in ξ' and the chain rule for the derivatives in x, we see that a and $b \in C^{4/3}$ imply $\lambda \in {}'\Sigma^0_{4/3}$ and $s \in {}'\Sigma^2_{4/3}$; since $q(x,\xi') = Q(a(x),\xi')$

and $r(x,\xi') = Q(b(x),\xi')$, we have $r = \lambda q + s$; at last, we also have for $x \in \delta W$ (maybe for a smaller $\delta > 0$) and $|\xi'| \geq 1$,

$$s(x,\xi') = \sum_{1\leq\nu\leq N} \varphi_\nu(\xi'/|\xi'|)\, r\Big(x, \Xi^\nu(a(x),\xi'/|\xi'|)\Big)\, |\xi'|^2 \geq 0$$

since $q(x,\Xi^\nu(a(x),\xi'/|\xi'|)) = Q(a(x),\Xi^\nu(a(x),\xi'/|\xi'|)) = 0$ implies $r(x,\Xi^\nu(a(x),\xi'/|\xi'|)) \geq 0$. □

We can now write our estimate for $(\phi' R^\varepsilon v, v)$.

COROLLARY 9. *Under the assumptions of proposition 2, we have an estimate*

$$(\phi' R^\varepsilon v, v) \leq C(\tau^{-1}\|J_1^\varepsilon v\|^2 + \tau^{-1}\|v\|_1^2 + \tau\|L^\varepsilon v\|^2 + \tau^2\|v\|^2)$$

for all $v \in C_0^\infty(\mathbb{R}^n)$, $\tau \geq 1$ and $\varepsilon \leq \delta/2$, where J_1^ε is the operator defined in corollary 4.

PROOF : Using the estimates (iii) and (iv) of lemma 6 and that of lemma 7, we have

$$\begin{aligned}
(\phi' R^\varepsilon v, v) &= (\phi' T'_{\lambda^\varepsilon q^\varepsilon + s^\varepsilon} v, v) + \Big(\phi'(R^\varepsilon - T'_{r^\varepsilon})v, v\Big) \\
&= \Re e(\phi' T'_{\lambda^\varepsilon} T'_{q^\varepsilon} v, v) + \Re e\Big(\phi'(T'_{\lambda^\varepsilon q^\varepsilon} - T'_{\lambda^\varepsilon} T'_{q^\varepsilon})v, v\Big) \\
&\quad + \Re e\Big(T'_{-s^\varepsilon}((-\phi')^{1/2} v), (-\phi')^{1/2} v\Big) + \Re e\Big(\phi'(R^\varepsilon - T'_{r^\varepsilon})v, v\Big) \\
&\leq \Re e(\phi' T'_{\lambda^\varepsilon} Q^\varepsilon v, v) + C\|v\|_1 \|v\| + C\|v\|_{2/3}^2 .
\end{aligned}$$

The product $\|v\|_1\|v\|$ can be estimated by $\tau^{-1}\|v\|_1^2 + \tau\|v\|^2$ while the interpolation estimate $\|v\|_{2/3}^2 \leq \tau^{-1}\|v\|_1^2 + \tau^2\|v\|^2$ directly follows from the easy inequality $(1+|\xi|^2)^{2/3} \leq \tau^{-1}(1+|\xi|^2) + \tau^2$; thus, the last two terms are estimated as required.

As for the first term, we have $Q^\varepsilon = J_1^\varepsilon - (L^\varepsilon)^2 - \tau + (\tau\phi')^2$, then

$$\begin{aligned}
|(\phi' T'_{\lambda^\varepsilon} Q^\varepsilon v, v)| &\leq |(\phi' T'_{\lambda^\varepsilon} J_1^\varepsilon v, v)| + \Big|\Big(\phi' T'_{\lambda^\varepsilon}(L^\varepsilon)^2 v, v\Big)\Big| + C\tau^2\|v\|^2 \\
&\leq \sum_{j\geq 1} \Big|\Big(\phi' T'_{\lambda^\varepsilon} D_j(a_j^\varepsilon L^\varepsilon v), v\Big)\Big| \\
&\qquad + C\|v\|\,(\|J_1^\varepsilon v\| + \|L^\varepsilon v\| + \tau^2\|v\|) \\
&\leq \sum_{j\geq 1} \Big\{ |(\phi' T'_{\lambda^\varepsilon}(a_j^\varepsilon L^\varepsilon v), D_j v)| + \varepsilon\Big|\Big(\phi' T'_{(D_j\lambda)^\varepsilon}(a_j^\varepsilon L^\varepsilon v), v\Big)\Big| \Big\} \\
&\qquad + |(T'_{\lambda^\varepsilon} L^\varepsilon v, v)| + C\|v\|\,(\|J_1^\varepsilon v\| + \|L^\varepsilon v\| + \tau^2\|v\|) \\
&\leq C(\tau^{-1}\|J_1^\varepsilon v\|^2 + \tau^{-1}\|v\|_1^2 + \tau\|L^\varepsilon v\|^2 + \tau^2\|v\|^2)
\end{aligned}$$

thanks to estimate (i) and formula (ii) of lemma 6. □

3.c. Proof of proposition 2. From corollaries 4 and 9 we get

$$\left\|e^{\tau\phi}((L^\varepsilon)^2+Q^\varepsilon)u\right\|^2 \geq \left(\frac{1}{2}-C\varepsilon\right)\|J_1^\varepsilon v\|^2+\left(\frac{1}{4}-C\varepsilon\right)\tau^3\|v\|^2+(1-C\varepsilon)\tau^2\|L^\varepsilon v\|^2$$

so that we have for sufficiently small $\varepsilon>0$

$$\left\|e^{\tau\phi}((L^\varepsilon)^2+Q^\varepsilon)u\right\|^2 \geq \frac{1}{5}\left(\|J_1^\varepsilon v\|^2+\tau^3\|v\|^2+\tau^2\|L^\varepsilon v\|^2\right) .$$

On the other hand, thanks to corollary 4 (ii) again, we have for $\tau \geq 4$

$$\begin{aligned}
&\tau^3\left\|e^{\tau\phi}u\right\|^2+\tau\left\|e^{\tau\phi}L^\varepsilon u\right\|^2+\sum_{j\geq 1}\left\|e^{\tau\phi}D_j u\right\|^2\\
&=\tau^3\|v\|^2+\tau\|(L^\varepsilon+i\tau\phi')v\|^2+\|(D_1+i\tau\phi')v\|^2+\sum_{j>1}\|D_j v\|^2\\
&\leq 3\tau^3\|v\|^2+3\tau\|L^\varepsilon v\|^2+2\|v\|_1^2\\
&\leq (3+2K)\tau^3\|v\|^2+(3+2K)\tau^2\|L^\varepsilon v\|^2+2K\|J_1^\varepsilon v\|^2\\
&\leq 5(3+2K)\left\|e^{\tau\phi}((L^\varepsilon)^2+Q^\varepsilon)u\right\|^2
\end{aligned}$$

as required. □

4. Variants of theorem 1

In this section, we want to show that the assumption $a_\alpha \in C^{7/3}$ of theorem 1 can be weakened in some special situations.

First, we recall that, thanks to a remark of LERNER and ROBBIANO [7, lemme 2.1.1], our general assumptions (see section 1.a) imply that

$$\Sigma=\left\{(x,\xi)\,;\,x=x_0\,,p(x_0,\xi)=H_p\varphi(x_0,\xi)=0\,,\ |\xi|=1\right\}$$

(for any choice of local coordinates) is an analytic manifold of dimension $n-3$. Since the pseudo-convexity assumption of theorem 1 implies that $H_p^2\varphi \leq 0$ on Σ, the zeroes of this analytic function are critical points. In our next result, we assume that these critical points are nondegenerate.

THEOREM 10. *Let P be, as above, a second order linear partial differential operator of real principal type at x_0, and $\varphi(x)=0$ be a smooth, noncharacteristic hypersurface through x_0. Assume that $a_\alpha \in C^{5/3}$ for $|\alpha|=2$, that the weak pseudo-convexity condition of theorem 1 holds near x_0 and that the zeroes of the function $H_p^2\varphi|_\Sigma$ are nondegenerate critical points. Then the same conclusion as in theorem 1 holds.*

PROOF : Using the same arguments as in section 2.b, we see that the problem is just to prove the estimate of proposition 2 under the assumptions that $\rho = 5/3$, $q(0,\xi')$ is nondegenerate, $q = 0 \Rightarrow r \geq 0$ on $\delta W \times \mathbf{R}^{n-1}$, and the zeroes of $r|_{\{0\}\times K}$, where $K = \{\xi' \in \mathbf{R}^{n-1}\,;\, q(0,\xi') = 0,\ |\xi'| = 1\}$, are nondegenerate critical points. We proceed in two steps.

Step 1 : decomposition of the quadratic form r. Using theorem A in the appendix we see that there is a real-valued function $\lambda(x)$ such that $r(x,\xi') - \lambda(x)\,q(x,\xi') \geq 0$ on $\delta W \times \mathbf{R}^{n-1}$. We claim that this function λ can be chosen so that $\lambda \in C^{2/3}$.

We first remark that we can assume that $q(x,\xi') \equiv (|\xi''|^2 - |\xi'''|^2)/2$ where $\xi' = (\xi'', \xi''')$; indeed, since the conclusion is just that $r - \lambda q \geq 0$, we can forget the link between x and ξ and calculate with ξ' expressed in any basis, even depending on x. Thus, using the standard process, we can reduce $q(0,\xi')$ to $(|\xi''|^2 - |\xi'''|^2)/2$, and it is then easy to see that, by continuity, there is a $C^{5/3}$ change of bases reducing q to $(|\xi''|^2 - |\xi'''|^2)/2$ everywhere near 0. In this new basis, $r(x,\xi')$ is still a quadratic form with $C^{2/3}$ coefficients, and we have $K = \{\xi' = (\xi'',\xi''')\,;\, |\xi''|^2 = |\xi'''|^2 = 1/2\}$.

Then, let us set $\mu(x) = \inf\limits_{\xi' \in K} r(x,\xi')$ (thanks to our assumptions, we have $\mu \geq 0$ on δW). If for every small x this infimum is reached at some point $\xi'_x \in K$, we can write

$$\begin{aligned}\mu(y) \leq r(y,\xi'_x) &= r(x,\xi'_x) + [r(y,\xi'_x) - r(x,\xi'_x)]\\ &\leq \mu(x) + C\,|y-x|^{2/3}\end{aligned}$$

so that $\mu \in C^{2/3}$. We now consider the quadratic form $s(x,\xi') = r(x,\xi') - \mu(x)\,|\xi'|^2$, which still has $C^{2/3}$ coefficients, satisfies $q = 0 \Rightarrow s \geq 0$ on $\mathbf{R}^n \times \mathbf{R}^{n-1}$ and is such that $K_x = \{\xi' \in K\,;\, s(x,\xi') = 0\} \neq \phi$. We can assume that $\mu(0) = 0$ since otherwise the surface is strongly pseudo-convex at 0 and the result is easier (see e.g. theorem 11 below), and this implies that the zeroes of $s|_{\{0\}\times K}$ are still nondegenerate critical points. Then it follows from an easy Morse lemma that the points of K_x are $C^{2/3}$ functions of x. Finally, using proposition B in the appendix, we see that the formula $\lambda(x) = \langle s'_{\xi'}(x,\xi'_x), q'_{\xi'}(\xi'_x)\rangle$, where ξ'_x is any point of K_x, defines a $C^{2/3}$ real valued function λ such that $s(x,\xi') - \lambda(x)\,q(x,\xi') \geq 0$, and this is our claim since $r(x,\xi') \geq s(x,\xi')$ on $\delta W \times \mathbf{R}^{n-1}$.

Step 2 : estimate of $(\phi' R^\varepsilon v, v)$. We complete the proof of theorem 10 by showing that the estimate of corollary 9 can be obtained thanks to the decomposition $r = \lambda q + s$ with $\lambda \in C^{2/3}$ and $s \geq 0$ on $\delta W \times \mathbf{R}^{n-1}$ we wrote in step 1.

Indeed, let us set $c_{jk} = b_{jk} - \lambda a_{jk}$ (we know that $\sum_{j>1,k>1} c_{jk}\xi_j\xi_k \geq 0$ for $x \in \delta W$ and $\xi' \in \mathbf{R}^{n-1}$). Since $(\phi' R^\varepsilon v, v) = \sum_{j>1,k>1} \Big(D_j(\phi' b^\varepsilon_{jk} D_k v), v\Big)$, we are led to estimate

$$\begin{aligned}
\Big(D_j(\phi' b^\varepsilon_{jk} D_k v), v\Big) &= \Big(D_j(\phi' \lambda^\varepsilon a^\varepsilon_{jk} D_k v), v\Big) + (\phi' c^\varepsilon_{jk} D_k v, D_j v)\\
&= \Big(\phi' a^\varepsilon_{jk} D_k v, \Big\{(\lambda^\varepsilon - T_{\lambda^\varepsilon}) D_j + [T_{\lambda^\varepsilon}, D_j]\Big\} v\Big)\\
&\quad + \Big(D_j(a^\varepsilon_{jk} D_k v), \phi' T_{\lambda^\varepsilon} v\Big) + (\phi' c^\varepsilon_{jk} D_k v, D_j v)\\
&\le \Re e\Big(D_j(a^\varepsilon_{jk} D_k v), \phi' T_{\lambda^\varepsilon} v\Big) + \Re e(\phi' c^\varepsilon_{jk} D_k v, D_j v)\\
&\quad + C \left\|\phi' a^\varepsilon_{jk} D_k v\right\|_{-1/3} \Big(\|(\lambda^\varepsilon - T_{\lambda^\varepsilon}) D_j v\|_{1/3} + \|[T_{\lambda^\varepsilon}, D_j] v\|_{1/3}\Big)\\
&\le \Re e\Big(D_j(a^\varepsilon_{jk} D_k v), \phi' T_{\lambda^\varepsilon} v\Big) + \Re e(\phi' c^\varepsilon_{jk} D_k v, D_j v) + C \|v\|^2_{2/3}
\end{aligned}$$

thanks to the estimates of lemma 5 and since $a_{jk} \in C^1$ implies

$$\begin{aligned}
\left\|\phi' a^\varepsilon_{jk} D_k v\right\|_{-1/3} &\le \left\|[\phi' a^\varepsilon_{jk}, D_k] v\right\| + \left\|\phi' a^\varepsilon_{jk} v\right\|_{2/3}\\
&\le C \|v\| + \left\|(\phi' a^\varepsilon_{jk} - T_{\phi' a^\varepsilon_{jk}}) v\right\|_{2/3} + \left\|T_{\phi' a^\varepsilon_{jk}} v\right\|_{2/3}\\
&\le C \|v\|_{2/3} \; .
\end{aligned}$$

After summing over $j > 1$ and $k > 1$ and using the nonpositivity of $\phi' s^\varepsilon$, we get

$$\begin{aligned}
(\phi' R^\varepsilon v, v) &\le \Re e(Q^\varepsilon v, \phi' T_{\lambda^\varepsilon} v) + C \|v\|^2_{2/3}\\
&\le \Re e(J^\varepsilon_1 v, \phi' T_{\lambda^\varepsilon} v) + C\tau^2 \|v\|^2 - \Re e\Big((L^\varepsilon)^2 v, \phi' T_{\lambda^\varepsilon} v\Big) + C \|v\|^2_{2/3}\\
&\le \tau^{-1} \|J^\varepsilon_1 v\|^2 - \Re e\Big(L^\varepsilon v, \sum_{j\ge 1} D_j(a^\varepsilon_j \phi' T_{\lambda^\varepsilon} v)\Big) + C(\tau^2 \|v\|^2 + \|v\|^2_{2/3})\\
&\le \tau^{-1} \|J^\varepsilon_1 v\|^2 + \tau^{-1} \|v\|^2_1 + C(\tau^2 \|v\|^2 + \tau \|L^\varepsilon v\|^2)
\end{aligned}$$

by using the same interpolation estimate as in the proof of corollary 9, and this is the estimate of corollary 9. □

REMARK. We believe that the decomposition of the quadratic form r in the form $\lambda(x) q(x, \xi') + s(x, \xi')$ with $\lambda \in C^{2/3}$ and $s \ge 0$ on $\delta W \times \mathbb{R}^{n-1}$ is still possible under the assumptions of theorem 1. The proof of theorem 1 using this method would have been complete if we had proved that $\lambda \in C^{2/3}$ when $\rho = 7/3$; however, all we have been able to prove in this direction is that $\lambda \in C^{(\rho-1)/2}$ when $\rho \le 2$.

We also point out that the same proof allows us to get the following result, which was first proved by HÖRMANDER [4].

THEOREM 11. *Let P be, as above, a second order linear partial differential operator of real principal type at x_0, and $\varphi(x) = 0$ be a smooth, noncharacteristic hypersurface through x_0. Assume that $a_\alpha \in C^1$ for $|\alpha| = 2$, and* *that the surface $\varphi(x)=0$ is strongly pseudo-convex at x_0 (see section 1.b). Then we have the following uniqueness property : any solution $u \in H^1_{\mathrm{loc}}(\mathbf{R}^n)$ of $Pu=0$ vanishing in $\{x \in \mathbf{R}^n \,;\, \varphi(x)<0\}$ must also vanish near x_0 in $\mathbf{R}^n$.*

PROOF : It is similar to that of theorem 10, but here we can find a constant λ such that $r(x,\xi')-\lambda q(x,\xi') \geq 0$. □

Finally we conclude our article with a corollary of theorems 10 and 11 which treats our problem in the case of equations in three or two variables.

COROLLARY 12. *Keep the same assumptions as in theorem 1 except "$a_\alpha \in C^{7/3}$ for $|\alpha|=2$". Assume that either $n=3$ and $a_\alpha \in C^{5/3}$ for $|\alpha|=2$, or $n=2$ and $a_\alpha \in C^1$ for $|\alpha|=2$. Then the conclusion of theorem 1 still holds.*

PROOF : Since $\dim \Sigma = n-3$, the zeroes of $H_p^2\varphi|_\Sigma$ are automatically nondegenerate critical points when $n=3$ (resp. $H_p^2\varphi$ does not vanish on Σ when $n=2$). Therefore, the assumptions of theorem 10 (resp. theorem 11) are fulfilled. □

Appendix. Results on pairs of real quadratic forms

Let q and r be two real quadratic forms on $\mathbf{R}^n$, $n \geq 1$. Our goal in this appendix is to determine the real numbers λ such that $r-\lambda q$ is positive semidefinite. Since we don't know if this problem was explicitly studied in the literature, we give here a short way to get the result needed in the proof of theorem 10 above. They are deduced from the "main theorem" in UHLIG [8] which states that if

$$\text{(P)} \qquad \forall \xi \in \mathbf{R}^n \setminus \{0\}\,, \quad q(\xi)=0 \Rightarrow r(\xi)>0\,,$$

then there exists a $\lambda \in \mathbf{R}$ such that the quadratic form $r-\lambda q$ is positive definite.

THEOREM A. *Let q and r be two real quadratic forms on $\mathbf{R}^n$ satisfying the nonnegativity condition*

$$\text{(NN)} \qquad \forall \xi \in \mathbf{R}^n\,, \quad q(\xi)=0 \Rightarrow r(\xi)\geq 0\,.$$

Assume in addition that if q does not change sign, then q is positive or negative definite. Then there exists a $\lambda \in \mathbf{R}$ such that the quadratic form $r-\lambda q$ is positive semi-definite.

PROOF : When q is positive or negative definite, we just have to take a sufficiently large λ such that $\lambda q \leq 0$.

When q changes sign, let us choose η and $\zeta \in \mathbf{R}^n$ such that $q(\eta)=1$ and $q(\zeta)=-1$. If s is a positive definite quadratic form, then condition (NN) implies that for $\varepsilon>0$,

$$\forall \xi \in \mathbf{R}^n \setminus \{0\}\,, \quad q(\xi)=0 \Rightarrow r(\xi)+\varepsilon s(\xi)>0\;;$$

thanks to the "main theorem" of [8] we then have $r+\varepsilon s-\lambda_\varepsilon q \geq 0$ for some $\lambda_\varepsilon \in \mathbb{R}$, and this also implies that

$$-r(\zeta)-\varepsilon s(\zeta) \leq \lambda_\varepsilon \leq r(\eta)+\varepsilon s(\eta) \ ;$$

it follows that λ_ε is bounded when ε tends to zero, and we thus get by compactness a sequence ε_k tending to zero such that λ_{ε_k} tends to a $\lambda \in \mathbb{R}$; this λ solves our problem since we can take the limit in the inequalities $r+\varepsilon_k s-\lambda_{\varepsilon_k} q \geq 0$. □

REMARK. Theorem A fails to hold when the quadratic form q is degenerate and does not change sign as in the following example in $\mathbb{R}^2$: $q(\xi) = \xi_1^2$, $r(\xi) = 2\xi_1\xi_2$.

The proof given above does not provide explicitly the values of λ solving $r-\lambda q \geq 0$. We get them in our next result.

PROPOSITION B. *Keep the assumptions of theorem A, then set* $C_+ = \{\xi \in \mathbb{R}^n \,; q(\xi) > 0\}$, $C_- = \{\xi \in \mathbb{R}^n \,; q(\xi) < 0\}$, $\lambda_+ = \inf_{C_+}(r/q)$ *and* $\lambda_- = \sup_{C_-}(r/q)$ *(with the convention :* $\lambda_+ = +\infty$ *if* $C_+ = \phi$, *and* $\lambda_- = -\infty$ *if* $C_- = \phi$*). Then* $\lambda_+ \geq \lambda_-$, *and* $r-\lambda q \geq 0$ *if and only if* $\lambda \in [\lambda_-, \lambda_+]$. *Moreover if* q *is nondegenerate,* $\lambda_+ = \lambda_-$ *if and only if* q *and* r *do not satisfy condition* (P), *and in this situation we have* $r'(\xi) = \lambda_+ q'(\xi) = \lambda_- q'(\xi)$ *for all solution* $\xi \in \mathbb{R}^n$ *of* $q(\xi) = r(\xi) = 0$ *(here,* q' *and* r' *denote the gradients of* q *and* r, *which are vectors in the dual of* $\mathbb{R}^n$*).*

PROOF : It follows from our definitions that $r-\lambda q \geq 0$ on C_+ (resp. on C_-) if and only if $\lambda \leq \lambda_+$ (resp. $\lambda \geq \lambda_-$); since by assumption (NN) we also have $r-\lambda q \geq 0$ on the zeroes of q for all $\lambda \in \mathbb{R}$, λ solves the problem if and only if $\lambda_- \leq \lambda \leq \lambda_+$. The existence result of theorem A thus implies that $\lambda_- \leq \lambda_+$.

If q and r satisfy condition (P) and if s is a positive definite quadratic form, then q and $r-\varepsilon s$ still satisfy condition (P) for some $\varepsilon > 0$, and $|q| \leq Cs$ for some $C > 0$. Then, if λ_0 is such that $r-\varepsilon s-\lambda_0 q \geq 0$, it follows that $\lambda_- \leq \lambda_0 - (\varepsilon/C) < \lambda_0 + (\varepsilon/C) \leq \lambda_+$.

Finally, if q is nondegenerate and $q(\xi) = r(\xi) = 0$ for some $\xi \in \mathbb{R}^n \backslash \{0\}$, then $q'(\xi) \neq 0$ and $r'(\xi) = \lambda_0 q'(\xi)$ since r has a minimum at ξ under the constraint $q = 0$. Therefore, if $\eta \in \mathbb{R}^n$ is chosen such that $\langle q'(\xi), \eta\rangle > 0$, then $\xi + t\eta$ belongs to C_+ for small $t > 0$ and to C_- for small $t < 0$, and since $\lambda_0 = \langle r'(\xi), \eta\rangle \,/\, \langle q'(\xi), \eta\rangle = \lim_{t \to 0} r(\xi + t\eta)/q(\xi + t\eta)$, we have $\lambda_+ \leq \lambda_0 \leq \lambda_-$; it follows that these three values are equal. □

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Appendix. Para–Differential Calculus

Following the idea of a calculus for operators having a symbol with minimal regularity, originally exposed in the book of Coifman and Meyer [CM] which prophetical title is "au–delà des opérateurs pseudo–différentiels", Bony [Bo] developed in the early 80's a theory for the operators he called "para–différentiels". This theory has found successful applications in the study of non–linear partial differential equations.

We want to give in this Appendix a rough survey of this theory, collecting here definitions and statements of all the fundamental results and sketching only few proofs: in doing so we will follow very closely the cited work of Bony. In the last part of the Appendix we outline some results for operators depending on a parameter and a "large parameter", for these are the operators used in the proofs of the uniqueness results of this thesis.

As said, the fundamental work on para–differential operators is [Bo]. Related topics can be found in [CM], [Zu3], [Mey], [Mét1], [Mét2], [Bou], [A4], [Hör4], [Hör5], [Hör6], [GR].

A.1. The Littlewood–Paley's Decomposition

Lemma A.1. *Let $k > 1$ be a real number. There exist two even functions ψ , $\varphi \in C_0^\infty(\mathbf{R})$, with the following properties:* $\operatorname{supp}\psi \subseteq [-1,1]$, $\operatorname{supp}\varphi \subseteq \{x \in \mathbf{R} \,|\, k^{-1} \leq |x| \leq 2k\}$ *and for all $x \in \mathbf{R}$:*

$$\psi(x) + \sum_{p=0}^{+\infty} \varphi(2^{-p}x) = 1 \quad , \qquad \psi(x) + \sum_{p=0}^{N-1} \varphi(2^{-p}x) = \psi(2^{-N}x) \quad .$$

Proof. See [Zu3, Prop. 1].

Definition A.2. *Let k, ψ, φ as in Lemma* A.1. *Let u be an element of $\mathcal{S}'(\mathbf{R}^n)$, the set of temperate distributions. We will call a Littlewood–Paley's decomposition (or dyadic decomposition) of u (with respect to k, ψ, φ) the series:*

$$\sum_{p=-1}^{+\infty} u_p \tag{A.1}$$

where $u_{-1} = \psi(|D|)u = \mathcal{F}^{-1}(\psi(|\xi|)\mathcal{F}u(\xi))$ and $u_p = \varphi(2^{-p}|D|)u = \mathcal{F}^{-1}(\varphi(2^{-p}|\xi|)\mathcal{F}u(\xi))$ for all $p \geq 0$ ($\mathcal{F}$, as well as $\hat{\cdot}$, denotes the Fourier transform).

Let us remark that, for all $p \geq -1$, u_p is an analytic entire function and the series (A.1) is convergent in $\mathcal{S}'$ to u.

Important function spaces are characterized in terms of the Littlewood–Paley's decomposition, i. e. in terms of the convergence properties of (A.1). Before showing some of these characterizations we are forced to give a detailed notation. Let s be a real number. H^s denotes the Sobolev space of index s:

$$H^s = \{u \in \mathcal{S}'(\mathbf{R}^n) \,|\, (1+|\xi|^2)^{s/2}\,\hat{u}(\xi) \in L^2(\mathbf{R}^n)\} \quad ,$$

and we set $\|u\|_s = ((2\pi)^{-n}\int(1+|\xi|^2)^s|\hat{u}(\xi)|^2 d\xi)^{1/2}$. Let σ be a real number, $\sigma > 0$. If $0 < \sigma < 1$, $\mathcal{C}^\sigma = \mathcal{C}^\sigma_b$ is the Hölder space of index σ:

$$\mathcal{C}^\sigma = \mathcal{C}^\sigma_b = \{u \in L^\infty(\mathbf{R}^n) \,|\, \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\sigma} = [u]_\sigma < +\infty\} \quad ,$$

and we set $|u|_\sigma = \|u\|_\infty + [u]_\sigma$. If $\sigma \in \mathbf{N}$, $\mathcal{C}^\sigma_b$ denotes the set of all the σ–continuously differentiable functions on $\mathbf{R}^n$, such that all the derivatives up to the order σ are bounded; in this case we set $|u|_\sigma = \sum_{|\alpha|\leq\sigma}\|D^\alpha u\|_\infty$. If $\sigma > 1$, $\sigma \notin \mathbf{N}$ we define $\mathcal{C}^\sigma = \mathcal{C}^\sigma_b$ as follows:

$$\mathcal{C}^\sigma = \mathcal{C}^\sigma_b = \{u \in \mathcal{C}^{[\sigma]}_b \,|\, D^\alpha u \in \mathcal{C}^{\sigma-[\sigma]} \quad \text{for all } \alpha \text{ such that } |\alpha| = [\sigma]\} \quad ,$$

and in this case we set $|u|_\sigma = |u|_{[\sigma]-1} + \sum_{|\alpha|=[\sigma]}|D^\alpha u|_{\sigma-[\sigma]}$. Finally, if $\sigma \in \mathbf{N}$, $\mathcal{C}^\sigma_*$ is the Zygmund space of index σ: for $\sigma = 1$:

$$\mathcal{C}^1_* = \{u \in L^\infty(\mathbf{R}^n) \,|\, \sup_{\substack{x,h\\ h\neq 0}} \frac{|u(x-h)+u(x+h)-2u(x)|}{|h|} = [u]_1 < +\infty\} \quad ,$$

and $|u|_{1,*} = \|u\|_\infty + [u]_1$; for $\sigma > 1$, $\sigma \in \mathbf{N}$,:

$$\mathcal{C}^\sigma_* = \{u \in \mathcal{C}^{\sigma-1}_b \,|\, D^\alpha u \in \mathcal{C}^1_* \quad \text{for all } \alpha \text{ such that } |\alpha| = \sigma - 1\} \quad ,$$

and $|u|_{\sigma,*} = |u|_{\sigma-1} + \sum_{|\alpha|=[\sigma]}|D^\alpha u|_{1,*}$.

We give now the characterization of the Sobolev spaces.

Theorem A.3. *Let s be a positive real number and let u be a temperate distribution. Let $k > 1$ be a real number. The following conditions are equivalent:*

(A.2) $$u \in H^s \quad ;$$

(A.3) $$u \underset{\mathcal{S}'}{=} \sum_{p=-1}^{+\infty} u_p, \quad u_p \in L^2, \quad \operatorname{supp}\hat{u}_p \subseteq C_p\,, \text{ and there exists } (c_p) \in l^2$$

such that $\|u_p\|_0 \leq c_p 2^{-ps}$, *for all* p,

(here and in the following $C_{-1} = \bar{B}_1$ and $C_p = \{\xi \in \mathbf{R}^n \,|\, k^{-1}2^p \leq |\xi| \leq k2^{p+1}\}$ if $p \geq 0$);

(A.4) $$u \underset{\mathcal{S}'}{=} \sum_{p=-1}^{+\infty} u_p, \quad u_p \in L^2, \text{ there exists } k' > 0 \text{ such that } \operatorname{supp}\hat{u}_p \subseteq \bar{B}_{k'2^p}\,,$$

and there exists $(c_p) \in l^2$ *such that* $\|u_p\|_0 \leq c_p 2^{-ps}$, *for all* p;

$$\text{(A.5)} \qquad u \underset{S'}{=} \sum_{p=-1}^{+\infty} u_p\,, \quad u_p \in \mathcal{C}^\infty, \text{ and there exists } (c_p) \in l^2 \text{ such that}$$
$$\|D^\alpha u_p\|_0 \leq c_p 2^{-ps+p|\alpha|}, \text{ for all } \alpha \in \mathrm{N}^n \text{ and for all } p\,.$$

Moreover there exist two positive constants C and C' not depending on u, such that:

$$C\|u\|_s \leq \|(c_p)\|_{l^2} \leq C'\|u\|_s \quad .$$

Proof. See [Zu3, Prop. 7].

Let us remark that (A.2) is equivalent to (A.3) also in the case of $s \leq 0$. We characterize now the Hölder and the Zygmund spaces.

Theorem A.4. *Let σ be a positive real number and let u be a temperate distribution. Let $k > 1$ be a real number. The following conditions are equivalent:*

$$\text{(A.6)} \qquad u \in \mathcal{C}^\sigma \text{ if } \sigma \notin \mathrm{N} \text{ and } u \in \mathcal{C}^\sigma_* \text{ if } \sigma \in \mathrm{N}\,.$$

$$\text{(A.7)} \qquad u \underset{S'}{=} \sum_{p=-1}^{+\infty} u_p, \quad u_p \in L^\infty, \quad \operatorname{supp} \hat{u}_p \subseteq C_p \text{ and there exist } K > 0$$
$$\text{such that } \|u_p\|_\infty \leq K2^{-p\sigma}, \text{ for all } p;$$

$$\text{(A.8)} \qquad u \underset{S'}{=} \sum_{p=-1}^{+\infty} u_p, \quad u_p \in L^\infty, \text{ and there exist } k', K > 0 \text{ such that}$$
$$\operatorname{supp} \hat{u}_p \subseteq \bar{B}_{k'2^p} \text{ and } \|u_p\|_\infty \leq K2^{-p\sigma}, \text{ for all } p;$$

$$\text{(A.9)} \qquad u \underset{S'}{=} \sum_{p=-1}^{+\infty} u_p, \quad u_p \in \mathcal{C}^\infty, \text{ and there exists } K > 0 \text{ such that}$$
$$\|D^\alpha u_p\|_\infty \leq K2^{-p\sigma+p\alpha}, \text{ for all } \alpha \text{ and for all } p\,.$$

Moreover there exist two positive constants C and C' not depending on u, such that, if $\sigma \notin \mathrm{N}$,:

$$C|u|_\sigma \leq K \leq C'|u|_\sigma \quad ,$$

and, respectively, if $\sigma \in \mathrm{N}$,:

$$C|u|_{\sigma,*} \leq K \leq C'|u|_{\sigma,*} \quad .$$

Proof. We prove the theorem in the case $\sigma = 1$, the other ones being similar (see also [Zu3, Th.6]). Suppose (A.6) is valid. Let $\sum u_p$ a Littlewood–Paley's decomposition for u. So:

$$u_{-1}(x) = (\mathcal{F}^{-1}(\psi(|\xi|)) * u)(x)\,,$$
$$u_p(x) = (\mathcal{F}^{-1}(\varphi(2^{-p}|\xi|)) * u)(x)\,,\ \text{if } p \geq 0 \quad .$$

Obviously $u_p \in L^\infty$ for all p. Moreover:

$$\|u_{-1}\|_\infty \leq \|\mathcal{F}^{-1}(\psi(|\xi|))\|_{L^1}\|u\|_\infty \quad . \tag{A.10}$$

For $p \geq 0$, $u_p(x) = \int \mathcal{F}^{-1}(\varphi(|\xi|))(z)u(x - 2^{-p}z)dz$. As φ is even and $\varphi(0) = 0$, we have:

$$|u_p(x)| = 2^{-p}\int |\frac{\mathcal{F}^{-1}(\varphi(|\xi|))(z)}{2}|\frac{|u(x + 2^{-p}z) + u(x - 2^{-p}z) - 2u(x)|}{2^{-p}|z|}|z|dz \quad ,$$

and from this:

$$\|u_p\|_\infty \leq 2^{-p}\int |\frac{\mathcal{F}^{-1}(\varphi(|\xi|))(z)}{2}z|dz\,[u]_{1,*} \quad . \tag{A.11}$$

(A.10) and (A.11) imply (A.7). Let us remark that $K \leq C|a|_{1,*}$, with C not depending on u. The implications (A.7) $\Rightarrow$ (A.8) and (A.8) $\Rightarrow$ (A.9) being evident, we show the next one, i.e. (A.9) $\Rightarrow$ (A.6). The series $\sum u_p$ is convergent in L^∞, so $u \in L^\infty$ and:

$$\|u\|_\infty \leq 4K \quad . \tag{A.12}$$

Let us consider:

$$\begin{aligned} &u(x-h) + u(x+h) - 2u(x) \\ &\quad = \sum_{p=-1}^{N}(u_p(x-h) + u_p(x+h) - 2u_p(x)) + \sum_{p=N+1}^{+\infty}(u_p(x-h) + u_p(x+h) - 2u_p(x))\,. \end{aligned}$$

Knowing that if $v \in C_b^2$ then:

$$\sup_{\substack{x,h\\ h\neq 0}} \frac{|v(x-h) + v(x+h) - 2v(x)|}{|h|^2} \leq \sum_{|\alpha|=2}\|D^\alpha v\|_\infty \quad ,$$

we deduce:

$$|u(x-h) + u(x+h) - 2u(x)| \leq \sum_{p=-1}^{N} KC2^p|h|^2 + \sum_{p=N+1}^{+\infty} 16K2^{-p} \quad ,$$

where C depends only on n, the dimension of the euclidean space $\mathbf{R}^n$. If now N is such that $2^{-N-1} \leq |h| \leq 2^{-N}$, we get:

$$|u(x-h)+u(x+h)-2u(x)| \leq \Big(\sum_{p=-1}^{N} KC2^{p-N}\Big)|h| + 32K|h| \leq C'K|h| \quad ,$$

and finally $[u]_1 \leq KC'$. From this and (A.12) we have that $u \in C^1_*$, with $|u|_{1,*} < KC'$. The proof is complete.

For some other function spaces characterized in term of Littlewood–Paley's decompositions see [Mey].

A.2. The Para–Multiplication

Definition A.5. *Let k, ψ, φ as in Lemma A.1. Let $N_0 > 1 + 2\log_2 k$. Let $a \in L^\infty$ and $u \in H^s$ and let $\sum a_p$, $\sum u_p$ the corresponding Littlewood–Paley's decompositions. We define:*

$$T_a u = \sum_{q \geq N_0 - 1} v_q = \sum_{q \geq N_0-1} \Big(u_q \sum_{p \leq q - N_0} a_p\Big) \quad . \tag{A.13}$$

The operator T_a is the para–multiplication operator of symbol a.

From the Theorem A.3 we immediately deduce the following result.

Theorem A.6. *T_a is a continuous operator from H^s to H^s, for all $s \in \mathbf{R}$, and the norm of T_a is $\leq C\|a\|_\infty$, where C depends only on n and s.*

Lemma A.7. *Let a be in C^ρ if $\rho \notin \mathbf{N}$, or in C^ρ_* if $\rho \in \mathbf{N}$. Let u be in H^s and let k, ψ, φ as in Lemma A.1. Suppose that there exist $k' > k$, $(c_q) \in l^2$ and a sequence (v_q) of L^2 functions such that:*

$$u = \sum_{q=-1}^{+\infty} v_q, \ \operatorname{supp} \hat{v}_q \subseteq C'_q, \text{ and } \|v_q\|_0 \leq c_q 2^{-qs}, \tag{A.14}$$

(where $C'_{-1} = \bar{B}_1$ and $C'_q = \{\xi \in \mathbf{R}^n \,|\, (k')^{-1}2^q \leq |\xi| \leq k'2^{q+1}\}$).
Suppose also that there exist $\varepsilon > 0$, with $\varepsilon < 1/2k'$, $K > 0$ and a sequence (A_q) of L^∞ functions such that:

$$\operatorname{supp} \hat{A}_q \subseteq B_{\varepsilon 2^q}, \ \|a\|_\infty < K, \text{ and } \|a - A_q\|_\infty \leq K2^{-q\rho}. \tag{A.15}$$

Then $T_a u - \sum A_q v_q \in H^{s+\rho}$ and there exists C not depending on a and u such that:

$$\|T_a u - \sum_{q=-1}^{+\infty} A_q v_q\|_{s+\rho} \leq CK\|(c_q)\|_{l^2} \quad . \tag{A.16}$$

Proof. see [Zu3, Th.27] and [Bo, Rem. 2.2]

We are now ready to state the theorem on the symbolic calculus for the para–mutiplication operator, and to verify the independence of it, up to a ρ–regularizing operator, from the Littlewood–Paley's decomposition chosen.

Theorem A.8. *Let a, b be in C^ρ, if $\rho \notin \mathbf{N}$, or in C^ρ_* if $\rho \in \mathbf{N}$.*

Then $T_aT_b - T_{ab}$ is a ρ–regularizing operator, i. e. it maps continuously H^s in $H^{s+\rho}$ for all $s \in \mathbf{R}$, and its norm is $\leq C|a|_\rho|b|_\rho$, or $\leq C|a|_{\rho,*}|b|_{\rho,*}$ respectively.

Let a be in C^ρ if $\rho \notin \mathbf{N}$, or in C^ρ_* if $\rho \in \mathbf{N}$.

Then $T_a^* - T_{\bar{a}}$ is a ρ–regularizing operator, with norm $\leq C|a|_\rho$, or $\leq C|a|_{\rho,*}$ respectively.

Proof. We prove the first part; for the second statement see [Bo, Th. 2.4]. From (A.13) we have that $T_bu = \sum v_q$ where $v_q = u_q \sum_{p\leq q-N_0} b_p$. The sequence (v_q) satisfies (A.14), where u is substituted by T_bu. So that, posing $A_q = \sum_{p\leq q-N_0} a_p$, we have:

$$\|T_a(T_bu) - \sum_{q=-1}^{+\infty} A_qv_q\|_{s+\rho} \leq C|a|_\rho\|T_bu\|_s \leq C|a|_\rho|b|_\rho\|u\|_s \quad .$$

Consider now:

$$\sum_{q=-1}^{+\infty} A_qv_q = \sum_{q=-1}^{+\infty}\Big(\sum_{\substack{j\leq q-N_0\\ k\leq q-N_0}} a_jb_k\Big)u_q = \sum_{q=-1}^{+\infty} Z_qu_q \quad .$$

We have that supp $\hat{Z}_q \subseteq B_{\varepsilon 2^q}$ and $\|ab - Z_q\|_\infty \leq C|a|_\rho|b|_\rho 2^{-q\rho}$. Using again the Lemma A.7 we deduce that:

$$\|\sum_{q=-1}^{+\infty} A_qv_q - T_{ab}u\|_{s+\rho} \leq C|a|_\rho|b|_\rho\|u\|_s \quad .$$

The proof is complete.

Theorem A.9. *Let a be in C^ρ if $\rho \notin \mathbf{N}$, or in C^ρ_* if $\rho \in \mathbf{N}$. Let $\{k, \psi, \varphi, N_0\}$, $\{k', \psi', \varphi', N_0'\}$ two possible sets of parameters for the definition of the para–multiplication. Let T_a and T_a' the corresponding para–multiplication operators of symbol a.*

Then $T_a - T_a'$ is a ρ–regularizing operator of norm $\leq C|a|_\rho$.

Proof. It is again a consequence of the Lemma A.7. See [Zu3, Cor. 27′]

Let us remark that some other interesting properties of the para–multiplication are collected in the Lemma 5 of the Chapter 3 of this thesis. Finally we want to state a result which gives the relation between the para–multiplication operators and the differential operators.

Theorem A.10. *Let a be in C_b^ρ. Let h be a C^∞ function, homogeneous of degree m, defined in $\mathbf{R}^n \setminus \{0\}$, and let s be a C^∞ function such that $s(x) = 1$ if $|x| \geq 1$ and $s(x) = 0$ if $|x| \leq 1/2$.*

Then:

$$s(|D|)h(D)T_a u = \sum_{|\alpha|<\rho} T_{\frac{1}{\alpha!}D_x^\alpha a}(\partial_\xi^\alpha h(D)s(|D|)u) + Ru \quad ,$$

where R is a $(\rho - m)$-regularizing operator with norm $\leq C\|h\|_{C^M(S^{n-1})}|a|_\rho$.

Proof. We prove the theorem for $\rho = 1$. We have that $T_a u = \sum v_q = \sum(u_q \sum_{p\leq q-N_0} a_p)$. We know that $\operatorname{supp} \hat{v}_q \subseteq C_q'$. In particular this implies that there exists $\bar{q}$ such that, for $q \geq \bar{q}$, we have $\operatorname{supp} \hat{v}_q \cap \bar{B}_1 = \emptyset$. Considering now a function $\phi \in C_0^\infty$ with $\operatorname{supp} \phi \subseteq C_0''$ and $\phi(\xi) = 1$ on C_0', we deduce that:

$$s(|D|)h(D)T_a u = \sum_{q<\bar{q}} s(|D|)h(D)v_q + \sum_{q\geq\bar{q}} \phi(2^{-q}D)h(D)v_q = ① + ② \quad .$$

Easily we obtain that for all s, σ, there exists $C_{s,\sigma} > 0$ such that:

$$\|①\|_{s+\sigma} \leq C_{s,\sigma}\|h\|_{L^\infty(S^{n-1})}|a|_1\|u\|_s \quad . \tag{A.17}$$

Let us come to ②. We get:

$$\phi(2^{-q}D)h(D)v_q = \mathcal{F}^{-1}(\phi(2^{-q}\xi)h(\xi)\hat{v}_q(\xi)) = 2^{qm}\mathcal{F}^{-1}(\phi(2^{-q}\xi)h(2^{-q}\xi)\hat{v}_q(\xi)) \quad .$$

Calling now $\phi(\xi)h(\xi) = H(\xi)$, we have that $H \in C_0^\infty$ and:

$$\phi(2^{-q}D)h(D)v_q = 2^{qm}\mathcal{F}^{-1}(H(2^{-q}\xi)) * v_q \quad .$$

Let now $r = \mathcal{F}^{-1}H$; so:

$$\begin{aligned}
\phi(2^{-q}D)h(D)v_q &= 2^{qm+qn}r(2^q x) * v_q \\
&= 2^{qm}\int r(z)v_q(x - 2^{-q}z)dz \\
&= 2^{qm}\int r(z)\Big(\sum_{p\leq q-N_0} u_q(x - 2^{-q}z)a_p(x - 2^{-q}z)\Big)dz \\
&= 2^{qm}\Big(\sum_{p\leq q-N_0} a_p\Big)\int r(z)u_q(x - 2^{-q}z)dz \\
&\qquad + 2^{qm}\int \sum_{p\leq q-N_0} (a_p(x - 2^{-q}z) - a_p(x))r(z)u_q(x - 2^{-q}z)dz \\
&= 2^{qm}\Big(\sum_{p\leq q-N_0} a_p\Big)H(2^{-q}D)u_q \\
&\qquad + 2^{qm-q}\int \sum_{p\leq q-N_0} \Big(\frac{a_p(x - 2^{-q}z) - a_p(x)}{2^{-q}|z|}\Big)|z|r(z)u_q(x - 2^{-q}z)dz \\
&= 2^{qm}(H(2^{-q}D)u_q)\Big(\sum_{p\leq q-N_0} a_p\Big) + f_q
\end{aligned}$$

We remark that supp $\hat{f}_q \subseteq C_q'''$ and that:

$$(A.18)\quad \|f_q\|_0 \leq 2^{qm-q}|\sum_{p\leq q-N_0} a_p|_1\|u_q\|_0 \int |z|\,|r(z)|dz \leq 2^{qm-q}\|h\|_{C^M(S^{n-1})}|a|_1\|u_q\|_0 \quad .$$

Finally:

$$s(|D|)h(D)T_a u = ① + \sum_{q\geq\bar{q}}((h(D)u)_q(\sum_{p\leq q-N_0} a_p)) + \sum_{q\geq\bar{q}} f_q \quad ,$$

from (A.17) and (A.18) we reach the conclusion.

A.3. Global Para–Differential Operators

Let $\Delta_{S^{n-1}}$ be the Laplace–Beltrami operator on S^{n-1}. Let $(\tilde{h}_\nu)$ be a hortonormal basis of $L^2(S^{n-1})$, such that $\Delta_{S^{n-1}}\tilde{h}_\nu = \lambda_\nu \tilde{h}_\nu$, for each ν. We know that there exist M, C_1, C_2, and $\bar{\nu}$ such that for $\nu > \bar{\nu}$ we have $0 < C_2 < |\lambda_\nu/\nu^M| < C_1$, and for all $N \in \mathbf{N}$ there exists $C_N > 0$ such that:

$$\|\tilde{h}_\nu\|_{C^N(S^{n-1})} \leq C_N \nu^{\frac{M}{2}(N+\frac{n}{2}+1)} \quad .$$

Definition A.11. *Let $l(x,\xi)$ be a function on $\mathbf{R}^n \times \mathbf{R}^n \setminus \{0\}$, with the following properties: l is C_b^ρ in x and C^∞ in ξ (i. e. for every fixed x, $f(x,\cdot) \in C^\infty(\mathbf{R}^n \setminus \{0\})$ and for every fixed $\beta \in \mathbf{N}^n$ and for every fixed ξ, $\partial_\xi^\beta f(\cdot,\xi) \in C_b^\rho(\mathbf{R}^n)$), l is homogeneous of degree m in ξ, and l has compact support in x. The set of such functions will be denoted by Γ_ρ^m.*

Let $\tilde{\rho}$ be the maximum between all the integer $< \rho$. Σ_ρ^m will indicate the set of all the functions l such that:

$$l = l_m + l_{m-1} + \ldots + l_{m-\tilde{\rho}} \quad ,$$

where $l_{m-j} \in \Gamma_{\rho-j}^{m-j}$. We call Σ_ρ^m (Γ_ρ^m) the set of the global para–differential (global homogeneous para–differential) symbols of degree m and regularity ρ.

To define the para–differential operators corresponding to the symbols of Γ_ρ^m and Σ_ρ^m, we need the following lemma.

Lemma A.12. *Let $l(x,\xi) \in \Gamma_\rho^m$, and let us define $h_\nu(\xi) = \tilde{h}_\nu(\frac{\xi}{|\xi|})$. There exists a sequence (a_ν) in C_b^ρ such that:*

$$(A.19)\quad l(x,\xi) = \sum_\nu a_\nu(x)h_\nu(\xi) \quad .$$

Moreover for every k, there exists C_k such that $|a_\nu|_\rho \leq C_k\nu^{-kM}$.

(A.19) is called the spherical harmonic decomposition of l.

Definition A.13. *Let $l \in \Gamma^m_\rho$, and let (A.19) be the spherical harmonic decomposition of l. Then the following operator:*

$$T_l u = \sum_\nu T_{a_\nu}(s(|D|)h_\nu(D)u) \tag{A.20}$$

is the global para–differential operator of symbol l (the function s is defined in the statement of the Theorem A.10).

If $l \in \Sigma^m_\rho$, and $l = l_m + l_{m-1} + \ldots + l_{m-\tilde{\rho}}$ with $l_{m-j} \in \Gamma^{m-j}_{\rho-j}$, then:

$$T_l = T_{l_m} + T_{l_{m-1}} + \ldots + T_{l_{m-\tilde{\rho}}} \quad .$$

Using the information we have on the norms of h_ν and a_ν it is easy to verify that for each s the series $\sum_\nu T_{a_\nu}(s(|D|)h_\nu(D))$ is convergent in the space of the linear continuous functionals from H^s to H^{s-m}, and the norm of T_l only depends on $\sup_{\xi\in S^{n-1}} |\partial^\alpha_\xi l(\cdot,\xi)|_\rho$.

Let $\chi(\theta,\eta)$ be a $\mathcal{C}^\infty$ function on $\mathbf{R}^n \times \mathbf{R}^n$, homogeneous of degree 0 in (θ,η) for $|\theta| + |\eta| > 1$, and such that there exist $0 < \varepsilon_1 < \varepsilon_2 < \frac{1}{10}$ such that:

$$\chi(\theta,\eta) = 1 \quad \text{for } |\theta| \le \varepsilon_1|\eta| \text{ and } |\theta| + |\eta| > 1 \quad ,$$
$$\chi(\theta,\eta) = 0 \quad \text{for } |\theta| \ge \varepsilon_2|\eta| \text{ or } |\theta| + |\eta| < \frac{1}{2} \quad .$$

We can verify that, defining T'_l in the following way:

$$(\widehat{T'_l u})(\xi) = (2\pi)^{-n}\int \chi(\xi-\eta,\eta)\mathcal{F}_x(l(x,\eta))(\xi-\eta)\hat{u}(\eta)d\eta \quad , \tag{A.21}$$

$T_l - T'_l$ is a $(\rho - m)$–regularizing operator. This essentially means that (A.21) is an alternative way to define the para–differential operator associate to l, and this also shows that T_l is independent, up to a $(\rho - m)$–regularizing operator, on the Littlewood–Paley's decomposition and on the spherical harmonic representation chosen. Moreover it can be shown in this way that $T_l \in \mathrm{Op}(\mathrm{S}^m_{1,1})$, where $\mathrm{S}^m_{1,1}$ is the Hörmander's set of pseudo–differential symbols of type $1,1$. For more on this subject see [Mey], [Mét1], [Mét2], [Hör4], [Hör5], [Hör6], [Bou], [Chin].

We give now the formula for the composition of homogeneous para–differential operators. From this formula it will be easy to obtain the general one.

Theorem A.13. *Let $l_1 \in \Gamma^{m_1}_\rho$, $l_2 \in \Gamma^{m_2}_\rho$. We set:*

$$l = l_1 \# l_2 = \sum_{|\alpha|<\rho} \frac{1}{\alpha!}\partial^\alpha_\xi l_1 D^\alpha_x l_2 \quad .$$

Then $T_{l_1}(T_{l_2}) = T_l + R$, where R is a $(\rho - m_1 - m_2)$–regularizing operator. Moreover the norm of R depends only on $\sup_{\xi\in S^{n-1}} |\partial^\alpha_\xi l_1(\cdot,\xi)|_\rho$ and $\sup_{\xi\in S^{n-1}} |\partial^\beta_\xi l_2(\cdot,\xi)|_\rho$.

Proof. It is a consequence of the Theorem A.10 and of (A.20). See [Bo, Th. 3.2].

Analogously the following result gives a formula for the adjoint operator.

Theorem A.14. *Let $l \in \Gamma^m_\rho$. We set:*

$$l^* = \sum_{|\alpha|<\rho} \frac{1}{\alpha!} \partial^\alpha_\xi D^\alpha_x \bar{l} \quad .$$

Then $T^*_l = T_{l^*} + R$, where R is a $(\rho - m)$–regularizing operator. Moreover the norm of R depends only on $\sup_{\xi \in S^{n-1}} |\partial^\beta_\xi l(\cdot, \xi)|_\rho$.

Proof. It is a consequence of the Theorems A.8 and A.10 and of (A.20). See [Bo, Th. 3.3].

Finally let us state two theorems, the first of them furnishes an estimation on the difference between a para–differential operator and the corresponding usual differential operator, and the second one will give the relation of para–differential operators with the pseudo–differential ones.

Theorem A.15. *Let $l \in \Gamma^m_\rho$. We denote by $l(x, D)$ the operator:*

$$l(x, D)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} l(x, \xi) s(|\xi|) \hat{u}(\xi) d\xi \quad .$$

Then for all $0 < \sigma < \rho$ there exists $C_\sigma > 0$ such that:

$$\|T_l u - l(x, D)u\|_0 \leq C_\sigma \|u\|_{m-\sigma} \quad .$$

Proof. Let $0 < \sigma < \rho$ and $u \in H^{m-\sigma}$. Then $s(D)h_\nu(D)u \in H^{-\sigma}$, and:

$$\|s(D)h_\nu(D)u\|_{-\sigma} \leq C\|h_\nu\|_{L^\infty(S^{n-1})}\|u\|_{m-\sigma} \quad .$$

Then by the claim **(iii)** of Lemma 5 in Chapter 3, we have that:

$$\begin{aligned} \|(T_{a_\nu} - a_\nu)(s(D)h_\nu(D)u)\|_0 &\leq \|(T_{a_\nu} - a_\nu)(s(D)h_\nu(D)u)\|_{\rho-\sigma} \\ &\leq C_\sigma |a_\nu|_\rho \|s(D)h_\nu(D)u\|_{-\sigma} \\ &\leq C_\sigma |a_\nu|_\rho \|h_\nu\|_{L^\infty(S^{n-1})}\|u\|_{m-\sigma} \quad . \end{aligned}$$

The proof is completed by using the information on the spherical harmonic decomposition.

Theorem A.16. *Let $l \in \Gamma^m_\rho$ for every ρ.*
Then $T_l - l(x, D)$ is a σ–regularizing for every σ.

Proof. The proof is analogous to that one of the Theorem A.15.

A.4. Para–Differential Operators on an Open Set

Let ω be an open set of $\mathbf{R}^n$.

Definition A.17. *Let $m, \rho \in \mathbf{R}$ with $\rho > 0$. $\Gamma^m_\rho(\omega)$ is the set of all the functions l defined on $\omega \times \mathbf{R}^n \setminus \{0\}$ such that: l is $\mathcal{C}^\rho_b$ in x and $\mathcal{C}^\infty$ in ξ, and l is homogeneous of degree m in ξ. $\Gamma^m_\rho(\omega)$ will be called the space of the homogeneous para–differential symbols, of degree m and regularity ρ, on ω. Let now $\tilde{\rho}$ be the maximum between all the integer $< \rho$. $\Sigma^m_\rho(\omega)$ is the set of all the functions l such that:*

$$l = l_m + l_{m-1} + \ldots + l_{m-\tilde{\rho}} \quad ,$$

where $l_{m-j} \in \Gamma^{m-j}_{\rho-j}(\omega)$. We call $\Sigma^m_\rho(\omega)$ the set of the para–differential symbols, of degree m and regularity ρ, on ω.

To each symbol we want to associate an operator. To do this let us remind a definition.

Definition A.18. *Let L be a linear map of $\mathcal{D}'(\omega)$ (the set of the distributions on ω) into itself. We say that L is properly supported if, for all K, compact set of ω, there exists $\hat{K}$, compact set of ω, such that if $u \in \mathcal{D}'(\omega)$ with $\operatorname{supp} u \subseteq K$ then $\operatorname{supp} Lu \subseteq \hat{K}$, and if $u \in \mathcal{D}'(\omega)$ with $\operatorname{supp} u \cap \hat{K} = \emptyset$ then $\operatorname{supp} Lu \cap K = \emptyset$.*

Definition A.19. *Let $l \in \Gamma^m_\rho(\omega)$. Let (U_j) be a locally finite open covering of ω and let $\varphi_j, \chi_j \in \mathcal{C}^\infty_0(U_j)$ such that: (φ_j) is a partition of unity and $\chi_j = 1$ on a neighborhood of the support of φ_j, for every j. We define the operator T_l in the following way:*

$$T_l u = \sum_j \chi_j T_{\chi_j l}(\varphi_j u) \quad . \tag{A.21}$$

If $l \in \Sigma^m_\rho(\omega)$, and $l = l_m + l_{m-1} + \ldots + l_{m-\tilde{\rho}}$ with $l_{m-j} \in \Gamma^{m-j}_{\rho-j}(\omega)$, then:

$$T_l = T_{l_m} + T_{l_{m-1}} + \ldots + T_{l_{m-\tilde{\rho}}} \quad .$$

T_l is a properly supported operator on $\mathcal{D}'(\omega)$ and it is easy to see that $T_l : H^s_{\mathrm{loc}}(\omega) \to H^{s-m}_{\mathrm{loc}}(\omega)$ and that for all K, compact set of ω, and for all $\chi \in \mathcal{C}^\infty_0(\omega)$, $\chi = 1$ on a neighborhood of K, the operator $T_l - \chi T_{\chi l}$ maps continuously the functions of H^s with support in K into $H^{s-m+\rho}$ and the norm of such an operator only depends on $\sup_{\xi \in S^{n-1}} |\partial^\alpha_\xi l(\cdot, \xi)|_\rho$ and on the sequences (φ_j) and (χ_j).

Let us finally remark that our definition is slightly more restrictive than Bony's one. This is caused by our aim to define para–differential operators depending on a parameter.

The fundamental result on para–differential operators on an open set are collected in the following theorem.

Theorem A.20. *Let l be in $\Sigma_\rho^m(\omega)$.*

Then T_l is independent from the open covering and the sequences (φ_j) and (χ_j) up to continuous operators from $H^s_{\mathrm{loc}}(\omega)$ to $H^{s-m+\rho}_{\mathrm{loc}}(\omega)$ (i.e. $(\rho-m)$–regularizing operators). Let $l_1 \in \Gamma_\rho^{m_1}(\omega)$ and $l_2 \in \Gamma_\rho^{m_2}(\omega)$.

Then $l = l_1 \# l_2 = \sum_{|\alpha|<\rho} \frac{1}{\alpha!}\partial_\xi^\alpha l_1 D_x^\alpha l_2$ is in $\Sigma_\rho^{m_1+m_2}(\omega)$, and $T_{l_1}(T_{l_2}) = T_l + R$, where R is a $(\rho - m_1 - m_2)$–regularizing operator. Let $l \in \Gamma_\rho^m(\omega)$.

Then $l^* = \sum_{|\alpha|<\rho} \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \bar{l}$ is in $\Sigma_\rho^m(\omega)$ and $T_l^* = T_{l^*} + R$, where R is a $(\rho - m)$–regularizing operator.

Proof. See [Bo, Th. 3.9].

Let us give some other results.

Theorem A.21. *Let $l \in \Gamma_\rho^m(\omega)$, with $l(x,\xi) \neq 0$ for all $x \in \omega$ and for all $\xi \in \mathbf{R}^n \setminus \{0\}$.*

Then there exists $h \in \Sigma_\rho^{-m}(\omega)$ such that $T_h(T_l) = \mathrm{Id} + R$, where R is ρ–regularizing.

Proof. It is enough to find $h \in \Sigma_\rho^{-m}(\omega)$ such that $h \# l = 1$. Such a function h is built using the homogeneity. Hence we apply the Theorem A.20.

The following result is a simple consequence of the Theorem A.21.

Theorem A.22. *Let $l \in \Gamma_\rho^m(\omega)$ with $\rho \geq 1$. suppose that $l(x,\xi) \neq 0$ for all $x \in \omega$ and for all $\xi \in \mathbf{R}^n \setminus \{0\}$.*

Then for each K compact set of ω there exist $C_K, c_K > 0$ such that:

$$\|T_l u\|_0 + C_K \|u\|_{m-1} \geq c_K \|u\|_m \quad ,$$

and:

$$\mathrm{Re}\,(T_l u\, ,\, u) + C_k \|u\|^2_{\frac{m-1}{2}} \geq c_K \|u\|^2_{\frac{m}{2}} \quad ,$$

for all $u \in C_0^\infty$ with $\operatorname{supp} u \subseteq K$.

Proof. The first claim being an immediate corollary of the Theorem A.21, let us sketch the proof of the second. We can suppose that $l(x,\xi) \in \mathbf{R}$ and $l(x,\xi) > \delta|\xi|^m$, with $\delta > 0$. Let $b = \sqrt{l}$. We have that $b \in \Gamma_\rho^{m/2}(\omega)$. Then:

$$\mathrm{Re}\,(T_l u\, ,\, u) = \mathrm{Re}\,(T_b u\, ,\, T_b u) + \mathrm{Re}\,(R u\, ,\, u) \quad ,$$

where R is $(1-m)$–regularizing. From this the conclusion is easy.

Theorem A.23. *Let $u \in H^s(\omega)$ with compact support in ω and such that* $\operatorname{supp}\hat{u} \subseteq \Gamma$, *where Γ is a conic set in $\mathbf{R}^n$. Suppose that $l \in \Sigma_\rho^m(\omega)$ is such that $l = 0$ on $\omega \times \Gamma'$, where Γ' is conic neighborhood of Γ.*

Then $T_l u \in H^{s-m+\rho}$.

Proof. Let $\psi \in \mathcal{C}_0^\infty(\mathbf{R}^n \setminus \{0\})$ be a function such that ψ is homogeneous of degree 0, $\operatorname{supp}\psi \subseteq \Gamma'$ and $\psi = 1$ on Γ. Let $a \in \mathcal{C}_0^\infty(\omega)$ be a function such that $a = 1$ on a neighborhood of the support of u. Then $u = T_{a\psi}u + Ru$ where R is σ–regularizing for every σ (see the Theorem A.16). So that:

$$T_l u = T_l(T_{a\psi}u) + T_l Ru = T_{l\#a\psi}u + R'u + T_l Ru \quad ,$$

where R' is a $(\rho - m)$–regularizing operator. The proof is concluded by observing that $l\#a\psi = 0$.

Let us finally state a fundamental result on para–differential operators: the so called sharp Gårding inequality.

Theorem A.24. *Let $l \in \Sigma_\rho^m(\omega)$. Suppose that $\operatorname{Re} l(x,\xi) \geq 0$ for all $x \in \omega$ and for all $\xi \in \mathbf{R}^n$ with $|\xi| > C$.*

Then for each K compact set of ω there exists $C_K > 0$ such that:

$$\operatorname{Re}(T_l u\,,\,u) \geq -C_K\|u\|^2_{\frac{2m-\rho}{4}} \qquad \text{if } 0 < \rho < 2 \quad ,$$

and

$$\operatorname{Re}(T_l u\,,\,u) \geq -C_K\|u\|^2_{\frac{m-1}{2}} \qquad \text{if } \rho \geq 2 \quad ,$$

for all $u \in \mathcal{C}_0^\infty(\omega)$ with $\operatorname{supp} u \subseteq K$.

Proof. See [Hör5, Th. 7.1], [Bo, Th. 6.8] and [Mét2, Prop. B.1.4]. This proof is highly non–trivial. Using the result of Cordoba and Fefferman [CF] together with a theorem due to Coifman and Meyer [CM, Th. 2.9], it is possible to see that the constant C_K depends on K, on $\sup_{\xi \in S^{n-1}} |\partial_\xi^\alpha l(\cdot,\xi)|_\rho$ and on the sequences (φ_j) and (χ_j).

A.5. Para–Differential Operators Depending on a Parameter

Let m, ρ, T be real numbers, with $\rho, T > 0$. Let ω be an open set of $\mathbf{R}^n$.

Definition A.25 *${}^t\Gamma_\rho^m(\omega)$ is the set of the functions $l : \omega \times [0,T] \times \mathbf{R}^n \setminus \{0\} \to \mathbf{C}$ with the following properties: l is $\mathcal{C}_b^\rho$ in (x,t) and $\mathcal{C}^\infty$ in ξ, and l is homogeneous of degree m in ξ.*

${}^t\Sigma_\rho^m(\omega)$ denotes the set of all the functions l such that, if $\tilde{\rho}$ is the maximum integer $< \rho$,:

$$l = l_m + l_{m-1} + \ldots + l_{m-\tilde{\rho}} \quad ,$$

where $l_{m-j} \in {}^t\Gamma^{m-j}_{\rho-j}(\omega)$.

Let $l \in {}^t\Gamma^m_\rho(\omega)$, we remark that for all $\beta \in \mathbf{N}^n$ there exists $C_\beta > 0$ such that:

$$\sup_{\xi \in S^{n-1}} |\partial^\beta_\xi l(\cdot, t, \xi)|_\rho \le C_\beta \quad , \tag{A.22}$$

for all $t \in [0, T]$. This fact has some remarkable consequences. Let us show some of them. Let $l \in {}^t\Sigma^m_\rho(\omega)$. For every fixed t, we can associate to l a para-differential operator on ω. We denote such an operator by $T_{l(t)}$. Let now $l_1 \in {}^t\Gamma^{m_1}_\rho(\omega)$ and $l_2 \in {}^t\Gamma^{m_2}_\rho(\omega)$; considering $(l_1(t)\#l_2(t))(x,\xi) = \sum_{|\alpha|<\rho} \frac{1}{\alpha!}\partial^\alpha_\xi l_1(x,t,\xi) D^\alpha_x l_2(x,t,\xi) = l(x,t,\xi)$, we have that $l \in {}^t\Sigma^{m_1+m_2}_\rho(\omega)$ and $T_{l_1(t)}(T_{l_2(t)}) = T_{l(t)} + R_t$, but for every K, compact set of ω, there exists $C_K > 0$ not depending on t, such that for all $u \in H^s(\omega)$ with $\operatorname{supp} u \subseteq K$ we have:

$$\|R_t u\|_{s-m+\rho} \le C_K \|u\|_s \quad .$$

Analogously let $l \in {}^t\Gamma^m_\rho(\omega)$. We consider $l^*(x,t,\xi) = \sum_{|\alpha|<\rho} \frac{1}{\alpha!}\partial^\alpha_\xi D^\alpha_x \bar{l}(x,t,\xi)$. We have that $l^* \in {}^t\Sigma^m_\rho(\omega)$ and $T^*_{l(t)} = T_{l^*(t)} + R'_t$, where R'_t has the same properties as R_t. Finally suppose that for every fixed t, $l(x,t,\xi)$ satisfies the hypotheses of the Theorem A.22 or of the Theorem A.24. Then the constants C_K and c_K don't depend on t.

Suppose now that $u \in C^\infty_0(\mathbf{R}^{n+1})$ has support in $\omega \times [0,T]$. Then, if $l \in {}^t\Sigma^m_\rho(\omega)$, for every fixed t, the function $x \mapsto T_{l(t)}u(x,t)$ is in C^∞_0. Let us consider the function $(x,t) \mapsto T_{l(t)}u(x,t)$. Such a function is measurable and in particular this implies that, for $l_1 \in {}^t\Sigma^{m_1}_\rho(\omega)$ and $l_2 \in {}^t\Sigma^{m_2}_\rho(\omega)$, the function $t \mapsto (T_{l_1(t)}u(x,t)\, , T_{l_2(t)}u(x,t)_{L^2,L^2}$ is measurable on $[0,T]$. This remark easily shows that, for instance, if K is a compact set of $\omega \times [0,T]$, then there exists $C_K > 0$ such that:

$$\Big(\int_0^T \|T_{l(t)}u(x,t)\|^2_{s-m} dt\Big)^{1/2} \le C_K \Big(\int_0^T \|u\|^2_s dt\Big)^{1/2} \quad ,$$

for all $u \in C^\infty_0(\mathbf{R}^{n+1})$ such that $\operatorname{supp} u \subseteq K$.

Finally we give a result on the commutator between $T_{l(t)}$ and D_t.

Theorem A.26. *Let $\rho \ge 1$ and suppose that $l \in {}^t\Sigma^m_\rho(\omega)$.*
Then $[D_t, T_{l(t)}]u = T_{D_t l(t)}u$ for all $u \in C^\infty_0(\mathbf{R}^{n+1})$ such that $\operatorname{supp} u \subseteq \omega \times [0,T]$.

Proof. Let $\rho \ge 1$ and let $a(x,t)$ be a function in $C^\rho_b(\omega \times [0,T])$ with compact support. Let $u \in C^\infty_0(\mathbf{R}^{n+1})$ such that $\operatorname{supp} u \subseteq \omega \times [0,T]$. We have that:

$$T_{a(t)}u(x,t) = \sum_{p \ge N_0 - 1} u_p(x,t)\Big(\sum_{q \le p - N_0} a_q(x,t)\Big) = \sum_{p \ge N_0 - 1} v_q(x,t) \quad .$$

For a fixed x we have that:

$$D_t(v_p(x,t)) = D_t(u_p(x,t)) \sum_{q \le p-N_0} a_q(x,t) + u_p(x,t) \sum_{q \le p - N_0} D_t a_q(x,t) \quad .$$

Moreover $D_t(u_p(x,t)) = (D_t u)_p(x,t)$ and $D_t(a_q(x,t)) = (D_t a)_q(x,t)$. As the series $\sum D_t(v_q)$ is uniformly convergent, we deduce that the function $t \mapsto T_{a(t)}u(x,t)$ is differentiable and:

$$D_t(T_{a(t)}u(x,t)) = T_{D_t a(t)}u(x,t) + T_{a(t)}(D_t u)(x,t) \quad .$$

Using the spherical harmonic decomposition and the partition of unity we reach the conclusion.

A.6. Para–Differential Operators Depending on a Large Parameter

Let k, ψ, φ as in the lemma A.1. Let $\lambda \in \mathbf{R}$, $\lambda \geq 1$. Let u be a temperate distribution in $\mathbf{R}^n$. We consider the series:

$$\sum_{p=-1}^{+\infty} u_p^\lambda \quad , \tag{A.23}$$

where $u_{-1}^\lambda = \mathcal{F}^{-1}(\psi(\sqrt{\lambda^2 + |\xi|^2})\hat{u}(\xi))$ and $u_p^\lambda = \mathcal{F}^{-1}(\psi(2^{-p}(\sqrt{\lambda^2 + |\xi|^2}))\hat{u}(\xi))$ for all $p \geq 0$. We call the series (A.23) a λ–Littlewood–Paley's decomposition for u.

Definition A.27. *Let $a \in L^\infty$. The operator:*

$$T_a^\lambda u = \sum_{q \geq N_0 - 1} (u_p^\lambda \sum_{p \leq q - N_0} a_p) \quad ,$$

is called the λ-para-multiplication operator of symbol a.

It is possible to show that $T_a^\lambda : H^s \to H^s$ for all s, and setting:

$$\|u\|_{s,\lambda} = ((2\pi)^{-n} \int (\lambda^2 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi)^{1/2} \quad ,$$

there exists C_s not depending on λ such that $\|T_a^\lambda u\|_{s,\lambda} \leq C_s \|a\|_\infty \|u\|_{s,\lambda}$.

Let now $h(\lambda, \xi)$ be a $\mathcal{C}^\infty$ function defined in $[1, +\infty[\times \mathbf{R}^n \setminus \{0\}$, homogeneous of degree m in (λ, ξ). We set, for $u \in \mathcal{S}'$,:

$$l(\lambda, D)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} l(\lambda, \xi) s(\sqrt{\lambda^2 + |\xi|^2}) \hat{u}(\xi) d\xi \quad .$$

Then $h(\lambda, D)$ maps H^s into H^{s-m}, continuously with respect to the norm $\|\cdot\|_{s,\lambda}$. Let finally $l(x, \lambda, \xi)$ be a $\mathcal{C}_b^\rho$ function in x and a $\mathcal{C}^\infty$ function in (λ, ξ), for $(\lambda, \xi) \in [1, +\infty[\times \mathbf{R}^n \setminus \{0\}$, homogeneous of degree m in (λ, ξ), and with compact support in x (the set of such functions will be denoted by ${}^\lambda\Gamma_\rho^m$). Let:

$$h(x, \lambda, \xi) = \sum_\nu a_\nu(x) h_\nu(\lambda, \xi) \quad ,$$

a spherical harmonic decomposition of l with respect to the variables (λ, ξ).

Definition A.28. *The operator:*

$$T_l^\lambda u = \sum_\nu T_{a_\nu}^\lambda (h_\nu(\lambda, D)u) \quad ,$$

is called the global λ–para–differential operator of symbol l.

As we have already done for the para–differential operators, we can define the set ${}^\lambda\Sigma_\rho^m$ with the sums of the elements of ${}^\lambda\Gamma_{\rho-j}^{m-j}$, for $j = 0, 1. \ldots, \tilde{\rho}$, and we can associate to these symbols the corresponding sum of homogeneous operators. As an analogous result of Theorem A.10 holds, we can obtain the symbolic calculus, which is summarized in Lemma 2.19. Also a result similar to the Theorem A.22 is valid. This is enough for our pourposes.

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