Investigating the Ultraviolet Properties of Gravity with a Wilsonian Renormalization Group Equation

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Abstract

We review and extend in several directions recent results on the asymptotic safety approach to quantum gravity. The central issue in this approach is the search of a Fixed Point having suitable properties, and the tool that is used is a type of Wilsonian renormalization group equation. This will be reviewed in chapter 2 after a general overview in the introductory chapter 1. Then we discuss various cutoff schemes, *i.e.* ways of implementing the Wilsonian cutoff procedure. We compare the beta functions of the gravitational couplings obtained with different schemes, studying first the contribution of matter fields and then the so–called Einstein–Hilbert truncation in chapter 3, where only the cosmological constant and Newton's constant are retained. In this context we make connection with old results, in particular we reproduce the results of the epsilon expansion and the perturbative one loop divergences. We discuss some possible phenomenological consequences leading to modified dispersion relations and show connections to phenomenological models where Lorentz invariance is either broken or deformed. We then apply the Renormalization Group to higher derivative gravity in chapter 4. In the case of a general action quadratic in curvature we recover, within certain approximations, the known asymptotic freedom of the four–derivative terms, while Newton's constant and the cosmological constant have a nontrivial fixed point. In the case of actions that are polynomials in the scalar curvature of degree up to eight we find that the theory has a fixed point with three UV–attractive directions, so that the requirement of having a continuum limit constrains the couplings to lie in a three–dimensional subspace, whose equation is explicitly given. We emphasize throughout the difference between scheme–dependent and scheme–independent results, and provide several examples of the fact that only dimensionless couplings can have "universal" behavior.

Collaborations

The research work presented in this thesis was carried out at SISSA-International School for Advanced Studies between November 2004 and November 2008. It is the result of the author's own work as well as the one in scientific collaborations if there is no reference to other sources. The content is based on the following research papers appeared in refereed journals or as preprints:

[1] F. Girelli, S. Liberati, R. Percacci and C. Rahmede, "Modified dispersion relations from the renormalization group of gravity," Class. Quant. Grav. **24** (2007) 3995 [arXiv:gr-qc/0607030].

[2] A. Codello, R. Percacci and C. Rahmede, "Ultraviolet properties of f(R)-gravity," Int. J. Mod. Phys. A **23** (2008) 143 [arXiv:0705.1769 [hep-th]].

[3] A. Codello, R. Percacci and C. Rahmede, "Investigating the Ultraviolet Properties of Gravity with a Wilsonian Renormalization Group Equation," arXiv:0805.2909 [hep-th]; accepted for publication in Annals of Physics.

After the first introductory section, in section 2 the Wilsonian Renormalization Group approach is reviewed. The original results from [1] are contained in section 3.10. Sections 2.3, 3, 4 contain original material from [2] and [3]. New material has been added in section 3.4 and an alternative version for the trace evaluation has been given in section A.7. The results are summarized in section 5.

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Trieste, 11/09/2008 **Christoph Rahmede**

Contents

1. Introduction

Of the four fundamental forces in nature, gravity was the first one described in mathematical terms by Newton and later essentially extended by Einstein to General Relativity (GR) describing successfully physics on macroscopic scales. The other three forces, electromagnetic, strong and weak, found their modern formulation in Quantum Field Theory (QFT) describing with precision the physics of the microcosmos and unifying all three of them into a common framework. The hope is that one might be able to unify all four forces in a single theory. This would however require the application of GR on microscopic scales where several severe problems occur limiting its range of validity.
In the microscopic regime, GR fails as a fundamental theory because it predicts spacetime singularities at the beginning of the universe evolving from a big bang as well as in the stellar collapse to black holes depending only on very general conditions for the energy-momentum tensor. This indicates a complete breakdown of GR, as then there do not exist any boundary conditions for the field equations at the singular points and the theory is incomplete as the continuation beyond these points is missing.
This singular behaviour happens however at scales where quantum effects should become important. The question is therefore if gravity can be coupled consistently to quantum matter while treating gravity classically, or if also gravity has to be quantized. The first option is taken in semiclassical gravity. The classical gravitational part is coupled to a source term given by the expectation value of the energy-momentum tensor of some quantum matter contribution. This formulation gives however again rise to several severe problems.
The expectation value will depend on some matter state which itself will depend on the metric. Due to the nonlinearity of the Einstein equations, the linear superposition of two matter states will in general not give a new solution, requiring therefore a modification of standard quantum mechanics. Inconsistencies seem to arise from the renormalization of the energy-momentum tensor. Classically equivalent theories give rise to different statements about renormalizability in the quantum treatment[1]. Even if no definitive conclusion has been obtained, it seems that such a treatment goes wrong and that consistency can only be obtained in quantizing the gravitational field as well.
How the definitive version of Quantum Gravity (QG) could look like, explaining the nature of singularities and the interactions between gravity and quantum matter, is so far an open question. Different research communities have different convictions of how much the theoretical frameworks have to be modified to obtain a fundamental theory[2]. The only way one can use so far to find this theory is physical intuition and general consid-

[1]See e.g. [4, 5] for further discussions on these issues.
[2]See e.g. [1, 2, 3] for further motivations for quantum gravity and introductions to the current approaches.

erations to try to construct a complete and consistent theory which has GR as a limit for macroscopic bodies and low curvatures of spacetime.
Experimental access to quantum gravitational effects is so far not possible as they are strongly suppressed. This is mainly due to the fundamental weakness of gravity. Comparing the gravitational attraction between two electrons to the Coulomb force between them, one sees that the gravitational attraction is by a factor 10^{41} weaker expressed by the size of Newton's gravitational constant. From dimensional analysis, one can argue on which scales quantum gravitational effects should become important. The constants of nature being important in that regime should be Newton's gravitational constant G, the Planck constant $\hbar$ for the contribution of quantum effects, and the speed of light c fundamental for causality. Out of these three constants one can form a length, a time, and a mass scale as

$$l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35}\mathrm{m} \tag{1.1}$$

$$t_p = \frac{l_p}{c} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.40 \times 10^{-44}\mathrm{s} \tag{1.2}$$

$$m_p = \frac{\hbar}{l_p c} = \sqrt{\frac{\hbar c}{G}} \approx 2.17 \times 10^{-8}\mathrm{kg} \approx 1.22 \times 10^{19}\mathrm{GeV} \tag{1.3}$$

which are called Planck length, time, and mass. In the units used here, where $\hbar = 1$ and $c = 1$, all are expressed by the mass scale simply referred to as the Planck scale. A further argument supports that this should be the relevant scale for quantum gravity effects. Comparing the Compton wavelength of a particle with mass m

$$\lambda_c = \frac{\hbar}{mc} , \tag{1.4}$$

which gives the wavelength for massive particles for which quantum effects are no longer negligible, to the Schwarzschild radius of an object with mass m,

$$R_S = \frac{2Gm}{c^2} , \tag{1.5}$$

which is the scale where curvature effects become strong and GR effects are no longer negligible, one sees that they equal for $m = M_p/\sqrt{2}$. As this is an extremely huge mass lying 15 orders of magnitude beyond the scales accessible in accelerator experiments, what physics is like beyond that scale is an open question.
Fortunately, it is not true that physicists are not at all able to calculate quantum gravitational corrections. In fact, general relativity can be treated very well as an effective quantum field theory [11, 12, 14]. This means that it is possible to compute quantum effects due to graviton loops as long as the momentum of the particles in the loops is cut off at some scale. For example, in this way it has been possible to unambiguously compute quantum corrections to the Newtonian potential [12]. The results are independent of the

structure of any "ultraviolet completion", and therefore constitute genuine low energy predictions of any quantum theory of gravity. When one tries to push this effective field theory to energy scales comparable to the Planck scale, or beyond, well-known difficulties appear.

It is convenient to distinguish two orders of problems. The first is that the strength of the gravitational coupling grows without bound. For a particle with energy p the effective strength of the gravitational coupling is measured by the dimensionless number $\sqrt{\tilde{G}}$, with $\tilde{G} = Gp^2$. This is because the gravitational couplings involve derivatives of the metric. The consequence of this is that if we let $p \to \infty$, also $\tilde{G}$ grows without bound. The second problem is the need of introducing new counterterms at each order of perturbation theory. Since each counterterm has to be fixed by an experiment, the ability of the theory to predict the outcome of experiments is severely limited.

To cure these problems, there are many different approaches. The attitude is either to postulate new gravitational degrees of freedom which have spacetime metrics only as a low-energy approximation like in string theory [6, 7] or emergent gravity(For recent proposals in this context see e.g. [8, **?**]), or that the calculus of QFT is not developed well-enough, making some more rigorous procedure necessary as in Loop Quantum Gravity [9, 10] or some discretized versions of gravity which make a numerical treatment possible [75, 76].

However, it could be possible that a QFT, taking the gravitational field as the carrier of the degrees of freedom seriously, got around these obstacles if one succeeded to tame the problems of the UV divergences in some way. These divergencies are related to the quantum fluctuations in a system. These can, within the uncertainty relation, in principle contain arbitrarily large contributions. A way to deal with them is the effective field theory approach where an average is taken over all quantum fluctuations. For gravity this works only as long as one sets a cutoff at the Planck scale and all terms from loop corrections are suppressed by the Planck scale. A method to see if one can extend a theory beyond a certain energy scale is given by Renormalization Group methods. One considers a theory with a cutoff at some scale and then tries to see what happens as soon as the cutoff is shifted to lower scales. So between old and new cutoff one takes into account only the average of the quantum fluctuations, they are integrated out. In this way one can estimate how much influence quantum fluctuations have on the system. The integral between old and new cutoff scales is limited and therefore finite. If one takes the step from one cutoff scale to the next one infinitesimally, one obtains a flow, the Renormalization Group (RG) flow, which will be determined by a set of differential equations, the beta functions. If the flow can be extended to infinite scales, the theory is renormalizable. Also Newton's constant, as any coupling constant in a QFT, must be subject to RG flow. It is conceivable that when $p \to \infty$, $G(p) \sim p^{-2}$, in which case $\tilde{G}$ would cease to grow and would reach a finite limit, thereby avoiding the first problem. If this is the case, we say that Newton's constant has an UV Fixed Point (FP). More generally, if we allow the action to contain several couplings g_i with canonical mass dimension d_i, we say that the theory has a FP if all the dimensionless parameters

$$\tilde{g}_i = g_i k^{-d_i} \tag{1.6}$$

tend to finite values in the UV limit [3]. This particular RG behaviour would therefore solve the first of the two problems mentioned above and would guarantee that the theory has a sensible UV limit.

In order to address the second problem we have to investigate the set of RG trajectories that have this good behaviour. We want to use the condition of having a good UV limit as a criterion for selecting a QFT of gravity. If all trajectories were attracted to the FP in the UV limit, we would encounter a variant of the second problem: the initial conditions for the RG flow would be arbitrary, so determining the RG trajectory of the real world would require in principle an infinite number of experiments and the theory would lose predictivity. At the other extreme, the theory would have maximal predictive power if there was a single trajectory ending at the FP in the UV. However, this may be too much to ask. An acceptable intermediate situation occurs when the trajectories ending at the FP in the UV are parametrized by a finite number of parameters. A theory with these properties is said to be "asymptotically safe" [15].

To better understand this property, imagine, in the spirit of effective field theories, a general QFT with all possible terms in the action which are allowed by the symmetries. We can parametrize the (generally infinite dimensional) "space of all theories", $\mathcal{Q}$, by the dimensionless couplings $\tilde{g}_i$. We assume that redundancies in the description of physics due to the freedom to perform field redefinitions have been eliminated, *i.e.* all couplings are "essential" (such couplings can be defined *e.g.* in terms of cross sections in scattering experiments). We then consider the RG flow in this space; it is given by the beta functions

$$\beta_i = k\frac{d\tilde{g}_i}{dk} \ . \tag{1.7}$$

If there is a FP, *i.e.* a point with coordinates $\tilde{g}_{i*}$ such that all $\beta_i(\tilde{g}_*) = 0$, we call $\mathcal{C}$ its "critical surface", defined as the locus of points that are attracted towards the FP when $k \to \infty$ [4]. One can determine the tangent space to the critical surface at the FP by studying the linearized flow

$$k\frac{d(\tilde{g}_i - \tilde{g}_{i*})}{dk} = B_{ij}(\tilde{g}_j - \tilde{g}_{j*}) \ , \tag{1.8}$$

where

$$B_{ij} = \frac{\partial \beta_i}{\partial \tilde{g}_j}\Big|_* . \tag{1.9}$$

[3]Strictly speaking only the essential couplings, *i.e.* those that cannot be eliminated by field redefinitions, need to reach a FP. See [13] for a related discussion in a gravitational context.

[4]RG transformations lead towards lower energies, and the trajectories lying in $\mathcal{C}$ are repelled by the FP under these transformations. For this reason, $\mathcal{C}$ is also called the "unstable manifold". Since we are interested in studying the UV limit, it is more convenient to study the flow for increasing k.

The attractivity properties of a FP are determined by the signs of the critical exponents ϑ_i, defined to be minus the eigenvalues of B. The couplings corresponding to negative eigenvalues (positive critical exponent) are called relevant and parametrize the UV critical surface; they are attracted towards the FP for $k \to \infty$ and can have arbitrary values. The ones that correspond to positive eigenvalues (negative critical exponents) are called irrelevant; they are repelled by the FP and must be set to zero.

A free theory (zero couplings) has vanishing beta functions, so the origin in $\mathcal{Q}$ is a FP, called the Gaußian FP. In the neighborhood of the Gaußian FP one can apply perturbation theory, and one can show that the critical exponents are then equal to the canonical dimensions ($\vartheta_i = d_i$), so the relevant couplings are the ones that are power–counting renormalizable [5]. In a local theory they are usually finite in number. Thus, a QFT is perturbatively renormalizable and asymptotically free if and only if the critical surface of the Gaußian FP is finite dimensional. Points outside $\mathcal{C}$ flow to infinity, or to other FPs.

A theory with these properties makes sense to arbitrarily high energies, because the couplings do not diverge in the UV, and is predictive, because all but a finite number of parameters are fixed by the condition of lying on $\mathcal{C}$. Asymptotic safety is a form of nonperturbative renormalizability. It generalizes this picture, replacing the Gaußian FP by an arbitrary FP. An asymptotically safe theory would have the same good properties of a renormalizable and asymptotically free one: the couplings would have a finite UV limit and the condition of lying on $\mathcal{C}$ would leave only a finite number of parameters to be determined by experiment. In general, studying the properties of such theories requires the use of nonperturbative tools. If the nontrivial FP is sufficiently close to the Gaußian one, its properties can also be studied in perturbation theory, but unlike in asymptotically free theories, the results of perturbation theory do not become better and better at higher energies.

For Newton's constant in particular, the beta function is given by

$$k\partial_k \tilde{G} = [d - 2 - \eta_N]\, G \tag{1.10}$$

with $\eta = k\partial_k \ln G$ being the anomalous dimension. One sees that a non-Gaußian fixed point can only exist for $\eta_N = d - 2$. The value of the anomalous dimension becomes therefore an essential ingredient for asymptotic safety. It leads to a corrected scaling behaviour of the propagator as $(p^2)^{-1+\eta_N/2}$ in momentum space or $\sqrt{x}^{2-d-\eta_N}$ in configuration space. For the fixed point to exist, the correction from the anomalous dimension has to be negative for $d > 2$, $\tilde{G}$ has to decrease at high energies leading to an antiscreening behaviour of gravity. In four dimensions, the requirement is $\eta_N = 2$, so the correction by the anomalous dimension is large and perturbation theory is not applicable.

In two dimensions instead, a Gaußian fixed point is obtained for $\eta_N = 0$, in its vicinity perturbation theory can be applied. That is why for first studies of asymptotic safety in gravity, several authors [15, 16, 17] applied the ϵ expansion around two dimensions which is the critical dimension where Newton's constant is dimensionless. The beta function of

[5] The behavior of dimensionless or marginal couplings may require a more sophisticated analysis.

Newton's constant then has the form

$$\beta_{\tilde{G}} = \epsilon\tilde{G} + B_1\tilde{G}^2 \,, \tag{1.11}$$

where $\tilde{G} = Gk^{-\epsilon}$ and $B_1 < 0$ [15, 16], so there is a FP at $\tilde{G} = -\epsilon/B_1 > 0$. Unfortunately this result is only reliable for small ϵ and it is not clear whether it will extend to four dimensions.

In the effective field theory approach (in $d = 4$), Bjerrum-Bohr, Donoghue and Holstein have proposed interpreting a class of one loop diagrams as giving the scale dependence of Newton's constant [14]. They calculate

$$G(r) = G_0 \left[1 - \frac{167}{30\pi}\frac{G_0}{r^2}\right] \,,$$

where r is the distance between two gravitating point particles. If we identify $k = 1/ar$, with a a constant of order one, this corresponds to a beta function

$$\beta_{\tilde{G}} = 2\tilde{G} - a^2\frac{167}{15\pi}\tilde{G}^2 \,. \tag{1.12}$$

This beta function has the same form as (1.11) in four dimensions, and, most important, the second term is again negative. This means that the dimensionful Newton constant G *decreases* towards lower distances or higher energies, *i.e.* gravity is *antiscreening*. This is the behavior that is necessary for a FP to exist. And indeed this beta function predicts a FP for $\tilde{G} = 30\pi/167a^2$. This calculation was based on perturbative methods and since the FP occurs at a not very small value of $\tilde{G}$, it is not clear that one can trust the result. What we can say with confidence is that the onset of the running of G has the right sign for asymptotic safety. Clearly in order to make progress on this issue we need different tools.

The approach to quantum gravity described here starts from an EFT picture trying to establish that the theory is not merely effective but applicable also to arbitrarily large UV scales due to the asymptotic safety behaviour. What is gained compared to the pure EFT approach is not so much the large energy range - already EFT is valid till the Planck scale. The advantage of a fully consistent theory is rather to get control over possible quantum gravity effects arising at the highest energies and being propagated from high to low scales by renormalization group trajectories. The question what are the truly fundamental degrees of freedom becomes then secondary. If a consistent picture based on metrical degrees of freedom exists, no quantum gravitational effects should escape by tracing their effects to lower energies where theories based on different degrees of freedom have to match by the correspondence principle.

In this thesis, the application of Wilsonian renormalization group methods to the UV behaviour of gravity will be discussed. In section 2, first the Wilsonian RG approach will be presented in section 2.1 mainly based on the review articles [15, 65, 66, 67], in section 2.2 a particularly convenient tool will be introduced called the "Exact Renormalization

Group Equation" (ERGE) which can be used to calculate the "beta functional" of a QFT. Renormalizability is not necessary and the theory may have infinitely many couplings. In section 2.3 we illustrate the use of the ERGE by calculating the contribution of minimally coupled matter fields to the gravitational beta functions. In this simple setting, the techniques that are used to extract from the full beta functional the beta functions of individual couplings will be reviewed, emphasizing those results that are "scheme independent" in the sense that they are the same irrespective of technical details of the calculation. In section 3 the same techniques are applied to the calculation of the beta functions for the cosmological constant and Newton's constant in Einstein's theory in arbitrary dimensions, extending in various ways the results of earlier studies [18, 19, 20, 21, 22]. It is also shown that the FP that is found in four–dimensional gravity is indeed the continuation for $\epsilon \to 2$ of the FP that is found in the $2 + \epsilon$ expansion. Various ways of defining the Wilsonian cutoff are compared and find the results to be qualitatively stable. In sections 3.9 and 4.1 connection with old results from perturbation theory is made. In section 3.9 the 't Hooft–Veltman one loop divergence in 4d is rederived from the ERGE and it is shown to be scheme–independent. It is also discussed why the Goroff–Sagnotti two loop divergence cannot be seen with this method and the significance of this fact. In section 4 higher derivative gravity is considered. In section 4.1 the existence of the FP is derived in the most general truncation involving four derivatives at one loop, and we highlight the differences between the Wilsonian procedure as in [23] and earlier calculations. In section 4.2 higher powers of curvature are considered but restricted to polynomials in the scalar curvature based on calculations in [24] and [25]. In section 5 the present status of the asymptotic safety approach to quantum gravity is assessed and the various open problems are discussed.

2. General techniques

2.1. Kadanoff-Wilson renormalization

In Quantum Field Theory, a distinctive role is played by those actions which are renormalizable. In such theories, divergences appear only in a few parameters which lead to a shift between bare and renormalized masses and coupling constants and can be expressed by the addition of a number of counterterms to the action. These values depend on the choice of the renormalization scale. They can be fixed at one renormalization scale so that apart from a few couplings all of them can be set to zero. Changing the scale will however bring all other terms in the action into play which are consistent with the symmetry principles of the theory. Their behaviour under renormalization scale changes can be studied with Renormalization Group (RG) methods. One finds however that non-renormalizable terms are suppressed with respect to the renormalizable ones explaining therefore the importance of renormalizable action in the description of physical phenomena. From this point of view, a theory is not defined by a specific form of action, but instead by its field content and its symmetry principles. That means that an action will in principal contain an infinite amount of interaction terms.
To handle such a presumably more complicated theory one performs the following steps. First one distinguishes between fast and slow field modes. The functional integral is performed piecewise such that fast modes are integrated out and slow modes are retained while the values of observables remain fixed. This process is called "coarse graining" and defines a flow in the space of action functionals depending on the way the "coarse graining" is performed, see section 2.1.1.
Practically one starts from some initial form of action including all possible interaction terms. The RG flow of the coupling constants of the interaction terms will determine which of the couplings are relevant (or essential) and which ones are irrelevant (or inessential) (section 2.1.2). If there exists a fixed point of the RG flow, the flow determines which couplings are relevant near the fixed point (section 2.1.3). The number of relevant couplings at the fixed point gives the degree of renormalizability (section 2.1.4). This section is essentially based on the review articles [15, 65, 66, 67].

2.1.1. The coarse graining procedure

The reason for performing the functional integral step after step is inherited from statistical physics and critical phenomena. These occur where fluctuations of the dynamical variables over a large (and possibly diverging) amount of length scales have to be taken into account. The coarse graining serves to break up the problem into many different

steps where at each coarse graining step only fluctuations in a narrow range have to be taken into account. As an example, a macroscopic observable $\langle \mathcal{O} \rangle$ which results from a functional average over a function $\mathcal{O}$ of the fields χ might be sensitive to the microscopic field fluctuations over a large range of scales.
The coarse graining is then done by introducing a kernel $K(\chi', \chi) = K_{k,\delta k}(\chi', \chi)$ with support on momenta $k - \delta k \leq p \leq k$ which is therefore limited to a certain momentum range δk and which is normalized to $\int \Pi_{p\leq\Lambda} d\chi_p K(\chi', \chi) = 1$. With this kernel one can integrate out field configurations with momentum modes below a UV scale Λ by performing

$$\begin{aligned} \langle \mathcal{O} \rangle &= \int \Pi_{p\leq\Lambda} d\chi_p \mathcal{O}(\chi) e^{-S[\chi]} = \int \Pi_{p\leq\Lambda-\delta k}\, d\chi'_p \int \Pi_{p\leq\Lambda} d\chi_p K_{\Lambda,\delta k}(\chi, \chi') \mathcal{O}(\chi) e^{-S[\chi]} \\ &= \int \Pi_{p\leq\Lambda-\delta k} d\chi'_p \mathcal{O}'(\chi') e^{-S'[\chi']} \end{aligned} \tag{2.1}$$

where the outer integration goes only till $p \leq \Lambda - \delta k$ and the last integration step is performed separately with the help of the kernel which is restricted to the last momentum step. In the final step the variables have been renamed so that the same expression is obtained with a different form of operator, action and field. This procedure gives the operator, action, and field at the scale $\Lambda - \delta k$ instead of Λ. Specifying $\langle \mathcal{O} \rangle = \mathbf{1}$, S' and $\mathcal{O}'$ are defined from the coarse graining procedure. One sees that at each scale of coarse graining only field configurations close to the coarse graining scale will contribute. The evaluation of the functional integral at each step should thus become easier.
One can then continue the coarse graining procedure to lower and lower resolution scales where at each step a further small amount of the degrees of freedom is integrated out. Thus at each coarse graining step the functional measure is modified as

$$d\mu_k[\chi] = \Pi_p d\chi_p e^{-S_k[\chi]} \,. \tag{2.2}$$

From this relation the modification of the measure at each step can be translated into a modification of the action at each step. One thus obtains a flow in the actions. For the specific form

$$K_{k,\delta k} = \delta(\chi, \chi') - \delta k \partial_{\delta k} K_{k,0}(\chi, \chi') + O((\delta k)^2)) \tag{2.3}$$

and $\mathcal{O} = 1$ one obtains the flow equations

$$\partial_k e^{-S_k[\chi]} = \int \Pi_{p\leq\Lambda} d\chi'_p \partial_{\delta_k} K_{k,0}(\chi, \chi') e^{-S_k[\chi']} \tag{2.4}$$

with the initial condition

$$\Pi_{p\leq\Lambda} d\chi'_p \partial_{\delta_k} K_{k,0}(\chi, \chi') = 0 \,. \tag{2.5}$$

For a specific form of kernel and initial action these flow equations, known as *functional renormalization group equations* of Wilsonian type, will determine the flow of actions $S_k[\chi]$. Results obtained in this way will of course depend on the form of the kernel. We will use here a concept differing a bit from the original approach and described in section 2.2. We do not specify a kernel for each integration step, but rather a scale-dependent mode-suppression function which is added to the action and is chosen in such a way that momentum modes χ_p with $p^2 \ll k^2$ are suppressed in the functional integral whereas the modes with $p^2 \gg k^2$ are left unsuppressed and therefore integrated out.

2.1.2. Relevant and irrelevant couplings

In general, the action will change very much from one coarse-graining step to the other. Therefore, the best strategy is to start from a scale dependent action functional Γ_k[1] which includes all possible (or at least as many as possible) interaction monomials which are compatible with the symmetry principles of the proposed theory

$$\Gamma_k(\phi_A, g_i) = \sum_{n=0}^{\infty} \sum_i g_i(k) \mathcal{O}_i\left(\phi_A\right) \tag{2.6}$$

which is a functional on $\mathcal{F} \times \mathcal{Q} \times R^+$ where $\mathcal{F}$ is the field configuration space, $\mathcal{Q}$ an infinite dimensional manifold spanned by all the coupling constants called coupling space, and R^+ is the positive real line parametrized by the RG scale k.
Dimensional analysis tells that this functional will be invariant under the rescaling

$$\Gamma_k(\phi_A, g_i) = \Gamma_{bk}(b^{d_A}\phi_A, b^{d_i} g_i) \tag{2.7}$$

where $b \in R^+$ and d_A is the dimension of the field A and d_i the dimension of the coupling g_i. As all the k-dependence is carried by the coupling constant, one has

$$\partial_t \Gamma_k(\phi_A, g_i) = \sum_{n=0}^{\infty} \sum_i \beta_i(k) \mathcal{O}_i\left(\phi\right) \ . \tag{2.8}$$

where the flow of the different couplings will be determined by the beta functions,

$$k\frac{d\tilde{g}_i}{dk} = \beta_i(\tilde{g}) \ . \tag{2.9}$$

and $t = \log k/k_0$ is the RG time. Each physically possible theory will be characterized by a trajectory in $\mathcal{Q}$, as obtained as a solution of the beta functions with some appropriate set of initial values. The beta functions can be calculated as a power series in the couplings, but only in a regime where higher order terms can be neglected. This is not necessarily the case in the UV.
The couplings g_i will in general be dimensionful and therefore also include power-counting nonrenormalizable ones. They will be subject to RG flow and depend on the coarse-graining scale k, the scale Λ where the bare action is defined, and a set of initial values for the couplings. Each coupling can be measured in units of the cutoff scale k so that one

[1] At some fixed UV scale Λ the functional will correspond to the bare action.

obtains the dimensionless quantities

$$\tilde{g}_i(k) = k^{-d_i} g_i(k) \ . \tag{2.10}$$

Rescaling as well the fields as

$$\tilde{\phi}_A = k^{-d_A} \phi_i(k) \tag{2.11}$$

and choosing $b = k^{-1}$ one obtains the dimensionless functional $\tilde{\Gamma}_k$ on $(\mathcal{F} \times \mathcal{Q} \times R^+)/R^+$ as

$$\tilde{\Gamma}(\tilde{\chi}_A, \tilde{g}_i) := \tilde{\Gamma}_1(\tilde{\chi}_A, \tilde{g}_i) = \Gamma_k(\chi_A, g_i) \sum_{n=0}^{\infty} \sum_i g_i(k) \mathcal{O}_i(\phi) \ . \tag{2.12}$$

Also the beta functions can be made dimensionless by rescaling

$$\beta_i(g_j, k) = k^{d_i} a_i(\tilde{g}_j) \tag{2.13}$$

where $a_i(g_j) = \beta_i(\tilde{g}_j, 1)$ so that

$$\tilde{\beta}_i(\tilde{g}_j, k) = \partial_t \tilde{g}_i = a_i(\tilde{g}_j) - d_i \tilde{g}_i \tag{2.14}$$

which is dimensionless and therefore does not depend on k explicitly any more, but only implicitly via $\tilde{g}_j(k)$.
In the path integral, the ϕ_A are the integration variables and a redefinition of them does not change the physical content of the theory which is expressed by the invariance of $\mathcal{F}$ under the group G of coordinate transformations in $\mathcal{F}$, the diffeomorphism group. A similar arbitrariness exists for the choice of coordinates on $\mathcal{Q}$ as the freedom to redefine the couplings.
If Γ_k is the most general functional on $\mathcal{F} \times \mathcal{Q}$, there exists a field redefinition $\phi' = \phi'(\phi)$ and a set g'_i of couplings with

$$\Gamma_k(\phi'_A(\phi_A), g_i) = \Gamma_k(\phi_A, g'_i) \tag{2.15}$$

defining an action of G on $\mathcal{Q}$. If one chooses a coordinate system adapted to these transformations, one can find a subset $g_{\hat{i}}$ of all couplings g_i which transforms nontrivially and can be used as coordinates for the orbits of G, and a subset $g_{\bar{i}}$ which is invariant under the action of G and therefore defining coordinates on $\mathcal{Q}$/G. The couplings from the first set are called *irrelevant or inessential or redundant* whereas the latter ones are called *relevant or essential or nonredundant*. This terminology comes from the fact that there exist locally field redefinitions $\bar{\phi}(\phi)$ giving fixed values $(g_{\hat{i}})_0$ to the inessential couplings from eq. (2.15) so that the new action

$$\bar{\Gamma}(\bar{\phi}_A, g_{\bar{i}}) := \Gamma_k(\bar{\phi}_A, g_{\bar{i}}, g_{\hat{i}}|_0) = \Gamma_k(\phi_A, g_{\bar{i}}, g_{\hat{i}}) \tag{2.16}$$

depends only on the essential couplings. Inessential couplings are not constrained to flow towards the fixed point. The part of the coupling space $\mathcal{Q}$ spanned by the essential

couplings is called the *unstable manifold*, the one spanned by the inessential couplings the *stable manifold*. The latter one is usually infinite dimensional and corresponds to the interaction monomials dying out under the coarse graining transformations. The dimension of the unstable manifold gives the degree of renormalizability as explained below.
The distinction between the two species of couplings can be illustrated with a scalar field in a gravitational background $g_{\mu\nu}$,

$$\Gamma_k(g_{\mu\nu}, Z_\phi, \lambda_{2i}) = \int d^4x\sqrt{g}\left[\frac{Z_\phi}{2}g_{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \lambda_2\phi^2 + \lambda_4\phi^4 + \dots\right] \tag{2.17}$$

which has the scaling invariance

$$\Gamma_k(c\phi, g_{\mu\nu}, c^{-2}Z_\phi, c^{-2i}\lambda_{2i}) = \Gamma_k(\phi, g_{\mu\nu}, Z_\phi, \lambda_{2i}) \ . \tag{2.18}$$

So there exists a parametrization where the wave function renormalization Z_ϕ is inessential whereas the couplings $\bar{\lambda}_{2i} = \lambda_{2i}Z_\phi^{-i}$ are essential. Choosing $c = \sqrt{Z_\phi\phi}$ leads to $Z_\phi = 1$ and the essential couplings are unaffected. More complicated, nonlinear field redefinitions are possible. The only restriction is that the field redefinitions have to be consistent with the symmetry principles. In general, there could be an infinite number of essential as well as inessential couplings.
Comparing equations (2.15) and (2.16) one sees that also the RG scale k can be regarded as an inessential coupling. The invariance of k and ϕ under rescalings can then be used to eliminate k and one other inessential coupling for each field multiplet, for which one chooses usually Z_{ϕ_A}. For the RG flow of the inessential couplings no fixed point conditions have to be imposed. Their flow is given by the so-called anomalous dimension $\eta_A = \partial_t \log Z_{\phi_A}$. The dimensionless effective action $\tilde{\Gamma}(\tilde{\phi}_A, \tilde{g}_{\bar{i}})$ will then also only depend on the essential couplings which have to reach the fixed point.
This can be reformulated in the following way. A coupling will be called irrelevant if the change of the couplings can be absorbed in the bare Lagrangian by a mere field redefinition. Under the change of any unrenormalized coupling parameter g_0 by an infinitesimal amount ϵ the Lagrangian changes by

$$\mathcal{L} \to \mathcal{L} + \epsilon\frac{\partial\mathcal{L}}{\partial g_0}. \tag{2.19}$$

A field redefinition of the kind $\phi_A(x) \to \phi_A(x) + \epsilon F_A(\phi_A(x), \partial_\mu\phi_A(x), \dots)$ will change the Lagrangian by the amount

$$\begin{aligned}\delta\mathcal{L} &= \epsilon\sum_A\left[\frac{\partial\mathcal{L}}{\partial\phi_A}F_A + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\partial_\mu F_A + \dots\right] \\ &= \epsilon\sum_A\left[\frac{\partial\mathcal{L}}{\partial\phi_A} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)} + \dots\right]F_A + \text{total derivative terms} \ . \end{aligned} \tag{2.20}$$

That means we can cancel the effect from the changes in the couplings if there exist F_n such that

$$\frac{\partial \mathcal{L}}{\partial g_0} = \epsilon \sum_A \left[\frac{\partial \mathcal{L}}{\partial \phi_A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} + \dots \right] F_A + \text{total derivative terms.} \tag{2.21}$$

Thus a coupling is inessential if and only if $\frac{\partial \mathcal{L}}{\partial g_0} = 0$, or, as $F_n \neq 0$ in general, is a total derivative when the Euler-Lagrange equations are used.
For gravity, one uses a derivative expansion of the form

$$\Gamma_k(\phi, g_i) = \sum_{n=0}^{\infty} \sum_i g_i^{(n)}(k) \mathcal{O}_i^{(2n)}(\phi) \ , \tag{2.22}$$

where $\mathcal{O}_i^{(2n)} = \int d^dx \sqrt{g} M_i^{(2n)}$ with $M_i^{(2n)}$ being monomials in the curvature tensor and its derivatives where each term contains $2n$ derivatives of the metric. The index i labels different operators with the same number of derivatives. The couplings $g_i^{(n)}$ have dimension $d_n = d - 2n$. For the first monomial with zero derivatives exists only one possibility $M^{(0)} = 1$ and $g^{(0)} = 2Z_g\Lambda$ with the cosmological constant Λ and the Newton constant G gives $Z_g = 1/(16\pi G)$. Also for the term containing two derivatives there is only one possibility, $M^{(1)} = R$, the Ricci scalar, and $g^{(1)} = -Z_g$. For the four derivative terms there are, neglecting the total derivative terms of the Ricci scalar and the Gauss-Bonnet term, two possibilities, $M_1^{(2)} = C^2$, the Weyl tensor squared, and $M_2^{(2)} = R^2$, the Ricci scalar squared. These terms can be reexpressed as a combination of Ricci scalar and Ricci tensor or Riemann tensor squared terms, one of them being eliminated by the Gauss-Bonnet identity. As in Yang-Mills theories, the coupling constants of the higher-derivative terms are defined to be the inverses of the coupling coefficients, $2\lambda = (g_1^{(2)})^{-1}$ and $\xi = (g_2^{(2)})^{-1}$. The coupling corresponding to the wave function renormalization is Z_g and can be eliminated by constant rescalings of $g_{\mu\nu}$,

$$\Gamma_k(g_{\mu\nu}, g_i^{(n)}) = \Gamma_{bk}(b^{-2} g_{\mu\nu}, b^{d-2n} g_i^{(n)}) \ . \tag{2.23}$$

At this point, one sees that gravity has the particularity that the two invariances under field reparametrizations and dimensional analysis are the same. So the RG scale k and Z_g cannot be eliminated at the same time, one of them has to be retained. This fixes the choice of units. The statements that we will make about dimensionful quantities will only be valid for a specific choice of units. The translation from one system of units to another might be cumbersome. In the next chapters, we will retain Z_g and eliminate k, but instead one could also fix $Z_g = 1$ and retain k thus working in Planck units.

2.1.3. Fixed point and ultraviolet critical surface

The scaling behaviour of the couplings will influence experimentally measurable quantities such as any sort of partial or total reaction rates $\mathcal{R}$. If they have a mass dimension D

they will scale as the cutoff scale,

$$\mathcal{R} = k^D f(\frac{E}{k}, X, \tilde{g}(k)) \tag{2.24}$$

where E is the energy characterizing the process, X are all other dimensionless variables like ratios of energies, angles et cetera. In the case of a cross section one will have $D = -2$. As $\mathcal{R}$ cannot depend on the arbitrary choice of the renormalization scale k where the couplings are defined one can choose $k = E$ and obtains therefore a scaling with respect to the energy,

$$\mathcal{R} = E^D f(1, X, \tilde{g}(E)) \tag{2.25}$$

whose UV behaviour will be fully determined by the UV behaviour of the couplings. For the on-shell case, this behaviour will depend only on the relevant couplings as on-shell the reaction rates do not depend on the field definitions. Off-shell instead the rates will also depend on the inessential coupling parameters. So the fundamental question is if the couplings have a well-defined behaviour up to the UV-range so that also observables like reaction rates remain well-defined. This translates to the condition that the beta functions should be well-behaved without poles or discontinuities. These will be absent if the beta functions approach a fixed point in the UV.
A point g^* in coupling space where the beta functions vanish, $\beta_i(g^*) = 0$, is a *fixed point* of the RG flow. It is called a *UV fixed point* if $\lim_{t\to\infty} g_i(t) = g_i^*$. Those couplings which lie on a trajectory $\tilde{g}_i(k)$ hitting the UV fixed point form a surface in the space of all couplings which is called the *UV critical surface*.
Such a fixed point will however not be approached by the inessential parameters. The RG equations cannot change their form when each field is multiplied by an independent constant. So if the inessential coupling parameters $Z_r(k)$ fulfill the RG equations, so they must do also after multiplication by some arbitrary constants. Therefore, their beta functions must be linear in $Z_r(k)$,

$$k\frac{dZ_r}{dk} = Z_r \gamma_r(\tilde{g}) \tag{2.26}$$

and γ_r can only depend on the essential coupling parameters. So, if γ_r does not vanish or diverge, the solution of this equation will be $Z_r(k) \propto k^{\gamma_r(g^*)}$. This will give corrections to the scaling of off-shell Green's-functions, but not to reaction rates.
Usually, a theory has at least one fixed point, the one where all coupling constants vanish, $g_i = 0$, so that also $\beta_i = 0$ as it is a function of all other coupling constants only. Then all interactions are going to zero, only the kinetic term is left in the Lagrangian, and all loop corrections vanish . This fixed point is called the *Gaußian fixed point*. In its vicinity, perturbation theory can be applied because the couplings are small. This is not the case at fixed points with non-zero couplings which are called *non–Gaußian fixed points*. The position of the fixed point depends on the kind of coarse-graining operation chosen.

2.1.4. Degree of renormalizability

The dimension of the stable and the unstable manifold and therefore of the UV critical surface will determine the degree of renormalizability. If the dimension of the UV critical surface is infinite, one has the same problem as in perturbation theory and nothing new is learned. Neither can the theory be asymptotically safe if the dimension of the UV critical surface is zero as one then has only irrelevant couplings and reaction rates will not depend on these parameters. So the hope is that one will end up with some finite number C so that one has $C-1$ free dimensionless parameters plus the RG scale k. Ideally C would be one, so that all couplings would be determined from the RG scale telling when which point on the RG trajectories is reached.
The dimension of the stable and the unstable manifold and the UV critical surface can be determined from the dimension of the corresponding tangent spaces at the fixed point which is obtained from linearization around the fixed point

$$\beta_i(k) = B_{ij}\left(\tilde{g}_j(k) - \tilde{g}_j^*\right) \, , \, B_{ij} = \frac{\partial \beta_i(\tilde{g})}{\partial \tilde{g}_j} \|_{\tilde{g}=\tilde{g}^*} \tag{2.27}$$

with solution

$$\tilde{g}_i(k) = \tilde{g}_i^* + C_k V_i^k k^{-\vartheta_k}$$

where C_k are some initial constants, V_i^k the eigenvectors, and $-\vartheta_k$ the eigenvalues of the stability matrix B_{ij}. The stability matrix can be degenerate and nonsymmetric so that the eigenvectors do not necessarily span the tangent space at the fixed point and the eigenvalues can be complex. The stable manifold is spanned by the eigenvectors with $\mathrm{Re}\ \vartheta_k > 0$ so that $|g_i(t) - g_i^*|$ decreases exponentially, and the unstable manifold is spanned by those with $\mathrm{Re}\ \vartheta_k < 0$ so that $|g_i(t) - g_i^*|$ grows exponentially in t. The couplings giving rise to $\mathrm{Re}\ \vartheta_k = 0$ are called marginal and cannot be decided to belong to either of the manifolds in the linearized approximation.
From $\tilde{g}_i(k) = k^{-d_i} g_i(k)$ one has

$$\beta_i = k\partial_k \tilde{g}_i = k^{-d_i}(-d_i g_i + k\partial_k g_i) = -d_i \tilde{g}_i + k^{-d_i+1}\partial_k g_i \tag{2.28}$$

and

$$B_{ij} = \frac{\partial \beta_i}{\partial \tilde{g}_j} = -d_i + k^{-d_i+1}\partial_k \frac{\partial g_i}{\partial \tilde{g}_j} \, . \tag{2.29}$$

Here the first term in both equations comes from the canonical coupling dimension whereas the second term, depending on higher powers of couplings, corresponds to all possible loop contributions. These loop corrections vanish at the Gaußian fixed point as all interactions go to zero when approaching it. Therefore at the Gaußian fixed point the couplings scale just according to their mass dimension. Interactions with more derivatives or more powers of fields will lower the dimension d_i of the respective coupling. Therefore only a finite number of d_i can be positive, and all but a finite number have dimension lower than any given negative value. So one can expect that the UV critical surface will

be finite dimensional.
When the stability matrix is degenerated, one has to be careful. At the Gaußian fixed point, this will be the case for the dimensionless couplings with $d_i = 0$. For these, the Gaußian fixed point will be approached in the UV only if $\frac{\beta_i}{\tilde{g}_i} < 0$ near $\tilde{g}_i = 0$. This means that the UV critical surface of the Gaußian fixed point corresponds to those theories which are renormalizable and asymptotically free. At a non-Gaußian fixed point the loop contributions will be important as they can change the sign of the eigenvalues compared to the value at the Gaußian fixed point, and this can lead to a different number of relevant parameters. At the non-Gaußian fixed point however perturbation theory will not be valid.
In general, trajectories not contained in the stable manifold will approach the fixed point in the ultraviolet before being driven away from it. How close they will come is determined by the initial value, finetuning can make the trajectories pass arbitrarily close to the fixed point. If they do come close to the fixed point, they will be for some time (nearly) part of the stable manifold before emanating from the fixed point. These trajectories are called renormalized.
One therefore comes to the two the definition of nonperturbative renormalizability and asymptotic safety:

- There exists a UV fixed point
- Its unstable manifold is finite-dimensional.

The so defined coupling-trajectories are independent of any ultraviolet cutoff-scale and independent of all irrelevant couplings (which are determined from the finite set of relevant couplings). The requirement for asymptotic safety can also be formulated in the following way. For the space $\tilde{\mathcal{Q}} = (\mathcal{Q} \times R^+)/(G \times R^+)$ of essential couplings the set $\mathcal{C}$ of all points in $\tilde{\mathcal{Q}}$ flowing towards a fixed point in the UV forms the UV critical surface. For an initial point on $\mathcal{C}$ the whole trajectory will remain on $\mathcal{C}$ and flow toward the UV fixed point. Points outside $\mathcal{C}$ flow either to infinity or to other fixed points. Trajectories lying in $\mathcal{C}$ certify a sensible UV limit. If the dimension of $\mathcal{C}$ is finite, there will only be a finite number of free parameters.
An asymptotically safe theory does not show unphysical singularities in the UV. The contrary is not necessarily true. A non-asymptotically safe theory does not necessarily show singularities. We will give an example of a non-asymptotically safe theory from [15]. Consider the RG equations of the form

$$k\partial_k \tilde{g}_i = a_i \sum_j (\tilde{g}_j - \tilde{g}_j^*)^2 \tag{2.30}$$

where a_i and $\tilde{g}_j^*$ are arbitrary constants. If a fixed point should be approached in the UV, it should be possible to parameterize the couplings as a function ξ being subtracted from the fixed point value and vanishing for $k \to \infty$, so that

$$\tilde{g}_i = \tilde{g}_i^* - a_i \xi \ . \tag{2.31}$$

This leads to a differential equation for ξ which can be solved to give

$$\xi = \frac{\xi_0}{1 + \xi_0 \ln \frac{k}{k_0} \sum_i a_i^2} \tag{2.32}$$

with initial values ξ_0 and k_0 being greater than zero. So $\xi \to 0$ for $k \to \infty$ and the theory is asymptotically safe. If instead $\tilde{g}_i$ does not lie on the line parameterized by ξ, for $\tilde{g}_i \to \infty$ the solution becomes

$$\tilde{g}_i \to a_i \left[- \sum_j a_j^2 \ln(k/k_\infty) \right]^{-1} \tag{2.33}$$

and $\tilde{g}_i \to \infty$ for $k \to k_\infty$, so trajectories not included in the UV critical surface parameterized by the line will lead to singularities at the energy scale $E = k_\infty$.
In this discussion one also has to certify that a redefinition of the couplings does not lead to problems. Defining for example $\tilde{g}'_i = (\tilde{g}_i - \tilde{g}^*_i)^{-1}$ will make R finite for $\tilde{g}_i$ being infinite. This problem can be avoided by defining the coupling constants as the coefficients in a power-series expansion of the reaction rates around some physical renormalization point.
This rises also the question if a redefinition of the couplings will not spoil the fixed point properties as the beta functions as well as the stability matrix depend on the definition of the couplings whereas the eigenvalues of the stability matrix do not. Suppose that a set of new couplings $\bar{g}$ is related to the old set $\tilde{g}$ by

$$\bar{g}_i(\bar{k}) = \bar{g}_i(\frac{\bar{k}}{k}, \tilde{g}_i(k)) . \tag{2.34}$$

As the new couplings cannot depend on the definition of the old couplings, one has

$$0 = k\frac{d\bar{g}_i}{dk} = -\bar{k}\frac{\partial \bar{g}_i}{\partial \bar{k}} + \sum \frac{\partial \bar{g}_i}{\partial \tilde{g}_j} k \frac{d\tilde{g}_j}{dk} \tag{2.35}$$

and one can define

$$\bar{\beta}_i(\bar{g}(\bar{k})) = \bar{k}\frac{\partial \bar{g}_i}{\partial \bar{k}} = \sum_j \frac{\partial \bar{g}_i}{\partial \tilde{g}_j} \beta_j(\tilde{g}) . \tag{2.36}$$

That shows that if $\beta_i = 0$ also $\bar{\beta}_i = 0$ and the fixed point remains. The beta functions and their derivatives, and therefore the stability matrix, do not stay invariant. For the stability matrix one has the transformation

$$\bar{B}_{ij} = \sum_{kl} A_{ik} B_{kl} A^{-1}_{lj} \tag{2.37}$$

with

$$A_{ik} = \frac{\partial \bar{g}_i}{\partial \tilde{g}_k} \|_{\tilde{g}=\tilde{g}^*} \tag{2.38}$$

which is just a similarity transformation and preserves the eigenvalues.

2.2. The Exact Renormalization Group Equation and its approximations

The central lesson of Wilson's analysis of QFT as described in the last section is that the "effective" (as in "effective field theory") action describing physical phenomena at a momentum scale k can be thought of as the result of having integrated out all fluctuations of the field with momenta larger than k [26]. It is therefore an interpolation between the bare action, valid at some UV scale, and the effective action where all field modes are integrated out corresponding to $k \to 0$. At this general level of discussion, it is not necessary to specify the physical meaning of k: for each application of the theory one will have to identify the physically relevant variable acting as k. A specific application where such a choice has to be taken is discussed in section 3.10 [2]. Since k can be regarded as the lower limit of some functional integration, it will usually be refered to as the infrared cutoff. The dependence of the "effective" action on k is the Wilsonian RG flow.
There are several ways of implementing this idea in practice, resulting in several forms of the RG equation. In the specific implementation that is used here, instead of introducing a sharp cutoff in the functional integral, the contribution of the field modes with momenta lower than k is suppressed. This is obtained by modifying the low momentum end of the propagator, and leaving all the interactions unaffected. Here, this procedure is described for a scalar field. One starts from a bare action $S[\phi]$, and adds to it a suppression term $\Delta S_k[\phi]$ that is quadratic in the field. In flat space this term can be written simply in momentum space. In order to have a procedure that works in an arbitrary curved spacetime one chooses a suitable differential operator $\mathcal{O}$ whose eigenfunctions φ_n, defined by $\mathcal{O}\varphi_n = \lambda_n\varphi_n$, can be taken as a basis in the functional space one integrates over,

$$\phi(x) = \sum_n \tilde{\phi}_n \varphi_n(x)$$

where $\tilde{\phi}_n$ are generalized Fourier components of the field. (A notation suitable for an operator with a discrete spectrum is used.) Then, the additional term can be written in either of the forms

$$\Delta S_k[\phi] = \frac{1}{2}\int dx\, \phi(x)\, R_k(\mathcal{O})\, \phi(x) = \frac{1}{2}\sum_n \tilde{\phi}_n^2 R_k(\lambda_n)\,. \tag{2.39}$$

The kernel $R_k(\mathcal{O})$ will also be called "the cutoff". It is arbitrary, except for the general requirements that $R_k(z)$ should be a monotonically decreasing function both in z and k, that $R_k(z) \to 0$ for $z \gg k$ and $R_k(z) \neq 0$ for $z \ll k$. These conditions are enough to guarantee that the contribution to the functional integral of field modes $\tilde{\phi}_n$ corresponding to

[2] In scattering experiments k is usually identified with some external momentum. See [27] for a discussion of this choice in concrete applications to gravity.

eigenvalues $\lambda_n \ll k^2$ are suppressed, while the contribution of field modes corresponding to eigenvalues $\lambda_n \gg k^2$ are unaffected. From here on one fixes $R_k(z) \to k^2$ for $k \to 0$. The distinction between high and low momentum modes is then implemented by defining a k-dependent generating functional of connected Green functions by

$$Z_k\,[J] = e^{-W_k[J]} = \int D\phi \exp\left(-S[\phi] - \Delta S_k[\phi] - \int dx\, J(x)\phi(x)\right) \tag{2.40}$$

which defines the expectation value of ϕ in the presence of ΔS and J as

$$\langle\phi(x)\rangle = \frac{\delta W_k\,[J]}{\delta J(x)} \tag{2.41}$$

and the k-dependent connected two-point function

$$G_k(x,y) = \frac{\delta^2 W_k}{\delta J(x)\delta J(y)} = \langle\phi(x)\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle\ . \tag{2.42}$$

where $\langle\phi(x)\phi(y)\rangle = Z_k^{-1}\int D\phi\phi(x)\phi(y)e^{-S[\phi]}$.
By a Legendre transformation one introduces

$$\tilde{\Gamma}_k[\phi] = W_k[J] - \int dx\, J(x)\phi(x) \tag{2.43}$$

such that

$$\frac{\delta\tilde{\Gamma}_k\,[\phi]}{\delta\phi(x)} = J(x)\ . \tag{2.44}$$

The differentiation of (2.41) with respect to ϕ and of (2.44) with respect to J gives the identity

$$\int dx G_k(x,y)\frac{\delta^2\tilde{\Gamma}_k}{\delta\phi(y)\delta\phi(z)} = \delta(x-z)\ . \tag{2.45}$$

This gives the RG scale dependence of $\tilde{\Gamma}_k$ for fixed ϕ as

$$\partial_t\tilde{\Gamma}_k|_\phi = -\partial_t W_k|_\phi = \partial_t\langle\Delta S\rangle = \frac{1}{2}\int dx\partial_t R_k\left[G_k(x,x) + \langle\phi(x)\rangle\langle\phi(x)\rangle\right] \tag{2.46}$$

To absorb the last term, one defines a modified k-dependent Legendre transform

$$\Gamma_k[\phi] = \tilde{\Gamma}_k - \Delta S_k[\phi] \tag{2.47}$$

where $\Delta S_k[\phi]$ has been subtracted. The functional Γ_k is sometimes called the *average effective action* because it can be interpreted as the effective action for fields that have been averaged over volumes of order k^{-d} (d being the dimension of spacetime) [28]. The "classical fields" $\delta W_k/\delta J$ are denoted ϕ without brackets for notational simplicity. In the limit $k \to 0$, Γ_k tends to the usual effective action $\Gamma[\phi]$, the generating functional of one-particle irreducible Green functions. Γ_k is similar in spirit to the Wilsonian effective action, but differs from it in the details of the implementation. The Legendre transformation of Γ_k can be inverted to obtain

$$J = \frac{\delta \Gamma_k}{\delta \phi} + \phi R_k. \tag{2.48}$$

Inserting this expression into Z_k, equation (2.40), and substituting ϕ by $\phi' + \chi$ one obtains

$$\exp(-\Gamma_k[\phi]) = \int D\phi' \exp\left(-S[\phi' + \chi] + \frac{\delta \Gamma_k}{\delta \chi}\phi' - \frac{1}{2}\phi' R_k \phi'\right) . \tag{2.49}$$

One sees that Γ_k approaches S for $k \to \Lambda$ if R_k diverges. The aim is to start with some UV action at Γ_k and to continue the integration till $k \to 0$. This would mean knowing the action at all microscopic and macroscopic scales and solve the whole theory.
Several properties of Γ_k are important to note. All symmetries of the theory which are also respected by the cutoff term ΔS_k are also symmetries of Γ_k. Therefore Γ_k can be expanded in all terms of invariants consistent with the symmetries of the theory, ordered for example according to the number of derivatives as we will do later on for gravity on basis of the different curvature invariants.
The functional $\tilde{\Gamma}_k[\phi] = \Gamma_k[\phi] + \Delta S_k[\phi]$ is the Legendre transform of $W_k[\phi]$. So it is convex and all second functional derivatives $\delta^2 \Gamma_k / \delta\phi\delta\phi + R_k$ are positive semi-definite. For the modified Γ_k this is only true in the limit $k \to 0$. This formalism can be easily generalized to the case of multicomponent fields Φ^A, where A may denote both internal and space-time (Lorentz) indices. In this case the cutoff term will have the form $\Phi^A R_{kAB} \Phi^B$, and the trace in the r.h.s. of the ERGE will also involve a finite trace over the indices A, B. The generalization to chiral fermionic fields is unproblematic and will occur in later sections. In the case of gauge theories there are further complications due to the fact that the cutoff interferes with gauge invariance. One way of dealing with this issue, which will be used here, is to use the background field method [18, 31] (for another approach see [32]). One defines a functional of two fields, the background field and the classical field, which is invariant under background gauge transformations. In the end the two fields are identified and one obtains a gauge invariant functional of one gauge field only.
Note also that Γ_k will be the generating functional for all 1PI n-point functions which will also be scale dependent. The interaction models will be different at different scales.
If the theory contains some reparametrization invariance of physical quantities under field rescalings $\phi(x) \to \alpha\phi(x)$ giving some wave function renormalization constant, also the infrared cutoff term should contain this constant. This will be important in gravity where the Newton constant plays the role of a wave function renormalization constant.
In principle one could start calculations with the integro-differential equation (2.49). However, from the RG scale dependence of $\tilde{\Gamma}_k$ follows also that Γ_k satisfies the following *Exact Renormalization Group Equation* (or ERGE) [29, 30]

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{Tr} G_k k \frac{dR_k}{dk} = \frac{1}{2}\mathrm{Tr}\left[\Gamma_k^{(2)} + R_k\right]^{-1} k\frac{dR_k}{dk} . \tag{2.50}$$

As a functional differential equation it will be easier to handle than the integro-differential equation. The trace in the r.h.s. is a sum over the eigenvalues of the operator $\mathcal{O}$ (in flat space it would correspond to a momentum integration $\mathrm{Tr} = \sum_A \int d^d q/(2\pi)^d$) and the notation $\Gamma_k^{(2)} = \frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi}$ is introduced for the inverse propagator of the field ϕ defined by the functional Γ_k. Together with the cutoff term, the r.h.s. involves the full propagator of the theory. As the equation holds exactly, effects of arbitrarily high loop order are included.
Flow equations for n-point functions can be obtained easily from equation (2.50) by differentiating with respect to the fields at the vertex. One can represent the connection between different n-point functions by modified Feynman-diagrams with insertion points from the cutoff function stressing the similarity of the exact equation to the perturbative expansion. It can be shown that the k-dependent n-point function will depend only on the n+1 and n+2 point function.
As the structure of the ERGE is similar to the perturbative expansion, the ERGE can be seen formally as a RG improved one loop equation. To see this, recall that given a bare action S (for a bosonic field), the one loop effective action $\Gamma^{(1)}$ is

$$\Gamma^{(1)} = S + \frac{1}{2}\mathrm{Tr}\log\left[\frac{\delta^2 S}{\delta\phi\delta\phi}\right] . \tag{2.51}$$

Then one adds to S the cutoff term (2.39); the functional

$$\Gamma_k^{(1)} = S + \frac{1}{2}\mathrm{Tr}\log\left[\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k\right] \tag{2.52}$$

may be called the "one loop average effective action". It satisfies the equation

$$k\frac{d\Gamma_k^{(1)}}{dk} = \frac{1}{2}\mathrm{Tr}\left[\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k\right]^{-1} k\frac{dR_k}{dk} \tag{2.53}$$

which is formally identical to (2.50) except that in the r.h.s. the renormalized running couplings $g_i(k)$ are replaced everywhere by the "bare" couplings g_i, appearing in S. Thus the "RG improvement" in the ERGE consists in replacing the bare couplings by the running renormalized couplings.
The formal derivation of the ERGE from a path integral makes use of a bare action S. To explore the relation between S and the (renormalized) average effective action Γ_k for any k requires that an ultraviolet regularization be defined (in addition to the infrared regularization provided by R_k). This point does not need to be discussed, since the bare action is an unphysical quantity and all the physics is encoded in the running renormalized action Γ_k. In this connection note that the trace in the r.h.s. of (2.50), which includes an integration over momenta, is perfectly ultraviolet convergent and does not require any UV regulator. This is because the function $k\frac{d}{dk}R_k$ in the r.h.s. goes to zero for momenta greater than k and makes the integration convergent. So, one can regard the derivations given above as merely formal manipulations that motivate the form of the ERGE, but then the ERGE itself is perfectly well defined, without the need of introducing an UV

regulator. If one assumes that at a given scale k physics is described by a renormalized action Γ_k, the ERGE gives a way of studying the dependence of this functional on k, and the behavior of the theory at high energy can be studied by taking the limit of Γ_k for $k \to \infty$ (which need not coincide with the bare action S).
The average effective action $\Gamma_k[\phi]$, used at tree level, gives an accurate description of processes occurring at momentum scales of order k. In the spirit of effective field theories, one assumes that Γ_k exists and is quasi–local in the sense that it admits a derivative expansion of the form 2.22.
The r.h.s. of (2.50) can be regarded as the "beta functional" of the theory, giving the k–dependence of all the couplings of the theory. In fact, taking the derivative of (2.22) one gets

$$k\frac{d\Gamma_k}{dk} = \sum_{n=0}^{\infty}\sum_i \beta_i^{(n)}\mathcal{O}_i^{(2n)} \tag{2.54}$$

where

$$\beta_i^{(n)}(g_j, k) = k\frac{dg_i^{(n)}}{dk} = \frac{dg_i^{(n)}}{dt} \tag{2.55}$$

are the beta functions of the (generally dimensionful) couplings. Here, the renormalization group time $t = \log(k/k_0)$ has been introduced, k_0 being an arbitrary initial value. If one expands the trace on the r.h.s. of (2.50) in operators $\mathcal{O}_i^{(2n)}$ and compares with (2.54), one can in principle read off the beta functions of the individual couplings.
In this connection, note that in general the cutoff function R_k may contain the couplings g_i and therefore the term $k\frac{d}{dk}R_k$ in the r.h.s. of (2.50) will itself contain the beta functions. Thus, extracting the beta functions from the ERGE generally implies solving an algebraic equation where the beta functions appear on both sides [3]. This complication can be avoided by choosing the cutoff in such a way that it does not contain any coupling. Then, the entire content of the ERGE is in the (RG–improved) one loop beta functions. The result is still "exact" insofar as one is able to keep track of *all* possible couplings of the theory.
There is an obvious problem as soon as one tries to solve the ERGE: one has to deal with an infinite system of coupled nonlinear partial differential equations. Most solution methods are based on the identification of a small expansion parameter. Using small couplings as an expansion parameter will of course not go beyond the usual perturbative calculations. Alternatives are the expansion in the number N of matter fields as used in section 2.3 or in the number of dimensions, especially the $2+\epsilon$-expansion around two dimension where gravity is renormalizable as used in section 3.7.
In most cases it is however impossible to follow the flow of infinitely many couplings and a common procedure is to consider a truncation of the theory, namely to retain only a finite subset of terms in the effective action Γ_k. There are different procedures to collect the retained terms. One can expand in the powers of fields and therefore in the n-point

[3]For example, in a scalar theory with action $\int d^4x\ \left[Z(\partial\phi)^2 + m^2\phi^2 + \lambda\phi^4\right]$ it is natural to choose the cutoff of the form $R_k(z) = Z_k k^2 r(z/k^2)$. Then, the r.h.s. of (2.50) will contain $\partial_t Z$.

functions, in the dimension of the coupling constants, or in the number of derivatives. We will mostly work with the latter form of expansion as in (2.22) and retain all terms up to some given order n. For each choice, one calculates the coefficients of the retained operators in the r.h.s. of (2.50) and in this way the corresponding beta functions are computed. In general the set of couplings that one chooses in this way will not be closed under RG evolution, so one is neglecting the potential effect of the excluded couplings on the ones that are retained. Still, in this way one can obtain genuine nonperturbative information, and this procedure has been applied to a variety of physical problems with good quantitative results. For reviews, see [33, 34, 35].

If one truncates the effective action in this way, there is usually no small parameter to allow to estimate the error one is making. One indirect way to estimate the quality of a truncation relies on an analysis of the cutoff scheme dependence. The effective action Γ_k obviously depends on the choice of the cutoff function R_k. This dependence is similar to the scheme dependence of the renormalized effective action in perturbative QFT; only physically observable quantities derived from Γ_k must be independent of R_k. This provides an indirect check on the quality of the truncation. For example, the critical exponents should be universal quantities and therefore cutoff–independent. In concrete calculations, usually involving a truncation of the action, critical exponents do depend on the cutoff scheme, and the observed dependence can be taken as a quantitative measure of the quality of the approximation. Ultimately, there is no substitute for performing calculations with truncations that contain more terms. Note that a good truncation is not necessarily one for which the new terms are small, but one for which the effect of the new terms on the old ones is small. In other words, in search of a nontrivial FP, one wants the addition of new terms not to affect too much the FP value of the "old" couplings, nor the "old" critical exponents.

2.3. Matter fields and cutoff schemes

Having introduced the Exact Renormalization Group Equation (ERGE) (2.50) in the last section, now the method used to compute the trace in the r.h.s. of (2.50) in a gravitational setting and to evaluate the beta functions of the gravitational couplings will be illustrated. Quite generally, one considers the contribution of fields whose inverse propagator $\Gamma_k^{(2)}$ is a differential operator of the form

$$\Delta = -\nabla^2 + \mathbf{E} \tag{2.56}$$

where ∇ is a covariant derivative, both with respect to the gravitational field and possibly also with respect to other gauge connections coupled to the internal degrees of freedom of the field, and $\mathbf{E}$ is a linear map acting on the quantum field. As this operator contains derivatives only in combinations of Laplacians, we will be able to use heat kernel methods for the trace evaluation in the ERGE as described in appendix A.1. In general, $\mathbf{E}$ could contain mass terms or terms linear in curvature. For example, in the case of a nonminimally coupled scalar, $\mathbf{E} = \xi R$, where ξ is a coupling. A priori, nothing will be assumed about the gravitational action and also the spacetime dimension d can be left arbitrary at this stage.

2.3.1. Cutoff types

In order to write the ERGE one has to define the cutoff. For the operator to be used in the definition of (2.39), several possible choices suggest themselves. One splits $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$, where $\mathbf{E}_1$ does not contain any couplings and $\mathbf{E}_2$ consists only of terms containing the couplings. A cutoff will be called
- *of type I*, if R_k is a function of the "bare Laplacian" $-\nabla^2$,
- *of type II* if it is a function of $-\nabla^2 + \mathbf{E}_1$ and
- *of type III* if it is a function of the full kinetic operator $\Delta = -\nabla^2 + \mathbf{E}$.

The substantial difference between the first two types and the third is that in the latter case, due to the running of the couplings, the spectrum changes along the flow. For this reason these cutoffs are said to be "spectrally adjusted" [36]. [4]
If one restricts to the case where $\mathbf{E}_2 = 0$, i.e. the kinetic operator does not depend on the couplings, there is only a choice between cutoffs of type I and II. The derivation of the beta functions is technically simpler with a *type II cutoff*. In this case one chooses a real function R_k with the properties listed in section 2.2 and defines a modified inverse propagator

$$P_k(\Delta) = \Delta + R_k(\Delta)\ . \tag{2.57}$$

If the operator $\mathbf{E}$ does not contain couplings, using (A.10) the trace in the r.h.s. of the ERGE reduces simply to

$$\mathrm{Tr}\frac{\partial_t R_k(\Delta)}{P_k(\Delta)} = \frac{1}{(4\pi)^{d/2}}\sum_{i=0}^{\infty} Q_{\frac{d}{2}-i}\left(\frac{\partial_t R_k}{P_k}\right) B_{2i}(\Delta) \tag{2.58}$$

where $B_{2i}(\Delta)$ are the heat kernel coefficients of the operator Δ and the Q-functionals, defined in (A.14,A.15) are the analogues of momentum integrals in this curved spacetime setting. The derivative with respect to the explicit dependence of R_k on k is denoted as $\partial_t R_k$; when the argument of R_k does not contain couplings this coincides with the total derivative dR_k/dt.
With a *type I cutoff* one uses the same profile function R_k but now with $-\nabla^2$ as its argument. This implies the replacement of the inverse propagator Δ by

$$\Delta + R_k(-\nabla^2) = P_k(-\nabla^2) + \mathbf{E}\ . \tag{2.59}$$

Therefore the r.h.s. of the ERGE will now contain the trace $\mathrm{Tr}\frac{\partial_t R_k(-\nabla^2)}{P_k(-\nabla^2)+\mathbf{E}}$. Since $\mathbf{E}$ is linear

[4] In (2.39) it was assumed for simplicity that the operator $\mathcal{O}$ appearing in the argument of the cutoff function is also the operator whose eigenfunctions are used as a basis in the evaluation of the functional trace. It is worth stressing that this need not be the case, as discussed in appendix A.1.

in curvature, in the limit when the components of the curvature tensor are uniformly much smaller than k^2, one can expand

$$\frac{\partial_t R_k}{P_k+\mathbf{E}} = \sum_{\ell=0}^{\infty}(-1)^\ell \mathbf{E}^\ell \frac{\partial_t R_k}{P_k^{\ell+1}} \, .$$

Each one of the terms on the r.h.s. can then be evaluated in a way analogous to (A.10), so in this case one gets a double series

$$\mathrm{Tr}\frac{\partial_t R_k(-\nabla^2)}{P_k(-\nabla^2)+\mathbf{E}} = \frac{1}{(4\pi)^{d/2}}\sum_{i=0}^{\infty}\sum_{\ell=0}^{\infty} Q_{\frac{d}{2}-i}\left(\frac{\partial_t R_k}{P_k^{\ell+1}}\right)\int dx\,\sqrt{g}(-1)^\ell \mathrm{tr}\mathbf{E}^\ell b_{2i}(-\nabla^2)\, . \quad (2.60)$$

In order to extract the beta functions of the gravitational couplings one has to collect terms with the same monomials in curvature. An example of this will be presented shortly.

Before discussing specific examples, however, it is interesting to consider the scheme–independent part of the trace. In general, on dimensional grounds, the functionals $Q_n\left(\partial_t R_k/P_k^m\right)$ appearing in (2.58) and (2.60) will be equal to $k^{2(n-m+1)}$ times a number depending on the profile function. As discussed in appendix A.1, the integrals with $m=n+1$ are independent of the shape of R_k. Thus, in even-dimensional spacetimes with a *cutoff of type II*, and using (A.19), the coefficient of the term in the sum (2.58) with $i=\frac{d}{2}$ is

$$Q_0\left(\frac{\partial_t R_k}{P_k}\right) B_d(\Delta) = 2B_d(\Delta)\, . \quad (2.61)$$

On the other hand with a *type I cutoff*, using (A.18), (A.19) and (A.5) the terms with $\ell = \frac{d}{2}-i$ add up to

$$\begin{aligned}
&\sum_{\ell=0}^{d/2} Q_\ell\left(\frac{\partial_t R_k}{P_k^{\ell+1}}\right)\int dx\,\sqrt{g}(-1)^\ell \mathrm{tr}\mathbf{E}^\ell b_{2i}(-\nabla^2) \\
=\;& 2\int dx\,\sqrt{g}\,\mathrm{tr}\left[b_d(-\nabla^2) - \mathbf{E}b_{d-2}(-\nabla^2) + \ldots + \frac{(-1)^{d/2}}{(d/2)!}\mathbf{E}^{d/2} b_0(-\nabla^2)\right] \\
=\;& 2B_d(-\nabla^2+\mathbf{E})\, .
\end{aligned}$$

Therefore, in addition to being independent of the shape of the cutoff function, these coefficients are also the same using type I or type II cutoffs.

2.3.2. Minimally coupled matter

As an example one now specializes to four-dimensional gravity coupled to n_S scalar fields, n_D Dirac fields, n_M gauge (Maxwell) fields, all massless and minimally coupled

$$\Gamma_k(g_{\mu\nu},\phi,\psi,A_\mu)=\int d^4x\,\sqrt{g}\left[\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi+\bar{\psi}\gamma^\mu\nabla_\mu\psi+\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right]. \tag{2.62}$$

In addition Γ_k must contain a generic action for gravity of the form (2.22), which is not written here. (For the terms with four derivatives one uses the parametrization given in (3.74) below.) The contribution of these matter fields to the gravitational beta functions will be computed here. The contribution of the gravitational field to its beta functions will be calculated in the next section using the same methods; as one will see, the details of the calculation are technically more involved, but conceptually there is no difference.
The field equation of each type of field defines a second order differential operator $\Delta^{(A)}=-\nabla^2+\mathbf{E}^{(A)}$, with $A=S,D,M,gh$ and

$$\mathbf{E}^{(S)}=0\,;\qquad \mathbf{E}^{(D)}=\frac{R}{d}\,;\qquad \mathbf{E}^{(M)}=\mathrm{Ricci}\,;\qquad \mathbf{E}^{(gh)}=0\,. \tag{2.63}$$

Here "Ricci" stands for the Ricci tensor regarded as a linear operator acting on vectors $\mathrm{Ricci}(v)_\mu=R_\mu{}^\nu v_\nu$. For the gauge fields, the Lorentz gauge is chosen, and $\Delta^{(gh)}$ is the operator acting on the scalar ghost. (It can be shown that the results do not depend on the choice of gauge [37].)
With a *type II cutoff*, for each type of field one defines the modified inverse propagator $P_k(\Delta^{(A)})=\Delta^{(A)}+R_k(\Delta^{(A)})$. Then, using the heat kernel coefficients for the different fields, the ERGE reduces simply to

$$\begin{aligned}
\frac{d\Gamma_k}{dt} &= \frac{n_S}{2}\mathrm{Tr}_{(S)}\left(\frac{\partial_t R_k(\Delta^{(S)})}{P_k(\Delta^{(S)})}\right)-\frac{n_D}{2}\mathrm{Tr}_{(D)}\left(\frac{\partial_t R_k(\Delta^{(D)})}{P_k(\Delta^{(D)})}\right)\\
&\quad+\frac{n_M}{2}\mathrm{Tr}_{(M)}\left(\frac{\partial_t R_k(\Delta^{(M)})}{P_k(\Delta^{(M)})}\right)-n_M\mathrm{Tr}_{(gh)}\left(\frac{\partial_t R_k(\Delta^{(gh)})}{P_k(\Delta^{(gh)})}\right)\\
&= \frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\,\sqrt{g}\Bigg[(n_S-4n_D+2n_M)\,Q_2\left(\frac{\partial_t R_k}{P_k}\right)\\
&\quad+\frac{1}{6}R\,(n_S-2n_D-4n_M)\,Q_1\left(\frac{\partial_t R_k}{P_k}\right)\\
&\quad+\frac{1}{180}\Bigg((3n_S+18n_D+36n_M)\,C^2-(n_S+11n_D+62n_M)\,E\\
&\quad+5n_S R^2+12\,(n_S+n_D-3n_M)\,\nabla^2 R\Bigg)+\ldots\Bigg]
\end{aligned} \tag{2.64}$$

where C^2 is the square of Weyl's tensor and $E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ (in four dimensions $\chi = \frac{1}{32\pi^2}\int d^4x\,\sqrt{g}E$ is Euler's topological invariant). The terms with zero and one power of R depend on the profile function R_k, but using (A.19) one sees that the coefficients of the four–derivative terms, *i.e.* the beta functions of $g_i^{(4)}$, are scheme–independent.
With *type I cutoffs*, the modified inverse propagators are $\Delta^{(A)} + R_k(-\nabla^2) = P_k(-\nabla^2) + \mathbf{E}^{(A)}$ and the ERGE then becomes:

$$\begin{aligned}\frac{d\Gamma_k}{dt} &= \frac{n_S}{2}\mathrm{Tr}_{(S)}\left(\frac{\partial_t P_k(-\nabla^2)}{P_k(-\nabla^2)}\right) - \frac{n_D}{2}\mathrm{Tr}_{(D)}\left(\frac{\partial_t R_k(-\nabla^2)}{P_k(-\nabla^2)+\frac{R}{4}}\right) \\ &+\frac{n_M}{2}\mathrm{Tr}_{(M)}\left(\frac{\partial_t R_k(-\nabla^2)}{P_k(-\nabla^2)+\mathrm{Ricci}}\right) - n_M\mathrm{Tr}_{(gh)}\left(\frac{\partial_t R_k(-\nabla^2)}{P_k(-\nabla^2)}\right) . \qquad (2.65)\end{aligned}$$

Expanding each trace as in (A.10), collecting terms with the same number of derivatives of the metric, and keeping terms up to four derivatives one gets

$$\begin{aligned}\frac{d\Gamma_k}{dt} &= \frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\,\sqrt{g}\Bigg[\left(n_S - 4n_D + 2n_M\right)Q_2\left(\frac{\partial_t R_k}{P_k}\right) \\ &+\Bigg[\frac{1}{6}Q_1\left(\frac{\partial_t R_k}{P_k}\right)n_S + \left(\frac{2}{3}Q_1\left(\frac{\partial_t R_k}{P_k}\right) - Q_2\left(\frac{\partial_t R_k}{P_k^2}\right)\right)n_D \\ &+\left(\frac{1}{3}Q_1\left(\frac{\partial_t R_k}{P_k}\right) - Q_2\left(\frac{\partial_t R_k}{P_k^2}\right)\right)n_M\Bigg]R \\ &+\frac{1}{180}\Bigg(\left(3n_S + 18n_D + 36n_M\right)C^2 - \left(n_S + 11n_D + 62n_M\right)E \\ &+5n_S R^2 + 12\left(n_S + n_D - 3n_M\right)\nabla^2 R\Bigg) + \ldots\Bigg] . \qquad (2.66)\end{aligned}$$

One sees that the terms linear in curvature, which contribute to the beta function of Newton's constant, have changed. However, the terms quadratic in curvature have the same coefficients as before, confirming that the beta functions of the dimensionless couplings are scheme–independent.
In order to have more explicit formulae, and in numerical work, one needs to calculate also the scheme–dependent Q-functionals. This requires fixing the profile R_k. Here, in most cases the so–called optimized cutoff (A.21) will be used in which the integrals are readily evaluated, see equations (A.22,A.23,A.24). This cutoff has the very convenient property that $Q_{-n}\left(\partial_t R_k/P_k\right) = 0$ for $n \geq 1$. Thus, the sum over heat kernel coefficients on the r.h.s. of (A.10) terminates. In particular, in four dimensions, there are no terms beyond those that are explicitly written in (2.64) or (2.66) [5]. For more general cutoffs a

[5]This had also been observed in a different context in [38].

calculation of beta functions for curvature–polynomials of cubic and higher order would require the knowledge of higher heat kernel coefficients.
One has to briefly comment on the spectrally adjusted (type III) cutoffs. These only occur when the kinetic operator Δ contains couplings, for example a mass term or, in the case of a scalar field, a nonminimal coupling of the form ξR. In this case the last factor in (2.50) is

$$\frac{dR_k}{dt} = \partial_t R_k + \sum_i R'_k \frac{\partial \mathbf{E}}{\partial g_i} \partial_t g_i$$

where R'_k denotes the derivative of $R_k(z)$ with respect to z and the sum extends over all couplings (this assumes that the derivative of the operator appearing in R_k commutes with the operator itself). This introduces further nonlinearities in the system. Since the beta functions of the couplings appear linearly in the r.h.s. of the equation, to obtain the beta functions one has to solve a system of linear equations.
In a consistent truncation one would have to add to the terms in (2.66) or (2.64) the contribution due to the gravitational field. This will be done in the next section. For the time being one observes that if the number of matter fields is of order $N \to \infty$, this is the dominant contribution and constitutes the leading order of a $1/N$ expansion [39]. The matter contributions by themselves have a form that leads to a gravitational FP. Comparing equation (2.54) with equations (2.66) or (2.64) one can read off the beta functions, which all have the form

$$\frac{dg_i^{(n)}}{dt} = a_i^{(n)} k^{4-n} \tag{2.67}$$

where $a_i^{(n)}$ are constants. Then, the beta functions of the dimensionless variables $\tilde{g}_i^{(n)} = k^{n-4} g_i^{(n)}$ are

$$\frac{d\tilde{g}_i^{(n)}}{dt} = (n-4)\tilde{g}_i^{(n)} + a_i^{(n)} \,. \tag{2.68}$$

This simple flow has indeed a FP for all couplings. For $n \neq 4$

$$\tilde{g}_{i*}^{(n)} = \frac{a_i^{(n)}}{4-n} \,, \tag{2.69}$$

in particular writing $g^{(0)} = 2Z\Lambda$ and $g^{(2)} = -Z = -\frac{1}{16\pi G}$, the flow of the cosmological constant and Newton's constant is given by

$$\begin{aligned} \frac{d\tilde{\Lambda}}{dt} &= -2\tilde{\Lambda} + 8\pi a^{(0)} \tilde{G} + 16\pi a^{(2)} \tilde{G}\tilde{\Lambda} \\ \frac{d\tilde{G}}{dt} &= 2\tilde{G} + 16\pi a^{(2)} \tilde{G}^2 \end{aligned} \tag{2.70}$$

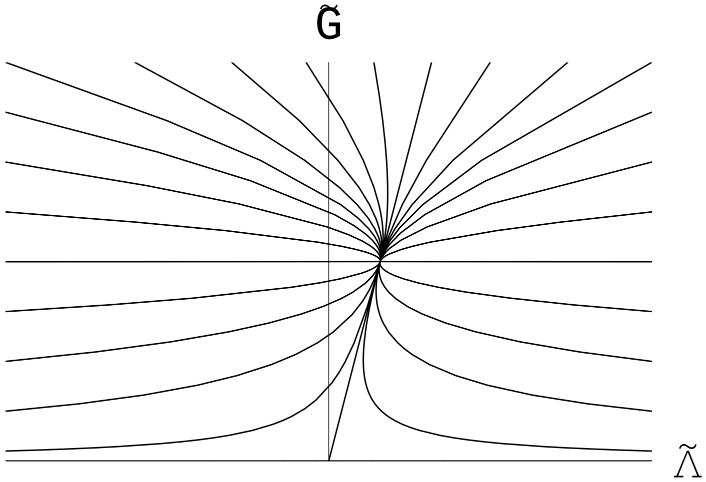

Figure 2.1.: The generic form of the flow induced by matter fields.

which, for a type II cutoff, has a FP at

$$\tilde{\Lambda}_* = -\frac{3}{4}\frac{n_S - 4n_D + 2n_M}{n_S - 2n_D - 4n_M}\ ,\quad \tilde{G}_* = \frac{12\pi}{-n_S + 2n_D + 4n_M}\ . \tag{2.71}$$

Note that the FP occurs for positive or negative $\tilde{\Lambda}$ depending on whether there are more bosonic or fermionic degrees of freedom. The FP value of $\tilde{G}$, on the other hand, will be positive provided there are not too many scalar fields. For $n = 4$, (2.68) gives a logarithmic running

$$g_i^{(4)}(k) = g_i^{(4)}(k_0) + a_i^{(4)}\ln(k/k_0)\ ,$$

implying asymptotic freedom for the couplings $1/g_i^{(4)}$. This is the same behavior that is observed in Yang–Mills theories and is in accordance with earlier perturbative calculations [40, 41]. As noted, it follows from (A.24) that with the optimized cutoff, for $n > 4$, $\tilde{g}_{i*}^{(n)} = 0$. The critical exponents at the nontrivial FP are equal to the canonical dimensions of the $g^{(n)}$s, so Λ and G are UV–relevant (attractive), $1/g_i^{(4)}$ are marginal and all the higher terms are UV–irrelevant. Note that in perturbation theory G is irrelevant. At the nontrivial FP the quantum corrections conspire with the classical dimensions of Λ and G to reconstruct the dimensions of $g^{(0)}$ and $g^{(2)}$. This must happen because the critical exponents for $g^{(0)}$ and $g^{(2)}$ are equal to their canonical dimensions and the critical exponents are invariant under regular coordinate transformations in the space of all couplings; the transformation between $\tilde{G}$ and $\tilde{g}^{(2)}$ is regular at the nontrivial FP, but it is singular at the Gaußian FP where the coupling vanishes.
This simple flow is exact in the limit $N \to \infty$, but is also a rough approximation when graviton effects are taken into account, as will be discussed in sections 3.8 and 4.1. It is shown in figure 2.1.

In this section we have reviewed the Wilsonian Renormalization Group approach and presented the idea of an asymptotically safe theory with the essential ingredient of a nontrivial Renormalization Group flow fixed point. After that, we introduced a form of Exact Renormalization Group Equation (ERGE) which is the necessary nonperturbative tool to calculate the beta functions of the couplings and to study if a theory possesses such a kind of fixed point. Then we presented the necessary techniques to calculate the beta functions in a curved spacetime setting. With these tools at hand, we are now able to consider gravitational actions and to see if a well-defined UV limit can exist for gravity. This will be the matter of the next sections.

3. Einstein-Hilbert truncation

With the preparation from the last section, we are now able to deal with gravity. As a first step towards the inclusion of quantum gravitational effects, we discuss in this section the Renormalization Group (RG) flow for Einstein's gravity, with or without cosmological constant. This truncation has been extensively discussed before [18, 20]. Here we will extend those results in various directions. Since the dependence of the results on the choice of gauge and profile function R_k has already been discussed in [20, 21] here we shall fix our attention on a particular gauge and profile function, and analyze instead the dependence of the results on different ways of implementing the cutoff procedure. The simplicity of the truncation will allow us to compare the results of different approximations and cutoff schemes, a luxury that is progressively reduced going to more complicated truncations.

The theory is parametrized by the cosmological constant Λ and Newton's constant $G = 1/(16\pi Z)$, so that we set $g^{(0)} = 2\Lambda Z$ and $g^{(2)} = -Z$ in equation (2.22). All higher couplings are neglected. Then the truncation takes the form

$$\Gamma_k = \int dx \sqrt{g}\,(2\Lambda Z - ZR(g)) + S_{GF} + S_{\mathrm{ghost}} \tag{3.1}$$

where S_{GF} is a gauge–fixing term and S_{ghost} is the ghost action. We decompose the metric into $g_{\mu\nu} = g^{(B)}_{\mu\nu} + h_{\mu\nu}$ where $g^{(B)}_{\mu\nu}$ is a background. We will refer to the field $h_{\mu\nu}$ as the graviton, even though it is not assumed to be a small perturbation. We consider background gauges of the type

$$S_{GF}(g^{(B)}, h) = \frac{Z}{2\alpha} \int dx \sqrt{g^{(B)}}\chi_\mu g^{(B)\mu\nu}\chi_\nu \tag{3.2}$$

where

$$\chi_\nu = \nabla^\mu h_{\mu\nu} - \frac{1+\rho}{d}\nabla_\nu h\,.$$

All covariant derivatives are with respect to the background metric. In the following all metrics will be background metrics, and we will omit the superscript (B) for notational simplicity. In this section we will restrict ourselves to the de Donder gauge with parameters $\alpha = 1$, $\rho = \frac{d}{2} - 1$, which leads to considerable simplification. The inverse propagator of $h_{\mu\nu}$, including the gauge fixing term, can be written in the form

$$\frac{1}{2}\int dx\,\sqrt{g}h_{\mu\nu}\Gamma_k^{(2)\mu\nu\rho\sigma}h_{\rho\sigma}$$

containing the minimal operator

$$\Gamma^{(2)\mu\nu}_{k\ \rho\sigma} = Z\left[K^{\mu\nu}_{\rho\sigma}(-\nabla^2 - 2\Lambda) + U^{\mu\nu}_{\rho\sigma}\right] \tag{3.3}$$

where [1]

$$\begin{aligned}
K^{\mu\nu}_{\rho\sigma} &= \frac{1}{2}\left(\delta^{\mu\nu}_{\rho\sigma} - \frac{d}{2}P^{\mu\nu}_{\rho\sigma}\right)\ ; \qquad \delta^{\mu\nu}_{\rho\sigma} = \frac{1}{2}\left(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho\right)\ ; \qquad P^{\mu\nu}_{\rho\sigma} = \frac{1}{d}g^{\mu\nu}g_{\rho\sigma}\ ;\\
U^{\mu\nu}_{\rho\sigma} &= R\,K^{\mu\nu}_{\rho\sigma} + \frac{1}{2}\left(g^{\mu\nu}R_{\rho\sigma} + R^{\mu\nu}g_{\rho\sigma}\right) - \delta^{(\mu}_{(\rho}R^{\nu)}_{\sigma)} - R^{(\mu}{}_{(\rho}{}^{\nu)}{}_{\sigma)}\ .
\end{aligned}$$

In the derivation of these expressions, the variations of the occurring tensors given in appendix A.6 are useful. In the following we will sometimes suppress indices for notational clarity; we will use boldface symbols to indicate linear operators on the space of symmetric tensors. For example, the objects defined above will be denoted $\mathbf{K}$, $\mathbf{1}$, $\mathbf{P}$, $\mathbf{U}$. Note that $\mathbf{P}$ and $\mathbf{1}-\mathbf{P}$ are projectors onto the trace and tracefree parts in the space of symmetric tensors, $h_{\mu\nu} = h^{(TF)}_{\mu\nu} + h^{(T)}_{\mu\nu}$ where $h^{(T)}_{\mu\nu} = P^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} = \frac{1}{d}g_{\mu\nu}h$. Using that $\mathbf{K} = \frac{1}{2}\left((\mathbf{1}-\mathbf{P}) + \frac{2-d}{2}\mathbf{P}\right)$, if $d \neq 2$ we can rewrite equation (3.3) in either of the following forms

$$\begin{aligned}
\mathbf{\Gamma}^{(2)}_k &= Z\,\mathbf{K}(-\nabla^2 - 2\Lambda\mathbf{1} + \mathbf{W})\\
&= \frac{Z}{2}\left[(\mathbf{1}-\mathbf{P})\left(-\nabla^2 - 2\Lambda\mathbf{1} + 2\mathbf{U}\right) - \frac{d-2}{2}\mathbf{P}\left(-\nabla^2 - 2\Lambda\mathbf{1} - \frac{4}{d-2}\mathbf{U}\right)\right]
\end{aligned} \tag{3.4}$$

where we have defined

$$W^{\mu\nu}_{\rho\sigma} = 2U^{\mu\nu}_{\rho\sigma} - \frac{(d-4)}{2(d-2)}\left(R_{\rho\sigma}g^{\mu\nu} + g_{\rho\sigma}R^{\mu\nu} - Rg_{\rho\sigma}g^{\mu\nu}\right)\ . \tag{3.5}$$

Note that the overall sign of the second term in the second line of (3.4) is negative when $d > 2$. This is the famous problem of the unboundedness of the Euclidean Einstein–Hilbert action. We shall see shortly how this is dealt with in the ERGE. Later on, we will need the traces

$$\begin{aligned}
\mathrm{tr}\,\mathbf{1} &= \frac{d(d+1)}{2}\ ;\ \ \mathrm{tr}\,\mathbf{P} = 1\ ;\ \ \mathrm{tr}\ (\mathbf{1}-\mathbf{P}) = \frac{d^2+d-2}{2}\ ;\ \ \mathrm{tr}\,\mathbf{W} = \frac{d(d-1)}{2}R\ ;\\
\mathrm{tr}\,\mathbf{W}^2 &= 3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{d^2-8d+4}{d-2}R_{\mu\nu}R^{\mu\nu} + \frac{d^3-5d^2+8d+4}{2(d-2)}R^2\ .
\end{aligned} \tag{3.6}$$

The ghost action is

$$S_{\text{ghost}} = -\int\sqrt{g}\,\bar{C}_\mu\left(-\nabla^2\delta^\mu_\nu - R^\mu{}_\nu\right)C^\nu\ . \tag{3.7}$$

[1] These definitions coincide with those of [18] except that the Λ term has been removed from $\mathbf{U}$.

On the d-dimensional sphere where

$$R_{\mu\nu} = \frac{R}{d} g_{\mu\nu} \quad ; R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) \tag{3.8}$$

we can write

$$\mathbf{U} = \frac{1}{2} \left[(\mathbf{1} - \mathbf{P}) \frac{d^2 - 3d + 4}{d(d-1)} R - \mathbf{P} \frac{d-2}{2} \frac{d-4}{d} R \right] . \tag{3.9}$$

Then, using the second line of (3.4), we have

$$\mathbf{\Gamma}_k^{(2)} = \frac{Z}{2} \left[(\mathbf{1} - \mathbf{P}) \left(-\nabla^2 - 2\Lambda + \frac{d^2 - 3d + 4}{d\,(d-1)} R \right) - \frac{d-2}{2} \mathbf{P} \left(-\nabla^2 - 2\Lambda + \frac{d-4}{d} R \right) \right] . \tag{3.10}$$

We will now discuss separately various types of cutoff schemes.

3.1. Cutoff of type Ia

This is the scheme that was used originally in [18] designed to just replace $-\nabla^2$ by $P_k(-\nabla^2)$ in the modified inverse propagator (3.10). It is defined by the cutoff term

$$\Delta S_k[h_{\mu\nu}] = \frac{1}{2} \int dx\, \sqrt{g}\, h_{\mu\nu} R_k(-\nabla^2)^{\mu\nu\rho\sigma}\, h_{\rho\sigma} - \int dx\, \sqrt{g}\, \bar{C}_\mu R_k^{(gh)}(-\nabla^2)^\mu{}_\nu C^\nu \tag{3.11}$$

where

$$\begin{aligned} \mathbf{R}_k(-\nabla^2) &= Z\mathbf{K} R_k(-\nabla^2) \\ R_k^{(gh)}(-\nabla^2)^\mu{}_\nu &= \delta^\mu{}_\nu R_k(-\nabla^2) \,. \end{aligned} \tag{3.12}$$

for gravitons and ghosts respectively. Defining the anomalous dimension by

$$\eta = \frac{1}{Z} \frac{dZ}{dt} \tag{3.13}$$

we then have

$$\frac{d\mathbf{R}_k}{dt} = Z\mathbf{K} \left[\partial_t R_k(-\nabla^2) + \eta R_k(-\nabla^2) \right] . \tag{3.14}$$

The calculation in [18] proceeded as follows. The background metric is chosen to be that of Euclidean de Sitter space. Using the properties of the projectors, its inversion is trivial

$$\left(\mathbf{\Gamma}_k^{(2)} + \mathbf{R}_k \right)^{-1} = \frac{2}{Z} \left[(\mathbf{1} - \mathbf{P}) \frac{1}{P_k - 2\Lambda + \frac{d^2-3d+4}{d(d-1)} R} - \frac{2}{d-2} \mathbf{P} \frac{1}{P_k - 2\Lambda + \frac{d-4}{d} R} \right] . \tag{3.15}$$

Decomposing in the same way the term $\frac{d}{dt}\mathbf{R}_k$, multiplying and tracing over spacetime indices one obtains

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\mathrm{Tr}_{xL}(\mathbf{1}-\mathbf{P})\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda + \frac{d^2-3d+4}{d(d-1)}R} + \frac{1}{2}\mathrm{Tr}_{xL}\mathbf{P}\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda + \frac{d-4}{d}R} - \mathrm{Tr}_{xL}\delta^\mu_\nu \frac{\partial_t R_k}{P_k - \frac{R}{d}} \, .$$

One can now expand to first order in the Ricci scalar R, use the traces (3.6) and formula (A.10) to obtain

$$\begin{aligned}\frac{d\Gamma_k}{dt} \;=\; & \frac{1}{(4\pi)^{d/2}}\int dx\,\sqrt{g}\Bigg\{\frac{d\,(d+1)}{4}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k+\eta R_k}{P_k-2\Lambda}\right) - dQ_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k}\right) \\ & + \Bigg[\frac{d\,(d+1)}{24}Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k+\eta R_k}{P_k-2\Lambda}\right) - \frac{d}{6}Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k}{P_k}\right) \\ & - \frac{d\,(d-1)}{4}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k+\eta R_k}{(P_k-2\Lambda)^2}\right) - Q_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k^2}\right)\Bigg] R + O(R^2)\Bigg\} \, . \end{aligned} \tag{3.16}$$

This derivation highlights two noteworthy facts. The first is that the negative sign of the kinetic term for the trace part of h is immaterial. With the chosen form for the cutoff, *any prefactor multiplying the kinetic operator in the inverse propagator cancels out between the two factors in the r.h.s. of the ERGE*. The second fact, which we will exploit in the following, is that *the singularity occurring in the kinetic operator for the trace part in $d = 2$ (see equation (3.4)) is actually made harmless by a hidden factor $d - 2$ occurring in* $\mathbf{U}$. So, the final result (3.16) is perfectly well defined also in two dimensions.
On the other hand, an issue that is sometimes raised in connection with this calculation is background dependence. The calculations in section 2.3 were done without choosing a specific background, so the question arises whether the same can be done here. The answer is positive, provided we do not decompose the field $h_{\mu\nu}$ into tracefree and trace parts, and we use for the inverse propagator the form given in the first line of (3.4). Then, the modified inverse propagator for gravitons is

$$\mathbf{\Gamma}^{(2)}_k + \mathbf{R}_k = Z\mathbf{K}\left(P_k(-\nabla^2) - 2\Lambda\mathbf{1} + \mathbf{W}\right) \, . \tag{3.17}$$

On a general background $\mathbf{\Gamma}^{(2)}_k + \mathbf{R}_k$ cannot be inverted exactly, but remembering that $\mathbf{W}$ is linear in curvature we can expand to first order

$$\begin{aligned}\left(\mathbf{\Gamma}^{(2)}_k + \mathbf{R}_k\right)^{-1} &= \frac{1}{P_k-2\Lambda}\left[\mathbf{1} - \frac{1}{P_k-2\Lambda}\mathbf{W} + O(R^2)\right] \\ \Gamma^{(2)}_{C\bar{C}}{}^{\mu}{}_{\nu} + R^{(gh)}_k{}^{\mu}{}_{\nu} &= \frac{1}{P_k}\left[\delta^\mu_\nu + \frac{1}{P_k}R^\mu{}_\nu + O(R^2)\right] \, . \end{aligned} \tag{3.18}$$

Then the ERGE becomes, up to terms of higher order in curvature,

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\mathrm{Tr}_{xL}\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda}\left[\mathbf{1} - \frac{1}{P_k - 2\Lambda}\mathbf{W}\right] - \mathrm{Tr}_{xL}\frac{\partial_t R_k}{P_k}\left[\delta^\mu_\nu + \frac{1}{P_k}R^\mu{}_\nu\right] .$$

From here, using (A.10) one arrives again at (3.16). This alternative derivation explicitly highlights the *background independence* of the results.

We are now ready to extract the beta functions. The first line of (3.16) gives the beta function of $2Z\Lambda$, while the other two lines give the beta function of $-Z$. Note the appearance of the beta function of Z in the η terms on the r.h.s. In a perturbative one loop calculation such terms would be absent; they are a result of the "renormalization group improvement" implicit in the ERGE. The beta functions can be written in the form

$$\begin{aligned} \frac{d}{dt}\left(\frac{2\Lambda}{16\pi G}\right) &= \frac{k^d}{16\pi}(A_1 + A_2\eta) \\ -\frac{d}{dt}\left(\frac{1}{16\pi G}\right) &= \frac{k^{d-2}}{16\pi}(B_1 + B_2\eta) \end{aligned} \tag{3.19}$$

where A_1, A_2, B_1 and B_2 are dimensionless functions of Λ, k and of d which, by dimensional analysis, can also be written as functions of $\tilde\Lambda = \Lambda k^{-2}$ and d. One can solve these equations for $\frac{d\tilde\Lambda}{dt}$ and $\frac{d\tilde G}{dt}$, obtaining

$$\begin{aligned} \frac{d\tilde\Lambda}{dt} &= -2\tilde\Lambda + \tilde G\,\frac{A_1 + 2B_1\tilde\Lambda + \tilde G(A_1B_2 - A_2B_1)}{2(1 + B_2\tilde G)} , \\ \frac{d\tilde G}{dt} &= (d-2)\tilde G + \frac{B_1\tilde G^2}{1 + B_2\tilde G} . \end{aligned} \tag{3.20}$$

The corresponding perturbative one loop beta functions are obtained by neglecting the η terms in (3.19), *i.e.* setting $A_2 = B_2 = 0$, and expanding A_1 and B_1 in $\tilde\Lambda$. The leading term is

$$\begin{aligned} \frac{d\tilde\Lambda}{dt} &= -2\tilde\Lambda + \frac{1}{2}A_1(0)\tilde G + B_1(0)\tilde G\tilde\Lambda \\ \frac{d\tilde G}{dt} &= (d-2)\tilde G + B_1(0)\tilde G^2 \end{aligned} \tag{3.21}$$

where A_1 and B_1 are evaluated at $\tilde\Lambda = 0$. This flow has the same structure as the one written in (2.70); we will refer to it as the "perturbative Einstein–Hilbert flow". We will discuss its solution in section 3.8.

The explicit form of the coefficients appearing in (3.19), with the optimized cutoff, is

$$A_1 = \frac{16\,\pi\,(d - 3 + 8\,\tilde\Lambda)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})\,(1 - 2\,\tilde\Lambda)}$$

$$
\begin{aligned}
A_2 &= \frac{16\pi\,(d+1)}{(4\pi)^{\frac{d}{2}}(d+2)\,\Gamma(\frac{d}{2})\,(1-2\,\tilde\Lambda)} \\
B_1 &= \frac{-4\pi(-d^3+15d^2-12d+48+(2d^3-14d^2-192)\tilde\Lambda+(16d^2+192)\tilde\Lambda^2)}{3(4\pi)^{\frac{d}{2}}\,d\,\Gamma(\frac{d}{2})\,(1-2\,\tilde\Lambda)^2} \\
B_2 &= \frac{4\pi\,(d^2-9\,d+14-2\,(d+1)\,(d+2)\,\tilde\Lambda)}{3\,(4\pi)^{\frac{d}{2}}(d+2)\,\Gamma(\frac{d}{2})\,(1-2\,\tilde\Lambda)^2}\,.
\end{aligned}
$$

A similar form of the beta functions had been given in [42] in another gauge. For the sake of clarity we write here the beta functions in four dimensions:

$$
\begin{aligned}
\beta_{\tilde\Lambda} &= -2\tilde\Lambda+\frac{\tilde G}{6\pi}\frac{3-4\tilde\Lambda-12\tilde\Lambda^2-56\tilde\Lambda^3+\frac{107-20\tilde\Lambda}{12\pi}\tilde G}{(1-2\tilde\Lambda)^2-\frac{1+10\tilde\Lambda}{12\pi}\tilde G} \\
\beta_{\tilde G} &= 2\tilde G-\frac{\tilde G^2}{3\pi}\frac{11-18\tilde\Lambda+28\tilde\Lambda^2}{(1-2\tilde\Lambda)^2-\frac{1+10\tilde\Lambda}{12\pi}\tilde G}\,.
\end{aligned}
\tag{3.22}
$$

Note the nontrivial denominators, which in a series expansion could be seen as resummations of infinitely many terms of perturbation theory. They are the result of the "RG improvement" in the ERGE.

3.2. Cutoff of type Ib

This type of cutoff was introduced in [43]. The fluctuation $h_{\mu\nu}$ and the ghosts are decomposed into their different spin components according to

$$
h_{\mu\nu}=h^T_{\mu\nu}+\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu+\nabla_\mu\nabla_\nu\sigma-\frac{1}{d}g_{\mu\nu}\nabla^2\sigma+\frac{1}{d}g_{\mu\nu}h \tag{3.23}
$$

and

$$
C^\mu=c^{T\,\mu}+\nabla^\mu c\ \ ,\bar C_\mu=\bar c^T_\mu+\nabla_\mu\bar c\,, \tag{3.24}
$$

where $h^T_{\mu\nu}$ is transverse and traceless, ξ is a transverse vector, σ and h are scalars, c^T and $\bar c^T$ are transverse vectors, c and $\bar c$ are scalars. These fields are subject to the differential constraints

$$
h^{T\mu}_\mu=0\,;\qquad \nabla^\nu h^T_{\mu\nu}=0\,;\qquad \nabla^\nu\xi_\nu=0\,;\qquad \nabla^\mu\bar c^T_\mu=0\,;\qquad \nabla_\mu c^{T\,\mu}=0\,.
$$

Using this decomposition can be advantageous in some cases because it can lead to a partial diagonalization of the kinetic operator and it allows an exact inversion. This is the case for example when the background is a maximally symmetric metric. In this section we will therefore assume that the background is a sphere; this is enough to extract exactly and unambiguously the beta functions of the cosmological constant and Newton's constant. Then the ERGE (2.50) can be written down for arbitrary gauge α and ρ. We refer to [43] for more details of the calculation. In the gauge $\alpha = 1$ and without making any approximation, the inverse propagators of the individual components are

$$
\begin{aligned}
\Gamma^{(2)}_{h^T_{\mu\nu}h^T_{\alpha\beta}} &= \frac{Z}{2}\left[-\nabla^2 + \frac{d^2-3d+4}{d(d-1)}R - 2\Lambda\right]\delta^{\mu\nu,\alpha\beta} \\
\Gamma^{(2)}_{\xi_\mu\xi_\nu} &= Z\left(-\nabla^2 - \frac{R}{d}\right)\left[-\nabla^2 + \frac{d-3}{d}R - 2\Lambda\right]g^{\mu\nu} \\
\Gamma^{(2)}_{hh} &= -Z\frac{d-2}{4d}\left[-\nabla^2 + \frac{d-4}{d}R - 2\Lambda\right] \\
\Gamma^{(2)}_{\sigma\sigma} &= Z\frac{d-1}{2d}\left(-\nabla^2\right)\left(-\nabla^2 - \frac{R}{d-1}\right)\left[-\nabla^2 + \frac{d-4}{d}R - 2\Lambda\right] \\
\Gamma^{(2)}_{\bar{c}^T_\mu c^T_\nu} &= \left[\nabla^2 + \frac{R}{d}\right]g^{\mu\nu} \\
\Gamma^{(2)}_{\bar{c}c} &= -\nabla^2\left[\nabla^2 + \frac{2}{d}R\right] .
\end{aligned}
\tag{3.25}
$$

The change of variables (3.23) and (3.24) leads to Jacobian determinants involving the operators

$$
J_V = -\nabla^2 - \frac{R}{d},\; J_S = -\nabla^2\left(-\nabla^2 - \frac{R}{d-1}\right),\; J_c = -\nabla^2 \tag{3.26}
$$

for the vector, scalar and ghost parts. The inverse propagators (3.25) contain four derivative terms. In [43, 20] this was avoided by making the field redefinitions

$$
\xi_\mu \;\to\; \sqrt{-\nabla^2 - \frac{R}{d}}\,\xi_\mu,\; \sigma \to \sqrt{-\nabla^2}\sqrt{-\nabla^2 - \frac{R}{d-1}}\sigma\,. \tag{3.27}
$$

At the same time, such redefinitions also eliminate the Jacobians. These field redefinitions work well for truncations containing up to two powers of curvature, but cause poles for higher truncations as the heat kernel expansion will involve derivatives of the trace arguments. Therefore, in later sections we will not perform the field redefinitions, but treat the Jacobians instead as further contribution to the ERGE by exponentiating them, introducing appropriate auxiliary fields and a cutoff on these variables. Here we describe the result of performing the field redefinitions. The ERGE is

$$
\begin{aligned}
\frac{d\Gamma_k}{dt} &= \frac{1}{2}\mathrm{Tr}_{(2)}\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda + \frac{d^2-3d+4}{d(d-1)}R} + \frac{1}{2}\mathrm{Tr}'_{(1)}\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda + \frac{d-3}{d}R} \\
&\quad + \frac{1}{2}\mathrm{Tr}_{(0)}\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda + \frac{d-4}{d}R} + \frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda + \frac{d-4}{d}R} \\
&\quad - \mathrm{Tr}_{(1)}\frac{\partial_t R_k}{P_k - \frac{R}{d}} - \mathrm{Tr}'_{(0)}\frac{\partial_t R_k}{P_k - \frac{2R}{d}} .
\end{aligned}
\tag{3.28}
$$

The first term comes from the spin–2, transverse traceless components, the second from the spin–1 transverse vector, the third and fourth from the scalars h and σ. The last two contributions come from the transverse and longitudinal components of the ghosts. A prime or a double prime indicate that the first or the first and second eigenvalues have to be omitted from the trace. The reason for this is explained in appendix A.2.
Expanding the denominators to first order in the Ricci scalar R, but keeping the exact dependence on Λ as in the case of a type Ia cutoff, and using the formula (A.10), one obtains

$$\begin{aligned}\frac{d\Gamma_k}{dt} &= \frac{1}{(4\pi)^{d/2}}\int dx\,\sqrt{g}\Bigg\{\frac{d(d+1)}{4}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k+\eta R_k}{P_k-2\Lambda}\right)-dQ_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k}\right)\\ &\quad +R\left[-\frac{d^4-2d^3-d^2-4d+2}{4d(d-1)}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k+\eta R_k}{(P_k-2\Lambda)^2}\right)-\frac{d+1}{d}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k^2}\right)\right.\\ &\quad \left.+\frac{d^4-13d^2-24d+12}{24d(d-1)}Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k+\eta R_k}{P_k-2\Lambda}\right)-\frac{d^2-6}{6d}Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k}{P_k}\right)\right]+O(R^2)\Bigg\}.\end{aligned} \tag{3.29}$$

In principle in two dimensions one has to subtract the contributions of some excluded modes. However, using the results in appendix A.2, the contributions of these isolated modes turn out to cancel. Thus, the ERGE is continuous in the dimension also at $d=2$.
The beta functions have again the form (3.20); the coefficients A_1 and A_2 are the same as for the type Ia cutoff but now the coefficients B_1 and B_2 are

$$\begin{aligned}B_1 &= 4\pi\Big(d(d-1)(d^3-15d^2-36)+24-2(d^5-8d^4-5d^3-72d^2-36d+96)\tilde\Lambda\\ &\quad -16(d-1)(d^3+6d+12)\tilde\Lambda^2\Big)\Big/3(4\pi)^{\frac{d}{2}}d(d-1)\Gamma\left(\frac{d}{2}\right)(1-2\tilde\Lambda)^2\\ B_2 &= 4\pi\frac{d(d^4-10d^3+11d^2-38d+12)-2(d+2)(d^4-13\,d^2-24d+12)\,\tilde\Lambda}{3(4\pi)^{\frac{d}{2}}(2+d)(d-1)d^2\,\Gamma(\frac{d}{2})(1-2\tilde\Lambda)^2}\,.\end{aligned} \tag{3.30}$$

In four dimensions, the beta functions are

$$\begin{aligned}\beta_{\tilde\Lambda} &= -2\tilde\Lambda+\frac{1}{24\pi}\frac{(12-33\tilde\Lambda+20\tilde\Lambda^2-200\tilde\Lambda^3)\tilde G+\frac{467-572\tilde\Lambda}{12\pi}\tilde G^2}{(1-2\tilde\Lambda)^2-\frac{29-9\tilde\Lambda}{72\pi}\tilde G}\\ \beta_{\tilde G} &= 2\tilde G-\frac{1}{24\pi}\frac{(105-212\tilde\Lambda+200\tilde\Lambda^2)\tilde G^2}{(1-2\tilde\Lambda)^2-\frac{29-9\tilde\Lambda}{72\pi}\tilde G}\,.\end{aligned} \tag{3.31}$$

In order to appreciate the numerical differences between this procedure and the one where the fields ξ_μ and σ are not redefined as in (3.27), we report in appendix A.4 the results of the alternative formulation.

3.3. Cutoff of type IIa

Let us define the following operators acting on gravitons and on ghosts

$$\begin{aligned} \Delta_2 &= -\nabla^2 + \mathbf{W} \quad (3.32)\\ \Delta_{(gh)} &= -\nabla^2 - \text{Ricci}\,. \quad (3.33) \end{aligned}$$

The traces of the $\mathbf{b}_2$–coefficients of the heat kernel expansion for these operators are

$$\begin{aligned} \mathbf{tr}\mathbf{b}_2(\Delta_2) &= \text{tr}\left(\frac{R}{6}\mathbf{1} - \mathbf{W}\right) = \frac{d(7-5d)}{12}R \\ \mathbf{tr}\mathbf{b}_2(\Delta_{gh}) &= \text{tr}\left(\frac{R}{6}\mathbf{1} + \text{Ricci}\right) = \frac{d+6}{6}R\,. \end{aligned} \quad (3.34)$$

The type II cutoff is defined to have as an argument the differential operator and the curvature terms without couplings and therefore leads to the choice

$$\begin{aligned} \mathbf{R}_k &= Z\mathbf{K}R_k(\Delta_2) \\ R_k^{(gh)\mu}{}_{\nu} &= \delta^\mu{}_\nu R_k(\Delta_{(gh)}) \end{aligned} \quad (3.35)$$

which results in

$$\begin{aligned} \mathbf{\Gamma}_k^{(2)} + \mathbf{R}_k &= Z\mathbf{K}\left(P_k(\Delta_2) - 2\Lambda\right) \\ \Gamma^{(2)}_{C\bar{C}} + R_k^{(gh)} &= P_k(\Delta_{(gh)}) \end{aligned} \quad (3.36)$$

and

$$\frac{d\mathbf{R}_k}{dt} = Z\mathbf{K}\left(\partial_t R_k(\Delta_2) + \eta R_k(\Delta_2)\right)\,. \quad (3.37)$$

Collecting all terms and evaluating the traces leads to

$$\begin{aligned} \frac{d\Gamma_k}{dt} &= \frac{1}{2}\text{Tr}_{xL}\frac{\partial_t R_k(\Delta_2) + \eta R_k(\Delta_2)}{P_k(\Delta_2) - 2\Lambda} - \text{Tr}_{xL}\frac{\partial_t R_k(\Delta_{(gh)})}{P_k(\Delta_{(gh)})} \\ &= \frac{1}{(4\pi)^{d/2}}\int dx\,\sqrt{g}\left\{\frac{d(d+1)}{4}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda}\right) - d\,Q_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k}\right)\right. \\ &\quad \left. + \left[\frac{d(7-5d)}{24}\,Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda}\right) - \frac{d+6}{6}Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k}{P_k}\right)\right]R + O(R^2)\right\}\,. \end{aligned} \quad (3.38)$$

The beta functions are again of the form (3.20), and the coefficients A_1 and A_2 are the same as in the case of the cutoffs of type I. The coefficients B_1 and B_2 are now

$$B_1 = -\frac{4\pi(5d^2 - 3d + 24 - 8(d+6)\tilde{\Lambda})}{3(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})(1 - 2\,\tilde{\Lambda})}$$

$$B_2 = -\frac{4\pi(5d-7)}{3(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})(1-2\tilde{\Lambda})} \,. \tag{3.39}$$

In four dimensions, the beta functions are

$$\begin{aligned}\beta_{\tilde{\Lambda}} &= -2\tilde{\Lambda} + \frac{1}{6\pi}\frac{(3-28\tilde{\Lambda}+84\tilde{\Lambda}^2-80\tilde{\Lambda}^3)\tilde{G} + \frac{191-512\tilde{\Lambda}}{12\pi}\tilde{G}^2}{(1-2\tilde{\Lambda})(1-2\tilde{\Lambda}-\frac{13}{12\pi}\tilde{G})} \\ \beta_{\tilde{G}} &= 2\tilde{G} - \frac{1}{3\pi}\frac{(23-20\tilde{\Lambda})\tilde{G}^2}{(1-2\tilde{\Lambda})-\frac{13}{12\pi}\tilde{G}} \,.\end{aligned} \tag{3.40}$$

3.4. Cutoff of type IIb

Now we decompose the fluctuation $h_{\mu\nu}$ and the ghosts as for the type Ib cutoff with the same field redefinitions, but apply the type II cutoff procedure afterwards, replacing therefore the Laplace operator and the term proportional to the scalar curvature together by the cutoff function with the same argument. Therefore we define the operators acting on tensor, vector, and scalar graviton components, and vector, and scalar ghost components as

$$\Delta_2^{TT} = -\nabla^2 + \frac{d^2-3d+4}{d(d-1)}R \tag{3.41}$$

$$\Delta_2^{\xi_\mu\xi_\mu} = -\nabla^2 + \frac{d-3}{d}R \tag{3.42}$$

$$\Delta_2^{hh} = -\nabla^2 + \frac{d-4}{d}R \tag{3.43}$$

$$\Delta_2^{\sigma\sigma} = -\nabla^2 + \frac{d-4}{d}R \tag{3.44}$$

$$\Delta_2^{\bar{c}_\mu^T c_\mu^T} = -\nabla^2 - \frac{1}{d}R \tag{3.45}$$

$$\Delta_2^{\bar{c}c} = -\nabla^2 - \frac{2}{d}R \,. \tag{3.46}$$

The traces of the $\mathbf{b}_2$–coefficients of the heat kernel expansion for these operators are

$$\begin{aligned}\mathrm{tr}\mathbf{b}_2(\Delta_2^{TT}) &= \mathrm{tr}\left(\frac{R}{6}\mathbf{1} - \frac{d^2-3d+4}{d(d-1)}R\mathbf{1}\right) = \frac{-5\,d^3+17d^2-26d-48}{12\,d} \\ \mathrm{tr}\mathbf{b}_2(\Delta_2^{\xi_\mu\xi_\mu}) &= \mathrm{tr}\left(\frac{R}{6}\mathbf{1} - \frac{d-3}{d}R\mathbf{1}\right) = \frac{-5\,d^2+23d-24}{6\,d} \\ \mathrm{tr}\mathbf{b}_2(\Delta_2^{hh}) &= \mathrm{tr}\left(\frac{R}{6}\mathbf{1} - \frac{d-4}{d}R\mathbf{1}\right) = \frac{-5d+24}{6\,d} \\ \mathrm{tr}\mathbf{b}_2(\Delta_2^{\sigma\sigma}) &= \mathrm{tr}\left(\frac{R}{6}\mathbf{1} - \frac{d-4}{d}R\mathbf{1}\right) = \frac{-5\,d+24}{6\,d}\end{aligned}$$

$$
\begin{aligned}
\mathrm{tr}\mathbf{b}_2(\Delta_2^{\bar{c}^T_\mu c^T_\mu}) &= \mathrm{tr}\left(\frac{R}{6}\mathbf{1} + \frac{1}{d}R\mathbf{1}\right) = \frac{d^2+5d-12}{6\,d} \\
\mathrm{tr}\mathbf{b}_2(\Delta_2^{\bar{c}c}) &= \mathrm{tr}\left(\frac{R}{6}\mathbf{1} + \frac{2}{d}R\mathbf{1}\right) = \frac{d+12}{6\,d}\,.
\end{aligned} \tag{3.47}
$$

The type II cutoff is defined to have as an argument the differential operator and the curvature terms without couplings and therefore leads to the choice

$$
\begin{aligned}
\mathbf{R}_k^{TT} &= ZR_k(\Delta_2^{TT}) \\
\mathbf{R}_k^{\xi_\mu\xi_\mu} &= ZR_k(\Delta_2^{\xi_\mu\xi_\mu}) \\
\mathbf{R}_k^{hh} &= ZR_k(\Delta_2^{hh}) \\
\mathbf{R}_k^{\sigma\sigma} &= ZR_k(\Delta_2^{\sigma\sigma}) \\
\mathbf{R}_k^{\bar{c}^T_\mu c^T_\mu} &= R_k(\Delta_2^{\bar{c}^T_\mu c^T_\mu}) \\
\mathbf{R}_k^{\bar{c}c} &= R_k(\Delta_2^{\bar{c}c})
\end{aligned} \tag{3.48}
$$

which results in

$$
\begin{aligned}
\mathbf{\Gamma}_k^{(2)TT} + \mathbf{R}_k^{TT} &= Z\left(P_k(\Delta_2^{TT}) - 2\Lambda\right) \\
\mathbf{\Gamma}_k^{(2)\xi_\mu\xi_\mu} + \mathbf{R}_k^{\xi_\mu\xi_\mu} &= Z\left(P_k(\Delta_2^{\xi_\mu\xi_\mu}) - 2\Lambda\right) \\
\mathbf{\Gamma}_k^{(2)hh} + \mathbf{R}_k^{hh} &= Z\left(P_k(\Delta_2^{hh}) - 2\Lambda\right) \\
\mathbf{\Gamma}_k^{(2)\sigma\sigma} + \mathbf{R}_k^{\sigma\sigma} &= Z\left(P_k(\Delta_2^{\sigma\sigma}) - 2\Lambda\right) \\
\mathbf{\Gamma}_k^{(2)\bar{c}^T_\mu c^T_\mu} + \mathbf{R}_k^{\bar{c}^T_\mu c^T_\mu} &= \left(P_k(\Delta_2^{\bar{c}^T_\mu c^T_\mu}) - 2\Lambda\right) \\
\mathbf{\Gamma}_k^{(2)\bar{c}c} + \mathbf{R}_k^{\bar{c}c} &= \left(P_k(\Delta_2^{\bar{c}c}) - 2\Lambda\right)
\end{aligned} \tag{3.49}
$$

and

$$
\begin{aligned}
\frac{d\mathbf{R}_k^{TT}}{dt} &= Z\left(\partial_t R_k(\Delta_2^{TT}) + \eta R_k(\Delta_2^{TT})\right) \\
\frac{d\mathbf{R}_k^{\xi_\mu\xi_\mu}}{dt} &= Z\left(\partial_t R_k(\Delta_2^{\xi_\mu\xi_\mu}) + \eta R_k(\Delta_2^{\xi_\mu\xi_\mu})\right) \\
\frac{d\mathbf{R}_k^{hh}}{dt} &= Z\left(\partial_t R_k(\Delta_2^{hh}) + \eta R_k(\Delta_2^{hh})\right) \\
\frac{d\mathbf{R}_k^{\sigma\sigma}}{dt} &= Z\left(\partial_t R_k(\Delta_2^{\sigma\sigma}) + \eta R_k(\Delta_2^{\sigma\sigma})\right) \\
&.
\end{aligned} \tag{3.50}
$$

Collecting all terms and evaluating the traces leads to

$$
\begin{aligned}
\frac{d\Gamma_k}{dt} &= \frac{1}{2}\mathrm{Tr}_{(2)}\frac{\partial_t R_k(\Delta_2^{TT}) + \eta R_k(\Delta_2^{TT})}{P_k(\Delta_2^{TT}) - 2\Lambda} + \frac{1}{2}\mathrm{Tr}'_{(1)}\frac{\partial_t R_k(\Delta_2^{\xi_\mu\xi_\mu}) + \eta R_k(\Delta_2^{\xi_\mu\xi_\mu})}{P_k(\Delta_2^{\xi_\mu\xi_\mu}) - 2\Lambda} \\
&+ \frac{1}{2}\mathrm{Tr}_{(0)}\frac{\partial_t R_k(\Delta_2^{hh}) + \eta R_k(\Delta_2^{hh})}{P_k(\Delta_2^{hh}) - 2\Lambda} + \frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\partial_t R_k(\Delta_2^{\sigma\sigma}) + \eta R_k(\Delta_2^{\sigma\sigma})}{P_k(\Delta_2^{\sigma\sigma}) - 2\Lambda} \\
&- \mathrm{Tr}_{(1)}\frac{\partial_t R_k(\Delta_2^{\bar{c}^T_\mu c^T_\mu})}{P_k(\Delta_2^{\bar{c}^T_\mu c^T_\mu})} - \mathrm{Tr}'_{(0)}\frac{\partial_t R_k(\Delta_2^{\bar{c}c})}{P_k(\Delta_2^{\bar{c}c})} \\
&= \frac{1}{(4\pi)^{d/2}}\int dx\,\sqrt{g}\left\{\frac{d(d+1)}{4}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda}\right) - d\,Q_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k}\right)\right. \\
&\left. + \left[\frac{d(7-5d)}{24}Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k + \eta R_k}{P_k - 2\Lambda}\right) - \frac{d+6}{6}Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k}{P_k}\right)\right] R + O(R^2)\right\}\,.
\end{aligned}
\tag{3.51}
$$

The last equation is exactly the same one as for type IIa cutoff. The different components added up to the same coefficient. This is due to the simple structure of the cutoff. The trace arguments are for all gravitational and ghost fields the same. What differs are only the heat kernel coefficients for the different spin components. So the Q-functionals containing the trace argument can be separated in the same way as for type IIa cutoff and the coefficients in front must therefore be the same. In the next sections we will therefore simply speak of type II cutoff without distinction.

3.5. Cutoff of type III

Finally we discuss the spectrally adjusted, or type III cutoff. This consists of defining the cutoff function as a function of the whole inverse propagator $\Gamma_k^{(2)}$, only stripped of the overall wave function renormalization constants. In the case of the graviton, $\mathbf{\Gamma}_k^{(2)} = Z\mathbf{K}(\Delta_2 - 2\Lambda\mathbf{1})$ while for the ghosts $\Gamma^{(2)}_{C\bar{C}} = \Delta_{gh}$, where Δ_2 and Δ_{gh} were defined in (3.32). Type III cutoff is defined by the choice

$$
\mathbf{R}_k = Z\mathbf{K}R_k(\Delta_2 - 2\Lambda) \tag{3.52}
$$

for gravitons, while for ghosts it is the same as in the case of type II cutoff. Since the operator in the graviton cutoff now contains the coupling Λ, the derivative of the graviton cutoff now involves an additional term

$$
\frac{d\mathbf{R}_k}{dt} = Z\mathbf{K}\left(\partial_t R_k(\Delta_2 - 2\Lambda) + \eta R_k(\Delta_2 - 2\Lambda) - 2R'_k(\Delta_2 - 2\Lambda)\partial_t\Lambda\right) \tag{3.53}
$$

where R'_k denotes the partial derivative of $R_k(z)$ with respect to z. Note that the use of the chain rule in the last term is only legitimate if the t-derivative of the operator appearing as the argument of R_k commutes with the operator itself. This is the case for the operator $\Delta_2 - 2\Lambda$, since its t-derivative is proportional to the identity. The modified inverse propagator is then simply

$$\mathbf{\Gamma}_k^{(2)} + \mathbf{R}_k = Z\mathbf{K}P_k(\Delta_2 - 2\Lambda)$$

for gravitons, while for ghosts it is again given by equation (3.36). Collecting,

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\mathrm{Tr}_{xL}\frac{\partial_t R_k(\Delta_2 - 2\Lambda) + \eta R_k(\Delta_2 - 2\Lambda) - 2R'_k(\Delta_2 - 2\Lambda)\partial_t\Lambda}{P_k(\Delta_2 - 2\Lambda)} - \mathrm{Tr}_{xL}\frac{\partial_t R_k(\Delta_{(gh)})}{P_k(\Delta_{(gh)})} . \tag{3.54}$$

The traces over the ghosts are exactly as in the case of a cutoff of type II. As in previous cases, one should now proceed to evaluate the trace over the tensors using equation (A.10) and the heat kernel coefficients of the operator $\Delta_2 - 2\Lambda$. However, the situation is now more complicated because the heat kernel coefficients $B_{2k}(\Delta_2 - 2\Lambda)$ contain terms proportional to Λ^k and $\Lambda^{k-1}R$, all of which contribute to the beta functions of $2\Lambda Z$ and $-Z$.

This is in contrast to the calculations with cutoffs of type I and II, where only the first two heat kernel coefficients contributed to the beta functions of $2\Lambda Z$ and $-Z$. In order to resum all these contributions, one can proceed as follows. We define the function

$$W(z) = \frac{\partial_t R_k(z) + \eta R_k(z) - 2R'_k(z)\partial_t\Lambda}{P_k(z)} \tag{3.55}$$

and the function

$$\bar{W}(z) = W(z - 2\Lambda) . \tag{3.56}$$

It is shown explicitly in the end of appendix A.1 (equation (A.33) and following) that $\mathrm{Tr}W = \mathrm{Tr}\bar{W}$. Then, the terms without R and the terms linear in R (which give the beta functions of $2\Lambda Z$ and $-Z$ respectively) correspond to the first two lines in (A.34). In this way we obtain

$$\begin{aligned}\frac{d\Gamma_k}{dt} &= \frac{1}{(4\pi)^{d/2}}\int dx\,\sqrt{g}\left\{\frac{d(d+1)}{4}\sum_{i=0}^{\infty}\frac{(2\Lambda)^i}{i!}Q_{\frac{d}{2}-i}\left(\frac{\partial_t R_k + \eta R_k - 2\partial_t\Lambda R'_k}{P_k}\right) - dQ_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k}\right)\right.\\ &\left.+\frac{d(7-5d)}{24}R\sum_{i=0}^{\infty}\frac{(2\Lambda)^i}{i!}Q_{\frac{d}{2}-1-i}\left(\frac{\partial_t R_k + \eta R_k - 2\partial_t\Lambda R'_k}{P_k}\right) - \frac{d+6}{6}Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k}{P_k}\right)R\right\} .\end{aligned} \tag{3.57}$$

The remarkable property of the optimized cutoff is that in even dimensions the sums in those expressions contain only a finite number of terms; in odd dimensions the sum involves an infinite number of terms but can still be evaluated analytically. Using the results (A.22, A.23, A.24, A.29, A.30, A.31, A.32) the first sum in (3.57) gives

$$\frac{1}{(4\pi)^{d/2}}\frac{d+1}{2}\frac{(k^2+2\Lambda)^{d/2}}{\Gamma(d/2)}\left(2 + \frac{\eta}{\frac{d}{2}+1}\frac{k^2+2\Lambda}{k^2} + 2\frac{\partial_t\Lambda}{k^2}\right)\int dx\,\sqrt{g} \tag{3.58}$$

whereas the second sum gives

$$\frac{1}{(4\pi)^{d/2}}\frac{d(7-5d)}{24}\frac{(k^2+2\Lambda)^{\frac{d-2}{2}}}{\Gamma(d/2)}\left(2+\frac{\eta}{d/2}\frac{k^2+2\Lambda}{k^2}+2\frac{\partial_t\Lambda}{k^2}\right)\int dx\,\sqrt{g}R\,. \tag{3.59}$$

This resummation can actually be done also with other cutoffs. An alternative derivation of these formulae, based on the proper time form of the ERGE is given in appendix A.3. The beta functions cannot be written in the form (3.20) anymore, because of the presence of the derivatives of Λ on the right hand side of the ERGE. Instead of (3.19) we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{2\Lambda}{16\pi G}\right) &= \frac{k^d}{16\pi}(A_1+A_2\eta+A_3\partial_t\tilde{\Lambda})\\ -\frac{d}{dt}\left(\frac{1}{16\pi G}\right) &= \frac{k^{d-2}}{16\pi}(B_1+B_2\eta+B_3\partial_t\tilde{\Lambda})\end{aligned} \tag{3.60}$$

where

$$\begin{aligned}A_1 &= \frac{16\pi(-4+(d+1)(1+2\tilde{\Lambda})^{\frac{d}{2}+1})}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\\ A_2 &= \frac{16\pi(d+1)(1+2\tilde{\Lambda})^{\frac{d}{2}+1}}{(4\pi)^{\frac{d}{2}}(d+2)\Gamma(\frac{d}{2})}\\ A_3 &= \frac{16\pi(d+1)(1+2\tilde{\Lambda})^{\frac{d}{2}}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\\ B_1 &= \frac{4\pi(-4(d+6)+d(7-5d)(1+2\tilde{\Lambda})^{\frac{d}{2}})}{3(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\\ B_2 &= \frac{4\pi(7-5d)(1+2\tilde{\Lambda})^{\frac{d}{2}}}{3(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\\ B_3 &= \frac{4\pi d(7-5d)(1+2\tilde{\Lambda})^{\frac{d}{2}-1}}{3(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\,.\end{aligned} \tag{3.61}$$

Solving (3.60) for $d\tilde{\Lambda}/dt$ and $d\tilde{G}/dt$ gives

$$\begin{aligned}\frac{d\tilde{\Lambda}}{dt} &= -2\tilde{\Lambda}+\frac{(A_1+2(B_1-A_3)\tilde{\Lambda}-4B_3\tilde{\Lambda}^2)\tilde{G}+(A_1B_2-A_2B_1+2(A_2B_3-A_3B_2)\tilde{\Lambda})\tilde{G}^2}{2+(2B_2-A_3-2B_3\tilde{\Lambda})\tilde{G}+(A_2B_3-A_3B_2)\tilde{G}^2}\\ \frac{d\tilde{G}}{dt} &= (d-2)\tilde{G}+\frac{2(B_1-2B_3\tilde{\Lambda})\tilde{G}^2+(A_1B_3-A_3B_1)\tilde{G}^3}{2+(2B_2-A_3-2B_3\tilde{\Lambda})\tilde{G}+(A_2B_3-A_3B_2)\tilde{G}^2}\,.\end{aligned} \tag{3.62}$$

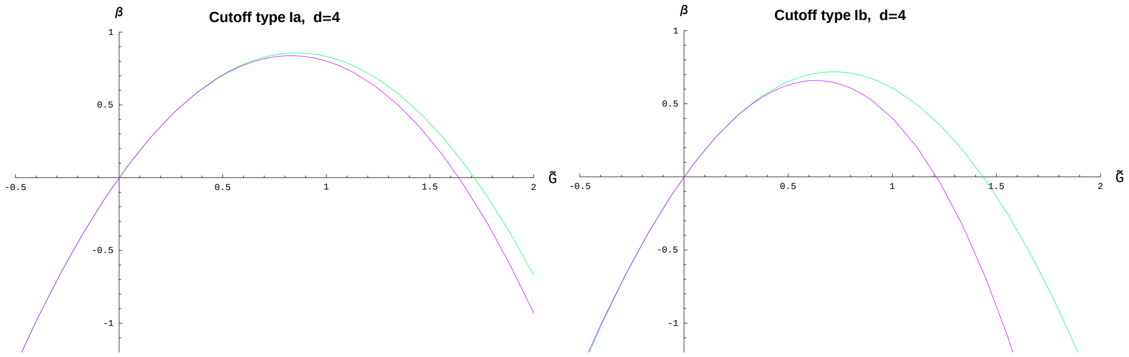

Figure 3.1.: The beta function of $\tilde{G}$ with $\tilde{\Lambda} = 0$ and cutoffs of type Ia and Ib. The perturbative one loop result in light gray, the RG improved one in darker color.

In four dimensions, the beta functions are

$$
\begin{aligned}
\beta_{\tilde{\Lambda}} &= -2\tilde{\Lambda} + \frac{1}{6\pi} \frac{(3 + 14\tilde{\Lambda} + 8\tilde{\Lambda}^2)\tilde{G} + \frac{(1+2\tilde{\Lambda})^2}{12\pi}(191 - 60\tilde{\Lambda} - 260\tilde{\Lambda}^2)\tilde{G}^2}{1 - \frac{1}{12\pi}(43 + 120\tilde{\Lambda} + 68\tilde{\Lambda}^2)\tilde{G} + \frac{65}{72\pi^2}(1 + 2\tilde{\Lambda})^4\tilde{G}^2} \\
\beta_{\tilde{G}} &= 2\tilde{G} - \frac{1}{3\pi} \frac{(23 + 26\tilde{\Lambda})\tilde{G}^2 - \frac{51+152\tilde{\Lambda}+100\tilde{\Lambda}^2}{\pi}\tilde{G}^3}{1 - \frac{1}{12\pi}(43 + 120\tilde{\Lambda} + 68\tilde{\Lambda}^2)\tilde{G} + \frac{65}{72\pi^2}(1 + 2\tilde{\Lambda})^4\tilde{G}^2} \,.
\end{aligned}
\tag{3.63}
$$

3.6. Gravity without cosmological constant

Before discussing the general case, it is instructive to consider the case $\Lambda = 0$. In terms of the dimensionless coupling $\tilde{G}$ the beta functions with cutoffs of type I and II have the form (3.20), where the constants B_1 and B_2 are evaluated at $\tilde{\Lambda} = 0$. The cutoff of type III leads instead to the more complicated beta function (3.62), with $\tilde{\Lambda} = 0$. The beta function of $\tilde{G}$ in four dimensions is shown in figures 3.1 and 3.2 (blue, dark lines) for different cutoff types. It always has a Gaußian FP in the origin and a nontrivial FP at

$$
\tilde{G}_* = -\frac{d-2}{B_1 + (d-2)B_2} \,. \tag{3.64}
$$

The Gaußian FP is always UV–repulsive (positive slope) whereas the non–Gaußian FP is UV–attractive. In a theory with a single coupling constant as in this case, the slope of the beta function at the nontrivial FP is related to ν, the mass critical exponent. For type I and II cutoffs it is given by

$$
\vartheta = \frac{1}{\nu} = -\left.\frac{\partial \beta_{\tilde{G}}}{\partial \tilde{G}}\right|_* = (d-2)\left(1 + (d-2)\frac{B_2}{B_1}\right) \,. \tag{3.65}
$$

The leading, classical term is universally equal to $d-2$, the correction is scheme–dependent.

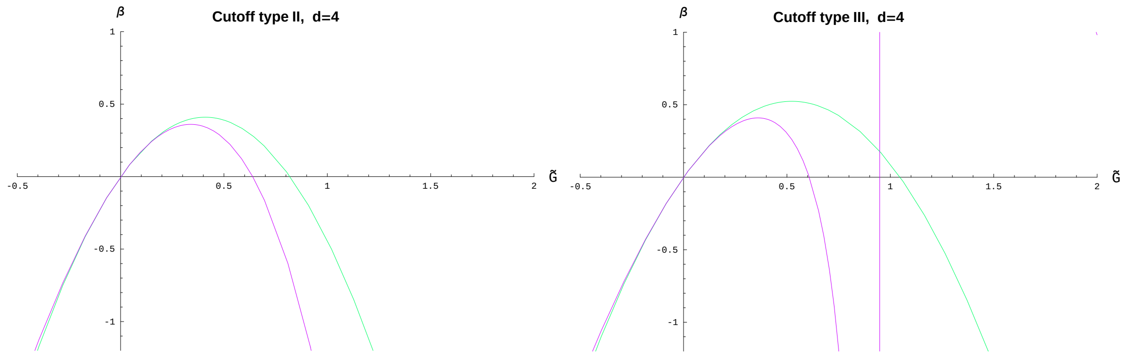

Figure 3.2.: The beta function of $\tilde{G}$ with $\tilde{\Lambda} = 0$ and cutoffs of type II and III. The perturbative one loop result in light gray, the RG improved one in darker color. In the case of cutoff type III the vertical line is the asymptote of the RG improved beta function.

It is instructive to compare these "RG improved" results to the respective "one loop approximations". As discussed in section 2.2, the "unimproved" perturbative one–loop beta functions are obtained by neglecting the derivatives of the couplings occurring in the r.h.s. of the ERGE, in practice setting $B_2 = 0$,

$$\beta_{\tilde{G}}|_{1loop} = (d-2)\tilde{G} + B_1\tilde{G}^2 \, . \tag{3.66}$$

These beta functions are shown as the gray, light lines in figures 3.1 and 3.2. Since $B_1 < 0$, they are inverted parabolas, with a Gaußian FP in the origin and a nontrivial FP at $\tilde{G}_* = -(d-2)/B_1$. The slope at the nontrivial FP is always the opposite of the one at the Gaußian FP, and therefore equal to -2. Note that both (1.11) and (1.12) are of this form, for specific values of the constant B_1.

Because $B_1 < B_2 < 0$, the "RG improved" FP occurs always at smaller values of G_* than the corresponding perturbative one. The Gaußian FP is always UV–repulsive (positive slope) whereas the non–Gaußian FP is UV–attractive. In table 3.1 we report the numerical values of the coefficients B_1 and B_2 and the position of the FP and the critical exponent in four dimensions.

One can see from figure 3.1 that for cutoffs of type I the one loop approximation is quite good up to the nontrivial FP and a little beyond. For larger values of $\tilde{G}$ the effect of the denominator in (3.20) becomes important; the beta function deviates strongly from the one loop approximation and has a negative pole at $\tilde{G} = -1/B_2 > \tilde{G}_*$. When one considers type II and III cutoffs, the RG improved beta functions deviate from the perturbative one loop beta functions sooner and the effect is stronger; in the case of the type III cutoff the pole occurs before the one loop beta function has the zero. We see that the RG improvement leads to stronger effects if more terms of the operator are taken into the definition of the cutoff. It is interesting to observe that the RG improvement always brings the nontrivial fixed point closer to the perturbative regime. Since at low energies

Approximation	$B_1(0)$	$B_2(0)$	$\tilde{G}_*$	ϑ
ϵ exp., leading order	$-\frac{19}{24\pi}$		0.158	2.000
Ia	$-\frac{11}{3\pi}$	$-\frac{1}{12\pi}$	1.639	2.091
Ia - 1 loop			1.714	2.000
Ib with f.r.	$-\frac{35}{8\pi}$	$-\frac{29}{72\pi}$	1.213	2.368
Ib with f.r. - 1 loop			1.436	2.000
Ib without f.r.	$-\frac{131}{45\pi}$	$-\frac{61}{720\pi}$	2.040	2.116
Ib without f.r. - 1 loop			2.158	2.000
II	$-\frac{23}{3\pi}$	$-\frac{13}{12\pi}$	0.954	2.565
II - 1 loop			0.820	2.000
III	$-\frac{23}{3\pi}$	$-\frac{13}{12\pi}$	0.424	3.870
III - 1 loop			0.820	2.000

Table 3.1.: The fixed point for Einstein's theory in $d = 4$ without cosmological constant. The leading beta function for the ϵ expansion is derived in section 3.7; the one for the cutoff of type Ib without field redefinitions is given in appendix A.4.

$\tilde{G}$ is close to zero, the region of physical interest is $0 < \tilde{G} < \tilde{G}_*$. Thus, the pole in the beta functions at finite values of $\tilde{G}$ should not worry us.

3.7. The ϵ expansion.

Before discussing the four dimensional case, it is useful and instructive to consider the situation in arbitrary dimensions. In particular, this will allow us to compare the results of the ERGE with those obtained in the ϵ–expansion. We have seen within the approximations of section 2.2 that in d dimensions the beta function of the dimensionless coefficient of the $R^{d/2}$ term is scheme–independent. Therefore in two dimensions one expects the beta function of Newton's constant, or at least its leading term, to be scheme–independent. This is confirmed by formula (3.66) for the one loop beta function, and the results listed in table 3.1,

$$-B_1 = \frac{38}{3} \tag{3.67}$$

for all types of cutoff. On the other hand the coefficient B_2 is scheme–dependent. This mirrors the well known fact that in perturbation theory the leading term of the beta functions of dimensionless couplings is scheme–independent and higher loop corrections are not.

The beta function (3.20) with $\tilde{\Lambda} = 0$ can be solved exactly for any d. The nontrivial FP occurs at

$$\tilde{G}_* = -\frac{d-2}{B_1 + (d-2)B_2} \, . \tag{3.68}$$

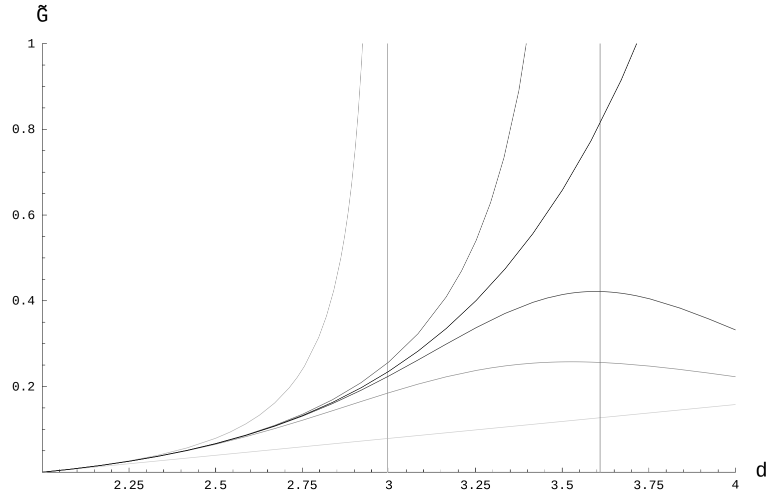

Figure 3.3.: The position of $\tilde{G}_*$ as a function of d in the one loop approximation with cutoff of type Ia (central line in black). The other lines are the first five orders of the ϵ expansion (order n means the beta function has been expanded to order n in ϵ.) The zeroth order is the light gray straight line. Higher orders are represented by darker shades of gray. Order 1 and order 3 have singularities at $d \approx 2.998$ and $d \approx 3.609$.

Knowing the solution in any dimension we can now check a posteriori how good the ϵ–expansion is. To this end we have to expand the beta function in powers of $\epsilon = d - 2$ and look for FPs of the approximated beta functions.
The leading term of the ϵ expansion consists in retaining only the constant, scheme–independent term (3.67). Then the beta function is given by equation (1.11) and the fixed point occurs (for any cutoff type) at

$$\tilde{G}_* = \frac{3}{38}\epsilon \, .$$

We see from table 3.1 that this value is quite small compared to the direct calculations in four dimensions. For the higher orders of this expansion it is necessary to specialize the discussion to a specific type of cutoff. For definiteness we will consider the case of a type Ia cutoff, for which (setting $\tilde{\Lambda}$ to zero)

$$B_1 = \frac{4\pi\left(d^3 - 15d^2 + 12d - 48\right)}{3(4\pi)^{d/2} d\Gamma(\frac{d}{2})} \, . \tag{3.69}$$

To order ϵ the beta function is

$$\beta_{\tilde{G}} = -\frac{38}{3}\tilde{G}^2 + \left[\tilde{G} + \left(\frac{1}{3} - \frac{19\gamma}{3} + \frac{38\log 2}{3} + \frac{19\log\pi}{3}\right)\tilde{G}^2\right]\epsilon + O(\epsilon^2) \, .$$

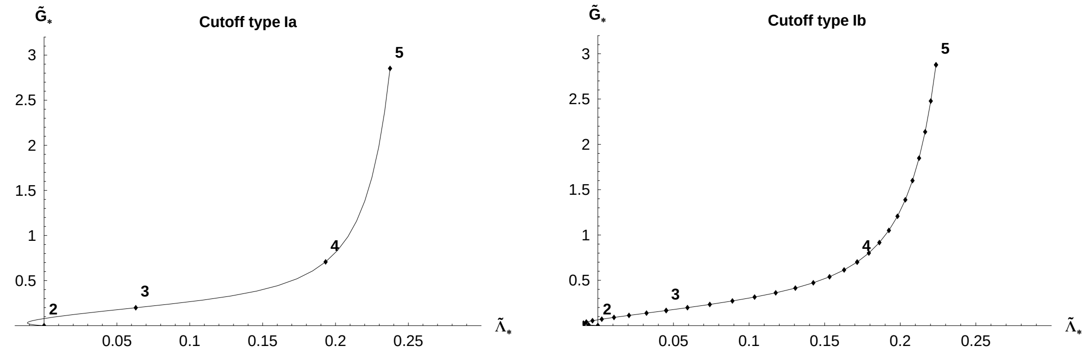

Figure 3.4.: The position of the nontrivial fixed point as a function of d for cutoffs of type Ia (left panel) and Ib (right panel).

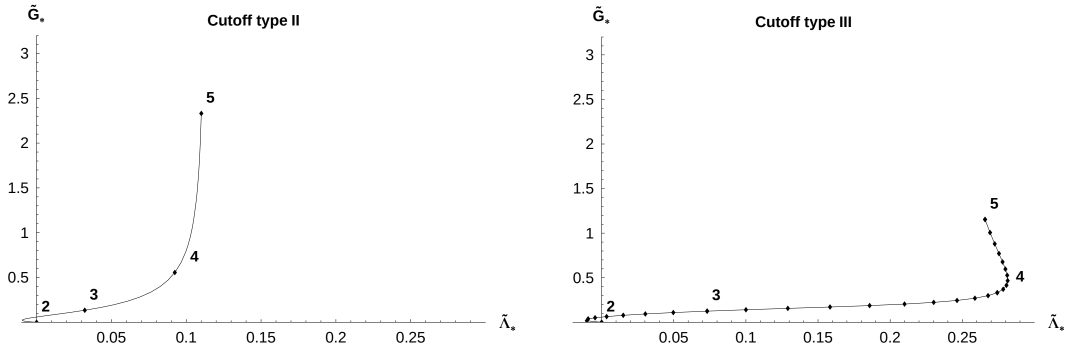

Figure 3.5.: The position of the nontrivial fixed point as a function of d for cutoffs of type II (left panel) and III (right panel).

Whereas to leading order the solution exists for all d, to first order the solution has a singularity for $\epsilon \approx 2.99679$. The occurrence of such singularities at finite values of ϵ is expected. The one loop solution together with some of its approximants is plotted in figure 3.3 for dimensions $2 < d < 4$. We see that when the beta function is expanded to even order in ϵ ($n = 0, 2, 4$ in the figure) the ϵ expansion significantly underestimates the value of $\tilde{G}_*$ in $d = 4$ whereas for odd order ($n = 1, 3$ in the figure) it has a positive pole at some value of the dimension. On the other hand from equations (3.68) and (3.69) one sees that for large d, $\tilde{G}_*$ grows faster than exponentially.

We conclude this discussion by mentioning that when the ϵ expansion is used in presence of a cosmological constant, there are several FPs and even for ϵ very small they have negative $\tilde{\Lambda}_*$. Thus the ϵ expansion is not very helpful in the presence of $\tilde{\Lambda}$. One can solve exactly the equations $d\tilde{\Lambda}/dt = 0$ and $d\tilde{G}/dt = 0$ for arbitrary d and plot the position of the FP in the $\tilde{\Lambda}$–$\tilde{G}$ plane as a function of d. This is shown in figures 3.4 and 3.5. The fixed point is in the origin at $d = 2$; as d grows, $\tilde{G}_*$ grows monotonically while $\tilde{\Lambda}_*$ is initially negative, then becomes positive. For moderately large dimensions (of order 10)

$\tilde{G}_*$ becomes very large (of the order 10^6) while $\tilde{\Lambda}_* < 1/2$ always.

3.8. Four dimensions

Let us now consider Einstein's theory with cosmological constant in $d = 4$. The beta functions for $\tilde{\Lambda}$ and $\tilde{G}$ for various cutoff types have been given in equations (3.22,3.31,3.40,3.63). All of these beta functions admit a trivial (Gaußian) FP at $\tilde{\Lambda} = 0$ and $\tilde{G} = 0$ and a nontrivial FP at positive values of $\tilde{\Lambda}$ and $\tilde{G}$. Let us discuss the Gaußian FP first. As usual, the perturbative critical exponents are equal to 2 and -2, the canonical mass dimensions of Λ and G. However, the corresponding eigenvectors are not aligned with the $\tilde{\Lambda}$ and $\tilde{G}$ axes. It is instructive to trace the origin of this fact. Since it can be already clearly seen in perturbation theory, we consider the perturbative one loop Einstein–Hilbert flow (3.21). The linearized flow is given by the matrix

$$M = \begin{pmatrix} \frac{\partial \beta_{\tilde{\Lambda}}}{\partial \tilde{\Lambda}} & \frac{\partial \beta_{\tilde{\Lambda}}}{\partial \tilde{G}} \\ \frac{\partial \beta_{\tilde{G}}}{\partial \tilde{\Lambda}} & \frac{\partial \beta_{\tilde{G}}}{\partial \tilde{G}} \end{pmatrix} = \begin{pmatrix} -2 + B_1 \tilde{G} + \frac{1}{2}\tilde{G}\frac{\partial A_1}{\partial \tilde{\Lambda}} + \tilde{\Lambda}\tilde{G}\frac{\partial B_1}{\partial \tilde{\Lambda}} & \frac{1}{2}A_1 + B_1 \tilde{\Lambda} \\ \tilde{G}^2 \frac{\partial B_1}{\partial \tilde{\Lambda}} & 2 + 2B_1 \tilde{G} \end{pmatrix} . \tag{3.70}$$

At the Gaußian FP this matrix becomes

$$M = \begin{pmatrix} -2 & \frac{1}{2}A_1(0) \\ 0 & 2 \end{pmatrix} \tag{3.71}$$

which has the canonical dimensions of Λ and G on the diagonal, as expected. However, the eigenvectors do not point along the Λ and G axes. At the Gaußian FP the "attractive" eigenvector is in the direction $(1, 0)$ but the "repulsive" one is in the direction $(A_1(0)/4, 1)$. The slant is proportional to $A_1(0)$ and can therefore be seen as a direct consequence of the running of the vacuum energy. This fact has a direct physical consequence: it is not consistent to study the ultraviolet limit of gravity neglecting the cosmological constant. One can set $\tilde{\Lambda} = 0$ at some energy scale, but if $\tilde{G} \neq 0$, as soon as one moves away from that scale the RG will generate a nontrivial cosmological constant. This fact persists when one considers the RG improved flow. The value of the constant $A_1(0)$ in four dimensions for various cutoff types is listed in table 3.2.

Let us now come to the nontrivial FP. We begin by making for a moment the drastic approximation of treating A_1 and B_1 as constants, independent of $\tilde{\Lambda}$ (this is the leading term in a series expansion in $\tilde{\Lambda}$). Thus we consider again the perturbative one loop Einstein–Hilbert flow (3.21). In this approximation the flow can be solved exactly:

$$\begin{aligned} \tilde{\Lambda}(t) &= \frac{(2\tilde{\Lambda}_0 - \frac{1}{4}A_1\tilde{G}_0(1 - e^{4t}))e^{-2t}}{2 + B_1\tilde{G}_0(1 - e^{2t})} , \\ \tilde{G}(t) &= \frac{2\tilde{G}_0 e^{2t}}{2 + B_1\tilde{G}_0(1 - e^{2t})} . \end{aligned} \tag{3.72}$$

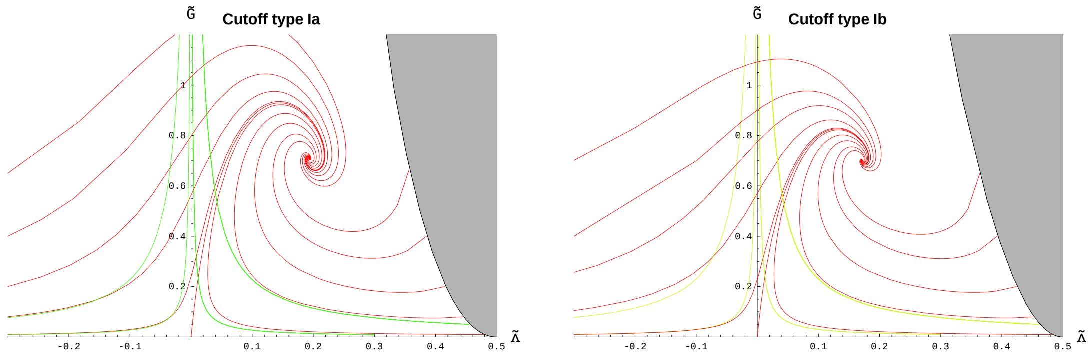

Figure 3.6.: The flow near the perturbative region with cutoffs of type Ia and Ib. The boundary of the shaded region is a singularity of the beta functions.

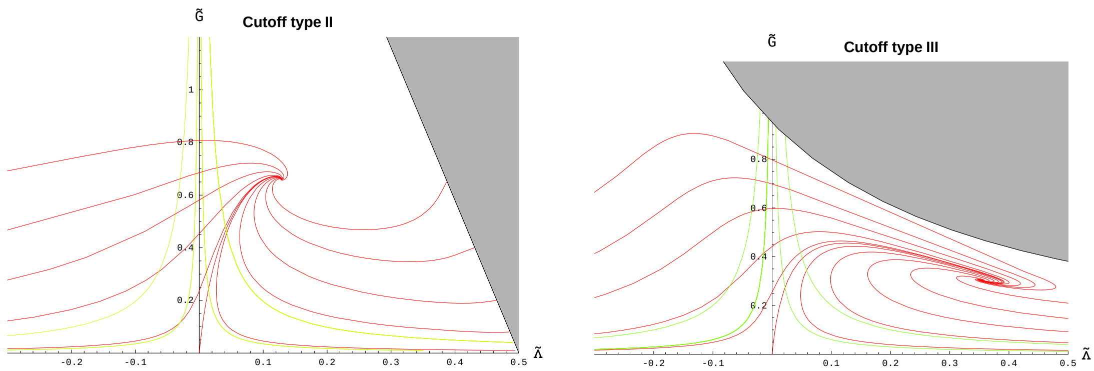

Figure 3.7.: The flow near the perturbative region with cutoffs of type II and III. The boundary of the shaded region is a singularity of the beta functions.

Scheme	$\tilde{\Lambda}_*$	$\tilde{G}_*$	$\tilde{\Lambda}_*\tilde{G}_*$	ϑ
Ia	0.1932	0.7073	0.1367	1.475±3.043i
Ia - 1 loop	0.1213	1.1718	0.1421	1.868±1.398i
Ib with fr	0.1715	0.7012	0.1203	1.689± 2.486i
Ib with fr - 1 loop	0.1012	1.1209	0.1134	1.903± 1.099 i
Ib without fr	0.2329	0.5634	0.1312	2.205 ±3.214 i
Ib without fr - 1 loop	0.2302	0.7450	0.1715	2.430±2.383 i
II	0.0924	0.5557	0.0513	2.425±1.270i
II - 1 loop	0.0467	0.7745	0.0362	2.310± 0.382 i
III	0.2742	0.3321	0.0910	1.752±2.069 i
III - 1 loop	0.0840	0.7484	0.0628	1.695± 0.504 i

Table 3.2.: The nontrivial fixed point for Einstein's theory in $d = 4$ with cosmological constant.

The FP would occur at $\tilde{\Lambda}_* = -A_1/4B_1$, $\tilde{G}_* = -2/B_1$, where the matrix (3.70) becomes

$$M = \begin{pmatrix} -4 & -\frac{1}{4}A_1 \\ 0 & -2 \end{pmatrix} . \tag{3.73}$$

It has real critical exponents 2 and 4, equal to the canonical dimensions of the constants $g^{(0)} = 2Z\Lambda$ and $g^{(2)} = -Z$. This should not come as a surprise, since the linearized flow matrix for the couplings $g^{(0)}$ and $g^{(2)}$ is diagonal, with eigenvalues equal to their canonical dimensions, and the eigenvalues are invariant under regular coordinate transformations in the space of the couplings. So we see that a nontrivial UV–attractive FP in the $\tilde{\Lambda}$–$\tilde{G}$ plane appears already at the lowest level of perturbation theory. It has the form shown in figure 2.1.

All the differences between the perturbative Einstein–Hilbert flow and the exact flow are due to the dependence of the constants A_1 and B_1 on $\tilde{\Lambda}$, and in more accurate treatments to the RG improvements incorporated in the flow through the functions A_2, B_2, A_3, B_3. Such improvements are responsible for the nonpolynomial form of the beta functions. In all these calculations the critical exponents at the nontrivial FP always turn out to be a complex conjugated pair, giving rise to a spiralling flow. The real part of these critical exponents is positive, corresponding to eigenvalues of the linearized flow matrix with negative real part. Therefore, the nontrivial FP is always UV-attractive in the $\tilde{\Lambda}$–$\tilde{G}$ plane. Conversely, an infinitesimal perturbation away from the FP will give rise to a renormalization group trajectory that flows towards lower energy scales away from the nontrivial FP. Among these trajectories there is a unique one that connects the nontrivial FP in the ultraviolet to the Gaußian FP in the infrared. This is called the "separatrix".

An important aspect of the flow equations in the Einstein–Hilbert truncation is the existence of a singularity of the beta functions. In section 3.6, when we neglected the cosmological constant, they appeared at some value $\tilde{G}_c > \tilde{G}_*$. Now, looking at equations

(3.22,3.31,3.40,3.63), we see that there are always choices of $\tilde{\Lambda}$ and $\tilde{G}$ for which the denominators vanish. The singularities are the boundaries of the shaded regions in figures 3.6 and 3.7. Of course the flow exists also beyond these singularities but those points cannot be joined continuously to the flow in the perturbative region near the Gaußian FP, which we know to be a good description of low energy gravity. When the trajectories emanating from the nontrivial FP approach these singularities, they reach it at finite values of t and the flow cannot be extended to $t \to -\infty$. The presence of these singularities can be interpreted as a failure of the Einstein–Hilbert truncation to capture all features of infrared physics and it is believed that they will be avoided by considering a more complete truncation. Let us note that for cutoffs of type I and II the singularities pass through the point $\tilde{\Lambda} = 1/2$, $\tilde{G} = 0$. Thus, there are no regular trajectories emanating from the nontrivial FP and reaching the region $\tilde{\Lambda} > 1/2$. However, for type III cutoffs the shaded region is not attached to the $\tilde{\Lambda}$ axis and there are trajectories that avoid it, reaching smoothly the region $\tilde{\Lambda} > 1/2$.
In table 3.2 we collect the main features of the UV–attractive FP for the Einstein–Hilbert truncation with cosmological constant for the different cutoff schemes.

3.9. Ultraviolet divergences

The Einstein–Hilbert truncation does not give a closed set of flow equations, in the sense that the beta functions of the higher couplings, which have been neglected in the previous section, are not zero. So, if we assume that the higher couplings vanish at some initial scale, they will immediately appear as one integrates the flow equations. Before discussing truncations that involve higher derivative terms, it will be instructive to see, using the ERGE, how such terms are generated in Einstein's theory and how this is related to the issue of ultraviolet divergences in perturbation theory.
A cautionary remark is in order here. In perturbation theory, the divergences appear in the formulae relating bare and renormalized couplings. We recall that in our approach we never talk of the bare action; instead, we follow the flow of the renormalized action Γ_k as $k \to \infty$. In this limit divergences can appear. However, the limit of Γ_k for $k \to \infty$ cannot be simply identified with the bare action. Exploring the relation between these two functionals would require introducing an ultraviolet cutoff. We are not going to do this here. In the following we will simply compare the divergences of Γ_k to the perturbative ones.
In the perturbative approach to quantum gravity, the analysis of ultraviolet divergences plays a central role. This issue is not so prominent in the modern literature on asymptotic safety, but this does not mean that divergences do not occur. In an asymptotically safe theory, the asymptotic behavior of every quantity is dictated simply by dimensional analysis. The dimensionless "couplings in cutoff units" $\tilde{g}_i$ defined in (1.6) tend to constant values, so the dimensionful couplings g_i must run like k^{d_i}. The couplings with positive mass dimension diverge, and those with negative mass dimension go to zero. So, for example, near the nontrivial FP in the Einstein–Hilbert truncation discussed in the

previous section, the graviton wave function renormalization $Z = (16\pi G)^{-1}$ diverges quadratically and the vacuum energy $2\Lambda Z$ diverges quartically at the FP.
This matches the powerlike divergences that one encounters in perturbation theory when one uses an ultraviolet cutoff. However, in the Wilsonian context these divergences have a different physical meaning. The parameter k has not been introduced in the functional integral as an UV regulator, rather as an IR cutoff, and in any physical application k corresponds to some externally prescribed scale. So, in the Wilsonian approach the divergences would seem to acquire almost a physical character: they are a manifestation of the dependence of the couplings on an external parameter, and it should not be too surprising that if an input parameter is allowed to tend to infinity also some output could tend to infinity.
At a deeper level, however, one should take into account the fact that a dimensionful quantity does not have an intrinsic value and therefore cannot be observable. In order to give a value to a dimensionful quantity q, one has to specify a unit u, and the result of any measurement gives only a value for the dimensionless ratio q/u. In an asymptotically safe theory, the divergence of a coupling g_i with positive mass dimension for $k \to \infty$ is just a restatement of the fact that g_i, measured in units of k, tends to a constant. If we choose another unit u, since u is also ultimately expressible in terms of other couplings, it will also be subject to RG flow. Then, the limit $q(k)/u(k)$ may tend asymptotically to zero, to a finite limit or to infinity depending on the behavior of u. This highlights that the divergence of a dimensionful coupling cannot have a direct physical meaning. The only intrinsic (unit-independent) statement that one can make about a dimensionful quantity is whether it is zero, positive or negative.
Furthermore, only dimensionless functions of the couplings have a chance of being observable. It is only for such combinations that the theory is required to give unambiguous answers, *i.e.* it is only such combinations that one could expect to be scheme–independent. Now, very often the choice of k which is appropriate to a specific experimental setup is not entirely unambiguous. Rather, k sets a characteristic scale of the problem and is usually known only up to a factor of order one (see for example section 3.10, the discussion of equation (1.12), or [27] for some concrete examples in a gravitational context). The reason why k is nevertheless a useful quantity in practice is that dimensionless functions of k tend to depend weakly on k and so an uncertainty of order one in the value of k produces only a very small uncertainty in the value of the observable (think for example of the logarithmic running of gauge coupling constants). However, if a coupling g_i is dimensionful, the corresponding dimensionless variable $\tilde{g}_i$ depends strongly on k, so one should not always expect the value of $\tilde{g}_i$ at a given scale to be precisely defined. In particular, one should not expect the value of $\tilde{g}_i$ at the FP to be scheme–independent.
These expectations are confirmed in the previous treatment of gravity in the Einstein–Hilbert truncation. In two dimensions Newton's constant is dimensionless; its value at the nontrivial FP (namely zero) and the slope of the beta function are scheme–independent. In four dimensions the position of the nontrivial FP in the $\tilde{\Lambda}$–$\tilde{G}$ plane is scheme–dependent. However, for all cutoff schemes that have been tried so far $\tilde{\Lambda}$ and $\tilde{G}$ are always positive:

the existence of the FP and the sign of the couplings seem to be robust features of the theory. In fact, in order for $\tilde{\Lambda}_*$ and $\tilde{G}_*$ to be zero, one would have to find a cutoff function such that the Q-functionals in front of the heat kernel coefficients B_2 and B_0 are zero. Since P_k and $\partial_t R_k$ are positive functions, one sees from (A.14) that no such choice exists. The dimensionless combination ΛG is related to the on-shell effective action and is known to be gauge–independent [16]. Numerical studies have also shown that the value of ΛG at the FP is only very weakly dependent on the cutoff function, much less so than the values of $\tilde{\Lambda}$ and $\tilde{G}$. It is expected that this residual weak dependence is only an effect of the truncation. In fact, it has been argued in [20] that the weakness of this dependence is a sign that the Einstein–Hilbert truncation must be stable against the inclusion of further terms in the truncation. We will see in section 4 that this is indeed the case.
The UV behavior of dimensionless couplings (*i.e.* those that are marginal in power counting) requires some additional clarification. According to the preceding discussion, they have a chance of being physically measurable and asymptotic safety requires that they have a finite limit. On the other hand in perturbation theory they generically present logarithmic divergences. How can these two behaviors be reconciled? It is necessary here to distinguish two possibilities: the limit could be finite and nonzero, or it could be zero. If in a certain theory all couplings have the former behavior, then there cannot be any logarithmic divergences. On the other hand if the coefficient g of some operator diverges logarithmically, its inverse will go to zero. So if the coupling is the inverse of g, it is asymptotically free. This is what happens in Yang–Mills theories, where the (square of the) asymptotically free Yang-Mills coupling is the inverse of the coefficient of F^2.
In the derivative expansion of four dimensional gravity it is the terms with four derivatives of the metric that have dimensionless coefficients. We can parametrize this part of the action as follows:

$$\sum_i g_i^{(4)} \mathcal{O}_i^{(4)}[g_{\mu\nu}] = \int d^4x \sqrt{g} \left[\frac{1}{2\lambda} C^2 + \frac{1}{\xi} R^2 + \frac{1}{\rho} E + \frac{1}{\tau} \nabla^2 R \right] \tag{3.74}$$

where we use the notation introduced in (2.64). The question then arises, what is the asymptotic behavior of these couplings, in particular what is the behavior of λ and ξ? This issue can be addressed at various levels, the most basic one being: if we start from Einstein's theory, do we encounter divergences proportional to these terms?
It was shown early on by 't Hooft and Veltman [44] using dimensional regularization that (neglecting total derivatives) the one loop effective action contains the following simple pole divergence

$$\frac{1}{\epsilon} \int d^4x \sqrt{g} \left[\frac{7}{20} R_{\mu\nu} R^{\mu\nu} + \frac{1}{120} R^2 \right] . \tag{3.75}$$

Can this result be seen within the ERGE? Let us return to the Einstein–Hilbert truncation. In the previous section we have expanded the r.h.s. of the ERGE using the heat kernel formula (A.10) and retained only the first two terms, which are sufficient to give the beta functions of the cosmological constant and Newton's contant. Keeping the same inverse propagators, we can now consider the next terms in the heat kernel expansion, which will give the beta functions of λ, ξ, ρ, τ or more precisely the dependence of these beta functions on Newton's constant and on the cosmological constant. We begin by considering a type II cutoff; the terms $O(R^2)$ which were not computed in (3.38) are

$$\int d^4x\,\sqrt{g}\left[\frac{1}{2}Q_0\left(\frac{\partial_t R_k+\eta R_k}{P_k-2\Lambda}\right)\mathrm{tr}b_4(\Delta_2)-Q_0\left(\frac{\partial_t R_k}{P_k}\right)\mathrm{tr}b_4(\Delta_{(gh)})\right]\,. \tag{3.76}$$

The b_4 coefficients for the relevant operators can be computed using equation (A.8) and the traces in (3.6):

$$\begin{aligned}\mathrm{tr}b_4(\Delta_2) &= \frac{7}{12}C^2+\frac{35}{36}R^2+\frac{17}{36}E-\frac{2}{3}\nabla^2 R &(3.77)\\ \mathrm{tr}b_4(\Delta_{gh}) &= \frac{7}{60}C^2+\frac{13}{36}R^2-\frac{8}{45}E+\frac{3}{10}\nabla^2 R\,. &(3.78)\end{aligned}$$

In order to compare to the 't Hooft–Veltman calculation we use the one loop approximation to the ERGE, which as explained in section 2.2 consists in neglecting η, and we also set $\Lambda=0$ in (3.76). This gives a contribution to the ERGE equal to

$$\begin{aligned}\left.\frac{d\Gamma_k}{dt}\right|_{\sim R^2} &= \frac{1}{16\pi^2}\int d^4x\,\sqrt{g}\left[\frac{7}{20}C^2+\frac{1}{4}R^2+\frac{149}{180}E-\frac{19}{15}\nabla^2 R\right]\\ &= \frac{1}{16\pi^2}\int d^4x\,\sqrt{g}\left[\frac{7}{10}R_{\mu\nu}R^{\mu\nu}+\frac{1}{60}R^2+\frac{53}{45}E-\frac{19}{15}\nabla^2 R\right] &(3.79)\end{aligned}$$

where in the last step we have used the identity $C^2=E+2\left(R_{\mu\nu}R^{\mu\nu}-\frac{1}{3}R^2\right)$. From here one can directly read off the beta functions. When one then solves for the flow, in the limit of large k each of these couplings diverges logarithmically, with a coefficient that can be read off (3.79). Recalling that for 't Hooft and Veltman $\frac{1}{\epsilon}$ corresponds to $\frac{1}{8\pi^2}\log\Lambda_{UV}$, where Λ_{UV} is an UV cutoff, we find agreement with their result in the topologically trivial case. On the other hand, if we assume $R_{\mu\nu}=0$, which is the on shell condition in perturbation theory without cosmological constant, (3.79) agrees with the one loop divergence computed in [45]. This provides an independent check on the coefficient of the Euler term. Furthermore, our calculation shows that all these terms, being proportional to $Q_0\left(\partial_t R_k/P_k\right)$, are independent of the choice of the profile function R_k. We have also verified that calculating these terms with cutoffs of type Ia and III leads to the same results. Thus, these divergences are indeed independent of the cutoff scheme [2].

In the case of cutoffs of type Ib there is a subtlety that needs some clarification. With these cutoffs (whether one performs a field redefinition, as in section 3.2, or not, as in Appendix A.4) it is only possible to perform the calculation on Euclidean de Sitter space (a 4–sphere). This provides a check on a single combination of the terms appearing in (3.74). Specializing to the sphere and using that the volume of the sphere is $384\pi^2/R^2$,

[2]In these calculations we stick to the de Donder gauge with $\alpha=1$.

(3.79) becomes

$$\left.\frac{d\Gamma_k}{dt}\right|_{\sim R^2} = \frac{419}{45} \,. \tag{3.80}$$

Using type Ib cutoffs one has to pay special attention to the contribution of some of the lowest modes in the traces. This is due to the fact that the traces over vector and scalar modes may have a prime or a double prime, meaning that some modes have to be left out. When evaluating the ERGE without redefining the fields ξ_μ and σ, as in equation (D1), the isolated modes give an overall contribution -14, which adds up to the contribution of the rest of the spectrum, which is equal to $\frac{1049}{45}$, to give the correct result. The -14 comes from the contributions to be subtracted from the ξ-vector field $10(\partial_t R_k(0)/P_k(0)) = 20$, from the σ-scalar field $6(\partial_t R_k(0)/P_k(0)) = 12$, and from the ghost field with only one prime $\partial_t R_k(0)/P_k(0) = 2$. The first two terms enter in the ERGE with a factor 1/2, the last one with a factor -1, adding up to -14. This is an important consistency check on the expression (D1). A similar result holds for the calculation when the fields ξ_μ and σ are redefined, as in equation (3.29).

It is interesting to consider also the case when $\Lambda \neq 0$. In the case of a type II cutoff, it appears from (3.76), with $\eta = 0$, and using (A.22,A.23,A.24), that the logarithmic divergence will be the same as in the case $\Lambda = 0$. This is due to the fact that expanding the fraction in Λ, terms containing Λ give rise to power–like divergences, so only the leading, Λ–independent term contributes to the logarithmic divergence. The same will be the case for type I cutoffs, since again Λ only appears in denominators. On the other hand for a type III cutoff, (3.76) should be replaced by

$$\int d^4x\,\sqrt{g}\left[\frac{1}{2}Q_0\left(\frac{\partial_t R_k}{P_k}\right)\mathrm{tr}b_4(\Delta_2 - 2\Lambda\mathbf{1}) - Q_0\left(\frac{\partial_t R_k}{P_k}\right)\mathrm{tr}b_4(\Delta_{(gh)})\right] \,. \tag{3.81}$$

Then, assuming $R_{\mu\nu} = \Lambda g_{\mu\nu}$, which is the on shell condition in perturbation theory with cosmological constant,

$$\left.\frac{d\Gamma_k}{dt}\right|_{\sim R^2} = \frac{1}{16\pi^2}\int d^4x\,\sqrt{g}\left[\frac{53}{45}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{58}{5}\Lambda^2\right] \,. \tag{3.82}$$

On the 4–sphere this gives, instead of (3.80)

$$\left.\frac{d\Gamma_k}{dt}\right|_{\sim R^2} = -\frac{571}{45} \,. \tag{3.83}$$

These results agree with those obtained in [46]. Note therefore that the Λ–dependent contributions to the logarithmic divergences *are scheme–dependent*. This should not come as a surprise, in view of the discussion above.

We observe here for future reference that certain authors *subtract* from the ghost term the contribution of the ten lowest eigenvalues of $-\nabla^2$ on vectors, which correspond to the ten Killing vectors of S^4 [47, 48, 49], see also [50, 51] for a discussion. This amounts to putting a prime on the ghost determinant (e.g. putting an extra prime on the last two terms in (3.28)). This can be motivated by the observation that Killing vectors generate global symmetries, and global symmetries are not gauge transformations. Since the ghost contribution has a minus sign, with a cutoff of type II this corresponds to *adding* to (3.83) the term $10(\partial_t R_k/P_k)$ evaluated on the lowest eigenvalue of the operator $-\nabla^4 - \frac{R}{4}$, which is equal to zero. This is equal to $10(\partial_t R_k(0)/P_k(0) = 20$. Thus, (3.83) would be replaced by

$$\left.\frac{d\Gamma_k}{dt}\right|_{\sim R^2} = \frac{329}{45} \,. \tag{3.84}$$

We will return to this issue in section 4, when we define the ERGE for $f(R)$–gravity.

Having found the expected agreement with earlier one loop calculations based on the Einstein-Hilbert action, the next level of sophistication would be to include the terms in (3.74) in the truncation, as is required in a more accurate approximation to the exact flow. We will discuss this in the next section. In the rest of this section we shall discuss the possible appearance of divergences that are cubic or of higher order in curvature, keeping the kinetic operator that comes from the Einstein–Hilbert action.

Among higher powers of curvature, of particular interest is the term cubic in the Riemann tensor. In perturbation theory, the divergence (3.75) can be absorbed into a redefinition of the metric and therefore does not affect the S matrix: pure Einstein theory is one loop renormalizable. The first divergence in the effective action that cannot be eliminated by a field redefinition in perturbation theory is proportional to $R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}$. The coefficient of this term was calculated by Goroff and Sagnotti in [52] at two loops.

Can this divergence be seen in the Einstein–Hilbert truncation of the ERGE, in the same way as we have seen the 't Hooft–Veltman divergence? Let g be the coefficient of this operator in the Lagrangian. In the one loop approximation, neglecting Λ and using a type II cutoff, the beta function of g will be proportional to $Q_{-1}\left(\partial_t R_k/P_k\right) = k^{-2}\tilde{Q}$ where $\tilde{Q}$ is a scheme–dependent dimensionless number. As explained in appendix A.1, one can choose the cutoff scheme in such a way that $\tilde{Q} = 0$ (this is the case, for example, with the optimized cutoff used in this paper). Then, the coupling $\tilde{g}$ will have a FP at $\tilde{g}_* = 0$ [3].

These facts are not surprising: they are a reflection of the fact that the ERGE has the structure of a one loop RG equation, and of the absence of a Riemann-cube divergence at one loop in perturbation theory. The Goroff–Sagnotti counterterm can only be seen in perturbation theory at two loops. On the other hand, if we truncate the ERGE at higher order, for example including all terms cubic in curvature, it is expected that the beta function of g, though scheme–dependent, cannot be set to zero just by a choice of cutoff. There are then two possibilities. The arguments given in the end of section 2.3 suggest that a FP will exist for all terms in the derivative expansion, including the Riemann–cube term; in this case the Goroff–Sagnotti divergence would be an artifact of perturbation

[3] If we chose another cutoff such that $\tilde{Q} \neq 0$, g will go asymptotically to zero; however, as discussed before, this is a unit–dependent statement and does not have much physical meaning.

theory. Alternatively, it is possible that the FP will cease to exist when the Riemann–cube term (or some other high order term) is added to the truncation. It is not yet known whether this is the case or not; in the conclusions we give some reasons why we believe in the former alternative.

3.10. Modified dispersion relations

3.10.1. Introduction

Having discussed in extent the possibility of finding a theory of gravity with a well-defined ultraviolet behaviour, an important question is of course if the running of the gravitational couplings has any consequences for current experiments. Some of the expected playgrounds for such kinds of effects have been discussed in the context of black hole physics [85], cosmology [86, 100, 101, 102, 103, 104, 105, 107, 112, 113], and Large Extra Dimension scenarios [27]. The running of Newton's constant will become significant near the Planck scale (or the size of compactified extra dimensions) $M_{\mathrm{P}} = G^{-1/2} = 1.22 \cdot 10^{19}$ GeV (we use units with $c = 1$, $\hbar = 1$) and might therefore seem to be out of observational reach. However this picture has recently been changed by the realization that several (though not all) quantum gravity models seem to predict a departure from exact Lorentz invariance due to ultraviolet physics at the Planck scale [89]. This could lead either to Effective Field Theories (EFTs) characterized by Planck suppressed Lorentz violations for elementary particles (see *e.g.* [90]) or to some new physics where the Lorentz symmetry is deformed in order to include an extra invariant scale (the Planck scale) apart from the speed of light [91]. The latter framework is generally called Deformed Special Relativity (DSR).

Common to all these models is that they seem to predict a modified form of the free particle dispersion relation, exhibiting extra momentum dependent terms, apart from the usual quadratic one occurring in the Lorentz invariant dispersion relation. In particular one most often considers violations or deformations of the boost subgroup, leaving rotational invariance unaffected and leading to an expansion of the dispersion relation in momentum dependent terms

$$\begin{aligned} E^2 &= m^2 + p^2 + F(p, \mu, M_{\mathrm{P}}) \\ &= m^2 + p^2 + \sum_{n=1}^{\infty} \alpha_n(\mu, M_{\mathrm{P}})\, p^n, \end{aligned} \tag{3.85}$$

where $p = \sqrt{||\vec{p}||^2}$ and μ is some particle physics mass scale. In the following, we assume it to be equal to m, the mass of the particle[4].

In the case of an EFT with Lorentz Invariance Violation (LIV), strong constraints on the coefficients α_n for the cases $n = 1, 2, 3$ have been obtained [89], and there is some hope

[4] Note that according to [88] all theories with modified dispersion relations can be formulated in a Finsler geometry.

that LIV with $n = 4$ will be constrained in the near future by forthcoming experiments and improved observations [90].

In the case of deformed special relativity [91] constraints are more uncertain, as we are still lacking a satisfactory understanding of the theory in coordinate space and hence of the corresponding EFT. There are however conjectures about the phenomenological implications of such a framework and some constraints have been tentatively provided (e. g., for the GZK threshold and $n = 3$ [92]).

In [87], a mechanism was explored that could lead to the emergence of such Modified Dispersion Relations (MDRs), based on the idea that the spacetime structure could be an emergent concept. If so, it would be natural to expect that at sufficiently high energies the effective spacetime metric could become energy dependent. We shall see that sizeable effects could occur well below the Planck scale. It is interesting to note that such a framework closely resembles that of emergent effective geometries and Lorentz symmetries characterizing many condensed matter systems (see *e.g.* [93, 94]). There, linearized perturbations propagate on Lorentzian geometries in the low energy (phononic) limit but do show MDRs of the kind of Eq. (3.85) as the perturbation's wavelength approaches the inter-molecular distance, or coherence length, below which the background cannot be considered a continuum. This phenomenon can be seen as an energy dependent background metric interpolating between a purely Lorentzian (and energy independent) form at very low energies, and a pre-geometric structure present at ultra-short length scales (sub-Planckian in the quantum gravity scenario).

Albeit intriguing, the above described phenomenology of condensed matter systems is still an analogy after all. So one might wonder if there is a framework within known quantum gravity models that could naturally produce such an energy dependence of the metric. Some EFT models with this property had been discussed in [95]. There it was argued that in certain gauge theories of gravity with torsion, the renormalization group (RG) flow of the couplings would produce a scale dependent metric. However, those calculations did not yield significant effects below the Planck scale. More recently, other ideas in this direction have been advanced where an interpretation of the MDR used in quantum gravity phenomenology was provided by arguing for an explicit dependence of the spacetime metric on the energy at which it is probed (see *e.g.* [96, 97, 98] for alternative, but related, frameworks). For example in [97] Magueijo and Smolin proposed a generalization of DSR where the metric becomes energy dependent. In [87] we followed the general ideas of [18] where it was shown that within the context of an EFT of gravity, based on the conventional Einstein–Hilbert action with a cosmological constant, it is indeed possible to derive an energy dependent metric from the RG flow of the couplings. From here, with some reasonable assumptions, we will arrive at MDRs for the propagation of massive particles. We shall then analyze some constraints that can be cast on such quantum gravity phenomenology.

3.10.2. Solving the RG flow near the Gaußian fixed point

Here, we restrict ourselves to the sub-Planckian ($k^2 \ll G^{-1}$) and small curvature ($k^2 \gg R$) regime, where it is usually believed that General Relativity is a good approximation (for different views see *e.g.* [99]). We therefore assume the validity of the Einstein–Hilbert action, possibly with a cosmological constant:

$$\Gamma_k [g] = \frac{1}{16\pi G_k} \int \mathrm{d}^4 x \sqrt{g}\, (2\Lambda_k - R) \,. \tag{3.86}$$

The running of the gravitational couplings can be computed with the ERGE as described earlier in the search of a UV fixed point [20, 39, 21, 42]. Let us stress, however, that, in this section, we do not need to commit ourselves to any specific model of Planck scale physics, in particular we do not need to assume the existence of an UV fixed point in spite of the strong evidence for its existence described above.
In the regime we are interested in, one obtains from the RG equation the following β-functions for Newton's constant and the cosmological constant [18]:

$$k\frac{\partial\Lambda}{\partial k} = A_1\, G_k k^4 \tag{3.87}$$

$$k\frac{\partial}{\partial k}\left(\frac{1}{G_k}\right) = B_1\, k^2 \tag{3.88}$$

where A_1 and B_1 are the order one, positive coefficients from eqs. 3.19, where however now a factor 1/2 has been absorbed in A_1 [5]. The β-functions for gravity have also been discussed in different approaches (see *e.g.* [100, 101, 102, 103, 104, 105]).
The solutions to these ordinary differential equations are

$$\Lambda_k = \Lambda_{k_0} + A_1 \frac{G_{k_0}}{4}\left(k^4 - k_0^4\right) \tag{3.89}$$

$$\frac{1}{G_k} = \frac{1}{G_{k_0}} + \frac{B_1}{2}\left(k^2 - k_0^2\right) \tag{3.90}$$

$$. \tag{3.91}$$

Here k_0 is the scale at which the initial conditions are set. In fact, G and Λ are measured on quite different scales. The value $G_{k_0}^{-1} = M_\mathrm{P}^2 \approx 1.49 \cdot 10^{38}\mathrm{GeV}^2$ is measured to be the same from laboratory up to planetary distance scales, whereas for the cosmological constant we have a value $\Lambda_{k_0} \approx 1.75 \cdot 10^{-123} M_\mathrm{P}^2$ measured at the Hubble scale $H_0 \approx 10^{-42}$ GeV [106] .
From equation (3.90) we see that the running of G is highly suppressed below M_P and hence will be neglected for the rest of this section. We see instead from Eq. (3.89) that the

[5] Note that if we allow for the presence of minimally coupled matter fields, the form of equations (3.88) and (3.87) will not change but the values of A_1 and B_1 are affected [43]. We shall discuss the possible relevance of this fact in the conclusions.

running of Λ becomes significant at energies of order $k_T \approx 10^{-31} M_{\rm P} \approx 10^{-3}$ eV or higher. (This corresponds to the "turning point", in the language of [107].) This significant running of the cosmological constant at relatively low scales will play a crucial role in our analysis.
We have not yet provided a prescription to determine k_0 as a function of some physical scale. Given however that there is no strong evidence for a present running of the cosmological constant at cosmological scales, we will assume, for the moment, that k_0 is placed below k_T far enough in the infrared to be always negligible. We shall check *a posteriori* that such an assumption is justified in the cases of our interest.
The equations of motion (EOM) at scale k are obtained varying the effective action with respect to the metric,

$$\frac{\delta \Gamma_k}{\delta g_{\mu\nu}} = 0. \tag{3.92}$$

The solutions of the EOM at scale k give the metric relevant for the physical process under consideration, with the couplings evaluated at k. In the theory with action (3.86) the EOM are

$$R^{\mu}{}_{\nu}\left[g_k\right] = \Lambda_k \delta^{\mu}{}_{\nu}. \tag{3.93}$$

Since $R^{\mu}{}_{\nu}\left[cg\right] = c^{-1} R^{\mu}{}_{\nu}\left(g\right)$ for any constant factor $c > 0$, equation (3.93) can be rewritten as

$$R^{\mu}{}_{\nu}\left[g_{k_0}\right] \;=\; \Lambda_{k_0}\delta^{\mu}{}_{\nu} = R^{\mu}{}_{\nu}\left[\frac{\Lambda_k}{\Lambda_{k_0}} g_k\right], \tag{3.94}$$

where we have used the coordinate independence of Λ_k. Therefore, for any solution of equation (3.93) the inverse metric scales with the cosmological constant as [109]

$$g_k^{\mu\nu} = \frac{\Lambda_k}{\Lambda_{k_0}} g_{k_0}^{\mu\nu}. \tag{3.95}$$

We want now to analyze the consequences of such a scaling behaviour of the spacetime metric on the propagation of a free particle.

3.10.3. Modified dispersion relations from a "running" metric

Starting from Eq. (3.95) we can derive a MDR by contracting it with the particle's four momentum and identifying k with a function of the three momentum. In the presence of an effective cosmological constant the solution of the EOM cannot be flat space. However, we want to work in a regime where the typical wavelength of the particle is much smaller than the characteristic curvature radius of spacetime, in our case $1/p \ll 1/\sqrt{R} \approx 1/\sqrt{\Lambda_k}$. We can then approximate $g^{\mu\nu}$ by a flat metric and equation (3.95) just results in an overall scaling of the latter. Of course, in order to check that the above condition holds, we need to know the relation between p and k. We shall check *a posteriori* that this is indeed the case for our choice of k.

A global rescaling of the metric can be eliminated by a choice of coordinates, but this can only be done at a particular scale. We choose $g_{k_0}^{\mu\nu} = \eta^{\mu\nu}$ (the usual Minkowski metric $(1, -1, -1, -1)$) for any $k = k_0 < k_T$. At scale $k > k_T$, we have the metric $g_k^{\mu\nu}$ defined by

$$g_k^{\mu\nu} = \frac{\Lambda_k}{\Lambda_{k_0}} \eta^{\mu\nu}; \tag{3.96}$$

and contracting both sides of Eq. (3.96) with the particle four momentum we then obtain

$$m^2 = \frac{\Lambda_k}{\Lambda_{k_0}} \eta^{\mu\nu} p_\mu p_\nu = \frac{\Lambda_k}{\Lambda_{k_0}} (E^2 - p^2) \,, \tag{3.97}$$

where we have defined the mass to be $m^2 = g_k^{\mu\nu} p_\mu p_\nu$ and $p_\mu = (E, -\vec{p})$. So, using Eq. (3.89), one finally gets

$$E^2 - p^2 = \frac{\Lambda_{k_0}}{\Lambda_k} m^2 = \left(1 + \frac{A_1}{4} X \frac{k^4}{M_{\rm P}^4}\right)^{-1} m^2 \tag{3.98}$$

where $X = M_P^2/\Lambda_0 \approx 6 \cdot 10^{122}$.
In order to proceed further in our analysis we now need to clarify the relation between the RG scale k and the particle momentum. Assuming that rotational invariance is preserved, one can predict that for a free particle the RG scale k will be generically determined by the modulus of the particle's three-momentum $p := \sqrt{\|\vec{p}\|^2}$ (or alternatively its energy given that they are practically the same, at first order, for high energy particles), its mass m, and possibly by the Planck scale. As we expect that any deviation from standard physics should be Planck suppressed, we can then write the following ansatz

$$k = \frac{p^\alpha m^\beta}{M_{\rm P}^{\alpha+\beta-1}}, \tag{3.99}$$

where α and β are chosen to be positive integers. The above ansatz is of course inspired by the standard framework adopted in most of the quantum gravity phenomenology literature (see *e.g.* [89]) and for any $\alpha \neq 0$ it will lead to dispersion relations characterized by higher order terms in the momentum of the particle suppressed by appropriate powers of the Planck mass.
For sufficiently low momenta the dispersion relations so obtained will take the form

$$E^2 = p^2 + m^2 \left(1 - \frac{A_1}{4} X \left(\frac{m}{M_{\rm P}}\right)^{4\beta} \left(\frac{p}{M_{\rm P}}\right)^{4\alpha}\right). \tag{3.100}$$

Note that due to the factor m^2 there is no modification of the dispersion relations for massless particles and that for a particle at rest $E = m$ as expected. Let us also emphasize that the above dispersion relation was derived assuming a point-like particle, as it is not clear at this stage which quantities might enter in the relation between k and the physical momentum for composite particles. For this reason, we shall in what follows focus on electrons/positrons.

In applications of the RG to high-energy physics, where one considers mainly scattering processes, k is usually identified with one of the Mandelstam variables of the process, a Lorentz invariant combination of the incoming particle momenta. From this point of view the most obvious and conservative choice would be to assume that k is the unique Lorentz invariant function of the particle momentum, namely its mass. This would imply $(\alpha, \beta) = (0, 1)$. Of course from the perspective of this work this is an uninteresting choice, as it implies that, for a given particle type, k is fixed once for all: an ultra–high–energy particle would "feel" the same spacetime as one at rest. We shall then consider the case of $\alpha \neq 0$.

Conversely one might wonder if there could be some strong motivation to rule out a priori the mass dependence of relation (3.99). One possible argument can be based on the requirement that the natural condition $\Lambda_{k=M_{\rm P}} \approx M_{\rm P}^2$ holds. This implies that the corresponding physical momenta will be $p = M_{\rm P}(M_{\rm P}/m)^{(\beta/\alpha)}$. The case $\beta = 0$ is then the only one for which k and p would coincide at the Planck scale. Albeit appealing this feature of the $\beta = 0$ class of models does not seem sufficient for excluding a priori the other kinds of dispersion relations. We shall hence, for the moment, consider all the possible values of α and β selecting them only on the basis of their phenomenological viability. However it is interesting to note that, in the end, such analysis will indeed select for us a dispersion relation belonging to the $\beta = 0$ class.

Before discussing the phenomenological viability of the above class of dispersion relations, it is perhaps important to stress that while the Planck scale dependence of (3.100) does imply a departure from standard GR at this scale, as generally expected, it does not conflict with the possible existence of an UV fixed point [20, 39, 21, 42]. In fact the fixed–point action will not be the Einstein–Hilbert action but some general diffeomorphism invariant action with extra degrees of freedom affecting local Lorentz invariance (like for example the well studied Einstein-aether theory [108]). Furthermore, note that since the gravitons are massless, their propagation is not affected by (3.100). Thus, the presence of Planck suppressed terms in the propagators of massive particles and their detectability through carefully chosen experiments and observations [89] does not imply a sizable departure from standard GR at these sub-Planckian energies. It only means that the extra degrees of freedom characterizing the UV theory are weakly coupled to matter fields through Planck–suppressed interactions.

Phenomenological viability

A good indicator of the phenomenological viability of the above class of dispersion relations is easily obtained by considering when, for some choice of the parameters α and β, the Lorentz violating term becomes of the same order as the mass term, so that the approximation taken in order to derive Eq. (3.100) breaks down. This would indicate where the Planck–suppressed term starts introducing a running mass term for the particle and hence producing a detectable phenomenology for example via threshold reactions. Re-

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
$\beta = 0$	$3 \cdot 10^{-15}$	6.1	$7 \cdot 10^{5}$	$3 \cdot 10^{8}$
$\beta = 1$	$7 \cdot 10^{7}$	$9 \cdot 10^{11}$	$2 \cdot 10^{13}$	$1 \cdot 10^{14}$
$\beta = 2$	$2 \cdot 10^{30}$	$1 \cdot 10^{23}$	$6 \cdot 10^{20}$	$4 \cdot 10^{19}$

Table 3.3.: The critical energies for order–one deviations from standard physics given in TeV for electrons/positrons at different combinations of α and β.

sults for such critical values of the particle momentum for different choices of α and β are given for the case of the electron (assuming $A_1 \approx O(1)$) in Table 3.3. The dispersion relations (3.100) will be of course phenomenologically acceptable if the modifications to standard physics arise only for very high energy particles. However we do also have to ask that the critical value of the momentum is not too high so that the corresponding MDR might lead to observable effects and consequently be subject to observational constraints. For example, in the case of QED, the most interesting MDRs will be those for which observable effects are expected at TeV energies, as this is the scale of the most energetic QED particles observed so far. Looking at Table 3.3 we see that the only case of phenomenological interest seems to be $(\alpha, \beta) = (2, 0)$. Higher values of (α, β) are not a priori incompatible with observations, but at the moment lie beyond observational reach. Before starting to consider the case $(\alpha, \beta) = (2, 0)$ let us note however that the cases with $\alpha = 1$ are particularly interesting from a theoretical point of view as they would lead to an MDR of the form

$$E^2 = p^2 + m^2 + \eta_{\alpha=1} \frac{p^4}{M_{\mathrm{P}}^2} \tag{3.101}$$

with $\eta_{\alpha=1} = -A_1/4\, X (m/M_{\mathrm{P}})^{2+4\beta}$. What is noticeable in our case is that the dimensionless coefficients η do indeed contain, as conjectured (see *e.g.* [90]), powers of the small ratio m/M_{P}. These are however not necessarily leading, in the present framework, to an overwhelming suppression of the LIV term. In contrast, the presence of the huge numerical factor, X, that we inherited from the initial conditions (the observed value of the cosmological constant on cosmological scales) basically allows us to rule out the most obvious case $(\alpha, \beta) = (1, 0)$ as this would lead to sizeable deviations from standard physics for any particle above $\approx 10^{-3}$ eV. If the observed Λ_{k_0} contained also the contribution of some quintessence-like fields, the "true" cosmological constant at k_0 would be smaller hence leading to an even larger value of X.
The choice of parameters $(\alpha, \beta) = (2, 0)$ gives

$$k = \frac{p^2}{M_{\mathrm{P}}}, \tag{3.102}$$

and

$$E^2 = m^2 + p^2 - \frac{A_1}{4} X \frac{m^2 p^8}{M_{\mathrm{P}}^8}, \tag{3.103}$$

which can be cast in the more suggestive form

$$E^2 = m^2 + p^2 + \eta \frac{p^8}{M_{\mathrm{P}}^6}, \tag{3.104}$$

with $\eta = -(A_1/4)X\,(m/M_{\mathrm{P}})^2$.
Let us start noting that Eq. (3.102), together with Eq. (3.89), implies that at sub-Planckian energies ($p \ll M_{\mathrm{P}}$) the de Broglie wavelength of the particle is always much smaller than the curvature radius of spacetime as we initially assumed. Furthermore it is also much smaller than the inverse of the RG parameter k. This can be interpreted as saying that the particle has a somewhat lower resolution in probing spacetime than naively expected. We shall discuss this at length in the next section. Passing to the dispersion relation Eq. (3.103) we already saw (see Table 3.3) that it leads to order–one deviations around 10 TeV. The reason for this lies again in the presence of the huge numerical factor X which is able to contrast the large Planck suppression. This feature makes the above dispersion relation compatible with current (low energy) observations while at the same time amenable to experimental constraints via high– energy astrophysics observations of QED phenomena. From the experimental point of view, $(\alpha, \beta) = (2, 0)$ is therefore the most interesting value. We shall argue now that theoretically it is the best motivated.

Physical motivation for the case $(\alpha, \beta) = (2, 0)$

In order to motivate physically the choice of the set of parameters $(\alpha, \beta) = (2, 0)$ we shall start by addressing the question of the influence of the effective cutoff on the fluctuations of the gravitational field, which is relevant for the propagation of a free particle. As mentioned above, k marks the distinction between those modes that are integrated over and those that have to be treated classically in the resulting EFT.
As before, we assume that, in the absence of the particle, spacetime would be effectively flat (or anyway would have a curvature much smaller than m^2). The particle produces a disturbance in the gravitational field in the form of local curvature and the fluctuations of the gravitational field will be affected by this curvature.
We have to consider the specific dynamics describing the effect of the particle on the gravitational field, which we have assumed to be given by Einstein's equations. Stripped of all indices, these equations tell us that the second derivatives of the metric, or the square of its first derivatives, are of the order of $G\rho$, where ρ is a typical component of $T_{\mu\nu}$. The solution depends on the distance from the particle, and for a classical particle becomes singular near the origin. These issues do not arise when we take into account the quantum nature of the particle. The position of the quantum particle cannot be determined with a precision greater than the de Broglie wavelength $\lambda = 1/p$. For an order of magnitude estimate, we can therefore spread the total energy and momentum p in a box of size $1/p$, so the order of magnitude of the diagonal components of the energy momentum tensor must be $\rho \approx p^4$. (We observe that this is also the vacuum energy density in a box

of size $1/p$.) Einstein's equations then give an estimate for the curvature

$$(\partial g)^2 \approx R \approx G\rho \approx \frac{p^4}{M_{\mathrm{P}}^2}. \tag{3.105}$$

Of course, the gravitational field will fall off away from the particle; this equation gives just a characteristic value for the curvature very near the position of the particle. It says that a particle of momentum p excites Fourier modes of the gravitational field with momenta up to p^2/M_{P}. Gravitational modes with higher momenta are essentially unaffected by the particle. It is therefore natural to assume that in the EFT these are the modes which have to be integrated over, meaning that the relevant cutoff is $k \approx p^2/M_{\mathrm{P}} \ll p$. [6]

It is interesting to observe that the same argument, applied to a charged particle in an electromagnetic field, would yield a completely different result. In fact, the order of magnitude of the charge and current density is p^3, and from Maxwell's equations one gets an estimate $F \approx p^2$. Since F has dimension of mass squared, we conclude that the characteristic momentum scale of the electromagnetic field generated by a particle of momentum p is $k \approx \sqrt{F} \approx p$. This corresponds to the naive estimate $\alpha = 1$. Clearly, the different behavior is determined by the fact that the coupling constant of electromagnetism is dimensionless, while that of gravity has the dimension of area.

In closing this section, we observe that in the case of a Friedmann–Lemaître–Robertson–Walker metric, our ansatz $k = \sqrt{G\rho}$ corresponds to the Hubble scale [7]. Interestingly this choice has also been advocated for applications to cosmology on the basis that the Bianchi-identity has to remain valid even in the presence of running coupling constants [102, 104, 105]. However, we stress that the framework presented here significantly differs from the one above. For example, in their argument the RG scale is assumed to be a function of the cosmological time $k = k(t)$. This would be incompatible with Eq. (3.94) which explicitly relies on the coordinate independence of the cosmological constant. The same is valid for the models discussed in [112, 113], where spherical symmetry of spacetime leads to a radial dependence of the couplings giving rise to a modified action from which also Brans–Dicke theory can be obtained. In our argument, the cutoff k is determined by the properties of the object/apparatus that probes the spacetime metric, it does not depend *a priori* on the characteristic scale of the universe.

3.10.4. Physical interpretation and phenomenology

The derived MDR has the form of a global scale transformation, so that a massless particle ends up probing the same spacetime no matter what energy it has. However, there will be phenomenological consequences for massive particles. To make some precise experimental predictions, we need to choose the framework in which we interpret this MDR. There

[6]Note that the same cutoff has been derived in other frameworks in [110, 111].

[7]Incidentally this confirms that our assumption of neglecting k_0 in Eq. (3.89) was justified. In fact in this case $k_0 \approx H_0 \approx 10^{-33}$ eV which is definitely negligible for any particle with energies well above few meV.

are two main (inequivalent) candidates: EFT with LIV, or DSR. Both frameworks seem to be a priori compatible with our MDR. The main difference between the two theories is in the way in which momenta add up and possibly in the spacetime (non)commutativity.

Effective field theory with Lorentz invariance violation

The EFT framework has several advantages when discussing phenomenological consequences of LIV. Apart from being a well known and versatile framework it is also able to make sharp predictions as it allows to use the standard energy-momentum conservation and requires for its applicability just locality and local spacetime translational invariance above some length scale. In this context, one can apply the usual QFT tools, bearing in mind that the effective action will contain explicit Lorentz breaking terms, and hence cast constraints using experimental or observational tests. In particular for Lorentz violations at order $O(p^3/M)$ and higher (*i.e.* induced by nonrenormalizable operators of mass dimension five or greater in the action) the most appropriate tests are coming from high–energy astrophysical observations (see *e.g.* [90]), as these are among the highest energy phenomena we can access nowadays.
Not all of the above–cited astrophysical tests can be applied to our framework. In particular cumulative effects based on the propagation of photons over cosmological distances [114, 90] are unavailable as no modification is induced in dispersion relations of massless particles. Similarly some anomalous reactions, like the vacuum Čerenkov $e^{\pm} \to e^{\pm}\gamma$ emission [115], are not allowed for "subluminal" dispersion relations of the leptons (and unmodified photons) like the one we found in Eq. (3.103). Finally one might consider possible constraints coming from the shifting of normally allowed threshold reactions as the photon pair production, $\gamma\gamma \to e^+e^-$, or the GZK reaction, $p\gamma \to p\pi^0$ (with γ a CMB photon, see *e.g.* [116, 115]). Apart from the previously mentioned caveats related to the application of our MDR to composite particles, such a route is again unfeasible within our framework. In fact the analysis of such scattering reactions would require a new derivation of the relation between the RG parameter k and the physical momenta which (missing a better understanding of the theory) would require more arbitrary choices and assumptions on our side. Hence, given the above theoretical and observational uncertainties, characterizing these scattering reactions is beyond the scope of this paper. We can however provide very strong constraints on the MDR provided in Eq. (3.103) by considering the so called "photon decay", $\gamma \to e^+e^-$, usually forbidden by momentum conservation, and the synchrotron emission.

Photon decay Within an EFT framework the photon decay process becomes possible above a certain threshold energy once one has dispersion relations violating Lorentz invariance. This threshold energy is given by the minimal momentum of the incoming photon such that the decay could happen preserving energy–momentum conservation. Following the steps from [115] and referring to eqs. (3.98) and (3.99), one obtains for the threshold momentum

$$p_{\rm th} = \left(\frac{10^2}{A_1 X}\right)^{1/8} \left(\frac{M_{\rm P}}{m_e}\right)^{\beta/2} M_{\rm P}\ . \tag{3.106}$$

In the case $(\alpha, \beta) = (2, 0)$, $A_1 = 1$ and using $m_e \approx 0.511$ MeV, the threshold energy for this process is $p_{\rm th} \approx 9.75$ TeV. Moreover, following the steps in [90], one can calculate the decay rate of the photon, which turns out to be extremely fast, so much that a photon would not be able to propagate on any distance of astrophysical relevance. We then see that the observations of photons with energies above $\approx$ 10 TeV propagating on astrophysical distances allow us to put upper bounds on the coefficient A_1.
A strong constraint comes from the observation of high-energy γ-rays emitted by the Crab nebula [90]. The γ-ray spectrum of this source is very well understood. It results from a high–energy wind of electrons (and possibly positrons) which leads to a combination of synchrotron emission and inverse Compton scattering of (mainly the synchrotron) photons. The inverse Compton γ-ray spectrum so produced extends up to energies of at least 50 TeV. This implies that for these photons the threshold energy for their decay must be above 50 TeV. We must have

$$A_1 \leq \left(\frac{9.75\,\text{TeV}}{p_{\rm obs}}\right)^8 , \tag{3.107}$$

which for $p_{\rm obs} = 50$ TeV gives a bound on A_1 of order 10^{-6}. This is clearly a very strong constraint since A_1 is naturally of order one.

Synchrotron radiation An even stronger constraint can be provided by the observation of high–energy synchrotron emission from the Crab nebula. Cycling electrons in a magnetic field B emit synchrotron radiation with a spectrum that sharply cuts off at a frequency ω_c given by the formula

$$\omega_c = \frac{3}{2} eB \frac{\gamma^3(E)}{E}\,, \tag{3.108}$$

where $\gamma(E) = (1 - v^2(E))^{-1/2}$ and $v(E)$ is the electron's group velocity. The formula (3.108) is based on the electron trajectory for a given energy in a given magnetic field, the radiation produced by a given current, and the relativistic relation between energy and velocity (see [117, 90] for a discussion about the validity of this formula in EFT with LIV and a detailed derivation of the constraint).
The maximum synchrotron frequency $\omega_c^{\rm max}$ is obtained by maximizing ω_c (3.108) with respect to the electron energy, which amounts to maximizing $\gamma^3(E)/E$. Using the MDR (3.103) one can easily calculate the modified group velocity of the electron and from that $\gamma(E)$. Then the maximization of the synchrotron frequency yields

$$\omega_c^{\rm max} = 0.47\,\frac{eB}{m_e}[-\eta\,(m_e/M_{\rm P})^6]^{-2/8}, \tag{3.109}$$

where $\eta = -A_1/4\,X\,(m_e/M_{\rm P})^2$. This maximum frequency is attained at the energy $E_{\rm max} = (-m_e^2 M_{\rm P}^6/35\eta)^{1/8} \approx 4.2\,A_1^{-1/8}$ TeV.
The rapid decay of synchrotron emission at frequencies larger than ω_c implies that most of the flux at a given frequency in a synchrotron spectrum is due to electrons for which ω_c is above that frequency. Thus $\omega_c^{\rm max}$ must be greater than the maximum observed synchrotron emission frequency $\omega_{\rm obs}$. This yields the constraint

$$A_1 < \frac{4}{X}\left(\frac{M_{\rm P}}{m_e}\right)^8 \left(\frac{0.47\,eB}{m_e\omega_{\rm obs}}\right)^{8/2} . \tag{3.110}$$

Using as in [117] the observation of synchrotron emission from the Crab nebula up to energies of about 100 MeV, and a conservative estimate of the magnetic field of 0.6 mG (this is the largest proposed value which yields the weakest constraint) we then infer[8] that $A_1 \lesssim 2 \times 10^{-22}$. This constraint is so strong that one has to conclude that dispersion relations like (3.103) for leptons are ruled out by current astrophysical observations within an EFT framework [9].

Deformed Special Relativity

It is less easy to provide experimental constraints in the DSR framework since this theory still lacks a clear understanding. In particular in this approach spacetime is in general noncommutative, although this non-commutativity can be absorbed by a coordinate transformation in phase space [118]. Hence the form of DSR in coordinate space is still debated. In momentum space DSR is described by a deformation of the Lorentz symmetry so that the p_μ carries a nonlinear representation of the Lorentz group. This nonlinear representation can be constructed from a linear representation carried by a "platonic" momentum π_μ via a non-linear map U so that $\pi_\mu = U^{-1}(p_\mu)$. The π_μ add linearly which implies a nonlinear addition for p_μ. It is still unclear why the latter nonlinear momenta should be the physically measured ones and there is a rich literature now devoted to the interpretation of DSR in momentum space. Such an interpretation should hopefully clarify some of the problems pointed out in the recent literature [119].
DSR can also be seen as a natural framework for our results if one recalls its interpretation as a new measurement theory [98]. Generally we measure the momentum π^μ of a particle in a given reference frame described by a tetrad field $e^\alpha{}_\mu$, so that the measurement outcomes are $p^\alpha = e^\alpha{}_\mu \pi^\mu$. If the metric is endowed with quantum gravity fluctuations, the theory of measurement will imply an averaging procedure at some given energy scale possibly provided by the test particle. This can naturally lead to an energy dependent

[8] Note that possible complications related to different MDRs between electrons and positrons in EFT with LIV are not present here as the breakdown of Lorentz invariance is a geometric effect in our framework and as such will not distinguish between leptons of same mass.

[9] One might wonder, given the strength of the synchrotron constraint for $(\alpha, \beta) = (2, 0)$, if this might also help constraining cases which require higher, but not totally unreasonable, critical energies like the cases $(\alpha, \beta) = (1, 1)$ and $(\alpha, \beta) = (3, 0)$. Unfortunately it is easy to check that this is not the case. For example, for $(\alpha, \beta) = (1, 1)$, the synchrotron constraint is just $b \lesssim 10^{17}$ and for $(\alpha, \beta) = (3, 0)$ it is $A_1 \lesssim 10^{29}$.

tetrad field and hence to a nonlinear relation between the measurement outcomes p^α and the particle momentum π^μ [98]. This is precisely what happens in Eq. (3.103). At the scale k_0, since the metric is just the Minkowski metric, π^μ and p^α coincide and transform linearly under Lorentz transformations. As soon as we get to higher energies the metric, and hence the tetrad field, becomes momentum dependent: $g^{\mu\nu}(p) = \eta^{\alpha\beta} e^\mu{}_\alpha(p) e^\nu{}_\beta(p)$. The nonlinear relation between π^μ and p^μ is then given by

$$\pi_\mu = \sqrt{\frac{\Lambda_k}{\Lambda_{k_0}}} p_\mu = \left(\sqrt{1 + \frac{A_1}{4} X \frac{p^8}{M_{\rm P}^8}} \right) p_\mu . \tag{3.111}$$

Clearly, if π^μ undergoes linear Lorentz transformations, p^μ is transformed nonlinearly, which is characteristic of the usual DSR framework.
With regard to phenomenological constraints, the main experimental prediction of DSR concerns γ-ray bursts [120], but photons are not affected in our framework. Even worse, in this case we cannot resort to anomalous (normally forbidden) threshold reactions as the latter are not allowed in DSR either. The reason for this is simply that a kinematically forbidden reaction in the "platonic" variables π_μ cannot be made viable just via a nonlinear redefinition of momenta. There have been attempts to consider constraints provided by shifts of normally allowed threshold reactions [92]. We note here that for such reactions the possible constraints are strongly dependent not just on kinematical considerations, but also on reaction rates which require some working framework for their derivation. Hence, missing a field theory description of DSR we cannot safely pose such constraints. Similar considerations hold for constraints based on the synchrotron emission [117].

3.10.5. Consequences

We have shown in this section how the RG of gravity could lead to MDRs for massive minimally coupled particles due to the effects of quantum fluctuations. We have argued that for a free particle the most plausible identification of the cutoff is $k = p^2/M_P$, where p is the particle three momentum, leading to sizeable effects in the region of $p \approx 10$ TeV for QED processes. To do this, we had to make several assumptions: we assumed the validity of Einstein's theory of gravity from cosmological to particle physics scales (*i.e.* still much below the Planck energy), we assumed that the cutoff is a function of three momentum squared rather than four momentum squared, and we took for granted the value $\Lambda_0 \approx 10^{-85}$ GeV^2 for the cosmological constant at cosmological scales. The desribed framework is detached from the existence of a gravitational fixed point which concerns physics at or beyond the Planck scale.
Another implicit assumption was the identification of the components of the four momentum $p_\mu = (E, -\vec{p})$, rather than $p^\mu = (E, \vec{p})$. The two identifications are not compatible if the metric is scale dependent. Had we chosen $p^\mu = g_k^{\mu\nu} p_\nu = (E, \vec{p})$, we would have

obtained a MDR of the type

$$E^2 = p^2 + m^2 \left(1 + \frac{A_1}{4}\frac{Xk^4}{M_{\mathrm{P}}^4}\right). \tag{3.112}$$

The main difference with respect to Eq. (3.103) is the positive sign in front of the correction. From an LIV EFT perspective, the immediate consequence is that the previously discussed synchrotron bound does not apply, so this MDR is not as constrained as the previous one (Eq. (3.103))[10]. There is no photon decay either, however a vacuum Čerenkov effect may occur [89]. It can be shown that the latter can cast constraints on A_1 of the same order as the photon decay case (cf Eq. (3.107)). Thus this case would also be quite constrained in the LIV interpretation. Note that a similar change in the sign of the LIV term in the MDR could also be due to a change in the sign of the coefficient A_1 which can be induced by considering in the RG analysis also minimally coupled fermion fields [43]. Different initial assumptions may lead to effects that are either too strong to be compatible with current data, or too small to be detectable in the foreseeable future, or, hopefully, they might produce some interesting phenomenology.

The RG has been applied in an LIV context in [122] whose authors proved that Lorentz invariance can arise as a low–energy symmetry in an otherwise non–Lorentz invariant theory. Since our MDRs reduce to the standard ones at sufficiently low energies, our results are in agreement with theirs on this point, even though the formalism is quite different.

From the theoretical point of view, we cannot say at this point if the RG approach to gravity prefers EFT with LIV or DSR. Let us stress that our model is not a priori equivalent to any of the above frameworks. In fact in the case of EFT with LIV one assumes the existence of some aether field which allows one to construct LIV operators in the matter Lagrangians. However in our case the departure from standard special relativistic dynamics of matter is induced uniquely via the k dependence of the background metric and the fact that k is chosen not to be a Lorentz invariant. This can be seen as a special case of EFT with an aether field but it is not equivalent to it. (Note that if the aether field is taken to be dynamical then it will affect the RG flow, but this will be equivalent to the presence of an extra matter field.) Similarly we cannot a priori completely identify our framework with a DSR one since we cannot say at this stage if our effective geometry will also be accompanied by some alternative rule for the addition of momenta.

Which of the above possibilities would be actually realized within our framework could be probably assessed only after gaining a better understanding of EFT on running geometries, something that is still lacking at this time. Following the intuition that comes from the analogue models (where the underlying microscopic physics indeed violates Lorentz invariance), one would probably need to have some deeper understanding of the physics above the Planck scale (quantum gravity regime) to be able to distinguish between the two. In this sense the phenomenological analysis we have performed has to be taken as a first try aimed at seeing what constraints could be cast once this discrimination is done.

[10] See however [121] for a derivation of constraints also in this case in a different way.

In this chapter, we have worked with a simple form of truncation including only the Newton constant and the Cosmological Constant. We applied different cutoff schemes in d dimensions and found in each of them a non-Gaußian fixed point. Performing the $2 + \epsilon$ approximation we saw that the result differs more and more from the exact calculation when increasing the ϵ-parameter. Based on the Einstein-Hilbert truncation we were able to calculate the curvature squared terms which are generated from this action and found them to be in agreement with the known one loop divergences in gravity. Einstein–Hilbert gravity had been found earlier to be renormalizable at one loop level and we found a non-Gaußian fixed point for this case. However, as soon as matter contributions are added, the loop-expansion leads to divergences which cannot be absorbed by field redefinitions. The calculations based on the Exact Renormalization Group Equation confirm instead the existence of a non-Gaußian fixed point also in this case. The approach therefore does not fail at the same level as perturbation theory. The question is now what happens for theories including more couplings. This will be the matter of the next section.

We gave here also a possible form of phenomenological consequences of the RG running of the gravitational constants which works as well in a pure effective field theory framework and is independent of the existence of a non-Gaußian fixed point. Due to its smallness, the gravitational coupling whose running has the most promising potential to lead to measurable phenomenological consequences is the cosmological constant. We analyzed the consequences of this running for a test particle moving on a de Sitter background and found that its dispersion relation will be modified. The connection to models of Lorentz invariance violation or deformed special relativity was studied and used to give constraints on several parameters.

4. Higher-derivative truncations

4.1. Curvature squared truncations

In the preceding section we have seen that starting from the Einstein–Hilbert action, the RG flow will generate terms quadratic in curvature. In particular, we have discussed the way in which these terms diverge logarithmically when $k \to \infty$. It is therefore not consistent to neglect these terms, and a more accurate treatment will take into account the contribution of these terms to the r.h.s. of the ERGE. In other words, we should include the terms (3.74) in the truncation. The resulting theory is perturbatively renormalizable [69] but has problems with unitarity. Furthermore, in the perturbative treatment the cosmological constant is set to zero, something we know can only be imposed at a given scale. The corresponding Wilsonian calculation has been done within the one loop approximation, and has been briefly reported in [23]. Here we review and extend those results.

We take the action Γ_k as the sum of the Einstein–Hilbert action (3.1) and the terms (3.74). The linearized wave operator is now a complicated fourth order operator. In order to simplify its form, following [53, 54, 55, 56] it is convenient to choose a gauge fixing of the form

$$S_{GF} = \int d^4x \sqrt{g}\, \chi_\mu Y^{\mu\nu} \chi_\nu$$

where $\chi_\nu = \nabla^\mu h_{\mu\nu} + \beta \nabla_\nu h$ (all covariant derivatives are with respect to the background metric) and

$$Y^{\mu\nu} = \frac{1}{\alpha}\left[g^{\mu\nu}\nabla^2 + \gamma \nabla^\mu \nabla^\nu - \delta \nabla^\nu \nabla^\mu\right] .$$

The ghost action contains the term

$$S_c = \int d^4x \sqrt{g}\, \bar{C}_\nu (\Delta_{gh})^\nu{}_\mu C^\mu$$

where

$$(\Delta_{gh})^\nu{}_\mu = -\delta^\nu_\mu \nabla^2 - (1+2\beta)\nabla_\mu \nabla^\nu + R^\nu{}_\mu$$

as well as a "third ghost" term

$$S_b = \frac{1}{2}\int d^4x \sqrt{g}\, b_\mu Y^{\mu\nu} b_\nu$$

due to the fact that the gauge averaging operator $\mathbf{Y}$ depends nontrivially on the metric. We follow earlier authors [55] in choosing the gauge fixing parameters α, β, γ and δ in such a way that the quadratic part of the action is:

$$\frac{1}{2}\int d^4x\sqrt{g}\,\delta g\mathbf{K}\mathbf{\Delta}^{(4)}\delta g\ , \tag{4.1}$$

where

$$\mathbf{\Delta}^{(4)} = (-\nabla^2)^2 + \mathbf{V}^{\rho\lambda}\nabla_\rho\nabla_\lambda + \mathbf{U}\ . \tag{4.2}$$

For details of the operators $\mathbf{K}$, $\mathbf{V}$ and $\mathbf{U}$ we refer the reader to [55] whose notation we mostly follow. We choose type III cutoffs:

$$\mathbf{R}^g_k(\mathbf{\Delta}^{(4)}) = \mathbf{K}R^{(4)}_k(\mathbf{\Delta}^{(4)})\ ; \quad R^c_k(\mathbf{\Delta}_{(gh)})^\mu{}_\nu = \delta^\mu_\nu R^{(2)}_k(\mathbf{\Delta}_{(gh)})\ ; \quad R^b_k(\mathbf{Y})^{\mu\nu} = g^{\mu\nu}R^{(2)}_k(\mathbf{Y})\ .$$

For higher derivative operators we will use a generalized optimized cutoff of the form $R^{(n)}_k(z) = (ak^n - z)\theta(ak^n - z)$, with $a = 1$ unless otherwise stated.
We restrict ourselves to the one loop approximation, as explained in previous sections. Furthermore, only the contributions of the heat kernel coefficients up to B_4 will be taken into account. In this way, we will essentially neglect all RG improvements on the right hand side of the ERGE. The beta functions of the dimensionless couplings appearing in (3.74) turn out to be:

$$\begin{aligned}
\beta_\lambda &= -\frac{1}{(4\pi)^2}\frac{133}{10}\lambda^2\ ,\\
\beta_\xi &= -\frac{1}{(4\pi)^2}\left(10\lambda^2 - 5\lambda\xi + \frac{5}{36}\right)\ ,\\
\beta_\rho &= \frac{1}{(4\pi)^2}\frac{196}{45}\rho^2\lambda\ .
\end{aligned} \tag{4.3}$$

They form a closed system and agree with those calculated in dimensional regularization [54, 55, 56]. It is convenient to define new couplings ω and θ by $\xi = -3\lambda/\omega$ and $\rho = \lambda/\theta$. In this way $1/\lambda$ is the overall strength of the terms quadratic in curvatures, and θ and ω give the relative strength of the different invariants. Then, the beta functions become

$$\begin{aligned}
\beta_\omega &= -\frac{1}{(4\pi)^2}\frac{25 + 1098\,\omega + 200\,\omega^2}{60}\lambda\ ,\\
\beta_\theta &= \frac{1}{(4\pi)^2}\frac{7(56 - 171\,\theta)}{90}\lambda\ .
\end{aligned} \tag{4.4}$$

The coupling λ has the usual logarithmic approach to asymptotic freedom, while the other two couplings have the FP values $\omega_* \approx (-5.467, -0.0228)$ and $\theta_* \approx 0.327$. Of the two roots for ω, the first turns out to be UV–repulsive, so the second has to be chosen [54, 55, 56].

The beta functions of $\tilde\Lambda$ and $\tilde G$ are:

$$\beta_{\tilde\Lambda} = -2\tilde\Lambda + \frac{1}{(4\pi)^2}\left[\frac{1+20\omega^2}{256\pi\tilde G\omega^2}\lambda^2 + \frac{1+86\omega+40\omega^2}{12\omega}\lambda\tilde\Lambda\right] - \frac{1+10\omega^2}{64\pi^2\omega}\lambda + \frac{2\tilde G}{\pi} - q(\omega)\tilde G\tilde\Lambda\,,$$
$$\beta_{\tilde G} = 2\tilde G - \frac{1}{(4\pi)^2}\frac{3+26\omega-40\omega^2}{12\omega}\lambda\tilde G - q(\omega)\tilde G^2 \tag{4.5}$$

where $q(\omega) = (83+70\omega+8\omega^2)/18\pi$. The first two terms in each beta function exactly reproduce the results of [54, 55, 56], the remaining ones are new. The origin of the new terms will be discussed below.

To study the flow of $\tilde\Lambda$ and $\tilde G$, we set the remaining variables to their FP values $\omega = \omega_*$, $\theta = \theta_*$, and $\lambda = \lambda_* = 0$. Then, the flow takes the form of the perturbative Einstein–Hilbert flow (3.20), with $A_1 = 4/\pi$, $B_1 = -q_* = -q(\omega_*) \approx -1.440$, $A_2 = B_2 = 0$ (see figure 2.1). It has two FPs: the Gaußian FP at $\tilde\Lambda = \tilde G = 0$ and another one at

$$\tilde\Lambda_* = \frac{1}{\pi q_*} \approx 0.221\,, \qquad \tilde G_* = \frac{2}{q_*} \approx 1.389\,. \tag{4.6}$$

Like all one loop flows with constant A_1 and B_1, the critical exponents are 2 and 4. This, however, is due to the approximation. If we were able to take into account the contribution of the heat kernel coefficients B_6, B_8 etc., they would contribute terms of order $\tilde\Lambda^2$ and higher to the beta functions. We expect that these terms would produce a complex conjugate pair of critical exponents, and the corresponding spiralling flow.

In the preceding calculation we have used, besides the truncation to four–derivative terms, also the one loop approximation, and contributions coming from the heat kernel coefficients B_6 and higher have been neglected. These are also the approximations made in earlier perturbative calculations [53, 54, 55, 56], so it is instructive to understand the origin of the additional terms in (4.5), which are essential in generating the nontrivial FP. The beta functions were originally derived as coefficients of $1/\epsilon$ poles in dimensional regularization, which correspond to logarithmic divergences in the effective action. In a heat kernel derivation these terms are given by the B_4 coefficient. In the old calculations only these terms were retained. The new terms that we find come from the B_2 and B_0 coefficients, which in a conventional calculation of the effective action would correspond to quadratic and quartic divergences. Such terms are discarded in dimensional regularization, but we see that proceeding in this way one would throw away essential physical information. In order to keep track of this information in dimensional regularization one would have to take into account the contribution of the pole occurring in dimension 2. This can be done with the $2+\epsilon$ expansion, and we have already discussed the way in which the FP then appears. The situation is entirely analogous to the Wilson–Fisher FP in three dimensional scalar theory, which can be seen either using a cutoff regularization or, if dimensional regularization is used, in the $4-\epsilon$ expansion [57]. It is appropriate to stress once more that our "Wilsonian" calculation of the beta functions does not require any UV regularization. Accordingly, there are no regularization/renormalization ambiguities; the only ambiguity is in the choice of the cutoff procedure, but we have already

seen that no choice of R_k could remove the B_2 and B_0 terms.
Having discussed the beta functions in the higher derivative truncation of pure gravity, it is a simple exercise to add to them the contributions of minimally coupled matter fields, which were discussed in section 2.3. From (4.3) and (2.64) we find

$$\begin{aligned}
\beta_\lambda &= -\frac{1}{(4\pi)^2}\frac{133}{10}\lambda^2 - 2\lambda^2 a_\lambda^{(2)} \, , \\
\beta_\xi &= -\frac{1}{(4\pi)^2}\left(10\lambda^2 - 5\lambda\xi + \frac{5}{36}\right) - \xi^2 a_\xi^{(2)} \, , \\
\beta_\rho &= \frac{1}{(4\pi)^2}\frac{196}{45}\rho^2\lambda - \rho^2 a_\rho^{(2)} \, ,
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
a_\lambda^{(2)} &= \frac{1}{2880\pi^2}\left(\frac{3}{2}n_S + 9n_D + 18n_M\right) \, , \\
a_\xi^{(2)} &= \frac{1}{2880\pi^2}\frac{5}{2}n_S \, , \\
a_\rho^{(2)} &= \frac{1}{2880\pi^2}\left(-\frac{1}{2}n_S - \frac{11}{2}n_D - 31n_M\right) \, .
\end{aligned} \tag{4.8}$$

For these couplings, the new terms simply change the direction and speed of the logarithmic approach to asymptotic freedom. In particular, the ratios of the couplings approach asymptotically the following FP values

$$\begin{aligned}
\omega_* &= \frac{1}{200}\left(-549 - 960\pi^2 a_\lambda^{(2)} \pm \sqrt{921600\pi^4 (a_\lambda^{(2)})^2 + 1054080\pi^2 a_\lambda^{(2)} - 576000 a_\xi^{(2)}\pi^2 + 296401}\right) \\
\theta_* &= \frac{8}{9}\frac{49 + 180\pi^2 a_\rho^{(2)}}{133 + 320\pi^2 a_\lambda^{(2)}}
\end{aligned} \tag{4.9}$$

As before, in ω_* the positive sign will have to be chosen for stability. On the other hand, the beta functions of $\tilde\Lambda$ and $\tilde G$ are modified by the addition of $8\pi a^{(0)}\tilde G + 32\pi a^{(2)}\tilde\Lambda\tilde G$ and $32\pi a^{(2)}\tilde G^2$ respectively, where, using a type II optimized cutoff,

$$\begin{aligned}
a^{(0)} &= \frac{1}{32\pi^2}\left(n_S - 4n_D + 2n_M\right) \, , \\
a^{(2)} &= \frac{1}{96\pi^2}\left(n_S - 2n_D - 4n_M\right) \, .
\end{aligned} \tag{4.10}$$

Let us consider again the flow in the $\tilde\Lambda$–$\tilde G$ plane. Putting ω and θ to their FP values and taking the limit $\lambda \to 0$, the flow takes again the form (3.20), with the coefficients $A_1 = 4/\pi + 16\pi a^{(0)}$, $B_1 = -q(\omega_*) + 16\pi a^{(2)}$, $A_2 = B_2 = 0$.
This calculation is of some interest for the following reason. Recall that whereas applying perturbation theory to pure Einstein theory, terms proportional to $R_{\mu\nu}R^{\mu\nu}$ and R^2 can be

absorbed by a field redefinition, this is no longer so when matter fields are present [44]. Thus, in perturbation theory, gravity coupled to matter is nonrenormalizable already at one loop. One may fear that the non-Gaußian FP ceases to exist when one includes in the truncation terms that correspond to nonrenormalizable divergences in the perturbative treatment of Einstein's theory. The preceding calculation shows, at least at one loop, that this is not the case. We will further comment on the significance of this point in the conclusions.

4.2. Truncation to polynomials in R

Following the scheme outlined in the previous two sections, we now extend the formalism to handle truncations in the form of the so–called "$f(R)$–gravity" theories, where the Lagrangian density in (2.22) is a function of the Ricci scalar only. Such theories rose strong interest recently in cosmological applications [58]. At one loop, the quantization of these theories has been discussed in [59]. Here we analyze the RG flow of this type of theories where f is a polynomial of order $n \leq 8$.
The (Euclidean) action is approximated by

$$\Gamma_k[\Phi] = \sum_{i=0}^{n} g_i(k) \int d^d x \, \sqrt{g} R^i + S_{GF} + S_{gh} \, , \tag{4.11}$$

where $\Phi = \{h_{\mu\nu}, C_\mu, \bar{C}_\nu\}$ and the last two terms correspond to the gauge fixing and the ghost sector [18, 43]. The gauge fixing will have the general form

$$S_{GF} = \frac{1}{2} \int d^d x \sqrt{g} \, \chi_\mu G^{\mu\nu} \chi_\nu \tag{4.12}$$

where $\chi_\nu = \nabla^\mu h_{\mu\nu} - \frac{1+\rho}{d} \nabla_\nu h^\mu_{\ \mu}$ and $G_{\mu\nu} = \left(\alpha + \beta \nabla^2\right) g_{\mu\nu}$. In the following four subsections we give the second variation of the action and the ghost action, then we insert these expressions into the ERGE and finally we discuss the choice of gauge.

4.2.1. Truncation ansatz

The second variation of $\int \mathrm{d}^d x \sqrt{g} f(R)$ gives [1]

$$\begin{aligned} \delta^2 \int d^d x \, \left(\sqrt{g} f(R)\right) \;&=\; \int d^d x \left\{ \delta^2 \sqrt{g} f(R) + 2 \, \delta \sqrt{g} \, \delta f(R) + \sqrt{g} \left(f'(R) \delta^2 R + f''(R) (\delta R)^2 \right) \right\} \\ &=\; \int d^d x \, \sqrt{g} \Bigg\{ \frac{1}{2} f''(R) \Big(h_{\alpha\beta} \nabla^\alpha \nabla^\beta \nabla^\mu \nabla^\nu h_{\mu\nu} - 2h \nabla^2 \nabla^\alpha \nabla^\beta h_{\alpha\beta} \end{aligned}$$

[1] In the derivation of these expressions, the variations of the occurring tensors given in appendix A.6 are useful.

$$
\begin{aligned}
&+h(\nabla^2)^2 h - 2\nabla^\alpha\nabla^\beta h_{\alpha\beta}R_{\mu\nu}h^{\mu\nu} + 2\nabla^2 h R_{\mu\nu}h^{\mu\nu} + R_{\mu\nu}R_{\alpha\beta}h^{\mu\nu}h^{\alpha\beta}\Big) \\
&+\frac{1}{2}f'(R)\Big(h_{\alpha\beta}\nabla^2 h^{\alpha\beta} + h_{\alpha\beta}\nabla^\alpha\nabla^\mu h^\beta_\mu - h_{\alpha\beta}\nabla^\mu\nabla^\alpha h^\beta_\mu - h\nabla^2 h \\
&+2R_{\mu\nu}h^{\mu\beta}h^\nu_\beta - R_{\mu\nu}h^{\mu\nu}h\Big) + f(R)\left(-\frac{1}{4}h_{\mu\nu}h^{\mu\nu} + \frac{1}{8}h^2\right)\Bigg\} .
\end{aligned}
$$

All covariant derivatives are with respect to the background metric, the trace $h^\mu{}_\mu$ is abbreviated as h, prime denotes derivative with respect to R. It is already assumed here that the curvature tensor of the background metric is covariantly constant. To this one has to add the gauge fixing terms (4.12). In order to diagonalize the complete expression, we choose a (Euclidean) de Sitter background and decompose into tensor, vector and scalar parts as we did in section 3.2 for the cutoff of type Ib. The decompositions of the different operators are listed in Appendix A.5. Then one obtains for the tensor part

$$
\Gamma^{(2)}_{h^T_{\mu\nu}h^T_{\alpha\beta}} = -\frac{1}{2}\left[f'\left(-\nabla^2 - \frac{2(d-2)}{d(d-1)}R\right) + f\right]\delta^{\mu\nu,\alpha\beta} , \tag{4.13}
$$

for the vector part

$$
\Gamma^{(2)}_{\xi_\mu\xi_\nu} = \left(-\nabla^2 - \frac{R}{d}\right)\left[(\alpha+\beta\nabla^2)\left(-\nabla^2 - \frac{R}{d}\right) + \frac{2R}{d}f' - f\right]g^{\mu\nu} , \tag{4.14}
$$

and for the scalar part (which contains a nontrivial mixing between h and σ)

$$
\begin{aligned}
\Gamma^{(2)}_{hh} &= \frac{d-2}{4d}\left[\frac{4(d-1)^2}{d(d-2)}f''\left(-\nabla^2 - \frac{R}{d-1}\right)^2 + \frac{2(d-1)}{d}f'\left(-\nabla^2 - \frac{R}{d-1}\right) - \frac{2R}{d}f' + f\right] \\
&\quad -\frac{\rho^2}{d^2}\left[\alpha + \beta\left(\nabla^2 + \frac{R}{d}\right)\right]\nabla^2 \\
\Gamma^{(2)}_{h\sigma} &= \frac{d-1}{d^2}\left[(d-1)f''\left(-\nabla^2 - \frac{R}{d-1}\right)\right. \\
&\quad \left. + \frac{d-2}{2}f' + \rho\left(\alpha + \beta\left(\nabla^2 + \frac{R}{d}\right)\right)\right]\nabla^2\left(\nabla^2 + \frac{R}{d-1}\right) \\
\Gamma^{(2)}_{\sigma\sigma} &= \frac{d-1}{2d}\left[\frac{2(d-1)}{d}f''\nabla^2\left(\nabla^2 + \frac{R}{d-1}\right) - \frac{d-2}{d}f'\nabla^2 + \frac{2R}{d}f' - f\right. \\
&\quad \left. + \frac{2(d-1)}{d}\left(-\nabla^2 - \frac{R}{d-1}\right)\left(\alpha + \beta\left(\nabla^2 + \frac{R}{d}\right)\right)\right]\nabla^2\left(\nabla^2 + \frac{R}{d-1}\right) .
\end{aligned} \tag{4.15}
$$

We have dropped the subscript k for typographical clarity.

4.2.2. Ghost terms

The Fadeev–Popov ghost consists of two parts. It is calculated in the usual way from the variation of the gauge condition and the generators of gauge transformations by

$$S_c = \int d^dx\sqrt{g}\,\bar{C}_\mu G^\mu{}_\rho \frac{\delta\chi^\rho}{\delta\epsilon^\nu} C^\nu. \tag{4.16}$$

From the infinitesimal gauge transformation of the gravitational field, $\delta h_{\mu\nu} = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu$, the variation of the gauge condition under gauge transformations is given by

$$\begin{aligned} \delta\chi_\nu &= \nabla^\mu\delta h_{\mu\nu} - \frac{1+\rho}{d}\nabla_\nu\delta h \\ &= \nabla^2\epsilon_\nu + R_{\mu\nu}\epsilon^\mu + \frac{d-2-2\rho}{d}\nabla_\nu\nabla_\mu\epsilon^\mu. \end{aligned} \tag{4.17}$$

This gives

$$S_c = \int d^dx\sqrt{g}\,\bar{C}_\mu\left(\alpha+\beta\nabla^2\right)\left[\delta^\mu_\nu\nabla^2 + R^\mu{}_\nu + \frac{(d-2-2\rho)}{d}\nabla^\mu\nabla_\nu\right]C^\nu \tag{4.18}$$

where $\bar{C}_\mu$ and C^μ are the ghost and anti-ghost fields.
As we want to treat also higher-derivative gravity, it is natural to assume the operator $G^\mu{}_\nu$ can contain derivatives (for $\beta \neq 0$). In that case, one obtains a nonconstant square root of a determinant in the Fadeev-Popov procedure which on exponentiation gives rise to the so–called third ghost term [56]

$$S_b = \frac{1}{2}\int d^dx\sqrt{g}\,b^\mu G_{\mu\nu} b^\nu. \tag{4.19}$$

For $\beta = 0$ this gives just a constant factor which can be absorbed in the normalization of the functional integral. The full ghost action is then $S_{gh} = S_c + S_b$. The ghost, anti-ghost, and third ghost fields are also decomposed into transverse and longitudinal parts as in (3.24). The decompositions of the different operators are listed in Appendix A.5. Then the operators acting on these fields are

$$\begin{aligned} \Gamma^{(2)}_{\bar{c}^T_\mu c^T_\nu} &= \left(\alpha+\beta\nabla^2\right)\left(\nabla^2+\frac{R}{d}\right)g^{\mu\nu} \\ \Gamma^{(2)}_{\bar{c}c} &= -\frac{2\,(d-1-\rho)}{d}\left(\alpha+\beta\left(\nabla^2+\frac{R}{d}\right)\right)\left(\nabla^2+\frac{1}{d-1-\rho}R\right)\nabla^2 \\ \Gamma^{(2)}_{b^T_\mu b^T_\nu} &= \left(\alpha+\beta\,\nabla^2\right)g^{\mu\nu} \\ \Gamma^{(2)}_{bb} &= -\left(\alpha+\beta\left(\nabla^2+\frac{R}{d}\right)\right)\nabla^2\,. \end{aligned} \tag{4.20}$$

Finally, the decomposition of the ghosts gives rise to Jacobian determinants involving the operators Jacobians

$$J_c = -\nabla^2 \; ; \qquad J_b = -\nabla^2 \; . \tag{4.21}$$

4.2.3. Inserting into the ERGE

We choose a cutoff of type Ib. The inverse propagators (4.13,4.14,4.15,4.20) are all functions of $-\nabla^2$. Then, for each type of tensor components, the (generally matrix-valued) cutoff function $\mathbf{R_k}$ is chosen to be a function of $-\nabla^2$ such that

$$\mathbf{\Gamma}^{(2)}(-\nabla^2) + \mathbf{R_k}(-\nabla^2) = \mathbf{\Gamma}^{(2)}(P_k) \; , \tag{4.22}$$

where P_k is defined as in (2.59) for some profile function R_k. Inserting everything into the ERGE (2.50) gives:

$$\begin{aligned}
\frac{d\Gamma_k}{dt} \;=\; & \frac{1}{2}\mathrm{Tr}_{(2)}\,\frac{\frac{d}{dt}\mathbf{R}_{h^Th^T}}{\mathbf{\Gamma}^{(2)}_{h^Th^T}+\mathbf{R}_{h^Th^T}} + \frac{1}{2}\mathrm{Tr}'_{(1)}\,\frac{\frac{d}{dt}R_{\xi\xi}}{\Gamma^{(2)}_{\xi\xi}+R_{\xi\xi}} + \\
& +\frac{1}{2}\mathrm{Tr}''_{(0)}\begin{pmatrix} \Gamma^{(2)}_{hh}+R_{hh} & \Gamma^{(2)}_{h\sigma}+R_{h\sigma} \\ \Gamma^{(2)}_{\sigma h}+R_{\sigma h} & \Gamma^{(2)}_{\sigma\sigma}+R_{\sigma\sigma}\end{pmatrix}^{-1}\begin{pmatrix} \frac{d}{dt}R_{hh} & \frac{d}{dt}R_{h\sigma} \\ \frac{d}{dt}R_{\sigma h} & \frac{d}{dt}R_{\sigma\sigma}\end{pmatrix} \\
& +\frac{1}{2}\sum_{j=0,1}\frac{\frac{d}{dt}R_{hh}(\lambda_j)}{\Gamma^{(2)}_{hh}(\lambda_j)+R_{hh}(\lambda_j)} \\
& -\mathrm{Tr}_{(1)}\frac{\frac{d}{dt}R_{\bar{c}^Tc^T}}{\Gamma^{(2)}_{\bar{c}^Tc^T}+R_{\bar{c}^Tc^T}} - \mathrm{Tr}'_{(0)}\frac{\frac{d}{dt}R_{\bar{c}c}}{\Gamma^{(2)}_{\bar{c}c}+R_{\bar{c}c}} \\
& +\frac{1}{2}\mathrm{Tr}_{(1)}\frac{\frac{d}{dt}R_{b^Tb^T}}{\Gamma^{(2)}_{b^Tb^T}+R_{b^Tb^T}} + \frac{1}{2}\mathrm{Tr}'_{(0)}\frac{\frac{d}{dt}R_{bb}}{\Gamma^{(2)}_{bb}+R_{bb}} \\
& -\frac{1}{2}\mathrm{Tr}'_{(1)}\frac{\frac{d}{dt}R_{J_V}}{J_V+R_{J_V}} - \frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\frac{d}{dt}R_{J_S}}{J_S+R_{J_S}} \\
& +\mathrm{Tr}'_{(0)}\frac{\frac{d}{dt}R_{J_c}}{J_c+R_{J_c}} - \frac{1}{2}\mathrm{Tr}'_{(0)}\frac{\frac{d}{dt}R_{J_b}}{J_b+R_{J_b}} \; .
\end{aligned} \tag{4.23}$$

The first three lines contain the contribution from the metric fluctuation $h_{\mu\nu}$, which has been decomposed into its irreducible parts according to (3.23). Note that the trace over the scalar components is doubly primed, since the first two modes of the σ field do not contribute to $h_{\mu\nu}$. However, the first two modes of h do contribute, and their contribution to the trace is added separately in the third line. The fourth and fifth lines contain the contributions of the ghosts and the third ghost, each decomposed into transverse and longitudinal parts. Note that the first mode of the scalar (longitudinal) parts is omitted, as it does not contribute to C^μ and b^μ respectively. The sixth line is the contribution of the Jacobians of the transformation (3.23). These have to carry the same number of primes as

the fields in (3.23). The same is valid for the traces over the Jacobians resulting from the split into vector and scalar parts of the ghost and third ghost field, given in the last line. Eliminating the Jacobians by a further field redefinition, as in section 3.2, would produce some technically undesirable poles in the heat kernel expansion. For this reason we shall proceed as in appendix A.4 and explicitly retain the Jacobians.
In equation (4.23) we have used the convention for the primes that was used in section 3.2 and in appendix A.4. This differs from the calculation we did in [24] in having less primes in the traces over the ghosts. We will discuss this point in some detail in the next section.

4.2.4. Discussion of gauge choices

In the gauge fixing term (4.12) we have allowed in principle three gauge fixing parameters ρ, β, and α. The choice $\rho = 1$, $\beta = 0$ corresponds to the familiar de Donder gauge; in the discussion of the Einstein–Hilbert truncation we have further chosen the gauge fixing parameter $\alpha = Z$, which produces a minimal kinetic operator. Other values of α have been treated in [20]. To avoid the issue of the RG running of α, the limit $\alpha/Z \to \infty$ is sometimes invoked. A gauge fixing with $\beta \neq 0$ contains terms with four derivatives and is natural in higher derivative gravity. We will only discuss gauge choices where either α or β are nonzero, and not both simultaneously. We will call these "α gauges" and "β gauges" respectively. Note that ρ and β are dimensionless but α has dimension of mass squared. There are two ways of turning it into a dimensionless parameter: the first is to proceed as with all other couplings and define $\tilde{\alpha} = \alpha k^{-2}$; then we can set $a = 1/\tilde{\alpha}$. The second is to proceed as in (3.2) for the Einstein–Hilbert truncation and set $\alpha = Z/a$. In the following, when we use α gauges we will always adopt the first method; the second method yields similar results (up to a rescaling of a) and will not be reported. We will also use the definition $b = 1/\beta$. We will always neglect the RG running of the dimensionless gauge parameters ρ, a and b.
To reach the highest degree of simplification, a convenient gauge choice is to set $\rho = 0$ and then take either $\beta = 0$ and $\alpha \to \infty$ or $\alpha = 0$ and $\beta \to \infty$. We will now show explicitly how this simplification works in the β gauges, treating tensor, vector and scalar components separately. To write the formulae in a more compact form, in this section we will denote $\Delta = -\nabla^2$ and we define the following shorthands

$$\Delta^{(n)} = \Delta - \frac{R}{n}, \; P_k^{(n)} = P_k - \frac{R}{n} \, .$$

The transverse traceless tensor part is gauge independent and therefore is not affected by the gauge choice. The vector part receives contributions from ξ, $\bar{c}^T_\mu$, $c^{T\mu}$, b_μ, J_V:

$$\frac{1}{2}\mathrm{Tr}'_{(1)} \frac{\frac{d}{dt}(\Gamma^{(2)}_{\xi\xi}(P_k) - \Gamma^{(2)}_{\xi\xi}(\Delta))}{\Gamma^{(2)}_{\xi\xi}(P_k)} - \mathrm{Tr}_{(1)} \frac{\frac{d}{dt}\left[(\alpha - \beta P_k)\, P_k^{(4)} - (\alpha - \beta\Delta)\, \Delta^{(4)}\right]}{(\alpha - \beta P_k)\, P_k^{(4)}}$$

$$+\frac{1}{2}\mathrm{Tr}_{(1)}\frac{\partial_t\left[-\beta\left(P_k-\Delta\right)\right]}{\left(\alpha-\beta P_k\right)}-\frac{1}{2}\mathrm{Tr}'_{(1)}\frac{\partial_t\left(P_k-\Delta\right)}{P_k^{(4)}}\ .$$

In the first term, looking at (4.14) one sees that in the limit $\beta\to\infty$ only the gauge fixing term matters. In this limit the first term becomes simply

$$\frac{1}{2}\mathrm{Tr}'_{(1)}\frac{\partial_t\left(P_k(P_k^{(4)})^2\right)}{P_k\left(P_k^{(4)}\right)^2}=\frac{1}{2}\mathrm{Tr}'_{(1)}\frac{\partial_t R_k}{P_k}+\mathrm{Tr}'_{(1)}\frac{\partial_t R_k}{P_k^{(4)}}\ .$$

Treating the other three terms in the same way, several simplifications occur. However, one has to pay attention to the fact that some of the traces are primed and some are not. Therefore, in the simplifications the contributions of some isolated modes are left out. Using the optimized cutoff and specializing to $d=4$, the final result for the vector terms is

$$-\frac{1}{2}\mathrm{Tr}'_{(1)}\frac{\partial_t R_k}{P_k^{(4)}}-5\frac{\partial_t R_k(\frac{R}{4})}{P_k(\frac{R}{4})}-10\frac{\partial_t R_k(\frac{R}{4})}{P_k^{(4)}(\frac{R}{4})} \tag{4.24}$$

where the argument of the last two terms, $R/4$, is the first eigenvalue of the Laplacian on transverse vectors (see table A.4).

Let us now come to the scalar part. It receives contributions from h, σ, $\bar{c}$, c, b, J_S, J_c and J_b. The contribution of h and σ is given by the second line in eq. (4.23). One sees from (4.15) that when we set $\rho=0$, only $\Gamma^{(2)}_{\sigma\sigma}$ and $R_{\sigma\sigma}$ contain β. Therefore in the limit $\beta\to\infty$ these terms become

$$\frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\Gamma^{(2)}_{\sigma\sigma}\partial_t R_{hh}-2\Gamma^{(2)}_{h\sigma}\partial_t R_{h\sigma}+\Gamma^{(2)}_{hh}\partial_t R_{\sigma\sigma}}{\Gamma^{(2)}_{\sigma\sigma}\Gamma^{(2)}_{hh}-\left(\Gamma^{(2)}_{h\sigma}\right)^2}=\frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\partial_t R_{hh}}{\Gamma^{(2)}_{hh}(P_k)}+\frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\partial_t R_{\sigma\sigma}}{\Gamma^{(2)}_{\sigma\sigma}(P_k)}\ .$$

The second term is equal to

$$\frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\partial_t R_{\sigma\sigma}}{\Gamma^{(2)}_{\sigma\sigma}(P_k)}=\frac{1}{2}\mathrm{Tr}''_{(0)}\left(\frac{\partial_t R_k}{P_k^{(4)}}+2\frac{\partial_t R_k}{P_k^{(3)}}+\frac{\partial_t R_k}{P_k}\right)\ .$$

The longitudinal ghost fields $\bar{c}$ and c give a contribution

$$-\mathrm{Tr}'_{(0)}\frac{\partial_t(P_kP_k^{(3)}P_k^{(4)}-\Delta\Delta^{(3)}\Delta^{(4)})}{P_kP_k^{(3)}P_k^{(4)}}=-\mathrm{Tr}'_{(0)}\left(\frac{\partial_t R_k}{P_k^{(4)}}+\frac{\partial_t R_k}{P_k^{(3)}}+\frac{\partial_t R_k}{P_k}\right)\ .$$

The third ghost and the Jacobians together contribute

$$\frac{1}{2}\mathrm{Tr}'_{(0)}\frac{\partial_t(P_kP_k^{(4)}-\Delta\Delta^{(4)})}{P_kP_k^{(4)}}-\frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\partial_t(P_kP_k^{(3)}-\Delta\Delta^{(3)})}{P_kP_k^{(3)}}$$

$$
\begin{aligned}
&+\mathrm{Tr}'_{(0)}\frac{\partial_t(P_k-\Delta)}{P_k}-\frac{1}{2}\mathrm{Tr}'_{(0)}\frac{\partial_t(P_k-\Delta)}{P_k}\\
=\;&\frac{1}{2}\mathrm{Tr}'_{(0)}\left(\frac{\partial_t R_k}{P_k^{(4)}}+\frac{\partial_t R_k}{P_k}\right)-\frac{1}{2}\mathrm{Tr}''_{(0)}\left(\frac{\partial_t R_k}{P_k^{(3)}}+\frac{\partial_t R_k}{P_k}\right)+\mathrm{Tr}'_{(0)}\frac{\partial_t R_k}{P_k}-\frac{1}{2}\mathrm{Tr}'_{(0)}\frac{\partial_t R_k}{P_k}\,.
\end{aligned}
$$

Thus the scalar terms together give

$$
\frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\partial_t R_{hh}}{\Gamma^{(2)}_{hh}(P_k)}-\frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\partial_t R_k}{P_k^{(3)}}-5\frac{\partial_t R_k(\frac{R}{3})}{P_k^{(3)}(\frac{R}{3})}-\frac{5}{2}\frac{\partial_t R_k(\frac{R}{3})}{P_k^{(4)}(\frac{R}{3})}\,. \tag{4.25}
$$

The last two terms are evaluated on the second eigenvalue of the Laplacian on scalars, $R/3$. Finally we can collect the tensor, vector and scalar contributions to obtain

$$
\begin{aligned}
\frac{d\Gamma_k}{dt}\;=\;&\frac{1}{2}\mathrm{Tr}_{(2)}\left\{\frac{\partial_t P_k f'+(P_k-\Delta)\partial_t f'}{(P_k-\frac{R}{3})f'+f}\right\}-\frac{1}{2}\mathrm{Tr}'_{(1)}\frac{\partial_t R_k}{P_k-\frac{R}{4}}-\frac{1}{2}\mathrm{Tr}''_{(0)}\frac{\partial_t R_k}{P_k-\frac{R}{3}} \qquad (4.26)\\
&+\frac{1}{2}\mathrm{Tr}''_{(0)}\left\{\frac{\partial_t P_k(f'+6(P_k-\frac{R}{3})f'')+(P_k-\Delta)(\partial_t f'+3(P_k+\Delta-\frac{2}{3}R)\partial_t f'')}{\frac{2}{3}f+(P_k-\frac{2}{3}R)f'-3f''(P_k-\frac{R}{3})^2}\right\}+\Sigma
\end{aligned}
$$

where Σ is the contribution of the isolated modes. In the gauge $\rho=0$, $\alpha=0$, $\beta\to\infty$ it is

$$
\Sigma=-5\frac{\partial_t R_k(\frac{R}{4})}{P_k(\frac{R}{4})}-10\frac{\partial_t R_k(\frac{R}{4})}{P_k^{(4)}(\frac{R}{4})}-5\frac{\partial_t R_k(\frac{R}{3})}{P_k^{(3)}(\frac{R}{3})}-\frac{5}{2}\frac{\partial_t R_k(\frac{R}{3})}{P_k^{(4)}(\frac{R}{3})}\,. \tag{4.27}
$$

The calculation in the gauge $\rho=0$, $\alpha\to\infty$, $\beta=0$ proceeds in a similar way. The final result is the same except for the isolated modes, which give

$$
\Sigma=-10\frac{\partial_t R_k(\frac{R}{4})}{P_k^{(4)}(\frac{R}{4})}-10\frac{\partial_t R_k(\frac{R}{3})P_k^{(6)}(\frac{R}{3})}{P_k(\frac{R}{3})P_k^{(3)}(\frac{R}{3})}+5\frac{\partial_t R_k(\frac{R}{3})}{P_k(\frac{R}{3})}\,. \tag{4.28}
$$

Using the formulae in the appendix for the trace evaluation, writing $\tilde{R}=k^{-2}R$ and $\tilde{f}=k^{-4}f$, with the optimized cutoff this equation becomes

$$
\begin{aligned}
\frac{d\Gamma_k}{dt}\;=\;&\frac{384\pi^2}{30240\tilde{R}^2}\Bigg\{-\frac{1008(511\tilde{R}^2-360\tilde{R}-1080)}{\tilde{R}-3}-\frac{2016(607\tilde{R}^2-360\tilde{R}-2160)}{\tilde{R}-4}\\
&+20\frac{(311\tilde{R}^3-126\tilde{R}^2-22680\tilde{R}+45360)\partial_t\tilde{f}'-252(\tilde{R}^2+360\tilde{R}-1080)\tilde{f}'}{3\tilde{f}-(\tilde{R}-3)\tilde{f}'}\\
&+\Big[1008(29\tilde{R}^2+360\tilde{R}+1080)\tilde{f}'+4(185\tilde{R}^3+3654\tilde{R}^2+22680\tilde{R}+45360)\partial_t\tilde{f}'\\
&-2016(29\tilde{R}^3+273\tilde{R}^2-3240)\tilde{f}''-9(181\tilde{R}^4+3248\tilde{R}^3+15288\tilde{R}^2-90720)\partial_t\tilde{f}''\Big]\Big/
\end{aligned}
$$

$$\left(\tilde{f}''(\tilde{R}-3)^2+2\tilde{f}+(3-2\tilde{R})\tilde{f}'\right)\Bigg\}+\Sigma\,. \tag{4.29}$$

where

$$\Sigma=-\frac{10(\tilde{R}^2-20\tilde{R}+54)\tilde{R}^2}{\tilde{R}^2-7\tilde{R}+12}\quad\text{or}\qquad\Sigma=\frac{10(11\tilde{R}-36)}{\tilde{R}^2-7\tilde{R}+12} \tag{4.30}$$

in the β– and α–gauge respectively. The (dimensionful) beta functions can be extracted from this function by taking derivatives:

$$\frac{dg_i}{dt}=\frac{1}{i!}\frac{1}{V}\frac{\partial^i}{\partial R^i}\frac{d\Gamma_k}{dt}\Bigg|_{R=0}, \tag{4.31}$$

where $V=\int d^4x\sqrt{g}$. This we have done using algebraic manipulation software.
As already mentioned in section 3.9, different authors have different prescriptions for the treatment of zero modes of the ghost operator. In [24] we have taken the point of view that since in the gauge $\rho=0$, $\alpha=0$, $\beta\to\infty$ there is a one-to-one correspondence between the modes of the unphysical components ξ_μ and σ and those of the ghost field, and since the ghost contribution is supposed to cancel the contribution of the gauge degrees of freedom of the field, the cancellation occurs mode by mode, so that the trace over the vector part of the ghost must have a prime and the trace over the scalar part of the ghost must have a double prime. A similar argument applies to the third ghost. Finally, when one makes this choice for the ghosts, then also the Jacobian determinants J_c and J_b must have a double prime. So, altogether, all vector traces would have a prime and all scalar traces would have a double prime. This amounts to putting $\Sigma=0$. In the next sections we shall begin by giving the results with this definition of the traces. Later we shall also describe the effect of having Σ as in equation (4.30).

4.2.5. Results

We can now state our results. Table 4.1 gives the position of the nontrivial FP and table 4.2 gives the critical exponents, for truncations ranging from $n=1$ (the Einstein–Hilbert truncation) to $n=8$. The same information is shown graphically in figure (4.1).

Some comments are in order. First of all, we see that a FP with the desired properties exists for all truncations considered. When a new coupling is added, new unphysical FPs tend to appear; this is due to the approximation of f by polynomials. A similar phenomenon is known to occur in scalar theory in the local potential approximation [60, 61]. However, among the FP's it has always been possible to find one for which the lower couplings and critical exponents have values that are close to those of the previous truncation. That FP is then identified as the nontrivial FP in the new truncation.
Looking at the columns of Tables 4.1 and 4.2 we see that in general the properties of the FP are remarkably stable under improvement of the truncation. In particular the projection of the flow in the $\tilde{\Lambda}$-$\tilde{G}$ plane agrees well with the case $n=1$. This confirms the claims

n	$\tilde{\Lambda}_*$	$\tilde{G}_*$	$\tilde{\Lambda}_*\tilde{G}_*$	$10^3\times$								
				$\tilde{g}_{0*}$	$\tilde{g}_{1*}$	$\tilde{g}_{2*}$	$\tilde{g}_{3*}$	$\tilde{g}_{4*}$	$\tilde{g}_{5*}$	$\tilde{g}_{6*}$	$\tilde{g}_{7*}$	$\tilde{g}_{8*}$
1	0.1297	0.9878	0.1282	5.226	-20.140							
2	0.1294	1.5633	0.2022	3.292	-12.726	1.514						
3	0.1323	1.0152	0.1343	5.184	-19.596	0.702	-9.682					
4	0.1229	0.9664	0.1188	5.059	-20.585	0.270	-10.967	-8.646				
5	0.1235	0.9686	0.1196	5.071	-20.538	0.269	-9.687	-8.034	-3.349			
6	0.1216	0.9583	0.1166	5.051	-20.760	0.141	-10.198	-9.567	-3.590	2.460		
7	0.1202	0.9488	0.1141	5.042	-20.969	0.034	-9.784	-10.521	-6.048	3.421	5.905	
8	0.1221	0.9589	0.1171	5.066	-20.748	0.088	-8.581	-8.926	-6.808	1.165	6.196	4.695

Table 4.1.: Position of the FP for increasing order n of the truncation. To avoid writing too many decimals, the values of $\tilde{g}_{i*}$ have been multiplied by 1000.

n	$Re\vartheta_1$	$Im\vartheta_1$	ϑ_2	ϑ_3	$Re\vartheta_4$	$Im\vartheta_4$	ϑ_6	ϑ_7	ϑ_8
1	2.382	2.168							
2	1.376	2.325	26.862						
3	2.711	2.275	2.068	-4.231					
4	2.864	2.446	1.546	-3.911	-5.216				
5	2.527	2.688	1.783	-4.359	-3.761	-4.880			
6	2.414	2.418	1.500	-4.106	-4.418	-5.975	-8.583		
7	2.507	2.435	1.239	-3.967	-4.568	-4.931	-7.572	-11.076	
8	2.407	2.545	1.398	-4.167	-3.519	-5.153	-7.464	-10.242	-12.298

Table 4.2.: Critical exponents for increasing order n of the truncation. The first two critical exponents ϑ_0 and ϑ_1 are a complex conjugate pair. The critical exponent ϑ_4 is real in the truncation $n = 4$ but for $n \geq 5$ it becomes complex and we have set $\vartheta_5 = \vartheta_4^*$.

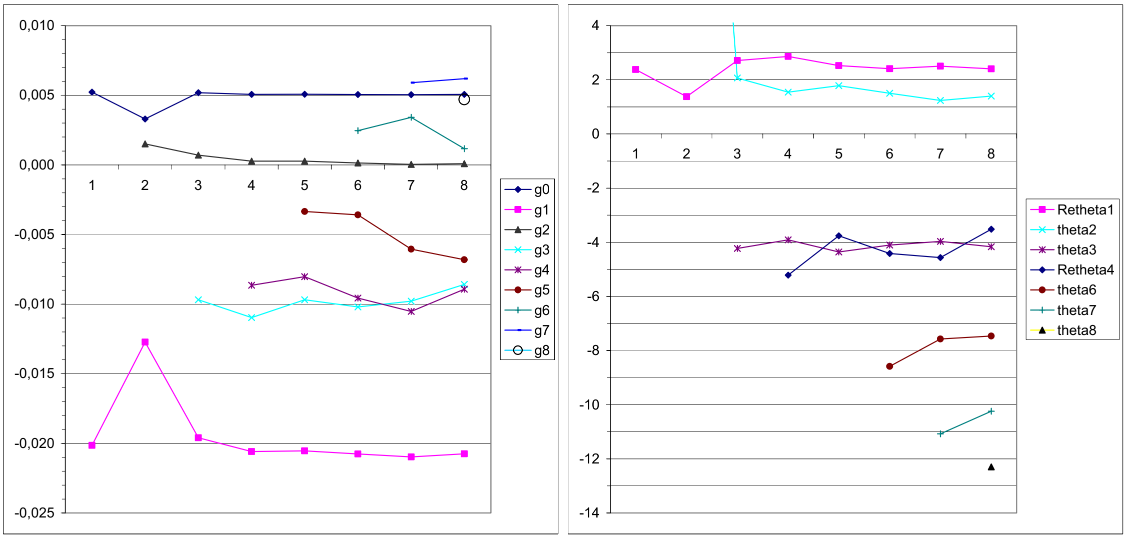

Figure 4.1.: The position of the fixed point (left panel) and the critical exponents (right panel) as functions of n, the order of the truncation.

made in [20] about the robustness of the Einstein–Hilbert truncation.
The greatest deviations seem to occur in the row $n = 2$, and in the columns g_2 and ϑ_2. The value of g_{2*} decreases steadily with the truncation. The critical exponent ϑ_2 appears for the first time in the truncation $n = 2$ with a very large value, but it decreases quickly and seems to converge around 1.5. This behaviour may be related to the fact that g_2 is classically a marginal variable.
The beta function of g_2 due to the Einstein–Hilbert action (in the spirit of section V) was considered first in [62]; the full truncation $n = 2$ has been studied in [63]. When comparing our results for the case $n = 2$ with those of [63], one has to keep in mind that they generally depend on the shape of the cutoff function. A significant quantity with very weak dependence on the cutoff function is the dimensionless product ΛG. The value $0.12 \div 0.14$ given in [63] for ΛG is very close to the value we find in all truncations except $n = 2$. Our value for $\tilde{g}_{2*}$ in the $n = 2$ truncation has the same sign but is between one half and one third of their value, depending on the cutoff function. This is another manifestation of the relatively unstable behaviour of this variable. The value given in [63] for the critical exponent ϑ' varies in the range $2.2 \div 3.2$ depending on the shape of the cutoff, and is in good agreement with our results, again with the exception of the $n = 2$ truncation. Finally, in [63] the critical exponent ϑ_2 has stably large values of the order of 25 with the compact support cutoffs, but varies between 28 and 8 with the exponential cutoffs. The values at the high end agree well with our result in the $n = 2$ truncation. The shape dependence that is observed with exponential cutoffs can be taken as a warning of the truncation–dependence of this quantity.
Tables 4.3 and 4.4 give the position of the FP and the critical exponents in the truncation

	$\tilde{\Lambda}_*$	$\tilde{G}_*$	$\Lambda_* G_*$	$10^3\times$								
				$\tilde{g}_{0*}$	$\tilde{g}_{1*}$	$\tilde{g}_{2*}$	$\tilde{g}_{3*}$	$\tilde{g}_{4*}$	$\tilde{g}_{5*}$	$\tilde{g}_{6*}$	$\tilde{g}_{7*}$	$\tilde{g}_{8*}$
α	0.1239	0.9674	0.1199	5.096	-20.564	0.153	-6.726	-5.722	-2.981	1.980	3.305	1.631
β	0.1242	0.9682	0.1202	5.103	-20.548	0.138	-6.133	-4.621	-1.407	2.240	2.207	0.610

Table 4.3.: Position of the FP for $n = 8$ taking into account the contribution of the isolated modes given in (4.30). To avoid writing too many decimals, the values of $\tilde{g}_{i*}$ have been multiplied by 1000.

gauge	$Re\vartheta_1$	$Im\vartheta_1$	ϑ_2	ϑ_3	$Re\vartheta_4$	$Im\vartheta_4$	ϑ_6	ϑ_7	ϑ_8
α	2.123	2.796	1.589	-4.212	-1.107	5.558	-7.321	-9.923	-12.223
β	2.049	2.511	1.438	-3.928	-0.102	7.320	-7.239	-9.664	-12.381

Table 4.4.: Critical exponents in the α– and β–gauge taking into account the contribution of the isolated modes given in equation (4.30).

$n = 8$, using the definition of the traces with less primes (*i.e.* using for Σ the values given in equation (4.30)). While the numerical results, especially for some of the higher couplings, do change sensibly, the overall qualitative picture is not affected. In this connection we also mention that in [64] the same calculation has been independently repeated for $n \leq 6$. The slight numerical differences between their results and those reported here is due entirely to the fact that they define the traces over the Jacobians J_c and J_b with a single prime; it has been checked that when the same definition is used, the results agree perfectly.

4.2.6. Ultraviolet critical surface

Possibly the most important result of this calculation is that in all truncations the operators from R^3 upwards are irrelevant. One can conclude that in this class of truncations the UV critical surface is three–dimensional. Its tangent space at the FP is spanned by the three eigenvectors corresponding to the eigenvalues with negative real part. In the parametrization (4.11), it is the three–dimensional subspace in $\mathbf{R}^9$ defined by the equation:

$$\begin{aligned}
\tilde{g}_3 &= +0.0006 + 0.0682\,\tilde{g}_0 + 0.4635\,\tilde{g}_1 + 0.8950\,\tilde{g}_2 \\
\tilde{g}_4 &= -0.0092 - 0.8365\,\tilde{g}_0 - 0.2089\,\tilde{g}_1 + 1.6208\,\tilde{g}_2 \\
\tilde{g}_5 &= -0.0157 - 1.2349\,\tilde{g}_0 - 0.7254\,\tilde{g}_1 + 1.0175\,\tilde{g}_2 \\
\tilde{g}_6 &= -0.0127 - 0.6226\,\tilde{g}_0 - 0.8240\,\tilde{g}_1 - 0.6468\,\tilde{g}_2 \\
\tilde{g}_7 &= -0.0008 + 0.8139\,\tilde{g}_0 - 0.1484\,\tilde{g}_1 - 2.0181\,\tilde{g}_2 \\
\tilde{g}_8 &= +0.0091 + 1.2543\,\tilde{g}_0 + 0.5085\,\tilde{g}_1 - 1.9012\,\tilde{g}_2
\end{aligned} \tag{4.32}$$

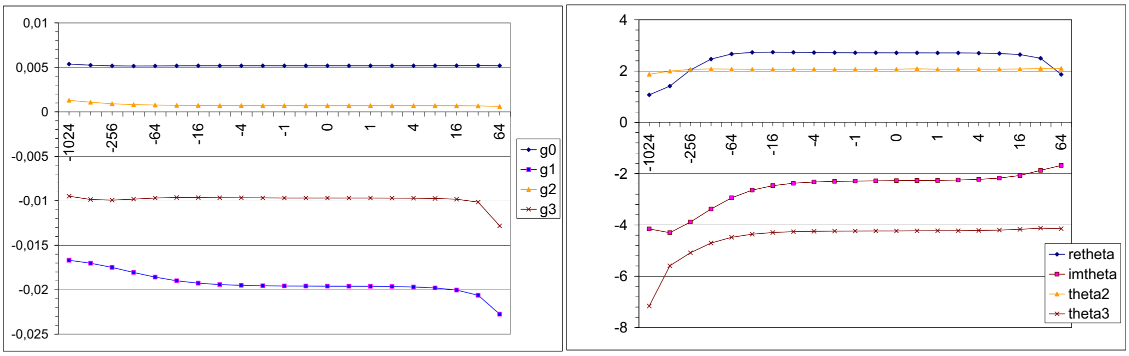

Figure 4.2.: The position of the nontrivial fixed point (left panel) and the critical exponents (right panel) in the R^3 truncation, in the gauge $\rho = 0$ and gauge parameter a variable in the range $-1024 < a < 64$. Note that the scales are logarithmic for $a > 0$ and $a < 0$ but not at $a = 0$.

Unfortunately, we cannot yet conclude from this calculation that the operators $\mathcal{O}_i^{(n)}$ with $n \geq 6$ would be irrelevant if one considered a more general truncation: the beta functions that we computed here are really mixtures of the beta functions for various combinations of powers of Riemann or Ricci tensors, which, in de Sitter space, are all indistinguishable. However, there is a clear trend for the eigenvalues to grow with the power of R. In fact, in the best available truncation, the real parts of the critical exponents differ from their classical values d_i by at most 2.1, and there is no tendency for this difference to grow for higher powers of R. This is what one expects to find in an asymptotically safe theory [15]. With a finite dimensional critical surface, one can make definite predictions in quantum gravity. The real world must correspond to one of the trajectories that emanate from the FP, in the direction of a relevant perturbation. Such trajectories lie entirely in the critical surface. Thus, at some sufficiently large but finite value of k one can choose arbitrarily three couplings, for example $\tilde{g}_0$, $\tilde{g}_1$, $\tilde{g}_2$ and the remaining four are then determined by (4.32). These couplings could then be used to compute the probabilities of physical processes, and the relations (4.32), in principle, could be tested by experiments. The linear approximation is valid only at very high energies, but it should be possible to numerically solve the flow equations and study the critical surface further away from the FP.

4.2.7. Scheme dependence in R^3–gravity

The gauge parameters cannot be chosen totally independently. For example, if one takes $\rho = 0$, α and β can be taken to infinity or zero, the limits exist and do commute. If one takes instead $\rho \neq 0$, either α or β has to stay finite.
It would be very cumbersome to check the stability of the results under changes in the gauge fixing parameters in high truncations. However, it is possible to do so for $n = 3$. As we shall see, this is enough to verify that g_3 is always irrelevant and therefore is a strong indication that the dimension of the critical surface is stable under such variations.

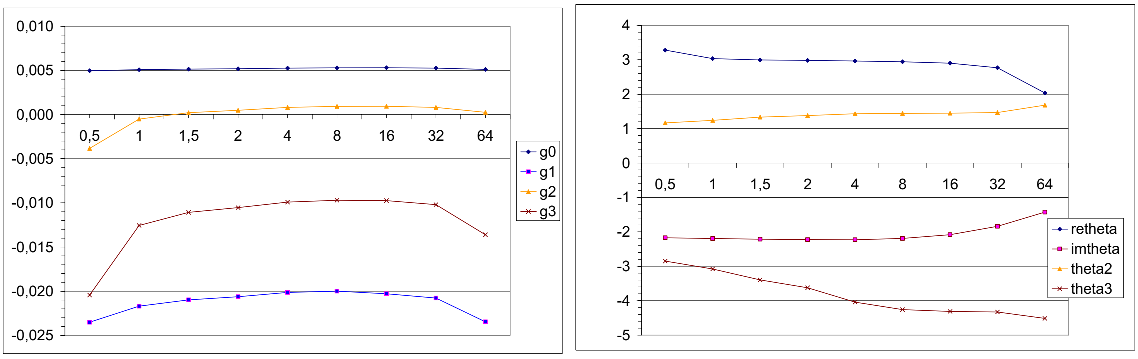

Figure 4.3.: The position of the nontrivial fixed point (left panel) and the critical exponents (right panel) in the R^3 truncation, in the gauge $\rho = 1$ and gauge parameter a variable in the range $0.5 < a < 64$.

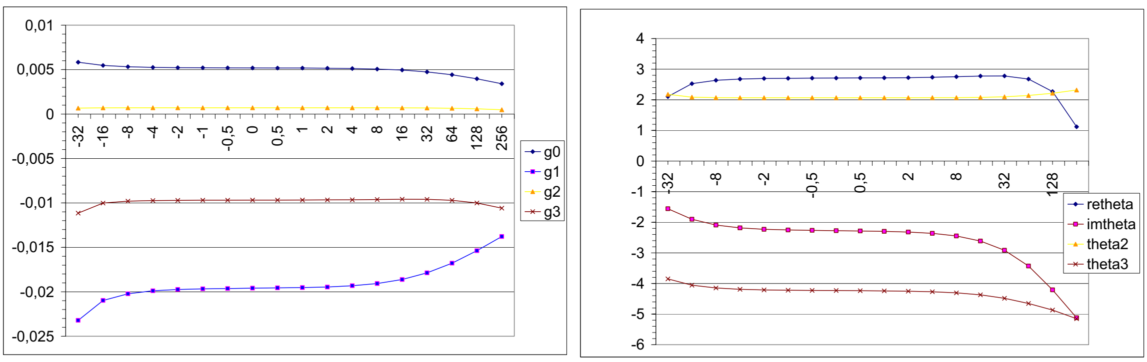

Figure 4.4.: The position of the nontrivial fixed point (left panel) and the critical exponents (right panel) in the R^3 truncation, in the gauge $\rho = 0$ and gauge parameter b variable in the range $-32 < b < 256$. Note that the scales are logarithmic for $b > 0$ and $b < 0$ but not at $b = 0$.

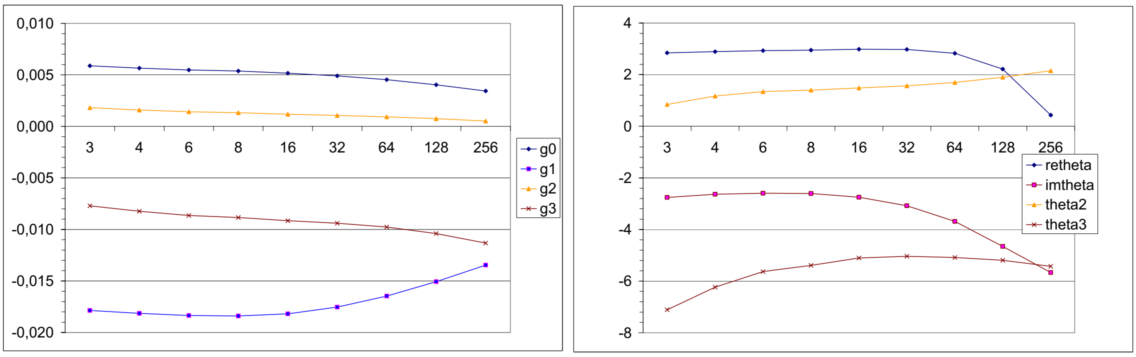

Figure 4.5.: The position of the nontrivial fixed point (left panel) and the critical exponents (right panel) in the R^3 truncation, in the gauge $\rho = 1$ and gauge parameter b variable in the range $3 < b < 256$.

Figures 4.2 and 4.4 show the position of the FP and the critical exponents in the gauges $\rho = 0$, $\beta = 0$, α variable and $\rho = 0$, $\alpha = 0$, β variable, respectively. It appears that one can take the limits $\alpha \to \pm\infty$ and $\beta \to \pm\infty$ without problems and that the results are continuous.
We have also checked the results in the gauge $\rho = 1$, which was discussed extensively in the case of the Einstein–Hilbert truncation. Figures 4.3 and 4.5 show the position of the FP and the critical exponents in the gauges $\rho = 1$, $\beta = 0$, α variable and $\rho = 0$, $\alpha = 0$, β variable, respectively. It appears that when $\rho = 1$ one cannot take the limits $\alpha \to \infty$ and $\beta \to \infty$, so we have limited ourselves to positive values of α and β.

In this section we have shown that also truncations including higher-derivative terms, namely curvature squared truncations with different curvature invariants and polynomials in the Ricci scalar till eighth order, possess a nontrivial UV fixed point. For the first case, one had to restrict to the one loop approximation, but was also able to include minimally coupled matter. It was found that all couplings are relevant and the curvature squared couplings are asymptotically free.
For the second case it was found that all couplings with positive mass dimension remain irrelevant at the non-Gaußian fixed point so that only three relevant couplings remained, the cosmological constant, Newton's constant and the curvature squared coupling. Therefore the UV critical surface remains three-dimensional in this case and has been determined at the fixed point. The parameter dependence of the different truncations has been studied and the results have been confirmed to be stable. Especially the fixed point values of the cosmological constant and Newton's constant have been found to be very stable under parameter variations and the inclusion of higher-derivative couplings. This indicates that the Einstein-Hilbert truncation is a very good approximation and leads to reliable results. In the next section we will summarize all results and discuss several open issues.

5. Conclusions

In this thesis we have mostly reviewed and extended recent work on the asymptotic safety approach to quantum gravity. In this approach, the metric is taken seriously as the carrier of the fundamental degrees of freedom, relying on Quantum Field Theory with its well-tested principles. The central hypothesis to make this procedure work in spite of the well-known difficulties is the existence of a nontrivial FP for gravity, having finitely many UV–attractive directions. In this way one can obtain
i) a well–defined UV scaling limit which is lost in perturbation theory because of the uncontrollable influence of higher–loop corrections, and
ii) a persistence of predictivity which is lost in perturbation theory due to the necessity of an infinite number of counterterms.
Accordingly, most of the work has gone towards proving the existence of such a FP. Necessary ingredients for its existence are that Newton's gravitational constant becomes antiscreening, that means diminishing, in the UV, and that it has an anomalous dimension leading to a propagator scaling as p^{-4} near the fixed point. With the possible existence of the fixed point the serious consideration of phenomenological consequences at and beyond the Planck has become important and has been strived also in this thesis. Let us summarize the evidence for the existence of a UV fixed point that has been obtained by applying Wilsonian RG methods to gravity (for general reviews see also [65, 66, 67]).

Large N limit In order to start from the simplest setting, we have begun by considering the contribution of minimally coupled matter fields to the beta functions of the gravitational couplings, which can be simply obtained from the heat kernel expansion for Laplace–type operators and give beta functions that are just constants (see equations (2.68,2.70)). Such beta functions produce a FP *for all gravitational couplings*.
This result is important for two reasons: the first reason is that in the limit when the number of matter fields is very large, this is the dominant contribution to the beta functions. Insofar as the number of matter fields in the real world is large, and matter couplings can be assumed to be asymptotically free, this may be a reasonably good approximation for some purposes. The second reason is that the contribution of gravitons to the gravitational beta functions, neglecting the RG improvements and considering only the leading term in the expansion in the cosmological constant, is essentially of the same form (compare equations (2.70), (3.21) and (4.5)). One may therefore see this as a bare skeleton that can be dressed by taking into account increasingly more subtle effects.

Truncations The simplest way of approximating the ERGE consists in truncating the form of the average effective action Γ_k, *i.e.* retaining only certain operators and neglect-

ing all others. It is known from experience with scalar field theory that the most reliable procedure is not to truncate on the power of the field but rather to truncate in the number of derivatives. In the case of gravity the lowest order of the derivative expansion is the so–called Einstein–Hilbert truncation, where one retains only the cosmological constant and Newton's constant. In this case one can treat the ERGE without any further approximation. The results are therefore "nonperturbative", in the sense that couplings are not required to be small.
In section 3 we have discussed in detail several ways of implementing the cutoff procedure of the ERGE and we have shown that the results are robust against such changes. This adds to earlier studies concerning the dependence of the results under changes of gauge and changes of the cutoff profile and increase our confidence that the FP is not an artifact of the truncation. The results have been obtained in arbitrary dimensions d. Since in this approach the dimension is not used as a regulator, one can follow the position of the FP as a function of d without encountering singularities, and compare with the results of the epsilon expansion. We have seen that, numerically, the epsilon expansion gives a rather poor approximation at $d = 4$.

UV divergences and asymptotic safety We have then shown how to recover the perturbative divergences from the ERGE. In particular, we have seen that the one loop divergences obtained by 't Hooft and Veltman [44] can be reproduced starting from the Einstein–Hilbert truncation of the ERGE, and that they are independent of the cutoff procedure and of the profile of the cutoff functions. We have also reproduced the known (scheme–dependent) one loop divergences in the presence of a cosmological constant.
A more accurate treatment of these divergences requires that the terms quadratic in curvature be retained from the start, *i.e.* that they are included in the truncations. Unfortunately, due to technical complications, it has been impossible so far to treat the curvature–squared truncation (which would constitute the second order in the derivative expansion) in the same way as the Einstein–Hilbert truncation. The most complete available treatment, which was described in section 4.1, requires that further approximations be made: essentially, one is just keeping the lowest terms in the perturbative expansion. Still, the results obtained from the ERGE differ from those that had been calculated before using more conventional methods.
The beta functions of the dimensionless couplings λ, ξ, ρ (defined in (3.74)) coincide with those that had been computed previously. In this approximation these couplings are asymptotically free, tending logarithmically to zero from a well–defined direction in (λ, ξ, ρ)–space. However, the beta functions of $\tilde{\Lambda}$ and $\tilde{G}$ contain, in addition to the terms that were known before, also some new terms that generate a nontrivial FP. When the other couplings are set to their FP values, the flow in the $(\tilde{\Lambda}, \tilde{G})$–plane has the same form as the perturbative Einstein–Hilbert flow (3.21).
In the perturbative approach to Einstein's theory, the one loop divergences are at most quadratic in the curvature. But the Euler term is a total derivative and the remaining terms vanish on shell, so, if we neglect the cosmological constant, all counterterms can

be eliminated by a field redefinition, up to terms of higher order. One could suspect that in pure gravity the existence of the FP in truncations involving at most terms quadratic in curvature is in some way related to the absence of corresponding genuine divergences in perturbation theory. If this was the case, one might expect that the FP ceases to exist as soon as one includes in the truncation terms that correspond to nonrenormalizable divergences in the perturbative treatment of Einstein's theory.
From the point of view of the ERGE, this looks quite unlikely, for various reasons. First of all, there exist examples of theories that are perturbatively nonrenormalizable while asymptotically safe at a nontrivial FP [70, 71]. Second, when one truncates the action Γ_k, in principle all terms that are retained could be equally important. The argument about eliminating terms that contain the Ricci tensor only applies when the higher order terms are considered as infinitesimal perturbations of the Hilbert action with zero cosmological constant, for then at leading order the on shell condition is simply $R_{\mu\nu} = 0$.
However, if the higher order terms are not infinitesimal, the on shell condition is much more complicated and there is no indication that they can still be eliminated by field redefinitions. In fact for certain classes of terms it is known that they can only be eliminated at the price of introducing a number of scalar fields with new interactions [68]. Finally, the calculations reported here already provide evidence to the contrary. In the presence of a cosmological term neither $R_{\mu\nu}R^{\mu\nu}$ nor R^2 vanish on shell, so, according to the perturbative reasoning, none of the FP's discussed in section 4 should exist. If for some reason one is willing to neglect the cosmological constant, in pure gravity the first genuine perturbative divergence is cubic in the Riemann tensor, but in the presence of matter fields already the one loop logarithmic divergences, which are quadratic in curvature, do not vanish on shell.
Does this imply that in the presence of matter the FP ceases to exist? We have seen explicitly in section 4.1 that at least at one loop this is not the case. Admittedly, this is only a partial result, and an "exact" calculation would be necessary to definitely settle this point, but it is very strong indication that the Wilsonian approach can handle terms that would be troublesome in perturbation theory. For these reasons we also believe that nothing special will happen when the Goroff–Sagnotti cubic term (which was used to prove that pure gravity is perturbatively nonrenormalizable) will be included in the truncation.

f(R) truncations While for the time being the systematic derivative expansion cannot be pushed beyond the fourth order, one can still consider different truncations that go beyond the Einstein–Hilbert one. So far, the truncation with the greatest number of free parameters that can be dealt with exactly is so–called $f(R)$ gravity, where f is a polynomial in the scalar curvature. The calculation of the beta functions of this theory was briefly reported in [24] for polynomials up to sixth order and has been described in greater detail in section 4.2 for polynomials of order up to eight. The most important results of these calculations are the relative stability of the results under the increase in the number of terms in the truncation, and the finite dimensionality of the critical surface. It appears that the critical exponents do not deviate very strongly from the classical dimensions, as

expected, so that the terms with six or more derivatives are irrelevant.
Further results involving infinitely many couplings were given in [72] in the two Killing vector reduction of gravity and more recently in [73] in the conformal reduction, where transverse degrees of freedom of the metric are ignored. There is some work showing that the FP exists also in the presence of some types of matter [74]. Some independent evidence in favor of a gravitational FP is also found in numerical simulations, both in the causal dynamical triangulation approach [75] and in Regge calculus [76]. This concludes our overview of the currently available evidence for a gravitational FP. Let us now make a few comments and discuss future prospects.

Scheme dependence The dependence of certain results on the choice of the cutoff scheme (which we called "scheme dependence") is sometimes the source of worries. For example, could the FP for Newton's constant disappear if we chose the cutoff function in a perverse way? This scheme dependence is the counterpart in the Wilsonian approach of the regularization and renormalization scheme ambiguities that are encountered in perturbation theory. As we have discussed in section 3.9, such scheme dependence is to be expected for all results that concern dimensionful couplings. We have seen that within certain approximations (which we called the "perturbative Einstein–Hilbert flow") all the terms with six or more derivatives can indeed be made to vanish by a choice of cutoff. However, as can be seen from equations (3.16,3.29,3.38,3.57), the general properties of the cutoff functions are such that one cannot similarly set to zero Newton's constant and the cosmological constant. This is the main reason for the robustness of the nontrivial FP.

Exactness of the RG equation Another point that is sometimes a source of misunderstandings is the use of the term "exact" in relation to the ERGE, and the "nonperturbative" character of these calculations. The beta functions that we have calculated in this thesis are rational functions of the couplings $\tilde{g}_i$. The appearance of the couplings in the denominators suggests that they could be regarded as resummations of infinitely many perturbative terms. Thus, the beta functions might still be considered "perturbative", in the sense that they are *analytic in the coupling constants*. We think that the ERGE is actually capable of obtaining also results that are nonanalytic in the coupling constant, but insofar as the FP is present already in the lowest order of perturbation theory (see equation (1.12)) such a degree of sophistication would be unnecessary. On the other hand, the beta functions obtained from the ERGE can be said to be "nonperturbative" in the sense that their validity is not limited to small couplings.
If we compare the one loop beta function (1.12) to the beta function (3.66) obtained from the ERGE, they are seen to have the same form. However, the former calculation by hypothesis is valid only for $\tilde{G} \ll 1$; while the latter was obtained from a completely different procedure and is not similarly limited. (The difference in the coefficient is not important because the result is scheme–dependent anyway.) This suggests that the perturbative result is at least qualitatively valid also for relatively large values of the coupling. Notice that the RG improvement modifies the one loop result and produces singularities of the

beta functions. The nontrivial FP always occurs on the side of the singularity that can be continuously connected to the Gaußian FP.

Unitarity The appearance of higher derivative terms in the action at the FP raises the old issue of unitarity. The Wilsonian point of view puts that problem in a slightly different perspective. A tree–level analysis of the action (3.1) plus (3.74) shows that it generally contains besides a massless spin two graviton other particles with a mass of the order of Planck's mass and a negative residue at the pole (ghosts) [69]. Some authors have suggested that, due to RG effects, these particles may not propagate [53, 77]. From a general Wilsonian point of view, the presence of a propagator pole at a given mass can only be reliably established by considering the effective action at a value of k comparable to that mass. The effective action in the FP regime is probably quite different from the effective action at the scale of the putative ghost mass, so any conclusion about the mass spectrum based on the FP–effective action is probably of little value.
This fact is clearly exemplified by QCD. The tree level analysis of the QCD action, which is known to be a good description of strong interactions at high energies (*i.e.* near the Gaußian FP), would suggest a spectrum of particles (quarks and gluons) none of which is observed in nature. Since the QCD coupling becomes stronger as the energy decreases, the description of strong interactions in terms of quarks and gluons becomes extremely complicated long before one reaches the scale of the quark masses, and it is believed that if one could actually do such calculations, quarks and gluons would be found not to propagate. It is conceivable that some similar phenomenon may occur in gravity, so the FP action (which is the analog of the QCD action) should not be expected to be a good guide to the particle spectrum of the theory. The confinement of quarks and gluons is one of the outstanding problems in particle physics and it is unfortunately possible that the analogous problem in gravity may prove equally hard.
This is related to the more general question about the low energy action corresponding to the FP action but this lies outside the scope of the present thesis. We refer the reader to [78] for some discussions of this point.

Einstein-Hilbert versus higher-derivative truncations Another remarkable aspect of these calculations is that the one loop flow in the $\tilde{\Lambda}$–$\tilde{G}$ plane is essentially the same (aside from nonuniversal numerical coefficients) in the Einstein–Hilbert truncation (section 3.8) and in the curvature squared truncation (section 4.1). In the latter, λ and ξ tend logarithmically to zero and the corresponding terms in the action diverge. Thus the dynamics becomes dominated by the four derivative terms, while in the Einstein–Hilbert truncation it is dominated by the two derivative term. It may be somewhat surprising that the structure of the flow should be so similar in spite of such differences in the dynamics.
This can be at least partly understood by the following argument. In gravity at low energies (hence in the perturbative regime) one can consider all couplings to be scale–independent, and therefore the relative importance of the terms in the action can be determined simply by counting the number of derivatives of the metric. For example, at

momentum scales $p^2 \ll Z$ (recall that Z is the square of the Planck mass), using standard arguments of effective field theories, the terms in the action (3.74) with four derivatives are suppressed relative to the term with two derivatives by a factor p^2/Z. This is not the case in the FP regime. If we consider phenomena occurring at an energy scale p, then also the couplings should be evaluated at $k \approx p$. But then, if there is a nontrivial FP, Z runs exactly as p^2 and therefore both terms are of order p^4. This argument can be generalized to all terms in the derivative expansion (2.22): at a FP, the running of each coefficient $g_i^{(n)}$, which is given by the canonical dimension, exactly matches the number of derivatives in the operator $\mathcal{O}_i^{(n)}$, so that all terms are of order p^4.
This may at least in part explain the robustness of the results. When many terms are taken into account in the truncation, it is hard to have an intuitive feeling for the mechanism that gives rise to the FP. For example, the beta functions which are obtained by taking derivatives of (4.29) with respect to curvature are exceedingly complicated. In fact, they are manipulated by the software and one does not even see them. This is why we have strived in the first few sections of this thesis to emphasize the simplest approximations. They give a clear and intuitive picture suggesting the emergence of a FP *to all orders in the derivative expansion*.
We would therefore like to conclude by overturning a common belief: the existence of a nontrivial FP does not require a delicate cancellation of terms. The FP appears essentially due to the dimensionful nature of the coupling constants, and it can be seen already in the perturbative Einstein–Hilbert flow (*i.e.* in the approximation where one considers just the contribution of gravitons or matter fields with kinetic operators of the form $-\nabla^2 + \mathbf{E}$, where $\mathbf{E}$ is linear in curvature). More advanced approximations dress up this simple result with RG improvements and with the contribution of additional couplings. The argument in the preceding paragraph suggests that the new couplings will not qualitatively change the results. And indeed, so far it seems that generically such dressing does not spoil the FP. So, to conclude on an optimistic note, one could say that it would actually require a special conspiracy by the new terms to undo the perturbative FP.

Phenomenology of RG in gravity There remains the question for the phenomenological consequences of the RG effects in gravity. In section 3.10 we have analyzed the case of a test particle in de Sitter space which would feel the scale dependence of the cosmological constant. This would lead to modified dispersion relations and therefore connect to the phenomenology of models with Lorentz invariance violation or deformed special relativity.
Other possible scenarios were considered in a number of papers covering the following topics. Black hole spacetimes and black hole formation were considered in [85]. Interesting questions in this context are if the running of the gravitational couplings, especially the weakening of the Newton constant, will change the black hole formation process significantly up to prohibiting the formation of a singularity, and how an object in a black hole spacetime will respond to the scale dependence of the metric. This, as in other concrete applications, involves the identification of the appropriate momentum scale corre-

sponding to the RG scale. Symmetry arguments suggest that the radial distance to the black hole center might play a significant role. As the formation of black holes might become possible in colliders if compactified additional space dimensions exist the influence of the RG effects could be studied in such scenarios as done in [27].

Cosmology was addressed by different groups where the considered scenarios are not stringently connected to the existence or nonexistence of a UV fixed point [78, 86, 100, 101, 102, 103, 104, 105]. RG effects could also arise in a general effective field theory limit. Of phenomenological importance could be especially the Cosmological Constant which, due to its tiny value, will start running significantly at very low RG scales. This could lead to RG effects on large distances and could modify the evolution of the universe. Exciting is the possibility to give rise to an inflationary phase from RG running in the early universe or possible imprints on the Cosmological Microwave Background.

The possible running of the Newton constant could also give rise to a modified Newtonian potential on galactic distances and therefore (partially) explain the motivation for the existence of dark matter [107, 113].

Further scenarios deal with the question what a scale dependent metric, giving rise to scale dependent distance measurements, will admit as possible length scales, if there could be a minimal length for example [123], or what are the consequences of the dimensional reduction by the huge anomalous dimension of Newton's constant at high energies [109].

Altogether, in the interpretation of the exciting possible phenomenology obtainable in the RG approach there is still much open space to explore and could push the view on the nature of quantum gravity to new frontiers.

A. Appendix

A.1. Trace technology

The right hand side of the ERGE (2.50) is the trace of a function of a differential operator. To illustrate the methods employed to evaluate such traces, we begin by considering the covariant Laplacian in a metric g, $-\nabla^2$. If the fields carry a representation of a gauge group G and are coupled to gauge fields for G, the covariant derivative ∇ contains also these fields. We will denote $\Delta = -\nabla^2 \mathbf{1} + \mathbf{E}$ a second order differential operator. $\mathbf{E}$ is a linear map acting on the spacetime and internal indices of the fields. In our applications to de Sitter space it will have the form $\mathbf{E} = qR\,\mathbf{1}$ where $\mathbf{1}$ is the identity in the space of the fields and q is a real number.
The trace of a function W of the operator Δ can be written as

$$\mathrm{Tr}W(\Delta) = \sum_i W(\lambda_i) \tag{A.1}$$

where λ_i are the eigenvalues of Δ. Introducing the Laplace anti-transform $\tilde{W}(s)$

$$W(z) = \int_0^\infty ds\, e^{-zs} \tilde{W}(s) \tag{A.2}$$

we can rewrite (A.1) as

$$\mathrm{Tr}W(\Delta) = \int_0^\infty ds \mathrm{Tr}K(s) \tilde{W}(s) \tag{A.3}$$

where $\mathrm{Tr}K(s) = \sum_i e^{-s\lambda_i}$ is the trace of the heat kernel of Δ. We assume that there are no negative and zero eigenvalues; if present, these will have to be dealt with separately. The trace of the heat kernel of Δ has the well-known asymptotic expansion for $s \to 0$

$$\mathrm{Tr}\left(e^{-s\Delta}\right) = \frac{1}{(4\pi)^{\frac{d}{2}}} \left[B_0(\Delta)\, s^{-\frac{d}{2}} + B_2(\Delta)\, s^{-\frac{d}{2}+1} + \ldots + B_d(\Delta) + B_{d+2}(\Delta)\, s + ...\right] \tag{A.4}$$

where $B_n = \int \mathrm{d}^d x \sqrt{g} \mathrm{tr} \mathbf{b}_n$ and $\mathbf{b}_n$ are linear combinations of curvature tensors and their covariant derivatives containing $2n$ derivatives of the metric.[1]
Assuming that $[\Delta, \mathbf{E}] = 0$, the heat kernel coefficients of Δ are related to those of $-\nabla^2$ by

[1] An alternative derivation of the heat kernel coefficients by directly suming the eigenvalues is given in appendix A.7.

$$\mathrm{Tr}e^{-s(-\nabla^2+\mathbf{E})} = \frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{k,\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\int \mathrm{d}^d x\sqrt{g}\,\mathrm{tr}\,\mathbf{b}_k(\Delta)\mathbf{E}^\ell s^{k+\ell-2}. \tag{A.5}$$

The first six coefficients have the form [79]

$$\begin{aligned}
\mathbf{b}_0 &= \mathbf{1} && \text{(A.6)}\\
\mathbf{b}_2 &= \frac{R}{6}\mathbf{1}-\mathbf{E} && \text{(A.7)}\\
\mathbf{b}_4 &= \frac{1}{180}\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}-R^{\mu\nu}R_{\mu\nu}+\frac{5}{2}R^2+6\nabla^2 R\right)\mathbf{1}\\
&\quad+\frac{1}{12}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu}-\frac{1}{6}R\mathbf{E}+\frac{1}{2}\mathbf{E}^2-\frac{1}{6}\nabla^2\mathbf{E} && \text{(A.8)}\\
\mathbf{b}_6 &= \frac{1}{180}R\mathbf{1}\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}-R^{\mu\nu}R_{\mu\nu}+\frac{5}{6}R^2+\frac{7}{2}\nabla^2 R\right)\\
&\quad+\frac{R}{2}\mathbf{E}^2+\mathbf{E}^3+\frac{1}{30}\mathbf{E}\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}-R^{\mu\nu}R_{\mu\nu}+\frac{5}{2}R^2+6\nabla^2 R\right)\\
&\quad+\frac{R}{12}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu}+\frac{1}{2}\mathbf{E}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu}+\frac{1}{2}\mathbf{E}\nabla^2\mathbf{E}-\frac{1}{2}\mathbf{J}_\mu\mathbf{J}^\mu\\
&\quad+\frac{1}{30}\left(2\mathbf{\Omega}^\mu{}_\nu\mathbf{\Omega}^\nu{}_\alpha\mathbf{\Omega}^\alpha{}_\mu-2R^\mu{}_\nu\mathbf{\Omega}_{\mu\alpha}\mathbf{\Omega}^{\alpha\nu}+R^{\mu\nu\alpha\beta}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}_{\alpha\beta}\right)\\
&\quad+\mathbf{1}\left[-\frac{1}{630}R\nabla^2 R+\frac{1}{140}R_{\mu\nu}\nabla^2 R^{\mu\nu}+\frac{1}{7560}\left(-64R^\mu{}_\nu R^\nu{}_\alpha R^\alpha{}_\mu+48R^{\mu\nu}R_{\alpha\beta}R^\alpha{}_\mu{}^\beta{}_\nu\right.\right.\\
&\quad\left.\left.+\,6R_{\mu\nu}R^\mu{}_{\rho\alpha\beta}R^{\nu\rho\alpha\beta}+17R_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta}{}^{\rho\sigma}R_{\rho\sigma}{}^{\mu\nu}-28R^\mu{}_\alpha{}^\nu{}_\beta R^\alpha{}_\rho{}^\beta{}_\sigma R^\rho{}_\mu{}^\sigma{}_\nu\right)\right] && \text{(A.9)}
\end{aligned}$$

where $\mathbf{\Omega}^{\mu\nu} = [\nabla^\mu, \nabla^\nu]$ is the curvature of the connection acting on a set of fields in a particular representation of the Lorentz and internal gauge group and $\mathbf{J}_\mu = \nabla_\alpha\mathbf{\Omega}^\alpha{}_\mu$. We neglect total derivative terms. The coefficient $\mathbf{b}_8$, which is also used in this work, is much too long to write here, and can be found in [80]. These coefficients are for unconstrained fields. The ones for fields satisfying differential constraints such as $h^T_{\mu\nu}$ and ξ_μ in the field decompositions (3.23) are given in appendix A.2.

Let us return to equation (A.3). If we are interested in the local behaviour of the theory (*i.e.* the behaviour at scales k much smaller than the typical curvature) we can use the asymptotic expansion (A.4) and then evaluate each integral separately. Then we get

$$\begin{aligned}
\mathrm{Tr}W(\Delta) = \frac{1}{(4\pi)^{\frac{d}{2}}}&\left[Q_{\frac{d}{2}}(W)B_0(\Delta)+Q_{\frac{d}{2}-1}(W)B_2(\Delta)+\ldots\right.\\
&\left.+\,Q_0(W)B_d(\Delta)+Q_{-1}(W)B_{d+2}(\Delta)+\ldots\right], \qquad \text{(A.10)}
\end{aligned}$$

where

$$Q_n(W) = \int_0^\infty ds s^{-n}\tilde{W}(s)\,. \tag{A.11}$$

In the case of four dimensional field theories, it is enough to consider integer values of n. However, in odd dimensions half-integer values of n are needed and we are also interested in the analytic continuation of results to arbitrary real dimensions. We will therefore need expressions for (A.11) that hold for any real n.
If we denote by $W^{(i)}$ the i-th derivative of W, we have from (A.2)

$$W^{(i)}(z) = (-1)^i \int_0^\infty ds\, s^i e^{-zs} \tilde{W}(s)\ . \tag{A.12}$$

This formula can be extended to the case when i is a real number to define a notion of "noninteger derivative". From this it follows that for any real i

$$Q_n(W^{(i)}) = (-1)^i Q_{n-i}(W)\ . \tag{A.13}$$

For n a positive integer one can use the definition of the Gamma function to rewrite (A.11) as a Mellin transform

$$Q_n(W) = \frac{1}{\Gamma(n)} \int_0^\infty dz\, z^{n-1} W(z) \tag{A.14}$$

while for m a positive integer or $m = 0$

$$Q_{-m}(W) = (-1)^m W^{(m)}(0)\ . \tag{A.15}$$

More generally, for n a positive real number we can define $Q_n(W)$ by equation (A.14), while for n real and negative we can choose a positive integer k such that $n + k > 0$; then we can write the general formula

$$Q_n(W) = \frac{(-1)^k}{\Gamma(n+k)} \int_0^\infty dz\, z^{n+k-1} W^{(k)}(z)\ . \tag{A.16}$$

This reduces to the two cases mentioned above when n is integer. In the case when n is a negative half integer $n = -\frac{2m+1}{2}$ we will set $k = m + 1$ so that we have

$$Q_{-\frac{2m+1}{2}}(W) = \frac{(-1)^{m+1}}{\sqrt{\pi}} \int_0^\infty dz\, z^{-1/2} f^{(m+1)}(z)\ . \tag{A.17}$$

Let us now consider some particular integrals that are needed in this paper. As discussed in section 2.3, there are two natural choices of cutoff function: type I cutoff is a function $R_k(-\nabla^2)$ such that the modified inverse propagator is $P_k(-\nabla^2) = -\nabla^2 + R_k(-\nabla^2)$; type II cutoff is the same function but its argument is now the entire inverse propagator: $R_k(\Delta)$, such that the modified inverse propagator is $P_k(\Delta) = \Delta + R_k(\Delta)$.
We now restrict ourselves to the case when $\mathbf{E} = q\mathbf{1}$, so that we can write $\Delta = -\nabla^2 + q\mathbf{1}$. The evaluation of the r.h.s. of the ERGE reduces to knowledge of the heat kernel coefficients and calculation of integrals of the form $Q_n\left(\partial_t R_k/(P_k + q)^\ell\right)$. It is convenient to measure everything in units of k^2. Let us define the dimensionless variable y by $z = k^2 y$; then $R_k(z) = k^2 r(y)$ for some dimensionless function r, $P_k(z) = k^2(y + r(y))$

and $\partial_t R_k(z) = 2k^2(r(y) - yr'(y))$.
In general the coefficients $Q_n(W)$ will depend on the details of the cutoff function. However, if $q = 0$ and $\ell = n + 1$ they turn out to be independent of the shape of the function. Note that they are all dimensionless. For $n > 0$, as long as $r(0) \neq 0$:

$$Q_n\left(\frac{\partial_t R_k}{P_k^{n+1}}\right) = \frac{2}{\Gamma(n)}\int_0^\infty dy \frac{d}{dy}\left[\frac{1}{n}\frac{y^n}{(y+r)^n}\right] = \frac{2}{n!}\ . \tag{A.18}$$

Similarly, if $r(0) \neq 0$ and $r'(0)$ is finite,

$$Q_0\left(\frac{\partial_t R_k}{P_k}\right) = 2\ . \tag{A.19}$$

Finally, for $n = -m < 0$

$$Q_n\left(\frac{\partial_t R_k}{P_k^{1-m}}\right)|_{y=0} = (-1)^m\left(\frac{\partial_t R_k}{P_k^{1-m}}\right)^{(m)}(0)\,|_{y=0} = \sum_{n=0}^{m}\binom{m}{n}\left(r - y\,r'\right)^{(n)}(y+r)^{(m-1)}\,|_{y=0} = 0 \tag{A.20}$$

as $\left(r - y\,r'\right)^{(n)} = r^{(n)} - y\,r^{(n+1)} - r^{(n)} = -y\,r^{(n+1)}$ which vanishes at $y = 0$. This concludes the proof that $Q_n\left(\partial_t R_k / P_k^{n+1}\right)$ are scheme–independent.
Regarding the other coefficients $Q_n\left(\partial_t R_k/(P_k + q)^\ell\right)$ whenever explicit evaluations are necessary, we will use the so-called “optimized cutoff function” [81]

$$R_k(z) = (k^2 - z)\theta(k^2 - z)\ . \tag{A.21}$$

With this cutoff $\partial_t R_k = 2k^2\theta(k^2 - z)$. Since the integrals are all cut off at $z = k^2$ by the theta function in the numerator, we can simply use $P_k(z) = k^2$ in the integrals. For $n > 0$ we have

$$Q_n\left(\frac{\partial_t R_k}{(P_k + q)^\ell}\right) = \frac{2}{n!}\frac{1}{(1+\tilde{q})^\ell}k^{2(n-\ell+1)} \tag{A.22}$$

where $\tilde{q} = qk^{-2}$. For $n = 0$ we have

$$Q_0\left(\frac{\partial_t R_k}{(P_k + q)^\ell}\right) = \left.\frac{\partial_t R_k}{(P_k + q)^\ell}\right|_{z=0} = \frac{2}{(1+\tilde{q})^\ell}k^{2(-\ell+1)}\ . \tag{A.23}$$

Finally, owing to the fact that the function $\frac{\partial_t R_k(z)}{(P_k(z)+q)^\ell}$ is constant in an open neighborhood of $z = 0$, we have

$$Q_n\left(\frac{\partial_t R_k}{(P_k + q)^\ell}\right) = 0 \ \text{ for } n < 0\ . \tag{A.24}$$

This has the remarkable consequence that with the optimized cutoff the trace in the ERGE consists of finitely many terms.

For noninteger indices of the Q-functional let us calculate

$$Q_{-\frac{2n+1}{2}}\left(\frac{\partial_t R_k}{P_k}\right) = \frac{(-1)^{n+1}}{\sqrt{\pi}} \int_0^\infty dz\, z^{-1/2} \frac{d^{n+1}}{dx^{n+1}} \frac{\partial_t R_k(z)}{P_k(z)} \tag{A.25}$$

where $P_k(z) = z + (k^2 - z)\theta(k^2 - z)$. We change the variable to $x = z/k^2$ so we have

$$Q_{-\frac{2n+1}{2}}\left(\frac{\partial_t R_k}{P_k}\right) = \frac{(-1)^{n+1} k^{-(2n+1)}}{\sqrt{\pi}} \int_0^\infty dx\, x^{-1/2} \frac{d^{n+1}}{dx^{n+1}} \frac{2x\theta(1-x)}{x + (1-x)\theta(1-x)} \,. \tag{A.26}$$

We find

$$\begin{aligned} \int_0^\infty dx\, x^{-1/2} \frac{d}{dx} f(x) &= 2 \\ \int_0^\infty dx\, x^{-1/2} \frac{d^2}{dx^2} f(x) &= -5 \end{aligned} \tag{A.27}$$

so that

$$\begin{aligned} Q_{-1/2}\left(\frac{\partial_t R_k}{P_k}\right) &= -\frac{2}{\sqrt{\pi} k} \\ Q_{-3/2}\left(\frac{\partial_t R_k}{P_k}\right) &= -\frac{5}{\sqrt{\pi} k^3} \,. \end{aligned} \tag{A.28}$$

We also need Q-functionals of $\frac{R_k}{(P_k+q)^\ell}$. For $n > 0$ we have

$$Q_n\left(\frac{R_k}{(P_k+q)^\ell}\right) = \frac{1}{(n+1)!} \frac{1}{(1+\tilde{q})^\ell} k^{2(n-\ell+1)} \,. \tag{A.29}$$

The function $\frac{R_k(z)}{(P_k(z)+q)^\ell}$ is equal to $\frac{k^2-z}{(k^2+q)^\ell}$ in an open neighborhood of $z = 0$; therefore

$$Q_0\left(\frac{R_k}{(P_k+q)^\ell}\right) = \left.\frac{R_k}{(P_k+q)^\ell}\right|_{z=0} = \frac{1}{(1+\tilde{q})^\ell} k^{2(-\ell+1)} \tag{A.30}$$

$$Q_{-1}\left(\frac{R_k}{(P_k+q)^\ell}\right) = \frac{1}{(1+\tilde{q})^\ell} k^{-2\ell} \,, \quad Q_n\left(\frac{R_k}{(P_k+q)^\ell}\right) = 0 \;\text{ for } n < -1 \,. \tag{A.31}$$

Finally, for the type III cutoff one also needs the following

$$Q_n\left(\frac{1}{(P_k+q)^\ell}\right) = \frac{1}{n!} \frac{k^{2(n-\ell)}}{(1+\tilde{q})^\ell} \;\text{ for } n \geq 0 \,;\; Q_n\left(\frac{1}{(P_k+q)^\ell}\right) = 0 \;\text{ for } n < 0 \,. \tag{A.32}$$

In conclusion let us address a general problem concerning the choice of the operator $\mathcal{O}$, whose eigenfunctions are taken as a basis in the functional space. In some calculations the r.h.s. of the ERGE takes the form $\frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta+q\mathbf{1})}{P_k(\Delta+q\mathbf{1})}$ where Δ is an operator and q is a constant. Equation (A.10) tells us how to compute the trace of this function, regarded as a function of the operator $\Delta + q\mathbf{1}$. In the derivation of this result it was implicitly assumed that $\mathcal{O} = \Delta + q\mathbf{1}$. However, the trace must be independent of the choice of basis in the functional space. It is instructing to see this explicitly, namely to evaluate the trace regarding $\frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k(\Delta+q\mathbf{1})}{P_k(\Delta+q\mathbf{1})}$ as a function of Δ. Given any function $W(z)$ we can define $\bar{W}(z) = W(z+q)$; in general, expanding in q we then have

$$\begin{aligned}
Q_n(\bar{W}) &= \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1} W(z+q) \\
&= \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}(W(z) + qW'(z) + \frac{1}{2!}q^2 W''(z) + \frac{1}{3!}q^3 W'''(z)\ldots) \\
&= Q_n(W) + qQ_n(W') + \frac{1}{2!}q^2 Q_n(W'') + \frac{1}{3!}q^3 Q_n(W''') + \ldots \\
&= Q_n(W) - qQ_{n-1}(W) + \frac{1}{2!}q^2 Q_{n-2}(W) - \frac{1}{3!}q^3 Q_{n-3}(W)\ldots
\end{aligned} \tag{A.33}$$

where in the last step we have used equation (A.13). Using (A.10) for the function $\bar{W}$ we then have

$$\begin{aligned}
\mathrm{Tr}\bar{W}[\Delta] &= \frac{1}{(4\pi)^{\frac{d}{2}}}\left[Q_{\frac{d}{2}}(\bar{W})B_0(\Delta) + Q_{\frac{d}{2}-1}(\bar{W})B_2(\Delta) + \ldots + Q_0(\bar{W})B_{2d}(\Delta) + \ldots\right] \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}}\Bigg[\left(Q_{\frac{d}{2}}(W) - qQ_{\frac{d}{2}-1}(W) + \frac{1}{2!}q^2 Q_{\frac{d}{2}-2}(W) - \frac{1}{3!}q^3 Q_{\frac{d}{2}-3}(W) + \ldots\right)B_0(\Delta) \\
&\quad + \left(Q_{\frac{d}{2}-1}(W) - qQ_{\frac{d}{2}-2}(W) + \frac{1}{2!}q^2 Q_{\frac{d}{2}-3}(W) - \frac{1}{3!}q^3 Q_{\frac{d}{2}-4}(W) + \ldots\right)B_2(\Delta) \\
&\quad + \qquad \ldots \\
&\quad + \left(Q_0(W) - qQ_{-1}(W) + \frac{1}{2!}q^2 Q_{-2}(W) - \frac{1}{3!}q^3 Q_{-3}(W) + \ldots\right)B_{2d}(\Delta) \\
&\quad + \qquad \ldots\Bigg]
\end{aligned} \tag{A.34}$$

We can now collect the terms that have the same Q-functions. They correspond to the anti-diagonal lines in (A.34). Using equation (A.5) one recognizes that the coefficient of $Q_{\frac{d}{2}-k}$ is $B_{2k}(\Delta + q\mathbf{1})$. Therefore

$$\begin{aligned}
\mathrm{Tr}\bar{W}[\Delta] &= \frac{1}{(4\pi)^{\frac{d}{2}}}\Big[Q_{\frac{d}{2}}(\bar{W})B_0(\Delta + q\mathbf{1}) + Q_{\frac{d}{2}+1}(\bar{W})B_2(\Delta + q\mathbf{1}) \\
&\quad + \ldots + Q_0(\bar{W})B_{2d}(\Delta + q\mathbf{1}) + \ldots\Big]
\end{aligned} \tag{A.35}$$

which coincides term by term with the expansion of $\mathrm{Tr}W[\Delta + q]$ using the basis of eigenfunctions of the operator $\mathcal{O} = \Delta + q\mathbf{1}$. This provides an explicit check, at least in this particular example, that the trace of this function is independent of the basis in the functional space.

A.2. Spectral geometry of differentially constrained fields

In this appendix we work on a sphere. Consider the decomposition of a vector field A_μ into its transverse and longitudinal parts

$$A_\mu \to A^T_\mu + \nabla_\mu \Phi \tag{A.36}$$

The spectrum of $-\nabla^2$ on vectors is the disjoint union of the spectrum on transverse and longitudinal vectors. The latter can be related to the spectrum of $-\nabla^2 - \frac{R}{d}$ on scalars using the formula

$$-\nabla^2 \nabla_\mu \Phi = -\nabla_\mu \left(\nabla^2 + \frac{R}{d}\right)\Phi\,. \tag{A.37}$$

Therefore one can write for the heat kernel

$$\mathrm{Tr}\,\mathrm{e}^{-s\left(-\nabla^2\right)}\,|_{A_\mu} = \mathrm{Tr}\,\mathrm{e}^{-s\left(-\nabla^2\right)}\,|_{A^T_\mu} + \mathrm{Tr}\,\mathrm{e}^{-s\left(-\nabla^2 - \frac{R}{d}\right)}\,|_\Phi - \mathrm{e}^{\left(s\frac{R}{d}\right)}\,. \tag{A.38}$$

The last term has to be subtracted because a constant scalar is an eigenfunction of $-\nabla^2 - \frac{R}{d}$ with negative eigenvalue, but does not correspond to an eigenfunction of $-\nabla^2$ on vectors. The spectrum of $-\nabla^2$ on scalars and transverse vectors is obtained from the representation theory of $SO(d+1)$ and is reported in table A.4.
A similar argument works for symmetric tensors, when using the decomposition (3.23). One can use equation

$$-\nabla^2\left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\right) = \nabla_\mu\left(-\nabla^2 - \frac{d+1}{d\,(d-1)}R\right)\xi_\nu + \nabla_\nu\left(-\nabla^2 - \frac{d+1}{d\,(d-1)}R\right)\xi_\mu \tag{A.39}$$

and equation

$$-\nabla^2\left(\nabla_\mu\nabla_\nu - \frac{1}{d}g_{\mu\nu}\nabla^2\right)\sigma = \left(\nabla_\mu\nabla_\nu - \frac{1}{d}g_{\mu\nu}\nabla^2\right)\left(-\nabla^2 - \frac{2}{d-1}R\right)\sigma \tag{A.40}$$

to relate the spectrum of various operators on vectors and scalars to the spectrum of $-\nabla^2$ on tensors. One has to observe that the $d(d+1)/2$ Killing vectors are eigenvectors of $-\nabla^2 - \frac{d+1}{d(d-1)}R$ on vectors but give a vanishing tensor $h_{\mu\nu}$, so they do not contribute to the spectrum of $-\nabla^2$ on tensors. Likewise, a constant scalar and the $d+1$ scalars proportional to the Cartesian coordinates of the embedding $\mathbf{R}^n$, which correspond to the two lowest eigenvalues of $-\nabla^2 - \frac{2}{d-1}R$, also do not contribute to the spectrum of tensors. So one has for the heat kernel on tensors

$$\mathrm{Tr}\,\mathrm{e}^{\left(-s\left(-\nabla^2\right)\right)}\Big|_{h_{\mu\nu}} = \mathrm{Tr}\,\mathrm{e}^{\left(-s\left(-\nabla^2\right)\right)}\Big|_{h^T_{\mu\nu}} + \mathrm{Tr}\,\mathrm{e}^{\left(-s\left(-\nabla^2 - \frac{(d+1)R}{d(d-1)}\right)\right)}\Big|_\xi + \mathrm{Tr}\,\mathrm{e}^{\left(-s\left(-\nabla^2\right)\right)}\Big|_h \tag{A.41}$$

	VT	TT
b_2	$\frac{R}{2}\delta_{d,2}$	$\frac{7R}{2}\delta_{d,2}$
b_4	$\frac{R^2}{4}\delta_{d,2}+\frac{R^2}{24}\delta_{d,4}$	$4R^2\delta_{d,2}+\frac{2R^2}{3}\delta_{d,4}$
b_6	$\frac{R^3}{16}\delta_{d,2}+\frac{R^3}{96}\delta_{d,4}+\frac{R^3}{450}\delta_{d,6}$	$\frac{5R^3}{2}\delta_{d,2}+\frac{R^3}{6}\delta_{d,4}+\frac{29R^3}{450}\delta_{d,6}$
b_8	$\frac{R^4}{96}\delta_{d,2}+\frac{R^4}{768}\delta_{d,4}+\frac{R^4}{2700}\delta_{d,6}+\frac{15R^4}{175616}\delta_{d,8}$	$\frac{7R^4}{6}\delta_{d,2}+\frac{23R^4}{864}\delta_{d,4}+\frac{8R^4}{1125}\delta_{d,6}+\frac{345R^4}{87808}\delta_{d,8}$

Table A.1.: Excluded modes for the heat kernel coefficients.

	Scalar	Vector	Tensor
b_0	1	d	$\frac{1}{2}d(d+1)$
b_2	$\frac{1}{6}R$	$\frac{d}{6}R$	$\frac{d(d+1)}{12}R$
b_4	$\frac{6-7d+5d^2}{360d(d-1)}R^2$	$\frac{-60+6d-7d^2+5d^3}{360d(d-1)}R^2$	$\frac{-240-114d-d^2-2d^3+5d^4}{720d(d-1)}R^2$

Table A.2.: Scalar, vector, and tensor b_0, b_2, and b_4 heat kernel coefficients.

$$+\mathrm{Tr}\,\mathrm{e}^{\left(-s\left(-\nabla^2-\frac{2}{d-1}R\right)\right)}\,|_\sigma\,-\mathrm{e}^{\left(\frac{2}{d-1}sR\right)}-(d+1)\,\mathrm{e}^{\left(\frac{1}{d-1}sR\right)}-\frac{d\,(d+1)}{2}\,\mathrm{e}^{\left(\frac{2}{d(d-1)}sR\right)}\,.$$

The last exponentials can be expanded in Taylor series as $\sum_{m=0}^{\infty} c_m R^m$ and these terms can be viewed as modifications of the heat kernel coefficients of $-\nabla^2$ acting on the differentially constrained fields. To see where these modifications enter, recall that the volume of the sphere is

$$V_{\mathrm{dS}}=(4\pi)^{\frac{d}{2}}\left(\frac{d\,(d-1)}{R}\right)^{\frac{d}{2}}\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(d\right)} \tag{A.42}$$

so that

$$\int \mathrm{d}^d x\sqrt{g}\,\mathrm{tr}\,\mathbf{b}_n \propto R^{\frac{n-d}{2}}\,. \tag{A.43}$$

This means a coefficent c_m from the Taylor series will contribute to a heat kernel coefficient for which $2m=n-d$. So there are contributions to b_n only if $n\geq d$. The contributions of these excluded modes to the heat kernel coefficients are listed in table (A.1) for the transverse vector and the transverse traceless tensor, in some specific dimensionality.

	T-Vector	TT-Tensor
b_0	$d-1$	$\frac{1}{2}(d-2)(d+1)$
b_2	$\frac{(d+2)(d-3)+6\delta_{d,2}}{6d}R$	$\frac{(d+1)(d+2)\left(d-5+3\delta_{d,2}\right)}{12(d-1)}R$
b_4	$\frac{5d^4-12d^3-47d^2-186d+180+360\delta_{d,2}+720\delta_{d,4}}{360d^2(d-1)}R^2$	$\frac{(d+1)\left(1440\delta_{d,2}+3240\delta_{d,4}-228-392d-83d^2-22d^3+5d^4\right)}{720d(d-1)^2}R^2$

Table A.3.: b_0, b_2, and b_4 heat kernel coefficients for transverse vector and transverse traceless tensor fields.

Spin s	Eigenvalue $\lambda_l(d,s)$	Multiplicity $D_l(d,s)$
0	$\frac{l(l+d-1)}{d(d-1)}R;\, l=0,1\ldots$	$\frac{(2l+d-1)(l+d-2)!}{l!(d-1)!}$
1	$\frac{l(l+d-1)-1}{d(d-1)}R;\, l=1,2\ldots$	$\frac{l(l+d-1)(2l+d-1)(l+d-3)!}{(d-2)!(l+1)!}$
2	$\frac{l(l+d-1)-2}{d(d-1)}R;\, l=2,3\ldots$	$\frac{(d+1)(d-2)(l+d)(l-1)(2l+d-1)(l+d-3)!}{2(d-1)!(l+1)!}$

Table A.4.: Eigenvalues and their multiplicities of the Laplacian on the d-sphere.

We have discussed how the negative and zero modes from constrained scalar and vector fields affect the heat kernel coefficients of the decomposed vector and tensor fields. These modes have to be excluded also from the traces over the constrained fields; this is denoted by one or two primes, depending on the number of excluded modes. This can be done by calculating the trace and subtracting the contributions to the operator trace from the excluded modes. Thus the trace with m primes is

$$\mathrm{Tr}^{'\cdots'}\left[W(-\nabla^2)\right] = \mathrm{Tr}\left[W(-\nabla^2)\right] - \sum_{l=1}^{m} D_l\,(d,s)\,W\left(\lambda_l\,(d,s)\right) \tag{A.44}$$

where $\lambda_l(d,s)$ are the eigenvalues, $D_l(d,s)$ their multiplicities, both depending on the dimension d and on the spin of the field, s. The eigenvalues and multiplicities for the m-th mode of the Laplacian on the sphere are given in table A.4.
The expressions that we will need are those for the cases where one mode is excluded from the transverse vector trace ($s=1$, $m=1$), or one or two modes from the scalar trace ($s=0$, $m=1,2$), each one in two and four dimensions. The results obtained by calculating the corresponding multiplicity and eigenvalue from table A.4 are given in table A.5. To see what is the relevant contribution to one of the heat-kernel coefficients, one can expand the obtained expression in R. For the case $s=0$, $d=4$, $m=2$ one has

	s=1	s=0
$m=1, d=2$	$3W\left(\frac{R}{2}\right)$	$W(0)$
$m=1, d=4$	$10W\left(\frac{R}{4}\right)$	$W(0)$
$m=2, d=2$		$W(0)+3W(R)$
$m=2, d=4$		$W(0)+5W\left(\frac{R}{3}\right)$

Table A.5.: $\sum_{l=1}^{m} D_l(d,s)\, W(\lambda_l(d,s))$ for $s=0,1$, $d=2,4$, $m=1,2$.

for example

$$\sum_{l=1}^{2} D_l(4,0)\, W(\lambda_l(4,0)) = W(0) + 5W\left(\frac{R}{3}\right) \tag{A.45}$$
$$= \frac{R^2}{4(4\pi)^2}\int \mathrm{d}x\sqrt{g}\left(W(0) + \frac{5R}{18}W'(0) + \frac{5}{108}R^2W''(0) + \frac{5}{36\cdot 27}R^3W'''(0) + \ldots\right) .$$

From this one sees that here the $\mathbf{b}_{2n}$ receive a correction for $n \geq 2$. In two dimensions, that would be already the case for $n \geq 1$. For the case of arbitrary dimension, the heat kernel coefficients b_0, b_2, b_4 are listed for unconstrained scalar, vector and tensor fields in table A.2, for transverse vector and transverse traceless tensor fields in table A.3, b_6 and b_8 in table A.6. For the four-dimensional case, the full list of heat kernel coefficients of $-\nabla^2$ in 4d is given in table A.7.

A.3. Proper time ERGE

Let us start from the ERGE for gravity in the Einstein–Hilbert truncation with a type III cutoff, written in equation (3.54). Define the functions

$$A_k(z) = \frac{\partial_t R_k(z)}{z+R_k(z)} \qquad B_k(z) = \frac{R_k(z)}{z+P_k(z)} \qquad C_k(z) = \frac{\partial_z R_k(z)}{z+R_k(z)} . \tag{A.46}$$

The term in equation (3.54) containing C is nontrivial. To rewrite it in a manageable form we take the Laplace transform

$$C_k(z) = \int_0^\infty ds\, \tilde{C}_k(s)\, e^{-sz} . \tag{A.47}$$

Since the operator $\partial_t(\Delta_2 - 2\Lambda)$ commutes with $\Delta_2 - 2\Lambda$, we can write

$$\begin{aligned} C_k(\Delta_2 - 2\Lambda)\, \partial_t(\Delta_2 - 2\Lambda) &= \int_0^\infty ds\, \tilde{C}_k(s)\, \partial_t(\Delta_2 - 2\Lambda)\, e^{-s(\Delta_2 - 2\Lambda)} \\ &= -\int_0^\infty \frac{ds}{s}\, \tilde{C}_k(s)\, \partial_t e^{-s(\Delta_2 - 2\Lambda)} . \end{aligned} \tag{A.48}$$

b_{6S}	$\frac{R^3}{45360d^2(d-1)^2}\left(96-110d+187d^2-112d^3+35d^4\right)$
b_{6V}	$\frac{R^3}{45360d^2(d-1)^2}\left(-1008+852d-1370d^2+187d^3-112d^4+35d^5\right)$
b_{6VT}	$\frac{R^3}{45360(d-1)^2d^3}\left(-7560+10992d-676d^2-3825d^3-331d^4-147d^5+35d^6\right)$ $+\left(\frac{1}{8}\delta_{d,2}+\frac{1}{96}\delta_{d,4}\right)R^3$
b_{6T}	$\frac{R^3}{90720d^2(d-1)^2}\left(-4032+1104d-3542d^2-2443d^3+75d^4-77d^5+35d^6\right)$
b_{6TT}	$\frac{(d+1)R^3}{90720d^2(d-1)^3}\left(35d^6-217d^5-667d^4\right.$ $\left.-7951d^3-13564d^2-10084d-28032\right)+\left(\frac{3}{2}\delta_{d,2}+\frac{5}{36}\delta_{d,4}\right)R^3$
b_{8S}	$\frac{R^4}{5443200(d-1)^3d^3}\left(2160-516d+3304d^2-3111d^3+2389d^4-945d^5+175d^6\right)$
b_{8V}	$\frac{R^4}{1360800(-1+d)^3d^3}\left(-13500+13884d-5426d^2+1929d^3-761d^4-945d^5+175d^6\right)$
b_{8VT}	$\frac{R^4}{75600(d-1)^3d^4}\left(75600-206400d+133924d^2+16144d^3-15911d^4\right.$ $\left.-8531d^5-2345d^6+175d^7\right)+\left(\frac{1}{96}\delta_{d,2}+\frac{1}{768}\delta_{d,4}+\frac{1}{2700}\delta_{d,6}+\frac{15}{175616}\delta_{d,8}\right)R^4$
b_{8T}	$\frac{R^4}{453600(d-1)^3d^3}\left(172800+1387440d-375636d^2-530732d^3\right.$ $\left.+554593d^4-126722d^5+1444d^6-770d^7+175d^8\right)$
b_{8TT}	$\frac{R^4}{453600(d-1)^4d^4}\left(-1814400-7018560d-10359960d^2-5191124d^3-2945774d^4\right.$ $\left.-2028005d^5+478295d^6-150566d^7+464d^8-945d^9+175d^{10}\right)$ $+\left(\frac{1}{2}\delta_{d,2}+\frac{5}{288}\delta_{d,4}+\frac{7}{1225}\delta_{d,6}+\frac{675}{175616}\delta_{d,8}\right)R^4$

Table A.6.: b_6 and b_8 heat kernel coefficients for scalar, vector, transverse vector, tensor, and transverse traceless tensor fields.

	S	V	VT	T	TT	TTT
$\mathrm{tr}\mathbf{b}_0$	1	4	3	10	9	5
$\mathrm{tr}\mathbf{b}_2$	$\frac{R}{6}$	$\frac{2R}{3}$	$\frac{R}{4}$	$\frac{5R}{3}$	$\frac{3R}{2}$	$-\frac{5R}{6}$
$\mathrm{tr}\mathbf{b}_4$	$\frac{29R^2}{2160}$	$\frac{43R^2}{1080}$	$-\frac{7R^2}{1440}$	$\frac{11R^2}{216}$	$\frac{81R^2}{2160}$	$-\frac{R^2}{432}$
$\mathrm{tr}\mathbf{b}_6$	$\frac{37R^3}{54432}$	$-\frac{R^3}{17010}$	$-\frac{541R^3}{362880}$	$-\frac{1343R^3}{136080}$	$\frac{-319R^3}{30240}$	$\frac{311R^3}{54432}$
$\mathrm{tr}\mathbf{b}_8$	$\frac{149R^4}{6531840}$	$-\frac{2039R^4}{13063680}$	$-\frac{157R^4}{2488320}$	$-\frac{2999R^4}{3265920}$	$\frac{683R^4}{725760}$	$\frac{109R^4}{1306368}$

Table A.7.: Heat kernel coefficients for S^4. The columns for the transverse vector (VT) and transverse traceless tensor (TTT) are obtained from equations (A.38) and (A.41) in $d = 4$. Note that the excluded modes contribute to $\mathrm{tr}\mathbf{b}_n$ only for $n \geq 4$.

Laplace transforming also A_k and B_k, the first term in equation (3.54) becomes

$$\frac{1}{2}\int_0^\infty ds \left[\tilde{A}_k(s) + \tilde{B}_k(s)\,\eta - \frac{1}{s}\,\tilde{C}_k(s)\,\partial_t\right] \mathrm{Tr}\, e^{-s(\Delta_2 - 2\Lambda)} \,. \tag{A.49}$$

This is the functional RG equation in "proper time" form [82, 83, 84]. Note that the first term corresponds precisely to the one loop approximation. The trace of the heat kernel can be expanded

$$\begin{aligned}\mathrm{Tr}\, e^{-s(\Delta_2 - 2\Lambda)} &= e^{-s(-2\Lambda)}\frac{1}{(4\pi)^{d/2}}\int dx\sqrt{g}\,\mathrm{tr}\left[\mathbf{1}s^{-\frac{d}{2}} + \left(\mathbf{1}\frac{R}{6} - \mathbf{W}\right)s^{-\frac{d}{2}+1} + O(R^2)\right] \\ &= e^{-s(-2\Lambda)}\frac{1}{(4\pi)^{d/2}}\int dx\sqrt{g}\left[\frac{d(d+1)}{2}s^{-\frac{d}{2}} + \frac{d(7-5d)}{12}R\,s^{-\frac{d}{2}+1} + O(R^2)\right]\end{aligned} \tag{A.50}$$

whereas for the ghosts

$$\begin{aligned}\mathrm{Tr}\, e^{-s(\delta^\mu_\nu \Delta - R^\mu_\nu)} &= \frac{1}{(4\pi)^{d/2}}\int dx\sqrt{g}\,\mathrm{tr}\left[\delta^\mu_\nu s^{-\frac{d}{2}} + \left(\delta^\mu_\nu\frac{R}{6} + R^\mu_\nu\right)s^{-\frac{d}{2}+1} + O(R^2)\right] \\ &= \frac{1}{(4\pi)^{d/2}}\int dx\sqrt{g}\left[d s^{-\frac{d}{2}} + \frac{d+6}{6}R\,s^{-\frac{d}{2}+1} + O(R^2)\right] .\end{aligned} \tag{A.51}$$

The ERGE then takes the form

$$\begin{aligned}\partial_t\Gamma_k &= \frac{1}{(4\pi)^{d/2}}\int dx\sqrt{g}\Big\{\frac{d(d+1)}{4}Q_{\frac{d}{2}}\left(\bar{A}_k+\eta\bar{B}_k-2\partial_t\Lambda\,\bar{C}_k\right)-dQ_{\frac{d}{2}}\left(A_k\right) \\ &+\left[\frac{d(7-5d)}{12}\,Q_{\frac{d}{2}-1}\left(\bar{A}_k+\eta\bar{B}_k-2\partial_t\Lambda\,\bar{C}_k\right)-\frac{d+6}{d}\,Q_{\frac{d}{2}-1}\left(A_k\right)\right]\,R+O(R^2)\Big\}\end{aligned} \tag{A.52}$$

where $\bar{W}(z)=W(z-2\Lambda)$. Using an optimized cutoff one can now reproduce equations (3.58) and (3.59). However, in this way the sums in equation (3.57) can be resumed for any type of cutoff shape.

A.4. Cutoff of type Ib without field redefinitions

We collect here the formulae for the beta functions of Λ and G in the Einstein–Hilbert truncation, using a cutoff of type Ib and without redefining the fields ξ_μ and σ. The ERGE, including the contributions of the Jacobians, is

$$\begin{aligned}\frac{d\Gamma_k}{dt} &= \frac{1}{2}\mathrm{Tr}_{(2)}\frac{\partial_t R_k+\eta R_k}{P_k-2\Lambda+\frac{d^2-3d+4}{d(d-1)}R} \\ &+\frac{1}{2}\mathrm{Tr}'_{(1)}\frac{\partial_t R_k\left(2P_k+\frac{d-4}{d}R-2\Lambda\right)+\eta R_k\left(P_k+z+\frac{d-4}{d}R-2\Lambda\right)-2\partial_t\Lambda R_k}{\left(P_k-\frac{R}{d}\right)\left(P_k+\frac{d-3}{d}R-2\Lambda\right)} \\ &+\frac{1}{2}\mathrm{Tr}_{(0)}\frac{\partial_t R_k+\eta R_k}{P_k-2\Lambda+\frac{d-4}{d}R}+\frac{1}{2}\mathrm{Tr}''_{(0)}\frac{1}{\left(P_k+\frac{d-4}{d}R-2\Lambda\right)P_k\left(P_k-\frac{R}{d-1}\right)}\times \\ &\Bigg\{\partial_t R_k\left[3P_k^2+2P_k\left(\frac{d^2-6d+4}{d(d-1)}R-2\Lambda\right)-\frac{R}{d-1}\left(\frac{d-4}{d}R-2\Lambda\right)\right] \\ &+\eta\left[(P_k^3-z^3)+\left(\frac{d^2-6d+4}{d(d-1)}R-2\Lambda\right)(P_k^2-z^2)-\frac{R}{d-1}\left(\frac{d-4}{d}R-2\Lambda\right)R_k\right] \\ &-2\partial_t\Lambda\left[(P_k^2-z^2)-\frac{R}{d-1}R_k\right]\Bigg\}-\mathrm{Tr}_{(1)}\frac{\partial_t R_k}{P_k-\frac{R}{d}}-\mathrm{Tr}'_{(0)}\frac{2\left(P_k-\frac{R}{d}\right)\partial_t R_k}{\left(P_k-\frac{2R}{d}\right)P_k} \\ &-\frac{1}{2}\mathrm{Tr}'_{(1)}\frac{\partial_t R_k}{P_k-\frac{R}{d}}-\frac{1}{2}\mathrm{Tr}''_{(0)}.\frac{2\left(P_k-\frac{R}{2(d-1)}\right)\partial_t R_k}{P_k(P_k-\frac{R}{d-1})}+\mathrm{Tr}'_{(0)}\frac{\partial_t R_k}{P_k}\end{aligned} \tag{A.53}$$

which gives

$$\begin{aligned}\frac{d\Gamma_k}{dt} &= \int dx\,\sqrt{g}\Bigg\{Q_{\frac{d}{2}}\left(\frac{(d+1)\,\left((d+2)\;P_k-4\,\Lambda+\right)\,\partial_t R_k}{4\,P_k\,\left(P_k-2\,\Lambda\right)}\right) \\ &+\eta Q_{\frac{d}{2}}\left(\frac{\left(+d\,(d+1)\,P_k^2+2d\,(z-2\Lambda)\,P_k+2z\,(z-2\Lambda)\right)R_k}{4\,{P_k}^2\,\left(P_k-2\,\Lambda\right)}\right)\end{aligned}$$

$$
\begin{aligned}
&-\partial_t\Lambda Q_{\frac{d}{2}}\left(\frac{(d\,P_k+z)\,R_k}{{P_k}^2\,(P_k-2\,\Lambda)}\right)\\
&+R\left[-Q_{\frac{d}{2}}\left(\frac{\left[(d^4-2d^3-7d^2+18d-16)P_k^2+8(d^2-d+1)\Lambda(P_k-\Lambda)\right]\partial_t R_k}{4d(d-1)P_k^2(P_k-2\Lambda)^2}\right)\right.\\
&+\eta\,Q_{\frac{d}{2}}\left(\left[(d^4-4d^3+3d^2+12d-16)P_k^3+2(d-1)^2(2\Lambda+(d-4)z)P_k^2\right.\right.\\
&+\left(2(d^2-6d+4)z^2+4(d^2-d+1)z\Lambda+8(d-1)^2\Lambda^2\right)P_k\\
&\left.\left.+4dz(z-2\Lambda)\Lambda\right]R_k\Big/4d(d-1)P_k^3\,(P_k-2\,\Lambda)^2\right)\\
&+\partial_t\Lambda\,Q_{\frac{d}{2}}\left(\frac{R_k\left(d\left(d^2-5d+4\right){P_k}^2+\left(\left(d^2-6d+4\right)z+2(d-1)^2\Lambda\right)P_k+2dz\Lambda\right)}{d(d-1)P_k^3(P_k-2\Lambda)^2}\right)\\
&+Q_{\frac{d}{2}-1}\left(\frac{\left(\left(d^4+2\,d^3-13\,d^2-38\,d+24\right)\,P_k-4\,\left(d^3-7d+6\right)\,\Lambda\right)\partial_t R_k}{24d\,(d-1)\,P_k\,(P_k-2\,\Lambda)}\right)\\
&+\eta\,Q_{\frac{d}{2}-1}\left(\left[R_k\left((d^4-13d^2-24d+12)P_k^2+2(d^3-d^2-6d+6)(z-2\,\Lambda)P_k\right.\right.\right.\\
&+\left.\left.\left.\;2d(d-1)z(z-2\,\Lambda)\right)\right]/24d(d-1)P_k^2(P_k-2\,\Lambda)\right)\\
&\left.\left.-\partial_t\Lambda\,Q_{\frac{d}{2}-1}\left(\frac{\left(\left(d^2-6\right)\,P_k+dz\right)R_k}{6\,d\,{P_k}^2\,(P_k-2\,\Lambda)}\right)\right]\right\}
\end{aligned}
\tag{A.54}
$$

With the optimized cutoff one obtains the coefficients

$$
\begin{aligned}
A_1 &= -2\pi\frac{d^4+5d^3+4d^2+4d+16+(4d^3+12d^2-32d-32)\tilde\Lambda}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2}+3)(2\tilde\Lambda-1)}\\
A_2 &= 16\pi\frac{d^3+13d^2+48d+28-(4d^2+44d+120)\tilde\Lambda}{(4\pi)^{\frac{d}{2}}(d+2)(d+4)(d+6)\Gamma(\frac{d}{2})(1-2\tilde\Lambda)}\\
A_3 &= -\frac{64\pi(d+5)}{(4\pi)^{\frac{d}{2}}(d^2+6d+8)\Gamma(\frac{d}{2})(1-2\tilde\Lambda)}\\
B_1 &= -\pi\Big(-d^7+8d^6+61d^5+48d^4+12d^3+16d^2-480d-384+(2d^7+4d^6\\
&\quad -34d^5-276d^4-848d^3+464d^2-1184d+2688)\tilde\Lambda+(8d^6+16d^5-72d^4\\
&\quad +208d^3+1216d^2+832d-3840)\tilde\Lambda^2\Big)\Big/6(4\pi)^{\frac{d}{2}}d(d-1)\Gamma(\frac{d}{2}+3)(1-2\tilde\Lambda)^2\\
B_2 &= \pi\Big(d^7+2d^6-49d^5-62d^4+84d^3-3432d^2+2688d+(-2d^7-32d^6-142d^5\\
&\quad +64d^4+1728d^3+3808d^2+1440d-2304)\tilde\Lambda+(8d^6+96d^5+360d^4\\
&\quad +336d^3-1280d^2-3456d+4608)\tilde\Lambda^2\Big)\Big/12(4\pi)^{\frac{d}{2}}d(d-1)\Gamma(\frac{d}{2}+4)(1-2\tilde\Lambda)^2
\end{aligned}
$$

$$B_3 = 2\pi\Big(-d^5+6d^4+3d^3-216d^2+244d-48+(2d^5+12d^4+18d^3-24d^2 -176d+192)\tilde{\Lambda}\Big)\Big/3(4\pi)^{\frac{d}{2}}d(d-1)\Gamma(\frac{d}{2}+3)(1-2\tilde{\Lambda})^2$$

and the beta functions in four dimensions

$$\begin{aligned} \beta_{\tilde{\Lambda}} &= -2\tilde{\Lambda}+\frac{1}{12\pi}\frac{(21-105\tilde{\Lambda}+142\tilde{\Lambda}^2-78\tilde{\Lambda}^3+92\tilde{\Lambda}^4)\tilde{G}+\frac{3334-13309\tilde{\Lambda}+12416\tilde{\Lambda}^2-780\tilde{\Lambda}^3}{240\pi^2}\tilde{G}^2}{(1-2\tilde{\Lambda})^3+\frac{61-779\tilde{\Lambda}+1314\tilde{\Lambda}^2}{720\pi}\tilde{G}+\frac{-1406+1945\tilde{\Lambda}}{5760\pi^2}\tilde{G}^2} \\ \beta_{\tilde{G}} &= 2\tilde{G}-\frac{1}{12\pi}\frac{(39-130\tilde{\Lambda}+150\tilde{\Lambda}^2-92\tilde{\Lambda}^3)\tilde{G}^2+\frac{242-373\tilde{\Lambda}+39\tilde{\Lambda}^2}{12\pi}\tilde{G}^3}{(1-2\tilde{\Lambda})^3+\frac{61-779\tilde{\Lambda}+1314\tilde{\Lambda}^2}{720\pi}\tilde{G}+\frac{-1406+1945\tilde{\Lambda}}{5760\pi^2}\tilde{G}^2}\,. \end{aligned} \tag{A.55}$$

A.5. Diagonalization of several operators

Here we list the decompositions of the operators necessary when decomposing the second variation of the gravitational action and its ghost part into their corresponding tensor, vector, and scalar parts.

$$\begin{aligned} h_{\mu\nu}h^{\mu\nu} &= {h^T}_{\mu\nu}h^{T\mu\nu}+\xi_\mu\left[-2\Box-\frac{2R}{\mathrm{d}}\right]\xi^\mu \\ &\quad +\sigma\left[\frac{R}{\mathrm{d}}\Box+\left(1-\frac{1}{\mathrm{d}}\right)\Box^2\right]\sigma+\frac{1}{\mathrm{d}}h^2 \end{aligned} \tag{A.56}$$

$$\begin{aligned} h_{\mu\nu}\Box h^{\mu\nu} &= {h^T}_{\mu\nu}\Box h^{T\mu\nu}+\xi_\mu\left[-2\Box^2-\frac{4R}{\mathrm{d}}\left(1+\frac{1}{d-1}\right)\Box-\frac{2R^2}{\mathrm{d}^2}\left(1+\frac{2}{d-1}\right)\right]\xi^\mu \\ &\quad +\sigma\left[\left(1-\frac{1}{\mathrm{d}}\right)\Box^3+\frac{3R}{\mathrm{d}}\Box^2+\frac{2R^2}{\mathrm{d}^2}\left(1+\frac{1}{\mathrm{d}-1}\right)\Box\right]\sigma+\frac{1}{\mathrm{d}}h\Box h \end{aligned} \tag{A.57}$$

$$h\nabla_\mu\nabla_\nu h^{\mu\nu} = \frac{1}{\mathrm{d}}h\Box h+h\left[\left(1-\frac{1}{\mathrm{d}}\right)\Box^2+\frac{R}{\mathrm{d}}\Box\right]\sigma \tag{A.58}$$

$$h\Box\nabla_\mu\nabla_\nu h^{\mu\nu} = \frac{1}{\mathrm{d}}h\Box^2 h+h\left[\left(1-\frac{1}{\mathrm{d}}\right)\Box^3+\frac{R}{\mathrm{d}}\Box^2\right]\sigma \tag{A.59}$$

$$\begin{aligned} h_{\mu\nu}\nabla^\mu\nabla^\rho h^\nu_\rho &= \xi_\mu\left[-\Box^2-\frac{2R}{\mathrm{d}}\Box-\frac{R^2}{\mathrm{d}^2}\right]\xi^\mu \\ &\quad +\sigma\left[\left(1+\frac{1}{\mathrm{d}^2}-\frac{2}{\mathrm{d}}\right)\Box^3+\frac{2R}{\mathrm{d}}\left(1-\frac{1}{\mathrm{d}}\right)\Box^2+\frac{R^2}{\mathrm{d}^2}\Box\right]\sigma \end{aligned}$$

$$+\frac{1}{\mathrm{d}^2}h\Box h + h\left[\frac{2}{\mathrm{d}}\left(1-\frac{1}{\mathrm{d}}\right)\Box^2 + \frac{2R}{\mathrm{d}^2}\Box\right]\sigma \tag{A.60}$$

$$\begin{aligned} h_{\mu\nu}\nabla^\mu\nabla^\nu\nabla^\alpha\nabla^\beta h_{\alpha\beta} &= \sigma\left[\left(1+\frac{1}{\mathrm{d}^2}-\frac{2}{\mathrm{d}}\right)\Box^4 + \frac{2R}{\mathrm{d}}\left(1-\frac{1}{\mathrm{d}}\right)\Box^3 + \frac{R^2}{\mathrm{d}^2}\Box^2\right]\sigma \\ &\quad +\frac{1}{\mathrm{d}^2}h\Box^2 h + h\left[\frac{2}{\mathrm{d}}\left(1-\frac{1}{\mathrm{d}}\right)\Box^3 + \frac{2R}{\mathrm{d}^2}\Box^2\right]\sigma \end{aligned} \tag{A.61}$$

$$\begin{aligned} h_{\mu\nu}\Box\nabla^\mu\nabla^\beta h^\nu_\beta &= \frac{1}{(-1+d)\,d^3}\xi_\mu\left[\left(d^3-d^4\right)\Box^3 + \left(d^2-3d^3\right)R\Box^2 - 3d^2R^2\Box - (1+d)\,R^3\right]\xi^\mu \\ &\quad +\frac{1}{(-1+d)\,d^3}\sigma\left[\left(-d+3d^2+3d^3+d^4\right)\Box^4\right. \\ &\quad \left.+\left(4d+8d^2+4d^3\right)R\Box^3 + \left(2-10d+8d^2\right)R^2\Box^2 + (-2+5d)\,R^3\Box\right]\sigma \\ &\quad +\frac{1}{d^2}h\Box^2 h + \frac{1}{(-1+d)\,d^2}h\left[\left(2-2d+d^2\right)\Box^3 + 2\,(-2+2d)\,R\Box^2 + 2R^2\Box\right]\sigma \end{aligned} \tag{A.62}$$

$$b^\mu\Box b_\mu = {b^T}^\mu\Box {b^T}_\mu - \theta\Box^2\theta - \frac{R}{\mathrm{d}}\theta\Box\theta \tag{A.63}$$

$$b^\mu b_\mu = {b^T}^\mu {b^T}_\mu - \theta\Box\theta \tag{A.64}$$

$$b_\mu\nabla^\mu\nabla^\nu b_\nu = -\theta\Box^2\theta \tag{A.65}$$

$$b_\mu\Box\nabla^\mu\nabla^\nu b_\nu = -\theta\Box^3\theta - \frac{R}{\mathrm{d}}\theta\Box^2\theta \tag{A.66}$$

$$b_\mu\Box^2 b^\mu = {b^T}^\mu\Box^2 {b^T}_\mu - \theta\Box^3\theta - \frac{2R}{\mathrm{d}}\theta\Box^2\theta - \frac{R^2}{\mathrm{d}^2}\theta\Box\theta \tag{A.67}$$

A.6. Variations

In section 3 and 4.2 we needed the second variation of the truncation ansatz after having split the metric according to the background gauge technique into

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{A.68}$$

Then the action can be expanded around the background field (without requiring that the fluctuations around the background are small)

$$\Gamma_k\left[g_{\mu\nu}\right] = \Gamma_k\left[\bar{g}_{\mu\nu}\right] + \frac{\delta\Gamma_k\left[g_{\mu\nu}\right]}{\delta g}\|_{\bar{g}} h + \frac{1}{2}\frac{\delta^2\Gamma_k\left[g_{\mu\nu}\right]}{\delta g\delta g}\|_{\bar{g}} hh + \dots \tag{A.69}$$

To facilitate the derivation of the second variation we find it useful to give the second variations of several tensors occurring in the calculation for reference. At least each of the single expressions can be calculated in a few steps.
For the metric one has

$$\delta g_{\mu\nu} = h_{\mu\nu} \tag{A.70}$$
$$\delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta} \tag{A.71}$$
$$\delta\delta g_{\mu\nu} = \delta g_{\mu\alpha}\delta g^{\alpha}{}_{\nu} \tag{A.72}$$
$$\delta\sqrt{g} = \sqrt{\bar{g}}\frac{1}{2}h \tag{A.73}$$
$$\delta\delta\sqrt{g} = \sqrt{\bar{g}}(\frac{1}{2}h^{\mu\nu}h_{\mu\nu} - \frac{1}{4}h\,h) \tag{A.74}$$

For the Christoffel symbols one has

$$\Gamma_{\mu}{}^{\lambda}{}_{\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{A.75}$$
$$\delta\Gamma_{\mu}{}^{\lambda}{}_{\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu \delta g_{\rho\nu} + \partial_\nu \delta g_{\rho\mu} - \partial_\rho \delta g_{\mu\nu}) \tag{A.76}$$
$$-\frac{1}{2}g^{\lambda\rho}g^{\alpha\beta}\delta g_{\alpha\beta}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{A.77}$$
$$\delta\delta\Gamma_{\mu}{}^{\lambda}{}_{\nu} = -\delta g^{\lambda\rho}(\nabla_\mu \delta g_{\rho\nu} + \nabla_\nu \delta g_{\rho\mu} - \nabla_\rho \delta g_{\mu\nu}) \tag{A.78}$$
$$\Gamma_{\mu}{}^{\mu}{}_{\nu} = -\delta g^{\mu\rho}\nabla_\nu g_{\mu\rho} \tag{A.79}$$
$$\Gamma^{\nu}{}_{\mu}{}^{\mu} = -2\delta g^{\nu\rho}\nabla_\lambda \delta g^{\lambda}{}_{\rho} + \nabla_\rho \delta g^{\mu}{}_{\mu}\delta g^{\nu\rho} \tag{A.80}$$

We use the Riemann tensor in the form

$$R_{\mu\nu}{}^{\alpha}{}_{\beta} = \partial_\mu\Gamma_{\nu}{}^{\alpha}{}_{\beta} - \partial_\nu\Gamma_{\mu}{}^{\alpha}{}_{\beta} + \Gamma_{\mu}{}^{\alpha}{}_{\gamma}\Gamma_{\nu}{}^{\gamma}{}_{\beta} - \Gamma_{\nu}{}^{\alpha}{}_{\gamma}\Gamma_{\mu}{}^{\gamma}{}_{\beta} \tag{A.81}$$

and the Ricci tensor is $R_{\nu\beta} = R_{\mu\nu}{}^{\mu}{}_{\beta}$ and the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$. Then the variations are

$$\delta R_{\mu\nu}{}^{\alpha}{}_{\beta} = \nabla_\mu\delta\Gamma_{\nu}{}^{\alpha}{}_{\beta} - \nabla_\nu\delta\Gamma_{\mu}{}^{\alpha}{}_{\beta} \tag{A.82}$$
$$\delta R_{\nu\beta} = \nabla_\mu\delta\Gamma_{\nu}{}^{\mu}{}_{\beta} - \nabla_\nu\delta\Gamma_{\mu}{}^{\mu}{}_{\beta} \tag{A.83}$$
$$g^{\nu\beta}\delta R_{\nu\beta} = \nabla_\mu\delta\Gamma^{\mu\nu}_{\nu} - \nabla_\nu\delta\Gamma^{\mu\nu}_{\mu} \tag{A.84}$$
$$\delta R = \delta(g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} = -R^{\mu\nu}\delta g_{\mu\nu} + \nabla^\mu\nabla^\nu\delta g_{\mu\nu} - g^{\mu\nu}\nabla^2\delta g_{\mu\nu} \tag{A.85}$$
$$\delta\delta R_{\nu\beta} = \nabla_\mu\delta\delta\Gamma_{\nu}{}^{\mu}{}_{\beta} - \nabla_\nu\delta\delta\Gamma_{\mu}{}^{\mu}{}_{\beta} + 2\delta\Gamma_{\mu}{}^{\mu}{}_{\gamma}\delta\Gamma_{\nu}{}^{\gamma}{}_{\beta} - 2\delta\Gamma_{\mu}{}^{\gamma}{}_{\beta}\delta\Gamma_{\nu}{}^{\gamma}{}_{\beta} \tag{A.86}$$

A.7. Euler-Maclaurin formula for trace evaluation

From equation A.6, the heat kernel coefficients for a scalar field on the sphere can be used to evaluate the trace using the optimized cutoff $R_k = (k^2 - \nabla^2)\theta(k^2 - \nabla^2)$ as

$$\mathrm{Tr}\frac{\partial_t R_k}{P_k} = \mathrm{Tr}2\theta(k^2 - \nabla^2) = 2Vol\,(1 + \frac{1}{6}R + \frac{29}{2160}R^2) \tag{A.87}$$

where the volume of the d-sphere is given by

$$Vol = 2^d\,\pi^{\frac{d}{2}}\left(\frac{d(d-1)}{R}\right)^{\frac{d}{2}}\frac{\Gamma(\frac{d}{2})}{\Gamma(d)} \tag{A.88}$$

and the radius r of the sphere is given by

$$r = \sqrt{\frac{d(d-1)}{R}}\,. \tag{A.89}$$

On the sphere, the spectrum of the Laplacian is known completely, so the trace evaluation can also be performed directly by summing over the eigenvalues weighted with the respective multiplicities. To do so, the Euler-Maclaurin formula is used.
The eigenvalues are given by

$$\lambda(i) = \frac{i\,(d+i-1)\,R}{d(d-1)} \tag{A.90}$$

and the multiplicities are

$$m(i) = \frac{(d+2\,i-1)\,(d+i-2)!}{(d-1)!\,i!}\,. \tag{A.91}$$

Then the trace is evaluated as

$$\mathrm{Tr}\frac{\partial_t R_k(-\nabla^2)}{P_k(-\nabla^2)} = \sum_{i=0}^{\infty}\frac{\partial_t R_k(\lambda(i))}{P_k(\lambda(i))}\,. \tag{A.92}$$

For the optimized cutoff the sum goes over $2\theta(\lambda(i) - \nabla^2)$. Due to the theta-function, one has to sum only until the n-th eigenvalue which is the last one to be smaller than k^2. Bringing the scalar curvature to the other side, one obtains $k^2/R = \tilde{R}$ as the limit which we fix at $\tilde{R} = 1$.
The sum over the first n eigenvalues gives

$$\frac{d\,\left((d+1)\,\Gamma(n) + 2\,\Gamma(n+1)\right)\,\Gamma(d+n) - 2\,\Gamma(n)\,\Gamma(d+n+1)}{(d-1)\,\Gamma(d+1)\,\Gamma(n)\,\Gamma(n+1)}\,. \tag{A.93}$$

Here we will now restrict to $d = 4$. Then this sum becomes

$$\frac{1}{12}(n+1)(n+2)^2(n+3)\ . \tag{A.94}$$

The leading power goes just as

$$\frac{n^4}{12} \tag{A.95}$$

which on the 4-sphere scales as $12/\tilde{R}^2$. Obviously the end of the sum has to be adjusted because the number of the highest eigenvalue is not an integer. For that reason we introduce parameters a and b and set

$$n = \frac{3\left(\sqrt{a+\frac{16}{3\,r}}-b\right)}{2} \tag{A.96}$$

so that the summation up to that n gives

$$\frac{\left(2-3\,b+\sqrt{9\,a+\frac{48}{\tilde{R}}}\right)\left(4-3\,b+\sqrt{9\,a+\frac{48}{\tilde{R}}}\right)^2\left(6-3\,b+\sqrt{9\,a+\frac{48}{\tilde{R}}}\right)\tilde{R}^2}{2304}\ . \tag{A.97}$$

Series expansion to second order gives

$$\begin{aligned}
&1+\frac{\left(4\,\sqrt{3}\,\left(48\,(2-3\,b)+96\,(4-3\,b)\right)+192\,\sqrt{3}\,(6-3\,b)\right)\,\sqrt{\tilde{R}}}{2304}\\
&+\frac{1}{2304}\left(216\,a+4\,\sqrt{3}\,\left(18\,\sqrt{3}\,a+4\,\sqrt{3}\,\left(9\,a+(4-3\,b)^2\right)+8\,\sqrt{3}\,(2-3\,b)\,(4-3\,b)\right)\right.\\
&\left.+\left(48\,(2-3\,b)+96\,(4-3\,b)\right)\,(6-3\,b)\right)\,\tilde{R}\\
&+\frac{1}{2304}\left(\frac{3\,\sqrt{3}\,a\,\left(48\,(2-3\,b)+96\,(4-3\,b)\right)}{8}\right.\\
&+4\,\sqrt{3}\,\left(\left(9\,a+(4-3\,b)^2\right)\,(2-3\,b)+18\,a\,(4-3\,b)\right)+\left(18\,\sqrt{3}\,a+4\,\sqrt{3}\,\left(9\,a+(4-3\,b)^2\right)\right.\\
&\left.\left.+8\,\sqrt{3}\,(2-3\,b)\,(4-3\,b)\right)\,(6-3\,b)\right)\,\tilde{R}^{\frac{3}{2}}\\
&+\frac{1}{2304}\left(\frac{-81\,a^2}{8}+\frac{3\,\sqrt{3}\,a\,\left(18\,\sqrt{3}\,a+4\,\sqrt{3}\,\left(9\,a+(4-3\,b)^2\right)+8\,\sqrt{3}\,(2-3\,b)\,(4-3\,b)\right)}{8}\right.\\
&+4\,\sqrt{3}\,\left(\frac{-27\,\sqrt{3}\,a^2}{32}+\frac{3\,\sqrt{3}\,a\,\left(9\,a+(4-3\,b)^2\right)}{8}+\frac{3\,\sqrt{3}\,a\,(2-3\,b)\,(4-3\,b)}{4}\right)\\
&\left.+\left(\left(9\,a+(4-3\,b)^2\right)\,(2-3\,b)+18\,a\,(4-3\,b)\right)\,(6-3\,b)\right)\,\tilde{R}^2+O(\tilde{R})^{\frac{5}{2}}\ .
\end{aligned} \tag{A.98}$$

This expansion includes terms proportional to proportional to $\tilde{R}^{1/2}$ and $\tilde{R}^{3/2}$ which are absent in the heat kernel expansion. They can be eliminated by choosing $b = 4/3$. Then the series simplifies to

$$1 + \frac{(9\,a-2)\,\tilde{R}}{24} + \frac{a\,(9\,a-4)\,\tilde{R}^2}{256}\,. \tag{A.99}$$

This will give the right first order heat kernel coefficient when choosing $a = 10/9$ but the second order one, 0.0260417, is a little bit different from the value $29/1080 \approx 0.0268519$,

$$1 + \frac{\tilde{R}}{3} + \frac{5\,\tilde{R}^2}{192}\,. \tag{A.100}$$

So this approach is too naive and one has to think about a more accurate method to calculate the correction terms. This is given by the Euler-Maclaurin formula.

$$\sum_{i=0}^{n} F(i) = \int_0^n dx F(x) + \frac{F(0)+F(n)}{2} + \sum_{j=1}^{k} \frac{\tilde{B}_{2j}}{(2j)!} F^{2j-1}(n)\|_0^n + \text{rest} \tag{A.101}$$

where $\tilde{B}_n$ are the Bernoulli numbers, $\tilde{B}_n(x)$ the Bernoulli polynomials, $[x]$ is the Gauß bracket (the next lower integer), and the rest term is given by

$$\text{rest} = \frac{1}{(2k+1)!} \int_0^n dx\, \tilde{B}_{2k+1}(x-[x]) F^{(2k+1)}(x)\,. \tag{A.102}$$

This works however only for continuous functions, so we have to approximate the step function of the optimized cutoff in the sum over all eigenvalues by the continuous form

$$\frac{(x+1)\,(x+2)\,(2\,x+3)}{6\left(1+e^{2a\left(\frac{1}{12}\tilde{R}x(x+3)-1\right)}\right)} \tag{A.103}$$

and take the limit $a \to \infty$ afterwards. This function has to be summed from zero to infinity in the formula. The integration can be performed replacing $x(x+3) = y$ or

$$x = \frac{\sqrt{4\,y+9}-3}{2}\,. \tag{A.104}$$

This gives a Jacobian $\frac{1}{\sqrt{4\,y+9}}$ and the integrand simplifies to $\frac{y+2}{6}$. Integration from 0 to $12/\tilde{R}$ gives

$$\frac{1}{6}\left(\frac{72}{\tilde{R}^2} + \frac{24}{\tilde{R}}\right) \tag{A.105}$$

reproducing the first two coefficients in the sum correctly. Now we have to calculate the correction terms for the higher order terms. Therefore one needs

$$F[0] = \frac{1}{1+e^{-2a}} \tag{A.106}$$

which in the limit $a \to \infty$ goes to 1. The correction from the first derivative term is

$$F'[0] = \frac{13}{6\left(1+e^{-2a}\right)} - \frac{ae^{-2a}r}{2\left(1+e^{-2a}\right)^2} \tag{A.107}$$

which in the limit $a \to \infty$ goes to $13/6$. The correction from the third derivative term is

$$\begin{aligned} F'''[0] \quad = \quad & -\frac{a^3e^{-2a}r^3}{8\left(1+e^{-2a}\right)^2} + \frac{3a^3e^{-4a}r^3}{4\left(1+e^{-2a}\right)^3} - \frac{3a^3e^{-6a}r^3}{4\left(1+e^{-2a}\right)^4} - \frac{17a^2e^{-2a}r^2}{8\left(1+e^{-2a}\right)^2} \\ & + \frac{17a^2e^{-4a}r^2}{4\left(1+e^{-2a}\right)^3} - \frac{20ae^{-2a}r}{3\left(1+e^{-2a}\right)^2} + \frac{2}{1+e^{-2a}} \end{aligned} \tag{A.108}$$

which in the limit $a \to \infty$ goes to 2. The higher derivative terms vanish in the limit $a \to \infty$ as it should be for the optimized cutoff. For these, each term contains a factor e^{-2a} driving the expression to zero in the limit. The sum over the correction terms gives

$$\frac{1}{2} - \frac{13\,\tilde{B}_2}{12} - \frac{\tilde{B}_4}{12} = \frac{29}{90} \tag{A.109}$$

agreeing with the second order heat kernel term.

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