Scuola Internazionale Superiore di Studi Avanzati - Trieste

Time-dependent Singular Interactions

MICHELE CORREGGI

Mathematical Physics Sector
International School for Advanced Studies

Supervisor
Prof. GIANFAUSTO DELL'ANTONIO

Thesis submitted for the degree of "Doctor Philosophiae"
in Mathematical Physics

Academic Year 2003-04

SISSA – Via Beirut 2-4 – 34014 TRIESTE – ITALY

Contents

Introduction

The aim of this thesis is the study of time-dependent singular interactions, namely singular time-dependent perturbations of the Laplacian supported on "small" sets and described by formal Schrödinger operators of the form

$$-\Delta + V_t(\vec{x})$$

for $\vec{x} \in \mathbb{R}^d$ with $d = 2, 3$, where the potential $V(\vec{x})$ is singular, i.e. is supported on sets of Lebesgue measure zero. Roughly speaking the potential can be thought to be $V(\vec{x}) = \alpha\,\delta(\vec{x} - S)$ in $L^2(\mathbb{R}^d, d\mu_{\mathcal{L}})$, where $\mu_{\mathcal{L}}(S) = 0$ and $\mu_{\mathcal{L}}$ is the Lebesgue measure. This kind of operators can be rigorously defined by means of the theory of self-adjoint extensions of symmetric operators.
The time-dependences we shall consider are of two kinds, time-dependent coupling constants[1] $\alpha(t)$ and moving supports[2] S_t. The Hamiltonians of such systems will be explicitly given in both cases, together with their resolvent, spectrum and unitary propagator.
Moreover, using Laplace transform techniques, we shall solve the problem of asymptotic complete ionization for systems with time-dependent point interactions. In the case of rotating singular perturbations we shall study also the asymptotic limit of large angular velocity, proving convergence in the strong sense of the time propagator to some one-parameter unitary group with time-independent generator.

Let us start with some historical remarks about point interactions.
Schrödinger operators with point interactions were first introduced by Kronig and Penney [37], as a model for one-dimensional crystals. At the same time point interactions were proposed as realistic potentials describing the two-body nuclear interaction at low energy by Bethe and Peierls [6], Thomas [53] and Fermi [29].
Up to now point interactions (also called Fermi pseudo-potentials) have been

[1]The precise physical and mathematical meaning of the coupling constant shall be clarified later on.

[2]Actually we shall only study uniformly rotating supports.

extensively used in nuclear physics and in condensed matter physics (see e.g. [7, 22, 39]).
Nevertheless in almost all the physical applications above the operator $-\Delta + \alpha\delta(\vec{x}-\vec{y})$ has only a formal meaning and is used to perform first order perturbation theory (i.e. the Born approximation) calculations. Indeed subsequent orders give rise to divergent terms and so perturbation theory can not be used.
The first rigorous definition of Hamiltonians with point interactions as self-adjoint operators is due to Berezin and Faddeev [5] in 1961: applying Krein's theory, they define such operators as self-adjoint extensions of the Laplacian restricted to smooth functions vanishing in a neighborhood of the point $\vec{y}$, where the singular perturbation is supported.
In fact many different methods can be employed to define rigorously the Schrödinger operator $-\Delta + \alpha\delta(\vec{x}-\vec{y})$ (see [3] and references therein). For instance the singular potential $\delta(\vec{x}-\vec{y})$, which is not an operator perturbation in dimension greater than one, even in terms of the associated quadratic forms, can be approximated by regular potentials $V_\varepsilon(\vec{x})$, as $\varepsilon \to 0$, with a suitable scaling in ε and a suitable renormalization, in such a way that the corresponding Schrödinger operators H_ε converge in strong resolvent sense.
Indeed let us consider the resolvent associated to the operator $H = -\Delta - \mu V$ in three dimensions: for $\lambda > 0$,

$$\big(H+\lambda\big)^{-1} = \big(-\Delta - \mu V + \lambda\big)^{-1} = \big(1-\mu(-\Delta+\lambda)^{-1}V\big)^{-1}(-\Delta+\lambda)^{-1} =$$

$$= (-\Delta+\lambda)^{-1} + \sum_{n=1}^{\infty}\big(\mu(-\Delta+\lambda)^{-1}V\big)^n(-\Delta+\lambda)^{-1}$$

and then, substituting $V(\vec{x}) = \delta(\vec{x})$,

$$\big(H+\lambda\big)^{-1}(\vec{x};\vec{x}') = G_\lambda(\vec{x}-\vec{x}') + G_\lambda(\vec{x})\,\big(\mu^{-1} - G_\lambda(0)\big)^{-1}G_\lambda(\vec{x}')$$

where

$$G_\lambda(\vec{x}) = \frac{e^{-\sqrt{\lambda}\,|\vec{x}|}}{4\pi|\vec{x}|}$$

is the Green function associated to the free Hamiltonian $-\Delta$.
Of course the expression above is formal because $G_\lambda(0)$ does not exist, but if we formally set

$$\mu^{-1} = G_0(0) + \alpha$$

for some $\alpha \in \mathbb{R}$, we shall see that the expression above coincide with the kernel of the resolvent associated to the Hamiltonian with a point interaction at the origin.

Indeed the usual way to define Hamiltonians with a point interaction at $\vec{y}$ is the analysis of self-adjoint extensions of the symmetric closable operator

$$H = -\Delta$$

on the domain of smooth function with support away from $\vec{y}$, i.e.

$$\mathcal{D}(H) = C_0^\infty(\mathbb{R}^d - \{\vec{y}\})$$

with $d = 2, 3$. The operator H has deficiency indexes equals to $(1, 1)$ and its self-adjoint extensions are given by the one-parameter family of operators H_α with domain

$$\mathcal{D}(H_\alpha) = \Big\{\Psi \in L^2(\mathbb{R}^d) \,\big|\, \exists \Phi_\lambda \in H^2(\mathbb{R}^d), \Psi = \Phi_\lambda(\vec{x}) + \\ + \big(\alpha + \sqrt{\lambda}/4\pi\big)^{-1} \Phi_\lambda(\vec{y})\, G_\lambda(\vec{x} - \vec{y})\Big\} \qquad (I.1)$$

where $\lambda > 0$ and $G_\lambda(\vec{x})$ is the Green function of $-\Delta$ in d dimensions, i.e.

$$G_\lambda(\vec{x} - \vec{x}') \equiv \big(- \Delta + \lambda\big)^{-1}(\vec{x}; \vec{x}') \qquad (I.2)$$

Moreover

$$\big(H_\alpha + \lambda\big)\Psi = \big(- \Delta + \lambda\big)\Phi_\lambda \qquad (I.3)$$

and

$$\big(H_\alpha + \lambda\big)^{-1}(\vec{x}; \vec{x}') = G_\lambda(\vec{x} - \vec{x}') + \frac{4\pi}{4\pi\alpha + \sqrt{\lambda}}\, G_\lambda(\vec{x} - \vec{y})\, G_\lambda(\vec{x}' - \vec{y}) \qquad (I.4)$$

We want to stress that the domain $(I.1)$ can be written in a slightly different way, having in mind an analogy with electrostatics,

$$\mathcal{D}(H_\alpha) = \big\{\Psi \in L^2(\mathbb{R}^d) \,\big|\, \exists q \in \mathbb{C}, \Psi - q\, G_\lambda(\vec{x} - \vec{y}) \in H^2(\mathbb{R}^d)\big\} \qquad (I.5)$$

where the "charge" q is fixed by the boundary condition,

$$\lim_{\vec{x} \to \vec{y}} \Big\{\Psi - q\, G_\lambda(\vec{x} - \vec{y})\Big\} = \left(\alpha + \frac{\sqrt{\lambda}}{4\pi}\right) q \qquad (I.6)$$

For any $-\infty < \alpha \leq \infty$ and $\vec{y} \in \mathbb{R}^d$, the essential spectrum of H_α is purely absolutely continuous and covers the nonnegative real axis, i.e.

$$\sigma_{\text{ess}}(H_\alpha) = \sigma_{\text{ac}}(H_\alpha) = [0, \infty) \qquad (I.7)$$

In three dimensions, if $\alpha < 0$, there is exactly one negative, simple eigenvalue, i.e.

$$\sigma_p(H_\alpha) = \{-(4\pi\alpha)^2\} \tag{I.8}$$

with strictly positive normalized eigenfunction

$$\varphi_\alpha(\vec{x}) = \sqrt{2|\alpha|}\,\frac{\exp\{4\pi\alpha|\vec{x}-\vec{y}|\}}{|\vec{x}-\vec{y}|} \tag{I.9}$$

while, if $\alpha \geq 0$,

$$\sigma_p(H_\alpha) = \emptyset \tag{I.10}$$

On the other hand in two dimensions, for any value of $\alpha \in \mathbb{R}$, there exists one negative, simple eigenvalue,

$$\sigma_p(H_\alpha) = \{-4\exp[-4\pi\alpha + 2\Psi(1)]\} \tag{I.11}$$

with normalized eigenfunction

$$\varphi_\alpha(\vec{x}) = (i/4)\, H_0^{(1)}[2i\exp\{-2\pi\alpha + \Psi(1)\}\,|\vec{x}-\vec{y}|] \tag{I.12}$$

Strictly speaking, the parameter α, obtained after the renormalization procedure described above, is not the strength of the interaction, but it is associated to the inverse of a scattering length, indeed it is easy to see that the scattering length of H_α is precisely $-1/4\pi\alpha$.
The previous discussion clarifies also the reason why point interactions do not exist in dimension greater that three: the operator $-\Delta$, restricted to the domain of smooth functions vanishing in a neighborhood of $\vec{y}$, is indeed essentially self-adjoint in $L^2(\mathbb{R}^d)$, for $d \geq 4$ and then it has a unique self-adjoint extension, which coincides with the free Hamiltonian.
By means of the same method, it is also possible to give a rigorous meaning in $L^2(\mathbb{R}^d)$ to Schrödinger operators of the form

$$-\Delta + \alpha\,\delta(\vec{x} - S)$$

where $S \subset \mathbb{R}^d$ is a set of lower codimension, e.g. a plane in three dimensions or a straight line in two dimensions. For a detailed account of the subject see [51, 52] and references therein.
Suppose for instance that S is a regular closed surface in three dimensions, the quadratic form associated to the previous formal operator is given by

$$F_{\gamma,S}(\Psi,\Psi) = \int_{\mathbb{R}^3} d^3\vec{x}\,|\nabla\Psi|^2 - \int_S d\Sigma\,\gamma(\vec{x})\,\left|\Psi(\vec{x})|_S\right|^2 \tag{I.13}$$

where $d\Sigma$ is the restriction of the Lebesgue measure to S and $\gamma(\vec{x})$ some suitable smooth, real valued function on S, which never vanishes. Setting

$$\xi_\Psi(\vec{x}) \equiv \gamma(\vec{x})\,\Psi(\vec{x})\big|_S \tag{I.14}$$

$$\phi_\lambda(\vec{x}) \equiv \Psi(\vec{x}) - \big(\tilde{G}_\lambda\xi\big)(\vec{x}) \tag{I.15}$$

and, for every $x \in \mathbb{R}^3$,

$$\big(\tilde{G}_\lambda\xi\big)(\vec{x}) \equiv \int_S d\Sigma(\vec{y})\, G_\lambda(\vec{x}-\vec{y})\,\xi(\vec{y}) \tag{I.16}$$

G_λ denoting the Green function (*I.2*), the quadratic form (*I.13*) can be rewritten in the following way

$$F_{\gamma,S}(\Psi,\Psi) = \mathcal{F}_S^\lambda(\Psi,\Psi) + \Phi_{\gamma,S}^\lambda(\xi_\Psi,\xi_\Psi)$$

where

$$\mathcal{F}_S^\lambda(\Psi,\Psi) \equiv \int_{\mathbb{R}^3} d^3\vec{x}\,\left|\nabla\phi_\lambda\right|^2 + \lambda\int_{\mathbb{R}^3} d^3\vec{x}\,|\phi_\lambda|^2 - \lambda\int_{\mathbb{R}^3} d^3\vec{x}\,|\Psi|^2 \tag{I.17}$$

$$\Phi_{\gamma,S}^\lambda(\xi_\Psi,\xi_\Psi) \equiv \int_S d\Sigma(\vec{x})\,\frac{|\xi_\Psi(\vec{x})|^2}{\gamma(\vec{x})} - \int_S d\Sigma(\vec{x})\,\xi_\Psi^*(\vec{x})\,\big(\tilde{G}_\lambda\xi\big)(\vec{x}) \tag{I.18}$$

Since the quadratic form (*I.18*) is bounded and then closed in $L^2(S,d\Sigma)$ (we denote the corresponding bounded self-adjoint operator with $\Gamma_{\gamma,S}^\lambda$) and the form (*I.17*) is bounded from below and closed in $L^2(\mathbb{R}^3)$, the whole form $F_{\gamma,S}(\Psi,\Psi)$ is closed. Hence it defines a self-adjoint operator $H_{\gamma,S}$ and it is not difficult to see that it is given by

$$\mathcal{D}(H_{\gamma,S}) = \Big\{\Psi \in L^2(\mathbb{R}^3)\mid \exists\, \xi_\Psi \in L^2(S,d\Sigma),\, \Psi - \tilde{G}_\lambda\xi_\Psi \in H^2(\mathbb{R}^3),$$

$$\Big[\Psi - \tilde{G}_\lambda\xi_\Psi\Big]\Big|_S = \Gamma_{\gamma,S}^\lambda\xi_\Psi\Big\} \tag{I.19}$$

$$\big(H_{\gamma,S}+\lambda\big)\Psi = \big(H_0+\lambda\big)\phi_\lambda \tag{I.20}$$

Up to now many models of point interactions has been suggested to study problems like the interaction between a quantum particle and a polymer [24], the constrained motion of a quantum particle inside a wave guide [28, 20], the quantum Rayleigh gas [21], the rigorous derivation of the Gross-Pitaevskii equation from a microscopic dynamics [2], justifying a certain interest in singular perturbations of the Laplacian.

However the main motivation in investigating such models from a mathematical and physical point of view is that Schrödinger operators with point interactions are "solvable", in the sense that their resolvent, spectrum, eigenfunctions, as well as their scattering data can be determined explicitly. Such nice features make point interactions very useful to build simplified physical models, that maintain all the relevant aspects of the problem and, at the same time, can be treated in an easier way. We are referring to what Simon [49] calls the "second level foundation" problems of quantum mechanics, which are of particular physical interest in condensed matter physics.
A remarkable example of such problems is the study of the long time behavior of a quantum system evolving under a time-dependent Hamiltonian of the form

$$H(t) = H_0 + H_I(t) \tag{I.21}$$

From a physical point of view, this problem is strictly related to the ionization of atoms and dissociation of molecules: in the usual setting a system is in a bound state at time $t = 0$ and then, switching on an external time-dependent potential $H_I(t)$, one would know the probability of survival $\mathcal{P}(t)$ of the bound states for $t > 0$. From the time-dependent Schödinger equation

$$i\frac{\partial \Psi_t}{\partial t} = \big(H_0 + H_I(t)\big)\Psi_t$$

one obtain the expression for $\mathcal{P}(t)$,

$$\mathcal{P}(t) = \sum_n \Big|\Big(\Psi_t\,,\, u_n\Big)\Big|^2$$

where the sum runs over all the bound states of the system.
This problem has been already investigated in the early years of quantum mechanics and, if the external potential is small enough to be treated as a time-dependent perturbation of the free Hamiltonian, a solution is given by the well known Fermi's golden rule (see e.g. [8, 30, 38]). Starting from the ground state of the system, the behavior at large time of the survival probability is given by

$$\mathcal{P}(t) \sim e^{-\Gamma_F t} \tag{I.22}$$

where Γ_F is proportional to (the squared modulus of) the matrix elements of H_I between bound and scattering states times the scattering state density. In deriving (*I.22*), one applies perturbation theory, assuming that the perturbation series converges uniformly for all times and that the limits $t \to \infty$ and $n \to \infty$ (n is the order of perturbation) can be interchanged. This assumption remains unproved, even for small smooth time-dependent potentials and

indeed, even if H_I is in some sense small with respect H_0, it is possible (see [50, 17]) to find special potentials, such that the long time behavior of the survival probability is an inverse power law decay.
The problem is much harder when the external potential is not small in any sense with respect H_0. For instance Bayfield and Koch (see [4] for a review), analysing experimentally the ionization of an atom by an intense electric field, found an unexpected nonlinear dependence of $\mathcal{P}(t)$ on the initial and external data. Such nonlinear behavior is also verified in experiments on strong laser field ionization of multielectron atoms and dissociation of molecules: in many cases increasing the strength of the field reduces the probability of ionization of the system[3].
It has been shown by Costin et al. [12, 13, 14, 15, 16], that the same features are present in a simplified model involving attractive point interactions. These authors consider a one-dimensional system with an Hamiltonian of the form (*I.21*), where $H_I(t)$ is given[4] by $-\alpha(t)\delta(x)$, for some $\alpha(t) \geq 0$:

$$H(t) = H_0 - \alpha(t)\delta(x)$$

Using non-perturbative methods, thanks to the solvability of point interactions models, they deal with the problem of asymptotic complete ionization and find that under suitable and very weak conditions on the function $\alpha(t)$, the survival probability of the bound state has a power law decay at large time. On the other hand, if the conditions are not satisfied by $\alpha(t)$, there is asymptotic partial ionization, i.e. the survival probability does not tend to zero. Moreover they show (see [16]) a connection between this behavior and the stabilization phenomenon cited above.
This analysis shows that point interactions models may be very useful as toy models in approaching complicated time-dependent problems, like the asymptotic complete ionization, the power law decay of the survival probability and nonlinear phenomena.
The first part of this thesis is devoted to the study of two such models.
In Chapter 1 we consider[5] the time evolution of a three dimensional system given by a quantum particle under a time-periodic zero-range interaction. The Hamiltonian of the system is then formally given by

$$H(t) = H_0 + \alpha(t)\delta(\vec{x})$$

[3]This phenomenon is often called stabilization. For a review on theoretical models about it see for example [23, 32].

[4]Since the model is one-dimensional, the distribution $\alpha(t)\delta(x)$ defines a quadratic form which is small with respect to the quadratic form associated with the Laplacian [3, 45].

[5]The contents of Chapter 1 are a more detailed version of [10].

or, more rigorously, by the the self-adjoint operator (*I.3*) with domain (*I.1*) and (*I.5*), where the parameter $\alpha = \alpha(t)$ is a continuous periodic function of t and $\vec{y} = 0$. Studying the time-evolution, we prove complete ionization of the system as $t \to \infty$, starting from a bound state at $t = 0$, under suitable generic conditions on $\alpha(t)$; namely we show that the survival probability of the bound state has a power law decay as $t \to \infty$ (with exponent -3).
A remarkable feature of this model is that the asymptotic ionization holds, whatever is the value of the parameter $\alpha(t)$, provided that it satisfies the genericity condition. In particular this means that, even if for every $t \in \mathbb{R}^+$ $\alpha(t) < 0$ and then there exists a bound state at any given time (see (*I.8*) and (*I.9*)), nevertheless the survival probability decay to zero. On the other hand, if $\alpha(t)$ is always positive (and then there is no bound state), the result does not require the genericity condition (see the Remark at the end of Chapter 1 and Appendix A).
Moreover, under the same genericity conditions, we derive a stronger version of the ionization statement, i.e. every state Ψ in the Hilbert space of the system is a scattering state. According to the geometric time-dependent scattering theory (for a review see [25, 34, 46]), the scattering states of a system with time-dependent Hamiltonian $H(t)$ are the Hilbert states Ψ, which satisfy the property,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t d\tau \int_S d^3\vec{x} \, \left|\Psi_\tau(\vec{x})\right|^2 = 0 \qquad (I.23)$$

for every compact set $S \subset \mathbb{R}^3$, $\Psi_t = U(t,0)\Psi$ denoting the time evolution of the state Ψ.
Another interesting feature of time-periodic scattering theory is related to the eigenvalues of the Floquet operator associated to $H(t)$ (see [33, 35, 57]): given a time-periodic Hamiltonian (with period T), one can define the Floquet operator

$$\mathcal{F} \equiv -i\frac{\partial}{\partial t} + H(t) \qquad (I.24)$$

acting on $L^2(\mathbb{T}_T) \otimes L^2(\mathbb{R}^3)$, where $\mathbb{T}_T$ is the torus $\mathbb{R}/T\mathbb{Z}$. It is usually claimed, but proved only for a small class of potentials, that there is a (one-to-one) correspondence between the bound states of the system and the eigenvalues of $\mathcal{F}$. As a consequence of absence of bound states, we show that the discrete spectrum of the Floquet operator is empty, although the explicit relation between the genericity condition and the discrete spectrum of $\mathcal{F}$ remains unknown.
The main ingredient in the proofs is the analysis of the time-evolution equation satisfied by the "charge" $q(t)$ (see (*I.5*) and (*I.6*) for its definition),

which is a Volterra-type integral equation with weakly singular kernel (see e.g. [18]). The asymptotic behavior of $q(t)$ for large time is obtained from the study of singularities of its Laplace transform, essentially applying the analytic Fredholm theorem to the transformed equation. The cited genericity condition on the Fourier coefficients of $\alpha(t)$ enters explicitly to exclude the existence of non-zero solutions of the associated homogeneous equation.

In Chapter 2 we study[6] a slightly more complicated model: we assume that the Hamiltonian of the three-dimensional quantum system has the form *(I.21)*, where the unperturbed Hamiltonian describes a particle interacting with a zero-range time-independent attractive perturbation, while $H(t)$ is a time-periodic point interaction, i.e. formally

$$H(t) = H_0 + a\delta(\vec{x}) + \alpha(t)\delta(\vec{x} - \vec{r})$$

The N point interactions Hamiltonians can be defined in the same way as *(I.3)* (see [3]), i.e. identifying the self-adjoint extensions of the symmetric operator given by the Laplacian on the domain of smooth functions with compact support away from the centers of the interaction. Such self-adjoint extensions form a N parameters family and their domain is given again by L^2-functions which can be decomposed in a regular part $\Phi_\lambda \in H^2(\mathbb{R}^3)$ and a singular part given by the sum

$$\sum_{i=1}^{N} q_i \, G_\lambda(\vec{x} - \vec{y}_i)$$

where $\vec{y}_i$ denotes the position of the i-th point interaction and the "charges" q_i solve a boundary condition similar to *(I.6)*.
Hence in our case one has to deal with two "charges", which solve two coupled integral equations and depend both on time. However the Laplace transform techniques used before work also for these equations. Indeed, by means of almost the same methods, we prove asymptotic complete ionization in the generic case, with the same decay for the survival probability, and absence of generalized bound states.
We want to stress that this result is somehow unexpected, since there is no requirement on both the "strength" and the frequency of the time-dependent perturbation. Indeed even if $\alpha(t)$ is very large (small) for every t, so that the perturbation is in fact small (large) in some sense, the system shows asymptotic complete ionization; moreover this is still true for a frequency so small that the perturbation can be considered time-adiabatic. These features

[6] A brief review of these results can be found in [11].

witness the highly non-perturbative nature of the results about ionization.

In the second part of this thesis, we deal with a simple case of moving singular perturbations of the Laplacian, namely the uniformly rotating case. The formal time-dependent Schrödinger operators we consider are

$$H(t) = H_0 + V_t = -\Delta + V_t \tag{I.25}$$

on $L^2(\mathbb{R}^n)$, $n = 2, 3$, with uniformly rotating potentials

$$V_t(\vec{x}) = V(\mathcal{R}^{-1}(t)\, \vec{x}) \tag{I.26}$$

where V is a singular potential (e.g. formally $V(\vec{x}) = \delta(\vec{x} - \vec{y}_0)$) and $\mathcal{R}(t)$ a rotation on the $x, y-$plane with period $2\pi/\omega$.
As pointed out by Enss et al. [25], uniformly rotating Hamiltonians can be studied in a simpler way than general time-dependent operators. Considering the time evolution $U_{\text{rot}}(t, s)$ of the system in a uniformly rotating frame around the $z-$axis, it is easy to see that the following relation with the time evolution in the inertial frame $U_{\text{inert}}(t, s)$ holds

$$U_{\text{inert}}(t, s) = R(t)\, U_{\text{rot}}(t - s)\, R^{\dagger}(s) \tag{I.27}$$

where $R(t)\Psi(\vec{x}) = \Psi(\mathcal{R}(t)^{-1}\, \vec{x})$ and $U_{\text{rot}}(t, s) = U_{\text{rot}}(t - s)$ is the one-parameter unitary group

$$U_{\text{rot}}(t - s) = e^{-iK(t-s)} \tag{I.28}$$

with a time-independent generator K, formally defined in the following way

$$K = H_0 - \omega J + V \tag{I.29}$$

Here J stands for the third component of the angular momentum and V is the time-independent potential (*I.26*).
Our purpose is to define in a rigorous way the time-dependent Hamiltonians (*I.25*) when the potential has a singular behavior: we study[7] rotating point perturbations of the Laplacian in 2 and 3 dimensions (Chapter 3) and rotating blades (Chapter 4), namely rotating singular potentials supported on sets of codimension 1 (a segment in 2 dimensions and an half-disc in 3 dimensions respectively).
Using this trick suggested by Enss et al., we define such time-dependent Hamiltonians considering the corresponding formal time-independent generators in the rotating frame and studying their self-adjoint extensions. This

[7] Chapters 3 and 4 refer to [9].

goal is achieved by means of Krein's theory of self-adjoint extensions and studying the associated quadratic forms for rotating blades respectively. In particular, since the operator $H_\omega = H_0 - \omega J$ has a spectrum unbounded from below, we need to use some limit procedure, in order to give a rigorous meaning to rotating blade.

In this way we give an explicitly expression for such Hamiltonians, their resolvent, spectrum and propagator: the use of the rotating frame is very useful to avoid the problem associated to the definition of time-dependent operators, as for regular potentials.

The last goal is the analysis of the asymptotic limit of the systems when the angular velocity $\omega \longrightarrow \infty$: by means of the explicit expression of resolvents we prove convergence in strong sense of $U_{\text{inert}}(t,s)$ to some one-parameter unitary group $U_{\text{asympt}}(t-s)$ with time-independent generator H_{asympt}. Moreover we show that, for point interactions, H_{asympt} is given by the Laplacian with singular perturbation on a circle, while the asymptotic limit of the rotating blade is simply a regular potential supported on a compact set.

Part I

Time-dependent Point Interactions

Chapter 1

Ionization for a Time-dependent Point Interaction

1.1 Introduction

This Chapter is devoted to the study of asymptotic complete ionization for a quantum particle subjected to a time-dependent zero-range potential, starting from a bound state of the system at the initial time.
In Section 1.2 we introduce the model and the associated time evolution, which is given in terms of a corresponding evolution equation for the "charge" (see (*I.5*), (*I.6*) and the following (1.6)). After having described the assumptions on the time-dependent parameter $\alpha(t)$, which identifies the Hamiltonian of the system, we focus our attention on the equation for the charge and we state two results about the large time behavior of the solution (Proposition 1.1) and the absence of a non-zero solution of the associated homogeneous equation (Proposition 1.3). The first result together the unitarity of time evolution guarantees existence and analyticity of the Laplace transform of the solution in the open right half-plane (Proposition 1.2). Therefore we introduce the Laplace transform of the equation (1.6), in order to infer the asymptotic behavior for large time of the "charge" from the singular behavior of its Laplace transform in a neighborhood of the origin.
The analysis of the Laplace transform of the "charge" is performed in three slightly different ways, according to the sign of the 0-th Fourier coefficient α_0 of $\alpha(t)$. In fact the results do not depend on the sign of α_0 but the proof has to be suitably modified. In Section 1.3 we consider the case $\alpha_0 < 0$, in Section 1.4, $\alpha_0 = 0$ and in Section 1.5, $\alpha_0 > 0$ respectively.

In all the three cases we study, we extend the transformed equation on the imaginary axis and we check the singularities of the solution there. By means of the analytic Fredholm theorem we prove analyticity of the solution on the imaginary axis for $p \neq i\omega n$, $n \in \mathbb{Z}$ (Propositions 1.5, 1.11, 1.13). In the case $\alpha_0 < 0$ we need an extra discussion, due to the presence of poles in the coefficients of the equation, but, using the genericity condition, we explicitly show that the solution is in fact analytic there (Propositions 1.6).
In the same way, we prove that the solution has branch point singularities at $p = i\omega n$, $n \in \mathbb{Z}$, where it behaves like the square root of p (Propositions 1.7, 1.12, 1.14). This point of the proof requires the genericity condition (1.19), except for $\alpha(t) \geq 0$, $\forall t \in \mathbb{R}^+$ (see the Remark at the end of the Chapter). If $\alpha_0 < 0$ an additional proof is needed in the resonant case, i.e. $\omega = 1/N$ for some integer N (Proposition 1.8).
A straightforward consequence of the above results is the power law decay of the survival probability of the bound state (Theorem 1.1 and Corollary 1.1, Theorem 1.3 and Corollary 1.3, Theorem 1.5). Moreover, extending the discussion to a large class of initial states, we prove that all the state of the system are scattering states (Theorems 1.2, 1.4, 1.6) and that, in the generic case, the Floquet operator has empty point spectrum.

1.2 The model

The model we are going to study is a quantum particle subjected to a time-dependent point interaction fixed at the origin in three dimensions, namely a system defined by the time-dependent self-adjoint Hamiltonian $H_{\alpha(t)}$,

$$\mathcal{D}(H_{\alpha(t)}) = \Big\{ \Psi \in L^2(\mathbb{R}^3) \;\big|\; \exists\, q_\lambda(t) \in \mathbb{C},\, \big(\Psi(\vec{x}) - q_\lambda(t)\, G^\lambda(\vec{x})\big) \in H^2(\mathbb{R}^3),$$

$$\big(\Psi - q_\lambda(t)\, G^\lambda\big)\big|_{\vec{x}=0} = \left(\alpha(t) + \frac{\sqrt{\lambda}}{4\pi}\right) q_\lambda(t) \Big\} \tag{1.1}$$

$$\big(H_{\alpha(t)} + \lambda\big)\Psi = \big(H_0 + \lambda\big)\big(\Psi - q_\lambda(t)\, G^\lambda\big) \tag{1.2}$$

where $\lambda \in \mathbb{R}$, $\lambda > 0$ and

$$G^\lambda(\vec{x} - \vec{x}') = \frac{e^{-\sqrt{\lambda}|\vec{x}-\vec{x}'|}}{4\pi|\vec{x} - \vec{x}'|}$$

is the Green function of the free Hamiltonian $H_0 = -\Delta$.
According to the discussion contained in the Introduction, the Schrödinger operator above can be interpreted as

$$-\Delta + H_I(t)$$

where formally

$$H_I(t) = \alpha(t)\delta(\vec{x})$$

Actually one should use this heuristic argument with some care, since the parameter $\alpha(t)$ is not the coupling constant, because of the renormalized required in order to define point interactions in three dimensions. In fact it is proportional to the inverse of the scattering length.
At any fixed time t, the operator (1.2) has absolutely continuous spectrum if $\alpha(t)$ is positive, while, when $\alpha(t) < 0$, there exists exactly one negative eigenvalue $-(4\pi\alpha(t))^2$, with normalized eigenfunction

$$\varphi_{\alpha(t)}(\vec{x}) \equiv \frac{\sqrt{2|\alpha(t)|}\, e^{4\pi\alpha(t)|\vec{x}|}}{|\vec{x}|} \tag{1.3}$$

It is well known (see [19, 31, 48, 56]) that the operator (1.2) defines a time propagation $U(t,s)$ given by a two-parameters unitary family, solving the time-dependent Shrödinger equation

$$i\frac{\partial \Psi_t}{\partial t} = H_{\alpha(t)}\Psi_t \tag{1.4}$$

and

$$\Psi_t(\vec{x}) = U(t,s)\,\Psi_s(\vec{x}) = U_0(t-s)\Psi_s(\vec{x}) + i\int_s^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x}) \tag{1.5}$$

where $U_0(t) = \exp(-iH_0t)$, $U_0(t;\vec{x})$ is the kernel associated to the free propagator and the charge $q(t)$ satisfies a Volterra integral equation for $t \geq s$,

$$q(t) + 4\sqrt{\pi i}\int_s^t d\tau\, \frac{\alpha(\tau)q(\tau)}{\sqrt{t-\tau}} = 4\sqrt{\pi i}\int_s^t d\tau\, \frac{(U_0(\tau)\Psi_s)(0)}{\sqrt{t-\tau}} \tag{1.6}$$

In order to prove the result above, it is sufficient to study the time evolution of the quadratic form associated to $H_{\alpha(t)}$, starting from a initial state in $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ (so that $q(0) = 0$) and using the ansatz (1.5) about the form of the state at time t. The result can be then extended to any initial state in $L^2(\mathbb{R}^3)$ by the unitarity of the evolution. For a complete review of the proof see [18], where the procedure is applied to a slightly different case, namely when the position of the point interaction moves along a smooth path while the parameter α does not depend on time.
We are interested in studying complete ionization of system defined by (1.2) and (1.4), starting from initial conditions

$$\Psi_0(\vec{x}) = \varphi_{\alpha(0)}(\vec{x}) \tag{1.7}$$

$\varphi_{\alpha(0)}(\vec{x})$ being the bound state[1] of $H_{\alpha(0)}$.
We assume that $\alpha(t)$ is a real periodic continuous function with period T. The meaningful parameter of the system is the negative lower bound of $\alpha(t)$. Indeed, if $\inf(\alpha(t)) \geq 0$, the wave operator associated to $(H_0, H_{\alpha(t)})$ is unitary (see [56]) so that any initial state evolves into a scattering state (see also the Remark at the end of Section 1.5). Hence we require that

$$1. \quad \alpha(0) < 0 \tag{1.8}$$

Continuity of $\alpha(t)$ guarantees that it can be decomposed in a Fourier series and the series converges uniformly on every compact subset of the real line. In terms of the Fourier coefficients of $\alpha(t)$, we assume

$$\begin{aligned} &2. \quad \alpha(t) = \sum_{n\in\mathbb{Z}} \alpha_n \, e^{-in\omega t}, \ \{\alpha_n\} \in \ell_1(\mathbb{Z}), \ \omega = \frac{2\pi}{T} \\ &3. \quad \alpha_n = \alpha^*_{-n} \end{aligned} \tag{1.9}$$

In order to apply the Laplace transform to (1.6), we are going to prove that the solution $q(t)$ is bounded for any finite time and does not diverge worse than e^{ct} as $t \to \infty$, for some positive constant c: indeed in that case the Laplace transform $\tilde{q}(p)$ of $q(t)$ exists (and is analytic) for $\Re(p)$ sufficiently large.

Proposition 1.1 *For any $t \in \mathbb{R}^+$, the solution $q(t)$ of (1.6) satisfies the following estimate,*

$$|q(t)| \leq C \, e^{ct}$$

where

$$c \equiv \left(4\pi \sup_{t\in\mathbb{R}^+} |\alpha(t)|\right)^2$$

and $C < \infty$.

Proof: We start noticing that from (1.6) we have

$$|q(t)| \leq 4\sqrt{\pi} \, \sup(|\alpha|) \int_0^t d\tau \, \frac{|q(\tau)|}{\sqrt{t-\tau}} + 4\sqrt{\pi} \int_0^t d\tau \, \frac{\left|\left(U_0(\tau)\Psi_0\right)\right|(0)}{\sqrt{t-\tau}} \tag{1.10}$$

from which we deduce that $\eta(t) - |q(t)| \geq 0$, if $\eta(t)$ is the unique solution of the equation

$$\eta(t) = 4\sqrt{\pi} \, \sup(|\alpha|) \int_0^t d\tau \, \frac{\eta(\tau)}{\sqrt{t-\tau}} + 4\sqrt{\pi} \int_0^t d\tau \, \frac{\left|\left(U_0(\tau)\Psi_0\right)\right|(0)}{\sqrt{t-\tau}} \tag{1.11}$$

[1]In order to do this analysis we shall require that $\alpha(0) < 0$.

Iterating (1.11) once and differentiating we obtain for η the differential equation

$$\frac{d\eta}{dt} = 16\pi^2\big(\sup(|\alpha|)\big)^2\eta + 16\pi^2\big|\big(U_0(t)\Psi_0\big)\big|(0) \tag{1.12}$$

where the inhomogeneous term is finite at each time t with, at most, an integrable singularity at $t = 0$. We conclude that

$$|q(t)| \leq \eta(t) \leq C\, e^{16\pi^2(\sup(|\alpha|))^2 t} \tag{1.13}$$

□

As a consequence the Laplace transform of $q(t)$, denoted by

$$\tilde{q}(p) \equiv \int_0^\infty dt\, e^{-pt} q(t)$$

exists analytic at least for $\Re(p) > 16\pi^2\big(\sup(|\alpha|)\big)^2$.
Applying the Laplace transform to equation (1.6), one has

$$\tilde{q}(p) = -4\pi\sqrt{\frac{i}{p}} \sum_{k\in\mathbb{Z}} \alpha_k\, \tilde{q}(p + i\omega k) + \tilde{f}(p) \tag{1.14}$$

where

$$\tilde{f}(p) \equiv \frac{2\sqrt{2|\alpha(0)|}}{\pi}\sqrt{\frac{i}{p}}\int_0^\infty dt\, e^{-pt}\int_{\mathbb{R}^3} d^3\vec{k}\, \frac{e^{-ik^2 t}}{k^2 + (4\pi\alpha(0))^2} =$$

$$= 8\sqrt{\frac{2|\alpha(0)|}{ip}}\int_0^\infty dk\, \frac{k^2}{(k^2 + (4\pi\alpha(0))^2)(k^2 - ip)} =$$

$$= 4\pi i\sqrt{\frac{2|\alpha(0)|}{-ip}}\, \frac{4\pi\alpha(0) + \sqrt{-ip}}{(4\pi\alpha(0))^2 + ip}$$

and with the choice of the branch cut for the square root along the negative real line: if $p = \varrho\, e^{i\vartheta}$,

$$\sqrt{p} = \sqrt{\varrho}\; e^{i\vartheta/2} \tag{1.15}$$

with $-\pi < \vartheta \leq \pi$.
By unitarity of the evolution (1.4), it follows that the Laplace transform of $q(t)$ is indeed analytic on the open right half plane:

Proposition 1.2 *The Laplace transform of $q(t)$, solution of (1.6), is analytic at least for $\Re(p) > 0$.*

Proof: Using the decomposition of the wave function at time t defined by (1.5), we can write the survival probability in the following way:

$$\theta(t) \equiv \Big(\varphi_{\alpha(0)}\,,\,\Psi_t\Big)_{L^2(\mathbb{R}^3)} = \Big(\varphi_{\alpha(0)}\,,\,e^{-iH_0t}\varphi_{\alpha(0)}\Big)_{L^2(\mathbb{R}^3)} + \tag{1.16}$$

$$+i\Big(\varphi_{\alpha(0)}(\vec{x})\,,\,\int_0^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x})\Big)_{L^2(\mathbb{R}^3)}$$

Let us define

$$Z_1(t) \equiv \Big(\varphi_{\alpha(0)}\,,\,e^{-iH_0t}\varphi_{\alpha(0)}\Big)_{L^2(\mathbb{R}^3)}$$

By the usual dissipative estimate for the free propagator, one has

$$\big|Z_1(t)\big| \leq c_1\, t^{-\frac{3}{2}}$$

as $t \to \infty$ for some constant $c_1 \in \mathbb{R}$. Hence $Z_1(t)$ belongs to $L^1(\mathbb{R}^+)$ and then its Laplace transform $\tilde{Z}_1(p)$ is analytic at least for $\Re(p) > 0$.
The second piece of the scalar product is given by

$$Z(t) \equiv i\Big(\varphi_{\alpha(0)}(\vec{x})\,,\,\int_0^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x})\Big)_{L^2(\mathbb{R}^3)} =$$

$$= i\int_0^t d\tau\, q(\tau)\,\Big(e^{-iH_0(t-\tau)}\varphi_{\alpha(0)}\Big)(0)$$

and taking the Laplace transform of $Z(t)$, we have

$$\tilde{Z}(p) = \tilde{Z}_2(p)\,\tilde{q}(p)$$

where

$$\tilde{Z}_2(p) \equiv -\frac{4\sqrt{2\pi|\alpha(0)|}}{4\pi\alpha(0) - \sqrt{-ip}}$$

is analytic for $\Re(p) > 0$ and never equal to 0, because of condition (1.8).
Hence the Laplace transform of $\theta(t)$ is given by

$$\tilde{\theta}(p) = \tilde{Z}_1(p) + \tilde{Z}_2(p)\,\tilde{q}(p)$$

But $\theta(t)$ is a bounded function[2], because of unitarity of the evolution (1.4), and then its Laplace transform is analytic on the open right half plane. The claim then follows from analyticity of $\tilde{Z}_1(p)$, $\tilde{Z}_2(p)$ and absence of zeros of $\tilde{Z}_2(p)$.

[2]Actually $|\theta(t)| \leq 1$, since the initial state is normalized.

□

A well known property of Volterra integral operators, with regular or weakly singular kernel, implies

Proposition 1.3 *The homogeneous equations associated to (1.6) has no non-zero solution in* $L^p_{\text{loc}}(\mathbb{R}^+)$*,* $1 \leq p \leq \infty$.

Proof: The proof (see e.g. [42]) exploits the fact that the n-fold iterated kernel is a contraction in any $L^p(0, T_n)$ with T_n increasing to infinity for increasing n.

□

In the following sections we shall prove asymptotic complete ionization of the system under generic conditions on $\alpha(t)$. Although the result does not depend on the sign of the mean α_0 of $\alpha(t)$, we have to discuss separately the case $\alpha_0 < 0$ and $\alpha_0 \geq 0$, because of the slightly different form of equation (1.14).

1.3 CASE I: $\alpha_0 < 0$

Since $\alpha(0) < 0$, changing the energy scale, it is always possible to assume that $\alpha(t)$ satisfies the normalization

$$4. \quad \alpha(0) = \sum_{n\in\mathbb{Z}} \alpha_n = -\frac{1}{4\pi} \tag{1.17}$$

Moreover we introduce another condition we shall use later on: let $\mathcal{T}$ the right shift operator on $\ell_1(\mathbb{N})$, i.e.

$$\left(\mathcal{T}a\right)_n \equiv a_{n+1} \tag{1.18}$$

we say that $\alpha = \{\alpha_n\} \in \ell_1(\mathbb{Z})$ is *generic* with respect to $\mathcal{T}$, if $\tilde{\alpha} \equiv \{\alpha_n\}_{n>0} \in \ell_1(\mathbb{N})$ satisfies the following condition

$$e_1 = (1, 0, 0, \dots) \in \overline{\bigvee_{n=0}^{\infty} \mathcal{T}^n \tilde{\alpha}} \tag{1.19}$$

For a detailed discussion of genericity condition see [12].
If (1.17) holds, equation (1.14) becomes (at least for $\Re(p) > 0$)

$$\tilde{q}(p) = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{-ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k\, \tilde{q}(p + i\omega k) - \frac{2i\sqrt{2\pi}}{4\pi\alpha_0 + \sqrt{-ip}} \frac{1 - \sqrt{-ip}}{1 + ip} \tag{1.20}$$

and by Proposition 1.2 its solution is analytic on the open right half plane. In the following Section we shall extend the equation (1.20) above to the imaginary axis and study the behavior of the solution there.

1.3.1 Behavior on the imaginary axis at $p \neq 0$

Setting $q_n(p) \equiv \tilde{q}(p + i\omega n)$, we obtain a sequence of functions on the strip $\mathcal{I} = \{p \in \mathbb{C},\ 0 \leq \Im(p) < \omega\}$,

$$q(p) \equiv \{q_n(p)\}_{n\in\mathbb{Z}}$$

so that equation (1.20) can be rewritten

$$q(p) = \mathcal{L}(p)\, q(p) + g(p) \tag{1.21}$$

where

$$\left(\mathcal{L}q\right)_n(p) \equiv -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k\, q_{n+k}(p) \tag{1.22}$$

and $g(p) = \{g_n(p)\}_{n\in\mathbb{Z}}$ with

$$g_n(p) \equiv -\frac{2i\sqrt{2\pi}}{4\pi\alpha_0 + \sqrt{\omega n - ip}}\, \frac{1 - \sqrt{\omega n - ip}}{1 + ip - \omega n} \tag{1.23}$$

From the explicit expression of the operator (1.22) and (1.23), it is clear that the coefficients of the equation fails to be analytic on the imaginary axis at $\bar{p} = ((4\pi\alpha_0)^2 - \omega\bar{n})i$, for some $\bar{n} \in \mathbb{Z}$ and then the solution may be singular there.
Since $\Im(p) \in [0, \omega)$, one has

$$\frac{(4\pi\alpha_0)^2}{\omega} - 1 < \bar{n} \leq \frac{(4\pi\alpha_0)^2}{\omega} \tag{1.24}$$

and then the singularity appears at most in the equation for $q_{\bar{n}}$ (there is only one integer[3] which satisfies the previous inequality) at $\bar{p} = ((4\pi\alpha_0)^2 - \omega\bar{n})i$. For instance, if $\omega > (4\pi\alpha_0)^2$, the pole may be at $\bar{p} = (4\pi\alpha_0)^2 i$ in the equation for q_0.
Actually we have to distinguish the so called (see [12]) resonant case, i.e. when

$$(4\pi\alpha_0)^2 = N\omega$$

for some $N \in \mathbb{N}$, because in that case we can have a pole only at $p = 0$ and then the solution is immediately seen to be analytic on the whole imaginary

[3] In fact $\bar{n}$ must be non negative.

axis except at most for $p = 0$.
Let us first consider the behavior of the solution on the imaginary axis for $p \neq 0, \bar{p}$. We are going to prove that the solution is in fact analytic there. We prove first an important property of the operator $\mathcal{L}$:

Proposition 1.4 *For $p \in \mathcal{I}$, $\Re(p) = 0$, $p \neq 0, \bar{p}$, $\mathcal{L}(p)$ is an analytic operator-valued function and $\mathcal{L}(p)$ is a compact operator on $\ell_2(\mathbb{Z})$.*

Proof: Analyticity on the imaginary axis for $p \neq 0, \bar{p}$ easily follows from the explicit expression of the operator.
Moreover $\mathcal{L}(p)$ can be written

$$\mathcal{L}(p) = b(p) \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \alpha_k \, \mathcal{T}^{n+k}$$

where $b(p)$ is the operator

$$(b\, q)_n(p) \equiv b_n(p)\, q_n(p) = -\frac{4\pi q_n(p)}{4\pi\alpha_0 + \sqrt{\omega n - ip}}$$

and $\mathcal{T}$ is the right shift operator in $\ell_2(\mathbb{Z})$.
Since $\|\mathcal{T}\| = 1$, the series converges strongly to a bounded operator. Moreover $b(p)$ is a compact operator on the imaginary axis for $p \neq 0, \bar{p}$: $b(p)$ is the norm limit of a sequence of finite rank operators, because $\lim_{n \to \infty} b_n(p) = 0$. Hence the result follows for example from Theorem VI.12 and VI.13 of [44].

□

Proposition 1.5 *There exists a unique solution $q_n(p) \in \ell_2(\mathbb{Z})$ of (1.21) and it is analytic on the imaginary axis for $p \neq 0, \bar{p}$.*

Proof: The key point will be the application of the analytic Fredholm theorem to the operator $\mathcal{L}(p)$ (Theorem VI.14 of [44]), in order to prove that $(I - \mathcal{L}(p))^{-1}$ exists for $p \neq 0, \bar{p}$.
Since there is no non-zero solution in $L^2_{\text{loc}}(\mathbb{R}^+)$ of the homogeneous equation associated to (1.6) (see the Proposition 1.3), then the homogeneous equation associated to (1.21) has only the trivial solution in $\ell_2(\mathbb{Z})$. Moreover the operator $\mathcal{L}$ is compact and thus analytic Fredholm theorem applies. The result easily follows, because $g(p) \in \ell_2(\mathbb{Z})$ and each $g_n(p)$ is analytic for $p \neq 0, \bar{p}$.

□

We can now study the equation (1.21) in a neighborhood of $\bar{p}$ (if $\bar{p} \neq 0$). An important preliminary result is the following

Lemma 1.1 *Let (1.9) and the genericity condition (1.19) be satisfied by* $\{\alpha_n\}$. *The system of equations*

$$r_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z} \\ k\neq n,\bar{n}}} \alpha_{k-n} r_k + h_n(p) \right\} \tag{1.25}$$

has a unique solution $\{r_n\} \in \ell_2(\mathbb{Z} \setminus \{\bar{n}\})$ *in a pure imaginary neighborhood of* $\bar{p}$, *where* $\bar{n} \in \mathbb{Z}$ *and* $\bar{p} \in \mathcal{I}$, $\Re(\bar{p}) = 0$, *are defined by (1.24), for every* $h_n(p)$ *such that*

$$h'_n(p) \equiv \frac{h_n(p)}{4\pi\alpha_0 + \sqrt{\omega n - ip}}$$

belongs to $\ell_2(\mathbb{Z} \setminus \{\bar{n}\})$.
Moreover, if $h_n(p)$ *is analytic in a neighborhood of* $\bar{p}$, *the solution is analytic in the same neighborhood.*

Proof: Equation (1.25) is of the form

$$r = \mathcal{L}' r + h'$$

where $h' \equiv \{h'_n\}$ belongs to $\ell_2(\mathbb{Z} \setminus \{\bar{n}\})$ and $\mathcal{L}'$ is a compact operator (see Proposition 1.4).
In order to apply analytic Fredholm theorem to the operator $\mathcal{L}'$, we need to prove that there is no non-zero solution in a neighborhood of $\bar{p}$ of the homogeneous equation. Suppose that the contrary is true, so that $\{R_n\} \in \ell_2(\mathbb{Z} \setminus \{\bar{n}\})$ is a non-zero solution of

$$R_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq n,\bar{n}}} \alpha_{k-n} R_n$$

Multiplying both sides of equation above by R_n^* and summing over $n \in \mathbb{Z} \setminus \{\bar{n}\}$, one has

$$\sum_{\substack{n\in\mathbb{Z} \\ n\neq\bar{n}}} \sqrt{\omega n - ip}\, |R_n|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z} \\ n,k\neq\bar{n}}} {R_n}^* \alpha_{k-n} R_k$$

and, since the right hand side is real,

$$\Im\left[\sum_{\substack{n\in\mathbb{Z} \\ n\neq\bar{n}}} \sqrt{\omega n - ip}\, |R_n|^2 \right] = 0$$

for $p = i\lambda$, $0 < \lambda < \omega$, and then $R_n = 0$ for $n < 0$. Now suppose that $R \neq 0$ and let $n_0 \in \mathbb{N}$ be such that $R_n = 0$, $n < n_0$, and $R_{n_0} \neq 0$ (hence $n_0 \geq 0$). Fixing $R_{\bar{n}} = 0$, for each $n < n_0$ the homogeneous equation gives

$$\sum_{k=n_0}^{\infty} \alpha_{k-n} R_k = 0$$

or, setting $k = n_0 - 1 + k'$, for $n \geq 0$,

$$\sum_{k'=1}^{\infty} \alpha_{k'+n} R_{n_0-1+k'} = 0$$

which implies (see (1.9)), for each $n \geq 0$,

$$\left(R' \, , \mathcal{T}^n \alpha\right)_{\ell_2(\mathbb{N})} = 0$$

where $R'_n = R^*_{n_0-1+n}$ and $(\cdot \, , \, \cdot)$ stands for the standard scalar product on $\ell_2(\mathbb{N})$.
If $\{\alpha_n\}$ satisfies the genericity condition (1.19), R' has to be orthogonal also to e_1 and then $R_{n_0} = 0$, which is a contradiction. Therefore $R = 0$.
The first part of the Lemma then follows from analyticity of $\mathcal{L}'(p)$ and analytic Fredholm theorem. Moreover if $\{h_n(p)\}$ is analytic in a neighborhood of $\bar{p}$, analyticity of the solution is a straightforward consequence.

□

Proposition 1.6 *If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition with respect to $\mathcal{T}$ (1.19), the unique solution $\{q_n\} \in \ell_2(\mathbb{Z})$ of (1.21) is analytic on the imaginary axis except at most for $p = 0$.*

Proof: If $(4\pi\alpha_0)^2 = N\omega$ for some $N \in \mathbb{N}$ (resonant case) there is nothing to prove, since the coefficients of (1.21) fails to be analytic only at $p = 0$. On the other hand, in the non resonant case, Proposition 1.5 guarantees analyticity on imaginary axis for $p \neq 0, \bar{p}$. Therefore it is sufficient to study the behavior of the solution in a neighborhood of $\bar{p}$, where the coefficients of (1.21) have a singularity. We are going to prove that in fact the solution is analytic at $\bar{p}$.
The strategy of the proof is to analyze separately the terms q_n, $n \neq \bar{n}$, $\bar{n}$ being defined in (1.24), and then prove that also $q_{\bar{n}}$ is analytic in a neighborhood of $\bar{p}$.
By Lemma 1.1 there is a unique solution of the system

$$t_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq n,\bar{n}}} \alpha_{k-n} t_k - \frac{4\pi\alpha_{\bar{n}-n}}{4\pi\alpha_0 + \sqrt{\omega n - ip}} \tag{1.26}$$

Setting $q_n = r_n + t_n q_{\bar{n}}$, $n \neq \bar{n}$, on (1.21), one has

$$r_n + t_n q_{\bar{n}} = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}}\left\{\alpha_{\bar{n}-n} q_{\bar{n}} + \sum_{\substack{k\in\mathbb{Z}\\ k\neq n,\bar{n}}} \alpha_{k-n}\big(r_k + t_k q_{\bar{n}}\big)\right\} +$$

$$-\frac{2i\sqrt{2\pi}}{4\pi\alpha_0 + \sqrt{\omega n - ip}}\frac{1-\sqrt{\omega n - ip}}{1+ip-\omega n}$$

and therefore the equation for $\{r_n\}$, $n \neq \bar{n}$, becomes

$$r_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n - ip}}\left\{\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n}} \alpha_k r_{n+k} + \frac{i}{\sqrt{2\pi}}\frac{1-\sqrt{\omega n - ip}}{1+ip-\omega n}\right\} \tag{1.27}$$

while $q_{\bar{n}}$ satisfies the equation

$$q_{\bar{n}} = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega\bar{n} - ip}}\left\{\sum_{\substack{k\in\mathbb{Z}\\ k\neq\bar{n}}} \alpha_{k-\bar{n}}\big(r_k + t_k q_{\bar{n}}\big) + \frac{i}{\sqrt{2\pi}}\frac{1-\sqrt{\omega\bar{n} - ip}}{1+ip-\omega\bar{n}}\right\}$$

or

$$\left[4\pi\alpha_0 + \sqrt{\omega\bar{n} - ip} + 4\pi\sum_{\substack{k\in\mathbb{Z}\\ k\neq\bar{n}}} \alpha_{k-\bar{n}} t_k\right] q_{\bar{n}} = -4\pi\sum_{\substack{k\in\mathbb{Z}\\ k\neq\bar{n}}} \alpha_{k-\bar{n}} r_k - \frac{2i\sqrt{2\pi}}{1+\sqrt{\omega\bar{n} - ip}}$$

Since the last term is analytic in a neighborhood of $\bar{p}$ and $\{t_n\}$, $\{r_n\} \in \ell_2(\mathbb{Z}\setminus\{\bar{n}\})$ are both analytic, as it follows applying Lemma 1.1 above to (1.26) and (1.27), it is sufficient to prove that

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq\bar{n}}} \alpha_{k-\bar{n}}\tilde{t}_k \neq 0$$

where

$$\tilde{t}_n \equiv t_n(p)\big|_{p=\bar{p}}$$

Assume that the contrary is true: from equation (1.26) we obtain

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq\bar{n}}}\left(4\pi\alpha_0 + \sqrt{\omega n - i\bar{p}}\right)\left|\tilde{t}_n\right|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z}\\ n,k\neq\bar{n},n\neq k}} \tilde{t}_n^*\alpha_{k-n}\tilde{t}_k - 4\pi\sum_{\substack{n\in\mathbb{Z}\\ n\neq\bar{n}}} \alpha_{n-\bar{n}}^*\tilde{t}_n^* =$$

$$= -4\pi \sum_{\substack{n,k\in\mathbb{Z}\\ n,k\neq\bar{n},n\neq k}} \tilde{t}_n^*\alpha_{k-n}\tilde{t}_k$$

where we have used condition 2 in (1.9). The previous equation implies (the right hand side is real) $\tilde{t}_n = 0$, $\forall n < \bar{N} = \frac{i\bar{p}}{\omega}$ and then, since $-1 < \bar{N} < 0$, $\tilde{t}_n = 0$, $\forall n < 0$. Hence from (1.26) we have, $\forall n < 0$,

$$\sum_{\substack{k \geq 0 \\ k \neq \bar{n}}} \alpha_{k-n}\tilde{t}_k + \alpha_{\bar{n}-n} = 0$$

Now supposing without loss of generality that $\tilde{t}_0 \neq 0$ and setting $T_n = \tilde{t}_{n-1}$, $n \neq \bar{n}+1$, and $T_{\bar{n}+1} = 1$, we obtain, $\forall n \geq 0$,

$$\sum_{k=1}^{\infty} \alpha_{k+n}T_k = 0$$

and using the genericity condition (1.19) (as in the proof of Lemma 1.1) we get $T_1 = t_0 = 0$, which is a contradiction.
In conclusion $q_{\bar{n}}$ is analytic in a neighborhood of $\bar{p}$: analyticity of q_n, $n \neq \bar{n}$ is then a straightforward consequence of analyticity of $\{r_n\}$, $\{t_n\}$ and decomposition $q_n = r_n + t_n q_{\bar{n}}$. The proof is then completed, since r_n and t_n belong to $\ell_2(\mathbb{Z} \setminus \{\bar{n}\})$ in a neighborhood of $p = \bar{p}$.

□

1.3.2 Behavior at $p = 0$

We shall now study the behavior of the solution of (1.21) on the imaginary axis at the origin. With the choice (1.15) for the branch cut of the square root, it is clear that we must expect branch points of $\tilde{q}(p)$, solution of (1.20), at $p = i\omega n$, $n \in \mathbb{Z}$, which should imply a branch point at $p = 0$ for each q_n in (1.21).
We are going to show that q_n, $n \in \mathbb{Z}$ has a branch point at $p = 0$. The non-resonant case and the resonant one will be treated separately.

Proposition 1.7 (non-resonant case)
If $(4\pi\alpha_0)^2 \neq N\omega$, $\forall N \in \mathbb{N}$ and $\{\alpha_n\}$ satisfies (1.9) and (1.19) (genericity condition), the solution of equation (1.21) has the form $q_n(p) = c_n(p) + d_n(p)\sqrt{p}$, $n \in \mathbb{Z}$, in an imaginary neighborhood of $p = 0$, where the functions $c_n(p)$ and $d_n(p)$ are analytic at $p = 0$.

Proof: Setting $q_n = r_n + t_n q_0$, $n \neq 0$ and choosing a solution $\{t_n\} \in \ell_2(\mathbb{Z} \setminus \{0\})$ of the system of equations (1.26) with $\bar{n} = 0$, we obtain that $\{r_n\}$ must satisfy (1.27). It is easy to see that the result of Lemma 1.1 holds also

in a neighborhood of $\bar{p} = 0$ with $\bar{n} = 0$, so that $\{r_n\}$, $\{t_n\} \in \ell_2(\mathbb{Z} \setminus \{0\})$ are unique and analytic at $p = 0$.
Thus it is sufficient to prove that q_0, which is solution of

$$\left[4\pi\alpha_0 + \sqrt{-ip} + 4\pi \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \alpha_k t_k\right] q_0 = -4\pi \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \alpha_k r_k - \frac{2i\sqrt{2\pi}(1-\sqrt{-ip})}{1+ip}$$

has the required behavior near $p = 0$.
First, setting $t_n^0 = t_n(p = 0)$, we have to prove that

$$\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \alpha_k t_k^0 \neq -\alpha_0$$

but, assuming that the contrary is true and multiplying both sides of equation (1.26) by t_n^{0*} and summing over $n \in \mathbb{Z}$, $n \neq 0$, one has

$$\sum_{n\in\mathbb{Z}} \sqrt{\omega n}\, |t_n^0|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z}\\n,k\neq 0}} t_n^{0*} \alpha_{k-n} t_k^0 + 4\pi\alpha_0$$

and then, because of genericity condition (1.19), $\{t_n^0\} = 0$, $\forall n \in \mathbb{Z} \setminus \{0\}$, which is impossible, since $\{t_n\}$ solves (1.26).
Now, calling

$$F \equiv 4\pi \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \alpha_k t_k$$

and

$$G \equiv -4\pi \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \alpha_k r_k$$

we have

$$\left[4\pi\alpha_0 + \sqrt{-ip} + F\right] q_0 = G + \frac{2i\sqrt{2\pi}(1-\sqrt{-ip})}{1+ip}$$

and

$$q_0 = F' + \sqrt{p}\ G'$$

where F' is analytic in a neighborhood of $p = 0$, because of analyticity of F and G, and

$$G' \equiv -\frac{2i\sqrt{-2\pi i}\,(4\pi\alpha_0 + F + 1) + \sqrt{-i}\,(1+ip)\,G}{(1+ip)[(4\pi\alpha_0 + F)^2 + ip]} \tag{1.28}$$

□

The resonant case, i.e. $4\pi\alpha_0 = -\sqrt{\omega N}$ for some $N \in \mathbb{N}$, is not so different from the non-resonant one and we shall prove that the solution has the same behavior at the origin. The proof is slightly different because we need to show the absence of a pole at $p = 0$: from (1.21) one has

$$q_N(p) = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega N - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k\, q_{n+k}(p) + \frac{i}{\sqrt{2\pi}} \frac{1-\sqrt{\omega N - ip}}{1+ip-\omega N} \right\}$$

and the coefficients have a singularity at $p = 0$.
We are going to prove that in fact the solution has no pole at the origin: proceeding as in the proof of Proposition 1.6, let us begin with a preliminary result, which take the place of Lemma 1.1:

Lemma 1.2 *Let (1.9) and the genericity condition (1.19) be satisfied by* $\{\alpha_n\}$. *The system of equations*

$$r_n = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega n - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n}} \alpha_k r_{n+k} + h_n(p) \right\} \tag{1.29}$$

has a unique solution $\{r_n\} \in \ell_2(\mathbb{Z}\setminus\{N\})$ *in a purely imaginary neighborhood of* $p = 0$, *for every* $h_n(p)$ *such that*

$$h_n'(p) \equiv \frac{h_n(p)}{\sqrt{\omega N} - \sqrt{\omega n - ip}}$$

belongs to $\ell_2(\mathbb{Z}\setminus\{N\})$.
Moreover, if $h_n(p)$ *is analytic in a neighborhood of* $p = 0$, *the solution is analytic in the same neighborhood.*

Proof: We shall proceed as in the proof of Proposition 1.6, separating the contribution of r_N, which may be singular: setting $r_n = u_n + v_n r_N$, $n \neq 0, N$, on (1.29), one has

$$u_n + v_n r_N = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega n - ip}} \left\{ \alpha_{N-n} r_N + \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0,-n,N-n}} \alpha_k \left(u_{n+k} + v_{n+k} r_N\right) \right\} +$$

$$+ \frac{2i\sqrt{2\pi}}{\sqrt{\omega N} - \sqrt{\omega n - ip}} \frac{1-\sqrt{\omega n - ip}}{1+ip-\omega n}$$

and requiring that $\{v_n\}$, $n \neq 0, N$, solves

$$v_n = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-n,N-n}} \alpha_k v_{n+k} + \frac{4\pi\alpha_{N-n}}{\sqrt{\omega N} - \sqrt{\omega n - ip}} \tag{1.30}$$

the equation for $\{u_n\}$, $n \neq 0, N$, becomes

$$u_n = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega n - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-n,N-n}} \alpha_k u_{n+k} + \frac{i}{\sqrt{2\pi}} \frac{1 - \sqrt{\omega n - ip}}{1 + ip - \omega n} \right\} \tag{1.31}$$

Moreover r_N satisfies

$$r_N = \frac{4\pi}{\sqrt{\omega N} - \sqrt{\omega N - ip}} \left\{ \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-N}} \alpha_k \big(u_k + v_k r_N\big) + \frac{i}{\sqrt{2\pi}} \frac{1 - \sqrt{\omega n - ip}}{1 + ip - \omega n} \right\}$$

or

$$\left[\sqrt{\omega N} - \sqrt{\omega N - ip} - 4\pi \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,N}} \alpha_{k-N} v_k \right] r_N =$$

$$= 4\pi \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,N}} \alpha_{k-N} u_k + \frac{i}{\sqrt{2\pi}} \frac{1 - \sqrt{\omega n - ip}}{1 + ip - \omega n}$$

Applying the discussion contained in the proof of Lemma 1.1, it is not difficult to see that the solutions of equations (1.31) and (1.30) are analytic in a neighborhood of the origin and belong to $\ell_2(\mathbb{Z}\setminus\{0, N\})$. Therefore it remains to prove that (setting $v_n^0 = v_n(p = 0)$)

$$\sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,N}} \alpha_{k-N} v_k^0 \neq 0$$

but the argument in the proof of Proposition 1.6 excludes this possibility, if $\{\alpha_n\}$ satisfies the genericity condition. The proof is then completed, because analyticity of r_N implies analyticity of all r_n, $n \neq 0, N$.

□

Proposition 1.8 (resonant case)
If $(4\pi\alpha_0)^2 = N\omega$, for some $N \in \mathbb{N}$ and $\{\alpha_n\}$ satisfies (1.9) and (1.19) (genericity condition), the solution of equation (1.21) has the form $q_n(p) = c_n(p) + d_n(p)\sqrt{p}$, $n \in \mathbb{Z}$, in an imaginary neighborhood of $p = 0$, where the functions $c_n(p)$ and $d_n(p)$ are analytic at $p = 0$.

Proof: See the proof of Proposition 1.7 and Lemma 1.2 above.

□

1.3.3 Complete ionization in the generic case

Summing up the results about the behavior of the Laplace transform $\tilde{q}(p)$ of $q(t)$ we can state the following

Theorem 1.1 *If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition (1.19) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$|q(t)| \leq A\, t^{-\frac{3}{2}} + R(t) \qquad (1.32)$$

where $A \in \mathbb{R}$ and $R(t)$ has an exponential decay, $R(t) \sim Ce^{-Bt}$ for some $B, C > 0$.

Proof: Propositions 1.5, 1.6 and 1.7 guarantee that $\tilde{q}(p)$ is analytic on the closed right half plane, except branch point singularities on the imaginary axis at $p = i\omega n$, $n \in \mathbb{Z}$.
Therefore we can chose a integration path for the inverse of Laplace transform of $\tilde{q}(q)$ along the imaginary axis like in [12].
Proposition 1.7 implies that the contribution of the branch point at $p = 0$ is given by the integral

$$2i \int_0^\infty dp\, \sqrt{p}\; G'(-p)\, e^{-pt}$$

where G', defined in (1.28), is a bounded analytic function on the negative real line: from explicit expression of F and G and equations (1.27) and (1.26), it is clear that G' is analytic and $\lim_{p\to\infty} G'(-p) = 0$ on the real line. So that the corresponding asymptotic behavior as $t \to \infty$ is

$$\left| \int_0^\infty dp\, \sqrt{p}\; G'(-p)\, e^{-pt} \right| \leq C \int_0^\infty dp\, \sqrt{p}\; e^{-pt} = A\, t^{-\frac{3}{2}}$$

Let us consider now the contribution of branch points at $p = i\omega n$, $n \neq 0$: from Propositions 1.7 and 1.8 it follows that, in a neighborhood of $p = 0$,

$$q_n(p) = c_n(p) + d_n(p)\, \sqrt{p}$$

where $c_n(p)$ and $d_n(p)$ are analytic at $p = 0$. Moreover using the decomposition $q_n = r_n + t_n q_0$, $n \neq 0$, as in the proof of Proposition 1.7 and 1.8, and studying the equation (1.26) for t_n, we immediately obtain $\{d_n\} \in \ell_1(\mathbb{Z}\backslash\{0\})$, because of condition 2 in (1.9). Since $q_n(p) = \tilde{q}(p + i\omega n)$, the contribution of singularities at $p = i\omega n$, $n \neq 0$, is then given by

$$2 \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \int_{i\omega n-\infty}^{i\omega n} dp\;\, d_n(p - i\omega n)\sqrt{p - i\omega n}\;\, e^{pt} =$$

$$= 2i \int_0^\infty dp \left\{ \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} d_n(-p)\, e^{i\omega n t} \right\} \sqrt{p}\ e^{-pt} =$$

and the series

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} d_n(-p)\, e^{i\omega n t}$$

converges uniformly to a bounded function of t, because $\{d_n\} \in \ell_1(\mathbb{Z} \setminus \{0\})$. Adding up the contributions of every branch cut, one obtain the required leading term in the asymptotic behavior. Indeed the rest function $R(t)$ is given by the contribution of poles outside the imaginary axis and then shows an exponential decay as $t \to \infty$.

□

A straightforward consequence of Theorem 1.1 is that the scalar product (and thus the survival probability of the bound state)

$$\theta(t) = \left(\varphi_{\alpha(0)} \, , \Psi_t \right)_{L^2(\mathbb{R}^3)}$$

tends to 0 when $t \to \infty$:

Corollary 1.1 *If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition (1.19) with respect to $\mathcal{T}$, the system shows asymptotic complete ionization and, as $t \to \infty$,*

$$|\theta(t)| \leq D\, t^{-\frac{3}{2}} + E(t)$$

where $D \in \mathbb{R}$ and $E(t)$ has an exponential decay.

Proof: The Laplace transform of $\theta(t)$ can be expressed in the following way (see the proof of Proposition 1.2)

$$\tilde{\theta}(p) = \tilde{Z}_1(p) + \tilde{Z}_2(p)\, \tilde{q}(p)$$

where $\tilde{Z}_1(p)$ is analytic on the closed right half plane and $\tilde{Z}_2(p)$ has only a branch point at the origin of the form $a_1 + a_2\sqrt{p}$.
Hence $\tilde{\theta}(p)$ has the same singularities as $\tilde{q}(p)$ and then its asymptotic behavior coincides with that of $q(t)$, i.e.

$$|\theta(t)| \leq D\, t^{-\frac{3}{2}} + E(t)$$

for some constant $D \in \mathbb{R}$ and for a bounded function $E(t)$ with exponential decay.

□

In the following we shall prove a stronger result about complete ionization of the system, namely that every state $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state[4] for the operator $H_{\alpha(t)}$, i.e.

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \, \left\| F(|\vec{x}| \leq R) U(\tau, 0) \Psi \right\|^2 = 0 \tag{1.33}$$

where $F(S)$ is the multiplication operator by the characteristic function of the set $S \subset \mathbb{R}^3$ and $U(t,s)$ the unitary two-parameters family associated to $H_{\alpha(t)}$ (see (1.4)).
In order to prove (1.33), we first need to study the evolution of a generic initial datum in a suitable dense subset of $L^2(\mathbb{R}^3)$ and then we shall extend the result to every state using the unitarity of the evolution defined by (1.4) (see e.g. [18]).

Proposition 1.9 *Let $\Psi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ a smooth function with compact support away from 0 and $q(t)$ be the solution of equation (1.6) with initial condition $\Psi_0 = \Psi$. If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition (1.19) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$|q(t)| \leq A\, t^{-\frac{3}{2}} + R(t) \tag{1.34}$$

where $A \in \mathbb{R}$ and $R(t)$ has an exponential decay, $R(t) \sim Ce^{-Bt}$ for some $B > 0$.

Proof: The proof of Proposition 1.2 still applies, considering

$$\theta'(t) \equiv \left(\Psi \,, \Psi_t \right)_{L^2(\mathbb{R}^3)}$$

instead of $\theta(t)$, so that $\tilde{q}(p)$, solution of (1.14) with initial condition $\Psi_0 = \Psi$, is analytic $\forall p$ with $\Re(p) > 0$.
Hence we can consider the Laplace transform of equation (1.6), which has the form (1.14) with

$$f(p) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{i}{p}} \int_0^\infty dt \; e^{-pt} \int_{\mathbb{R}^3} d^3\vec{k} \;\; \hat{\Psi}(\vec{k}) \; e^{-ik^2 t}$$

[4]For the definition of scattering states of a time-dependent operator see e.g. [25, 34].

where $\hat{\Psi}(\vec{k})$ is the Fourier transform of $\Psi(\vec{x})$.
The equation for $\tilde{q}(p)$ is then given by

$$\tilde{q}(p) = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{-ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k \, \tilde{q}(p + i\omega k) + \frac{g(p)}{4\pi\alpha_0 + \sqrt{-ip}}$$

where

$$g(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty dt \, e^{-pt} \int_{\mathbb{R}^3} d^3\vec{k} \; \hat{\Psi}(\vec{k}) \, e^{-ik^2 t}$$

It is now sufficient to show that the solution $\tilde{q}(p)$ is also analytic on the imaginary axis except at most square root branch points at $p = i\omega n$ as in the discussion of section 3.2 and 3.3.
For every smooth function Ψ with compact support, $\hat{\Psi}(\vec{k})$ is a smooth function with an exponential decay as $k \to \infty$, so that

$$g(is) = \lim_{r\to 0^+} \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^3} d^3\vec{k} \; \frac{\hat{\Psi}(\vec{k})}{r + (s + k^2)i} = -i\sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^3} d^3\vec{k} \; \frac{\hat{\Psi}(\vec{k})}{s + k^2}$$

is a bounded function for $s > 0$. Hence the function $g(p)$ has no pole for $\Im(p) \in (0, \omega)$ and therefore the result contained in Proposition 1.6 still holds. Moreover

$$g(0) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^3} d^3\vec{k} \; \hat{\Psi}(\vec{k}) \int_0^\infty dt \, e^{-ik^2 t} = -i\sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^3} d^3\vec{k} \; \frac{\hat{\Psi}(\vec{k})}{k^2}$$

which is again bounded, so that $g(p)$ has at the origin at most a branch point singularity of the form $a(p) + b(p)\sqrt{p}$: following the proofs of Proposition 1.7 and 1.8, we can show that $\tilde{q}(p)$ has the same behavior at the origin.
In conclusion the solution is analytic on the closed right half plane except for branch points at $p = i\omega n$, $n \in \mathbb{Z}$, of the form $a(p) + b(p)\sqrt{p - i\omega n}$. The proof of Theorem 1.1 then implies that $q(t)$ has the prescribed behavior as $t \to \infty$.

□

Theorem 1.2 *If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition (1.19) with respect to $\mathcal{T}$, every $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state of $H_{\alpha(t)}$, i.e.*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \, \left\| F(|\vec{x}| \leq R) U(\tau, 0) \Psi \right\|^2 = 0$$

Proof: We shall restrict the proof to the dense subset of $L^2(\mathbb{R}^3)$ given by smooth functions with compact support and then we shall extend the result to every state using the unitarity of the evolution defined by (1.5) (see e.g. [18]). Actually we are going to prove a slightly stronger statement, i.e. $\forall \varepsilon > 0$, there exists t_0 such that $\forall t > t_0$,

$$\left\| F(|\vec{x}| \leq R) U(t,0) \Psi \right\| \leq \varepsilon$$

The evolution of an initial state Ψ according to (1.5) is given by

$$\Psi_t(\vec{x}) = U(t,s)\Psi_s(\vec{x}) = U_0(t-s)\Psi_s(\vec{x}) + i \int_s^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x}) \quad (1.35)$$

Moreover, since $\Psi_t \in \mathcal{D}(H_{\alpha(t)})$, the following decomposition holds

$$\Psi_t(\vec{x}) = \varphi_t(\vec{x}) + \frac{q(t)}{4\pi|\vec{x}|} \quad (1.36)$$

where $q(t)$ is the solution of (1.6), $\varphi_t \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ and

$$\varphi_t(0) = \alpha(t) q(t)$$

We are going to show that, if $q(t) \in L^1(\mathbb{R}^+)$, Ψ_t satisfies the required property. Let us start analyzing the second term in (1.35): imposing the unitarity condition of the evolution we have

$$\|\Psi_s\|^2 = \|\Psi_t\|^2 = \left\| U_0(t-s)\Psi_s(\vec{x}) + i \int_s^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x}) \right\|^2$$

and then

$$\left\| \int_s^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x}) \right\|^2 = 2\Im\left(\int_s^t d\tau\, q(\tau)\, U_0(t-\tau;\vec{x})\,,\, U_0(t-s)\Psi_s(\vec{x}) \right) =$$

$$= 2\Im\left[\int_s^t d\tau\, q^*(\tau) \left(e^{-iH_0(\tau-s)}\Psi_s \right)(0) \right]$$

but, using the decomposition (1.36),

$$\left(e^{-iH_0(s-\tau)}\Psi_s \right)(0) = \left(e^{-iH_0(s-\tau)}\varphi_s \right)(0) + \int_{\mathbb{R}^3} d^3\vec{k}\, e^{-ik^2(\tau-s)} \frac{q(s)}{(2\pi)^3 k^2} =$$

$$= \left(e^{-iH_0(s-\tau)}\varphi_s \right)(0) + \frac{q(s)}{4\pi\sqrt{\pi i}\sqrt{\tau-s}}$$

Since $\varphi_s \in H^2_{\text{loc}}(\mathbb{R}^3)$, the absolute value of the first term on the right hand side is bounded by a constant $c(\tau, s) < \infty$ such that $c(s,s) = q(s)$ and

$$\lim_{\tau \to \infty} c(\tau, s) = 0$$

Hence there exists $s_1(\varepsilon) > 0$ such that, $\forall s > s_1$,

$$2\left| \int_s^t d\tau\, q^*(\tau) \left(e^{-iH_0(s-\tau)}\varphi_s\right)(0) \right| \leq \frac{2\varepsilon^2}{9}$$

if $q(t) \in L^1(\mathbb{R}^+)$. Moreover by the same reason there exists $s_2(\varepsilon) > 0$ such that $\forall s > s_2$,

$$2\left| \int_s^t d\tau\, q^*(\tau) \frac{q(s)}{4\pi\sqrt{\pi i}\sqrt{\tau - s}} \right| \leq \frac{2\varepsilon^2}{9}$$

Setting $s_0(\varepsilon) = \max(s_1(\varepsilon), s_2(\varepsilon))$, one has $\forall s > s_0$

$$\left\| \int_s^t d\tau\, q(\tau)\, U_0(t - \tau; \vec{x}) \right\| \leq \frac{2\varepsilon}{3} \tag{1.37}$$

so that the whole L^2–norm of the second term in decomposition (1.35) is suitably small for $s > s_0$.
On the other hand the first term in (1.35) is the free evolution of a L^2–function and hence there exists $\delta(\varepsilon) > 0$ such that $\forall t > s + \delta$ and $\forall R < \infty$,

$$\left\| F(|\vec{x}| \leq R) U(t - s) \Psi_s \right\| \leq \frac{\varepsilon}{3} \tag{1.38}$$

Setting $t_0(\varepsilon) = s_0(\varepsilon) + \delta(\varepsilon)$, from (1.35), (1.37) and (1.38) one has

$$\left\| F(|\vec{x}| \leq R) \Psi_t \right\| \leq \varepsilon$$

$\forall t > t_0$, if $q(t) \in L^1(\mathbb{R}^+)$.
By Proposition 1.9 the inequality is then satisfied by every $\Psi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$: unitarity of the family $U(t,s)$ allows to extend the result to the whole Hilbert space $L^2(\mathbb{R}^3)$.

□

Corollary 1.2 *If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition with respect to T (1.19), the discrete spectrum of the Floquet operator associated to $H_{\alpha(t)}$,*

$$K \equiv -i\frac{\partial}{\partial t} + H_{\alpha(t)}$$

is empty.

Proof: The result is a straightforward consequence of Theorem 1.2: every eigenvector of K differs from a periodic function by a phase factor and hence can not satisfy (1.33).

□

1.4 CASE II: $\alpha_0 = 0$

If $\alpha(t) = \alpha_0 = 0$ does not depend on time, the problem has a simple solution: the spectrum of $H_{\alpha(t)}$ is absolutely continuous and equal to the positive real line, with a resonance at the origin; hence there is no bound state and the system shows complete ionization independently on the initial datum.
On the other hand if $\alpha(t)$ is a zero mean function, we shall see that the genericity condition (1.19) is still needed to prove complete ionization.
So let us assume that $\alpha_0 = 0$, the normalization (1.17) holds and the initial datum is given by (1.7): equation (1.14) then becomes

$$\tilde{q}(p) = -4\pi\sqrt{\frac{i}{p}} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \alpha_k \, \tilde{q}(p + i\omega k) - 2i\sqrt{\frac{2\pi i}{p}} \, \frac{1 - \sqrt{-ip}}{1 + ip} \tag{1.39}$$

with the choice (1.15) for the branch cut of $\sqrt{p}$. By Proposition 1.2 the solution is analytic in the open right half plane. In the following section we shall study the singularities on the imaginary axis.

1.4.1 Singularities on the imaginary axis

Setting $q_n(p) \equiv \tilde{q}(p + i\omega n)$, $p \in \mathcal{I} = [0, \omega)$, as in Section 3.1, equation (1.39) assumes the form (1.21),

$$q(p) = \mathcal{M}(p) \, q(p) + o(p) \tag{1.40}$$

with

$$\left(\mathcal{M}q\right)_n(p) \equiv -\frac{4\pi}{\sqrt{\omega n - ip}} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \alpha_k \, q_{n+k}(p) \tag{1.41}$$

and $o(p) = \{o_n(p)\}_{n \in \mathbb{Z}}$,

$$o_n(p) \equiv -\frac{2i\sqrt{2\pi}}{\sqrt{\omega n - ip} \, \left(1 + \sqrt{\omega n - ip}\right)} \tag{1.42}$$

Proposition 1.10 *For $p \in \mathcal{I}$, $\Re(p) = 0$, $p \neq 0$, $\mathcal{M}(p)$ is an analytic operator-valued function and $\mathcal{M}(p)$ is a compact operator on $\ell_2(\mathbb{Z})$.*

Proof: See the proof of Proposition 1.4.

□

Proposition 1.11 *There exists a unique solution $q_n(p) \in \ell_2(\mathbb{Z})$ of (1.40) and it is analytic on the imaginary axis for $p \neq 0$.*

Proof: See the proof of Proposition 1.5.

□

Proposition 1.12 *If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition (1.19), the solution of equation (1.40) has the form $q_n(p) = c_n(p) + d_n(p)\sqrt{p}$, $n \in \mathbb{Z}$, in a neighborhood of $p = 0$, where the functions $c_n(p)$ and $d_n(p)$ are analytic at $p = 0$.*

Proof: Let us proceed as in the proof of Proposition 1.7: setting $q_n = r_n + t_n q_0$, $n \in \mathbb{Z} \setminus \{0\}$, where $\{t_n\}$ is the solution of

$$t_n = -\frac{4\pi}{\sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-n}} \alpha_k t_{n+k} - \frac{4\pi\alpha_{-n}}{\sqrt{\omega n - ip}} \tag{1.43}$$

A slightly different version of Lemma 1.1 guarantees that the solution $\{t_n\} \in \ell_2(\mathbb{Z} \setminus \{0\})$ is unique and analytic at $p = 0$.
By means of this substitution we obtain

$$r_n = -\frac{4\pi}{\sqrt{\omega n - ip}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-n}} \alpha_k r_{n+k} - \frac{2i\sqrt{2\pi}}{\sqrt{\omega n - ip}\ (1 + \sqrt{\omega n - ip})} \tag{1.44}$$

and

$$q_0 = -\frac{4\pi}{\sqrt{-ip}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k \left(r_k + t_k q_0\right) - \frac{2i\sqrt{2\pi}}{\sqrt{-ip}\ (1 + \sqrt{-ip})}$$

or

$$\left(\sqrt{-ip} + F\right) q_0 = G - \frac{2\sqrt{2\pi}}{1 + \sqrt{-ip}}$$

where (like in the proof of Proposition 1.7)

$$F \equiv 4\pi \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k t_k$$

and

$$G \equiv -4\pi \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k r_k$$

Moreover $F(0) \neq 0$, because of genericity condition (1.19) (see the proof of Proposition 1.7), F and G are analytic in a neighborhood of $p = 0$ (see Lemma 1.1), so that

$$q_0 = F' + \sqrt{p}\ G'$$

where F' and G' are analytic and

$$G' \equiv \frac{2\sqrt{-2\pi i}(F+1) - \sqrt{-i}(1+ip)G}{(1+ip)(F^2+ip)}$$

□

1.4.2 Complete ionization in the generic case

As in section 3 we can now state the main result:

Theorem 1.3 *If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition (1.19) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$|q(t)| \leq A\, t^{-\frac{3}{2}} + R(t) \tag{1.45}$$

where $A \in \mathbb{R}$ and $R(t)$ has an exponential decay, $R(t) \sim Ce^{-Bt}$ for some $B > 0$.

Proof: See the proof of Theorem 1.1.

□

Corollary 1.3 *If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition (1.19) with respect to $\mathcal{T}$, the system shows asymptotic complete ionization and, as $t \to \infty$,*

$$|\theta(t)| \leq D\, t^{-\frac{3}{2}} + E(t)$$

where $D \in \mathbb{R}$ and $E(t)$ has an exponential decay.

Proof: See the proof of Corollary 1.1.

□

Theorem 1.4 *If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition (1.19) with respect to $\mathcal{T}$, every $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state of $H_{\alpha(t)}$, i.e.*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau\, \left\|F(|\vec{x}| \leq R)U(\tau,0)\Psi\right\|^2 = 0$$

Moreover the discrete spectrum of the Floquet operator is empty.

Proof: See the proof of Proposition 1.9 and Theorem 1.2.

□

1.5 CASE III: $\alpha_0 > 0$

To complete the analysis of the problem, we shall consider the case of mean greater than 0: taking the normalization (1.17) and the initial condition (1.7), (1.14) assumes the form (1.20):

$$\tilde{q}(p) = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{-ip}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k \, \tilde{q}(p + i\omega k) - \frac{2i\sqrt{2\pi}}{4\pi\alpha_0 + \sqrt{-ip}} \frac{1-\sqrt{-ip}}{1+ip} \tag{1.46}$$

Analyticity of the solution on the open right half plane is a consequence of Proposition 1.2.
Moreover, following the discussion contained in section 3 and setting $q_n(p) \equiv \tilde{q}(p + i\omega n)$, $\Im(p) \in [0, \omega)$, the equation assumes the form (1.21).
Let us now consider the behavior on the imaginary axis: singularities for $\Re(p) = 0$ are associated to zeros of $4\pi\alpha_0 + \sqrt{\omega n + s}$, $s \in [0, \omega)$, but, since $\alpha_0 > 0$, it is clear that the expression can not have zeros on the imaginary axis. Hence the proof of Proposition 1.5 can be extended to the closed right half plane except the origin:

Proposition 1.13 *If $\{\alpha_n\}$ satisfies (1.9), the solution $\tilde{q}(p)$ of (1.46) is unique and analytic for $\Re(p) \geq 0$, $p \neq i\omega n$, $n \in \mathbb{Z}$.*

Proof: See the proof of Proposition 1.5, Propositions 1.4 and 1.3 and the previous discussion.

□

Moreover the behavior at the origin is described by the following

Proposition 1.14 *If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition with respect to $\mathcal{T}$ (1.19), then, in an imaginary neighborhood of $p = i\omega n$, $n \in \mathbb{Z}$, the solution of equation (1.46) has the form $\tilde{q}(p) = c_n(p) + d_n(p)\sqrt{p - i\omega n}$, where the functions $c_n(p)$ and $d_n(p)$ are analytic at $p = i\omega n$.*

Proof: The proof of Proposition 1.7 still applies with only one difference: since, independently of ω, the solution can not have a pole on the imaginary axis, we need not to distinguish between the resonant case and the non-resonant one.

□

We can now prove asymptotic complete ionization of the system:

Theorem 1.5 *If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition (1.19) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$|q(t)| \le A\, t^{-\frac{3}{2}} + R(t) \tag{1.47}$$

where $A \in \mathbb{R}$ and $R(t)$ has an exponential decay, $R(t) \sim Ce^{-Bt}$ for some $B > 0$.
Moreover the system shows asymptotic complete ionization and, as $t \to \infty$,

$$|\theta(t)| \le D\, t^{-\frac{3}{2}} + E(t)$$

where $D \in \mathbb{R}$ and $E(t)$ has an exponential decay.

Proof: See the proof of Theorem 1.1 and Corollary 1.1.

□

Theorem 1.6 *If $\{\alpha_n\}$ satisfies (1.9) and the genericity condition (1.19) with respect to $\mathcal{T}$, every $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state of $H_{\alpha(t)}$, i.e.*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \, \left\| F(|\vec{x}| \le R) U(\tau, 0)\Psi \right\|^2 = 0$$

Moreover the discrete spectrum of the Floquet operator is empty.

Proof: See the proof of Proposition 1.9 and Theorem 1.2.

□

Remark: If $\alpha(t) \ge 0$, $\forall t \in \mathbb{R}^+$, Proposition 1.14 holds without the genericity condition on the Fourier coefficients of $\alpha(t)$: for instance the genericity condition enters (see the proof of Proposition 1.7) in the proof of absence of non-zero solutions of the homogeneous equation

$$t_n = -\frac{4\pi}{4\pi\alpha_0 + \sqrt{\omega n + s}} \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0,-n}} \alpha_k \, t_{n+k}$$

where $s \in [0, \omega)$. Let us suppose that there exists a non-zero solution $\{T_n\} \in \ell_2(\mathbb{Z})$. Multiplying both sides of the equation by T_n^*, one has

$$\sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \sqrt{\omega n + s}\, |T_n|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z} \\ n,k\neq 0}} T_n^* \alpha_{k-n} T_k$$

Since the right hand side is real, $T_n = 0$, $\forall n < 0$. Moreover, fixing $T_0 = 0$ and setting

$$T(t) \equiv \sum_{n \in \mathbb{Z}} T_n \, e^{-i\omega n t}$$

it follows that

$$-4\pi \sum_{n,k \in \mathbb{Z}} T_n^* \alpha_{k-n} T_k = -4\pi \Big(T(t), \alpha(t) T(t) \Big)_{L^2([0,T])} \leq 0$$

because $\alpha(t) \geq 0$, $\forall t \in [0, T]$, but the left hand side is positive and then $T_n = 0$, $\forall n \in \mathbb{Z}$.

Chapter 2

Decay of a Bound State under a Time-dependent Singular Perturbation

2.1 Introduction

In this Chapter we discuss the decay of a bound state under a time-periodic perturbation, studying a toy model which involves time-dependent point interactions. Indeed we consider a zero-range time-dependent perturbation of a system given by a quantum particle interacting with a zero-range (time-independent) potential (see Section 2.2). Assuming that the unperturbed Hamiltonian has a bound state and the system is in that state at the initial time, we investigate the problem of asymptotic complete ionization (see Chapter 1).

Following the same procedure of Chapter 1, that is applying the Laplace transform to the evolution equation of the "charges" and studying the singularities of the solutions in a neighborhood of the origin (Sections 2.3, 2.4 and 2.5), we prove that the system shows asymptotic complete ionization (Section 2.6) under the same genericity conditions introduced in Chapter 1 (see (1.19)) on the parameter $\alpha(t)$, identifying the time-dependent perturbation. In that case we prove that the survival probability of the bound state has a power law decay to zero as $t \to \infty$ and that every state of the system is a scattering state (Section 2.6).

2.2 The model

The model we are going to study describes a quantum particle with a zero range interactions fixed at the origin and of strength $-1/4\pi$, subject to a periodic force, which we shall take of zero range and placed in a point of coordinates $\vec{r}$ and with strength $\alpha(t)$ of nonnegative mean. If the time-dependent part of the interaction were not present, the system would have a bound state, with normalized wave function

$$\Psi_0(\vec{x}) = \frac{e^{-|\vec{x}|}}{\sqrt{4\pi}|\vec{x}|} \tag{2.1}$$

and the remaining part of the spectrum would be absolutely continuous conciding with $[0, \infty)$.
The entire system is described by the time-dependent self-adjoint Hamiltonian $H_{\alpha(t)}$,

$$\mathcal{D}(H_{\alpha(t)}) = \Big\{ \Psi \in L^2(\mathbb{R}^3) \mid \exists\, q^{(1)}(t),\ q^{(2)}(t) \in \mathbb{C},$$

$$\varphi_\lambda(\vec{x}) \equiv \Big(\Psi(\vec{x}) - q^{(1)}(t)\, \mathcal{G}_\lambda(\vec{x}) - q^{(2)}(t)\, \mathcal{G}_\lambda(\vec{x} - \vec{r}) \Big) \in H^2(\mathbb{R}^3) \Big\} \tag{2.2}$$

$$\big(H_{\alpha(t)} + \lambda\big)\Psi(\vec{x}) = \big(H_0 + \lambda\big) \Big[\Psi(\vec{x}) - q^{(1)}(t)\, \mathcal{G}_\lambda(\vec{x}) - q^{(2)}(t)\, \mathcal{G}_\lambda(\vec{x} - \vec{r}) \Big] \tag{2.3}$$

where λ is an arbitrary positive parameter. The "charges" $q^{(i)}(t)$ are determined by the boundary conditions

$$\begin{aligned} \varphi_\lambda(0) &= -\frac{1 + i\lambda}{4\pi}\, q^{(1)}(t) - \mathcal{G}_\lambda(\vec{r})\, q^{(2)}(t) \\ \varphi_\lambda(\vec{r}) &= \frac{4\pi\alpha(t) - i\lambda}{4\pi}\, q^{(2)}(t) - \mathcal{G}_\lambda(\vec{r})\, q^{(1)}(t) \end{aligned} \tag{2.4}$$

and

$$\mathcal{G}_\lambda(\vec{x} - \vec{x}') = \frac{e^{-\sqrt{\lambda}|\vec{x} - \vec{x}'|}}{4\pi|\vec{x} - \vec{x}'|}$$

is the Green function of the free Hamiltonian $H_0 = -\Delta$.
The essential spectrum of the operator (2.3) is purely absolutely continuous and equals the positive real line $[0, \infty)$. Moreover there exist at most 2 negative eigenvalues (counting multiplicity) and $-k^2 \in \sigma_{\mathrm{p}}(H_{\alpha(t)})$, $k > 0$, if

and only if $\det[\Gamma_{\alpha(t)}(k)] = 0$, where $\Gamma_{\alpha(t)}(k)$ is the 2×2 matrix

$$\Gamma_{\alpha(t)}(k) \equiv \begin{pmatrix} -\dfrac{1-k}{4\pi} & -\mathcal{G}_k(\vec{r}) \\ & \\ -\mathcal{G}_k(\vec{r}) & \dfrac{4\pi\alpha(t)+k}{4\pi} \end{pmatrix} \tag{2.5}$$

The multiplicity of the eigenvalue k^2 is equal to the multiplicity of the eigenvalue 0 of the matrix. The corresponding eigenfunction can be expressed in the form

$$\varphi_k(\vec{x}) = c_1\, \mathcal{G}_k(\vec{x}) + c_2\, \mathcal{G}_k(\vec{x}-\vec{r})$$

where (c_1, c_2) are the eigenvectors with eigenvalue 0 of $\Gamma_\alpha(k)$.
It is well known (see [18, 19, 31, 48, 56]) that the solution of the time-dependent Schrödinger equation

$$i\frac{\partial \Psi_t}{\partial t} = H_{\alpha(t)}\Psi_t \tag{2.6}$$

associated to the operator (2.3) is given by

$$\Psi_t(\vec{x}) = U_0(t-s)\Psi_s(\vec{x}) + i\int_s^t d\tau \left[q^{(1)}(\tau)\, U_0(t-\tau;\vec{x}) + \right.$$

$$\left. + q^{(2)}(\tau)\, U_0(t-\tau;\vec{x}-\vec{r}) \right] \tag{2.7}$$

where $U_0(t) = \exp(-iH_0 t)$, $U_0(t;\vec{x})$ is the kernel associated to the free propagator and the charges $q^{(j)}(t)$ satisfy a system of Volterra integral equations for $t \geq s$,

$$q^{(1)}(t) + \frac{\sqrt{-2i}}{\pi}\int_s^t d\tau\, q^{(2)}(\tau) \int_\tau^t d\sigma\, \frac{U_0(\sigma-\tau;\vec{r})}{\sqrt{t-\sigma}} - \frac{1}{\sqrt{-\pi i}}\int_s^t d\tau\, \frac{q^{(1)}(\tau)}{\sqrt{t-\tau}} =$$

$$= 4\sqrt{\pi i}\int_s^t d\tau\, \frac{\left(U_0(\tau)\Psi_s\right)(0)}{\sqrt{t-\tau}} \tag{2.8}$$

$$q^{(2)}(t) + \frac{\sqrt{-2i}}{\pi}\int_s^t d\tau\, q^{(1)}(\tau) \int_\tau^t d\sigma\, \frac{U_0(\sigma-\tau;\vec{r})}{\sqrt{t-\sigma}} + 4\sqrt{\pi i}\int_s^t d\tau\, \frac{\alpha(\tau)\, q^{(2)}(\tau)}{\sqrt{t-\tau}} =$$

$$= 4\sqrt{\pi i}\int_s^t d\tau\, \frac{\left(U_0(\tau)\Psi_s\right)(\vec{r})}{\sqrt{t-\tau}} \tag{2.9}$$

We are interested in studying complete ionization of the system defined by (2.3) and (2.6), starting by the normalized bound state (2.1) at time $t = 0$.

Moreover we require that $\alpha(t)$ is a real continuous periodic function with period T, so that it can be decomposed in a Fourier series, for each $t \in \mathbb{R}^+$, and the series converges uniformly on every compact subset of the real line. Then, in terms of Fourier coefficients of $\alpha(t)$, we assume

$$\begin{aligned} &1. \quad \alpha(t) = \sum_{n\in\mathbb{Z}} \alpha_n \, e^{-in\omega t} \, , \; \{\alpha_n\} \in \ell_1(\mathbb{Z}), \; \alpha_0 \geq 0 \\ &2. \quad \alpha_n = \alpha^*_{-n} \end{aligned} \tag{2.10}$$

We recall the definition of the genericity condition (1.19): if $\mathcal{T}$ denotes the right shift operator on $\ell_2(\mathbb{N})$, we say that $\alpha = \{\alpha_n\} \in \ell_2(\mathbb{Z})$ is *generic* with respect to $\mathcal{T}$, if $\tilde{\alpha} \equiv \{\alpha_n\}_{n>0} \in \ell_2(\mathbb{N})$ satisfies the following condition

$$e_1 = (1, 0, 0, \dots) \in \overline{\bigvee_{n=0}^{\infty} \mathcal{T}^n \tilde{\alpha}} \tag{2.11}$$

By simple estimates on the sup norm of $r_j(t) \equiv q^{(j)}(t)\, e^{-bt}$, it is easy to prove that the charges $q^{(j)}(t)$ have at most an exponential behavior as $t \to \infty$, i.e. asymptotically $|q^{(j)}(t)| \leq A_j e^{b_j t}$.

Proposition 2.1 *There exist $A_j > 0$, $j = 1, 2$, such that, $\forall\, b > b_0 \in \mathbb{R}^+$ and $\forall\, t \in \mathbb{R}^+$,*

$$|q^{(j)}(t)| \leq A_j e^{bt} \tag{2.12}$$

Proof: Let us do the substitution $q^{(j)}(t) = r_j(t) e^{bt}$ in equations (2.8) and (2.9):

$$r_1(t) = \frac{1}{\sqrt{-\pi i}} \int_0^t d\tau \, \frac{r_1(\tau)\, e^{-b(t-\tau)}}{\sqrt{t-\tau}} + \int_0^t d\tau \, r_2(\tau) D(t-\tau) + f_1(t) \tag{2.13}$$

$$r_2(t) = -4\sqrt{\pi i} \int_0^t d\tau \, \frac{\alpha(\tau)\, r_2(\tau)\, e^{-b(t-\tau)}}{\sqrt{t-\tau}} + \int_0^t d\tau \, r_1(\tau) D(t-\tau) + f_2(t) \tag{2.14}$$

where

$$D(t-\tau) \equiv -\frac{\sqrt{-2i}}{\pi} \int_\tau^t d\sigma \, \frac{U_0(\sigma - \tau; \vec{r})}{\sqrt{t-\sigma}}$$

and

$$f_1(t) \equiv 4\sqrt{\pi i}\, e^{-bt} \int_0^t d\tau \, \frac{\big(U_0(\tau)\Psi_0\big)(0)}{\sqrt{t-\tau}}$$

$$f_2(t) \equiv 4\sqrt{\pi i}\, e^{-bt} \int_0^t d\tau \, \frac{\big(U_0(\tau)\Psi_0\big)(\vec{r})}{\sqrt{t-\tau}}$$

It is easy to see that, for each $b > 0$, the functions $f_i(t)$ are bounded for each $t \in \mathbb{R}^+$, i.e.

$$\|f_i\|_\infty = c_i(b) < \infty$$

at least for initial states in the dense subset given by the domain $\mathcal{D}(H_{\alpha(0)})$. Moreover

$$\sup_{t\in\mathbb{R}^+} \left| \frac{1}{\sqrt{-\pi i}} \int_0^t d\tau \, \frac{r_1(\tau)\, e^{-b(t-\tau)}}{\sqrt{t-\tau}} \right| \leq \frac{\|r_1\|_\infty}{\sqrt{\pi}} \sup_{t\in\mathbb{R}^+} \left| \int_0^t d\tau \, \frac{e^{-b\tau}}{\sqrt{\tau}} \right| = \frac{\|r_1\|_\infty}{\sqrt{b}}$$

$$\sup_{t\in\mathbb{R}^+} \left| 4\sqrt{\pi i} \int_0^t d\tau \, \frac{\alpha(\tau) r_2(\tau)\, e^{-b(t-\tau)}}{\sqrt{t-\tau}} \right| \leq \frac{4\pi \|\alpha\|_\infty \|r_2\|_\infty}{\sqrt{b}}$$

and

$$\sup_{t\in\mathbb{R}^+} \left| \int_0^t d\tau \; r_j(\tau) D(t-\tau) \right| \leq c(b) \|r_j\|_\infty$$

where

$$c(b) = \sup_{t\in\mathbb{R}^+} \int_0^t d\tau |D(\tau)|$$

is finite $\forall b > 0$ and goes to 0 as $b \longrightarrow \infty$.
Therefore, taking the norm $\| \; \|_\infty$ of (2.13) and (2.14) and using the triangular inequality, one has

$$\frac{\sqrt{b}-1}{\sqrt{b}} \|r_1\|_\infty \leq c(b)\|r_2\|_\infty + c_1(b)$$

$$\|r_2\|_\infty \leq \frac{\|r_2\|_\infty}{\sqrt{b}} + c(b)\|r_1\|_\infty + c_2(b)$$

and

$$\left[1 - \frac{1}{\sqrt{b}} - \frac{c^2(b)\sqrt{b}}{\sqrt{b}-1} \right] \|r_2\|_\infty \leq \frac{c(b)c_1(b)\sqrt{b}}{\sqrt{b}-1} + c_2(b)$$

Now, since

$$\lim_{b\longrightarrow\infty} c(b) = \lim_{b\longrightarrow\infty} c_i(b) = 0$$

it is always possible to find a $b_0 > 1$ such that the claim is true.

□

Therefore the Laplace transform of $q^{(j)}(t)$, denoted by

$$\tilde{q}^{(j)}(p) \equiv \int_0^\infty dt \; e^{-pt} q^{(j)}(t)$$

exists analytic at least for $\Re(p) > b_0$. Hence, applying the Laplace transform to equations (2.8) and (2.9), one has

$$\tilde{q}^{(1)}(p) = -\frac{1}{(2\pi)^{\frac{3}{2}}r}\frac{e^{-r\sqrt{-ip}}}{1-\sqrt{-ip}}\,\tilde{q}^{(2)}(p) + F_1(p) \tag{2.15}$$

$$\tilde{q}^{(2)}(p) = -\frac{4\pi}{\sqrt{-ip}}\sum_{k\in\mathbb{Z}}\alpha_k\tilde{q}^{(2)}(p+i\omega k) + \frac{e^{-\sqrt{-ip}r}}{2\pi r\sqrt{-2\pi ip}}\,\tilde{q}^{(1)}(p) + F_2(p) \tag{2.16}$$

where the explicit expression of $F_i(p)$ for the initial datum (2.1) is given by

$$F_1(p) \equiv -\frac{2i\sqrt{2\pi}}{1+ip}$$

$$F_2(p) \equiv -\frac{2i\sqrt{2\pi}}{\sqrt{-ip}}\,\frac{e^{-\sqrt{-ip}r}-e^{-r}}{r(1+ip)}$$

Let us start considering the system of equations (2.15) and (2.16), for the specific initial datum (2.1): analyticity at least for for $\Re(p) > b_0$ suggest to choose the branch cut of the square root along the negative real line: if $p = \varrho\, e^{i\vartheta}$,

$$\sqrt{p} = \sqrt{\varrho}\; e^{i\vartheta/2} \tag{2.17}$$

with $-\pi < \vartheta \leq \pi$.

Before dealing with the behavior of the solution, let us simplify the problem: setting $q_n^{(j)}(p) \equiv \tilde{q}^{(j)}(p+i\omega n)$ we obtain a sequence of functions on the strip $\mathcal{I} = \{p \in \mathbb{C},\ 0 \leq \Im(p) < \omega\}$ and setting $q_j(p) \equiv \{q_n^{(j)}(p)\}_{n\in\mathbb{Z}}$, equations (2.15) and (2.16) can be rewritten

$$q_1(p) = \mathcal{M}_1\, q_2(p) + G_1(p) \tag{2.18}$$

$$q_2(p) = \mathcal{L}\, q_2(p) + \mathcal{M}_2\, q_1(p) + G_2(p) \tag{2.19}$$

where

$$\left(\mathcal{M}_1\, q\right)_n(p) \equiv -\frac{1}{(2\pi)^{\frac{3}{2}}r}\frac{e^{-r\sqrt{\omega n-ip}}}{1-\sqrt{\omega n-ip}}\; q_n(p) \tag{2.20}$$

$$\left(\mathcal{M}_2\, q\right)_n(p) \equiv \frac{1}{(2\pi)^{\frac{3}{2}}r}\frac{e^{-r\sqrt{\omega n-ip}}}{4\pi\alpha_0+\sqrt{\omega n-ip}}\; q_n(p) \tag{2.21}$$

$$\left(\mathcal{L}\, q\right)_n(p) \equiv -\frac{4\pi}{4\pi\alpha_0+\sqrt{\omega n-ip}}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\alpha_k\; q_{n+k}(p) \tag{2.22}$$

and $G_j(p) = \{g_n^{(j)}(p)\}_{n\in\mathbb{Z}}$ with

$$g_n^{(1)}(p) \equiv \frac{2i\sqrt{2\pi}}{1-\omega n+ip} \tag{2.23}$$

$$g_n^{(2)}(p) \equiv -\frac{2i\sqrt{2\pi}}{r}\frac{e^{-r\sqrt{\omega n-ip}}-e^{-r}}{(4\pi\alpha_0+\sqrt{\omega n-ip})(1-\omega n+ip)} \tag{2.24}$$

2.3 Analyticity on the (open) right half plane

We are going to prove that the solution of equations (2.18) and (2.19) exists and is analytic for $\Re(p) > 0$.
Let us start with some preliminary results:

Proposition 2.2 *For $p \in \mathcal{I}$, $\Re(p) > 0$, $\mathcal{M}_j(p)$ are analytic operator-valued functions and $\mathcal{M}_1(p)$ are compact operators on $\ell_2(\mathbb{Z})$.*

Proof: Let us consider only $\mathcal{M}_1$, since the argument does apply to $\mathcal{M}_2$ too.
The analyticity of the operator is a straightforward consequence of the explicit expression (2.20). Moreover the operator $\mathcal{M}_1(p)$ is a multiplication operator in $\ell_2(\mathbb{Z})$ and it is bounded and compact since

$$\left\{\frac{1}{(2\pi)^{\frac{3}{2}}r}\frac{e^{-r\sqrt{\omega n-ip}}}{1-\sqrt{\omega n-ip}}\right\} \in \ell_2(\mathbb{Z})$$

on the open right half plane: indeed the choice (2.17) for the branch cut of the square root implies $\Re(\sqrt{\omega n-ip}) > 0$, if $\Re(p) > 0$.

□

Proposition 2.3 *For $p \in \mathcal{I}$, $\Re(p) > 0$, $\mathcal{L}(p)$ is an analytic operator-valued function and $\mathcal{L}(p)$ is a compact operator on $\ell_2(\mathbb{Z})$.*

Proof: Analyticity and compactness follow from the explicit expression (2.22) as in the proof of Proposition 1.4.

□

Lemma 2.1 *For each $r, \omega \in \mathbb{R}^+$ and for $\Re(p) > 0$*

$$\Im\left[\sqrt{\omega n-ip}+\frac{1}{(2\pi)^3r^2}\frac{e^{-2r\sqrt{\omega n-ip}}}{1-\sqrt{\omega n-ip}}\right] < 0$$

$\forall n \in \mathbb{Z}$.

Proof: First of all we want to stress that the choice (2.17) for the branch cut implies that $\Re(\sqrt{\omega n - ip}) > 0$ and $\Im(\sqrt{\omega n - ip}) < 0$, if $\Re(p) > 0$. Calling $x \equiv \Re(\sqrt{\omega n - ip})$, $y \equiv \Im(\sqrt{\omega n - ip})$ and

$$f_r(x,y) \equiv \Im\left[x + iy + \frac{1}{(2\pi)^3 r^2}\frac{e^{-2r(x+iy)}}{1-x-iy}\right]$$

one has

$$\left|\frac{1}{(2\pi)^3 r^2}\frac{e^{-2r(x+iy)}}{1-x-iy}\right| < \frac{1}{(2\pi)^3 r^2 |y|}$$

and then $f_r(x,y) \leq 0$, if $|y| \geq \frac{1}{(2\pi)^{\frac{3}{2}} r}$. Moreover

$$f_r(x,y) = \frac{(2\pi)^3 r^2\left[(1-x)^2 + y^2\right]y + e^{-2rx}\left[y\cos(2ry) - (1-x)\sin(2ry)\right]}{(2\pi)^3 r^2[(1-x)^2 + y^2]}$$

and the claim is true if $x \geq 1$, since $\sin(2ry) < 0$ and $\cos(2ry) > 0$, for $y > -\frac{1}{(2\pi)^{\frac{3}{2}} r}$.
Hence it is sufficient to prove that $f_r(x,y) < 0$ on the set

$$R = \left\{(x,y) \in \mathbb{R}^2 \mid x < 1, -1/[(2\pi)^{\frac{3}{2}} r] < y < 0\right\}$$

Now set

$$g_r(x,y) \equiv \frac{(2\pi)^3 r^2[(1-x)^2 + y^2]\, f_r(x,y)}{y}$$

and consider

$$\frac{\partial g_r}{\partial y} = 2(2\pi)^3 r^2 y - 2e^{-2rx}\left[r\sin(2ry) + \frac{r(1-x)\cos(2ry)}{y} - \frac{(1-x)\sin(2ry)}{2y^2}\right]$$

Since, for $(x,y) \in R$,

$$2e^{-2rx} r \sin(2r|y|) < 2(2\pi)^3 r^2 |y|$$

and

$$2ry\cos(2ry) \leq \sin(2ry)$$

the partial derivative of g_r with respect to y is always negative in R.
Thus

$$g_r(x,y) \geq g_r(x,0) > 0$$

In conclusion $g_r(x,y) > 0$ and then $f_r(x,y) < 0$, $\forall (x,y) \in R$.

□

Proposition 2.4 *The solution $\tilde{q}^{(j)}(p)$, $j = 1, 2$, of (2.15) and (2.16) exists unique and analytic for $\Re(p) > 0$.*

Proof: Since $G_1(p) \in \ell_2(\mathbb{Z})$ is analytic on the right half plane and thanks to Proposition 2.2, we can substitute (2.18) in (2.19) and consider only the second equation. So that (2.19) now read

$$q_2(p) = \left[\mathcal{L} + \mathcal{M}_2\mathcal{M}_1\right] q_2(p) + \mathcal{M}_2\, G_1(p) + G_2(p) \qquad (2.25)$$

Then the key point will be the application of the analytic Fredholm theorem (Theorem VI.14 of [44]) to the operator $\mathcal{L}'(p) \equiv \mathcal{L} + \mathcal{M}_2\mathcal{M}_1$, in order to prove that $(I - \mathcal{L}'(p))^{-1}$ exists for $\Re(p) > 0$.
So let us begin with the analysis of the homogeneous equation associated to (2.25),

$$q(p) = \mathcal{L}'(p)\, q(p)$$

and suppose that there exists a nonzero solution $Q(p) = \{Q_n(p)\}_{n\in\mathbb{Z}}$. Multiplying both sides of the equation by Q_n^* and summing over $n \in \mathbb{Z}$, we have

$$\sum_{n\in\mathbb{Z}} \left[\sqrt{\omega n - ip} + \frac{1}{(2\pi)^3 r^2} \frac{e^{-2r\sqrt{\omega n - ip}}}{1 - \sqrt{\omega n - ip}}\right] |Q_n|^2 = -4\pi \sum_{n,k\in\mathbb{Z}} Q_n^*\, \alpha_{k-n}\, Q_k$$

but, since the right hand side is real, because of condition 2 in (2.10), it follows that

$$\Im\left[\sum_{n\in\mathbb{Z}} \left(\sqrt{\omega n - ip} + \frac{1}{(2\pi)^3 r^2} \frac{e^{-2r\sqrt{\omega n - ip}}}{1 - \sqrt{\omega n - ip}}\right) |Q_n|^2\right] = 0$$

and then, by Lemma 2.1, $Q_n = 0$, $\forall n \in \mathbb{Z}$.
Since there is no nonzero solution of the homogeneous equation associated to (2.25) and $\mathcal{L}$ is compact on the whole open right half plane, analytic Fredholm theorem applies and the result then easily follows, because $\mathcal{M}_2 G_1(p) + G_2(p) \in \ell_2(\mathbb{Z})$ and, for each $n \in \mathbb{Z}$, $\left[\mathcal{M}_2 G_1(p) + G_2(p)\right]_n$ is analytic for $\Re(p) > 0$.

□

2.4 Behavior on the imaginary axis at $p \neq 0$

The equation for $q_2(p)$ can be written

$$(4\pi\alpha_0 + c_n(p))\, q_n^{(2)}(p) = -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_k q_{n+k}^{(2)}(p) + f_n^{(2)}(p) \qquad (2.26)$$

where

$$c_n(p) \equiv \sqrt{\omega n - ip} + \frac{e^{-2r\sqrt{\omega n - ip}}}{(2\pi)^3 r^2 (1 - \sqrt{\omega n - ip})} \tag{2.27}$$

$$f_n^{(2)}(p) \equiv -\frac{2i\sqrt{2\pi}}{r(1 - \omega n + ip)} \left[\frac{(2\pi)^{\frac{3}{2}} - 1}{(2\pi)^{\frac{3}{2}}} e^{-r\sqrt{\omega n - ip}} - e^{-r} \right] \tag{2.28}$$

and it is clear that the solution may have a pole where

$$4\pi\alpha_0 + \sqrt{\omega n - ip} + \frac{e^{-2r\sqrt{\omega n - ip}}}{(2\pi)^3 r^2 (1 - \sqrt{\omega n - ip})} = 0$$

and that the coefficients of the equation for $q_0^{(2)}$ fail to be analytic at $p = i$: for $p \in \mathcal{I}$, $\Re(p) = 0$, and $n \in \mathbb{Z}$, the unique solution of $1 - \sqrt{\omega n - ip} = 0$ is $p = i$, $n = 0$.
In the following we shall see that in fact the solution is analytic on the imaginary axis except at most some singularities at $p = 0$. Let us start considering the position of the eventual pole:

Lemma 2.2 *There exists a unique $\bar{n} \in \mathbb{N}$ and a unique $\bar{p} \in \mathcal{I}$, $\Re(p) = 0$, such that*

$$4\pi\alpha_0 + \sqrt{\omega \bar{n} - i\bar{p}} + \frac{e^{-2r\sqrt{\omega \bar{n} - i\bar{p}}}}{(2\pi)^3 r^2 (1 - \sqrt{\omega \bar{n} - i\bar{p}})} = 0$$

Moreover $\forall n \in \mathbb{Z}$, $n < 0$ *and* $\forall p \in \mathcal{I}$, $\Re(p) = 0$,

$$\Im\left[4\pi\alpha_0 + \sqrt{\omega n - ip} + \frac{e^{-2r\sqrt{\omega n - ip}}}{(2\pi)^3 r^2 (1 - \sqrt{\omega n - ip})}\right] > 0$$

Proof: Let us first consider the second statement: on the strip $\mathcal{I}$ and for $n < 0$, $\sqrt{\omega n - ip} \equiv i\lambda$, with $\lambda \in \mathbb{R}$, $\lambda > 0$. Hence

$$\Im\big(c_n(i\lambda)\big) = \frac{(2\pi)^3 r^2 (1 + \lambda^2)\lambda + \lambda\cos(2r\lambda) - \sin(2r\lambda)}{(2\pi)^3 r^2 (1 + \lambda^2)}$$

and following the proof of Lemma 2.1, it can be easily proved that the expression above is positive $\forall \lambda \in \mathbb{R}^+$. On the other hand, if $n \geq 0$ and $p \in \mathcal{I}$, $\Re(p) = 0$, $\sqrt{\omega n - ip} = \lambda$, with $\lambda > 0$, and, $\forall r, \omega \in \mathbb{R}^+$, the equation

$$(2\pi)^3 r^2 (4\pi\alpha_0 + \lambda)(\lambda - 1) = e^{-2r\lambda}$$

has a unique solution for $\lambda \in \mathbb{R}^+$. Then, since there exists a unique $\bar{p} \in \mathcal{I}$, $\Re(\bar{p}) = 0$, such that, for fixed $\lambda \in \mathbb{R}^+$, the equation $\bar{p} = i(\lambda^2 - \omega\bar{n})$ is satisfied for some $\bar{n} \in \mathbb{N}$, the proof is complete.

□

Lemma 2.3 *If $\{\alpha_n\}$ satisfies (2.10) and the genericity condition with respect to $\mathcal{T}$ (2.11), the solution of (2.15) and (2.16) is unique and analytic on the imaginary axis for $p \neq 0, i, \bar{p}$.*

Proof: Since for $p \in \mathcal{I}$, $\Re(p) = 0$, and $p \neq 0, i, \bar{p}$, the coefficient of equation (2.18) and (2.19) are analytic (see Lemma 2.2) and belong to $\ell_2(\mathbb{Z})$ and since the operators $\mathcal{L}, \mathcal{M}_1$ and $\mathcal{M}_2$ are still compact on the same region, it is sufficient to show that the homogeneous equation associated to (2.26) has no non zero solution, in order to apply analytic Fredholm theorem.
If Q_n is such a non zero solution, following the proof of Proposition 2.4, we immediately obtain the condition

$$\sum_{n\in\mathbb{Z}} \left[\sqrt{\omega n - ip} + \frac{1}{(2\pi)^3 r^2} \frac{e^{-2r\sqrt{\omega n - ip}}}{1 - \sqrt{\omega n - ip}} \right] |Q_n|^2 \in \mathbb{R}$$

and then Lemma 2.2 guarantees that $Q_n = 0$, $\forall n < 0$. Now suppose that there exists $n_0 \in \mathbb{N}$, such that $Q_{n_0} \neq 0$. For $n < n_0$, one has $\sum_{k=n_0}^{\infty} \alpha_{k-n} Q_k = 0$ or, setting $k = n_0 - 1 + k'$, for $n \geq 0$,

$$\sum_{k'=1}^{\infty} \alpha_{k'+n} Q_{n_0-1+k'} = 0$$

and then, for each $n \geq 0$,

$$\left(Q' \, , \mathcal{T}^n \alpha \right)_{\ell_2(\mathbb{N})} = 0$$

where $Q'_n = Q^*_{n_0-1+n}$ and $(\cdot \, , \, \cdot)$ stands for the standard scalar product on $\ell_2(\mathbb{N})$. Finally the genericity condition (2.11) implies that $Q'_1 = Q^*_{n_0} = 0$ and then $Q_n = 0$, $\forall n \in \mathbb{Z}$.

□

Proposition 2.5 *If $\{\alpha_n\}$ satisfies (2.10) and the genericity condition with respect to $\mathcal{T}$ (2.11), the solution of (2.15) and (2.16) is unique and analytic on the imaginary axis except at most at $p = 0$.*

Proof: In the first part of the proof we are going to consider only the equation (2.26) for $q_2(p)$ and we shall extend then the results to $q_1(p)$.
In order to prove analyticity of the solution we need to analyze the behavior of the solution of (2.26) in a neighborhood of $p = \bar{p}$ (see Lemma 2.2) and $p = i$

separately and show that it has no singularity, while, for $p \in \mathcal{I}$, $\Re(p) = 0$, and $p \neq i, \bar{p}$ the proof of Proposition 2.4 still applies (see Lemma 2.3).
Let us look for a solution of (2.26) of the form (for simplicity we are going to omit the index 2)

$$q_n = u_n + v_n q_{\bar{n}}$$

for $n \neq \bar{n}$: q_n satisfies (2.26) if and only if $\{u_n\} \in \ell_2(\mathbb{Z} \setminus \{\bar{n}\})$ and $\{v_n\} \in \ell_2(\mathbb{Z} \setminus \{\bar{n}\})$ are solutions of

$$c_n(p)\, u_n = -4\pi \sum_{\substack{k \in \mathbb{Z} \\ k \neq \bar{n}}} \alpha_{k-n} u_k + f_n^{(2)}(p) \tag{2.29}$$

$$c_n(p)\, v_n = -4\pi \sum_{\substack{k \in \mathbb{Z} \\ k \neq \bar{n}}} \alpha_{k-n} v_k - 4\pi \alpha_{\bar{n}-n} \tag{2.30}$$

Existence of non-zero solutions of the homogeneous equations associated to (2.29) and (2.30) can be excluded because of the genericity condition as in the proof of Lemma 2.3 and then, since the coefficients of the equations above are analytic in a neighborhood of $\bar{p}$ and belong to $\ell_2(\mathbb{Z} \setminus \{\bar{n}\})$, $\{u_n\}, \{v_n\} \in \ell_2(\mathbb{Z} \setminus \{\bar{n}\})$ are analytic in the same neighborhood.
Moreover $q_{\bar{n}}$ satisfies the equation

$$\left\{ 4\pi\alpha_0 + c_{\bar{n}}(p) + 4\pi \sum_{\substack{k \in \mathbb{Z} \\ k \neq \bar{n}}} \alpha_{k-\bar{n}} v_k \right\} q_{\bar{n}} = -4\pi \sum_{\substack{k \in \mathbb{Z} \\ k \neq \bar{n}}} \alpha_{k-\bar{n}} u_k + f_{\bar{n}}^{(2)}(p)$$

It is then sufficient to show that

$$\sum_{\substack{k \in \mathbb{Z} \\ k \neq \bar{n}}} \alpha_{k-\bar{n}} v_k(\bar{p}) \neq 0$$

Let us suppose that the contrary is true: calling $V_n \equiv v_n(\bar{p})$, multiplying equation (2.30) at $p = \bar{p}$ by V_n^* and summing over $n \in \mathbb{Z}$, $n \neq \bar{n}$, one has

$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq \bar{n}}} \left\{ \sqrt{\omega n - i\bar{p}} + \frac{e^{-2r\sqrt{\omega n - i\bar{p}}}}{(2\pi)^3 r^2 (1 - \sqrt{\omega n - i\bar{p}})} \right\} |V_n|^2 = -4\pi \sum_{\substack{n,k \in \mathbb{Z} \\ n,k \neq \bar{n}}} V_n^* \alpha_{k-n} V_k$$

Using condition (2.10) and the genericity condition (2.11), as in the proof of Lemma 2.3, one obtain $V_n = 0$, $\forall n \in \mathbb{Z} \setminus \{\bar{n}\}$, but this is impossible since V_n satisfies equation (2.30).
This concludes the proof of analyticity of $q_2(p)$ in a neighborhood of $p = \bar{p}$.

In the same way it is possible to conclude that $q_2(p)$ is also analytic at $p = i$. It remains to study the behavior of $q_1(p)$ and in particular to analyze $q_0^{(1)}(p)$ in a neighborhood of $p = i$, where it may have a pole (see equation (2.18)): from (2.26) we have

$$\frac{e^{-2r\sqrt{\omega n+1}}}{(2\pi)^3 r^2}\, q_n^{(2)}(i) = -\frac{2i\sqrt{2\pi}}{r}\left[\frac{(2\pi)^{\frac{3}{2}}-1}{(2\pi)^{\frac{3}{2}}}\, e^{-r\sqrt{\omega n+1}} - e^{-r}\right]$$

and then $q_0^{(1)}(i) = i\sqrt{2\pi}$.

□

2.5 Behavior at $p = 0$

We shall now study the behavior of the solution of (2.18) and (2.19) on the imaginary axis at the origin. With the choice (2.17) for the branch cut of the square root, it is clear that we must expect branch points of $\tilde{q}^{(j)}(p)$, solutions of (2.15) and (2.16), at $p = i\omega n$, $n \in \mathbb{Z}$, which should imply a branch point at $p = 0$ for each $q_n^{(j)}$.
We are going to show that the solutions of (2.18) and (2.19) have a branch point singularity at the origin.

Proposition 2.6
If $\{\alpha_n\}$ satisfies (2.10) and (2.11) (genericity condition), the solution of the system (2.15), (2.16) has the form $\tilde{q}^{(j)}(p) = c_j(p) + d_j(p)\sqrt{p}$, $j = 1, 2$, in an imaginary neighborhood of $p = 0$, where the functions $c_j(p)$ and $d_j(p)$ are analytic at $p = 0$.

Proof: The resonant case, namely if, for some $N \in \mathbb{N}$, $\omega = 1/N$, and the non-resonant one will be treated separately.
1) Non-resonant case

Setting $q_n = u_n + v_n q_0$, $n \neq 0$ in (2.26), one obtains the following equations for $\{u_n\}$, $\{v_n\} \in \ell_2(\mathbb{Z} \setminus \{0\})$,

$$c_n(p)\, u_n = -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_{k-n} u_k + g_n^{(2)}(p) \tag{2.31}$$

$$c_n(p)\, v_n = -4\pi \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \alpha_{k-n} v_k - 4\pi\alpha_{-n} \tag{2.32}$$

If, for every $n \in \mathbb{Z}$, $c_n(0) \neq -4\pi\alpha_0$, using the genericity condition, it is easy to prove that $\{u_n\}$, $\{v_n\} \in \ell_2(\mathbb{Z} \setminus \{0\})$ are unique and analytic at $p = 0$. On the other hand if the condition above is not satisfied and there exists $N_1 \in \mathbb{Z}$ such that

$$4\pi\alpha_0 + \sqrt{\omega N_1} + \frac{e^{-2r\sqrt{\omega N_1}}}{(2\pi)^3 r^2 (1 - \sqrt{\omega N_1})} = 0$$

one can repeat the trick, setting for example $v_n = u_n' + v_n' v_{N_1}$ for $n \neq N_1$, and prove that in fact $\{u_n\}$ and $\{v_n\}$ are still analytic in a neighborhood of $p = 0$.

Thus it is sufficient to prove that q_0, which is solution of

$$\left\{ 4\pi\alpha_0 + c_0(p) + 4\pi \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k v_k \right\} q_0(p) = -4\pi \sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k u_k + f_0^{(2)}(p)$$

has the required behavior near $p = 0$. First, setting $v_n^0 = v_n(p = 0)$, we have to prove that

$$\sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \alpha_k v_k^0 \neq -\alpha_0 - \frac{1}{4\pi(2\pi)^3 r^2}$$

but, assuming that the contrary is true and multiplying both sides of equation (2.32), with $\bar{n} = 0$, by v_n^{0*} and summing over $n \in \mathbb{Z}$, $n \neq 0$, one has

$$\sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \sqrt{\omega n}\, |v_n^0|^2 = -4\pi \sum_{\substack{n,k\in\mathbb{Z} \\ n,k\neq 0}} v_n^{0*} \alpha_{k-n} v_k^0 + 4\pi\alpha_0 + \frac{1}{(2\pi)^3 r^2}$$

The right hand side is still real so that, assuming that the genericity condition is satisfied by $\{\alpha_n\}$ and applying the argument contained in the proof of Proposition 2.5, we immediately obtain $\{v_n^0\} = 0$, which is a contradiction, since $\{v_n^0\}$ solves (2.32).

The result for $\tilde{q}^{(2)}$ follows then directly from the equation for q_0, since $e^{-2r\sqrt{-ip}}$ has a branch cut along the negative real line. The extension to $q^{(1)}$ is thus trivial.

2) Resonant case

As before let us look for a solution of (2.26) of the form $q_n = u_n + v_n q_0$, $n \neq 0$, so that $\{u_n\}$, $\{v_n\} \in \ell_2(\mathbb{Z}\setminus\{0\})$ solve (2.31) and (2.32) with $\omega = 1/N$. Multiplying both sides of (2.31) and (2.32) for $n = N$ by $1 - n/N - ip$, one sees that u_N and v_N have no pole singularity at $p = 0$. On the other, if there exists $N_1 \in \mathbb{Z}$ such that

$$4\pi\alpha_0 + \sqrt{\frac{N_1}{N}} + \frac{e^{-2r\sqrt{N_1/N}}}{(2\pi)^3 r^2 (1 - \sqrt{N_1/N})} = 0$$

the solutions could have a pole at $p = 0$, for $n = N_1$ (the expression above guarantees that $N_1 \neq N$). Nevertheless, repeating the above procedure for $n = N_1$, it is easily seen that in fact $\{u_n\}$, $\{v_n\} \in \ell_2(\mathbb{Z}\setminus\{0\})$ are both analytic in a neighborhood of $p = 0$.
The behavior of $q^{(2)}$ near $p = 0$ is then proved like in the non-resonant case, but we have now to take care about $q^{(1)}$, since the coefficient in $\mathcal{M}_1$ (see the definition (2.20)) has a pole at $p = 0$ for $n = N$. But from (2.26) one has

$$\frac{e^{-2r\sqrt{n/N-ip}}}{(2\pi)^3r^2}\, q_n^{(2)}(0) = -\frac{2i\sqrt{2\pi}}{r(1+\sqrt{n/N})}\left[\frac{(2\pi)^{\frac{3}{2}}-1}{(2\pi)^{\frac{3}{2}}}\, e^{-r\sqrt{n/N}} - e^{-r}\right]$$

so that $q_N^{(1)}(0) = i\sqrt{2\pi}$.

□

2.6 Complete ionization in the generic case

Summing up the results about the behavior of the Laplace transforms $\tilde{q}^{(j)}(p)$, $j = 1, 2$, we can state the following

Theorem 2.1 *If $\{\alpha_n\}$ satisfies (2.10) and the genericity condition (2.11) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$|q^{(j)}(t)| \leq A_j\, t^{-\frac{3}{2}} + R_j(t) \tag{2.33}$$

where $A_j \in \mathbb{R}$ and $R_j(t)$ has an exponential decay, $R_j(t) \sim C_j e^{-B_j t}$ for some $B_j > 0$.
Moreover the system shows asymptotic complete ionization and, as $t \to \infty$,

$$|\theta(t)| = \left|\left(\varphi_{\alpha(0)}\,,\Psi_t\right)\right| \leq D\, t^{-\frac{3}{2}} + E(t)$$

where $D \in \mathbb{R}$ and $E(t)$ has an exponential decay.

Proof: Propositions 2.4, 2.5 and 2.6 guarantee that $\tilde{q}(p)$ is analytic on the closed right half plane, except branch point singularities on the imaginary axis at $p = i\omega n$, $n \in \mathbb{Z}$. Therefore we can chose a integration path for the inverse of Laplace transform of $\tilde{q}(q)$ along the imaginary axis like in [12] and the result is a straightforward consequence of the behavior of $q^{(j)}(p)$ around the branch points given by Proposition 2.6 (see e.g. the proof of Theorem 1.1 in Chapter 1).

The Laplace transform of $\theta(t)$ can be expressed in the following way (see e.g. Proposition 1.2 and Corollary 1.1)

$$\tilde{\theta}(p) = \tilde{Z}(p) + \tilde{Z}_1(p)\, \tilde{q}^{(1)}(p) + \tilde{Z}_2(p)\, \tilde{q}^{(2)}(p)$$

where $\tilde{Z}(p)$ is analytic on the closed right half plane and $\tilde{Z}_j(p)$ has only a branch point at the origin of the form $a_j + b_j\sqrt{p}$. Hence $\tilde{\theta}(p)$ has the same singularities as $\tilde{q}(p)$ and then its asymptotic behavior coincides with that of $q(t)$.

□

In the following we shall prove a stronger result about complete ionization of the system, namely that every state $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state for the operator $H_{\alpha(t)}$, i.e.

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \, \left\| F(|\vec{x}| \leq R) U(\tau, 0) \Psi \right\|^2 = 0 \tag{2.34}$$

where $F(S)$ is the multiplication operator by the characteristic function of the set $S \subset \mathbb{R}^3$ and $U(t,s)$ the unitary two-parameters family associated to $H_{\alpha(t)}$ (see (2.6)).
In order to prove (2.34), we first need to study the evolution of a generic initial datum in a suitable dense subset of $L^2(\mathbb{R}^3)$ and then we shall extend the result to every state using the unitarity of the evolution defined by (2.6) (see e.g. [18]).

Proposition 2.7 *Let $\Psi \in C_0^\infty(\mathbb{R}^3 \setminus \{0, \vec{r}\})$ a smooth function with compact support away from $0, \vec{r}$ and $q^{(j)}(t)$ be the solutions of equations (2.8) and (2.9) with initial condition $\Psi_0 = \Psi$. If $\{\alpha_n\}$ satisfies (2.10) and the genericity condition (2.11) with respect to $\mathcal{T}$, as $t \to \infty$,*

$$|q^{(j)}(t)| \leq A_j\, t^{-\frac{3}{2}} + R_j(t) \tag{2.35}$$

where $A_j \in \mathbb{R}$ and $R_j(t)$ has an exponential decay, $R_j(t) \sim C_j e^{-B_j t}$ for some $B_j > 0$.

Proof: The estimate on the behavior for large time contained in Section 2.2 still applies, so that $\tilde{q}^{(j)}(p)$ is analytic $\forall p$ with $\Re(p) > b_0$.
Hence we can consider the Laplace transforms of equations (2.8) and (2.9), which have the form (2.18) and (2.19) with

$$G_1(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty dt\, e^{-pt} \int_{\mathbb{R}^3} d^3\vec{k}\ \hat{\Psi}(\vec{k})\, e^{-ik^2 t}$$

$$G_2(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty dt\, e^{-pt} \int_{\mathbb{R}^3} d^3\vec{k}\; \hat{\Psi}(\vec{k})\, e^{-i(k^2 - \vec{k}\cdot\vec{r})\,t}$$

where $\hat{\Psi}(\vec{k})$ is the Fourier transform of Ψ.
Since for every smooth function Ψ with compact support, $\hat{\Psi}(\vec{k})$ is a smooth function with an exponential decay as $k \to \infty$, so that $G_j(p)$ has the same singularities as in the case already studied, i.e. a branch point singularity at the origin of the form $a(p) + b(p)\sqrt{p}$.

□

Theorem 2.2 *If $\{\alpha_n\}$ satisfies (2.10) and the genericity condition (2.11) with respect to $\mathcal{T}$, every $\Psi \in L^2(\mathbb{R}^3)$ is a scattering state of $H_{\alpha(t)}$, i.e.*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau\, \left\|F(|\vec{x}| \leq R)U(\tau,0)\Psi\right\|^2 = 0$$

Moreover the discrete spectrum of the Floquet operator associated to $H_{\alpha(t)}$,

$$K \equiv -i\frac{\partial}{\partial t} + H_{\alpha(t)}$$

is empty.

Proof: The proof follows from unitarity of the evolution and the explicit expression (2.7), together with Proposition 2.7 (see the proof of Theorem 1.2). The absence of eigenvalues of the Floquet operator is a straightforward consequence: every eigenvector of K differs from a periodic function by a phase factor and hence can not satisfy (2.34).

□

Remark: All the results about asymptotic complete ionization still hold for $\alpha_0 < 0$. The proof can be given in the same way but it is slightly more complicated, because $4\pi\alpha_0 + c_n(p)$ in Lemma 4.1 could vanish in two points instead of one. Nevertheless the argument contained in Proposition 4.1 can applied once more, in order to exclude the presence of the corresponding singularity of the solution.

Part II

Rotating Singular Interactions

Chapter 1

Rotating Point Interactions

1.1 Introduction

In this Chapter we consider a simple kind of moving point interactions, namely zero-range perturbations of the Laplacian supported on a uniformly rotating point in three and two dimensions.
We first give an heuristic definition of the models in terms of "pseudo-potentials" and then we classify the Hamiltonians of such systems (Sections 1.2.2 and 1.3.1) as a family of suitable self-adjoint extensions of some symmetric operator: studying the system in a uniformly rotating frame we eliminate the time-dependence of the Schrödinger operator and, by means of the Krein's theory, we show the explicit expression of its self-adjoint extensions (Theorems 1.2 and 1.6). As a straightforward consequence, the integral kernel of the resolvent and the spectrum of such self-adjoint extensions is obtained (Theorem 1.1, 1.5, 1.3, 1.7).
In Sections 1.2.3 and 1.3.2 we study the asymptotic limit of large angular velocity and we prove, in both the three dimensional and the two dimensional case, that the unitary family describing the time evolution converge in strong sense to some one-parameter unitary group (Theorems 1.4 and 1.8). The generator of such group is proved to be a time-independent self-adjoint operator given by a zero-range perturbation of the Laplacian supported on a circle.

1.2 Rotating Point Interactions in Three Dimensions

1.2.1 The Hamiltonian

The system we consider is defined by a formal time-dependent Schrödinger operator on $L^2(\mathbb{R}^n)$, $n = 2, 3$, of the form

$$H(t) = H_0 + V_t = -\Delta + V_t \tag{1.1}$$

with a uniformly rotating potential

$$V_t(\vec{x}) = V(\mathcal{R}^{-1}(t)\,\vec{x}) \tag{1.2}$$

where V is the formal zero-range perturbation

$$V(\vec{x}) = a\,\delta^{(3)}(\vec{x} - \vec{y}_0) \tag{1.3}$$

and $\mathcal{R}(t)$ a rotation on the x, y–plane with period $2\pi/\omega$:

$$\mathcal{R}(t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & & 1 \end{pmatrix}$$

Our purpose is to define in a rigorous way the time-dependent Hamiltonians (1.1) when the potential has the singular behavior (1.3).
Following the discussion contained in the Introduction and using the relation

$$U_{\text{inert}}(t, s) = R(t)\,U_{\text{rot}}(t - s)\,R^\dagger(s) \tag{1.4}$$

where $R(t)\Psi(\vec{x}) = \Psi(\mathcal{R}(t)^{-1}\,\vec{x})$, it is clear that the generator of unitary time evolution $U_{\text{rot}}(t - s)$ in the rotating frame is given by the formal time-independent operator

$$K = H_0 - \omega J + a\,\delta^{(3)}(\vec{x} - \vec{y}_0)$$

i.e., at least formally,

$$U_{\text{rot}}(t - s) = e^{-iK(t-s)} \tag{1.5}$$

Therefore the Hamiltonian of the system (in the rotating frame) is a self-adjoint extension of the operator

$$K_{y_0} = H_\omega$$

$$\mathcal{D}(K_{y_0}) = C_0^\infty(\mathbb{R}^3 - \{\vec{y}_0\})$$

The operator K_{y_0} is symmetric and then closable; let $\dot{K}_{y_0}$ be its closure, with domain $\mathcal{D}(\dot{K}_{y_0})$.
The function

$$\mathcal{G}_z(\vec{x},\vec{y}_0) = \int_0^\infty dk \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{k^2 - m\omega - z}\,\varphi^*_{klm}(\vec{y}_0)\,\varphi_{klm}(\vec{x}) \tag{1.6}$$

for $\vec{x} \in \mathbb{R}^3 - \{\vec{y}_0\}$ and $z \in \mathbb{C} - \mathbb{R}$, is the unique solution of

$$\dot{K}^*_{y_0}\Psi_z(\vec{x}) = z\Psi_z(\vec{x})$$

with $\Psi \in \mathcal{D}(\dot{K}^*_{y_0})$ (see Proposition B.1).
The operator $\dot{K}_{y_0}$ has then deficiency indexes $(1,1)$ and its self-adjoint extensions are given by the one-parameter family of operators K_{α,y_0}, $\alpha \in [0,2\pi)$:

$$\mathcal{D}(K_{\alpha,y_0}) = \{f + c\mathcal{G}_+ + ce^{i\alpha}\mathcal{G}_- \mid g \in \mathcal{D}(\dot{K}_{y_0}), c \in \mathbb{C}\} \tag{1.7}$$

$$K_{\alpha,y_0}(f + c\mathcal{G}_+ + ce^{i\alpha}\mathcal{G}_-) = \dot{K}_{y_0}g + ic\mathcal{G}_+ - ice^{i\alpha}\mathcal{G}_- \tag{1.8}$$

where

$$\mathcal{G}_\pm(\vec{x}) = \mathcal{G}_{\pm i}(\vec{x},\vec{y}_0) = \int_0^\infty dk \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{k^2 - m\omega \mp i}\,\varphi^*_{klm}(\vec{y}_0)\,\varphi_{klm}(\vec{x})$$

for $\vec{x} \in \mathbb{R}^3 - \{\vec{y}_0\}$.
Moreover the self-adjoint extension K_{π,y_0} corresponds to the "free" Hamiltonian $\dot{H}_\omega$: indeed, if $\Psi \in \mathcal{D}(K_{\pi,y_0})$,

$$\Psi = f + c(\mathcal{G}_+ - \mathcal{G}_-)$$

and the difference $\mathcal{G}_+ - \mathcal{G}_-$ is a continuous function at $\vec{x} = \vec{y}_0$, which belongs to the domain of H_ω, so that K_{π,y_0} becomes exactly the operator $\dot{H}_\omega$.
Using this result and applying the Krein's theory of self-adjoint extensions, it is easy to obtain the following

Theorem 1.1 *The resolvent of K_{α,y_0} has integral kernel given by*

$$(K_{\alpha,y_0} - z)^{-1}(\vec{x},\vec{x}') = \mathcal{G}_z(\vec{x},\vec{x}') + \lambda(z,\alpha)\mathcal{G}^*_{\bar{z}}(\vec{x}',\vec{y}_0)\mathcal{G}_z(\vec{x},\vec{y}_0) \tag{1.9}$$

with $z \in \varrho(K_{\alpha,y_0})$, $\vec{x},\vec{x}' \in \mathbb{R}^3$, $\vec{x} \neq \vec{x}'$, $\vec{x},\vec{x}' \neq \vec{y}_0$ and

$$\frac{1}{\lambda(z,\alpha)} = \frac{1}{\lambda(-i,\alpha)} - (z+i)\big(\mathcal{G}_{\bar{z}}(\vec{x}),\mathcal{G}_-(\vec{x})\big) \tag{1.10}$$

$$\lambda(-i,\alpha) = \frac{1+e^{i\alpha}}{2i\|\mathcal{G}_-(\vec{x})\|^2} \tag{1.11}$$

Proof: Since $\dot{K}_{y_0}$ is a densely defined, closed, symmetric operator with deficiency indexes $(1, 1)$, we can apply Krein's theory (cfr. [3, 43]) to classify all its self-adjoint extensions: from Krein's formula we immediately obtain

$$(K_{\alpha,y_0} - z)^{-1} - (K_{\pi,y_0} - z)^{-1} = \lambda(z, \alpha)\big(\mathcal{G}_{\bar{z}}(\vec{x}), \cdot\big)\mathcal{G}_z(\vec{x})$$

for $z \in \varrho(K_{\alpha,y_0}) \cap \varrho(H_\omega)$. It follows that $(K_{\alpha,y_0} - z)^{-1}$ has integral kernel given by

$$(K_{\alpha,y_0} - z)^{-1}(\vec{x}, \vec{x}') = (\dot{H}_\omega - z)^{-1}(\vec{x}, \vec{x}') + \lambda(z, \alpha)\mathcal{G}^*_{\bar{z}}(\vec{x}', \vec{y}_0)\mathcal{G}_z(\vec{x}, \vec{y}_0)$$

Moreover $\lambda(z, \alpha)$ satisfies the following equation

$$\frac{1}{\lambda(z, \alpha)} = \frac{1}{\lambda(z', \alpha)} - (z - z')\big(\mathcal{G}_{\bar{z}}(\vec{x}), \mathcal{G}_{z'}(\vec{x})\big)$$

The explicit expression of the factor $\lambda(-i, \alpha)$ is given in the following Theorem.

□

Theorem 1.2 *The domain $\mathcal{D}(K_{\alpha,y_0})$, $\alpha \in [0, 2\pi)$, consists of all elements $\Psi \in \mathbb{R}^3$ which can be decomposed in the following way*

$$\Psi(\vec{x}) = \Phi_z(\vec{x}) + \lambda(z, \alpha)\Phi_z(\vec{y}_0)\mathcal{G}_z(\vec{x}, \vec{y}_0)$$

for $\vec{x} \neq \vec{y}_0$, $\Phi_z \in \mathcal{D}(\dot{H}_\omega)$ and $z \in \varrho(K_{\alpha,y_0})$. The previous decomposition is unique and on every Ψ of this form

$$(K_{\alpha,y_0} - z)\Psi = (H_\omega - z)\Phi_z$$

Proof: First of all we observe that functions belonging to $\mathcal{D}(\dot{H}_\omega)$ are Hölder continuous with exponent smaller than $1/2$ in every compact subset of $\mathbb{R}^3$. Indeed the domain of self-adjointness of $\dot{H}_\omega$ contains functions in $H^2_{\text{loc}}(\mathbb{R}^3)$: on every compact set $S \subset \mathbb{R}^3$, the domain[1] of H^S_0 is strictly contained on the domain of J^S, since J^S is a bounded operator on $\mathcal{D}(\dot{H}^S_0) = H^2(S)$, therefore $\mathcal{D}(\dot{H}^S_\omega) = \mathcal{D}(\dot{H}^S_0) = H^2(S)$. Hence it makes sense to write $\Phi(\vec{y}_0)$ for every $\Phi \in \mathcal{D}(\dot{H}_\omega)$ and $\vec{y}_0 \in \mathbb{R}^3$.
Moreover

$$\mathcal{D}(K_{\alpha,y_0}) = (K_{\alpha,y_0} - z)^{-1}(\dot{H}_\omega - z)\mathcal{D}(\dot{H}_\omega)$$

[1]The notation A^S denotes the restriction of the operator A to the Hilbert space $L^2(S)$.

and the claim follows from the expression of the resolvent given in the previous Theorem 1.6.
To prove the uniqueness of the decomposition let $\Psi = 0$, so that

$$\Phi_z(\vec{x}) = -\frac{1+e^{i\alpha}}{2i\|\mathcal{G}_-(\vec{x})\|^2}\Phi_z(\vec{y}_0)\mathcal{G}_z(\vec{x})$$

but $\Phi_z(\vec{x})$ must be continuous at $\vec{x} = \vec{y}_0$: it follows that $\Phi_z(\vec{y}_0) = 0$ and then $\Phi_z = 0$.
Finally the last equality of the Theorem easily follows from

$$(K_{\alpha,y_0} - z)^{-1}(\dot{H}_\omega - z)\Phi_z = \Phi_z + \lambda(z,\alpha)\big(\mathcal{G}_{\bar{z}}(\vec{x}), (\dot{H}_\omega - z)\Phi_z(\vec{x})\big)\mathcal{G}_z = \Psi$$

To find the explicit expression of $\lambda(-i,\alpha)$ it is sufficient to study the behavior of functions in $\mathcal{D}(K_{\alpha,y_0})$ at $\vec{y}_0$. Let $\Psi(\vec{x}) \in \mathcal{D}(K_{\alpha,y_0})$,

$$\Psi(\vec{x}) = f(\vec{x}) + c\mathcal{G}_+(\vec{x}) + ce^{i\alpha}\mathcal{G}_-(\vec{x})$$

with $f \in \mathcal{D}(\dot{H}_{y_0})$ and $c \in \mathbb{C}$.
Since

$$\mathcal{G}_+(\vec{x}) = \int_0^\infty dk \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{k^2 - m\omega + i} + \frac{2i}{|k^2 - m\omega - i|^2}\right]\varphi^*_{klm}(\vec{y}_0)\,\varphi_{klm}(\vec{x}) =$$

$$= \mathcal{G}_-(\vec{x}) + 2ig(\vec{x},\vec{y}_0)$$

where

$$g(\vec{x},\vec{y}_0) = \int_0^\infty dk \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{|k^2 - m\omega - i|^2}\,\varphi^*_{klm}(\vec{y}_0)\,\varphi_{klm}(\vec{x})$$

belongs to $\mathcal{D}(\dot{H}_\omega)$, $\forall \vec{y}_0 \in \mathbb{R}^3$, we obtain

$$\Psi(\vec{x}) = f(\vec{x}) + 2ic\,g(\vec{x},\vec{y}_0) + c(1+e^{i\alpha})\mathcal{G}_-(\vec{x})$$

and

$$\lim_{\vec{x}\to\vec{y}_0}\Big[\Psi(\vec{x}) - c(1+e^{i\alpha})\mathcal{G}_-(\vec{x})\Big] = 2ic\|\mathcal{G}_-(\vec{x})\|^2_{L^2}$$

Thus Ψ can be uniquely decomposed in

$$\Psi(\vec{x}) = \Phi(\vec{x}) + \lambda(-i,\alpha)\Phi(\vec{y}_0)\mathcal{G}_-(\vec{x})$$

with $\Phi \in \mathcal{D}(\dot{H}_\omega)$ and boundary condition

$$\lim_{\vec{x}\to\vec{y}_0}\Big[\Psi(\vec{x}) - \lambda(-i,\alpha)\Phi(\vec{y}_0)\mathcal{G}_-(\vec{x})\Big] = \Phi(\vec{y}_0)$$

Comparing the two boundary conditions we obtain

$$\Phi(\vec{y}_0) = 2ic||\mathcal{G}_-(\vec{x})||_{L^2}^2$$

$$c(1+e^{i\alpha}) = \lambda(-i,\alpha)\Phi(\vec{y}_0)$$

and then

$$c = \frac{\Phi(\vec{y}_0)}{2i||\mathcal{G}_-(\vec{x})||_{L^2}^2}$$

$$\lambda(-i,\alpha) = \frac{1+e^{i\alpha}}{2i||\mathcal{G}_-(\vec{x})||_{L^2}^2}$$

□

Theorem 1.3 *The spectrum $\sigma(K_{\alpha,y_0})$ is purely absolutely continuous and*

$$\sigma(K_{\alpha,y_0}) = \sigma_{\mathrm{ac}}(K_{\alpha,y_0}) = \sigma(H_\omega) = \mathbb{R} \tag{1.12}$$

Proof: Considering the explicit expression of the resolvent given in Theorem 1.1, we immediately see that $\sigma(K_{\alpha,y_0}) = \sigma(H_\omega) = \mathbb{R}$: indeed, since $(K_{\alpha,y_0} - z)^{-1} - (H_\omega - z)^{-1}$ is of rank 1 for each $z \in \mathbb{R}$ and $\alpha \in [0, 2\pi)$, Weyl's Theorem (see for example Theorem XIII.14 in [44]) implies $\sigma_{\mathrm{ess}}(K_{\alpha,y_0}) = \sigma_{\mathrm{ess}}(H_\omega)$.
In order to prove absence of pure point and singular spectrum, we are going to apply the limiting absorption principle (see Theorem XIII.19 in [44]): to this purpose we need to prove that the following inequality is satisfied for every interval $[a, b] \subset \mathbb{R}$,

$$\sup_{0<\varepsilon<1} \int_a^b dx \, \left|\Im\left[\left(\Psi \, , \, \left(K_{\alpha,y_0} - x - i\varepsilon\right)^{-1}\Psi\right)\right]\right|^p < \infty$$

with Ψ in a dense subset of $L^2(\mathbb{R}^3)$ and $p > 1$.
Since the operator H_ω has no singular spectrum, the inequality is easily satisfied if $\alpha = \pi$. So, let $\alpha \neq \pi$, from Theorem 1.1 one has

$$\left(\Psi \, , \, \left(K_{\alpha,y_0} - x - i\varepsilon\right)^{-1}\Psi\right) = \left(\Psi \, , \, \left(H_\omega - x - i\varepsilon\right)^{-1}\Psi\right) +$$

$$+\lambda(\alpha, x+i\varepsilon)\left(\mathcal{G}_{x-i\varepsilon} \, , \, \Psi\right)\left(\Psi \, , \, \mathcal{G}_{x+i\varepsilon}\right)$$

and again the inequality holds for the first term. It is very easy to see that the second term is a bounded function of x if $\varepsilon > 0$, so that we have only to control the limit when $\varepsilon \to 0$. Since the singular spectrum of H_ω is empty, we can choose the dense subset of $L^2(\mathbb{R}^3)$ given by functions of the form $(H_\omega - x)\varphi$ where $\varphi \in \mathcal{D}(H_\omega)$:

$$\left(\mathcal{G}_{x-i\varepsilon}\,,\,\Psi\right)\left(\Psi\,,\,\mathcal{G}_{x+i\varepsilon}\right) = \left[\left(H_\omega - x - i\varepsilon\right)^{-1}\left(H_\omega - x\right)\varphi\right](\vec{y}_0)\,\cdot$$

$$\cdot\left[\left(H_\omega - x - i\varepsilon\right)^{-1}\left(H_\omega - x\right)\varphi^*\right](\vec{y}_0) \underset{\varepsilon\to 0}{\longrightarrow} \left|\varphi(\vec{y}_0)\right|^2 < \infty$$

since functions in $\mathcal{D}(H_\omega)$ are continuous and because

$$\left[\left(H_\omega - x - i\varepsilon\right)^{-1}\left(H_\omega - x\right)\varphi\right](\vec{y}_0) = \varphi(\vec{y}_0) + i\varepsilon\left[\left(H_\omega - x - i\varepsilon\right)^{-1}\varphi\right](\vec{y}_0)$$

and

$$\lim_{\varepsilon\to 0}\left|\varepsilon\left[\left(H_\omega - x - i\varepsilon\right)^{-1}\varphi\right](\vec{y}_0)\right| \le \lim_{\varepsilon\to 0}\varepsilon\,\left\|\mathcal{G}_{x-i\varepsilon}\right\|\,\left\|\varphi\right\| = 0$$

Indeed from Proposition B.1 we can easily extract the following upper bound for $\left\|\mathcal{G}_{x-i\varepsilon}\right\|$,

$$\left\|\mathcal{G}_{x-i\varepsilon}\right\| \le \frac{C}{\sqrt{\varepsilon}}$$

Finally from equation (1.10) it follows that

$$\left|\lambda(\alpha, x + i\varepsilon)\right| \underset{\varepsilon\to 0}{\longrightarrow} 0$$

Since the previous argument applies for each interval $[a, b] \subset \mathbb{R}$, the proof is completed.

□

1.2.2 Asymptotic Limit of Rapid Rotation

Let $U_{\mathrm{rot}}(t-s)$ be the unitary group generated by K_{α,y_0} for some $\alpha \in [0, 2\pi)$, i.e.

$$U_{\mathrm{inert}}(t,s) = R(t)\,U_{\mathrm{rot}}(t-s)\,R^\dagger(s)$$

In the following, we are going to prove that

$$\mathrm{s-}\lim_{\omega\to\infty} U_{\mathrm{inert}}(t,s) = e^{-iH_{\gamma,C}(t-s)}$$

where $H_{\gamma,C}$ is an appropriate self-adjoint extension of H_C, a singular perturbation of the Laplacian supported over a circle of radius y_0 in the $x, y-$plane[2].

[2]See (*I.13*)-(*I.20*) in the Introduction for the definition of singular interactions supported on set with lower codimension.

Let C the curve $\vec{y}(\varphi) = (y_0, \frac{\pi}{2}, \varphi)$, $\varphi \in [0, 2\pi]$, and $\dot{H}_C$ the closure of the operator

$$H_C = H_0$$
$$\mathcal{D}(H_C) = C_0^\infty(\mathbb{R}^3 - C)$$

we first classify the self-adjoint extensions of $\dot{H}_C$:

Proposition 1.1 *The self-adjoint extensions of the operator $\dot{H}_C$, that are invariant under rotations around the $z-$axis, are given by the one-parameter family $H_{\gamma,C}$, $\gamma \in \mathbb{R}$, with domain*

$$\mathcal{D}(H_{\gamma,C}) = \{\Psi \in L^2(\mathbb{R}^3) \,|\, \exists\, \xi_\Psi \in \mathcal{D}(\Gamma_{\gamma,C}(z)), \Psi - \tilde{G}_z\xi_\Psi \in H^2(\mathbb{R}^3),$$
$$\left(\Psi - \tilde{G}_z\xi_\Psi\right)\Big|_C = \Gamma_{\gamma,C}(z)\xi_\Psi\} \tag{1.13}$$
$$\left(H_{\gamma,C} - z\right)\Psi = \left(H_0 - z\right)\left(\Psi - \tilde{G}_z\xi_\Psi\right) \tag{1.14}$$

where $z \in \mathbb{C}$, $\Im(z) > 0$,

$$\mathcal{D}(\Gamma_{\gamma,C}(z)) = \{\xi \in L^2([0, 2\pi]) \,|\, \Gamma_{\gamma,C}(z)_m\xi_m \in l^2\} \tag{1.15}$$
$$\xi_m \equiv \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} d\phi\, \xi(\phi)e^{-im\phi}$$
$$\left(\Gamma_{\gamma,C}(z)\xi\right)(\phi) = \gamma\xi(\phi) - \int_0^{2\pi} d\phi' \frac{e^{i\sqrt{z}|\vec{y}(\phi)-\vec{y}(\phi')|}}{4\pi|\vec{y}(\phi) - \vec{y}(\phi')|}\xi(\phi') \tag{1.16}$$
$$\Gamma_{\gamma,C}(z)_m = \gamma - 2\pi\int_0^\infty dk \sum_{l=|m|}^\infty \frac{1}{k^2 - z}\left|\varphi_{klm}(\vec{y}_0)\right|^2 \tag{1.17}$$

and

$$\left(\tilde{G}_z\xi\right)(\vec{x}) \equiv \int_0^{2\pi} d\phi\, \frac{e^{i\sqrt{z}|\vec{x}-\vec{y}(\phi)|}}{4\pi|\vec{x} - \vec{y}(\phi)|}\xi(\phi)$$

Proof: See [51, 52]. The formula for $\Gamma_{\alpha,C}(\lambda)_m$ is obtained expressing the free resolvent in terms of spherical waves.

□

Proposition 1.2 *For every $\Psi \in L^2(\mathbb{R}^3)$, $z \in \varrho(H_{\gamma,C})$, $\Im(z) > 0$ and $\vec{y}_0 = (0, y_0, 0)$,*

$$\left(H_{\gamma,C} - z\right)^{-1}\Psi(\vec{x}) = \left(H_0 - z\right)^{-1}\Psi(\vec{x}) +$$
$$+ \sum_{m=-\infty}^{+\infty} \frac{2\pi}{\Gamma_{\gamma,C}(z)_m}\, G_z^m(\vec{x}, \vec{y}_0)\left(G_z^{m*}(\vec{x}', \vec{y}_0)\,,\Psi(\vec{x}')\right)_{L^2(\mathbb{R}^3)}$$

where

$$G_z^m(\vec{x},\vec{y}_0) \equiv \int_0^\infty dk \sum_{l=|m|}^{\infty} \frac{1}{k^2-z}\, \varphi^*_{klm}(\vec{y}_0)\, \varphi_{klm}(\vec{x})$$

Proof: The expression for the resolvent of $H_{\gamma,C}$ for a generic curve C is given in [51, 52]:

$$\left(H_{\gamma,C}-z\right)^{-1}\Psi(\vec{x}) = \left(H_0-z\right)^{-1}\Psi(\vec{x}) + \tilde{G}_z\left[\Gamma^{-1}_{\gamma,C}(z)\left(\left(H_0-z\right)^{-1}\Psi\right)\Big|_C\right]$$

Since $\Gamma_{\gamma,C}(z)$ is diagonal in the basis $e_m(\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi}$ of $L^2([0,2\pi],d\phi)$,

$$\left(\Gamma^{-1}_{\gamma,C}(z)\xi\right)(\phi) = \sum_{m=-\infty}^{\infty} \frac{1}{\Gamma_{\gamma,C}(z)_m}\, \xi_m\, e_m(\phi)$$

and therefore

$$\Gamma^{-1}_{\gamma,C}(z)\left(\left(H_0-z\right)^{-1}\Psi\right)\Big|_C = \sum_{m=-\infty}^{\infty} \frac{\left[\left((H_0-z)^{-1}\Psi\right)\Big|_C\right]_m}{\Gamma_{\gamma,C}(z)_m}\, e_m(\phi)$$

where

$$\left[\left((H_0-z)^{-1}\Psi\right)\Big|_C\right]_m = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} d\phi\, e^{-im\phi} \int_{\mathbb{R}^3} d^3\vec{x}' \int_0^\infty dk \sum_{l=0}^{\infty}\sum_{m'=-l}^{l} \frac{1}{k^2-z}\cdot$$

$$\cdot\, \varphi^*_{klm'}(\vec{x}')\, \varphi_{klm'}(\vec{y}(\phi))\, \Psi(\vec{x}') = \sqrt{2\pi}\, e^{im\frac{\pi}{2}} \int_{\mathbb{R}^3} d^3\vec{x}'\, G_z^m(\vec{x}',\vec{y}_0)\, \Psi(\vec{x}')$$

Finally

$$(\tilde{G}_z\, e_m)(\vec{x}) = \int_0^{2\pi} \frac{d\phi}{\sqrt{2\pi}} \int_0^\infty dk \sum_{l=0}^{\infty}\sum_{m'=-l}^{l} \frac{1}{k^2-z}\, \varphi^*_{klm'}(\vec{y}(\phi))\, \varphi_{klm'}(\vec{x})\, e^{im\phi} =$$

$$= \sqrt{2\pi}\, e^{-im\frac{\pi}{2}}\, G_z^m(\vec{x},\vec{y}_0)$$

□

Corollary 1.1 *If* $\Psi(\vec{x}) \in L^2(\mathbb{R}^3)$, $\Psi(\vec{x}) = \chi(r)Y_{l_0}^{m_0}(\theta,\phi)$ *and* $z \in \varrho(H_{\gamma,C})$, $\Im(z) > 0$,

$$\left(\left(H_{\gamma,C}-z\right)^{-1}\Psi\right)(\vec{x}) = \int_0^\infty dr' r'^2\, g_z^{l_0}(r,r')\chi(r')\; Y_{l_0}^{m_0}(\theta,\phi) +$$

$$+\frac{2\pi\, Y_{l_0}^{m_0}(\pi/2,0)}{\Gamma_{\gamma,C}(z)_{m_0}}\, G_z^{m_0}(\vec{x},\vec{y}_0)\int_0^\infty dr' r'^2\, g_z^{l_0}(y_0,r')\chi(r')$$

where

$$g_z^{l_0}(r,r')\equiv\frac{2}{\pi}\int_0^\infty dk\,\frac{k^2}{k^2-z}\,j_{l_0}(kr)j_{l_0}(kr')=\big(H_0-z\big)^{-1}\big|_{\mathcal{H}_{l_0}^{m_0}}(r,r')$$

and $\mathcal{H}_{l_0}^{m_0}$ *is the subspace of* $L^2(\mathbb{R}^3)$ *spanned by* $\chi(r)Y_{l_0}^{m_0}(\theta,\phi)$.

Proof: The result follows from a straightforward calculation: indeed, if $\Psi(\vec{x})=\chi(r)Y_{l_0}^{m_0}(\theta,\phi)$,

$$\Big(G_z^{m*}(\vec{x}',\vec{y}_0)\,,\Psi(\vec{x}')\Big)=\delta_{m,m_0}\,Y_{l_0}^{m_0}(\pi/2,0)\int_0^\infty dr' r'^2\, g_z^{l_0}(y_0,r')\chi(r')$$

and

$$\Big(\big(H_0-z\big)^{-1}\Psi\Big)(\vec{x})=\int_0^\infty dr' r'^2\, g_z^{l_0}(r,r')\chi(r')\;Y_{l_0}^{m_0}(\theta,\phi)$$

□

Now we can state the main result:

Theorem 1.4 *For every* $t,s\in\mathbb{R}$,

$$\text{s-}\lim_{\omega\to\infty}U_{\text{inert}}(t,s)=e^{-iH_{\gamma,C}(t-s)}$$

where $\gamma(\alpha,y_0)\in\mathbb{R}$ *and*

$$\gamma(\alpha,y_0)=2\pi\int_0^\infty dk\sum_{l=0}^{\infty}\left[\frac{2i}{(1+e^{i\alpha})|k^2+i|^2}+\frac{1}{k^2+i}\right]\left|\varphi_{kl0}(\vec{y}_0)\right|^2$$

Proof: First we observe that (see Lemma 1.1 below)

$$\text{s-}\lim_{\omega\to\infty}\int_{-\infty}^0 dt\, e^{-izt}\,U^*_{\text{inert}}(t,0)=-i\big(H_{\gamma,C}-z\big)^{-1}=\int_{-\infty}^0 dt\, e^{-izt}\,e^{iH_{\gamma,C}t}$$

and, since the previous equality holds for every $z\in\mathbb{C}$, $\Im(z)>0$, we obtain

$$\text{s-}\lim_{\omega\to\infty}U^*_{\text{inert}}(t,0)=e^{iH_{\gamma,C}t}$$

and therefore

$$\text{s-}\lim_{\omega\to\infty}U_{\text{inert}}(t,0)=e^{-iH_{\gamma,C}t}$$

The result then follows from the property of the 2-parameters unitary group $U_{\text{inert}}(t,s)$:

$$\text{s-}\lim_{\omega\to\infty}U_{\text{inert}}(t,s)=\text{s-}\lim_{\omega\to\infty}\Big[U_{\text{inert}}(t,0)\,U^*_{\text{inert}}(s,0)\Big]=e^{-iH_{\gamma,C}(t-s)}$$

□

The explicit expression of the parameter $\gamma(\alpha, y_0)$ is given in the following Lemma 1.1.

Lemma 1.1 *For every $z \in \mathbb{C}$, $\Im(z) > 0$,*

$$\text{s}-\lim_{\omega\to\infty} \int_{-\infty}^{0} dt\, e^{-izt}\, U^*_{\text{inert}}(t,0) = -i\big(H_{\gamma,C} - z\big)^{-1}$$

Proof: We shall verify the equality on the dense subset of $L^2(\mathbb{R}^3)$ given by functions of the form $\Psi(\vec{x}) = \chi(r) Y_{l_0}^{m_0}(\theta, \phi)$, with $l_0 = 0, \ldots \infty$ and $m_0 = -l_0, \ldots, l_0$,

$$U^*_{\text{inert}}(t,0)\Psi(\vec{x}) = e^{iK_{\alpha,y_0}t} R^*(t)\Psi(\vec{x}) = e^{i(K_{\alpha,y_0}+m_0\omega)t}\Psi(\vec{x})$$

Therefore

$$\int_{-\infty}^{0} dt\, e^{-izt}\, U^*_{\text{inert}}(t,0)\Psi(\vec{x}) = \int_{-\infty}^{0} dt\, e^{-izt}\, e^{i(K_{\alpha,y_0}+m_0\omega)t}\Psi(\vec{x}) =$$

$$\int_{-\infty}^{0} dt\, e^{-i(z-m_0\omega)t}\, e^{iK_{\alpha,y_0}t}\Psi(\vec{x}) = -i\big(K_{\alpha,y_0} + m_0\omega - z\big)^{-1}\Psi(\vec{x})$$

Hence we have now to prove that

$$\lim_{\omega\to\infty} \big(K_{\alpha,y_0} + m_0\omega - z\big)^{-1}\Psi(\vec{x}) = \big(H_{\gamma,C} - z\big)^{-1}\Psi(\vec{x})$$

First of all we observe that, for each $z \in \mathbb{C}$, $\Im(z) > 0$, $m_0 \in \mathbb{Z}$ and $\vec{y}_0 = (0, y_0, 0)$,

$$\lim_{\omega\to\infty} \mathcal{G}_{z-m_0\omega}(\vec{x}, \vec{y}_0) = G_z^{m_0}(\vec{x}, \vec{y}_0)$$

in the norm topology of $L^2(\mathbb{R}^3)$: indeed, since

$$\mathcal{G}_{z-m_0\omega}(\vec{x}, \vec{y}_0) = G_z^{m_0}(\vec{x}, \vec{y}_0) + R_z^{m_0}(\vec{x}, \vec{y}_0)$$

with

$$R_z^{m_0}(\vec{x}, \vec{y}_0) = \int_0^{\infty} dk \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m\neq m_0}}^{l} \frac{1}{k^2 - (m-m_0)\omega - z}\, \varphi^*_{klm}(\vec{y}_0)\, \varphi_{klm}(\vec{x})$$

it is sufficient to prove that

$$\lim_{\omega\to\infty} \big\| R_z^{m_0}(\vec{x}, \vec{y}_0) \big\|_{L^2(\mathbb{R}^3)} = 0$$

but

$$\left\|R_z^{m_0}(\vec{x},\vec{y}_0)\right\|^2_{L^2(\mathbb{R}^3)} = \int_0^\infty dk \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m\neq m_0}}^{l} \frac{1}{|k^2-(m-m_0)\omega - z|^2} |\varphi_{klm}(\vec{y}_0)|^2$$

and the right hand side is bounded for each $\omega \in \mathbb{R}$ (see Proposition B.1), so that we can exchange the limit with the integration

$$\lim_{\omega\to\infty} \int_0^\infty dk \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m\neq m_0}}^{l} \frac{1}{|k^2-(m-m_0)\omega - z|^2} |\varphi_{klm}(\vec{y}_0)|^2 =$$

$$= \int_0^\infty dk \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m\neq m_0}}^{l} |\varphi_{klm}(\vec{y}_0)|^2 \lim_{\omega\to\infty} \frac{1}{|k^2-(m-m_0)\omega - z|^2} = 0$$

Now, since (see Theorem 1.1)

$$\left[\left(K_{\alpha,y_0} + m_0\omega - z\right)^{-1}\Psi\right](\vec{x}) = \left(\mathcal{G}^*_{z-m_0\omega}(\vec{x},\vec{x}'), \Psi(\vec{x}')\right)_{L^2(\mathbb{R}^3)} +$$

$$+\lambda(z-m_0\omega,\alpha)\left(\mathcal{G}_{\bar{z}-m_0\omega}(\vec{x}',\vec{y}_0), \Psi(\vec{x}')\right)_{L^2(\mathbb{R}^3)} \mathcal{G}_{z-m_0\omega}(\vec{x},\vec{y}_0)$$

and

$$\lim_{\omega\to\infty} \left(\mathcal{G}^*_{z-m_0\omega}(\vec{x},\vec{x}'), \Psi(\vec{x}')\right)_{L^2(\mathbb{R}^3)} = e^{im_0\frac{\pi}{2}} \left(G_z^{m_0 *}(\vec{x},\vec{x}'), \Psi(\vec{x}')\right)_{L^2(\mathbb{R}^3)} =$$

$$= \int_0^\infty dr' r'^2 g_z^{l_0}(y_0, r')\, \chi(r')\ Y_{l_0}^{m_0}(\pi/2, \pi/2)$$

$$\lim_{\omega\to\infty} \left(\mathcal{G}_{\bar{z}-m_0\omega}(\vec{x}',\vec{y}_0), \Psi(\vec{x}')\right)_{L^2(\mathbb{R}^3)} = e^{-im_0\frac{\pi}{2}} \left(G_{\bar{z}}^{m_0}(\vec{x}',\vec{y}_0), \Psi(\vec{x}')\right)_{L^2(\mathbb{R}^3)}$$

$$\lim_{\omega\to\infty} \mathcal{G}_{z-m_0\omega}(\vec{x},\vec{y}_0) = e^{im_0\frac{\pi}{2}} G_z^{m_0}(\vec{x},\vec{y}_0)$$

we obtain

$$\lim_{\omega\to\infty} \left(K_{\alpha,y_0} + m_0\omega - z\right)^{-1}\Psi(\vec{x}) = \int_0^\infty dr' r'^2\, g_z^{l_0}(r,r')\chi(r')\ Y_{l_0}^{m_0}(\theta,\phi) +$$

$$+ \beta(z,\alpha)\, G_z^{m_0}(\vec{x},\vec{y}_0) \int_0^\infty dr' r'^2\, g_z^{l_0}(y_0,r')\chi(r') = \left(H_{\gamma,C} - z\right)^{-1}\Psi(\vec{x})$$

with[3]

$$\beta(z,\alpha) = \lim_{\omega\to\infty} \lambda(z - m_0\omega, \alpha)$$

and

$$\frac{\Gamma_{\gamma,C}(z)_{m_0}}{2\pi} = \frac{1}{\beta(z,\alpha)}$$

It remains to find the explicit expression of $\gamma(\alpha, y_0)$: using the relation (see Theorem 1.1)

$$\frac{1}{\lambda(z - m_0\omega, \alpha)} = \frac{1}{\lambda(-i,\alpha)} - (z - m_0\omega + i)\Big(\mathcal{G}_{-m_0\omega+\bar{z}}(\vec{x}),\, \mathcal{G}_{-}(\vec{x})\Big)$$

we obtain

$$\frac{1}{\beta(z,\alpha)} = \lim_{\omega\to\infty}\left[\frac{1}{\lambda(-i,\alpha)} - (z - m_0\omega + i)\Big(\mathcal{G}_{-m_0\omega+\bar{z}}(\vec{x}),\, \mathcal{G}_{-}(\vec{x})\Big)\right] =$$

$$= \frac{2i}{1+e^{i\alpha}}\int_0^\infty dk \sum_{l=0}^{\infty} \frac{1}{|k^2+i|^2}\left|\varphi_{kl0}(\vec{y}_0)\right|^2 + \int_0^\infty dk \sum_{l=0}^{\infty}\frac{1}{k^2+i}\left|\varphi_{kl0}(\vec{y}_0)\right|^2 +$$
$$- \int_0^\infty dk \sum_{l=|m_0|}^{\infty} \frac{1}{k^2 - z}\left|\varphi_{klm_0}(\vec{y}_0)\right|^2$$

and hence the result. We want to stress that, as it was expected, $\gamma \in \mathbb{R}$:

$$\Im\left\{\frac{2i}{(1+e^{i\alpha})|k^2+i|^2} + \frac{1}{k^2+i}\right\} = \frac{1}{|k^2+i|^2}\left\{\Im\left[\frac{2i}{1+e^{i\alpha}}\right] - 1\right\} =$$

$$= \frac{1}{|k^2+i|^2}\left\{\frac{\Im\left[2i + 2ie^{-i\alpha}\right]}{2+2\cos\alpha} - 1\right\} = 0$$

□

1.3 Rotating Point Interactions in Two Dimensions

1.3.1 The Hamiltonian

As in the previous Section, the formal time-dependent Hamiltonian we consider is given by

$$H(t) = H_0 + a\,\delta^{(2)}(\vec{x} - \vec{y}(t)) \tag{1.18}$$

[3]Actually λ is a function separately of $z - m_0\omega$ and ω, since the Green's function $\mathcal{G}_{-}(\vec{x})$ depends on ω.

where $\vec{y}(t) = \mathcal{R}(t)\vec{y}_0$. Hence the corresponding formal generator in the uniformly rotating frame is

$$K = H_0 - \omega J + a\,\delta^{(2)}(\vec{x} - \vec{y}_0)$$

The Hamiltonian of the system is then a self-adjoint extension of the symmetric operator

$$K_{y_0} = H_\omega$$

$$\mathcal{D}(K_{y_0}) = C_0^\infty(\mathbb{R}^2 - \{\vec{y}_0\})$$

According to the discussion of Section 2, such self-adjoint extensions are given by the one-parameter family of operators

$$\mathcal{D}(K_{\alpha,y_0}) = \{f + c\mathcal{G}_+ + ce^{i\alpha}\mathcal{G}_- | g \in \mathcal{D}(\dot{K}_{y_0}), c \in \mathbb{C}\} \tag{1.19}$$

$$K_{\alpha,y_0}(f + c\mathcal{G}_+ + ce^{i\alpha}\mathcal{G}_-) = \dot{K}_{y_0}g + ic\mathcal{G}_+ - ice^{i\alpha}\mathcal{G}_- \tag{1.20}$$

with $\alpha \in [0, 2\pi)$ and where

$$\mathcal{G}_\pm(\vec{x}) = \mathcal{G}_{\pm i}(\vec{x}, \vec{y}_0)$$

$$\mathcal{G}_z(\vec{x}, \vec{y}_0) = \int_0^\infty dk \sum_{n=-\infty}^{\infty} \frac{1}{k^2 - n\omega - z}\, \varphi_{kn}^*(\vec{y}_0)\, \varphi_{kn}(\vec{x}) \tag{1.21}$$

for $\vec{x} \in \mathbb{R}^2 - \{\vec{y}_0\}$.
As in the 3D case, the self-adjoint extension K_{π,y_0} corresponds to the "free" Hamiltonian $\dot{H}_\omega$ and

Theorem 1.5 *The resolvent of K_{α,y_0} has integral kernel given by*

$$(K_{\alpha,y_0} - z)^{-1}(\vec{x}, \vec{x}') = \mathcal{G}_z(\vec{x}, \vec{x}') + \lambda(z, \alpha)\mathcal{G}_{\bar{z}}^*(\vec{x}', \vec{y}_0)\mathcal{G}_z(\vec{x}, \vec{y}_0) \tag{1.22}$$

with $z \in \varrho(K_{\alpha,y_0})$, $\vec{x}, \vec{x}' \in \mathbb{R}^2$, $\vec{x} \neq \vec{x}'$, $\vec{x}, \vec{x}' \neq \vec{y}_0$ and

$$\frac{1}{\lambda(z, \alpha)} = \frac{1}{\lambda(-i, \alpha)} - (z + i)\big(\mathcal{G}_{\bar{z}}(\vec{x}), \mathcal{G}_-(\vec{x})\big) \tag{1.23}$$

$$\lambda(-i, \alpha) = \frac{1 + e^{i\alpha}}{2i\|\mathcal{G}_-(\vec{x})\|^2} \tag{1.24}$$

Proof: See the Proof of Theorem 1.1 and Proposition B.2.

□

Theorem 1.6 *The domain $\mathcal{D}(K_{\alpha,y_0})$, $\alpha \in [0,2\pi)$, consists of all elements $\Psi \in \mathbb{R}^3$ which can be decomposed in the following way*

$$\Psi(\vec{x}) = \Phi_z(\vec{x}) + \lambda(z,\alpha)\Phi_z(\vec{y}_0)\mathcal{G}_z(\vec{x},\vec{y}_0)$$

for $\vec{x} \neq \vec{y}_0$, $\Phi_z \in \mathcal{D}(\dot{H}_\omega)$ and $z \in \varrho(K_{\alpha,y_0})$. The previous decomposition is unique and on every Ψ of this form we obtain

$$(K_{\alpha,y_0} - z)\Psi = (H_\omega - z)\Phi_z$$

Proof: See the Proof of Theorem 1.2.

□

Theorem 1.7 *The spectrum $\sigma(K_{\alpha,y_0})$ is purely absolutely continuous and*

$$\sigma(K_{\alpha,y_0}) = \sigma_{\mathrm{ac}}(K_{\alpha,y_0}) = \sigma(H_\omega) = \mathbb{R} \tag{1.25}$$

Proof: See the Proof of Theorem 1.3, Theorem 1.1 and Proposition B.2.

□

1.3.2 Asymptotic Limit of Rapid Rotation

As in the 3D case, we are going to prove that

$$\mathrm{s}-\lim_{\omega\longrightarrow\infty} U_{\mathrm{inert}}(t,s) = e^{-iH_{\gamma,C}(t-s)}$$

where $H_{\gamma,C}$ is an appropriate self adjoint extension of H_C, a singular perturbation of the Laplacian supported over a circle of radius y_0: let C the curve $\vec{y}(\theta) = (y_0,\theta)$, $\theta \in [0,2\pi]$, and $\dot{H}_C$ the closure of the operator

$$H_C = H_0$$

$$\mathcal{D}(H_C) = C_0^\infty(\mathbb{R}^2 - C)$$

Proposition 1.3 *The self-adjoint extensions of the operator $\dot{H}_C$, that are invariant under rotations around the $z-$axis, are given by the one-parameter family of operators $H_{\gamma,C}$, $\gamma \in \mathbb{R}$, with domain*

$$\mathcal{D}(H_{\gamma,C}) = \{\Psi \in L^2(\mathbb{R}^2) \mid \exists\, \xi_\Psi \in \mathcal{D}(\Gamma_{\gamma,C}(z)), \Psi - \tilde{G}_z\xi_\Psi \in H^2(\mathbb{R}^2),$$

$$\left(\Psi - \tilde{G}_z\xi_\Psi\right)\big|_C = \Gamma_{\gamma,C}(z)\xi_\Psi\} \tag{1.26}$$

$$\left(H_{\gamma,C} - z\right)\Psi = \left(H_0 - z\right)\left(\Psi - \tilde{G}_z\xi_\Psi\right) \tag{1.27}$$

where $z \in \mathbb{C}$, $\Im(z) > 0$,

$$\mathcal{D}(\Gamma_{\gamma,C}(z)) = \{\xi \in L^2([0,2\pi]) \mid \Gamma_{\gamma,C}(z)_n \xi_n \in l^2\} \tag{1.28}$$

$$\xi_n \equiv \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta\, \xi(\theta) e^{-in\theta} = \Big(e_n \,,\, \xi_\Psi \Big)_{L^2([0,2\pi],d\theta)}$$

$$\big(\Gamma_{\gamma,C}(z)\xi\big)(\theta) \equiv \frac{\xi(\theta)}{\gamma} - \int_0^{2\pi} d\theta' \frac{e^{i\sqrt{z}|\vec{y}(\theta)-\vec{y}(\theta')|}}{4\pi|\vec{y}(\theta)-\vec{y}(\theta')|}\xi(\theta') \tag{1.29}$$

$$\Gamma_{\gamma,C}(z)_n = \frac{1}{\gamma} - 2\pi \int_0^\infty dk \frac{1}{k^2 - z} \big|\varphi_{kn}(\vec{y}_0)\big|^2 \tag{1.30}$$

and

$$\big(\tilde{G}_z\xi\big)(\vec{x}) \equiv \int_0^{2\pi} d\theta \frac{e^{i\sqrt{z}|\vec{x}-\vec{y}(\theta)|}}{4\pi|\vec{x}-\vec{y}(\theta)|}\xi(\theta)$$

Proof: Singular perturbations of the Laplacian supported on a curve in $\mathbb{R}^2$ are analogous to singular perturbations supported on a surface in $\mathbb{R}^3$: indeed the quadratic form

$$\mathcal{F}(\Psi,\Psi) \equiv \int_{\mathbb{R}^2} d^2\vec{x}\, \big|\nabla\Psi\big|^2 - \int_C d\theta\, \gamma(\theta) \big|\Psi(\vec{y}(\theta))\big|^2$$

is easily seen to be a closed semibounded quadratic form (see e.g. [51, 52] and the discussion contained in the next Chapter) on

$$\mathcal{D}(\mathcal{F}) = \big\{\Psi \in L^2(\mathbb{R}^2) \mid \exists\, \xi_\Psi \in L^2(C), \Psi - \tilde{G}_z\xi_\Psi \in H^1(\mathbb{R}^2)\big\}$$

and it can be proved that it is associated to the self-adjoint operator $H_{\gamma,C}$.

□

Proposition 1.4 *If* $\Psi(\vec{x}) \in L^2(\mathbb{R}^2)$, $\Psi(\vec{x}) = \chi(r) e_{n_0}(\theta)$ *and* $z \in \varrho(H_{\gamma,C})$, $\Im(z) > 0$,

$$\Big(\big(H_{\gamma,C} - z\big)^{-1}\Psi\Big)(\vec{x}) = \int_0^\infty dr' r'\, g_z^{n_0}(r,r')\chi(r') +$$

$$+ \frac{2\pi}{\Gamma_{\gamma,C}(z)_{n_0}}\, G_z^{n_0}(\vec{x},\vec{y}_0) \int_0^\infty dr' r'\, g_z^{n_0}(y_0,r')\chi(r')$$

where

$$g_z^{n_0}(r,r') \equiv \int_0^\infty dk\, \frac{k}{k^2 - z}\, J_{|n_0|}(kr) J_{|n_0|}(kr') = \big(H_0 - z\big)^{-1}\big|_{\mathcal{H}_{n_0}}(r,r')$$

and

$$G_z^n(\vec{x},\vec{y}_0) \equiv \int_0^\infty dk \frac{1}{k^2 - z}\, \varphi_{kn}^*(\vec{y}_0)\, \varphi_{kn}(\vec{x})$$

Proof: See the Proof of Proposition 1.2 and Corollary 1.1.

□

Theorem 1.8 *For every $t, s \in \mathbb{R}$,*

$$\text{s-}\lim_{\omega \to \infty} U_{\text{inert}}(t,s) = e^{-iH_{\gamma,C}(t-s)}$$

where $\gamma(\alpha, y_0) \in \mathbb{R}$ and

$$\gamma(\alpha, y_0) = \int_0^\infty dk\, k \left[\frac{2i}{(1+e^{i\alpha})|k^2+i|^2} + \frac{1}{k^2+i} \right] J_0^2(ky_0)$$

Proof: See the Proof of Theorem 1.4 and the following Lemma 1.2.

□

Lemma 1.2 *For every $z \in \mathbb{C}$, $\Im(z) > 0$,*

$$\text{s-}\lim_{\omega \to \infty} \int_{-\infty}^0 dt\, e^{-izt}\, U^*_{\text{inert}}(t,0) = -i\big(H_{\gamma,C} - z\big)^{-1}$$

Proof: The first part of the proof is analogous to the Proof of Lemma 1.1 (the only difference is the dense subset of $L^2(\mathbb{R}^2)$ given by functions of the form $\Psi(\vec{x}) = \chi(r) e_{n_0}(\theta)$, with $n_0 \in \mathbb{Z}$).
Hence it remains to prove that

$$\lim_{\omega \to \infty} \big(K_{\alpha,y_0} + n_0\omega - z\big)^{-1} \Psi(\vec{x}) = \big(H_{\gamma,C} - z\big)^{-1} \Psi(\vec{x})$$

Now, for each $z \in \mathbb{C}$, $\Im(z) > 0$, $n_0 \in \mathbb{Z}$ and $\vec{y}_0 = (0, y_0)$,

$$\lim_{\omega \to \infty} \mathcal{G}_{z-n_0\omega}(\vec{x}, \vec{y}_0) = G_z^{n_0}(\vec{x}, \vec{y}_0)$$

in the norm topology of $L^2(\mathbb{R}^2)$: since

$$\mathcal{G}_{z-n_0\omega}(\vec{x}, \vec{y}_0) = G_z^{n_0}(\vec{x}, \vec{y}_0) + R_z^{n_0}(\vec{x}, \vec{y}_0)$$

with

$$R_z^{n_0}(\vec{x}, \vec{y}_0) = \int_0^\infty dk \sum_{\substack{n=-\infty \\ n \neq n_0}}^{\infty} \frac{1}{k^2 - (n-n_0)\omega - z}\, \varphi^*_{kn}(\vec{y}_0)\, \varphi_{kn}(\vec{x})$$

it is sufficient to prove that

$$\lim_{\omega \to \infty} \left\| R_z^{n_0}(\vec{x}, \vec{y}_0) \right\|_{L^2(\mathbb{R}^2)} = 0$$

But

$$\left\| R_z^{n_0}(\vec{x},\vec{y}_0)\right\|^2_{L^2(\mathbb{R}^2)} = \int_0^\infty dk \sum_{\substack{n=-\infty \\ n\neq n_0}}^{\infty} \frac{1}{|k^2-(n-n_0)\omega - z|^2} |\varphi_{kn}(\vec{y}_0)|^2$$

and the right hand side is bounded (see Proposition B.2) for each $\omega \in \mathbb{R}$, so that exchanging the limit with the integration, we obtain the result.
Now, substituting in the expression of the resolvent (see Theorem 1.5),

$$\left[\left(K_{\alpha,y_0} + m_0\omega - z\right)^{-1}\Psi\right](\vec{x}) = \left(\mathcal{G}^*_{z-m_0\omega}(\vec{x},\vec{x}'),\Psi(\vec{x}')\right)_{L^2(\mathbb{R}^3)} +$$

$$+\lambda(z-m_0\omega,\alpha)\left(\mathcal{G}_{\bar{z}-m_0\omega}(\vec{x}',\vec{y}_0),\Psi(\vec{x}')\right)_{L^2(\mathbb{R}^3)}\mathcal{G}_{z-m_0\omega}(\vec{x},\vec{y}_0)$$

the result follows from a straightforward calculation. Moreover we obtain the same relation between γ and α:

$$\frac{\Gamma_{\gamma,C}(z)_{n_0}}{2\pi} = \frac{1}{\beta(z,\alpha)}$$

where

$$\beta(z,\alpha) = \lim_{\omega\to\infty} \lambda(z-n_0\omega,\alpha)$$

but

$$\frac{1}{\lambda(z-n_0\omega,\alpha)} = \frac{1}{\lambda(-i,\alpha)} - (z-n_0\omega+i)\left(\mathcal{G}_{-n_0\omega+\bar{z}}(\vec{x}),\ \mathcal{G}_-(\vec{x})\right)$$

and then

$$\frac{1}{\beta(z,\alpha)} = \lim_{\omega\to\infty}\left[\frac{1}{\lambda(-i,\alpha)} - (z-n_0\omega+i)\left(\mathcal{G}_{-n_0\omega+\bar{z}}(\vec{x}),\ \mathcal{G}_-(\vec{x})\right)\right] =$$

$$= \frac{2i}{1+e^{i\alpha}}\int_0^\infty dk \frac{1}{|k^2+i|^2}|\varphi_{k0}(\vec{y}_0)|^2 + \int_0^\infty dk\frac{1}{k^2+i}|\varphi_{k0}(\vec{y}_0)|^2 +$$

$$-\int_0^\infty dk\frac{1}{k^2-z}|\varphi_{kn_0}(\vec{y}_0)|^2$$

□

Chapter 2

Rotating Singular Interactions Supported on Sets of Lower Codimension

2.1 Introduction

In this Chapter we extend the results of Chapter 3 to uniformly rotating singular perturbations of the Laplacian supported on sets of lower codimension: we study the so called rotating blade, namely a singular perturbation supported on a rotating surface in three dimensions and on a rotating segment in two dimensions respectively.
In Sections 2.2.1 and 2.3.1 we identify the Hamiltonians of such systems in the rotating frame with families of suitable self-adjoint operators. The starting point is the procedure defined in the Introduction ((*I.13*) and following), which allows us to split the quadratic form into the "free" form (*I.17*) and the "charges" form (*I.18*). Nevertheless, since the operator H_ω is not bounded from below, it is not possible to go further: the "free" part of the form is evidently not bounded from below and then we can not easily conclude that the whole form is closed. Therefore we introduce a cut-off on the spectrum of H_ω, so that the entire form is closed and defines a self-adjoint operator which is now bounded from below. The last step is then the removal of the cut-off (Theorems 2.2 and 2.6), which is done proving the converge of the cut-off Hamiltonians in strong resolvent sense to a self-adjoint operator.
As in Chapter 3 we also study the asymptotic limit of large angular velocity and, by means of an explicit expression of the resolvents, we prove that the unitary family describing the time evolution converge strongly to a one-parameter unitary group (Theorems 2.4 and 2.8), whose generator is a

time-independent Schrödinger operator with a regular potential.

2.2 The Rotating Blade in 3D

2.2.1 The Hamiltonian

Let D be the half-disc $D \equiv \{(r, \theta, \phi) \in \mathbb{R}^3 \,|\, 0 \le r \le A,\, 0 \le \theta \le \pi,\, \phi = 0\}$ and $\Theta_D(x, z)$ its characteristic function. The formal time-dependent Hamiltonian of the system is given by

$$H(t) = H_0 + \alpha(x, z)\, R(t)\, \Theta_D(x, z)\, \delta(y) \tag{2.1}$$

where $R(t)\Psi(\vec{x}) = \Psi(\mathcal{R}(t)^{-1}\,\vec{x})$ and $\|\alpha\|_\infty < \infty$. Therefore in the rotating frame the formal generator of time evolution is

$$K = H_0 - \omega J + \alpha\, \Theta_D(x, z)\, \delta(y)$$

or more rigorously a self-adjoint extension of the symmetric operator

$$K_D = H_\omega$$

$$\mathcal{D}(K_D) = C_0^\infty(\mathbb{R}^3 - D)$$

The Hamiltonian cannot be easily defined with the method of quadratic form, because of its unboundedness from below. Hence are going to pursue a different strategy: we define a sequence of cut-off Hamiltonians which converge to the operator H_ω in the strong resolvent sense and that are self-adjoint and bounded from below; then we add the singular perturbation and prove that the so obtained operators are self-adjoint. Finally prove that the limit (in the strong resolvent sense) of the sequence of cut-off perturbed Hamiltonians is a self-adjoint operator that we identify with the Hamiltonian of the system. So let

$$H_\omega^L = H_\omega\, \Pi_L \tag{2.2}$$

where Π_L is the projector on the subspace of $L^2(\mathbb{R}^3)$ generated by functions of the form $\chi(r)Y_l^m(\theta, \phi)$, with $l \le L$. It is very easy to prove that the operator H_ω^L is self-adjoint on the domain $H^2(\mathbb{R}^3)$: the operator J is bounded on the domain of the projector Π_L and therefore it is an infinitesimally bounded perturbation of H_0, so that we can apply the Kato Theorem [36]. Moreover for each $z \in \varrho(H_\omega^L)$ the resolvent $(H_\omega^L - z)^{-1}$ is given by an integral operator with kernel

$$\mathcal{G}_z^L(\vec{x}, \vec{x}') = \int_0^\infty dk \sum_{l=0}^{L} \sum_{m=-l}^{l} \frac{\varphi_{klm}^*(\vec{x}')\, \varphi_{klm}(\vec{x})}{k^2 - m\omega - z} \tag{2.3}$$

Proposition 2.1 *The sequence of cut-off Hamiltonians converge as $L \to \infty$ in the strong resolvent sense to the self-adjoint operator H_ω.*

Proof: For each $L \in \mathbb{N}$ and $z \in \mathbb{C} - \mathbb{R}$, the function $\mathcal{G}_z^L(\vec{x}, \vec{x}')$ belongs to $L^2(\mathbb{R}^3, d^3\vec{x})$:

$$\left\|\mathcal{G}_z^L(\vec{x}, \vec{x}')\right\|^2 \leq \left\|\mathcal{G}_z(\vec{x}, \vec{x}')\right\|^2 < \infty$$

and then the result is a straightforward consequence of Proposition B.1. The operator H_ω was studied in [25, 54].

□

Now we can defined the perturbed cut-off Hamiltonians with the method of quadratic form: let[1]

$$\mathcal{F}_{\alpha,L}(\Psi, \Psi) = F_{\omega,L}(\Psi, \Psi) - \int_D d\mu_D(\vec{r})\, \alpha(\vec{r})\, \left|\Psi\big|_D(\vec{r})\right|^2 \tag{2.4}$$

where $F_{\omega,L}$ is the closed[2] semibounded quadratic form associated to H_ω^L. The form $\mathcal{F}_{\alpha,L}$ is well defined if $\Psi \in \mathcal{D}(F_{\omega,L})$ and α is a smooth real function on D bounded away from 0.

Proposition 2.2 *Let $z \in \mathbb{C} - \mathbb{R}$, the form $\mathcal{F}_{\alpha,L}$ can be written in the following way,*

$$\mathcal{F}_{\alpha,L}(\Psi, \Psi) = \mathcal{F}_{\omega,L}^z(\Psi, \Psi) + \Phi_{\alpha,L}^z(\xi_\Psi, \xi_\Psi) - 2\Im(z)\, \Im\Big[\big(\Psi\,,\, \tilde{\mathcal{G}}_z^L \xi_\Psi\big)\Big] \tag{2.5}$$

where

$$\mathcal{F}_{\omega,L}^z(\Psi, \Psi) = F_{\omega,L}(\Psi - \tilde{\mathcal{G}}_z^L \xi_\Psi, \Psi - \tilde{\mathcal{G}}_z^L \xi_\Psi) - \Re(z)\|\Psi - \tilde{\mathcal{G}}_z^L \xi_\Psi\|^2 + \Re(z)\|\Psi\|^2 \tag{2.6}$$

$$\Phi_{\alpha,L}^z(\xi_\Psi, \xi_\Psi) = \Re\Big[\big(\xi_\Psi\,,\, \Gamma_\alpha^L(z)\, \xi_\Psi\big)_{L^2(D, d\mu_D)}\Big] \tag{2.7}$$

and

$$\Big[\Gamma_\alpha^L(z)\, \xi_\Psi\Big](\vec{r}) = \frac{\xi_\Psi(\vec{r})}{\alpha(\vec{r})} - \int_D d\mu_D(\vec{r}')\; \mathcal{G}_z^L(\vec{x}, \vec{x}')\big|_{\vec{x}, \vec{x}' \in D}\, \xi_\Psi(\vec{r}') \tag{2.8}$$

$$(\tilde{\mathcal{G}}_z^L \xi)(\vec{x}) \equiv \int_D d\mu_D(\vec{r}')\; \mathcal{G}_z^L(\vec{x}, \vec{x}')\big|_{\vec{x}' \in D}\, \xi(\vec{r}')$$

---

[1]Here $d\mu_D(\vec{r})$ stands for the restriction of the Lebesgue measure to D, namely $d\mu_D(\vec{r}) \equiv r^2\, dr\, d\cos\theta$ for $\vec{r} = (r, \theta) \in D$; $\vec{r}$ denotes the restriction of $\vec{x} \in \mathbb{R}^3$ to D, i.e. $\vec{r} \equiv (r, \theta)$.

[2]The form $F_{\omega,L}$ is closed on the domain $\mathcal{D}(F_{\omega,L}) = H^1(\mathbb{R}^3)$.

Proof: The result follows from a simple calculation: setting

$$\xi_\Psi(\vec{r}) = \alpha(\vec{r})\ \Psi\big|_D(\vec{r}) \tag{2.9}$$

one has

$$\mathcal{F}_{\alpha,L}(\Psi,\Psi) - F_{\omega,L}(\Psi - \tilde{\mathcal{G}}_z^L\xi_\Psi, \Psi - \tilde{\mathcal{G}}_z^L\xi_\Psi) = \left(\tilde{\mathcal{G}}_z^L\xi\,, H_\omega^L(\Psi - \tilde{\mathcal{G}}_z^L\xi)\right) +$$

$$+\left(\Psi\,,\ H_\omega^L\tilde{\mathcal{G}}_z^L\xi\right) - \int_D d\mu_D\, \frac{|\xi_\Psi|^2}{\alpha} =$$

$$= \int_D d\mu_D\, \frac{|\xi_\Psi|^2}{\alpha} - \left(\tilde{\mathcal{G}}_z^L\xi\,,\ (H_\omega^L - z^*)\tilde{\mathcal{G}}_z^L\xi\right) - z^*\left\|\tilde{\mathcal{G}}_z^L\xi\right\|^2 + 2\Re\Big[z\left(\Psi\,,\ \tilde{\mathcal{G}}_z^L\xi\right)\Big] =$$

$$= \Phi_{\alpha,L}^z(\xi_\Psi,\xi_\Psi) - \Re(z)\left\|\tilde{\mathcal{G}}_z^L\xi\right\|^2 + 2\Re\Big[z\left(\Psi\,,\ \tilde{\mathcal{G}}_z^L\xi\right)\Big]$$

since

$$\Im(z)\ \left\|\tilde{\mathcal{G}}_z^L\xi\right\|^2 = \Im\Big[\left(\tilde{\mathcal{G}}_z^L\xi\,,\ (H_\omega^L - z^*)\tilde{\mathcal{G}}_z^L\xi\right)\Big]$$

but

$$\left\|\tilde{\mathcal{G}}_z^L\xi\right\|^2 = \left\|\Psi - \tilde{\mathcal{G}}_z^L\xi\right\|^2 - \left\|\Psi\right\|^2 + 2\Re\Big[\left(\Psi\,,\ \tilde{\mathcal{G}}_z^L\xi\right)\Big]$$

so that we obtain the result.

□

Of course the form $\mathcal{F}_{\alpha,L}$ is independent on z and the decomposition $\Psi = \varphi_z + \tilde{\mathcal{G}}_z^L\xi_\Psi$ is unique, since $\tilde{\mathcal{G}}_z^L\xi_\Psi \notin \mathcal{D}(F_{\omega,L})$ if $\xi_\Psi \in L^2(D, d\mu_D)$. Moreover the form $\Phi_{\alpha,L}^z(\xi,\xi)$ is bounded and one can choose $z \in \mathbb{C}$ such that the form satisfies another useful inequality:

Proposition 2.3 *The form $\Phi_{\alpha,L}^z(\xi,\xi)$ is bounded for each $\xi \in L^2(D, d\mu_D)$.*

Proof: The first term of the form is of course bounded if $\xi \in L^2(D, d\mu_D)$ and

$$\left|\int_D d\mu_D\ \xi\ (\tilde{\mathcal{G}}_z^L\xi)^*\big|_D\right| \leq \|\xi\|_{L^2(D,d\mu_D)}\ \|(\tilde{\mathcal{G}}_z^L\xi)\big|_D\|_{L^2(D,d\mu_D)}$$

but we are going to prove that the function $(\tilde{\mathcal{G}}_z^L\xi)|_D(\vec{r})$ is bounded $\forall\, \vec{r} \in D$, so that

$$\|(\tilde{\mathcal{G}}_z^L\xi)\big|_D\|_{L^2(D,d\mu_D)} < C(A)\ \|\xi\|^2_{L^2(D,d\mu_D)}$$

and hence the result. Indeed

$$\left|(\tilde{\mathcal{G}}_z^L\xi)\big|_D(\vec{r})\right|^2 = \left|\left(\mathcal{G}_z^L(\vec{x}',\vec{x})\big|_{\vec{x},\vec{x}'\in D}\,,\ \xi(\vec{r}')\right)_{L^2(D,d\mu_D)}\right|^2 \leq$$

$$\leq \left\|\mathcal{G}_z^L(\vec{x}',\vec{x})\right\|^2_{L^2(D,d\mu_D(\vec{r}'))}\ \left\|\xi\right\|^2_{L^2(D,d\mu_D)} \leq C\ \ \left\|\xi\right\|^2_{L^2(D,d\mu_D)}$$

since the Green's function $\mathcal{G}_z^L(\vec{x},\vec{y}_0)$ belongs to $L^2(\mathbb{R}^3)$, for each $z \in \mathbb{C} - \mathbb{R}$ and $\vec{y}_0 \in \mathbb{R}^3$.

□

Proposition 2.4 *For each smooth real function α on D bounded away from 0, there exists $\zeta \in \mathbb{R}$, $\zeta < 0$ such that, for each $z \in \mathbb{C} - \mathbb{R}$, $\Re(z) < \zeta$, the following inequality holds*

$$\Phi^z_{\alpha,L}(\xi,\xi) - 2\Im(z)\,\Im\Big[\big(\Psi\,,\,\tilde{\mathcal{G}}^L_z\xi_\Psi\big)\Big] - \big(\Re(z) + \omega L\big)\,\|\Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi\|^2 > 0 \quad (2.10)$$

Proof: We first point out that (see Proposition B.1)

$$\lim_{\Re(z)\to\infty} \big\|\mathcal{G}^L_z(\vec{x},\vec{y}_0)\big\| \leq C(\Im(z)) < \infty$$

Thus, since the form $\Phi^z_{\alpha,L}(\xi,\xi)$ remains bounded for each $z \in \mathbb{C}-\mathbb{R}$, $\Im(z) \neq 0$, and

$$\lim_{\Re(z)\to\infty} \Re(z)\|\Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi\|^2 = \infty$$

$$\Big|\Im(z)\,\Im\Big[\big(\Psi\,,\,\tilde{\mathcal{G}}^L_z\xi_\Psi\big)\Big]\Big| \leq C(\Im(z))\,\|\xi\|^2$$

we can always found a ζ satisfying the requirement.

□

But now we can prove that the complete form $\mathcal{F}_{\alpha,L}$ is closed and bounded from below:

Theorem 2.1 *The form $\mathcal{F}_{\alpha,L}$ is bounded from below and closed on the domain*

$$\mathcal{D}(\mathcal{F}_{\alpha,L}) = \big\{\Psi \in L^2(\mathbb{R}^3) \mid \exists\xi_\Psi \in L^2(D, d\mu_D), \Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi \in H^1(\mathbb{R}^3)\big\} \quad (2.11)$$

where $z \in \mathbb{C} - \mathbb{R}$.

Proof: Semiboundedness is trivial thanks to Proposition 2.4: since the form $\mathcal{F}_{\alpha,L}$ does not depend on z, we can choose $z \in \mathbb{C} - \mathbb{R}$, $\Re(z) < \zeta$, so that the inequality (2.10) applies and

$$\mathcal{F}_{\alpha,L}(\Psi,\Psi) \geq F_{\omega,L}(\Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi, \Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi) + \omega L\,\|\Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi\|^2 + \Re(z)\,\|\Psi\|^2 \geq$$
$$\geq F_0(\Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi, \Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi) + \Re(z)\,\|\Psi\|^2 \geq \Re(z)\,\|\Psi\|^2$$

So it remains to prove closure. Let $\Psi_n = \varphi_n + \tilde{\mathcal{G}}_z\xi_n$ be a sequence in $\mathcal{D}(\mathcal{F}_{\alpha,L})$ converging to Ψ in the norm topology of $L^2(\mathbb{R}^3)$, such that[3]

$$\lim_{n,m\to\infty} \big(\mathcal{F}_{\alpha,L} - \Re(z)\big)(\Psi_n - \Psi_m) = 0$$

[3] F_0 is simply the form associated to the free Hamiltonian, i.e. $F_0(\Psi,\Psi) = \int |\nabla\Psi|^2$.

$$\lim_{n,m\to\infty}\left(\mathcal{F}_{\alpha,L}-\Re(z)\right)(\Psi_n-\Psi_m)\geq\lim_{n,m\to\infty}F_0(\varphi_n-\varphi_m)\geq 0$$

so that

$$\lim_{n,m\to\infty}F_0(\varphi_n-\varphi_m)=0$$

and

$$\lim_{n,m\to\infty}\Phi^z_{\alpha,L}(\xi_n-\xi_m)=0$$

The result easily follows, because F_0 and $\Phi^z_{\alpha,L}$ are closed forms (see Proposition 2.3).

□

Thus the form $\mathcal{F}_{\alpha,L}$ defines a semibounded self-adjoint operator:

Proposition 2.5 *The operators K^L_α defined below are self-adjoint:*

$$\mathcal{D}(K^L_\alpha)=\left\{\Psi\in L^2(\mathbb{R}^3)\mid\exists\xi_\Psi\in L^2(D,d\mu_D),\Psi-\tilde{\mathcal{G}}^L_z\xi_\Psi\in\mathcal{D}(H^L_\omega),\right.$$

$$\left.\left(\Psi-\tilde{\mathcal{G}}^L_z\xi_\Psi\right)\big|_D=\Gamma^L_\alpha(z)\xi_\Psi\right\}\tag{2.12}$$

$$\left(K^L_\alpha-z\right)\Psi=\left(H^L_\omega-z\right)\left(\Psi-\tilde{\mathcal{G}}^L_z\xi_\Psi\right)\tag{2.13}$$

where $\alpha\in C(D)$, $\alpha(\vec{r})\neq 0$, for each $\vec{r}\in D$.
Moreover

$$\left[\left(K^L_\alpha-z\right)^{-1}\Psi\right](\vec{x})=\left[\left(H^L_\omega-z\right)^{-1}\Psi\right](\vec{x})+$$

$$+\int_D d^2\vec{r}'\ \left[\Gamma^L_\alpha(z)\right]^{-1}\left[\left(H^L_\omega-z\right)^{-1}\Psi\right]\Big|_D(\vec{r}')\ \mathcal{G}^L_z(\vec{x},\vec{x}')\big|_{\vec{x}'\in D}\tag{2.14}$$

for each $z\in\varrho(K_\alpha)$.

Proof: The result easily follows from Theorem 2.1. The explicit expression of the resolvent is a direct consequence of the equation (2.13). We want only to remark that the operator $\Gamma^L_\alpha(z)$ is invertible if $\Im(z)\neq 0$: the form $\Phi^z_{\alpha,L}$ can be written in the following way

$$\Phi^z_{\alpha,L}(\xi,\xi)\equiv\int_D d\mu_D\ \frac{|\xi|^2}{\alpha}-\Re(z)\left\|\tilde{\mathcal{G}}^L_z\xi\right\|^2$$

Since $\left\|\tilde{\mathcal{G}}^L_z\xi\right\|^2$ is bounded by $C(\Im(z))\left\|\xi\right\|^2$, if $\Im(z)\neq 0$, we can always choose the real part of z is such a way that the form is positive.

□

At last we can remove the cut-off in the angular momentum and define the Hamiltonian of the system:

Theorem 2.2 *For each $\alpha \in C(D)$, $\alpha(\vec{r}) \neq 0$, $\forall\, \vec{r} \in D$, the sequence of semibounded self-adjoint operators K_α^L converge as $L \to \infty$ in the strong resolvent sense to the self-adjoint (unbounded from below) operator K_α:*

$$\mathcal{D}(K_\alpha) = \big\{\Psi \in L^2(\mathbb{R}^3) \mid \exists \xi_\Psi \in L^2(D, d\mu_D), \Psi - \tilde{\mathcal{G}}_z\xi_\Psi \in \mathcal{D}(H_\omega),$$

$$\big(\Psi - \tilde{\mathcal{G}}_z\xi_\Psi\big)\big|_D = \Gamma_\alpha(z)\xi_\Psi\big\} \tag{2.15}$$

$$\big(K_\alpha - z\big)\Psi = \big(H_\omega - z\big)\big(\Psi - \tilde{\mathcal{G}}_z\xi_\Psi\big) \tag{2.16}$$

where

$$\Big[\Gamma_\alpha(z)\, \xi_\Psi\Big](\vec{r}) = \frac{\xi_\Psi(\vec{r})}{\alpha(\vec{r})} - \int_D d\mu_D(\vec{r}\,')\ \mathcal{G}_z(\vec{x}, \vec{x}\,')\big|_{\vec{x},\vec{x}\,' \in D}\, \xi_\Psi(\vec{r}\,') \tag{2.17}$$

$$\big(\tilde{\mathcal{G}}_z\xi\big)(\vec{x}) \equiv \int_D d\mu_D(\vec{r}\,')\ \mathcal{G}_z(\vec{x}, \vec{x}\,')\big|_{\vec{x}\,' \in D}\, \xi(\vec{r}\,')$$

Moreover the resolvent of K_α is

$$\Big[\big(K_\alpha - z\big)^{-1}\Psi\Big](\vec{x}) = \Big[\big(H_\omega - z\big)^{-1}\Psi\Big](\vec{x}) +$$

$$+ \int_D d^2\vec{r}\,'\ \Gamma_\alpha^{-1}(z)\Big[\big(H_\omega - z\big)^{-1}\Psi\Big]\Big|_D(\vec{r}\,')\, \mathcal{G}_z(\vec{x}, \vec{x}\,')\big|_{\vec{x}\,' \in D} \tag{2.18}$$

for each $z \in \varrho(K_\alpha)$.

Proof: The key point of the proof is the application of the Trotter-Kato Theorem (see Theorem VIII.22 in [45]) to the sequence of self-adjoint operators K_α^L: we shall prove that $(K_\alpha^L - z)^{-1}$ converge in the strong sense for all $z \in \mathbb{C} - \mathbb{R}$ to the operator $(K_\alpha - z)^{-1}$, then the Trotter-Kato Theorem guarantees that there exists a self-adjoint operator T such that K_α^L converges in the strong resolvent sense to T. The identification of T with K_α is then trivial.
So we shall start with the analysis of the sequence of bounded operators $(K_\alpha - z)^{-1}$, $z \in \mathbb{C} - \mathbb{R}$, defined in (2.14): thanks to Proposition 2.1, the first part of the resolvent converges in the strong sense to $(H_\omega - z)^{-1}$, so that, in order to prove convergence of the whole operator, we need to consider the second part,

$$\int_D d^2\vec{r}\,'\ \big[\Gamma_\alpha^L(z)\big]^{-1}\Big[\big(H_\omega^L - z\big)^{-1}\Psi\Big]\Big|_D(\vec{r}\,')\, \mathcal{G}_z^L(\vec{x}, \vec{x}\,')\big|_{\vec{x}\,' \in D}$$

but, for the same reason,

$$\lim_{L\to\infty} \mathcal{G}_z^L(\vec{x},\vec{x}')\big|_{\vec{x}'\in D} = \mathcal{G}_z(\vec{x},\vec{x}')\big|_{\vec{x}'\in D}$$

in $L^2(\mathbb{R}^3)$ and

$$\lim_{L\to\infty} \left[\left(H_\omega^L - z\right)^{-1}\Psi\right]\Big|_D(\vec{r}') = \left[\left(H_\omega - z\right)^{-1}\Psi\right]\Big|_D(\vec{r}')$$

in $L^2(D, d\mu_D)$, for all $\Psi \in L^2(\mathbb{R}^3)$. Hence, to complete the first part of the proof, it is sufficient to show that

$$\lim_{L\to\infty} \left[\Gamma_\alpha^L(z)\right]^{-1} = \Gamma_\alpha^{-1}(z)$$

in the norm topology of $L^2(D, d\mu_D)$, but this is again a consequence of Proposition 2.1: for each L the operator $\Gamma_\alpha^L(z)$ is invertible (see the Proof of Proposition 2.5) and, in the same way, we can prove that $\Gamma_\alpha^{-1}(z)$ is bounded and well defined, if $\Im(z) \neq 0$; moreover it is easy to see that

$$\lim_{L\to\infty} \Gamma_\alpha^L(z) = \Gamma_\alpha(z)$$

We have then proved that, for each $z \in \mathbb{C} - \mathbb{R}$,

$$\text{s-}\lim_{L\to\infty} \left(K_\alpha^L - z\right)^{-1} = \left(K_\alpha - z\right)^{-1}$$

and the operator $(K_\alpha - z)^{-1}$ has of course a dense range. Thus the Trotter-Kato Theorem applies and the limiting self-adjoint operator T is immediately identified with K_α: the domain of K_α is given by functions of the form $(K_\alpha - z)^{-1}\Psi$, $\Psi \in L^2(\mathbb{R}^3)$, and the action of the operator on its domain follows from (2.18).

□

Theorem 2.3 *The spectrum of K_α is purely absolutely continuous and*

$$\sigma(K_\alpha) = \sigma_{\text{ac}}(K_\alpha) = \sigma(H_\omega) = \mathbb{R}$$

Proof: First of all we shall prove that the operator

$$\mathcal{R}_\alpha^z \equiv \left(K_\alpha - z\right)^{-1} - \left(H_\omega - z\right)^{-1}$$

is a compact operator $\forall\, z \in \mathbb{C} - \mathbb{R}$. Let Ψ_n be a weakly convergent sequence in $L^2(\mathbb{R}^3)$, namely $(\varphi\, , \Psi_n - \Psi_m) \to 0$ when $n, m \to \infty$ for each $\varphi \in L^2(\mathbb{R}^3)$,

$$\mathcal{R}_\alpha^z\,(\Psi_n - \Psi_m) = \int_D d^2\vec{r}'\ \Gamma_\alpha^{-1}(z)\left[\left(H_\omega - z\right)^{-1}(\Psi_n - \Psi_m)\right]\Big|_D \mathcal{G}_z(\vec{x},\vec{x}')\big|_{\vec{x}'\in D}$$

and

$$\left\|\mathcal{R}^z_\alpha\,(\Psi_n - \Psi_m)\right\| \leq \left\|\mathcal{G}_z\right\| \left\|\Gamma^{-1}_\alpha(z)\right\| \left|\left(\mathcal{G}^*_{z^*}\,,\;\Psi_n - \Psi_m\right)\right| \leq$$

$$\leq C \left|\left(\mathcal{G}_{z^*}\,,\;\Psi_n - \Psi_m\right)\right| \underset{n,m\to\infty}{\longrightarrow} 0$$

since the operator $\Gamma^{-1}_\alpha(z)$ is bounded (see the Proof of Theorem 2.2). Therefore we can apply Weyl's theorem and thus

$$\sigma_{\text{ess}}(K_\alpha) = \sigma_{\text{ess}}(H_\omega) = \mathbb{R}$$

To prove that the singular and pure points spectrum of K_α are empty, we refer again to the limiting absorption principle. To show that the condition of the principle is satisfied, we have to consider the scalar product (where $z = x + i\varepsilon$)

$$\left|\left(\Psi\,,\mathcal{R}^z_\alpha\Psi\right)\right| = \left|\int_D d^2\vec{r}'\;\Gamma^{-1}_\alpha(z)\left[\left(H_\omega - z\right)^{-1}\Psi\right]\Big|_D \left(\Psi\,,\;\mathcal{G}_z(\vec{x},\vec{x}')\big|_{\vec{x}'\in D}\right)\right| \leq$$

$$\leq \left\|\Gamma^{-1}_\alpha(z)\right\| \left|\int_D d^2\vec{r}'\;\left(\mathcal{G}_{z^*}(\vec{x},\vec{x}')\big|_{\vec{x}'\in D}\,,\;\Psi\right)\left(\Psi\,,\;\mathcal{G}_z(\vec{x},\vec{x}')\big|_{\vec{x}'\in D}\right)\right|$$

The operator $\Gamma^{-1}_\alpha(z)$ remains bounded when $\varepsilon \to 0$ and, applying the same trick used in the Proof of Theorem 1.3, one has

$$\lim_{\varepsilon\to 0}\left(\mathcal{G}_{x-i\varepsilon}(\vec{x},\vec{x}')\big|_{\vec{x}'\in D}\,,\;\Psi\right)\left(\Psi\,,\;\mathcal{G}_{x+i\varepsilon}(\vec{x},\vec{x}')\big|_{\vec{x}'\in D}\right) = \left|\varphi(\vec{r}')\right|^2 < \infty$$

where $\Psi = (H_\omega - x)\varphi$ and $\varphi \in \mathcal{D}(H_\omega)$, so that

$$\sup_{0<\varepsilon<1}\int_a^b dx\;\left|\left(\Psi\,,\;\mathcal{R}^{x+i\varepsilon}_\alpha\Psi\right)\right|^p < \infty$$

for some $p > 1$ and for each interval $[a, b] \subset \mathbb{R}$.

□

2.2.2 Asymptotic Limit of Rapid Rotation

In this Section we shall study the asymptotic limit of rapid rotation of the unitary group

$$U_{\text{inert}}(t, s) = R(t)\,U_{\text{rot}}(t - s)\,R^\dagger(s)$$

which represents the time evolution in the inertial frame associated to the formal time-dependent Hamiltonian defined in (2.1), while $U_{\text{rot}}(t - s)$ is the unitary group associated to the self-adjoint generator K_α: our main goal will be the proof of the following result,

$$\text{s}-\lim_{\omega\to\infty} U_{\text{inert}}(t,s) = e^{-iH_\alpha(t-s)}$$

where H_α is the self-adjoint generator[4]

$$H_\alpha = H_0 - \alpha(\vec{r})\,\Theta_S(\vec{r}) \tag{2.19}$$

and $\Theta_S(\vec{r})$ is the characteristic function of a sphere S of radius A centered at the origin.

Theorem 2.4 *For every $t, s \in \mathbb{R}$,*

$$\text{s}-\lim_{\omega\to\infty} U_{\text{inert}}(t,s) = e^{-iH_\alpha(t-s)}$$

where

$$H_\alpha = H_0 - \alpha(\vec{r})\,\Theta_S(\vec{r})$$

Proof: See the Proof of Theorem 1.4 and the following Lemma 2.1.

□

Lemma 2.1 *For every $z \in \mathbb{C}$, $\Im(z) > 0$,*

$$\text{s}-\lim_{\omega\to\infty}\int_{-\infty}^{0} dt\, e^{-izt}\, U^*_{\text{inert}}(t,0) = -i\big(H_\alpha - z\big)^{-1}$$

Proof: Like in the Proof of Lemma 1.1, we shall prove the result on the dense subset of $L^2(\mathbb{R}^3)$ given by functions of the form $\Psi(\vec{x}) = \chi(r)Y_{l_0}^{m_0}(\theta,\phi)$, with $l_0 = 0,\ldots\infty$ and $m_0 = -l_0,\ldots,l_0$. The first part of the Proof of Lemma 1.1 still applies, so that it is sufficient to prove that

$$\lim_{\omega\to\infty}\big(K_\alpha + m_0\omega - z\big)^{-1}\Psi(\vec{x}) = \big(H_\alpha - z\big)^{-1}\Psi(\vec{x})$$

First of all we observe that

$$\big(K_\alpha + m_0\omega - z\big)^{-1}\Psi = \big(H_\omega + m_0\omega - z\big)^{-1}\Psi +$$

[4]The operator H_α is easily defined with the method of quadratic form (see for example [45]): since the potential $\alpha(r)$ is bounded, it is associated to a form infinitesimally bounded w.r.t. the free Hamiltonian H_0. Hence the operator $H_0 + \alpha(r)\,\Theta_D(\vec{r})$ is self-adjoint on the domain of H_0.

$$+\left(\Gamma_\alpha^{-1}(z^*-m_0\omega)\left[\left(H_\omega+m_0\omega-z^*\right)^{-1}\Psi\right]\Big|_D\,,\ \mathcal{G}_{z-m_0\omega}(\vec{x},\vec{x}')\big|_{\vec{x}'\in D}\right)_{L^2(D,d\mu_D)}$$

and

$$\lim_{\omega\to\infty}\left(H_\omega+m_0\omega-z\right)^{-1}\Psi=\left(H_0-z\right)^{-1}\Psi$$

as we have proved in Lemma 1.1.
Therefore we need only to study the second part of the resolvent: it is easy to see that

$$\lim_{\omega\to\infty}\left[\left(H_\omega+m_0\omega-z\right)^{-1}\Psi\right]\Big|_D=\left[\left(H_0-z\right)^{-1}\Psi\right]\Big|_D$$

in $L^2(D,d\mu_D)$. Moreover, since $\left[\left(H_0-z\right)^{-1}\Psi\right]\big|_D(\vec{r})$ is a function of the form $\chi(r)Y_{l_0}^{m_0}(\theta,0)$, we can apply the result found in the following Lemma 2.2:

$$\lim_{\omega\to\infty}\Gamma_\alpha^{-1}(z-m_0\omega)\left[\left(H_\omega+m_0\omega-z\right)^{-1}\Psi\right]\Big|_D=$$

$$=\Xi_\alpha(z)\left[\left(H_0-z\right)^{-1}\Psi\right]\Big|_D=$$

$$=\alpha(\vec{r})\Theta_D(\vec{r})\left(H_0-\alpha(\vec{r})\,\Theta_S(\vec{r})-z\right)^{-1}\Psi$$

In conclusion we obtain

$$\lim_{\omega\to\infty}\left(K_\alpha+m_0\omega-z\right)^{-1}\Psi=\left(H_0-z\right)^{-1}\left[1+\alpha\Theta_D\left(H_0-\alpha\,\Theta_S-z\right)^{-1}\right]\Psi=$$

$$=\left(H_0-\alpha\,\Theta_S-z\right)^{-1}\Psi$$

□

Lemma 2.2 *Let $\Gamma_\alpha(z)$ the operator defined in (2.17) and $\Psi(\vec{x})\in L^2(\mathbb{R}^3)$ of the form $\Psi(\vec{x})=\chi(r)Y_{l_0}^{m_0}(\theta,\phi)$,*

$$\lim_{\omega\to\infty}\Gamma_\alpha^{-1}(z-m_0\omega)\,\Psi|_D=\Xi_\alpha(z)\,\Psi|_D$$

in $L^2(D,d\mu_D)$, where

$$\left(\Xi_\alpha(z)\Psi|_D\right)(\vec{r})\equiv\left[\alpha(\vec{r})\left(H_0-\alpha(\vec{r})\,\Theta_S(\vec{r})-z\right)^{-1}\left(H_0-z\right)\,\Psi|_D\right](\vec{r})\qquad(2.20)$$

Proof: First of all we are going to prove that

$$\text{norm-}\lim_{\omega\to\infty}\Gamma_\alpha(z-m_0\omega)=\Lambda_\alpha(z)$$

where

$$\left(\Lambda_\alpha(z)\,\xi\right)=\frac{\xi}{\alpha}-\int_D d\mu_D(\vec{r}')\;G_z^{m_0}(\vec{x},\vec{x}')\big|_{\vec{x},\vec{x}'\in D}\xi(\vec{r}')$$

for the definition of $G_z^{m_0}$ see Proposition 1.2.
Indeed

$$\Gamma_\alpha(z - m_0\omega) = \Lambda_\alpha(z) + R_z^{m_0}$$

where $R_z^{m_0}$ is a bounded integral operator on $L^2(D, d\mu_D)$ with kernel

$$R_z^{m_0}(\vec{r}, \vec{r}') \equiv \int_0^\infty \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m \neq m_0}}^{l} \frac{\varphi_{klm}(\vec{r})\, \varphi_{klm}(\vec{r}')}{k^2 - (m - m_0)\omega - z}$$

that goes to 0 when $\omega \to \infty$ (see the Proof of Lemma 1.1).
Moreover $\forall \omega \in \mathbb{R}^+$ the operator $\Gamma_\alpha(z)$ is invertible if $\Im(z) \neq 0$ (see the Proof of Theorem 2.2) and, for each $l_0 \in \mathbb{N}$, $m_0 = -l_0, \dots, l_0$, $z \in \mathbb{C} - \mathbb{R}$ it can be seen that the operator Λ_α is also invertible: indeed, let Ψ is the dense subset of $L^2(D, d\mu_D)$ given by functions of the form $\chi(r) Y_{l_0}^{m_0}(\theta, 0)$,

$$\big(\Lambda_\alpha(z)\; \Psi|_D\big)(r, \theta) = \frac{\Psi|_D}{\alpha}(r, \theta) - \frac{Y_{l_0}^{m_0}(\theta, 0)}{2\pi} \int_0^A dr'\, r'^2\; g_z^{l_0}(\vec{r}, \vec{r}')\, \chi(\vec{r}')$$

and

$$\Big[\big(H_0 - z\big)\Lambda_\alpha(z)\; \Psi|_D\Big](\vec{r}) = \left[\big(H_0 - z\big)\frac{\Psi|_D}{\alpha}\right](\vec{r}) - \Theta_D(\vec{r})\; \Psi|_D(\vec{r})$$

so that $\Lambda_\alpha^{-1}(z)\Psi|_D = \Xi_\alpha(z)\Psi|_D$.

□

2.3 The Rotating Blade in 2D

2.3.1 The Hamiltonian

The formal time-dependent Hamiltonian of the system is given by the operator

$$H(t) = H_0 + \alpha(x)\; R(t)\; \Theta_A(x)\; \delta(y) \qquad (2.21)$$

where $\Theta_A(x)$ is the characteristic function of the segment $0 \leq x \leq A$. In the rotating frame the generator of time evolution is a self-adjoint extension of the symmetric operator

$$K_S = H_\omega$$

$$\mathcal{D}(K_S) = C_0^\infty(\mathbb{R}^2 - S)$$

where S is the segment $S \equiv \{(x, 0) \in \mathbb{R}^2 \mid 0 \leq x \leq A\}$.
In order to rigorously define the self-adjoint extensions of the operator K_S,

we shall proceed like in the 3D case, namely we shall introduce a sequence of cut-off perturbed Hamiltonians and then we shall identify their limit with the Hamiltonian of the system.
So let

$$H_\omega^N = H_\omega \, \Pi_N \tag{2.22}$$

where Π_N is the projector on the subspace of $L^2(\mathbb{R}^2)$ generated by functions of the form $\chi(r)e_n(\theta)$, with $|n| \leq N$. The operator H_ω^N is self-adjoint on the domain $H^2(\mathbb{R}^2)$ (see the discussion at the beginning of Section 4) and, for each $z \in \varrho(H_\omega^N)$, the resolvent $(H_\omega^N - z)^{-1}$ is given by an integral operator with kernel

$$\mathcal{G}_z^N(\vec{x}, \vec{x}') = \int_0^\infty dk \sum_{n=-N}^{N} \frac{\varphi_{kn}^*(\vec{x}') \, \varphi_{kn}(\vec{x})}{k^2 - \omega n - z} \tag{2.23}$$

Proposition 2.6 *The sequence of cut-off Hamiltonians converge as $N \to \infty$ in the strong resolvent sense to the self-adjoint operator H_ω.*

Proof: See the Proof of Proposition 2.1 and Proposition B.2.

□

The perturbed cut-off Hamiltonian is associated to the form

$$\mathcal{F}_{\alpha,N}(\Psi, \Psi) = F_{\omega,N}(\Psi, \Psi) - \int_S d\mu_S \, \alpha(r) \left| \Psi\big|_S(r) \right|^2 \tag{2.24}$$

which is well defined[5] if $\Psi \in \mathcal{D}(F_{\omega,N})$, $F_{\omega,N}$ being the closed semibounded form associated to the self-adjoint operator H_ω^N, and $\alpha \in C(S)$, $\alpha(r) \neq 0$, $\forall r \in S$.

Proposition 2.7 *Let $z \in \mathbb{C} - \mathbb{R}$, the form $\mathcal{F}_{\alpha,N}$ can be written in the following way,*

$$\mathcal{F}_{\alpha,N}(\Psi, \Psi) = \mathcal{F}_{\omega,N}^z(\Psi, \Psi) + \Phi_{\alpha,N}^z(\xi_\Psi, \xi_\Psi) - 2\Im(z) \, \Im\Big[(\Psi \, , \, \tilde{\mathcal{G}}_z^N \xi_\Psi) \Big] \tag{2.25}$$

where

$$\mathcal{F}_{\omega,N}^z(\Psi, \Psi) = F_{\omega,N}(\Psi - \tilde{\mathcal{G}}_z^N \xi_\Psi, \Psi - \tilde{\mathcal{G}}_z^N \xi_\Psi) - \Re(z) \|\Psi - \tilde{\mathcal{G}}_z^N \xi_\Psi\|^2 + \Re(z) \|\Psi\|^2 \tag{2.26}$$

$$\Phi_{\alpha,N}^z(\xi_\Psi, \xi_\Psi) = \Re\Big[(\xi_\Psi \, , \, \Gamma_\alpha^N(z) \, \xi_\Psi)_{L^2(S, d\mu_S)} \Big] \tag{2.27}$$

[5]In the 2D case, the measure $d\mu_S$ is given by $r \, dr$.

and

$$\left[\Gamma^N_\alpha(z)\,\xi_\Psi\right](r) = \frac{\xi_\Psi(r)}{\alpha(r)} - \int_S d\mu_S(r')\;\mathcal{G}^N_z(\vec{x},\vec{x}')\big|_{\vec{x},\vec{x}'\in S}\,\xi_\Psi(r') \tag{2.28}$$

$$(\tilde{\mathcal{G}}^N_z\xi)(\vec{x}) \equiv \int_S d\mu_S(r')\;\mathcal{G}^N_z(\vec{x},\vec{x}')\big|_{\vec{x}'\in S}\,\xi(r')$$

Proof: See the Proof of Proposition 2.2.

□

Now we shall prove that the properties of the form $\Phi^z_{\alpha,N}$ still hold:

Proposition 2.8 *The form $\Phi^z_{\alpha,N}(\xi,\xi)$ is bounded for each $\xi \in L^2(S,d\mu_S)$.*

Proof: Using the result proved in Proposition B.2, we can follow the Proof of Proposition 2.3.

□

Proposition 2.9 *For each smooth real function α on S bounded away from 0, there exists $\zeta \in \mathbb{R}$, $\zeta < 0$ such that, for each $z \in \mathbb{C}-\mathbb{R}$, $\Re(z) < \zeta$, the following inequality holds*

$$\Phi^z_{\alpha,N}(\xi,\xi) - 2\Im(z)\,\Im\Big[\big(\Psi\,,\,\tilde{\mathcal{G}}^N_z\xi_\Psi\big)\Big] - \big(\Re(z)+\omega N\big)\,\|\Psi - \tilde{\mathcal{G}}^N_z\xi_\Psi\|^2 > 0$$

Proof: See the Proof of Proposition 2.4 and Proposition B.2.

□

We can now state the following Theorem,

Theorem 2.5 *The form $\mathcal{F}_{\alpha,N}$ is bounded from below and closed on the domain*

$$\mathcal{D}(\mathcal{F}_{\alpha,N}) = \left\{\Psi \in L^2(\mathbb{R}^2)\,|\,\exists\xi_\Psi \in L^2(S,rdr), \Psi - \tilde{\mathcal{G}}^N_z\xi_\Psi \in H^1(\mathbb{R}^2)\right\} \tag{2.29}$$

Proof: See the Proof of Theorem 2.1.

□

Proposition 2.10 *The operators K_α^N defined below are self-adjoint:*

$$\mathcal{D}(K_\alpha^N) = \{\Psi \in L^2(\mathbb{R}^2) \mid \exists \xi_\Psi \in L^2(S, d\mu_S), \Psi - \tilde{\mathcal{G}}_z^N \xi_\Psi \in \mathcal{D}(H_\omega^N),$$

$$(\Psi - \tilde{\mathcal{G}}_z^N \xi_\Psi)\big|_D = \Gamma_\alpha^N(z)\xi_\Psi\} \tag{2.30}$$

$$(K_\alpha^N - z)\Psi = (H_\omega^N - z)(\Psi - \tilde{\mathcal{G}}_z^N \xi_\Psi) \tag{2.31}$$

where $\alpha \in C(D)$, $\alpha(\vec{r}) \neq 0$, for each $\vec{r} \in D$.
Moreover

$$\Big[(K_\alpha^N - z)^{-1}\Psi\Big](\vec{x}) = \Big[(H_\omega^N - z)^{-1}\Psi\Big](\vec{x}) +$$

$$+ \int_D d^2\vec{r}' \; [\Gamma_\alpha^N(z)]^{-1}\Big[(H_\omega^N - z)^{-1}\Psi\Big]\Big|_D(\vec{r}')\, \mathcal{G}_z^N(\vec{x}, \vec{x}')\big|_{\vec{x}' \in D} \tag{2.32}$$

for each $z \in \varrho(K_\alpha)$.

Proof: The result follows from Theorem 2.5. Like in the 3D case it is possible to prove that the operator $\Gamma_\alpha^N(z)$ is invertible if $\Im(z) \neq 0$.

□

Theorem 2.6 *For each $\alpha \in C(S)$, $\alpha(r) \neq 0$, $\forall\, r \in S$, the sequence of semibounded self-adjoint operators K_α^N converge as $N \to \infty$ in the strong resolvent sense to the self-adjoint (unbounded from below) operator K_α:*

$$\mathcal{D}(K_\alpha) = \{\Psi \in L^2(\mathbb{R}^2) \mid \exists \xi_\Psi \in L^2(S, d\mu_S), \Psi - \tilde{\mathcal{G}}_z \xi_\Psi \in \mathcal{D}(H_\omega),$$

$$(\Psi - \tilde{\mathcal{G}}_z \xi_\Psi)\big|_S = \Gamma_\alpha(z)\xi_\Psi\} \tag{2.33}$$

$$(K_\alpha - z)\Psi = (H_\omega - z)(\Psi - \tilde{\mathcal{G}}_z \xi_\Psi) \tag{2.34}$$

where

$$\Big[\Gamma_\alpha(z)\, \xi_\Psi\Big](r) = \frac{\xi_\Psi(r)}{\alpha(r)} - \int_S d\mu_S(r')\; \mathcal{G}_z(\vec{x}, \vec{x}')\big|_{\vec{x}, \vec{x}' \in S}\, \xi_\Psi(r') \tag{2.35}$$

$$(\tilde{\mathcal{G}}_z \xi)(\vec{x}) \equiv \int_S d\mu_S(r')\; \mathcal{G}_z(\vec{x}, \vec{x}')\big|_{\vec{x}' \in D}\, \xi(r')$$

Moreover the resolvent of K_α is

$$\Big[(K_\alpha - z)^{-1}\Psi\Big](\vec{x}) = \Big[(H_\omega - z)^{-1}\Psi\Big](\vec{x}) +$$

$$+ \int_S dr'\, r'\, \Gamma_\alpha^{-1}(z)\Big[(H_\omega - z)^{-1}\Psi\Big]\Big|_S(r')\, \mathcal{G}_z(\vec{x}, \vec{x}')\big|_{\vec{x}' \in S} \tag{2.36}$$

for each $z \in \varrho(K_\alpha)$.

Proof: See the Proof of Theorem 2.2.

□

Theorem 2.7 *The spectrum of K_α is purely absolutely continuous and*

$$\sigma(K_\alpha) = \sigma_{ac}(K_\alpha) = \sigma(H_\omega) = \mathbb{R}$$

Proof: See the Proof of Theorem 2.3, Theorem 2.6 and Proposition B.2.

□

Remark. An interesting application of previous results is the study of the 3D rotating needle, i.e. a singular rotating perturbation of the Laplacian supported on a (finite) segment. Indeed the system can be easily reduced to a 2D rotating blade on the plane of rotation and a free motion on its perpendicular axis: the Hamiltonian is formally given by

$$H = H_0^{x,y} + \alpha(x)\, R(t)\, \Theta_A(x)\, \delta(y) + H_0^z$$

where $\Theta_A(x)$ is the characteristic function of the segment $0 \leq x \leq A$. According to the previous discussion, the self-adjoint extensions of H are given by the family of operators $K_\alpha^{x,y} + H_0^z$, where $K_\alpha^{x,y}$ denotes the Hamiltonians of the 2D rotating blade defined in (2.34). Moreover the domain of self-adjointness can be identified with the set of functions $\Psi(\vec{x}) = f(x,y)\, g(z)$ such that $f \in \mathcal{D}(K_\alpha)$ and $g \in H^2(\mathbb{R})$.

2.3.2 Asymptotic Limit of Rapid Rotation

In this Section, we shall prove that

$$\text{s}-\lim_{\omega\longrightarrow\infty} U_{\text{inert}}(t,s) = e^{-iH_\alpha(t-s)}$$

where H_α is the self-adjoint generator

$$H_\alpha = H_0 - \alpha(r)\, \Theta_C(r) \tag{2.37}$$

and $\Theta_C(r)$ is the characteristic function of a circle C of radius A centered at the origin.

Theorem 2.8 *For every $t, s \in \mathbb{R}$,*

$$\text{s}-\lim_{\omega\longrightarrow\infty} U_{\text{inert}}(t,s) = e^{-iH_\alpha(t-s)}$$

where

$$H_\alpha = H_0 - \alpha(r)\, \Theta_C(r)$$

Proof: See the Proof of Theorem 1.4 and the following Lemma 2.3.

□

Lemma 2.3 *For every $z \in \mathbb{C}$, $\Im(z) > 0$,*

$$\text{s-}\lim_{\omega \to \infty} \int_{-\infty}^{0} dt\, e^{-izt}\, U^*_{\text{inert}}(t,0) = -i\big(H_\alpha - z\big)^{-1}$$

Proof: Like in the Proof of Lemma 1.2, we shall prove the result on the dense subset of $L^2(\mathbb{R}^2)$ given by functions of the form $\Psi(\vec{x}) = \chi(r)e_{n_0}(\theta)$, $n_0 \in \mathbb{Z}$. Following the Proof of Lemma 1.2, it remains to prove that

$$\lim_{\omega \to \infty} \big(K_\alpha + n_0\omega - z\big)^{-1}\Psi(\vec{x}) = \big(H_\alpha - z\big)^{-1}\Psi(\vec{x})$$

but

$$\big(K_\alpha + n_0\omega - z\big)^{-1}\Psi = \big(H_\omega + n_0\omega - z\big)^{-1}\Psi +$$

$$+ \Big(\Gamma_\alpha^{-1}(z^* - n_0\omega)\Big[\big(H_\omega + n_0\omega - z^*\big)^{-1}\Psi\Big]\Big|_S,\, \mathcal{G}_{z - n_0\omega}(\vec{x},\vec{x}')\big|_{\vec{x}' \in S}\Big)_{L^2(S,d\mu_S)}$$

and

$$\lim_{\omega \to \infty} \big(H_\omega + n_0\omega - z\big)^{-1}\Psi = \big(H_0 - z\big)^{-1}\Psi$$

as we have proved in Lemma 1.2. Moreover

$$\lim_{\omega \to \infty} \Big[\big(H_\omega + n_0\omega - z\big)^{-1}\Psi\Big]\Big|_S = \Big[\big(H_0 - z\big)^{-1}\Psi\Big]\Big|_S$$

in $L^2(S, d\mu_S)$ and, applying the result found in the following Lemma 2.4,

$$\lim_{\omega \to \infty} \Gamma_\alpha^{-1}(z - n_0\omega)\Big[\big(H_\omega + n_0\omega - z\big)^{-1}\Psi\Big]\Big|_S =$$

$$= \Xi_\alpha(z)\Big[\big(H_0 - z\big)^{-1}\Psi\Big]\Big|_S =$$

$$= \alpha(r)\Theta_S(r)\big(H_0 - \alpha(r)\,\Theta_C(r) - z\big)^{-1}\Psi$$

In conclusion we obtain

$$\lim_{\omega \to \infty} \big(K_\alpha + n_0\omega - z\big)^{-1}\Psi = \big(H_0 - z\big)^{-1}\Big[1 + \alpha\Theta_S\big(H_0 - \alpha\,\Theta_C - z\big)^{-1}\Big]\Psi =$$

$$= \big(H_0 - \alpha\,\Theta_C - z\big)^{-1}\Psi$$

□

Lemma 2.4 *Let $\Gamma_\alpha(z)$ the operator defined in (2.35),*

$$\lim_{\omega \to \infty} \Gamma_\alpha^{-1}(z - n_0\omega) = \Xi_\alpha(z)$$

in $L^2(S, d\mu_S)$, where

$$\left(\Xi_\alpha(z)\xi\right)(r) \equiv \left[\alpha(r)\Big(H_0 - \alpha(r)\,\Theta_C(r) - z\Big)^{-1}(H_0 - z)\,\xi\right](r) \qquad (2.38)$$

Proof: First of all we are going to prove that

$$\text{norm}-\lim_{\omega \to \infty} \Gamma_\alpha(z - n_0\omega) = \Lambda_\alpha(z)$$

where

$$\Lambda_\alpha(z)\,\xi = \frac{\xi}{\alpha} - \frac{1}{2\pi}\int_S d\mu_S(r')\; g_z^{n_0}(r, r')\xi(r')$$

for the definition of $g_z^{n_0}$ see Proposition 1.4.
Indeed

$$\Gamma_\alpha(z - n_0\omega) = \Lambda_\alpha(z) + R_z^{n_0}$$

where $R_z^{n_0}$ is a bounded integral operator on $L^2(S, d\mu_S)$ with kernel

$$R_z^{n_0}(r, r') \equiv \int_0^\infty \sum_{\substack{n=-\infty \\ n \neq n_0}}^{\infty} \frac{\varphi_{kn}(r)\,\varphi_{kn}(r')}{k^2 - (n - n_0)\omega - z} \longrightarrow 0$$

as $\omega \to \infty$ (see the Proof of Lemma 1.2).
Moreover for each $n_0 \in \mathbb{Z}$ and $z \in \mathbb{C} - \mathbb{R}$ it can be seen that the operator Λ_α is invertible: indeed

$$\left[(H_0 - z)\Lambda_\alpha(z)\,\xi\right](\vec{r}) = \left[(H_0 - z)\frac{\xi}{\alpha}\right](r) - \Theta_S(r)\,\xi(r)$$

so that $\Lambda_\alpha^{-1}(z) = \Xi_\alpha(z)$.

□

Appendix A

The Genericity Condition

This Appendix is devoted to the analysis of the *genericity condition* we have defined and used in Chapter 1 and 2. Let us recall its definition (see (1.19)): we say that a sequence $a \equiv \{a_n\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{N})$ satisfies the genericity condition if

$$e_1 = (1, 0, 0, \ldots) \in \overline{\bigvee_{n=0}^{\infty} \mathcal{T}^n a} \tag{A.1}$$

where

$$\left(\mathcal{T} a\right)_n \equiv a_{n+1} \tag{A.2}$$

is the right shift operator on $\ell_2(\mathbb{N})$.
In the following discussion we often refer to [12], for a more detailed study of the condition and to [41], for the properties of the right shift operator.
In order to understand if a sequence a satisfies the genericity condition, it is interesting to look for nontrivial solutions in $\ell_2(\mathbb{N})$ of the system of equations

$$\left(b\,,\, \mathcal{T}^j a\right) = 0 \tag{A.3}$$

$\forall j \in \mathbb{N}$. As suggested in [12], such solutions define analytic L^2-functions on the unit disk $\{x \in \mathbb{C} \mid |x| \leq 1\}$, setting

$$A(x) = \sum_{n=0}^{\infty} a_n\, x^n \tag{A.4}$$

$$B(x) = \sum_{n=0}^{\infty} b_n\, x^n \tag{A.5}$$

So that (A.3) becomes

$$b_0 A(x) + b_1 \frac{A(x) - A(0)}{x} + \ldots + b_n\, x^{-n} \left[A(x) - \sum_{k=0}^{\infty} \frac{x^k A^{(k)}(0)}{k!}\right] + \ldots = 0$$

or

$$\oint_{|z|=1} \frac{A(z)B(1/z)}{z-x}\, dz = 0 \tag{A.6}$$

where we have used the Cauchy's formula,

$$\frac{1}{x^n}\left[A(x) - \sum_{k=0}^{\infty} \frac{x^k A^{(k)}(0)}{k!}\right] = \frac{1}{2\pi i} \oint_{|z|=1} \frac{A(z)}{z^n(z-x)}\, dz$$

The analytic functions $A(z)$ such that equation (A.6) has nontrivial solutions are very special and they relate to the Beurling inner functions (see [41]). However in the following we check the genericity condition for some simple functions and we discuss the relation with asymptotic complete ionization for a time-dependent point interaction (see Chapter 1).

A.1 Examples

(i) In order to apply the discussion above, we prove that the sequence

$$a_n = \frac{1}{n}$$

satisfies the genericity condition. The associated function on the disc is then given by $A(x) = \ln(1-x)$ and equation (A.6) becomes, setting $t \equiv 1/z$,

$$\frac{1}{x} \oint_{|t|=1} \frac{B(t)\ln(t-1)}{t-x^{-1}}\, dt - \frac{1}{x} \oint_{|t|=1} \frac{B(t)\ln(t)}{t(t-x^{-1})}\, dt = 0$$

Taking the cut of the log on $[1, \infty)$, the first integral vanishes since the function is analytic on the unit circle, and then one has

$$\frac{1}{x} \oint_{|t|=1} \frac{B(t)\ln(t)}{t}\, dt \frac{1}{x} - \oint_{|t|=1} \frac{B(t)\ln(t)}{t-x^{-1}}\, dt = 0$$

Studying the analyticity properties of the second integral (see [12] for the details), it is possible to prove that the existence of the cut implies $B = 0$.

(ii) A simple case in which the genericity condition fails is given by geometric sequences, that is

$$a_n = \lambda^n \tag{A.7}$$

for some $|\lambda| < 1$. Indeed $\mathcal{T}^j a = \lambda^j a$, so that the subspace generated by $\mathcal{T}^j a$ for any $j \in \mathbb{N}$ is in fact one dimensional.

A.2 Is the Genericity Condition Necessary?

In this Section we want to investigate the role of the genericity condition in the proof of asymptotic complete ionization for a time-dependent point interaction[1] (see Chapter 1). The main question is then: is such condition necessary or only sufficient?

A key point about the meaning of the genericity condition is the Remark at the end of Chapter 1: if the function $\alpha(t)$ is positive at any time $t \in \mathbb{R}^+$, the proof of asymptotic complete ionization does not require any condition at all.

Having a closer look to the condition (1.19), one can see that it does not involve the 0-th Fourier coefficient α_0. Let $\alpha(t)$ be a continuous real function and α_n its Fourier coefficients[2], we say that $\alpha(t)$ satisfies the genericity condition if the sequence $a_n \equiv \alpha_n$, $n > 0$ is generic with respect to $\mathcal{T}$ (see (1.19)). Hence the coefficient α_0 does not enter in the condition and it can be chosen freely.

On the other hand, since we can always take α_0 sufficiently large so that the function $\alpha(t)$ is positive at any time, one can easily construct examples of non-generic positive functions $\alpha(t)$, for which the system still shows asymptotic complete ionization. Therefore such a condition can not be necessary.

Nevertheless one can expect that the condition could be necessary, excluding positive functions. It should therefore be investigated the asymptotic behavior of the system when the function $\alpha(t)$ does not satisfy the genericity condition and is yet negative for a very short time. We conjecture that for such a small class of functions, it should be also taken into account the frequency and perhaps other parameters like e.g. the L^1-norm of the negative part of $\alpha(t)$.

It is then clear that the problem is very subtle and it could be solved only by a detailed analysis of the point spectrum of the Floquet operator.

---

[1] A similar discussion holds also for the decay problem studied in Chapter 2.

[2] Since the function is assumed to be real, $\alpha_n = \alpha^*_{-n}$ and we can only consider the coefficient α_n for $n \geq 0$.

Appendix B

The Green's Function of H_ω

In this Appendix we shall study the Green's function $\mathcal{G}_z(\vec{x}, \vec{y}_0)$ of H_ω and we shall prove that it belongs to $L^2(\mathbb{R}^n, d^n\vec{x})$, $\forall \vec{y}_0 \in \mathbb{R}^n$ with $n = 2, 3$.
We shall start from the 3D case:

Proposition B.1 *The resolvent $(H_\omega - z)^{-1}$, $z \in \mathbb{C} - \mathbb{R}$, has the following integral representation*

$$(H_\omega - z)^{-1}\Psi(\vec{x}) = \int_{\mathbb{R}^3} d^3x' \mathcal{G}_z(\vec{x}, \vec{x}')\Psi(\vec{x}')$$

with $\Psi(\vec{x}) \in L^2(\mathbb{R}^3, d^3x)$ and

$$\mathcal{G}_z(\vec{x}, \vec{x}') = \int_0^\infty dk \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{k^2 - m\omega - z} \varphi^*_{klm}(\vec{x}') \varphi_{klm}(\vec{x}) \tag{B.1}$$

The functions $\varphi_{klm}(\vec{x})$ are the spherical waves[1]*:*

$$\varphi_{klm}(\vec{x}) = \sqrt{\frac{2k^2}{\pi}} j_l(kr) Y_l^m(\theta, \varphi)$$

Moreover, for every $\vec{y}_0 \in \mathbb{R}^3$ and $z \in \mathbb{C} - \mathbb{R}$, $\mathcal{G}_z(\vec{x}, \vec{y}_0) \in L^2(\mathbb{R}^3, d^3\vec{x})$.

Proof: The integral representation of the Green's function of H_ω is a straightforward consequence of the eigenvectors decomposition of H_ω. Moreover in the following we shall prove that, for each $\Psi \in L^2(\mathbb{R}^3)$, $z \in \mathbb{C} - \mathbb{R}$ and $\vec{y}_0 \in \mathbb{R}^3$,

$$\left| \Big(\mathcal{G}_z(\vec{x}, \vec{y}_0) \, , \, \Psi(\vec{x}) \Big)_{L^2(\mathbb{R}^3, d^3\vec{x})} \right| < \infty$$

[1]Here $j_l(r)$ denotes the spherical Bessel function of order l (see [40, 55]) and $Y_l^m(\theta, \phi)$, with $l \in \mathbb{N}$ and $m = -l, \ldots, l$, the spherical harmonics.

Every function $\Psi \in L^2(\mathbb{R}^3)$ can be decomposed in terms of spherical harmonics:

$$\Psi(\vec{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Psi_{lm}(r)\, Y_l^m(\theta, \phi)$$

with the L^2-condition

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left|\left|\Psi_{lm}(r)\right|\right|^2_{L^2(\mathbb{R}^+, r^2 dr)} < \infty$$

Thus

$$\left|\Big(\mathcal{G}_z(\vec{x}, \vec{y}_0)\,,\,\Psi(\vec{x})\Big)\right|^2 \leq \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left|\Big(G_{z+m\omega}(\vec{x}, \vec{y}_0)\,,\,\Psi_{lm}(r) Y_l^m(\theta, \phi)\Big)\right|^2 \leq$$

$$\leq \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left|\left|G_{z+m\omega}(\vec{x}, \vec{y}_0)\right|\right|^2_{L^2(\mathbb{R}^3, d^3\vec{x})} \left|\left|\Psi_{lm}(r) Y_l^m(\theta, \phi)\right|\right|^2 \leq$$

$$\leq C(\Im(z)) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left|\left|\Psi_{lm}(r)\right|\right|^2_{L^2(\mathbb{R}^+, r^2 dr)} < \infty$$

because the Green's function of the free Hamiltonian

$$G_{z+m\omega}(\vec{x}, \vec{x}) = \frac{e^{i\sqrt{z+m\omega}|\vec{x}-\vec{y}_0|}}{4\pi|\vec{x} - \vec{x}|}$$

belongs to $L^2(\mathbb{R}^3, d^3\vec{x})$ for each $z \in \mathbb{C} - \mathbb{R}$ and $\vec{y}_0 \in \mathbb{R}^3$: we have to choose the root of $z + m\omega$ with imaginary part

$$\Im\big(\sqrt{z + m\omega}\big) = \sqrt{\frac{\left[(\Re(z) + m\omega)^2 + \Im(z)^2\right]^{\frac{1}{2}} - \Re(z) - m\omega}{2}} \geq \sqrt{\frac{|\Im(z)|}{2}} > 0$$

so that $G_{z+m\omega} \in L^2$ independently on $m \in \mathbb{Z}$.

□

An analogous result can be proved in the 2D case:

Proposition B.2 *The resolvent* $(H_\omega - z)^{-1}$, $z \in \mathbb{C} - \mathbb{R}$, *has the following integral representation*

$$(H_\omega - z)^{-1}\Psi(\vec{x}) = \int_{\mathbb{R}^2} d^2x' \mathcal{G}_z(\vec{x}, \vec{x}')\Psi(\vec{x}')$$

with $\Psi(\vec{x}) \in L^2(\mathbb{R}^2, d^2x)$ *and*[2]

$$\mathcal{G}_z(\vec{x}, \vec{x}') \equiv \int_0^{\infty} dk \sum_{n=-\infty}^{\infty} \frac{1}{k^2 - \omega n - z}\, \varphi^*_{kn}(\vec{x}')\, \varphi_{kn}(\vec{x}) \tag{B.2}$$

$$\varphi_{kn}(\vec{x}) = \sqrt{\frac{k}{2\pi}} J_{|n|}(kr)\, e^{in\theta}$$

Moreover, for every $\vec{y}_0 \in \mathbb{R}^2$ *and* $z \in \mathbb{C} - \mathbb{R}$, $\mathcal{G}_z(\vec{x}, \vec{y}_0) \in L^2(\mathbb{R}^2, d^2\vec{x})$.

Proof: Following the Proof of Proposition $B.1$, we shall consider the scalar product

$$\Big(\mathcal{G}_z(\vec{x}, \vec{y}_0)\,,\, \Psi(\vec{x})\Big)_{L^2(\mathbb{R}^3, d^3\vec{x})}$$

with

$$\Psi(\vec{x}) = \sum_{n=-\infty}^{\infty} \Psi_n(r)\, \frac{e^{in\theta}}{2\pi}$$

and we obtain

$$\Big|\Big(\mathcal{G}_z(\vec{x}, \vec{y}_0)\,,\, \Psi(\vec{x})\Big)\Big|^2 \le \sum_{n=-\infty}^{\infty} \big\|G_{z+m\omega}(\vec{x}, \vec{y}_0)\big\|^2_{L^2(\mathbb{R}^3, d^3\vec{x})} \big\|\Psi_n(r)\big\|^2_{L^2(\mathbb{R}^+, r^2 dr)} < \infty$$

since[3]

$$G_{z+n\omega}(\vec{x}, \vec{y}_0) = \frac{i}{4}\, H_0^{(1)}(\sqrt{z + n\omega}\, |\vec{x} - \vec{y}_0|)$$

belongs to $L^2(\mathbb{R}^2, d^2\vec{x})$, for each $z \in \mathbb{C} - \mathbb{R}$ and $\Im(\sqrt{z + n\omega}) > 0$.

□

[2] $J_n(r)$ stands for the Bessel function of order $n \in \mathbb{N}$.

[3] $H_0^{(1)}$ denotes the Hankel function of first kind and order zero (see [1]).

Acknowledgements

First of all I want to thank Gianfausto Dell'Antonio, who taught me to carry on a research project and supervised my work with continual encouragements and suggestions. I am also grateful to the members of the "Delta Team", namely Riccardo Adami, Rodolfo Figari, Domenico Finco and Sandro Teta, for many fruitful discussions. A special thanks to Rodolfo Figari and Andrea Mantile, who collaborated with me on part of this work.
I am also indebted to my former teacher, Giovanni Morchio, who first introduced me to the world of quantum mechanics and constantly ecouraged me.
Finally I want to thank Ludwik Dabrowski and the INTAS Research Project nr. 00-257 of European Community, "Spectral Problems for Schrödinger-Type Operators", for the financial support.

Ad essere onesto, devo ammettere che non avrei mai potuto scrivere questa tesi senza l'aiuto di tutti i miei amici. Per cominciare un grazie speciale a Javarnanda Balor per l'amicizia e i B movies, a Mimì per gli scacchi e la pazienza, al Cinghiale, a Fabbio, Guido, Paolo, Mmbuto, il Bestiame (Subcomandante, Dottore, Vacca, Carabiniere Motociclista, Interlocutrice ecc.) e tutti gli assidui frequentatori del Centro Sociale (Stanza 30) per aver ripetutamente tentato di sabotare il mio lavoro e per avere reso molto "interessante" la mia permanenza a Trieste...
Un grazie sentito anche ai miei vecchi amici (Berto, Cinese, Davide, Pai) per avermi sempre sostenuto (a modo loro...), a Giulia e Lorenzo (e alle loro mamme) per esserci, ai miei amici fisici Psycho e Giacomino per il supporto negli anni pisani, al Signore Anziano per la musica, a Vincent per Pulp, a Martini per la Scozia e a tutti i ragazzi di Busana e La Spezia...

Bibliography

[1] M. Abramovitz, I. A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1965.

[2] R. Adami, C. Bardos, F. Golse, A. Teta, Toward a Rigorous Derivation of the Cubic NLSE in Dimension One, preprint *mp-arc/03-347*, 2003.

[3] S.A. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden, *Solvable Models in Quantum Mechanics*, Springer-Verlag, New York, 1988.

[4] P.M. Koch, K.A.H. Van Leeuven, The Importance of Resonances in Microwave "Ionization" of Excited Hydrogen Atoms, *Physics Reports* **255**, 289-403, 1995.

[5] F.A. Berezin, L.D. Faddeev, A Remark on Schrödinger Equation with a Singular Potential, *Sov. Math. Dokl.* **2**, 372-375, 1961.

[6] H. Bethe, R. Peierls, Quantum Theory of Diplon, *Proc. Roy. Soc.* **148**A, 146-156, 1945.

[7] J.M. Blatt, V.F. Weisskopf, *Theoretical Nuclear Physics*, Springer-Verlag, New York, 1979.

[8] C. Cohen-Tannoudji, J. Duport-Roc, G. Arynberg, *Atom-Photon Interactions*, Wiley, 1992.

[9] M. Correggi, G.F. Dell'Antonio, Rotating Singular Perturbations of the Laplacian, *Ann. Henri Poincaré* **5**, 773-808, 2004.

[10] M. Correggi, G.F. Dell'Antonio, R. Figari, A. Mantile, Ionization for Three Dimensional Time-dependent Point Interactions, preprint *math-ph/0402011*, to appear in *Comm. Math. Phys.*, 2004.

[11] M. CORREGGI, G.F. DELL'ANTONIO, Decay of a Bound State under a Time-periodic Perturbation: a Toy Case, preprint *math-ph/0408003*, submitted to *J. Phys. A: Math. Gen.*, 2004.

[12] O. COSTIN, R.D. COSTIN, J.L. LEBOWITZ, A. ROKHLENKO, Evolution of a Model Quantum System under Time Periodic Forcing: Conditions for Complete Ionization, *Comm. Math. Phys.* **221**, no.1 1-26, 2001.

[13] O. COSTIN, J.L. LEBOWITZ, A. ROKHLENKO, Decay versus Survival of a Localized State Subjected to Harmonic Forcing: Exact Results, *J. Phys. A: Math. Gen.* **35**, 8943-8951, 2002.

[14] O. COSTIN, J.L. LEBOWITZ, A. ROKHLENKO, Exact Results for the Ionization of a Model Quantum System, *J. Phys. A: Math. Gen.* **33**, 6311-6319, 2000.

[15] O. COSTIN, R.D. COSTIN, J.L. LEBOWITZ, Transition to the Continuum of a Particle in Time-Periodic Potentials, in *Advances in Differential Equations and Mathematical Physics, Birmingham, AL, 2002*, 75-86, *Contemp. Math.* **327**, AMS, Providence, 2003.

[16] O. COSTIN, J.L. LEBOWITZ, A. ROKHLENKO, Ionization of a Model Atom: Exact Results and Connection with Experiment, preprint *physics/9905038*, 1999.

[17] O. COSTIN, A. SOFFER, Resonance Theory for Schrödinger Operators, *Comm. Math. Phys.* **224**, 133-152, 2001.

[18] G.F. DELL'ANTONIO, R. FIGARI, A. TETA, Schrödinger Equation with Moving Point Interactions in Three Dimensions, in *Stochastic Processes, Physics and Geometry: New Interplays, Leipzig, 1999*, 99-113, *CMS Conference Proceedings* **28**, AMS, Providence, 2000.

[19] G.F. DELL'ANTONIO, Point Interactions, in *Mathematical Physics in Mathematics and Physics, Siena, 2000*, 139-150, *Fields Institute Communications* **30**, AMS, Providence, 2001.

[20] G.F. DELL'ANTONIO, L. TENUTA, Semiclassical Analysis of Constrained Quantum Systems, *J. Phys. A: Math. Gen.* **37**, 5605-5624, 2004.

[21] G.F. DELL'ANTONIO, D. FINCO, A. TETA, Singularly Perturbed Hamiltonians of a Quantum Rayleigh Gas Defined as Quadratic Forms, preprint *mp-arc/03-233*, 2003.

[22] Y.N. DEMKOV, V.N. OSTROWSKII, *The Use of Zero-Range Potentials in Atomic Physics*, Nauke, Moscow, transl. Plenum Press, 1975.

[23] J.M. EBERLY, K.C. KULANDER, Atomic-Stabilization by Super-Intense Lasers, *Science* **262**, 1033, 1993.

[24] S.F. EDWARDS, Functional Problems in the Theory of Polymers, in *Functional Integration and Its Applications*, Clarendon Press, Oxford, 1975.

[25] V. ENSS, K. VESELIC, Bound States and Propagating States for Time-dependent Hamiltonians, *Ann. Inst. H. Poincaré A* **39**, 159-191, 1983.

[26] V. ENSS, V. KONSTRYKIN, R. SCHRADER, Energy Transfer In Scattering By Rotating Potential, in *Proceedings of the Workshop on Spectral and Inverse Spectral Problems for Schröedinger Operators, Goa, India*, 55-70, *Proceedings of the Indian Academic of Science, Mathematical Science* **112**, 2002.

[27] V. ENSS, V. KONSTRYKIN, R. SCHRADER, Perturbation Theory for the Quantum Time-Evolution in Rotating Potentials, in *Proceedings of the Conference QMath-8 "Mathematics Results in Quantum Mechanics", Taxco, Mexico*, 113-127, *Contemp. Math.* **307**, AMS, Providence, 2002.

[28] P. EXNER, Spectral Properties of Schrödinger Operators with a Strongly Attractive δ Interaction Supported by a Surface, in *Waves in Periodic and Random Media, South Hadley, MA, 2002*, 25-36, *Contemp. Math.* **339**, AMS, Providence, 2003.

[29] E. FERMI, Sul Moto dei Neutroni nelle Sostanze Idrogenate, *Ricerca Scientifica* **7**, 13-52, 1936.

[30] E. FERMI, *Notes on Quantum Mechanics*, The University of Chicago Press, Chicago, 1961.

[31] R. FIGARI, Time Dependent and Non Linear Point Interactions, in *Proceedings of Mathematical Physics and Stochastic Analysis, Lisbon, 1998*, 184-197, World Scientific Publisher, New York, 2000.

[32] A. FRING, V. KOSTRYKIN, A. SCHRADER, Ionization Probabilities through Ultra-Intense Fields in the Extreme Limit, *J. Phys. A* **30**, 8599-8610, 1997.

[33] S. GRAFFI, V. GRECCHI, H.J. SILVERSTONE, Resonances and Convergence of Perturbative Theory for N-body Atomic Systems in External AC-electric Field, *Ann. Inst. H. Poincaré A*, **42**, 215-234, 1985.

[34] J.S. HOWLAND, Stationary Scattering Theory for Time-dependent Hamiltonians, *Math. Ann.* **207**, 315-335, 1974.

[35] J.S. HOWLAND, Scattering Theory for Hamiltonians Periodic in Time, *Indiana Univ. Math. J.* **28** No.3, 471-494, 1979.

[36] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1976.

[37] R.L. KRONIG, W.G. PENNEY, Quantum Mechanics of Electrons in Crystal Lattices, *Proc. Roy. Soc.* **130**A, 499-513, 1931.

[38] L.D. LANDAU, E.M. LIFSCHITZ, *Quantum Mechanics - Nonrelativistic Theory*, Pergamon, New York, 1965.

[39] S.W. LOVESEY, *Theory of Neutron Scattering form Condensed Matter*, Clarendon Press, Oxford, 1984.

[40] A.F. NIKIFOROV, V.B. UVAROV, *Special Functions of Mathematical Physics*, Birkhäuser, Basel, 1988.

[41] N.K. NIKOL'SKII, *Treatise on the Shift Operator*, Springer-Verlag, New York, 1986.

[42] D. PORTER, D.S.G. STIRLING, *Integral Equations*, Cambridge University Press, Cambridge, 1990.

[43] A. POSILICANO, A Krein-like Formula for Singular Perturbation of Self-Adjoint Operators and Applications, *J. Funct. Anal.* **183**, 109-147, 2001.

[44] M. REED, B. SIMON, *Methods of Modern Mathematical Physics, Vol.I: Functional Analysis*, Academic Press, San Diego, 1975.

[45] M. REED, B. SIMON, *Methods of Modern Mathematical Physics, Vol.II: Self-Adjointness and Existence of Dynamics*, Academic Press, San Diego, 1975.

[46] M. REED, B. SIMON, *Methods of Modern Mathematical Physics, Vol.III: Scattering Theory*, Academic Press, San Diego, 1975.

[47] M. REED, B. SIMON, *Methods of Modern Mathematical Physics, Vol.IV: Analysis of Operators*, Academic Press, San Diego, 1975.

[48] M.R. SAYAPOVA, D.R. YAFAEV, The Evolution Operator for Time-dependent Potentials of Zero Radius, *Proc. Stek. Inst. Math.* **2**, 173-180, 1984.

[49] B. SIMON, Schrödinger Operators in the Twentieth Century, *Jour. Math. Phys.* **41**, 3523, 2000.

[50] A. SOFFER, M.I. WEINSTEIN, Nonautonomous Hamiltonians, *Journ. Stat. Phys.* 93, 359-391, 1998.

[51] A. TETA, *Singular Perturbations of the Laplacian and Connections with Models of Random Media*, Ph.D Thesis SISSA/ISAS, 1989.

[52] A. TETA, Quadratic Forms for Singular Perturbations of the Laplacian, *Publ. RIMS, Kyoto University*, **26**, 803-817, 1990.

[53] L.H. THOMAS, The Interaction between a Neutron and a Proton and the Structure of H^3, *Phys. Rev.* **47**, 903-909, 1935.

[54] A. TIP, Atoms in Circularly Polarized Fields: the Dilatation-Analytic Approach, *J. Phys. A: Math. Gen.* **16**, 3237-3259, 1983.

[55] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, University Press, Cambridge, 1944.

[56] D.R. YAFAEV, Scattering Theory for Time-dependent Zero-range Potentials, *Ann. Inst. H. Poincaré A* **40**, 343-359, 1984.

[57] K. YAJIMA, H. KITADA, Bound States and Scattering States for Time Periodic Hamiltonians, *Ann. Inst. H. Poincaré A* **39**, 145-157, 1983.