ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Energy and Rotation in Relativistic Astrophysics

Thesis of "Doctor Philosophiæ"

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Ph.D. Thesis

Abstract

This thesis is devoted to the study of a number of slightly independent topics. The material herein have therefore been divided into three chapters. The first chapter represents a collection of topics studied during the first two years of the SISSA PhD course. Here I will present a brief overview of the possibility of having non-ordinary extremely compact objects. These are mainly the strange stars, Q-stars and boson stars whose study extend over many different sectors in physics, from condensed matter to high energy nuclear physics. The remaining two chapters are devoted to the study of the Hartle-Thorne rotating axisymmetric exterior space time and the localization and definition of gravitational energy, respectively. The former is particularly interesting as it provides the first analytical solution to these spacetimes which can give tighter constraints on many orbital parameters. In fact two of these parameters, namely the radial and orthogonal epicyclic frequencies, may be used as a possible explanation of QPOs. In the last chapter the problem of energy localization in GR has yet again been attacked, using the optical geometry framework of GR, which has given us some additional insight but with a negative outcome in the sense that the definition of a gravitational energy density still remains obscure if even possible at all.

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Notation

Throughout this thesis we will use the standard *geometrized* units, for which $G = c = 1$, unless otherwise stated. I provide a brief overview of these units and their conversion factors, see Table (as derived from Wald [24].

Quantity	Dimension	Geometrized
Mass	M	L
Length	L	L
Time	T	L
Angular Momentum	ML^2T^{-1}	L^2
Energy	ML^2T^{-2}	L
Energy Density	$ML^{-1}T^{-2}$	L^{-2}
Force	MLT^{-2}	1
Mass Density	$ML^{-3}M$	L^{-2}
Pressure	$ML^{-1}T^{-2}$	L^{-2}
Quadrupole Moment	ML^2	L^3
Velocity	MLT^{-1}	1

In addition the following values are used:

$$\begin{aligned} c &= 2.9979 \cdot 10^{10}\ cm.s^{-1} \\ G &= 6.6730 \cdot 10^{-8}\ cm^3.g^{-1}.s^{-1} \\ M_\odot &= 1.989 \cdot 10^{33}\ g \\ \hbar &= 1.05 \cdot 10^{-27}\ erg.s \end{aligned}$$

Unfortunately there are also many other conventions of units used in literature. In particle theory it is most common to use *natural* units. In natural units, $\hbar = c = k = 1$, wich means that length, time and all other quantities are measured in the same units (such as Fermi = fm). The conversion factors are then:

$$\begin{aligned} c &= 3 \cdot 10^{23}\ \mathrm{fm} \\ \hbar c &= 197.32\ \mathrm{MeV} \cdot \mathrm{fm} \\ G &= 1.166 \cdot 10^{-11}\ \mathrm{MeV}^{-2} \end{aligned}$$

Thus length, time and reciprocal mass have the same units:

$$\begin{aligned} 1\mathrm{fm} &= (197\ \mathrm{MeV})^{-1} &\sim\ & 5\ \mathrm{GeV}^{-1} \\ 1s &= 3 \cdot 10^{23}\ \mathrm{fm} &=\ & 1.52 \cdot 10^{21}\ \mathrm{MeV}^{-1} \end{aligned}$$

Example (1)
The Compton wavelength of the electron is $\hbar/(mc)$. And so in Planck units its just m^{-1}. To convert back to standard units we just have to put in the right factors:

$$m^{-1} = \hbar c(mc^2)^{-1} = (197\ MeV.fm)\,(.511\ MeV)^{-1} = 386\ fm.$$

Example (2)
The lifetime of a muon is: $\tau_\mu = 192\pi^3(G^2 m_\mu^5)^{-1}$. Using the above conversion factor for G and the mass of the muon, we then have in standard units:

$$\begin{aligned}\tau_\mu &= 192\pi^3\ (1.166\cdot 10^{-11})^{-2}\ (105.66)^{-5}\ MeV^{-1}\\ &= 3.3\cdot 10^{15}\ MeV^{-1}.\\ &= 2.2\cdot 10^{-6}\ s.\end{aligned}$$

Finally we use the index notation that the lowercase greek and lowercase latin (a,b,c) indices take values $0\ldots 3$ while the uppercase latin indices denote spatial directions and range over $1\ldots 3$. (However, in chapter 3, the lowercase greek indices will be reserved for use in some chart.) Furthermore, for *tetrads* we use a subset of the lowercase latins, $(i, j, k$, etc.)

As for the metric signature, we mainly use the relativists standard signature of $S = +2 = (-,+++)$. However, some formulas are better stated in the $S = -2 = (+,---)$ convention used by particle physicists. In those cases, all attempts have been made to clearly indicate the convention used.

The following table may also be useful to decipher the thesis...

$*$		Complex conjugate
$\dagger$		Hermitian conjugate
Tr		Trace
∂_a,	$X_{,a}$	Partial derivative
∇_a,	$X_{;a}$	Covariant derivative
$\Box$	$:= g^{ab}\nabla_a\nabla_b$	D'Alambertian operator
$\bar{g}$	$:= \sqrt{-g_{ab}}$	In $(-+++)$ signature
$[A,B]$	$= AB - BA$	Commutator
$\{A,B\}$	$= AB + BA$	Anti-commutator

Chapter 1

Extreme Compact Objects

1.1 Introduction

There is an alternative viewpoint of very compact objects, in which there is a belief that there could exist ultra compact stellar objects in addition to blackholes with maximum masses different from the commonly accepted, $3M_\odot$. The details of some of these models are highly speculative, although the general consensus is that there may be some truths behind these ideas. Some common motivations for these studies are that: a) Serious use of QCD may yield stable ultracompact models due to various conserved quantities. b) Some blackhole candidates may actually be ultracompact objects with $2M < R < 3M$. c) The observed clustering around $1.4M_\odot$ may just reflect stellar evolution rather than limiting the existence of extreme stellar objects of masses $M > M_\odot$.

In this chapter I will present my studies from the first two years at the SISSA/ISAS PhD program. During those two years I tried to get a better understanding of the problems and solutions with regard to the ultracompact stellar objects. I attempt to clarify some of the ideas regarding compactness, conservation laws, quantum field theory (QFT), quantum chromodynamics (QCD), gravity, vacuum energy and stability. As an immediate result, we get into models of *strange stars, soliton stars, Q-stars* and *boson stars.*

First I will make some comments on compactness. Then go on to introduce the notation and definition of conservation laws, followed by a very brief review of the quantization of scalar fields and how vacuum effects may arise from the different zero point energies of the fields. Then another short review of non topological solitons and Q-balls, is followed by the section on boson stars and their importance and differences. As a consequence of the Witten hypothesis I will introduce the QCD phase diagram and then some phenomenology of color superconductivity, color flavor locking and strange stars. Finally there will be some comments on Q-balls in the minimally super symmetric theory.

In this chapter, I will use the $(-+++)$ signature with the standard geometrized units, as described in the frontmatter pages.

1.1.1 Compactness

As just mentioned, part of my PhD topic is ultracompact stars and their structure and existence. But what do we really mean by ultracompact? In our context we simply mean something which has an average density greater than that of a "classical" neutron star. Typically this kind neutron star has a mass of not more than 3 solar masses and a radius of about 10 Km. This means that its average density is about 10^{15} $g.cm^{-3}$.

But in GR it is well known that non-rotating and spherically symmetric stars, can not have a surface radius R smaller than 9/4 times its gravitational mass M (in geometrized units, where $G = c = 1$). Any such object would immediately collapse into a Schwarzschild blackhole with radius $R = 2M$.

Therefore we usually write these two quantities as one by defining the *compactness* parameter $\beta := M/R$. However the compactness is usually also given as its inverse, the *tenuity*; $\alpha := R/M$. It is believed that neutron stars have tenuities in the range $3 - 11$. Some authors then say that a star is ultracompact if it has a tenuity of 9/4 to 3. The lower limit is known as the *Buchdahl limit* which represents the maximum compactness for any static star for which the energy density is increasing inward. Beyond that the star is unstable and would collapse to form a Schwarzschild blackhole. However, recently it was realized that one could still have ultracompact stars with tenuities greater than 3, by considering equations of state (EOS) which posesses closed photon orbits in the stellar interior [64]. Other authors like to denote *ultracompact* as those objects which may posses closed photon orbits in their interior. In this respect I will use the notation of "extreme object" to denote any compact stellar object that could give rise to an "ultracompact" object. Therefore the extreme object class would include neutron stars, strange stars, boson stars and soliton stars. But what are limiting these objects to these numbers?

Essentially it is the equation of state (EOS). The EOS describes the relation of density and pressure as you go from the center to the surface of the star. The complication occurs when we realize that nucleons at extremely high pressures and energies needs to be described by quantum field theory (QFT), in particular quantum chromodynamics (QCD). (Why? Essentially because at these high densities the wavefunctions of the constitutional particles overlap so badly, so that there is no longer a simple way to express the interparticle forces.) This leads to quite complicated EOS's in most models but a few extremely simple and slightly useless toymodels.

However, there has been some recent progress in the QCD description of high density quark matter. It is seen that in most cases, this type of matter should posses both color superconductivity (CSC) and color flavor locking (CFL). In addition these two QCD characteristics are phenomenologically much simpler than that found in normal superconductivity (eg. Landau-sc and the 2 phase superfluidity of ^{3}He-A and ^{3}He-B.) In particular this may play an important role in the properties and existence of a solid stellar crust of extreme objects such as neutron stars and strange stars.

But before we even get to this point, we have to ensure that the bulk of the star is stable. In the stellar examples I will be discussing below, this stability is primarily due to some conservation law.

1.1.2 Conservation Laws

The stability of any stellar object depends strongly and primarily on two types of conservation laws. *Strongly* because one of them is required and *primarily* because there are many other reasons for in-

stabilities (gravitational, rotational, chemical, nuclear etc.) These conservation laws are the **Noether conservation**, which depends on the symmetry of the theory and the **topological conservation**, which depends on the boundary conditions at infinity. As will be shown, these in turn depends on the dimensionality of space and the vacuum structure of the fields involved. (These are of course those of the standard model particle physics.) Clearly the discussion is most clear in the cases where the stellar object is of the extreme type, such as strange stars, soliton stars, Q-stars and neutron stars. Another interesting point is that it has been shown that certain solitons (those of the non-linear Korteweg-de Vries (KdV) equation) can have an infinite number of conserved charges [65].

Noether Conservation

Noethers theorem [68] states that in a field theory, each continuous symmetry transformation (gauge choice) leads to a conservation law. The conserved quantity Q can be obtained from the Lagrangian density using the Euler-Lagrange equation (ELE) to define a current density j^μ and then performing a volume integration. I will here outline the steps to show how a symmetry gives rise to a conserved quantity.

If we assume a gauge invariant (globally symmetric) scalar theory, we have $\phi \to \phi\, \mathrm{e}^{-i\theta}$, where ϕ is a scalar field and θ is the phase. This means that if we multiply the original field by a phase factor the theory remains invariant and independent of that phase. So that given the Lagrangian of the theory we use the ELE to define a 4-current, j^μ. Without further complications such as the inclusion sources and sinks, this current is conserved with, $\partial_\mu j^\mu = 0$. Usually the j^0 component is identified with the particle number density, so that the space integral gives the number of particles. But because of the current conservation the particle number is conserved as well, which can be seen by integrating by parts. We then have to integrate a $3D$ surface term which is normally taken to vanish at the spatial boundary, which leaves the integral of the $\partial_0 j^0$ term to be identified with the particle number derivative, $\dot{N}$ The schematic derivation for identifying the conserved charge with conserved particle number goes like this.

$$\boxed{j^\mu := -i\bar{g} g^{\mu\nu}\left(\phi^\dagger\, \phi_{,\nu} - \phi^\dagger_{,\nu}\, \phi\right)} \qquad \text{(Define the current)}$$

$$\Downarrow$$

$$\boxed{\partial_\mu j^\mu(x) = 0} \qquad \text{(Continuity equation)}$$

$$\Downarrow$$

$$\boxed{Q := \int j^0\, \mathrm{d}^3x} \qquad \text{(Define the charge)}$$

$$\Downarrow$$

$$\boxed{\frac{\mathrm{d}Q}{\mathrm{d}t} = 0} \qquad \text{(Charge conservation)}$$

$$\Downarrow$$

$$\boxed{\int \partial_0 j^0\, \mathrm{d}^3x = \int \partial_\mu j^\mu\, \mathrm{d}^3x = 0} \qquad \text{(Because surface term at } \infty = 0\text{, Identify } Q \text{ with } N.\text{)}$$

$$\Downarrow$$

$$\boxed{\dot{N} = 0} \qquad \text{Particle number conservation}$$

Topological Conservation

Topological conservation is perhaps best visualized by a simple example. One such popular example is the $\lambda\phi^4$ scalar theory in $(1+1)$ dimensions.

Example (1):

The Lagrangian of a typical $\lambda\phi^4$ theory is given by:

$$\mathcal{L} = \tfrac{1}{2}\,\phi_{,\mu}\phi_{,}{}^{\mu} - V(\phi) \tag{1.1}$$

where:

$$V(\phi) := \tfrac{1}{2}\lambda\,(\phi^2 - a^2)^2 \qquad \textit{with:} \qquad a^2 = \frac{\alpha^2}{\lambda} \tag{1.2}$$

The solutions to the field equations of this theory are then given by:

$$\phi_{\pm}(x) = a\,\tanh(\alpha x)\,. \tag{1.3}$$

Therefore,

$$\boxed{\phi_{+} = \pm a \quad \textit{at} \quad x \to \pm\infty\,.} \tag{1.4}$$

But requiring finite energy:

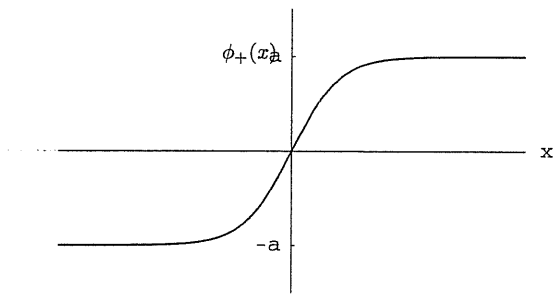

Figure 1.1: The *kink* (+) soliton solution. The *anti-kink* (-) solution would be its mirror image.

$$\phi(\infty) - \phi(-\infty) = 2an \qquad \begin{cases} n = 0 & \textit{vacuum} \\ n = \pm 1 & \textit{kink/anti-kink} \end{cases} \tag{1.5}$$

Then if we could define a current as $j_\mu := \epsilon_{\mu\nu}\partial^\nu\phi$, we can rewrite eq (1.5) as an integral:

$$\int_{-\infty}^{\infty} \partial_x\phi(x)\,dx = \int_{-\infty}^{\infty} j_0\,dx = 2an \tag{1.6}$$

But this is exactly the form of a topological charge, Q. Therefore we have identified the topological charge with: $Q = 2an$. But where is this topology? Well, consider the previous 2d theory with one spatial and one temporal dimension. Then spatial infinity consists of two discrete points:

$$S := \{x : -\infty, \infty\} \tag{1.7}$$

and the set of minima of $V(\phi)$ is located at $\phi = \pm a$.

$$M_0 = \{\phi : V(\phi) = 0\} \quad = \quad \{\phi : \pm a\} \tag{1.8}$$

Requiring that the energy is finite

$$\lim_{x\to\infty} \phi(x) = \phi \in M_0 \tag{1.9}$$

so that the mappings: $S \to M_0$ are:

$$\begin{aligned} n &= 0 & \pm\infty &\to a \\ n &= +1 & \pm\infty &\to \pm a \\ n &= -1 & \pm\infty &\to \mp a\,. \end{aligned}$$

And these mappings are topologically distinct!

□

What we just saw was a typical example of a **soliton solution.** So a soliton is a solution to the ELE's which in addition is characterized by the following:

⋆ Is spatially finite.

⋆ Has finite energy.

⋆ Stable. (It would require ∞ energy to convert the n's, since the barrier is ∞ in extent.)

Generally these soliton solutions can be divided into two different **homotopy classes** depending on their **topological charge**, Q.

Non-topological solitons (NTS) $Q = 0$
These objects depends on an additional conserved Noether charge, and their boundary conditions at ∞ are the same as the vacuum. They include NTS stars and Q-balls.

Topological solitons $Q \neq 0$
These needs some kind of degenerate vaccua. Their BC's at ∞ are different from the vacuum. They describe objects such as Vortices, Monopoles and Skyrmions.

But for very large non-topological charges (Noether) of lets say, $N > 10^{57}$, gravity could become important. Actually one should be careful when making this statement, as the charge Q can not always be identified with the particle number N. For example in EM it is the electric charge. Here I shall focus on Q-balls as they give rise to Q-stars when gravity is included. Whereas for the topological solitons, apart the Skyrmions, I shall leave these for the mathematical physicists. [See my short note on Skyrmions in the appendix.]

In any case I will first make a diversion and review how different scalar fields are quantized to get a better feeling of the important points. The most relevant cases of a real and a complex scalar field will be presented in the next two sections.

1.2 Scalar Field Quantization

The quantization procedure of a scalar field depends on whether the scalar field to be quantized is real or complex. It is easy to show that a real-valued Klein-Gordon field describes identical (neutral) spin-0 particles. Whereas the generalization with a complex-valued field introduces an additional *internal degree* of freedom which can be interpreted as a *doublet* of particles and antiparticles depending on their *charge*. We will discuss the quantization of each. The details of the derivations are best presented in the excellent books by Grenier [21, 22].

Another reason for including these rather standard derivations is for completeness, as they also show how the plane wave and spherical wave basis differ. In relativistic quantum mechanics it is easily shown that states constructed in the *Fock space* of *plane waves* do not have well defined angular momentum, leading to a non-diagonal angular momentum tensor. However with a *spherical wave* basis, angular momentum is well behaved. The question that arises is therefore if the NTS "wave" also gives good angular momentum. Or better, how would we define a NTS basis with good angular momentum?

1.2.1 The neutral KG field

We choose to call this field $\phi = \phi(x) = \phi(\mathbf{x}, t)$. Then if we want it to be neutral, it has to satisfy $(\phi = \phi^*, \hat{\phi} = \hat{\phi}^\dagger)$. Then if $\mathcal{L}$ is the Lagrange density for this field, we can apply the Euler-Lagrange equation, which immediately leads to the Klein-Gordon (KG) equation:

$$\mathcal{L} = \tfrac{1}{2}\left(\phi_{,\mu}\phi_{,}{}^{\mu} - m^2\,\phi^2\right) \tag{1.10}$$

$$\Box\phi + m^2\phi = 0 \qquad \textbf{KG equation} \tag{1.11}$$

where m is the mass of the spin-0 field and we are assuming a Minkowskian background with metric $\eta_{\mu\nu}$. We can then define the *canonical conjugate field* and the Hamiltonian density by:

$$\pi(x) := \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi} \tag{1.12}$$

$$\mathcal{H} := \pi\dot{\phi} - \mathcal{L}\,. \tag{1.13}$$

Next we apply the *equal time commutation relations* (ETCR):

$$[\phi(x), \pi(x')] = i\delta^3(x - x') \tag{1.14}$$

$$[\phi(x), \phi(x')] = [\pi(x), \pi(x')] = 0 \tag{1.15}$$

with continuous normalization. We choose a continuous normalization because we know that the phenomenological field needs to vanish asymptotically, but with a finite value at the boundary where $x = R$. [1] The introduction of the creation and annihilation operators, $\hat{a}_{\mathbf{p}}$ and $\hat{a}^\dagger_{\mathbf{p}}$ and their commutation relations

$$[a_p, a^\dagger_{p'}] = \delta^3(p - p') \tag{1.16}$$

$$[a_p, a_{p'}] = \left[a^\dagger_p, a^\dagger_{p'}\right] = 0 \tag{1.17}$$

then gives the Hamiltonian operator

$$\hat{H} = \tfrac{1}{2}\int \omega_{\mathbf{p}}\,(a^\dagger a + aa^\dagger)\mathrm{d}^3p \tag{1.18}$$

where $\omega^2_{\mathbf{p}} = \mathbf{p}^2 + m^2$. The last term is equivalent to the vacuum energy and thus when integrated gives infinity. The trick to get rid of this term is by *normal ordering*. First split the operator into its positive and negative frequency parts: $\phi = \phi^+ + \phi^-$, then apply normal ordering by moving all negative frequency parts to the left. (The "positive" part is the one with the term $\mathrm{e}^{-i\omega t}$ with $\omega > 0$.)

$$:\phi\chi: = \phi^-\chi^- + \phi^-\chi^+ + \chi^-\phi^+ + \phi^+\chi^+ \tag{1.19}$$

Then the normal ordered Hamiltonian density is

$$H = \int \omega_{\mathbf{p}}\, a^\dagger a\, d^3p \tag{1.20}$$

Then the vacuum energy can be written as $E_0 = H - \hat{H}$. One could also have energy differences between different vacuum configurations such that

$$\Delta E = E_0[1] - E_0[2] \tag{1.21}$$

[1] Box-type field normalization does not generally give good (simple) behaviour of the fields for spherical symmetries and vice versa.

where 1 and 2 refer to different boundary conditions of ϕ. In some cases where $\Delta E \geq 0$, we can have observable effects such as the Casimir effect [84, 85]. In fact eq.(1.21) is usually referred to as the "Casimir Subtraction".

This derivation is usually done by using field operators $\hat{\phi}$ expanded in terms of plane waves. But then the field modes are classified according to their longitudinal momentum ($\mathbf{p}$), which does not give a well defined angular momentum operator. However, if one instead expands $\hat{\phi}$ in terms of spherical waves, one does get well behaved angular momentum operators. I purposely left out any reference to the expanded forms of the field operators, for exactly this reason. After some algebra we have the plane-wave orbital angular momentum operator:

$$\mathbf{L} = i \int \mathrm{d}^3p \; a^\dagger_{\mathbf{p}} \left(\mathbf{p} \times \nabla_{\mathbf{p}}\right) a_{\mathbf{p}} \tag{1.22}$$

However, the presence of the operator $\nabla_{\mathbf{p}}$ shows that the angular momentum operator in the plane-wave representation is non-diagonal. The solution is to expand the field operator into spherical waves instead of plane-waves. Then the field modes are classified by their angular momentum quantum numbers; l and m. We proceed along the lines of separation of variables by writing the spherical wave function as:

$$\phi_{plm}(x) = N_p \, u_{pl}(r) \, Y_{lm}(\Omega) \, . \tag{1.23}$$

Here $N_p = (2\omega_p)^{-1/2}$ is a normalization factor and Y_{lm} are the spherical harmonics, with l, m being the angular momentum and azimuthal quantum numbers, respectively. p is the absolute value of the radial momentum. The basis functions $u_{pl}(r)$ has to satisfy the radial part of the KG equation. The real valued solution that is also regular at the origin is given in terms of a spherical Bessel function $j_l(pr)$:

$$u_{pl}(r) = (2/\pi)^{1/2} \, p \, j_l(pr) \tag{1.24}$$

The remaining useful expansions are (with the operator notation left out):

$$\ddot{\phi} = \sum_{plm} N_p \left(-p^2 - m^2\right) u_{pl}(r) \, Y_{lm}(\Omega) \, a_{plm}(t) \tag{1.25}$$

$$a_{plm} = i^l p \int \mathrm{d}\Omega_p \, Y^*_{lm}(\Omega_p) \, a_{\mathbf{p}} \tag{1.26}$$

$$\hat{H} = \tfrac{1}{2} \int \mathrm{d}^3x : \left(\pi^2 - \phi\ddot{\phi}\right) : \, . \tag{1.27}$$

Where we also used the Rayleigh expansion of planewaves:

$$\mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}} = 4\pi \sum_{lm} i^l \, j_l(pr) \, Y^*_{lm}(\Omega_p) \, Y_{lm}(\Omega_r) \, . \tag{1.28}$$

At the end of the day we obtain the normal ordered Hamiltonian and angular momentum operators:

$$\hat{H} = \int_0^\infty \mathrm{d}p \sum_{lm} \omega_{\mathbf{p}} \, \hat{a}^\dagger_{plm} \, \hat{a}_{plm} \tag{1.29}$$

$$\hat{L}_3 = \int_0^\infty \mathrm{d}p \sum_{lm} m \, \hat{a}^\dagger_{plm} \, \hat{a}_{plm} \tag{1.30}$$

1.2.2 The charged KG field

It is easy to generalize the previous derivation to particles having an internal degree of freedom, the charge. The most simple of these theories can be described by using complex fields and therefore

non-hermitian field operators,such that $\phi \neq \phi^*$ and $\hat{\phi} \neq \hat{\phi}^\dagger$, respectively. Then the Lagrangian reads:

$$\mathcal{L} = \phi^*_{,\mu}\phi_{,}{}^{\mu} - m^2\,\phi^*\phi \tag{1.31}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^* \tag{1.32}$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi} \tag{1.33}$$

And the Hamiltonian density:

$$\mathcal{H} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L} \tag{1.34}$$

$$= \pi^*\pi + \nabla\phi^*\,\nabla\phi + m^2\,\phi^*\phi \tag{1.35}$$

The quantization is then achieved by going over to field operators, which are required to satisfy the ETCR. The Fourier decomposition of the field operators ϕ and $\phi^\dagger$ now contains terms with the additional coefficients $b_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$. Since these are no longer hermitian, there are two independent sets of creation and annihilation operators, with similar commutation relations. Again after some work and after introducing the number operator; $n = a^\dagger a$, we find the normal ordered Hamiltonian:

$$\hat{\mathcal{H}} = \int \mathrm{d}^3p\,\omega_{\mathbf{p}}\,[n^{(a)}_{\mathbf{p}} + n^{(b)}_{\mathbf{p}}] \tag{1.36}$$

Obviously this theory describes two independent particles a and b having the same mass. In addition, closer examination of the Lagrangian uncovers a connection between the two particles. It exhibits a symmetry under phase transformations, where $(\phi \to \phi e^{i\alpha}, \phi^* \to \phi^* e^{-i\alpha})$. Noether's theorem then tells us that this symmetry leads to a conserved quantity, the charge.

$$Q = -i\int \mathrm{d}^3x\,(\pi\phi - \phi^*\pi^*) \tag{1.37}$$

But since $\pi = \dot{\pi}^*$ and $\pi^* = \dot{\phi}$ we can write Q in the compact notation of the scalar product (ϕ, ϕ):

$$Q = (\phi, \phi) := i\int \mathrm{d}^3x\;\phi^*\overleftrightarrow{\partial_0}\phi \tag{1.38}$$

In the quantized theory this becomes:

$$\hat{Q} = \int \mathrm{d}^3p\,\omega_{\mathbf{p}}\,[n^{(a)}_{\mathbf{p}} - n^{(b)}_{\mathbf{p}}]\,. \tag{1.39}$$

For any type of conserved charge, it has to satisfy: $\dot{Q} = -i\,[Q, H] = 0$, where again Q and H are operators. Thus the charge remains a conserved quantity also in the quantized theory. It is then understood that this theory describes particles and anti-particles. There are also other types of charges that can arise from similar internal symmetries.

1.2.3 The KG-Schrödinger connction

Although they look the same, there are some fundamental and important differences between the Klein-Gordon and the Schrödinger fields. For one thing, the field variables are functionally different. In the Schrödinger equation it is a simple field (ψ), whereas in the KG equation they are field operators $(\hat{\psi})$. These does not necessarily satisfy the same differential equations. Another important difference is that

unlike the schrödinger functions, the relativistic KG functions have both positive and negative frequency contributions, giving rise to antiparticles. This difference is primarily due to the inclusion of Lorentz invariance in the KG fields. As we shall see later, it is important to understand this connection, since some types of boson stars can be thought of as enormous atoms, with its boson field satisfying a single Schrödinger wavefunction.

1.2.4 The gravitationally coupled scalar field

At some point in our calculations we need to recognize that these fields will eventually live in a gravitational background. However, this is not only a passive background, but is very much connected to the fields themselves. In general realativity matter and fields are nonlinearly coupled to the curvature. This is almost expected since GR is a geometric theory, where spacetime is completely represented by a 4D geometry. To couple the scalar fields we have to agree on some coupling formula. That is, to which type of scalar quantity representing the curvature. There are several types of these ($R^{abcd}R_{abcd}, R^{ab}R_{ab}$) but the general consensus is that it should be sufficient with the Ricci scalar R, unless one is interested in higher derivative metric theories.

We introduce a real scalar field with curvature coupling and a self interaction potential modeled by the following lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\left[\phi_{;\mu}\phi_{;}{}^{\mu} - \xi R\phi^2\right] - V(\phi) \tag{1.40}$$

where we define the potential as: $V(\phi) := \frac{1}{2}m^2\phi^2 + U(\phi)$. This allows us to apply previous results without a potential by identifying the mass terms with $V(\phi)$. The *Klein-Gordon* equation is then obtained by the Euler-Lagrange equations by variation with respect to the field.

$$\Box\phi + \xi R\phi + V'(\phi) = 0 \tag{1.41}$$

The prime denotes partial differentiation with respect to the argument. In a similar manner the stress-energy tensor (SET) is calculated as usual from the variation of the action with respect to the metric. One then obtains [15]

$$\begin{aligned} T_{ab} \;=\; & (1-2\xi)\phi_{;\mu}\phi_{;\nu} + (2\xi - \tfrac{1}{2})g_{ab}\, g^{\rho\sigma}\, \phi_{;\rho}\phi_{;\sigma} \\ & +(1-3\xi)\, g_{ab}\left[\tfrac{1}{2}m^2\phi^2 + U(\phi)\right] \\ & -2\xi\phi\phi_{;\mu\nu} + \tfrac{1}{2}\xi g_{ab}\,\phi\Box\phi \\ & -\xi\phi^2 G_{\mu\nu} - \tfrac{3}{2}g_{ab}\xi^2\phi^2 R \end{aligned}$$

To further simplify this expression we may try using the Klein-Gordon equation. But the only way to simplify is by removing either the $\phi\Box\phi$ term or the $\xi^2\phi^2 R$ term at the expense of introducing an additional $U'(\phi)$ term. Thus the only improvement may be at most computational by calculating a simple derivative of $U(\phi)$ rather than R. In fact when $V(\phi) = 0$ with conformal coupling $\xi = \frac{1}{6}$, the Klein-Gordon equation is redundant anyway due to the contracted Bianchi identities.

But we have to remember that this real scalar field cannot be used to model static boson stars, since there is no global symmetry to provide a conserved charge. Regular boundary conditions can only be satisfied by time dependent solutions, corresponding to expanding/contracting or oscillating boson stars.

1.3 Vacuum effects of geometry & surface tension

From a classical point of view it seems obvious that one has to include the energy due to surface tension when calculating the total energy of a body. In particular when two phases are separated by a boundary area. Consider for example a water drop whose mere existence is strongly dependent of its surface tension. When the surface tension vanishes (at the triple point) the drop vaporizes. As we shall see, this importance also remains at the quantum level where different phases gives rise to different definitions of the quantum vacuum.

The notion of surface tension appears in most aspects of physics, showing its generality. The definition is usually in terms of an energy which depends on an intrinsic variable differing across the boundary. Thus the actual formula may vary widely between the different fields of physics. However, the most intuitive interpretation of surface tension is probably that of a two-dimensional analogue of pressure, the pressure parallel to the interface. The units are then; *energy per unit area* or *force per unit length*, giving: $\gamma = [E/L^2] = [F/L]$. But before we go on with GR, we shall give some common examples in physics.

Classical Thermodynamics [14]

Here the surface tension is defined in terms of a change in the *Helmholtz free energy* (F) with respect to a small change in the area (A) of the boundary. This process is reversible at constant temperature (T), volume (V) and particle number (η_i).

$$\gamma := \left(\frac{\partial F}{\partial A}\right)_{T,V,\eta_i} \tag{1.42}$$

(One could also write the Helmholtz energy in terms of an *order parameter*, η which could then be used to mimic the scalar potential in soliton stars, as will be discussed later.) From the dependence of the area one may foresee a dependence of the curvature of the interface. In fact a shell has only half the surface tension as that of a ball because of the second inner surface, leading to the statement: *more surface, less tension.*

$$\begin{array}{rcll} J & := & \frac{1}{r_1} + \frac{1}{r_2} = 2/r & \text{(“mean curvature” of a sphereoid)} \\ \Delta p & = & p_1 - p_2 = \gamma J & \text{(Hydrostatic pressure difference)} \end{array} \tag{1.43}$$

Statistical Mechanics [14, pp.67]

Without going into detail we simply state the results in 3D given in terms of a distribution function and an interaction term.

$$\gamma = \frac{\pi}{8}\,\rho^2 \int_0^\infty g(r)\,\epsilon'(r)\,r^4 dr \tag{1.44}$$

$$E_s = \frac{-\pi}{2}\,\rho^2 \int_0^\infty g(r)\,\epsilon(r)\,r^3 dr \tag{1.45}$$

where E_s is the total surface energy and

$$\begin{array}{rcl} \rho & = & \text{number density, } [\langle N\rangle / V], \\ g(r) & = & \text{radial probability distribution,} \\ \epsilon(r) & = & \text{potential of interaction, with } \epsilon'(r) = d\epsilon/dr\,. \end{array}$$

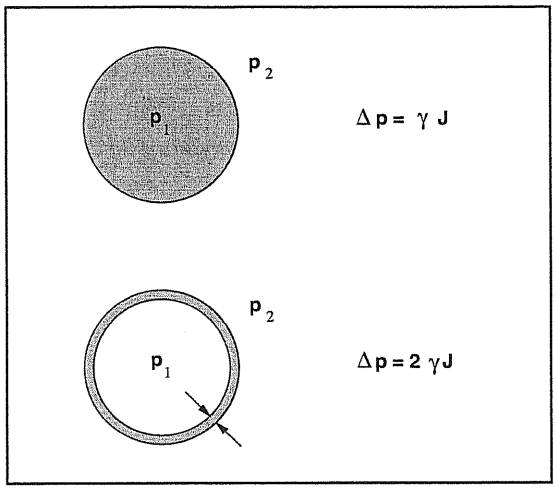

Figure 1.2: Difference in surface tension of a ball and a shell.

Quantum mechanics [2, 12, 13]

The vacuum energy can be calculated in various ways depending on the properties of the introduced medium. We give some examples like conductors and dielectrics. The vacuum energy is related to the geometry by the following expansion

$$E_0 = \rho V + \gamma S + \gamma^C \int (\kappa_1 + \kappa_2) dS + \gamma_1^C \int (\kappa_1 - \kappa_2)^2 dS + \gamma_2^C \int \kappa_1 \kappa_2 dS + \dots \tag{1.46}$$

where κ_1 and κ_2 are the principal curvatures of the surface and the various γ's are shape tensions. These coefficients can be identified with the parameters of the *bag model*, which is a particular model for the nucleon, where the nucleons are thought to be contained in a "bag" whose surface tension would then correspond to the binding energy or alternatively to the difference in the vacuum energy inside versus outside the bag.

The Bag Model

In the bag model, the analog of a normal ground state is called the *perturbative* vacuum. The vacuum expectation value (VEV) of the quark bilinear $q\bar{q}$ ($q = u, d, s$) plays the role of an order parameter by distinguishing between the two vacua,

$$\begin{aligned} {}_{QCD}\langle 0|q\bar{q}|0\rangle_{QCD} &< 0 \\ {}_{\text{pert}}\langle 0|q\bar{q}|0\rangle_{\text{pert}} &= 0\,. \end{aligned}$$

Hadrons are then represented as color singlet bags of perturbative vacuum occupied by quarks and gluons. The starting point of this model is the Lagrangian density:

$$\mathcal{L}_{\text{bag}} = (\mathcal{L}_{QCD} - B)\,\Theta(q\bar{q}) \tag{1.47}$$

where $\Theta(q\bar{q} < 0) = 0$ defines the spatial volume encompassed by the perturbative vacuum and B is the *bag constant* generally believed to be ~ 65 $MeV.fm^{-3}$.

Quantum Field Theory in a gravitational field

Eventually we will be interested to what happens to a ball of *quarkmatter*. The first thing to notice is that the surface is not a normal (at the large scale discontinuous) phase boundary, but rather a

continuous profile with a sharp transition between the two phases (fig.1.3). In fact the proper name for this kind of phase boundary is "interface". We shall adopt the scalar field σ to describe these phases as a type of order parameter describing the phase transition. One could then write the *Helmholtz free energy* as: $f = -\frac{1}{2}\epsilon\sigma^2 + \frac{1}{3}b\sigma^3 + \frac{1}{4}c\sigma^4 + \ldots$ This looks very similar to the scalar potential, $U(\sigma)$, used in soliton bag models, in particular the Friedberg-Lee model for the hadrons in [9, pp.298], where they use: $U(\sigma) = \frac{1}{2!}a\sigma^2 + \frac{1}{3!}b\sigma^3 + \frac{1}{4!}c\sigma^4 + p$ to describe a perturbative vacuum.

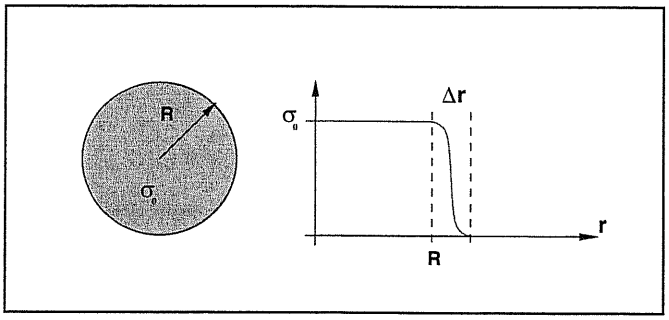

Figure 1.3: The scalar field of a ball of radius R.

However, the most important point is that the mere presence of fields in a particular region of space will change the notion of the quantum vacuum state. In the language of QFT, one would say that the field would curve spacetime and this curvature destroys the uniqueness of the solutions of the wave equations. In the extreme case of changing gravitational field, this may cause gravitational particle creation out of the vacuum [6]. In the case of presence of an EM field, classically speaking, the introduced EM fields may be thought of as a dielectric medium embedded in a vacuum. This induces a change in the dispersion relation and in the density of states, resulting in a change in the zero-point energy of the EM field. This change in the total zero-point energy is called the *Casimir* energy [12]. A similar phenomenon occurs for Fermion fields where a change in the vacuum structure induces a *shift in the infinite Dirac sea.* (See Fig. [1.5]) However, this kind of language is far from clear and should be improved by specific examples.

It is clear from all this that the surface tension does not tell the whole story. Rather one should calculate the Casimir energy in addition to the kinetic and potential parts, remembering that the Casimir energy includes terms from both the bulk and higher order curvatures in addition to the surface term, like in eq.(1.46) from [3]. Depending on the size of the object each of these terms can be more or less important.

1.3.1 The Casimir energy

This discussion of the Casimir effect will follow very closely that of the PhD thesis of Liberati [82]. As the thesis concerns various aspects of vacuum effects like that of Casimir, I have chosen to summarize one of these relevant parts.

In 1948 the Dutch physicist H.B.G Casimir correctly predicted that two parallel conducting plates in a vacuum environment should attract each other by a very weak force that varies as the inverse fourth power of their separation [84],

$$F = -\frac{\pi^2}{240}\frac{\hbar c}{a^4} \tag{1.48}$$

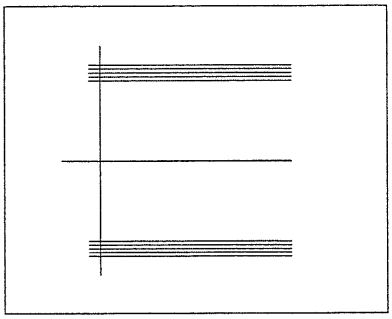

Figure 1.4: The normal fermion energy levels of the *Dirac sea.*

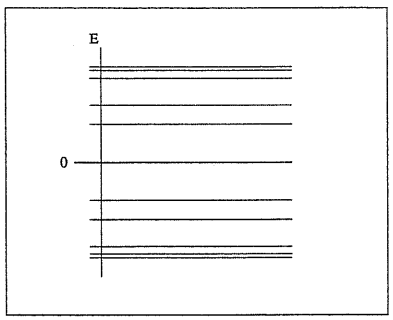

Figure 1.5: The shift in energy levels due to the Casimir effect.

For plates with an area of 1 cm^2 and a separation of about half a micron the force was $\approx 0.2 \cdot 10^{-5}$ N in agreement with the theory.

What is understood today, is that the Casimir effect is just a phenomena leading to vacuum polarization via some special boundary conditions that makes the quantization manifold different from the Minkowskian one. Generally speaking, this is divided into two classes. One where the effects are due to physical boundaries, the *Casimir effects* and the other in which the effects are due to nontrivial topology of the spacetime, the *topological Casimir effects.*

In our context we are mainly interested in the first case, where most phenomenology is taking place in a nearly Minkowskian space with some boundary structure. The only things that change here are the field and the geometry of the boundaries (parallel plates and spheres etc.) For this case I will present an example calculation.

In the other class, the quantization is performed in a flat or curved spacetime endowed with a nontrivial topology (anything different from the $R \times R^3$ Minkowskian manifold). The topology is usually reflected in periodic or anti-periodic boundary conditions of the quantized fields. For other examples of this, see [82] and references therein. There it is also shown that the Casimir energy is really due to a redistribution of the field modes, and not due to boundary conditions reducing the number of allowed modes and hence leading to an energy shift. We now show a simplified calculation of the Casimir effect using a massless scalar field and working in 1+1 dimensions.

Example (2):

The massless Klein-Gordon equation takes the form

$$\Box\phi(t,x) = 0 \tag{1.49}$$

We assume that the internal product and canonical commutation relations are given as in the standard quantization procedure previously discussed in section 1.2. Generally the solutions are plane waves:

$$\phi_k^{(\pm)}(t,x) \quad = \quad \frac{1}{\sqrt{4\pi\omega}} e^{\pm i(\omega t - kx)} \qquad \textit{where:} \quad \omega^2 = k^2 \quad (-\infty < k < +\infty)$$

We can now consider the one dimensional analogue of the Casimir effect, that is a boundary in the x direction that imposes Dirichlet boundary conditions on the field.

$$\phi(t,0) = \phi(t,a) = 0 \tag{1.50}$$

In this case we have again that there is a discrete set of allowed wavenumbers and that the wavefunctions are of the form

$$\phi_k^{(\pm)}(t,x) \quad = \quad \frac{1}{\sqrt{4\pi\omega}} e^{\pm i(\omega_n t)} \sin(k_n x) \qquad \textit{where:} \quad \omega_n^2 = k_n^2, \quad k_n = \pi n/a \quad (n = 1, 2, \ldots)$$

From the above forms of the normal modes it is clear that the creation and annihilation operators will be different in the two configurations (free and bounded space) and hence the vacuum states, $|0\rangle_{\text{Mink}}$ *and* $|0\rangle$*, will differ as well.*

Using the fact that the Hamiltonian of the field has the form (1.29) we can find the expectation value of the field energy in the vacuum state for the two configurations

$$\begin{aligned} E_{\text{free}} = \quad & \langle 0|\mathcal{H}|0\rangle_{mink} &= \tfrac{1}{2}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}|k|L \\ E_{\text{bound}} = \quad & \langle 0|\mathcal{H}|0\rangle &= \tfrac{1}{2}\sum_{n=1}^{\infty}\frac{\pi n}{a} \end{aligned}$$

where in the first line we introduced a normalization length $L \to \infty$*. (This comes from the box boundary condition that* $\phi(L/2) = 0$ *so that* $kL = 2\pi n$*.) Both of the above quantities are divergent so, in order to compute them and correctly execute the "Casimir subtraction" of eq.(1.21) we should adopt some regularization scheme. This can be done using an exponential cutoff of the kind* $e^{-\alpha\omega}$ *and by looking at the Minkowskian case at the energy in a region of length* a*. To compute this last quantity we shall look at the energy density* E_{free}/L *times* a*. The results are then*

$$E^a_{\text{free}} = \frac{E_{\text{free}}\cdot a}{L} = \tfrac{1}{2}\int_{-\infty}^{\infty} dk|k|a\,\exp\left(-\alpha|k|\right) = \frac{a}{2\pi\alpha^2} \tag{1.51}$$

$$E_{\text{bound}} = \tfrac{1}{2}\sum_{n=1}^{\infty}\frac{\pi n}{a}\exp\left(-\alpha\pi n/a\right) = \frac{a}{2\pi\alpha^2} - \frac{\pi}{24a} + O(\alpha^2) \tag{1.52}$$

So we see that in the limit $\alpha \to \infty$ *the subtraction* $E_{\text{bound}} - E^a_{\text{free}}$ *gives a finite quantity* $E_{\text{Cas}} = -\pi/(24a)$ *to which corresponds to an attractive force*

$$F_{\text{Cas}} = -\frac{\partial E_{\text{Cas}}}{\partial a} = -\frac{\pi}{24a^2} \tag{1.53}$$

□

Thus the Casimir energy can be thought of as the change in the energy of the vacuum of a (quantized) field in the presence of an external field. The external field plays the role of the parameters specifying the boundary conditions imposed on the quantized field. The full parallel plate Casimir calculation, performed in three spatial dimensions and with the electromagnetic field yields:

$$F_{\text{Cas}} = \frac{\pi hc}{480a^4} \tag{1.54}$$

The calculation just sketched has been performed for different geometrical configurations and notably it turns out that the value and even the sign of the Casimir energy is a non trivial function of the chosen geometry. In particular it is interesting to note that the Casimir energy in a box cavity is negative whereas that in a spherical cavity is positive. In addition, there are a few more important points as stated by Liberati.

1. The Casimir effect could be erroneously interpreted as a manifestation of van der Waals forces of molecular attraction. However the force (1.48) has the property of being independent of the details of the material of the conducting plates. The independence from the microscopic structure of the plates proves that the effect is just a byproduct of the nature of the quantum vacuum and of the global structure of the manifold.

2. The fact that for different geometries also positive energy densities can be obtained implies that the heuristic interpretation, that the presence of the boundaries "takes away" some modes and hence leads to a decrement of the total energy density with respect to the unbound vacuum, is

wrong. What actually happens is that the number of modes allowed between the (ideal) plates is still infinite, what changes is the distribution of the vacuum field modes. What we see as the appearance of a repulsive or attractive vacuum pressure is an effect of this (geometry dependent) redistribution of modes.

3. A common terminology used for describing the shift in the vacuum energy appearing in static Casimir effects is that of "vacuum polarization". The vacuum is described as a sort of dielectric material in which at small distances virtual particle-antiparticle pairs are present, analogously to the bounded charges in dielectrics. The presence of boundaries or external forces "polarizes" the vacuum by distorting the virtual particle-antiparticle pairs. If the force is strong enough it can eventually break the pairs and "the bound charges of the vacuum dielectric" become free.

   Although this has turned out to be a useful concept, at the same time it should be stressed that it is strictly speaking incorrect. The vacuum state is an eigenstate of the number operator which is a global operator (because it is defined by the integral of the particle density over the whole 3-space). On the other hand, the field operator, the current density and the other typical operators describing the presence of particles are local, so there is a sort of complementarity between the observation of a (global) vacuum state and (local) particle-antiparticle pairs.

1.4 Non-Topological Solitons and Q-balls

At first sight these two objects may seem equivalent, but strictly speaking this is not true. Both objects require an unbroken continuous symmetry which leads to a conserved charge Q in order to remain stable. In the Q-ball case one can identify the conserved charge with the particle number N. However the NTS case depends on a second spontaneously broken symmetry, and this time Q can be identified with various other quantities depending on the symmetries involved. Lets then discuss these in more detail.

According to Coleman [38], a Q-ball is an NTS with some conserved (global) charge in the scalar field theory, where the conserved charge is now to be identified with the particle number. A necessary condition for the existence for a Q-ball is to first have a NTS where we can make this identification. There is also a more recent existence criterion by Kusenko [83] who states that a Q-ball solution exists if there is an energy minimum at a non-zero field value ϕ_0 with a fixed charge Q. Or as he puts it, given a potential $V(\phi)$, it has to satisfy:

$$V(\phi)/\phi^2 = Min, \qquad \text{for} \qquad \phi = \phi_0 \,. \tag{1.55}$$

But this is less clear, as the potential may involve more than one scalar field. In any case, since the difference between a NTS and a Q-ball is very slight, it should be enough to discuss the NTS case in detail, and then compare the two in the discussion of stars.

As briefly mentioned before, NTS's are the basis of many of the hypothetical extreme stellar models. Some of these are the strange stars, boson stars, mini-boson stars and the fermion soliton stars. Where in this context we mean that *something* "star" = fields + gravity. But there is an important point here. Unlike other stellar models, soliton stars can exist without gravity. For educational purposes we will take another look at our previous example in Section (1.1.2). By changeing the meaning of the variables,

we can reinterpret the problem as equivalent to a particle moving in a well, or equivalently tunneling through a potential. We use:

$$\mathcal{L} = \tfrac{1}{2}\,\phi_{,\mu}\phi_{,}{}^{\mu} - V(\phi) \tag{1.56}$$

$$\mathcal{H} = \tfrac{1}{2}\,\dot{\phi}^2 + \tfrac{1}{2}\,(\phi')^2 + V(\phi) \tag{1.57}$$

$$V(\phi) := \tfrac{1}{2}\lambda\,(\phi^2 - a^2)^2 \tag{1.58}$$

Then $\delta\mathcal{L} = 0$ so that the action integral can be written as

$$\delta \int [\tfrac{1}{2}\,(\phi')^2 + V(\phi)]\,\mathrm{d}x = 0 \tag{1.59}$$

Changing the variable with the transformations $x \to t$ and $\phi \to x$, we have the action integral equivalent to that of a particle moving in a potential: $-V(x)$.

$$\delta \int [\tfrac{1}{2}\,\dot{x}^2 + V(x)]\,\mathrm{d}t = 0 \tag{1.60}$$

The two interpretations can be summarized:

$$\text{NTS:} \qquad \phi \to \{\phi : V(\phi) = 0\} \quad \text{as} \quad (x \to \pm\infty) \tag{1.61}$$

$$\text{particle:} \qquad x \to \{x : V(x) = 0\} \quad \text{as} \quad (t \to \pm\infty) \tag{1.62}$$

with their corresponding energy conservation:

$$\text{NTS:} \qquad \tfrac{1}{2}\,(\phi')^2 - V(\phi) = 0 \tag{1.63}$$

$$\text{particle:} \qquad \tfrac{1}{2}\,\dot{x}^2 - V(x) = 0 \tag{1.64}$$

Then the motion has stationary points at $\pm a$ and the particle rolls (tunnels) between these two points which are the degenerate vacua's as shown in the figure.

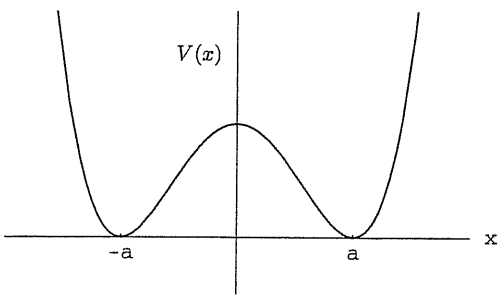

Figure 1.6: The particle analogy to the NTS, showing the stable minima corresponding to degenerate vacua's wherein the particle may tunnel.

1.4.1 Plane Waves and NTS Stability

The functional time dependence for plane waves (PW) and non-topological solitons (NTS) are given by:

$$\begin{array}{llllll} \text{PW:} & \phi \sim A\mathrm{e}^{i(kx-\omega t)} & \text{with:} & A = A[\Omega^{-1/2}] & \text{and} & (\omega^2 > m^2) \\ \text{NTS:} & \phi \sim B\mathrm{e}^{-i\omega t} & \text{with:} & B = B[\Omega] & \text{and} & (\omega^2 < m^2) \end{array}$$

Thus as the volume enclosing the system become infinite ($\Omega \to \infty$), the amplitude become infinitesimal in the PW and very large in the NTS case. Therefore it is not true that *weak coupling* implies *weak amplitude*. Furthermore in this sense, an NTS can be interpreted as the analytical continuation of a PW.

The nice feature of NTSs is that they get more stable the bigger they get [9]. This can be easily seen from the binding energy versus particle number plots in 2 and 3 spatial dimensions respectively. To clarify the main features of the schematic plots, they have actually been redrawn by hand to emphasize the cusp behavior and the generality of the analysis. In $2D$ the E(N) curve for a NTS falls entirely below that of a planewave which is just the straight line, $E = Nm$. Therefore it remains stable for all N and thus any size. However, we live in 3 spatial dimensions and here things gets slightly more complicated. This time NTS curve has a cusp in it at N_c which lies above the PW solutions. The upper part of the curve then follows the PW curve asymptotically, whereas the lower part crosses at N_s and goes off into the stable region. These different regions are shown in the two plots below, and can be summarized as:

$N > N_s$	:	Absolutely Stable
$N_c < N < N_s$	:	QM: Meta Stable
		Class: Stable
Upper Branch	:	Always Unstable!

Thus, in the upper branch, $N_c < N < N_s$ the PW's are the lowest energy solutions, whereas in the $N > N_s$ lower branch, the NTS is the lower energy solution.

The stability criteria are found from a combination of energy minimization and (in the 3D case) some lengthy calculations involving the solution of an eigenvalue problem, where Ω_N are the roots of a quadratic equation, which are then shown to be real. The authors then conclude that the lower branch is classically stable. Generally, the appearance of a cusp indicates the onset of a new mode of instability. Actually this is most easily seen in a *bifurcation diagram* also known as a *Whitney surface*. But unfortunately I do not have a picture of one...

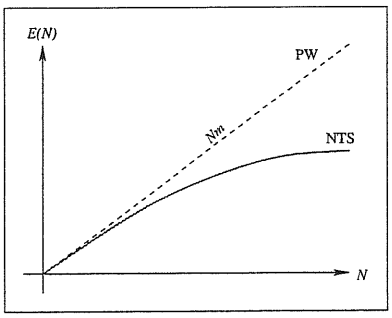

Figure 1.7: In the $2d$ case all NTSs are absolutely stable for all N.

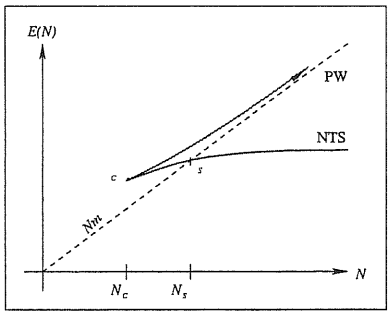

Figure 1.8: In $3d$ the NTSs are only stable where $N > N_s$ and quantum mechanically meta-stable in $N_c < N < N_s$.

Comment: In order to have a soliton solution in two or more dimensions, one must either: include gauge fields of nonzero spin or consider time-dependent but non-dispersive solutions. Hence with scalar fields alone, there are no topologically stable finite-energy solutions in 4 dimensions. This is usually referred to as *Derricks Theorem* [87].

1.5 Generalized Boson Stars

We attempt to derive the field equations and their numerical solutions for a complex massive scalar field with arbitrary curvature coupling and an arbitrary self interaction. However, for simplicity we start with a real Hermitian scalar field ($\phi = \phi^*, \phi = \phi^\dagger$) for purposes of introducing the notation. We then expand these ideas to include a $U(1)$ symmetry in order to account for a conserved charge by generalizing to the complex non-Hermitian field in the next section.

1.5.1 A not so perfect fluid

What is a perfect fluid? One way to describe a perfect fluid is that of Weinberg [23] who define it as a fluid which has at each point a velocity v, such that each observer moving along at v, sees the fluid as *isotropic*, whereas on the molecular level, a fluid is considered perfect if the mean free path (λ_{mfp}) between collisions is much smaller than the length scales used by the observer. Thus a sound wave may propagate if its $\lambda \gg \lambda_{mfp}$. In a compact stellar object (CSO) we can play with the mean free path by giving the constituent particles more or less momentum. However this is not a simple task since different species has different speeds at a given momentum.

As was realized by Kaup [79], the stress energy tensor of a boson star is in general *anisotropic*. This is in contrast to neutron stars, where the ideal fluid approximation demands an isotropic symmetry for the pressure, whereas for spherically symmetric boson stars there are different stresses in the radial p_r and tangential $p_\perp$ directions, respectively. The non-rotating boson star (BS) stress energy tensor (SET) is:

$$T_\mu{}^\nu(\phi) = \mathrm{diag}[\rho,\, -p_r,\, -p_\perp,\, -p_\perp] \tag{1.65}$$

with

$$\begin{aligned} \rho &= \tfrac{1}{2}(P^2\,\omega^2\,\mathrm{e}^{-\nu} + P'^2\mathrm{e}^{-\lambda} + U) \\ p_r &= \rho - U \\ p_\perp &= p_r - P'^2\,\mathrm{e}^{-\lambda} \end{aligned}$$

These differences are due to the fact that when we include gravity, the radial functions $P(r)$ describing the boson star are perturbed by the gravitational curvature, which gives different signs of the $(P')^2$ terms . If the BS was also rotating we would in addition have non-diagonal terms in the SET.

1.5.2 Rotating superfluids

The behavior of superfluid ^{3}He can be very intricate even though its structure is that of a simple liquid composed of identical and inert rare gas atoms [77]. This combination of the simple and the complex

makes superfluid ^{3}He an ideal substance in which to study condensed-matter phenomena ranging from neutron stars to superconductors.

Helium at low temperatures is a *quantum liquid.* Essentially this is an effect of Heisenberg's uncertainty principle. This means that, if the atoms are very light and interact only weakly, their momenta are well known and as a result their positions are quite uncertain even at absolute zero. Therefore they cannot be kept stationary enough to form a solid at low pressures, because of their large zero point motion.

A ^{4}He atom consists of two electrons, two protons and two neutrons, each with half integer spin. As a result the atom is a boson. Therefore as a fluid of ^{4}He is cooled below the lambda point (at $2.17K$) it condenses to the lowest energy state. At this temperature the entire liquid is in this same state, so that a single wavefunction can be used to describe the entire macroscopic liquid. But ^{3}He are fermions and as a result they need to form a cooper pair before becoming superfluid. This occurs at a much lower temperature than for ^{4}He. (Indeed, ^{3}He form two super fluids, ^{3}He-A and ^{3}He-B, whose theory is much more complicated.)

The superfluid state posses characteristics very different from normal fluids, in particular when one tries to rotate the superfluid. In a normal liquid, if one rotates it in a bucket, the liquid eventually spins at the same angular velocity as the bucket, as if it were a solid body. In this case the momentum and the velocity is proportional to the radial distance from the axis of rotation. But superfluid cannot be made to rotate in this way. This is because uniform rotation requires that the velocity and the momentum increase linearly with the distance from the axis. But momentum and wavelength are inversely proportional, and so the wavefunctions of the atoms in the outer parts have have shorter wavelengths than those located near the axis. But in the superfluid state all atoms are described by the same wavefunction and therefore its impossible for the fluid to rotate uniformly. It is therefore believed that superfluid helium exists in a state of non-rotation, with respect to the universe as a whole. This means that the total angular momentum of the fluid is zero, with respect to the "fixed" stars.

So what happens then, when we rotate a superfluid? What is possible is to have a wavefunction whose wavelength increases with increasing radius. This kind of behavior is typical for liquid motions around vorticies. Indeed, if one begins to rotate a container of SF helium, the stationary state breaks down and tiny vortices form. Rotation, rather than being uniformly distributed throughout the liquid, gets redistributed into a number of vortices. The interaction between the vortices and their neighbors and walls creates some friction, and so the liquid is no longer completely superfluid. The circulating flow of these vortices repels each other, so that they form a regular hexagonal lattice. Furthermore these vortices contain a discontinuous core, where a thread of ordinary fluid runs through its center.

For example, consider a rotating SF in a cylinder. For a nonuniform velocity we assume that the macroscopic wavefunction of the liquid is:

$$\psi(r) = \psi_0 \, \mathrm{e}^{i\phi(r)} \tag{1.66}$$

where r is the radius (vector) from the rotational axis and $\psi_0 = \sqrt{\rho}$ and $\rho(r)$ is the density. Then the velocity field $v(r)$ is found from $\phi(r)$ by using:

$$v(r) = \frac{\hbar}{m} \nabla\phi(r) \,. \tag{1.67}$$

But from gauge invariance of the wavefunction in eq. (1.66), we have that the phase function $\phi(r)$ can

only take on the discrete values, $2\pi n$. And therefore we see that the SF velocity profile around the vortex is quantized by the unit quantization $2\pi\,\hbar/m$. Then the contour integral of $v(r)$ around some loop is given by:

$$K = \oint_L v(r)\cdot \mathrm{d}\mathbf{r} = \oint_A \nabla\times v(r)\,\mathrm{d}S = 2\pi r\, v(r) = nh/m \tag{1.68}$$

Then with a *singly connected* system, where one goes around a closed contour not containing a vortex, one obtains zero. Or using Stokes' theorem, we can say that $\nabla\times v(r) = 0$. However, if the path is *multiply connected* (we do enclose vortices), we get $\nabla\times v(r) = 2\omega$, where ω is the angular momentum. ("Connectedness" is a topological term for describing objects which have holes in them.) As an example, for ^{4}He, we can calculate the number density of these vortices, given the ^{4}He mass, m and the angular velocity ω.

$$2\omega\,\frac{m}{2\pi\hbar} = 2.1\cdot 10^3\omega \qquad [\text{vortices/cm}^2]\,. \tag{1.69}$$

It would be interesting to see how these vortices would behave like in the analog case of a boson or NTS star. (Given that there would be a superfluid phase.) In fact it is not even clear in the Helium case that there would be any vortices in a spherical enclosure.

1.5.3 Modelling stars from a field point of view

For simplicity we start with a complex scalar field ϕ. This is not a paradox, but rather reflects that a $U(1)$ symmetry is needed in order to have a conserved charge. From this field we then try to build a boson star with a NTS solution. But in order for the NTS solutions to be stable, they have to exist also without gravity. However, a potential $U(\phi)$ with only up to ϕ^4 terms does not admit NTS solutions without gravity. To get such solutions we have to add new terms to the interaction potential.

If the scalar field ϕ is to be considered as a fundamental field, we may introduce an additional real valued scalar field σ in order for the theory to be renormalizable. (A ϕ^4 theory is renormalizable but does not admit NTS solutions.) On the other hand, if the ϕ-field is purely phenomenological and thus does not require renormalizability, we may obtain NTS solutions by just adding a ϕ^6-like term to the potential. We are interested in a fundamental field that is renormalizable and therefore choose to include an additional real scalar field σ. However, it should be noted that in a more complicated theory including vector fields we may achieve renormalization without this extra scalar field. A better motivation for introducing this new field would then be to model the degenerate vacuum. That is, we can use the σ-field to describe the global properties of the spacetime containing the ϕ-field and/or possibly a fermion ψ-field. In fact we can then model this vacuum in two different ways depending on our choice of renormalization.

$$\begin{aligned} U(\sigma) &= \tfrac{1}{2}m_\sigma^2\,\sigma^2\,(1-\sigma/\sigma_0)^2 && (\phi \text{ is fundamental}) \\ U(\phi^*\phi) &= m^2\phi^*\phi(1-\phi^*\phi/\phi_0^2)^2 && (\phi \text{ is phenomenological}) \end{aligned} \tag{1.70}$$

(Where it is understood that these have to be added to the kinetic terms.) Then we need to know how these fields couple to each other and to gravity. Immediately one needs to address the following questions:

Does σ couple to gravity through the Ricci curvature?
Does it have a mass?
Does it interact with the ϕ-field?

Does it need to be quantized?
Does it interact with itself?

We start with gravity. It is well known that scalar fields couple to gravity through the Ricci curvature term $\xi\phi^2 R$. And in order for the ϕ field to remain fundamental, we have to choose conformal coupling with $\xi = \frac{1}{6}$, so that the excitations cannot propagate outside the light cone. (Actually it has been shown that by applying the equivalence principle to the propagation of scalar waves in a curved spacetime, the coupling constant has to be $\frac{1}{6}$, and that no other coupling to higher-order scalars constructed from the Riemann tensor is allowed [5].) However we may think that this coupling is not needed for the real field (σ), since it is only phenomenological and already couples to gravity indirectly since it represents the perturbative vacuum which is due to gravity itself in the first place. However, this is a trap because gravitational coupling is intrinsically nonlinear so we need to leave the coupling on.

$$\mathcal{L}_\sigma = \tfrac{1}{2}[\sigma_{;\mu}\sigma_{;}{}^{\mu} - \xi R\sigma^2\,] + U(\sigma) \tag{1.71}$$

Next we would like to know if there is a coupling between the fields. It is well known that scalar particles such as the neutral Pion (π^0) or Kaon (K^0) interact strongly with all other fields present. Besides, if σ is to modify ϕ in any way due to the different vacuum expectation values (VEV), it surely has to couple to it as well.

It is then a good idea to include this kind of interaction in the model with an additional interaction Lagrangian:

$$\mathcal{L}_{int} = f\sigma\phi^*\phi\,. \tag{1.72}$$

As we are not interested in doing any quantum mechanics with the σ field, it seems unnecessary to quantize it. But we could use it just as a phenomenological parameter that is assumed to not contradict QM. The same reasoning goes for the question of whether it should interact with itself, and it should not. The remaining question of mass seems a little more tricky. After all m seems to be a sort of strength parameter, that could be anything, except zero! Otherwise there would be no degeneracy of the vacuum, which was the main reason for including the σ field in the first place.

But why do we need a renormalizable theory (Lagrangian)? There are two very good reasons for this. For one, QED can be derived by imposing *renormalizability* and *gauge invariance*. The other is that, in a renormalizable theory, asymptotic freedom cannot be realized without non-Abelian gauge fields. Okay, this is fine but why does it have to be gauge invariant? (Gauge invariance means invariance under space-time phase transformations. It is equivalent to that in differential geometry, but operating in the *internal* charge space.) Well, according to the standard model, the fundamental interactions are described by gauge theory. The gauge symmetries then generate the dynamics of these theories. Another good reason is that QCD is gauge invariant and we know QCD is a good theory so we certainly want to keep that feature.

Comment: Why it is usually not a good idea to use Effective Field Theories (EFTs) to model ultracompact stellar objects:

At the level of pions, mesons and nucleons, EFTs can only be used at energies low enough for the composite particles to appear point-like (at $\rho < 7\rho_{nuclear}$.) Thus these theories do not need to be renormalizable or even unitary.

Then the EFTs are good only at neutron star densitites and therefore any Q-star model in the EFT approximation would just give another neutron star and nothing more interesting.

1.5.4 Rotating Boson Stars

A non-rotating boson star model would be more or less useless, because of the additional important phenomena occurring due to rotation. A rotating boson star would essentially behave as a *macroscopic H-atom.* Its angular momentum would be quantized and the mass density would be concentrated in an oblique shell.

Even more unusual is that along the rotational axis, the energy density is exactly zero. Thus a boson star would have a hole in it. With some afterthought this may not be as incredible as it may seem. Consider a bath of rotating superfluid helium. As we have showed earlier, in condensed matter theory it is well known and understood that when superfluids are put into rotation they set up vortices in order to achieve a state of lower energy for a given angular momentum. Thus the corresponding velocity fields are not continuous, so that any rotating boson distribution would have at least one vortex hole in it.

Furthermore, from the observed glitches of pulsars, it may be inferred that a superfluid of paired neutrons exists within the crust of a rotating neutron stars. The dominant component of this superfluid would then obeys Bose-Einstein statistics, so that it would seem reasonable to model it by an average gravity coupled scalar field. If this is so, the inner parts of a neutron star may resemble a boson star to some extent.

The NTS boson star

The action for a *non-topological* soliton boson star is then:

$$S = \int \mathrm{d}^4x\, \bar{g} \left[-\frac{R}{2\kappa} + g^{ab}\phi^*_{;\mu}\phi_{;\nu} + \tfrac{1}{2}g^{ab}\sigma_\mu\sigma_\nu - U(\sigma, \phi^*\phi) \right] \tag{1.73}$$

where if the theory is to be renormalizable, the potential has to be of the form:

$$U(\sigma, \phi^*\phi) = m^2\phi^*\phi + \tfrac{1}{2}\lambda(\phi^*\phi)^2 + U(\sigma) \tag{1.74}$$

where $U(\sigma)$ is given by eq.(1.70). Usually for the most general potential, there is also a scalar-scalar coupling term $f^2\sigma^2\phi^*\phi$ included [9]. This coupling can be thought of as giving mass to the ϕ field through the σ field. The mass would then be $m_\phi = f\sigma_{vac}$, which is similar to the way the Higgs scalar gives mass to the other fields. In fact the Higgs potential has the similar form: $U_H(|\sigma|) = -\mu^2|\sigma|^2 + h|\sigma|^4$. The conserved current is defined by the above action to be

$$j^\mu = i\,\bar{g}g^{ab}\,(\phi^*\phi_{,\nu} - \phi\phi^*_{,\nu}). \tag{1.75}$$

The conserved charge is given by the j^0 component of the current which turns out to be the particle number density. We can then identify the charge with the particle number, N.

$$N = \int j^0 \mathrm{d}^3 x \tag{1.76}$$

The Rotating NTS Boson Star

For a rotating object we can apply the axisymmetric line element of the form:

$$ds^2 = f(r,\theta)\, dt^2 - 2k(r,\theta)\, dt d\phi - l(r,\theta)\, d\phi^2 - \mathrm{e}^{\mu(r,\theta)}\, (dr^2 - r^2 d\theta^2) \tag{1.77}$$

From the previous discussion of superfluids, we could in principle consider a boson star as a *macroscopic quantum state.* An addition we also know that since a free Klein–Gordon equation for a complex scalar field is a just the relativistic generalization of the Schrödinger equation, we can consider for the ground state a generalization of the wave function of the hydrogen atom. This is exactly what Schunck and Mielke does in [81]. As such we make the ansatz:

$$\phi(r,\theta,\varphi,t) := P(r,\theta)\, \mathrm{e}^{-i(a\varphi+\omega t)} \tag{1.78}$$

where $P(r,\theta)$ is not separable into a product of radial and angular wavefunctions in the axisymmetric spacetime. Then uniqueness of the scalar field under rotations such that $\phi(\varphi) = \phi(\varphi + 2\pi)$, requires that $a = 0, \pm 1, \pm 2, \ldots$ In addition, because of the exponential phase factor, our solutions are only stationary. Furthermore, since boson stars are *macroscopic quantum states*, they are prevented from complete gravitational collapse by the Heisenberg uncertainty principle.

The Conserved Quantities

The constant of motion (conserved quantity) is given by a Noether symmetry. From the Noether current j^0 in [81] we can find the total number of particles N, when we integrate over the whole space

$$N = \int_S j^0 \mathrm{d}S = 4\pi \int_0^{\pi/2} \int_0^{\infty} (\omega\ell - ak)\, P^2\, E^{-1/2}\, \mathrm{e}^{\mu}\, r \mathrm{d}r \mathrm{d}\theta\,. \tag{1.79}$$

But more generally, it was shown by Komar [80] that the conserverd quantities in asymptotically flat spacetimes are given by the generator

$$K := \int \xi^\alpha n_\beta\, (T_\alpha{}^\beta - \tfrac{1}{2}\delta_\alpha^\beta\, T_\mu{}^\mu)\, \bar{g}\, \mathrm{d}^3 x \tag{1.80}$$

where $n_\beta = \delta_\beta^0$ is a unit timelike vector and ξ^α is a Killing vector field. This is then used to express the remaining conserved quantities of mass and angular momentum, depending on the choice for ξ^α. For a timelike Killing vector $\xi^\alpha = 2n^\alpha$ one obtains the *Tolman mass* for the total gravitational mass:

$$M = \int (2T_0{}^0 - T_\mu{}^\mu)\, \bar{g}\, \mathrm{d}^3 x \tag{1.81}$$

$$= 4\pi \int\int \left[2P^2 E^{-1/2}\, (\omega^2\ell - a\omega k) - E^{1/2} U\right] \mathrm{e}^{\mu}\, r\, \mathrm{d}r\, \mathrm{d}\theta \tag{1.82}$$

To derive the total angular momentum we use $\xi^\alpha = \varphi^\alpha = \delta_3^\alpha$, with the fact that from (1.78) we have in addition $\partial_\varphi \phi = -ia\phi$. Then the integrand in (1.80) is:

$$\delta_\beta^0\, (T_3{}^\beta - \tfrac{1}{2}\delta_3^\beta\, T_\mu{}^\mu)\, \bar{g} = T_3{}^0\, \bar{g} = a j^0\,. \tag{1.83}$$

So that

$$J = \int_S T_3{}^0 \, \bar{g} \, \mathrm{d}^3 x = aN \tag{1.84}$$

Thus the total angular momentum is proportional to the total number of particles and the analog of the magnetic quantum number a, which is quantized as seen above.

At first sight, this may seem quite crazy. But when one consider to what happens in a rotating superfluid, things starts to make better sense. The big hole in the particle distribution is like the vortex core in superfluid helium. There the centrifugal force keeps particles out of the core region. One may then wonder if this phenomena is the same as in this case, only on a bigger scale! (After all we used the ansatz of a hydrogen like atom.)

1.5.5 Stability of Boson Stars

A non stable boson star would be quite uninteresting for the astronomer, since it would merely decay into smaller fragments or not even form in the first place. However, there are some exceptions with respect to supersymmetric theories as will be discussed briefly later in the section of "Q-stars in SUSY theories". But fortunately the stability issues has been relatively well studied in a number of sources. For example, most of the different types of boson stars and their stability criteria can be found in the unclear review paper by Jetzer in [8]. I will briefly summarize some of the main points in that review.

When it comes to the dynamical stability analysis of boson stars, these have already been done for most cases in the linear regime. There one has to analyze the time evolution of infinitesimal radial oscillations, which conserve the total number of particles. Following the method used for stars described in the perfect fluid approximation, one finds an eigenvalue equation, known as the *pulsation equation*, which determines the normal modes of the radial oscillations. One can then use a variational principle to determine the eigenvalues. And from there it can be shown that for perturbations with particle number conservation, the eigenvalues are real. Therefore the stability depends crucially on the sign of the lowest eigenvalue. If it is negative the boson star would be unstable.

Later it was also shown by Lee and Pang [49] that the stability theorem used for stars in the perfect fluid approximation, is also valid for boson stars, given that the total mass and particle number are parametrized by a single variable, usually the central density.

Dynamical stability issues have also been analyzed in the fully relativistic but static regime, using the coupled Einstein-Klein-Gordon equations. Here some authors also apply results from *catastrophe theory*. [See My short notes.] There have also been some models made for mixed fermion-boson stars. However, this analysis is more complicated because the equilibrium configurations are parametrized by two variables.

1.6 The QCD Phase Diagram

From simple geometrical considerations, it follows that nuclei (with radius $r \sim 1$ fm) begin to touch each other at densities of about $(4\pi r^3/3)^{-1} \simeq 0.24$ fm^{-3}, which is less than twice nuclear density (ρ_0) which is easily surpassed in rotating neutron stars. Above this density, one may then expect that the nuclear boundaries of the hadrons begin to dissolve and the formerly confined quarks populate free

states outside the hadrons.

In fact for a collection of a few hundred quarks, the energy per baryon can be less than that of the most stable atomic nucleus, ^{56}Fe. The energy per baryon in ^{56}Fe is

$$E/A = m_{Fe56}/56 = 930.4 \text{ MeV} \tag{1.85}$$

whereas in strange quark matter it is less than 915 MeV, depending on the bag constant. Therefore the ground state for the strong interaction would be strange quark matter as we shall see later.

But the quark matter can exist in many different phases. These are of course governed by the details of the underlying theory of QCD. But for us it should be enough to know that these phases depends mostly on the density and temperature of the matter.

Figure 1.9 shows a possible phase diagram of temperature as a function of chemical potential. The lower left is the region for normal nuclear matter. Then at high density and low temperature, the ground state of QCD exhibits *color superconductivity* (CSC), resulting from quark quasiparticles Cooper pairing near the fermi surface. This phenomena in turn has two phases, the two-flavored CSC (2SC) and the color flavor-locked (CFL) phase. Then at higher temperatures there is also the quark-gluon plasma or three-flavored CSC phase (3SC). I will only discuss the first two, which are more relevant for extreme stellar objects.

1.6.1 Color Superconductivity

In contrast to normal superconductivity, the occurrence of *color superconductivity* is a relatively straight forward phenomenon. This is because the fundamental interaction between two quarks, unlike that between two electrons, is already attractive. Quarks form triplet representations of color $SU(3)$. A pair of quarks, in the antisymmetric color state, form an antitriplet. This is just particle physics language for saying that in any quark-antiquark pair, the color of one has to be the anticolor of the other, assuming that there are only three colors to choose from (r,g,b), thereby using the word *triplet*.

So if two quarks in this arrangement are brought together, the effective color charge is reduced by a factor of two compared to when they are separated. The color flux emerging from them is reduced, and this means the energy in the color field is less, which implies an attractive force. So we should consider very carefully what color superconductivity can do for us.

The two central phenomena of ordinary superconductivity are the Meissner effect and the energy gap. The Meissner effect is the phenomenon that magnetic fields cannot penetrate far into the body of a superconductor, since supercurrents arise to cancel them out. Of course, electric fields are also screened, by the motion of charges. Thus electromagnetic fields in general become short-ranged. Effectively, it appears as if the photon has acquired a mass. We can therefore anticipate that in a color superconductor, the colored gluons will acquire a mass.

The energy gap means that it costs a finite amount of energy to excite electrons from their superconducting ground state.

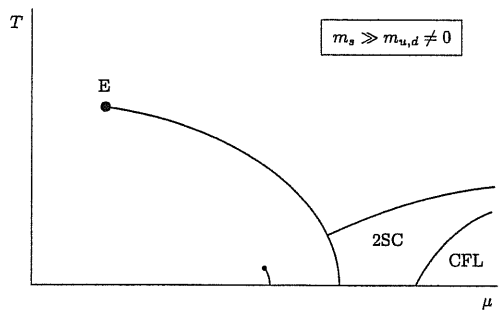

Figure 1.9: A simplified QCD phase diagram in the density-temperature plane, for realistic quark masses.[Credit [72] p.67]

1.6.2 The CFL phase

The simplest and most elegant form of color superconductivity is predicted for a slightly idealized version of real-world QCD, in which we imagine there are exactly three flavors of massless quarks. (At extremely high density it is an excellent approximation to neglect quark masses, however these densities are probably higher than those we encounter in extreme stellar objects.) Here we discover the remarkable phenomenon of color-flavor locking. Whereas ordinarily the symmetry among different colors of quarks is quite distinct and separate from the symmetry among different flavors of quarks, in the color-flavor locked state they become correlated. Both color symmetry and flavor symmetry, as separate entities are spontaneously broken, and only a certain mixture of them survives unscathed.

Color-flavor locking in high-density QCD drastically affects the properties of quarks and gluons. As we have already seen, the gluons become massive. Due to the commingling of color and flavor, the electric charges of particles, which originally depended only on their flavor, are modified. Specifically, some of the gluons become electrically charged, and the quark charges are shifted. The charges of these particles all turn out to be integer multiples of the electron's charge! Thus the most striking features of confinement, the absence of long-range color forces, and integer charge for all physical excitations – emerge as simple, rigorous consequences of color superconductivity. Also, since both left- and right-handed flavor symmetries are locked to color, they are effectively locked to one another. Thus chiral symmetry, which was the freedom to make independent transformations among the left and right handed quarks, remains spontaneously broken as in the QCD ground state.

(Remember, chiral symmetry has to do with unitary rotations among the different flavors of quarks: $q_i^{LR} \rightarrow U_{ij}\, q_j^{L,R}$ where $i = (u, d, s)$, U is a unitary matrix and L, R labels the chirality. Chiral symmetry then reflects the fact that, at high densities, the gluons cannot distinguish between the different flavors of left and right handed quarks. These symmetries are broken in by quark condensation in the QCD ground state, but are restored in the 2SC phase and the quark-gluon plasma.)

1.7 The Most Interesting Stellar Models

Now we have everything we need to review the three most interesting extreme stellar models; strange stars, soliton Stars and Q-stars, each of which has its own advantages and drawbacks. But why do I choose *these* three? First of all, they all contain neutron stars and white dwarfs in their respective

effective field theory limits. Second, they all contain fermions which is what ordinary matter is made from. And finally, because all of them can be quantized.

Conventional neutron star models can be found in very large variety. The structure of the most common of these are shown in the figure below. Our three models are interesting because they may tell something about the stability, size and structure, rather independently of the exact distribution of the baryons. Essentially they could provide for a quark level description of the star, thus having all those different models already included into the theory.

Finally there is the possibility that these objects could exist without any crust. This would have profound effects on observability.

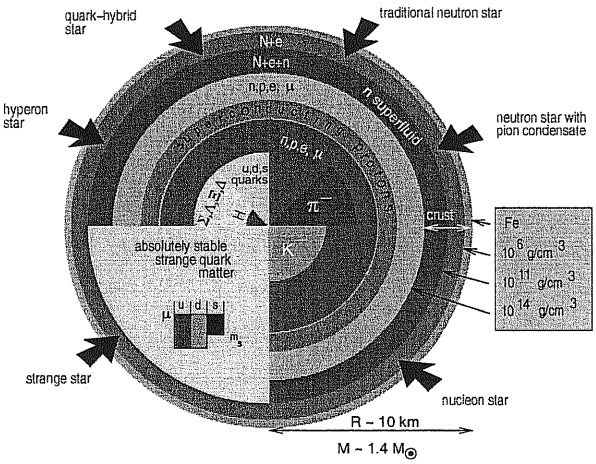

Figure 1.10: Competing structures and novel phases of subatomic matter predicted by theory to make their appearance in the cores ($R \sim 8$ km) of neutron stars. Credit [11], Fig.1.

Strange Stars

At high pressures and densities, a normal baryon containing u and d quarks can convert one of its quarks into either c, s, t or b flavor. Why? Because of the Pauli Exclusion principle. With a given number of quarks, there will be fewer fermions in the same state if another flavor is present, thus decreasing the degeneracy. This is known as *Wittens Conjecture*: 3 flavor quarkmatter have lower energy per baryon than 2-flavor quarkmatter at zero pressure (or any other matter at all.) Stability is achieved if the decease in Fermi energy is greater than the increase of energy due to the non-ordinary quark mass, $m_{c,s,t,b}$. But the t, b and c quarks have too large masses. Thus only the strange quark remains important and we call this type of quarkmatter, *strangematter*. The remaining properties of strangematter is determined by the *thermodynamic potential* Ω. In statistical mechanics language this is equivalent to the number of microstates of a statistical ensemble.

Conserved baryon number constraint can be implemented by introducing a *lagrange multiplier* and minimizing the thermodynamic potential. The lagrange multiplier is usually taken to be the *chemical potential* μ_x, in the case of Q-stars and strangematter.

As discussed above, a strange star is just made of strange matter plus the addition of gravity. Where strangematter is taken to be any number of bound (u, d, s) quarks. This type of matter is generated by mainly two mechanisms. Relief of Fermi pressure [34, 39, 184] and in Kaon condensation [47]. In the first case, at high pressure and density, it is energetically favored to convert a nucleon (n) to a hyperon (Λ), by converting one of the constituent d-quarks to an s-quark, thereby reducing the number of particles in the same state. Thus for large N, the Pauli exclusion principle states that there should be fewer fermions in the same state for 3 flavored quarkmatter than for 2 flavored. This kind of matter should then be stable if the change in Fermi-energy due to the conversion is such that:

$$-\Delta E_F > \Delta m_{(c,s,t,b)}$$

But as discussed before, the (c, t, b) quark are too heavy, so we are left with only the strange ones, the strange matter. However, one should be careful, because one may still have to overcome a potential barrier due to the breaking of the nuclei.

Once the reaction $e^- \rightarrow K^- + \nu$ becomes possible in a neutron star, it becomes energetically favorable for the star to replace electrons with the bosonic K^- mesons. Whether or not this happens depends on the effective mass of the Kaon in dense matter. The threshold for Kaon condensation is believed to be $\rho \geq 3\rho_0$. Similar to the π^- (pion) condensate, a negative Kaon condensate would also soften the equation of state.

Another possibility of strange matter creation is through the formation of *H-dibaryons.* This is a doubly strange six-quark composite with baryon number two and spin and isospin zero. In neutron stars, which may contain a significant fraction of Λ hyperons, the Λ's could combine to form H-dibaryons. This H-matter would form at densities between $3 - 6\rho_0$, depending on the properties of the H-dibaryon when contained in the star. This could again cause a significant softening of the equation of state. But H-matter is unstable to compression, so if indeed formed, H-matter could trigger the conversion of neutron stars into strange stars.

How would it look?

This is a question with many answers. The uncertainty appears because we do not know whether the strange star would be made purely of strange quarks or a mixture and if it would be in some particular QCD phase (which is more likely.) As a consequence it is currently not known whether a strange star would have any crust, but it does seem more likely.

Pure strange matter

Since a strange star is stable at zero pressure, the surface is very different from any other stellar surface [35]. Here the density changes abruptly from zero to 10^{14} g.cm^3. This abrupt change occurs because the surface material is bound by the strong force and not by gravity. As a consequence the *Eddington limit* for the maximum attainable luminosity of standard stars, does not apply. There has even been speculation that this may have something to do with gamma ray bursts (GRB). There is also another important effect on the photon emissivity. Since both the down and strange quarks are fermions with a charge of $-1/3$, This means that if you have a deficit of s-quarks, we would have a positive bulk which needs to be neutralized by a large number of electrons. And because of the higher mass of the s-quark, we do have this deficit and the corresponding electron number. This plasma of electrons would

modify the propagation of the EM waves (photons), just like in any other plasma, whose dispersion relation is

$$\omega = \omega_p^2 + k^2 \tag{1.86}$$

where the plasma frequency is $\omega_p \sim 20$ MeV [36]. Then only modes with $\omega > \omega_p$ will propagate and in the limit $\omega \to \infty$ we get photon propagation in the vacuum. If $\omega < \omega_p$, the incoming photon cannot penetrate the surface and is reflected. The reflectivity for photons $\omega \ll \omega_p$ is close to one, and so for x-ray photons, the star would look like a mirrored sphere. Correspondingly the emissivity of the surface is also very low. In fact it has been estimated to be 4 orders of magnitude less than that of an equilibrium black body.

Strange matter with crust

A crust of normal neutron matter could exist, because a coulomb barrier develops between the strange matter and the ions in the crust. The origin of this potential is the bulk chemical potential of the electrons ($\sim$ 20 MeV.) At the surface the electrons are free to move around up to 10^3 Fermi above the quark surface. Here the electrostatic potential drops to about 3/4 of its value in the bulk. This creates a very large electric field ($\sim 10^{17}$ V.cm^{-1}), which is capable of supporting a crust against the gravitational attraction of the underlying star. The only requirements for this crust is that it doesn't react with the strange matter below. These conditions are:

i) The weight of the above crust must not close the gap.
ii) The density of the crust must not exceed neutron drip. ($4.3 \cdot 10^{11}$ g.cm^{-3})

This new surface is again subject to the Eddington limit and can again emit soft photons, like a black body.

Strange matter in QCD

We have seen that the presence of electrons in strange quark matter is crucial for the possible existence of a nuclear crust on such object However, recently it has been argued that strange quark matter is a color superconductor (see Sect. 1.6.1) which, at extremely high densities, is in the Color-Flavor-Locked (CFL) phase. This phase is rigorously electrically neutral with no electrons required. If the CFL phase would extend all the way to the surface of a strange star, then strange stars would not be able to carry nuclear crusts because of the missing electric dipole layer. However, for sufficiently large strange quark masses, the "low" density regime of strange matter is rather expected to form a 2-flavor color superconductor (2SC) in which electrons are present [72, 71, 67].

However, if the surface is in the CFL phase, then the quarks would be effectively massless as well as some of the charged Nambu-Goldstone bosons. These bosonic modes would then dominate and change the EM response. As a result the CFL material becomes a transparent insulator, with no charged excitations at zero temperature and would resemble more of a diamond. An ordinary light wave incident upon it would be partially reflected, but some fraction would be admitted which would travel through the transparent "diamond", and would partially emerge and be partially internally reflected on the far side. Turning on small but unequal quark masses would darken the diamond somewhat.

Although a quantitative calculation of light reflecting off the facets of a CFL-diamond has not yet been done, the effect of color superconducting quark matter (whether in the CFL phase or in the less symmetric phase in which only up and down quarks pair) in a static magnetic field has been described in complete detail in [73].

The critical temperature below which quark matter is a CSC is high enough that any quark matter which occurs within neutron stars that are more than a few seconds old is in a CSC state.

Fermion Soliton Stars

The existence conditions for Fermion soliton stars are:

i) That there is conservation of Fermion number, N.
ii) That there are NTS solutions also without gravity.

The simplest model to consider is the Friedberg-Lee [43, 44, 45] hadron model without gluons. This model is similar to that of a bag model with a square well potential. This model contains only the fields σ and ψ, with σ being a scalar field with self interaction with the degenerate vacuum, and coupling to the ψ field. We consider the slight modification to the original potential as

$$U(\sigma) = \tfrac{1}{2} m_\sigma^2 \sigma^2 (1 - \sigma/\sigma_0)^2 \,, \tag{1.87}$$

$$\text{with:} \quad \begin{cases} \sigma = 0 & \text{Normal} & (\textit{outside}) \\ \sigma = \sigma_0 & \text{Degenerate} & (\textit{inside}). \end{cases} \tag{1.88}$$

The Lagrangian density for this model is:

$$\mathcal{L} = \mathcal{L}_q + \mathcal{L}_\sigma + \mathcal{L}_{q\sigma} + \mathcal{L}_g \tag{1.89}$$

where:

$$\begin{aligned} \mathcal{L}_q &= \bar{\psi}\left(i\gamma^\mu D_\mu - m_\psi\right)\psi && \text{(Fermions)} \\ \mathcal{L}_\sigma &= \tfrac{1}{2}\sigma_{,\mu}\sigma_{,}{}^{\mu} - U(\sigma) && \text{(Scalars)} \\ \mathcal{L}_{q\sigma} &= -f\sigma\bar{\psi}\psi && (\sigma\psi \text{ coupling}) \\ \mathcal{L}_g &= -\tfrac{1}{4}\kappa(\sigma) F_c{}^{\mu\nu} F^c{}_{\mu\nu} = 0 && \text{(gluons)} \end{aligned}$$

Here the $\mathcal{L}_{q\sigma}$ term acts indirectly like a mass term for the quarks so that the effective mass can be defined as:

$$m^* := m - f\sigma = \begin{cases} m & \text{(outside)} \\ 0. & \text{(inside)} \end{cases} \tag{1.90}$$

Thus outside the condensate, the quarks would have a large effective mass which would make it energetcially unfavourable for them to exsist there. In addition, in order to guarantee color confinement, the dielectric function $\kappa(\sigma)$ have to satisfy the following conditions:

$$\kappa(0) = 1 \qquad \kappa(\sigma_0) = 0 \quad \text{and} \quad \kappa'(\sigma_0) = 0 \tag{1.91}$$

Then because $c = 1$, the product of the color dielectric constant κ and the color magnetic susceptibility μ must be 1, so that as $\kappa \to \infty$, $\mu \to 0$ so the degenerate vacuum is also a perfect color "anti-diamagnet".

Comment: When we put fermions in a soliton background, and then quantize the fermion field one can show [70] that the soliton has two states with fermion number: $n = \pm 1/2$, where n is the expectation value of the conserved charge in these states: $\hat{Q}_0 = \int \mathrm{d}x : \hat{\bar{\psi}}\gamma^0\hat{\psi} :$. Therefore it has no spin and does not obey Fermi-Dirac statistics!

Fermion Q-star

What is it that makes the Fermion Q-star interesting? One good reason is that these models allows for two classes of Fermion Q-stars. One in which one is working on the quark-level and which yields normal strange stars with maximum masses of $\sim 2M_\odot$. In the other class one considers the hadrons as the basic constituents. This case then changes the low density equation of state so radically that it allows for models in which $M \gg M_\odot$. These models could potentially be used as an alternative description of certain low-mass black-hole candidates [62]. The Fermion Q-star model was analyzed in [40]. The existence conditions to have a stable Q-star are the following:

i) Again with a conserved particle number N.

ii) An effective potential $V_{eff} = P_\psi - U$ with:

$$\begin{cases} V_{eff}(\sigma_{in}) & \leq & 0 \\ V'_{eff}(\sigma_{in}) & = & 0\,. \end{cases}$$

iii) Validity of the Thomas-Fermi approximation: $\frac{\mathrm{d}m}{\mathrm{d}r} \ll m^2$.

Then if we are ignoring the details of QCD and taking the strange quark mass to be zero, we can obtain the hadron EOS from the following procedure:

$$\boxed{\nabla^2\sigma = -\frac{\mathrm{d}V_{eff}}{\mathrm{d}\sigma}}$$
$$\Downarrow$$
$$\boxed{\sigma = \sigma(k_F)}$$
$$\Downarrow$$
$$\boxed{\rho(\sigma, U_0), P(\sigma, U_0)}$$
$$\Downarrow$$
$$\boxed{\text{It is chiral: } m = 0}$$
$$\Downarrow$$
$$\boxed{\rho - 3P - 4U_0 + \alpha_v(\rho - P - 2U_0)^{3/2} = 0} \quad \text{(hadron EOS)}$$

where ρ is the density, p is the pressure, U_0 is the energy density of the confining scalar field and α_v measures the strength of the repulsive interaction between nucleons. This represents chiral Q-matter (for which the particles have zero mass within the false vacuum) but the results obtained are only marginally different for non-chiral Q-matter.

Then there is the high density version obtained from the quark level description. The limitations of this theory are:

⋆ Only valid for a quark bulk with a baryon shell.
⇒ Effective Field theory of baryon and mesons are not good.
⋆ The general non-renormalizable chiral potential implies that U_0 is unknown.
⋆ Should be ok for $\rho \sim 6\rho_0$.

<u>Other features:</u>

⋆ P_ψ is balanced by the attractive boson pressure and gravity.
⋆ The conventional neutron star phase is present.

⋆ σ_{in} = const. throughout the star.
⋆ They are really Q-balls which has the same EOS as MIT bags with $B := 4U(\sigma_{in})$.

Thus in conclusion we see that in the Fermion Q-star, we have an effective lagrangian (Q-ball) with a scalar field whose dynamics is determined by the effective potential. Most of the energy of this object is in the bulk, with a critical mass on the order of $M_c \sim 10M_\odot$ when $\sigma_0 = 100$ MeV. Whereas in the Fermion NTS case the potential is governed by some parameter tuning such that $U(\sigma_{in}) = 0$, only this time there is also much energy in the surface. Here the critical mass is sensitive to $U(\sigma)$ and $M_c \sim (1 - 10^{15})M_\odot$.

Some Mass Limits

It is rather easy to estimate the maximum masses, and it is surprising how correct the estimations turn out to be. I will estimate 2 configurations with different properties. These are:

a) Boson stars → Heisenbergs uncertainty principle: $p \cdot x = \hbar/2$.
b) Fermion stars → Pauli exclusion principle.

So for boson stars we understand that for a quantum state confined to a region R, the momentum is: $p = \hbar/(2R) \sim 1/R$. But in a relativistic boson star, the speed is close to light and since we are using units where $c = \hbar = 1$, $p \simeq mc \sim m$, so that $R \simeq 1/m$. However, the maximum mass occurs at $R = 2GM_{max}$ where $G = m_{\rm P}^{-2}$, so we have:

$$M_{max} = \frac{R}{2G} \sim \frac{m_{\rm P}^{-2}}{m}. \tag{1.92}$$

Whereas for fermion NTS stars we need the surface tension, $s \simeq \frac{1}{6} m_\sigma \sigma_0^2$. This quantity is non-trivial and the calculations of this may be very complicated. (In fact most authors have consciously ignored this point.) However, with this quantity known, we find the surface contribution to the energy: $E_{surf} \simeq s\, 4\pi R^2$. But $E = E_{\rm KE} + E_{surf}$ and for a minimum in E, it is found that $E_{\rm KE} = 2E_{surf}$. Now,

$$M = E = 3E_{surf} = 12 s \pi R^2 \tag{1.93}$$

and since $N = \rho\, 4/3\pi R^3 \simeq R^3$, we find that $M \sim N^{2/3}$. Therefore in the extreme case, where $R \sim 2GM_{\rm c}$ we have:

$$M_{\rm c} \sim (G^2 s)^{-1} \sim \frac{m_{\rm P}^4}{m_\sigma \sigma_0^2}. \tag{1.94}$$

We summarize these and other results as:

Compact Object	Other	Critical Mass, $M_{\rm c}$
Normal Star		$M_{\rm Ch} := m_{\rm P}^3/m^2$
Q-star	$Q \equiv N$	$M_{\rm Ch}$
Boson NTS		**See below.**
Mini-boson NTS	$p = \rho = 0$	$M_{\rm Kaup} = 0.633\, m_{\rm P}^2/m$
Fermion NTS	Depends on $U(\sigma)$	$10^{-2}\, m_{\rm P}^4/(m\phi_0^2)$

For the different Boson stars studied we have the following table of critical masses, M_c:

Interaction	Author	Critical Mass, M_c	
$\lambda\phi^4$	Colpi	$0.06\lambda^{1/2}\,\frac{m_P^3}{m}$	
$m^2\phi^2$	Kaup	$0.633\,\frac{m_P^3}{m}$	
$\xi\phi^2 R$	Gleiser	$0.6\,\frac{m_P^2}{m}$	$(\xi = 0)$
		$\xi^{1/2}\,\frac{m_P^2}{m}$	$(\xi \neq 0)$
$F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\tilde{\lambda}\lvert\phi\rvert^4$	Jetzer	$0.440(e_c - e)^{-1/2}\,\frac{m_P^2}{m}$	$(\tilde{\lambda} = 0)$
		$0.226(e_c - e)^{-1/2}\,(\frac{\tilde{\lambda}}{8\pi})^{1/2}\,\frac{m_P^3}{m^2}$	$(\tilde{\lambda} \neq 0)$

Where $e_c = 1/\sqrt{2}$ in units of $[m_{\rm P}\,(8\pi m^2)^{-1/2}]$. For $m \simeq 1$ GeV/c^2 of the order of the proton mass and $\Lambda \simeq 1$, this is in the range of the Chandrasekhar limiting mass $M_{\rm Ch} := m_{\rm P}^3/m^2 \simeq 1.5 M_\odot$.

1.8 Q-stars in SUSY

1.8.1 Q-ball phenomenology

The properties of q-balls depends strongly on whether or not supersymmetry is an exact symmetry in nature. Because of the current uncertainty, the standard model (SM) has been extended to include supersymmetry (SUSY). The simplest of these extensions is called the minimal supersymmetric standard model (MSSM). These models are particularly interesting from the point of view of baryogenesis, where inflation leads to an Affleck-Dine (AD) condensate. What is interesting here is that this condensate is unstable and normally breaks up into non-topological solitons (NTS) or *q-balls*, which may carry conserved charges such as baryon and/or lepton number, and are therefore called *B-balls* and *L-balls*. Furthermore these q-balls come in two classes:

- **SECS** Sypersymmetric Electrically Charged Solitons
- **SENS** Sypersymmetric Electrically Neutral Solitons

The interaction between q-balls and other matter is then intuitively different depending on which class we are dealing with.

SECS are q-balls with a net positive charge inside. This charge originates from the unequal absorption rate in the the condensate of quarks (squarks) and electrons (selectrons). This positive charge is neutralized by a surrounding cloud of electrons.

SENS are q-balls whose existence depends on a large interior vacuum expectation value (VEV) of squarks, sleptons and Higgs fields. Inside the q-ball, the $SU(3)_c$ color symmetry is broken into $SU(2)$ and deconfinement takes place. Thus a nucleon entering this region will dissociate into its constituent quarks and become part of the baryon condensate of charge Q. Typically this occurs via the quark reaction $qq \to \tilde{q}\tilde{q}$, or in terms of the condensate reaction:

$$Q + Nucleon \;\to\; (Q+1) + 2(3)\pi \tag{1.95}$$

$$Q + Nucleon \;\to\; (Q+1) + 2K \tag{1.96}$$

Where the energy difference is ~ 1 GeV (the mass equivalent of a neutron or proton.) Thus a SENS can cause proton decay, while a SECS would produce massive ionization.

1.8.2 SUSY Light [93, 91]

What is supersymmetry? Why do we need it? How do we see it? These are the main questions which I will try to answer in this extremely brief introduction to SUSY. The first thing to ask is probably why we need it. The three main motivations for SUSY are: (i) It is a solution to the "hierarchy" problem, (ii) Its intrinsic connection to gravity and finally (iii) It can explain the darkmatter problem.

The hierarchy problem is just another name for wondering why the electroweak energy scale ($M_{W^\pm} \sim M_{Z^0} \sim 10^2$ GeV) is so very different from the grand unification energy ($M_U \sim 10^{15}$ GeV). The second motivation is the fact that if SUSY is formulated as a *local* symmetry, then a spin-2 (graviton) field must be introduced. Thus it automatically incorporates general relativity in the appropriate limit. Theories with local supersymmetry are therefore called supergravity (SUGRA) theories. Finally SUSY incorporates an entire spectrum of supersymmetric particles (sparticles) which are the counterparts to the standard model particle spectrum. This would give a wide range of new possibilities for darkmatter candidates. From 100 GeV scale lightest supersymmetric particles (LSP) to generation of q-balls through the *Affleck-Dine* mechanism of nucleosynthesis. This also answers the question, how do we see it? Through a complex mixture of reactions and couplings between the following particles:

$$\begin{array}{cccccc} [u,d,c,s,t,b]_{L,R} & [e,\mu,\tau]_{L,R} & [\nu_{e,\mu,\tau}]_L & g & \underbrace{W^\pm, H^\pm} & \underbrace{\gamma, Z, H_1^0, H_2^0} \\ [\tilde{u},\tilde{d},\tilde{c},\tilde{s},\tilde{t},\tilde{b}]_{L,R} & [\tilde{e},\tilde{\mu},\tilde{\tau}]_{L,R} & [\tilde{\nu}_{e,\mu,\tau}]_L & \tilde{g} & \tilde{\chi}^\pm_{1,2} & \tilde{\chi}^0_{1,2,3,4}. \end{array}$$

The details this is obviously far beyond this thesis. Of more importance is a brief understanding of the phenomenology of SUSY.

Supersymmetry differs from all other symmetries in that it relates two very different classes of elementary particles, those of fermions and bosons. Accordingly, every SM particle has its own superpartner, which is only differing in mass and half a unit of spin, thus reversing SM fermions into SUSY bosons and vice versa. Generally the interactions in a SUSY version of the standard model can be found from the standard couplings by replacing any two particles by their superpartners. The "two" in order to conserve spin angular momentum. However, we don't see any of these reactions at our present experimental energy levels, implying that the SUSY particles must have different masses (higher) than those of the SM family. We therefore conclude that SUSY must be a *broken* symmetry, giving the SUSY particles a higher mass and therefore jeopardizing the solution hierarchy problem. With SUSY breaking, the contribution is now proportional to the difference in mass between the SUSY and SM particles. In order for the electroweak scale to be *natural* at about 10^2 GeV, this mass difference should be less than 1 TeV:

$$|m^2_{SM} - m^2_{SUSY}| < (1\text{TeV})^2 \tag{1.97}$$

These energy requirements are usually referred to as a "soft" symmetry breaking. Another nice thing with broken SUSY is that the VEV will always be positive.

There is possibly one more symmetry of the theory which is discrete and exact, called R-parity. This R-parity can be thought of as a new conservation law, where all SM particles have $R = +$ (even) while all SUSY particles have $R = -$ (odd). Then any physical process must always involve an even number of sparticles. In other words, sparticles must be produced in pairs and must decay in odd numbers. A direct result of this is that the LSP(usually one of $\tilde{\gamma}, \tilde{Z}, \tilde{H}_{1,2}$) must be a stable particle. The inclusion of SUSY into a grand unified theory (GUT) also leads to some new mechanisms for proton decay. The

dominant decay mode would be the $p \to K^+ + \bar{\nu}_\mu$ reaction. However for the MSSM GUT with $SU(5)$ symmetry, this leads to a proton lifetime of $\tau = 10^{26} - 10^{31}$ years, which is thus ruled out by the current experimental limit of $\tau > 7 \times 10^{31}$ years.

SUSY Light formalism

Basically the most simple SUSY lagrangian consists of boson (ϕ) and fermion (ψ) fields according to:

$$\mathcal{L}_{free} = (\partial_\mu \phi)^2 + \bar{\psi}(i\partial)\psi + ... \tag{1.98}$$

These satisfy the Klein-Gordon and Dirac equations:

$$\Box\phi = 0 \tag{1.99}$$

$$\partial\psi = 0 \tag{1.100}$$

We can then add interaction terms to that lagrangian.

$$\mathcal{L}_{int} = a_1\phi^4 + a_2\phi\bar{\psi}\psi + ... \tag{1.101}$$

The following transformations then specify the theory and give the relations between a_1 and a_2 required to satisfy the SUSY requirement.

$$\delta\phi = \epsilon\psi \tag{1.102}$$

$$\delta\psi = \epsilon\partial\phi \tag{1.103}$$

The more proper definition of SUSY is then that the variation of the action; $\delta S[\psi, \phi]$ is zero. However, when one includes more fields and more complicated interactions it is a matter of simplification to introduce a *superfield* $\Phi[x, \theta] := \phi(x) + \theta\psi(x)$, which in addition requires the introduction of a single spinorial dimension to space, denoted θ. With this introduction the symmetries and calculations become more transparent. The different types of SUSY theories are characterized by N, where the simplest one is the $N = 1$ theory [2] . The taylor expansion of the superfield then take the form

$$\Phi(x, \theta) = \phi(x) + \sqrt{2}\,\theta\psi(x) + (\theta\theta)F(x) \tag{1.104}$$

(with higher orders in θ being zero.) Here ψ is a fermion field (a complex left-handed Weyl spinor), while $\phi(x)$ and $F(x)$ are complex scalar fields. Out of these superfields one can also construct the *superpotential*, W.

A natural feature of this potential is the existence of flat directions in field space in which there are no renormalizable contributions beyond the soft SUSY breaking terms. This flatness of the potential can be best visualized as the valleys on a mountain terrain.

SUSY breaking light

There are two ways in which we can break supersymmetry spontaneously. One is to simply add the gauge superfield V directly directly to the Lagrangian: $\mathcal{L} \to \mathcal{L} + kV$. Then the integration of this V

[2] In the 4 dimensional Weyl spinor notation, N is the number of supersymmetry generators which are denoted: Q^A_α, where $A = (1, \ldots, N)$. These are actually the conserved Noether charges arising from 2 separate supercurrents.

field generates a *D-term*, whose variation is zero and thus we are still left with a SUSY theory. The other way of SUSY breaking is to add an *F-term* to the action.

The most important point is that the breaking terms should be *soft*, meaning that most of the nice features of SUSY still remains after breaking. Those terms are generally the ϕ^2 and $\bar{\psi}\psi$ terms.

Symmetry breaking is best visualized by two simple examples. We consider two different forms of the superpotential W and investigate the behavior of the (energy) potential V in each case.

We know that the scalar potential has two contributions; one from the F-term and the other from the D-term (representing a gauge choice). The D-terms originate from the kinetic part of the SUSY lagrangian, and are actually just the superspace covariant derivatives. [3] Whereas there is an F-term for each superfield Φ appearing in the superpotential, W. In fact the potential is defined as

$$V := |F|^2 + \tfrac{1}{2}g^2 D^a D^a \tag{1.105}$$

$$F := \frac{\partial W}{\partial \Phi} \tag{1.106}$$

Ignoring the much more complicated D-terms we have

$$\left.\begin{aligned} W_1 &= \tfrac{1}{3}\Phi^3 + \Phi \\ W_2 &= \tfrac{1}{3}\Phi^3 - \Phi \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} V_1 &= |\Phi^2 + 1|^2 \\ V_2 &= |\Phi^2 - 1|^2 \end{aligned} \tag{1.107}$$

It is clear (see figs.1.11) that SUSY is broken by positive terms and that the minima are degenerate in

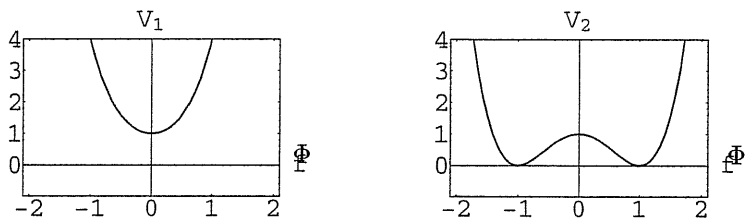

Figure 1.11: a) Broken SUSY potential V_1 with $E \neq 0$. b) SUSY potential V_2 with degenerate $E = 0$.

the SUSY preserving case. In this particular case the *flat* part are the two discrete points $\Phi = \pm 1$. If Φ would be a complex valued field we would have a continuous set of flat points, which would then be called a *flat direction.*

In fact these ideas can then be simply extended further for multiple fields. One could make another example of the 2-field potential of hybrid inflation

$$V(\phi, \chi) = \tfrac{1}{4}\lambda(\chi^2 - M^2)^2 + \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{2}\lambda'\phi^2\chi^2 \tag{1.108}$$

as plotted in fig.1.12. Here the interesting physics is that when ϕ is large, the minimum of the potential is at $\chi = 0$. But as the field rolls down and approaches $\phi^2 = \lambda/\lambda' M^2$, χ becomes unstable and rolls down to one of the minima at $\chi = \pm M$. In this kind of potential the ϕ field can be rolling extremely slowly before settling into the ground state.

[3] Using M and A to denote the superspace and super-tangent space indices, we can define the covariant derivative as: $D_M := E_M^A\,(\partial_A + \frac{1}{2}W_A^{mn}\,M_{mn})$, where E_M^A and W_A^{mn} are the veilbein and spin connection, and $\partial_A = (\partial_m, \partial_\alpha, \bar{\partial}_{\dot{\alpha}})$.

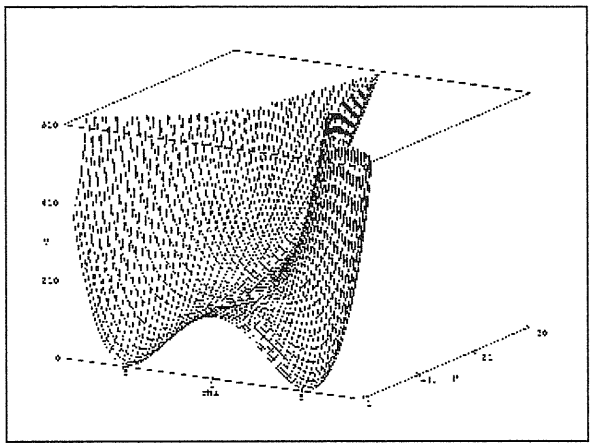

Figure 1.12: SUSY potential $V(\phi, \sigma)$ with degenerate $E = 0$.

Affleck-Dine Light

A new mechanism for baryogenesis was proposed by Affleck and Dine [91], in which there is a significant baryon asymmetry. The mechanism is based on the observation that SUSY theories have a many-parameter set of vacuum states known as *flat directions*. These are really directions in field space in which the scalar potential vanishes. In general these states have a non-zero vacuum expectation value (VEV) for squarks and sleptons, which may be arbitrarily large. When SUSY is broken, the degeneracy is lifted by an amount on the order of $m^2|v|^2$ where v is the VEV and m is a typical scalar mass on the order of m_W. Even if v is as large as M_P, this is a small amount of energy compared to that of inflation ($\sim M_G^4$).

Thus it is possible that the squarks and sleptons fields are far from the origin after an inflationary period, producing a large baryon number.

1.8.3 Some Comments

Matter fermions cannot penetrate inside some q-balls, because their masses inside would increase by the large Higgs VEV, as well as through their mixing with gauge fermions.

However the outer region has a boundary layer where:

(1) The quark masses are less than Λ_{QCD}.

(2) The gauge $SU(3)$ symmetry is broken spontaneously by the VEV's of the squarks.
Thus when a nucleon enters this outer region it dissociates into quarks, emitting the difference in energy by pions or kaons.

For energetic reasons, large q-balls consists of a neutral scalar condensate.

1.9 Conclusion

As we have seen in this chapter, there is a wide variety of models for creating extreme compact objects. However, at the end they are all connected, since in some limit of their various parameters, they have to model something that we observe.

We learned about the importance of conserved charges and topology, and how topology may change the vacuum energy of matter fields. This was then shown to be crucial for the stability and existence of the different extreme objects considered, the soliton, strange, boson stars and Q-stars. We also discovered the great similarity of boson stars with superfluids and how the high energy QCD phases are phenomenologically connected to superconductivity, through color superconductivity. These QCD phases were then shown to be very important for the optical properties of strange stars. I then briefly mentioned the importance of how Q-stars and in particular Q-balls would differ in nature if super symmetry (SUSY) would be an exact symmetry in nature and therefore also a valid extension of the standard model.

However, since this chapter was based purely on introductory research, I have indeed left out some of the perhaps most interesting aspects of these extreme objects. The possibility for near future observational confirmation using recent astrophysical instrumentation and methods. Some of which is based on gravitational spin-up/down of millisecond pulsars, and their pulsar glitches. The spectra of low mass x-ray binaries (LMXB) and new understanding of quasi-periodic oscillations (QPO) in these.

In fact in the next chapter I will briefly describe a new possible explanation of QPO's. Where we have a new idea that QPOs may perhaps be explained by orbital resonance and epicyclic frequencies.

Finally I would like to mention that part of my initial research also includes studies of cosmology in higher dimensions, i.e. D-brane and string cosmology. Neither of which have a place in this thesis.

Chapter 2

Rotating Extreme Objects

– The question is: what is a "mahna mahna"?
– The question is: who cares?!

The Muppet Show

Ever since the discovery of the Kerr Solution to Einsteins field equations, which is a blackhole solution only, people have sought some more general external analytic solution for the spacetime outside rotating axisymmetric stars. For without such solutions, how could we ever expect to properly describe particle and body dynamics around planets and stars? However, there is always the possibility for numerical solutions, but these are usually not enough general in case one would like to change some parameter unless one want to redo the entire numerical calculations. Other solutions are at best some series expansion of some rotational parameters. Perhaps the closest such solution to an exact analytical solution, is the one by Hartle and Thorne.

In this chapter we will look at some analytical and numerical solutions, describing the exterior spacetime of some general stellar objects that will be assumed to be stationary and axially symmetric. With these solutions in hand I will then calculate a number of extremely useful quantities related to circular and orbital motion. The way to derive these quantities will follow closely the powerful formalism of [Abramowicz et al.], which allows for quick and transparent derivations, which when condensed in one place, may provide a very useful toolbox for anyone dealing with rotation. One could then in principle repeat most of these example calculations for the Hartle-Thorne metric, describing analytically a rotating axially symmetric spheroid with some equation of state. However I will just apply the relevant formulas to obtain the relevant parameters algebraically using computer algebra. The HT metric is good up to second order in the angular momentum and first order in the quadrupole moment. We then proceed to compare these analytical results with some numerical results using the RNS code.

2.1 Introduction and Overview

As a start I will mention the different meanings of "radius" that can and will be found in the literature and in this thesis. The next section will then be an introduction to particle dynamics. This means

that we will briefly discuss the metric symmetries for stationary and axially symmetric spacetimes and introduce some basic conserved quantities, such as the energy and specific angular momentum. Then in the following section I will derive a number of useful quantities related to circular motion, the acceleration, the effective potential and finally the epicyclic frequency. The use of effective potentials is a powerful tool in the derivations and comprehension of many of the following orbital quantities. However, there are different ways to obtain an effective potential, and I will show these and when they are applied.

The introduction of the kinematic invariants, then allows for the calculation of vorticity, shear and gyroscope precession through the use of the Fermi-Walker derivative. Shear and vorticity is important in understanding what forces acts on a conglomerate of dust while orbiting some massive object. For example whether the conglomerate would deform in some way like changing shape or volume. Gyroscope precession is of course important if one wants to understand how small spinning dust conglomerates would behave in relation to their orbits.

I will then explain what is holonomy invariance and how it can be used togther with the epicyclic frequencies to obtain "quantized" orbits, followed by a new possible description and explanation for quasi periodic objects (QPO's.) The radial condition for having parametric resonance will then be derived in the next section on the Hartle-Thorne metric.

Since all orbital quantities will be derived using Mathematica, we will start by showing how to put the Hartle-Thorne metric in the right form good for input into this software, followed by an explanation on how to take the limit of the HT-metric to obtain equivalent results as in the Kerr metric. I mention the method for obtaining some of the radial solutions, followed a presentation of all the formulas for the orbital quantities.

In the last part I use the RNS code to make some constant rest mass sequencies of stars in order to compare how well the Hartle-Thorne metric describes stars with higher rotational speeds.

In this section we shall use the $(+---)$ signature. With this convention, the vector fields become timelike when $x^a x_a = 1$, and spacelike when $x^a x_a = -1$. Further we shall use the shorthand notion of a scalar product as: $(xx) = x^a x_a$.

The Notion of Radii in GR

In this chapter we will eventually encounter some different *radii* that will be used throughout the rest of the thesis. However, quite generally in GR there are a number of different notations and definitions of what is meant by a "radius". I will briefly mention some of these here.

Let us start by considering the arctic circle around earth's north pole. The *geodesic radius* would then simply be the distance measured by walking from the arctic circle to the north pole. That is, we are confined to a surface enclosing an inaccessible volume. In fact this geodesic radius is better known as the *proper radius*. Of course this is not the same as the *radius of gyration*, which is a straight line measurement through that (inaccessible) volume. (Actually this definition is more subtle as this radius is really a distance from the axis of rotation in rotating spacetimes.) Another useful measurement is the *circumferential radius* which is the radius one would obtain by measuring the circumference of the arctic circle and dividing the result by 2π. For earth this would be nearly equivalent to the radius of gyration, whereas for an extremely compact object, the result would be very different. This is a purely GR effect,

since these bodies enclose more volume within a given sphere. Or in other words, in Newtons's theory with Euclidean geometry, one has $\mathcal{R}_{geodesic} = \mathcal{R}_{circumferential} = \mathcal{R}_{curvature}$, but one may also use a non Euclidean geometry in which all of these would be different. Now, all that is left, is to mentally generalize all these to a $4D$ spacetime...

These are then defined as:

$$\begin{aligned}
\mathcal{R}_{proper} &= \int_0^R \sqrt{-g_{rr}}\,\mathrm{d}r \\
\mathcal{R}_{gyration} &= \left(\tfrac{\ell}{\Omega}\right)^{1/2} = \left(-\tfrac{(\xi\xi)}{(\eta\eta)}\right)^{1/2} && \text{(static spacetimes)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{circumferential} &= \tfrac{1}{2\pi}\int_0^{2\pi} \sqrt{-g_{\phi\phi}}\,\mathrm{d}\phi \\
\mathcal{R}_{curvature} &= 1/r_1 + 1/r_2 + \ldots && \text{(Euclidean)} \\
&= R = R^a{}_a && \text{(Non-Euclidean)}
\end{aligned}$$

where r_n and are the radii of the circles in n orthogonal directions. (So that for a perfect sphere, $r_1 = r_2$ so that $\mathcal{R}_{curv.} = 2/r$.) As shown in [186] and references therein, the radius of gyration can also be interpreted as *luminosity radius* or *embedding radius.* In this chapter we will be mainly concerned with the radius of gyration and the curvature radius. Thus we will just write $\mathcal{R}$ for the radius of gyration and R for the Ricci curvature.

2.2 Particle Dynamics

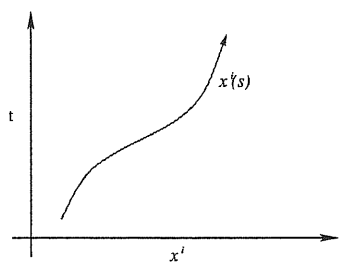

Figure 2.1: A simple curve in spacetime, parametrized by the length along the curve s, and where all other directions (x, y, z) are shown as x^i.

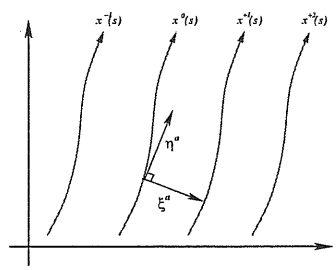

Figure 2.2: A Killing vector field, showing the timelike and spacelike Killing vectors for one curve.

Consider a particle moving in a spacetime as in Fig.2.1. Then the particle trajectory can be described by the equation $x^i = x^i(s)$, where s is some parameter along its path, usually the lenth. We can then define the 4-velocities of the particle and observer as:

$$u^a := \frac{dx^a}{ds} \qquad \text{(4-velocity of particle)} \tag{2.1}$$

$$n^a := \frac{dx'^a}{ds} \qquad \text{(4-velocity of observer)} \tag{2.2}$$

We then know from Noethers Theorem, that in order to have conserved quantities, there have to exsist some symmetries. (See Chapter 1.) For a $4d$ geodesic, these symmetries are expressed through the orthogonal Killing vectors. A congruence of Killing vectors form a Killing vector field (Kvf). (See

Fig.2.2.)

$$\eta^a := \delta^a_{\ t} \qquad \text{(timelike Kvf.)} \tag{2.3 a}$$

$$\xi^a := \delta^a_{\ \phi} \qquad \text{(spacelike Kvf.)} \tag{2.3 b}$$

These in turn can be used to define the *unit vectors*:

$$n^a := \eta^a \left(\eta^b \eta_b\right)^{-1/2} \qquad (n^a n_a = +1) \tag{2.4}$$

$$\tau^a := \xi^a \left(\xi^b \xi_b\right)^{-1/2} \qquad (\tau^a \tau_a = -1) \tag{2.5}$$

The spacelike unit vector τ^a is then orthogonal to n^a as $\tau^a n_a = 0$.

To see how the observer and particle frames are related, we consider an example, where we are in the local rest frame of the observer, where $n^a = (1, 0, 0, 0)$. In the $3+1$ language (where we have split spacetime to a $3d$ hypersurface and a temporal part), we have that

$$u^a n_a = u^t = \frac{dx^t}{ds} = \left(\frac{dx^t}{dx'^t}\right)\left(\frac{dx'^t}{ds}\right) = \gamma\, 1\,. \tag{2.6}$$

So that that temporal considerations gives; $u^a n_a = \gamma$. This is just the Lorentz factor relating the two frames. In considering spatial parts, we use the projection tensor that takes a $4d$ vector and projects it down onto the $3d$ hypersurface as in Fig.2.3.

$$\begin{aligned} h^a_{\ b} &:= \delta^a_{\ b} - n^a n_b \qquad \text{(projection tensor)} \\ \Rightarrow x_a h^a_{\ b} &= x_b - n_b (x^a n_a) = x^{\perp}_{\ b} \end{aligned} \tag{2.7}$$

The spatial part then tells us that $n_a h^a_{\ b} = 0$, which means that n^a is orthogonal to the hypersurface. (See Fig.2.4.)

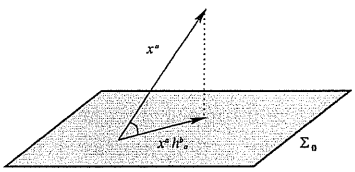

Figure 2.3: The $4d$ vector x^a projected onto the hypersurface Σ_0.

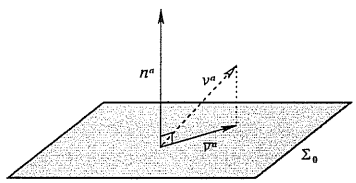

Figure 2.4: The orthogonal vector n^a and the projected vector $\bar{v}^a$ in the hypersurface.

Now consider another vector living on the hypersurface, we can call it $\bar{v}^a$. Then we could imagine this vector as being the projected part of another vector living in spacetime, v^a. Then the spacelike part $\bar{v}^a$ is automatically orthogonal to n^a.

$$\bar{v}_b = v_a\, h^a_{\ b} \tag{2.8}$$

We could then in principle *define* $\bar{v}_b := \gamma v_b$ In fact, we could even define the spacelike part $(\bar{v}_b)$ to be proportional to some spacelike unit vector τ^a also living on the hyperplane: $\bar{v}_b := \beta \tau_b$ where γ and β would be proportionality constants.

The particle 4-velocity is then the vectorial sum of the observers 4-velocity and the proper 3-velocity, both multiplied by the Lorentz factor. The Lorentz transformation for the particle 4-velocity can then be written as:

$$\begin{aligned} u^a &= \gamma(n^a + \bar{v}^a) \\ &= \gamma(n^a + \beta\tau^a) \end{aligned}$$

where $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = \frac{v}{c}$. It is then easy to show that u^a is timelike: $u^a u_a = \gamma^2(1-\beta^2) = 1$.

2.2.1 Symmetries in Spacetime

As we already mentioned, symmetries of a spacetime are expressed through the Killing vectors. But there is an invariant way to express these symmetry properties of a given metric, through the Killing equation.

$$2\,\nabla_{(a}\xi_{b)} = \nabla_a\xi_b + \nabla_b\xi_a = 0 \qquad \textbf{(Killing Equation)} \tag{2.9}$$

where ξ^a is some Killing vector field (Kvf.) Considering a *stationary* and *axially symmetric* spacetime, we will then have two Killing vectors, one timelike and one spacelike, η and ξ, respectively. Therefore we also have two Killing equations.

$$\nabla_a\eta_b + \nabla_b\eta_a = 0 \tag{2.10 a}$$

$$\nabla_a\xi_b + \nabla_b\xi_a = 0 \tag{2.10 b}$$

Now, if we assume that they commute, we also have

$$\xi^a\nabla_a\eta_b - \eta^a\nabla_a\xi_b = 0 \tag{2.11}$$

We can then derive a very useful relation. Let x^a, y^a be commuting Killing vectors. Then:

$$\begin{aligned} \nabla_b(x^a y_a) &= x^a\nabla_b y_a + y_a\nabla_b x^a \\ &= x^a\nabla_b y_a + y^a\nabla_b x_a \\ &\overset{(1)}{=} x^a(-\nabla_a y_b) + y^a(-\nabla_a x_b) \\ &\overset{(2)}{=} -x^a\nabla_a y_b - x^a\nabla_a y_b \\ &= -2\,x^a\nabla_a y_b \end{aligned}$$

$$\Rightarrow \quad \boxed{x^a\nabla_a y_b = -\tfrac{1}{2}\,\nabla_b(x^a y_a)} \tag{2.12}$$

2.2.2 Conserved Quantities

Here I will show that there are conserved quantities along geodesics. Generally geodesics represent the motion of free particles. Thus, if u^a is the particle 4-velocity, geodesic motion in spacetime is represented by:

$$u^b\,\nabla_b u^a = 0 \qquad \textbf{(Geodesic Line Equation)} \tag{2.13}$$

But conserved quantities come from constants of motion, which essentially means that the derivative of the scalar product of two vectors is zero at all points. So that

$$\begin{aligned} u^b \nabla_b(\eta^a u_a) &= u^b u_a \nabla_b \eta^a + u^b \eta^a \nabla_b u_a \\ &= u^b u^a \nabla_b \eta_a + \eta^a u^a \nabla_b u_a \\ &= \underbrace{u^b u^a}_{\text{sym}} \underbrace{\nabla_b \eta_a}_{\text{anti-sym}} + \eta^a \underbrace{u^a \nabla_b u_a}_{\text{geodesic}} = 0 \end{aligned}$$

where we used the fact that the product of a symmetric and a anti-symmetric tensor is zero and that $\nabla_a g_{ab} = 0$.

Similarly one will obtain one conserved quantity for each independent killing vector. As we have stationary axial symmetry, we have only two Killing vectors summarized below.

$$\boxed{\begin{array}{lcll} \eta^a u_a & = & E & \text{Energy/mass} \\ \xi^a u_a & = & -L & \text{Angular Momentum/mass} \end{array}} \tag{2.14}$$

In fact we usually define a new constant of motion; the **specific angular momentum.**

$$\ell := \frac{L}{E} = \frac{-u^a \xi_a}{u^a \eta_a} \tag{2.15}$$

2.2.3 Circular Motion

In this section we will derive some orbital parameters for some general stationary and axisymmetric spacetime. With this ansatz, we know immediately that the spacetime posses two Killing vectors as shown in equations (2.9-2.14). We can then identify the Killing vectors with the covariant components of the metric, which allows us to write:

$$\begin{aligned} (\eta\eta) &= \eta^a \eta^b g_{ab} &= g_{tt} \\ (\eta\xi) &= \eta^a \xi^b g_{ab} &= g_{t\phi} \\ (\xi\xi) &= \xi^a \xi^b g_{ab} &= g_{\phi\phi} \end{aligned}$$

The metric can then be written in the form:

$$\boxed{\mathrm{d}s^2 = (\eta\eta)\,\mathrm{d}t^2 + 2\,(\eta\xi)\,\mathrm{d}t\,\mathrm{d}\phi + (\xi\xi)\,\mathrm{d}\phi^2 + g_{rr}\,\mathrm{d}r^2 + g_{\phi\phi}\,\mathrm{d}\theta^2} \tag{2.16}$$

With the condition of stationarity and axisymmetry given by:

$$g_{it} = g_{i\phi} = 0 \qquad i \in \{r, \theta\}$$

The 4-velocity of a particle orbit is:

$$u^a = A\,(\eta^a + \Omega\,\xi^a) = \frac{dx^a}{ds}\ , \tag{2.17}$$

where (A, Ω) are scalars such that,

$$\begin{aligned} u^t &= A &= dt/ds \\ u^\phi &= A\,\Omega &= d\phi/ds\ . \end{aligned}$$

We then see that Ω defines the **angular velocity** since

$$\Omega := \frac{u^\phi}{u^t} = \left(\frac{d\phi}{ds}\right) \Big/ \left(\frac{dt}{ds}\right) = \frac{d\phi}{dt} \,. \tag{2.18}$$

We might as well evaluate the specific angular momentum also:

$$\ell := -\frac{u_\phi}{u_t} = \frac{-u^a \xi_a}{u^a \eta_a} = \frac{(\eta\xi) + \Omega\,(\xi\xi)}{(\eta\eta) + \Omega\,(\eta\xi)} \,. \tag{2.19}$$

Comment: Note that this is not the only definition for the specific agular momentum used in the literature. Often, in fact, the specific angular momentum is defined as $\ell \equiv u_\phi$ because this is a constant of geodesic (i.e. zero-pressure) motion in axially symmetric spacetimes. When the pressure is non-zero, on the other hand, u_ϕ is a constant of motion, while ℓ is not. For axially symmetric, stationary spacetimes $\ell \equiv -u_\phi/u_t$ is constant for both geodesic and perfect fluid motion.

Inverting eq (2.19), gives us $\Omega(\ell)$ as well.

$$\Omega = \frac{(\eta\xi) + \ell\,(\eta\eta)}{(\xi\xi) + \ell\,(\eta\xi)} \tag{2.20}$$

We know that dragging of the inertial frames comes from the $(\eta\xi)$ term. Then to get the angular velocity, we divide by the distance from the axis of rotation $-(\xi\xi) = r^2 \sin^2\theta$. And we obtain the *angular velocity of inertially dragged frames.*

$$\omega := -\frac{(\eta\xi)}{(\xi\xi)} \tag{2.21}$$

In general for axisymmetric spacetimes we can immediately see from ω whether or not the spacetime is static or stationary.

$$\begin{array}{lll} \text{stationary} & \rightarrow & \omega = constant \neq 0 \\ \text{static} & \rightarrow & \omega = 0 \end{array}$$

The Acceleration

Next, if we consider a special circular orbit in which $(\theta, R) = const$, we can directly obtain from the metric, a quantity that will be very useful later.

$$\begin{aligned} \mathrm{d}s^2 &= (\eta\eta)\,\mathrm{d}t^2 + 2\,(\eta\xi)\,\mathrm{d}t\,\mathrm{d}\phi + (\xi\xi)\,\mathrm{d}\phi^2 \\ 1 &= (\eta\eta)\,(dt/ds)^2 + 2\,(\eta\xi)\,(dt/ds)\,(d\phi/ds) + (\xi\xi)\,(d\phi/ds)^2 \\ &= A^2\left[(\eta\eta) + 2\,(\eta\xi)\,\Omega + (\xi\xi)\,\Omega^2\right] \\ &\Rightarrow \quad A^{-2} = (\eta\eta) + 2\,(\eta\xi)\,\Omega + (\xi\xi)\,\Omega^2 \end{aligned} \tag{2.22}$$

We then want to find the acceleration, as this will be important later when determining the effective potential and the angular velocity in terms of the metric components and its derivatives.

$$\begin{aligned} a_b &= u^a \nabla_a u_b \\ &= A\,(\eta^a + \Omega\xi^a)\,\nabla_a\left[A\,(\eta_b + \Omega\xi_b)\right] \\ & \quad \text{Assume } A, \Omega \text{ are } t, \phi \text{ independent:} \end{aligned}$$

$$
\begin{aligned}
&= A^2\left[\eta^a\nabla_a\eta_b + \Omega\eta^a\nabla_a\xi_b + \Omega\xi^a\nabla_a\eta_b + \Omega^2\xi^a\nabla\xi_b\right] \\
&\quad \text{Using the Killing symmetry of eq.(2.12)} \\
&= A^2\left[-\tfrac{1}{2}\nabla_b(\eta^a\eta_a) - \tfrac{1}{2}\Omega\nabla_b(\eta^a\xi_a) - \tfrac{1}{2}\Omega\nabla_b(\xi^a\eta_a) - \tfrac{1}{2}\Omega^2\nabla_b(\xi^a\xi_a)\right] \\
&= -\tfrac{1}{2}A^2\left[\nabla_b(\eta\eta) + \Omega\nabla_b(\eta\xi) + \Omega\nabla_b(\xi\eta) + \Omega^2\nabla_b(\xi\xi)\right] \\
&= -\tfrac{1}{2}A^2\left[\nabla_b(\eta\eta) + \underbrace{2\Omega\nabla_b(\eta\xi)}_{\text{dragging}} + \Omega^2\nabla_b(\xi\xi)\right] \qquad (2.23) \\
&= -\tfrac{1}{2}A^2\left\{\nabla_b\left[(\eta\eta) + 2\Omega(\eta\xi) + \Omega^2(\xi\xi)\right] - (\eta\xi)\nabla_b(2\Omega) - (\xi\xi)\nabla_b(\Omega^2)\right\} \\
&\quad \text{Using eq.(2.22)} \\
&= -\tfrac{1}{2}A^2\left\{\nabla_b(A^{-2}) - 2(\eta\xi)\nabla_b\Omega - 2\Omega(\xi\xi)\nabla_b\Omega\right\} \\
&= -\tfrac{1}{2}A^2\left\{\nabla_b(A^{-2}) - 2\nabla_b\Omega\left[(\eta\xi) + \Omega(\xi\xi)\right]\right\} \\
&= A^{-1}\nabla_b A + \left(\frac{(\eta\xi) + \Omega(\xi\xi)}{(\eta\eta) + 2\Omega(\eta\xi) + (\xi\xi)\Omega^2}\right)\nabla_b\Omega \\
&\quad \text{Divide top and bottom of second term by: } \left[(\eta\eta) + \Omega(\eta\xi)\right]. \\
&= \nabla_b \ln A + \frac{B}{1+\Omega B}\nabla_b\Omega \qquad (2.24) \\
&\quad \text{Identify } B \text{ with the negative specific angular momentum in eq. (2.19).} \\
&= \nabla_b \ln A - \frac{\ell}{1-\Omega\ell}\nabla_b\Omega \qquad (2.25)
\end{aligned}
$$

From this derivation we see that the $2\Omega\nabla_b(\eta\xi)$ term is caused by the **dragging**[1] of inertial frames are, which for obvious reasons also explains why co-rotation and counter rotation does not have equal rotational periods. Furthermore, for circular geodesic motion, $a_b = 0$ and we can use eq.(2.23) to derive $\Omega = \Omega[g_{ab}, \partial_c g_{ab}]$.

$$
\begin{aligned}
0 &= \nabla_b(\eta\eta) + 2\Omega\nabla_b(\eta\xi) + \Omega^2\nabla_b(\xi\xi) \\
&\quad \text{But these are all scalar functions of } r \text{ only,} \\
&\quad \text{if we are in the equatorial plane } \theta = \pi/2. \\
&= \partial_b(\eta\eta) + 2\Omega\partial_b(\eta\xi) + \Omega^2\partial_b(\xi\xi) \\
&= g_{tt,r} + 2\Omega g_{t\phi,r} + \Omega^2 g_{\phi\phi,r} \\
&= a\Omega^2 + 2b\Omega + c
\end{aligned}
$$

Then we have the quadratic solution for the angular velocity:

$$
\boxed{\Omega_\pm = \frac{-b \pm (b^2 - ac)^{1/2}}{a}} \qquad (2.26)
$$

where: $a = g_{\phi\phi,r}$, $b = g_{t\phi,r}$ and $c = g_{tt,r}$.

The Effective Potentials

The introduction of various effective potentials can be a very useful tool for understanding orbital mechanics. I will derive three slightly different potentials. Each which will serve its own purpose to shed light on some particular feature. In mechanics one can often write a force as the gradient of some

[1] Dragging of inertial frames is also known as the Lense-Thirring or gravo-magnetic effect.

scalar potential. In the first example, we will use this fact to understand what is the corresponding gravitational potential for a particle orbiting in a $4d$ spacetime. In the second case we will derive the energy conservation equation for the static spacetime case, again showing the exact form of the gravitational potential. In the last case we will derive another effective potential which will be used in the derivation for the epicyclic frequencies.

In fact realizing that ℓ is a constant of motion, we can take the analysis of eq.(2.25) further by rewriting it as:

$$\begin{aligned} a_b &= \nabla_b \ln A - \frac{\ell\, \nabla_b \Omega}{1-\Omega\,\ell} \\ &= \nabla_b \ln A + \frac{\nabla_b\,(1-\Omega\,\ell)}{(1-\Omega\,\ell)} \\ &= \nabla_b \ln\left[A\,(1-\Omega\,\ell)\right] \\ &=: \nabla_b \psi\,. \end{aligned} \tag{2.27}$$

In this light, we have discovered that we can derive an **effective potential**, ψ, which in turn can be used to extract a wealth of information, especially in the Newtonian limit. So let us start by reworking the effective potential.

$$\begin{aligned} \mathrm{e}^{\psi} &= (1-\Omega\,\ell)\left\{(\eta\eta) + 2\,\Omega\,(\eta\xi) + \Omega^2\,(\xi\xi)\right\}^{-1/2} \\ &= (1-\Omega\,\ell)\left\{\left[(\eta\eta) + \Omega\,(\eta\xi)\right] + \Omega\left[(\eta\xi) + \Omega\,(\xi\xi)\right]\right\}^{-1/2} \\ &= (1-\Omega\,\ell)\left\{\left[(\eta\eta) + \Omega\,(\eta\xi)\right]\left[1-\Omega\,\ell\right]\right\}^{-1/2} \\ &= (1-\Omega\,\ell)^{1/2}\left[(\eta\eta) + \Omega\,(\eta\xi)\right]^{-1/2} \\ &\quad \text{Inserting eq.(2.20) for } \Omega. \\ &= (1-\Omega\,\ell)^{1/2}\left[\frac{(\eta\eta)\,(\xi\xi) - (\eta\xi)^2}{(\xi\xi) + \ell\,(\eta\xi)}\right]^{-1/2} \\ &\quad \text{Inserting eq.(2.21) for } \omega. \\ &= (1-\Omega\,\ell)^{1/2}\left[\frac{(\eta\eta)\,(\xi\xi) - (\eta\xi)^2}{(\xi\xi)\,(1-\omega\,\ell)}\right]^{-1/2} \\ &= \left[\frac{(1-\Omega\,\ell)\,(1-\omega\,\ell)}{(\eta\eta) + \omega\,(\eta\xi)}\right]^{1/2} \end{aligned}$$

We can then recover the separate terms of the effective potential.

$$\begin{aligned} \psi &= \tfrac{1}{2}\ln[(1-\Omega\,\ell)\,(1-\omega\,\ell)] \quad \underbrace{-\tfrac{1}{2}\ln[(\eta\eta) + \omega\,(\eta\xi)]}_{\Phi} \\ &= \Phi + \tfrac{1}{2}\ln(1-\Omega\,\ell) + \tfrac{1}{2}\ln(1-\omega\,\ell) \end{aligned}$$

But in the Newtonian limit we have low velcities as compared to that of light, so $(\Omega\ell, \omega\ell \ll 1)$ and no frame dragging $\omega = 0$. Then

$$\Omega = \ell/r^2 = v/r \quad \Rightarrow \quad v^2 = \Omega\ell \tag{2.28}$$

$$\begin{aligned} \psi &\simeq \Phi - \tfrac{1}{2}\Omega\,\ell - \tfrac{1}{2}\omega\,\ell \\ &\simeq \Phi - \tfrac{1}{2}\,v^2 \end{aligned}$$

And we can thus understand Φ as the gravitational potential, with

$$\Phi = -\tfrac{1}{2}\ln[(\eta\eta) + \omega\,(\eta\xi)]\,. \tag{2.29}$$

Next we consider more general particle motion in a **static** gravitational field. The 4-velocity of such a particle can be described by

$$u^a = A\,(\eta^a + \Omega\,\xi^a + v\,\tau^a) \tag{2.30}$$

where v is the particle 3-velocity such that $\{v \in (-1,1)\}$ and

$$\begin{aligned} \eta^a &= \text{timelike Kvf.} \\ \xi^a &= \text{spacelike Kvf.} \\ \tau^a &= \text{spacelike unit vector} \perp \text{to } (\eta,\xi). \end{aligned}$$

Then in a static spacetime, we have the following conserved quantities.

$$\begin{aligned} E &= u^a\eta_a &&= A\,(\eta\eta) \\ L &= -u^a\xi_a &&= -A\Omega\,(\xi\xi) \end{aligned}$$

Then we can use the normalization of u^a to find the effective potential.

$$\begin{aligned} 1 &= u^a u_a \qquad\qquad (2.31) \\ &= A^2\,[(\eta\eta) + \Omega^2\,(\xi\xi) + v^2\,(\tau\tau)] \quad + \text{other terms} \\ &= A^2\,(\eta\eta) + A^2\,\Omega^2\,(\xi\xi) - A^2\,v^2 \\ &\qquad \text{Multiply both sides by } (\eta\eta). \\ (\eta\eta) &= E^2 + L^2\left[\frac{(\eta\eta)}{(\xi\xi)}\right] - A^2\,v^2\,(\eta\eta) \qquad\qquad (2.32) \end{aligned}$$

Then if we define:

$$\begin{aligned} V^2 &:= A^2\,v^2\,(\eta\eta) && \text{(Gravitational Potential)} \\ \psi &:= (\eta\eta) - L^2\tfrac{(\eta\eta)}{(\xi\xi)} && \text{(Effective Potential)} \\ &= (\eta\eta)\,[1 - L^2(\xi\xi)^{-1}] \end{aligned}$$

Moving the terms around in (2.32) we see that the effective potential is equal to the energy minus the potential energy, $\psi = E^2 - V^2$.

Yet another potential can be derived by assuming only a **stationary** spacetime with circular orbits. We then follow a slightly different route, by using

$$u^a = g^{ab}u_b\,. \tag{2.33}$$

So that:

$$\begin{aligned} u^t &= g^{tb}u_b = g^{tt}u_t + g^{t\phi}u_\phi \\ u^\phi &= g^{\phi b}u_b = g^{t\phi}u_t + g^{\phi\phi}u_\phi \\ u^r &= g^{rr}u_r \\ u^\theta &= g^{\theta\theta}u_\theta \end{aligned}$$

In addition we need

$$\begin{aligned} u^r &= \frac{\mathrm{d}r}{\mathrm{d}s} = \delta r \\ u^\theta &= \frac{\mathrm{d}\theta}{\mathrm{d}s} = \delta\theta\,. \end{aligned}$$

Then from the normalization for a timelike trajectory,

$$\begin{aligned}
1 &= u^a u_a = u^t u_t + u^r u_r + u^\theta u_\theta + u^\phi u_\phi \\
&= (g^{tt}u_t + g^{t\phi}u_\phi)u_t + g^{rr}(u_r)^2 + g^{\theta\theta}(u_\theta)^2 + (g^{t\phi}u_t + g^{\phi\phi}u_\phi)u_\phi \\
&= g^{tt}(u_t)^2 + 2g^{t\phi}(u_t u_\phi) + g^{\phi\phi}(u_\phi)^2 + g^{rr}(u_r)^2 + g^{\theta\theta}(u_\theta)^2 \\
&\quad \text{But:} \quad -L = u_\phi \text{ and } E = u_t. \\
&= g^{tt}E^2 + 2g^{t\phi}(-LE) + g^{\phi\phi}(-L)^2 + g^{rr}(u_r)^2 + g^{\theta\theta}(u_\theta)^2 \\
&\quad \text{But:} \quad \ell = L/E, \text{ and } g^{AA}(u_A)^2 = g_{AA}(u^A)^2, \text{ where } A \in \{r, \theta\}. \\
&= E^2\left[g^{tt} - 2\ell\, g^{t\phi} + \ell^2\, g^{\phi\phi}\right] + g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 \\
&= E^2\left[g^{tt} - 2\ell\, g^{t\phi} + \ell^2\, g^{\phi\phi}\right] + g_{rr}(\delta\dot{r})^2 + g_{\theta\theta}(\delta\dot{\theta})^2
\end{aligned}$$

Dividing through by E^2 we have

$$1/E^2 = U + E^{-2}\left[g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2\right] \tag{2.34}$$

Where we have defined an effective potential as

$$U(r, \theta, \ell) := g^{tt} - 2\ell\, g^{t\phi} + \ell^2\, g^{\phi\phi} \tag{2.35}$$

since for circular motion, $\delta r = 0$ and $\delta\theta = 0$.

The Epicyclic Frequencies

If one consider small perturbations of circular orbits, we can choose two different perturbations. One in the radial direction and one in the transverse (θ in spherical coordinates) direction. It will be shown that these kind of perturbations give rise and even define the **epicyclic frequencies**. We will first show the derivation for the radial perturbation and then see that the derivation of the transverse is nearly equivalent.

From Newtonian kinematics, we know that energy is the sum of potential and kinetic energy ($E = U + \frac{1}{2}v^2$.) But in orbital kinematics we also know that stable orbits take place where the effective potential has a minimum. (See Fig.2.8.) But since it is a minimum, one could imagine that for small perturbations, the effective potential may oscillate around this minima, resulting in a similar perturbation of the orbits. We can therefore Taylor expand the effective potential around r or θ for small oscillations, up to 2nd order. Next we make use of the fact that for circular orbits

$$\left.\begin{aligned}
E - U &= 0 \\
\partial U/\partial r &= 0 \\
\partial^2 U/\partial r^2 &> 0
\end{aligned}\right\} \quad \text{(circular motion)} \tag{2.36}$$

which will give us the energy equation for the perturbation. Taking the time derivative of this equation we obtain the equation of motion.

The radial perturbation

For motion in the radial direction (δr) the velocity is

$$v = \frac{\mathrm{d}}{\mathrm{d}t}(\delta r) = \delta\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right) = \delta\dot{r}\,. \tag{2.37}$$

The Taylor expansion of the effective potential around r for small oscillations δr, up to 2nd order, is

$$U \to U' = U + \frac{\partial U}{\partial r}\,\delta r + \frac{1}{2}\,\frac{\partial^2 U}{\partial r^2}\,(\delta r)^2 = 0 \tag{2.38}$$

And so our energy equation take the form:

$$\begin{aligned} E - U - \tfrac{1}{2}v^2 &= 0 \\ E - U - \frac{\partial U}{\partial r}\,\delta r - \tfrac{1}{2}\,\frac{\partial^2 U}{\partial r^2}\,(\delta r)^2 - \tfrac{1}{2}(\delta\dot{r})^2 &= 0 \end{aligned}$$

Then upon applying the constraints (2.36) for circular motion,

$$\boxed{\tfrac{1}{2}\,\tfrac{\partial^2 U}{\partial r^2}\,(\delta r)^2 + \tfrac{1}{2}(\delta\dot{r})^2 = 0}$$

Converting to a force equation by taking the time derivative and dividing out common factors.

$$\boxed{\tfrac{\partial^2 U}{\partial r^2}\,(\delta r) + \delta\ddot{r} = 0}$$

The coefficient in front of δr is what we define as the square of the epicyclic frequency:

$$\kappa_r^2 := \frac{\partial^2 U}{\partial r^2}\,. \tag{2.39}$$

The transverse perturbation

For circular motion in the transverse direction ($\delta\theta$) the velocity is

$$v = r\omega = r\,\frac{\mathrm{d}}{\mathrm{d}s}(\delta\theta) = r\,\delta(\frac{\mathrm{d}\theta}{\mathrm{d}s}) = r\,\delta\dot{\theta}\,. \tag{2.40}$$

Similarly

$$U \to U' = U + \frac{\partial U}{\partial \theta}\,\delta\theta + \tfrac{1}{2}\,\frac{\partial^2 U}{\partial \theta^2}\,(\delta\theta)^2 = 0 \tag{2.41}$$

But this time:

$$\boxed{\tfrac{1}{2r^2}\,\tfrac{\partial^2 U}{\partial \theta^2}\,(\delta\theta)^2 + \tfrac{1}{2}(\delta\dot{\theta})^2 = 0}$$

Converting to a force equation by taking the time derivative and dividing out common factors.

$$\boxed{\tfrac{1}{r^2}\,\tfrac{\partial^2 U}{\partial \theta^2}\,(\delta\theta) + \delta\ddot{\theta} = 0}$$

Which defines the transverse epicyclic frequency:

$$\kappa_\theta^2 := \frac{1}{r^2}\,\frac{\partial^2 U}{\partial \theta^2}\,. \tag{2.42}$$

The general formula

We are now ready to see what happens to the effective potential on the equatorial plane. We begin by checking the constraints for circular motion in eq (2.34). On the equatorial plane $\delta\dot{\theta} = 0$ and together with the constraints for circular orbits, we rewrite as

$$\frac{1}{2}\,\frac{\partial^2 U}{\partial r^2}\,(\delta r)^2 + E^{-2}\,g_{rr}(\delta\dot{r})^2 = 0 \tag{2.43}$$

So:

$$\begin{aligned} \frac{\partial^2 U}{\partial r^2}(\delta r) + 2\,E^{-2}\,g_{rr}(\delta\ddot{r})^2 &= 0 \\ \Rightarrow \qquad \frac{E^2}{2g_{rr}}\frac{\partial^2 U}{\partial r^2}(\delta r) + \delta\ddot{r} &= 0 \end{aligned}$$

we then have an equation of the form:

$$\ddot{x} + \kappa_r^2\, x = 0 \tag{2.44}$$

where we have defined the coefficient as:

$$\kappa_r^2 := \frac{E^2}{2\,g_{rr}}\frac{\partial^2 U}{\partial r^2}\,. \tag{2.45}$$

But for an observer at infinity, epicyclic frequency will be affected by the redshift factor $1/u_t^2$. Therefore he will measure

$$\widetilde{\kappa}_r^2 = \frac{E^2}{2\,A^2\,g_{rr}}\left(\frac{\partial^2 U}{\partial r^2}\right)_\ell \tag{2.46}$$

But

$$\frac{E}{A} = \frac{u^a\eta_a}{A} = (\eta\eta) + \Omega(\eta\xi) = g_{tt} + \Omega\, g_{t\phi} \tag{2.47}$$

Similarly we can derive the formula for the transverse epicyclic frequency. The final formulas for the epicyclic frequencies in any axisymmetric and stationary spacetime are

$$\widetilde{\kappa}_r^2 = \frac{(g_{tt} + \Omega_\pm\, g_{t\phi})^2}{2g_{rr}}\left(\frac{\partial^2 U}{\partial r^2}\right)_\ell \tag{2.48}$$

$$\widetilde{\kappa}_\theta^2 = \frac{(g_{tt} + \Omega_\pm\, g_{t\phi})^2}{2g_{\theta\theta}}\left(\frac{\partial^2 U}{\partial \theta^2}\right)_\ell \tag{2.49}$$

There have been some objections to these formulas, since it can be easily proved that in Newtonian theory, if the specific angular is held constant, the epicyclic frequency is zero. This since

$$\kappa_r^2 := \frac{2\Omega}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\Omega\right) \tag{2.50}$$

and that $\ell = r^2\Omega =$ const. This is true also in GR, but the point is that our formulas does *not* hold ℓ constant, all we are doing is saying that we can approximate the the equatorial orbits as Keplerian, where $\Omega = 1/r^3$.

2.3 Kinematics

In this section I derive the *kinematic invariants*, the shear, vorticity and expansion. In addition I shall study the Fermi-Walker derivative and its importance in gyroscope precession.

Let u^a be the particle 4-velocity. Then we define a projection tensor $h^a{}_b$ and a velocity derivative $B^a{}_b$.

$$\begin{aligned} h^a{}_b &:= \delta^a{}_b - u^a u_b \\ B^a{}_b &:= \nabla_b u^a \end{aligned}$$

Project the velocity derivative into the hyperplane.

$$\begin{aligned} h^a{}_b \nabla_b u^a &= (\delta^a{}_b - u^a u_b)\nabla_b u^a \\ &= \nabla_b u^a - u^a u_b \nabla_a u^b \\ & \quad \text{But } u^a \nabla_a u^b = a^b. \\ &= \nabla_b u^a - u_b a^c \end{aligned} \tag{2.51}$$

Then

$$B^c_{\perp b} := h^a{}_b B^a{}_b = B^a{}_b - u_b a^c \tag{2.52}$$

The kinematic invariants are then the different projections of the $4d$ velocity derivative onto the $3d$ hyperplane. In particular the trace, symmetric traceless and antisymmetric parts are summarized in the table below.

θ	$:=$	$B^a_{\perp a}$	Volume Expansion	(Trace)
σ_{ab}	$:=$	$B_{\perp(ab)} - \frac{1}{3}\theta g_{ab}$	Shear Tensor	(Symmetric Traceless)
ω_{ab}	$:=$	$B_{\perp[ab]}$	Vorticity Tensor	(Antisymmetric)

Then we can write $B_{\perp ab}$ as a sum of the above parts of the projected acceleration.

$$\begin{aligned} B_{\perp ab} &= \sigma_{ab} + \omega_{ab} + \tfrac{1}{3}\theta g_{ab} \\ & \quad \text{Substituting in eq. (2.51).} \\ \Rightarrow \quad \nabla_a u_b &= u_a a_b + \sigma_{ab} + \omega_{ab} + \tfrac{1}{3}\theta g_{ab} \end{aligned}$$

Circular Motion

For circular motion:

$$\begin{aligned} \theta &= B^a_{\perp a} = \nabla^a u_b \equiv 0 \\ 2\sigma_{ab} &= 2\sigma_{(ab)} = \sigma_{ab} + \sigma_{ba} \\ &= B_{\perp(ab)} - \tfrac{1}{3}\theta g_{ab} \\ &= g_{ac} B^c_{\perp b} \\ &= g_{ac}\left(\nabla_b u^a - u_b a^c\right) \\ &= \nabla_b u_c + \nabla_c u_b - u_b a_c - u_a a_b \end{aligned} \tag{2.53}$$

But since $\Gamma^a{}_{bc}$ is symmetric, we can also write

$$\nabla_b u_c + \nabla_c u_b = \partial_b u_c + \partial_c u_b - 2\Gamma^a{}_{bc} u_a \,. \tag{2.54}$$

Introducing the special indicies:

$$\begin{aligned} \{A, B, C\} &\in \{r, \theta\} \\ \{\Phi, \Psi\} &\in \{\phi, t\} \end{aligned}$$

We easily find that

$$\begin{aligned} \sigma_{AB} &= \partial_A u_B + \partial_B u_A - \Gamma^\Phi{}_{AB} u_\Phi - \Gamma^C{}_{AB} u_C \equiv 0 \\ \sigma_{\Phi\Psi} &= \partial_\Phi u_\Psi + \partial_\Psi u_\Phi - \Gamma^A{}_{\Psi\Phi} u_A - \Gamma^\theta{}_{\Phi\Psi} u_\theta \equiv 0 \end{aligned} \tag{2.55}$$

The Shear

Therefore, the only non-zero components of the shear are those of $\sigma_{A\Phi}$. However, we can make use of the fact that there are in addition two invariant vectors, $\xi^c\sigma_{ac}$ and $\eta^c\sigma_{ac}$. On contracting σ_{ab} with the first one of these, we can follow a calculation similar to that of the acceleration, to find the shear. (The other one can be used in a similar way to derive the vorticity, but as will be shown in the next section, we shall rather make use of a mathematical trick for this derivation.)

How can we see this invariance?

For a static spacetime with $u_a = A\,(\eta_a + \Omega\,\xi_a)$, we can proceed by using (2.53) and splitting σ_{ab} like above.

$$2\xi^c\sigma_{bc} = \underbrace{\xi^c\nabla_c u_b}_{(i)} + \underbrace{\xi^c\nabla_b u_c}_{(ii)} - \underbrace{(\xi u)}_{(iii)} - \underbrace{\xi^c u_b a_c}_{=\,0} \tag{2.56}$$

From previous calculations we already have that:

$$\begin{aligned}
(i) &= \xi^c\nabla_c u_b = -\tfrac{1}{2}A\,\Omega\,\nabla_b(\xi\xi) \\
(ii) &= \xi^c\nabla_b u_c = \xi^c\,\nabla_b[A\,(\eta_c + \Omega\,\xi_c)] \\
&= (\xi\xi)\,\nabla_b(A\,\Omega) + \tfrac{1}{2}\,A\,\Omega\,\nabla_c(\xi\xi) \\
(iii) &= -A\,\Omega\,(\xi\xi)\left[\frac{\nabla_b A}{A} - \frac{\ell\nabla_b\Omega}{1-\Omega\ell}\right]
\end{aligned}$$

$$\begin{aligned}
(i)+(ii)+(iii) &= (\xi\xi)\left\{\nabla_b(A\,\Omega) - \Omega\nabla_a A + \frac{A\,\Omega\,\ell\nabla_b\Omega}{1-\Omega\ell}\right\} \\
&= A\,(\xi\xi)\,(\nabla_a\Omega)\left[1 + \frac{\Omega\ell}{1-\Omega\ell}\right] \\
&= A\,(\xi\xi)\,(\nabla_a\Omega)\,\frac{1}{1-\Omega\ell} \\
&= \gamma^2\,A\,(\xi\xi)\,(\nabla_a\Omega) \\
&= \gamma^3\,(\eta\eta)^{-1/2}\,(\xi\xi)\,(\nabla_a\Omega) \\
&= -(-\xi\xi)^{1/2}(-\xi\xi)^{1/2}\,\gamma^3\,(\eta\eta)^{-1/2}\,(\nabla_a\Omega)
\end{aligned}$$

Dividing both sides by $-(-\xi\xi)^{1/2}$, we have

$$-2\,\frac{\xi^c}{(-\xi\xi)^{1/2}}\,\sigma_{bc} = \frac{(-\xi\xi)^{1/2}}{(\eta\eta)^{-1/2}}\,\gamma^3\,\nabla_a\Omega \tag{2.57}$$

But $\tau^a = \xi^c/(-\xi\xi)^{1/2}$, and $\mathcal{R} := (-\xi\xi)^{1/2}(\eta\eta)^{-1/2}$, so

$$-2\tau^c\sigma_{bc} = \gamma^3\,\mathcal{R}\,\nabla_b\Omega\,. \tag{2.58}$$

But since,

$$\begin{aligned}
A^2 &= \frac{1}{(\eta\eta)\,(1-\beta^2)} \\
\Rightarrow\quad A &= (\eta\eta)^{-1/2}\,\gamma
\end{aligned}$$

$$\boxed{\tau^c\sigma_{bc} = -\tfrac{1}{2}\gamma^3\,\mathcal{R}\,\nabla_a\Omega} \tag{2.59}$$

So when we say that we calculate the shear, we actually mean to calculate the contraction $\sigma^{ab}\sigma_{ab}$.

$$\begin{aligned}\sigma^2 &= \sigma^{ac}\sigma_{ac} \\ &= \sigma^{\phi A}\sigma_{\phi A} + \sigma^{tA}\sigma_{tA}\end{aligned}$$

But contraction of the shear with the 4-velocity is zero since they are orthogonal.

$$\begin{aligned}\sigma_{ac}u^c &= \sigma_{ac}\eta^c + \sigma_{ac}\xi^c\Omega = 0 \\ \Rightarrow \quad \sigma_{ac}\eta^c &= -\sigma_{ac}\xi^c\Omega \end{aligned} \tag{2.60}$$

But

$$\begin{aligned}\sigma^{b\phi} &= -\tfrac{1}{2}(-\xi\xi)^{1/2}g^{\phi\phi}\,\mathcal{R}\,\gamma^3\,\nabla^{\,b}\Omega \qquad & \sigma^{bt} &= \tfrac{1}{2}(-\xi\xi)^{1/2}g^{tt}\,\mathcal{R}\,\gamma^3\,\Omega\nabla^{\,b}\Omega \\ \sigma_{b\phi} &= -\tfrac{1}{2}(-\xi\xi)^{1/2}\mathcal{R}\,\gamma^3\,\nabla_b\Omega \qquad & \sigma_{bt} &= \tfrac{1}{2}(-\xi\xi)^{1/2}\mathcal{R}\,\gamma^3\,\Omega\nabla_b\Omega\end{aligned}$$

Then

$$\begin{aligned}\sigma^2 &= \tfrac{1}{4}\left(\nabla_b\Omega\right)\left(\nabla^b\Omega\right)\left[\mathcal{R}\,\gamma^3\right]\left[1+\Omega^2\left(-\xi\xi\right)\left(\eta\eta\right)\right] \\ &= \tfrac{1}{4}\left(\nabla_b\Omega\right)\left(\nabla^b\Omega\right)\left[\mathcal{R}\,\gamma^3\right]\left[\mathcal{R}\,\gamma\right] \\ &= \tfrac{1}{4}\,\mathcal{R}^2\,\gamma^4\left(\nabla_b\Omega\right)\left(\nabla^b\Omega\right)\end{aligned} \tag{2.61}$$

We can then check these results by considering the Newtonian limit. In Newtonian hydrodynamics we know that motion with constant angular momentum $\ell = const$, the shear is equal to angular velocity. With $\Omega = \ell/r^2$, we can check this by taking the Newtonian limit, which reduces to making $\mathcal{R} = r$ and $\gamma = 1$. So that

$$\begin{aligned}\sigma^2 &= \tfrac{1}{4}\,\mathcal{R}^2\,\gamma^4\left(\nabla_b\Omega\right)\left(\nabla^b\Omega\right) \\ &\simeq \tfrac{1}{4}\,r^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2 = \tfrac{1}{4}\,r^2\left(\frac{-2\ell}{r^3}\right)^2 = \frac{\ell^2}{r^4} = \Omega^2 \qquad \textbf{Ok!}\end{aligned}$$

Vorticity and Duality

If one where to follow the same procedure as previously, but using the other invariant vector $\eta^c\sigma_{ac}$ we would obtain the vorticity. However, there is a symmetry which allows you to directly write down the equation for the expansion scalar ω. The manifestation of this symmetry is known as a duality transform, which in this case is obtained by just replacing the following quantities in the previous calculation.

$$\begin{aligned}\eta &\rightarrow \xi \\ \mathcal{R} &\rightarrow \tfrac{1}{\mathcal{R}} \\ \Omega &\rightarrow \ell\end{aligned}$$

so that

$$\omega^2 = \tfrac{1}{4}\,\mathcal{R}^{-2}\,\gamma^4\left(\nabla_b\ell\right)\left(\nabla^b\ell\right)\,. \tag{2.62}$$

This duality is particularly apparent in the shear-free and vorticity-free cases as briefly discussed in [118].

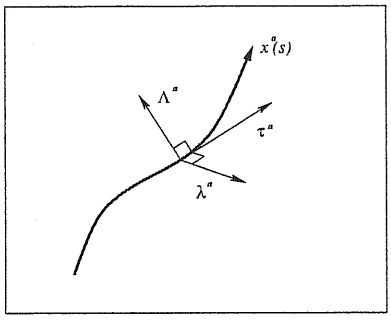

Figure 2.5: The Frenet construction, where τ^a is the tangent, λ^a and Λ^a are the first and second normals, respectively, all mutually orthogonal.

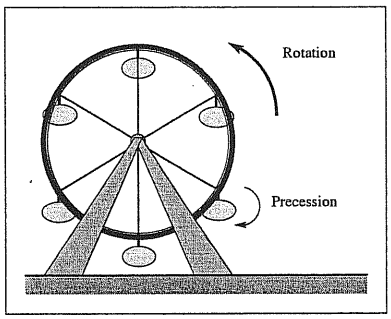

Figure 2.6: The Fermi-Walker derivative measures the precession of vectors during spacetime rotations.

The Fermi-Walker Derivative

The Fermi-Walker (FW) derivative can be intuitively better known as the *Ferris-Wheel* derivative. The reason for this is that it is a measure of the angular precession during rotation. This is exactly what happens to the precession angle of the baskets in a ferris-wheel rotating counter-clockwise as in Fig.2.6, given that it is rotating slowly enough that the baskets remain paralell. We define the *Fermi-Walker derivative* of a vector τ_c along the particle trajectory (4-velocity) as:

$$\mathcal{D}_u^{FW}\tau_c := u^a\,\nabla_a\tau_c + (a^a u_c - u^a a_c)\,\tau_a \tag{2.63}$$

The (first) sign depends on the signature used. The correct sign is determined by the requirement that $\mathcal{D}_u^{FW}u_b = 0$. With our convention $(uu) = +1$, we have that $\mathcal{D}_u^{FW}u_b = a_b - a_b = 0$, and so its ok.

Static spacetime

Starting with the usual ansatz that $u^a = A(\eta^a + \Omega\xi^a)$ and that $\tau^a = \xi^a/(-\xi\xi)^{1/2}$ is spacelike unit vector tangent to u^a, we can derive the Fermi-Walker derivative for circular motion in a static spacetime. In that case we can take $(\eta\xi) = 0$, in the acceleration formula eq.(2.23). In addition we define $B := \beta^2/(1-\beta^2)$. And so

$$\begin{aligned} a_c &= \nabla_c\Phi + \frac{\beta^2}{1-\beta^2}\frac{1}{\mathcal{R}}\nabla_c\mathcal{R} \\ &= \nabla_c\Phi + B\,\nabla_c(\ln\mathcal{R}) \end{aligned}$$

We procced with

$$\begin{aligned} \mathcal{D}_u^{FW}\tau_c &:= u^a\,\nabla_a\tau_c + (a^a u_c - u^a a_c)\,\tau_a \\ &= u^a\,\nabla_a\tau_c - a_c\,(u_c\tau^c) \end{aligned}$$

With:

$$\begin{aligned} u_c\tau^c &= A\,(\eta^a + \Omega\xi^a)\,\xi_a\,(-\xi\xi)^{-1/2} \\ &= A\,\Omega\,(\xi\xi)\,(-\xi\xi)^{-1/2} \\ &= -A\,\Omega\,(-\xi\xi)^{1/2} \end{aligned}$$

$$\text{and:}\quad u^a\,\nabla_a\tau_c = A\,(\eta^a + \Omega\xi^a)\,\nabla_a[\xi_c\,(-\xi\xi)^{-1/2}]$$

$$
\begin{aligned}
&= A\,(\eta^a + \Omega\xi^a)\,[\xi_c\,\nabla_a(-\xi\xi)^{-1/2} + (-\xi\xi)^{-1/2}\,\nabla_a\xi_c] \\
&\quad \text{But: } \eta \perp \xi \text{ and } \xi^a\xi_c = 0. \\
&= A\,\Omega\,(-\xi\xi)^{-1/2}\,\xi^a\,\nabla_a\xi_c \\
&\quad \text{But from eq.(2.12): } \xi^a\nabla_a\xi_c = -\tfrac{1}{2}\nabla_c(\xi\xi). \\
&= -\tfrac{1}{2}\,A\,\Omega\,(-\xi\xi)^{-1/2}\,\nabla_c(\xi\xi)
\end{aligned}
$$

So that

$$
\begin{aligned}
\mathcal{D}_u^{FW}\tau_c &= -\tfrac{1}{2}\,A\,\Omega\,(-\xi\xi)^{-1/2}\,\nabla_c(\xi\xi) + [\nabla_c\Phi + B\,\nabla_c(\ln\mathcal{R})]\;A\,\Omega\,(-\xi\xi)^{1/2} \\
&= A\,\Omega\,(-\xi\xi)^{-1/2}\{-\tfrac{1}{2}\,\nabla_c(\xi\xi) - (\xi\xi)\,[\nabla_c\Phi + B\,\nabla_c(\ln\mathcal{R})]\} \\
&\quad \text{But } 2\nabla_c\ln\mathcal{R} = \nabla_c\ln\mathcal{R}^2. \\
&= -\tfrac{1}{2}\,A\,\Omega\,(-\xi\xi)^{-1/2}\,\{\nabla_c(\xi\xi) - (\xi\xi)\,\nabla_c(-2\,\Phi) + (\xi\xi)\,B\,\nabla_c(\ln\mathcal{R}^2)\} \\
&\quad \text{But } \Phi = -\tfrac{1}{2}\ln(\eta\eta). \\
&= -\tfrac{1}{2}\,A\,\Omega\,(-\xi\xi)^{-1/2}\,\{\nabla_c(\xi\xi) - (\xi\xi)\,\nabla_c\ln(\eta\eta) + (\xi\xi)\,B\,\nabla_c(\ln\mathcal{R}^2)\} \\
&= -\tfrac{1}{2}\,A\,\Omega\,(-\xi\xi)^{-1/2}\,(\eta\eta)\left\{\frac{(\eta\eta)\nabla_c(\xi\xi) - (\xi\xi)\,\nabla_c(\eta\eta)}{(\eta\eta)^2} + \frac{(\xi\xi)}{(\eta\eta)}\,B\,\nabla_c(\ln\mathcal{R}^2)\right\} \\
&\quad \text{But } \mathcal{R}^2 = -(\xi\xi)/(\eta\eta) \\
&= -\tfrac{1}{2}\,A\,\Omega\,(-\xi\xi)^{-1/2}\,(\eta\eta)\left\{\nabla_c\left[\frac{(\xi\xi)}{(\eta\eta)}\right] - B\,\nabla_c(\mathcal{R}^2)\right\} \\
&= \tfrac{1}{2}\,A\,\Omega\,(-\xi\xi)^{-1/2}\,(\eta\eta)\,\nabla_c(\mathcal{R}^2)\,(1+B) \\
&\quad \text{But } 1+B = (1-\beta^2)^{-1} = \gamma^2 \text{ and } A = (\eta\eta)^{-1/2}\gamma. \\
&= \tfrac{1}{2}\,\gamma^3\,\Omega\,(-\xi\xi)^{-1/2}\,(\eta\eta)^{1/2}\,\nabla_c(\mathcal{R}^2) \\
&= \gamma^3\,\Omega\,\nabla_c\mathcal{R}
\end{aligned}
$$

Thus finally, the Fermi-Walker derivative with respect to the particle 4-velocity along the tangential direction is given by:

$$
\boxed{\mathcal{D}_u^{FW}\tau_c = \gamma^3\,\Omega\,\nabla_c\mathcal{R}} \tag{2.64}
$$

2.3.1 Shear and Vorticity

The shear and vorticity plays an important role in general relativity, as it describes what happens to nearby particles in a rotating spacetime. For obvious reasons, this is particularly important in the theory of accretion disks, where it plays a fundamental role in friction and transfer of angular momentum.

These formulas were given in [118] for a stationary spacetime.

$$
\sigma^2 = -\tfrac{1}{4}\,(1-\Omega\ell)^{-2}\,\mathcal{R}^2\,(\nabla_a\Omega)(\nabla^a\Omega) \tag{2.65}
$$

$$
\omega^2 = -\tfrac{1}{4}\,(1-\Omega\ell)^{-2}\,\mathcal{R}^{-2}\,(\nabla_a\ell)(\nabla^a\ell) \tag{2.66}
$$

$$
\mathcal{R}_\pm^2 := \left(\frac{U}{E}\right)^2 (g_{t\phi}^2 - g_{tt}g_{\phi\phi}) = \frac{(\ell_\pm g_{t\phi} + g_{\phi\phi})^2}{g_{t\phi}^2 - g_{tt}\,g_{\phi\phi}} \tag{2.67}
$$

These we can simplify using the fact that on the equatorial plane, all metric functions are scalar functions of r only.

$$\sigma_\pm^2 = -\tfrac{1}{4}\left(1-\Omega_\pm\ell_\pm\right)^{-2}\mathcal{R}_\pm^2\, g^{rr}\left[\partial_r\Omega_\pm\right]^2 \tag{2.68}$$

$$\omega_\pm^2 = -\tfrac{1}{4}\left(1-\Omega_\pm\ell_\pm\right)^{-2}\mathcal{R}_\pm^{-2}\, g^{rr}\left[\partial_r\ell_\pm\right]^2 \tag{2.69}$$

2.3.2 Precession

The precession of orbiting gyroscopes is interesting for the understanding of the centrifugal forces. In some very extreme cases, for example inside the photon horizon of a static ultradense object, the centrifugal force may change sign (direction.) Here we will derive the formula describing the precession of a (very small) gyroscope on an equatorial and circular orbit. The original formula for the precession of a gyroscope moving along a circular orbit was given by Abramowicz in [119]. The precession rate with respect to a unit tangent vector, τ^a was then given as:

$$\frac{\Omega_i}{\widetilde{\Omega}} = \frac{1}{2}\tilde{r}e^{3(A-\Phi)}\left[\nabla_i \ln \tilde{r}^2 - (1+\tilde{v}^2)\frac{\nabla_i\omega}{\widetilde{\Omega}}\right], \tag{2.70}$$

where $\tilde{r}$ is the radius of gyration, $\tilde{v} = \gamma v$ the 4-velocity, $\Phi = \frac{1}{2}\ln[-(\eta\eta) + \omega^2(\xi\xi)]$ the gravitational potential and $\widetilde{\Omega} = \Omega - \omega$ the angular velocity of a particle with respect to a locally non-rotating observer.

Then the redshift factor is given by the u^t component of u^a, which we wil write as simply,

$$u = \left[g_{tt} + 2\Omega_\pm g_{t\phi} + \Omega_\pm^2 g_{\phi\phi}\right]^{-1/2}. \tag{2.71}$$

We can then rewrite eq (2.70) in our notation as,

$$\Omega_a = \tfrac{1}{2}u^3\left(\frac{\ell_\pm(\Omega_\pm-\omega_f)}{(g_{tt}+\omega_f^2 g_{\phi\phi})^3}\right)^{1/2}\left[\nabla_a \ln\left(\frac{\ell_\pm}{\Omega_\pm-\omega_f}\right) - \frac{[1+\ell_\pm(\Omega_\pm-\omega_f)]}{(\Omega_\pm-\omega_f)}\nabla_a\omega_f\right] \tag{2.72}$$

where, $\omega_f = 2J/r^3$. But since all the terms in the square bracket are scalar functions of r only, we can simplify the covariant derivatives.

$$\nabla_a\omega_f = \partial_r\omega_f = -6J/r^4 \tag{2.73}$$

But if $A := \ell_\pm$ and $B := \Omega_\pm - \omega_f$, we have

$$\begin{aligned}\nabla_a \ln\left(\frac{A(r)}{B(r)}\right) &= \partial_r\left[\ln\left(\frac{A}{B}\right)\right] = \left(\frac{B}{A}\right)\partial_r\left(\frac{A}{B}\right)\\ &= A^{-1}B\left[B^{-1}\partial_r A - AB^{-2}\partial_r B\right]\\ &= A^{-1}\partial_r A - B^{-1}\partial_r B\end{aligned}$$

so the coefficient of the second term in the brackets reads:

$$\frac{1+\ell_\pm(\Omega_\pm-\omega_f)}{\Omega_\pm-\omega_f} = \frac{1+AB}{B} = B^{-1}+A \tag{2.74}$$

But

$$\partial_r A = \partial_r\ell_\pm \tag{2.75}$$

$$\partial_r B = \partial_r\Omega_\pm - \partial_r\omega_f = \partial_r\Omega_\pm + 6J/r^4 \tag{2.76}$$

So the whole bracket reads:

$$\begin{aligned}
[...] &= A^{-1}\,\partial_r A - B^{-1}\partial_r B - (B^{-1} + A)\,\partial_r\omega_f \\
&= A^{-1}\,\partial_r A - B^{-1}(\partial_r\Omega_\pm - \partial_r\omega_f) - (B^{-1} + A)\,\partial_r\omega_f \\
&= A^{-1}\,\partial_r A - A\,\partial_r\omega_f - B^{-1}\,\partial_r\Omega_\pm \\
&= \ell_\pm^{-1}\,\partial_r\ell_\pm + 6J/r^4\,\ell_\pm + (\omega_f - \Omega_\pm)^{-1}\,\partial_r\Omega_\pm
\end{aligned} \tag{2.77}$$

2.4 Holonomy Invariance

What is **holonomy**? In general it describes what happens when we perform a cycle of transformations. In particular, holonomy transformations measure the change in direction acquired by a vector under parallel transport around a closed loop, or between two distinct points via different paths. Thus holonomy can be quantified by the *deficit angle*, which would be the angular difference between the original vector and the one parallel transported around the loop. However, this is a global property of the manifold, and can therefore be used to classify spacetimes just like the Petrov and Segre classifications. (In fact, holonomy is even more general. Properties of holonomy can be found in a wide range of disciplines, such as gauge theories, quantum mechanical path integration, in cylindrically symmetric spacetimes and in cosmic string backgrounds, among others.)

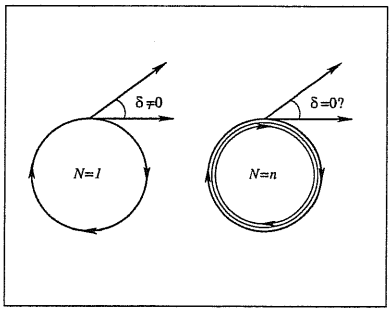

Figure 2.7: The deficit angle δ for N number of circular orbits.

The deficit angle, δ as shown in Fig. 2.7, for two cases. One in which it is non-zero and the other which may be zero after some fractional $N = m/n$ number of loops.

It is rather intuitive then, that in the flat Euclidean space, parallel transport around a closed circle S^1, has a zero deficit angle. [See comment on parallel transport.] Then in a curved spacetime, one would always expect a non-zero deficit angle. However, one may consider the possibility of winding modes, such that after N number of loops around, the deficit angle may once again add up to zero. This is what we mean with *holonomy invariance.*

It should be mentioned that the results of holonomy invariance of the spherically symmetric Schwarzschild metric does not yield a zero deficit angle as is easily mistaken by a non relativist. This error is probably done because it is easy to get fooled into thinking that spherical symmetry also means that all vectors on a circular orbit are equivalent. In fact the detailed holonomy calculations for the general Schwarzschild

and Kerr spacetimes, can be found in [115] and [117], respectively. So when can we have holonomy invariance? Lets look at the Schwarzschild geometry.

Comment: Parallel Transport
On a differentiable manifold there is no intrinsic notion of parallelism between vectors defined at different points. The *affine connection* is merely a rule whereby some notion of parallelism can be defined. For example, on the 3-dimensional S^2 sphere, a vector on the north-pole can be parallel transported along two different routes to the south-pole, ending up pointing in opposite directions. This does not happen on the 2-dimensional S^1 circle. Therefore we can conclude that there is no global notion of parallelism in dimensions higher than 2.

Example (3):

Consider a loop around the equatorial plane with $r = const$ in the Schwarzschild exterior spacetime.

$$\begin{aligned} N &= mn \qquad \textit{with:} \quad (1 \leq n < m) \\ r &= 3M\left(1 - \frac{n^2}{m^2}\right)^{-1} \end{aligned} \tag{2.78}$$

Thus we have holonomy invariance at any radius satisfying the above condition. However, the condition that $r > 2M$ leads to restrictions on the allowed values of n and m. Or to reverse the argument, holonomy invariance leads to a "quantization" of the orbits.

□

2.4.1 Quantization of orbits

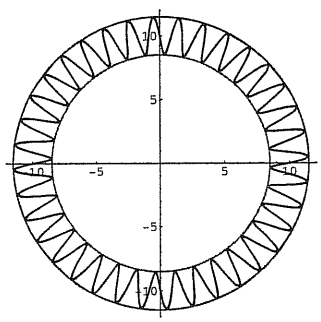

Figure 2.8: The epiciclic motion for a small perturbation of r.

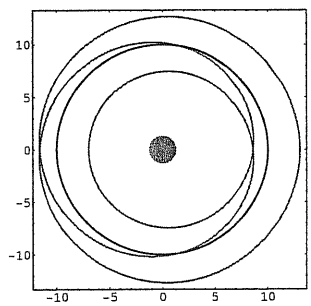

Figure 2.9: The epiciclic motion for $m/n = 2/3$ whith $R = 10$ and $r = 3$.

Now consider the case of an orbit with orbital frequency Ω, where r is not constant, but perturbed radially by some function $\alpha(t)$, where $\alpha(t) \ll R$.

$$r = R + \alpha(t) \qquad (\alpha(t) \ll R) \tag{2.79}$$

Then the equation for the perturbation is given by:

$$\ddot{\alpha} + \omega_r^2\,\alpha = 0 \tag{2.80}$$

where ω_r is the radial epicyclic frequency. The stability of the orbit due to the perturbation is:

$$\begin{aligned} \omega_r^2 > 0 \quad & \textbf{stable} \\ \omega_r^2 < 0 \quad & \textbf{unstable}. \end{aligned}$$

Thus for any spherically symmetric gravitational potential we have 2 cases:

1) The Euclidean Space: $\Omega^2 = \omega_r^2 > 0$

$\Rightarrow$ $N \equiv 1$ Give ordinary Keplerian Orbits

2) Non-Euclidean Space: $\Omega^2 \neq \omega_r^2$

$\Rightarrow$ $N = n$ Gives closed orbits if N=n, but

$\Rightarrow$ There could be a range of radii,

where $\omega_r^2 < 0$ and thus unstable.

For example, in Schwarzschild:

$$\begin{aligned} \omega_r &= \left(1 - \frac{6M}{r}\right)^{1/2} \Omega \\ \Omega &= (M/r)^{1/2}\, c/r \\ \Rightarrow \quad \omega_r^2 &= (r - 6M)\,(M/r^4)\, c^4 \end{aligned}$$

So that for $r < 6M$ we have unstable orbits as is known since the marginally stable orbit is $r_{ms} = 6M$. When $r \geq 6M$ we are in the stable regime, but this time with $N = n \neq 1$. A good example of this would be the perihelion precession of mercury. So for slightly perturbed circular orbits in Schwarzschild geometry , the closure condition is:

$$n\,\Omega = m\,\omega_r \qquad (1 \leq n < m) \tag{2.81}$$

or better

$$r = 6M\left(1 - \frac{n^2}{m^2}\right)^{-1} \tag{2.82}$$

One might worry that this is different from eq.(2.78). However, that would be an error as eq (2.78) is for $r = const$, whereas eq (2.82) is for $r = R + \alpha(t)$.

2.4.2 Parametric Resonance

Parametric resonance occurs when the radial and azimuthal epicyclic frequencies match up in such a way to close the orbits with integer number of rotations of each. The condition for resonance is then given by: $m\omega_r = n\omega_\theta$. But for each value of (m, n), there is an associated radius, r_{mn} wich gives ...

$$m\omega_r = n\omega_\theta \tag{2.83}$$

We can then write an equation for the radius, by first defining

$$z_{mn} := \frac{m^2}{n^2} \tag{2.84}$$

and then using the function

$$F(M,r,j,q) = \frac{\omega_\theta^2}{\omega_r^2}\,. \tag{2.85}$$

Then the relation from above gives the equation:

$$F(M,r,j,q) - z_{mn} = 0 \tag{2.86}$$

In the section presenting the Hartle-Thorne formulas, we have used this in Mathematica to solve for r_{mn} using the ansatz:

$$r_{mn}(M,j,q) = K + Aj + Bj^2 + Cq\,. \tag{2.87}$$

2.5 QPO's

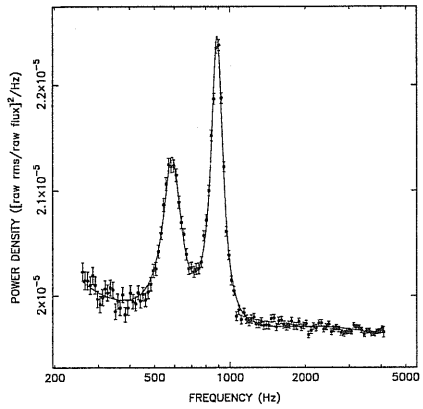

Figure 2.10: A typical kHz QPO power spectrum from *Sco X-1*. [Courtesy van der Klis.]

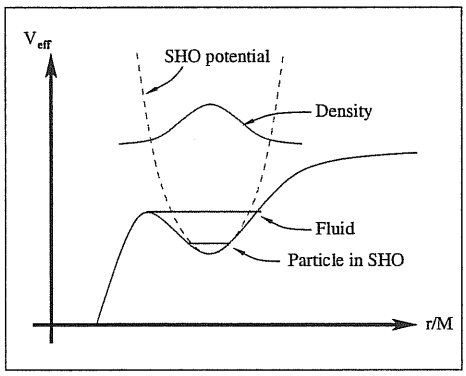

Figure 2.11: Since most matter in a fluid are located near the center of the SHO potential, fluids are also expected to admit epicyclic behaviour. [Shapiro & Teulkowski]

What is a QPO? QPO stands for Quasi Periodic Oscillation. QPO's are usually observed in the peaks of the power-spectrum of the X-ray flux of accreting neutron stars (NS) or in low mass x-ray binaries (LMXB's). Consider the power-spectrum (Fourier transform) of of an oscillator containing 2 constant frequencies, ω_1 and ω_2. If we were to plot the power versus frequency (Fourier transform) we would see that in this case there would be two delta functions centered at ω_1 and ω_2. On the other hand, if the 2 frequencies are not constant, these 2 peaks would be widened. This is in fact what we observe for these objects (as shown in Fig. 2.10.) In addition it is observed that the QPO frequencies scale as $1/M$, which suggests a general relativity origin of QPOs. We are here presenting such a model based on the ideas of holonomy invariance and orbital resonance [114].

What is special about the 3:2 resonance? The answer is that if it is parametric resonance between two frequencies ω_1 and ω_2. Then the resonance condition is that

$$\omega_1 = \frac{2}{n}\,\omega_2\,. \tag{2.88}$$

If in addition $\omega_1 < \omega_2$, we then have that $n \geq 3$. Thus the fastest growing resonance (and therefore also the strongest) is the one in the ratio 2:3. We know that for blackholes in GR, $\omega_r < \omega_\perp$, so that this resonance should always be the dominant one.

If we see multiple frequencies, this is an indication of very strong-field gravity.

In numerous sources that astronomers directly observe, matter is being accreted onto the central black black hole through a vertically thin accretion disk: the vertical thickness H is much smaller than the corresponding radius r. For inner parts of accretion disks that are relevant in the present context, typically $h = H/r \approx 0.1$. In such disks, and in the relevant range of radii (a few times r_{ms}), matter moves on orbits not much different from circular geodesics: pressure, viscosity, radiation, magnetic field, and other effects could be considered as being very small perturbations. Therefore, in thin accretion disks, there should then be present oscillatory modes close to the epicyclic frequencies, ω_r and $\omega_\perp$. In Schwarzschild and Kerr geometry, $\omega_\perp = \Omega$. The functions $\Omega(r)$, $\omega_r(r)$, $\omega_\perp(r)$ are explicitly known for both Schwarzschild and Kerr geometry (see e.g. [122]). Therefore if we know the mass and the QPO frequencies of the central object, we can put a constraint on the angular momentum $j = J/M^2 = J_* c/(M_*^2 G)$, given that we believe that it is a blackhole.

Example (4):

Thus for example [120, 121] in the blackhole candidate GRO J1655-40 we are given:

$$M = 5.5 - 7.9 M_\odot \tag{2.89}$$
$$\omega_1 = 300 Hz. \tag{2.90}$$
$$\omega_2 = 450 Hz. \tag{2.91}$$

There are only two possible resonaces for that case: (3:1) and (2:1).

$$\begin{array}{lll} (3:1) \quad \textit{resonance:} & \Omega_K & = 450 Hz. \\ & \Omega_K - \omega & = 300 Hz. \\ (2:1) \quad \textit{resonance:} & \Omega_K + \omega & = 450 Hz. \\ & \Omega_K & = 300 Hz. \end{array} \tag{2.92}$$

The resulting constraints are then:

$$0.20 \leq j \leq 0.60 \qquad \textit{for} \quad (2:1) \tag{2.93}$$
$$0.36 \leq j \leq 0.67 \qquad \textit{for} \quad (3:1) \tag{2.94}$$

□

The analysis can be taken even further. It may be possible that the above analysis can be used also for fluids. The reason is that a given fluid generally have a certain density distribution which is peaked at the center of the fluid, such that most of the mass would be located in the inner parts. Now, if we consider the effective potential V_{eff} of particles moving around in a Schwarzschild geometry, we see that the circular orbits are located at a radius where there is a local minima. The particle oscillations near this minima can then be approximated by a simple harmonic oscillator (SHO) potential as shown in the qualitative figure 2.11. But this is also true for a fluid since most of the particles in the fluid are located in the central parts where density is high.

As we have seen these features are quite general for accreting black holes. It is then only natural to ask whether these features are also present in general accreting neutron stars. The answer is yes, but in this case the orbital frequency is no longer similar to any of the radial or orthogonal epicyclic

frequencies, respectively. This makes it difficult to test this theory without a more accurate description of a rotating stellar spacetime. One such is the Hartle-Thorne metric. Next we shall derive all relevant orbital parameters for this metric.

2.6 The Hartle-Thorne Metric

The Hartle-Thorne metric was first derived by J.B.Hartle and K.S.Thorne in 1967 [105, 106]. They used a model in which the main assumptions were that the star was rigidly and slowly rotating, together with the approximation that terms greater than second order in the angular velocity (or first order in the quadrupole moment) were neglected. In these two papers they calculated the equilibrium configurations of white dwarfs and neutron stars using the Harrison-Wheeler and the Tsuruta-Cameron V_γ equations of state. As the explicit derivation of these results are rather long and not giving any additional insight, I will merely state the results.

The only assumptions used in the derivation for the external Hartle-Thorne metric were:

- Stationary. A configuration can only be in equilibrium if it does not radiate angular momentum.
- Axially symmetric.
- Slow rotation. The original criterion for rotation to be considered "slow" was that:

$$\Omega \left(\frac{R^3}{GM}\right)^{1/2} \ll 1 . \tag{2.95}$$

The HT metric is then shown as the $l = 2$ solution of a perturbed, non-rotating sphere, up to 2nd order in the angular velocity. And thus up to that order it is an exact relativistic representation of the structure.

We derive the orbital formulas for the Hartle-Thorne metric describing the spacetime outside any axisymmetric stationary rotating object. The formulas are derived analytically up to second order in the angular momentum and quadrupole moment. The results gives the equivalent Kerr values in the limit as: $q \to j^2$, given that one uses the transformation from the HT coordinates to the standard Boyer-Lindquist coordinates. These are related by the transformations:

$$r_{BL} = r - a^2/(2r^3)\,(r+2M)\,(r-M) \tag{2.96}$$

$$\theta_{BL} = \theta - a^2/(2r^3)\,(r+2M)\,\cos\theta\,\sin\theta \tag{2.97}$$

$$a = Mj$$

The formulas are given in geometrized units, $c = G = 1$, and with the further simplification that we are using the dimensionless units: $j = J/M^2$ and $q = Q/M^3$. Where $M = GM_*/c^2$ is the mass of the central object in geometrical units, and $J = GJ_*/c^3$ is its angular momentum, also expressed in geometrical units. In addition we employ the signature convention $(+---)$. The positive/negative signs in the formulas refer to the *corotating* and *counter rotating* cases, respectively.

2.6.1 The Metric

In order to simplify for computational use we will rewrite the original formula somewhat. The original formula for the Hartle-Thorne metric was:

$$\begin{aligned} \mathrm{d}s^2 &= -A\,[1+2P_2\,(C+B\,Q_2)]\,\mathrm{d}t^2 \\ &\quad +A^{-1}\left[1-2P_2\left(C+B\,Q_2-6\,\frac{J^2}{r^4}\right)\right]\mathrm{d}r^2 \\ &\quad +r^2\left[1-2P_2\left(C+\frac{J^2}{r^4}-B\,\{D\,Q_1-Q_2\}\right)\right]\cdot\left[\mathrm{d}\theta^2+\sin^2\theta\,(\mathrm{d}\phi-f\,\mathrm{d}t)^2\right] \end{aligned}$$

Where for simplicity, we have defined:

$$\begin{aligned} A &:= \left(1-\frac{2M}{r}+\frac{2J^2}{r^4}\right) \\ B &:= \frac{5}{8}\left(\frac{Q}{M^3}-\frac{J^2}{M^4}\right) \\ C &:= \frac{J^2}{Mr^3}\left(1+\frac{M}{r}\right) \\ D &:= \frac{2M}{r}\left(1-\frac{2M}{r}\right)^{-1/2} \\ f &:= \frac{2J}{r^3} \\ g &:= \frac{J^2}{r^4} \end{aligned}$$

The other functions are related to the solutions of the Legendre type differential equations.

The legendre polynomial: $P_2 := P_2\,[\cos\theta] = \frac{1}{2}\,(3\cos^2\theta-1)$

The associated Legendre polynomials of 2nd kind: $Q_1 := Q_2^1\,[\frac{r}{M}-1]$, $Q_2 := Q_2^2\,[\frac{r}{M}-1]$

(See Appendix B for the exact formulas for Q_1 and Q_2.) Also we rewrite:

$$(\mathrm{d}\phi-f\,\mathrm{d}t)^2 = \mathrm{d}\phi^2-2\,f\,\mathrm{d}\phi\mathrm{d}t+f^2\,\mathrm{d}t^2 \tag{2.98}$$

The metric could then be rewritten in a slightly simpler component form, more suitable for computer algebraic manipulation. (Dropping the subscript on P_2.)

$$\begin{aligned} g_{tt} &= -A\,[1+2P\,U_1]+f^2\,g_{\theta\theta}\,\sin^2\theta \\ g_{rr} &= A^{-1}\,[1-2P\,U_2] \\ g_{\theta\theta} &= r^2\,[1-2P\,U_3] \\ g_{\phi\phi} &= g_{\theta\theta}\,\sin^2\theta \\ g_{t\phi} &= g_{\phi t} = -f\,g_{\theta\theta}\,\sin^2\theta \end{aligned}$$

where

$$\begin{array}{lcl} U_1 := C + B\,Q_2 & = & J^2/(Mr^3) + J^2/r^4 + BQ_2 \\ U_2 := C + B\,Q_2 - 6\,J^2/r^4 & = & J^2/(Mr^3) - 5J^2/r^4 + BQ_2 \\ U_3 := C + B\,Q_2 + J^2/r^4 - B\,D\,Q_1 & = & J^2/(Mr^3) + 2J^2/r^4 + BH \\ H := Q_2 - D\,Q_1 & & \end{array}$$

However, computer algebra packages are terribly ineffective when trying to simplify anything involving trigonometric functions. The trick is to transform away all types of trigonometry. In the case of the HT-metric, the transformation is $u := \cos[\theta]$. Then all $\sin^2[\theta] = 1 - u^2$, so that we have the final input structure for our Mathematica notebooks.

$$\left.\begin{array}{lcl} P & = & \frac{1}{2}\,(3u^2 - 1) \\ g_{tt} & = & [1 - 2P\,U_3]\;(1 - u^2)\,(4J^2/r^4) - A\;[1 + 2P\,U_1] \\ g_{rr} & = & [1 - 2P\,U_2]\,/A \\ g_{uu} & = & [1 - 2P\,U_3]\,/(1 - u^2)\,r^2 \\ g_{\phi\phi} & = & [1 - 2P\,U_3]\;(1 - u^2)\,r^2 \\ g_{t\phi} & = & [1 - 2P\,U_3]\;(1 - u^2)\,(-2J/r) \end{array}\right\} \tag{2.99}$$

Now, with these definitions, the Hartle-Thorne line element reads:

$$\mathrm{d}s^2 = g_{tt}\,\mathrm{d}t^2 + g_{rr}\,\mathrm{d}r^2 + g_{uu}\,\mathrm{d}u^2 + g_{\phi\phi}\,\mathrm{d}\phi^2 + g_{\phi t}\,\mathrm{d}\phi\,\mathrm{d}t + g_{t\phi}\,\mathrm{d}t\,\mathrm{d}\phi\,, \tag{2.100}$$

where $g_{t\phi} = g_{\phi t}$.

2.6.2 The Kerr Limit

In this section we merely state the orbital parameters for the external Kerr metric as a reference to compare the Hartle-Thorne results in the Kerr limit. This comparison will work as a guide to check our results up to first order in the angular velocity. However, all these have been calculated elsewhere but with slightly different notation, which may cause confusion if one is not very observant. Here I will give the different results, with their respective authors definitions. Furthermore, it is crucial for successful comparison that we:

1) Take the Kerr limit ($q \to j^2$) of our results.
2) Convert our results from HT coordinates into the Boyer-Lindquist coordinates using (2.96).
3) Convert the "exact" Kerr formulas to 2nd order approximations.

Different Author, Different Formula

As some of our most interesting results depends on the formulas for the epicyclic frequencies, we shall show the differences among different authors for these formulas in particular, being understood that it applies to the other quantities as well. In Newtonian theory, the epicyclic frequencies are equal to the angular frequency of rotation. So that

$$\kappa_N^2 = \Omega_N^2 = M/r^3\,. \qquad \textbf{Newton} \tag{2.101}$$

However in general relativity, for a static spacetime, we obtain the Schwarzschild result.

$$\kappa_{GR}^2 = \Omega_N^2\,(1 - 6M/r) = M\,(r - 6M)/r^4 \qquad \textbf{Schwarzschild} \tag{2.102}$$

From which it is clear that the minimum epicyclic frequency occurs at $r = 6M$ which corresponds to the marginally stable orbit. However, for the more general stationary Kerr metric, we have to apply more general formulas. The derivation of these can be found in various books and papers. I will just state the results for the Kerr metric found in three of these references without further explanation.

Kato et.al. [122]

Uses $(+---)$ signature with $a = J/M = jM$ and $a_* := a/M = j$. However, they only give the radial component of the epicyclic frequency.

$$\kappa_r^2 = M/r^2\,(r^2 - 6Mr \pm 8jM^{3/2}r^{1/2} - 3j^2M^2)\,(r^{3/2} \pm jM^{3/2})^{-2} \tag{2.103}$$

The other quantities are: the Keplerian angular velocity, the specific angular momentum and the specific energy,

$$\Omega_K = \pm\frac{\sqrt{M}}{r^{3/2} \pm a\sqrt{M}} \tag{2.104}$$

$$\ell_K = \pm\frac{\sqrt{M}\,(r^2 \mp 2a\sqrt{Mr} + a^2)}{r^{1/2}(r^2 - 3Mr \pm 2a\sqrt{Mr})^{1/2}} \tag{2.105}$$

$$e_K = \pm\frac{r^2 - 2Mr\sqrt{Mr} + a\sqrt{Mr})}{r(r^2 - 3Mr \pm 2a\sqrt{Mr})^{1/2}}\,. \tag{2.106}$$

In addition the location for the *marginally stable* orbit, the *marginally bound* orbit and the *photon orbit* are given by:

$$r_{ms}^{\pm} = M\left[3 + Z_2 \mp \{(3 - Z_1)(3 + Z_1 + 2Z_2)\}^{1/2}\right] \tag{2.107}$$

$$r_{mb}^{\pm} = r_g\,(1 \mp j/2 + \sqrt{1 \mp j}) \tag{2.108}$$

$$r_{ph}^{\pm} = 2M\,(1 + \cos[2/3\cos^{-1}(\mp j)])\,, \tag{2.109}$$

and where

$$Z_1 := 1 + (1 - j^2)^{1/3}\left[(1 + j)^{1/3} + (1 - j)^{1/3}\right] \tag{2.110}$$

$$Z_2 := (3j^2 + Z_1^2)^{1/2}\,. \tag{2.111}$$

Nowak and Lehr [124]

They use $(-+++)$ signature with $a := J/M^2 = j$ and $\hat{r} := r/M$, so that

$$\Omega_K = \frac{\sqrt{M}}{a\sqrt{M} \pm r^{3/2}} \tag{2.112}$$

$$\kappa_r^2 = \Omega_K^2\,(1 - 6/r + 8ar^{-3/2} - 3a^2r^{-2}) \tag{2.113}$$

$$\kappa_\theta^2 = \Omega_K^2\,(1 - 4ar^{-3/2} - 3a^2r^{-2}) \tag{2.114}$$

Merloni et.al.[123]

They also use the $(-+++)$ signature with $a := Mj$.

$$\kappa_\theta^2 = \Omega_K^2\,(1 \mp 4aM^{1/2}r^{-3/2} - 3a^2r^{-2}) \tag{2.115}$$

Thus the only difference is the way they use the quantity a and whether or not j is a specific quantity.

In addition we mention the formula for Frame-Dragging, which is given in terms of a where $a = J/M = j$. In the Kerr spacetime the frame-dragging is given by:

$$\omega(r,\theta) = 2Mra\left[(r^2+a^2)^2 - (r^2-2Mr+a^2)\,a^2\sin^2\theta\right]^{-1}. \tag{2.116}$$

2.6.3 The Mathematica Solutions

All formulas presented in this section have been derived using Mathematica 4.0 [164] and the two free computer algebra packages TTC and GRT [165, 166], running on a PIII 800 MHz/256MB under Linux Red Hat 7.2. The Mathematica notebooks containing the calculations can be found in [167]. By no means are these notebooks maximally effective in the calculations as the main formulas have been generalized to simplify in case one would like to find formulas also good out of the horizontal plane with $\theta \neq \pi/2$, or should higher order terms ever be needed. The following notebooks are then considered part of the thesis:

`HTricci.nb`	(Calculation of the Ricci tensor)
`HThorizon.nb`	(Calculation of the would-be horizon)
`HTinfred.nb`	(Calculation of the infinite redshift surface)
`HTomega.nb`	(Calculation of the angular velocity)
`HTmomentum.nb`	(Calculation of the specific angular momentum, r_{ms})
`HTenergy.nb`	(Calculation of the energy, r_{ph}, r_{mb})
`HTepiciclic.nb`	(Calculation of the epiciclic frequencies)
`HTresonance.nb`	(Calculation of the resonance condition)
`HTshearvort.nb`	(Calculation of the shear and vorticity)
`HTPrecession.nb`	(Calculation of the gyroscope precession formula)

where in addition the HT metric in the $(-+++)$ signature has been defined in the GRTensor file: `HTb.g`.

Several parameters presented below requires the solution of higher (> 3) order algebraic equation. In particular those which are some kind of radius, since most other formulas are given as a power series in terms of r, j, j^2, q. (Throughout it will be understood that our q arises from rotation and has nothing to do with other types of quadrupole moments, such as those involed in radiation.) The trick is then to recognize that we already know some of the coefficients of these radial functions, because of the already known Schwarzschild and Kerr solutions.

So for example when finding the photon orbital radius r_{ph}, we set the resulting polynomial equation containing r to zero.

$$F(r,j,j^2,q) = k(r) + a(r)j + b(r)j^2 + c(r)q = 0 \tag{2.117}$$

We then assume a solution of the form:

$$r_{ph} = 3M + Aj + Bj^2 + Cq\,, \tag{2.118}$$

where A, B, C are functions of r only. We can then substitute this into the original equation for $F = 0$ and we see that we will have many terms of order higher than j^2 and q. But since we are only interested in up to 2nd order terms, we can set A, B and C to zero when multiplying with j, j^2 and q, respectively. Then its possible to find A by keeping just 1st order terms, because of the fact that if a polynomial is zero, then the coefficients vanish identically. Similarly we find B and C by keeping just 2nd order terms.

Of course, in practice, this is slightly more complicated as we have to make series expansions of square roots and divisions, should they occur in the equations. The last step is then to convert the solution into Boyer-Lindquist coordinates and take the Kerr limit, and to compare this to the Taylor expanded Kerr formulas.

2.6.4 The HT orbital quantities

The Horizon

The radius of the would-be horizon is obtained by setting $g_{\phi t}^2 - g_{tt} g_{\phi\phi} = 0$ and solving for r.

$$r_h = 2M \left[1 - \frac{1}{16} j^2 (7 - 15u^2) + \frac{5}{16} q (1 - 3u^2)\right] . \tag{2.119}$$

The Infinite Redshift Surface

Similarly the radius of surface of infinite redshift, can be easily obtained by setting $g_{tt} = 0$.

$$r_{ir} = 2M \left[1 - \frac{1}{16} j^2 (3 - 11\, u^2) + \frac{5}{16} q (1 - 3\, u^2)\right] . \tag{2.120}$$

Dragging of Inertial Frames

This is the angular velocity of an inertial observer near the rotating star as observed by someone at infinity. The formula is trivially given by:

$$\omega_{FD} = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2J}{r^3} = \frac{2M^2 j}{r^3} \tag{2.121}$$

which can be seen from eq (2.99). As such, it is independent of the quadrupole moment and the second order terms as well as the angle of inclination.

The Angular Velocity

The angular velocity for corotating/counterrotating circular particle orbits is given by:

$$\Omega_\pm = \pm\Omega_0 \left[1 \mp j \frac{M^{3/2}}{r^{3/2}} + j^2 F_1(r) + q F_2(r)\right] \tag{2.122}$$

where

$$\Omega_0 := \frac{M^{1/2}}{r^{3/2}} \tag{2.123}$$

$$F_1(r) = \frac{48\, M^7 - 80\, M^6 r + 4\, M^5 r^2 - 18\, M^4 r^3 + 40\, M^3 r^4 + 10\, M^2 r^5 + 15\, M\, r^6 - 15\, r^7}{16\, M^2\, (r - 2M)\, r^4} + A(r) \tag{2.124}$$

$$F_2(r) = \frac{5\, \left(6\, M^4 - 8\, M^3 r - 2\, M^2 r^2 - 3\, M\, r^3 + 3\, r^4\right)}{16\, M^2\, (r - 2M)\, r} - A(r) \tag{2.125}$$

$$A(r) = \frac{15\, \left(r^3 - 2\, M^3\right)}{32\, M^3} \ln\left(\frac{r}{r - 2M}\right) \tag{2.126}$$

The Specific Angular Momentum

$$\ell_\pm = \pm\ell_0 \left[1 \mp j\,\frac{M^{3/2}(3r-4M)}{r^{3/2}(r-2M)} + j^2\,F_1(r) - q\,F_2(r)\right] \tag{2.127}$$

where

$$\ell_0 := \frac{M^{1/2}r^{3/2}}{r-2M} \tag{2.128}$$

$$\begin{aligned} F_1(r) &= \left[16M^2\,r^4\,(r-2M)^2\right]^{-1}\left(96M^8 - 112M^7r - 8M^6r^2 - 48M^5r^3\right. \\ &\quad \left. +42M^4r^4 + 220M^3r^5 - 260M^2r^6 + 105Mr^7 - 15r^8\right) + A(r) \end{aligned} \tag{2.129}$$

$$F_2(r) = \left[16M^2\,r\,(r-2M)\right]^{-1}\left(5\,\left(6M^4 - 22M^2r^2 + 15Mr^3 - 3r^4\right)\right) + A(r) \tag{2.130}$$

$$A(r) = \frac{15}{32M^3}\left(2M^3 + 4M^2r - 4Mr^2 + r^3\right)\ln\left(\frac{r}{r-2M}\right) \tag{2.131}$$

Then by setting $d\ell/dr = 0$ we can solve for the radius of the *marginally stable* orbit:

$$r_{ms} = 6\,M\left[1 \mp j\,\frac{2}{3}\sqrt{\frac{2}{3}} + j^2\left(\frac{251647}{2592} - 240\ln\frac{3}{2}\right) + q\left(-\frac{9325}{96} + 240\ln\frac{3}{2}\right)\right] \tag{2.132}$$

The Specific Energy

The specific energy ($\varepsilon = E/M$) is defined in terms of the redshift factor, which is given by the temporal part of the particle 4-velocity, u^t. It is then convenient to define a redshift potential:

$$A := g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}\,. \tag{2.133}$$

such that $u^t := A^{-1/2}$ and the specific energy is given by:

$$\varepsilon = u^t(g_{tt} + \Omega g_{t\phi})\,. \tag{2.134}$$

This gives the formula

$$\varepsilon = E_0\left[1 \mp j\,F_1(r) + j^2\,F_2(r) + q\,F_3(r)\right] \tag{2.135}$$

where

$$E_0 := \frac{r-2M}{r^{1/2}(r-3M)^{1/2}} \tag{2.136}$$

$$F_1(r) = \frac{M^{5/2}}{r^{1/2}\,(r-2M)(r-3M)} \tag{2.137}$$

$$\begin{aligned} F_2(r) &= \left[16M\,r^4\,(r-2M)\,(r-3M)^2\right]^{-1}\left(144M^8 - 144M^7r - 28M^6r^2 - 58M^5r^3\right. \\ &\quad \left. -176M^4r^4 + 685M^3r^5 - 610M^2r^6 + 225Mr^7 - 30r^8\right) + B(r) \end{aligned} \tag{2.138}$$

$$F_3(r) = \frac{5\,(r-M)\,(6M^3 - 20M^2r - 21Mr^2 + 6r^3)}{16M\,r\,(r-2M)\,(r-3M)} - B(r) \tag{2.139}$$

$$B(r) = \frac{15r\,\left(8M^2 - 7Mr + 2r^2\right)}{32M^2\,(r-3M)}\ln\left(\frac{r}{r-2M}\right) \tag{2.140}$$

Together with the two conditions that; $A = 0$ and $\varepsilon = 1$, we obtain the radii of the *photon orbits* and the *marginally bound* orbits, respectively.

$$r_{ph} = 3M\left[1 \mp j\,\frac{2\sqrt{3}}{9} - j^2\left(\frac{7036 - 6075\ln 3}{1296}\right) + q\left(\frac{7020 - 6075\ln 3}{1296}\right)\right] \tag{2.141}$$

$$r_{mb} = 4M\left[1 \mp \tfrac{1}{2}j - j^2\,(8047/256 - 45\ln 2) + q\,(1005/32 - 45\ln 2)\right] \tag{2.142}$$

The Epicyclic Frequencies

By considering the formulas in eqs (2.48) and (2.48) for the Hartle-Thorne metric on the equatorial plane, we also derived the two formulas for the epiciclic frequency. The tilde indicates that the measurements were made by an observer at infinity.

$$\tilde{\kappa}_r^2 = K_r\left[1 \pm j\,F_1(r) - j^2\,F_2(r) - q\,F_3(r)\right] \tag{2.143}$$

$$\tilde{\kappa}_\theta^2 = K_\theta\left[1 \mp j\,G_1(r) + j^2\,G_2(r) + q\,G_3(r)\right] \tag{2.144}$$

where

$$K_r := M(r-6M)r^{-4} \tag{2.145}$$

$$K_\theta := Mr^{-3} \tag{2.146}$$

and

$$F_1(r) = \frac{6\,M^{3/2}\,(r+2M)}{r^{3/2}\,(r-6M)} \tag{2.147}$$

$$\begin{aligned} F_2(r) = {} & \left[8M^2\,r^4\,(r-2M)\,(r-6M)\right]^{-1}\left[384M^8 - 720M^7r - 112M^6r^2 - 76M^5r^3\right. \\ & \left. - 138M^4r^4 - 130M^3r^5 + 635M^2r^6 - 375Mr^7 + 60r^8\right] + A(r) \end{aligned} \tag{2.148}$$

$$F_3(r) = \frac{5\,(48M^5 + 30M^4r + 26M^3r^2 - 127M^2r^3 + 75Mr^4 - 12r^5)}{8M^2\,r\,(r-2M)\,(r-6M)} - A(r) \tag{2.149}$$

$$A(r) = \frac{15r\,(r-2M)\,(2M^2 + 13Mr - 4r^2)}{16M^3\,(r-6M)}\ln\left(\frac{r}{r-2M}\right) \tag{2.150}$$

$$G_1(r) = \frac{6\,M^{3/2}}{r^{3/2}} \tag{2.151}$$

$$\begin{aligned} G_2(r) = {} & \left[8M^2\,r^4\,(r-2M)\right]^{-1}\left[48M^7 - 224M^6r + 28M^5r^2\right. \\ & \left. + 6M^4r^3 - 170M^3r^4 + 295M^2r^5 - 165Mr^6 + 30r^7\right] - B(r) \end{aligned} \tag{2.152}$$

$$G_3(r) = \frac{5\,\left(6M^4 + 34M^3r - 59M^2r^2 + 33Mr^3 - 6r^4\right)}{8M^2\,r\,(r-2M)} + B(r) \tag{2.153}$$

$$B(r) = \frac{15\,(2r-M)\,(r-2M)^2}{16M^3}\ln\left(\frac{r}{r-2M}\right) \tag{2.154}$$

Shear and Vorticity

The shear σ^2 and vorticity ω^2 were derived by first calculating the radius of gyration for a stationary spacetime. Then this result was used in the formulas of equations (2.65) and (2.66). The results for the HT metric are:

$$\sigma_\pm^2 = S_0\left[1 \pm j\,F_1(r) + j^2\,F_2(r) + q\,F_3(r)\right] \tag{2.155}$$

$$\omega_\pm^2 = V_0\left[1 \mp j\,G_1(r) + j^2\,G_2(r) + q\,G_3(r)\right] \tag{2.156}$$

where

$$S_0 := \frac{9M}{16r^3}\frac{(r-2M)^2}{(r-3M)^2} \tag{2.157}$$

$$V_0 := \frac{M}{16r^3}\frac{(r-6M)^2}{(r-3M)^2} \tag{2.158}$$

and

$$F_1(r) = \frac{4M^{3/2}}{r^{1/2}(r-3M)} \tag{2.159}$$

$$\begin{aligned} F_2(r) &= \left[8M^2r^4\,(r-2M)^2(r-3M)^2\right]^{-1}\cdot \\ &\quad -\left(864M^{10}-1776M^9r+1048M^8r^2-592M^7r^3-10M^6r^4+542M^5r^5\right. \\ &\quad \left.-693M^4r^6+820M^3r^7-530M^2r^8+150Mr^9-15r^{10}\right)-A(r) \end{aligned} \tag{2.160}$$

$$F_3(r) = \frac{5(36M^6-2M^5r-28M^4r^2+35M^3r^3-43M^2r^4+21Mr^5-3r^6)}{8M^2r\,(r-2M)^2(r-3M)}+A(r) \tag{2.161}$$

$$A(r) = \frac{15r\,(r-4M)(M^2+r^2)}{16M^3\,(r-3M)}\ln\left(\frac{r}{r-2M}\right) \tag{2.162}$$

$$G_1(r) = \frac{12M^{3/2}\,(r-2M)}{r^{1/2}\,(r-3M)(r-6M)} \tag{2.163}$$

$$\begin{aligned} G_2(r) &= -\left(7776M^{10}-12528M^9r+3672M^8r^2-4080M^7r^3+150M^6r^4-12594M^5r^5\right. \\ &\quad \left.+30891M^4r^6-25620M^3r^7+9590M^2r^8-1650Mr^9+105r^{10}\right)\cdot \\ &\quad \left[8M^2r^4\,(r-3M)^2\,(r-6M)^2\right]^{-1}+B(r) \end{aligned} \tag{2.164}$$

$$G_3(r) = \frac{-5(54M^5+30M^4r+147M^3r^2-271M^2r^3+141Mr^4-21r^5)}{8M^2r\,(r-3M)\,(r-6M)}-B(r) \tag{2.165}$$

$$B(r) = \frac{15\,r\,\left(24M^4-126M^3r+135M^2r^2-54Mr^3+7r^4\right)}{16M^3\,(r-3M)\,(r-6M)}\ln\left(\frac{r}{r-2M}\right) \tag{2.166}$$

Needs a serious check...

Precession

The precession for a gyroscope on a circular orbit in the equatorial plane.

$$\Pi_\pm = \Pi_0\,\left[-1\mp j\,F_1(r)+j^2\,F_2(r)+q\,F_3(r)\right] \tag{2.167}$$

$$\Pi_0 := \frac{M^{1/2}\,r^2}{(r-3M)^{1/2}\,(r-2M)^3} \tag{2.168}$$

$$F_1(r) = \frac{3M^{5/2}\,(r-4M)}{2\,r^{3/2}\,(r-2M)\,(r-3M)} \tag{2.169}$$

$$\begin{aligned} F_2(r) &= \left(576M^{10}-1704M^9r+2904M^8r^2-1280M^7r^3+1604M^6r^4-8060M^5r^5\right. \\ &\quad \left.+10688M^4r^6-6290M^3\,r^7+1865M^2r^8-270Mr^9+15r^{10}\right)\cdot \\ &\quad \left[16M^2\,r^4\,(r-2M)^2\,(r-3M)^2\right]^{-1}-A(r) \end{aligned} \tag{2.170}$$

$$F_3(r) = \frac{5\,(12M^4 + 36M^3 r - 94M^2 r^2 + 33Mr^3 - 3r^4)}{16M^2\,r\,(r-3M)} + A(r) \tag{2.171}$$

$$A(r) = \frac{15r\,(r-4M)\,(10M^2 - 8Mr + r^2)}{32M^3\,(r-3M)} \ln\left(\frac{r}{r-2M}\right) \tag{2.172}$$

Parametric Resonance

Given that $m\omega_r = n\omega_\theta$ and $z := m^2/n^2$ we can obtain the formula for the radius.

$$r_{mn} = r_0\,\left[1 - j\,F_1(m,n) + j^2\,F_2(m,n) + q\,F_3(m,n)\right] \tag{2.173}$$

where

$$r_0 = 6\,M\,\left(1 - \frac{n^2}{m^2}\right)^{-1} \tag{2.174}$$

and

$$F_1 = \frac{1}{m^3}\left(\frac{2}{3^3}\right)^{1/2} (m^2 - n^2)^{1/2}\,(2m^2 + n^2) \tag{2.175}$$

$$F_2 = \frac{1 - 50\,z - 11\,z^2 + 385\,z^3 + 10612\,z^4 + 123286\,z^5 + 496927\,z^6 + 691843\,z^7 + 251647\,z^8}{1296\,z^4\,(z-1)^3\,(1+2\,z)} - A(z) \tag{2.176}$$

$$F_3 = \frac{-5\,(1+5\,z)\,\left(1 + 74\,z + 546\,z^2 + 950\,z^3 + 373\,z^4\right)}{48\,(z-1)^3\,z\,(1+2\,z)} + A(z) \tag{2.177}$$

$$A(z) = \frac{15\,(1+2\,z)\,\left(1 + 12\,z + 63\,z^2 + 32\,z^3\right)}{4\,(z-1)^4} \ln\left(\frac{3\,z}{1+2\,z}\right) \tag{2.178}$$

2.7 The HT-RNS comparison

In a series of papers in the 1960s, Hartle and collaborators introduced and used a formalism for calculating models of rotating neutron stars with the effects of the rotation being considered as small perturbations with respect to a comparison non-rotating model. Metric and fluid variables were expanded in terms of the parameter $\Omega\sqrt{R^3/M}$ (where Ω is the rotation velocity as measured by a distant observer and R and M are the radius and gravitational mass of the comparison non-rotating model. To first order in this parameter, the only change with respect to the non-rotating model is the appearance of frame-dragging; at second order, centrifugal distortions of the shape, internal structure and metric appear. Hartle and his collaborators retained terms up to and including second order but neglected those beyond that. It is convenient to write the second-order perturbations in terms of harmonic functions (only $l = 0$ and $l = 2$ being needed) and the problem then reduces to solution of a set of ordinary differential equations. For the external space-time, these can be solved analytically but for the interior a numerical solution is necessary even in the simplest case of constant density.

The slow-rotation approach (as this is known) has turned out to be surprisingly powerful and has been very widely used. Clearly, it is satisfactory only when the rotation is sufficiently slow, but it has turned out that it is rather accurate for all pulsars so-far observed (the fastest being PSR 1937+214 with a period of 1.558 ms). If some newly-born neutron stars or re-cycled pulsars (spun up by accretion from a binary companion) are rotating at near to the break-up speed (as is widely supposed) then the

slow-rotation approach would not be appropriate for them and an approach using numerical integration of the full Einstein equations without approximations should then be used. However, we note that these full calculations are intrinsically less accurate than results from the slow rotation approach within the regime where the latter is appropriate, because of the difficulty of accurately resolving small deformations on a finite-difference grid whereas ordinary differential equations can be solved essentially to machine accuracy. (Spectral methods can give more accurate representations of small deformations however.) Another advantage of the slow-rotation approach is that it gives an analytic form for the metric in the vacuum outside the star. It is then of interest to make a comparison of the regimes of validity of the slow-rotation approach and the full numerical calculations for neutron stars with arbitrarily rapid rotation and we address this in the present section.

In Hartle's approach, the rotating models were taken to have the same central density as the comparison non-rotating ones giving a sequence of models with varying rotation speeds but the same central density. This produces a mathematical simplification but has the disadvantage that the sequence cannot be viewed as representing the behavior of a specific neutron star as it is spun up (or down) keeping its rest-mass constant. The rest-mass varies along the Hartle sequences as the rotation speed is increased. In the following, we calculate constant rest-mass sequences which are more useful both for having a conceptual understanding of the results and also for making comparison with the full numerical calculations for arbitrary rotation speeds. This involves making a small but important modification in the implementation of Hartle's scheme as we outline below. For ease of comparison, we follow the notation of Hartle & Thorne [106].

The method proceeds by first calculating the non-rotating comparison model by integrating the Tolman-Oppenheimer-Volkoff equations:

$$\frac{dP}{dr} = -(E+P)\frac{(M+4\pi r^3 P)}{r(r-2M)} \tag{2.179}$$

$$\frac{dM}{dr} = 4\pi r^2 E \tag{2.180}$$

where E and P are the energy density and pressure of the stellar material, r is the radial coordinate and M is the mass-energy contained within a sphere of radius r. These equations are integrated out from the center, where $r = 0$ and $E = E_c$, to the surface where $P = 0$. The space-time metric for the spherical non-rotating object is taken to have the form

$$ds^2 = -e^{\nu(r)}\mathrm{d}t^2 + [1 - 2M(r)/r]^{-1}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \tag{2.181}$$

with $\nu(r)$ being determined by integrating out from the center the equation

$$\frac{d\nu}{dr} = -2(E+P)^{-1}(dP/dr) \tag{2.182}$$

with the boundary condition $\nu(\infty) = 0$. We will also be concerned with the baryon number (related to the rest mass) which is given by

$$\frac{dA}{dr} = 4\pi r^2 N[1 - 2M(r)/r]^{-1/2} \tag{2.183}$$

where A is the total baryon number contained within a sphere of radius r and N is the baryon number density.

Having calculated the non-rotating model, one then proceeds to the rotational deformations. The form used for the perturbed metric is

$$\begin{aligned} ds^2 &= -e^{\nu}\left[1+2(h_0+h_2P_2)\right]\mathrm{d}t^2 + \frac{[1+2(m_0+m_2P_2)/(r-2M)]}{1-2M/r}\,\mathrm{d}r^2 \\ &\quad +r^2\left[1+2(v_2-h_2)P_2\right]\left[\mathrm{d}\theta^2+\sin^2\theta(\mathrm{d}\phi-\omega\mathrm{d}t)^2\right]+O(\Omega^3) \end{aligned} \tag{2.184}$$

where $P_2 = P_2(\cos\theta) = (3\cos^2\theta - 1)/2$ is the second-order Legendre polynomial; ω (which is a function of r) represents dragging of the local inertial frames and is of order Ω while h_0, h_2, m_0, m_2 and v_2 (also functions of r) represent small metric perturbations and are of order Ω^2. A full discussion of this has been given by Hartle [105]. Throughout the discussion of this section, we will be taking the rotation to be uniform so that Ω is a constant through the star. It is relevant to note here that Hartle's second-order slow-rotation approximation includes quadrupole deformations but neglects higher-order multipoles.

The rotational perturbations for the fluid variables are represented in terms of a "pressure perturbation factor" p^* defined by

$$p^* := \frac{\Delta P}{(E+P)} \tag{2.185}$$

where ΔP is the Eulerian pressure perturbation produced by the rotation within the slow rotation approximation. The quantity p^* is expanded in terms of Legendre functions as

$$p^* = p_0^* + p_2^* P_2(\cos\theta) + O(\Omega^4) \tag{2.186}$$

with p_0^* and p_2^* being dimensionless functions of r, of order Ω^2. (The subscripts refer to the different l's. The rotational perturbations of the energy density E and baryon number density N are then written as

$$\Delta E = (E+P)(dE/dP)(p_0^* + p_2^* P_2) + O(\Omega^4) \tag{2.187}$$

$$\Delta N = (E+P)(dN/dP)(p_0^* + p_2^* P_2) + O(\Omega^4) \tag{2.188}$$

Since the pressure is taken to be locally isotropic, p_2^* is clearly zero at the center of the star $r = 0$. Hartle also took p_0^* to be zero at the center, thus fixing the central density of the rotating models to be the same as that of the comparison non-rotating one. (This simplifies the calculation but we will be making the alternative choice of fixing the central value of p_0^* by requiring a zero change in the baryon number A, designated by δA.

The equations used for calculating the rotational perturbations have been derived and discussed in Hartle's papers and we just list them here for completeness together with a few comments which are necessary for understanding the line taken in the calculation.

The magnitude of centrifugal effects in a rotating star depends on the angular velocity of the fluid relative with respect to the local inertial frame (as measured from infinity) which will be denoted by $\widetilde{\omega}$ with $\widetilde{\omega} = \Omega - \omega$. This quantity is of first order in Ω and is found by integrating the equation

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4 j\frac{d\widetilde{\omega}}{dr}\right) + \frac{4}{r}\frac{dj}{dr}\widetilde{\omega} = 0 \tag{2.189}$$

where

$$j(r) = e^{-\nu(r)/2}[1-2M(r)/r]^{1/2} \tag{2.190}$$

For making the integration, it is convenient to measure $\widetilde{\omega}$ in the dimensionless units of its central value $\widetilde{\omega}_c$, such that $\widetilde{\omega} = \omega/\widetilde{\omega}_c$. Outside the star $\widetilde{\omega}$ is given by

$$\widetilde{\omega} = \Omega - \frac{2J}{r^3} \tag{2.191}$$

where J is the total angular momentum of the star, and so by matching the computed values of $\widetilde{\omega}$ and $d\widetilde{\omega}/dr$ at the surface with the analytic exterior solution, values can be found for $\Omega/\widetilde{\omega}_c$ and $J/\widetilde{\omega}_c$. This then allows the results to be presented in terms of Ω or a/m $(= J/M^2)$.

Equation (2.189) is integrated together with the equations for the second order spherical ($l = 0$) and quadrupole ($l = 2$) deformations in order to obtain the complete solution. For the spherical deformations, the basic equations concerned are

$$\frac{dm_0}{dr} = 4\pi r^2 \frac{dE}{dP}(E+P)p_0^* + \frac{1}{12}j^2 r^4 \left(\frac{d\widetilde{\omega}}{dr}\right)^2 - \frac{1}{3}r^3 \frac{dj^2}{dr}\widetilde{\omega}^2 \tag{2.192}$$

$$\frac{dp_0^*}{dr} = -\frac{m_0(1+8\pi r^2 P)}{(r-2M)^2} - \frac{4\pi(E+P)r^2}{(r-2M)}p_0^* + \frac{1}{12}\frac{r^4 j^2}{(r-2M)}\left(\frac{d\widetilde{\omega}}{dr}\right)^2 + \frac{1}{3}\frac{d}{dr}\left(\frac{r^3 j^2 \widetilde{\omega}^2}{r-2M}\right) \tag{2.193}$$

The "mass perturbation factor" m_0 clearly goes to zero at the center of the star. Equations (2.192) and (2.193) are integrated out from the center where $m_0 = 0$ and p_0^* taking values which are iterated so as to converge to giving zero change in the total baryon number A. This change is evaluated by

$$\delta A = 4\pi r^2 \left(1 - \frac{2M}{r}\right)^{-3/2} \left[(E+P)\frac{dN}{dP}p_0^* \left(1 - \frac{2M}{r}\right) + \frac{Nm_0}{r} + \frac{1}{3}Nj^2 r^2 \widetilde{\omega}^2\right] \tag{2.194}$$

The remaining $l = 0$ metric quantity h_0 can then be calculated from associated equations of the original paper [See equations (15c,16,17).]

The quadrupole deformations are calculated by integrating the equations

$$\frac{dv_2}{dr} = -\frac{d\nu}{dr}h_2 + \left(\frac{1}{r} + \frac{1}{2}\frac{d\nu}{dr}\right)\left[-\frac{1}{3}r^3\frac{dj^2}{dr}\widetilde{\omega}^2 + \frac{1}{6}j^2 r^4 \left(\frac{d\widetilde{\omega}}{dr}\right)^2\right] \tag{2.195}$$

and

$$\begin{aligned}\frac{dh_2}{dr} = & \left\{-\frac{d\nu}{dr} + \frac{r}{r-2M}\left(\frac{d\nu}{dr}\right)^{-1}\left[8\pi(E+P) - \frac{4M}{r^3}\right]\right\}h_2 - \frac{4v_2}{r(r-2M)}\left(\frac{d\nu}{dr}\right)^{-1} \\ & + \frac{1}{6}\left[\frac{1}{2}\frac{d\nu}{dr}r - \frac{1}{r-2M}\left(\frac{d\nu}{dr}\right)^{-1}\right] r^3 j^2 \left(\frac{d\widetilde{\omega}}{dr}\right)^2 \\ & - \frac{1}{3}\left[\frac{1}{2}\frac{d\nu}{dr}r + \frac{1}{r-2M}\left(\frac{d\nu}{dr}\right)^{-1}\right] r^2 \frac{dj^2}{dr}\widetilde{\omega}^2\end{aligned} \tag{2.196}$$

Doing this requires some care because of the complicated behavior of the equations near to $r = 0$, as described in the earlier papers, but we have followed standard procedures in the present work and so we will not enter into these details here. By matching the computed solutions for v_2 and h_2 onto their analytic exterior solutions, the value of the quadrupole moment Q can be obtained and m_2 and p_2^* can be obtained with the aid of associated algebraic equations as mentioned above for h_0. Finally, the shape of the isobaric surfaces $r^*(\theta)$ can be calculated from

$$r^*(\theta) = r + \xi_0(r) + \{\xi_2(r) + r[v_2(r) - h_2(r)]\}P_2 \tag{2.197}$$

where r is the radius of the corresponding isobaric surface in the comparison non-rotating model and ξ_0 and ξ_2 are given by

$$\xi_0 = -p_0^*(E+P)/(dP/dr) \tag{2.198}$$

$$\xi_2 = -p_2^*(E+P)/(dP/dr)\,. \tag{2.199}$$

Our main interest here is in the shape of the surface of the rotating star, i.e. the isobaric surface on which the pressure is zero, so we are only interested in the parameters measuring the quadrupole moment, the proper (circumferential) equatorial radius and the eccentricity. These will be denoted by q, R_e, e, respectively.

In the following section, we present results comparing the output from calculations using the slow-rotation approach outlined above with ones carried out using a finite-difference code solving the full equations for models rotating arbitrarily fast. We choose some standard models for rapidly rotating neutron stars with 3 different equations of state. Namely the Pandharipande [125], the Wibringa-Fikes-Fabrocini [126] and the Walecka [127] EOS's, also known as *eos B, eos UU and eos W*, respectively. These describe soft, intermediate and stiff matter, respectively. We then plot q, R_e, e versus the angular momentum $j = a/M$, using models of constant rest mass, corresponding to a star with $M_G = 1.4M_\odot$, in the static case.

2.7.1 The RNS Code

The code used for constructing the rapidly rotating models is the Rapidly rotating NeutronStar code (RNS Version 1.1d) based on the Komatsu-Eriguchi-Hachisu [108] method with modifications by Cook-Shapiro-Teukolsky [112], assuming equilibrium and uniform rotation. This code takes as input a tabulated zero temperature EOS, containing the energy density, pressure, enthalpy and baryon number density.

It is then useful to understand how the code works. Consider a typical plot of the gravitational mass versus the central density of some given stellar EOS. (See Fig.2.12.) In that plot the outermost curve represents the mass-density function for a star rotating with the maximal angular velcity (at the mass-shed limit.) The inner most curve is where $\Omega = 0$. Thus RNS can create a constant restmass sequence between these two limits. Such a sequence would be shown as a straight but sloped line between two points, one located at $\Omega = 0$ and the other at $\Omega = \Omega_{max}$ where mass shedding occurs. These two points would of course correspond to two different central densities, which are then used as input parameters for RNS, when creating the sequence.

We will focus on sequencies for which the non-rotating model has a gravitational mass of $M_G = 1.4M_\odot$.

So we need to do two things before we can make this sequence of models. First determine at which central density ρ_2 we have $M_G = 1.4M_\odot$ in the static case. This is done by an iterative procedure. Also given in this result is the rest mass M_0. We then calculate to find the maximally rotating model which have this same restmass. In other words we find by an iterative procedure the new central density ρ_1 corresponding to the shedding limit.

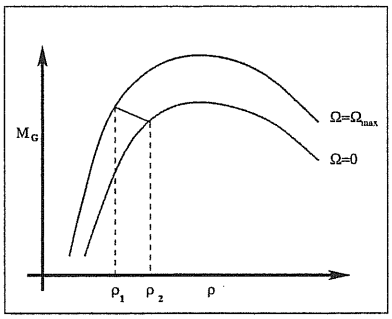

Figure 2.12: A typical polytropic mass-density function.

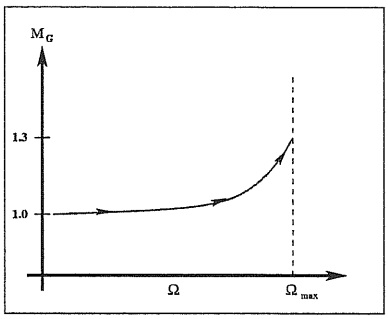

Figure 2.13: The gravitational mass dependence become non-linear as one approaches Ω_{max}.

Example Procedure

The procedure can be outlined in terms of the commands used with RNS. For example, using the *eosUU* EOS compiled with a high resolution grid of MDIVxSDIV $= 101 \times 201$, we have the following.

The required value is found by an iterative procedure. Anticipating the result, we choose a value for the central density, that will correspond with $M_G = 1.4M_\odot$:

```
./rns -f eos/eosUU -t static -e 1.038e15
```

This gives a table containing an entry with $M_G = 1.4M_\odot$ and $M_0 = 1.560M_\odot$. Next we want to find the central density, giving a maximally rotating star model with the same value for M_0. Similarly as above, we do this by making the (anticipated) attempt

```
./rns -f eos/eosUU -t kepler -e 0.8852e15 -d 0
```

Thus we can make a constant restmass sequence of $M_0 = 1.560M_\odot$, for the range of central densities of $\rho_c = 0.8852 - 1.038 \quad (\times 10^{15})$, which gives a sequance of rotating models with different rotational speed.

```
./rns -f eos/eosUU -t rmass -z 1.56 -e 0.9e15 -l 1.03e15 -n 3 -d 0 -p 7
```

We modified the program output switch to get only the parameters we are interested in. With the above switch -p 7 we get a sequence of stars each with the following parameters.

Quantity	Units	Meaning
rho_c	[10^15 gr.cm^-3]	Central density
M	[M_sun]	Mass-energy
M_0	[M_sun]	Rest mass
R_e	[km]	Equatorial radius
Omega	[10^4 s^-1]	Angular velocity
omega_FD	[10^4 s^-1]	Central Ang. vel. of frame dragging
j	[GM_sun^2/c]	Angular momentum

```
q                [10^42 gr.cm^2]        Quadrupole moment
e                [1]                    Eccentricity
-------------------------------------------------------------------
```

(Note. The j and q in the table above are not the usual ones we use throughout the text, but rather J and Q as output by RNS.)

However, we are only interested in comparing the parameters q, R_e, e, which we plot against j. But the plotting actually involves some number shuffling, as the output units of RNS are geometrized. In particular the output for the angular momentum J and the quadrupole moment Q. Using the table in the front matter, we can find the correct factors to use in the plots.

$$\begin{aligned}[\mathrm{Q}] &= M.L^2 \\ [\mathrm{J}] &= M.L^2.T^{-1}\end{aligned}$$

The output of RNS is:

$$\begin{aligned}M_{rns} &= \frac{M}{M_\odot} \\ Q_{rns} &= \frac{Q}{10^{42}} \\ J_{rns} &= \frac{cJ}{GM_\odot^2}\end{aligned}$$

In the formulas for the HT metric we have used units such that:

$$[q_{our}] = \frac{QM}{J^2}$$

where q_{our} is the q we have used throughout the chapter. In order to make q_{plot} dimensionless we need to multiply it by some factor x wich turnout to be c^2.

$$\begin{aligned}[q_{plot}] &= x.[q_{our}] = c^2\left(\frac{QM}{J^2}\right) \\ &= \frac{c^2(10^{42}Q_{rns})(M_{rns}M_\odot)}{J_{rns}^2\left(\frac{GM_\odot^2}{c}\right)^2} \\ &= 10^{42}\left(\frac{c^4}{G^2M_\odot^3}\right)\left(\frac{Q_{rns}M_{rns}}{J_{rns}^2}\right)\end{aligned}$$

The results of these numerical calculations are shown and discussed below. But before showing these, I will briefly explain the output of the code we used for the plots.

2.7.2 The HT-Miller Code

As discussed in detail above, we have together with and thanks to Prof. J. Miller, also developed a code to generate constant restmass stellar models based on the Hartle-Thorne approach. In the example below we show and explain the output from this code.

Example (5):

Since the only parameter needed (the gravitaional mass M_G) is hard-coded into the program, it only allows for changing the equation of state by putting the EOS data into a file called eos.dat. *Running*

the code then produces:

```
EOS UU:

  MASS =  1.4000
  M_0 =  1.5722
  RADIUS = 11.1154
  EC = 0.10447E+16


  DEC/EC = -0.3415
  DR/R =  0.4301
  DM/M =  0.0462
  ELLIPTICITY =  0.8445
  KERR FACTOR =  4.6312
  MIF =  0.3722
  W0 =  0.5354
  W1 =  0.8615
```

The first group of values represents non-rotating models, where the different values are: total gravitational mass, rest mass, equatorial radius and central energy density, respectively. The second group of values are for the rotating HT star. The explicit functional dependence of these variables are:

$$
\begin{aligned}
DEC/EC &= \frac{\delta e_c}{e_c} & [(\tfrac{a}{m})^2] \\
DR/R &= \frac{\delta R_e}{R_e} & [(\tfrac{a}{m})^2] \\
DM/M &= \frac{\delta M_0}{M_0} & [(\tfrac{a}{m})^2] \\
W0 &= \left(\frac{\tilde{\omega}}{\Omega}\right)_{r=0} & [\tfrac{a}{m}] \\
W1 &= \left(\frac{\tilde{\omega}}{\Omega}\right)_{r=R_{surf}} & [\tfrac{a}{m}]
\end{aligned}
$$

where $\tilde{\omega} = \Omega - \omega_{FD}(r = R_s) = 1 - \omega_{rns}(r = R_s)$. *The Kerr factor is simply* QM/J^2 *and* MIF *is the moment of inertia factor which is* $M\mathcal{R}^2_{gyration}$. *We could also express the ellipticity in terms of the eccentricity as is output by* RNS, *in the slow rotation limit.*

$$
\begin{aligned}
\epsilon &:= \frac{a-b}{\bar{R}} & \textit{ellipticity} \\
e &:= \left(1 - \frac{b^2}{a^2}\right)^{1/2} & \textit{eccentricity} \\
&\simeq \sqrt{2\epsilon} & \textit{(for slow rotation)}
\end{aligned}
$$

where a and b are the semi-major and semi-minor elliptical radii, respectively, and $\bar{R}$ *is an average radius close to the non-rotating radius in the slow-rotation limit.*

□

2.7.3 The Results

Each plot below shows q, R_e, e for each of the three eos's used. However, eventhough the gravitational mass $M_G = 1.400 M_\odot$ for all models, the stellar rest mass differs:

EOS	M_0	ρ_1-ρ_2
EosW	1.66829	6.4505 – 7.40905
EosB	1.62556	2.7984 – 4.755
EosUU	1.55893	0.88544 – 1.03825

In the beginning we wanted to show also how the code behaved for the angular velocity of frame-dragging, but since *RNS* was not providing the output for this quantity at the surface, we have ignored it for the moment due to time constraints. In the first plot we see that the quadrupole moment for the softer EOS B is accurate in a wider range than the other EOS's. Whereas the plots of the equatorial radius are surprisingly accurate even at very high rotation speeds and for all different EOS's. The last plot shows the eccentricity, but here the HT plotting formula is only good at slow rotation speeds, which may be the reason for the larger deviation in this variable, eventhough they show good agreement at lower values of j. Again there are less deviation for softer equations of state.

2.8 Conclusion

The work shown in this chapter, began as an exercise as proposed by professor Abramowicz, who suggested that I should just take the Hartle-Thorne metric from the appendix of [106] and calculate the epicyclic frequencies using his formulas. This seemed like a brilliant idea in the beginning, but soon I ran into many problems. First of all I realized that since the HT metric is a second order object, any mathematical manipulation other than addition would require complicated series expansions, making the expression growing out of hand of what is normally considered humanly doable. The chances for errors would have been too great and every error would most likely have been a large setback since all following calculations would rely on it. The solution was of course to use a computer algebraic package, such as Mathematica or Maple. But this was just the beginning, as calculations proceeded I was constantly running out of memory and the computer was always very slow, even though Fermi-Pasta-Ulam [2] may not have thought so. In any case, what ever one thinks about computer mathematics, a computer is never equally efficient at spotting simplifications. This means a large amount of time was spent in gaining experience in how the software would most efficiently deal with calculations. Other more normal problems were such as, trying to decipher the non-standard definitions of the associated Legendre polynomials and trying to understand the relation to the Kerr metric. With the non-existent previous experience I had with these kinds of calculations, this project then seemed to become the longest one. As I was starting to get worried about my other projects, I just worked on this one on and off for a few months at the time. In the end, together with Dr. A. Thampan, we have checked all calculations and obtained all the orbital parameters sought in the beginning.

Thus we have obtained formulas for all orbital parameters with respect to the equatorial plane for the Hartle-Thorne metric, and in some cases even for a general inclined orbit. These formulas are all good up to second order in the angular momentum and first order in the quadrupole moment (since it is a second order quantity.) We have also found a formula for the parametric resonance, depending on the different n and m's. We mentioned the importance of this holonomy invariance as a possible explanation for QPO's. We then made a comparison of the Hartle-Thorne metric to the RNS computer code, too find the limits of both, for rapid and slow rotation. We discovered that the Hartle-Thorne metric can be used reasonable well even up to rapid rotation but not quite at the maximal rotation speed (at the mass shedding limit.) Furthermore, softer equations of state seems more forgiving with a wider range of accuracy.

[2] For those who do not remember, Fermi, Pasta and Ulam where interested in how energy was distributed among a chain of connected (coupled) oscillators. Fermi then alone programmed one of the worlds first computers with a Runge-Kutta type algorithm, using pure machine code.

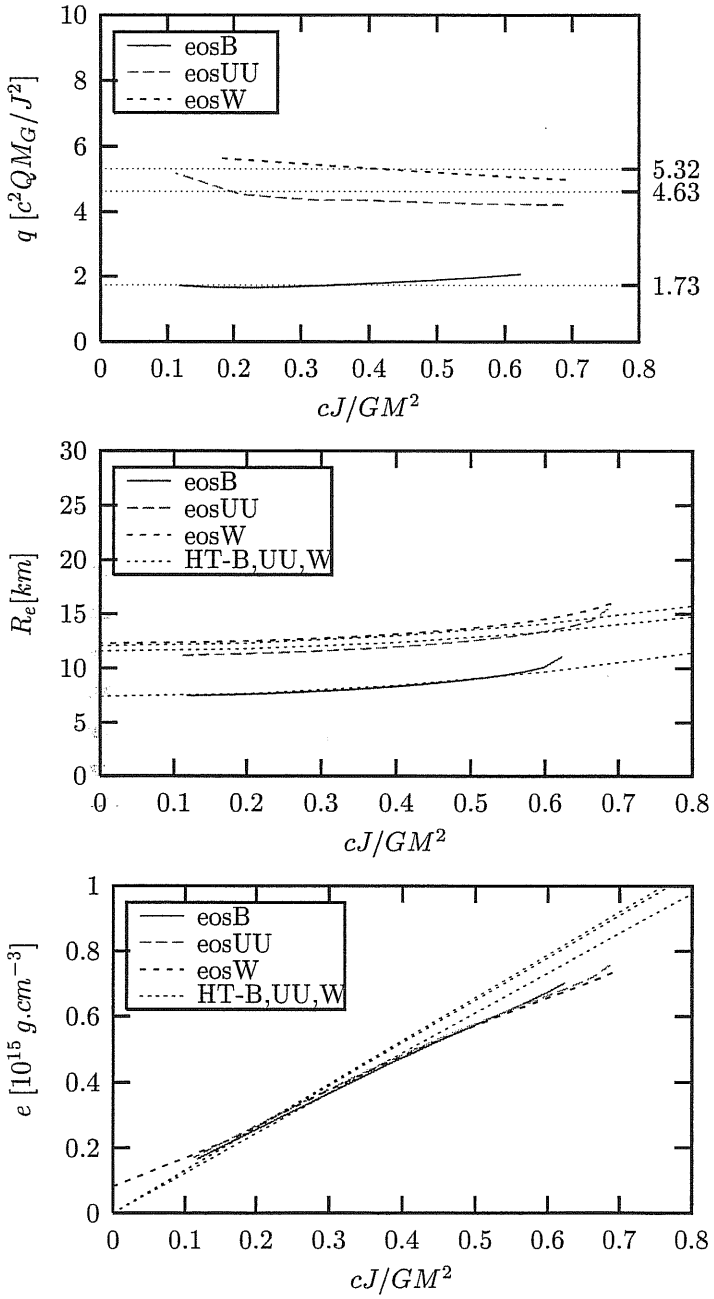

Figure 2.14: From top to bottom, we have plotted the quadrupole moment, equatorial radius and eccentricity for each EOS (B, UU, W).

Chapter 3

Gravitational Energy

Problems worthy
of attack
prove their worth
by hitting back.

Piet Hein[1]

The search for a clear and concise definition and notion of gravitational energy, is perhaps one of the most illusive problems of general relativity today. The problem of course, concerns the idea that the gravitational field itself should contribute to the total energy of an object, and thus one should be able to somehow split the energy into a part due to matter and another due to gravity. But, as we shall see in this chapter, this idea is slightly miss-formulated.

A brief scan of the literature on this subject, could make you very worried, as you realize the enormous quantity. Of course some of the material is out of date, but it merely demonstrates that this energy problem, has kept scientists occupied since the very beginning. Needless to say, I will not even attempt to make a review of this material, but rather go straight to the points of the most recent ideas and developments. As such, it is quite likely that I have missed some references and ignored some others.

This chapter then provides the current status of the work I started less than two years ago, on the problem of energy in general relativity. Many other topics studied have again been left out since they do not have a proper placement with regard to this thesis. However, it is with interest and also a slight disappointment that I have to conclude by saying that much of this material is still work in progress. And from this, it is clear that "this one" is surely hitting back!

[1]In the book by J. Wheeler, "Geons Black-Holes and Quantum Foam", NY, W.W.Norton, (1998)

3.1 Introduction and Overview

The notion of gravitational energy has been plagued by misunderstandings, controversies and problems since the very creation of general relativity [129]. But why do we even care about energy and its localization in in General Relativity (GR)? There are several reasons but mainly two. For one, it is known that in the post Newtonian approximation (PNA) of certain observables, there is a location dependent effect of the gravitational energy. For example if one takes into account the curvature of space in a Newtonian theory of gravity one can actually derive the perihelion advance of mercury as shown in [186]. (This reference is interesting and recommended further reading.) There, the presence of curvature introduces an additional location dependent term. Second, to understand better several phenomena such as:

⋆ The energy transfer by gravitational waves.
⋆ Mass calculations of stars, galaxies and blackholes.
⋆ Blackhole thermodynamics where $\delta m = \kappa/(8\pi)\,\delta A + \Omega\,\delta J$.
⋆ The cosmological constant Λ.

But are not all these phenomena already understood? Well, this is the point. It is not clear that our current methods are completely correct on astrophysical scales, as these calculations usually involve either frame-dependent or coordinate dependent quantities, which in turn relies on the asymptotic structure, where hopefully the spacetime is close to flat.

In fact, the main difficulty is due to a very basic feature of the theory — the equivalence principle — and is common to all purely metric theories of gravity. Roughly, it can be expressed as follows. A candidate for "gravitational energy density at a point" should be somehow constructed from the gravitational field strength at that point. However, the equivalence principle implies that a gravitational field can always be locally gauged away by a suitable choice of reference frame. Therefore, gravitational energy density cannot be associated with any covariant object — in particular, there is no stress-energy-momentum tensor for gravity (see also Ref. [130, p.467].)

Another side of the same problem concerns energy conservation. Generally energy definitions involve a conserved quantity, much like we have seen in chapter one. For example in a continuous system in the absence of gravity, the stress-energy-momentum (SEM) tensor [2] T_{ab} satisfies the differential equation

$$\partial^b T_{ab} = f_a \,, \tag{3.1}$$

where f_a is the external force density. Then for an isolated system without external forces, this becomes

$$\partial^b T_{ab} = 0 \qquad \text{(vacuum)}\,. \tag{3.2}$$

From the discussion of conserved charges in chapter one, this could equally well be thought of as a differential conservation law. The problem then appears when we want to generalize this to a curved spacetime. To do this we usually and sometimes naively apply the rule that $\partial_a \to \nabla_a$ and then expect that:

$$\nabla_a T^{ab} = 0 \tag{3.3}$$

[2] Latin letters a, b, ... from the beginning of the alphabet are used as abstract indices (see, e.g., Ref. [24], pp. 24–25), which just indicate the tensorial nature of an object without requiring the specification of a coordinate system. Greek letters μ, ν, ..., and latin letters from the middle of the alphabet, i, j, ..., run from 0 to 3 and from 1 to 3, respectively, and denote components in some chart. We choose +2 as signature of the metric and work in units in which $c = 1$. The conventions for the curvature tensors are those of Ref. [24].

should be a conserved quantity as well. However, this is a big mistake. This is not a conservation law, because Gauss' divergence theorem only applies to ordinary derivatives and not to covariant ones. Therefore the last relation can not be used to yield the conserved charges. This means that there is no covariant local law of energy conservation in general relativity [129].

Choosing a chart, one can nevertheless rewrite Eq. (3.3) as

$$\partial^\nu T_{\mu\nu} = g^{\nu\sigma}\Gamma^\rho{}_{\nu\mu}T_{\rho\sigma} + g^{\nu\sigma}\Gamma^\rho{}_{\nu\sigma}T_{\mu\rho} \ , \tag{3.4}$$

which can be interpreted, in analogy with Eq. (3.1), identifying the right hand side as the gravitational force density. Using Einstein's equation

$$\boxed{G_{ab} + \Lambda\, g_{ab} = 8\pi G\, T_{ab}} \tag{3.5}$$

(Λ and G denote the cosmological constant and Newton's gravitational constant, respectively), this quantity can be expressed as a combination of the connection coefficients $\Gamma^\mu{}_{\nu\rho}$ and of their first derivatives. Back-stepping, one may instead expect

$$\partial^\nu \left(T_{\mu\nu} + t_{\mu\nu}\right) = 0$$

to be well behaved, by defining a *pseudotensor*, $t_{\mu\nu} \sim \Sigma(g\cdot g\cdot\Gamma\cdot\Gamma)$ to take into account for the curvature dependent derivatives. However, $t_{\mu\nu}$ is coordinate dependent and as such can be made to vanish at any point in spacetime. In any case, $t^{\mu\nu}$ does provide a conserved charge which is good non-locally.

$$P^\mu := \int_\Sigma (T^{\mu\nu} + t^{\mu\nu})\, \bar{g}\, \mathrm{d}\Sigma_\nu$$

where Σ_ν is a 3d hypersurface. Then the energy is $E := P^0$. This energy is known as a *quasi-local* energy, since it is associated with a closed 2-surface.

So what is the problem? Essentially, the origin of the difficulties lies in the fact that the notion of gravitational force — hence of gravitational field — is ill-defined in general relativity, where gravity becomes just a manifestation of spacetime geometry. However, while there is a meaningful notion of energy for physical fields, there is nothing like an "energy of geometry." Life is much easier in Newtonian gravity, where the gravitational field

$$\mathbf{g}(\mathbf{x}, t) = -\nabla\phi(\mathbf{x}, t) \tag{3.6}$$

is a well-defined quantity, and one can write the energy density as [138]

$$u(\mathbf{x}, t) = -\frac{1}{8\pi G}\, \mathbf{g}(\mathbf{x}, t)^2 \ . \tag{3.7}$$

These considerations suggest that, if we could somehow introduce a notion of gravitational force in general relativity, it would then be much easier to arrive at a satisfactory definition of gravitational energy.

Since force is a frame-dependent quantity, in order to define it in a unique way we must be able to select a preferred frame. In Newtonian mechanics and special relativity one uses inertial frames for this purpose. An inertial frame can be characterized as a family of observers that is non-accelerating, non-rotating and non-deforming. If n^a is a four-velocity field with orthogonal projector $h_a{}^b := \delta_a{}^b + n_a n^b$, we can define its acceleration a_a, vorticity ω_{ab}, shear σ_{ab} and expansion θ through the irreducible decomposition

$$\nabla_b n_a = -n_b a_a + \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\,\theta h_{ab} \ . \tag{3.8}$$

For an inertial frame all quantities must vanish, so an alternative definition of such a frame is $\nabla_a n^b = 0$. Unfortunately, it can be shown [139] that vector fields satisfying this condition only exist in the so-called *ultrastatic* spacetimes [140], whose metric can be written as

$$g = -\mathrm{d}t^2 + g_{ij}\,\mathrm{d}x^i \mathrm{d}x^j \ , \tag{3.9}$$

with $\partial g_{ij}/\partial t = 0$. (From hereon we shall use the notation of "g" to represent the line element $\mathrm{d}s^2$.) These metrics are rather dull, describing spacetimes without gravitational forces, which differ from Minkowski spacetime only by having curved $t = \text{const}$ spatial sections. However, it is possible to extend the notion of inertial frames to spacetimes which are only conformally stationary [141]. Hence, for this class of spacetimes, one can reformulate general relativity in such a way that gravitational force is a well-defined notion.

In this thesis, we restrict ourselves to consider the narrower class of *conformally static* spacetimes — those which possess a timelike conformal Killing vector field or, equivalently, that can be mapped into an ultrastatic spacetime $(\mathcal{M}, \widetilde{g}_{ab})$ through a conformal transformation [3]

$$g_{ab} = \sigma^2\, \widetilde{g}_{ab} \ , \tag{3.10}$$

where σ is a nowhere vanishing function on $\mathcal{M}$. Since the spacetime $(\mathcal{M}, \widetilde{g}_{ab})$ is ultrastatic, its unit timelike Killing vector field η^a satisfies the criterion for an inertial frame: $\widetilde{\nabla}_a \eta^b = 0$, where $\widetilde{\nabla}_a$ is the Riemannian covariant derivative associated with the metric $\widetilde{g}_{ab}$. We shall then define the inertial frame in $(\mathcal{M}, g_{ab})$ as the four-velocity field n^a which is parallel to η^a,

$$n^a := \frac{\eta^a}{(-\eta_b \eta^b)^{1/2}} = \frac{1}{\sigma}\,\eta^a \ . \tag{3.11}$$

The observers with four-velocity n^a are analogous to those of a Newtonian rest frame. Their four-acceleration is

$$a_a = n^b \nabla_b n_a = h_a{}^b \nabla_b \Phi \ , \tag{3.12}$$

where $\Phi = \ln \sigma$ (see Ref. [142], App. A, for a proof). Hence, they see freely-falling particles as moving with the four-acceleration $-h_a{}^b \nabla_b \Phi$, in close analogy with the Newtonian equation (3.6). The scalar field Φ can thus be considered as the relativistic generalization of the Newtonian potential ϕ.

The parallel between n^a and an inertial frame in Newtonian physics can be pushed much further. One can show that many equations of physics take their simplest form when rewritten in the spacetime $(\mathcal{M}, \widetilde{g}_{ab})$ and interpreted according to the frame η^a [143]. In particular, massless particles (hence light) propagate along geodesics of the spatial metric $\widetilde{h}_{ab}$. For this reason, $\widetilde{g}_{ab}$ is called *optical metric* [144]. Correspondingly, we shall refer to $(\mathcal{M}, \widetilde{g}_{ab})$ as the *optical spacetime*. The use of this paradigm allows one to give simple, intuitively appealing descriptions of phenomena that may look obscure in the usual formalism of general relativity [142, 145, 146, 147, 148].

To summarize, if one has a conformally static spacetime $(\mathcal{M}, g_{ab}, \psi)$ in general relativity (ψ denotes, collectively, the matter fields), it is convenient to redefine the degrees of freedom by "extracting" a scalar

[3] The conversion to the more common notation $\widetilde{g}_{ab} = \Omega^2 g_{ab}$ is, of course, $\Omega = 1/\sigma$. Hereafter, indices of geometrical objects pertaining to the spacetime $(\mathcal{M}, \widetilde{g}_{ab})$ are lowered and raised using $\widetilde{g}_{ab}$ and its inverse $\widetilde{g}^{ab}$. Thus we have, for example, $\widetilde{X}^a := \widetilde{g}^{ab}\widetilde{X}_b$. Also, $\Box := g^{ab}\nabla_a\nabla_b$ and $\widetilde{\Box} := \widetilde{g}^{ab}\widetilde{\nabla}_a\widetilde{\nabla}_b$ denote the D'Alembertian operator in the two spacetimes.

field σ from the spacetime geometry. The spacetime model then becomes $(\mathcal{M}, \widetilde{g}_{ab}, \sigma, \psi)$, where $\widetilde{g}_{ab}$ is ultrastatic and σ is a new scalar field describing the relativistic counterpart of Newtonian gravity [4]

Since, in the optical spacetime, gravity is associated with a covariantly defined scalar field σ, we have a new possibility to define a gravitational energy: We can identify it with the energy of σ. Thus, while the "energy of g_{ab}" is not defined , we isolate a part which *is* defined (the one involving σ) from one which *is not* (associated with the metric $\widetilde{g}_{ab}$).

The first section will begin with an overview of the material covered in this chapter, followed by an introduction concerning some important background material. This means the variational principle in the Lagrangian formulation, and the definition of stress-energy-momentum tensors, including conserved quantities. In the following section I will discuss and summarize some of the most common approaches for localization of gravitational energy. This include discussions of pseudotensors, quasilocality, Noether charges and surface integrals. I will then introduce optical geometry from the point of view of conformal transformations and the dependence of local inertial frames. This will then be used as a stepping stone to the field equations and the derivation of the gravitational energy density in the optical geometry frame, finally concluding with two examples, the Schwarzschild spacetime and the Friedmann-Lemaitre-Robertson-Walker cosmological model, and their corresponding discussions.

3.2 The Variational Principle

There are a number of different ways one can obtain the Einstein field equations (EFEs). The most straightforward procedures are those following the Lagrangian and Hamiltonian formulations of a classical field theory. The idea is to generalize the classical point mechanics to continuous variables. Essentially letting the fields ψ_i and their time derivatives at each point in spacetime, play the roles of the familiar canonical conjugate variables, the p's and q's of Hamiltonian mechanics. (See, e.g. [173].)

In the following sections I will derive the Euler-Lagrange equations and the canonical energy-momentum tensor as well as the more standard symmetrical stress-energy-momentum tensor. This will be done in Minkowski spacetime and then generalized to a curved spacetime. I will then outline the connection between these two derivations of the energy-momentum tensors. This allows us to clarify the problem in our optical geometry approach.

Most of the material herein can be found in the standard textbooks of GR and classical field theory [24, 174, 176, 172]. However the most transparent derivations (of the Lagrange formalism) was that of Glendenning [175] and the one by Babak and Grishchuk [181].

3.2.1 The Lagrangian Formulation

From Hamilton's variational principle we know that the equations of motion are those for which the classical action is an extrema. It is then believed that the same applies to field theory, where the fields represent the various particles, with minor modifications in the mathematical formalism. For simple

[4] Of course, the total number of degrees of freedom is the same in the two descriptions. In general relativity, we have the ten variables associated with the independent coefficients of the metric (which could be reduced to six by exploiting gauge freedom). In optical geometry, one of these is "frozen" (in conformally static spacetimes, one chooses σ such that $\widetilde{g}_{00} = -1$), so $\widetilde{g}_{ab}$ has only nine non-trivial components, the tenth variable being represented by σ.

notation it is convenient to define the following quantities

$$\begin{aligned} \bar{g} &:= \sqrt{-g} \\ \mathcal{L} &:= \bar{g}\, L \\ S &:= \int \mathcal{L}\, d^4x = \int L\, dt \end{aligned}$$

Where g is the absolute value of the metric determinant, and $\mathcal{L}$ and S are the Lagrangian density and the action, respectively. The variational principle is then used in such a way that $\delta S = 0$, which is known as the *principle of least action.* Thus the action integral becomes

$$\delta S = \int_{\mathcal{M}} \left(\frac{\delta \mathcal{L}}{\delta g^{ab}} \right) \delta g^{ab}\, \mathrm{d}^4x = 0 \tag{3.13}$$

where Ω is the entire 4-volume. But the integrand can be defined as a *functional derivative.* In case the functional derivative is set to zero, $\delta\mathcal{L}/\delta g^{ab} = 0$, the resulting equations are called the *Euler-Lagrange equations* or simply, the field equations. However, when gravity is considered, we usually separate the total Lagrangian density into a free gravity part ($\mathcal{L}_G = \bar{g}R$) and a matter part ($\mathcal{L}_m$) representing all other fields.

$$S = \int_{\Omega} (\mathcal{L}_G + \mathcal{L}_m) \mathrm{d}\,\Omega \tag{3.14}$$

The stress energy tensor is then defined through the variation of the action in the following way

$$\frac{\delta S}{\delta g^{ab}} := \frac{\partial S}{\partial g^{ab}} - \left(\frac{\partial S}{\partial (g^{ab}{}_{,\lambda})} \right)_{,\lambda} \tag{3.15}$$

$$T_{ab} := \frac{2}{\bar{g}} \frac{\delta S}{\delta g^{ab}} \,. \tag{3.16}$$

Applying this definition to eq (3.13) and (3.14), we can write

$$\left.\begin{aligned} \tfrac{\delta \mathcal{L}_G}{\delta g^{ab}} &:= \bar{g}\, G_{\mu\nu} \quad \text{(Gravity)} \\ \tfrac{\delta \mathcal{L}_m}{\delta g^{ab}} &:= \bar{g}\, T_{ab} \quad \text{(Matter \& Fields)} \end{aligned}\right\} \Rightarrow \quad G_{\mu\nu} = \kappa\, T_{ab} \tag{3.17}$$

This defines the Einstein-equation (3.17) through its matter and gravitational field content, where $\kappa = 8\pi G$ in natural units (where $c = 1, G \neq 1$.) But how do we actually derive eq (3.17) from (3.15)? We shall show the details of this derivation below.

The Euler-Lagrange Equations

Consider a first order Lagrangian of some arbitrary field and linear in the field derivatives, $\mathcal{L}(\phi, \partial_a\phi)$. The variation of this can then be thought of as a first order Taylor expansion, where we will use the short hand notation: $\phi' := \partial_a\phi$.

$$\delta\mathcal{L} \simeq \frac{\partial \mathcal{L}}{\partial \phi}\, \delta\phi + \frac{\partial \mathcal{L}}{\partial \phi'}\, \delta\phi' \tag{3.18}$$

But we can expand this slightly by using the following.

$$\begin{aligned} \partial_a(X\,\delta\phi) &= X\,\partial_a(\delta\phi) + \delta\phi\,\partial_a X \\ &= X\,\delta(\partial_a\phi) + \delta\phi\,\partial_a X \quad \textbf{[Since: } \delta(\partial_a\phi) = \partial_a(\delta\phi)\textbf{.]} \\ \Rightarrow \quad X\,\delta\phi' &= \partial_a(X\,\delta\phi) - \delta\phi\,\partial_a X \end{aligned}$$

$$\delta\mathcal{L} \simeq \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \partial_a\left(\frac{\partial\mathcal{L}}{\partial\phi'}\,\delta\phi\right) - \partial_a\left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)\delta\phi \tag{3.19}$$

The variational principle is then used in such a way to extremize the action $\delta S = 0$, to yield the minima. Thus when integrating, we can drop the approximation and write the varied action as:

$$\begin{aligned}
\delta S &= \int_{\mathcal{M}} \delta\mathcal{L}\,\mathrm{d}^4x \\
&= \int_{\mathcal{M}} \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_a\left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)\right]\delta\phi\,\mathrm{d}^4x + \int_{\mathcal{M}} \partial_a\left(\frac{\partial\mathcal{L}}{\partial\phi'}\,\delta\phi\right)\mathrm{d}^4x \\
&= \int_{\mathcal{M}} \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_a\left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)\right]\delta\phi\,\mathrm{d}^4x + \int_{\partial\mathcal{M}} \left(\frac{\partial\mathcal{L}}{\partial\phi'}\,\delta\phi\right)\mathrm{d}^3x
\end{aligned}$$

Where we applied the divergence theorem to the last term. If we then consider variations (of ϕ or ϕ') that vanish on the boundary, this term can be ignored and set to zero. In fact, we *must* set it to zero to obtain the standard Euler-Lagrange equations (ELE's).

Additionally, varying the action with respect to some scalar field ϕ, with vanishing boundary terms, one obtains the **Euler-Lagrange derivative** also known as the **functional** or **variational** derivatives. Then since $\delta S = 0$, the integrand has to be zero as well and the functional derivatives become the Euler-Lagrange equations.

$$\frac{\delta S}{\delta\phi} := \frac{\partial\mathcal{L}}{\partial\phi} + \partial_a\left[\frac{\partial\mathcal{L}}{\partial(\partial_a\,\phi)}\right] = 0 \tag{3.20}$$

As a note, one may certainly wonder whether or not one should keep the boundary terms in this formula, for more general variations. This procedure was satisfactory in Minkowski spacetime, but what about in some general curvilinear spacetime?

The Canonical SEM Tensor (Flat)

We shall repeat the whole procedure of the variation, with a slight twist. This will enable us to obtain the **canonical** stress-energy-momentum (SEM) tensor, for an arbitrary field in Minkowski spacetime. In the next section we will then generalize this into a curved spacetime and thus showing some difficulties regarding the differences between canonical and **symmetrical** (ordinary) SEM tensors.

We start out as before with a Lagrangian which also depends on the Minkowski metric,

$$\mathcal{L}(\phi,\, \partial_a\phi,\, \eta_{ab})\,.$$

The twist will be that the variation will this time be produced by a one-parameter family of diffeomorphisms generated by a vector field, ξ^a. In other words the variation will be an infinitesimal coordinate transformation, where: $x'^a = x^a + \epsilon\xi^a(x)$. Some prefer to interpret this as a field redistribution [178, 155]. The corresponding change in the Lagrangian density is given by:

$$\begin{aligned}
\delta\mathcal{L} &:= \mathcal{L}(x'^b) - \mathcal{L}(x^b) \\
&= \mathcal{L}[\phi(x'),\, \partial_a\phi(x'),\, \eta_{ab}(x')] - \mathcal{L}[\phi(x),\, \partial_a\phi(x),\, \eta_{ab}(x)] \\
&= \epsilon\,\xi^a\,\frac{\partial\mathcal{L}}{\partial x^a} = \epsilon\,\xi^a\,\partial_a\mathcal{L} = \epsilon\,\pounds_\xi\mathcal{L}. \\
&= \epsilon\,\xi_b\,\partial^b\mathcal{L}
\end{aligned}$$

Where we made a first order Taylor expansion of the first term. We then see that our variational operator δ is essentially equivalent to the Lie derivative based on ξ^a of the transformation above, (i.e. $\delta/\epsilon \simeq \pounds_\xi$). Similarly we can Taylor expand also for the field variables:

$$\begin{aligned}
\delta\phi &:= \phi(x^b + \epsilon\xi^b) - \phi(x^b) \\
&= \epsilon\xi^b\partial_b\phi = \epsilon\xi_b\partial^b\phi \\
\delta(\partial_a\phi) &= \partial_a\delta\phi = \epsilon\,\partial_a(\xi^b\partial_b\phi) \\
&= \epsilon\,(\partial_a\xi^b)(\partial_b\phi) + \epsilon\,\xi^b\partial_a\partial_b\phi \\
&= \epsilon\,\xi_b\partial_a\partial^b\phi
\end{aligned}$$

However we could equally well have written the variation like:

$$\begin{aligned}
\delta\mathcal{L} &:= \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi'}\,\delta\phi' + \frac{\partial\mathcal{L}}{\partial\eta_{ab}}\,\delta\eta_{ab} \\
&= \left[\frac{\partial\mathcal{L}}{\partial\phi}\,\partial^b\phi + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\,\partial_a\partial^b\phi\right]\epsilon\,\xi_b + \frac{\partial\mathcal{L}}{\partial\eta_{ab}}\,\pounds_\xi\eta_{ab} \\
&\overset{\text{(ELE)}}{=} \left[\partial_a\left(\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\right)\partial^b\phi + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\,\partial_a\partial^b\phi\right]\epsilon\,\xi_b \\
&= \partial_a\left[\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\partial^b\phi\right]\epsilon\,\xi_b
\end{aligned}$$

Where in the second row we used the fact that ξ^a is a *Killing vector field* (Kvf), so that the Lie derivative of the metric vanishes. With this assumption in hand, we can rewrite the first expression for $\delta\mathcal{L}$ as well, using

$$\begin{aligned}
\partial_a(\eta^{ab}\,\mathcal{L}) &= \eta^{ab}\,\partial_a\mathcal{L} + \mathcal{L}\,\partial_a\,\eta^{ab} \\
\Rightarrow\quad \xi_b\,\eta^{ab}\,\partial_a\mathcal{L} &= \xi_b\,\partial_a(\eta^{ab}\,\mathcal{L})
\end{aligned}$$

$$\delta\mathcal{L} = \partial_a(\eta^{ab}\,\mathcal{L})\,\epsilon\,\xi_b \tag{3.21}$$

Then equating the two expressions for $\delta\mathcal{L}$, we have

$$\partial_a\left[\frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\partial^b\phi - \eta^{ab}\,\mathcal{L}\right]\epsilon\,\xi_b = 0\,. \tag{3.22}$$

This result is known as **Noethers Theorem**, which states that for every symmetry of the Lagrangian there exists a conservation law, such that invariance under spacetime transformations leads to a tensor which obeys; $\partial_a\,S^{ab} = 0$. In general we say that a diffeomorphism ϕ is a symmetry of some tensor S if the tensor is invariant after being pulled back under ϕ. If we then have a one parameter family of symmetries generated by a vector field ξ, then invariance is implied by $\pounds_\xi\,S = 0$. And in fact contracting (3.22) with η^{ab}, we get exactly this. Therefore we can define the contents of the square brackets as the *canonical energy-momentum* tensor.

$$S^{ab} := \frac{\partial\mathcal{L}}{\partial(\partial_a\phi)}\partial^b\phi - \eta^{ab}\,\mathcal{L} \tag{3.23}$$

However, this is not considered in general as *the* energy-momentum tensor for several reasons: a) It is *not* what appears in the Einstein equations, when derived from an matter-action in the Lagrangian formulation. b) In general it is not symmetric. c) It does not generalize to a conserved tensor in a

curved spacetime. However the second requirement can be satisfied by adding a divergence term Ψ^{ab} to (3.23), so that

$$S^{ab} := \frac{\partial \mathcal{L}}{\partial(\partial_a \phi)} \partial^b \phi - \eta^{ab}\, \mathcal{L} + \Psi^{ab} \tag{3.24}$$

is equally good under the condition that

$$\Psi^{ab}{}_{,b} = 0\,. \tag{3.25}$$

This condition can be satisfied either identically or by the ELEs. To satisfy (3.25) identically, it is sufficient to have $\Psi_a{}^b = \psi_a{}^{bc}{}_{,c}$ where $\psi_a{}^{bc}$ is antisymmetric in b and c. so that $\psi_a{}^{bc}{}_{,cb} \equiv 0$. The function $\psi_a{}^{bc}$ is usually called a **superpotential**. By an appropriate choice of $\Psi_a{}^b$ one can make S^{ab} symmetric in its components. However, this is not needed in a Lagrangian describing a scalar field, since it is automatically symmetric. The transformation properties of S^{ab} under coordinate transformations are not defined until the transformation properties of the field variables and $\mathcal{L}$ are defined.

As stated by [172, p.117], when thinking about the EFEs without specifying the theory from which T_{ab} is derived, one realizes that without constraints on T_{ab}, any metric will satisfy the EFEs. (Just, take any metric and compute $G_{\mu\nu}$ and set it equal to T_{ab}. Then it will be automatically conserved by the Bianchi identities.)

The Canonical SEM Tensor (Curved)

The generalization of the derivation of the canonical SEM tensor, for a curved spacetime is merely a mathematical exercise very similar to the previous derivation. The difference this time is that the Lagrangian density also depends on the metric tensor g_{ab} and the covariant derivatives of the fields involved,

$$\mathcal{L}(\phi,\, \nabla_a \phi,\, g_{ab})\,. \tag{3.26}$$

Actually one could equally well derive the curved canonical SEM, using the flat Minkowski metric, and a tensorfield and its covariant derivative as independent variables, such that $\mathcal{L} = \mathcal{L}\,(\gamma_{ab}, h^{ab}, h^{ab}{}_{;c})$. In that case one would also have to generalize the ELE for a curved background. This could be of interest in a field-theoretical formulation of general relativity, where gravity is treated as a non-linear tensor field $h^{ab}(x^\alpha)$ in a Minkowski spacetime [181]. Without complicating things further we will just state the result of the canonical SEM in a curved spacetime.

$$S^{ab} := \frac{\partial \mathcal{L}}{\partial(\partial_a \phi)} \partial^b \phi - \eta^{ab}\, \mathcal{L} + \Psi_\alpha{}^\beta \tag{3.27}$$

Here we have again added a non-symmetric divergence that can be used to make S^{ab} symmetric, with the condition that $\Psi^{ab}{}_{;b} = 0$.

The Symmetric and Canonical SEM Relation

For Lagrangian-based theories, the derivation of the conserved energy-momentum object is closely related to the variational procedure by which the equations of motion are being derived. In fact, there are two routes of derivation. One produces a "canonical" tensor, and another produces a "metrical" tensor (usually known as the symmetric stress-energy momentum tensor.)

The first route takes its origin from Euler-Lagrange approach and hence the name of the resulting equations. This route does not care about transformation properties of the field variables and the

Lagrangian itself. But what is important is whether the Lagrangian contains explicitly (in a manner other than through the field variables) the independent variables (coordinates) x^α. If such dependence is present, one should not expect that there are any conserved quantities. If there is no such dependence, some sort of conservation laws are guaranteed as a consequence of the equations of motion.

The second route is associated with Noethers theorem. Here one exploits from the very beginning the transformation properties of fields and Lagrangians. One requires the action to be a quantity independent of any coordinate transformations and, hence that the Lagrangian is a scalar density. (This means a scalar function times the square root of the metric determinant.) This route produces what I shall call a "metrical" tensor (not to be confused with the metric), which is essentially the variational derivative of the Lagrangian with respect to the metric tensor. This object is automatically a conserved symmetric tensor, if the equations of motion are satisfied. The conserved tensors are usually understood in the sense that they obey differential conservation equations, but one can also derive from them the integral conserved quantities if the system is isolated. Actually many standard derivations mix these two routes either to shorten the calculation or because of ignorance of their difference.

Both of these tensors, canonical and metrical, are defined up to certain additive terms which do not violate equations of motion. These terms are a generalization of the additive constant which arises even in a simplest one-dimensional mechanical problem, when the Lagrangian does not depend on time explicitly. It is known that the first integral of the equation of motion, which we interpret as energy, can be shifted by a constant. In field theories, the additive terms can be used to our advantage. For instance, the canonical tensor can be made symmetric, and the metrical can be made free of second derivatives, if they did not originally have these characteristics. Despite the different routes of derivation, the canonical and metrical tensors are deeply related. If they are derived from the same Lagrangian, explicitly containing the metric tensor in addition to field variables, they are equal to each other, up to a certain well defined term.

In traditional field theories, one arrives, after some work, at the energy-momentum tensor which is: 1) derivable from the Lagrangian in a regular prescribed way, 2) a proper tensor under arbitrary coordinate transformations, 3) symmetric in its components, 4) conserved due to the equations of motion obtained from the same Lagrangian, 5) free of the second (highest) derivatives of the field variables, and 6) is unique up to trivial modifications not containing the field variables. There is nothing else, in addition to these 6 conditions, that we could demand from an acceptable energy-momentum object, both on physical and mathematical grounds.

When it comes to the gravitational field, as described by the geometrical formulation of general relativity, things become more complicated. It is often argued that the equivalence principle forbids any gravitational energy-momentum tensor. What is meant in practice is that the all first derivatives of any metric tensor $g_{ab}(x^\mu)$ can be made, by an appropriate choice of coordinates x^μ, equal to zero along the world line of a freely falling observer (along a timelike geodesic line). But the first derivatives of g_{ab} can be eliminated along any world line, not necessarily of a freely falling observer. And this is true independently of the presence and form of coupling of g_{ab} to other fields, and independently of whether the g_{ab} obeys any equations. Since all components of a tensor can not be eliminated by a coordinate transformation, this reference to a physical principle is regarded to be an argument against having a gravitational energy-momentum tensor, but the argument sounds more like a fact from the differential geometry. Despite this argument, one usually notices that it is desirable, nevertheless, to construct at

least an "effective" gravitational energy-momentum tensor.

If the symmetric tensor (3.16) and the canonical tensor (3.27) are derived from the same Lagrangian (3.26), one expects them to be related. Assuming that the field equations are satisfied we conclude that

$$T^{ab} = S^{ab} + \psi^{abc}{}_{;c} \ . \tag{3.28}$$

Thus, the metrical and canonical tensors are related by a superpotential whose explicit form is given by eq. (C.14). (This derivation is similar to the one given in [183].) The conservation laws are satisfied because $\psi^{abc}{}_{;cb} \equiv 0$.

3.3 Some Different Approaches

There have been a number of other more or less successful approaches to defining gravitational energy. These differ greatly in their approach and formalism, as they rely on different ideas of where the energy should be localized. To mention just a few of these ideas, we start with that of Witten [184, 185], who obtained an expression for the total gravitational energy, motivated by his work on supergravity. The expression is a natural result of a Hamiltonian treatment of classical supergravity, wherein one does not consider quantum theory. As such, it involves spinors integrated over a 3-surface, and is not considered a local energy density. Later, Penrose [132] presented his quasilocal energy expression, which depends on the construction of a twistor space $\mathbb{T}^{\alpha}(S)$ associated with any spacelike 2-sphere S. His quasilocal mass expression is then given in terms of a sum of the components of the 4d Weyl tensor, the Ricci tensor and the Ricci scalar. However, as twistor theory is not generally well understood, the meaning of this mass may be quite unclear at its best. In complete contrast to this is the very simple *coordinate independent* method of Lynden-Bell and Katz [133]. They show by purely physical arguments that static spherical systems have a truly coordinate independent gravitational field energy density. They are led to this expression by enclosing any static spherically symmetric spacetime by a sphere. Then removing the interior and adding a surface distribution of mass in such a way as not to change the exterior spacetime. In this way they find the field energy simply by subtracting the matter energy from the total energy. The energy density is then obtained by changing the radius and dividing the energy by the volume difference. Another approach is to simply accept that one has to be satisfied with the global conservation of SEM tensor components in some preferred class of reference frames. These are the ideas of Nissani and Liebowitz [134]. They call these *non-rotating* since (as they state) in them, the individual components of the 4-momentum are separately conserved. (However, it is unclear why this should be so. Perhaps because rotation would introduce mixing of the terms.) This can be done by solving a differential equation for the non-rotating coordinate transformation. Another radical idea that has not been very well accepted in the GR community is that of Cooperstock [135], who claims that the conservation equation for the symmetric SEM tensor in vacuum is meaningless, since all components of the SEM vanishes identically. His resolution is then to use the coordinate dependence of a pseudotensor to localize the energy and to resurrect the validity of the symmetrical SEM tensor and its corresponding conservation laws. Another interesting method for defining gravitational energy is that of Wald [24, 168, 169], who uses a Noether current in the boundary term in the gravitational action to obtain a conserved Noether charge which is then taken to be the energy. Because the boundary integral is over a spatially compact 2-surface and not a hypersurface, this energy is again quasilocal. But perhaps the most interesting point is that with this method one can obtain many previous results,

just by choosing a different timelike vectors [136]. One can then obtain the Misner-Sharp, Sachs-Bondi and Komar energy expressions by choosing the timelike killing vector to be either a stationary Killing vector, an asymptotic time translation vector or the Kodama vector, respectively. A closely related method is what is generally called the quasilocal energy method. This has been extensively clarified and studied by Nester and Chen [170], who in addition have shown that any pseudotensor corresponds to a particular Hamiltonian boundary term and vice versa. In particular that the Hamiltonian density determines the evolution and constraint equations, whereas the boundary term determines the boundary conditions and the quasilocal energy-momentum.

As for the quasilocal energy, there are a great number of different definitions, and they do not coincide in general. The differences are coming from the characteristics of their respective pseudotensors. Some of the most popular energies are:

ADM	($\mathcal{I}^+$, all)
Bondi	(i^0, asymptotically flat)
Komar	(quasilocal, stationary)
Misner-Sharp	(quasilocal, spherically symmetric)
Penrose	(quasilocal, all)

where I have listed in parenthesis where they are defined and for which spacetimes they are defined. Some of these energies are known as *quasi-local* energy. Those are the ones which are associated with a closed 2-surface. It has been shown that the lack of consensus of quasilocal energies is due to the different choices of a preferred timelike vector [136].

In addition it should be mentioned that it has been recently shown [162] that there exists an infinite number of expressions that satisfy the criteria of ADM, Bondi, and the flat space, weak field limits. Clearly, then there is a need for additional criteria and principles. Apparently this can be provided by the Hamiltonian formalism, where a Hamiltonian boundary term provides both the quasilocal quantities and the boundary conditions. In fact it was shown by [161] that this quasilocal Hamiltonian formalism can be used to derive all other pseudotensors.

In the following descriptions of other approaches I have tried to keep the original notation of the authors, apart some minor changes in the tensor indices to conform to my own notation as stated in the front pages. In addition one should take care to the authors metric signature which I have not changed.

3.3.1 Wittens Gravitational Energy

In 1981, E. Witten published [184] a new expression for the total gravitational energy of an isolated system. This formula however, involves spinors which satisfy the spatial Dirac equation. His use of spinors was motivated by his work on a quantum theory of supergravity (SUGRA). However, the expression arises naturally from a Hamiltonian treatment of *classical supergravity* (without considering quantum theory) which is an extension of general relativity which admits an enlarged group of symmetry transformations. These symmetries enable one to see relations in pure GR that might otherwise remain obscure. In particular, *diffeomorphisms* can be expressed as the square of supersymmetry (SUSY) transformations. In the Hamiltonian formulation, one obtains a classical equation relating the generator of diffeomorphisms to the square of the generator of SUSY transformations. For the case that this diffeomorphism is a global time translation, one obtains Witten's expression for the total energy. His

original expression was

$$E = \int_\Sigma [T_{ab}\xi^a t^b + 2(D_m\alpha)^\dagger(D^m\alpha)]\,\mathrm{d}\Sigma \tag{3.29}$$

where α is a spinor, D_a is the 4-d covariant derivative projected onto Σ which is an asymptotically flat 3-surface. ξ^a is the Dirac current to α and t^a is the unit normal to Σ.

This expression was then slightly modified by Horowitz and Strominger [185] to more clearly account for conformal invariance. They show that ξ^a is everywhere orthogonal to Σ if and only if the trace of the extrinsic curvature of Σ vanishes (i.e. $\pi = \pi^m{}_m = 0$.) In this case the gravitational contribution can be divided into a conformally invariant (μ_i), a non conformally invariant (μ_{ni}) and a matter part (μ_m) as

$$E = \int_\Sigma \omega^n\,[\mu_m + \mu_i + (n-1)\mu_{ni}]\,\mathrm{d}\Sigma \qquad (\pi^m{}_m = 0) \tag{3.30}$$

where

$$\mu_m = \mu \tag{3.31}$$

$$\mu_i = 2\omega^4\,(\widetilde{\mathcal{D}}^{AB}\,\widetilde{\beta}^{jC})^\dagger(\widetilde{\mathcal{D}}_{AB}\,\widetilde{\beta}^j{}_C) \tag{3.32}$$

$$\mu_{ni} = 4\omega^{-2}\,D_a\omega\,D^a\omega\,. \tag{3.33}$$

and β^{jC} is a spinor with $j = \{1,2\}$. (For further detail on notation, see their article.) Here ω^n can be thought of as an overall red-shift factor, such that increasing n, one increases the effective red-shift, but compensates by adding more binding energy. If $n = 1$, μ_{ni} does not contribute, so for a static spherically symmetric star, the integrand vanishes identically outside the star.

Can we interpret the integrand of (3.30) as an energy density for a gravitating system? For the case $n = 4$ which corresponds to eq (3.29) the answer according to the authors is no. They claim that the difficulty lies in the fact that this energy density would be extremely nonlocal since it would be spread out over the entire 3-surface (*slice* in their language.)

Comment: The SUSY extension of GR to classical supergravity can be viewed like the extension of real functions to the complex plane, where many properties of the real functions are most easily discovered by considering their analytical extensions. Similarly the SUSY extension of GR allows a greatly expanded group of symmetry transformations. The most important consequence of this is that diffeomorphisms of the spacetime may be represented as repeated SUSY transformations.

3.3.2 Penrose's Quasilocal Energy

The Penrose approach [132] to defining energy-momentum and angular momentum depends on the construction of a *twistor space* $\mathbb{T}^\alpha(S)$ associated with any spacelike topological 2-sphere S, and thus it is a quasilocal quantity. Furthermore it also reproduces the Bondi-Sachs mass m_B at spatial infinity ($\mathcal{I}^+$) and the Hawking mass m_H for all spherically symmetric spacetimes. It also agrees with the Komar mass m_K for stationary vacuum spacetimes. His quasilocal mass is given by:

$$m_P = \frac{c^2}{4\pi G}\,N\left(\frac{\Delta^3}{4\pi}\right)^{1/2} \tag{3.34}$$

where $N := \phi_{11} + \Lambda - \psi_2$ and $\Delta := 4\pi R^3$ is the area, with the further notation that: ψ_i are components of the 4d Weyl tensor, ϕ_{ij} are components of the trace-free Ricci tensor and Λ is proportional to the Ricci scalar.

3.3.3 Cooperstock's Idea

Cooperstock's energy localization hypothesis [135] is based on the idea that the conservation equation of the stress energy tensor in vacuum is meaningless, since all components of the SEM tensor vanish identically. His resolution is to use the coordinate dependence of a pseudotensor to localize the energy, to resurrect the validity of the SEM tensor.

Taking for granted that

$$\nabla_b T^{ab} = 0\,. \tag{3.35}$$

we can always rewrite it as

$$\partial_b(\bar{g}T^b{}_a) - \tfrac{1}{2}\bar{g}(\partial_c g_{bc})\,T^{bc} = 0 \tag{3.36}$$

But as mentioned earlier, eq (3.35) is not a valid conservation law. The standard way for recovering this conservation law is by instead defining

$$\partial_b[-g(T^{ab} + t^{ab})] = 0\,. \tag{3.37}$$

We could then fix t^{ab} in such a way that the total 4-momentum has a definite value

$$P^a = \tfrac{1}{c}\int -g(T^{ab} + t^{ab})\,\mathrm{d}S_b \tag{3.38}$$

He then claims that in vacuum where T^{ab} is identically zero, equation (3.35) is devoid of content, since $0 = 0$, whereas eq. (3.37) do have a meaning in vacuum, namely that $\partial_k(g\,t^{ab}) = 0$. Then to be consistent with this idea, he propose the coordinate condition, that $t^{0b} = 0$. This allows him to express the vacuum form of eq (3.37) in integral form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V -g\,T^{00}\mathrm{d}V = -c\oint_S -g\,T^{0\alpha}\mathrm{d}S_\alpha \tag{3.39}$$

which is very similar to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int T^{00}\mathrm{d}V = -c\oint_S T^{0\alpha}\mathrm{d}S_\alpha\,, \tag{3.40}$$

the difference being the weight $(-g)$. Note, however, that even in coordinate systems where $(-g) = 1$, in the presence of gravity, eq (3.39) no longer refers to a conservation of purely non-gravitational energy, since this would mean conservation of $\int \sqrt{{}^3g}\,T^{00}\,dV$ alone.

Then what is conserved in presence of gravity is the amalgam of matter and gravitational field contributions to the energy. As noticed by the author himself, the ambiguity lies in the fact that one could equally well have defined eq (3.39) with the factor $\sqrt{-g}$ instead of $(-g)$. We can then regard a gravitational system as one in which there is energy transfer of the various energy forms between themselves. As the interacting parts respond and move, there is a redistribution of the different forms of energy within the system, and one can use eq (3.39) to determine the composite at any time-slice. This may sound simple in principle, but may be more difficult in practice as one has to ensure that $t^{0b} = 0$.

In this formalism the role of the pseudotensor is drastically altered. Rather than embodying the gravitational field energy, it is seen as a guide for the coordinate conditions which localize energy in accordance with the conservation laws. Therefore T^{ab} regains its status as a true SEM tensor as in SR.

Comment: It has been shown by Rosen [182] that it is possible to express coordinate conditions in a covariant form if a supplementary background metric is introduced. Covariance is then achieved by replacing all partial derivatives by covariant ones with respect to the background metric. If this background metric is flat, we have ordinary general relativity. These theories are know as *dual metric theories* of gravity. For a nice introduction with a slightly different point of view, see [177].

3.3.4 The NL Non-Rotating Frame Approach

The Nissani and Liebowitz [134] approach to the gravitational energy problem is to simply accept that one has to be satisfied with the global conservation of SEM components in a preferred class of reference frames. They call these frames *non-rotating* since in them, the individual components of the 4-momentum are separately conserved, in the sense that they are not mixed with the passage of time. (That is, because rotation would introduce a mixing of the spacelike and timelike terms terms in the metric.)

Their method is based on considering the covariant divergence-free condition that:

$$\bar{g}T^{ab}{}_{;b} = (\bar{g}\,T^{ab})_{,b} + \bar{g}\,\Gamma^{a}{}_{cb}T^{cb} = 0\,. \tag{3.41}$$

This is accomplished by finding coordinates in which the non-conservative term vanishes,

$$\Gamma'^{a}{}_{cb}\,T'^{ab} = 0\,. \tag{3.42}$$

In this coordinate system, eq (3.41) transforms into the continuity equation:

$$(\bar{g}\,T^{ab})_{,b} = 0 \tag{3.43}$$

This leads to a differential equation for the non-rotating coordinate transformation:

$$\Gamma^{d}{}_{ab}\,G^{ab}\left(\frac{\partial x'^{c}}{\partial x^{d}}\right) - G^{ab}\left(\frac{\partial^2 x'^{c}}{\partial x^{a}\partial x^{b}}\right) = 0\,. \tag{3.44}$$

Then the transformation function from any ordinary geodesic system to a non-rotating geodesic system is obtained by solving eq (3.44) with the following boundary conditions; as imposed on the world line of an observer A.

$$\left.\frac{\partial x'^{a}}{\partial x^{b}}\right|_{A} = \delta^{a}_{b}, \qquad \text{and} \qquad \left.\frac{\partial^2 x'^{a}}{\partial x^{b}\partial x^{c}}\right|_{A} = 0 \tag{3.45}$$

Once this is accomplished, we can consider a region of 3-space, on whose surface, $T^{ab} = 0$. Between the times t_1 and t_2, this region traces a 4-volume in spacetime. Applying Gauss's theorem, one obtains in a non-rotating frame, where $\int(\bar{g}T^{ab})_{,b}\,\mathrm{d}^4x = 0$, that $P^{\alpha}(t_1) = P^{\alpha}(t_2)$, where

$$P^{\alpha}(t) = \int_{t} \bar{g}\,T^{\alpha 0}\mathrm{d}^3x\,. \tag{3.46}$$

Hence they where able to obtain four conserved integrals that can be interpreted as the energy-momentum 4-vector content of the 3-space region, which is not restricted to be local.

3.3.5 The LBK Coordinate Independent Energy

The only truly coordinate independent method to date, is the simple construction of Lynden-Bell and Katz [133]. They show by physical arguments that static spherical systems have a coordinate independent field energy density. Using the special relativity expression for matter energy E_m

$$E_m = \int_\Sigma \bar{g}\, \xi_b\, T_s^{ab}\, \mathrm{d}\Sigma_b\,, \tag{3.47}$$

together with the expression for the total energy,

$$E = Mc^2 = E_m + E_f \tag{3.48}$$

they find the gravitational field energy E_f in the Schwarzschild spacetime. (Here Σ is a timelike hypersurface, and T_s^{ab} is the stress energy tensor on the shell.)

The physical arguments leading to this coordinate independent result are as follows. First consider any static spherically symmetric spacetime. Then enclose the center of symmetry by a sphere of radius a, in such a way that the exterior metric is unchanged. Replace the interior by a flat spacetime, by introducing a suitable surface distribution of matter on the sphere. Since the exterior spacetime is unchanged, the total energy as perceived by the Schwarzschild metric at infinity is Mc^2. We could then evaluate the field energy by taking the difference between its total energy and the matter energy. The difficulty then lies in evaluating E_m which would have contributions both from the matter exterior and the from the surface distribution.

$$E_f(a) = Mc^2 - E_m(a) \tag{3.49}$$

Now change the radius of the sphere from a to $a + da$. Then $\mathrm{d}E_f/\mathrm{d}a$ is the field energy in the volume contained between the spheres a and $a + da$. But from spherical symmetry the field energy must be uniformly distributed over spheres, so if we divide it by the enclosed volume between the spheres we obtain the gravitational energy density. Thus for static spheres, this energy density is *physically* defined and therefore the authors call this "coordinate independent" eventhough the better word would be coordinate *invariant*. For a test particle moving in a stationary metric with conserved energy, we may take the particle to consist of a localized distribution of matter with stress energy tensor T^{ab}. Then eq (3.47) could be written as:

$$E_m = \int \bar{g}\, T^{00}\, \mathrm{d}^3x\,. \tag{3.50}$$

The result for the Schwarzschild spacetime is:

$$u = \frac{1}{8\pi G}\left[\frac{GM}{a^2(1+m/(2a))^3}\right]^2 \tag{3.51}$$

where a is the isotropic radius. The total energy is

$$E = \tfrac{1}{2}\, GM^2/a\,. \tag{3.52}$$

For a general spherical distribution with a metric $(+---)$

$$g_{ab} = \mathrm{e}^{2\nu}\mathrm{d}t^2 - \mathrm{e}^{2\mu}\left[\mathrm{d}\bar{r}^2 + \bar{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right)\right] \tag{3.53}$$

the gravitational energy density is given by

$$u = \frac{c^4}{2\pi G}\mathrm{e}^{-3\mu}\left[\mathrm{e}^{\mu/2}\right]'\left[\mathrm{e}^{\nu+\mu/2}\right]' \tag{3.54}$$

where the prime denotes a derivative with respect to the isotropic radius $\bar{r}$.

This energy density is also defined only in a particular coordinate system, since LBK has not addressed the question on how the total energy would transform for a non-static or free-falling observer. However, this may not pose a problem as one may realize by considering the example of measuring the hydrogen line ($H - \alpha$). Different observers measure different values, but this is not a problem, because upon closer scrutiny (i.e. by calculation) they find it invariant.

3.3.6 Gravitational Energy as Noether Charge

Another interesting method for defining gravitational energy is that of Wald [24, 168, 169], who uses a Noether current in the boundary term in the gravitational action to obtain a conserved Noether charge which is then taken to be energy. Because the boundary integral is over a spatially compact 2-surface and not a hypersurface, this energy is again quasilocal. But perhaps the most interesting point is that with this method one can obtain many previous results, just by choosing a different timelike vectors [136]. One can then obtain the Misner-Sharp, Sachs-Bondi and Komar energy expressions by choosing the timelike killing vector to be either a stationary Killing vector, an asymptotic time translation vector or the Kodama vector, respectively. As his method is presented in the slightly unusual (but powerful) formalism of differential forms, I will merely state the results below.

Usually in Einstein gravity, one *defines* the SEM tensor of matter by the variation of a matter Lagrangian with respect to the metric. Similarly, one defines the SEM of gravity by varying the Einstein-Hilbert Lagrangian, which is then known as the Einstein tensor. The Einstein field equations then expresses a zero combined energy. But this just gives a definition of the mass of the matter without any contribution from the gravitational energy. As a result, any variation of any Lagrangian will similarly give zero combined energy by the EFE's.

However, this result involves the assumption that the boundary terms in the variations all vanish. Instead one may expect that they contribute to the energy and need to be added to the vanishing bulk terms. A general framework for studying boundary terms in diffeomorphism invariant actions has been developed by Wald & Iyer in [168, 169].

This approach also answers the Einstein equivalence principle objection to gravitational energy: generally there is no preferred time, different choices give different energies. In particular we obtain the following energies.

Vector Used	Energy Obtained
Usual asymptotic time translation	Bondi (in asymp.flat spacetimes)
Stationary Killing vector	Komar
Spherically symmetric Kodama vector	Misner-Sharp

Because of this time dependence of the energy, Hayward also suggests that there may be a classical relationship between time and gravitational energy, similar to that in quantum theory.

3.3.7 Where is the energy?

For comparison with our own results, we are interested in knowing where the energy density is located. The most simple case to investigate is the Schwarzschild spacetime. However it is possible that this case

may be too simple. The next slightly more complicated cases would then be be the Reissner-Nordstrom or FLRW spacetimes. In the table below we summarize the different approaches for the Schwarzschild case.

Energy	How?	Where is the energy?
ADM		
Bondi-Sachs		
Cooperstock	Localizing matter	Where $T^{ab} \neq 0$.
Komar		
LBK	Coordinate Independent	Outside the matter
LN	Preferred Ref. Frame	
Penrose	Twistorspace $\mathbb{T}^{\alpha}(S)$	Inside the matter
Witten	Classical SUGRA	3-surface (Inside & Outside)
AS	Optical Geometry	Inside & ???

Here we use the shorthand notation for the authors as: ADM is Arnowitt-Deser-Misner, LBK is LyndenBell-Katz, LN is Liebowiz-Nissani and AS is Almergren-Sonego. The last line is of course our work which will be presented below.

3.4 Optical Geometry

In this section I will describe optical geometry at two levels and then introduce the Local Inertial Frame (LIF) and its justification. The first level is a brief description for the non specialist. The second is a fully developed technical definition of optical geometry.

3.4.1 Brief Description

What is it? *Optical geometry* (OG) also known as *optical reference geometry* is a concept which provides an accurate and correct framework for understanding physics in strong gravitational fields. In contrast to what one may guess, optical geometry has little to do with (geometric) optics, although geometric optics is also simple in optical geometry.

Optical geometry was slowly developed through the resolution of a number of apparent paradoxes. Beginning in 1985, with the resolution of what is sometimes referred to as "the centrifugal force paradox", which states that the force on an object that orbits a black hole along the same trajectory as a circular light ray, is independent of the orbital speed. This is a rather counter-intuitive result. But the resolution is simply a matter of how to define directions such as *inward* and *outward* in a spacetime [188].

Optical geometry depends on measurements made with light rays. Thus in OG, the distance between two points in space is defined as half the time it takes the light to make a round trip from one point to the other and then back, where the 'time' we refer to is a timelike Killing vector, at least conformally, and therefore it depends on the stationarity of the spacetime used. For example: In the static Schwarzschild geometry, it is the Schwarzschild coordinate t. On the other hand, if the spacetime is only *conformally static*, it would be the conformal Killing parameter in that particular spacetime.

One of the most important points in OG is that light trajectories are geodesic lines in space with conformal geometry. This can be easily seen from *Fermats principle*, which states that among all possible

paths between two points p and q in space, light moves on the one that extremizes time.

$$\delta \int \mathrm{d}t = \delta \int_p^q \left(\tilde{h}_{ij}\, \mathrm{d}x^i \mathrm{d}x^j\right)^{1/2} = 0 \tag{3.55}$$

where we immediately notice that the conformal geodesic expression is zero. So in a general static spacetime we can use Newtonian theory, if we use a conformal geometry, instead of full GR. In that case all formulas will have the same meaning and the same form. Thus OG can be thought of as a convenient map of a curved space, where true distances are distorted, but where light rays are geodesics. For an excellent review of this, look at the article by Abramowicz [187].

Why is it useful? Because of the close analogy with Newtonian theory, OG can provide the necessary intuitive and calculational tools to greatly reduce and simplify advanced calculations in curved spacetime. So far it has been used successfully in *static, stationary* and even *non-stationary* spacetimes [142]. There is also an extension for spacetimes without shear [178, 179].

How do we use it? In OG we are looking at spacetime in the view of a $3 +$ *gravity* formulation rather than the typical $3 + 1$ formulation commonly used in general relativity. The idea is to separate out the gravitational field from the spacetime geometry into a separate field and a curved space. Once this is done one can in principle separate out all inertial forces. However, this requires the introduction of a Newtonian-like gravitational potential, together with the optical metric.

Before entering into the technical definition of OG, it will be very useful to clarify the matter of what is meant with an *inertial* frame.

3.4.2 Inertial Frames in General Relativity

In this section we introduce the concept of a local inertial frame (LIF). The idea is that there is no such thing as a global inertial frame. This is so mainly because there is no such thing as a 'fixed star' reference frame. It is just an idealization that is only valid locally in our stellar neighborhood. Thus it would be more convenient to introduce a local inertial frame.

Local Inertial Frames

So what is meant by a LIF in special relativity (SR), general relativity (GR) and OG? Well, in special relativity we have a flat Minkowski spacetime. So when one is accelerating or moving around, one is really doing Lorentz transformations from one local inertial frame to another. In GR, however, space is not flat, so we want to choose a frame in such a way that physics looks the same as in SR (think of an elevator in the context of the equivalence principle.) We then say, that locally in a small neighborhood, we may take space as flat. We call this neighborhood a local inertial frame. Then one uses the *connection* to relate these LIF to different points in spacetime. Optical geometry does not provide for LIF, because LIF really means freefalling and in OG one is not freefalling. Thus, the frames you generally use in Newtonian theory are not LIF either. As an example, consider yourself sitting here on earth, accelerating against gravity, you are obviously not inertial here. However, LIFs provides for simpler calculations in GR.

What is an inertial frame?

This depends on the particular model used. However, the main purpose of an inertial frame is to separate the different components of a force, into those related to rotation, deformation and acceleration. This is exactly what inertial means in SR. In order to do the same thing for GR, we first have to find an ultrastatic spacetime. For the study of dynamics in Schwarzschild spacetime, it has been shown [189] that it is useful to introduce a notion of *gravitational* and *centrifugal* forces. This is an example of how the centrifugal force is accounted for in optical geometry. In particular, the notion of centrifugal force is one example of an inertial force (i.e. an apparent force associated to motion that is non inertial!) Therefore, we must first introduce a notion of inertial frames in GR. So how does such an inertial frame look like? We will present two examples. The Minkowski and the Schwarzschild inertial reference frame.

The Minkowski and Schwarzschild Inertial Frames

So, if we let n^a be the velocity field of a congruence of observers, then the Minkowski inertial frame must have the following properties:

$$\begin{array}{lll} \text{Non-accelerating:} & a_a & = & n^b\nabla_b n_a = 0 \\ \text{Non-rotating:} & \omega_{ab} & = & h_{[a}{}^c h_{b]}{}^d \nabla_d n_a = 0 \\ \text{Non-deforming:} & \sigma_{ab} & = & \theta = 0 \end{array}$$

where ω_{ab} is the vorticity tensor, $\sigma_{ab} = h_{(a}{}^c h_{b)}{}^d \nabla_d n_a = 0$ the shear tensor and $\theta = \nabla_a n^a$ the expansion scalar. We may now ask how n^a changes by taking the covariant derivative.

$$\nabla_b n_a = -n_b a_a + \omega_{ab} + \underbrace{\sigma_{ab} + \tfrac{1}{3}\,\theta\, h_{ab}}_{\theta_{ab}} = 0 \tag{3.56}$$

Can we use this as the definition of inertial frames in a curved spacetime? Only in those spacetimes that admit a timelike vector field such that $\nabla_b n_a = 0$. But these belong to a very narrow class of *ultrastatic* spacetimes. They are spacetimes in which there is no gravity, only curved *space*, with $\Gamma^i_{jk} \neq 0$ and $\Gamma^\mu{}_{tt} = \Gamma^\mu{}_{tj} = 0$. Otherwise, these spacetimes are just like the Minkowski ones. However, we are interested in spacetimes that are conformally ultrastatic.

We conclude that the definition $\nabla_b n_a = 0$ is too restrictive. It does not even allow us to recover the static observers of Schwarzschild spacetime as inertial ones, because they have $a_a = n^b\nabla_b n_a \neq 0$, since this static observer have to use an engine to remain at rest. So what shall we do? Well, remember that in a static spacetime with a Killing vector field η^a, we had $a_a = \nabla_a \Phi$, where $\Phi = \frac{1}{2}\ln(-\eta_a\eta^a)$. If we now introduce a new metric $\tilde{g}_{ab} = \mathrm{e}^{-2\Phi} g_{ab}$, we see that $\widetilde{\nabla}_a \tilde{n}^b = 0$. Where $\widetilde{\nabla}_a$ is the Riemannian connection associated to $\tilde{g}_{ab}$, and where $\tilde{n}^a = \mathrm{e}^{\Phi} n^a$ is such that $\tilde{g}_{ab}\,\tilde{n}^a\tilde{n}^b = -1$.

Thus, we can relax the condition $\nabla_a n_b = 0$ by defining n^a inertial iff $\exists\ \Phi \mid \tilde{n}^a = e^{\Phi} n^a$ is non-accelerating etc., in $(\mathcal{M}, \tilde{g}_{ab})$. This is possible if $(\mathcal{M}, g_{ab})$ is conformally static. Again, this is a restricted class, but it includes the Schwarzschild, the Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmological models, among others. An extension is possible, but more complicated. Thus, the idea of this definition of local inertial frames is that physics should take the Minkowskian form when formulated in $(\mathcal{M},\ \tilde{g}_{ab})$, with respect to the reference frame $\tilde{n}^a$. In fact, this is exactly what happens as seen in [192].

3.4.3 Technical Definition

A *conformally static* spacetime $(\mathcal{M}, g_{ab})$ admits a privileged congruence of timelike curves, corresponding to the flow lines of conformal Killing time t. Consequently, one can define a family of privileged observers with four-velocity $n^a = \eta^a/(-\eta_b\eta^b)^{1/2}$, where η^a is the conformal Killing vector field. The set of these observers can be thought of as a generalization of the Newtonian concept of rest frame. Their acceleration can be expressed as the projection of a gradient,

$$a_a = n^b \nabla_b n_a = h_a{}^b \nabla_b \Phi \tag{3.57}$$

(see Appendix C for a proof), where $h_a{}^b = \delta_a{}^b + n_a n^b$ and

$$\Phi = \tfrac{1}{2} \ln(-\eta_a \eta^a) \ ; \tag{3.58}$$

thus, Φ is a suitable general-relativistic counterpart of the gravitational potential [141]. One can define the *ultrastatic* [140] metric,

$$\begin{aligned} \tilde{g}_{ab} &:= (-\eta_a\eta^a)^{-1} g_{ab} \\ &= \mathrm{e}^{-2\Phi} g_{ab} \\ &= -\nabla_a t \nabla_b t + \tilde{h}_{ab}, \end{aligned} \tag{3.59}$$

where $\tilde{h}_{ab} = \mathrm{e}^{-2\Phi} h_{ab}$. The hypersurfaces $t = const$ of $\mathcal{M}$ are all diffeomorphic to some three-dimensional manifold $\mathcal{S}$. If the spacetime is static, it follows from Fermat's principle that light rays coincide with the geodesics on $\mathcal{S}$ according to $\tilde{h}_{ab}$ [143]. For this reason, $(\mathcal{M}, \tilde{g}_{ab})$ is referred to as the *optical geometry*, and $(\mathcal{S}, \tilde{h}_{ab})$ as the *optical space*.

There is a simple operational definition of the optical metric. Suppose that all the observers n^a agree to construct a set of synchronized devices that measure the Killing time t (of course, these 'clocks' will not agree with those based on local physical processes, but this is irrelevant in this case). As mentioned before, in the case of Schwarzschild geometry, t would simply be the Schwarzschild time, $t = C(r)^{-1/2}\ \tau$. Then, they use a light radar to define the distance between two points $P, Q \in \mathcal{S}$ as $t_{PQP}/2$, where $t_{PQP} = t_{QPQ}$ is the lapse of Killing time corresponding to the round trip of the signal (see fig.3.4.3) between the observers based at P and Q. [5] In this way, they attribute to $\mathcal{S}$ the metric $\tilde{h}_{ab}$.

In addition to the conformally static cases, we have also studied the non-stationary case of a spacetime describing the collapse of a spherically symmetric configuration of matter [142]. This problem was interesting for two reasons: First, it represents one of the simplest cases in which the requirement of conformal staticity is relaxed. This is particularly evident if we consider a situation in which the collapsing matter is distributed on an infinitely thin shell. In this case, the spacetime is composed of two regions, corresponding to the interior and the exterior of the shell, joined through a timelike hypersurface that corresponds to the history of the shell. Both these regions are, considered separately, static, their metrics being the Minkowski and the Schwarzschild ones, respectively. However, the fields n^a associated to these two metrics do not match in a satisfactory way at the surface of the shell. This failure can be seen as a consequence of the fact that the spacetime is not conformally static in any region containing the shell. An interesting side effect of this is that if one calculates the gravitational field in the center of the collapsing shell, one obtains a non-zero value, in contrast to Newtonian theory.

[5] There is a one-to-one correspondence between conformally static observers and points of $\mathcal{S}$.

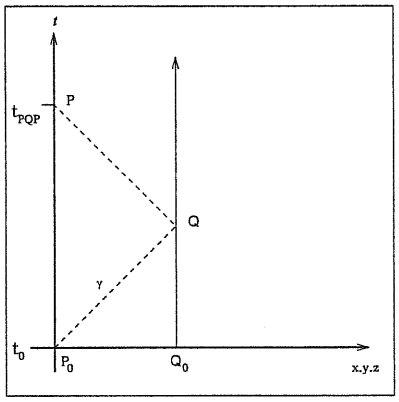

Figure 3.1: The distance between two points $P, Q \in \mathcal{S}$ are given by the lapse of Killing time, t_{PQP} .

A second motivation for studying this class of spacetimes is that they lead to the Hawking effect [194], where OG has given some new insights about black hole evaporation. However, this thesis does not concern any of these results, they are only mentioned here for reference.

3.4.4 More About Ultrastatic Spacetimes

A more technical definition and description of the relevant features of ultrastatic spacetimes is due to Sonego in [180].

A spacetime $(\mathcal{M}, \widetilde{g}_{ab})$ is called *ultrastatic* if and only if it possesses a hypersurface-orthogonal timelike Killing vector field (Kvf) η^a, such that $\widetilde{g}_{ab}\,\eta^a\eta^b = -1$. Thus, a spacetime is ultrastatic if and only if it is static and, in addition, the timelike Killing vector field has unit norm [140].

In order to rewrite the metric of an ultrastatic spacetime in a more explicit way, let us first prove that if η^a is a hypersurface-orthogonal timelike vector field of unit norm, there exists, at least locally, a real-valued function t on $\mathcal{M}$ such that $\eta_a = -\widetilde{\nabla}_a t$. This follows from the fact that if η^a is hypersurface-orthogonal, then there exist, locally on $\mathcal{M}$, two real-valued functions λ and t such that $\eta_a = \lambda\widetilde{\nabla}_a t$ [24]. Since $\eta^a\eta_a \neq 0$, the function t can be chosen as a parameter along the integral curves of η^a, namely, such that $\eta^a\widetilde{\nabla}_a t = 1$. Then $\lambda = -1$, which completes the proof.

Physically, the function t represents the time associated with the reference frame η^a. This result allows us to write $\widetilde{g}_{ab}$ in the form

$$\widetilde{g}_{ab} = -\widetilde{\nabla}_a t\,\widetilde{\nabla}_b t + \widetilde{h}_{ab}\ , \tag{3.60}$$

where $\widetilde{h}_{ab}$ is transverse to η^a, i.e., $\eta^a\widetilde{h}_{ab} = 0$. This is as far as one can go without exploiting the symmetry properties of spacetime. If, however, the spacetime is ultrastatic, one has also that $\pounds_\eta\widetilde{g}_{ab} = 0$, where $\pounds_\eta$ denotes the Lie derivative along the vector field η^a. Then, from the property $\pounds_\eta\eta_a = \eta^b\pounds_\eta\widetilde{g}_{ab}$ it follows that, for an ultrastatic spacetime, $\pounds_\eta\widetilde{h}_{ab} = 0$. This implies that one can choose a chart such that the components of $\widetilde{h}_{ab}$ do not depend on t. Thus, the hypersurfaces $t =$ const are all isometric to a three-dimensional manifold $\mathcal{S}$ with metric $\widetilde{h}_{ab}$, and we can call $(\mathcal{S}, \widetilde{h}_{ab})$ the *space*. This is an invariant definition based on the symmetries of $(\mathcal{M}, \widetilde{g}_{ab})$, and it is unique if η^a is. Notice that η^a is the unit normal to $\mathcal{S}$ in $\mathcal{M}$, so $(\mathcal{S}, \widetilde{h}_{ab})$ can be regarded as the "rest space" of the observers with four-velocity

η^a.

The geometrical properties of ultrastatic spacetimes are completely determined by the spatial metric $\widetilde{h}_{ab}$. This is not surprising if we notice that, (i) in coordinates $(t, \{x^i\})$ such that $\partial\widetilde{g}_{ij}/\partial t = 0$, Eq. (3.60) reduces to Eq. (3.9), and (ii) the space $(\mathcal{S}, \widetilde{h}_{ab})$ has vanishing extrinsic curvature into $(\mathcal{M}, \widetilde{g}_{ab})$, as follows immediately from the property [24]

$$\widetilde{K}_{ab} = \tfrac{1}{2}\, \pounds_\eta \widetilde{h}_{ab} = 0 \,. \tag{3.61}$$

As a consequence of (ii), one of the Gauss-Codazzi equations implies that the Riemann tensor of $(\mathcal{S}, \widetilde{h}_{ab})$ is simply given by ${}^{(3)}\widetilde{R}_{abc}{}^d = \widetilde{h}_a{}^{a'}\, \widetilde{h}_b{}^{b'}\, \widetilde{h}_c{}^{c'}\, \widetilde{h}_{d'}{}^d\, \widetilde{R}_{a'b'c'}{}^{d'}$. Actually, the relationship between the two curvature tensors is even stronger than this, because ${}^{(3)}\widetilde{R}_{abc}{}^d$ and $\widetilde{R}_{abc}{}^d$ do in fact coincide. To see this, let us first notice that $\widetilde{\nabla}_a\eta_b = -\widetilde{\nabla}_a\widetilde{\nabla}_b t = -\widetilde{\nabla}_b\widetilde{\nabla}_a t = \widetilde{\nabla}_b\eta_a = -\widetilde{\nabla}_a\eta_b$, from which it follows that $\widetilde{\nabla}_a\eta_b = 0$ [198]. This gives $\widetilde{R}_{abc}{}^d\,\eta_d = \widetilde{\nabla}_a\widetilde{\nabla}_b\eta_c - \widetilde{\nabla}_b\widetilde{\nabla}_a\eta_c = 0$ which, together with the symmetry properties of the Riemann tensor, implies that *any* contraction of $\widetilde{R}_{abc}{}^d$ with η^a must vanish. Then we have simply $\widetilde{R}_{abc}{}^d = \widetilde{h}_a{}^{a'}\, \widetilde{h}_b{}^{b'}\, \widetilde{h}_c{}^{c'}\, \widetilde{h}_{d'}{}^d\, \widetilde{R}_{a'b'c'}{}^{d'} = {}^{(3)}\widetilde{R}_{abc}{}^d$. Since this implies $\widetilde{R}_{ab} = {}^{(3)}\widetilde{R}_{ab}$, the second Gauss-Codazzi equation is automatically satisfied. All these properties could also have been derived using the $(t, \{x^i\})$ coordinates of Eq. (3.9), in which $\widetilde{\Gamma}^0{}_{\mu\nu} = \widetilde{\Gamma}^\mu{}_{\nu 0} = \partial\widetilde{\Gamma}^i{}_{jk}/\partial t = 0$. From these expressions for curvature in an ultrastatic spacetime, one can write the Einstein tensor as

$$\begin{aligned} \widetilde{G}_{ab} &= \tfrac{1}{2}\, {}^{(3)}\widetilde{R}\, \widetilde{\nabla}_a t\, \widetilde{\nabla}_b t + {}^{(3)}\widetilde{R}_{ab} - \tfrac{1}{2}\, {}^{(3)}\widetilde{R}\, \widetilde{h}_{ab} \\ &= \tfrac{1}{2}\, {}^{(3)}\widetilde{R}\, \widetilde{\nabla}_a t\, \widetilde{\nabla}_b t + {}^{(3)}\widetilde{G}_{ab} \,, \end{aligned} \tag{3.62}$$

which decomposes into a part "parallel" to η^a and one transverse to it.

Finally, it is useful to write the expression for the Weyl tensor $\widetilde{C}_{abcd}$ in an ultrastatic spacetime. From Eq. (3.60) and the definition of $\widetilde{C}_{abcd}$ one finds

$$\widetilde{C}_{abcd} = \widetilde{\nabla}_a t\, \widetilde{\nabla}_{[c} t\, {}^{(3)}\widetilde{R}_{d]b} - \widetilde{\nabla}_b t\, \widetilde{\nabla}_{[c} t\, {}^{(3)}\widetilde{R}_{d]a} - \tfrac{1}{3}\, {}^{(3)}\widetilde{R} \left(\widetilde{\nabla}_a t\, \widetilde{\nabla}_{[c} t\, \widetilde{h}_{d]b} + \widetilde{h}_{a[c}\, \widetilde{\nabla}_{d]} t\, \widetilde{\nabla}_b t \right) \,, \tag{3.63}$$

where the fact that the Weyl tensor of any three-dimensional manifold – hence, in particular $(\mathcal{S}, \widetilde{h}_{ab})$ – is equal to zero has been used.

3.5 The Field Equations

As a necessary preliminary to the discussion of gravitational energy which follows later,we will now discuss the derivation of field equations, and the Newtonian limit (NL) of these. We start by deriving the field equations by three different routes, from the Einstein equation, from the variational principle and from the trace split. In the following we also include a possible cosmological constant λ, for an easy transition when considering cosmological models.

The trace split approach is based on the possibility of splitting the Einstein equation into a traceless and a "tracefull" piece, with the additional requirement that the trace of the "would be" SEM tensor for the gravitational field would be zero.

3.5.1 From Einstein's equation

The easiest way to derive field equations for $\widetilde{g}_{ab}$ and σ, is by rewriting Einstein's equation (3.5) in terms of the quantities pertaining to the model $(\mathcal{M}, \widetilde{g}_{ab}, \sigma, \psi)$. We start by using the conformal relation of

G_{ab} to $\widetilde{G}_{ab}$ (as shown in Appendix C.)

$$G_{ab} = \widetilde{G}_{ab} - \frac{2}{\sigma}\,\widetilde{\nabla}_a\widetilde{\nabla}_b\sigma + \frac{2}{\sigma}\,\widetilde{g}_{ab}\,\widetilde{\Box}\sigma + \frac{4}{\sigma^2}\,\widetilde{\nabla}_a\sigma\widetilde{\nabla}_b\sigma - \frac{1}{\sigma^2}\,\widetilde{g}_{ab}\,\widetilde{g}^{cd}\,\widetilde{\nabla}_c\sigma\widetilde{\nabla}_d\sigma\,. \tag{3.64}$$

Then we have

$$G_{ab} = \widetilde{G}_{ab} - 8\pi G\,\widetilde{\theta}_{ab}\,, \tag{3.65}$$

where we have defined

$$\begin{aligned}\widetilde{\theta}_{ab} &:= \frac{1}{8\pi G\,\sigma^2}\left[\widetilde{\nabla}_a\widetilde{\nabla}_b\sigma^2 - 6\,\widetilde{\nabla}_a\sigma\widetilde{\nabla}_b\sigma + \widetilde{g}_{ab}\left(3\,\widetilde{g}^{cd}\widetilde{\nabla}_c\sigma\widetilde{\nabla}_d\sigma - \widetilde{\Box}\sigma^2\right)\right]\\ &= \frac{1}{8\pi G\,\sigma^2}\left[2\,\sigma\widetilde{\nabla}_a\widetilde{\nabla}_b\sigma - 4\,\widetilde{\nabla}_a\sigma\widetilde{\nabla}_b\sigma + \widetilde{g}_{ab}\left(\widetilde{g}^{cd}\widetilde{\nabla}_c\sigma\widetilde{\nabla}_d\sigma - 2\,\sigma\widetilde{\Box}\sigma\right)\right]\,,\end{aligned} \tag{3.66}$$

which is proportional to the so-called "improved" stress-energy-momentum tensor of a scalar field [149]. But the stress-energy-momentum tensor $\widetilde{T}_{ab}$ for the matter fields ψ in $(\mathcal{M}, \widetilde{g}_{ab}, \sigma, \psi)$, is a defined object which is conformally related to T_{ab} by $\widetilde{T}_{ab} = \sigma^2\,T_{ab}$, as we shall see below. Einstein's equation in the optical geometry then reads,

$$\widetilde{G}_{ab} + \sigma^2\,\Lambda\,\widetilde{g}_{ab} = 8\pi G\left(\frac{\widetilde{T}_{ab}}{\sigma^2} + \widetilde{\theta}_{ab}\right)\,. \tag{3.67}$$

In addition the field equation for σ can be obtained by simply taking the trace of Eq. (3.67):

$$\widetilde{\Box}\sigma - \tfrac{1}{6}\,\widetilde{R}\,\sigma + \tfrac{2}{3}\,\Lambda\,\sigma^3 = \frac{4\pi G}{3\,\sigma}\,\widetilde{T}\,. \tag{3.68}$$

3.5.2 From The Variational Principle

The action describing pure geometry in a spacetime $(\mathcal{M}, g_{ab})$ is

$$S_G[g_{ab}] := \frac{1}{16\pi G}\left(\int_{\mathcal{U}} (R - 2\,\Lambda)\;\mathrm{d}V + 2\oint_{\partial\mathcal{U}} K\,\mathrm{d}S\right)\,, \tag{3.69}$$

where $\mathrm{d}V$ and $\mathrm{d}S$ are the volume elements on a compact domain $\mathcal{U} \subseteq \mathcal{M}$ and on its boundary $\partial\mathcal{U}$, respectively, R is the scalar curvature in $\mathcal{U}$, and K is the extrinsic scalar curvature of $\partial\mathcal{U}$ in $(\mathcal{M}, g_{ab})$. The boundary integral in (3.69) serves to compensate a corresponding term coming from the integral over $\mathcal{U}$ when variations of the metric are considered; this allows one to require only that $\delta g_{ab} = 0$ on $\partial\mathcal{U}$, without further conditions on $\nabla_c\delta g_{ab}$, when imposing a variational principle (see Ref. [24], p. 454).

If $(\mathcal{M}, \widetilde{g}_{ab})$ is the spacetime related to $(\mathcal{M}, g_{ab})$ by the conformal transformation (3.10), we can rewrite $S_G[g_{ab}]$ as the action of a scalar-tensor theory:

$$S_G[g_{ab}] = -\frac{3}{4\pi G}\,S_{ST}[\widetilde{g}_{ab}, \sigma]\,, \tag{3.70}$$

where

$$S_{ST}[\widetilde{g}_{ab}, \sigma] = -\tfrac{1}{2}\int_{\mathcal{U}}\left(\widetilde{g}^{ab}\widetilde{\nabla}_a\sigma\widetilde{\nabla}_b\sigma + \tfrac{1}{6}\,\widetilde{R}\,\sigma^2 - \tfrac{1}{3}\,\Lambda\,\sigma^4\right)\mathrm{d}\widetilde{V} - \tfrac{1}{6}\oint_{\partial\mathcal{U}}\widetilde{K}\,\sigma^2\,\mathrm{d}\widetilde{S}\,. \tag{3.71}$$

The proof of this claim is straightforward and runs as follows. Under the conformal transformation (3.10) the determinants g and $\widetilde{g}$ of the metrics are related as $g = \sigma^8\widetilde{g}$; consequently, $\mathrm{d}V = \sigma^4\mathrm{d}\widetilde{V}$, which

trivially accounts for the term containing the cosmological constant. Similarly, $\mathrm{d}S = \sigma^3 \mathrm{d}\widetilde{S}$. Using Eq. (C.11) of Appendix C we have

$$R\,\mathrm{d}V = \sigma^2 \widetilde{R}\,\mathrm{d}\widetilde{V} + 6\,\widetilde{g}^{\,ab}\widetilde{\nabla}_a\sigma\,\widetilde{\nabla}_b\sigma\,\mathrm{d}\widetilde{V} - 6\,\widetilde{g}^{\,ab}\widetilde{\nabla}_a(\sigma\widetilde{\nabla}_b\sigma)\mathrm{d}\widetilde{V}\ . \tag{3.72}$$

As far as the boundary term is concerned, let $\widetilde{u}^a$ and u^a be vectors normal to $\partial\mathcal{U}$ and such that $\widetilde{g}_{ab}\widetilde{u}^a\widetilde{u}^b = g_{ab}u^a u^b = \pm 1$; then $\widetilde{u}^a = \sigma u^a$ and $\widetilde{u}_a = \sigma^{-1}u_a$. It follows that the metrics on $\partial\mathcal{U}$, $\widetilde{k}_{ab} = \widetilde{g}_{ab} \mp \widetilde{u}_a\widetilde{u}_b$ and $k_{ab} = g_{ab} \mp u_a u_b$, are related in the same way as those in $\mathcal{U}$: $\widetilde{k}_{ab} = \sigma^{-2}k_{ab}$. Therefore,

$$\nabla_a u^b = \frac{1}{\sigma}\left(\widetilde{\nabla}_a\widetilde{u}^b + \frac{1}{\sigma}\,\delta^b{}_a\widetilde{u}^c\widetilde{\nabla}_c\sigma - \frac{1}{\sigma}\,\widetilde{u}_a\,\widetilde{g}^{\,bc}\,\widetilde{\nabla}_c\sigma\right)\ , \tag{3.73}$$

and

$$K = k^a{}_b\nabla_a u^b = \frac{1}{\sigma}\left(\widetilde{K} + \frac{3}{\sigma}\,\widetilde{u}^a\widetilde{\nabla}_a\sigma\right)\ . \tag{3.74}$$

On substituting into Eq. (3.69), the last terms on the right hand sides of Eqs. (3.72) and (3.74) contribute by boundary terms that cancel each other exactly. Thus, we arrive at the identity (3.70), q.e.d.

Assuming now that Einstein's equation holds in $(\mathcal{M}, g_{ab}, \psi)$, we want to find the field equations in $(\mathcal{M}, \widetilde{g}_{ab}, \sigma, \psi)$. The total action for matter and geometry is $S[g_{ab}, \psi] = S_M[g_{ab}, \psi] + S_G[g_{ab}]$, whose variation with respect to the metric yields the standard Einstein's equation (3.5). Similarly, since the two actions $S_G[g_{ab}]$ and $S_{ST}[\widetilde{g}_{ab}, \sigma]$ are functionally equivalent, we write in optical geometry

$$S[\widetilde{g}_{ab}, \sigma, \psi] = S_M[\widetilde{g}_{ab}, \sigma, \psi] - \frac{3}{4\pi G}\,S_{ST}[\widetilde{g}_{ab}, \sigma]\ . \tag{3.75}$$

Variation of the action (3.71) yields

$$\begin{aligned}\frac{\delta S_{ST}[\widetilde{g}_{ab}, \sigma]}{\delta\widetilde{g}^{\,ab}} \;=\; & -\frac{1}{12}\sqrt{-\widetilde{g}}\,\Big[\Big(\widetilde{G}_{ab} + \sigma^2\,\Lambda\,\widetilde{g}_{ab}\Big)\,\sigma^2 + 6\,\widetilde{\nabla}_a\sigma\widetilde{\nabla}_b\sigma \\ & -\widetilde{\nabla}_a\widetilde{\nabla}_b\sigma^2 + \widetilde{g}_{ab}\left(\widetilde{\Box}\sigma^2 - 3\,\widetilde{g}^{cd}\widetilde{\nabla}_c\sigma\widetilde{\nabla}_d\sigma\right)\Big]\ .\end{aligned} \tag{3.76}$$

On the other hand, the variations of S_M with respect to g^{ab} and $\widetilde{g}^{ab}$ define the stress-energy-momentum tensors T_{ab} and $\widetilde{T}_{ab}$ of the matter fields ψ in $(\mathcal{M}, g_{ab}, \psi)$ and $(\mathcal{M}, \widetilde{g}_{ab}, \sigma, \psi)$, respectively:

$$T_{ab} := -\frac{2}{\sqrt{-g}}\,\frac{\delta S_M}{\delta g^{ab}}\ ; \tag{3.77}$$

$$\widetilde{T}_{ab} := -\frac{2}{\sqrt{-\widetilde{g}}}\,\frac{\delta S_M}{\delta\widetilde{g}^{ab}}\ . \tag{3.78}$$

Clearly,

$$\widetilde{T}_{ab} = \sigma^2\,T_{ab}\ . \tag{3.79}$$

On requiring that $\delta S/\delta\widetilde{g}^{ab} = 0$ we then obtain the field equation in $(\mathcal{M}, \widetilde{g}_{ab}, \sigma, \psi)$. Using Eq. (3.76), we can write it simply in the form (3.67).

We now consider variations of S with respect to the scalar field σ. This can be done either directly, or by considering variations of g_{ab} in which $\widetilde{g}_{ab}$ is kept fixed, so $\delta g^{ab} = -2\,g^{ab}\,\delta\sigma/\sigma$. Using the latter technique

$$\frac{\delta S_{ST}[\widetilde{g}_{ab}, \sigma]}{\delta\sigma} = -\frac{4\pi G}{3}\,\frac{\delta S_G[g_{ab}]}{\delta g^{ab}}\,\frac{\delta g^{ab}}{\delta\sigma} = \frac{\sqrt{-g}}{\sigma}\,(R + 4\,\Lambda) = \sqrt{-\widetilde{g}}\left(\widetilde{\Box}\sigma - \frac{1}{6}\,\widetilde{R}\,\sigma + \frac{2}{3}\,\Lambda\,\sigma^3\right)\ . \tag{3.80}$$

From the variation of the action for matter we get

$$\frac{\delta S_M[\widetilde{g}_{ab},\sigma,\psi]}{\delta\sigma} = \frac{\delta S_M[g_{ab},\psi]}{\delta g^{ab}}\frac{\delta g^{ab}}{\delta\sigma} = \frac{1}{\sigma}\sqrt{-g}\,T = \frac{1}{\sigma}\sqrt{-\widetilde{g}}\,\widetilde{T}\,, \tag{3.81}$$

where T and $\widetilde{T}$ are the traces of T_{ab} and $\widetilde{T}_{ab}$ in the respective spacetimes. (It turns out that $\widetilde{T} = \sigma^4\,T$, so $\sqrt{-g}\,T$ is conformally invariant.) Putting Eqs. (3.80) and (3.81) together according to Eq. (3.75), and requiring that $\delta S[\widetilde{g}_{ab},\sigma,\psi]/\delta\sigma = 0$ (which corresponds to studying the "propagation" of σ on an otherwise fixed background) we finally find the field equation for σ, Eq. (3.68).

In summary, we have shown that we can replace a spacetime $(\mathcal{M}, g_{ab}, \psi)$ by a conformally related one $(\mathcal{M}, \widetilde{g}_{ab}, \sigma, \psi)$ containing, in addition to the usual matter fields ψ, also a massless, conformally coupled scalar field, carrying a contribution $\widetilde{\theta}_{ab}$ to the right hand side of Eq. (3.65) for $\widetilde{G}_{ab}$ [150].

For the sake of completeness, we also present Eq. (3.68) in terms of the scalar potential Φ:

$$\widetilde{\Box}\Phi + \widetilde{g}^{\,ab}\,\widetilde{\nabla}_a\Phi\,\widetilde{\nabla}_b\Phi - \frac{1}{6}\,\widetilde{R} + \frac{2}{3}\,\Lambda\,\mathrm{e}^{2\Phi} = \frac{4\pi G}{3}\,\mathrm{e}^{-2\Phi}\,\widetilde{T}\,. \tag{3.82}$$

From this equation, we can clearly identify the non-Gaussian behavior, as due to the non linear term and the curvature term which both acts like sources for the gravitational field.

3.5.3 The Trace Split

As already noticed at the end of Sec. 3.5.1, Eq. (3.68) can be obtained just by taking the trace of Eq. (3.67). Thus, Eq. (3.68) for the scalar field is actually *not* independent of Eq. (3.67) for the optical metric. This fact is related to the number of degrees of freedom of the theory [197].

In order to identify a stress-energy-momentum tensor of gravity which behaves correctly, let us first decompose [151] Einstein's equation (3.5) into its trace,

$$-R + 4\,\Lambda = 8\pi G\,T\,, \tag{3.83}$$

and a trace-free part:

$$R_{ab} - \frac{1}{4}\,R\,g_{ab} = 8\pi G\left(T_{ab} - \frac{1}{4}\,T\,g_{ab}\right)\,. \tag{3.84}$$

Clearly, (3.84) contains nine independent equations, which, together with (3.83), reproduce the full content of Einstein's equation (3.5). From (C.11), it is clear that (3.83) gives the field equation (3.68) for σ. On the other hand, (3.84) is *not* equivalent to (3.67) and (3.66) alone. Instead, it gives the trace-free part of (3.67):

$$\widetilde{R}_{ab} - \frac{1}{4}\,\widetilde{R}\,\widetilde{g}_{ab} = 8\pi\left(T_{ab} - \frac{1}{4}\,T\,g_{ab}\right) + 8\pi\left(\widetilde{\theta}_{ab} - \frac{1}{4}\,\widetilde{g}_{ab}\widetilde{g}^{\,cd}\widetilde{\theta}_{cd}\right)\,. \tag{3.85}$$

if we now call $\widetilde{\mathcal{T}}_{ab}$ the stress-energy-momentum tensor of gravity, and require that the second term on the right hand side of (3.85) be the trace-free part of $\widetilde{\mathcal{T}}_{ab}$, we find the condition

$$\widetilde{\mathcal{T}}_{ab} - \frac{1}{4}\,\widetilde{g}_{ab}\widetilde{g}^{\,cd}\widetilde{\mathcal{T}}_{cd} = \widetilde{\theta}_{ab} - \frac{1}{4}\,\widetilde{g}_{ab}\widetilde{g}^{\,cd}\widetilde{\theta}_{cd}\,, \tag{3.86}$$

which has the general solution

$$\begin{aligned}\widetilde{\mathcal{T}}_{ab} &= \widetilde{\theta}_{ab} - \frac{1}{4}\,\widetilde{g}_{ab}\widetilde{g}^{\,cd}\widetilde{\theta}_{cd} + \lambda\,\widetilde{g}_{ab} \\ &= \frac{1}{8\pi\,\sigma^2}\left[2\,\sigma\left(\widetilde{\nabla}_a\widetilde{\nabla}_b\sigma - \frac{1}{4}\,\widetilde{g}_{ab}\,\widetilde{\Box}\sigma\right) - 4\,\widetilde{\nabla}_a\sigma\widetilde{\nabla}_b\sigma + \widetilde{g}_{ab}\widetilde{g}^{\,cd}\widetilde{\nabla}_c\sigma\widetilde{\nabla}_d\sigma\right] + \lambda\,\widetilde{g}_{ab}\,,\end{aligned} \tag{3.87}$$

where λ is an unspecified constant.

The tensors $\widetilde{\mathcal{T}}_{ab}$ and $\widetilde{\theta}_{ab}$ differ from each other by a multiple of $\widetilde{g}_{ab}$. On the other hand, the sum $T_{ab}+\widetilde{\theta}_{ab}$ is fixed (it is proportional to $\widetilde{G}_{ab}$), so what we are saying is, essentially, that if we call $\widetilde{T}_{ab}$ and $\widetilde{\mathcal{T}}_{ab}$ the stress-energy-momentum tensors of matter and gravity, respectively, they are related to T_{ab} and $\widetilde{\theta}_{ab}$ as

$$\widetilde{T}_{ab} = T_{ab} - \mu\,\widetilde{g}_{ab}\ , \tag{3.88}$$

$$\widetilde{\mathcal{T}}_{ab} = \widetilde{\theta}_{ab} + \mu\,\widetilde{g}_{ab}\ . \tag{3.89}$$

In fact, we can split $T_{ab}+\widetilde{\theta}_{ab}$ as:

$$T_{ab} \ \rightarrow\ T_{ab} - \xi\widetilde{g}_{ab} \tag{3.90}$$

$$\widetilde{\theta}_{ab} \ \rightarrow\ \widetilde{\theta}_{ab} + \xi\widetilde{g}_{ab}\ . \tag{3.91}$$

Thus when spacetime is empty ($T_{ab}=0$), $\widetilde{G}_{ab} = 8\pi\widetilde{\theta}_{ab}$ gives unambiguously

$$\widetilde{T}_{ab}[\sigma] = \widetilde{\theta}_{ab}\ . \tag{3.92}$$

We cannot introduce an artificial $\widetilde{T}_{ab}[\phi]$ just to compensate for the $\xi\widetilde{g}_{ab}$ term! A necessary condition on ξ is therefore that $\xi=0$ for an empty spacetime. In general, this means that: $\xi = A^{cd}T_{cd} + \alpha T$. If we do not want higher order terms in the stress-energy-momentum tensor, we can tighten the requirements to $A^{cd} = \beta g^{cd}$. Then,

$$\xi = \gamma T = \lambda\left(\frac{\widetilde{\Box}\sigma}{\sigma} - \frac{\widetilde{R}}{6}\right)\ . \tag{3.93}$$

And thus we may guess that: $\lambda = c\sigma^n$...

We can check this for the Schwarzschild interior. For a very diluted body ($\sigma\approx 1$ and $M \ll R$), we must recover, by the correspondence principle,

$$\widetilde{\Theta}_{00} + c\sigma^n\left(\frac{\widetilde{\Box}\sigma}{\sigma} - \frac{\widetilde{R}}{6}\right) \approx -\frac{GM^2}{8\pi R^6}r^2\ . \tag{3.94}$$

Can we find c and n that fulfill this condition?

This is work in progress!

3.5.4 The Newtonian Limit

As already mentioned in the introduction to this chapter, the field $\Phi = \ln\sigma$ can be regarded as a generalization, in the optical spacetime $(\mathcal{M},\widetilde{g}_{ab})$, of the Newtonian potential ϕ. In the absence of a cosmological constant, ϕ obeys the Poisson equation in flat space,

$$\nabla^2\phi = 4\pi G\,\rho\ , \tag{3.95}$$

where ρ is the matter density of mass, while $\sigma = \mathrm{e}^{\Phi}$ obeys Eq. (3.68) in $(\mathcal{M},\widetilde{g}_{ab})$. We now investigate the way in which Eq. (3.95) can be regarded as a limit of Eq. (3.68). Since there is no cosmological constant in Eq. (3.95), we shall consider only the case $\Lambda\equiv 0$.

In the Newtonian limit of general relativity, the metric can be rewritten as a small perturbation of the Minkowski metric:

$$g \approx -(1+2\phi)\,\mathrm{d}t^2 + (1-2\phi)\,\mathrm{d}\mathbf{x}^2 \tag{3.96}$$

(see, e.g., Ref. [130], p. 445, or Ref. [24], p. 77). Since time derivatives are negligible in this approximation, this metric is static.

Working to first order in ϕ, we can choose

$$\sigma \approx 1 + \phi\,, \tag{3.97}$$

and use Eq. (3.10) to obtain the optical metric:

$$\widetilde{g} \approx -\mathrm{d}t^2 + (1-4\phi)\,\mathrm{d}\mathbf{x}^2\,. \tag{3.98}$$

Notice that Eq. (3.97) is consistent with the interpretation of Φ as a gravitational potential, because it entails $\Phi \approx \phi$.

Having identified σ, we can work out the right hand side of Eq. (3.68). For Newtonian matter, speeds and internal stresses are negligible, so we can use the stress-energy-momentum of dust, $\widetilde{T}_{ab} = \widetilde{\rho}\,\widetilde{v}_a\widetilde{v}_b$, where all quantities refer to the optical spacetime. In particular, $\widetilde{\rho} = \sigma^{-2}\rho$, and the four-velocity $\widetilde{v}^a$ is normalized with respect to $\widetilde{g}_{ab}$, so

$$\frac{4\pi G}{3\,\sigma}\,\widetilde{T} = \frac{4\pi G}{3\,\sigma}\,\widetilde{g}^{\,ab}\,\widetilde{T}_{ab} = -\frac{4\pi G}{3\,\sigma}\,\widetilde{\rho} \approx -\frac{4\pi G}{3}\,\rho\,, \tag{3.99}$$

because $\sigma \approx 1$. The left hand side of Eq. (3.68) can be evaluated straightforwardly to first order, again remembering that $\partial\phi/\partial t \approx 0$. The resulting equation for ϕ is

$$\widetilde{h}^{ab}\,\widetilde{\mathrm{D}}_a\widetilde{\mathrm{D}}_b\phi - \frac{1}{6}\,\widetilde{R} \approx -\frac{4\pi G}{3}\,\rho\,, \tag{3.100}$$

where

$$\widetilde{h}_{ab} = (1-4\,\phi)\,e_{ab} \tag{3.101}$$

is the metric on the spacelike hypersurfaces $\mathcal{S}$ of $(\mathcal{M}, \widetilde{g}_{ab})$ defined by the condition $t = \text{const}$, e_{ab} is the Euclidean metric on $\mathcal{S}$, and $\widetilde{\mathrm{D}}_a$ is the covariant derivative on $(\mathcal{S}, \widetilde{h}_{ab})$.

Equation (3.100) is obviously different from the Poisson equation (3.95). This fact seems, at first, to violate the correspondence principle. However, Newton's theory is formulated on $(\mathcal{S}, e_{ab})$, not on $(\mathcal{S}, \widetilde{h}_{ab})$, while Eq. (3.100) refers to quantities defined on $(\mathcal{S}, \widetilde{h}_{ab})$ and should not, thus, be expected to directly coincide with Eq. (3.95). In order to compare Eqs. (3.95) and (3.100) we still need to perform a further step, which is to express all the quantities appearing in Eq. (3.100) in terms of quantities in the Euclidean space $(\mathcal{S}, e_{ab})$. This is not particularly difficult, because $(\mathcal{S}, \widetilde{h}_{ab})$ and $(\mathcal{S}, e_{ab})$ are related by the conformal transformation (3.101). Setting $\Omega \approx 1 - 2\,\phi$ for the corresponding conformal factor, we can make use of standard formulas (see, e.g., Appendix D of Ref. [24]) to write

$$\widetilde{h}^{ab}\,\widetilde{\mathrm{D}}_a\widetilde{\mathrm{D}}_b\phi = \frac{1}{\Omega^2}\,\nabla^2\phi + \frac{1}{\Omega^3}\,\boldsymbol{\nabla}\Omega\cdot\boldsymbol{\nabla}\phi \approx \nabla^2\phi \tag{3.102}$$

and

$$\widetilde{R} = \frac{1}{\Omega^2}\left[-4\,\nabla^2\ln\Omega - 2\,(\boldsymbol{\nabla}\ln\Omega)^2\right] \approx 8\,\nabla^2\phi\,, \tag{3.103}$$

where again we have retained only first-order terms. Replacing these expressions into Eq. (3.100) we finally recover Eq. (3.95), in agreement with the correspondence principle.

To first order in ϕ (a restriction necessary for the metric (3.96) to make sense), Poisson's equation (3.95) in the flat space $(\mathcal{S}, e_{ab})$ and Eq. (3.100) in the optical space $(\mathcal{S}, \widetilde{h}_{ab})$ are completely equivalent. Hence, there is an ambiguity in the very notion of "Newtonian limit." We are accustomed to think of Newton's theory of gravity as Poisson's equation in flat space. However, under the hypotheses of slow motions for matter and weak gravitational fields one does not recover this description, but the equivalent one expressed by Eq. (3.100) in the optical space. Thus, from this point of view, the Poisson equation (3.95) is rather misleading, and should be written instead as

$$\nabla^2\phi - \frac{4}{3}\nabla^2\phi = -\frac{4\pi G}{3}\rho\,, \tag{3.104}$$

where the second term on the left hand side actually represents the curvature of $(\mathcal{S}, \widetilde{h}_{ab})$. This curvature is non-negligible *even in the slow-motion, weak-field approximation*, since it has the same order of magnitude of the D'Alembertian term.

Mathematically, (3.104) is just the same as (3.95), rearranged in a more complicated fashion. However, rewriting the Poisson equation in the way suggested by optical geometry is not a sterile formal exercise: It allows one to account for post-Newtonian effects (perihelion advance [147], deflection of light) within an almost-Newtonian context. Indeed, these effects are geometrical in character, i.e., they depend on the metric $\widetilde{h}_{ab}$ but do not require changing ϕ from its Newtonian expression. Hence, they can be discussed consistently within the slow-motion, weak-field limit, just by allowing space to deviate from flatness in the way expressed by Eq. (3.101).

3.6 The Gravitational Energy

3.6.1 Problems of $\widetilde{\theta}_{ab}$

Looking at Eq. (3.67) in the case $\Lambda = 0$, one would naively identify the two terms on its right hand side, $\widetilde{T}_{ab}$ and $\widetilde{\theta}_{ab}$, respectively as the stress-energy-momentum tensors of matter and gravity (the latter being described by the scalar field σ) in the spacetime $(\mathcal{M}, \widetilde{g}_{ab})$. This proposal is, however, untenable for several reasons.

The most severe problem from a physical point of view, emerges in the presence of matter. One way to realize this, is by taking the trace of $\widetilde{\theta}_{ab}$,

$$\widetilde{g}^{ab}\widetilde{\theta}_{ab} = -\frac{3}{4\pi G\,\sigma}\widetilde{\Box}\sigma = -\frac{\widetilde{T}}{\sigma^2} - \frac{1}{8\pi G}\widetilde{R} + \frac{\Lambda}{2\pi G}\sigma^2\,, \tag{3.105}$$

where we have used Eq. (3.68). Within matter, the trace of $\widetilde{\theta}_{ab}$ (hence $\widetilde{\theta}_{ab}$ itself) is comparable with $\widetilde{T}$. This is unacceptable if one requires that, in the Newtonian limit, the energy density should be proportional to the square of the gravitational field, as in Eq. (3.7). Consider, for example, the gravitational field in the vicinity of the Earth, where $|\nabla\phi| \sim 10^3$ cm/s^2, so $u \sim 10^{12}$ erg/cm^3. This energy density is only 10^{-8} times the energy density associated with matter at ordinary density of 1 g/cm^3. If $\widetilde{\theta}_{ab}$ were the correct description of gravitational stress-energy-momentum, the gravitational energy would contain contributions comparable to those of matter. Hence, the gravitational energy density inside, say, the

reader's brain would be 10^8 times larger than the one outside, while the gravitational fields inside and outside would be hardly distinguishable. This, of course, is an unacceptable property for a quantity that should be identified as the energy of the gravitational field.

Below is work in progress....

In fact, in the Newtonian limit $\widetilde{\theta}_{00}$ contains terms like $\nabla^2\phi$, while the left hand side of the field equation is extremely small. Then we have $0 \approx 4\pi\rho - \nabla^2\phi + (\nabla\phi)^2$. Clearly, $\nabla^2\phi$ should belong on the left hand side of this equation!

The reason for this behavior is that we have $\widetilde{g}_{00} \equiv -1$, so a (huge) part of $\widetilde{\theta}_{ab}$ serves just to compensate T_{ab} in order to "keep straight" the spacetime $(\mathcal{M}, \widetilde{g}_{ab})$ along the direction of η^a. There really is no deep reason to identify whatever appears on the right hand side of an equation which has G_{ab} on the left hand side as stress-energy-momentum. In Einstein's theory with no cosmological constant it is true that all that contributes to curvature is stress-energy-momentum, but to export this belief into other theories (e.g. scalar tensor theories like that of Brans-Dicke) would amount to nothing else than an unjustified prejudice. In optical geometry, curvature is partly fixed *a priori*, and stress-energy-momentum contributes also to generate σ.

One cannot define the *total* stress-energy-momentum tensor in the model $(\mathcal{M}, \widetilde{g}_{ab}, \sigma, \psi)$ as in Eq. (3.78), because $S_M[\widetilde{g}_{ab}, \sigma, \psi]$ depends also on σ, so now "matter" includes also the scalar field σ, in addition to ψ. Thus, there are "matter" contributions (those of σ) also in S_G. However, it would be meaningless to define the total stress-energy-momentum tensor as $\delta(S_M + S_G)/\delta\widetilde{g}^{ab}$, because this would vanish identically by the field equation. This situation arises because in the total action in $(\mathcal{M}, \widetilde{g}_{ab}, \sigma, \psi)$ one cannot isolate a purely geometric term. This is a common feature of all scalar-tensor theories, which therefore present the same difficulty in defining a stress-energy-momentum tensor as the theory here investigated [152].

3.6.2 The Canonical SEM Tensor

Looking at Eqs. (3.70) and (3.71), one can identify the Lagrangian density of the scalar field σ in $(\mathcal{M}, \widetilde{g}_{ab})$ as the quantity

$$\widetilde{\mathcal{L}} = \frac{3}{8\pi G}\left(\widetilde{g}^{ab}\,\widetilde{\nabla}_a\sigma\widetilde{\nabla}_b\sigma + \frac{1}{6}\,\widetilde{R}\,\sigma^2 - \frac{1}{3}\,\Lambda\,\sigma^4\right) . \tag{3.106}$$

The canonical stress-energy-momentum tensor associated to this Lagrangian density is

$$\begin{aligned}\widetilde{\Sigma}_{ab} &:= -\widetilde{\nabla}_a\sigma\,\widetilde{g}_{bc}\,\frac{\partial\widetilde{\mathcal{L}}}{\partial\widetilde{\nabla}_c\sigma} + \widetilde{\mathcal{L}}\,\widetilde{g}_{ab} \\ &= -\frac{3}{4\pi G}\left[\widetilde{\nabla}_a\sigma\widetilde{\nabla}_b\sigma - \frac{1}{2}\,\widetilde{g}_{ab}\left(\widetilde{g}^{cd}\,\widetilde{\nabla}_c\sigma\widetilde{\nabla}_d\sigma + \frac{1}{6}\,\widetilde{R}\,\sigma^2 - \frac{1}{3}\,\Lambda\,\sigma^4\right)\right] .\end{aligned} \tag{3.107}$$

Using the expression (3.107), the field equation (3.67) can be rewritten as

$$\sigma^2\,\widetilde{R}_{ab} + \sigma^4\,\Lambda\,\widetilde{g}_{ab} - \widetilde{\nabla}_a\widetilde{\nabla}_b\sigma^2 + \widetilde{g}_{ab}\,\widetilde{\Box}\sigma^2 = 8\pi G\left(\widetilde{T}_{ab} + \widetilde{\Sigma}_{ab}\right) . \tag{3.108}$$

In the spacetime $(\mathcal{M}, \widetilde{g}_{ab})$ there is a preferred time, corresponding with the Killing parameter t. We can then rewrite $\widetilde{\mathcal{L}}$ by isolating the time derivative of σ,

$$\widetilde{\mathcal{L}} = \frac{3}{8\pi G}\left(-\dot{\sigma}^2 + \widetilde{h}^{ab}\,\widetilde{\nabla}_a\sigma\widetilde{\nabla}_b\sigma + \frac{1}{6}\,\widetilde{R}\,\sigma^2 - \frac{1}{3}\,\Lambda\,\sigma^4\right) , \tag{3.109}$$

where $\dot{\sigma} := \partial\sigma/\partial t$, and define the canonical momentum density

$$\widetilde{\Pi} := \frac{\partial\widetilde{\mathcal{L}}}{\partial\dot{\sigma}} = -\frac{3}{4\pi G}\,\dot{\sigma}\,. \tag{3.110}$$

The Hamiltonian density is then

$$\widetilde{\mathcal{H}} = -\frac{3}{8\pi G}\left(\dot{\sigma}^2 + \widetilde{h}^{ab}\,\widetilde{\nabla}_a\sigma\widetilde{\nabla}_b\sigma + \frac{1}{6}\,\widetilde{R}\,\sigma^2 - \frac{1}{3}\,\Lambda\,\sigma^4\right)\,, \tag{3.111}$$

where we have chosen to use $\dot{\sigma}$ as a variable, instead of Π as it would perhaps be more appropriate, since it is what usually appears in the definition of Hamiltonian densities.

3.6.3 The Gravitational Force Density

In $(\mathcal{M}, g_{ab}, \psi)$, matter behaves in such a way that Eq. (3.3) holds. On the contrary, in $(\mathcal{M}, \widetilde{g}_{ab}, \sigma, \psi)$ it will be, in general, $\widetilde{\nabla}_b\widetilde{T}^{ab} \neq 0$, because of the additional forces exerted by the gravitational field σ. For point particles these forces are known, since a geodesic motion in $(\mathcal{M}, g_{ab})$ is non-geodesic in $(\mathcal{M}, \widetilde{g}_{ab})$, and the corresponding four-force for a particle with mass m in $(\mathcal{M}, g_{ab})$ is

$$\widetilde{F}_a = -\widetilde{\nabla}_a\widetilde{m} = -\widetilde{m}\,\frac{\widetilde{\nabla}_a\sigma}{\sigma} = -\widetilde{m}\,\widetilde{\nabla}_a\Phi\,, \tag{3.112}$$

where $\widetilde{m} = \sigma\,m$ is the particle mass in $(\mathcal{M}, \widetilde{g}_{ab})$ [153]. It is therefore interesting to evaluate $\widetilde{\nabla}_b\widetilde{T}^{ab}$, which can be identified with the gravitational force density (see Eq. (3.1) for a special-relativistic analog).

Let us start from Eq. (3.3) in the form $g^{bc}\nabla_b T_{ac} = 0$ or, more conveniently for our purpose, $\widetilde{g}^{bc}\nabla_b T_{ac} = 0$. On rewriting the left hand side in terms of the covariant derivative in $(\mathcal{M}, \widetilde{g}_{ab})$, and using the generalization of Eq. (C.6) for a second-rank tensor, we find

$$\begin{aligned}\widetilde{\nabla}^b T_{ab} &= \widetilde{g}^{bc}\,\widetilde{C}^d{}_{ba}\,T_{dc} + \widetilde{g}^{bc}\,\widetilde{C}^d{}_{bc}\,T_{ad} \\ &= \frac{1}{\sigma}\,\widetilde{g}^{cd}\,T_{cd}\widetilde{\nabla}_a\sigma - \frac{2}{\sigma}\,T_{ab}\,\widetilde{\nabla}^b\sigma\,,\end{aligned} \tag{3.113}$$

where we have used Eq. (C.7). Recalling now Eq. (3.87) we get [154]

$$\widetilde{\nabla}^b\widetilde{T}_{ab} = \widetilde{T}\,\frac{\widetilde{\nabla}_a\sigma}{\sigma} = \widetilde{T}\,\widetilde{\nabla}_a\Phi\,, \tag{3.114}$$

which allows us to identify $\widetilde{T}\,\widetilde{\nabla}_a\Phi$ as the gravitational force density. This can be regarded as the generalization of (3.112) for a continuum.

It is natural to try to rewrite the right hand side of Eq. (3.114) as the divergence of a symmetric rank-two tensor, which could then be identified with (minus) the stress-energy-momentum tensor of the gravitational field. In order to carry on this programme, it is convenient to express the ratio $\widetilde{T}/\sigma$ using Eq. (3.68), and to replace it into Eq. (3.114). It is then a trivial calculation to show that

$$\widetilde{\nabla}^b\left(\widetilde{T}_{ab} + \widetilde{\Sigma}_{ab}\right) = \frac{1}{16\pi G}\,\sigma^2\,\widetilde{\nabla}_a\widetilde{R}\,, \tag{3.115}$$

where $\widetilde{\Sigma}_{ab}$ is given by Eq. (3.107). Thus, although we do not completely succeed in rewriting Eq. (3.114) purely in terms of divergences, we nevertheless recover the canonical stress-energy-momentum tensor for σ from a very different road than the one followed in Sec. 3.6.2.

With the benefit of hindsight, the appearance of the term on the right hand side of Eq. (3.115) is not surprising at all. If the curvature scalar $\widetilde{R}$ is not constant on $(\mathcal{M}, \widetilde{g}_{ab})$, the Lagrangian density for matter (that is, "ordinary" matter *and* the field σ) is not invariant under spacetime translations. Hence, the hypothesis under which one usually proves that the canonical stress-energy-momentum tensor is divergence-free is not satisfied, and it is not surprising to see an extra force density appear. Physically, the term $\sigma^2\,\widetilde{\nabla}_a\widetilde{R}/(16\pi G)$, which is due to the inhomogeneities of $(\mathcal{M}, \widetilde{g}_{ab})$, can be thought of as an analogue of the external forces that act, in elementary mechanics, on non-isolated systems, whose Lagrangian is not explicitly time-independent.

3.6.4 The Gravitational Energy

The four-velocities of inertial observers in $(\mathcal{M}, \widetilde{g}_{ab})$ coincide with the timelike Killing vectors, $\widetilde{n}^a = \eta^a$. The energy density of the gravitational field is then simply defined as

$$\tilde{u} = \widetilde{\Sigma}_{ab}\,\widetilde{n}^a\widetilde{n}^b = \widetilde{\Sigma}_{ab}\,\eta^a\eta^b = \widetilde{\Sigma}_{00}\,, \tag{3.116}$$

and coincides, obviously, with the Hamiltonian density (3.111). Hence, the total gravitational energy contained within a region $\mathcal{D}$ of the three-space $\mathcal{S}$ is

$$\widetilde{E} = \int_{\mathcal{D}} \tilde{u}\,\mathrm{d}\widetilde{S} = \int_{\mathcal{D}} \widetilde{\Sigma}_{00}\,\mathrm{d}\widetilde{S}\,, \tag{3.117}$$

where the volume element $\mathrm{d}\widetilde{S}$ on $(\mathcal{S}, \widetilde{h}_{ab})$ is evaluated using the optical spatial metric $\widetilde{h}_{ab}$.

Of course, $\widetilde{\Sigma}_{ab}$ contains much more than just the gravitational energy density. In general, we can use it to define also gravitational momentum density and gravitational field stresses. Also, we are not restricted, in principle, to evaluate these quantities in the frame $\widetilde{n}^a$.

We have chosen to calculate energy density in $(\mathcal{M}, \widetilde{g}_{ab})$ simply because $\widetilde{\Sigma}_{ab}$ is a naturally defined object there, but not in $(\mathcal{M}, g_{ab})$, where it is merely part of the Einstein tensor G_{ab}. Indeed, one must appreciate that, although in $(\mathcal{M}, \widetilde{g}_{ab}, \sigma, \psi)$ there is now a meaningful definition of a gravitational stress-energy-momentum tensor, in $(\mathcal{M}, g_{ab}, \psi)$ such a concept is still meaningless.

What about negative energy density? This is no problem for a field, in spite of what some people might say [155]. The two frameworks are perfectly equivalent [156], so if there is no difficulty in general relativity, there cannot be any in the optical geometry.

3.7 Examples

We now calculate the gravitational energy density in two different cases of physical interest. First, we consider a simple static spacetime describing an isolated body, obtained by matching the interior Schwarzschild solution with an incompressible fluid and the exterior Schwarzschild solution, and with $\Lambda = 0$. Then, we shall compute the energy density for the Friedman-Lemaitre-Robertson-Walker (FLRW) cosmological models, also with a nonzero cosmological constant Λ.

3.7.1 Schwarzschild spacetime

Using eq (3.10) we can rewrite the first two metrics in a more general form, which allows us to shorten the calculations somewhat.

$$g = -\sigma^2\,\mathrm{d}t^2 + B^2\,\mathrm{d}r^2 + r^2\,\omega \tag{3.118}$$

$$\widetilde{g} = -\mathrm{d}t^2 + B^2\,\sigma^{-2}\,\mathrm{d}r^2 + r^2\,\sigma^{-2}\,\omega\,. \tag{3.119}$$

Where $\omega = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$ is the metric on the unit 2-sphere, while σ and B are functions of r. To summarize, we have:

$$\begin{aligned}
\widetilde{g}_{ab} &= \mathrm{diag}\,[-1, B^2\,\sigma^{-2}, r^2\,\sigma^{-2}, r^2\,\sigma^{-2}\,\sin^2\theta] \\
\widetilde{g}^{ab} &= \mathrm{diag}\,[-1, B^{-2}\,\sigma^{2}, r^{-2}\,\sigma^{2}, r^{-2}\,\sigma^{2}\,\sin^{-2}\theta]
\end{aligned}$$

$$\begin{aligned}
-\widetilde{g} &= \sigma^{-6}\,r^4\,B^2\,\sin^2\theta \\
\sqrt{-\widetilde{g}} &= \sigma^{-3}\,r^2\,B\,\sin\theta
\end{aligned}$$

Now the gravitational energy density is given by the $\widetilde{\Sigma}_{00}$ component of the canonical stress energy tensor in eq.(3.107), where we have also made the additional assumption that $\Lambda = 0$. Since we are considering conformally static spacetimes, where σ is a function of r only, we can simplify the derivative

$$\widetilde{g}^{ab}\widetilde{\nabla}_a\sigma\,\widetilde{\nabla}_b\sigma = \widetilde{g}^{rr}\,\partial_r\sigma\,\partial_r\sigma = \widetilde{g}^{rr}\,\sigma'(r)^2 = B^{-2}\,\sigma^2\,\sigma'^2\,. \tag{3.120}$$

Then in addition, in optical geometry $\widetilde{g}_{00} = -1$, which allows us to write

$$\widetilde{\Sigma}_{00} = \alpha\left[\tfrac{1}{2}\,\widetilde{g}^{rr}\,\sigma'(r)^2 + \tfrac{1}{12}\,\widetilde{R}\,\sigma^2\right] \tag{3.121}$$

$$= \tfrac{1}{12}\,\alpha\,\sigma^2\left[6\,B^{-2}\,\sigma'^2 + \widetilde{R}\right], \tag{3.122}$$

where $\alpha := -3/(4\pi G)$. The scalar curvature term can then be easily calculated using a few lines of Mathematica code (see appendix), with the result

$$\widetilde{R} = 2r^{-2}\,B^{-3}\left[B^3\sigma^2 + 2r\sigma B'(\sigma - r\sigma') - B(\sigma^2 + 3r^2\sigma'^2 - 2r\sigma(2\sigma' + r\sigma''))\right] \tag{3.123}$$

$$= 2r^{-2}\,B^{-3}\,\sigma^2\left[B^3 - B + 2rB' + 2r\sigma^{-1}\,(\sigma'(2B - rB') + r\sigma''B)\right] - 6B^{-2}\sigma'^2\,, \tag{3.124}$$

with obvious notation. The last term on the last line is very interesting, because that term cancels exactly the first one in eq.(3.122). Therefore it seems that all the energy is coming from the curvature and not from the dynamics of the scalar field. (But one has to be careful with such a statement, since one could in principle construct arbitrary Ricci like objects by just adding and subtracting various derivative terms of the scalar field.) We are now ready to treat the specific cases, starting with the interior and working our way out.

Schwarzschild internal spacetime

The metric inside a static incompressible fluid sphere corresponds to

$$\sigma(r) = \tfrac{1}{2}(3\sigma_{\mathrm{R}} - B^{-1}) \tag{3.125}$$

$$B(r) = \left(1 - \frac{r^2}{a^2}\right)^{-1/2} \tag{3.126}$$

$$a := (R^3/(2M))^{1/2} \tag{3.127}$$

$$\sigma_R := (1 - 2M/R)^{1/2}\,. \tag{3.128}$$

With these definitions we again obtain the form of eq.(3.119), which allows us to recycle the previous formulas. For the interior spacetime, the calculation of $\widetilde{R}$ is slightly complicated. However, making frequent use of the following substitutions makes the calculation rather short.

$$\begin{aligned} \sigma'(r) &= \tfrac{r}{2a^2}\,B & B' &= \tfrac{r}{a^2}\,B^3 \\ \sigma''(r) &= \tfrac{1}{2a^2}\,B^3 & 1-B^{-2} &= \tfrac{r^2}{a^2} \end{aligned}$$

So from eq.(3.124) we find after some algebra that

$$\widetilde{R} = 6\,a^{-2}\,\sigma\,(\sigma + B^{-1}) - 6\,B^{-2}\,\sigma'^2\,. \tag{3.129}$$

The gravitational energy density is then

$$\begin{aligned} \tilde{u} &= \tfrac{1}{2}\,\alpha\,\sigma^2\left[B^{-2}\,\sigma'^2 + \tfrac{1}{6}\,\widetilde{R}\right] && (3.130)\\ &= \tfrac{1}{2}\,\alpha\,a^{-2}\,\sigma^3\,(\sigma + B^{-1}) && (3.131)\\ &= \tfrac{1}{4}\,\alpha\,a^{-2}\,\sigma^2\,(3\sigma_R + B^{-1})\,. && (3.132) \end{aligned}$$

Expanding out all the functions in terms of r and R we have:

$$\tilde{u} = -\frac{3M}{8\pi G R^3}\left[3\left(1-\frac{2M}{R}\right)^{1/2} + \left(1-\frac{2Mr^2}{R^3}\right)^{1/2}\right]^2\left(1-\frac{9M}{4R}+\frac{Mr^2}{4R^3}\right)\,. \tag{3.133}$$

This could then be plotted in terms of the coordinate r, however this plot would not be very useful, since that coordinate will not be the same as the one in the exterior, and thus can not be used for comparisons to anything astrophysical. In addition, we have to remember that this energy density is still defined in the optical space, which means that the coordinate r is less useful here. However, as we shall see below, there is a more useful coordinate for plotting. But before going into that, let us evaluate the exterior energy density as well. Moreover, attempting to calculate the total energy for this case would also be rather meaningless, since integrating over the total space generally gives infinity.

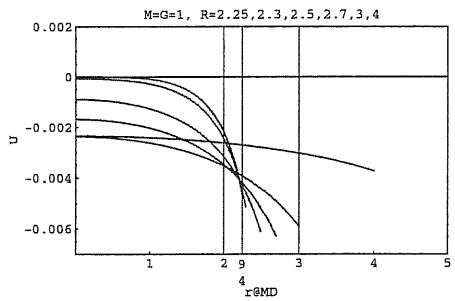

Figure 3.2: The conformal energy density without the Ricci scalar contribution for some different sizes R but with fixed mass $M = 1$.

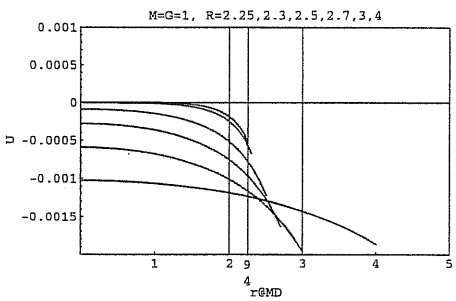

Figure 3.3: Similar plot, but this time including the Ricci scalar.

However, we will atempt this anyway for educational purposes.

$$M_{gr} = \int_0^R \int_0^\pi \int_0^{2\pi} \sqrt{\tilde{h}}\,\tilde{u}\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \tag{3.134}$$

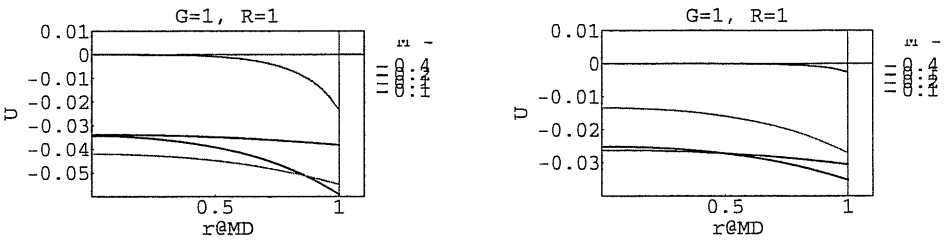

Figure 3.4: The conformal gravitational field energy density $\tilde{u}$ for different compactness by varying M but with fixed $R = 1$. The left plot is without the Ricci scalar contribution whereas the right plot includes it.

But

$$\sqrt{\tilde{h}}\,\tilde{u} = \tfrac{1}{2}\,\alpha\, a^{-2}\,(r^2 \sin\theta)[1 + B\sigma] \tag{3.135}$$

so that

$$M_{gr} = 2\pi\alpha\, a^{-2} \int_0^R (r^2 + r^2 B\sigma) \tag{3.136}$$

$$= \pi\alpha\, a^{-2} \int_0^R (r^2 + 3\sigma_R r^2 B) \tag{3.137}$$

Schwarzschild external spacetime

For the external Schwarzschild spacetime we have $B = \sigma^{-1}$, so its optical metric is simply

$$\tilde{g} = -\mathrm{d}t^2 + \sigma^{-4}\,\mathrm{d}r^2 + r^2\sigma^{-2}\,\omega\,. \tag{3.138}$$

In case one would like to extend the analysis to include also the Reissner-Nordström spacetime, it is useful to define the following quantities and their derivatives.

$$\sigma(r) = (1 - 2M/r + Q^2/r^2)^{1/2} \tag{3.139}$$

$$\sigma'(r) = -\sigma^{-1}\,A \tag{3.140}$$

$$\sigma''(r) = -\sigma^{-3}\,A^2 - \sigma^{-1}\,A' \tag{3.141}$$

$$A(r) := -M/r^2 + Q^2/r^3 \tag{3.142}$$

$$A'(r) = 2M/r^3 - 3Q^2/r^4\,. \tag{3.143}$$

But for our purposes we will set $Q = 0$. Using the above relations it is then easy to calculate $\tilde{R}$

$$\tilde{R} = -6A^2 = -6M^2\,r^{-4}\,. \tag{3.144}$$

It is then straightforward to obtain:

$$\tilde{u} = \tfrac{1}{12}\,\alpha\,\sigma^2\,\left[6A^2 + \tilde{R}\right] = 0 \tag{3.145}$$

For the exterior Schwarzschild case, the total gravitational energy in the optical frame is given by evaluating the integral in Eq.(3.117), where now

$$\mathrm{d}\tilde{S} = \sqrt{\tilde{h}}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi = \sigma^{-4}\,r^2\,\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi \tag{3.146}$$

so the total energy is given by

$$\widetilde{E}_{sch} = -\frac{3}{8}\int_{\sigma_0}^{1} \mathrm{d}\sigma\, \frac{(1-\sigma^2)^2}{\sigma^4}\,. \tag{3.147}$$

Coordinate Matching

Since we would like to continuously cover the entire Schwarzschild spacetime we have to use a common radial coordinate. (That we choose to represent the radial coordinate by "r" in both the internal and external metrics does not mean that they are the same coordinate. In fact they are not.) The matching is most easily done by using the Regge-Wheeler coordinate, x. This has a very special meaning in optical geometry where it represents the optical distance. It is obtained by integrating the line element of a radial lightray, with $\theta = \phi = const$,

$$\mathrm{d}s^2 = 0 = -\mathrm{d}t^2 + B^2\,\sigma^{-2}\,\mathrm{d}r^2\,. \tag{3.148}$$

Then we can define the Regge-Wheeler coordinate as:

$$x(r) := \int \mathrm{d}t = \int B\sigma^{-1}\mathrm{d}r \tag{3.149}$$

<u>Internal</u>

In the case of the interior Schwarzschild spacetime, the integrand is rather ugly. However, it can be integrated more easily by using the fact that in the optical geometry, the Schwarzschild interior solution is equivalent to the Einstein static universe in (M, g) [148], as shown in [148]. The result is that the *radius of gyration* $\widetilde{r}$ can be simply expressed in both coordinates r and x.

$$\widetilde{r} = r\mathrm{e}^{-\Phi(r)} = \tilde{a}\sin(x/\tilde{a}) \tag{3.150}$$

where

$$\widetilde{a} = \frac{R}{2}\left(\frac{R}{M}\right)^{1/2}\left(1-\frac{9M}{4R}\right)^{-1/2}\,. \tag{3.151}$$

Then the Regge-Wheeler coordinate is:

$$x(r) = \widetilde{a}\arcsin\left[\frac{r}{\widetilde{a}}\mathrm{e}^{-\Phi(r)}\right]\,. \tag{3.152}$$

So at the surface, we have the matching condition that:

$$X = x(R) = \widetilde{a}\arcsin\left[\frac{R}{\widetilde{a}\sigma_R}\right] = const\,. \tag{3.153}$$

<u>External</u>

Similarly, for the exterior spacetime where $B = \sigma^{-1}$ we have:

$$\begin{aligned} x(r) &= \int (1-2M/r)^{-1}\mathrm{d}r \\ &= r + 2M\ln(r-2M) + const \\ &= r + 2M\ln\left(\frac{r-2M}{k}\right)\,. \end{aligned} \tag{3.154}$$

Then applying the matching condition of eq.(3.153):

$$X = R + 2M\ln\left(\frac{R-2M}{k}\right)\,, \tag{3.155}$$

we can solve for the constant k.

$$k = (R - 2M)\,\mathrm{e}^{\frac{R-X}{2M}} \,. \tag{3.156}$$

Now in order to plot our previous derivations of $\tilde{u}$ which are all given in terms of r, with the new coordinate x, we need to first invert the two functions for $x(r)$. This inversion may at first seem impossible, but there actually exist a very useful function which does exactly this, the Lambert w-function [160], $w^{-1}(\xi) = \xi \mathrm{e}^{\xi}$. After some logarithmic algebra, we rewrite eq. (3.154) as:

$$\frac{x}{2M} = \frac{r}{2M} + \ln\left(\frac{2M}{k}\right) + \ln\left(\frac{r}{2M} - 1\right) \tag{3.157}$$

Then exponentiating,

$$\begin{aligned} \mathrm{e}^{x/(2M)} &= (2M/k)\,\mathrm{e}\,\underbrace{\left(\frac{r}{2M} - 1\right)\,\mathrm{e}^{r/(2M)-1}} \\ &= (2M/k)\,\mathrm{e}\,w^{-1}\left(\frac{r}{2M} - 1\right) \end{aligned}$$

Inverting the Lambert w-function, we get the final result:

$$r(x) = 2M\left[w\left(\frac{k}{2M}\mathrm{e}^{x/(2M)-1}\right) + 1\right] \tag{3.158}$$

3.7.2 FLRW cosmological models

The FLRW cosmological models are described by the class of metrics

$$g = -\mathrm{d}\tau^2 + a(\tau)^2\left(\mathrm{d}\chi^2 + f(\chi)^2\,\omega\right) \,, \tag{3.159}$$

where ω is again the metric of the unit 2-sphere, and the function $f(\chi)$ depends on the curvature of the spatial sections as:

$$f(\chi) = \begin{cases} \sinh\chi & \text{for} \quad k = -1\,, \\ \chi & \text{for} \quad k = 0\,, \\ \sin\chi & \text{for} \quad k = +1\,. \end{cases} \tag{3.160}$$

On defining the purely spatial metric

$$\widetilde{h} := \mathrm{d}\chi^2 + f(\chi)^2\,\omega \,, \tag{3.161}$$

and the so-called conformal time t, such that $\mathrm{d}t := \mathrm{d}\tau/a(\tau)$, the FLRW metric (3.159) can be rewritten as

$$g = a^2\left(-\mathrm{d}t^2 + \widetilde{h}\right) \,, \tag{3.162}$$

where a is now considered as a function of t. Thus, g_{ab} is conformal to the ultrastatic metric

$$\widetilde{g} = -\mathrm{d}t^2 + \widetilde{h} \,, \tag{3.163}$$

with the conformal factor σ depending only on time. Since in the FLRW spacetime, $\sigma = a(t)$, the optical metric has a simple form which makes it easy to evaluate the Ricci scalar.

$$\widetilde{g}_{ab} = \mathrm{diag}\left[-1, 1, f(\chi)^2, f(\chi)^2\,\sin^2\theta\right] \tag{3.164}$$

$$\sqrt{\widetilde{h}} = f(\chi)^2\,\sin\theta \tag{3.165}$$

The Ricci scalar is then

$$\widetilde{R} = -2f(\chi)^{-2}\left[-1 + f'(\chi)^2 + 2\,f(\chi)\,f''(\chi)\right] = \begin{cases} -6 & \text{for} \quad k = -1\,, \\ 0 & \text{for} \quad k = \ \ 0\,, \\ 6 & \text{for} \quad k = +1\,. \end{cases} \tag{3.166}$$

The gravitational energy density is then even simpler to evaluate from the gravitational stress energy tensor of eq.(3.107),

$$\begin{aligned} \tilde{u} &= \alpha\left(\dot{a}^2 - \tfrac{1}{2}\,\widetilde{g}_{00}\widetilde{g}^{00}\,\dot{a}^2 - \tfrac{1}{12}\widetilde{g}_{00}\widetilde{R}\,a^2\right) && (3.167) \\ &= \alpha/2\,(\dot{a}^2 + k\,a^2) && (3.168) \end{aligned}$$

where $\alpha := -3/(4\pi G)$. But we notice that $\widetilde{u}$ is a function of t only, so that the total gravitational energy in the universe at the conformal time t is

$$\widetilde{E}(t) = \int_{\mathcal{S}} \tilde{u}\,\mathrm{d}\widetilde{S} = \tilde{u}\,\widetilde{S}\,. \tag{3.169}$$

Of course, for $k = 0$ and $k = -1$ this quantity is infinite, due to the fact that the universe has an infinite volume. For a spatially closed model, $k = +1$, we have instead

$$\begin{aligned} \widetilde{E}(t) &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \sqrt{\widetilde{h}}\,\tilde{u}\,\mathrm{d}\chi\,\mathrm{d}\theta\,\mathrm{d}\varphi && (3.170) \\ &= 4\pi\,\tilde{u}\int_0^{\pi} \sin^2\chi\,\mathrm{d}\chi && (3.171) \\ &= \pi^2\alpha\,(\dot{a}^2 + a^2) && (3.172) \\ &= -\frac{3\pi}{4G}\,(\dot{a}^2 + a^2) && (3.173) \end{aligned}$$

This may be important because one could then give a precise meaning to theories in which cosmogenesis is based on an energy transfer between gravity and matter [157]. These theories are based on having an initial vacuum scalar field in a Minkowski background. Then there is some initial scalar fluctuation which then gives rise to curvature. However, there are several weak points which still needs to be adressed in this theory. However I will not discuss these further here.

3.8 Conclusion

In this work we have followed a well motivated approach for attempting to define a gravitational energy density, based upon the ideas from field theory and applying them to the optical geometry framework of GR.

At the moment we have run into problems because in the example of the exterior Schwarzschild, we have found that the energy density as currently defined by $\widetilde{\Sigma}_{ab}$, is zero. This is new and completely unexpected. In fact this reintroduces the problem we had with $\widetilde{\theta}_{ab}$ in section 3.6.1. However, it does agree with the results of both Penrose and Cooperstock, which unfortunately are both incomprehensible to us.

The problem seems to be the conformal Ricci scalar which cancels exactly with the dynamics of the conformal scalar field. However, if we choose to put $\widetilde{R}$ on the "other" side of the field equations, we run into other problems, because then the new field equation would be non-zero, i.e. $\widetilde{\nabla}_b(\widetilde{\theta}^{ab} + \widetilde{\Sigma}'_{ab}) = -(16\pi G)^{-1}\,\widetilde{R}\,\widetilde{\nabla}_a\sigma^2$.

At this point we do not know of any other or different formulations which may get around this problem. The only other possiblity is perhaps to reinvestigate the boundary term in the variational priciple, which is constructed in such a way as to cancel terms comming from the bulk. (This method seems similar to that which is known as the "trace-K" approach of the Brown-York formalism, something which we have not yet investigated.)

Furthermore, since much of the OG formalism can be exported directly to scalar-tensor theories, any problems *we* have, will be present also in those theories...

Acknowledgments

Warning, unless you think you will appear here, it is probably better you don't read this. It is not healthy reading and it is certainly not very interesting. If you just have a catastrophic social life I rather recommend you to go buy yourself a book instead, or better, go down to the local cafe and speak to the customers there.

As last words I would like to thank a whole bunch of people who have all helped me in one way or another, consciously or not, to survive these four long years at SISSA. Apart myself, the next most important person to thank must be my old girlfriend Valentina Viganò, who have cooked, cleaned, and defrosted my cold swedish hart, apart from writing some of my talks! Then there are the rest of you: Alex who got me into the latin-american circuit and who have tortured me since day one to start writing my thesis, Tino who showed me how to be happy using μ's and ν's, and the other "jungle cats" at SISSA and ICTP. (You know who you are!) Davide from Guatemala, with our long after lunch philosophical discussions about sex, drugs and rock'n roll, and the bitterness of life. Hamish who got me into climbing and out of town for crazy hitch-hiking trips to Slovenia. Sebastiano Sonego, who actually introduced me to GR and to Trieste as well as to more serious mountain hikes. Stefano and Paola Liberati who have taken me around to eat in many strange places in Friuli and Slovenia. Giuliano and Valentina for fixing my computers and feeding me, respectively. My office mates Pasquale and Olindo, for fixing my computer and supporting my daily complaints about the SISSA computer facilities, the decline of the roman empire, etc and doing exactly all those things that I would have hated to do myself... Then there are Elena and Luisa (ICTP) who have shown me that lunch conversations can actually be very interesting and fun. Daniele my ex-room-mate who gives me free drinks in bar *Naima*. My swedish friend Viktor, who introduced me to the wonderful world and philosophy of Jaques Costeu, including his diving lessons. All the family at restaurant *Siora Rosa* who have fed me for months with various german/venezian food combinations. The friendly guys at *Mascalzone Latino* who have both fed me and experimented on me with their food. Then of course a very varm and special thanks to all the other people I forgot to mention. And then last but not least all the nice girls I got to know here...

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Appendix A

This appendix belongs primarily to chapter 1. However, chapter 1 is self contained and this appendix should not be needed for understanding that chapter. These are merely a few reading notes collected during my studies, but which do not have a natural place within the text.

Vector Bosons (spin-1)

Spin-1 particles behave like vectors. That is, they have 3 components of polarization. These are usually taken to be: 2 transverse (perpendicular) and 1 longitudinal (parallel) to the direction of the momentum. However there is a difference depending on whether the particle is massive or massless. This difference become important at very high momenta like this; massive particles have 3 components with the longitudinal component approaching a constant $1/m^2$ while the massless ones have only 2 which are both transverse and approaching zero like $1/p^2$. Thus massive vector bosons lead to trouble (in that they are non-renormalizable) in gauge field theories for the weak interactions at high energies.

Boundary conditions

It is important to notice the difference when fields are subjected to boundary conditions in conductors and dielectrics, respectively. In fact for dielectrics these are not really boundary conditions, but rather *junction* conditions, as the field and its derivatives have to be continous across the junction. Below we list some of the most common boundary conditions with the boundary at R and where primes denote the normal partial derivatives.

- Conductors:
  Dirichlet: $\phi(R) = 0$
  Neumann: $\phi'(R) = 0$
  Robin: $\phi'(R) = k\phi$
- Dielectrics:
  acoustic junction: $\rho_1\phi_1 = \rho_2\phi_2$ and $\partial_n\phi_1 = \partial_n\phi_2$.
  dielectric junction: See [13]

Asymptotic Freedom

In 4d, the Yang-Mills theories are the only asymptotically free theories [18, p. 287]. The explanation is that non-Abelian gauge fields have integer spin and carry the gauge symmetry charges themselves. Asymptotic freedom means that the vacuum anti-shields charges, and thus acts like a dielectric medium

with dielectric constant, $K < 1$. But since the QFT vacuum is relativistically invariant we have to obey $\mu K = c = 1$, where μ is the magnetic permeability, translating the electric responses into magnetic ones. In EM theory we have two types of responses:

- Landau diamagnetism: $\mu < 1$ Induction of opposing magnetic field.
- Pauli paramagnetism: $\mu > 1$ Material has aligned magnetic moments.

Thus the Yang-Mills vacuum acts like a paramagnetic medium.

Catastrophe Theory

Catastrophe theory does not predict the end of life on earth, but is rather a very useful tool for describing functions of certain physical phenomena whose precise physical laws are unknown. This is something which has been particularly useful in stability analysis of boson stars. I shall give a short introduction to catastrophe theory followed by an example of its usage.

So what is catastrophe theory? It is simply a theory of the local behavior of typical smooth functions near critical points [102]. As such, its is really a classification of singularities of differential maps. What one then does, is to investigate the critical points (CP's) of a mapping and construct the corresponding *bifurcation diagrams.*

Let $V_c(x)$ be a family of real valued functions, where x are the generalized coordinates (state parameters) while c are control parameters labeling the family. Then we have

$$V_c(x) = V(x, c), \qquad \text{where:} \quad (x \in R^n, c \in R^k)\,. \tag{A.1}$$

One then studies the behavior of the critical points of these functions, while varying the control parameters. Discontinuous changes in their properties during a smooth variation of c are called *catastrophes.* We can then define a subset M of the space $R^n \times R^k$ as the space of catastrophes of V, to represent the set of all critical points of the family of functions V_c.

$$M = \{(x, c) \in R^n \times R^k | DV_c(x) = 0\} \tag{A.2}$$

We can then define the catastrophe map χ, by the restriction to M of the natural projection onto the space of control parameters.

$$\pi : R^n \times R^k \quad \rightarrow \quad R^k : (x, c) \mapsto c \tag{A.3}$$

with $\chi = \pi|_M$. In addition there are two more important subsets of M, the singular point set S the bifurcation set B (the projection of S), both of the map χ.

$$S = \{(x, c) \in M | Dim(Range(D_\chi)) < k\} \tag{A.4}$$

$$B = \chi(S) \subset R^k \tag{A.5}$$

The local properties of a single function or a family of functions $V(x, c)$ are then determined by a number of theorems from functional analysis. Let us see two examples, one of which is for a single function and the other for a family of functions.

(A single function)

Consider a function $V(x)$ at a non-critical point x, such that $V'(x) \neq 0$. From the *implicit function theorem*, one can make a smooth change of coordinates, $y = y(x)$, so that $V \doteq y$. Then at a critical point x, for which:

$$V'(x) = 0, \quad V''(x) \neq 0, \tag{A.6}$$

Then the *Morse theorem* states that $V \doteq y^2$. This type of critical point is called a *non-degenerate* or Morse critical point, and these are very typical since all other types are reduced to this one under small changes of the function.

(A family of functions)

...

The degenerate cases $V'(x) = V''(x) = 0$ are no longer removable.

One can then classify the types of critical points, using Arnold's classification and Whitney's Theorem. Then the key result is given by Thom's theorem: In the typical case, the class of real-valued functions on R^n depending on k parameters, is structurally stable and in the neighborhood of a given point, this function is equivalent to one of the following functions:

⋆ Non-critcal point:	x_1
⋆ Non-degenerate critical point:	$x_1 + ... + x_i^2 - x_{i+1}^2 - ... - x_n^2$
⋆ Degenerate critical point:	See below.

The degenerate critical points:

Fold (A_2)	$\frac{1}{3}x^3 + ax$	with CPs:	$V'(x,a) = x^2 + a = 0$
Cusp (A_3)	$\frac{1}{4}x^4 + \frac{1}{2}ax^2 + bx$	with CPs:	$V'(x,a,b) = x^3 + ax + b = 0$
Double Cusp (A_{-3})	$-\frac{1}{4}x^4 - \frac{1}{2}ax^2 - bx$		
Swallowtail (A_4)	$\frac{1}{5}x^5 + \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx$		
Elliptic umbilics (D_{+4})	$x^2y + y^3 + ay^2 + by + cx$		
Hyperbolic umbilics (D_{-4})	$x^2y - y^3 + ay^2 + by + cx$		

Example (1):

To investigate the stability against radial perturbations of a boson star, one considers a mapping from a 2d subspace of the dynamical field variables into the "integrals of motion", which are the total mass M and the particle number N. As variables of the boson field we take a scaling parameter of the stellar radius k and the frequency eigenvalue ω. The parameter k induces a scaling of the metric, the frequency and the scalar field, such that the particle number is fixed.

$$ds^2 \to k^2 ds^2, \qquad \omega \to \omega/k, \qquad \phi(r) \to k\phi(kr) \tag{A.7}$$

Then for each pair (k, ω) one can compute the corresponding (M, N). This correspondence is called a smooth mapping.

$$F : (k, \omega) \mapsto (M, N) \tag{A.8}$$

But according to Whitney's theorem, a two-parameter smooth mapping, has only two types of singularity: either a fold or a cusp singularity. Therefore the corresponding physical system is completely determined by the behavior of the singularity points. The singularities of the mapping are defined by the Jacobi matrix.

$$J = \begin{pmatrix} \frac{\partial M}{\partial k} & \frac{\partial M}{\partial \omega} \\ \frac{\partial N}{\partial k} & \frac{\partial N}{\partial \omega} \end{pmatrix} \tag{A.9}$$

If the rank of J is equal to 2, then J describes arbitrary wave packets which do not correspond to equilibrium solutions of the boson star. Rank 1 does give equilibrium solutions, which are extrema of the manifold. In that case: $\partial M/\partial k = \partial N/\partial k = 1$, defines the stationary points of M for a

fixed value of N. With a rank of 0, J describes the bifurcation points of the stability. One can then make a bifurcation diagram of $M = M(N)$ or equivalently the binding energy $E = E(N)$ since $E = M - Nm$. The stellar stability can only change at the cuspoidal points of this diagram.

□

Skyrmions

What is a Skyrmion? In 1962 Skyrme introduced a modification to the non-linear sigma model in order to approximate QCD at low energies. What he did, was to use the following Lagrangian,

$$\mathcal{L}_S = \tfrac{1}{4} f_\pi^2 \, Tr(\partial_\mu U \, \partial^\mu U^\dagger) + \tfrac{1}{32g^2} \, Tr[U^\dagger \partial_\mu U, U^\dagger \, \partial_\nu U]^2 \tag{A.10}$$

where f_π is the pion decay constant and g is the $\rho\pi\pi$ coupling. The first term is the usual non-linear σ model and the second term is the modification due to Skyrme. He then found a family of classical static solutions to the equations of motion (ELEs) derived from $\mathcal{L}_S$, given by:

$$U = \mathrm{e}^{i\tau\cdot\hat{r}F(r)} \tag{A.11}$$

where $F(r)$ is a radial function which must satisfy the boundary conditions:

$$F(r \to \infty) \quad \to \quad 0 \tag{A.12}$$

$$F(r = 0) \quad = \quad N\pi \, . \tag{A.13}$$

Here N is the topological charge (winding number), which has been identified as the baryon number, provided that the Wess-Zumino effective action is included to eliminate the unphysical symmetry inherent in the non-linear sigma model. But there there is no direct link of a skyrmion to a nucleon. Indeed apart from the baryon number, a skyrmion has no other quantum numbers, whereas baryons have in addition spin and isospin. However it was recently realized [66] that a skyrmion is really a coherent state of baryons and higher resonances.

Appendix B

This appendix belongs to chapter 2. Here I show that holonomy is a useful concept also outside relativity, such as in the method of Lagrange multipliers. Then I show the origin for the associated Legendre functions, that we needed in the Hartle-Thorne metric, with an example of how to obtain them in Mathematica. This is followed by the GnuPlot file used to make some of the plots found at the end of chapter 2.

Lagrange Multipliers

In classical mechanics one usually treat constraints by either directly eliminating the extra degrees of freedom, using a constraint equation, or introducing suitable forces maintaining the constrains. In the first case one obtains a problem with less degrees of freedom than initially. Whereas in the second case, one gets a free variational problem with all degrees of freedom including complete determination of the forces of constraint [174].

Lagrange Multipliers then provide a powerful method for dealing with mechanical problems involving both *holonomic* and *non-holonomic* constraints. Then the equations of motion including N constraint force terms, take the form

$$\frac{d}{d\tau}\left(\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial x^i} + \sum_{a=1}^{N} \lambda_{(a)} C^i_{(a)} = Q_i$$

where $\lambda_{(a)}$ are the undetermined multipliers, wheras Q_i are the components of the external force and $\dot{x}^i$ the velocity components. The two different constraints can then be summarized as

Non-holonomic:	$C_{(a)i}\delta x^i = 0$	$\Rightarrow$	The metric has only local significance.
Holonomic:	$C(x^i, ..., x^n) = 0$	$\Rightarrow$	The geometry of the problem is Riemannian.

The reason for this interpretation in the non-holonomic case, is that the equations on a hypersurface (in $n-1$ dimensions) are not integrable in n dimensions.

The Legendre Differential Equations

The **Legendre Differential Equation** reads:

$$(1-z^2)\,\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + n(n+1)\,z = 0$$

with solutions given by:

$$w = \begin{cases} A\,P_n(z) + B\,Q_n(z) & |z| < 1 \\ A\,P_n(z) + B\,D_n(z) & |z| > 1 \end{cases}$$

where $Q_n(z)$ are the Legendere functions of the *first* kind.
The **Associated Legendre Differential Equation** is obtained by letting

$$u := (1-z^2)^{m/2}\,\frac{d^m w}{dz^m}\,.$$

So that

$$(1-z^2)\,\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + \left[n(n+1) - \frac{m^2}{1-z^2}\right]u = 0$$

where the solutions are again given by:

$$u = A\,P_n^m(z) + B\,Q_n^m(z) \qquad |z| < 1$$

However, this time the solutions are called "the associated Legendre functions of the second kind". They are in turn obtained from the associated Legendere functions of the *first* kind, $Q_n(z)$.

$$Q_n^m(z) = (1-z^2)^{m/2}\,\frac{d^m Q_n(z)}{dz^m}$$

Example: _____

In our case we want Q_2^1 and Q_2^2, so that $n = 2$ and $m = 1, 2$, with $z = (r/M - 1)$. Then using the Mathematica function:

```
Q1[x_] := LegendreQ[2, 1, x]
Q2[x_] := LegendreQ[2, 2, x]
```

We obtain:

$$\begin{aligned} Q1 &= \frac{\sqrt{1-x^2}\,(-2+3\,x^2)}{-1+x^2} - \frac{3\,x\,\sqrt{1-x^2}\,\log(\frac{1+x}{1-x})}{2} \\ Q2 &= \frac{(1-x^2)\,(5\,x-3\,x^3)}{(-1+x^2)^2} + \frac{3\,(1-x^2)\,\log(\frac{1+x}{1-x})}{2} \end{aligned}$$

The Hartle form of these are obtained by a transformation rule of $Q(-x)$:

```
HQ1[x_] := Sqrt[x^2-1] ((3 x^2-2)/(x^2-1) - 3/2 x Log[(x+1)/(x-1)])x
HQ2[x_] := 3/2 (x^2-1) Log[(x+1)/(x-1)] - (3x^3-5x)/(x^2-1)
```

Substituting in the z's we then have:

$$\begin{aligned} Q1 &= (Sqrt[((2M-r)r)/M^2]((2(M^2-6Mr+3r^2))/(r(-2M+r)) + (3-(3r)/M)L[r]))/2 \\ Q2 &= -((2M^3+4M^2r-9Mr^2+3r^3)/(M(2M-r)r)) + (3(2M-r)rL[r])/(2M^2) \end{aligned}$$

$$\begin{aligned} HQ1 &= Sqrt[(r(-2M+r))/M^2](3 + M^2/(-2Mr+r^2) + (3(M-r)Log[r/(-2M+r)])/(2M)) \\ HQ2 &= (2M^3+4M^2r-9Mr^2+3r^3)/(M(2M-r)r) - (3(2M-r)rLog[r/(-2M+r)])/(2M^2) \end{aligned}$$

GnuPlot Files

As an example we show the file used to plot q Vs. j. The others are very similar.

```
%# Simple gnuplot command file.
%reset
%clear
%set terminal x11
%set size    .75,.75
%set xlabel  '$cJ/GM^2$'
%set ylabel  '$q\ [c^2 Q M_G/J^2]$' 0, 1.5
%set ytics border nomirror norotate
%set y2tics border nomirror norotate (1.73,4.63,5.32)
%set key left top Left samplen 2 width 2 box -1 reverse title
%#=== Defining the constants ================================
%a  = 1.0e42
%c  = 2.9979e10
%G  = 6.673e-8
%Ms = 1.988e33
%K = a*(c**4)/(G**2*Ms**3)
%#=== Some different plots...================================
%#  rho_c  M  M_0  R_e  Omega  omega_FD  j  q  e
%#   $1    2   3    4     5      6       7  8  9
%#-----------------------------------------------------------
%plot [0.0:0.8][0.0:10] 'DatB' using ($7/($2**2)):($8*$2/($7**2)*K)
% title "eosB" with lines
%replot 'DatUU' using ($7/($2**2)):($8*$2/($7**2)*K) title "eosUU" with lines
%replot 'DatW'  using ($7/($2**2)):($8*$2/($7**2)*K) title "eosW"  with lines
%replot 1.73 notitle with dots 0
%replot 4.63 notitle with dots 0
%replot 5.32 notitle with dots 0
%set terminal pstex color rotate
%set output  "figC2.2.q.tex"
%replot
```

Appendix C

This appendix belongs to chapter 3. It is needed in order to decipher some of the conformally related equations. Further I provide the formulas for the 6 most common energy momentum pseudotensors, all in our notation. This is followed by a short piece of Mathematica code to calculate the conformally related Ricci scalar, which was used in one of the examples.

Conformal Identities

An excellent reference for identities about conformal transformations can be found in Wald's book [24, App. D]. For convenience, we here restate the main identities in their original form as well as in the way we use them. In addition we also give the two additional relations; $\nabla_a \nabla_b \sigma$ and $\nabla_a u^b$. When using those identities in this thesis, however, one has to switch notation, since we are interested in expressing the quantities in $(\mathcal{M}, g_{ab})$ in terms of those in $(\mathcal{M}, \tilde{g}_{ab})$. For this conversion, it may be useful to know that

$$\nabla_a(\ln \sigma^{-1}) = -\sigma^{-1}\nabla_a \sigma \tag{C.1}$$

$$\nabla_a \nabla_c(\ln \sigma^{-1}) = \sigma^{-2}\nabla_a \sigma \nabla_c \sigma - \sigma^{-1}\nabla_a \nabla_c \sigma \tag{C.2}$$

$$\nabla_a(\ln \sigma^{-1})\nabla_c(\ln \sigma^{-1}) = \sigma^{-2}\nabla_a \sigma \nabla_c \sigma \tag{C.3}$$

Restated in another way, the covariant derivatives $\tilde{\nabla}_a$ in $(\mathcal{M}, \tilde{g}_{ab})$ and ∇_a in $(\mathcal{M}, g_{ab})$ differ by a linear transformation of the objects on which they act. Thus for scalars, vectors and one-forms we have

$$\nabla_a \sigma = \tilde{\nabla}_a \sigma \tag{C.4}$$

$$\nabla_a u^b = \tilde{\nabla}_a u^b + \tilde{C}^b{}_{ac}\, u^c \tag{C.5}$$

$$\nabla_a \omega_b = \tilde{\nabla}_a \omega_b - \tilde{C}^c{}_{ab}\, \omega_c \,. \tag{C.6}$$

respectively, while the derivatives of tensors of different rank is obtained from the last two relations by using the Leibniz rule. The tensor $\tilde{C}^c{}_{ab}$ can be expressed as

$$\begin{aligned} \tilde{C}^c{}_{ab} &= \frac{1}{2} g^{cd} \left(\tilde{\nabla}_a g_{db} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab} \right) \\ &= \frac{1}{\sigma} \left(\delta^c{}_a \tilde{\nabla}_b \sigma + \delta^c{}_b \tilde{\nabla}_a \sigma - \tilde{g}_{ab}\, \tilde{g}^{cd}\, \tilde{\nabla}_d \sigma \right) \,. \end{aligned} \tag{C.7}$$

Then

$$\nabla_a \nabla_b \sigma = \tilde{\nabla}_a \tilde{\nabla}_b \sigma - \frac{2}{\sigma} \tilde{\nabla}_a \sigma \tilde{\nabla}_b \sigma + \frac{1}{\sigma} \tilde{g}_{ab} \tilde{g}^{cd} \tilde{\nabla}_c \sigma \tilde{\nabla}_d \sigma \tag{C.8}$$

$$\nabla_a u^b = \sigma^{-2} \left(\sigma\, \tilde{\nabla}_a \tilde{u}^b - \delta^b{}_a \tilde{u}^c \tilde{\nabla}_c \sigma - \sigma\, \tilde{u}_a \tilde{g}^{bc} \tilde{\nabla}_c \sigma \right) \,. \tag{C.9}$$

Using these we easily obtain the conformal relations for R_{ab}, R, K and G^{ab} in the two conformally related spacetimes.

$$R_{ab} = \widetilde{R}_{ab} + \sigma^{-2}\left(4\,\widetilde{\nabla}_a\sigma\widetilde{\nabla}_b\sigma - 2\,\sigma\widetilde{\nabla}_a\widetilde{\nabla}_b\sigma - \widetilde{g}_{ab}\left[\sigma\,\widetilde{\Box}\sigma + \widetilde{g}^{cd}\widetilde{\nabla}_c\sigma\widetilde{\nabla}_d\sigma\right]\right) \tag{C.10}$$

$$R = \sigma^{-3}\left(\sigma\,\widetilde{R} - 6\,\widetilde{\Box}\sigma\right) \tag{C.11}$$

$$K = k^a{}_b\nabla_a u^b = \sigma^{-2}\left(\sigma\,\widetilde{K} + 3\,\widetilde{u}^a\widetilde{\nabla}_a\sigma\right) \tag{C.12}$$

$$G_{ab} = \widetilde{G}_{ab} - \frac{2}{\sigma}\,\widetilde{\nabla}_a\widetilde{\nabla}_b\sigma + \frac{2}{\sigma}\,\widetilde{g}_{ab}\,\widetilde{\Box}\sigma + \frac{4}{\sigma^2}\,\widetilde{\nabla}_a\sigma\widetilde{\nabla}_b\sigma - \frac{1}{\sigma^2}\,\widetilde{g}_{ab}\,\widetilde{g}^{cd}\,\widetilde{\nabla}_c\sigma\widetilde{\nabla}_d\sigma\,. \tag{C.13}$$

where we have used the D'Alembertian definition: $\widetilde{\Box} := \widetilde{g}^{ab}\widetilde{\nabla}_a\widetilde{\nabla}_b$, with implied dummy indices.

Energy Momentum Pseudotensors

The six most commonly used pseudotensors [131].

- Einstein
  $\Theta_a{}^b = \frac{1}{16\pi}\left\{\frac{g_{ae}}{\sqrt{-g}}\left[-g\left(g^{be}g^{cd} - g^{ac}g^{bd}\right)\right]_{,d}\right\}_{,c}$
- Landau-Lifshitz [p.280]
  $L^{ab} = \frac{1}{16\pi}\left[-g\left(g^{ab}g^{cd} - g^{ac}g^{bd}\right)\right]_{,cd}$
- Möller
  $\mathcal{S}_a{}^b = \frac{1}{8\pi}\chi_a{}^{bc}{}_{,c}$ with local conservation law: $\frac{\partial\mathcal{S}_a{}^b}{\partial x^b} = 0$
  $\chi_a{}^{bc} = \sqrt{-g}\left[g_{ae,d} - g_{ad,e}\right]g^{bd}g^{ec}$
- Papapetrou
  $\Sigma^{ab} = \frac{1}{16\pi}\left[\sqrt{-g}\left(g^{ab}\eta^{cd} - g^{ac}\eta^{bd} + g^{cd}\eta^{ab} - g^{cb}\eta^{ad}\right)\right]_{,cd}$
- Tolman [p.227]
  $\mathcal{T}_a{}^b = \frac{1}{8\pi}\left\{\sqrt{-g}\left[-g^{eb}V_{ae}{}^c + \frac{1}{2}\,g_a{}^b g^{ed}V_{ed}{}^c\right]\right\}_{,d}$
  $V_{fb}{}^a := -\Gamma^a{}_{fb} + \frac{1}{2}\,g_f{}^a\Gamma^d{}_{db} + \frac{1}{2}\,g_b{}^a\Gamma^d{}_{df}$
- Weinberg [p.165]
  $W^{ab} = \frac{1}{16\pi}\left[\frac{\partial h^d_d}{\partial x_c}\eta^{ab} - \frac{\partial h^d_d}{\partial x_a}\eta^{cb} - \frac{\partial h^{dc}}{\partial x^d}\eta^{ab} + \frac{\partial h^{da}}{\partial x^d}\eta^{cb} + \frac{\partial h^{cb}}{\partial x_a} - \frac{\partial h^{ab}}{\partial x_c}\right]_{,c}$

Mathematica Code

The Mathematica code used to calculate the Ricci scalar $\widetilde{R}$ for the interior Schwarzschild spacetime, requires the additional (free) package *TTC* [165].

```
<<~/TTC/ttc.m
Compact[Off];
InputCoordinates[sph, {t, r, th, ph}]
e = ZZ[sph];
g = - e[t,t] + S[r]^(-2) B[r]^2 e[r,r]
             + S[r]^(-2) r^2 (e[th,th] + Sin[th]^2 e[ph,ph]);
```

```
InputMetric[ogIS, sph, g]
InputSimplifyLevel[4];
InputTTCSimplify[{Expand, Factor}]
Curvature[ogIS, sph]
FullSimplify[%]
```

The Superpotential

$$
\begin{aligned}
\sqrt{-\gamma}\psi_{\sigma}{}^{\tau\lambda} &= -4\frac{\partial L}{\partial h^{\alpha\beta}{}_{;\rho}}h^{\phi\beta}\frac{\partial C^{\alpha}{}_{\phi\rho}}{\partial \gamma^{\sigma\nu}{}_{,\lambda}}\gamma^{\nu\tau} - 2\frac{\partial L}{\partial h^{\alpha\sigma}{}_{;\lambda}}h^{\tau\alpha} \\
&= \left(\frac{\partial L}{\partial h^{\sigma\beta}{}_{;\tau}}h^{\lambda\beta} - \frac{\partial L}{\partial h^{\sigma\beta}{}_{;\lambda}}h^{\tau\beta}\right) + \frac{\partial L}{\partial h^{\alpha\beta}{}_{;\phi}}(h^{\lambda\beta}\gamma^{\alpha\tau} - h^{\tau\beta}\gamma^{\alpha\lambda})\gamma_{\phi\sigma} \\
&\quad +\gamma_{\phi\sigma}h^{\phi\beta}\left(\frac{\partial L}{\partial h^{\alpha\beta}{}_{;\lambda}}\gamma^{\alpha\tau} - \frac{\partial L}{\partial h^{\alpha\beta}{}_{;\tau}}\gamma^{\alpha\lambda}\right)
\end{aligned}
\tag{C.14}
$$

It is now clear that $\psi_{\sigma}{}^{\tau\lambda} = -\psi_{\sigma}{}^{\lambda\tau}$.

Dictionary

This glossary is intended to provide a short reference for the reader, rather than an in-depth explanation of each term.

Analytic Extension Suppose that we started with a spacetime $(\mathcal{M}, g)$, and discovered that this actually was only a portion of a larger spacetime $(\bar{\mathcal{M}}, \bar{g})$, with $\bar{\mathcal{M}} \supset \mathcal{M}$, and $\bar{g}_{|\mathcal{M}} = g$. Then, we consider $(\bar{\mathcal{M}}, \bar{g})$ as the spacetime. This procedure can be continued until one encounters a true singularity, and is called analytic extension of $(\mathcal{M}, g)$. Kruskal spacetime is the maximal analytic extension of Schwarzschild; it is a solution of Einstein's field equation in a vacuum.

B-balls are Q-balls where the conserved charge is carried by the baryon number. Furthermore they are thin-walled, which implies [101] it has no upper limit of charge.

Bag model A model of QCD as a two phase vacuum with one phase inside a *bag* and the other outside, where the ordinary gluon-free vacuum is outside and the other inside phase contains freely propagating gluons. The idea is that the two phases of vacuua differ by an energy B per unit volume, plus other terms of the shape tensions. The energy of the bag is then $E = BV + \gamma S - Z/R$. Where Z contains a collection of shape dependent terms. The most popular model is the MIT bag with $\gamma = 0$.

Casimir effect is the effect of degenerate vacuua because of the creation of different phases due to the presence and non-presence of external fields or matter. There are two different types

- Static: The externally applied fields varies slowly in time $1 \gg \frac{\dot{\omega}}{\omega}$.
- Dynamic: The fields are changing rapidly and may produce particles.

Calculations of Casimir energy usually involves the index of refraction of the media $n := kv/\omega = (\epsilon\mu)^{1/2}$.

Chemical potential Just as temperature governs the flow of energy between systems, the chemical potential governs the flow of particles. The chemical potential is equivalent to a true potential energy. In terms of the thermodynamic potentials: $\mu(T, V, N) := (\frac{\partial F}{\partial N})_{T,V} = -\frac{1}{T}(\frac{\partial S}{\partial N})_{U,V} = (\frac{\partial U}{\partial N})_{S,V}$, where the derivative subscripts denote what is held constant.

Chirality Chirality appears when considering handedness of spinors when $m = 0$. In fact "chiral" is the greek word for "hand" as in "chiropractor". The chiral transformation is $\psi \to \gamma_5\psi$, where all $m = 0$ particles are γ_5 eigenstates. A soliton solution obeying this transformation is called a *Skyrmion*. Also in literature, chiral matter is used to denote baryon matter in which the nucleons are assumed to be massless in the interior.

Conformal Factor is the factor Ω used in a conformal transformation.

Conformally Flat is any metric which can be reduced by a conformal transformation, $g_{ab} = \Omega^2 \eta_{ab}$, such that η_{ab} is a flat metric.

Conformal Transformation

If we let $(\mathcal{M},\ g_{ab})$ be a spacetime, and $\Omega : \mathcal{M} \to I\!R$ be a nowhere vanishing, smooth function on $\mathcal{M}$, then the metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$ is said to be conformally related to g_{ab}. The replacement $(\mathcal{M},\ g_{ab}) \to (\mathcal{M},\ \tilde{g}_{ab})$ is called a conformal transformation (CT). A CT has the following properties:

Vectors: Let (X, Y) be two vectors at $P \in \mathcal{M}$, then
(1) In general the norm of a vector is altered by a CT.
(2) Angles, when they can be defined, are *not* altered.
(3) Null vectors remain null vectors. In particular, lightcones and the causal structure are the same for $(\mathcal{M},\ g)$ and $(\mathcal{M},\ \tilde{g})$.
(4) If we can find an Ω such that $\Omega(\infty) = 0$, we can represent $(\mathcal{M},\ g)$ in a compact region, without losing information about causal relations.

Conformally Static is any spacetime satisfying $\tilde{\nabla}_a \tilde{n}^b = 0$, where $\tilde{\nabla}_a$ is the Riemannian connection associated to $\tilde{g}_{ab}$.

Conformal curvature coupling is really the true "minimal" coupling, since it is needed in all metric theories of gravity in order to preserve the Einstein equivalence principle (EEP) and thus keep massive particles from propagating outside the lightcone. It is given by $\xi = \frac{1}{6}$.

Connection

Given a derivative operator ∇_a, we can define the notion of parallel transport of a vector along a curve C. We can then identify the tangent spaces T_p and T_q of points p and q given the derivative operator and a curve *connecting* p and q. The mathematical structure arising from such a curve dependent identification of the tangent spaces of different points is called the connection. Conversely one could start with the general notion of a connection and use it to define the derivative operator. We only use Riemannian connections, which implies that the Christoffel symbol, $\Gamma^c{}_{ab} = \frac{1}{2} g^{cd}[\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}]$.

Dynamical symmetry breaking is a symmetry that breaks due to an asymptotically free gauge theory becoming strong. The scale of the symmetry breaking is naturally much smaller than the fundamental energy scale.

Euler-Lagrange equations are defined by the Lagrange derivative being set equal to zero. For m dynamical variables $y_a(x)$ on an n-dimensional manifold

$$\frac{\delta L}{\delta y_a} := \frac{\partial S}{\partial y_a} - \left(\frac{\partial S}{\partial y_{a,\lambda}}\right)_{,\lambda} = 0 \qquad (a = 1, 2, \ldots, m)$$

Fermi-energy is the energy of the highest filled orbital at absolute zero temperature, the energy below, which there are just enough orbitals to hold the number of particles assigned to the system. $\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$, where n is the particle number density, N/V.

Fiber bundles provide the proper framework for the geometric interpretation of *gauge fields*. An elementary view is that gauge fields (A_μ) describe parallel transport in charge space with the field intensity ($F_{\mu\nu}$) being identified with the curvature tensor.

Flat potentials are potentials $U(\phi)$ that grow more slowly than the square of the scalar VEV ϕ. They arise naturally in theories with low-energy supersymmetry breaking.

Fock space is equivalent to a Hilbert space in the *particle number representation* of a second quantized theory.

Frobenius Condition is satisfied when: $\eta_{[a}\nabla_b\eta_{c]} = 0$.

Goldstone bosons See *Goldstone theorem*

Goldstone theorem: Spontaneous breaking of a continuous *global* symmetry is always accompanied by the appearance of one or more massless scalar particles, called *Goldstone bosons*. However, it may be evaded in gauge theories, since its validity requires: Lorentz invariance and a positive norm.

Instanton is not a particle, but rather represents vacuum fluctuations of gauge fields which may lead to observable effects on nearby particles.

Killing Time is defined as the function t such that $\eta^a\nabla_a f = \partial f/\partial t$, for any differentiable function f on $\mathcal{M}$.

Killing Vector Field

Is a vector field η^a that satisfies the following:

(1) $\pounds_\eta\, g_{ab} = 0$ (Vanishing of the Lie derivative)

(2) $\nabla_{[a}\eta_{b]} = 0$ (The Killing equation).

In fact these are equivalent if the *connection* is Riemannian.

L-balls are Q-balls in which the conserved charge are Lepton number. They have a maximum charge and a fixed radius.

Lagrange multiplier By introducing a constraint equation, one can obtain local solutions to a system of equations containing more than one maximum or minimum within its domain. The additional artificially introduced variable is called the Lagrange multiplier, λ. The Lagrange equations are:

$$\begin{aligned} \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial c_j}{\partial x_i} &= 0 \\ c_j &= k_j \end{aligned}$$

Legendre transformation is used to determine the relationship between the Euler-Lagrange equations and a first order partial differential equation. Thus it is simply the procedure for replacing the Lagrangian by the Hamiltonian by introducing the new variables p_i according to:

$$\begin{aligned} p_i &= \frac{\partial L}{\partial \dot{q}_i} \qquad (i = 1, ..., n) \\ H &= \sum_{i=1}^{n} p_i \dot{q}_i - L \end{aligned}$$

Then we can solve for: $\dot{q}_i = f(q_i, p_i, t)$ and use this to define $H := H(q_i, p_i, t)$.

Noethers Theorem Each continuous symmetry transformation leads to a conservation law. The conserved quantity Q (the Noether charge) can be obtained from the Lagrange density using eq.(1.37) and then performing a volume integration.

Optical Geometry can be considered as $(\mathcal{M}, \tilde{g})$.

Optical Space can be considered as $(\mathcal{M}, \tilde{h})$.

Skyrmion is a *chiral* soliton and is the most popular model of topological solitons. The Skyrme (Hedgehog) ansatz is the unitary matrix defined by:

$$U(\mathbf{r}) := \mathrm{e}^{iF(r)\,\tau\cdot\hat{\mathbf{r}}}$$

where $F(r)$ is the Skyrmion profile function which is determined by the differential equation obtained from the variation $\delta E/\delta F = 0$. The Skyrme model does not represent a rigorous model of QCD, because it uses chiral Lagrangian's outside the validity of the energy expansion [16].

Solitons are solutions of the Euler-Lagrange equations which are localized (spatially finite extension - thus the "-on" in the name) and stable with finite energy. The stability is due to a conserved current, which in turn gives a conserved charge known as the *topological charge* (Q), because of its topological invariance. Hence there are different classes of solitons depending on whether $Q = 0, \pm 1$. The former are known as *non-topological* solitons. The existence criteria for the latter *topological* solitons are:

1. More than one vacuum state (degenerate)
2. Different boundary conditions at infinity

In a model with $O(3)$ symmetry, we can identify Q with the *winding number*, $\nu \in \mathbb{Z}$, which is also known as the *Pontryargin index* that characterizes the homotopic class. Furthermore it should be noted that *geons* are simply gravitational solitons.

Sphaleron Unstable field configurations of finite energy. The word derives from the Greek word *sphaleros* meaning unstable. Essentially it is like an unstable classical particle.

Static

The condition for staticity is satisfied when there exists a vector field, η^a that is both *Stationary* and *Frobenius*.

Stationary

A spacetime in which there exist a timelike Killing vector field, η^a, with coordinates such that $\eta = \partial_t$ and $\frac{\partial g_{\mu\nu}}{\partial t} = 0$.

Symmetries of a theory are related to the transformation properties of the fields. Thus a symmetry describes what is needed in order that the Lagrangian remains invariant during a change of gauge. In fact *gauge symmetry* is used to denote a local symmetry which means that the fields transform differently at different spacetime points. These gauge symmetries can then generate the dynamics, the gauge interactions. The most common types of these phase transformations are:

1. Global $\psi(x) \to e^{i\theta}\,\psi(x)$

2. Local $\psi(x) \to e^{i\theta(x)}\,\psi(x)$
   - Abelian: No gauge invariant self coupling. (QED)
   - Non-Abelian: Self coupling and highly non-linear. (Yang-Mills)
3. Discrete $\to$ Finite number of ground states.
4. Continuous $\to$ Infinite number of ground states.

The most simple way to impose local gauge invariance of a given field is to replace all partial derivatives (∂_μ) with the gauge covariant derivatives (D_μ) in the Lagrangian. The covariant derivative in QFT is defined as $D_\mu := \partial_\mu + iqA_\mu$ in terms of a *gauge field* A^μ. Thus in QED, $q = e$, whereas in Yang-Mills theory it is defined as $D_\mu := \partial_\mu - \frac{1}{2}ig\tau \cdot A_\mu$.

Thermodynamic potential of a system is the number of possible *microstates* it can support, the number of different ways in which the total energy E of the system can be distributed among the N particles constituting it. $\Omega = \Omega(N, V, E)$

Thomas-Fermi approximation Given a massive scalar field that varies slowly enough (compared with the fermion Compton wavelength) that one can assume that there is a (spherical) Fermi-sea of plane-wave states at each point in space. Thus for a system of many fermions we can make the approximation that $\frac{dm}{dr} \ll m^2$.

Ultra Static

A spacetime $(\mathcal{M}, g_{ab})$ is called *ultrastatic* iff it has a hypersurface-orthogonal timelike Killing vector field η^a, such that $g_{ab}\eta^a\eta^b = -1$. Thus, a spacetime is ultrastatic iff it is static and, in addition, the timelike Killing vector field has unit norm [140]. These spacetimes all have: $g = -dt^2 + h_{ij}dx^i dx^j$, $\frac{\partial h_{ij}}{\partial t} = 0$, where $i, j = 1, 2, 3$.

Q-axiton is stupid name for a q-ball in a higher energy state.

Quasi particles The Pauli exclusion principle states that fermions occupy distinct quantum states, hence at zero temperature the ground state consists of filled levels up to some energy (the Fermi surface). The lowest energy excitations of this system are called *quasi particles*. These are states just above the Fermi surface.

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[195] Latin letters a, b, ... from the beginning of the alphabet are used as abstract indices (see, e.g., Ref. [24], pp. 24–25), which just indicate the tensorial nature of an object without requiring the specification of a coordinate system. Greek letters μ, ν, ..., and latin letters from the middle of the alphabet, i, j, ..., run from 0 to 3 and from 1 to 3, respectively, and denote components in some chart. We choose +2 as signature of the metric and work in units in which $c = 1$. The conventions for the curvature tensors are those of Ref. [24].

[196] The conversion to the more common notation $\widetilde{g}_{ab} = \Omega^2 g_{ab}$ is, of course, $\Omega = 1/\sigma$. Hereafter, indices of geometrical objects pertaining to the spacetime $(\mathcal{M}, \widetilde{g}_{ab})$ are lowered and raised using $\widetilde{g}_{ab}$ and its inverse $\widetilde{g}^{ab}$. Thus we have, for example, $\widetilde{X}^a := \widetilde{g}^{ab}\widetilde{X}_b$. Also, $\Box := g^{ab}\nabla_a\nabla_b$ and $\widetilde{\Box} := \widetilde{g}^{ab}\widetilde{\nabla}_a\widetilde{\nabla}_b$ denote the D'Alembertian operator in the two spacetimes.

[197] Of course, the total number of degrees of freedom is the same in the two descriptions. In general relativity, we have the ten variables associated with the independent coefficients of the metric (which could be reduced to six by exploiting gauge freedom). In optical geometry, one of these is "frozen" (in conformally static spacetimes, one chooses σ such that $\widetilde{g}_{00} = -1$), so $\widetilde{g}_{ab}$ has only nine non-trivial components, the tenth variable being represented by σ.

[198] This is related to the vanishing of $\widetilde{K}_{ab}$. Indeed, if η^a is a Killing vector field of unit norm, then $\widetilde{K}_{ab} = \widetilde{h}_a{}^c\, \widetilde{\nabla}_c \eta_b = \widetilde{\nabla}_a \eta_b + \eta_a \eta^c\, \widetilde{\nabla}_c \eta_b = \widetilde{\nabla}_a \eta_b - \eta_a \eta^c\, \widetilde{\nabla}_b \eta_c = \widetilde{\nabla}_a \eta_b$.