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Boundedness problem for semistable G-bundles in positive characteristic

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Contents

Introduction

One of the main problem in the construction of moduli spaces using geometric invariant theory is to prove the boundedness of the set that we want to parametrize.

This thesis is devoted to the proof of boundedness of the set of semistable G-bundles (with given Chern classes) on a projective algebraic variety over a field of positive characteristic and is based on [11]. We give an overview of the main results and of the techniques use to prove them.

Let G be a connected reductive algebraic group over an algebraically closed field k of arbitrary characteristic. Let X be a smooth projective variety over k with a fixed polarization H. In this paper we address the question of what happens to semistability of principal G-bundles under the extension of structure groups.

Recall the definition of a rational G-bundle E as a principal G-bundle over a big open subscheme (whose complement is of codimension at least 2). A rational G-bundle E over X is semistable with respect to the polarization H if for any reduction to a parabolic subgroup P of G over any big open subscheme, the line bundle associated to any dominant character on P has degree ≤ 0.

One notes that restrictions of torsion free sheaves to suitable open sets define rational $\mathrm{GL}(V)$-bundles and in this case the above definition of semistability coincides with usual μ-semistability.

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a representation of G on a vector space V which sends the connected component of the center of G to that of $\mathrm{GL}(V)$. For any rational G-bundle E we denote by $E(V)$ the associated rational vector bundle.

When the characteristic of the field is zero, it is proved in [36] that the bundle $E(V)$ is semistable. If the characteristic of the field is a prime p which is sufficiently large (quantified by the height of the representation) then the semistability of $E(V)$ is proved in [29].

In the case of arbitrary characteristic it is known that the bundle $E(V)$ need not be semistable. Let $\mu_{\max}(E(V))$ (and $\mu_{\min}(E(V))$) be the slopes of the first (and the last) term in the Harder-Narasimhan filtration of $E(V)$. We prove the following theorem.

Theorem 0.0.1. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation which sends the connected component of the center of G to that of $\mathrm{GL}(V)$. Then there exists a constant $C(X\,,\,\rho)$ (depending only on X and ρ) such that for each rational semistable G-bundle E over X we have*

$$\mu_{\max}(E(V)) - \mu_{\min}(E(V)) \leq C(X\,,\,\rho)$$

We briefly describe the proof. Let $E(G)$ be the group scheme associated to E and $E(G)_0$ be the group scheme at the generic point $\mathrm{Spec}(k(X)) \to X$. Let P be a maximal parabolic subgroup of $\mathrm{GL}(V)$ and let σ be a rational reduction of structure group of $E(\mathrm{GL}(V))$. There is an action of $E(G)_0$ on the smooth projective variety $E(\mathrm{GL}(V)/P)_0$ over $k(X)$ which is linearized by a suitable line bundle. The section σ gives a $k(X)$-valued point σ_0 of $E(\mathrm{GL}(V)/P)_0$. It is known that if σ_0 is a semistable point for the above action then the reduction σ does not violate the semistability of $E(\mathrm{GL}(V))$ (see Proposition (1.3.8)). Also, if σ_0 is not semistable and its instability parabolic $P(\sigma_0)$ (see Section 3 for the definition) is defined over the function field $k(X)$ of X then again σ does not violate the semistability of $E(\mathrm{GL}(V))$ (see Proposition (1.3.9)). This argument in characteristic zero proves that $E(V)$ is semistable because $P(\sigma_0)$ is defined over $\overline{k(X)}$ and by its uniqueness it is invariant by the Galois group hence it is defined over $k(X)$.

In the case of characteristic p one of the important points in our proof is to show that there is an integer N (independent of the G-bundle E) such that if σ_0 is not semistable then its instability parabolic is defined over the field $k(X)^{p^{-N}}$ (see Proposition 2.1.5). This part is achieved by repeated use of an algebraic result which enables us to get uniform bounds for non-reducedness of fibers of morphisms of algebraic varieties (see Proposition 2.1.2).

Once the instability parabolic is defined over $K^{p^{-N}}(X)$, this parabolic gives rise to a reduction of structure group of the Frobenius pull-back $(F^N)^*E$. Using this reduction and some geometric invariant theory arguments, we reduce the problem to proving the following Theorem which bounds instability of Frobenius pull-backs.

Theorem 0.0.2. *There exists a constant $C(X\,,\,G)$ and a constant $N(G)$ such that for any rational G-bundle E we have*

$$\mathrm{Ideg}(F^*E) \geq p\,N(G)\,\mathrm{Ideg}(E) + C(X\,,\,G)$$

(The instability degree Ideg is defined by equation (1.1) (in Section 2).

In the case of vector bundles, the above result was proved by X. Sun (see [43]) and Shephard-Barron (see [42]).

We use the Theorem (0.0.1) for groups of lower semisimple rank to prove the Theorem (0.0.2). In fact we prove a generalization of Theorem (0.0.1) where we replace $\mathrm{GL}(V)$ by an arbitrary reductive group and (0.0.2) will then be a special case of this result (see Remark (2.2.5)).

Let $\widetilde{c}_i \in A^*(X)$ for $i = 1 \ldots n$ be elements with $n = \dim(X)$. Let $S_b(r\,; \widetilde{c}_1 \ldots, \widetilde{c}_n)$ be the set of isomorphism classes of torsion free sheaves V of rank r and $c_i(V) = \widetilde{c}_i$ satisfying $\mu_{\max}(V) - \mu_{\min}(V) \leq b$.

Let $c_i \in A^i(X)$ for $2 \leq i \leq n$ be fixed. We also fix a homomorphism $d \in \mathrm{Hom}(\mathcal{X}(G)\,, A^1(X))$. Here $A^k(X)$'s are the Chow groups and $\mathcal{X}(G)$ is the group of characters.

In the last section, we use the above results to show the following result on boundedness of semistable G-bundles (which are defined on all of X).

Theorem 0.0.3. *Assume that the set $S_b(r\,; \widetilde{c}_1, \ldots, \widetilde{c}_n)$ is bounded for all choices of $\widetilde{c}_i$, b and r. Then the set $S_G(d\,; c_2, \ldots, c_n)$ of isomorphism classes of semistable G-bundles $\{E\}$ with degree $d_E = d$ and $c_i(\mathrm{ad}(E)) = c_i$ is bounded.*

Here the degree of a principal bundle E is an element $d_E \in \mathrm{Hom}(\mathcal{X}(G)\,, A^1(X))$ defined by $d_E(\chi) = c_1(\chi_*(E))$ for any character χ of the group G. Here $\chi_*(E)$ is the line bundle associated to E via χ.

In characteristic 0, the boundedness of the set $S_b(r\,; \widetilde{c}_1, \ldots, \widetilde{c}_n)$ is well known (see [20] for example). In the case of positive characteristic, for surfaces, this is due to Gieseker [12] and Maruyama [30]. For higher dimensional varieties this is recently claimed by Langer [28]. This along with Theorem (0.0.3) would then prove boundedness of semistable G-bundles over X with fixed Chern classes.

When X is a smooth projective curve in characteristic 0, the boundedness of the semistable G-bundles with fixed degree is due to Ramanathan [39]; in the case of positive characteristic, it is proved in [19].

The last part of this work contain some results on orthogonal and symplectic bundles. In particular we prove the existence and uniqueness of the Harder-Narasimhan reduction for G-bundles as defined by Ramanathan in [37] and we give a proof of Behrend conjecture [4] in these cases.

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Chapter 1

Preliminaries

1.1 Basic facts about semistable G-bundles

In this section we recall and prove some basic facts about principal G-bundles over varieties. Let k be an algebraically closed field. Let G be a connected reductive algebraic group over k. Let T be a maximal torus and B a Borel subgroup containing T. Let $R_u(B)$ be the unipotent radical of B. Then B is a semi-direct product $R_u(B) \cdot T$. We denote by $\mathcal{X}_*(T)$, the group of 1-parameter subgroups of T (denote by 1-PS). $\mathcal{X}^*(T)$ denotes the group of characters of T. We have a perfect pairing $\mathcal{X}_*(T) \otimes \mathcal{X}^*(T) \longrightarrow \mathbb{Z}$ which will be denoted by $(\cdot\,,\,\cdot)$. Let $\Phi \subset \mathcal{X}^*(T)$ be the set of roots of G, Φ^+ be the set of positive roots and Δ be the set of simple roots corresponding to B. For any $\alpha \in \Phi$, let T_α be the connected component of $\ker(\alpha)$ and Z_α the centralizer of T_α in G. Then the derived group $[Z_\alpha,\, Z_\alpha]$ is of rank one and there is a unique 1-PS $\hat{\alpha} : \mathbb{G}_m \longrightarrow T \cap [Z_\alpha,\, Z_\alpha]$ such that $T = (\operatorname{im}\hat{\alpha}) \cdot T_\alpha$ and $(\hat{\alpha},\, \alpha) = 2$. This $\hat{\alpha}$ is the coroot corresponding to α. We denote by $\hat{\Phi}$ the set of coroots. The quadruple $\{\mathcal{X}^*(T),\, \Phi,\, \mathcal{X}_*(T),\, \hat{\Phi}\}$ defines a root system. For each $\alpha \in \Delta$ we have the fundamental dominant weight $w_\alpha \in \mathcal{X}^*(T) \otimes \mathbb{Q}$ defined by $(\hat{\beta}, w_\alpha) = \delta_{\alpha,\beta}$ for $\beta \in \Delta$ and $(\gamma, w_\alpha) = 0$ for any 1-parameter group in the connected component of the center of G. Let $W = N(T)/T$ be the Weyl group. We fix a W-invariant inner product on $\mathcal{X}_*(T) \otimes \mathbb{Q}$ (hence on $\mathcal{X}^*(T) \otimes \mathbb{Q}$).

Let P be a parabolic subgroup of G containing B. Let U be its unipotent radical. Then there is a subset $\Pi \subset \Delta$ such that $P = P_\Pi$. Let $Z_\Pi = (\bigcap_{\alpha \in \Delta - \Pi} \ker\alpha)^0$ be the connected component of the intersection of kernels of roots in $\Delta - \Pi$. By taking the centralizer of Z_Π one obtains a splitting $P \to P/U = L$ with Z_Π being the connected component of the center of L.

Let X be a smooth projective variety over k of dimension n and let $k(X) = K$ be the function field of X.

We recall the definition of algebraic principal G-bundle (G-bundle in brief) introduced by Serre [41]. A principal G-bundle over X is a scheme E on which G acts on the right and a G-invariant isotrivial morphism $\pi : E \to X$. This means that for any $x \in X$ there exists an étale morphism $\varphi : X' \to X$ such that the pull-back $\varphi^* E$ is isomorphic to $X' \times G$.

Let $\pi : E \to X$ be a G-bundle. Let F be a quasi projective sheme on which G operates from the left. The group G acts on the product $E \times F$ by $g(e, f) = (eg, g^{-1}f)$, $g \in G$, $e \in E$, $f \in F$. We denote by $E(F)$ the associated bundle obtained by the quotient of $E \times F$ under the above action of the group G.

If G' is a subgroup of G we denote by E/G' the fiber bundle associated to E for the action of G on G/G' by left translations.

Let $\rho : G \to G'$ be an homomorphism of algebraic groups and let G act on G' by left multiplication $g \cdot g' = \rho(g)g'$, $g \in G$, $g' \in G'$. The associated bundle $E(H)$ has a natural structure of H-bundle and sometimes we will denote it by $\rho_* E$. We will say that $\rho_* E$ is obtained from E by **extension of the structure group**.

IF G acts on F_1 and F_2 and $F_1 \to F_2$ is a G-equivariant morphism then there is a natural morphism $E(F_1) \to E(F_2)$.

Let G' be an algebraic subgroup of G $\rho : G' \hookrightarrow G$. A pair (E', ϕ), where E' is a G'-bundle and ϕ is an isomorphism $\phi : \rho_* \to E$ of G-bundle is said to give a **reduction of structure group** of E to G'. Two reduction of structure group (E_1, ϕ_1) and (E_2, ϕ_2) are isomorphic if there is a G'-bundle isomorphism $\psi : E_1 \to E_2$ such that the following diagram commute:

$$\begin{tikzcd}
\rho_* E_1 \arrow[r] \arrow[dr] & \rho_* E_2 \arrow[d] \\
& E
\end{tikzcd}$$

The quotient E/G' is naturally isomorphic to the associated bundle$E(G/G')$. Further $E \to F/G'$ is a G'-bundle and a section $\sigma X \to E/G'$ gives the G'-bundle $\sigma^* E$ on X with a natural isomorphism $\rho_* \sigma^* E \simeq E$. Hence equivalence classes of reductions of structure group of E to G' are in bijective correspondence with section of $E/G' \to X$.

In the case of higher dimensional variety we introduced the following concept. We recall from [36] that a **rational G-bundle** E is a principal G-bundle over a big open subscheme of X. For $G = \mathrm{GL}(V)$ this defines a vector bundle over a big open subscheme and we call it rational vector bundle.

Let $E \longrightarrow U \subset X$ be a rational G-bundle with U a big open subscheme. By **rational reduction** of structure group σ of E to P we mean a reduction of structure group over a big open subscheme $U' \subset U$. More precisely it is a pair (E_σ, ϕ) with E_σ a P-bundle over U' and an isomorphism $\phi : E_\sigma(G) \longrightarrow E|_{U'}$. This is equivalent to giving a section σ of the fiber bundle $\pi : E/P \longrightarrow U$ over U'. Here E/P denotes the extended rational fiber bundle $E(G/P)$ over X.

Let T_π be the tangent bundle along the fibers of the map π. Then T_π is a rational vector bundle. For a reduction of structure group σ we will denote by T_σ the rational vector bundle defined by the pull-back of T_π under σ. We will also fix notations for the Lie algebras by putting $\mathfrak{g}$, $\mathfrak{p}$ and $\mathfrak{l}$ for the Lie algebras of G, P and L respectively. Then it can be verified that T_σ is the rational vector bundle on X associated to E_σ for the natural representation of P on $\mathfrak{g}/\mathfrak{p}$.

Let H be a fixed polarization on X. Since any line bundle $\mathcal{L}$ over a big open subscheme U (whose complement is of codimension at least 2) admits a unique extension to all of X, its first Chern class makes sense. Recall the definition of the degree of the line bundle to be $\deg(L) = c_1(L) \cdot H^{n-1}$. Hence for any torsion free sheaf its first Chern class and the degree with respect to H makes sense.

Recall the following definition of semistability from Ramanan-Ramanathan [36].

Definition 1.1.1. *A rational G-bundle $E \longrightarrow U \subset X$, with U a big open set is **semistable** with respect to polarization H if for any reduction of E to any parabolic subgroup P of G over any big open set $U' \subset U$, the line bundle associated to any dominant character on P has degree ≤ 0.*

This definition is equivalent to the fact that $\deg(T_\sigma) > 0$ for each rational parabolic reduction.

If V is a torsion free sheaf then over a big open set U the restriction $V|_U$ is a vector bundle. The above definition of semistability is equivalent to the μ-semistability of V.

Instability degree and Harder-Narasimhan reduction

If the rational G-bundle is not semistable then there is a notion of Harder-Narasimhan reduction which we recall here. For a rational G-bundle E which is not semistable we define the **instability degree** to be:

$$\mathrm{Ideg}(E) = \mathrm{Min}_{\{P,\sigma\}} \deg(T_\sigma) \tag{1.1}$$

where the minimum is taken over all parabolic subgroups P and rational reductions σ. If the rational G-bundle is semistable then we say its instability degree is 0. The following lemma shows that the instability degree makes sense, and is an analogue of Lemma 2.1 of [19] for the higher dimensional varieties.

Lemma 1.1.2. *There exists a constant A_E such that for any rational reduction σ of E to any parabolic P we have* $\deg(T_\sigma) > A_E$

Proof. It is enough to show that the degree of the rational vector subbundle $\mathrm{ad}(E_\sigma) \subset \mathrm{ad}(E)$ is bounded above.

We can first extend the bundle $\mathrm{ad}(E)$ to get a torsion free sheaf $\mathcal{E}$ on X. Then we can extend $\mathrm{ad}(E_\sigma)$ inside $\mathcal{E}$ to obtain a torsion free subsheaf. There exists a constant A'_E such that for any curve C in the class $|H^{n-1}|$, we have a bound $h^0(C, \mathcal{E}|_C) \leq A'_E$. Let g be the maximum of the genus of smooth curves in $|H^{n-1}|$. Now if C is a smooth projective curve which sits in the domain of definition of $\mathrm{ad}(E_\sigma)$ and $\mathrm{ad}(E)$ then we get $\deg(E_\sigma) \leq A'_E + (g-1)\,\mathrm{rank}(\mathrm{ad}(E))$. This proves the lemma. □

Definition 1.1.3. *A rational reduction of structure group σ of E to a parabolic P is said to be a* ***Harder-Narasimhan reduction*** *if* $\deg(T_\sigma) = \mathrm{Ideg}(E)$ *and P is maximal among parabolic subgroups of G containing B for which the above equality holds.*

The Harder-Narasimhan reductions as defined above satisfy the following properties stated in Ramanathan [37]. (see [5] for a proof).

1. If L is the Levi quotient of P, then the principal L-bundle $E_\sigma(L)$ obtained by extending the structure group is semistable;

2. After fixing a Borel subgroup $B \subset P$ of G, for any nontrivial character χ of P which is a non-negative linear combination of simple roots, the associated rational line bundle $\chi_*((E_\sigma))$ over X is of positive degree.

It is proved in Behrend [4] that over a smooth projective curve there is a unique reduction to a parabolic subgroup containing B satisfying the above properties. In the case when X is higher dimensional the uniqueness is known only when the characteristic of the field is 0 or it is a large prime p.

We will not have the occasion to use the uniqueness of the above reduction. We will only use its existence.

For the case $G = \mathrm{GL}(V)$ the above reduction defines the Harder-Narasimhan filtration and the uniqueness is then immediate. We have the following lemma which compares the instability degree with the $\mu_{\max} - \mu_{\min}$ of the rational vector bundle.

Lemma 1.1.4. *Let E be a rational principal* $\mathrm{GL}(V)$ *bundle of rank r over X which is not semistable (we will denote by $E(V)$, the associated vector bundle). Then we have the following.*

$$\mu_{\max}(E(V)) - \mu_{\min}(E(V)) \leq -\frac{2}{r^2}\,\mathrm{Ideg}(E)$$

Proof. For the proof one first notices that if $F \subset E(V)$ is a rational subbundle of rank r_1 and F_1 is the quotient, then it defines a rational reduction of structure group σ of E to a maximal parabolic P_1 of $\mathrm{GL}(V)$. One further has an isomorphism of rational bundles $T_\sigma \cong \mathcal{H}om(F\,,\,F_1)$.

This implies that $\mu(F)-\mu(F_1) = -\mu(\mathcal{H}om(F\,,\,F_1)) \leq -\mathrm{Ideg}(E)/(\,r_1\,(r-r_1))$.

This inequality can also be written by eliminating F_1 or F, we get $\mu(F)-\mu(E(V)) \leq -\mathrm{Ideg}(E)/\,(r\,r_1\,)$ and $\mu(E(V)) - \mu(F_1) \leq -\mathrm{Ideg}(E)/\,(r\,(r-r_1))$.

Now if we take F to be the rational subbundle which is maximal destabilizing then we have $\mu_{\max}(E(V)) = \mu(F)$. This implies that $\mu_{\max}(E(V)) - \mu(E(V)) \leq -\mathrm{Ideg}(E)/\,r^2$. Similarly one has $\mu(E(V)) - \mu_{\min}(E(V)) \leq -\mathrm{Ideg}(E)/\,r^2$. Combining these we have the proof of the lemma. $\square$

1.2 Frobenius map

Let k a field of characteristic $p > 0$ and $n > 0$ an integer and consider a scheme over k, $\phi : X \longrightarrow Spec(k)$. The p^n-th power map $\mathcal{O}_\mathcal{X} \longrightarrow \mathcal{O}_\mathcal{X}$ given by $f \longrightarrow f^{p^n}$ is a homomorphism and gives rise to a morphism $F_X : X \longrightarrow X$ called the absolute Frobenius morphism, but this morphism is not k-linear. Let $f_k : Spec(k) \longrightarrow Spec(k)$ be the morphism induced by $k \longrightarrow k$, if we call $X^{(1)} = F_k^* X$ we have the following diagram

$$\begin{tikzcd}
X \arrow[r, "F_X"] \arrow[d] & X^{(1)} \arrow[r, "A"] \arrow[d, "f_k^*\phi"] & X \arrow[d, "\phi"] \\
Spec(k) \arrow[r, equal] & Spec(k) \arrow[r, equal] & Spec(k)
\end{tikzcd}$$

If k is a perfect field f_k and A are isomorphism and we call $F_X : X \longrightarrow X^{(1)}$ the geometric Frobenius. In our case we are working with an algebraic close field hence perfect so we can use the geometric Frobenius instead of the absolute one's.

Let $\pi : E \to X$ be a G-bundle, pulling back by the Frobenius we get a G-bundle $F_n^*(E) \to X_F$ (where we take the k-structure on $F_n^*(E)$ to be the one defined by the composite $F_n^*(E) \to X_F \xrightarrow{f_k \cdot \phi} Spec(k)$). If k is a perfect field we can change the k-structure of $F_n^*(E)$, X_F and G by composing their structure morphisms with $f_k^{-1} : Spec(k) \to Spec(k)$ to get bundle $F_n^*(E) \to X$ with structure group $f_k^*(G)$ (in the diagram replacing X by G we see that A gives a k-isomorphism of f_k^*G with G, the latter having the k-structure changed by f_k^{-1}). Let a group scheme $\mathbf{G} \to Spec(\mathbb{F}_q)$ where $q = p^n$ over $\mathbb{F}_q$ such that $G = \mathbf{G} \times_{\mathbb{F}_q} Spec(k)$, then $f_k^*(G) = G$ and so the $f_k^*(G)$-bundle $F_n^*(E) \to X$ gives G-bundle.

One of the main problem which arise in the study of G-bundles over algebraic variety in positive characteristic is the fact that we loose the semistabilty under extension of the structure group. To be more precise if $\rho : G \to G'$ is a group homomorphism sending the connected component of the center of G to that of G' and E is a semistable G-bundle then the associated G'-bundle $E(G')$ could be non-semistable [13] (Recall that the Frobenius morphism is a particular representation). This phenomenon doesn't appear in characteristic zero as proven in [38], using a Narasimhan-Seshadri Theorem for principal bundles, and in [36] using a G.I.T. argument.

Remark 1.2.1 If we consider **strongly semistable** G-bundles i.e. such that $F^{n*}E$, where F is the Frobenius morphism, is semistable for any n than is proven in [36] that if if $\rho : G \to G'$ is a group homomorphism sending the connected component of the center of G to that of G' and E is a strongly semistable G-bundle then the associated G'-bundle $E(G')$ is semistable.

We want to prove a result about Frobenius morphism which is yet proved in [43] or [31].

Recall that a connection on a principal bundle is defined using the Atiyah sequence [1]. Take a principal G-bundle E over X we denote by π the projection of E on X. For any open subset U of X we consider the space of G-invariant vector fields on $\pi^{-1}(U)$ for the natural action of G on the fibers of π, this gives rise to a vector bundle on X called the Atiyah bundle and denoted by $At(E)$. Now we consider the adjoint bundle $ad(E)$ corresponding to the sheaf of G- invariant vertical vector fields on E and we have the following exact sequence of vector bundle on X:

$$0 \to ad(E) \to At(E) \to T_X \to 0$$

Definition 1.2.2. *A connection in E is a splitting homomorphism $T_X \longrightarrow At(E)$ of the exact sequence 1.2.1.*

As in the vector bundle case a flat connection is a homomorphism sheaf of Lie algebras and in particular a p-algebra homomorphism defines a flat p-connection.

Lemma 1.2.3. *Let E a principal G-bundle over X with a p-connection. Consider a reduction of the structure group $P \subset G$, if the map $T_X \longrightarrow T_\sigma$ (the second fundamental form) is zero then the connection descents to P.*

Proof. We give a proof using local coordinate systems. Let g_{ij} transiction function with value in P and α_j a connection one form with value in $\mathfrak{g}$ the Lie algebra of G then

$$\alpha_j - ad(g_{ij}^{-1})\alpha_i = g_{ij}^{-1}dg_{ij} \tag{1.2}$$

Put $\overline{\alpha}_j = \alpha_j mod(\mathfrak{p}) \in \mathfrak{g}/\mathfrak{p}$ in (1.2) we have:

$$\overline{\alpha}_j = ad(g_{ij}^{-1})\overline{\alpha}_i \tag{1.3}$$

because $g_{ij}^{-1}dg_{ij}$ it is with value in P so is zero mod($\mathfrak{p}$). This means that $\overline{\alpha}_j \in H^0(X, \Omega^1 \otimes \mathfrak{g}/\mathfrak{p})$ hence if this is zero then α_j is a connection form with value in the lie algebra of P.

Using [7] pag. 65 the connection defined over P is still a p-connection. □

Lemma 1.2.4. *Let $\alpha : X \longrightarrow Y$ be a Frobenius morphism of smooth schemes and $f : X \longrightarrow Z$ be a morphism of scheme such that $df = 0$ then there exist an unique map $\beta : Y \longrightarrow Z$ such that $f = \beta \circ \alpha$*

Proof. The problem is local so we can consider $X = \text{spec}(A), Y = \text{spec}(B), Z = \text{spec}(R)$ affine schemes. $df = 0$ if and only if Im$R \subset$ Ker$(d : A \longrightarrow \Omega_{A/k})$. So it enough to prove that Ker$(d : A \longrightarrow \Omega_{A/k}) = A^p$ and this follows from a Theorem due to Cartier (see (3.5) of [21]) applied to $Y = \text{Spec}(k)$ with k algebraic closed field. Uniqueness follows from the fact that $\alpha^* : B \longrightarrow A$ is injective.

□

The above Lemmas give the following Proposition.

Proposition 1.2.5. *Let E be a rational G-bundle over X. Then there exists a p-connection ∇ on the G-bundle $F^*(E)$ which satisfies the following property: for any rational reduction of structure group σ of $F^*(E)$ to a parabolic P there is a vector bundle map (second fundamental form) $\nabla_\sigma : T_X \to T_\sigma$ (wherever T_σ is defined) such that the following are equivalent.*

1. *There exists a rational reduction σ_0 of E to P such that $F^*(\sigma_0) = \sigma$.*

2. *∇_σ is zero.*

Proof. Consider the following diagram:

$$\begin{tikzcd} F^*E/P \arrow[r] \arrow[d, "\eta"] & X \arrow[d, "F"] \\ E/P \arrow[r] & X \end{tikzcd}$$

and let $\sigma : X \to F^*E/P$ be a section. Consider the following map

$$(\eta \circ \sigma) : X \to E/P$$

whose tangent map is

$$d(\eta \circ \sigma) : T_X \to (\eta \circ \sigma)^* T_{E/P} \subset \sigma^* T_{F^*E/P}$$

The fact that the second fundamental form is zero is equivalent to the fact that $d(\eta \circ \sigma)$ is zero and so we can apply lemma 1.2.4 to get our result. □

Remark 1.2.6 Related with the problem of descent of inseparable morphism Behrend in [4] gives the following Conjecture:

Proposition 1.2.7 (Conjecture). *Let E be a G-bundle over X and let σ be the Harder-Narasimhan reduction of E then:*

$$H^0(X\,, T_\sigma) = 0$$

The above conjecture is proven in [27] in characteristic zero and is also true for special case of variety in any characteristic such as elliptic curves [43] and $\mathbb{P}_n$. In [31] is proven, using Proposition (1.2.5), that any G-bundle E on $\mathbb{P}_n$ is strongly semistable. Then by Remark (1.2.1) and using the same proof given in [27] Behrend's conjecture follows.

In section three we will prove the conjecture for orthogonal an symplectic bundles in positive characteristic (characteristic different from 2 in the orthogonal case).

1.3 Geometric invariant theory and the method of Ramanan and Ramanathan

In this section we will describe some basic facts about geometric invariant theory and briefly explain the results of Ramanan and Ramanathan [36].

The instability parabolic

Let K be a field and let $\overline{K}$ be a fixed algebraic closure. Let G be a connected reductive algebraic group over K. Let V be a finite dimensional representation of G we get an induced action of G on the projective space $\mathbb{P}(V)$ of lines in V. For a point $v \in \mathbb{P}(V)$ we will denote, by abuse of notation, a representative in V again by v.

Firstly we will describe the theory when $K = \overline{K}$ is algebraically closed and later extend the theory to non-algebraically closed fields.

Recall that a point $0 \neq v \in V$ is **semistable** for the G-action if the closure $\overline{Gv}$ of the orbit of v does not contain 0. One knows that this definition is equivalent to existence of a G-invariant element $\phi \in S^n(V^*)$ for some $n > 0$ such that $\phi(v) \neq 0$.

If $v \in V$ is not semistable then recall the following notions.

For a 1-PS λ of G, consider the decomposition of $V = \bigoplus V_i$ with $V_i = \{v \in V \mid \lambda(t)v = t^i v\}$. For an $v \in V$ one defines the invariant

$$m(v\,,\lambda) \;=\; \inf\{i\,|v \text{ has a non-zero component in } V_i\}$$

Using the W-invariant inner product on a fixed maximal torus T one defines the slope $\nu(v\,,\lambda) \;=\; m(v\,,\lambda)/||\lambda||$ for all 1-PS in the maximal torus T. Since maximal tori are conjugates this definition can be extended to all 1-PS in G.

The following lemma will be used in the sequel.

Lemma 1.3.1. *With the above notations there exists a constant C (independent of $v \in V$ and 1-PS λ) such that $\nu(v\,,\lambda) \leq C$.*

Proof. See Proposition (2.17) p. 64 of [33] for a proof.

We define the **instability 1-PS** for a given $v \in V$ (which is not semistable) as one for which $\nu(v\,,\lambda)$ attains its maximum among all 1-PS of G (see Theorem (1.5, a) of [36]).

For a 1-PS λ, recall the definition of the parabolic $P(\lambda)$ whose valued points are characterized by elements $g \in G$ for which the limit $\mathrm{Lim}_{t\longrightarrow 0}\lambda(t)\, g\, \lambda(t)^{-1}$ exists.

In the following Proposition we summarize the basic facts in geometric invariant theory.

Proposition 1.3.2. *Let $v \in V$ be a non-zero element which is non-semistable.*

1. *There is a unique parabolic subgroup $P(v)$ with the property that for any instability 1-PS λ for v we have $P(v) = P(\lambda)$.*

2. *For any maximal torus $T \subset P(v)$ there is a unique 1-PS $\lambda_T \subset T$ such that it is an instability 1-PS for v.*

Proof. See Theorem (1.5, b and c) [36].

The above uniquely defined parabolic $P(v)$ will be called the **instability parabolic** for v. Here the uniqueness of λ is as a subgroup of T rather than a morphism $\mathbb{G}_m \to T$.

If G acts on projective variety M which is linearized by an ample line bundle $\mathcal{L}$ then by taking some power of $\mathcal{L}$ we get a representation V of G and a G-equivariant embedding $i : M \to \mathbb{P}(V)$ with $i^*\mathcal{O}(1)$ being some power of $\mathcal{L}$. In this setup we say a point $m \in M$ is semistable for G action if the corresponding point in V is semistable.

Let $v \neq 0$ be a non-semistable point. Let $P = P(v)$ be its instability parabolic and let $\lambda \subset T \subset P$ be a chosen tuple of instability 1-PS and a maximal torus T. Let $V = \bigoplus_i V_i$ be the decomposition of V with respect to λ. Let $j = m(v, \lambda)$. Using this we have a decomposition $v = \sum_{i \geq 0} v_i$, where $v_i \in V_{i+j}$.

Here one notes that $V^j = \bigoplus_{i \geq j} V_i$ is preserved under the action of $P = P(v)$ and the unipotent radical $U \subset P$ pushes V^j to V^{j+1}, thus giving an action of the Levi quotient $L = P/U$ on V^j/V^{j+1}.

The W-invariant inner product on a fixed maximal torus of G naturally gives rise to a W-invariant inner product on $T \subset P(v)$. Let $l_\lambda \in \mathcal{X}^*(T) \otimes \mathbb{Q}$ be the dual of λ. Let $r_1 \in \mathbb{Z}^+$ such that $r_1 l_\lambda$ defines a character of T. The restriction of this character to the connected component $Z^0(L)$of the center of L, and taking a further multiple, extends to give a character of L. Hence given a λ we get a character χ of $P(v)$ which is well defined up to a positive integral multiple.

In the following proposition we describe the basic result of Ramanan-Ramanathan [36] concerning the behaviour of $v_0 \in V^j/V^{j+1}$ under the induced action of L on V^j/V^{j+1}.

Proposition 1.3.3. *Assume that the group $Z^0(G)$ acts trivially on V. Then there exists a positive integer r and dominant character χ of P such that the point $v_0 \in \mathbb{P}(V^j/V^{j+1})$ is semistable for the natural action of L with respect to linearization given by $\mathcal{O}(r) \otimes \mathcal{O}_{\chi^{-1}}$, where $\mathcal{O}_{\chi^{-1}}$ is the trivial line bundle with L acting on it by χ^{-1}*

The proof of the above result (as given in Proposition (1.12) in [36]) also gives a recipe to find the integer r and the character χ and they are related by the following Lemma.

Lemma 1.3.4. *There is a character χ' of the maximal torus $T \subset P$ such that the following holds.*

1. $\chi'|_{Z^0(L)} = \chi|_{Z^0(L)}$
2. $\chi' = r\,\nu(v\,,\lambda)\,||\lambda||\,l_\lambda$, *where l_λ is the dual of λ.*

The above Lemma will be used in our proof of the main Theorem.

The Rationality of the instability parabolic

We will now assume that the ground field K is not algebraically closed. Let G be a connected reductive group over K which acts on a projective K-variety M, linearized by an ample line bundle $\mathcal{L}$, thus giving a G-equivariant embedding $i : M \to \mathbb{P}(V)$ as before.

We will call a K-valued point $v \in V$ semistable if it is so after a base change to algebraic closure. In this way we will avoid the confusion of which field the semistability definition is used.

Let m be a K-rational point of M which is not semistable. Let $P(m)$ be its instability parabolic defined over $\overline{K}$.

Remark 1.3.5 Note that if $P(m)$ is defined over K_s then it is already defined over K. This is because of the uniqueness of $P(m)$ (see Proposition 1.3.2 (1)) and the Galois descent argument. Also note that if $P(m)$ is defined over K then it contains a maximal torus over K which splits over K_s. Then the instability 1-PS of m which is contained in the maximal torus over K_s, by uniqueness (Proposition 1.3.2 (2)) is Galois invariant and hence it is defined over K.

Let $O(m)$ be the (reduced) orbit of G at m. Since m is defined over K the orbit $O(m)$ is also defined over K.

We briefly recall the construction of a scheme $M(P)_{x_m}$ which will be used later in an important way.

We can find a $g \in G$ such that $g\,P(m)\,g^{-1} = P$ is defined over K_s (as the variety of all parabolics conjugate to $P(m)$ is defined over K_s, being absolutely reduced, has a K_s-rational point (see [9])). If $x_m = g\,m$ then $P = P(x_m)$ is the instability parabolic of x_m.

Since P is defined over K_s and over this field G splits, we have a maximal torus in P which splits over K_s. Hence there is a instability 1-PS λ of x_m in this maximal torus over K_s.

The representation V of G decomposes as $V = \bigoplus_{i\in\mathbb{Z}} V_i$ for the action of λ, where $V_i = \{v \in V | \lambda(t)\cdot v = t^i\,v, t \in \mathbb{G}_m\}$. Let $j = m(x_m, \lambda)$ and $V^j = \bigoplus_{i\geq j} V_i$.

Recall the definition of the K_s-scheme $M(P)_{x_m}$ as the scheme theoretic intersection of the K_s-subschemes $\mathbb{P}(V^j)$ and $O(m)$ of $\mathbb{P}(V)$.

The following two results summarizes the basic properties of the scheme $M(P)_{x_m}$.

Lemma 1.3.6. *The $\overline{K}$-rational points of the K_s-subscheme $M(P)_{x_m}$ of the K-scheme $O(m)$ are precisely those points which have $P(x_m)$ as their instability parabolic. Moreover, when the G action on m is strongly separable then $M(P)_{x_m}$ is absolutely reduced.*

Proof. See Lemma (2.4) of [36].

Recall that the G-action at $m \in M(\overline{K})$ is said to be strongly separable if the isotropy subgroup scheme G_x is reduced at every point $x \in M(\overline{K})$ which is in the closure of the orbit $O(m)$.

Lemma 1.3.7. *Suppose that $y \in M(P)_{x_m} \subset O(m)$ is a K_s-rational point and that there is an $h \in G(K_s)$ such that h maps to y under the orbit map $G \to O(m)$. Then $P(m)$ is defined over K.*

Proof. By lemma (1.3.6), the point $y = h\,m$ has the property that $P(y) = P(x_m)$, hence $P(y)$ is defined over K_s. This implies that $P(m) = h\,P(y)\,h^{-1}$ is also defined over K_s, hence by Remark (1.3.5) we conclude the proof of the Lemma. $\square$

The above lemma has the consequence that if the action of G is strongly separable at m then the parabolic $P(m)$ is already defined over K (also see Proposition 2.4, [36]).

The argument of Ramanan and Ramanathan

Let E be a rational G-bundle over X. Let $\rho : G \longrightarrow G_1$ be a representation of G which takes the connected component of the center of G to the center of G_1. Let P_1 be a parabolic subgroup of G_1. We fix a representation $G_1 \longrightarrow \mathrm{GL}(V_{P_1})$ such that it defines an embedding of $G_1/P_1 \subset \mathbb{P}(V_{P_1})$ with the property that the character of P_1 on V_{P_1} is a positive multiple m_{P_1} of the character of χ_{P_1} associated to the restriction of the adjoint representation of P_1 on the vector space $\mathfrak{g}_1/\mathfrak{p}_1$.

The line bundle $\mathcal{O}(1)$ on the projective variety $\mathbb{P}(V)$ gives rise to a line bundle $\mathcal{L}$ over G_1/P_1.

For the rational G-bundle E over X we have the group scheme $E(G)$ using the conjugation action of G on itself. Let $E(\mathcal{L})$ over $E(G_1/P_1)$ be the associated line bundle over the associated rational fiber bundle over X.

Let $E(G)_0$ be the group scheme defined over the function field K of X. We also have the action of $E(G)_0$ on the projective variety $E(G_1/P_1)_0 \subset E(\mathbb{P}(V))_0$ over K, linearized by the line bundle $E(\mathcal{L})_0$.

Let σ be a reduction of structure group of $E(G_1)$ to P_1. Let σ_0 be the associated K-rational point of $E(G_1/P_1)_0$.

In the following two propositions we summarise the basic argument of Ramanan-Ramanathan.

Proposition 1.3.8. *Let σ_0 be semistable for the action of $E(G)_0$ on $E(G_1/P_1)_0$ (over $\overline{K}$) for the polarization $E(\mathcal{L})_0$. Then the section σ has the property that* $\deg(T_\sigma) \geq 0$.

Proof. See Proposition (3.10, (1)) of [36].

Suppose that σ_0 is not semistable for the above action and that the instability parabolic P' for σ_0 is defined over the field K. Then the parabolic $P' \subset E(G)_0$ gives rise to a rational reduction of the structure group τ of E to the parabolic P such that $(E_\tau(P))_0 = P'$.

The following result is slightly more general than the Proposition (3.13) of [36] (without the semistability assumption on E) and its proof is along the same lines.

Proposition 1.3.9. *Suppose σ_0 is not semistable and its instability parabolic is defined over the field K then there exists a positive integer r and a dominant character χ of P (related by the Lemma 1.3.4) such that the following inequality holds*

$$-(r\, m_P)\deg(T_\sigma) \leq \deg(\chi_*(E_\tau)) \tag{1.4}$$

The above result when E is semistable implies that $\deg(T_\sigma) \geq 0$ and this along with Proposition 1.3.8 is used in characteristic zero to show that $E(G_1)$ is semistable.

We will use Proposition 1.3.9 in this generality because the instability parabolic will be defined after a suitable Frobenius pull-back of E which may not be semistable (see proof of Theorem 0.0.1).

Chapter 2

Boudedness for semistable G-bundles on variety

2.1 A result on instability parabolics

One of the main steps in our proof of Theorem (0.0.1) is a result (Proposition (2.1.5)) which gives uniform bounds for the domain of definition of instability parabolics. For proving this result we need to estimate the non-reducedness of the fibers of morphisms of algebraic varieties and we will do this part first.

We start with some definitions which will be used later. Let K be a field and let $\overline{K}$ be its algebraic closure.

We define the **radical index** $\mathrm{Ri}(A)$ of an affine algebra A over K to be the smallest integer n such that for any f in the radical $\mathrm{Rad}(A)$ of A we have $f^n = 0$. For an affine morphism $f : Y \longrightarrow X$ of finite type $\overline{K}$-schemes we define the radical index $\mathrm{Ri}(x)$ of a point $x \in X$ to be $\mathrm{Ri}(Y_x)$ where Y_x is the fiber of f at the point $x \in X$.

Proposition 2.1.1. *Let $f : Y \longrightarrow X$ be a morphism of finite type affine schemes over $\overline{K}$. There exists an integer n such that $\mathrm{Ri}(x) \leq n$ for each $x \in X$.*

Proof. The proof of this proposition is a series of reductions from the case of arbitrary X and Y to very specific ones using the induction on the dimension of X. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ and i be the homomorphism $A \longrightarrow B$. For any prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ (or $\mathrm{Spec}(B)$) we write $\mathrm{Ri}(\mathfrak{p}\,, B)$ for the radical index of $B/\mathfrak{p}B$.

First we may assume that A is integral. This part is an elementary check.

We use induction on the $\dim(A)$. So, successively we reduce to the situations where we need to bound $\mathrm{Ri}(\mathfrak{m}, B)$ for maximal ideals $\mathfrak{m} \in \mathrm{Spec}(A_f)$ for suitable choices of $f \in A$.

We make some reductions on B. We may assume that B is reduced. For this if $\mathfrak{m}$ is a maximal ideal of A then we can check that

$$\mathrm{Ri}(\mathfrak{m}, B) \leq \mathrm{Ri}(\mathfrak{m}, B_{\mathrm{red}}) + \mathrm{Ri}(B)$$

Next we may assume that B is irreducible. Let $\mathfrak{p}_i$, for $i = 1 \ldots m$, be the set of minimal prime ideals in B. Let $\mathfrak{m}$ be a maximal ideal in A. We will show that

$$\mathrm{Ri}(\mathfrak{m}, B) \leq m \, \mathrm{Max}_{i=1}^{m} \mathrm{Ri}(\mathfrak{m}, B/\mathfrak{p}_i)$$

For this one observes that if $x \in \mathrm{Rad}(B/\mathfrak{m}B)$ then the image of x in each of $B/\mathfrak{p}_i \otimes A/\mathfrak{m}$ lies in $\mathrm{Rad}((B/\mathfrak{p}_i)/\mathfrak{m}(B/\mathfrak{p}_i))$. Hence if $n = \mathrm{Max}_{i=1}^{m}\mathrm{Ri}(\mathfrak{m}, B/\mathfrak{p})$ then x has the property that $x^n \in \cap_{i=1}^{m}(\mathfrak{p}_i + \mathfrak{m}B)$. We can write $x^n = y_i + z_i$ such that $y_i \in \mathfrak{p}_i$ and $z_i \in \mathfrak{m}B$. Since B is reduced we have $\Pi_{i=1}^{m}(x^n - z_i) = \Pi y_i = 0$, and so $x^{nm} \in \mathfrak{m}B$. This proves the assertion.

Hence from now on we may assume that A and B are integral domains.

We reduce this problem to an open subscheme of $Y = \mathrm{Spec}(B)$. Let $b \in B$. Then there exists an element $a \in A$ such that $(B/bB)_a$ is flat over A_a. Hence for any maximal ideal $\mathfrak{m}$ in A_a we have $\mathrm{Tor}_1^A((B/bB)_a, A/\mathfrak{m}) = 0$.

Consider the map $B_a \longrightarrow B_{ab}$. We will show that $\mathrm{Ri}(\mathfrak{m}, B_a) \leq \mathrm{Ri}(\mathfrak{m}, B_{ab})$. If $x \in B$ is such that some power of it lies in $B_a/\mathfrak{m}B_a$ then the image of x also has the same property over B_{ab}. Then by clearing denominators if $r = \mathrm{Ri}(\mathfrak{m}, B_{ab})$ then there is an m such that $b^m x^r \in \mathfrak{m}B_a$. Using the exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_1^A((B/bB)_a, A/\mathfrak{m}) \longrightarrow B_a/\mathfrak{m}B_a \longrightarrow B_a/\mathfrak{m}B_a$$

with the last map being multiplication by b, we conclude, by the vanishing of the $\mathrm{Tor}_1^A((B/bB)_a, A/\mathfrak{m})$, that $x^r \in \mathfrak{m}B_a$. Hence it is enough to bound radical index of fibers of some open set of the type B_b over A.

We use Noether Normalization (and inverting an element of A) to get an inclusion $A \hookrightarrow A[x_1, \ldots, x_r] = A' \hookrightarrow B$ such that B is finite over A'. Let K_A(respectively K_B and $K_{A'}$) be the function fields of A (respectively B and A').

Let L be the separable closure of $K_{A'}$ in K_B. The extension $L \subset K_B$ is purely inseparable. Hence there is an integer n_1 such that for any $x \in K_B$, we have $x^{p^{n_1}} \in L$.

Let $C = B \cap L$. Then we observe that B is integral over C and is a finitely generated A-algebra. This implies that C is also a finitely generated A-algebra. Again by localizing A at an element we can assume that the A module B/C is flat over A and hence for any $\mathfrak{m}$ in A we have $\mathrm{Tor}_1^A(B/C\,,\,A/\mathfrak{m}) = 0$. This has the effect that for each maximal ideal $\mathfrak{m}$ in A we have an injection $C/\mathfrak{m}C \longrightarrow B/\mathfrak{m}B$. Using this and the fact that $x^{p^{n_1}} \in C$ for any $x \in B$ we have

$$\mathrm{Ri}(\mathfrak{m}\,,\,B) \leq \mathrm{Ri}(\mathfrak{m}\,,\,C) + p^{n_1}$$

The problem now reduces to proving the proposition for the case when $A \hookrightarrow B$ is an extension of finitely generated domains such that the function field extension is separable. Further, it is enough to prove the result for the case $A \hookrightarrow B_b$ for some $b \in B$.

We may now assume that A and B are smooth domains. Hence we conclude that there exist elements $b \in B$ and $a \in A$ such that the morphism $\mathrm{Spec}(B_{ab}) \longrightarrow \mathrm{Spec}(A_a)$ is smooth. This implies that the fibers here are reduced and hence proof of the proposition is complete. $\square$

Let T be a finite type scheme over $\overline{K}$. Let $\mathcal{N}$ be its radical ideal sheaf. This has the property that for any closed point of T, the stalk of $\mathcal{N}$ is the radical of the local ring. We define the radical index Ri(T) of T to be the smallest integer n such that for any open subset $U \subset T$ and for any $g \in \Gamma(U\,,\,\mathcal{N})$ we have $g^n = 0 \in \Gamma(U\,,\,\mathcal{N}^n)$.

Let $f : Y \longrightarrow X$ be a morphism of finite type $\overline{K}$-schemes. In this general setting we define the radical index of the closed point $x \in X$ by $\mathrm{Ri}(x) = \mathrm{Ri}(Y_x)$, where Y_x is the fiber at x. In this case we have the following result which generalizes Proposition (2.1.1) and this will also be used in our proof of main results.

Proposition 2.1.2. *Let $f : Y \longrightarrow X$ be a morphism of finite type schemes over $\overline{K}$ There exists an integer n such that $\mathrm{Ri}(x) \leq n$ for all closed points $x \in X$*

Proof. Let $\{U_i\}_{i=1}^r$ be an open cover of X by affine open subschemes with $U_i = \mathrm{Spec}(A_i)$. Then it is enough to prove the result for the case of each of U_i, hence we may assume that $X = \mathrm{Spec}(A)$ is an affine scheme.

Let $\{V_i\}_{i=1}^{r'}$ be an open cover of Y by a finite number of affine open subschemes. We write $V_i = \mathrm{Spec}(B_i)$. By Proposition (2.1.1), we have positive integers n_i such that for each maximal ideal $\mathfrak{m}$ of A we have $\mathrm{Ri}(\mathfrak{m}\,,\,B_i) \leq n_i$. Let $n = \mathrm{Max}\{n_i\}$.

Then we have $\mathrm{Ri}(x, Y) \le n$ for each closed point $x \in X$. This is because the fiber Y_x can be covered by affine open subschemes $\{V_{i,x}\}$. Here $V_{i,x}$ is the fiber of x in V_i. If $U \longrightarrow Y_x$ is any open immersion and $\nu \in \Gamma(U\,, \mathcal{N})$ then restriction ν_i of ν to $U \cap V_{i,x}$ lies in $\Gamma(U \cap V_{i,x}\,, \mathcal{N})$. Hence the result would follow if we show that for any K-algebra B and an element $b \in B$ we have $\mathrm{Ri}(B_b) \le \mathrm{Ri}(B)$. The last statement is a straight forward verification. This completes the proof of Proposition (2.1.2). $\square$

Remark 2.1.3 One notes that the constant n as defined in the above proposition depends on X, Y, and f but not on the $\overline{K}$-valued points of X.

Let G be a reductive algebraic group acting on a variety M (over $\overline{K}$). For any $x \in M(\overline{K})$ we denote the isotropy subgroup scheme at x by G_x. The following result is a consequence of the above proposition.

Proposition 2.1.4. *There exists an N_1 such that $\mathrm{Ri}(G_x) \le N_1$ for each $x \in M(\overline{K})$.*

Proof. Consider the map $G \times M \longrightarrow M \times M$ defined by $(\rho\,, pr_2)$ where ρ is the action map and pr_2 is the second projection map. Let Δ_M be the diagonal map $M \longrightarrow M \times M$. Let $H = (G \times M) \times_{M \times M} M$. Then we have a natural projection map $\pi : H \longrightarrow M$ which has the property that for any $x \in M(\overline{k})$ the fiber of the map π at x is the isotropy subscheme G_x. The result follows from Proposition (2.1.2). $\square$

Let K be an arbitrary field and K_s and $\overline{K}$ be its separable closure and the algebraic closure respectively (in fact K will be the function field of the smooth projective variety X).

In this case the radical index of a finite type scheme T over K is defined to be the radical index of the scheme $\overline{T} = T \otimes_K \overline{K}$.

Let G be a reductive group over K. Let M and V be as defined before (in Section 3).

Let m be a non-semistable K valued point of M. Let $P(m)$ be the instability parabolic defined over $\overline{K}$. Recall from Remark (1.3.5) that if $P(m)$ is defined over K_s then it is already defined over K. Hence $P(m)$ is always defined over a finite purely inseparable extension of K.

The following Proposition is the main result of this Section.

Proposition 2.1.5. *There exists an integer N such that for any K-rational point m of M which is not semistable, the instability flag $P(m)$ is defined over $K^{p^{-N}}$.*

Proof. It is enough to show that there exists an N such that the instability parabolic for any non-semistable K_s rational point of M is defined over $K_s^{p^{-N}}$ (see Remark (1.3.5). This enables us to assume that all our objects are defined over the field K_s.

Let m be a K_s-valued point of M which is not semistable for the action of G on M. Let $O(m)$ be the (reduced) orbit of G at m. Let $P(m)$ be its instability parabolic over $\overline{K}$. We can find a $g \in G$ such that $g\,P(m)\,g^{-1} = P$ is defined over K_s (this was seen before). If $x_m = g.m$ then $P = P(x_m)$ is the instability flag of x_m.

To this setup we have the scheme $M(P)_{x_m}$ as defined in Section 3 satisfying the properties of the Lemmas (1.3.6) and (1.3.7).

Our main goal is to estimate the non-reducedness of the scheme $M(P)_{x_m}$ independent of m, and this will enable us to prove that the parabolic $P(m)$ is defined over a fixed purely inseparable extension of K_s for all m.

Note that we have made a choice of x_m and this choice fixes the instability 1–PS λ of x_m which is defined over K_s. Hence the point x_m determines the vector subspace $V^j \subset V$ (as defined in Section 3) .

We will show that there exists a positive integer N_2 such that for any non-semistable point $m \in M$ and any choice of x_m as above, the radical index $\mathrm{Ri}(M(P)_{x_m}) \leq N_2$. The basic idea of the proof is to prove that the spaces $M(P)_{x_m}$ occurs as suitable subschemes of the fibers of a fixed morphism $Y \longrightarrow X$ and then apply Proposition (2.1.2) to bound the radical index. For this analysis we may assume that we are working over the algebraic closure $\overline{K}$ of K.

Consider the map $\widetilde{\rho} : G \times \mathbb{P}(V) \longrightarrow \mathbb{P}(V) \times \mathbb{P}(V)$ defined by $\widetilde{\rho} = (\rho\,, \mathrm{pr}_2)$, where ρ is the action map and pr_2 is the second projection. Let $\mathcal{Y}$ be the schematic image of $\widetilde{\rho}$. In this case $\mathcal{Y}$ gets the reduced induced scheme structure from the product and its points are the closure of the image of the map $\widetilde{\rho}$. We have the map $h : \mathcal{Y} \longrightarrow \mathbb{P}(V)$ which is the composition of the inclusion map to $\mathbb{P}(V) \times \mathbb{P}(V)$ with the second projection. Let $\mathcal{Y}_x$ be the fiber of h at a point $x \in \mathbb{P}(V)$. One observes that $\mathcal{Y}_x$ contains the closure of the orbit $O(x)$ of x.

For any integral locally closed subscheme $Z \subset \mathbb{P}(V)$ we have the restriction of the map $\widetilde{\rho}$ which we denote by $\widetilde{\rho}^Z$ from $G \times Z \longrightarrow \mathbb{P}(V) \times Z$. Let $\mathcal{Y}^Z$ be its schematic image. We again get the induced map $h^Z : \mathcal{Y}^Z \longrightarrow Z$ and we denote by $\mathcal{Y}_x^Z$ the fiber of the surjective map h^Z at $x \in Z$.

Note that since Z is reduced we have an open subscheme $U_1 \subset Z$ where the map h^Z is flat. One also observes that the schematic image of $\widetilde{\rho}^Z|_{(G \times U_1)}$ is exactly $(h^Z)^{-1}(U_1)$. Hence we can restrict the setup to U_1.

Since the actual image set $\mathcal{Z}' = \widetilde{\rho}^Z(G \times U_1)$ is constructible, it contains an open subset $U' \subset \mathcal{Y}^{U_1}$. The map $U' \longrightarrow U_1$ is flat, hence if we define $U = U' \cap U_1$ and restrict the whole setup to U, the image set $\mathcal{Z} = \widetilde{\rho}^Z(G \times U)$ contains an open subset $U' \subset Y^U$ which maps surjectively onto U. Now we can further translate the open set U' by an element of g which acts on the first factor to obtain that $\mathcal{Z}$ is open in $\mathcal{Y}^U$. The upshot of this analysis is that for any subvariety $Z \subset \mathbb{P}(V)$ we can find an open subscheme $U \subset Z$ such that we can define an open reduced subscheme $\mathcal{Z}^U$ of $\mathcal{Y}^U$ whose points are exactly the image of the map $\widetilde{\rho}^U$. This also has the property that $O(x)$ is exactly the reduced part $(\mathcal{Z}_x^U)_{\mathrm{red}}$ of the fiber $\mathcal{Z}_x^U$ at $x \in U$.

Let $0 < l < \dim(V)$ be an integer. Let $\mathrm{Gr}_l(V)$ be the Grassmannian of l-dimensional planes in $\mathbb{P}(V)$. We have a universal subscheme $i : H_l \subset \mathrm{Gr}_l(V) \times \mathbb{P}(V)$. This has the property that for each $y \in \mathrm{Gr}_l(V)$ the inverse image $(H_l)_y = i^{-1}(\{y\} \times \mathbb{P}(V)) \subset \mathbb{P}(V)$ is $\mathbb{P}(W_y)$, where W_y is the l dimensional subspace of V associated to the point y.

Now we consider the relative version of the setup to get a suitable intersection of the orbits with projective subspaces of $\mathbb{P}(V)$. Let Z and U be defined as before.

Let $\widetilde{\rho}_l^U : \mathrm{Gr}_l(V) \times G \times U \longrightarrow \mathrm{Gr}_l(V) \times \mathbb{P}(V) \times U$ be the map defined by $(\mathrm{id}, \widetilde{\rho}^U)$. Inside $\mathrm{Gr}_l(V) \times \mathbb{P}(V) \times U$ we have two subschemes namely $\mathrm{Gr}_l(V) \times \mathcal{Z}^U$ and $H_l \times U$. Let Y be the scheme theoretic intersection of these schemes. We have a map $f : Y \longrightarrow X = \mathrm{Gr}_l(V) \times U$ which is obtained by composing the inclusion map with the product of the first and the last projection map.

The map f has the property that for any $x \in U$ and $y \in \mathrm{Gr}_l(V)$, $M'_{y,x} = f^{-1}(y, x)$ is the scheme theoretic intersection of $\mathcal{Z}_x^U$ and $\mathbb{P}(W_y)$. Hence by Proposition (2.1.2), there exists an integer n such that $\mathrm{Ri}(M'_{y,x}) \leq n$, for each $(y, x) \in X$.

If $M_{y,x}$ is the schematic intersection of $O(x)$ and $M'_{y,x}$ then we have an inclusion $(M_{y,x})_{\mathrm{red}} \subset M_{y,x} \subset M'_{y,x}$. Hence we conclude that for each $(y, x) \in \mathrm{Gr}_l(V) \times U$, we have $\mathrm{Ri}(\mathbb{P}(W_y) \cap O(x)) \leq n$.

Take $Z = \mathbb{P}(V)$ and we obtain an open subscheme U with a bound n for the radical index of $\mathbb{P}(W_y) \cap O(x)$ for every $x \in U$ and $y \in \mathrm{Gr}_l(V)$. Then take the complement of U in $\mathbb{P}(V)$ and so on by induction there exists an integer n_l such that $\mathrm{Ri}(\mathbb{P}(W_y) \cap O(x)) \leq n_l$ for each $y \in \mathrm{Gr}_l(V)$ and $x \in \mathbb{P}(V)$.

We choose $N_2 = \mathrm{Max}_{l=1}^r\{n_l\}$. Then for any $m \in M(K_s)$ which is not semistable and any choice of x_m as before the subscheme $M(P)_{x_m}$ occurs as one such $M_{y,x}$ for some $y \in \mathrm{Gr}_l(V)$ for some l and $x = x_m$. This proves that for all choices of x_m and m we have $\mathrm{Ri}(M(P)_{x_m}) \leq N_2$.

As a next step in the proof of the Proposition (2.1.5) we will show that for any $m \in M(K)$ the K_s-scheme $M(P)_{x_m}$ admits a $K_s^{p^{-N_2}}$-rational point. For this we can first find an affine open subscheme U_{x_m} of $M(P)_{x_m}$ which is defined over K_s. As the radical index of U_{x_m} is $\leq N_2$, the existence of rational points on U_{x_m} follows from the Lemma below.

Lemma 2.1.6. *Let A be an affine K_s-algebra with radical index $\leq p^n$. Then A admits a $K_s^{p^{-n}}$-rational point.*

Proof. We will denote by A_n the $K_s^{p^{-n}}$-algebra $A \otimes_{K_s} K_s^{p^{-n}}$ and by $\overline{A}$ the $\overline{K}_s$-algebra defined by $A \otimes_{K_s} \overline{K}_s$. Since the radical $\mathrm{Rad}(A) \otimes_{K_s} \overline{K}_s \subset \mathrm{Rad}(\overline{A})$ we may assume that A is reduced. We show that the natural inclusion

$$\mathrm{Rad}(A_n) \otimes_{K_s} \overline{K}_s \subset \mathrm{Rad}(\overline{A})$$

is an isomorphism. This will prove that the $K_s^{p^{-n}}$-algebra $A_n/\mathrm{Rad}(A_n)$ is absolutely reduced and hence will admit a $K_s^{p^{-n}}$-rational point.

Let $f \in \mathrm{Rad}(\overline{A})$. Then we can write $f = \sum_{i=1}^{l} f_i' \otimes a_i$ with $f_i \in A$ and $a_i \in \overline{K}_s$. If each of $a_i \in K_s^{p^{-n}}$ then already we have $f \in \mathrm{Rad}(A_n) \otimes_{K_s} \overline{K}_s$.

Let $n_1 > n$ be so chosen that each of $a_i \in K_s^{p^{-n_1}}$. Using the identity $\overline{A} = A_n \otimes_{K_s^{p^{-n}}} \overline{K}_s$, we have an expansion of $f = \sum_{i=1}^{j} f_i \otimes b_i$ where $f_i \in A_n$ and $b_i \in K_s^{p^{-n_1}}$ such that $\{b_1, \ldots, b_j\}$ is linearly independent over $K_s^{p^{-n}}$.

We claim that $b_i^{p^n}$'s are linearly independent over K_s. Suppose that $\sum c_i b_i^{p^n} = 0$ with $c_i \in K_s$. Let $d_i \in K_s^{p^{-n}}$ be such that $d_i^{p^n} = c_i$. The above implies that $(\sum d_i b_i)^{p^n} = 0$. This proves the claim.

The radical index of A is n we have $f^{p^n} = 0$ and this gives us $0 = f^{p^n} = \sum_{i=1}^{j} f_i^{p^n} \otimes b_i^{p^n}$. Since $b_i^{p^n}$'s are linearly independent over K_s and A is flat over K_s, we have $f_i^{p^n} = 0$. Hence $f_i \in \mathrm{Rad}(A_n)$. This proves the lemma (2.1.6). □

We have the orbit morphism $G \longrightarrow O(m)$ which is defined over K_s. Also we have a $K_s^{p^{-N_2}}$-valued point of $M(P)_{x_m}$ which is K_s-subscheme of $O(m)$. Hence we obtain a $K_s^{p^{-N_2}}$-valued point y of $O(m)$. Let G_m be the isotropy subgroup scheme of G at m. Then we have a K_s isomorphism $G/G_m \cong O(m)$. Let N_1 be as defined in Proposition (2.1.4). We show that if $N = N_1 + N_2$ then the instability parabolic $P(m)$ of m is defined over $K_s^{p^{-N}}$. For this we first show that there is a $K_s^{p^{-N}}$-valued point h of G such that it maps to y. This statement follows from the fact that if Y is the fiber of the map $G \longrightarrow O(m)$ at the point y, then Y defines a principal G_m-bundle over $S = \mathrm{Spec}(K_s^{p^{-N_2}})$. Now it follows from Proposition (2.1.4) that the scheme Y is an finite type affine scheme over S whose radical index is $\leq N_1$. Hence by lemma (2.1.6) we conclude that Y admits a $K_s^{p^{-N}}$-rational point. This proves that there is a $K_s^{p^{-N}}$-valued point h of G such that $h\,m = y$.

The Lemma (1.3.7) now implies that the instability parabolic $P(m)$ of m is defined over $K_s^{p^{-N}}$ and this completes the proof of the Proposition (2.1.5). $\square$

2.2 The proof of the main Theorems

In this Section we will prove the main Theorems (0.0.1) and (0.0.2) stated in the introduction. The basic strategy of the proof is to assume (0.0.1) for the case of lower semisimple rank groups and prove (0.0.2). Finally prove the Theorem (0.0.1) using (0.0.2) and the Proposition (2.1.5).

Proof of Theorem (0.0.2)

(Assuming Theorem (0.0.1) for lower semisimple rank)

We fix the Borel subgroup B and a maximal torus $T \subset B$ and the root datum as before. Let Δ denote the set of simple roots. Let Q_α be the maximal parabolic subgroup of G containing B corresponding to the simple root $\alpha \in \Delta$. We will denote by $\mathfrak{q}_\alpha$ and $\mathfrak{g}$ the lie algebras of Q_α and G respectively. Let P be a parabolic subgroup of G containing B. Let L be its Levi quotient. We will use the following lemma which is proved in Biswas-Gomez (see proof of Theorem 4.1, in page 783, [6]) .

Lemma 2.2.1. *Let $P \subset Q_\alpha$ be an inclusion of parabolic subgroups. There exists a filtration*

$$0 = V_0^\alpha \subset V_1^\alpha \subset \ldots \subset V_{a_\alpha}^\alpha = \mathfrak{g}/\mathfrak{q}_\alpha$$

of P modules such that the following holds.

1. *The unipotent radical $R_u(P)$ acts trivially on each successive quotients $W_j^\alpha = V_j^\alpha / V_{j-1}^\alpha$ and (hence) the identity connected component of the center of the Levi quotient L acts by a scalar on the induced representation (denoted by ρ_j^α) of L.*

2. *The character χ_j^α of P on the action of P on W_j^α has the property that its restriction to the maximal torus T can be written as a non-positive linear combination of simple roots $\sum_{\beta\in\Delta} n_{j,\beta}^\alpha \beta$ with $n_{j,\beta}^\alpha \leq 0$ and $n_{j,\alpha}^\alpha < 0$.*

Let Π be the subset of simple roots defined by the property that $\alpha \in \Pi$ if and only if $P \subset Q_\alpha$. Here we can look at P as P_Π.

For an $\alpha \in \Pi$, let χ_0^α be the character of P which is defined by the representation of P on $\mathfrak{g}/\mathfrak{q}_\alpha$. One can check that the restriction of χ_0^α to the maximal torus is a non-positive linear combination of simple roots with the coefficients of α being negative. Let χ_P be the character of P defined by the representation of P on $\mathfrak{g}/\mathfrak{p}$.

Let $\mathcal{T}$ be the finite set $\Pi_{\alpha\in\Pi}[0\,,\,a_\alpha)$. For an element $\overline{z} \in \mathcal{T}$, the Lemma (2.2.1) and the observation above imply that there exists positive integers $n(\overline{z})$ and $m^\alpha(\overline{z})$ (for each $\alpha \in \Pi$) with the property that the restriction of the character $n(\overline{z})\chi_P - \sum_{\alpha\in\Pi} m^\alpha(\overline{z})\chi^\alpha_{\overline{z}_\alpha}$ to the maximal torus T is a linear combination of simple roots in $\Delta - \Pi$. This automatically implies that

$$n(\overline{z})\chi_P = \sum_{\alpha\in\Pi} m^\alpha(\overline{z})\chi^\alpha_{\overline{z}_\alpha} \tag{2.1}$$

We will define the constant N_P by

$$N_P = \mathrm{Max}_{\overline{z}\in T}\{\frac{1}{n(\overline{z})}\sum_{\alpha\in\Pi} m^\alpha(\overline{z})\} \tag{2.2}$$

Lemma (2.2.1) also implies that representation ρ_j^α of L takes the identity connected component of the center of L to connected component of the center of $GL(W_j^\alpha)$. Hence by the induction assumption (on Theorem 0.0.1) there exists a constant $C^L(X\,,\,\rho_j^\alpha)$ such that for any rational semistable L-bundle E_L we have

$$\mu_{\max}(E_L(W_j^\alpha)) - \mu_{\min}(E_L(W_j^\alpha)) \leq C^L(X, \rho_j^\alpha) \tag{2.3}$$

Let $C_P = \mathrm{Max}\,\{C^L(X, \rho_j^\alpha)\}$, where the maximum is taken over all $\alpha \in \Pi$ and $1 \leq j < a_\alpha$.

We also define the constant M_P by setting $M_P = \mathrm{Max}_{\alpha\in\Pi,j}\{\dim(W_j^\alpha)\}$

Note that the constants N_P, M_P and C_P depend on the parabolic P and since there are only finitely many choices of parabolic subgroups containing B, we will define the constant N (respectively M and C) to be the maximum of each N_P (respectively M_P and C_P) over all parabolics containing the Borel subgroup B.

We take a rational G-bundle E over X. Let $\mathrm{Ideg}(E)$ be its instability degree. Let $(P'\,,\sigma')$ be a Harder-Narasimhan reduction. One notes here that the reduction σ' satisfies the properties (1) and (2) stated in Section 2.

Let $F = F^*(E)$ be the Frobenius pull-back of E. Let σ be its Harder-Narasimhan reduction to a parabolic P containing B. We will denote by F_σ the P-bundle defined by σ and T_σ the tangent bundle along the fibers of X. We will denote by $F_{\sigma,L}$ the L-bundle obtained by extension of F_σ to L.

We need to bound the slope of T_σ in terms of the slope of $T_{\sigma'}$.

For each $\alpha \in \Pi$, we have an inclusion $P \subset Q_\alpha$. This gives rise to a reduction σ_α of F to Q_α and we will denote by F_{σ_α} the Q_α-bundle determined by this reduction.

By Lemma (2.2.1) we have representations of P in $\mathfrak{g}/\mathfrak{q}_\alpha$ and representations ρ_j^α for $1 \leq j < a_\alpha$. These give rise to a vector bundle T_{σ_α} and a filtration $F_\sigma(V_j^\alpha)$ of T_{σ_α} with the property that successive quotients are isomorphic to $F_{\sigma,L}(W_j^\alpha)$.

The bundle F (being the Frobenius pull-back) admits a p-connection ∇ satisfying the properties defined in Proposition (1.2.5).

We apply this to the reduction σ_α to get the map of vector bundles $\nabla_{\sigma_\alpha} : T_X \longrightarrow T_{\sigma_\alpha}$.

First we consider the case when the above map is zero. Then there is a reduction $\widetilde{\sigma}$ of E to Q_α such that $\sigma_\alpha = F^*(\widetilde{\sigma})$. This has the effect that $\deg(T_{\sigma_\alpha}) = p\deg(T_{\widetilde{\sigma}})$. Hence we have the inequality

$$\deg(T_{\sigma_\alpha}) \geq p\,\mathrm{Ideg}(E) \tag{2.4}$$

Now suppose that map ∇_{σ_α} is not zero. Then there is a j such that the image of ∇_{σ_α} is contained in $F_\sigma(V_j^\alpha)$ and not in $F_\sigma(V_{j-1}^\alpha)$. Hence we get a non-trivial map $T_X \longrightarrow F_{\sigma,L}(W_j^\alpha)$. This implies that $\mu_{\min}(T_X) \leq \mu_{\max}(F_{\sigma,L}(W_j^\alpha))$. ss Let $C_X = \mu_{\min}(T_X)$. This combined with equation (2.3) and the fact that $C_X - C$ can be made to be negative implies that

$$\deg(F_{\sigma,L}(W_j^\alpha)) \geq (C_X - C)\,M \tag{2.5}$$

Further the right hand side in the inequalities (2.4) and (2.5), being negative, can be summed up to get a common right hand side, namely $p\,\mathrm{Ideg}(E) + (C_X - C)\,M$

Hence for any $\alpha \in \Pi$ either the inequality (2.4) holds or the inequality (2.5) holds for some choice of j. This implies that if we vary $\alpha \in \Pi$ we obtain an element $\overline{z} \in \mathcal{T}$. Hence for the element $\overline{z}$ using the formula (2.1) we get

$$n(\overline{z})\deg(T_\sigma) \geq \sum_{\alpha\in\Pi}(m^\alpha(\overline{z}))(p\,\mathrm{Ideg}(E) + (C_X - C)\,M)$$

This implies that $\deg(T_\sigma) \geq N(p\,\mathrm{Ideg}(E) + (C_X - C)\,M)$. This completes the proof of Theorem (0.0.2). $\square$

The above theorem and an induction argument also proves the following.

Corollary 2.2.2. *There exists constants C and N (independent of E) such that*

$$\mathrm{Ideg}((F^n)^*E) \geq p^n\, N\, \mathrm{Ideg}(E) + C$$

.

Proof of the Theorem (0.0.1)

We fix a Borel subgroup B_1 of $\mathrm{GL}(V)$. For a parabolic P_1 of $\mathrm{GL}(V)$ containing B_1, we have an action of G on $M_{P_1} = \mathrm{GL}(V)/P_1$. We fix a representation $\mathrm{GL}(V) \longrightarrow \mathrm{GL}(V_{P_1})$ such that it defines an embedding of $\mathrm{GL}(V)/P_1 \subset \mathbb{P}(V_{P_1})$ with the property that the character of P_1 on V_{P_1} is a positive multiple m_{P_1} of the character of χ_{P_1} associated to the restriction of the adjoint representation of P_1 on the vector space $\mathfrak{gl}(V)/\mathfrak{p}_1$.

The line bundle $\mathcal{O}(-1)$ on $\mathbb{P}(V_{P_1})$ when restricted to $\mathrm{GL}(V)/P_1$ defines an anti ample line bundle $\mathcal{L}_{P_1}^{-1}$ which is also defined by the character $-m_{P_1}\,\chi_{P_1}$.

For a rational G-bundle E over X we have a rational fiber bundle $E(M_{P_1})$ and the line bundle $\mathcal{O}(-1)$ gives a rational line bundle $E(\mathcal{L}_{P_1}^{-1})$.

Let x_0 be the generic point of X. Let $E(G)_0$ the group scheme over $K = k(X)$ associated to E at the generic point x_0. Then we have an action of $E(G)_0$ on $E(M_{P_1})_0$ which is linearized with respect to the line bundle $E(\mathcal{L}_{P_1})_0$ over K.

Lemma 2.2.3. *There exists a constant N, depending only on G and X, such that for any rational G-bundle E and for any parabolic P_1 containing B_1 the instability parabolic for any K-valued non-semistable point of $E(M_{P_1})$ (for the above action of $E(G)_0$) is defined over $K^{p^{-N}}$.*

Proof. Let E_0 be the principal G-bundle over K obtained by restriction of E to the generic point of X. One observes that E_0 becomes trivial over a finite separable extension of K, hence when we change the base to K_s, the separable closure of K, we get an isomorphism $E_0 \otimes_K K_s \cong G \otimes_k K_s$. This isomorphism now canonically extends to give an isomorphism of $E(G)_0 \otimes_K K_s \cong G \otimes_k K_s$ and $E(M_{P_1})_0 \otimes_K K_s \cong M_{P_1} \otimes_k K_s$, and the last one being compatible with group actions, and also of the isomorphisms between the ample line bundles on these spaces.

For the induced action of $G \otimes_k K_s$ on $M_{P_1} \otimes_k K_s$ which is linearised by $\mathcal{L}_{P_1} \otimes_k K_s$, by Proposition (2.1.5), it follows that there is a positive integer N such that for any non-semistable point m of $M_{P_1} \otimes_k K_s$ the instability is defined over $K_s^{p^{-N}}$. Since there are only finitely many parabolic subgroups containing B_1 we can find a constant N which works for all these parabolic subgroups.

This implies that the group scheme $E(G)_0 \otimes_K K_s$ for any E also has the same property. The Galois descent argument implies that instability parabolic of a K-valued point of $E(M_{P_1})_0$ is defined over the field extension $K^{p^{-N}}$, with N being independent of the rational G-bundle E and the reduction σ. This proves the lemma. $\square$

A rational reduction σ of $\rho_* E$ to P_1 gives a K-rational point $\sigma(x_0)$ of $E(M_{P_1})_0$. If this point is semistable then by Proposition (1.3.8), we have

$$\deg(\sigma^* E(\mathcal{L}_{P_1})) \; = \; m_{P_1} \deg(T_\sigma) \geq 0 \tag{2.6}$$

If the point $\sigma(x_0)$ is not semistable we have an integer N prescribed by Lemma (2.2.3) such that its instability parabolic P_0' is defined over $K^{p^{-N}}$.

One observes that pull-back by the N-th Frobenius morphism F^N of X, the action of the generic fibre $((F^N)^* E(G))_0 = (F^N)^*_K(E(G)_0)$ on $((F^N)^* E(M_{P_1}))_0 = (F^N)^*_K(E(M)_0)$ is the base change by the Frobenius $F^N_K : \mathit{Spec}(K) \to \mathit{Spec}(K)$ of the action of $E(G)_0$ on $E(M)_0$. Hence the point $((F^N)^*\sigma)(x_0)$ has an instability parabolic P_0' for the action of $(F^N)^*_K(E(G)_0)$ defined over $K^{p^{-N}}$ (see proof of Theorem (3.23) of [36]).

The parabolic subgroup P_0' defines a rational reduction of the structure group τ of $(F^N)^* E$ to a parabolic subgroup $P' \subset G$ with $P_0' = (E_\tau(P'))_0$.

Since P_0' is defined over $K^{p^{-N}}$, by Remark (1.3.5), the instability 1-PS for $\sigma(x_0)$ is also defined over $K^{p^{-N}}$.

The Proposition (1.3.9) (applied to bundle $(F^N)^* E$ and the point $((F^N)^*\sigma)(x_0)$) implies that there is a positive integer r and a dominant character χ of P' such that the following inequality holds.

$$-r \deg(((F^N)^*\sigma)^*(((F^N)^* E)(\mathcal{L}_{P_1}))) \; \leq \; \deg(\chi_*(((F^N)^* E)_\tau)) \tag{2.7}$$

Let $\Pi \subset \Delta$ be the subset defining the parabolic P'. Let Q_α be the maximal parabolic subgroup of G containing P' defined by α. Let χ_α be the dominant character of Q_α defined by the representation $\mathfrak{g}/\mathfrak{q}_\alpha$. There is a positive integer m_α such that we have $\chi_\alpha|_T = -m_\alpha w_\alpha$ where w_α are the fundamental weights of G with respect to a fixed maximal torus contained in P'.

Let $L' = P'/R_u(P')$ and $Z_0(L')$ be the connected component of the center of the Levi L'.

By Lemma (1.3.4) we have a character χ' of the maximal torus $T \subset P'$ such that $\chi'|_{Z^0(L')} = \chi|_{Z^0(L')}$ and $\chi' = r\,\nu(v\,,\lambda)\,||\lambda||\,l_\lambda$. Here l_λ is the dual of λ.

The above equality can be rewritten as $\chi' = r\,\nu(\sigma_N\,,\,\lambda)l_a$ where $a \in \mathcal{X}_*(T)\otimes\mathbb{Q}$ is the element in the unit sphere defined by $a = \lambda/\|\lambda\|$.

By Lemma (1.3.1) there exists a constant B_G such that for every point $m \in M_{P_1}\otimes_k \overline{K}$ and $\lambda \in \mathcal{X}_*(T)\otimes\mathbb{Q}$ we have

$$\nu(m\,,\,\lambda) \leq B_G \tag{2.8}$$

The Weyl group invariant scalar product on $\mathcal{X}_*(T)\otimes\mathbb{Q}$ induces a scalar product on $\mathcal{X}^*(T)\otimes\mathbb{Q}$. The following lemma is an elementary calculation.

Lemma 2.2.4. *Let $S = \{l \in \mathcal{X}^*(T)\otimes\mathbb{Q} \mid \|l\| = 1\}$. Then there exists a constant A_G such that for each $l \in S$ if $l = \sum_{\alpha\in\Delta} r_\alpha w_\alpha$ then $|r_\alpha| \leq A_G$ for each $\alpha \in \Delta$.*

One notes that under the scalar product we have $\|l_a\| = 1$. Since l_a is trivial on the center of G, the Lemma (2.2.4) implies that $l_a = \sum_{\alpha\in\Delta} r_\alpha w_\alpha$ with $|r_\alpha| \leq A_G$.

This along with the above description of r and χ we get

$$\chi|_{Z_0(M')} = r\,\nu(\sigma_N\,,\,\lambda)\sum_{\alpha\in\Pi} r_\alpha w_\alpha \tag{2.9}$$

the last equality can be rewritten in terms of χ_α as follows.

$$-\chi = r\,\nu(\sigma_N\,,\,\lambda)\sum_{\alpha\in\Pi}(r_\alpha/m_\alpha)\chi_\alpha$$

Using the fact that $\deg(\chi_{\alpha*}(((F^N)^*E)_\tau)) \geq \mathrm{Ideg}((F^N)^*E)$ and combining it with the inequality (2.8) we obtain

$$\deg(\chi_*(((F^N)^*E)_\tau)) \leq -r\,B_G\,A_G\,|\Delta|\mathrm{Ideg}((F^N)^*E).$$

This along with (2.7) implies that there exists a constant $C(G)$ depending only on G such that $\deg(((F^N)^*(\sigma))^*E(\mathcal{L}_{P_1})) \geq C(G)\mathrm{Ideg}((F^N)^*E)$.

Since $((F^N)^*\sigma)^*(((F^N)^*E)(\mathcal{L}_{P_1})) = (\sigma^*(E(\mathcal{L}_{P_1})))^{p^N}$ it follows that $\deg(\sigma^*(E(\mathcal{L}_{P_1})) \geq (C(G)/p^N)\mathrm{Ideg}((F^N)^*E)$. By (2.6) we have $\deg T_\sigma \geq (C(G)/(m_{P_1}\,p^N))\mathrm{Ideg}((F^N)^*E)$ for every rational reduction σ of ρ_*E to P_1 which has the property that $\sigma(x_0)$ is not semistable.

This implies that we have a constant $C(G\,,\rho)$ such that

$$\mathrm{Ideg}(\rho_*E) \geq C(G\,,\rho)\frac{\mathrm{Ideg}((F^N)^*E)}{p^N}$$

By Corollary (2.2.2) and Lemma (1.1.4) we are through with the proof of the Theorem (0.0.1). □

Remark 2.2.5 The proof actually shows the following more general result. Let $\rho : G \to G'$ be a homomorphism of connected reductive groups which takes the identity connected component of the center of G to the center of G'. Then there exists constants C and C' (depending only on X, G, and ρ) such that for any rational G-bundle E over X we have

$$\mathrm{Ideg}(\rho_* E) \geq C\,\mathrm{Ideg}(E) + C'.$$

With this formulation, Theorem (0.0.2) is a special case of this result when applied to the Frobenius homomorphism of G.

2.3 Boundedness of semistable bundles

In this section we prove the boundedness of semistable G-bundles on X under the assumption stated in the introduction. From now on we work with G-bundles on X and not the rational ones.

Let $\mathcal{X}(G)$ be the group of characters of G. Let $A^k(X)$ be the k-th Chow group. To a principal G-bundle E, recall the definition of the degree $d_E \in \mathrm{Hom}(\mathcal{X}(G)\,,\, A^1(X))$ of a principal G bundle E from the introduction.

We fix a collection of elements $c_i \in A^i(X)$ for $2 \leq i \leq n = \dim(X)$ and also we fix an element $d \in \mathrm{Hom}(\mathcal{X}(G)\,,\, A^1(X))$.

Under the assumptions in Theorem (0.0.3), we will show that the set $\mathcal{S}_G(d\,;\, c_2\,, \ldots, c_n)$ of isomorphism classes of semistable G-bundles $\{E\}$ with degree $d_E = d$ and the Chern classes $c_i(\mathrm{ad}(E)) = c_i$ is bounded. Our proof is based on Proposition (4.12) of [39]

We begin with an elementary lemma which allows us to use representations.

Lemma 2.3.1. *There is a faithful completely reducible rational representation of G.*

Proof. For any irreducible representation ρ of G in a vector space V let $\ker(\rho)$ be the kernel. We first show that $N = \bigcap_\rho \ker(\rho)$, over all irreducible representations, is trivial. This is because if ρ_1 is a faithful representation of G (hence of N) on a vector space W then there is a filtration of W such that successive quotients are irreducible. This implies that $\rho_1(N) \subset \mathrm{GL}(W)$ lies in a parabolic subgroup and its image when composed with the projection to the Levi quotient is trivial. Hence N is a unipotent normal subgroup scheme. Let N_0 be the identity component of N. Since G is connected, using the conjugation map $G \times N_0 \longrightarrow N$ defined by $(g\,, n) \mapsto gng^{-1}$, we check that N_0 is normal. Again using the conjugation map, this time from $G \times (N_0)_{\mathrm{red}} \longrightarrow N$ we see that $(N_0)_{\mathrm{red}}$ is also normal. Since G is reductive this proves that N is a finite subgroup scheme of G. Now using the conjugation map for the third time we get that N is central and hence it is diagonalizable. Now we see that the representation ρ_1 restricted to N is trivial which is a contradiction.

Now by dimension and length count we can find finitely many irreducible representations ρ_i, for $i = 1, \ldots, m$, of G such that $\bigcap_{i=1}^m \ker(\rho_i) = 0$. This proves the lemma. □

We also have another general lemma.

Lemma 2.3.2. *Let $\rho : \mathrm{GL}(V) \longrightarrow \mathrm{GL}(W)$ be a representation of $\mathrm{GL}(V)$. Let E_1 and E_2 be two $\mathrm{GL}(V)$ bundles over X such that $c_i(E_1) = c_i(E_2)$ for each i. Then we have $c_i(E_1(W)) = c_i(E_2(W))$ for each i.*

Proof. Let $A^*(\mathrm{BGL}(V))$ (respectively $A^*(\mathrm{BGL}(W))$) be the Chow ring of $\mathrm{BGL}(V)$ (respectively $\mathrm{BGL}(W)$). Then one knows that $A^*(\mathrm{BGL}(V)) \cong \mathbb{Z}[c_1, \ldots, c_n]$(see [45]). The representation ρ gives rise to the map $\rho^* : A^*(\mathrm{BGL}(W)) \longrightarrow A^*(\mathrm{BGL}(V))$. Now we have the classifying maps $f_{E_i} : X \longrightarrow \mathrm{BGL}(V)$ for $i = 1, 2$. The conditions of the lemma imply that the induced maps $f_{E_i}^* : A^*(\mathrm{BGL}(V)) \longrightarrow A^*(X)$ are equal. This implies that the maps $f_{E_1}^* \circ \rho^* = f_{E_2}^* \circ \rho^*$. Hence the Lemma follows. □

We continue with the proof of the Theorem (0.0.3). By lemma (2.3.1), we have a completely reducible representation $\rho = \bigoplus \rho_i$ on $V = \bigoplus V_i$ of G which is faithful. Here V_i are the irreducible components.

For a fixed i, the representation ρ_i takes the identity connected component of the center of G to the center of $\mathrm{GL}(V_i)$. Hence for E and E' in $\mathcal{S}_G(d\,; c_2, \ldots, c_a)$, we have $c_1(E(V_i)) = c_1(E'(V_i))$.

The representation ρ_i induces a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathrm{End}(V_i)$. The lemma (2.3.2) now proves that the two vector bundles $E(\mathrm{End}(V_i)) = \mathrm{End}(E(V_i))$ and $E'(\mathrm{End}(V_i)) = \mathrm{End}(E'(V_i))$ have same Chern classes. Since the Chern classes of a vector bundle are completely determined by the first Chern class and the Chern classes of the endomorphism bundle we conclude that the Chern classes of $E(V_i)$ are are independent of the individual members E in $\mathcal{S}_G(d\,; c_2, \ldots, c_n)$.

By Theorem (0.0.1) there are constants $C_i = C(X, \rho_i)$ for the representation ρ_i.

Now the assumptions in Theorem (0.0.3) imply that there is a finite type scheme S_i and a family U_i of vector bundles over $S_i \times X$ which contains every member of $\mathcal{S}_{C_i}(r_i\,; c_1(E(V_i)), \ldots c_n(E(V_i)))$ occurs.

Now we take the scheme $S = \Pi_{i=1}^{m} S_i$ and $U = U_1 \times_X U_2 \times_X \ldots U_m$. Then Theorem (0.0.3) follows from the arguments in the last part of the Proposition 3.1 in [19].

Lemma 2.3.3. *Let $\mathcal{E}$ be a set of GL-bundles and $\mathcal{E}'$ be the set of $GL(V)$-bundles obtained from $\mathcal{E}$ by extension of structure group. Let $\mathcal{P} \to X \times W$ be a family of principal $GL(V)$-bundles on X parametrised by a scheme W of finite type over k such that all the bundles in $\mathcal{E}'$ occurs then there exists a scheme S of finite type over k which parametrised a family of G-bundles over X.*

Proof. The idea of the proof is construct a family of G-bundles parametrized by a scheme of finite type over k starting from P.

By the results of Grothendieck [14] there exists a W-scheme $S = \prod_{S\times X/W}((\mathcal{P}/G)/W \times X)$ which has the following universal property: for any W-scheme $Z \to W$, the set of sections of $(\mathcal{P}/G)_Z \to X \times_W Z$ is in bijective corrispondence with the set of sections of S_Z over Z. In particular, for $w \in W$, the fibers of $S \to W$ consists of the sections of the fibers bundle $\mathcal{P}_w/G \to X$, where $\mathcal{P}_w \times X$, and these are exactly the reductions of the $GL(V)$-bundle $\mathcal{P}_w$ to G.

Hence, the universal section of $(\mathcal{P}/G)_S \to X \times_W S$ gives a family of G-bundles parametrized by S, in which each bundle from the set $\mathcal{E}$ occurs. Finally, as G and $GL(V)$ are reductive groups, $GL(V)/G$ is affine and the vector space V can be chosen such that $GL(V)/G \to V$ is a $GL(V)$-equivariant closed embedding (see ([39])). The scheme S is a closed subscheme of the scheme $S' = \prod_{W\times X/W}(\tilde{V}/W \times X)$ where $\tilde{V}$ is the vector bundle associated to $\mathcal{P}$. Hence , S is of finite type over k. □

□

Remark 2.3.4 It should be possible to prove a version of the above Theorem for the case of principal G-sheaves in the sense of [16].

Chapter 3

Orthogonal and Symplectic bundles

In this chapter we want to study orthogonal and symplectic bundle. These G-bundles can be seen as vector bundles with some extra structure and using this we will prove the uniqueness of a canonical filtration in the sense of Ramanathan [37] and Behrend's conjecture [4] (characteristic different from 2 in the orthogonal case). We will also give an explicit description of the Harder-Narasimhan filtration in these cases.

In this chapter X will denote a smooth projective algebraic curve, but everything goes through for the general case by taking a codimension 2 open subscheme.

Let k be an algebraically closed field of positive characteristic and G an orthogonal or symplectic algebraic group over k (characteristic $\neq 2$ for the orthogonal group). Let E be principal G-bundle over X. In this case E is equivalent to a pair (V, b) where V is a vector bundle and $b : V \otimes_{\mathcal{O}_X} V \longrightarrow \mathcal{O}_X$ is a non-degenerate bilinear form which is either symmetric or skew-symmetric. Recall $F \subset E$ is isotropic subbundle of E if the restriction map $b : F \otimes_{\mathcal{O}_X} F \longrightarrow \mathcal{O}_X$ is identically zero. We will identify flags of isotropic subspaces with reduction to parabolic subgroups $P \subset G$ and we will formulate a equivalent definition of semistability for orthogonal and symplectic bundles.

We start with some notation. Let W be the vector space k^{2n} (or k^{2n+1}) with the standard basis (e_i). We denote $u_i = e_i$, $v_i = e_{i+n}$ for $1 \leq i \leq n$ and $w = e_{2n+1}$. The bilinear form is one of the following:

$$\begin{array}{rrl} b(u_i, v_i) = & b(v_i, u_i) = & b(w, w) = 1 \quad \text{for} \quad SO_{2n+1} \\ b(u_i, v_i) = & b(v_i, u_i) = & 1 \quad \text{for} \quad SO_{2n} \\ b(u_i, v_i) = & -b(v_i, u_i) = & 1 \quad \text{for} \quad Sp_{2n} \end{array}$$

It is a well known fact that parabolic subgroup of G are in bijective correspondence with isotropic flags. To be more precise parabolics subgroups are stabilizers of isotropic flags $V_1 \subset \cdots V_n$ where V_i is of dimension i(see for example [7] pag. 255). We will give an alternative definition of semistablity in the case of orthogonal and symplectic bundle. We will follow a proof due to Nitsure [34] that is more elementary than the original one due to Ramanathan [40].

Let $G \subset SL(W)$ be the subgroup preserving b. Let $P_r \subset G$ be the subgroup which carries the r-dimensional linear subspace $W_r =< u_1, \ldots, u_r >$ into itself $r < n$. Then P_r is a maximal parabolic subgroup of G. Let $T \subset G$ be the standard maximal torus, consisting of matrices of the form

$$diag(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1})$$

when W is even dimensional (that is, G is SO_{2n} or Sp_{2n}), and of the form

$$diag(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}, 1)$$

when W is odd dimensional (that is, G is SO_{2n+1}). Let for $1 \leq i \leq n$, the multiplicative character

$$\lambda_i : T \to \mathbf{G}_m$$

be defined to take the value a_i on the above diagonal matrices.

Let $\mathfrak{g}$ and $\mathfrak{p}_r$ be the Lie algebras of G and P_r respectively. Then T acts by adjoint action on the quotient $\mathfrak{g}/\mathfrak{p}_r$, and T acts on W_r by restriction of the defining representation of G on W.

The following lemma can be proved by an elementary calculation, which we omit.

Lemma 3.0.5. *(i) The torus T acts on* $\det(W_r)$ *by the character* $\alpha_r = \lambda_1 + \ldots + \lambda_r$.

(ii) The torus T acts on $\det(\mathfrak{g}/\mathfrak{p}_r)$ *by the character* χ_r*, given case by case as follows.*

$$\chi_r = \begin{cases} -(2n-r)\alpha_r & \text{if } G = SO_{2n+1} \\ -(2n-r+1)\alpha_r & \text{if } G = Sp_{2n} \\ -(2n-r-1)\alpha_r & \text{if } G = SO_{2n} \end{cases}$$

(iii) In particular, in each case χ_r is a negative multiple of the character α_r on $\det(W_r)$*, which we write as*

$$\chi_r = -C(n,r)\alpha_r$$

where $C(n,r) > 0$ is an integer.

Now let E be a principal G-bundle, where G is as above. Let (V, b) be the associated vector bundle together with a bilinear form b. Then for any isotropic subbundle $F_r \subset V$ of rank r, we get a reduction of structure group to (up to isomorphism) the parabolic subgroup P_r. If the transition functions are contained in the torus T, then it follows from the above lemma that the associated $\mathfrak{g}/\mathfrak{p}_r$-bundle $\mathcal{F}_r$ has degree equal to the strictly negative multiple

$$\deg(\mathcal{F}_r) = -C(n,r)\deg(F_r) \tag{3.1}$$

of the degree of F_r. Note that the bundle $\mathcal{F}_r$ is canonically isomorphis to the pullback $\sigma^*(T_\pi)$ of the relative tangent bundle of $\pi : E/P \longrightarrow X$ by the parabolic reduction σ. Hence if E is semistable, we must have $\deg(\mathcal{F}_r) \geq 0$. Hence the equation (3.1) now completes the proof of the 'only if' part of the following proposition.

Proposition 3.0.6. *The principal bundle corresponding to a pair (V, b) is semistable if and only if $\deg(F) \leq 0$ for any isotropic subbundle F of (V, b).*

For the 'if' part, just observe that any maximal parabolic subgroup of G is conjugate to one of the subgroups P_r that we have explicitly described.

If F is a subbundle of E we define $F^\perp$ as the kernel of $E \longrightarrow F^*$ where F^* denote the dual bundle with respect to the bilinear form b. Let $i : G \longrightarrow GL(V)$ be the underlying representation then, using the above fact, we have the following Proposition due to Ramanan [35].

Proposition 3.0.7. *A principal G-bundle E is semistable as an orthogonal or symplectic bundle (characteristic of $k \neq 2$ for orthogonal bundles) if and only if the underlying vector bundle is semistable.*

Proof. Let F be a subbundle of E. Consider the sheaf $N = F \cap F^\perp$. Let N' be the subbundle generated by N and consider the following exact sequence;

$$0 \longrightarrow F \oplus F^\perp \longrightarrow M' \longrightarrow 0,$$

where M' is the subbundle generated by $F + F^\perp$ and $M' = (N')^\perp$. Hence $\deg(F \otimes F^\perp) = deg(N') + deg((N')^\perp) = 2\deg(N')$. From the sequence

$$0 \longrightarrow (N')^\perp \oplus E \longrightarrow (N')^* \longrightarrow 0$$

we obtain $\deg(F) = \deg(N') \leq 0$ since N' is an isotropic subbundle by construction and using semistability. The converse of the proposition is straightforward. $\square$

Now we want to prove the existence and uniqueness of a canonical reduction. The method is similar to [2] and [3].

Definition 3.0.8. *A special filtration of a principal G-bundle $E = (V, b)$ is a filtration of V given by*

$$0 \subset V_0 \subset V_1 \subset \cdots \subset V_m \subset V_m^{\perp} \subset \cdots \subset V_1^{\perp} \subset V \tag{3.2}$$

such that $V_1, \ldots, V_m$ are isotropic subbundle of V and

1. *$\frac{V_i}{V_{i-1}}$ is semistable for any i.*

2. *$\left(\frac{V_m^{\perp}}{V_m}, \bar{b}\right)$ is a semistable orthogonal or symplectic bundle ($\bar{b}$ is the restriction of the bilinear form b to $V_m^{\perp}/V_m$).*

3. *$\mu(V_1) > \mu(\frac{V_2}{V_1}) > \cdots > \mu(\frac{V_m}{V_{m-1}}) > 0 = \mu(\frac{V_m^{\perp}}{V_m}) > \cdots > \mu(V_1^{\perp})$*

Proposition 3.0.9. *Any principal G-bundle $E = (V, b)$ admits an unique special filtration.*

Proof. If E is semistable there is nothing to prove. Let E non semistable and $V_1 \subset V$ with V_1 isotropic and such that $\mu(V_1)$ is maximal among all isotropic subbundle of V and V_1 is not contained in any isotropic subbundle with the same μ so V_1 is a semistable vector bundle. Consider the following sequence

$$0 \subset V_1 \subset V_1^{\perp} \subset V$$

Take $V_1^{\perp}/V_1$ and $\bar{b}$ the non-degenerate form on it. Suppose $V_1^{\perp}/V_1$ is not semistable then there exists $\overline{V_2} \subset V_1^{\perp}/V_1$ with $\overline{V_2}$, as before, is isotropic with maximal μ among all isotropic subbundle. We define V_2 as the inverse image of $\overline{V_2}$ under the map $p : V_2^{\perp} \to V_1^{\perp}/V_1$ and this imply that V_2 is isotropic by construction. By induction we get a filtration:

$$0 \subset V_0 \subset V_1 \subset \cdots \subset V_m \subset V_m^{\perp} \subset \cdots \subset V_1^{\perp} \subset V$$

such that V_i/V_{i-1} is semistable and since $(V_m^{\perp}/V_m, \bar{b})$ has no isotropic subbundle of positive degree (otherwise we must do a one step more in the filtration) is a semistable orthogonal or symplectic bundle.

Now we need to prove property (3). It is enough to prove that $\mu(V_1) > \mu(V_2/V_1) > \cdots$ because the other part follows immediately because $\mu(V_i^{\perp}/V_{i+1}^{\perp}) =$

$-\mu(V_{i+1}/V_i)$. First $\mu(V_1) > \mu(V_2)$, which follows from the maximality of $\mu(V_1)$ as V_2 is isotropic, implies $\mu(V_1) > \mu(V_2/V_1)$.

For the general case $\mu(V_i/V_{i-1}) > \mu(V_{i+1}/V_i)$ we consider the following sequence generated by $V_{i-1} \subset V_i \subset V_{i+1}$:

$$0 \longrightarrow \frac{V_i}{V_{i-1}} \longrightarrow \frac{V_{i+1}}{V_{i-1}} \longrightarrow \frac{V_{i+1}}{V_i} \longrightarrow 0$$

So it is enough to show $\mu(V_i/V_{i-1}) > \mu(V_{i+1}/V_{i-1})$. Consider the sequence $V_{i-1} \subset V_i \subset V_{i+1} \subset V_i^{\perp} \subset V_{i-1}^{\perp}$, since V_i/V_{i-1} has the property that $\mu(V_i/V_{i-1})$ is maximal among isotropic subbundle of $V_{i-1}^{\perp}/V_{i-1}$ hence $\mu(V_i/V_{i-1}) > \mu(V_{i+1}/V_{i-1})$ and this prove existence.

The proof of uniqueness is the same as in the vector bundles case. □

Lemma 3.0.10. *If $E = (V, b)$ is a G-bundle then a special filtration $0 \subset V_0 \subset V_1 \subset \cdots \subset V_m \subset V_m^{\perp} \subset \cdots \subset V_1^{\perp} \subset V$ is the Harder-Narasimhan filtration of the underlying vector bundle.*

Proof. This is a conseguence of the definition of special filtration and Proposition (3.0.7). □

We want to prove that a special filtration is the Harder-Narasimhan reduction of a G-bundle in the usual sense given by Ramanathan in [37]. To be more precise we will prove the following:

Theorem 3.0.11. *Let E_G a principal G-bundle on X. Then there exists a unique standard parabolic subgroup P of G and an unique reduction E_P of E_G to the subgroup P such that:*

1. *If U is the unipotent radical of P then the associated Levi bundle E_L, given by extension of the structure group, is semistable.*

2. *For any non-trivial character χ of P, which is a non-negative linear combination of simple roots of B, the line bundle on X associated to E_P by χ has strictly positive degree i.e. :*

$$deg(\chi_* \sigma^* E_P) > 0$$

in the case of orthogonal or symplectic bundle.

To do this we need some result taken from Biswas-Holla ([5]) We start with a general lemma on principal bundle.

Lemma 3.0.12. *Let E a principal G-bundle on X and $\rho : G \to G_1$ be a surjective homomorphism. Let H_1 be a closed subgroup of G_1. If the principal G_1-bundle $\rho_* E$ admits a reduction of the stucture group σ_1 to H_1 then the principla G-bundle E admits a reduction to the stucture group σ to $H = \rho^{-1}(H_1)$ such that $\rho'_* \sigma^* E = \sigma_1^* \rho_* E$, where $\rho' = \rho|_H$.*

Proof. By construction we have that $G/H \cong G_1/H_1$ so a reduction of $\rho_* E$ to H_1 gives a reduction of the structure group of E to H. □

Let P a parabolic group of G and consider the projection of P to its Levi subgroup $f : P \to L$. Let $\overline{P}$ a parabolic subgroup of the Levi then we have the following diagram

$$\begin{array}{ccc} P_1 & \longrightarrow & P \\ \downarrow & & \downarrow \\ \overline{P} & \longrightarrow & L \end{array}$$

where P_1 is the inverse image of $\overline{P}$ under the projection f such that $P/P_1 \cong L/\overline{P}$. Since the projection f is surjective we can apply Lemma (3.0.12) hence we have that a reduction of $\overline{P}$ to $\overline{\sigma}$ gives a reduction of the structure group of the principal P-bundle to P_1.

We apply the above fact to the following situation. Let (P, σ) be the canonical reduction of E and let P_2 be a parabolic subgroup of G containing P. We denote by L_2 the Levi factor of P_2 and by P' the image of P in L_2. E_{L_2} denotes the principal L_2-bundle $\sigma^* E(L_2)$ and (P', σ') denotes the reduction $\sigma^* E(P')$ of E_{L_2} to P' then

Proposition 3.0.13. *The reduction (P', σ') of the Levi-bundle E_{L_2} is its canonical reduction.*

Proof. The Levi factor of the parabolic subgroup P' of L_2, which we will denote by L_3, is the Levi factor L of P itself. This identifies the principal L-bundle $E_L := \sigma^* E(L)$ with the principal L_3-bundle $\sigma'^* E_{L_2}(L_3)$. Hence the semistability condition of E_L ensures that $\sigma'^* E_{L_2}(L_3)$ is semistable. □

Recall that if P is a maximal parabolic subgroup of $GL(V)$ the homogeneous space G/P can be identify with the Grassmannian $\mathrm{Gr}(W, V)$ of the $r = dim(W)$-dimensional subspace of V. Similarly If G is a orthogonal or symplectic group and P, as before, a maximal parabolic subgroup of G then G/P can be identify with the isotropic Grassmannian $\mathrm{IGr}(F, V)$ of isotropic subspaces of dimension $r = dim(F)$ of V. It is proven (see for example [25]) that $\mathrm{IGr}(r, V)$ is a closed subscheme of $\mathrm{Gr}(r, V)$ hence $T_{\mathrm{IGr}(r,V)} \hookrightarrow T_{\mathrm{Gr}(r,V)}$.

It is a well know fact that Zariski tangent space of the Grassmannian $\mathrm{Gr}(W, V)$ is isomorphic to $\mathrm{Hom}(W, V/W) = W^* \otimes V/W$. Let V_1 be an isotropic subbundle of V related to a maximal parabolic $P \subset G$. The inclusion $V_1 \subset V_1^{\perp} \subset V$ gives the exact sequence $0 \to V_1^{\perp}/V \to V/V_1 \to V/V_1^{\perp} \to 0$ hence $V/V_1 = V/V_1^{\perp} \oplus V_1^{\perp}/V_1 = V_1^* \oplus V_1^{\perp}/V_1$. We have:

$$T_{\mathrm{IGr}(V_1,V)} \; = \; V_1^* \otimes \frac{V}{V_1} = V_1^* \otimes V_1^* \oplus V_1^* \otimes \frac{V_1^{\perp}}{V_1}. \tag{3.3}$$

V_1 is isotropic so by definition $V_1^* \otimes V_1^*$ is identically zero hence

$$T_{\mathrm{IGr}(V_1,V)} \; = \; V_1^* \otimes \frac{V_1^{\perp}}{V_1} \tag{3.4}$$

Using the above fact we can prove the following Lemma:

Lemma 3.0.14. *The negativity of $deg(T_\sigma)$ imply the positivity of the slope of V_1 and $deg(T_\sigma)$ can be completely determinated by the slopes of V_1 and $V_1^{\perp}$.*

Proof. Recall $T_\sigma = \sigma^* T_{E/P}$ hence by equation (3.4) we have $T_\sigma = V_1^* \otimes \frac{V_1^{\perp}}{V_1}$. By an easy calculation we have the Lemma. □

Proposition 3.0.15. *A Harder-Narasimhan reduction of $E = (V, b)$ gives rise to a special filtration of V.*

Proof. Case1: Suppose the Harder-Narasimhan reduction is defined on the maximal parabolic subgroup of G then there is an isotropic subbundle V_1 of V which gives rise to the following sequence:

$$0 \subset V_1 \subset V_1^{\perp} \subset V$$

By the first property of the Harder-Narasimhan reduction we see that V_1 and $(V_1^{\perp}/V_1, \bar{b})$ are semistable because the Levi bundle decompose has $GL_{n_1} \times \overline{G}$ where n_1 is the rank of V_1 and $\overline{G}$ is a symplectic or orthogonal bundle.

We must show that $\mu(V_1) > 0$ but this follow from the second condition of the HN-reduction. i.e. $deg(T_\sigma) < 0$ and Lemma (3.0.14).

Case 2: we must extend the proof when P is not a maximal parabolic subgroup. A parabolic subgroup P of G defined an invariant flag $V_1, \ldots V_m$ of isotropic subbundle and hence we have a filtration:

$$0 \subset V_0 \subset V_1 \subset \cdots \subset V_m \subset V_m^{\perp} \subset \cdots \subset V_1^{\perp} \subset V.$$

We have to show that this is a special filtration. Let $V_1 \subset \cdots \subset V_n = \mathfrak{g}/\mathfrak{p}$ be a Jordan-Holder filtration of $\mathfrak{g}/\mathfrak{p}$ as a P-module. Then each V_i/V_{i-1} is an L-module and if we denote by abuse of notation with V_i/V_{i-1} the associated vector bundle of the Levi-bundle we have, using the second property of the Harder-Narasimhan reduction and Proposition (3.0.7), that V_i/V_{i-1} and $(V_m^{\perp}/V_m, \overline{b})$ are semistable.

We need to prove:

$$\mu(V_1) > \mu(\frac{V_2}{V_1}) > \cdots > \mu(\frac{V_m}{V_{m-1}}) > 0.$$

Let $P \subset P_1$ be the maximal parabolic subgroup on which there are a reduction σ_1 given by

$$0 \subset V_{i+1} \subset V_{i+1}^{\perp} \subset V$$

by Proposition (3.0.13) we see that if L_1 is the Levi of P_1 then the L_1-bundle associated to the reduction σ_1 is $V_{i+1} \times (V_{i+1}^{\perp}/V_{i+1}, \overline{b})$ and the Harder-Narasimhan reduction of this bundle is the extension of the Harder-Narasimhan reduction σ to P i.e. we take the Harder-Narasimhan filtation of the vector bundle V_{i+1} and the Harder-Narasimhan reduction of the bundle $(V_{i+1}^{\perp}/V_{i+1}, \overline{b})$. Because $0 \subset V_1 \subset \cdots \subset V_{i-1} \subset V_i \subset V_{i+1}$ is the Harder-Narasimhan reduction of V_{i+1} we get:

$$\mu(\frac{V_i}{V_{i-1}}) > \mu(\frac{V_{i+1}}{V_i}).$$

Using a similar argument and Lemma (3.4) the fact that $\mu(V_m/V_{m-1}) > 0$ follows and this concludes the proof. □

If we reverse the above argument we have the following

Proposition 3.0.16. *A special filtration of $E = (V, b)$ defines a Harder-Narasimhan reduction of E.*

Proof. The proof is similar to that of Proposition(3.0.15). □

This two Propositions relate the canonical reduction of an orthogonal or symplectic bundle with a Harder-Narasimhan fitration of the underlying vector bundle (special filtration). Using existence and uniqueness of the special filtration we have the following Corollary.

Corollary 3.0.17. *Let $E = (V, b)$ an orthogonal or symplectic bundle on an algebraic smooth variety X then there exists a Harder-Narasimahn reduction.*

Proof. By proposition(3.0.16) a special filtration defines an Harder-Narasimhan reduction so by the existence of the special filtration we get the result. □

Corollary 3.0.18. *Let $E = (V, b)$ an orthogonal or symplectic bundle on an algebraic smooth variety X then the Harder-Narasimhan reduction is unique.*

Proof. Using proposition (3.0.15) Harder-Narasimhan reduction defines a special filtration which is unique so we get the result. □

Corollary 3.0.19. *If $\rho : G \to GL(V)$ is an underlying representation then the Harder-Narasimhan reduction of a G-bundle $E = (V, b)$ extends to the Harder-Narasimhan filtration of the underlying vector bundle.*

Proof. This corollary follows immediately from the fact that the Harder-Narasimhan reduction gives rise to a special filtration of $V$and this is the Harder-Narasimhan filtration of E. □

We want to use these result to prove Behrend's conjecture [4] for orthogonal and symplectic bundles and then extend a result of Sun [43]. In particular we will give a weak form of Theorem (0.0.2).

Proposition 3.0.20 (Behrend's conjecture). *Let $E = (V, b)$ be a G-bundle (characteristic of $k \neq 2$ for orthogonal bundles) and let σ be the Harder-Narasimhan reduction of E to a parabolic subgroup P of G, then*

$$H^0(X, T_\sigma) = 0$$

To prove this Proposition we use the following Lemma:

Lemma 3.0.21. *Let $\rho : G \to G_1$ an injective homomorphism of groups which sends Z^0 to Z_1^0 where Z^0 and Z_1^0 are the connected component of the center. Let (P, σ) be the Harder-Narasimhan filtration of a G-bundle E and let P_1 a subgroup of G_1 such that $P_1 \cap G = P$. Suppose that the extension σ_1 of σ from a P-bundle to P_1 defines the Harder-Narasimhan reduction of the $E(G_1) = E_1$ and that Behrend's conjecture it is true for G_1 then is true for G.*

Proof. We have an injection $G/P \hookrightarrow G_1/P_1$ and this gives us the following commutative diagram:

$$\begin{tikzcd} E/P \arrow[r, "i"] \arrow[dr] & E_1/P_1 \arrow[d] \\ & X \end{tikzcd}$$

we denote by σ_1 and σ a Harder-Narasimhan reduction of E_1 and E. Now we have $T_{E/P} \hookrightarrow i^*T_{E_1/P_1}$ hence

$$T_\sigma = \sigma^* T_{E/P} \hookrightarrow \sigma^*(i^* T_{E_1/P_1}) = \sigma_1^* T_{E_1/P_1} = T_{\sigma_1}$$

Behrend's conjecture is true fo G_1 so $H^0(X, T_{\sigma_1}) = 0$ so it is true also for G. $\square$

From this Lemma the proof of Proposition (3.0.20) follows immediately if we take $G_1 = GL(V)$ using Corollary (3.0.19) and the fact that Behrend's conjecture is true for vector bundle [27].

Proposition 3.0.22. *If $E = (V, b)$ is a semistable G-bundle (characteristiic of $k \neq 2$ for orthogonal bundles) on smooth projective curve of genus $g \geq 2$ and let $F : X \to X$ be the absolute Frobenius morphism then, if we denote $\tilde{E} = F^*E$*

$$\mu_{max}(\tilde{E}) - \mu_{min}(\tilde{E}) \leq 2(n-1)(2g-2)$$

Proof. With the notation of Sun [43] let $\tilde{E} = F_X^*E$. We consider the Harder-Narasimhan filtration of the associated vector bundle given by a special filtation. Hence we consider the following exact sequences $0 \to \tilde{E}_i \to \tilde{E} \to \tilde{E}/\tilde{E}_1 \to 0$ and $0 \to \tilde{E}_i^{\perp} \to \tilde{E} \to \tilde{E}/\tilde{E}_i^{\perp} \to 0$. By Cartier (A.0.24) theorem we obtain that $Hom(T_X \otimes \tilde{E}_i, \tilde{E}/\tilde{E}_i) \neq 0$ and $Hom(T_X \otimes \tilde{E}_i^{\perp}, \tilde{E}/\tilde{E}_i^{\perp}) \neq 0$.

Thus we have $\mu_{min}(T_X \otimes \tilde{E}_i) - \mu_{max}(\tilde{E}/\tilde{E}_i) \leq 0$ and $\mu_{min}(T_X \otimes \tilde{E}_i^{\perp}) - \mu_{max}(\tilde{E}/\tilde{E}_i^{\perp}) \leq 0$. As in Sun we have that

$$\mu_{min}(T_X \otimes \tilde{E}_i) = \mu(\tilde{E}_i/\tilde{E}_{i-1}) + 2 - 2g$$

$$\mu_{max}(\tilde{E}/\tilde{E}_i) = \mu(\tilde{E}_{i+1}/\tilde{E}_i)$$

and

$$\mu_{min}(T_X \otimes \tilde{E}_i^{\perp}) = \mu(\tilde{E}_i^{\perp}/\tilde{E}_{i+1}^{\perp}) + 2 - 2g$$

$$\mu_{max}(\tilde{E}/\tilde{E}_i^{\perp}) = \mu(\tilde{E}_{i-1}^{\perp}/\tilde{E}_i^{\perp})$$

Now using the sequence $0 \to \tilde{E}_i^{\perp} \to \tilde{E} \to \tilde{E}/\tilde{E}_i^{\perp} \cong \tilde{E}^* \to 0$ and using the fact that $\mu(\tilde{E}) = -\mu(\tilde{E}^{\vee})$ we obtain that $\mu(\tilde{E}_i^{\perp})/\tilde{E}_{i+1}^{\perp} = -\mu(\tilde{E}_{i+1}/\tilde{E}_i)$ and $\mu(\tilde{E}_{i-1}^{\perp}/\tilde{E}_i^{\perp}) = -\mu(\tilde{E}_i/\tilde{E}_{i-1})$ from which we have that

$$\mu_{max}(\tilde{E}) - \mu_{min}(\tilde{E}) \leq 2(n-1)(2g-2)$$

$\square$

Appendix A

Brief Introduction to descent theory

We want briefly to describe Grothendieck's descent theory [15]. Let $\alpha : S' \to S$ be a morphism of schemes and consider the functor $\mathcal{F} \to \alpha^*\mathcal{F}$ from quasi-coherent S-module to quasi-coherent S'-module. The characterization of the image of the above functor is performed using Grothendieck's descent theory.

We denote by $S'' = S' \times_S S'$ and let $\mathbf{p}_i : S'' \to S'$ be the projection onto the i-th factor ($i = 1, 2$). For any quasi-coherent S'-module $\mathcal{F}'$ we call **covering datum** of $\mathcal{F}'$ an S'' -isomorphism $\phi : p_1^*\mathcal{F}' \to p_2^*\mathcal{F}'$. A pairs $(\mathcal{F}', \phi)$ of quasi-coherent S'-module with covering datum form a category whose morphism between two object $(\mathcal{F}', \phi)$ and $(\mathcal{G}', \psi)$ consists of an S'-morphism $f : \mathcal{F}' \to \mathcal{G}'$ which is compatible with the covering data ϕ and ψ i.e. the following diagram:

$$\begin{array}{ccc} p_1^*\mathcal{F}' & \xrightarrow{\phi} & p_2^*\mathcal{F}' \\ \downarrow{\scriptstyle p_1^* f} & & \downarrow{\scriptstyle p_2^* f} \\ p_1^*\mathcal{G}' & \xrightarrow{\psi} & p_2^*\mathcal{G}' \end{array}$$

is commutative.

A natural covering datum on $\alpha^*\mathcal{F}$, where $\mathcal{F}$ is a quasi-coherent S-module, is given by the following canonical isomorphism:

$$p_1^*(\alpha^*\mathcal{F}) \cong (\alpha \circ p_1)^*\mathcal{F} = (p \circ p_2)^*\mathcal{F} \cong p_2^*(\alpha^*\mathcal{F}).$$

So the functor $\mathcal{F} \to \alpha^*\mathcal{F}$ can be interpreted as a functor into the category of quasi-coherent S'-module with covering data. Recall that a covering datum is **effective** if the pair $(\mathcal{F}, \phi)$ is isomorphic to the pull-back $\alpha^*\mathcal{F}$ of a quasi-coherent S-module $\mathcal{F}$.

If $\alpha : S' \to S$ is fully faithfull and quasi-compact the functor $\mathcal{F} \to \alpha^*\mathcal{F}$ is an equivalence of category if we introduce a cocycle condition for descent data. More precisely we denote by $S''' = S' \times_S S' \times_S S'$ and let $p_{ij} : S''' \to S''$ be the projections onto the factors with indices i and j for $i < j$; $i, j = 1, 2, 3$. A quasi-coherent S'-module $\mathcal{F}'$ with covering datum $\phi : p_1^*\mathcal{F}' \to p_2^*\mathcal{F}'$ is a **descent datum** if the following cocycle condition is satisfied:

$$p_{13}^*\phi = p_{23}^*\phi \circ p_{12}^*\phi.$$

Recall that if $\alpha : S' \to S$ is faithfully flat and quasi-compact then the functor $\mathcal{F} \to \alpha^*\mathcal{F}$ from the category of quasi-coherent S-modules to quasi-coherent S'-modules with descent data is an equivalence of category (see [15] Theorem 1 pag. 17 or [10]).

One can generalized the above result to S-schemes considering quasi-coherent algebra (in particular structure sheaves of schemes over S) instead of quasi-coherent modules. More precisely we consider the functor $X \mapsto \alpha^*X$, where $\alpha : S' \to S$, from the category of S-schemes to S'-schemes with descent data. In this framework Grothendieck proves the following:

Theorem A.0.23 (Grothendieck). *Let $\alpha : S' \to S$ a faithtfully flat and quasi-compact morphism of schemes. If α is purely inseparable then the functor $X \mapsto \alpha^*X$ from the category of S-schemes to S'-schemes with descent data is an equivalence of category. If α is of finite type then is an equivalence with respect to the category of projective schemes.*

Proof. See [15] Theorem 3 pag. 20. □

The purely inseparability of the morphism α in necessary to ensure effectiveness.

We want to give an application of the above Theorem. We start introducing some notions. Let $\alpha : S' \to S$ a smooth morphism of schemes and $\mathcal{E}$ a quasi-coherent sheaf of $\mathcal{O}_{S'}$-modules. A **S-connection** on $\mathcal{E}$ is a homorphism ∇ of abelian sheaves:

$$\nabla : \mathcal{E} \longrightarrow \Omega^1_{S'/S} \otimes_{\mathcal{O}_{S'}} \mathcal{E}$$

such that

$$\nabla(ge) = g\nabla(e) + dg \otimes e$$

where g and e are sections of $\mathcal{O}_{S'}$ and $\mathcal{E}$ respectively over an open subset of S' and $d : \mathcal{O}_{S'} \to \Omega^1_{S'/S}$. Recall the curvature $R = R(\mathcal{E}, \nabla)$ of the S-connection is an $\mathcal{O}_{S'}$-linear map

$$R = \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_S} \Omega^2_{S'/S}.$$

A S-connection ∇ is called **integrable** if $R = 0$.

Let $\mathcal{D}er(S'/S)$ denote the sheaf of germs of S-derivations of $\mathcal{O}_{S'}$ into itself. $\mathcal{D}er(S'/S)$ is naturally a sheaf of $\alpha^{-1}(\mathcal{O}_S)$-Lie algebras, while as $\mathcal{O}_{S'}$-module it is isomorphic to $\mathrm{Hom}_{\mathcal{O}_S}(\Omega^1_{S'/S}, \mathcal{O}_S)$.

Let $\mathcal{E}nd_S(\mathcal{E})$ denote the sheaf of germs of $\alpha^{-1}(\mathcal{O}_S)$-linear endomorphism of $\mathcal{E}$and is naturally a sheaf of $\alpha^{-1}(\mathcal{O}_{S'})$-Lie algebras. Fixing a S-connection ∇ on $\mathcal{E}$ we have the following $\mathcal{O}'_S$-linear map:

$$\nabla : \mathcal{D}er(S'/S) \longrightarrow \mathcal{E}nd_S(\mathcal{E})$$

sending D to $\nabla(D)$, where $\nabla(D)$ is given by

$$\mathcal{E} \xrightarrow{\ \nabla\ } \Omega^1_{S'/S} \otimes_{\mathcal{O}_{S'}} \mathcal{E} \xrightarrow{D\otimes 1} \mathcal{O}_{S'} \otimes_{\mathcal{O}_{S'}} \mathcal{E} \cong \mathcal{E}.$$

We have

$$\nabla(D)(fe) = D(f)e + f\nabla(D)(e)$$

where D, f and e are sections of $\mathcal{D}er(S'/S)$, $\mathcal{O}_{S'}$ and $\mathcal{E}$ respectively over an open subset of S'.

It is easy to see that the map $\nabla : \mathcal{D}er(S'/S) \to \mathcal{E}nd_S(\mathcal{E})$ is a Lie-algebra homomorphism when ∇ is a integrable S-connection i.e.

$$[\nabla(D_1)\, ,\, \nabla(D_2)] = \nabla([D_1\, ,\, D_2])$$

where D_1 and D_2 are sections of $\mathcal{D}er(S'/S)$.

We consider now connections in characteristic p. Let S be a scheme of characteristic $p > 0$, i.e. $p\mathcal{O}_S = 0$. Suppose that $\alpha : S' \to S$ is a morphism purely inseparable of finite type. Let $\mathcal{D}er(S'/S)$ be the sheaf of S derivation of S'. Recall $\mathcal{D}er(S'/S)$ is a sheaf of Lie-algebra with respect to S, but not to S' and the following condition holds:

$$[X\, ,\, fY] = X(f)Y + f[X\, ,\, Y]$$

where $X, Y \in \mathcal{D}er(S'/S)$ and f is a section of $\mathcal{O}_{S'}$.

Recall (see [7] pag . 62) is a sheaf of restricted p-Lie algebra:

$$\begin{aligned} XY - YX &= [X\, ,\, Y] \\ X^p &= X^{[p]} \end{aligned}$$

Since $\mathcal{E}nd(\mathcal{E})$ is also a sheaf of restricted p-Lie algebra its natural to ask when the following map

$$\nabla : \mathcal{D}er \longrightarrow \mathcal{E}nd_S(\mathcal{E}) \tag{A.1}$$

is an homomorphism of restricted p-Lie algebra and this is equivalent to give a descent datum on a quasi-coherent sheaf $\mathcal{E}$ on S' ([15] pag. 21). The map (A.1) is an homomorphism if the S-connection is integrable, as seen before, and is compatible with the p-structure i.e.

$$\nabla(D^p) = (\nabla(D))^p.$$

If we apply Theorem (A.0.23) we see that there is an equivalence of category between quasi-coherent sheaves over S and quasi-coherent sheaves over S' with an linear, integrable p-connection with zero p-curvature.

We can apply the above result to the Frobenius map defined in the previous section and we obtain the following:

Theorem A.0.24 (Cartier). *There is an equivalence of categories between the category of quasi-coherent sheaves on $X^{(1)}$ and the category of $\mathcal{O}_X$-modules with integrable k-connections, whose p-curvature is zero. This equivalence is given by $E \to (F_X^* E, \nabla_{can})$ and $(G, \nabla) \to Ker\nabla$.*

In [23] the above Theorem is proven directly without using descent theory.

As first apllication of the above result we want to prove that the Harder-Narasimhan filtration is stable under a purely inseparable base field extension $K \subset L$. Recall the definition of Harder-Narasimhan filtration for vector bundles:

Definition A.0.25. *Let E be a vector bundle of rank r and degree d over X. An Harder-Narasimhan filtration is an increasing filtration of subbbundles $V_0 \subset \cdots \subset V_r = V$ such that:*

1. *V_i/V_{i-1} is semistable for any i*

2. *$\mu(V_i) \geq \mu(V_{i+1}/V_i)$ for any i*

We will denote by $HN(E)$ the Harder-Narasimhan filtration. It is a classical result the following Lemma

Lemma A.0.26. *If F and E are vector bundles over X and $\mu_{\min}(F) > \mu_{max}(E)$ then $\mathrm{Hom}(F, E) = 0$.*

Proposition A.0.27. *Let E a vector bundle of rank r and degree d on a smooth algebraic variety S and let L a purely inseparable field extension of K then:*

$$HN_{\bullet}(E \otimes_K L) = HN_{\bullet}(E) \otimes_K L \tag{A.2}$$

Proof. If $F \subset E$ is a destabilizing subbundle then so is $F \otimes_K L \subset E \otimes_K L$. Hence if $E \otimes_K L$ is semistable, then E is also semistable. We must prove that there exists a filtration of $E_{\bullet}$ of E such that $HN_i(E \otimes_K L) = E_i \otimes_K L$. Consider the morphism $\alpha : S' \to S$ where S' and S are respectively a L and K schemes and let E be a vector bundle on S. We denote by E' the pull back of E by α. By Cartier Theorem (A.0.24) there exist a canonical connection on E'. Let F' be a proper subbundle of E' and consider the map

$$\phi : F' \to \frac{E'}{F'} \otimes \Omega^1_{S'/S} \tag{A.3}$$

Using Theorem A.0.23 if the map ϕ is zero then the differential descents on F' and there exists a subbundle F of E such that $\alpha^* F = F'$. Consider the following cartesian diagram:

$$\begin{tikzcd}
S' \arrow[r] \arrow[d, "h"'] & S \arrow[d] \\
Spec(L) \arrow[r] & Spec(K)
\end{tikzcd}$$

from which follows $\Omega^1_{S'/S} \simeq h^*(\Omega^1_{L/K})$ and so it is a trivial vector bundle. We get

$$\begin{aligned}
Hom\left(F', \frac{E'}{F'} \otimes \Omega^1_{S'/S}\right) &= H^0\left(F'^* \otimes \frac{E'}{F'} \otimes \Omega^1_{S'/S}\right) \\
&= H^0\left(F'^* \otimes \frac{E'}{F'}\right) \otimes V = 0
\end{aligned}$$

from the second property of the Harder-Narasimhan filtration and Lemma A.0.26 we have $H^0(F'^* \otimes E'/F') = 0$.

□

Appendix B

Boundedness of semistable vector bundles on surface

Since our problem of boundedness is reduced to a problem on vector bundles for reason of completition we give a proof of boundedness of set of semistable vector bundle with fixed Hilbert polynomial following Gieseker [12].

This problem was solved by Takemoto [44] for a rank 2 vector bundles and then by Gieseker and Maruyama.

. Let E be a coherent torsion free sheaf on X of rank r at a generic point of X. Let p_E the polynomial given by

$$p_E(n) = \frac{\chi(E(n))}{r} \tag{B.1}$$

where $\chi(E)$ is the Hilbert polynomial of E

$$\chi_E(n) = \chi(E(n)) = h^0(F(n)) - h^1(E(n)) + h^2(E(n))$$

Recall E is said to be semistable with respect to an ample divisor H if for any coherent subsheaf F of E we have $p_F \leq_E$. We want to prove boundedness of set of semistable coherent torsion free sheaf on X with fixed Hilbert polynomial. Recall this means that the topological data $deg(E), c_1(E), c_2(E)$ are fixed.

We will use a Theorem of Kleiman [26]:

Theorem B.0.28. *Let $\mathcal{E}$ be a family of coherent sheaves on X with the same Hilbert polynomial P. Then this family is bounded if and only if there are constant A and B for any $E \in \mathcal{E}$ $h^0(E) \leq A$ and $h^0(E \otimes \mathcal{O}_D) \leq B$ for some divisor D linearly equivalent to H.*

Lemma B.0.29. *Let E a H semistable coherent torsion free sheafon X. If $\chi(E)$ is positive and $deg(E^{\vee} \otimes K, H) < 0$ then $H^0(X, E) \neq 0$.*

Proof. Let $\chi(E) = h^0(E) - h^1(E) + h^2(E)$. We apply Serre duality at $H^2(E)$.

$$H^2(E) = H^0(Hom_{\mathcal{O}_X}(E, K)) = H^0(E^{\vee} \otimes K)$$

Using semistability and the fact that $deg(E^{\vee} \otimes K, H) < 0$ we get that $H^2(E) = 0$, so $0 < \chi(E) = h^0(E) - h^1(E) < h^0(E) \Rightarrow H^0(E) \neq 0$. □

Corollary B.0.30. *Let S be a set of H-semistable coherent torsion free sheaves of rank r on X with fixed Chern classes. Then there exists an integer n such that $H^0(E(n)) \neq 0$ for any $E \in S$.*

Proof. Let $E \in S$,

$$\chi(E(n)) = \frac{H^2}{2} n^2 + (\frac{degE}{rkE} - \frac{degk}{2})n + const.$$

where $degE = H.c_1(E)$ and K is the canonical sheaf of X. For $n \gg 0$ $\chi(E(n)) > 0$ because is dominated by the term $\frac{H^2}{2}n^2$.

$$deg(E^{\vee}(n) \otimes K) = -degE - rnH^2 + rdegK < 0$$

for

$$n_1 > \frac{degE + rdegK}{rH^2}$$

Because the Hilbert polynomial is fixed for $n > n_1$and lemma 1 $H^0(E(n)) \neq 0$. □

Let $P = \chi_E$ and denote by $\triangle P(n) = P(n) - P(n-1)$. Without loss of generality we can assume that E has H degree negative hence $h^0(E) < 0$.

Is easy to check that P satisfied the following:

1. Every non-trivial subsheaf of E has negative degree
2. $\triangle P \geq \triangle p_E$
3. $h^0(E(n)) \geq r\,P(n)$ for some $n \geq N$ (B.0.29)

Theorem B.0.31. *Let E a semistable torsion free sheaf of rank r with Hilbert polynomial $P = \chi_E$ satisfying the above properties then there exists a costant B such that $h^0(E \otimes \mathcal{O}_D) < B$ for some divisor D linearly equivalent to H.*

Proof. Let E be a semistable coherent torsion free sheaf. Let H_n be the smallest coherent subsheaf of $E(n)$ such that $H^0(X, H_n) = H^0(X, E(n))$ and $E(m)/H_n$ is torsion free. First note that $deg(H_n) \geq 0$ so $H_0 = 0$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^0(H_n) \arrow[r, "\cong"] & H^0(E(n)) \arrow[r] & H^0(E(n)/H_n) \\
0 \arrow[r] & H^0(H_n(-p)) \arrow[r] \arrow[u] & H^0(E(n-p)) \arrow[r] \arrow[u] & H^0(E(n-p)/H_n(-p)) \arrow[u] \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

which gives $H^0(H_n(-p)) = H^0(E(n-p))$ for all non-negative integers.

From the above fact we have that the subsheaf H'_n of $E(n)$ generated by $H^0(E(n)$ is that of $H_{n+p}(-p)$ generated by $H^0(H_{n+p}(-p))$. Let H''_n be the inverse image of the torsion part of $H_{n+p}(-p)/H'_n$ by the natural homomorphism $H_{n+p}(-p) \to H_{n+p}(-p)/H''_m$. Then $E(n)/H''_m$ is torsion free because so are $E(n)/H_{n+p}(-p)$ and $H_{n+p}(-p)/H''_n$. Since $H^0(H''_m) = H^0(H_{n+p}(-p)) = H^0(E(n))$, we have $H''_n = H_n$ and this means that $H_n(p)$ is a subsheaf of H_{n+p}.

Let D be a smooth curve linearly equivalent to H so that all H_n's are locally free on D and so that for some point $x \in D$, the stalk oh H_n at x is generated by global sections. From the exact sequence

$$0 \longrightarrow H_n(-1) \longrightarrow H_n \longrightarrow H_n \otimes \mathcal{O}_D \longrightarrow 0 \tag{B.2}$$

we have

$$\begin{aligned} h^0(E(n)) &= h^0(H_n) \leq h^0(H_n(-1)) + h^0(H_n \otimes \mathcal{O}_D) \\ &= h^0(E(n-1)) + h^0(H_n \otimes \mathcal{O}_D). \end{aligned} \tag{B.3}$$

Summing up this inequality we have;

$$h^0(E(n)) \leq \sum_{i=0}^{n} h^0(H_n \otimes \mathcal{O}_D) \tag{B.4}$$

Let g the genus of D and $n_1, \cdots, n_k$ be the integers such that $H_{n_k} \neq H_{n_k-1}(1)$ with $k < r$. This means that for $n > n_k$ $H_n = E_n$ is generated by global sections. If we denote by r_n the rank of H_n then from the following exact sequence:

$$0 \longrightarrow \mathcal{O}_D^{r_n} \longrightarrow H_n \otimes \mathcal{O}_D \longrightarrow Q \longrightarrow 0$$

where Q is supported at a finite number of points, we have

$$h^1(H_n \otimes \mathcal{O}_D) \le r_n\, g \tag{B.5}$$

hence

$$h^0(H_n \otimes \mathcal{O}_D) \le \chi(H_n \otimes \mathcal{O}_D) + r_n\, g \tag{B.6}$$

We note that except for a finite number of points we can estimate $h^0(H_n \otimes \mathcal{O}_D)$ by $\chi(H_n \otimes \mathcal{O}_D)$. Using (B.6) we get

$$h^0(E(n)) \le \sum_{i=0}^{n} h^0(H_i \otimes \mathcal{O}_D) \le \sum_{i=0}^{n} \chi(H_i \otimes \mathcal{O}_D) + K$$

where K is some constant depending on r , g and H^2. If $n < n_k$ there exists a function $g(n)$ (see [12] for its construction) depending on r , n and H^2 such that

$$\sum_{i=0}^{n_k-1} \chi(H_i \otimes \mathcal{O}_D) \le g(n).$$

Let L be a constant such that if $n \le L$

$$g(n) - rP(n) \le -K$$

If $n < n_k$

$$h^0(E(n)) - rP(n) \le g(n) - rP(n) + K.$$

hence if $n > L$ and $h^0(E(n)) > rP(n)$ we must have $n \ge n_k$.

If $n \ge n_k$ hence $E(n_k) \otimes \mathcal{O}_D$ is generated by its global sections hence

$$\chi(H_n \otimes \mathcal{O}_D) \;=\; \triangle\chi_E(n) \le r\triangle P(n) \tag{B.7}$$

from which we get

$$\begin{aligned} \sum_{i=n_k}^{n} \chi((H_i \otimes \mathcal{O}_D) \;&\le\; \sum_{i=n_k}^{n} r\triangle P(n) \\ &\le\; r(P(n) - P(n_k - 1)). \end{aligned} \tag{B.8}$$

Suppose $n_k \ge L$ then

$$h^0(E(n)) - rP(n) \le g(n_k) - rP(n_k - 1) + K < 0$$

so we obtain that n_k is bounded above by a constant L.

If $m \geq n_k + (2g-2)/H^2$ then $h^1(E(m) \otimes \mathcal{O}_D) = 0$ because is generated by its global sections. Fixed an integer $k \geq N$ and let $B = r(P(k) - P(k-1))$ then

$$\begin{aligned} h^0(E \otimes \mathcal{O}_D) &\leq h^0(E(k) \otimes \mathcal{O}_D) = \chi(E(k) \otimes \mathcal{O}_D) \\ &\leq r(P(k) - P(k-1)) = B \end{aligned} \tag{B.9}$$

and this complete the proof.

$\square$

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