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Existence and regularity of solutions to evolutionary problems in perfect plasticity

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Contents

Notation

Throughout the text we adopt the following notation:

$\mathbb{R}^n$ denotes the n-dimensional Euclidian space,

We use the summation convention with respect to the repeating indices i, j, l which, however, *does not apply* to the index $k = 1, \dots, n$, which denotes single partial derivative with respect to x_k, and to indices $m, r = 1, \dots, N$, which index incremental problems,

$\mathbb{M}^{n\times n}_{sym}$ denotes the space of all $n \times n$ symmetric matrices, equipped with a Hilbert-Schmidt scalar product $\sigma : \xi = \sigma_{ij}\xi_{ij}$,

$\mathbf{1}$ denotes the $n \times n$ identity matrix,

$\mathbb{I}$ stands for the identity tensor, which is a linear operator from $\mathbb{M}^{n\times n}_{sym}$ to itself, such that $\mathbb{I}(M) = M$ for all $M \in \mathbb{M}^{n\times n}_{sym}$,

$a \odot b$ stands for the symmetrized tensor product of two vectors $a, b \in \mathbb{R}^n$, given by the formula $(a \odot b)_{ij} = \frac{1}{2}(a_i b_j + a_j b_i)$,

$L^p(\Omega;\mathbb{R}^m)$ is the Lebesgue space of functions from Ω into $\mathbb{R}^m$, having the finite norm

$$\|f\|_{p;\Omega} := \Big(\int_\Omega |f|^p\,dx\Big)^{1/p},$$

$W^{l,p}(\Omega;\mathbb{R}^m)$ is the Sobolev space of all functions from Ω into $\mathbb{R}^m$ with the norm

$$\|f\|_{l,p,\Omega} := \Big(\int_\Omega \sum_{\alpha=0}^{l} |\nabla^\alpha f|^p\Big)^{1/p},$$

$\mathcal{L}^n$ stands for the n-dimensional Lebesgue measure on $\mathbb{R}^n$,

$\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure,

$M_b(\Omega;\mathbb{R}^m)$ is the space of all bounded Radon measures on Ω with values in $\mathbb{R}^m$,

For a measure $\mu \in M_b(\Omega;\mathbb{R}^m)$, by μ^a and μ^s we denote its absolutely continuous and singular parts with respect to a corresponding Lebesgue measure $\mathcal{L}^n$,

For $\mu \in M_b(\Omega;\mathbb{R}^m)$, by $|\mu|$ we denote its total variation, which is an element of $M_b(\Omega)$, and we consider the norm $\|\mu\|_{1;\Omega} := |\mu|(\Omega)$,

$BV(\Omega)$ is the space of all functions in $L^1(\Omega;\mathbb{R}^n)$ such that $Du \in M_b(\Omega;\mathbb{M}^{n\times n}_{sym})$, equipped with the norm $\|u\|_{1,1;\Omega} = \|u\|_{1;\Omega} + \|Du\|_{1;\Omega}$,

$BD(\Omega)$ is the space of all functions in $L^1(\Omega;\mathbb{R}^n)$ such that $\varepsilon(u) \in M_b(\Omega;\mathbb{M}^{n\times n}_{sym})$, where $\varepsilon(u)$ is the symmetrized gradient of u, $\varepsilon(u) = \frac{\nabla u + \nabla u^T}{2}$; the norm in BD is defined as $\|u\|_{1,1;\Omega} = \|u\|_{1;\Omega} + \|\varepsilon(u)\|_{1;\Omega}$, For $\Omega \subset \mathbb{R}^2$ we denote by $BH(\Omega)$ the space of all functions in $L^1(\Omega)$ such that $Du \in BV(\Omega;\mathbb{R}^2)$, equipped with the norm $\|u\|_{2,1;\Omega} = \|u\|_{1,1;\Omega} + \|D^2u\|_{1;\Omega}$,

$\langle\cdot|\cdot\rangle$ denotes the duality between two objects, where the duality relation depends upon the context.

Chapter 1

Introduction

In this work we develop a rigorous mathematical analysis of variational problems describing the quasistatic evolutionary problems in plasticity. The common feature of the problems under consideration is the *variational energy formulation*, where the mathematical difficulties arise due to the presence of a term with a *linear growth* in the symmetric part of the gradient of the unknown vector-valued functions in a volumetric case, or on the Hessian of the unknown scalar function in two-dimensional problems for plates. In the applications these functions represent displacement fields of a body.

The issue of the correct formulation of plasticity problems in the mechanics of the continuum media from the mathematical viewpoint was addressed, for example, in [DL76, Joh76, Suq81, Tem85]. The usual way to treat the problems of this kind is to understand the corresponding physical background, to introduce a concept of an appropriate weak solution to the problem, and to prove existence theorems in spaces of generalized functions.

Apart from the existence results for quasistatic problems in plasticity, for which, by now, have been developed a number of quite standard approaches (see, for example [DL76, Suq81]), the most interesting and difficult problem concerns the regularity properties of these weak solutions. There are surprisingly few works concerning the higher differentiability of weak solutions to problems in plasticity (see [BF93, BF96, FS00, Kne06, Ser93c, Ser94, Ser85, Ser87, Suq82] for the essential contributions): due to the linear growth of objective functionals one is forced to use spaces like BV and BH, where it is relatively easy to get the existence of weak solutions, but establishing further differentiability turns out to be an extremely hard task.

In the present work we study the two aspects of mathematical formulation of evolutionary problems in perfect plasticity: *existence* and *smoothness* of these generalized solutions.

Precisely, we focus on the following problems:

1. *Existence* of weak solutions to evolution problems for pressure-sensitive materials. Based on the author's work [DDD07] in collaboration with G. Dal Maso and A. DeSimone, we treat the problem of existence of weak solutions to quasistatic evolution problems for pressure-sensitive materials.

   The class of materials under consideration includes concrete, granular media, metallic foams, and porous metals. Following the energy formulation of quasistatic problems (see [Mie02] for a general discussion of this approach), we prove the existence result under very mild technical assumptions. We also obtain some fine pointwise properties of generalized solutions.

2. *Existence* of weak solutions to evolution problems for elasto-plastic plates. Based on the author's work [Dem09], we present the existence result for quasistatic evolution problems for clamped perfect elasto-plastic plates.

   Following the general scheme for proving the existence of weak solutions of the continuous-time energy formulation of rate-independent processes (see [Mie02]), we prove the existence result and establish fine pointwise properties of solutions.

3. *Differentiability properties* of weak solutions in the Prandt-Reuss perfect plasticity. We present the results of author's work [Dem08b].

   We study the smoothness of the stress tensor in the Prandtl-Reuss model of perfect plasticity. We develop a general approach to proving the Sobolev differentiability of the stress tensor, which can be applied to other models in plasticity as well (the author has used it to study regularity of bending moments in an evolutionary problem for perfect elastoplastic plates, see [Dem08a]). By using the regularity results of G. Seregin (see, for example, [Ser87, FS00, Ser93c, Ser96]) and the approximation procedure of [DDM06] (which leads to an existence result, according to the energy approach), we get the $W^{1,2}_{loc}$-regularity of the stress tensor.

   We also discuss the issue of regularity for displacements, by giving the regularity counterexamples.

4. *Differentiability properties* of weak solutions to quasistatic evolution problems for clamped perfect elasto-plastic plates. We develop on the differentiability results for bending moments of the vertical displacement in evolutionary problems for perfect elasto-plastic plates, obtained in [Dem08a].

   Following the methodology, developed in [Dem08b] for the Prandtl-Reuss plasticity, we use the regularity results of [Ser93a, Ser87] for the static case and estimates for approximate solutions of [Dem09], to prove $W^{1,2}_{loc}$-smoothness of bending moments tensor.

We mention the author's previous works, concerning the existence of solutions to phase transition problems of mechanics of two-phase elastic medium: necessary and sufficient conditions for lower semicontinuity, leading to the existence of solutions, were established in [Dem04, Dem06], while relaxation and Γ-convergence were studied in [Dem05].

1.1 Weak solutions and their further differential properties

As usual in the Calculus of Variations and the theory of Partial Differential Equations, the strategy of solving a problem is the following.

1. First one introduces a notion of an appropriate generalized solution of the problem, which on the one hand should be weak enough for one to be able to prove its existence, and on the other hand, it should capture the essential features of the problem.

   Thus, this step consists in identifying a space of generalized functions and formulating initial problem in a weak form to obtain the following relations:

$$\begin{gathered}\text{classical solution } \textit{is a} \text{ weak solution,}\\ \text{a sufficiently smooth weak solution } \textit{is a} \text{ classical solutuion.}\end{gathered}\tag{1.1}$$

2. Then, hopefully, one shows that the weak solution found possesses an additional regularity, and thus, thanks to (1.1), it *is actually a strong solution* to the original problem. Therefore, the problem of differentiability properties of weak solutions attracts a lot of interest.

   The higher regularity of weak solutions is an important problem even if, in many cases, due to considerable technical complexity caused by the nonlinearity, it is not always possible to show that weak solutions possess *sufficient differentiability* to be strong ones.

1.2 Existence of weak solutions

As already mentioned, the first step in dealing with a problem is to introduce a suitable notion of weak solution and prove its existence.

There are several equivalent ways of formulating an original problem in perfect plasticity in a relaxed form, the most popular of them being:

- *variational inequalities* formulation in rate form, proposed in [Joh76] and [Suq81], where the existence result is proved by a visco-plastic approximation;

- *continuous-time energy formulation* for rate-independent processes (see [Mie02] for a review of the approach), where the existence is proved by time-discretization and piecewise approximation by solutions of the corresponding incremental problems.

Below we adopt the second approach for proving the existence of weak solutions for evolutionary problems for pressure sensitive materials (following [DDD07], see Chapter 2) and for evolutionary problems for perfect elasto-plastic plates, proved in [Dem09] and presented in Chapter 3.

Both results are obtained by applying a standard machinery of rate-independent processes (in a way, used in [DDM06] for the Prandtl-Reuss perfect plasticity) and defining appropriate dualities between measures and weakly differentiable function, corresponding to the pairing between stress and plastic part of the strain (in the case of perfect plasticity for pressure-sensitive materials) and bending moments and plastic curvatures in the case of plates.

Concerning the history of the problem, the existence result for pressure-sensitive materials [DDD07] *is new*, as well as the fine properties of the stress tensor.

The existence problem for plates was already studied by many authors (see, for example, [BK00] for a similar problem with different boundary conditions, obtained by means of parabolic regularization), but, to our best knowledge, it *has never been studied under a quasistatic evolution framework* and the *fine pointwise properties* of the solution (Theorem 1.2.8) *were not known previously.* Moreover, the way we construct approximations with piecewise constant functions, obtained by solving iteratively incremental problems (1.3) allows one to apply the methods of [Dem08b] (see Chapter 4) for *studying further differential properties of bending moments*, as it was done in [Dem08a] (see Chapter 5).

1.2.1 Quasistatic evolution for pressure-sensitive elastic materials

With reference to a domain $\Omega \subset \mathbb{R}^n$, the problem can be formulated as follows. The linearized strain $\varepsilon(u)$, defined as the symmetric part of the spatial gradient of the displacement u, is decomposed as the sum $\varepsilon(u) = e + p$, where e and p are the elastic and plastic strains. The stress σ is determined only by e, through the formula $\sigma = \mathbb{C}e$, where $\mathbb{C}$ is the elasticity tensor. It is constrained to lie in a prescribed convex subset $\mathbb{K}$ of the space $\mathbb{M}^{n\times n}_{sym}$ of $n{\times}n$ symmetric matrices, whose boundary $\partial\mathbb{K}$ is referred to as the yield surface. In this context, pressure sensitivity of the yield criterion leads to the hypothesis that $\mathbb{K}$ is bounded.

The data of our problem are a time-dependent body force $f(t,x)$, defined for $t \in [0,T]$ and $x \in \Omega$, a time-dependent surface force $g(t,x)$ acting on a portion Γ_1 of the boundary $\partial\Omega$, and a time-dependent displacement prescribed on the complementary porion Γ_0 of $\partial\Omega$. The classical formulation of the quasistatic evolution problem consists in finding functions $u(t,x)$, $e(t,x)$, $p(t,x)$, $\sigma(t,x)$ satisfying the following conditions for every $t \in [0,T]$ and every $x \in \Omega$:

1. additive decomposition: $\varepsilon(u)(t,x) = e(t,x) + p(t,x)$,
2. constitutive equation: $\sigma(t,x) = \mathbb{C}e(t,x)$,
3. equilibrium: $-\mathrm{div}\ \sigma(t,x) = f(t,x)$,
4. associative flow rule: $\dot{p}(t,x) \in N_{\mathbb{K}}(\sigma(t,x))$

where $N_{\mathbb{K}}(\xi)$ is the normal cone to $\mathbb{K}$ at ξ. The problem is supplemented by initial conditions at time $t=0$, by displacement boundary conditions $u(t,x) = w(t,x)$ for $t \in [0,T]$ and $x \in \Gamma_0$, and traction boundary conditions $\sigma(t,x)\nu(x) = g(t,x)$ for $t \in [0,T]$ and $x \in \Gamma_1$, where $\nu(x)$ is the outer unit normal to $\partial\Omega$.

In recent work [DDM06], a similar problem was considered for the pressure-insensitive case where $\mathbb{K}$ is a cylinder in $\mathbb{M}^{n\times n}_{sym}$ containing all scalar multiples of the identity matrix. There, the existence of a suitably defined weak solution was obtained by time-discretization. According to a general energy approach, see e.g. [Mie02], the discrete time formulation consists in solving a chain of incremental minimum problems which are quadratic in e and have linear growth in p.

Namely, the entire time interval $[0,T]$ is divided into N subintervals by means of points

$$0 = t_0^N < t_1^N < \cdots < t_{N-1}^N < t_N^N = T\,,$$

and the approximate solution u_i^N, e_i^N, p_i^N at time t_i^N is defined, inductively, as a minimizer of the incremental problem

$$\min_{(u,e,p)\in A(w(t_i^N))} \left\{ \mathcal{Q}(e) + \mathcal{H}(p - p_{i-1}^N) - \mathcal{F}[t_i^N]u \right\}, \tag{1.2}$$

where for the moment $A(w(t))$ denotes the set of all triples (u,e,p), such that $\varepsilon(u(x)) = e(x)+p(x)$ for $x \in \Omega$ and $u(x) = w(t,x)$ for $x \in \Gamma_0$, the quadratic form $\mathcal{Q}$, corresponding to the stored elastic energy, is defined by

$$\mathcal{Q}(e) := \frac{1}{2}\int_\Omega \mathbb{C}e : e\,dx,$$

the functional $\mathcal{H}$ is given by

$$\mathcal{H}(p) := \int_\Omega H(p(x))\,dx,$$

with $H : \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ being the support function to the set $\mathbb{K}$, and the total load $\mathcal{F}[t]$ is defined by

$$\mathcal{F}[t]u = \int_\Omega f(t)\cdot u\,dx + \int_{\Gamma_1} F(t)\cdot u\,d\mathcal{H}^{n-1}.$$

Since $\mathcal{H}$ has linear growth, problem (1.3) has, in general, no solution in Sobolev spaces. Thus, the direct methods of Calculus of Variations lead to a weak formulation, with a displacement u

being an element of $BD(\Omega)$, the space of functions with bounded deformation, whose theory was developed in [KT83, Tem85, TS80], the elastic strain e lying in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and the plastic strain p belonging to $M_b(\Omega\cup\Gamma_0,\mathbb{M}^{n\times n}_{sym})$, the space of bounded Radon measures on $\Omega\cup\Gamma_0$ with values in $\mathbb{M}^{n\times n}_{sym}$.

According to the theory of convex functionals of measures (see [GS64] and [Tem85, Chapter II]), define the functional $\mathcal{H}(p) : M_b(\Omega\cup\Gamma_0,\mathbb{M}^{n\times n}_{sym})\to\mathbb{R}$ as

$$\mathcal{H}(p) := \int_{\Omega\cup\Gamma_0} H(p/|p|)\, d|p|,$$

extending in an appropriate way the definition of the set of admissible triples $A(w(t))$ (see Section 2.2.2 for the details).

We define the piecewise constant interpolations

$$u^N(t) := u_i^N\,,\quad e^N(t) := e_i^N\,,\quad p^N(t) := p_i^N\,,\quad \sigma^N(t) := \sigma_i^N\,,$$

where i is the largest integer such that $t_i^N \le t$.

We introduce a definition of continuous-time quasistatic evolution in the functional framework $u\in BD(\Omega)$, $e\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p\in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$, $\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and prove that, up to a subsequence, the discrete-time solutions $u^N(t)$, $e^N(t)$, $p^N(t)$, $\sigma^N(t)$, obtained by solving the weak formulations of problems (1.3), converge to a continuous-time solution $u(t)$, $e(t)$, $p(t)$, $\sigma(t)$, provided $\max_i(t_i^N - t_{i-1}^N)\to 0$ as $N\to\infty$.

For every time interval $[s,t]\subset[0,T]$ introduce the dissipation associated with $\mathcal{H}$ given by

$$\mathcal{D}_{\mathcal{H}}(p;s,t) = \sup\left\{\sum_{j=1}^M \mathcal{H}(p(t_j)-p(t_{j-1}))\ :\ s=t_0\le\cdots\le t_M=t,\ M\in\mathbb{N}\right\}.$$

A variational formulation of the quasistatic problem is expressed by the following definition.

Definition 1.2.1. A quasistatic evolution is a function

$$(u,e,p) : [0,T]\to BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym}),$$

which satisfies the following conditions

(qs1) (global stability): For every $t\in[0,T]$ the triple $(u,e,p)(t)\in A(w(t))$ and

$$\mathcal{Q}(e(t)) - \mathcal{F}[t]u(t) \le \mathcal{Q}(\eta) + \mathcal{H}(q-p(t)) - \mathcal{F}[t]v$$

for every $(v,\eta,q)\in A(w(t))$,

(qs2) (energy balance): $p : [0,T]\to M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$ has bounded variation and for every $t\in[0,T]$

$$\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p;0,t) - \mathcal{F}[t]u(t) =$$
$$= \mathcal{Q}(e(0)) - \mathcal{F}[0]u(0) + \int_0^t \Big[\langle\sigma(s),\varepsilon(\dot w(s))\rangle_{L^2;L^2} - \mathcal{F}[s]\dot w(s) - \dot{\mathcal{F}}[s]u(s)\Big]\,ds$$

where $\sigma(t) = \mathbb{C}e(t)$.

The existence and time-regularity result is summarized in the following theorem, which establishes the existence of a solution to the quasistatic problem in perfect plasticity, satisfying the prescribed initial conditions, provided a uniform safe-load condition holds.

Theorem 1.2.2 (Theorems 2.4.3, 2.4.5 and 2.4.7)**.** *Let initial data* $(u_0, e_0, p_0) \in A(w(0))$ *satisfy the stability condition*

$$Q(e_0) - \mathcal{F}[0]u_0 \leq Q(\eta) + \mathcal{H}(q - p_0) - \mathcal{F}[0]v,$$

for every $(v, \eta, q) \in A(w(0))$. *Then there exists a quasistatic evolution*

$$(u(t), e(t), p(t)),$$

such that

$$u(0) = u_0, \ e(0) = e_0, \ p(0) = p_0.$$

Moreover, the elastic part of the symmetrized gradient $t \mapsto e(t)$ *is unique and a quasistatic evolution* (u, e, p) *as a function from* $[0, T]$ *to* $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ *is absolutely continuous in time.*

To investigate the relation between a weak solution in the form of a quasistatic evolution and a variational inequalities formulation in the rate form, we establish the following result.

Theorem 1.2.3 (Theorem 2.4.8)**.** *Let* $(u, e, p) : [0, T] \to BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ *and let* $\sigma(t) = \mathbb{C}e(t)$. *Then the following conditions are equivalent:*

(a) $t \mapsto (u(t), e(t), p(t))$ *is a quasistatic evolution;*

(b) $t \mapsto (u(t), e(t), p(t))$ *is absolutely continuous and*

(b1) for every $t \in [0, T]$ *we have* $(u(t), e(t), p(t)) \in A(w(t))$, $\sigma(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\operatorname{div} \sigma(t) = f(t)$ *in* Ω, *and* $[\sigma(t)\,\nu] = g(t)$ *on* Γ_1,

(b2) for a.e. $t \in [0, T]$ *we have*

$$\langle \sigma(t) - \tau | \dot{p}(t) \rangle \geq 0$$

for every $\tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ *with* $[\tau\,\nu] = g(t)$ *on* Γ_1.

Here the pairing $\langle \cdot, \cdot \rangle$ *corresponds to the duality between stress and a plastic part of the strain, defined in Section 2.3.2, and the set* $\Sigma(\Omega) \cap \mathcal{K}(\Omega)$ *stands for the mechanically admissible stress tensors.*

As in [DDM06] one can investigate further properties of the quasistatic evolution and establish the following pointwise version of a flow rule.

Theorem 1.2.4 (Theorems 2.4.10 and 2.4.11)**.** *Let* $(u, e, p) : [0, T] \to BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$, $\sigma(t) = \mathbb{C}e(t)$ *and let* $\mu(t) = \mathcal{L}^n + |\dot{p}(t)|$. *Then* $t \mapsto (u(t), e(t), p(t))$ *is a quasistatic evolution if and only if*

(d) $t \mapsto (u(t), e(t), p(t))$ *is absolutely continuous and*

(d1) for every $t \in [0, T]$ *we have* $(u(t), e(t), p(t)) \in A(w(t))$, $\sigma(t) \in \Sigma(\Omega) \cup \mathcal{K}(\Omega)$, $-\operatorname{div} \sigma(t) = f(t)$ *on* Ω, *and* $[\sigma(t)\nu] = g(t)$ *on* Γ_1,

(d2) for a.e. $t \in [0,T]$ *there exists* $\hat\sigma(t) \in L^\infty_{\mu(t)}(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ *such that*

$$\hat\sigma(t) = \sigma(t) \quad \mathcal{L}^n - a.e.\ on\ \Omega,$$

$$[\sigma(t) : \dot p(t)] = \hat\sigma(t) : \frac{\dot p(t)}{|\dot p(t)|} |\dot p(t)| \quad in\ M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym}),$$

$$\frac{\dot p(t)}{|\dot p(t)|}(x) \in N_{\mathbb{K}}(\hat\sigma(t,x)) \quad for\ |\dot p(t)| - a.e.\ x \in \Omega \cup \Gamma_0.$$

Moreover, if the set $\mathbb{K}$ *is strictly convex, then we can refine the definition of* $\hat\sigma$ *, by showing, that it is a strong* $\mu(t)$ *-limit of the mean values of* σ *. Namely, the following holds:*

$$\sigma^r(t,x) := \frac{1}{\mathcal{L}^n(B(x,r)\cap\Omega)} \int_{B(x,r)\cap\Omega} \sigma(t,y)\,dy \to \hat\sigma(t,x) \quad in\ L^1_{\mu(t)}(\Omega; \mathbb{M}^{n\times n}_{sym}),$$

for a.e. $t \in [0,T]$ *.*

1.2.2 Quasistatic evolution for elasto-plastic plates

Here the reference configuration is a bounded open set $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary and the elastic domain $\mathbb{K}$ is a bounded closed convex subset of $\mathbb{M}^{2\times 2}_{sym}$ (the space of symmetric 2×2 matrices) with the nonempty interior, whose boundary $\partial\mathbb{K}$ plays the role of the yield surface.

Given a scalar valued function $f(t,x)$ defined for $t \in [0,T]$ and $x \in \Omega$, which represents the transversal body force, the strong formulation of the evolution problem consists in finding a scalar valued function $u(t,x)$ (the vertical displacement) and three matrix-valued functions $e(t,x)$, $p(t,x)$ and $M(t,x)$ (the elastic and plastic curvatures and the bending moments) such that for every $t \in [0,T]$, for every $x \in \Omega$ the following conditions hold:

1. kinematic admissibility: $D^2 u(t,x) = e(t,x) + p(t,x)$ in Ω,
   $u(t,x) = 0$, $\frac{\partial u}{\partial \nu}(t,x) = 0$ on $\partial\Omega$
2. constitutive equation: $M(t,x) = \mathbb{C}\, e(t,x)$,
3. equilibrium: $\operatorname{div}\operatorname{div} M(t,x) = f(t,x)$ in Ω,
4. moment constraint: $M(t,x) \in \mathbb{K}$,
5. associative flow rule: $\dot p(t,x) \in N_{\mathbb{K}}(M(t,x))$,

where $\nu(x)$ is the outer unit normal to $\partial\Omega$ and $\mathbb{C}$ is the rigidity tensor. The symbol $N_{\mathbb{K}}(\xi)$ denotes the normal cone to the set $\mathbb{K}$ at the point ξ in the sense of convex analysis. The problem is supplemented by initial conditions at time $t = 0$.

The boundary conditions $u = 0$ and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ reflect the mechanical assumption that the plate is clamped.

Dealing with quasistatic evolution problems of the type considered above, we approximate them numerically by solving a finite number of incremental variational problems. As discussed in the previous section, the time interval $[0,T]$ is divided into N subintervals by means of points

$$0 = t^N_0 < t^N_1 < \cdots < t^N_{N-1} < t^N_N = T\,,$$

and to get the updated values u_i^N, e_i^N and p_i^N, we solve the incremental problem

$$\min_{u,e,p} \Big\{ \mathcal{Q}(e) + \mathcal{H}(p - p_{i-1}^N) - \mathcal{F}[t_i^N]u \Big\}, \tag{1.3}$$

where the minimization is carried out over the set of all kinematically admissible triples (u, e, p), such that $D^2u(x) = e(x) + p(x)$ for $x \in \Omega$, $u(x) = 0$ and $\frac{\partial u}{\partial \nu}$ for $x \in \Gamma_0$, the quadratic form $\mathcal{Q}$, corresponding to the stored elastic energy, is defined by

$$\mathcal{Q}(e) := \frac{1}{2} \int_\Omega \mathbb{C}e : e \, dx,$$

the functional $\mathcal{H}$ is given by

$$\mathcal{H}(p) := \int_\Omega H(p(x)) \, dx,$$

with $H : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{R}$ being the support function to the set $\mathbb{K}$, and the total load $\mathcal{F}[t]$ is defined by

$$\mathcal{F}[t]u = \int_\Omega f(t) \, u \, dx.$$

Due to the linear growth of $\mathcal{H}$, one has to relax the problem to look for vertical displacements in the space $BH(\Omega)$ of functions with bounded deformation (the reader is referred to [Tem85, Chapter III] for the definition and basic properties of $BH(\Omega)$), for elastic curvatures in the space $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ and for plastic curvatures in the space $M_b(\overline{\Omega})$ of $\mathbb{M}^{2\times 2}_{sym}$-valued bounded Radon measures.

The relaxed version of $\mathcal{H} : M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym}) \to \mathbb{R}$ takes the form

$$\mathcal{H}(p) := \int_{\overline{\Omega}} H(p/|p|) d|p|.$$

For $t \in [0, T]$ define piecewise constant interpolations

$$u^N(t) := u_i^N, \quad e^N(t) := e_i^N, \quad p^N(t) := p_i^N, \quad \sigma^N(t) := \sigma_i^N,$$

where i is the largest integer such that $t_i^N \le t$.

Working in the functional framework $u \in BH(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$, $p \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{2\times 2}_{sym})$, $\sigma \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$, we introduce a notion of continuous-time quasistatic evolution and prove that the approximate solutions $u^N(t)$, $e^N(t)$, $p^N(t)$, $\sigma^N(t)$ converge to a continuous-time solution $u(t)$, $e(t)$, $p(t)$, $\sigma(t)$, provided $\max_i(t_i^N - t_{i-1}^N) \to 0$ as $N \to \infty$.

For every interval $[s, t] \subset [0, T]$ the $\mathcal{H}$ variation of p on $[s, t]$ is defined as

$$\mathcal{D}_\mathcal{H}(p; s, t) = \sup \Big\{ \sum_{i=1}^N \mathcal{H}(p(t_i) - p(t_{i-1})) : \; s = t_0 < \cdots < t_N = t, \; N \in \mathbb{N} \Big\}.$$

A weak solution to the problem is defined as the following quasistatic evolution. Note, that in the definition below $\langle \cdot, \cdot \rangle$ stands for the scalar product in $L^2(\Omega.)$

Definition 1.2.5. A quasistatic evolution is a function $t \mapsto (u(t), e(t), p(t))$ from $[0, T]$ into $BH(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ which satisfies the following conditions (qs1) for every $t \in [0,T]$ the triple $(u(t), e(t), p(t))$ is kinematically admissible and

$$\mathcal{Q}(e(t)) - \langle f(t)|u(t)\rangle \leq \mathcal{Q}(\eta) + \mathcal{H}(q - p(t)) - \langle f(t)|v\rangle \tag{1.4}$$

for every kinematically admissible (v, η, q);

(qs2) the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ has bounded variation and for every $t \in [0,T]$

$$\begin{aligned} &\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle f(t)|u(t)\rangle = \\ &= \mathcal{Q}(e(0)) - \langle f(0)|u(0)\rangle - \int_0^t \langle \dot{f}(s)|u(s)\rangle\, ds. \end{aligned} \tag{1.5}$$

The following theorem states the main existence and time-regularity result for weak solutions of the quasistatic problem for perfect elasto-plastic plates. Note, that it is implicitly assumed that an appropriate safe-load condition is satisfied.

Theorem 1.2.6 (Theorems 3.4.3, 3.5.1, and 3.5.6)**.** *Let initial data* (u_0, e_0, p_0) *be kinematically admissible and satisfy the stability condition*

$$Q(e_0) - \langle f(0), u_0\rangle \leq Q(\eta) + \mathcal{H}(q - p_0) - \langle f(0), v\rangle,$$

for every kinematically admissible triple (v, η, q)*. Then there exists a quasistatic evolution*

$$(u(t), e(t), p(t)),$$

such that

$$u(0) = u_0,\ e(0) = e_0,\ p(0) = p_0.$$

Moreover, the elastic curvatures tensor $t \mapsto e(t)$ *is unique and a quasistatic evolution* (u, e, p) *as a function from* $[0,T]$ *to* $BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ *is absolutely continuous in time.*

The following result relates the quasistatic evolution properties with the classical formulation of the flow rule.

Theorem 1.2.7 (Theorem 3.6.1)**.** *Let* $t \mapsto (u(t), e(t), p(t))$ *be a function from* $[0,T]$ *into* $BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ *and let* $M(t) := \mathbb{C}e(t)$*. Then the following conditions are equivalent:*

(a) $t \mapsto (u(t), e(t), p(t))$ *is a quasistatic evolution;*

(b) $t \mapsto (u(t), e(t), p(t))$ *is absolutely continuous and*

(b1) for every $t \in [0,T]$ *the triple* $(u(t), e(t), p(t))$ *is kinematically admissible,* $M(t) \in S(\Omega) \cap \mathcal{K}(\Omega)$ *and* $\operatorname{div}\operatorname{div} M(t) = f(t)$ *in* Ω*,*

(b2) for a.e. $t \in [0,T]$ *we have*

$$\langle M(t) - m|\dot{p}(t)\rangle \geq 0$$

for every $m \in S(\Omega) \cap \mathcal{K}(\Omega)$*.*

Here the pairing $\langle\cdot,\cdot\rangle$ *corresponds to the duality between bending moments tensor and a plastic curvatures, defined in Section 3.2.2, and the set* $S(\Omega) \cap \mathcal{K}(\Omega)$ *stands for the mechanically admissible bending moments tensors.*

As in [DDM06] and [DDD07], we can study the pointwise properties of a bending moments tensor and formulate a pointwise version of a flow rule.

Theorem 1.2.8 (Theorem 3.6.3)**.** *Let $t \mapsto (u(t), e(t), p(t))$ be a function from $[0,T]$ into $BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$, let $M(t) := \mathbb{C}e(t)$, and let $\mu(t) := \mathcal{L}^2 + |\dot p(t)|$. Then $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution if and only if*

- *(e) $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous and*
  - *(e1) for every $t \in [0,T]$ the triple $(u(t), e(t), p(t))$ is kinematically admissible, $M(t) \in S(\Omega) \cap \mathcal{K}(\Omega)$, and $\operatorname{div}\operatorname{div} M(t) = f(t)$ in Ω,*
  - *(e2) for a.e. $t \in [0,T]$ there exists $\hat M(t) \in L^\infty_{\mu(t)}(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ such that*

$$\hat M(t) = M(t) \quad \mathcal{L}^2\text{-a.e. on } \Omega,$$
$$[M(t) : \dot p(t)] = \Big(\hat M(t) : \frac{\dot p(t)}{|\dot p(t)|}\Big)|\dot p(t)| \quad \text{in } M_b(\overline{\Omega}),$$
$$\frac{\dot p(t)}{|\dot p(t)|}(x) \in N_{\mathbb{K}}(\hat M(t,x)) \quad \text{for } |\dot p(t)|\text{-a.e. } x \in \overline{\Omega}.$$

Moreover, if the set $\mathbb{K}$ is strictly convex, then one can refine the definition of $\hat M$, by showing, that it is a strong $\mu(t)$-limit of the mean values of M. Namely, the following holds

$$M^r(t,x) := \frac{1}{\mathcal{L}^2(B(x,r)\cap\Omega)} \int_{B(x,r)\cap\Omega} M(t,y)\,dy \to \hat M(t,x) \quad \text{in } L^1_{\mu(t)}(\Omega; \mathbb{M}^{2\times 2}_{sym}),$$

for a.e. $t \in [0,T]$.

1.3 Regularity of weak solutions

As already discussed above, the issue of regularity of solutions of (systems of) PDEs and variational problems is very important and presents a considerable interest.

The question of regularity is already technically difficult, with many astonishing counterexamples, even for quite simple systems of PDEs. In this particular situation of perfect plasticity, the problem becomes even more complicated, due to the arising nonlinearities.

First let us discuss the *static case* of Hencky perfect plasticity. The main local regularity result (see [Ser87, Ser93c, BF93]) is that the stress tensor, which is known to be an $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ function, is actually $W^{1,2}_{loc}(\Omega; \mathbb{M}^{n\times n}_{sym})$. Further partial regularity results are due to G. Seregin, and we refer the reader to [FS00, Ser87, Ser93c] for the details. The only global regularity result known to the author is [Kne06], where it is shown that, under appropriate technical assumptions, the stress tensor is in $W^{1/2-\delta}(\Omega; \mathbb{M}^{n\times n}_{sym})$ for any $\delta > 0$.

The main approaches to investigating the Sobolev differentiability of stresses in Hencky perfect plasticity consist in approximating the original problem (P) with a sequence of "more regular" ones, (P_α) in a way, that

$$\text{Solutions } \sigma_\alpha \text{ of } (P_\alpha) \text{ are } W^{1,2}_{loc}(\Omega; \mathbb{M}^{n\times n}_{sym}); \tag{1.6}$$
$$\sigma_\alpha \text{ converge to the solution } \sigma^* \text{ of problem } (P) \text{ in some weak topology;} \tag{1.7}$$
$$\sup_\alpha \|\sigma_\alpha\|_{1,2;\Omega'} \le C(\Omega') \quad \text{holds, for every } \Omega' \subset\subset \Omega. \tag{1.8}$$

If one constructs a sequence of problems (P_α), satisfying (1.6)-(1.8), then the solution σ^* of problem (P) is also $W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$.

In [BF93] the authors used the Norton-Hoff regularization and a method of translations (as in [Suq82]), while in [FS00, Ser87, Ser93c] the original minimum problem is approximated by adding a "coercive" term, that guarantees the existence of a minimum in $W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$.

The main observation of G. Seregin is that working with a dual variational problem for the stress tensor gives more information, needed for investigating differentiability properties.

This approach towards regularity is based on the idea that the stress tensor is the most important quantity from the physical viewpoint, since it determines the elastic and plastic zones within the body. Therefore, the main object to study is the dual variational problem for the stress tensor, which has a unique solution. The form of this problem differs from the standard problems in calculus of variations: the functional does not involve any derivative of the unknown functions, the yield condition acts as a pointwise constraint, and the equilibrium equations for the stresses have to be incorporated in the class of admissible functions. Despite these difficulties one obtains additional regularity for the stress tensor.

Afterwards, by using the duality relations, which can be regarded as a weak form of the constitutive equations, one also establishes some regularity for the displacement field.

Recall, once again, that these results concern the *static* Hencky plasticity.

As for the quasistatic situation, for the Prandtl-Reuss case Norton-Hoff approximations were used in [BF96] to establish that the stress tensor satisfies

$$\sigma \in L^\infty([0,T];W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})).$$

In the sequel we present two regularity results: the smoothness of the stress tensor for the *Prandtl-Reuss perfect plasticity* (following [Dem08b], see Chapter 4) and the differentiability of the bending moments in *quasistatic problems for perfect elasto-plastic plates* (according to [Dem08a], see Chapter 5). We use the ideas of [FS00, Ser87, Ser93c, Ser93a], developed for static problems and the piecewise-constant approximations constructed in [DDM06, Dem09], that were briefly described in Sections 1.2.1 and 1.2.2.

Shortly, for a solution (u_i^N, e_i^N, p_i^N) of each incremental problem of the type (1.2), (1.3) we prove that

$$e_i^N \in W^{1,2}_{loc}(\Omega)$$

by an estimates, similar to that of [FS00, Ser93c, Ser93a]. Then we perform some analytical work to make this estimate uniform in i and N. Namely, we obtain

$$\sup_{N\in\mathbb{N}} \max_{i=0,\dots,N} \|e_i^N\|_{1,2;\Omega'} \le C(\Omega'), \tag{1.9}$$

for any $\Omega' \subset\subset \Omega$. This is done by using a Gronwall-type iterative estimate for e_i^N, for $i = 1,\dots,N$. Finally, one uses (1.9) and pointwise convergence of the piecewise constant approximations $(u^N(t), e^N(t), p^N(t))$ to the solutions of the original problem to get

$$\sup_{t\in[0,T]} \|e(t)\|_{1,2;\Omega'} \le C(\Omega')$$

for any $\Omega' \subset\subset \Omega$.

Observe that, even though the Prandtl-Reuss case was already studied in [BF96], the method proposed in [Dem08b] is of *completely different nature*, *imposes different restrictions on the data of* *the problem*, and, what is more important, *provides a general methodology for proving regularity* of solutions to various problems in perfect plasticity. It has proved to be useful for establishing $W^{1,2}_{loc}$ regularity of bending moments in the case of clamped perfect elasto-plastic plates, see [Dem08a] and Chapter 5, which is a *new result*.

1.3.1 Regularity of stresses in Prandtl-Reuss perfect plasticity

A strong formulation of the Prandtl-Reuss model of perfect plasticity is the following: given a domain $\Omega \subset \mathbb{R}^n$,

body force $f(t,x) : [0,T] \times \Omega \to \mathbb{R}^n$,
boundary displacement $w(t,x) : [0,T] \times \Gamma_0 \to \mathbb{R}^n$,
surface force $F(t,x) : [0,T] \times \Gamma_1 \to \mathbb{R}^n$,

the problem is to find functions

$$u(t,x),\ e(t,x),\ p(t,x) \quad \text{and } \sigma(t,x)$$

such that for every $t \in [0,T]$, for every $x \in \Omega$ the following hold:

1. kinematic admissibility: $\varepsilon(u)(t,x) = e(t,x) + p(t,x)$ in Ω, $u(t,x) = w(t,x)$ on Γ_0
2. constitutive equation: $\sigma(t,x) = \mathbb{A}^{-1}\, e(t,x)$,
3. equilibrium: $\operatorname{div}_x \sigma(t,x) = -f(t,x)$ in Ω, $\sigma(t,x)\,\nu(x) = F(t,x)$ on Γ_1,
4. stress constraint $\sigma(t,x) \in \mathbb{K}$,
5. associative flow rule: $(\xi - \sigma(t,x)) : \dot{p}(t,x) \le 0$ for every $\xi \in \mathbb{K}$,

where

$$\varepsilon(u) = \frac{\nabla u + \nabla u^T}{2},$$

$$\mathbb{K} = \{\tau \in \mathbb{M}^{n\times n}_{sym} \ :\ |\tau^D| \le \sqrt{2}k_*\}$$

and $\mathbb{A}$ is the compliance tensor (the inverse of the elasticity tensor), which in the isotropic case has the form

$$\mathbb{A}\sigma = \frac{\operatorname{tr}\sigma}{n^2 K_0}\,\mathbf{1} + \frac{1}{2\mu}\,\sigma^D, \tag{1.10}$$

where nK_0 is the first Lamé constant, and μ is the shear modulus. The problem is supplemented by initial conditions at time $t = 0$.

Under some technical conditions on volume and surface forces and on the geometry of the boundary $\partial\Omega$, described in detail in Section 4.2, the regularity result is expressed by the following Theorem.

Theorem 1.3.1 (Theorem 4.2.1)**.** *Suppose that* $n = 2, 3$, $\partial\Omega \in C^2$, $\mathbb{A}$ *has the form* (1.10) *and the assumptions* (4.3)-(4.5) *are satisfied. Then for the solution* (u, e, p) *of the quasistatic problem, see Definition 4.3.6, we have*

$$\sigma \in L^\infty([0,T]; W^{1,2}_{loc}(\Omega; \mathbb{M}^{n\times n}_{sym})),$$

with $\sigma(t,x) = \mathbb{A}^{-1}e(t,x)$.

We use the construction of piecewise constant approximations, obtained by discretizing the time interval $[0,T]$ (as outlined in Sections 1.2.1 and 1.2.2). The details are presented in Sections 4.3.1 and 4.3.2.

To prove the regularity theorem, we should obtain estimate (1.9) for the solutions $(u_i^N, e_i^N,\ p_i^N, \sigma_i^N)$ of the incremental problems of the form (1.2). Fixed N, for every $i = 1, \dots, N$ we consider a certain minimax program similar to that proposed in [FS00, Ser93c] for the Hencky plasticity (see Section 4.5 for details). We define a Lagrangian $L(\cdot\,,\cdot)$, and consider the following problem

$$\begin{gathered}\text{find a pair } (\delta u_i^N, \sigma_i^N), \text{ such that}\\ L_i^N(\delta u_i^N, \tau) \le L_i^N(\delta u_i^N, \sigma_i^N) \le L_i^N(v, \sigma_i^N) \text{ for all admissible } (v,\tau),\end{gathered}$$

such that its saddle points $(\delta u_i^N, \sigma_i^N)$ give rise to solutions of the corresponding incremental problems of type (1.2). Observe that, since we work in nonreflexive spaces, we actually consider a relaxed version of this problem, working with an extended Lagrangian $\widetilde{L}_i^N$, rather than L_i^N.

As usual, saddle points $(\delta u_i^N, \sigma_i^N)$ of $\widetilde{L}_i^N$ solve the corresponding primal and dual problems, respectively. The primal problem takes the form

$$\text{find } \delta u_i^N, \text{ such that } I_i^N(\delta u_i^N) = \inf_v I_i^N(v), \quad \text{with } I_i(v) := \sup_\tau \widetilde{L}_i^N(v,\tau), \tag{1.11}$$

while the dual one is

$$\text{find } \sigma_i^N, \text{ such that } R_i^N(\sigma_i^N) = \sup_\tau R_i^N(\tau), \quad \text{with } R_i^N(\tau) := \inf_v \widetilde{L}_i^N(v,\tau).$$

The main difficulty is that the functional I_i^N in (1.11) has a linear growth with respect to $\varepsilon(v)$, and thus has a minimum in the space BD only. To attain a better regularity, for every $\alpha \in (0,1)$ we consider the regularized problem of the form

$$\min_v \Big\{ \frac{\alpha}{2} \|\varepsilon(v)\|_2^2 + I_i^N(v) \Big\},$$

which, due to the Korn inequality, has a solution u_i^α in space $W^{1,2}(\Omega; \mathbb{M}^{n\times n}_{sym})$.

By introducing auxiliary functions σ_i^α, defined by adding correction terms to the functions

$$\alpha\, \varepsilon(u_i^\alpha) + \partial I_i^N(u_i^\alpha),$$

we write the Euler equation as

$$\operatorname{div}(\sigma_i^\alpha) = f_i^N. \tag{1.12}$$

Then we establish the following convergence properties:

$$\begin{aligned} u_i^\alpha &\overset{*}{\rightharpoonup} \delta u_i^N \quad \text{weakly}^* \text{ in } BD(\Omega;\mathbb{R}^n),\\ \sigma_i^\alpha &\rightharpoonup \sigma_i^N \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}),\end{aligned} \tag{1.13}$$

as $\alpha \to 0$.

By local regularity results we have that

$$u_i^\alpha \in W^{2,2}_{loc}(\Omega; \mathbb{M}^{n\times n}_{sym}),$$

that is

$$\sigma_i^\alpha \in W^{1,2}_{loc}(\Omega; \mathbb{M}^{n\times n}_{sym}). \tag{1.14}$$

Using Euler equation (1.12) and (1.14) we then prove the estimate

$$\sup_{\alpha\in(0,1)} \|\sigma_i^\alpha\|_{1,2;\Omega'} \le C(i,N;\Omega') \tag{1.15}$$

for every subdomain $\Omega' \subset\subset \Omega$. Thus, (1.15) and the convergence (1.14) immediately imply that

$$\sigma_i^\alpha \rightharpoonup \sigma_i^N \quad \text{as } \alpha\to 0 \text{ weakly in } W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym}), \tag{1.16}$$

and

$$\|\sigma_i^N\|_{1,2;\Omega'} \le C(i,N;\Omega').$$

It remains to make the last estimate uniform with respect to i and N. By using the improved convergence (1.16) (instead of (1.13)) and some quite technical transformations, we get the uniform estimate (1.9), which implies Theorem 1.3.1.

Thus, we have established the $W^{1,2}$ regularity of stresses $\sigma(t)$. Another interesting problem to investigate concerns possible regularity for displacements $u(t)$, similar to the static case of the Hencky plasticity (see [FS00] and [Ser85]). In Section 4.10 we present two counterexamples with C^∞ data, where the quasistatic evolution has a unique solution $(u(t),e(t),p(t))$. In the first one the displacement $u(t)$ develops a jump after a prescribed time t^*. In the second one we fix a singular diffuse measure μ and arrange the data such that $\varepsilon^s(u(t^*))=\mu$ for some time t^*. These examples show that one cannot expect that $u(t)\in W^{1,2}_{loc}(\Omega;\mathbb{R}^n)$ for every t, even with C^∞ data.

1.3.2 Regularity of bending moments for quasistatic evolution problems for perfect elasto-plastic plates

Given a scalar valued function $f(t,x)$ defined for $t\in[0,T]$ and $x\in\Omega$, which represents the transversal body force, the strong formulation of the evolution problem consists in finding a scalar valued function $u(t,x)$ (the vertical displacement) and three matrix-valued functions $e(t,x)$, $p(t,x)$ and $M(t,x)$ (the elastic and plastic curvatures and the bending moments) such that for every $t\in[0,T]$, for every $x\in\Omega$ the following conditions hold:

1. kinematic admissibility: $D^2u(t,x)=e(t,x)+p(t,x)$ in Ω, $u(t,x)=0$, $\frac{\partial u}{\partial\nu}(t,x)=0$ on $\partial\Omega$
2. constitutive equation: $M(t,x)=\mathbb{C}\,e(t,x)$,
3. equilibrium: $\operatorname{div}\operatorname{div} M(t,x)=f(t,x)$ in Ω,
4. moment constraint: $M(t,x)\in\mathbb{K}$,
5. associative flow rule: $\dot p(t,x)\in N_{\mathbb{K}}(M(t,x))$,

where $\nu(x)$ is the outer unit normal to $\partial\Omega$ and $\mathbb{C}$ is the rigidity tensor. The symbol $N_{\mathbb{K}}(\xi)$ denotes the normal cone to the set $\mathbb{K}$ at the point ξ in the sense of convex analysis. The problem is supplemented by initial conditions at time $t=0$.

The boundary conditions $u=0$ and $\frac{\partial u}{\partial\nu}=0$ on $\partial\Omega$ reflect the mechanical assumption that the plate is clamped.

For the regularity we restrict ourselves to the isotropic case where $\mathbb{K}$ is a *ball centered at the origin*, and $\mathbb{A}$ is the *multiple of the identity tensor* $\mathbb{I}$, which can be reduced to considering

$$\mathbb{K}=B_1(0),\quad \mathbb{C}=\mathbb{I}.$$

Under some technical conditions on the volume force and the safe-load condition, described in detail in Section 4.2, the regularity result is expressed by the following Theorem.

Theorem 1.3.2 (Theorem 5.2.2)**.** *Suppose that $\mathbb{C}$ is a multiple of the identity tensor, the set $\mathbb{K}$ is a ball, centered at the origin, and the assumptions* (5.3), (5.4) *are satisfied. Then for the solution (u, e, p) of the quasistatic problem (see Definition 5.3.4) we have*

$$M \in L^\infty([0,T]; W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym})),$$

with $M(t,x) = \mathbb{C}\, e(t,x)$.

We use the construction of piecewise constant approximations, obtained by discretizing the time interval $[0,T]$ (as outlined in Section 1.2.2). The details are presented in Section 5.3.

To prove the regularity Theorem, we should obtain uniform estimate (1.9) for the solutions $(u_i^N, e_i^N, p_i^N, M_i^N)$ of the incremental problems (1.3). Fixed N, for every $i = 1, \dots, N$ we consider a certain minimax program similar to that, proposed in [Ser93a] for the static problem for perfect elastoplastic plates (see Section 5.4 for details). We define a Lagrangian $L(\cdot\,,\cdot)$, and consider the following problem

$$\text{find a pair } (\delta u_i^N, M_i^N), \text{ such that}$$
$$L_i^N(\delta u_i^N, m) \le L_i^N(\delta u_i^N, M_i^N) \le L_i^N(v, M_i^N) \text{ for all admissible } (v, m),$$

such that its saddle points $(\delta u_i^N, M_i^N)$ give rise to solutions of the corresponding incremental problems (1.3). Note that, since we work in nonreflexive spaces, we actually consider a relaxed version of this problem, working with an extended Lagrangian $\widetilde{L}_i^N$, rather than L_i^N.

As usual, saddle points $(\delta u_i^N, M_i^N)$ of $\widetilde{L}_i^N$ solve the corresponding primal and dual problems, respectively. The primal problem takes the form

$$\text{find } \delta u_i^N, \text{ such that } I_i^N(\delta u_i^N) = \inf_v I_i^N(v), \quad \text{with } I_i^N(v) := \sup_m \widetilde{L}_i^N(v, m), \tag{1.17}$$

while the dual one is

$$\text{find } M_i^N, \text{ such that } R_i^N(M_i^N) = \sup_\tau R_i^N(m), \quad \text{with } R_i^N(m) := \inf_v \widetilde{L}_i^N(v, m).$$

The main difficulty is that the functional I_i^N in (1.17) has a linear growth with respect to $D^2 v$, and thus has a minimum in the space BH only. To attain a better regularity, for every $\alpha \in (0,1)$ we consider the regularized problem of the form

$$\min_v \Big\{ \frac{\alpha}{2} \|D^2 v\|^2_{2;\Omega} + I_i^N(v) \Big\},$$

which has a solution u_i^α in space $W^{1,2}(\Omega)$.

By introducing auxiliary functions M_i^α, defined by adding some correction terms to the functions

$$\alpha\, D^2 u_i^\alpha + \partial I_i^N(u_i^\alpha),$$

we rewrite the Euler equation as

$$\operatorname{div}\operatorname{div} M_i^\alpha = f_i^N. \tag{1.18}$$

Then we establish the following convergence properties

$$\begin{aligned} u_i^\alpha &\overset{*}{\rightharpoonup} \delta u_i^N \quad \text{weakly}^* \text{ in } BH(\Omega), \\ M_i^\alpha &\rightharpoonup M_i^N \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}), \end{aligned} \tag{1.19}$$

as $\alpha \to 0$.

By local regularity results we have that

$$u_i^\alpha \in W^{3,2}_{loc}(\Omega),$$

that is

$$M_i^\alpha \in W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym}). \tag{1.20}$$

Using the Euler equation (1.18) and (1.20) we then prove the estimate

$$\sup_{\alpha \in (0,1)} \|M_i^\alpha\|_{1,2;\Omega'} \leq C(i, N; \Omega') \tag{1.21}$$

for every subdomain $\Omega' \subset\subset \Omega$. Thus, (1.21) and the convergence (1.20) immediately imply that

$$M_i^\alpha \rightharpoonup M_i^N \quad \text{as } \alpha \to 0 \text{ weakly in } W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym}), \tag{1.22}$$

and

$$\|M_i^N\|_{1,2;\Omega'} \leq C(i, N; \Omega').$$

It remains to make the last estimate uniform with respect to i and N. By using the improved convergence (1.22) (instead of (1.19)) and some quite technical transformations, we get the uniform estimate (1.9), which implies Theorem 1.3.2.

Chapter 2

Quasistatic evolution problems for pressure-sensitive plastic materials

2.1 Introduction

Several materials of interest for applications, such as concrete, granular media, metallic foams, and porous metals, exhibit a pressure-sensitive yield behavior. There exists a large literature focusing on yield criteria for these materials, which identify the onset of irreversible inelastic behavior with the fact that a suitable measure of the state of internal stress reaches a threshold. Examples include the Gurson criterion for porous ductile materials [Gur77], the criterion of Ottosen for concrete [Ott77], the Desphande-Fleck criterion for metallic foams [DF00], and, for soils, Cam-Clay and the many subsequent variants (see, e.g., [DT05] and the references quoted therein). These criteria and several others are discussed in detail in [BP04].

Following the engineering literature, we work for simplicity in the framework of associative elasto-plasticity. Moreover we limit our analysis to the case of no hardening (perfect plasticity). With reference to a domain $\Omega \subset \mathbb{R}^n$, the problem can be formulated as follows. The linearized strain Eu, defined as the symmetric part of the spatial gradient of the displacement u, is decomposed as the sum $Eu = e + p$, where e and p are the elastic and plastic strains. The stress σ is determined only by e, through the formula $\sigma = \mathbb{C}e$, where $\mathbb{C}$ is the elasticity tensor. It is constrained to lie in a prescribed convex subset $\mathbb{K}$ of the space $\mathbb{M}^{n\times n}_{sym}$ of $n{\times}n$ symmetric matrices, whose boundary $\partial\mathbb{K}$ is referred to as the yield surface. In this context, pressure sensitivity of the yield criterion leads to the hypothesis that $\mathbb{K}$ is bounded.

The data of our problem are a time-dependent body force $f(t,x)$, defined for $t \in [0,T]$ and $x \in \Omega$, a time-dependent surface force $g(t,x)$ acting on a portion Γ_1 of the boundary $\partial\Omega$, and a time-dependent displacement prescribed on the complementary porion Γ_0 of $\partial\Omega$. The classical formulation of the quasistatic evolution problem consists in finding functions $u(t,x)$, $e(t,x)$, $p(t,x)$, $\sigma(t,x)$ satisfying the following conditions for every $t \in [0,T]$ and every $x \in \Omega$:

$$\begin{gathered}\text{additive decomposition: } Eu(t,x) = e(t,x) + p(t,x)\,,\\ \text{constitutive equation: } \sigma(t,x) = \mathbb{C}e(t,x)\,,\\ \text{equilibrium: } -\operatorname{div}\sigma(t,x) = f(t,x)\,,\\ \text{associative flow rule: } \dot p(t,x) \in N_{\mathbb{K}}(\sigma(t,x))\,,\end{gathered} \tag{2.1}$$

where $N_{\mathbb{K}}(\xi)$ is the normal cone to $\mathbb{K}$ at ξ. The problem is supplemented by initial conditions at time $t = 0$, by displacement boundary conditions $u(t,x) = w(t,x)$ for $t \in [0,T]$ and $x \in \Gamma_0$, and traction boundary conditions $\sigma(t,x)\nu(x) = g(t,x)$ for $t \in [0,T]$ and $x \in \Gamma_1$, where $\nu(x)$ is the outer unit normal to $\partial\Omega$.

In recent work [DDM06], a similar problem was considered for the pressure-insensitive case where $\mathbb{K}$ is a cylinder in $\mathbb{M}^{n\times n}_{sym}$ containing all scalar multiples of the identity matrix. There, the existence of a suitably defined weak solution was obtained by time-discretization. According to a general energy approach, see e.g. [Mie02], the discrete time formulation consists in solving a chain of incremental minimum problems which are quadratic in e and have linear growth in p. Thus, the direct methods of the calculus of variations lead to a weak formulation with $u \in BD(\Omega)$, the space of functions with bounded deformation, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, and $p \in M_b(\Omega \cup \Gamma_0, \mathbb{M}^{n\times n}_{sym})$, the space of bounded Radon measures on $\Omega \cup \Gamma_0$ with values in $\mathbb{M}^{n\times n}_{sym}$.

Notice that allowing for measure-valued plastic strains is also natural from the point of view of mechanics, see [Suq81]: indeed, localization of plastic deformation and formation of shear bands are often observed experimentally in the materials to which the models we analyze should apply.

In this work, we extend this approach to the case where $\mathbb{K}$ is an arbitrary convex bounded subset of $\mathbb{M}^{n\times n}_{sym}$ with nonempty interior. To adapt the technique to the new situation, we have to introduce a suitable duality product $\langle \sigma, p \rangle$, between stress and plastic strain, defined for every $\sigma \in L^\infty(\Omega; \mathbb{M}^{n\times n}_{sym})$ with $\operatorname{div}\sigma \in L^n(\Omega; \mathbb{R}^n)$ and for every $p \in M_b(\Omega \cup \Gamma_0, \mathbb{M}^{n\times n}_{sym})$ of the form $p = Eu - e$ with $u \in BD(\Omega)$ and $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. This is done in Section 2.3, using results from [KT83, Anz83].

After the properties of this duality have been established, we follow the lines of the proof of [DDM06], and obtain, under suitable hypotheses on the data f, g, and w, an existence result (Theorem 2.4.3) for a weak formulation (Definition 2.4.1) of problem (2.1), with $u \in AC([0,T]; BD(\Omega))$, $e \in AC([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$, and $p \in AC([0,T]; M_b(\Omega \cup \Gamma_0, \mathbb{M}^{n\times n}_{sym}))$. Moreover, we prove that e, and hence σ, are uniquely determined by the initial conditions.

We emphasize that our results are obtained under very general qualitative hypotheses on the yield surfaces $\partial\mathbb{K}$ and on the elasticity tensor $\mathbb{C}$. Namely, we just assume that $\mathbb{K}$ is a convex, bounded set with nonempty interior, and that $\mathbb{C}$, regarded as a linear operator acting on $\mathbb{M}^{n\times n}_{sym}$ is symmetric and positive definite. In particular no assumption of isotropy is required.

2.2 Preliminaries

2.2.1 Mathematical preliminaries

Given a locally compact subset X of $\mathbb{R}^n$ and a finite dimensional Hilbert space Ξ, the space of bounded Ξ-valued Borel measures on X is denoted by $M_b(X;\Xi)$ and is endowed with the norm $\|\mu\|_1 := |\mu|(X)$, where $|\mu| \in M_b(X;\mathbb{R})$ is the variation of the measure μ. By Riesz representation theorem (see, e.g., [Rud66, Theorem 6.19]) $M_b(X;\Xi)$ can be identified with the dual of $C_0(X;\Xi)$, the space of continuous functions $\varphi\colon X \to \Xi$ such that $\{|\varphi| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$. This defines the weak* topology in $M_b(X;\Xi)$.

For every $\mu \in M_b(X;\Xi)$ we consider the Lebesgue decomposition $\mu = \mu^a + \mu^s$, where μ^a is absolutely continuous and μ^s is singular with respect to Lebesgue measure $\mathcal{L}^n$. The space $L^1(X;\Xi)$ of Ξ-valued $\mathcal{L}^n$-integrable functions is regarded as a subspace of $M_b(X;\Xi)$, with the induced norm. When $\Xi = \mathbb{R}$, the indication of the space Ξ is omitted.

The L^p norm, $1 \leq p \leq \infty$, is denoted by $\|\cdot\|_p$. The brackets $\langle\cdot|\cdot\rangle$ denote the duality product between conjugate L^p spaces, as well as between other pairs of spaces, according to the context.

The space of *symmetric* $n\times n$ *matrices* is denoted by $\mathbb{M}^{n\times n}_{sym}$; it is endowed with the euclidean scalar product $\xi:\zeta := \mathrm{tr}(\xi\zeta) = \sum_{ij}\xi_{ij}\zeta_{ij}$ and with the corresponding euclidean norm $|\xi| := (\xi:\xi)^{1/2}$. The *symmetrized tensor product* $a\odot b$ of two vectors a, $b\in\mathbb{R}^n$ is the symmetric matrix with entries $(a_ib_j+a_jb_i)/2$.

For every $u\in L^1(U;\mathbb{R}^n)$, with U open in $\mathbb{R}^n$, let Eu be the $\mathbb{M}^{n\times n}_{sym}$-valued distribution on U, whose components are defined by $E_{ij}u = \frac{1}{2}(D_ju_i + D_iu_j)$. The space $BD(U)$ of functions with *bounded deformation* is the space of all $u\in L^1(U;\mathbb{R}^n)$ such that $Eu\in M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BD(U)$ is a Banach space with the norm $\|u\|_1 + \|Eu\|_1$. It is possible to prove that $BD(U)$ is the dual of a normed space (see [MSE79] and [TS80]), and this defines the weak* topology of $BD(U)$. A sequence u_k converges to u weakly* in $BD(U)$ if and only if $u_k\rightharpoonup u$ weakly in $L^1(U;\mathbb{R}^n)$ and $Eu_k\overset{*}{\rightharpoonup} Eu$ weakly* in $M_b(U;\mathbb{M}^{n\times n}_{sym})$. For the general properties of $BD(U)$ we refer to [Tem85].

In our problem $u\in BD(U)$ represents the *displacement* of an elasto-plastic body and Eu is the corresponding linearized *strain*.

We recall that a function f from $[0,T]$ into a Banach space Y is said to be absolutely continuous if for every $\varepsilon>0$ there exists $\delta>0$ such that $\sum_i\|f(t_i)-f(s_i)\|_Y<\varepsilon$, whenever $\sum_i(t_i-s_i)<\delta$ and $0\le s_1<t_1\le s_2<t_2\le\cdots\le s_k<t_k\le T$. The space of these functions is denoted by $AC([0,T];Y)$. For the general properties of absolutely continuous functions with values in reflexive Banach spaces we refer to [Bre, Appendix]. When Y is the dual of a separable Banach space, one can prove (see [DDM06, Theorem 7.1]) that for a.e. $t\in[0,T]$ there exists the weak*-limit

$$\dot f(t) := w^*\text{-}\lim_{s\to t}\frac{f(s)-f(t)}{s-t}.$$

Note that in this general situation it may happen that $\dot f$ is not Bochner integrable.

2.2.2 Mechanical preliminaries

The reference configuration. Throughout the paper the *reference configuration* Ω is a *bounded connected open set* in $\mathbb{R}^n$, with *Lipschitz boundary* $\partial\Omega=\Gamma_0\cup\Gamma_1$. We assume that $\Gamma_0\neq\emptyset$, Γ_1 is closed, and $\Gamma_0\cap\Gamma_1=\emptyset$.

The constraint and its support function. The *constraint on the stress* is given by a closed convex set $\mathbb{K}\subset\mathbb{M}^{n\times n}_{sym}$ with nonempty interior. Its boundary $\partial\mathbb{K}$ plays the role of *yield suface*. For the energy formulation of problem (2.1) it is convenient to introduce the support function $H:\mathbb{M}^{n\times n}_{sym}\to\mathbb{R}$ of $\mathbb{K}$ defined by

$$H(\xi) = \sup_{\zeta\in\mathbb{K}}\xi:\zeta\,. \tag{2.2}$$

H is convex and positively homogeneous of degree one.

For every $\mu\in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$, let $\mu/|\mu|$ be the Radon-Nikodym derivative of μ with respect to its total variation $|\mu|$. According to the general theory of convex functions of measures, we introduce the nonnegative Radon measure $H(\mu)\in M_b(\Omega\cup\Gamma_0)$ defined by

$$H(\mu)(B) = \int_B H\left(\frac{\mu}{|\mu|}\right)d|\mu| \tag{2.3}$$

for every Borel set $B\subset\Omega\cup\Gamma_0$. Finally we consider the functional $\mathcal{H}\colon M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})\to\mathbb{R}$ defined by

$$\mathcal{H}(\mu) := H(\mu)(\Omega\cup\Gamma_0)\,. \tag{2.4}$$

We refer to [GS64] and [Tem85, Chapter II, Section 4] for the properties of $H(\mu)$ and $\mathcal{H}(\mu)$.

The data of the problem. Let us fix a time interval $[0,T]$. We assume that the body force f, the surface force g and the prescribed boundary displacement w satisfy the following assumptions:

$$\begin{aligned} f &\in AC([0,T];L^n(\Omega;\mathbb{R}^n)),\\ g &\in AC([0,T];L^\infty(\Gamma_1;\mathbb{R}^n)),\\ w &\in AC([0,T];H^1(\Omega;\mathbb{R}^n)). \end{aligned} \tag{2.5}$$

Stress and strain. For a given displacement $u \in BD(\Omega)$ and a boundary datum $w \in H^1(\Omega;\mathbb{R}^n)$, the elastic and plastic strains $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$ satisfy the relation

$$Eu = e + p \text{ in } \Omega, \tag{2.6}$$

$$p = (w-u)\odot\nu\,\mathcal{H}^{n-1} \text{ on } \Gamma_0, \tag{2.7}$$

so that $e = E^a u - p^a$ a.e. on Ω and $p^s = E^s u$ on Ω. The stress $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ is defined by

$$\sigma := \mathbb{C}e. \tag{2.8}$$

The stored elastic energy $\mathcal{Q} : L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \to \mathbb{R}$ is given by

$$\mathcal{Q}(e) = \frac{1}{2}\int_\Omega \mathbb{C}e : e\,dx = \frac{1}{2}\int_\Omega \sigma : e\,dx.$$

For a $w \in H^1(\Omega;\mathbb{R}^n)$, the set of admissible displacements for the boundary datum w on Γ_0 is denoted by $A(w)$ and it is defined as:

$$A(w) := \Big\{(u,e,p) \in BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym}) : \ (2.6),\ (2.7) \text{ hold}\Big\}. \tag{2.9}$$

The space $\Pi_{\Gamma_0}(\Omega)$ of admissible plastic strains is the set of all $p \in M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym})$ for which there exist $u \in BD(\Omega), w \in H^1(\Omega,\mathbb{R}^n)$ and $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, such that $(u,e,p) \in A(w)$.

The following lemma, that can be proved as in [DDM06, Lemma 2.1] shows, that the multi-valued map $w \mapsto A(w)$ is closed.

Lemma 2.2.1. *Let $w_k \in H^1(\Omega;\mathbb{R}^n)$ and let $(u_k,e_k,p_k) \in A(w_k)$. If*

$$u_k \overset{*}{\rightharpoonup} u_\infty \ \textit{weakly}^* \textit{ in } BD(\Omega), \quad e_k \rightharpoonup e_\infty \ \textit{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}),$$

$$p_k \overset{*}{\rightharpoonup} p_\infty \ \textit{weakly}^* \textit{ in } M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym}), \quad w_k \rightharpoonup w_\infty \ \textit{weakly in } H^1(\mathbb{R}^n;\mathbb{R}^n),$$

then $(u_\infty,e_\infty,p_\infty) \in A(w_\infty)$.

The traces of the stress. If $\sigma \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n)$, then one can define a distribution $[\sigma\nu]$ on $\partial\Omega$ by

$$\langle[\sigma\nu]|\psi\rangle = \int_\Omega \operatorname{div}\sigma\cdot\psi\,dx + \int_\Omega \sigma : E\psi\,dx \tag{2.10}$$

for $\psi \in W^{1,1}(\Omega;\mathbb{R}^n)$. By Gagliardo's extension result [Gag05, Theorem 1.II], it is easy to see that $[\sigma\nu] \in L^\infty(\partial\Omega;\mathbb{R}^n)$ and that

$$[\sigma_k\nu] \xrightharpoonup{*} [\sigma\nu] \quad \text{weakly}^* \text{ in } L^\infty(\partial\Omega;\mathbb{R}^n), \tag{2.11}$$

whenever $\sigma_k \xrightharpoonup{*} \sigma$ weakly* in $L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^n(\Omega;\mathbb{R}^n)$.

Uniform safe-load condition. We assume that there exist a function ϱ in the space $AC([0,T];L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}))$ and a compact set $\mathbb{K}_0 \subset \operatorname{int}\mathbb{K}$, such that for every $t \in [0,T]$

$$\operatorname{div}\,\varrho(t) = -f(t) \text{ in } \Omega, \quad [\varrho(t)\nu] = g(t) \text{ on } \Gamma_1, \quad \varrho(t,x) \in \mathbb{K}_0 \text{ in } \Omega. \tag{2.12}$$

2.3 Stress-strain duality

In this section we develop the notion of duality between the stress and the plastic part of the strain. We begin with the definition and properties of the duality between stress and strain in the spirit of [KT83], where only the deviatoric part of the stress is bounded, and [Anz83], where a similar problem is studied in $BV(\Omega)$.

In the sequel we will make use of the following space

$$\Sigma(\Omega) = \{\sigma \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}) :\ \operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n)\}.$$

2.3.1 Duality between stress and strain

For every $u \in BD(\Omega)$ and $\sigma \in \Sigma(\Omega)$ we can define a distribution $[\sigma : Eu]$ on Ω by

$$\langle[\sigma : Eu]|\varphi\rangle = -\int_\Omega \varphi\, u \cdot \operatorname{div}\sigma\, dx - \int_\Omega \sigma : (u \odot \nabla\varphi)\, dx \tag{2.13}$$

for every $\varphi \in C^\infty_c(\Omega)$. Arguing as in [KT83, Theorem 3.2] one can prove that the distribution $[\sigma : Eu]$ is a bounded measure on Ω and its variation satisfies

$$|[\sigma : Eu]| \le \|\sigma\|_\infty\, |Eu| \quad \text{in } M_b(\Omega). \tag{2.14}$$

Moreover [Anz84, Corollary 3.2], with obvious changes, implies that

$$[\sigma : Eu]^a = \sigma : E^a u \quad \text{a.e. in } \Omega. \tag{2.15}$$

From the definition (2.13) it follows that

$$[\psi\,\sigma : Eu] = \psi[\sigma : Eu] \quad \text{in } M_b(\Omega) \tag{2.16}$$

for every $\psi \in C^1(\overline{\Omega})$.

We define the measure $[\sigma : E^s u]$ on Ω by putting

$$[\sigma : E^s u] := [\sigma : Eu]^s = [\sigma : Eu] - \sigma : E^a u. \tag{2.17}$$

Inequality (2.14) yields

$$|[\sigma : E^s u]| \le \|\sigma\|_\infty\, |E^s u| \quad \text{in } M_b(\Omega). \tag{2.18}$$

Remark 2.3.1. This inequality implies that $[\sigma_1 : E^s u_1] = [\sigma_2 : E^s u_2]$ in $M_b(\Omega)$ whenever $\sigma_1 = \sigma_2$ a.e. in Ω and $E^s u_1 = E^s u_2$.

As in [KT83, Theorem 3.2] one can prove the following stability property: if

$$\sigma_k \overset{*}{\rightharpoonup} \sigma \quad \text{weakly}^* \text{ in } L^\infty(\Omega; \mathbb{M}^{n\times n}_{sym}),$$

$$\operatorname{div} \sigma_k \rightharpoonup \operatorname{div} \sigma \quad \text{weakly in } L^n(\Omega; \mathbb{R}^n),$$

then for every $u \in BD(\Omega)$

$$[\sigma_k : Eu] \overset{*}{\rightharpoonup} [\sigma : Eu] \quad \text{and} \quad [\sigma_k : E^s u] \overset{*}{\rightharpoonup} [\sigma : E^s u] \quad \text{weakly}^* \text{ in } (C_b(\Omega))'$$

that is, for each bounded continuous function $\varphi\colon \Omega \to \mathbb{R}$ one has

$$\int_\Omega \varphi \, d[\sigma_k : Eu] \to \int_\Omega \varphi \, d[\sigma : Eu], \qquad \int_\Omega \varphi \, d[\sigma_k : E^s u] \to \int_\Omega \varphi \, d[\sigma : E^s u]. \tag{2.19}$$

2.3.2 Duality between stress and plastic strain

Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$, fix $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $w \in H^1(\Omega; \mathbb{R}^n)$, satisfying (2.6) and (2.7). Then we define a measure $[\sigma : p] \in M_b(\Omega \cup \Gamma_0)$ by setting

$$[\sigma : p] := \sigma : p^a + [\sigma : E^s u] = [\sigma : Eu] - \sigma : e \quad \text{on } \Omega,$$

$$[\sigma : p] := [\sigma\nu] \cdot (w - u)\mathcal{H}^{n-1} \quad \text{on } \Gamma_0,$$

so that

$$\int_{\Omega\cup\Gamma_0} \varphi \, d[\sigma : p] = \int_\Omega \varphi \, d[\sigma : Eu] - \int_\Omega \varphi \, \sigma : e \, dx + \int_{\Gamma_0} \varphi \, [\sigma\nu] \cdot (w - u) \, d\mathcal{H}^{n-1} \tag{2.20}$$

for every $\varphi \in C_b(\Omega \cup \Gamma_0)$, the space of bounded continuous functions on $\Omega \cup \Gamma_0$. In this case Remark 2.3.1 shows that the measure $[\sigma : p]$ is well defined, that is, it does not depend upon the particular choice of u, e and w.

It follows from the definition that

$$[\sigma : p]^a = \sigma : p^a \quad \text{a.e. on } \Omega, \qquad [\sigma : p]^s = [\sigma : E^s u] \quad \text{in } M_b(\Omega)$$

and

$$|[\sigma : p]| \le \|\sigma\|_\infty |p| \quad \text{in } M_b(\Omega \cup \Gamma_0), \qquad |[\sigma : p]^s| \le \|\sigma\|_\infty |p^s| \quad \text{in } M_b(\Omega \cup \Gamma_0). \tag{2.21}$$

Moreover (2.16) implies that

$$[\psi\sigma : p] = \psi[\sigma : p] \quad \text{in } M_b(\Omega \cup \Gamma_0)$$

for every $\psi \in C^1(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ and using the definitions one can deduce that

$$\int_{\Omega\cup\Gamma_0} \varphi \, d[\sigma : p] = \int_{\Omega\cup\Gamma_0} \varphi \, \sigma \, dp \tag{2.22}$$

for every $\sigma \in C^1(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ and every $\varphi \in C^1(\overline{\Omega})$. By (2.21) we deduce, that (2.22) holds for all $\sigma \in C(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ and $\varphi \in C(\overline{\Omega})$. Therefore for every $\sigma \in C(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ and $p \in \Pi_{\Gamma_0}(\Omega)$ we have

$$[\sigma : p] = \sigma : p \quad \text{in } M_b(\Omega \cup \Gamma_0),$$

where the right-hand side denotes the measure defined by

$$(\sigma : p)(B) = \int_B \sigma_{ij}\, dp_{ij}$$

for every Borel set $B \subset \Omega \cup \Gamma_0$.

Also it is easy to see that the relation

$$[\sigma : p] = (\sigma : p) \cdot \mathcal{L}^n \quad \text{in } M_b(\Omega)$$

holds in the case $\sigma,\, p \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$.

It follows from the definition and from (2.11) and (2.19) that

$$[\sigma_k : p] \overset{*}{\rightharpoonup} [\sigma : p] \quad \text{weakly}^* \text{ in } (C_b(\Omega \cup \Gamma_0))' \tag{2.23}$$

whenever $\sigma_k \overset{*}{\rightharpoonup} \sigma$ weakly* in $L^\infty(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $\operatorname{div}\sigma_k \rightharpoonup \operatorname{div}\sigma$ weakly in $L^n(\Omega; \mathbb{R}^n)$.

Finally, for every $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$, we define

$$\begin{aligned}\langle \sigma | p \rangle_{\Sigma,\Pi} :&= [\sigma : p](\Omega \cup \Gamma_0) = \\ &= \int_\Omega \sigma : p^a\, dx + [\sigma : E^s u](\Omega) + \int_{\Gamma_0} [\sigma\nu] \cdot (w - u)\, d\mathcal{H}^{n-1} = \\ &= [\sigma : Eu](\Omega) - \int_\Omega \sigma : e\, dx + \int_{\Gamma_0} [\sigma\nu] \cdot (w - u)\, d\mathcal{H}^{n-1}.\end{aligned}$$

where $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $w \in H^1(\Omega; \mathbb{R}^n)$ satisfy (2.6) and (2.7).

Let us now prove the integration by parts formula for stresses and displacements:

Proposition 2.3.2. *Let $\sigma \in \Sigma(\Omega)$, $w \in H^1(\Omega; \mathbb{R}^n)$, $f \in L^n(\Omega; \mathbb{R}^n)$, $g \in L^\infty(\Gamma_1; \mathbb{R}^n)$ and let $(u, e, p) \in A(w)$. Assume that $-\operatorname{div}\sigma = f$ in Ω and $[\sigma\nu] = g$ on Γ_1. Then*

$$\begin{aligned}\int_{\Omega\cup\Gamma_0} \varphi\, d[\sigma : p] + \int_\Omega \varphi\, \sigma : (e - Ew)\, dx + \\ + \int_\Omega \sigma : \Big((u - w) \odot \nabla\varphi\Big)\, dx = \\ = \int_\Omega \varphi\, f \cdot (u - w)\, dx + \int_{\Gamma_1} \varphi\, g \cdot (u - w)\, d\mathcal{H}^{n-1}\end{aligned} \tag{2.24}$$

for every $\varphi \in C^1(\overline{\Omega})$.

Proof: First, let us establish the following formula for $\sigma \in \Sigma(\Omega)$, $v \in BD(\Omega)$ and $\varphi \in C^1(\overline{\Omega})$:

$$\int_{\partial\Omega} \varphi\, [\sigma\nu] \cdot v\, d\mathcal{H}^{n-1} = \int_\Omega \varphi \operatorname{div}\sigma \cdot v\, dx + \int_\Omega \sigma : (v \odot \nabla\varphi)\, dx + \int_\Omega \varphi\, d[\sigma : Ev]. \tag{2.25}$$

Arguing as in [DDM06, Lemma 2.3] we can find a sequence σ_k in $C^\infty(\overline{\Omega})$, such that

$$\sigma_k \to \sigma \quad \text{strongly in } L^p(\Omega; \mathbb{M}^{n\times n}_{sym}), \quad \operatorname{div}\sigma_k \to \operatorname{div}\sigma \quad \text{strongly in } L^n(\Omega; \mathbb{R}^n)$$

for every $1 \leq p < \infty$. By the integration by parts formula for $BD(\Omega)$, formula (2.25) holds for every σ_k. The left-hand side converges to that of (2.25) by (2.11), while the convergence of the right-hand side follows from (2.19). This proves (2.25).

By the assumptions of the theorem, for $v = u - w \in BD(\Omega)$ formula (2.25) takes the form:

$$\begin{aligned} -\int_\Omega \varphi\, f\cdot(u-w)\,dx + \int_\Omega \sigma : \Big((u-w)\odot\nabla\varphi\Big)\,dx + \int_\Omega \varphi\, d[\sigma : E(u-w)] = \\ = \int_{\Gamma_0} \varphi\,[\sigma\nu]\cdot(u-w)\,d\mathcal{H}^{n-1} + \int_{\Gamma_1} \varphi\, g\cdot(u-w)\,d\mathcal{H}^{n-1}. \end{aligned} \tag{2.26}$$

On the other hand, (2.20) gives

$$\begin{aligned} \int_{\Omega\cup\Gamma_0} \varphi\, d[\sigma : p] + \int_\Omega \varphi\,\sigma : (e - Ew)\,dx + \int_\Omega \sigma : \Big((u-w)\odot\nabla\varphi\Big)\,dx = \\ = \int_\Omega \varphi\, d[\sigma : E(u-w)] + \int_\Omega \sigma : \Big((u-w)\odot\nabla\varphi\Big)\,dx - \int_{\Gamma_0} \varphi\,[\sigma\nu]\cdot(u-w)\,d\mathcal{H}^{n-1}. \end{aligned}$$

Thus, the last relation together with (2.26) yields (2.24). □

Let

$$\mathcal{K}(\Omega) := \{\sigma \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}) : \sigma(x) \in \mathbb{K} \text{ for a.e. } x \in \Omega\}.$$

The following proposition can pe proved as in [DDM06, Proposition 2.2].

Proposition 2.3.3. *Let $p \in \Pi_{\Gamma_0}(\Omega)$. Then*

$$H(p) \geq [\sigma : p] \quad \text{in } M_b(\Omega\cup\Gamma_0) \tag{2.27}$$

for every $\sigma \in \Sigma(\Omega)\cap\mathcal{K}(\Omega)$, and

$$\mathcal{H}(p) = \sup\{\langle\sigma|p\rangle \ : \ \sigma \in \Sigma(\Omega)\cap\mathcal{K}(\Omega)\}. \tag{2.28}$$

Moreover, if $g \in L^\infty(\Gamma_1;\mathbb{R}^n)$ and there exists $\varrho \in \Sigma(\Omega)\cap\mathcal{K}(\Omega)$ such that $[\varrho\nu] = g$ on Γ_1, then

$$\mathcal{H}(p) = \sup\{\langle\sigma|p\rangle \ : \ \sigma \in \Sigma(\Omega)\cap\mathcal{K}(\Omega),\ [\sigma\nu] = g \text{ on } \Gamma_1\}. \tag{2.29}$$

2.4 Quasistatic evolution

2.4.1 Definition and existence result

From assumptions (2.5) it follows, that for the functional $\mathcal{F}(t) \in BD(\Omega)'$, defined by

$$\langle\mathcal{F}(t)|u\rangle = \int_\Omega f(t)\,u\,dx + \int_{\Gamma_1} g(t)\,u, \tag{2.30}$$

the weak* limit

$$\dot{\mathcal{F}}(t) = w^*\text{-}\lim_{s\to t} \frac{\mathcal{F}(s)-\mathcal{F}(t)}{s-t}$$

exists in $BD(\Omega)'$ for a.e. $t \in [0,T]$, and that

$$\langle\dot{\mathcal{F}}(t)|u\rangle = \int_\Omega \dot{f}(t)\,u\,dx + \int_{\Gamma_1} \dot{g}(t)\,u. \tag{2.31}$$

Therefore the function $t \mapsto \langle \dot{\mathcal{F}}(t)|u(t)\rangle$ belongs to $L^1([0,T])$ whenever $t \mapsto u(t)$ is in $L^\infty([0,T]; BD(\Omega))$.

A function $p : [0,T] \to M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ will be regarded as a function defined on the time interval $[0,T]$ with values in the dual of the separable Banach space $C_0(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_{sym})$. Its variation $\mathcal{V}$ and $\mathcal{H}-$variation $\mathcal{D}_\mathcal{H}$ are defined as

$$\mathcal{V}(p;s,t) = \sup\left\{ \sum_{j=1}^{N} \|p(t_j) - p(t_{j-1})\|_1 \ : \ s = t_0 \le \cdots \le t_N = t,\ N \in \mathbb{N} \right\},$$

$$\mathcal{D}_\mathcal{H}(p;s,t) = \sup\left\{ \sum_{j=1}^{N} \mathcal{H}(p(t_j) - p(t_{j-1})) \ : \ s = t_0 \le \cdots \le t_N = t,\ N \in \mathbb{N} \right\}.$$

The notation $D_\mathcal{H}$ for the $\mathcal{H}$-variation is motivated by the more standard case in which the set $\mathbb{K}$ of admissible stresses contains the origin in its interior. In this case, $\mathcal{H}$ is positive and the $\mathcal{H}$-variation of p has the physical interpretation of plastic dissipation in the time interval (s,t).

Next we give a variational formulation of the quasistatic problem.

Definition 2.4.1. A quasistatic evolution is a function $t \mapsto (u(t), e(t), p(t))$ from $[0,T]$ into $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ which satisfies the following conditions:

(qs1) global stability: for every $t \in [0,T]$ we have $(u(t), e(t), p(t)) \in A(w(t))$ and

$$\mathcal{Q}(e(t)) - \langle \mathcal{F}(t)|u(t)\rangle \le \mathcal{Q}(\eta) + \mathcal{H}(q - p(t)) - \langle \mathcal{F}(t)|v\rangle \tag{2.32}$$

for every $(v, \eta, q) \in A(0)$.

(qs2) energy balance: the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ has bounded variation and for every $t \in [0,T]$

$$\begin{aligned} \mathcal{Q}(e(t)) + \mathcal{D}_\mathcal{H}(p;0,t) - \langle \mathcal{F}(t)|u(t)\rangle = \mathcal{Q}(e(0)) + \langle \mathcal{F}(0)|u(0)\rangle + \\ + \int_0^t \Big(\langle \sigma(s)|E\dot{w}(s)\rangle - \langle \mathcal{F}(s)|\dot{w}(s)\rangle - \langle \dot{\mathcal{F}}(s)|u(s)\rangle \Big)\, ds, \end{aligned} \tag{2.33}$$

where $\sigma(t) = \mathbb{C}e(t)$.

Remark 2.4.2. Since the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ has bounded variation, it is bounded and the set of its discontinuity points (in the strong topology) is at most countable. As the estimates of [DDM06, Theorem 3.8] are true also in this case, the same continuity property holds for $t \mapsto e(t)$ and $t \mapsto \sigma(t)$ from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and for $t \mapsto u(t)$ from $[0,T]$ into $BD(\Omega)$. Therefore

$$e,\ \sigma \in L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})) \quad \text{and } u \in L^\infty([0,T]; BD(\Omega)).$$

Finally, as $\dot{w} \in L^1([0,T]; W^{1,2}(\Omega; \mathbb{R}^n))$ and $E\dot{w} \in L^1([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$, the integral in the right-hand side of (2.33) is well-defined.

Theorem 2.4.3. *Assume* (2.5) *and* (2.12)*. If* $(u_0, e_0, p_0) \in A(w(0))$ *satisfy the stability condition*

$$\mathcal{Q}(e(0)) - \langle \mathcal{F}(0)|u_0\rangle \le \mathcal{Q}(e) + \mathcal{H}(p - p_0) - \langle \mathcal{F}(0)|u\rangle$$

for every $(u,e,p) \in A(w(0))$*, then there exists a quasistatic evolution* $t \mapsto (u(t), e(t), p(t))$ *such that* $u(0) = u_0,\ e(0) = e_0,\ p(0) = p_0$.

PROOF: The proof can be obtained by time discretization. For every $k \in \mathbb{N}$ we fix a subdivision $0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = T$, satisfying (4.11) of [DDM06]. At each time step we solve the incremental minimum problem (4.12) of [DDM06], adopting the definitions of $A(w)$ and $\mathcal{H}$ of the present paper. Then we define the piecewise constant interpolations $u_k(t)$, $e_k(t)$, $p_k(t)$, $\sigma_k(t)$ as in (4.15) of [DDM06], and we prove that for every $t \in [0,T]$ $u_k(t) \overset{*}{\rightharpoonup} u(t)$ weakly* in $BD(\Omega)$, $e_k(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $p_k(t) \overset{*}{\rightharpoonup} p(t)$ weakly* in $M_b(\Omega \cup \Gamma_0;\mathbb{M}^{n\times n}_D)$, where $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution.

The details can be recovered by repeating the arguments of [DDM06, Section 4], with obvious modifications due to the new definitions introduced in Section 3 of the present paper. □

The next theorem shows, that the convergence of elastic strains and stresses takes place in the strong topology of $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. See [DDM06, Theorem 4.8] for the proof.

Theorem 2.4.4. *Assume that*

$$p_k(t) \overset{*}{\rightharpoonup} p(t) \quad \textit{weakly}^* \textit{ in } M_b(\Omega \cup \Gamma_0;\mathbb{M}^{n\times n}_{sym}) \tag{2.34}$$

for every $t \in [0,T]$. *Then* $e_k(t) \to e(t)$ *and* $\sigma_k(t) \to \sigma(t)$ *strongly in* $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. *Moreover,*

$$\lim_k \sum_{0<t_k^r\le t} \Big\{\mathcal{H}(p_k(t_k^r) - p_k(t_k^{r-1})) - \langle \varrho(t_k^r) | p_k(t_k^r) - p_k(t_k^{r-1})\rangle\Big\} =$$
$$= \mathcal{D}_{\mathcal{H}}(p;0,t) - \langle \varrho(t)|p(t)\rangle + \langle \varrho(0)|p(0)\rangle + \int_0^t \langle \dot\varrho(s)|p(s)\rangle\, ds$$

for every $t \in [0,T]$.

2.4.2 Regularity and uniqueness

The next statement shows that the quasistatic evolution is absolutely continuous with respect to time. We refer to [DDM06, Theorem 5.2] for the proof.

Theorem 2.4.5. *Let* $t \mapsto (u(t), e(t), p(t))$ *be a quasistatic evolution. Then*

$$e \in AC([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \quad p \in AC([0,T];M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_{sym}), \quad u \in AC([0,T];BD(\Omega)).$$

Moreover, for a.e. $t \in [0,T]$

$$\|\dot e(t)\|_2 \le C_1(\|\dot\varrho(t)\|_\infty + \|E\dot w(t)\|_2), \tag{2.35}$$

$$\|\dot p(t)\|_1 \le C_2(\|\dot\varrho(t)\|_\infty + \|E\dot w(t)\|_2), \tag{2.36}$$

$$\|E\dot u(t)\|_1 \le C_1(\|\dot\varrho(t)\|_\infty + \|E\dot w(t)\|_2), \tag{2.37}$$

$$\|\dot u(t)\|_1 \le C_1(\|\dot\varrho(t)\|_\infty + \|E\dot w(t)\|_2 + \|\dot w(t)\|_2). \tag{2.38}$$

Remark 2.4.6. Assume that $u \in AC([0,T];BD(\Omega))$, $e \in AC([0,T];L^2(\Omega\mathbb{M}^{n\times n}_{sym}))$, and $p \in AC([0,T];M_b(\Omega\cup\Gamma_0);\mathbb{M}^{n\times n}_{sym})$. Assume that $(u(t), e(t), p(t)) \in A(w(t))$ for every $t \in [0,T]$. Then $(\dot u(t), \dot e(t), \dot p(t)) \in A(\dot w(t))$ for a.e. $t \in [0,T]$. Indeed, it is enough to apply Lemma 2.2.1 to the difference quotients.

As in [DDM06, Theorem 5.9] we can prove that $t \mapsto e(t)$ (and, consequently, $t \mapsto \sigma(t)$) is uniquely determined by its initial condition.

Theorem 2.4.7. *Let* $t \mapsto (u(t), e(t), p(t))$ *and* $t \mapsto (v(t), \eta(t), q(t))$ *be two quasistatic evolutions and let* $\sigma(t) := \mathbb{C}e(t)$ *and* $\tau(t) := \mathbb{C}\eta(t)$. *If* $e(0) = \eta(0)$, *then* $e(t) = \eta(t)$ *for every* $t \in [0,T]$. *Equivalently, if* $\sigma(0) = \tau(0)$, *then* $\sigma(t) = \tau(t)$ *for every* $t \in [0,T]$.

2.4.3 Equivalent formulations in rate form

Let $t \mapsto (u(t), e(t), p(t))$ be a quasistatic evolution. Suppose for now that $\dot{p}(t) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. In this section we want to prove that

$$\dot{p}(t,x) \in N_{\mathbb{K}}(\sigma(t,x)) \quad \text{for a.e.} x \in \Omega, \tag{2.39}$$

which represents the classical formulation of the flow rule. By the definition of $N_{\mathbb{K}}$ it is easy to see that (2.39) is equivalent to saying that

$$\langle \sigma(t) - \tau(t) | \dot{p}(t) \rangle \geq 0 \tag{2.40}$$

for every $\tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ with $[\tau\, \nu] = g(t)$ on Γ_1. Indeed, the implication (2.39)$\Rightarrow$(2.40) is straightforward, while the converse one is obtained by considering the test functions of the form $\tau = \varphi\xi + (1-\varphi)\sigma$, with a cut-off $\varphi \in C^\infty_c(\Omega)$, $0 \leq \varphi \leq 1$ and arbitrary $\xi \in \mathbb{K}$.

Note, that the variational inequality (2.40) makes sense even if one considers the duality between $\Sigma(\Omega)$ and $\Pi_{\Gamma_0}(\Omega)$, defined in Section 2.3, since $\dot{p}(t) \in \Pi_{\Gamma_0}(\Omega)$ by Remark 2.4.6. We will regard (2.40) as the weak formulation of the inclusion (2.39) when $\dot{p}(t) \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$.

The following theorem collects three different sets of conditions, including (2.40) and expressed in terms of the time derivatives $\dot{p}(t)$, $\dot{e}(t)$, and $\dot{u}(t)$, which are equivalent to the conditions considered in Definition 2.4.1. For its proof we refer to [DDM06, Theorem 6.1], with obvious modifications.

Theorem 2.4.8. *Let* $(u,e,p) : [0,T] \to BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ *and let* $\sigma(t) = \mathbb{C}e(t)$*. Then the following conditions are equivalent:*

(a) $t \mapsto (u(t), e(t), p(t))$ *is a quasistatic evolution;*

(b) $t \mapsto (u(t), e(t), p(t))$ *is absolutely continuous and*

  (b1) for every $t \in [0,T]$ *we have* $(u(t), e(t), p(t)) \in A(w(t))$, $\sigma(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\operatorname{div}\sigma(t) = f(t)$ *in* Ω, *and* $[\sigma(t)\,\nu] = g(t)$ *on* Γ_1,

  (b2) for a.e. $t \in [0,T]$ *we have*
$$\langle \sigma(t) - \tau | \dot{p}(t) \rangle \geq 0$$
  for every $\tau \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$ *with* $[\tau\, \nu] = g(t)$ *on* Γ_1;

(c) $t \mapsto (u(t), e(t), p(t))$ *is absolutely continuous and*

  (c1) for every $t \in [0,T]$ *we have* $(u(t), e(t), p(t)) \in A(w(t))$, $\sigma(t) \in \Sigma(\Omega) \cap \mathcal{K}(\Omega)$, $-\operatorname{div}\sigma(t) = f(t)$ *in* Ω, *and* $[\sigma(t)\,\nu] = g(t)$ *on* Γ_1,

  (c2) for a.e. $t \in [0,T]$ *we have*
$$\mathcal{H}(\dot{p}(t)) = \langle \sigma(t) | \dot{p}(t) \rangle;$$

Remark 2.4.9. By Proposition 2.3.3 the measure $H(\dot{p}(t)) - [\sigma(t) : \dot{p}(t)]$ is nonnegative on $\Omega \cup \Gamma_0$, so that (b2) implies that

$$H(\dot{p}(t)) = [\sigma(t) : \dot{p}(t)] \quad \text{on } \Omega \cup \Gamma_0. \tag{2.41}$$

Let us return to the classical formulation of the flow rule, which makes sense for $\dot{p}(t) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. It can be written equivalently in the form

$$\frac{\dot{p}(t,x)}{|\dot{p}(t,x)|} \in N_K(\sigma(t,x)) \quad \text{for } \mathcal{L}^n - \text{a.e. } x \in \{|\dot{p}(t)| > 0\}. \tag{2.42}$$

When $\dot{p}(t) \in M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$, we can consider the Radon-Nikodym derivative $\dot{p}(t)/|\dot{p}(t)|$ of $\dot{p}(t)$ with respect to its variation $|\dot{p}(t)|$, which is a function defined $|\dot{p}(t)|$-a.e. on $\Omega \cup \Gamma_0$.

We notice that

$$\frac{\dot{p}(t)}{|\dot{p}(t)|}(x) = \frac{\dot{p}(t,x)}{|\dot{p}(t,x)|} \quad \text{for } \mathcal{L}^n - \text{a.e. } x \in \{|\dot{p}(t)| > 0\}$$

when $p \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. Unfortunately, when $p \in M_b(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_{sym})$ one cannot prove the inclusion

$$\frac{\dot{p}(t)}{|\dot{p}(t)|}(x) \in N_K(\sigma(t,x)), \tag{2.43}$$

that is the natural generalization of (2.42), as a pointwise formulation of the flow rule, since its left-hand side is defined $|\dot{p}(t)|$-a.e. on $\Omega \cup \Gamma_0$, while its right-hand side is defined only $\mathcal{L}^n$-a.e. on Ω. In the following Theorem this difficulty is overcome by introducing a representative $\hat{\sigma}(t)$ of $\sigma(t)$, which is defined $\dot{p}(t)$-a.e. on $\Omega \cup \Gamma_0$. For the proof see [DDM06, Theorem 6.4].

Theorem 2.4.10. *Let $(u, e, p) : [0, T] \to BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_{sym})$, $\sigma(t) = \mathbb{C}e(t)$ and let $\mu(t) = \mathcal{L}^n + |\dot{p}(t)|$. Then $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution if and only if*

(d) $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous and

(d1) for every $t \in [0, T]$ we have $(u(t), e(t), p(t)) \in A(w(t))$, $\sigma(t) \in \Sigma(\Omega)\cup\mathcal{K}(\Omega)$, $-\operatorname{div}\sigma(t) = f(t)$ on Ω, and $[\sigma(t)\nu] = g(t)$ on Γ_1,

(d2) for a.e. $t \in [0, T]$ there exists $\hat{\sigma}(t) \in L^\infty_{\mu(t)}(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym})$ such that

$$\hat{\sigma}(t) = \sigma(t) \quad \mathcal{L}^n - a.e. \text{ on } \Omega, \tag{2.44}$$

$$[\sigma(t) : \dot{p}(t)] = \hat{\sigma}(t) : \frac{\dot{p}(t)}{|\dot{p}(t)|}|\dot{p}(t)| \quad \text{in } M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_{sym}), \tag{2.45}$$

$$\frac{\dot{p}(t)}{|\dot{p}(t)|}(x) \in N_{\mathbb{K}}(\hat{\sigma}(t,x)) \quad \text{for } |\dot{p}(t)| - a.e.\ x \in \Omega \cup \Gamma_0. \tag{2.46}$$

For every $r > 0$ and every $t \in [0, T]$ we consider the function $\sigma^r(t) \in C(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ defined by

$$\sigma^r(t,x) = \frac{1}{\mathcal{L}^n(B(x,r) \cap \Omega)} \int_{B(x,r)\cap\Omega} \sigma(t,y)\, dy. \tag{2.47}$$

When $\mathbb{K}$ is strictly convex, the previous result can be improved by making the definition of $\hat{\sigma}$ more precise. We refer to [DDM06, Theorem 6.6] for the proof.

Theorem 2.4.11. *Assume that $\mathbb{K}$ is strictly convex. Let $t \mapsto (u(t), e(t), p(t))$ be a quasistatic evolution, let $\mu(t) = \mathcal{L}^n + |\dot{p}(t)|$, let $\sigma(t) = \mathbb{C}e(t)$, and let $\sigma^r(t)$ be defined by* (2.47) *. Then $\sigma^r(t) \to \hat{\sigma}(t)$ strongly in $L^1_{\mu(t)}(\Omega; \mathbb{M}^{n\times n}_{sym})$ for a.e. $t \in [0, T]$, where $\hat{\sigma}(t)$ satisfies* (2.44)-(2.46).

Chapter 3

Quasistatic evolution in the theory of perfectly elasto-plastic plates: existence of a weak solution.

In this chapter we combine

3.1 Introduction

In this paper we study the quasistatic evolution of clamped perfectly elasto-plastic plates under the action of a time-dependent transversal body force. The reference configuration is a bounded open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary and the elastic domain $\mathbb{K}$ is a bounded closed convex subset of $\mathbb{M}^{2\times 2}_{sym}$ (the space of symmetric 2×2 matrices) with nonempty interior, whose boundary $\partial\mathbb{K}$ plays the role of the yield surface.

Given a scalar valued function $f(t,x)$ defined for $t \in [0,T]$ and $x \in \Omega$, which represents the transversal body force, the strong formulation of the evolution problem consists in finding a scalar valued function $u(t,x)$ (the vertical displacement) and three matrix-valued functions $e(t,x)$, $p(t,x)$ and $M(t,x)$ (the elastic and plastic curvatures and the bending moments) such that for every $t \in [0,T]$, for every $x \in \Omega$ the following conditions hold:

1. kinematic admissibility: $D^2u(t,x) = e(t,x) + p(t,x)$ in Ω,
   $u(t,x) = 0$, $\frac{\partial u}{\partial \nu}(t,x) = 0$ on $\partial\Omega$
2. constitutive equation: $M(t,x) = \mathbb{C}\, e(t,x)$,
3. equilibrium: $\operatorname{div}\operatorname{div} M(t,x) = f(t,x)$ in Ω,
4. moment constraint: $M(t,x) \in \mathbb{K}$,
5. associative flow rule: $\dot p(t,x) \in N_{\mathbb{K}}(M(t,x))$,

where $\nu(x)$ is the outer unit normal to $\partial\Omega$ and $\mathbb{C}$ is the rigidity tensor. The symbol $N_{\mathbb{K}}(\xi)$ denotes the normal cone to the set $\mathbb{K}$ at the point ξ in the sense of convex analysis. The problem is supplemented by initial conditions at time $t = 0$.

The boundary conditions $u = 0$ and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ reflect the mechanical assumption the plate is clamped.

The existence of weak solutions for variational problems in the theory of perfect plasticity was extensively studied during last decades (see, for example, [Anz84], [AG82], [DDM06], [Dem89], [FS00], [Ser94], [Suq81], [Tem85]). In this paper we develop an energy approach to the existence of weak solutions of this problem (Definition (3.4.1) below), which turns out to be particularly useful for studying their further differential properties (see Ref. [Dem08b]). The particular case of perfectly elasto-plastic plates has been studied by many authors, subject to various boundary conditions (see, for example, [BK00], [Dem83], [Tem85]). We examine here the quasistatic analogue of static problem, studied in [Dem83], [Ser93a], [Tem85].

The aim of this paper is to develop a new approach to the existence of weak solution to problem (1)-(5) (see Definition 3.4.1 below) in the spirit of the energy formulation of rate-independent problems, studied in [Mie02]. The advantage of this general approach is twofold. On the one hand it allows to obtain a weak formulation of the flow rule (5) in a measure-theoretic sense, on the other hand it is crucial in the proof of further differentiability properties of $M(t,x)$, that will be obtained in [Dem08a].

As usually in the energy approach (see [DDD07], [DDM06]) we obtain the existence of solutions by a time-discretization procedure: first, we consider a sequence of incremental minimum problems and show that an appropriately constructed sequence of piecewise-constant approximations has a bounded variation and satisfies the so-called discrete energy inequality. Then, by using a version of Helly theorem, we extract a converging subsequence, whose limit satisfies (2)-(4), a relaxed form of (1) and an an energy equality. These conditions are considered a weak formulation of the original problem.

By construction this weak solution has bounded variation with respect to time. the energy equality allows to prove that it is actually absolutely continuous.

At the end of the paper we follow the arguments of [DDM06] to investigate some fine pointwise properties of the tensor of moments M and we prove a weak formulation of the flow rule (5).

This paper is the first step of a program of proving higher differentiability of the tensor of moments M. In fact, the use of piecewise constant approximations constructed below will allow us to use the results of [FS00], [Ser93a] for proving that we have not only the regularity with respect to time (see Theorem (3.5.1) below)

$$M \in AC([0,T]; L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})),$$

but actually we are able to say something about the spatial properties of M. Precisely, the expected result is:

$$M \in L^\infty([0,T]; W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym}))$$

(see [Dem08b] for the analogous result in the case of Prandtl-Reuss perfect plasticity).

The paper is organized as follows: in Section 2 we state some preliminary notions and results. Every single incremental problem is considered in details in Section 3. In Section 4 the definition of a weak solution is given, and its existence is proved. Absolute continuity of weak solutions with respect to time and the uniqueness of the tensor of moments are obtained in Section 5. Equivalent definitions in rate form and some fine properties of the tensor of moments are discussed in Section 6.

3.2 Preliminaries

The set of admissible moments

Let $\mathbb{K}$ be a closed convex set in $\mathbb{M}^{2\times 2}_{sym}$, such that $B_{r_{\mathbb{K}}}(0) \subset \mathbb{K} \subset B_{R_{\mathbb{K}}}(0)$ for some positive constants $r_{\mathbb{K}}$ and $R_{\mathbb{K}}$. The set $\mathbb{K}$ plays the role of a constraint on the tensor of bending moments. The boundary $\partial\mathbb{K}$ is interpreted as the yield surface.
The set of admissible moments $\mathcal{K}(\Omega)$ is defined as

$$\mathcal{K}(\Omega) = \Big\{ M \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) : M(x) \in \mathbb{K} \quad \text{for a.e. } x \in \Omega \Big\}.$$

The support function $H : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{R}$ of $\mathbb{K}$ is given by

$$H(m) := \sup_{M \in \mathbb{K}} m : M$$

and satisfies

$$r_k |\xi| \leq H(\xi) \leq R_{\mathbb{K}} |\xi|, \quad \text{for all } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

For every $\mu \in M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ we introduce the nonnegative Radon measure $H(\mu) \in M_b(\overline{\Omega})$ defined by

$$H(\mu)(B) := \int_B H(\mu/|\mu|)\, d|\mu| \quad \text{for every Borel set } B \subset \overline{\Omega}.$$

We consider the convex functional $\mathcal{H}(\mu) : M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym}) \to \mathbb{R}$ defined by the formula

$$\mathcal{H}(\mu) := H(\mu)(\Omega) = \int_{\overline{\Omega}} H(\mu/|\mu|)\, d|\mu|.$$

As well-known (see, for example, [Tem85], Chapter II), $\mathcal{H}$ is lower semicontinuous on $M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ and the following holds

$$\mathcal{H}(\mu) = \sup \Big\{ \langle m, \mu \rangle : m \in C(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym}) \cap \mathcal{K}(\Omega) \Big\}. \tag{3.1}$$

The rigidity tensor

$\mathbb{C}$ is the rigidity tensor, considered as a symmetric positive definite linear operator $\mathbb{C} : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{M}^{2\times 2}_{sym}$. We introduce the quadratic form $Q : \mathbb{M}^{2\times 2}_{sym} \to [0, +\infty)$ of $\mathbb{C}$ by putting

$$Q(\xi) := \frac{1}{2} \mathbb{C}\xi : \xi.$$

Thus, there exist two constants $\alpha_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$ with $0 < \alpha_{\mathbb{C}} \leq \beta_{\mathbb{C}} < +\infty$, such that

$$\alpha_{\mathbb{C}} |\xi|^2 \leq Q(\xi) \leq \beta_{\mathbb{C}} |\xi|^2. \tag{3.2}$$

The stored elastic energy functional $\mathcal{Q} : L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \to \mathbb{R}$ is given by

$$\mathcal{Q}(e) = \int_\Omega Q(e)\, dx = \frac{1}{2} \int_\Omega \mathbb{C}e : e\, dx. \tag{3.3}$$

3.2.1 Transversal body force

Suppose, that a transversal body force $f : [0,T] \to L^2(\Omega)$ is given, such that f is absolutely continuous as a map from $[0,T]$ in $L^2(\Omega)$.

We also assume the uniform safe-load condition: there exists a function $m^1 : [0,T] \to L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ and a constant $\alpha > 0$, such that for every $t \in [0,T]$

$$\operatorname{div}\operatorname{div} m^1(t,x) = f \quad \text{in } \Omega$$

and $m^1(t,x) + \xi \in \mathbb{K}$ for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{M}^{2\times 2}_{sym}$ with $|\xi| < \alpha$.

Concerning the regularity of $t \mapsto m^1(t)$, we assume it to be absolutely continuous as a map from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$.

Kinematic admissibility

Now we give a definition of a kinematically admissible triple. Remark, that the first condition is responsible for an additive decomposition, the second one reflects the boundary conditions for u, while the third one is the relaxed form of the boundary conditions, which are typical in the variational theory of functionals with linear growth.

Definition 3.2.1. A triple $(u,e,p) \in BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ is called kinematically admissible, if the following conditions hold

$$\begin{gathered} D^2 u = e + p \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ p = -\nabla u \odot \nu\, \mathcal{H}^{n-1} \quad \text{on } \partial\Omega. \end{gathered}$$

Definition 3.2.2. The space $\Pi(\Omega)$ of admissible plastic curvatures is defined as the set of all $p \in M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ for which there exist $u \in BH(\Omega)$ and $e \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$, such that the triple (u,e,p) is kinematically admissible.

It is easy to see that the definition of a kinematically admissible triple is stable under weak convergence in the appropriate topologies.

Lemma 3.2.3. *Let a triple (u_k, e_k, p_k) be kinematically admissible. Assume that $u_k \overset{*}{\rightharpoonup} u_\infty$ in $BH(\Omega)$, $e_k \rightharpoonup e_\infty$ in $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ and $p_k \overset{*}{\rightharpoonup} p_\infty$ in $M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$. Then $(u_\infty, e_\infty, p_\infty)$ is also kinematically admissible.*

3.2.2 Moments and curvatures

Let us introduce the set:

$$S(\Omega) = \Big\{ M \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) : \operatorname{div}\operatorname{div} M \in L^2(\Omega) \Big\}. \tag{3.4}$$

We note, that the following approximation result holds for the functions from $S(\Omega)$. Remark, that the proof presented in [Tem85], Chapter III, contains an error. A correct proof was proposed by G. Seregin ([Ser]), and we present it here for completeness.

Lemma 3.2.4. *Let Ω be a bounded Lipshitz domain in $\mathbb{R}^2$ and let $M \in S(\Omega)\cap\mathcal{K}(\Omega)$. Then there exists a sequence $M_k \in C^\infty(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})\cap\mathcal{K}(\Omega)$ satisfying*

$$\begin{aligned} M_k \to M \quad &\text{in } L^p(\Omega;\mathbb{M}^{2\times 2}_{sym}), \text{ for any } p<\infty,\\ \operatorname{div}\operatorname{div} M_k \to \operatorname{div}\operatorname{div} M \quad &\text{in } L^2(\Omega),\\ \|M_k\|_{L^\infty} \le C\,\|M\|_{L^\infty}. \end{aligned} \tag{3.5}$$

Proof: Denote the space $L^2(\Omega; M^{2\times 2}_{sym}\times\mathbb{R})$ of vector-valued functions by L. Let D be a subspace of $L^2(\Omega)$ such that $(m, \operatorname{div}\operatorname{div} m)\in L$. Let D_0 be the closure of $C^\infty(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$ in the norm of the space L.

Assume that there exists an element $(m_*, \operatorname{div}\operatorname{div} m_*)\in D\setminus D_0$. As D_0 is closed and convex in L, by Hahn-Banach theorem there exists a pair $(u_1,u)\in L^2(\Omega;M^{2\times 2}_{sym})\times L^2(\Omega,\mathbb{R})$ (i.e., simply from L) such that

$$\int_\Omega (m_* : u_1 + u\operatorname{div}\operatorname{div} m_*)\,dx = 1$$

and

$$\int_\Omega (m : u_1 + u\operatorname{div}\operatorname{div} m)\,dx = 0$$

for any $m\in D_0$. The last identity shows that $u_1 = u$. So, $u\in W^2_2(\Omega)$ and u has usual traces on $\partial\Omega$ (u and $\nu\cdot\nabla u$), where ν is the normal to $\partial\Omega$. Those traces of u are zero that follows from the second identity. So, if the domain Ω is not bad, for example Lipshitz, u belongs to the closure of $C^\infty_0(\Omega)$ in $W^2_2(\Omega)$ (see [Maz85]). This means that there exist function $v\in C^\infty_0(\Omega)$ such that

$$\int_\Omega (m_* : \nabla^2 v + v\operatorname{div}\operatorname{div} m_*)\,dx > 1/2$$

The left hand side vanishes by definition of div div , which leads to a contradiction. □

Traces of the bending moments

For $M\in S(\Omega)$ one can define distributions $M_{ij}\nu_i\nu_j$ and $b_0(M)$ being the elements of $H^{-1/2}(\partial\Omega)$ and $H^{-3/2}(\partial\Omega)$ respectively as

$$\begin{aligned} \langle b_0(M), v\rangle_{H^{-3/2};H^{3/2}} - \langle M_{ij}\nu_i\nu_j, \frac{\partial v}{\partial\nu}\rangle_{H^{-1/2};H^{1/2}} &=\\ = \int_\Omega v\operatorname{div}\operatorname{div} M\,dx - \int_\Omega D^2 v : M\,dx& \end{aligned} \tag{3.6}$$

for $v\in W^{2,2}(\Omega)$. To see this it is enough to observe, that $H^{3/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)$ is precisely the space of traces and of normal derivatives of functions from $W^{2,2}(\Omega)$.

Remark 3.2.5. Arguing, as in [KT83], Lemma 2.4, one easily shows that $M_{ij}\nu_i\nu_j\in L^\infty(\partial\Omega)$ whenever $M\in S(\Omega)\cap L^\infty(\Omega;\mathbb{M}^{2\times 2}_{sym})$, and the estimate

$$\|M_{ij}\nu_i\nu_j\|_\infty \le C(\Omega)\|M\|_\infty$$

holds.

Moments-curvatures duality

Below we introduce an analogue of the duality between bending moments and plastic curvatures (defined in (3.8)), following the scheme, proposed in [DDM06] for studying weak solutions of quasistatic problems in perfect plasticity. As usually, the definition is constructed in a way to provide immediately the integration by parts formula (Proposition 3.2.8).

First of all, we recall the following construction: for $u \in BH(\Omega)$ and $M \in S(\Omega)$ we define a distribution $[D^2u : M]$ by

$$\langle [D^2u : M], \varphi\rangle = \int_\Omega u\,\varphi \operatorname{div}\operatorname{div} M\,dx - 2\int_\Omega (\nabla u \otimes \nabla\varphi) : M\,dx - \int_\Omega u(D^2\varphi : M)\,dx$$

for every $\varphi \in C_c^\infty(\Omega)$. The next statement (see [Dem89, Proposition 2.3]) describes some of its properties.

Proposition 3.2.6. *Let $M \in S(\Omega) \cap L^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym})$, $u \in BH(\Omega)$ and $[D^2u : M]$ be the distribution defined above. Then $[D^2u : M]$ can be extended to a bounded measure on Ω satisfying*

$$|[D^2u : M]| \le |D^2u|\,\|M\|_{L^\infty} \quad \text{in } M_b(\Omega).$$

Moreover the following integration by parts formula holds

$$\begin{aligned}\int_\Omega \varphi\, d\,[D^2u : M] &= \\ = \int_\Omega u\,\varphi \operatorname{div}\operatorname{div} M\,dx &- 2\int_\Omega (\nabla u\otimes\nabla\varphi) : M\,dx - \int_\Omega u(D^2\varphi : M)\,dx + \\ &+ \int_{\partial\Omega} M_{ij}\nu_i\nu_j \Big[\varphi\frac{\partial u}{\partial\nu} + u\frac{\partial\varphi}{\partial\nu}\Big]\,d\mathcal{H}^{n-1} - \int_{\partial\Omega} b_0(M)u\varphi\,d\mathcal{H}^{n-1},\end{aligned} \tag{3.7}$$

for every $\varphi \in C^2(\overline{\Omega})$.

Note, that

$$[\psi M : D^2u] = \psi[M : D^2u]$$

for every $\psi \in C^2(\overline{\Omega})$ and as in [Anz84], Corollary 3.2, one can show, that the following holds:

$$[D^2u : M]^a = D^2u^a : M \quad \text{a.e. in } \Omega.$$

We define the measure $[M : p] \in M_b(\overline{\Omega})$ by putting

$$\begin{aligned}[M : p] = M : p^a + [M : D^2u]^s &= [M : D^2u] - M : e \quad \text{on } \Omega,\\ [M : p] &= -\frac{\partial u}{\partial\nu} M_{ij}\nu_i\nu_j\,d\,\mathcal{H}^{n-1} \quad \text{on } \partial\Omega.\end{aligned} \tag{3.8}$$

Thus, the following duality between $S(\Omega)$ and $\Pi(\Omega)$ is well defined

$$\langle M|p\rangle := [M : p](\overline{\Omega}). \tag{3.9}$$

Then we have

$$|[M : p]| \le C|p| \quad \text{in } M_b(\overline{\Omega}),$$

which implies, that the definition of the duality does not depend on a particular choice of u and e.

Remark 3.2.7. Given $M \in C^2(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ and an admissible triple (u, e, p), the following holds

$$\int_{\overline{\Omega}} \varphi \, d[M : p] =: \langle [M : p] | \varphi \rangle = \langle \varphi M | p \rangle := \int_{\overline{\Omega}} \varphi M_{ij} \, dp_{ij},$$

for every $\varphi \in C^2(\overline{\Omega})$, where in both sides we have a duality pairing between $M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ and $M \in C(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$. By the definition of p and $[M : p]$ we are left to verify the equality of boundary integrals

$$\int_{\partial\Omega} \varphi M : (\nabla u \odot \nu) \, d\mathcal{H}^{n-1} = \int_{\partial\Omega} \varphi \, \frac{\partial u}{\partial \nu} \, M_{ik} \nu_i \nu_k \, d\mathcal{H}^{n-1}.$$

In the following calculations $\tau(x)$ stands for the tangential vector to $\partial\Omega$ at the point $x \in \partial\Omega$

$$\begin{aligned}
\int_{\partial\Omega} \varphi M : (\nabla u \odot \nu) \, d\mathcal{H}^{n-1} &= \int_{\partial\Omega} \varphi (M\nu) \cdot \nabla u \, d\mathcal{H}^{n-1} = \\
= \int_{\partial\Omega} \varphi (M_{ik} \nu_i \nu_k) \nu_j u_{,j} \, d\mathcal{H}^{n-1} &+ \int_{\partial\Omega} \varphi (M_{ik} \nu_i \tau_k) \tau_j u_{,j} \, d\mathcal{H}^{n-1} = \\
= \int_{\partial\Omega} \varphi (M_{ik} \nu_i \nu_k) \nu_j u_{,j} \, d\mathcal{H}^{n-1} - \int_{\partial\Omega} u \, \frac{\partial}{\partial \tau} (\varphi \, M_{ik} \nu_i \tau_k) \, d\mathcal{H}^{n-1} &= \int_{\partial\Omega} \varphi \, \frac{\partial u}{\partial \nu} \, M_{ik} \nu_i \nu_k \, d\mathcal{H}^{n-1},
\end{aligned}$$

as $u = 0$ on $\partial\Omega$.

From the very definition of measure $[M : p]$ and Proposition 3.2.6 we deduce the integration by parts formula for $M \in S(\Omega)$ and displacements $u \in BH(\Omega)$, involving the elastic and plastic curvatures e and p.

Proposition 3.2.8. *Let $M \in S(\Omega) \cap L^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym})$, $f \in L^2(\Omega)$ and let (u, e, p) be kinematically admissible. Assume that* $\operatorname{div}\operatorname{div} M = f$ *on Ω. Then*

$$\begin{aligned}
&\int_{\Omega} \varphi \, d[M : p] = \\
= \int_{\Omega} u \, \varphi \, f \, dx - \int_{\Omega} \varphi \, (M : e) \, dx &- \int_{\Omega} u \, (D^2\varphi : M) \, dx - 2 \int_{\Omega} (\nabla u \otimes \nabla \varphi) : M \, dx
\end{aligned} \tag{3.10}$$

for every $\varphi \in C^2(\overline{\Omega})$. In particular,

$$[M : p](\overline{\Omega}) = \int_{\Omega} u \, f \, dx - \int_{\Omega} (M : e) \, dx. \tag{3.11}$$

The following proposition makes the representation formula (3.1) more precise, expressing it by means of duality (3.9).

Proposition 3.2.9. *Let $p \in \Pi(\Omega)$. Then*

$$H(p) \geq [M : p] \quad \text{in } M_b(\Omega) \tag{3.12}$$

for every $M \in S(\Omega) \cap \mathcal{K}(\Omega)$, and

$$\mathcal{H}(p) = \sup\{\langle M | p \rangle : M \in S(\Omega) \cap \mathcal{K}(\Omega)\}. \tag{3.13}$$

Proof: Let $M \in S(\Omega) \cap \mathcal{K}(\Omega)$. First, we will show that

$$\langle H(p)|\varphi\rangle \geq \langle [M : p]|\varphi\rangle \tag{3.14}$$

for every $\varphi \in C(\overline{\Omega})$ with $\varphi \geq 0$ on $\overline{\Omega}$. By Lemma 3.2.4 there exists a sequence $M_k \in C^\infty(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ $\cap \mathcal{K}(\Omega)$ such that (3.5) holds. From the very definition of convex functional of a measure and be Remark 3.2.7 we have that

$$\langle H(p)|\varphi\rangle \geq \langle [M_k : p]|\varphi\rangle. \tag{3.15}$$

The integration by parts formula (3.7) and the convergence (3.5) permits us to pass to the limit in the right-hand side, so that (3.12) is proved.

We remark, that in (3.1) one can restrict the set of test functions to $C^\infty(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$, which implies (3.13). □

3.3 The minimum problem

In this section we show that each incremental problem has a solution, study the necessary conditions for minimality, which turn out to be also sufficient, due to convexity of the problem, and prove continuity estimates for the solutions.

Given p_0, to get the updated values u, e and p of displacement, elastic and plastic curvatures we will solve the minimum problem

$$\min_{(u,e,p)} \{\mathcal{Q}(e) + \mathcal{H}(p - p_0) - \langle f|u\rangle\}, \tag{3.16}$$

where the minimum is taken over all kinematically admissible triples (u, e, p). Note, that in the sequel we often write this minimization problem without mentioning explicitly that only kinematically admissible triples participate.

For the existence result we will assume the safe-load condition: there exists $m^1 \in S(\Omega)$ and $\alpha > 0$ such that

$$\begin{aligned} &\operatorname{div}\operatorname{div} m^1 = f \quad \text{on } \Omega, \\ &m^1(x) + \xi \in \mathbb{K} \text{ for a.e. } x \in \Omega \text{ and for every } \xi \in \mathbb{M}^{2\times 2}_{sym} \text{ with } |\xi| \leq \alpha. \end{aligned} \tag{3.17}$$

3.3.1 Existence of a minimizer

Lemma 3.3.1. *Let $f \in L^2(\Omega)$ and assume (3.17). Then*

$$\langle f|u\rangle = \langle m^1|e\rangle + \langle m^1|p\rangle$$

for every admissible triple (u, e, p).

Proof: Follows from the integration by parts formula (3.11). □

Lemma 3.3.2. *Let $f \in L^2(\Omega)$, $m^1 \in S(\Omega)$ and $\alpha > 0$. Assume that condition (3.17) holds. Then*

$$\mathcal{H}(p) - \langle m^1|p\rangle \geq \alpha\|p\|_1$$

for every $p \in \Pi(\Omega)$.

PROOF: By Proposition 3.2.9 we have

$$\mathcal{H}(p) - \langle m^1|p\rangle = \sup\Big\{\langle M - m^1|p\rangle : M \in S(\Omega)\cap\mathcal{K}(\Omega)\Big\} \geq$$
$$\geq \sup\Big\{\langle m|p\rangle : m \in S(\Omega),\ \|m\|_\infty \leq \alpha\Big\} \geq \alpha\|p\|_1.$$

□

Theorem 3.3.3. *Let $p_0 \in \Pi(\Omega)$, $f \in L^2(\Omega)$ and assume* (3.17)*. Then minimum problem* (3.16) *has a solution.*

PROOF: By Lemma 3.3.1 minimum problem (3.16) is equivalent to

$$\min_{(u,e,p)} \Big\{\mathcal{Q}(e) - \langle m^1|e\rangle + \mathcal{H}(p - p_0) - \langle m^1|p - p_0\rangle\Big\}. \tag{3.18}$$

Let (u_k, e_k, p_k) be a minimizing sequence of kinematically admissible triples. By Lemma 3.3.2 we have

$$\mathcal{H}(p - p_0) - \langle m^1|p - p_0\rangle \geq \alpha\|p_k - p_0\|_1,$$

and (3.2) implies, that

$$\mathcal{Q}(e_k) - \langle m^1|e_k\rangle \geq \frac{\alpha_{\mathbb{C}}}{2}\|e_k\|_2^2 - \frac{1}{2\alpha_{\mathbb{C}}}\|m^1\|_2^2.$$

Therefore the sequences e_k and p_k are bounded in $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$ and $M_b(\overline{\Omega};\mathbb{M}^{2\times2}_{sym})$ respectively. As $D^2u_k = e_k + p_k$ in Ω, it follows that D^2u_k is bounded in $M_b(\Omega;\mathbb{M}^{2\times2}_{sym})$. As $u_k = 0$ on $\partial\Omega$, u_k are bounded in $BH(\Omega)$. So we may assume that $u_k \xrightharpoonup{*} u$ in $BH(\Omega)$, $e_k \rightharpoonup e$ in $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$ and $p_k \xrightharpoonup{*} p$ in $M_b(\overline{\Omega};\mathbb{M}^{2\times2}_{sym})$. By Lemma 3.2.3 we know that the limit (u, e, p) is kinematically admissible. The lower semicontinuity of $\mathcal{Q}$ gives

$$\mathcal{Q}(e) - \langle m^1|e\rangle \leq \liminf_{k\to\infty} \Big\{\mathcal{Q}(e_k) - \langle m^1|e_k\rangle\Big\}, \tag{3.19}$$

so it remains to show that

$$\mathcal{H}(p - p_0) - \langle m^1|p - p_0\rangle \leq \liminf_k \Big\{\mathcal{H}(p_k - p_0) - \langle m^1|p_k - p_0\rangle\Big\}. \tag{3.20}$$

The integration by parts formula (3.10) implies

$$\langle m^1|p_k - p_0\rangle = \langle f|u_k\rangle - \langle m^1|e_k\rangle - \langle m^1|p_0\rangle.$$

Passing to the limit as $k \to \infty$ and using (3.10) again, we conclude that

$$\lim_{k\to\infty} \langle m^1|p_k - p_0\rangle = \langle m^1|p - p_0\rangle.$$

So, the latter relation and lower semicontinuity of $\mathcal{H}$ yield

$$\mathcal{H}(p - p_0) - \langle m^1|p - p_0\rangle \leq \liminf_k \{\mathcal{H}(p_k - p_0) - \langle m^1|p_k - p_0\rangle\}.$$

Now, as (u, e, p) is kinematically admissible, the inequalities (3.19) and (3.20) guarantee that (u, e, p) is a minimizer of (3.18). □

3.3.2 The Euler conditions

Below we derive the necessary and sufficient conditions for a triple (u, e, p) to be a solution to problem (3.16) with $p = p_0$.

Theorem 3.3.4. *Let $f \in L^2(\Omega)$, let (u, e, p) be kinematically admissible and let $M := \mathbb{C}e$. Then the following conditions are equivalent:*

(a) (u, e, p) is a solution of (3.16) with $p = p_0$;

(b) $-\mathcal{H}(q) \leq \langle M|\eta\rangle - \langle f|v\rangle \leq \mathcal{H}(-q)$ for every kinematically admissible (v, η, q);

(c) $M \in S(\Omega) \cap \mathcal{K}(\Omega)$ and $\operatorname{div}\operatorname{div} M = f$.

Proof: First, let us prove the implication (a)⇒(b). Take a kinematically admissible triple (v, η, q). As the triple $(u + \varepsilon v, e + \varepsilon\eta, p + \varepsilon q)$ is also kinematically admissible for every $\varepsilon \in \mathbb{R}$, the minimality condition yields

$$\mathcal{Q}(e + \varepsilon v) + \mathcal{H}(\varepsilon q) - \varepsilon\langle f|v\rangle \geq \mathcal{Q}(e) \quad \text{for every } \varepsilon \in \mathbb{R}.$$

Then, by the positive homogeneity of H

$$\mathcal{Q}(e \pm \varepsilon\eta) + \varepsilon(\pm q) \mp \varepsilon\langle f|v\rangle \geq \mathcal{Q}(e) \quad \text{for every } \varepsilon > 0.$$

Now, taking the derivative with respect to ε at $\varepsilon = 0$ we get (b).

The implication (b)⇒(a) holds true by convexity.

Let us prove (b)⇒(c) Assume (b) and let $v \in C_c^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym})$. Since the triple $(v, D^2v, 0)$ is kinematically admissible, we obtain

$$\langle M|D^2v\rangle = \langle f|v\rangle, \tag{3.21}$$

which implies that $\operatorname{div}\operatorname{div} M = f$.

Now let $\eta \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$. Then the triple $(0, \eta, -\eta)$ is kinematically admissible, and (b) yields

$$-\mathcal{H}(-\eta) \leq \langle M|\eta\rangle \leq \mathcal{H}(\eta).$$

Fix an arbitrary matrix $\xi \in \mathbb{M}^{2\times 2}_{sym}$, a Borel set $B \subset \Omega$ and take $\eta(x) = 1_B(x)\xi$. Then the latter relation yields

$$-H(-\xi) \leq M(x) : \xi \leq H(\xi) \quad \text{for a.e. } x \in \Omega.$$

Therefore $M(x) \in \partial H(0) = \mathbb{K}$ for a.e. $x \in \Omega$. Thus, $M \in S(\Omega) \cap \mathcal{K}(\Omega)$.

Now, let us show (c)⇒(b). Assume (b) and let (v, η, q) be kinematically admissible. By Proposition 3.2.9 we have

$$-\mathcal{H}(-q) \leq \langle M|q\rangle \leq \mathcal{H}(q). \tag{3.22}$$

Using the integration by parts formula (3.11) we get

$$\langle M|q\rangle = \langle f|v\rangle - \langle M|\eta\rangle,$$

and (b) follows from (3.22). □

By using Theorem 3.3.4 one can easily establish the stability property, expressed in Theorem below.

Theorem 3.3.5. *Let* (u_k, e_k, p_k) *be a sequence of kinematically admissible triples. Assume that* $u_k \xrightarrow{*} u_\infty$ *in* $BH(\Omega)$, $e_k \rightharpoonup e_\infty$ *in* $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$, $p_k \xrightarrow{*} p_\infty$ *in* $M_b(\Omega;\mathbb{M}^{2\times 2}_{sym})$ *and* $f_k \rightharpoonup f_\infty$ *in* $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$.

If

$$\mathcal{Q}(e_k) - \langle f_k|u_k\rangle \le \mathcal{Q}(\eta) + \mathcal{H}(q - p_k) - \langle f_k|v\rangle$$

for every k *and for every kinematically admissible* (v,η,q), *then* $(u_\infty, e_\infty, p_\infty)$ *is also kinematically admissible and*

$$\mathcal{Q}(e_\infty) - \langle f_\infty|u_\infty\rangle \le \mathcal{Q}(\eta) + \mathcal{H}(q - p_\infty) - \langle f_\infty|v\rangle$$

for every kinematically admissible (v,η,q).

3.3.3 Continuous dependence on the data

Next we will show that solutions (u,e) to minimum problem (3.16) depend continuously on f and p.

Theorem 3.3.6. *For* $i = 1,2$, *let* $f_i \in L^2(\Omega)$. *Suppose, that* (u_i, e_i, p_i) *is a solution of* (3.16) *with* $p_0 = p_i$. *Then*

$$\begin{aligned}&\|e_2 - e_1\|_2 + \|D^2u_2 - D^2u_1\|_1 + \|u_2 - u_1\|_1 \le \\ &\le C\Big(\|p_2 - p_1\|_1 + \|p_2 - p_1\|_1^{1/2} + \|f_2 - f_1\|_2\Big).\end{aligned} \tag{3.23}$$

Proof: Let $v = u_2 - u_1$, $\eta = e_2 - e_1$ and $q = p_2 - p_1$. As (v,η,q) is kinematically admissible, by Theorem 3.3.4 one has

$$\begin{aligned}-\mathcal{H}(p_2 - p_1) &\le \langle \mathbb{C}e_1|\eta\rangle - \langle f_1|v\rangle \\ \langle \mathbb{C}e_2|\eta\rangle - \langle f_2|v\rangle &\le \mathcal{H}(p_1 - p_2).\end{aligned}$$

Adding these two inequalities we obtain

$$\langle \mathbb{C}(e_2 - e_1)|e_2 - e_1\rangle \le \langle f_2 - f_1|v\rangle + 2R_{\mathbb{K}}\|p_2 - p_1\|.$$

Therefore

$$2\alpha_{\mathbb{C}}\|e_2 - e_1\|_2^2 \le \|f_2 - f_1\|_2\,\|v\|_2 + 2R_{\mathbb{K}}\|p_2 - p_1\|_1.$$

As $v = 0$ on $\partial\Omega$, we have

$$\|v\|_2 \le C\|D^2v\|_1 \le C\Big(\|e_2 - e_1\|_2 + \|p_2 - p_1\|_1\Big). \tag{3.24}$$

The last two inequalities imply

$$\|e_2 - e_1\|_2 \le C\Big(\|f_2 - f_1\|_2 + \|p_2 - p_1\|_1 + \|p_2 - p_1\|_1^{1/2}\Big).$$

Now, as $D^2u_i = e_i + p_i$, we get

$$\|D^2u_2 - D^2u_1\|_1 \le C\Big(\|f_2 - f_1\|_2 + \|p_2 - p_1\|_1 + \|p_2 - p_1\|_1^{1/2}\Big).$$

Finally, as

$$\|v\|_1 \le C\|D^2v\|,$$

the latter inequality guarantees (3.23). □

3.4 Quasistatic evolution

In this section we define a concept of quasistatic evolution - a weak solution to our initial problem, formulated in Section 1. As usually in the calculus of variations, strong solution is a weak solution, and vice versa, a sufficiently regular weak solution is a classical one.

3.4.1 Definition of quasistatic evolution

Below, we will apply the results of [DDM06, Section 7], with $X = M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$, $Y = C_0(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$ and $B = \mathcal{K}(\Omega) \cap C_0(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$. For a function $p : [0,T] \to M_b(\overline{\Omega},\mathbb{M}^{2\times 2}_{sym})$ with values in the dual of a separable Banach space $C_0(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$, for every $s,t \in [0,T]$ with $s < t$, the total variation of p on $[s,t]$ is defined as

$$\mathcal{V}(p;s,t) = \sup\Big\{\sum_{i=1}^{N} \|p(t_i) - p(t_{i-1})\|_{M_b(\overline{\Omega})} : \ s = t_0 < \cdots < t_N = t,\ N \in \mathbb{N}\Big\}.$$

The $\mathcal{H}$ variation of p is defined as

$$\mathcal{V}_{\mathcal{H}}(p;s,t) = \sup\Big\{\sum_{i=1}^{N} \mathcal{H}(p(t_i) - p(t_{i-1})) : \ s = t_0 < \cdots < t_N = t,\ N \in \mathbb{N}\Big\}$$

Definition 3.4.1. A quasistatic evolution is a function $t \mapsto (u(t), e(t), p(t))$ from $[0,T]$ into $BH(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$ which satisfies the following conditions

(qs1) for every $t \in [0,T]$ the triple $(u(t), e(t), p(t))$ is kinematically admissible and

$$\mathcal{Q}(e(t)) - \langle f(t)|u(t)\rangle \leq \mathcal{Q}(\eta) + \mathcal{H}(q - p(t)) - \langle f(t)|v\rangle \tag{3.25}$$

for every kinematically admissible (v,η,q);

(qs2) the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\Omega;\mathbb{M}^{2\times 2}_{sym})$ has bounded variation and for every $t \in [0,T]$

$$\begin{aligned} &\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p;0,t) - \langle f(t)|u(t)\rangle = \\ &= \mathcal{Q}(e(0)) - \langle f(0)|u(0)\rangle - \int_0^t \langle \dot{f}(s)|u(s)\rangle\, ds. \end{aligned} \tag{3.26}$$

The following theorem uses the integration by parts formula (3.11) to express conditions (qs1) and (qs2) in an equivalent form, involving the safe-load condition $t \mapsto m^1(t)$.

Theorem 3.4.2. *A function $t \mapsto (u(t), e(t), p(t))$ from $[0,T]$ into $BH(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$ is a quasistatic evolution if and only if it satisfies the following conditions:*

(qs1') for every $t \in [0,T]$ the triple $(u(t), e(t), p(t))$ is kinematically admissible and

$$\mathcal{Q}(e(t)) - \langle m^1(t)|e(t)\rangle \leq \mathcal{Q}(\eta) - \langle m^1(t)|\eta\rangle + \mathcal{H}(q - p(t)) - \langle m^1(t)|q - p(t)\rangle \tag{3.27}$$

for every kinematically admissible (v,η,q).

(qs2') the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$ has bounded variation and for every $t \in [0,T]$

$$\begin{aligned} &\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p;0,t) - \langle m^1(t)|e(t)\rangle - \langle m^1(t)|p(t)\rangle = \\ &= \mathcal{Q}(e(0)) - \langle m^1(0)|e(0)\rangle - \langle m^1(0)|p(0)\rangle - \int_0^t \Big[\langle \dot m^1(s)|e(s)\rangle + \langle \dot m^1(s)|p(s)\rangle\Big] ds. \end{aligned} \tag{3.28}$$

Proof: The equivalence of (qs1) and (qs1') follows from Lemma 3.3.1.

By Lemma 3.3.1 we have

$$\langle f(t)|v\rangle = \langle m^1(t)|\eta\rangle + \langle m^1(t)|q\rangle \tag{3.29}$$

for every $t \in [0,T]$ and every kinematically admissible triple (v,η,q). Hence

$$\langle \dot f(t)|v\rangle = \langle \dot m^1(t)|\eta\rangle + \langle \dot m^1(t)|q\rangle$$

for a.e. $t \in [0,T]$ and every kinematically admissible (v,η,q).

If conditions (qs1) or (qs1') hold, then

$$(u,e,p) \in L^\infty([0,T];BH(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})).$$

As $(u(t),e(t),p(t))$ is kinematically admissible for every $t \in [0,T]$, we have

$$\langle \dot f(t)|u(t)\rangle = \langle \dot m^1(t)|e(t)\rangle + \langle \dot m^1(t)|p(t)\rangle.$$

Thus

$$\int_0^t \langle \dot f(s)|u(s)\rangle\, ds = \int_0^t \Big[\langle \dot m^1(s)|e(s)\rangle + \langle \dot m^1(s)|p(s)\rangle\Big]\, ds. \tag{3.30}$$

Now the relations (3.29) and (3.30) imply the equivalence of (qs2) and (qs2'). □

3.4.2 The existence result

The following theorem states the existence of a weak solution.

Theorem 3.4.3. *Let (u_0,e_0,p_0) be kinematically admissible and satisfy the stability condition*

$$\mathcal{Q}(e_0) - \langle f(0)|u_0\rangle \le \mathcal{Q}(e) + \mathcal{H}(p-p_0) - \langle f(0)|u\rangle \tag{3.31}$$

for every kinematically admissible (u,e,p). Then there exists a quasistatic evolution $t \mapsto (u(t), e(t), p(t))$ such that $u(0)=u_0$, $e(0)=e_0$ and $p(0)=p_0$.

We prove Theorem 3.4.3 by a time discretization process. Fix a sequence of subdivisions $(t^i_k)_{0\le i\le k}$ of the interval $[0,T]$, with

$$0 = t^0_k < t^1_k < \cdots < t^{k-1}_k < t^k_k = T, \tag{3.32}$$

$$\lim_{k\to\infty} \max_{1\le i\le k} (t^i_k - t^{i-1}_k) = 0. \tag{3.33}$$

For $i = 0,\dots,k$ we set $f^i_k := f(t^i_k)$ and $(m^1)^i_k := m^1(t^i_k)$.

For every k we define u_k^i, e_k^i and p_k^i by induction. We set $(u_k^0, e_k^0, p_k^0) := (u_0, e_0, p_0)$, which, by assumption, is kinematically admissible, and for $i = 1, \dots, k$ we define (u_k^i, e_k^i, p_k^i) as a solution to the incremental problem

$$\min_{(u,e,p)} \Big\{ \mathcal{Q}(e) + \mathcal{H}(p - p_k^{i-1}) - \langle f_k^i | u \rangle \Big\}, \tag{3.34}$$

where minimum is sought among all kinematically admissible triples (u, e, p).

The existence of a solution to this problem is established in Theorem 3.3.3. By Lemma 3.3.1 the minimum problem (3.34) is equivalent to

$$\min_{(u,e,p)} \Big\{ \mathcal{Q}(e) - \langle (m^1)_k^i | e \rangle + \mathcal{H}(p - p_k^{i-1}) - \langle (m^1)_k^i | p - p_k^{i-1} \rangle \Big\}. \tag{3.35}$$

Moreover, by the triangle inequality the triple (u_k, e_k, p_k) is also a solution of the problem

$$\min_{(u,e,p)} \Big\{ \mathcal{Q}(e) + \mathcal{H}(p - p_k^{i}) - \langle f_k^i | u \rangle \Big\}. \tag{3.36}$$

For $i = 0, \dots, k$ we set $M_k^i := \mathbb{C} e_k^i$ and for every $t \in [0, T]$ we define the piecewise constant interpolations

$$\begin{gathered} u_k(t) := u_k^i, \quad e_k(t) := e_k^i, \quad p_k(t) := p_k^i, \quad M_k(t) := M_k^i, \\ f_k(t) := f_k^i, \quad m_k^1(t) = (m^1)_k^i, \end{gathered} \tag{3.37}$$

where i is the largest integer such that $t_k^i \le t$. By definition $(u_k(t), e_k(t), p_k(t))$ is kinematically admissible and by (3.36) we have

$$\mathcal{Q}(e_k(t)) - \langle f_k(t) | u_k(t) \rangle \le \mathcal{Q}(\eta) + \mathcal{H}(q - p_k(t)) - \langle f_k(t) | v \rangle \tag{3.38}$$

for every admissible triple (v, η, q).

3.4.3 The discrete energy inequality

Now we derive an energy estimates for solutions of the incremental problems which is an essential ingredient of the proof of Theorem 3.4.3.

Lemma 3.4.4. *For every k and every $t \in [0, T]$ the following holds true*

$$\begin{aligned} \mathcal{Q}(e_k(t)) - \langle m_k^1(t) | e_k(t) \rangle + \sum_{0 < t_k^r \le t} \Big\{ \mathcal{H}(p_k^r - p_k^{r-1}) - \langle m^1(t_k^r) | p_k^r - p_k^{r-1} \rangle \Big\} \le \\ \le \mathcal{Q}(e_0) - \langle m^1(0) | e_0 \rangle - \int_0^{t_k^i} \langle \dot{m}^1(s) | e_k(s) \rangle \, ds, \end{aligned} \tag{3.39}$$

where i is the largest integer such that $t_k^i \le t$.

Proof: We have to prove that It is enough to adapt the proof of [DDM06, Lemma 4.6]. □

3.4.4 Proof of the existence theorem

Having Lemma 3.4.4 we now prove Theorem 3.4.3.
PROOF OF THEOREM 3.4.3: Let us prove the following estimates

$$\sup_{t\in[0,T]} \|e_k(t)\|_2 \le C \qquad \text{and} \qquad \mathrm{Var}(p_k;0,T) \le C \tag{3.40}$$

for every k, where the constant C depends only on the constants $\alpha_{\mathbb{C}}$, $\beta_{\mathbb{C}}$ and α, and on the functions e_0 and $t \mapsto m^1(t)$.

From (3.39) we deduce

$$\begin{aligned} \alpha_{\mathbb{C}}\|e_k(t)\|_2^2 + \alpha \sum_{0<t_k^r\le t} \|p_k^r - p_k^{r-1}\|_1 \le \\ \le \beta_{\mathbb{C}}\|e_0\|_2^2 + \|m^1(0)\|_2\, \|e_0\|_2 + \\ + \sup_{t\in[0,T]} \|e_k(t)\|_2 \Big(\sup_{t\in[0,T]} \|m^1(t)\|_2 + \int_0^T \|\dot m^1(s)\|_2\, ds\Big) \end{aligned} \tag{3.41}$$

for every k and every $t \in [0,T]$.

Now we deduce the former estimate in (3.40) by using Cauchy inequality. As for the latter, by (3.41) and the first estimate in (3.40) we conclude that

$$\sum_{0<t_k^r\le t} \|p_k^r - p_k^{r-1}\|_1 \le C$$

for every k and every $t \in [0,T]$. Thus, as $t \mapsto p_k(t)$ is constant on the intervals $[t_k^{r-1}, t_k^r)$, we deduce the second inequality in (3.40).

By the generalized version of the classical Helly Theorem (see [DDM06, Lemma 7.2]), there exists a subsequence, still denoted by p_k, and a function $p : [0,T] \to M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ with bounded variation on $[0,T]$, such that $p_k(t) \overset{*}{\rightharpoonup} p(t)$ in $M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ for every $t \in [0,T]$.

From (3.40) it follows that $\|u_k(t)\|_{BH(\Omega)} \le C$ uniformly with respect to k and t. Let us fix $t \in [0,T]$. There exists a subsequence k_j, depending on t, and two functions $u(t) \in BH(\Omega)$ and $e(t) \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ such that $u_{k_j}(t) \overset{*}{\rightharpoonup} u(t)$ in $BH(\Omega)$ and $e_{k_j}(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$.

By (3.38) we can apply Theorem 3.3.5 to get that $(u(t), e(t), p(t))$ is a solution of the minimum problem

$$\min_{(v,\eta,q)} \Big\{ \mathcal{Q}(\eta) + \mathcal{H}(q - p(t)) - \langle f(t)|v\rangle \Big\}. \tag{3.42}$$

Theorem 3.3.6 implies that there exists the unique $(u,e) \in BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ such that $(u, e, p(t))$ is a solution to (3.42). Therefore, the convergence holds for the whole sequence, that is $u_k(t) \overset{*}{\rightharpoonup} u$ in $BH(\Omega)$ and $e_k(t) \rightharpoonup e(t)$ in $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$.

Let us show that the function $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution satisfying $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$. The initial condition is fulfilled. We observe, that the condition (3.25) is fulfilled by (3.42).

To prove the energy balance (3.26), or, equivalently (3.28), by Theorem 3.4.5 below, it is enough to prove the energy inequality

$$\begin{aligned} \mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p;0,t) - \langle m^1(t)|e(t)\rangle - \langle m^1(t)|p(t)\rangle \le \\ \le \mathcal{Q}(e(0)) - \langle m^1(0)|e(0)\rangle - \langle m^1(0)|p(0)\rangle - \int_0^t \Big[\langle \dot m^1(s)|e(s)\rangle + \langle \dot m^1(s)|p(s)\rangle \Big] ds. \end{aligned} \tag{3.43}$$

Let us fix $t \in [0,T]$. Since $t \mapsto p_k(t)$ is constant on the intervals $[t_k^{r-1}, t_k^r)$, we have

$$\mathcal{D}_{\mathcal{H}}(p_k; 0, t) = \sum_{0<t_k^r \le t} \mathcal{H}(p_k^r - p_k^{r-1}),$$

so by the lower semicontinuity of the dissipation one obtains

$$\mathcal{D}_{\mathcal{H}}(p; 0, t) \le \liminf_{k\to\infty} \sum_{0<t_k^r \le t} \mathcal{H}(p_k^r - p_k^{r-1}). \tag{3.44}$$

Let us write

$$\begin{aligned}\sum_{r=1}^{i} \langle m^1(t_k^r) | p_k^r - p_k^{r-1}\rangle = -\sum_{r=1}^{i} \langle m^1(t_k^r) - m^1(t_k^{r-1}) | p_k^{r-1}\rangle + \\ + \langle m^1(t_k^i) | p_k^i\rangle - \langle m^1(0)|p_0\rangle.\end{aligned} \tag{3.45}$$

Since $t \mapsto m^1(t)$ and $t \mapsto f(t)$ are absolutely continuous from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ and $L^2(\Omega)$ respectively, by (3.11) we have that

$$\sum_{r=1}^{i} \langle (m^1(t_k^r) - m^1(t_k^{r-1}) | p_k^{r-1}\rangle = \int_0^{t_k^i} \langle \dot f(s) | u_k(s)\rangle - \int_0^{t^i} \langle \dot m^1(s) | e_k(s)\rangle \, ds.$$

Passing to the limit as $k \to \infty$ and using (3.10) again we obtain

$$\lim_{k\to\infty} \sum_{r=1}^{i} \langle m^1(t_k^r) - m^1(t_k^{r-1}) | p_k^{r-1}\rangle = \int_0^t \langle \dot m^1(s) | p(s)\rangle \, ds. \tag{3.46}$$

Analogously we can show that

$$\lim_{k\to\infty} \langle m^1(t_k^i) | p_k^i\rangle = \langle m^1(t) | p(t)\rangle. \tag{3.47}$$

Combining together (3.44)-(3.47) we obtain that

$$\begin{aligned}\mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle m^1(t)|p(t)\rangle + \langle m^1(0)|p(0)\rangle + \int_0^t \langle \dot m^1(s)|p(s)\rangle \, ds \le \\ \le \liminf_{k\to\infty} \sum_{r=1}^{i} \Big\{ \mathcal{H}(p_k^r - p_k^{r-1}) - \langle m^1(t_k^r) | p_k^r - p_k^{r-1}\rangle \Big\}.\end{aligned} \tag{3.48}$$

Finally (3.43) follows from the last inequality, the weak convergence $e_k(s) \rightharpoonup e(s)$ for every $s \in [0,T]$ and the lower semicontinuity of $\mathcal{Q}$. □

As usually in the energy approach to rate-independent processes the inequality, opposite to (3.43) is obtained automatically by the construction of approximate solutions.

Theorem 3.4.5. *Let $t \mapsto (u(t), e(t), p(t))$ be a function from $[0,T]$ into $BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ which satisfies the stability condition* (3.27) *in Theorem 3.4.2. Assume that $t \mapsto p(t)$ from $[0,T]$ into $M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ has bounded variation. Then for every $t \in [0,T]$ we have*

$$\begin{aligned}\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle m^1(t)|e(t)\rangle - \langle m^1(t)|p(t)\rangle \ge \\ \ge \mathcal{Q}(e(0)) - \langle m^1(0)|e(0)\rangle - \langle m^1(0)|p(0)\rangle - \int_0^t \Big[\langle \dot m^1(s)|e(s)\rangle + \langle \dot m^1(s)|p(s)\rangle \Big] ds.\end{aligned} \tag{3.49}$$

If, in addition, (3.43) *is satisfied, then the exact energy balance* (3.28) *holds.*

Proof: It is enough to adapt the proof of [DDM06, Theorem 4.7]. □

3.4.5 Convergence of the approximate solutions

The next theorem states that, chosen a sequence of approximate solutions, such that $p_k(t) \overset{*}{\rightharpoonup} p(t)$ for every $t \in [0,T]$, the curvatures $e_k(t)$ converge to $e(t)$ strongly in $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$.

Theorem 3.4.6. *Assume that the plastic curvatures of the approximate solutions satisfy*

$$p_k(t) \overset{*}{\rightharpoonup} p(t) \quad \textit{weakly}^* \ \textit{in} \ M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym}).$$

Then $e_k(t) \to e(t)$ *and* $M_k(t) \to M(t)$ *in* $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$. *Moreover,*

$$\begin{aligned} &\lim_{k\to\infty} \sum_{0<t_k^r \le t} \Big\{ \mathcal{H}(p_k^r - p_k^{r-1}) - \langle m^1(t_k^r) | p_k^r - p_k^{r-1} \rangle \Big\} = \\ &= \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle m^1(t) | p(t) \rangle + \langle m^1(0) | p(0) \rangle + \int_0^t \langle \dot{m}^1(s) | p(s) \rangle \, ds \end{aligned} \tag{3.50}$$

for every $t \in [0,T]$.

Proof: By the discrete energy inequality (3.39) for every $t \in [0,T]$ we have

$$\begin{aligned} &\mathcal{Q}(e_k(t)) + \sum_{0<t_k^r \le t} \Big\{ \mathcal{H}(p_k^r - p_k^{r-1}) - \langle m^1(t_k^r) | p_k^r - p_k^{r-1} \rangle \Big\} \le \\ &\le \mathcal{Q}(e_0) - \langle m^1(0) | e_0 \rangle + \langle m_k^1(t) | e_k(t) \rangle - \int_0^{t_k^i} \langle \dot{m}^1(s) | e_k(s) \rangle \, ds, \end{aligned} \tag{3.51}$$

where i is the largest integer, such that $t_k^i \le t$. By the energy balance (3.28) we have also

$$\begin{aligned} &\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle m^1(t) | p(t) \rangle + \langle m^1(0) | p(0) \rangle + \int_0^t \langle \dot{m}^1(s) | p(s) \rangle \, ds = \\ &= \mathcal{Q}(e_0) - \langle m^1(0) | e_0 \rangle + \langle m^1(t) | e(t) \rangle - \int_0^t \langle \dot{m}^1(s) | e(s) \rangle \, ds. \end{aligned} \tag{3.52}$$

As the right-hand side of (3.51) converges to that of (3.52),

$$\begin{aligned} &\limsup \bigg\{ \mathcal{Q}(e_k(t)) + \sum_{0<t_k^r \le t} \Big\{ \mathcal{H}(p_k^r - p_k^{r-1}) - \langle m^1(t_k^r) | p_k^r - p_k^{r-1} \rangle \Big\} \bigg\} \le \\ &\le \mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p; 0, t) - \langle m^1(t) | p(t) \rangle + \langle m^1(0) | p(0) \rangle + \int_0^t \langle \dot{m}^1(s) | p(s) \rangle \, ds. \end{aligned}$$

By the lower semicontinuity of $\mathcal{Q}$ and by (3.48) we obtain that

$$\mathcal{Q}(e_k(t)) \to \mathcal{Q}(e(t)),$$

which gives strong convergence of $e_k(t)$. □

3.5 Regularity with respect to time and uniqueness result

In this section we prove that every quasistatic evolution $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous with respect to time, and that the functions $t \mapsto e(t)$ and $t \mapsto M(t)$ are uniquely determined by their initial conditions.

3.5.1 Regularity with respect to time

In the following proposition we establish the absolute continuity of the quasistatic evolution.

Theorem 3.5.1. *Let $t \mapsto (u(t), e(t), p(t))$ be a quasistatic evolution. Then the functions $t \mapsto e(t)$, $t \mapsto p(t)$ and $t \mapsto u(t)$ are absolutely continuous from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$, $M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ and $BH(\Omega)$ respectively. Moreover, for a.e. $t \in [0,T]$ we have*

$$\|\dot{e}(t)\|_2 + \|\dot{p}(t)\|_1 + \|D^2 \dot{u}(t)\|_1 + \|\dot{u}(t)\|_1 \le C \|\dot{m}^1(t)\|_\infty. \tag{3.53}$$

Proof: Since $\mathcal{H}(p(t_2) - p(t_1)) \le \mathcal{D}_{\mathcal{H}}(p; t_1, t_2)$ by the energy equality (3.28) we obtain, after integration by parts,

$$\begin{aligned} &\tfrac{1}{2}\langle M(t_2)|e(t_2)\rangle - \tfrac{1}{2}\langle M(t_1)|e(t_1)\rangle + \mathcal{H}(p(t_2)-p(t_1)) \le \\ &\le \langle m^1(t_2)|e(t_2)\rangle - \langle m^1(t_1)|e(t_1)\rangle + \langle m^1(t_2)|p(t_2)\rangle - \langle m^1(t_1)|p(t_1)\rangle - \\ &- \int_{t_1}^{t_2} \Big\{ \langle \dot{m}^1(s)|e(s)\rangle + \langle \dot{m}^1(s)|p(s)\rangle \Big\}\, ds \end{aligned} \tag{3.54}$$

for every t_1, $t_2 \in [0,T]$ with $t_1 < t_2$. Consider now the functions $v := u(t_2) - u(t_1)$, $\eta := e(t_2) - e(t_1)$, and the measure $q := p(t_2) - p(t_1)$. As (v, η, q) is kinematically admissible and $(u(t_1), e(t_1), p(t_1))$ is a solution of the minimum problem (3.16) with $p_0 = p(t_1)$ and $f = f(t_1)$, by Theorem 3.3.4 and Lemma 3.3.1 we obtain

$$\begin{aligned} -\langle M(t_1)|e(t_2) - e(t_1)\rangle + \langle m^1(t_1)|e(t_2) - e(t_1)\rangle + \langle m^1(t_1)|p(t_2) - p(t_1)\rangle + \\ \le \mathcal{H}(p(t_2) - p(t_1)), \end{aligned}$$

so that (3.54) implies

$$\begin{aligned} \tfrac{1}{2}\langle M(t_2)|e(t_2)\rangle - \tfrac{1}{2}\langle M(t_1)|e(t_1)\rangle - \langle M(t_1)|e(t_2) - e(t_1)\rangle \le \langle m^1(t_2) - m^1(t_1)|e(t_2)\rangle + \\ \langle m^1(t_2) - m^1(t_1)|p(t_2)\rangle - \int_{t_1}^{t_2} \Big\{ \langle \dot{m}^1(s)|e(s)\rangle - \langle \dot{m}^1(s)|p(s)\rangle \Big\}\, ds. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{2}\langle \mathbb{C}(e(t_2) - e(t_1))|e(t_2) - e(t_1)\rangle \le \\ &\le \int_{t_1}^{t_2} \Big\{ \langle \dot{m}^1(s)|e(t_2) - e(s)\rangle + \langle \dot{m}^1(s)|p(t_2) - p(s)\rangle \Big\}\, ds. \end{aligned}$$

Thus,

$$\begin{aligned} &\alpha_{\mathbb{C}} \|e(t_2) - e(t_1)\|_2^2 \le \\ &\le \int_{t_1}^{t_2} \|\dot{m}^1(s)\|_2 \|e(t_2) - e(s)\|_2\, ds + \int_{t_1}^{t_2} \|\dot{m}^1(s)\|_\infty \|p(t_2) - p(t_1)\|_1\, ds. \end{aligned} \tag{3.55}$$

By Lemma 3.2 we have that for every $t_1 \le s \le t_2$

$$\alpha \|p(t_2) - p(s)\|_1 \le \mathcal{H}(p(t_2) - p(s)) - \langle m^1(t_2)|p(t_2) - p(s)\rangle,$$

therefore (3.54) with $t_1 = s$ implies

$$\alpha\|p(t_2) - p(s)\|_1 \le \tfrac{1}{2}\langle M(s)|e(s)\rangle - \tfrac{1}{2}\langle M(t_2)|p(t_2)\rangle +$$
$$+\langle m^1(t_2)|e(t_2) - e(s)\rangle + \langle m^1(t_2) - m^1(s)|e(s)\rangle + \langle m^1(t_2) - m^1(s)|p(s)\rangle -$$
$$-\int_s^{t_2} \Big\{\langle \dot m^1(t)|e(t)\rangle + \langle \dot m^1(t)|p(t)\rangle\Big\}\, dt.$$

Observe that $\sup_t \|m^1(t)\|_\infty$, $\sup_t \|e(t)\|_2$ and $\sup_t \|p(t)\|_1$ are finite. The previous inequality implies that

$$\|p(t_2) - p(t_1)\|_1 \le C\Big(\|e(t_2) - e(s)\|_2 + \|m^1(t_2) - m^1(s)\|_\infty\Big) + C\int_s^{t_2} \|\dot m^1(t)\|_\infty\, dt.$$

Therefore, for every $t_1 \le s \le t_2$

$$\|p(t_2) - p(s)\|_1 \le C\,\|e(t_2) - e(s)\|_2 + C\int_{t_1}^{t_2} \|\dot m^1(t)\|_\infty\, dt. \tag{3.56}$$

By (3.55) and (3.56) and the triangle inequality $\|e(t_2)-e(t_1)\|_2 \le \|e(t_2)-e(t_1)\|_2 + \|e(s)-e(t_1)\|$ we deduce that

$$\|e(t_2) - e(t_1)\|_2^2 \le C\|e(t_2) - e(t_1)\|_2 \int_{t_1}^{t_2} \|\dot m^1(s)\|_\infty\, ds +$$
$$+C\int_{t_1}^{t_2} \|\dot m^1(s)\|_\infty \|e(s) - e(t_1)\|_2\, ds + C\Big(\int_{t_1}^{t_2} \|\dot m^1(s)\|_\infty\, ds\Big)^2,$$

that is

$$\|e(t_2) - e(t_1)\|_2^2 \le \int_{t_1}^{t_2} \psi(s)\|e(s) - e(t_1)\|_2\, ds + \Big(\int_{t_1}^{t_2} \psi(s)\, ds\Big)^2,$$

where

$$\psi(s) := C\|\dot m^1(s)\|_\infty.$$

By a version of Gronwall lemma, stated in Lemma 3.5.2 we have

$$\|e(t_2) - e(t_1)\|_2 \le \frac{3}{2}\int_{t_1}^{t_2} \psi(s)\, ds \le C\int_{t_1}^{t_2} \|\dot m^1(s)\|_\infty\, ds \tag{3.57}$$

Thus, $t \mapsto e(t)$ is absolutely continuous from $[0,T]$ into $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ and

$$\|\dot e(t)\|_2 \le C\|\dot m^1(t)\|_\infty. \tag{3.58}$$

So, (3.56) implies the absolute continuity of $t \mapsto p(t)$ and the estimate

$$\|\dot p(t)\|_1 \le C\|\dot m^1(t)\|_\infty. \tag{3.59}$$

Now, the additive decomposition $D^2u(t) = e(t) + p(t)$ yields that $t \mapsto D^2u(t)$ is absolutely continuous from $[0,T]$ into $M_b(\Omega;\mathbb{M}^{2\times 2}_{sym})$ and $D^2\dot u(t) = \dot e(t) + \dot p(t)$ for a.e. $t \in [0,T]$. Finally, as $u = 0$ on $\partial\Omega$, we have that

$$\|u(t_2) - u(t_1)\|_{BH} \le C\|D^2u(t_2) - D^2u(t_1)\|_1,$$

so $t \mapsto u(t)$ is absolutely continuous from $[0,T]$ into $BH(\Omega)$ and (3.53) holds. □

Lemma 3.5.2. *([DDM06, Lemma 5.3]) Let $\phi : [0,T] \to [0,+\infty)$ be a bounded domain and let $\psi : [0,T] \to [0,+\infty)$ be an integrable function. Suppose that*

$$\phi(t)^2 \leq \int_0^t \phi(s)\psi(s)\,ds + \Big(\int_0^t \psi(s)\,ds\Big)^2$$

for every $t \in [0,T]$. Then

$$\phi(t) \leq \frac{3}{2}\int_0^t \psi(s)\,ds$$

for every $t \in [0,T]$.

Lemma 3.5.3. *Let $t \mapsto u(t)$, $t \mapsto e(t)$, $t \mapsto p(t)$ be absolutely continuous function from $[0,T]$ to $BH(\Omega)$, $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ and $M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$, respectively. Assume that $(u(t), e(t), p(t))$ is kinematically admissible for every $t \in [0,T]$. Then $(\dot u(t), \dot e(t), \dot p(t))$ is also kinematically admissible for a.e. $t \in [0,T]$.*

Proof: The proof follows from Lemma 3.5, applied to difference quotients. □

Proposition 3.5.4. *Let $t \mapsto (u(t), e(t), p(t))$ be an absolutely continuous function from $[0,T]$ into $BH(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$ and let $M(t) := \mathbb{C}e(t)$. Then the following conditions are equivalent:*

(a) for every $t \in [0,T]$

$$\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p;0,t) - \langle f(t)|u(t)\rangle = \mathcal{Q}(e(0)) - \langle f(0)|u(0)\rangle + \int_0^t \langle \dot f(s)|u(s)\rangle\,ds;$$

(b) for a.e. $t \in [0,T]$

$$\langle M(t)|\dot e(t)\rangle + \mathcal{H}(\dot p(t)) = \langle f(t)|\dot u(t)\rangle$$

(c) for a.e. $t \in [0,T]$

$$\langle M(t) - m^1(t)|\dot e(t)\rangle + \mathcal{H}(\dot p(t)) = \langle m^1(t)|\dot p(t)\rangle$$

(d) for every $t \in [0,T]$

$$\mathcal{Q}(e(t)) + \int_0^t \Big\{\mathcal{H}(\dot p(s)) - \langle m^1(s)|\dot p(s)\rangle\Big\}\,ds =$$
$$= \mathcal{Q}(e(0)) + \int_0^t \langle m^1(s)|\dot e(s)\rangle\,ds.$$

Proof: It is a matter of differentiation and integration by parts, as in Lemma 3.3.1. □

Proposition 3.5.5. *Let $t \mapsto (u(t), e(t), p(t))$ be a quasistatic evolution. Then*

$$\sup_{t\in[0,T]} \|e(t)\|_2 \leq C\Big\{\|e(0)\|_2 + \sup_{t\in[0,T]}\|m^1(t)\|_2 + \int_0^T \|\dot m^1(t)\|_2\,dt\Big\}, \tag{3.60}$$

and

$$\sup_{t\in[0,T]} \|p(t)\|_1 \leq \|p(0)\|_1 + C\Big\{\|e(0)\|_2^2 + \sup_{t\in[0,T]}\|m^1(t)\|_2 + \big(\int_0^T \|\dot m^1(t)\|_2\,dt\big)^2\Big\}. \tag{3.61}$$

PROOF: By Theorem 3.5.1 the function $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous from $[0,T]$ into $BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$. As $t \mapsto (u(t), e(t), p(t))$ satisfies (qs2) in Definition 3.4.1, it satisfies conditions (a) and (d) of Proposition 3.5.4. After an integration by parts, we obtain from (d)

$$\mathcal{Q}(e(t)) + \int_0^t \Big\{ \mathcal{H}(\dot p(s)) - \langle m^1(s) | \dot p(s) \rangle \Big\}\, ds - \langle m^1(t) | e(t) \rangle =$$
$$= \mathcal{Q}(e(0)) - \int_0^t \langle \dot m^1(s) | e(s) \rangle \, ds - \langle m^1(0) | e(0) \rangle.$$

Thus, for every $t \in [0,T]$ we have

$$\alpha_{\mathbb{C}} \|e(t)\|_2^2 + \alpha \int_0^t \|\dot m^1(s)\|_1 \, ds \le \beta_{\mathbb{C}} \|e(0)\|_2^2 + \\ +2 \sup_{t\in[0,T]} \|m^1(t)\|_2 \sup_{t\in[0,T]} \|e(t)\|_2 + \sup \|e(t)\|_2 \int_0^T \|\dot m^1(s)\|_2 \, ds, \tag{3.62}$$

which yields (3.60) and (3.61). □

3.5.2 Uniqueness of bending moments and elastic curvatures

Our next aim is to prove that $t \mapsto e(t)$ and $t \mapsto M(t)$ are uniquely determined by their initial conditions.

Theorem 3.5.6. *Let $t \mapsto (u(t), e(t), p(t))$ and $t \mapsto (v(t), \eta(t), q(t))$ be two quasistatic evolutions and let $M(t) := \mathbb{C}e(t)$ and $m(t) := \mathbb{C}\eta(t)$. If $e(0) = \eta(0)$, then $e(t) = \eta(t)$ for every $t \in [0,T]$.*

PROOF: By Theorem 3.5.1 a quasistatic evolution is absolutely continuous with respect to time. By condition (c) of Proposition 3.5.4 we have

$$\langle M(t) - m^1(t) | \dot e(t) \rangle + \mathcal{H}(\dot p(t)) = \langle m^1(t) | \dot p(t) \rangle, \tag{3.63}$$
$$\langle m(t) - m^1(t) | \dot \eta(t) \rangle + \mathcal{H}(\dot q(t)) = \langle m^1(t) | \dot q(t) \rangle. \tag{3.64}$$

By the global stability condition (3.25) and Theorem 3.3.4 for every $t \in [0,T]$ we have that $m(t) \in S(\Omega) \cap \mathcal{K}(\Omega)$ and $\operatorname{div}\operatorname{div} m(t) = f(t)$ in Ω. Lemma 3.5.3 implies that $(\dot u(t), \dot e(t), \dot p(t))$ is kinematically admissible for a.e. $t \in [0,T]$. Therefore Proposition 3.2.9 gives $\mathcal{H}(\dot p(t)) \ge \langle m(t) | \dot p(t) \rangle$. By (3.63) this implies

$$\langle M(t) - m^1(t) | \dot e(t) \rangle + \langle m(t) - m^1(t) | \dot p(t) \rangle \le 0.$$

As $\operatorname{div}\operatorname{div}(m(t) - m^1(t)) = 0$, by using the integration by parts formula (3.11) we deduce

$$\langle M(t) - m(t) | \dot e(t) \rangle \le 0.$$

Analogously, from (3.64) we obtain

$$\langle m(t) - M(t) | \dot \eta(t) \rangle \le 0.$$

Thus, summing these two inequalities one concludes that

$$\langle \mathbb{C}(e(t) - \eta(t)) | \dot e(t) - \dot \eta(t) \rangle \le 0,$$

hence

$$\frac{d}{dt} \langle \mathbb{C}(e(t) - \eta(t)) | e(t) - \eta(t) \rangle \le 0.$$

As $e(0) = \eta(0)$ by the assumption, the theorem is proved. □

3.6 Equivalent formulations in rate form

Recall the classical formulation of the flow rule:

$$\dot{p}(t,x) \in N_{\mathbb{K}}(M(t,x)) \quad \text{for a.e. } x \in \Omega\,. \tag{3.65}$$

Unfortunately, this condition makes no sense whenever $\dot{p}$ is a bounded Radon measure, and $M \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$. However, instead of (3.65) we can consider the inequality

$$\langle M(t) - m | \dot{p}(t) \rangle \geq 0 \tag{3.66}$$

valid for every $m \in S(\Omega) \cap \mathcal{K}(\Omega)$.

For $\dot{p}$ sufficiently regular ($\dot{p} \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$) conditions (3.65) and (3.66) are equivalent, while (3.66) has an advantage of being defined also for $\dot{p} \in M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$ by means of duality (3.9).

Thus, (3.66) is considered the weak form of (3.65) when $\dot{p} \in M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$

3.6.1 Weak formulation

Theorem 3.6.1. *Let $t \mapsto (u(t), e(t), p(t))$ be a function from $[0,T]$ into $BH(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$ and let $M(t) := \mathbb{C}e(t)$. Then the following conditions are equivalent:*

(a) $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution;

(b) $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous and

(b1) for every $t \in [0,T]$ we have $(u(t), e(t), p(t))$ is kinematically admissible, $M(t) \in S(\Omega) \cap \mathcal{K}(\Omega)$ and $\operatorname{div}\operatorname{div} M(t) = f(t)$ in Ω,

(b2) for a.e. $t \in [0,T]$ we have

$$\langle M(t) - m | \dot{p}(t) \rangle \geq 0$$

for every $m \in S(\Omega) \cap \mathcal{K}(\Omega)$.

(c) $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous and

(c1) for every $t \in [0,T]$ we have $(u(t), e(t), p(t))$ is kinematically admissible, $M(t) \in S(\Omega) \cap \mathcal{K}(\Omega)$ and $\operatorname{div}\operatorname{div} M(t) = f(t)$ in Ω,

(c2) for a.e. $t \in [0,T]$ we have

$$\mathcal{H}(\dot{p}(t)) = \langle M(t) | \dot{p}(t) \rangle;$$

Proof: Let us first establish the equivalence (a)$\Leftrightarrow$(c). By Theorem 3.5.1 a quasistatic evolution is absolutely continuous. Theorem 3.3.4 guarantees, that (c1) is equivalent to the global stability. By Proposition 3.5.4 it is enough to prove, that for an absolutely continuous function $t \mapsto (u(t), e(t), p(t))$, satisfying either (c1) or (qs1), condition (c2) is equivalent to the balance of powers

$$\langle M(t) | \dot{e}(t) \rangle + \mathcal{H}(\dot{p}(t)) = \langle f(t) | \dot{u}(t) \rangle \tag{3.67}$$

for a.e. $t \in [0,T]$. As $(\dot{u}(t), \dot{e}(t), \dot{p}(t))$ is kinematically admissible for a.e. $t \in [0,T]$ by Lemma 3.5.3, condition (c2) is equivalent to (3.67) in view of the integration by parts formula (3.11).

To prove (b)$\Leftrightarrow$(c) it is enough to show that, if (c1) is satisfied, then (b2)$\Leftrightarrow$(c2). Condition (b2) is equivalent to

$$\langle M(t) | \dot{p}(t) \rangle = \sup \Big\{ \langle m | \dot{p}(t) : m \in S(\Omega) \cap \mathcal{K}(\Omega) \rangle \Big\},$$

which is equivalent to (c2) by Proposition 3.2.9. □

Remark 3.6.2. As the measure $H(\dot p(t)) - [M(t) : \dot p(t)]$ is nonnegative on $\overline{\Omega}$, the condition (b2) implies that

$$H(\dot p(t)) = [M(t) : \dot p(t)] \quad \text{in } M_b(\overline{\Omega}). \tag{3.68}$$

3.6.2 Strong formulation and precise definition of the bending moments

Below (Theorem 3.6.3) a precise representative $\hat M(t,x)$ of $M(t,x)$ is defined almost everywhere with respect to the measure $\mu(t) = \mathcal{L}^2 + |\dot p(t)|$. Theorem 3.6.5 states, that if $\mathbb{K}$ is strictly convex, this representative is unique and can be obtained as limit of the averages of M.

Theorem 3.6.3. *Let $t \mapsto (u(t), e(t), p(t))$ be a function from $[0,T]$ into $BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$, let $M(t) := \mathbb{C}e(t)$, and let $\mu(t) := \mathcal{L}^2 + |\dot p(t)|$. Then $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution if and only if*

- *(e) $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous and*
  - *(e1) for every $t \in [0,T]$ we have that $(u(t), e(t), p(t))$ is kinematically admissible, $M(t) \in S(\Omega) \cap \mathcal{K}(\Omega)$, and $\operatorname{div}\operatorname{div} M(t) = f(t)$ in Ω,*
  - *(e2) for a.e. $t \in [0,T]$ there exists $\hat M(t) \in L^\infty_{\mu(t)}(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ such that*

$$\hat M(t) = M(t) \quad \mathcal{L}^2\text{-a.e. on } \Omega, \tag{3.69}$$

$$[M(t) : \dot p(t)] = \Big(\hat M(t) : \frac{\dot p(t)}{|\dot p(t)|}\Big) |\dot p(t)| \quad \text{in } M_b(\overline{\Omega}), \tag{3.70}$$

$$\frac{\dot p(t)}{|\dot p(t)|}(x) \in N_{\mathbb{K}}(\hat M(t,x)) \quad \text{for } |\dot p(t)|\text{-a.e. } x \in \overline{\Omega}. \tag{3.71}$$

Remark 3.6.4. Assume that $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous. If (e1) holds, then from (3.10) it follows, that condition (3.70) of Theorem 3.6.3 is equivalent to the following integration by parts formula: for every $\varphi \in C^2(\overline{\Omega})$ we have

$$\begin{aligned}\int_{\overline{\Omega}} \varphi \big(\hat M : \frac{\dot p(t)}{|\dot p(t)|}\big)\, d|\dot p(t)| &= \langle \varphi \hat M(t) | \dot p(t)\rangle = -\langle M(t) | \varphi\, \dot e(t)\rangle - \langle \dot u\, D^2\varphi | M(t)\rangle - \\ &-2\langle M(t) | \nabla \dot u(t) \otimes \nabla\varphi\rangle - \langle f(t) | \varphi \dot u(t)\rangle = \langle [M : \dot p] | \varphi\rangle,\end{aligned} \tag{3.72}$$

where in the left-hand side we have the duality between the measure $\dot p(t)$ and a bounded measurable (with respect to this measure) function $\varphi \hat M(t)$.

As the matrix $\dot p(t)/|\dot p(t)|$ has the unit norm $|\dot p(t)|$-a.e. on $\overline{\Omega}$ and $N_{\mathbb{K}}(\xi) = 0$ if ξ is in the interior of $\mathbb{K}$, we deduce from (3.71) that for a.e. $t \in [0,T]$

$$\hat M(t,x) \in \partial\mathbb{K} \quad \text{for } |\dot p(t)|\text{-a.e. } x \in \overline{\Omega}. \tag{3.73}$$

By the formulas from the convex analysis we can prove that condition (3.71) is equivalent to

$$\hat M(t,x) \in \partial H\Big(\frac{\dot p(t)}{|\dot p(t)|}(x)\Big) \quad \text{for } |\dot p(t)|\text{-a.e. } x \in \overline{\Omega}. \tag{3.74}$$

Since ∂H is positively homogeneous of degree 0, this is equivalent to the fact, that both of the following two inclusions are satisfied:

$$\hat{M}(t,x) \in \partial H(\dot{p}^a(t)(x)) \quad \text{for } \mathcal{L}^2\text{-a.e. } x \in \{|\dot{p}^a(t)| > 0\}, \tag{3.75}$$

$$\hat{M}(t,x) \in \partial H\Big(\frac{\dot{p}(t)}{|\dot{p}(t)|}(x)\Big) \quad \text{for } |\dot{p}^s(t)|\text{-a.e. } x \in \overline{\Omega}. \tag{3.76}$$

Proof: We adopt the proof of [DDM06, Theorem 6.5]. □

For every $r > 0$ and every $t \in [0,T]$ we consider the function $M^r(t) \in C(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ defined by

$$M^r(t,x) := \frac{1}{\mathcal{L}^2(B(x,r)\cap\Omega)} \int_{B(x,r)\cap\Omega} M(t,y)\,dy. \tag{3.77}$$

As $\mathbb{K}$ is convex, it follows that $M^r(t,x) \in \mathbb{K}$ for every $x \in \Omega$.

If $\mathbb{K}$ is strictly convex, then H is differentiable at all points $\xi \neq 0$. Thus, for a.e. $t \in [0,T]$ the function $\hat{M}(t)$ is uniqiely determined $\mu(t)$-a.e. on $\Omega \cup \Gamma_0$ by (3.69) and (3.70)

$$\hat{M}(t) = M(t) \quad \mathcal{L}^2\text{-a.e. on } \Omega, \tag{3.78}$$

$$\hat{M}(t) = \partial H\left(\frac{\dot{p}(t)}{|\dot{p}(t)|}\right) \tag{3.79}$$

The following theorem states, that $\hat{M}(t,x)$ can be obtained in Ω as the limit of $M^r(t)$ as $r \to 0$. It reflects the intrinsic character of the precise representative introduced in Theorem 3.6.3.

Theorem 3.6.5. *Assume that $\mathbb{K}$ is strictly convex. Let $t \mapsto (u(t), e(t), p(t))$ be a quasistatic evolution, let $\mu(t) = \mathcal{L}^2 + |\dot{p}(t)|$, let $M(t) = \mathbb{C}e(t)$ and let $M^r(t)$ and $\hat{M}(t)$ be defined by (3.77) and (3.78)-(3.79). Then $M^r(t) \to \hat{M}(t)$ strongly in $L^1_{\mu(t)}(\Omega; \mathbb{M}^{2\times 2}_{sym})$ for a.e. $t \in [0,T]$.*

Proof: We refer to [DDM06, Theorem 6.6], for the proof. □

Chapter 4

Regularity of stresses in Prandtl-Reuss perfect plasticity

An application of the main result

4.1 Introduction

A strong formulation of the Prandtl-Reuss model of perfect plasticity is the following: given a domain $\Omega \subset \mathbb{R}^n$,

body force $f(t,x) : [0,T] \times \Omega \to \mathbb{R}^n$,
boundary displacement $w(t,x) : [0,T] \times \Gamma_0 \to \mathbb{R}^n$,
surface force $F(t,x) : [0,T] \times \Gamma_1 \to \mathbb{R}^n$,

the problem is to find functions

$$u(t,x),\ e(t,x),\ p(t,x) \quad \text{and } \sigma(t,x)$$

such that for every $t \in [0,T]$, for every $x \in \Omega$ the following hold:

1. kinematic admissibility: $\varepsilon(u)(t,x) = e(t,x) + p(t,x)$ in Ω, $u(t,x) = w(t,x)$ on Γ_0
2. constitutive equation: $\sigma(t,x) = \mathbb{A}^{-1} e(t,x)$,
3. equilibrium: $\operatorname{div}_x \sigma(t,x) = -f(t,x)$ in Ω, $\sigma(t,x)\,\nu(x) = F(t,x)$ on Γ_1,
4. stress constraint $\sigma(t,x) \in \mathbb{K}$,
5. associative flow rule: $(\xi - \sigma(t,x)) : \dot{p}(t,x) \leq 0$ for every $\xi \in \mathbb{K}$,

where

$$\varepsilon(u) = \frac{\nabla u + \nabla u^T}{2},$$

$$\mathbb{K} = \{\tau \in \mathbb{M}^{n\times n}_{sym} \ : \ |\tau^D| \leq \sqrt{2}k_*\}$$

and $\mathbb{A}$ is the compliance tensor (the inverse of the elasticity tensor), which in the isotropic case has the form

$$\mathbb{A}\sigma = \frac{\operatorname{tr}\sigma}{n^2 K_0}\,\mathbf{1} + \frac{1}{2\mu}\,\sigma^D, \tag{4.1}$$

where nK_0 is the first Lamé constant, and μ is the shear modulus. The problem is supplemented by initial conditions at time $t=0$.

During the last decades there was an extensive study of this problem in its weak formulation (see e.g. [DDD07, DDM06, Joh76, Suq81]). Due to the linear growth of the functional with respect to $\varepsilon(u)$, arising in this problem, one looks for displacements u in the space $BD(\Omega)$ and for stresses σ in the space $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. However, one can expect a better regularity of the stress tensor σ. Namely, as it was shown in [Ser87, Ser96, Ser93c, Ser93a, Ser93b], in some static situations the stress belongs to the space $W^{1,2}_{loc}(\Omega;\mathbb{R}^n)$.

In this paper we address the issue of a higher regularity of the stress tensor $\sigma(t)$ with respect to spatial variables. The main result (see Theorem 4.2.1 below) states that for the Prandtl-Reuss model one has

$$\sigma \in L^\infty([0,T];W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym}))$$

for $n=2,3$.

A similar result was obtained in [BF96] for arbitrary n, using Norton-Hoff approximations and the dual theory of elliptic equations. However, our proof is based on a completely different approach, developed by G. Seregin for proving regularity of stresses in the case of Hencky perfect plasticity (see [Ser87, FS00, Ser96, Ser94]). Observe that, due to this fact, our assumptions on the data of the problem are different from those of [BF96].

The method proposed in this paper will be used for proving the differentiability of stresses for other models occurring in plasticity (see [Dem09, Dem08a]).

Shortly, the strategy for proving Theorem 4.2.1 consists in refining the proof of the existence of a solution to the quasistatic problem, carried out in [DDM06], by generalizing the estimates obtained in [FS00] for proving the regularity of stresses in the case of Hencky perfect plasticity.

More precisely, we follow the general scheme for proving the existence of weak solutions of the continuous-time energy formulation of rate-independent processes (see e.g. [Mie02] and the references contained therein). Our arguments are similar to the ones used in [Ser94] for the case of plasticity with hardening. Note, that in [Joh76, Suq81] the existence was proved by visco-plastic approximations, while in order to use the methods of [FS00] one needs to have some analogue of the static problem. This is why we follow the proof of the existence given in [DDM06], where a quasistatic problem in perfect plasticity was solved by time discretization. In this case the incremental problems one has to solve to get the updated values of solutions, play the role of the static problem, where one can use the machinery of [Ser87, FS00].

We perform the standard time-discretization procedure, and for suitably defined approximate solutions $(u_N(t),e_N(t),p_N(t),\sigma_N(t))$, converging to a weak solution of the quasistatic problem, we obtain the estimate

$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\|\sigma_N(t)\|_{W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})} \le C, \tag{4.2}$$

which yields Theorem 4.2.1.

To get (4.2), one looks for solutions of the incremental problems, regarded as saddle points of some minimax problem, similar to the one considered in [FS00, Ser96] for the static case of Hencky perfect plasticity. The main difference is the presence of a term which takes into account the preceding history of plastic deformation. Then we approximate every incremental problem with a sequence of regularized problems and show, that their solutions σ^α_m, with $\alpha\in(0,1)$ converge to σ_m^N, a solution to the corresponding incremental problem, weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, as $\alpha \to 0$. Then we show, that for every incremental problem the bound

$$\sup_{\alpha>0} \|\sigma_m^\alpha\|_{W^{1,2}(\Omega';\mathbb{M}^{n\times n}_{sym})} \le C_m$$

holds for any domain $\Omega' \subset\subset \Omega$, where the constant C_m depends on the discretization step and on Ω'. This guarantees us, that σ_m^N is itself in $W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$, and that the convergence of σ_m^α to σ_m^N is actually better, and depends on the critical Sobolev exponent. Afterwards, we manage to make this estimate uniform, to get (4.2).

Let us note that Theorem 4.2.1 does not give any information about the behavior of the stress tensor near the boundary. As it was observed in [Ser99], the method we use is not suitable for the investigation of regularity up to the boundary, at least in the case of a nonconvex domain Ω. The issue of boundary regularity was discussed also in [FM99].

To our best knowledge, the only global regularity result for the stress in the case of Hencky perfect plasticity is contained in [Kne06], where under appropriate assumptions it is proved that $\sigma \in W^{1/2-\delta,2}(\Omega)$ for every $\delta > 0$.

The paper is organized as follows: in Section 4.2 we introduce the definitions and state the main result. In Section 4.3 we present a weak formulation of the quasistatic problem, outline the proof of existence of the quasistatic evolution and obtain some time-continuity estimates for the approximate solutions. In Section 4.4 an abstract scheme of relaxation of convex functionals in non reflexive spaces is described. A minimax formulation of the incremental problems is given in Section 4.5. In Section 4.6 we formulate the regularized problems, which are used for obtaining the differentiability of stresses, and show the convergence properties of their solutions. Section 4.7 contains the estimates of the $W^{1,2}_{loc}$ norms of the solutions to the regularized problems, which imply that for every approximate solution we actually have

$$\sup_{t\in[0,T]} \|\sigma_N\|_{W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})} \le C(N),$$

however, without any uniformity with respect to N. The uniform estimates (4.2) and the proof of Theorem 4.2.1 are contained in Section 4.9. In Section 4.10 we consider the examples which show that there is no analogue of regularity theorem, as in [Ser87, FS00, Ser96, Ser93c], for the displacement u and the plastic strain p.

4.2 Preliminary definitions and the main result

We use the following notation:

$\mathbb{R}^n$ denotes the n-dimensional Euclidian space,

$\mathbb{M}^{n\times n}_{sym}$ denotes the space of all $n \times n$ symmetric matrices, equipped with a Hilbert-Schmidt scalar product $\sigma : \xi = \sum_{i,j} \sigma_{ij}\xi_{ij}$,

$\mathbf{1}$ stands for the identity matrix, and we consider the orthogonal decomposition $\mathbb{M}^{n\times n}_{sym} = \mathbb{M}^{n\times n}_D \oplus \mathbb{R}\mathbf{1}$ of the space $\mathbb{M}^{n\times n}_{sym}$ into the subspace of trace-free matrices $\mathbb{M}^{n\times n}_D$ and of the multiples of identity $\mathbb{R}\mathbf{1}$,

$a \odot b$ stands for the symmetrized tensor product of two vectors $a, b \in \mathbb{R}^n$, given by the formula $(a \odot b)_{ij} = \frac{1}{2}(a_i b_j + a_j b_i)$,

$L^p(\Omega;\mathbb{R}^m)$ is the Lebesgue space of all functions from Ω into $\mathbb{R}^m$, having the finite norm

$$(\int_\Omega |f|^p\,dx)^{1/p},$$

$W^{l,p}(\Omega;\mathbb{R}^m)$ is the Sobolev space of all functions from Ω into $\mathbb{R}^m$ with the norm

$$\|f\|_{l,p,\Omega} := \Big(\int_\Omega \sum_{\alpha=0}^{l} |\nabla^\alpha f|^r\Big)^{1/r},$$

$M_b(\Omega;\mathbb{R}^m)$ is the space of all bounded Radon measures on Ω with values in $\mathbb{R}^m$,
$BD(\Omega)$ is the space of all functions in $L^1(\Omega;\mathbb{R}^n)$ such that $\varepsilon(u)\in M_b(\Omega;\mathbb{M}^{n\times n}_{sym})$,
$\mathcal{L}^n$ stands for the Lebesgue measure on $\mathbb{R}^n$,
$\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorf meausure.

In the sequel we will make use of the spaces

$$D^{2,1}(\Omega) = \Big\{v\in L^1(\Omega;\mathbb{R}^n) : \|v\|_{2,1} = \|\operatorname{div} v\|_{L^2(\Omega)} + \|v\|_{L_1(\Omega)} + \|\varepsilon^D(v)\|_{L^1(\Omega)} < +\infty\Big\},$$
$$D^{2,1}_{\Gamma_0}(\Omega) = \Big\{v\in D^{2,1}(\Omega) : v=0 \quad \text{on } \Gamma_0\Big\},$$

which are well-known spaces of weakly differentiable vector-valued functions. For their properties we refer to [FS00, Appendix A.2]. Let us introduce the notation

$$\Sigma = \Big\{\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \ : \ \operatorname{div}\sigma\in L^n(\Omega;\mathbb{R}^n),\ \sigma^D\in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)\Big\},$$
$$\mathcal{K} = \Big\{\sigma\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \ : \ \sigma(x)\in\mathbb{K} \text{ for a.e. } x\in\Omega\Big\}.$$

4.2.1 The main result

We impose the following assumptions on the data of the problem

$$\begin{aligned} &f\in AC([0,T];L^n(\Omega;\mathbb{R}^n))\cap L^\infty([0,T];C^1_{loc}(\Omega;\mathbb{R}^n))\\ &F\in AC([0,T];L^\infty(\Gamma_1))\\ &w\in AC([0,T];W^{1,2}(\Omega;\mathbb{R}^n)).\end{aligned} \tag{4.3}$$

We also assume the so-called uniform safe-load condition:

$$\begin{aligned} &\text{there exists a function } \varrho\in AC([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \text{ such that}\\ &\operatorname{div}_x\varrho(t) = -f(t) \text{ in } \Omega \text{ and } [\varrho\nu] = F(t) \text{ on } \Gamma_1 \text{ for every } t\in[0,T],\\ &|\varrho^D(t,x)|\le (1-\lambda)\sqrt{2}k_* \text{ for some } 0<\lambda<1, \text{ a.e. } x\in\Omega, \text{ for every } t\in[0,T],\\ &\text{and } \varrho^D\in AC([0,T];L^\infty(\Omega;\mathbb{M}^{n\times n}_D)).\end{aligned} \tag{4.4}$$

Suppose that $\partial\Omega\in C^2$ is partitioned into two disjoint open sets Γ_0, Γ_1 and their common interface $\gamma=\partial\Gamma_0=\partial\Gamma_1$:

$$\partial\Omega = \Gamma_0\cup\gamma\cup\Gamma_1.$$

Further, assume that

for each $x \in \gamma$, there exists a C^2 diffeomorphism defined in a neighbourhood of x which maps $\partial\Omega$ to an $(n-1)-$dimensional hyperplane, and γ to an $(n-2)$-dimensional plane. (4.5)

Finally, following [FS00] we require the following condition on the domain Ω and the partitioning Γ_0, Γ_1 that guarantees the density of smooth functions in anisotropic spaces:

$$D^{2,1}_{\Gamma_0}(\Omega) \cap C^\infty(\overline{\Omega}; \mathbb{R}^n) \text{ is dense in } D^{2,1}_{\Gamma_0}(\Omega). \tag{4.6}$$

Remark, that there are number of cases, which satisfy this condition, among them pure Dirichlet ($\Gamma_0 = \partial\Omega$) and Neumann ($\Gamma_1 = \partial\Omega$) cases, as well as numerous cases of mixed boundary conditions. We refer to [FS00, Appendix A.2] for some examples.

The main result of this paper is the following theorem.

Theorem 4.2.1. *Suppose that* $n = 2, 3$, $\partial\Omega \in C^2$, $\mathbb{A}$ *has the form* (4.1) *and the assumptions* (4.3)-(4.6) *are satisfied. Then for the solution* (u, e, p) *of the quasistatic problem, see Definition 4.3.6, we have*

$$\sigma \in L^\infty([0,T]; W^{1,2}_{loc}(\Omega; \mathbb{M}^{n\times n}_{sym})),$$

with $\sigma(t,x) = \mathbb{A}^{-1}e(t,x)$.

4.3 Weak formulation of the quasistatic problem

There are several equivalent ways to state the original problem in a weak form. In this section we present a formulation, expressed in terms of energy balance and energy dissipation, presented in [DDM06]. Then we state the existence and regularity results for this quasistatic problem and briefly discuss the method of the proof, which consists in time-discretization procedure. Finally, in the end of the section, we obtain a discrete version of the absolute continuity with respect to time, which holds also at the level of incremental problems.

4.3.1 Weak formulation: quasistatic evolution

The variational formulation of rate-independent processes expresses the evolution in terms of energy balance and dissipation. In the rest of this section we follow the exposition of [DDM06]. First, we recall two definitions, which are needed to deal with boundary conditions in a relaxed form and to have the duality between the plastic part of the strain and functions from the set Σ, defined above. We note that the latter definition generalizes the well-known stress-strain duality, studied in [KT83].

Definition 4.3.1. A triple $(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ is said to be admissible for a given boundary data $w \in W^{1,2}(\Omega; \mathbb{R}^n)$, if

1. $\varepsilon(u) = e + p$ in Ω,

2. $p = (w - u) \odot \nu \mathcal{H}^{n-1}$ on Γ_0.

The set of all admissible triples for a given w is denoted by $A(w)$.

Remark 4.3.2. We point out that the first part of this definition is responsible for the additive decomposition, while the second condition reflects the weak form of the boundary conditions, which are typical in the variational theory of functionals with linear growth.

Definition 4.3.3. For $w \in W^{1,2}(\Omega; \mathbb{M}^{n\times n}_{sym})$, an admissible triple $(u, e, p) \in A(w)$ and $\sigma \in \Sigma$ we define a measure $[\sigma^D : p] \in M_b(\Omega \cup \Gamma_0)$ by

$$\int_{\Omega\cup\Gamma_0} \varphi \, d[\sigma^D : p] = \int_\Omega \varphi \, d[\sigma^D : \varepsilon^D(u)] - \int_\Omega \varphi \sigma^D : e^D \, dx + \int_{\Gamma_0} \varphi(w - u) \cdot [\sigma\nu]^\perp \, d\mathcal{H}^{n-1},$$

for every $\varphi \in C(\Omega \cup \Gamma_0)$. Thus, the following duality is well-defined:

$$\langle \sigma^D : p \rangle_{\Sigma,\Pi} = [\sigma^D : p](\Omega \cup \Gamma_0).$$

Remark 4.3.4. Here $[\sigma^D : \varepsilon^D(u)]$ is the measure, defined in [KT83]. As in the case of stress-strain duality, here the difficulty is due to the fact, that σ^D is an L^∞ function, while p is just a bounded Radon measure.

One can show, that for the duality defined in this way, the usual integration by parts formula holds:

Proposition 4.3.5. *Let $\sigma \in \Sigma$, $f \in L^n(\Omega; \mathbb{R}^n)$, $F \in L^\infty(\Gamma_1; \mathbb{R}^n)$ and let $(u, e, p) \in A(w)$ with $w \in H^1(\Omega; \mathbb{R}^n)$. Assume that $\operatorname{div} \sigma = -f$ a.e. in Ω and $[\sigma\nu] = F$ on Γ_1. Then*

$$\langle \sigma^D, p \rangle_{\Sigma;\Pi} + \int_\Omega \sigma : (e - \varepsilon(w)) \, dx = \int_\Omega f \cdot (u - w) \, dx + \int_{\Gamma_1} F \cdot (u - w) \, d\mathcal{H}^{n-1}. \qquad (4.7)$$

Now let us define the functionals which appear in the energy formulation of the problem. We start by defining the quadratic form $\mathcal{Q} : L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \to \mathbb{R}$, corresponding to the stored elastic energy, by

$$\mathcal{Q}(e) = \frac{1}{2} \int_\Omega \mathbb{A}^{-1} e : e \, dx.$$

Denoting by $H : \mathbb{M}^{n\times n}_D \to \mathbb{R}$ the support function to the sections of $\mathbb{K}$, which in the case of Prandtl-Reuss perfect plasticity has a very simple form, we introduce in the usual way the convex functional of measures $\mathcal{H} : M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D) \to \mathbb{R}$. Then the dissipation associated with $\mathcal{H}$ in any time interval $[s,t] \subset [0,T]$ is given by

$$\mathcal{D}_{\mathcal{H}}(p; s, t) = \sup\left\{ \sum_{j=1}^{M} \mathcal{H}(p(t_j) - p(t_{j-1})) \ : \ s = t_0 \leq \cdots \leq t_M = t, \ M \in \mathbb{N} \right\}.$$

Finally, we define the total load $\mathcal{F} : [0,T] \to BD(\Omega)'$ by

$$\mathcal{F}[t]u = \int_{\Omega} f(t) \cdot u \, dx + \int_{\Gamma_1} F(t) \cdot u \, d\mathcal{H}^{n-1}. \tag{4.8}$$

Now we are in a position to give a variational formulation of the quasistatic problem.

Definition 4.3.6. A quasistatic evolution is a function

$$(u, e, p) : [0,T] \to BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D),$$

which satisfies the following conditions

(qs1) (global stability): For every $t \in [0,T]$ the triple $(u,e,p)(t) \in A(w(t))$ and

$$\mathcal{Q}(e(t)) - \mathcal{F}[t]u(t) \leq \mathcal{Q}(\eta) + \mathcal{H}(q - p(t)) - \mathcal{F}[t]v$$

for every $(v, \eta, q) \in A(w(t))$,

(qs2) (energy balance): $p : [0,T] \to M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t \in [0,T]$

$$\mathcal{Q}(e(t)) + \mathcal{D}_{\mathcal{H}}(p; 0, t) - \mathcal{F}[t]u(t) =$$
$$= \mathcal{Q}(e(0)) - \mathcal{F}[0]u(0) + \int_0^t \Big[\langle \sigma(s), \varepsilon(\dot{w}(s)) \rangle_{L^2;L^2} - \mathcal{F}[s]\dot{w}(s) - \dot{M}[s]u(s) \Big] \, ds$$

where $\sigma(t) = \mathbb{A}^{-1} e(t)$.

4.3.2 Existence result and time-discretization

The following theorem establishes the existence of a solution to the quasistatic problem in perfect plasticity.

Theorem 4.3.7. *Let $(u_0, e_0, p_0) \in A(w(0))$ satisfy the stability condition*

$$Q(e_0) - \mathcal{F}[0]u_0 \leq Q(\eta) + \mathcal{H}(q - p_0) - \mathcal{F}[0]v,$$

for every $(v, \eta, q) \in A(w(0))$. *Then there exists a quasistatic evolution*

$$(u(t), e(t), p(t)),$$

such that

$$u(0) = u_0, \; e(0) = e_0, \; p(0) = p_0.$$

Moreover, the elastic part of the symmetrized gradient $t \mapsto e(t)$ *is unique and a quasistatic evolution* (u, e, p) *as a function from* $[0, T]$ *to* $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ *is absolutely continuous in time.*

In [DDM06] this theorem is proved by a discretization of time. We divide the interval $[0, T]$ into N equal parts of length T/N by points $(t^N_m)_{m=0,\dots,N}$. For $m = 0, \dots, N$ we set

$$w^N_m = w(t^N_m), \; f^m_N = f(t^N_m), \; F^m_N = F(t^N_m), \; \mathcal{F}^N_m = \mathcal{F}[t^N_m], \quad \text{and } \varrho^N_m = \varrho(t^N_m). \tag{4.9}$$

For every N we define u^N_m, e^N_m and p^N_m by induction. We set

$$(u^N_0, e^N_0, p^N_0) = (u_0, e_0, p_0) \in A(w(0)),$$

while for every $m = 1, \dots, N$ we define (u^N_m, e^N_m, p^N_m) as a solution to the incremental problem

$$\min_{(u,e,p)\in A(w^N_m)} \Big\{ \mathcal{Q}(e) + \mathcal{H}(p - p^N_{m-1}) - \mathcal{F}^N_m(u) \Big\}. \tag{4.10}$$

Remark 4.3.8. We note, that (u, e, p) is a solution to (4.10) if and only if one of the following conditions holds for $\sigma := \mathbb{A}^{-1} e$:

1. $-\mathcal{H}(q) \le \langle \sigma | \eta \rangle_{L^2;L^2} - \langle f^N_m | v \rangle_{L^n;L^{n'}} \le \mathcal{H}(-q)$ for every $(v, \eta, q) \in A(0)$.

2. $\sigma \in \Sigma \cap \mathcal{K}$ with $\operatorname{div} \sigma = -f^N_m$ and $[\sigma\nu] = F^m_N$.

For $m = 0, \dots, N$ we set $\sigma^N_m = \mathbb{A}^{-1} e^N_m$ and for every $t \in [0, T]$ we define piecewise constant interpolations

$$u_N(t) = u^N_m, \quad e_N(t) = e^N_m, \quad p_N(t) = p^N_m, \quad \sigma_N(t) = \sigma^N_m,$$
$$w_N(t) = w^N_m, \quad f_N(t) = f^N_m, \quad F_N(t) = F^m_N, \quad \mathcal{F}_N(t) = \mathcal{F}^N_m, \quad \varrho_N(t) = \varrho^N_m,$$

where m is the largest integer such that $t^N_m \le t$. By definition $(u_N(t), e_N(t), p_N(t)) \in A(w_N(t))$.

In the proof of the existence, it was shown that for approximate solutions one has the estimate

$$\sup_{t\in[0,T]} \|e_N(t)\|_{L^2} + \operatorname{Var}(p_N; 0, T) + \sup_{t\in[0,T]} \|u_N\|_{BD} \le C, \tag{4.11}$$

which is uniform with respect to N, and it was established that these functions converge pointwise (with respect to t) to a solution of the quasistatic evolution problem.

4.3.3 Continuity estimates of solutions of the incremental problems

In [DDM06] it was established that the quasistatic evolution is absolutely continuous in time. However, as we will deal precisely with the solutions of the time-discretized problems, we would need the continuity estimates of solutions at the level of incremental problems.

The following notation will be often used below: given a function $h : [0,T] \to X$,

$$\delta h_m^N := h(t_m^N) - h(t_{m-1}^N). \tag{4.12}$$

We also consider the increment of the data of the problem, defined by

$$D_m^N := \|\delta\varrho_m^N\|_{L^2} + \|\delta\varrho_m^{ND}\|_{L^\infty} + \|\delta w_m^N\|_{W^{1,2}} + \|\delta f_m^N\|_{L^n} + \|\delta F_m^N\|_{L^\infty}. \tag{4.13}$$

We note, that by (4.3), we may assume the data of the problem to be Lipschitz with respect to time. Indeed, every absolutely continuous functions can be made Lipschitz just by time reparametrization, which leads to a corresponding reparametrization of the solutions, the problem being rate-independent. In other words, we may suppose, that

$$D_m^N \le \frac{C}{N}. \tag{4.14}$$

Theorem 4.3.9. *For solutions of the incremental problems* (u_m^N, e_m^N, p_m^N) *the following inequality holds:*

$$\|\delta e_m^N\|_{L^2} + \|\delta p_m^N\|_{M_b} + \|\varepsilon(\delta u_m^N)\|_{M_b} + \|\delta u_m^N\|_{L^1} \le D_m^N, \tag{4.15}$$

where δh_m^N *in understood as in* (4.12) *and* D_m^N *denotes the increment of the data of the problem, defined by* (4.13).

Proof: As the triple

$$(u_{m-1}^N + w_N^m - w_{m-1}^N, e_{m-1}^N + \varepsilon(w_m^N) - \varepsilon(w_{m-1}^N), p_{m-1}^N) \in A(w_m^N),$$

the minimality condition (4.10) and the integration by parts formula (4.7) imply

$$\mathcal{Q}(e_m^N) - \int_\Omega \varrho_m^N : e_m^N\,dx + \mathcal{H}(p_m^N - p_{m-1}^N) - \langle \varrho_m^N, p_m^N - p_{m-1}^N\rangle_{\Sigma;\Pi} \le$$
$$\le \mathcal{Q}(e_{m-1}^N + \varepsilon(w_m^N) - \varepsilon(w_{m-1}^N)) - \int_\Omega \varrho_m^N : (e_{m-1}^N + \varepsilon(w_m^N) - \varepsilon(w_{m-1}^N))\,dx$$

Developing the quadratic form in the right-hand side we arrive at:

$$\begin{aligned}\frac{1}{2}\int_\Omega \sigma_m^N : e_m^N\,dx - \frac{1}{2}\int_\Omega \sigma_{m-1}^N : e_{m-1}^N\,dx + \mathcal{H}(p_m^N - p_{m-1}^N) \le \\ \le \mathcal{Q}(\varepsilon(w_m^N) - \varepsilon(w_{m-1}^N)) + \int_\Omega \sigma_{m-1}^N : (\varepsilon(w_m^N) - \varepsilon(w_{m-1}^N))\,dx + \\ + \langle \varrho_m^N, p_m^N - p_{m-1}^N\rangle_{\Sigma;\Pi} - \int_\Omega \varrho_m^N : (e_{m-1}^N + \varepsilon(w_m^N) - \varepsilon(w_{m-1}^N))\,dx + \int_\Omega \varrho_m^N : e_m^N\,dx.\end{aligned} \tag{4.16}$$

Now consider the functions

$$v = u_m^N - u_{m-1}^N - (w_m^N - w_{m-1}^N),\ \eta = e_m^N - e_{m-1}^N - (\varepsilon(w_m^N) - \varepsilon(w_{m-1}^N)),$$
$$q = p_m^N - p_{m-1}^N.$$

Since $(v, \eta, q) \in A(0)$ and $(u^N_{m-1}, e^N_{m-1}, p^N_{m-1})$ is a solution of the corresponding minimum problem at the previous step, we obtain, by means of Remark 4.3.8 and the integration by parts formula (4.7)

$$\begin{gathered} -\int_\Omega \sigma^N_{m-1} : (e^N_m - e^N_{m-1})\, dx + \int_\Omega \varrho^N_{m-1} : (e^N_m - e^N_{m-1})\, dx + \\ \langle \varrho^{N\,D}_{m-1}, p^N_m - p^N_{m-1} \rangle_{\Sigma;\Pi} + \int_\Omega (\sigma^N_{m-1} - \varrho^N_{m-1}) : (\varepsilon(w^N_m) - \varepsilon(w^N_{m-1})) \le \\ \le \mathcal{H}(p^N_m - p^N_{m-1}). \end{gathered} \tag{4.17}$$

By combining (4.16) and (4.17) we get the following

$$\begin{gathered} \mathcal{Q}(e^N_m - e^N_{m-1}) = \frac{1}{2}\int_\Omega \sigma^N_m : e^N_m\, dx - \frac{1}{2}\int_\Omega \sigma^N_{m-1} : e^N_{m-1}\, dx - \\ -\int_\Omega \sigma^N_{m-1} : (e^N_m - e^N_{m-1})\, dx \le \mathcal{Q}(\varepsilon(w^N_m) - \varepsilon(w^N_{m-1})) + \int_\Omega \sigma^N_{m-1} : (\varepsilon(w^N_m) - \\ \varepsilon(w^N_{m-1}))\, dx + \langle \varrho^N_m, p^N_m - p^N_{m-1} \rangle_{\Sigma;\Pi} - \\ -\int_\Omega \varrho^N_m : (e^N_{m-1} + \varepsilon(w^N_m) - \varepsilon(w^N_{m-1}))\, dx + \int_\Omega \varrho^N_m : e^N_m\, dx - \\ -\int_\Omega \varrho^N_{m-1} : (e^N_m - e^N_{m-1})\, dx - \\ -\langle \varrho^{ND}_{m-1}, p^N_m - p^N_{m-1} \rangle_{\Sigma;\Pi} - \int_\Omega (\sigma^N_{m-1} - \varrho^N_{m-1}) : (\varepsilon(w^N_m) - \varepsilon(w^N_{m-1})). \end{gathered} \tag{4.18}$$

Let us apply the integration by parts formula (4.7) to compute $\langle \varrho^N_m, p^N_m - p^N_{m-1} \rangle_{\Sigma;\Pi}$:

$$\begin{aligned} \langle \varrho^N_m, p^N_m - p^N_{m-1} \rangle_{\Sigma;\Pi} = &-\int_\Omega \varrho^N_m : (e^N_m - \varepsilon(w^N_m) - e^N_{m-1} + \varepsilon(w^N_{m-1}))\, dx + \\ &+\int_\Omega f^N_m \cdot (u^N_m - w^N_m - u^N_{m-1} + w^N_{m-1})\, dx + \\ &+\int_{\Gamma_1} F^N_m \cdot (u^N_m - w^N_m - u^N_{m-1} + w^N_{m-1})\, d\mathcal{H}^{n-1}, \end{aligned} \tag{4.19}$$

with the analogous expression for $\langle \varrho^N_{m-1}, p^N_m - p^N_{m-1} \rangle_{\Sigma;\Pi}$.

Putting the identity (4.19) into the inequality (4.18) we end up with the estimate

$$\begin{gathered} \mathcal{Q}(e^N_m - e^N_{m-1}) \le \mathcal{Q}(\varepsilon(w^N_m) - \varepsilon(w^N_{m-1})) + \\ +\int_\Omega (f^N_m - f^N_{m-1}) \cdot (u^N_m - u^N_{m-1} - (w^N_m - w^N_{m-1}))\, dx + \\ +\int_{\Gamma_1} (F^N_m - F^N_{m-1}) \cdot (u^N_m - u^N_{m-1} - (w^N_m - w^N_{m-1}))\, d\mathcal{H}^{n-1} \le \\ \le C\, \|\varepsilon(w^N_m) - \varepsilon(w^N_{m-1})\|^2_{L^2} + \\ +\Big(\|f^N_m - f^N_{m-1}\|_{L^n} + \|F^N_m - F^N_{m-1}\|_{L^\infty} \Big)\, \|u^N_m - w^N_m - (u^N_{m-1} - w^N_{m-1})\|_{BD}. \end{gathered} \tag{4.20}$$

Now let us estimate $\|p^N_m - p^N_{m-1}\|_1$ in terms of the data of the problem. First of all, the safe load condition yields

$$\alpha \|p^N_m - p^N_{m-1}\|_1 \le \mathcal{H}(p^N_m - p^N_{m-1}) - \langle \varrho^{N\,D}_m, p^N_m - p^N_{m-1} \rangle.$$

Now, the relation (4.16) and the boundedness of $\|\varrho^N_m\|_{L^2}$, $\|\varrho^{N\,D}_m\|_{L^\infty}$, $\|e^N_m\|_{L^2}$ and $\|p^N_m\|_1$ imply

$$\|p^N_m - p^N_{m-1}\|_1 \le C(\|e^N_m - e^N_{m-1}\|_{L^2} + D^N_m) \tag{4.21}$$

Taking into account the inequality

$$\|u_m^N - w_m^N - (u_{m-1}^N - w_{m-1}^N)\|_{BD} \le C(\|e_m^N - e_{m-1}^N\|_{L^2} + \|p_m^N - p_{m-1}^N\|_1 + \|\varepsilon(w_m^N) - \varepsilon(w_{m-1}^N)\|_{L^2}),$$

proved in [DDM06, relations (3.24) and (3.25) in Theorem 3.8], the estimate

$$\|p_m^N - p_{m-1}^N\|_{M_b} + \|e_m^N - e_{m-1}^N\|_{L^2} \le CD_m^N \tag{4.22}$$

follows now from (4.20), (4.21) and the application of the Cauchy inequality.

To prove

$$\|\varepsilon(u_m^N) - \varepsilon(u_{m-1}^N)\|_{M_b} \le CD_m^N, \tag{4.23}$$

we recall the additive decomposition $\varepsilon(u) = e + p$ and make use of (4.22).

Finally to show the validity of (4.15), it remains to estimate $\|u_m^N - u_{m-1}^N\|_{L^1}$. By the Poincare inequality for BD it suffices to estimate $\|u_m^N - u_{m-1}^N\|_{L^1(\Gamma_0)}$:

$$\|u_m^N - u_{m-1}^N\|_{L^1(\Gamma_0)} \le \sqrt{2}\|p_m^N - p_{m-1}^N\|_1 + C\|w_m^N - w_{m-1}^N\|_{W^{1,2}},$$

so the result follows from (4.14), (4.22), (4.23) and the latter inequality. □

4.4 Relaxation of convex variational problems in non-reflexive spaces

For the reader's convenience, here we state the general construction of the relaxed convex variational problems in non-reflexive spaces, which is well-suited for studying the problems in plasticity theory. For the details, we refer to [FS00, Chapter 1]. We remark that, by abuse of notations, in this section the symbol u_0 stands for the boundary data of a saddle-point problem, which corresponds to w_m^N, the boundary data of the incremental problems, and has nothing to do with the initial data u_0 of the quasistatic problem.

Let V, U and P be Banach spaces, $V \subset U$, and let V_0 be a subspace of V. Let $A : V \to P$ denote a linear bounded operator, and suppose that $G : P \to \overline{\mathbb{R}}$ and $\widehat{M} : U \to \overline{\mathbb{R}}$ are convex, proper, lower semicontinuous functionals. We denote the dual spaces to P and U by P^* and U^*, and the duality relations between the corresponding spaces by $\langle \cdot, \cdot \rangle_{P,P^*}$ and $\langle \cdot, \cdot \rangle_{U,U^*}$.

By G^* we denote the conjugate functional to G, i.e. $G^*(p^*) = \sup\{\langle p^*, p\rangle_{P,P^*} - G(p) \ : \ p \in P\}$, for $p^* \in P^*$. Let us consider the variational problem

$$\text{find } u \in u_0 + V_0 \text{ such that } I(u) = \inf\{I(v) \ : \ v \in u_0 + V_0\}, \tag{4.24}$$

where $u_0 \in V$ is fixed, and

$$I(v) = G(Av) + \widehat{M}(v).$$

Let us intoduce the Lagrangian ℓ by letting

$$\ell(v, q^*) = \langle q^*, Av\rangle_{P^*,P} - G^*(q^*) + \widehat{M}(v). \tag{4.25}$$

The dual problem thus takes the form

$$\text{find } p^* \in P^* \text{ such that } R(p^*) = \sup\{R(q^*) \ : \ q^* \in P^*\}, \tag{4.26}$$

where $R(q^*) = \inf\{\ell(v, q^*) \ : \ v \in u_0 + V_0\}$. The following theorem (see [FS00, Chapter 1]) states that the problem (4.26) has a solution.

Theorem 4.4.1. *Suppose that the following two conditions hold:*

$$\widehat{C} := \inf\{I(v) \ : \ v \in u_0 + V_0\} \in \mathbb{R}; \tag{4.27}$$

$$\begin{cases} \textit{there exists } u_1 \in u_0 + V_0 \textit{ such that } G(Au_1) < +\infty, \ \widehat{M}(u_1) < +\infty \\ \textit{and the function } p \mapsto G(Au_1 + p) \textit{ is continuous at zero.} \end{cases} \tag{4.28}$$

Then problem (4.26) has at least one solution and the identity

$$\widehat{C} = \sup\{R(q^*) \ : \ q^* \in P^*\} \tag{4.29}$$

is valid.

Together with problems (4.24) and (4.26) let us consider the following minimax problem

$$\begin{cases} \text{find a pair } (u, p^*) \in (u_0 + V_0) \times P^* \text{ such that} \\ \ell(u, q^*) \le \ell(u, p^*) \le \ell(v, p^*), \ \text{for all } v \in u_0 + V_0, \ q^* \in P^*. \end{cases} \tag{4.30}$$

Since $G : P \to \mathbb{R}$ is a proper, convex, l.s.c. functional, then $G = G^{**}$, and therefore

$$I(v) = \sup\{\ell(v, q^*) \ : \ q^* \in P^*\}. \tag{4.31}$$

Thus under conditions (4.27) and (4.28) we have the identity

$$\inf_{v \in u_0 + V_0} \sup_{q^* \in P^*} \ell(v, q^*) = \widehat{C} = \sup_{q^* \in P^*} \inf_{v \in u_0 + V_0} \ell(v, q^*) \tag{4.32}$$

and the general duality theory provides the following statement:

$$\begin{cases} \text{a pair } (u, p^*) \in (u_0 + V_0) \times P^* \\ \text{is a saddle point of the minimax problem (4.30) if and only if} \\ u \in u_0 + V_0 \text{ is a minimizer of problem (4.24) and} \\ p^* \in P^* \text{ is a maximizer of problem (4.26).} \end{cases} \tag{4.33}$$

So by Theorem 4.4.1 and (4.33), the solvability of problem (4.24) is equivalent to the solvability of the minimax problem (4.30).

Let us assume the following additional properties:

$$\begin{cases} \text{the embedding } V \hookrightarrow U \text{ is continuous;} \\ V_0 \text{ is dense in } U; \\ U \ \text{ is a reflexive space;} \end{cases} \tag{4.34}$$

$$\begin{cases} \text{there exists } u_2 \in u_0 + V_0, \ \text{such that } u_2 \in \operatorname{int}\operatorname{dom} \widehat{M}, \\ \text{where } \operatorname{dom} \widehat{M} = \{u \in U \ : \ \widehat{M}(u) < +\infty\}. \end{cases} \tag{4.35}$$

$$I(v) \to +\infty \text{ if } \|v\|_V \to +\infty \text{ and } v \in u_0 + V_0. \tag{4.36}$$

If the space V is nonreflexive, in general, problems (4.24) and (4.30) have no solutions. Thus, we need to relax our problem, and the desired relaxation should satisfy the following two requirements:

1. conservation of the greatest lower bound for problem (4.24),
2. conservation of the dual problem.

Remark 4.4.2. The first requirement needs no explanations: speaking about relaxation, we should not change the infimum of the problem. While the second point is due to the fact, that in many physical applications the solution of the dual problem is unique and has a clear geometrical or mechanical interpretation, so there is no necessity to change the dual problem. In the case of perfect plasticity the stress tensor is responsible for the distribution of elastic and plastic zones.

In order to extend the domain of definition of the functional G, we should construct a suitable extension of the operator A. We begin by introducing an auxiliary operator A^* with a domain $D(A^*)$ defined as

$$\begin{cases} D(A^*) = \{p^* \in P^* \ : \ \text{there exists } u^* \in U^*, \text{ such that} \\ \langle p^*, Au\rangle_{P^*;P} = \langle u^*, u\rangle_{U^*;U} \text{ for all } u \in V_0\}. \end{cases} \tag{4.37}$$

The density condition (4.34) implies that for each $p^* \in D(A^*)$ there exists only one element $u^* \in U^*$ satisfying the identity $\langle p^*, Au\rangle_{P^*;P} = \langle u^*, u\rangle_{U^*;U}$ on V_0. Thus we can define the linear operator $A^* : D(A^*) \to U^*$ through the relation

$$\langle p^*, Au\rangle_{P^*;P} = \langle A^*p^*, u\rangle_{U^*;U} \quad \text{for every } p^* \in D(A^*),\ u \in V_0.$$

If u_0 is a fixed element from V, then we have the identity

$$\langle p^*, Au\rangle_{P^*;P} = \mathcal{E}(u_0, p^*) + \langle A^*p^*, u\rangle_{U^*;U}, \quad \text{for all } u \in u_0 + V_0,\ p^* \in D(A^*), \tag{4.38}$$

where

$$\mathcal{E}(u_0, p^*) = \langle p^*, Au_0\rangle_{P^*;P} - \langle A^*p^*, u_0\rangle_{U^*;U}.$$

We enlarge the set $u_0 + V_0$ by letting

$$V_+ = \left\{ u \in U \ : \sup_{p^* \in D(A^*),\ \|p^*\|_{P^*} \le 1} |\mathcal{E}(u_0, p^*) + \langle A^*p^*, u\rangle_{U^*;U}| < +\infty \right\}, \tag{4.39}$$

and introduce a relaxation Φ of the functional I by means of the Lagrangian L:

$$\begin{cases} L(v, q^*) = \mathcal{E}(u_0, q^*) + \langle A^*q^*, v\rangle_{U^*;U} - G^*(q^*) + \widehat{M}(v) \\ q^* \in D(A^*),\ v \in V_+; \\ \Phi(v) = \sup\limits_{q^* \in D(A^*)} L(v, q^*), \quad \Phi : V_+ \to \mathbb{R}. \end{cases} \tag{4.40}$$

Let us collect some consequences of these definitions.

Remark 4.4.3. The following relations hold:

$$u_0 + V_0 \subset V_+, \tag{4.41}$$

$$\Phi(v) \le I(v), \quad \text{for all } v \in u_0 + V_0. \tag{4.42}$$

Moreover, under certain hypotheses the equality holds in (4.42).

Remark 4.4.4. Suppose that for any $p^* \in \operatorname{dom} G^*$ there exists a sequence $p^*_m \in D(A^*)$ such that

$$\begin{cases} p^*_m \overset{*}{\rightharpoonup} p^* \quad \text{in } P^*, \\ G^*(p^*_m) \to G^*(p^*). \end{cases} \tag{4.43}$$

Then the identity

$$\Phi(v) = I(v) \quad \text{for all } v \in u_0 + V_0 \tag{4.44}$$

is valid.

The following remark clarifies the meaning of the relaxation considered:

Remark 4.4.5. Consider a sequence $u_m \in u_0 + V_0$, bounded in the norm of the space V and converging to u weakly in U. Then

$$u \in V_+,$$

$$\liminf_{m\to+\infty} I(u_m) \geq \Phi(u).$$

Now we consider the minimax problem

$$\begin{cases} \text{find a pair } (u^*, p) \in V_+ \times D(A^*) \text{ such that} \\ L(u, q^*) \leq L(u, p^*) \leq L(v, p^*), \quad \text{for all } v \in V_+,\ q^* \in D(A^*). \end{cases} \tag{4.45}$$

This minimax problem generates two variational problems being in duality:

$$\begin{cases} \text{find } u \in V_+ \text{ such that} \\ \Phi(u) = \inf\{\Phi(v) \ :\ v \in V_+\}, \end{cases} \tag{4.46}$$

where $\Phi(v) = \sup\{L(v, q^*) \ :\ q^* \in D(A^*)\}$, and

$$\begin{cases} \text{find } p^* \in D(A^*) \text{ such that} \\ \tilde{R}(p^*) = \sup\{\tilde{R}(q^*) \ :\ q^* \in D(A^*)\}), \end{cases} \tag{4.47}$$

with $\tilde{R}(q^*) = \inf\{L(v, q^*) \ :\ v \in V_+\}$.

Remark 4.4.6. Lemma 4.4.5 shows that there is a hope to apply the direct methods: the coercivity implies the boundedness of a minimizing sequence of the problem (4.24) in U and the potential minimizer of (4.46) will be a weak cluster point of this sequence, which belongs to the set V_+ and such that the $\liminf$ inequality is satisfied.

Indeed, this remark leads us to the following conclusion, that we state without proof:

Theorem 4.4.7. *Suppose that conditions (4.27), (4.28) and (4.34)-(4.36) hold. Then:*

1. *Problems* (4.46) *and* (4.47) *are solvable. Moreover, if* $u \in V_+$ *is a solution to problem* (4.46) *and* $p^* \in D(A^*)$ *is a solution to problem* (4.47), *then the identity*

$$\Phi(u) = \widehat{C} = \tilde{R}(p^*) \tag{4.48}$$

holds true.

2. *Problems* (4.26) *and* (4.47) *are equivalent, i.e. they have the same set of solutions.*

3. *A pair* $(u, p^*) \in V_+ \times D(A^*)$ *is a saddle point of the minimax problem* (4.45) *if and only if* $u \in V_+$ *is a minimizer of problem* (4.46) *and* $p^* \in D(A^*)$ *is a maximizer of problem* (4.47).

4. *Any minimizing sequence of problem* (4.24) *contains a subsequence converging to some solution of problem* (4.46) *weakly in* U.

4.5 Minimax formulation of the incremental problem

Recall that, during the proof of existence of a weak solution to the quasistatic evolution problem of perfect plasticity, the time-discretization procedure leads one to solving the following incremental problem at every step (see (4.10)):

$$\min_{(u,e,p)\in A(w_m^N)} \{\mathcal{Q}(e) + \mathcal{H}(p - p_{m-1}^N) - \mathcal{F}_m^N(u)\}, \tag{4.49}$$

with p_{m-1}^N be a solution of the corresponding incremental problem, obtained at the previous step.

In the rest of this section, to simplify the notations, we will omit writing the indices m and N when dealing with some functionals and spaces. So, in what follows the functionals G, $\widehat{M}$, $\mathcal{F}$, ℓ, L, I, R, Φ and the space V_+ should be understood as G_m^N, $\widehat{M}_m^N$, $\mathcal{F}_m^N$, ℓ_m^N, L_m^N, I_m^N, R_m^N, Φ_m^N and $(V_+)_m^N$, written, however, without an explicit dependence on t_m^N.

We state the minimax formulation of the incremental problem and briefly sketch the ideas, leading to the notion of a weak solution. More precisely we define a Lagrangian such that the saddle points of the corresponding relaxed problem are given by $(\delta u_m^N, \sigma_m^N)$, where $\delta u_m^N = u_m^N - u_{m-1}^N$, $\sigma_m^N = \mathbb{A}^{-1} e_m^N$ and the triple (u_m^N, e_m^N, p_m^N) is a suitable solution of (4.49) (see Theorem 4.5.5).

Note that this is a generalization of the functional formulation of the classical boundary value problem describing the equilibrium of a perfect elastoplastic body (see [FS00, Ser96, Ser93c, Ser93a, Ser93b, Ser94]).

First (Subsection 4.5.1) we introduce the functional spaces and define the functionals of the minimax problem. Then (Subsection 4.5.2) we define the Lagrangian and state the primal and dual problems. In Subsection 4.5.3 we check conditions (4.27), (4.28) and (4.34)-(4.36), that allow us to apply the abstract theory from Section 4.4. The relaxed problem and the properties of its solutions are presented in the same subsection. In subsection 4.5.4 we show, that every saddle point of the relaxed minimax problem generates a solution to the incremental problem (4.49).

4.5.1 Functional formulation

In order to handle this problem using the abstract relaxation scheme described in Section 4.4 we set

$$V = D^{2,1}(\Omega),\ V_0 = D^{2,1}_{\Gamma_0}(\Omega),\ U = L^{n/(n-1)}(\Omega;\mathbb{R}^n),$$

$$\begin{cases} P = \{p = \{\tau, a\} \in L^1(\Omega;\mathbb{M}^{n\times n}_{sym}) \times L^1(\Gamma_1;\mathbb{R}^n) : \\ \|p\|^2_P = \|\tau^D\|^2_{L^1(\Omega)} + \frac{1}{n}\|\mathrm{tr}\,\tau\|^2_{L^2(\Omega)} + \|a\|^2_{L^1(\Gamma_1)} < +\infty\}. \end{cases} \tag{4.50}$$

Then

$$\begin{cases} P^* = \{p^* = \{\sigma, b\} : \sigma^D \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}),\ \mathrm{tr}\,\sigma \in L^2(\Omega), \\ b \in L^\infty(\Gamma_1;\mathbb{R}^n)\} \end{cases} \tag{4.51}$$

Next, let us introduce the functionals $G : P \to \mathbb{R}$ and $\widehat{M} : U \to \mathbb{R}$

$$\begin{gathered} G(p) = \int_\Omega g(\tau + e^N_{m-1}) + \int_{\Gamma_1} F^N_m \cdot a\, d\mathcal{H}^{n-1}, \quad p = \{\tau, a\} \in P, \\ \widehat{M}(v) = -\int_\Omega f^N_m \cdot v\, dx, \quad v \in U, \end{gathered} \tag{4.52}$$

where $g : \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ is defined by (4.54) below.

Then it is easy to see, that for $p^* = \{\sigma, b\} \in P^*$ its Legendre transform G^* takes the form

$$G^*(p^*) = \begin{cases} \int_\Omega \left(g^*(\sigma) - \sigma : e^N_{m-1}\right) dx, & \text{if } b \equiv F^N_m \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.53}$$

Here

$$g^*(\sigma) = \frac{1}{2n^2K_0}\mathrm{tr}^2\tau + g_0^*(|\sigma^D|) = \sup_{\varkappa\in\mathbb{M}^{n\times n}_{sym}} \left\{\sigma : \varkappa - g(\varkappa)\right\}$$

is the Legendre transform of

$$g : \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}, \quad g(\varkappa) = \frac{1}{2}K_0\mathrm{tr}^2\varkappa + g_0(|\varkappa^D|), \quad \varkappa \in \mathbb{M}^{n\times n}_{sym}. \tag{4.54}$$

In the case of Hencky and Prandtl-Reuss models of plasticity $g_0 : \mathbb{R} \to \mathbb{R}$ has the form:

$$g_0(t) = \begin{cases} \mu t^2, & |t| \le t_0 = \frac{k_*}{\sqrt{2}\mu}, \\ k_*(\sqrt{2}|t| - \frac{k_*}{2\mu}), & |t| > t_0, \end{cases} \tag{4.55}$$

while its Legendre transform $g_0^* : \mathbb{R} \to \overline{\mathbb{R}}$ is given by

$$g_0^*(s) = \begin{cases} \frac{s^2}{4\mu}, & |s| \le \sqrt{2}k_*, \\ +\infty, & |s| > \sqrt{2}k_*, \end{cases}$$

4.5.2 Lagrangian and a saddle-point problem

The linear operator $A : V \to P$ is introduced as follows:

$$Av = \{\varepsilon(v),\ -v_{|\Gamma_1}\}, \quad v \in V,$$

and in view of the estimate

$$\|Av\|_P = \left(\frac{1}{n}\|\operatorname{div} v\|^2_{L^2(\Omega)} + \|\varepsilon^D(v)\|^2_{L^1(\Omega)} + \|v\|^2_{L^1(\Gamma_1)}\right)^{1/2} \leq c(\Omega, n)\|v\|_{2,1},$$

one concludes that A is continuous.

Following the ideas outlined in Section 4.4 (see (4.30)), the minimax problem is

$$\begin{cases}\text{find a pair } (u,\sigma) \in (\delta w_m^N + V_0) \times \mathcal{K}, \text{ such that} \\ \ell(u,\tau) \leq \ell(u,\sigma) \leq \ell(v,\sigma), \quad \text{for all } v \in \delta w_m^N + V_0,\ \tau \in \mathcal{K},\end{cases} \tag{4.56}$$

where the Lagrangian, according to (4.25), is given by

$$\ell(v,\tau) = \int_\Omega \Big(\varepsilon(v) : \tau + \tau : e_{m-1}^N\Big)\, dx - \int_\Omega g^*(\tau)\, dx - \int_{\Gamma_1} F_m^N \cdot v\, d\mathcal{H}^{n-1} - \int_\Omega f_m^N \cdot v\, dx,$$

and δw_m^N is defined according to (4.12). The functional I takes the form

$$I(v) = G(Av) + \widehat{M}(v) = \int_\Omega g(\varepsilon(v) + e_{m-1}^N) - \int_{\Gamma_1} F_m^N \cdot v\, d\mathcal{H}^{n-1} - \int_\Omega f_m^N \cdot v\, dx.$$

Recall that the functions f_m^N, F_m^N and δw_m^N satisfy the following conditions:

$$f_m^N \in L^n(\Omega;\mathbb{R}^n), \quad F_m^N \in L^\infty(\Gamma_1;\mathbb{R}^n), \quad \delta w_m^N \in W^{1,2}(\Omega;\mathbb{R}^n). \tag{4.57}$$

The minimax problem (4.56) generates two dual variational problems:

$$\begin{cases}\text{find } u \in \delta w_m^N + V_0 \text{ such that} \\ I(u) = \inf\{I(v)\ :\ v \in \delta w_m^N + V_0\},\end{cases} \tag{4.58}$$

and

$$\begin{cases}\text{find } \sigma_m^N \in Q_{f_m^N} \cap \mathcal{K} \text{ such that} \\ R(\sigma) = \sup\{R(\tau)\ :\ \tau \in Q_{f_m^N} \cap \mathcal{K},\}\end{cases} \tag{4.59}$$

where

$$R(\tau) = \begin{cases}\ell(\delta w_m^N, \tau), & \tau \in Q_{f_m^N} \cap \mathcal{K} \\ -\infty, & \tau \notin Q_{f_m^N} \cap \mathcal{K}\end{cases} \quad \text{for } \tau \in \mathcal{K},$$

with $Q_{f_m^N}$ defined as

$$Q_{f_m^N} = \left\{\tau \in \Sigma\ :\ \int_\Omega \tau : \varepsilon(v)\, dx = \mathcal{F}_m^N(v), \quad \text{for all } v \in V_0\right\},$$

where we refer to (4.8) and (4.9) for the definition of $\mathcal{F}_m^N$. We note that

$$\tau \in Q_{f_m^N} \Leftrightarrow \operatorname{div}\tau = -f_m^N \text{ in } \Omega, \quad [\tau\nu] = F_m^N \text{ on } \Gamma_1.$$

4.5.3 The relaxed problem

Let us check conditions (4.27), (4.28) and (4.34)-(4.36). Since the functional G is convex and finite, that is $\operatorname{dom} G = P$, the function $p \mapsto G(Au_1 + p)$ is continuous at zero for any $u_1 \in \delta w_m^N + V_0$. By the finiteness of the functional M, condition (4.28) is fulfilled. Conditions (4.34) and (4.35) are obviously satisfied.

The conditions (4.27) and (4.36) are guaranteed by the safe-load condition (4.4):

$$\begin{gathered} I(v) = \frac{K_0}{2} \int_\Omega |\operatorname{div} v + \operatorname{tr} e_{m-1}^N|^2\, dx + \\ + \sup_{\sigma \in \mathcal{K}} \left\{ \int_\Omega \sigma^D : (\varepsilon^D(v) + e_{m-1}^{ND}) - g^*(\sigma^D)\, dx \right\} - \\ - \int_\Omega \varrho_m^N : (\varepsilon(v) - \varepsilon(\delta w_m^N))\, dx + M(\delta w_m^N) \geq \frac{K_0}{2} \int_\Omega |\operatorname{div} v + \operatorname{tr} e_{m-1}^N|^2\, dx + \\ + \sup_{\sigma \in \mathcal{K}} \left\{ \int_\Omega (\sigma^D - \varrho_m^{ND}) : (\varepsilon^D(v) + e_{m-1}^{ND}) - g^*(\sigma^D)\, dx \right\} - \\ - C \int_\Omega \operatorname{tr} \varrho_m^N \operatorname{div} v\, dx + \int_\Omega \varrho_m^{ND} : e_{m-1}^N\, dx + \\ + \int_\Omega \varrho_m^N : \varepsilon(\delta w_m^N)\, dx + M(\delta w_m^N) \geq C_1 \int_\Omega \big(|\operatorname{div} v|^2 + |\varepsilon^D(v)|\big)\, dx - C \to \infty \end{gathered} \tag{4.60}$$

whenever $\|v\|_V \to \infty,\ v \in \delta w_m^N + V_0$. So the coercivity is established. Finally, the condition (4.27) is provided by the estimate

$$\widehat{C} = \inf\{I(v) \ :\ v \in \delta w_m^N + V_0\} \geq R(\varrho_m^N) > -\infty.$$

Thus, according to Theorem 4.4.1 we can state that problem (4.59) has at least one solution $\sigma \in Q_{f_m^N} \cap \mathcal{K}$, that identity (4.32) holds and the statement (4.33) is valid. Due to the non-reflexivity of V the variational problem (4.58) in general has no solutions. We construct relaxations of these variational problems following the scheme described above.

Define the operator $A^* : D(A^*) \to U^*$. As in (4.37), a pair $p^* = \{\sigma, b\} \in D(A^*)$ if and only if there exists $u^* \in U^* = L^n(\Omega; \mathbb{R}^n)$, such that

$$\int_\Omega u^* \cdot v\, dx = \int_\Omega \sigma : \varepsilon(v)\, dx - \int_{\Gamma_1} b \cdot v\ d\mathcal{H}^{n-1} \quad \text{for all } v \in V_0,$$

that is $A^* p^* := u^* = -\operatorname{div} \sigma \in L^n(\Omega; \mathbb{R}^n)$. Therefore

$$\begin{gathered} D(A^*) = \Big\{ p^* = \{\sigma, b\} \in P^* \ :\ \operatorname{div} \sigma \in L^n(\Omega; \mathbb{R}^n), \\ \int_{\Gamma_1} b \cdot v\, d\mathcal{H}^{n-1} = \int_\Omega (\sigma : \varepsilon(v) + v \cdot \operatorname{div} \sigma)\, dx, \quad \text{for all } v \in V_0 \Big\}. \end{gathered}$$

According to (4.39) the extension V_+ of the set $\delta w_m^N + V_0$ is

$$\begin{aligned} V_+ = \Big\{ & v \in L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n) : \\ & \sup_{\|p^*\|_{P^*} \leq 1,\ p^* = \{\sigma, b\} \in D(A^*)} \Big\langle - \int_{\Gamma_1} b \cdot \delta w_m^N\, d\mathcal{H}^{n-1} + \int_\Omega (\sigma : \varepsilon(\delta w_m^N) + (\delta w_m^N - v) \cdot \operatorname{div} \sigma)\, dx \Big\rangle < +\infty \Big\}. \end{aligned}$$

The important properties of this space are summarized below. In particular, the following proposition shows that a triple (u, e, p), constructed from a solution $(\delta u^m, \sigma^m)$ of a relaxed minimax problem in an obvious way (see Theorem 4.5.5 below), is kinematically admissible for the boundary data δw_m^N.

Proposition 4.5.1. *The following relations hold:*

$$V_+ \subset BD(\Omega), \tag{4.61}$$

and for every $v \in V_+$

$$\operatorname{div} v \in L^2(\Omega), \tag{4.62}$$

$$(v - \delta w_N^m) \cdot \nu = 0 \quad \text{on } \Gamma_0. \tag{4.63}$$

PROOF: The definition of V_+ implies that

$$\sup_{\sigma \in C_0^\infty(\Omega \cup \Gamma_0)} \left\langle \int_\Omega (\sigma : \varepsilon(\delta w_m^N) + (\delta w_m^N - v) \cdot \operatorname{div} \sigma)\, dx \right\rangle \le C(\|\operatorname{tr} \sigma\|_{L^2(\Omega)} + \|\sigma^D\|_{L^\infty(\Omega)}). \tag{4.64}$$

This estimate and the fact that $\delta w_m^N \in W^{1,2}(\Omega; \mathbb{R}^n)$ ensures the estimate

$$\sup_{\sigma \in C_c^\infty(\Omega; \mathbb{M}^{n\times n}_{sym})} \int_\Omega v \cdot \operatorname{div} \sigma\, dx \le C\|\sigma\|_{L^\infty(\Omega; \mathbb{M}^{n\times n}_{sym})}.$$

So the claim (4.61) is established.

By taking the test vector fields in (4.64) with $\sigma^D = 0$ we conclude that $\operatorname{div} v \in L^2(\Omega)$, thus (4.62) is proved.

As for the last claim, by taking arbitrary $\varphi \in C_c^\infty(\Omega \cup \Gamma_0)$ and taking $\sigma = \varphi I$ we get by the integration by parts formula the following inequality:

$$\begin{aligned} \int_{\Gamma_0} \varphi(\delta w_m^N - v) \cdot \nu\, d\mathcal{H}^{n-1} &= \int_{\partial\Omega} (\delta w_m^N - v) \cdot [\sigma\nu]\, d\mathcal{H}^{n-1} = \\ = \int_\Omega (\delta w_m^N - v) \cdot \operatorname{div} \sigma\, dx &+ \int_\Omega \operatorname{tr} \sigma \operatorname{div}(\delta w_m^N - v)\, dx \le C\|\varphi\|_{L^2(\Omega)}. \end{aligned}$$

This estimate, in its turn, implies that $(\delta w_m^N - v) \cdot \nu = 0$ on Γ_0. □

By the properties of g_0^* and by (4.53) we have that $G^*(p^*) = G^*(\{\tau, b\}) = +\infty$ if $b \neq F$ on Γ_1 or $\tau \notin \mathcal{K}$. Introduce the relaxed Lagrangian, as in (4.40):

$$\begin{aligned} L(v, q^*) = \mathcal{E}(\delta w_m^N, q^*) + \langle A^* q^*, v\rangle - G^*(q^*) + \widehat{M}(v) = \\ = -\int_{\Gamma_1} F_m^N \cdot \delta w_m^N\, d\mathcal{H}^{n-1} + \\ + \int_\Omega \left[\varepsilon(\delta w_m^N) : \tau + (\delta w_m^N - v) \cdot \operatorname{div} \tau - g^*(\tau) - f_m^N \cdot v + \tau : e_{m-1}^N\right] dx \end{aligned}$$

for all $v \in V_+$ and $q^* = \{\tau, F_m^N\} \in D(A^*)$, such that $\tau \in \mathcal{K}$. Now we introduce the set

$$Q = \{\tau \in \Sigma \ : \ \{\tau, F_m^N\} \in D(A^*)\} \tag{4.65}$$

and a new Lagrangian on $V_+ \times (Q \cap \mathcal{K})$ defined as

$$\widetilde{L}(v,\tau) = L(v,q^*) \tag{4.66}$$

where

$$q^* = \{\tau, F_m^N\} \in D(A^*), \quad \tau \in \mathcal{K}.$$

Now, instead of the minimax problem (4.56) we consider its relaxation

$$\begin{cases} \text{find a pair } (u,\sigma) \in V_+ \times (Q \cap \mathcal{K}) \text{ such that} \\ \widetilde{L}(u,\tau) \le \widetilde{L}(u,\sigma) \le \widetilde{L}(v,\sigma), \quad \text{for all } v \in V_+,\ \tau \in Q \cap \mathcal{K}. \end{cases} \tag{4.67}$$

For the functional $\Phi : V_+ \to \mathbb{R}$ we have the formula

$$\Phi(v) = \sup_{q^* \in D(A^*)} L(v,q^*) = \sup_{q^* = \{\tau, F_m^N\} \in D(A^*),\ \tau \in \mathcal{K}} L(v,q^*) = \sup_{\tau \in Q \cap \mathcal{K}} \widetilde{L}(v,\tau), \tag{4.68}$$

and the relaxation of the variational problem (4.58) takes the form

$$\text{find } u \in V_+ \text{ such that } \Phi(u) = \inf_{v \in V_+} \Phi(v) \tag{4.69}$$

As in [FS00, Lemma 1.3.1] one can show, that (4.43) holds, and thus Lemma 4.4.4 reads as follows.

Lemma 4.5.2. *We have*

$$\Phi(v) = I(v), \quad \text{for all } v \in \delta w_m^N + V_0.$$

Lemma 4.5.3. *For $u \in V_+$ we have*

$$\Phi(u) = \int_\Omega g(\varepsilon(u) + e_{m-1}^N) - \int_\Omega f_m^N \cdot v\,dx - \int_{\Gamma_1} F_m^N \cdot u\,d\mathcal{H}^{n-1} =$$
$$= \tfrac{K_0}{2} \int_\Omega |\operatorname{div} u + \operatorname{tr} e_{m-1}^N| + \int_\Omega g_0(|\varepsilon^D(u) + e_{m-1}^{ND}|) - \int_\Omega f_m^N \cdot v\,dx - \int_{\Gamma_1} F_m^N \cdot u\,d\mathcal{H}^{n-1},$$

where the corresponding integrals are understood as functionals of measures.

Proof: According to (4.68)

$$\begin{aligned} \Phi(u) = \sup_{\tau \in Q \cap \mathcal{K}} \Bigg\{ & -\int_{\Gamma_1} F_m^N \cdot \delta w_m^N\,d\mathcal{H}^{n-1} - \int_\Omega f_m^N \cdot u\,dx + \\ & + \int_\Omega \Big(\big(\varepsilon(\delta w_m^N) + e_{m-1}^N\big) : \tau + (\delta w_m^N - u) \cdot \operatorname{div} \tau - g^*(\tau) \Big)\,dx \Bigg\}. \end{aligned} \tag{4.70}$$

First, apply [KT83, relations (3.8) and (3.16)]:

$$\begin{aligned} \int_\Omega \varepsilon(\delta w_m^N) : \tau\,dx + \int_\Omega \delta w_m^N \cdot \operatorname{div} \tau\,dx &= \int_{\partial\Omega} [\tau\nu] \cdot \delta w_m^N\,d\mathcal{H}^{n-1} = \\ &= \int_{\Gamma_1} F_m^N \cdot \delta w_m^N\,d\mathcal{H}^{n-1} + \int_{\Gamma_0} [\tau\nu] \cdot \delta w_m^N\,d\mathcal{H}^{n-1}. \end{aligned} \tag{4.71}$$

On the other hand, by [KT83, relations (3.8), (3.16) and (3.23)]

$$
\begin{gathered}
-\int_\Omega u \cdot \operatorname{div} \tau \, dx = \int_\Omega d[\tau^D : \varepsilon^D(u)] + \tfrac{1}{n} \int_\Omega \operatorname{div} u \operatorname{tr} \tau \, dx - \\
-\int_{\Gamma_0} [\tau\nu]_\nu \cdot (\delta w_m^N)_\nu \, d\mathcal{H}^{n-1} - \int_{\Gamma_0} [\tau\nu]_\nu^\perp \cdot (\delta w_m^N)_\nu^\perp \, d\mathcal{H}^{n-1} - \int_{\Gamma_1} F_m^N \cdot u \, d\mathcal{H}^{n-1}
\end{gathered}
\tag{4.72}
$$

Using (4.71), (4.72) transform (4.70) as follows:

$$
\begin{gathered}
\Phi(u) = -\int_{\Gamma_1} F_m^N \cdot u \, d\mathcal{H}^{n-1} - \int_\Omega f_m^N \cdot u \, dx + \\
+ \sup_{\tau \in Q \cap \mathcal{K}} \left\{ \int_\Omega d[\varepsilon^D(u) : \tau^D] + \int_\Omega \tfrac{1}{n} \operatorname{div} u \operatorname{tr} \tau \, dx + \int_\Omega \big(e_{m-1}^N : \tau - g^*(\tau)\big) \, dx \right\}.
\end{gathered}
$$

Remark, that, approximating arbitrary $\tau \in Q \cap \mathcal{K}$ with $\tau_k \in C^\infty(\overline{\Omega}) \cap \mathcal{K}$ (see, e.g. [DDM06, Lemma 2.3] and [KT83, Theorem 3.2]), the former equality becomes

$$
\begin{gathered}
\Phi(u) = -\int_{\Gamma_1} F_m^N \cdot u \, d\mathcal{H}^{n-1} - \int_\Omega f_m^N \cdot u \, dx + \\
+ \sup_{\tau \in C^\infty(\overline{\Omega}) \cap \mathcal{K}} \left\{ \int_\Omega \Big(\big(\varepsilon(u) + e_{m-1}^N\big) : \tau - g^*(\tau) \Big) \, dx \right\} = \\
= -\int_{\Gamma_1} F_m^N \cdot u \, d\mathcal{H}^{n-1} - \int_\Omega f_m^N \cdot u \, dx + \int_\Omega g\big(\varepsilon(u) + e_{m-1}^N\big).
\end{gathered}
$$

The statement is proved. □

Finally, we can state Theorem 4.4.7, which in this case takes the following form.

Theorem 4.5.4. *Suppose that conditions* (4.4) *and* (4.57) *hold. Then there exists at least a solution* $(\delta u^m, \sigma^m)$ *of the minimax problem* (4.67) *in* $V_+ \times (Q \cap \mathcal{K})$. *Moreover,* σ^m *is the unique solution to the dual variational problem* (4.59), δu^m *is a solution of the relaxed variational problem* (4.69) *and the identity*

$$
\Phi(\delta u^m) = \inf\{I(v) \ : \ v \in \delta w_m^N + V_0\} = \widetilde{L}(\delta u_m^N, \sigma_m^N) = R(\sigma_m^N)
$$

holds.

Furthermore,

$$
\Phi(v) = I(v) \quad \text{for all } v \in \delta w_m^N + V_0.
$$

Finally, any minimizing sequence of problem (4.58) *converges strongly in* $L^1(\Omega;\mathbb{R}^n)$ *and weakly in* $L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ *to some solution of problem* (4.69).

4.5.4 Saddle points generate solutions to the incremental problem

Let us show, that if we interpret a saddle point $(\delta u_m^N, \sigma_m^N)$ of (4.67) as the increment of u and the updated value of σ, then we get a solution to the incremental problem (4.49).

Theorem 4.5.5. *Let $(\delta u_m^N, \sigma_m^N) \in V_+ \times (Q \cap \mathcal{K})$ be a saddle point for the relaxed Lagrangian $\widetilde{L}$. Then the triple (u_m^N, e_m^N, p_M^N), constructed as*

$$\begin{aligned}
&u_m^N = u_{m-1}^N + \delta u_m^N,\\
&e_m^N = \mathbb{A}\sigma_m^N,\\
&p_m^N = \varepsilon(u_m^N) - e_m^N \quad \text{in } \Omega,\\
&p_m^N = (w_m^N - u_m^N) \odot \nu\, \mathcal{H}^{n-1} \quad \text{on } \Gamma_0,
\end{aligned}$$

is admissible for the boundary data w_m^N, in the sense of Definition 4.3.1 and is a solution to the incremental problem (4.49).

Proof: Let $(\delta u_m^N, \sigma_m^N) \in V_+ \times (Q \cap \mathcal{K})$ be a saddle point of $\widetilde{L}$:

$$\widetilde{L}(\delta u_m^N, \tau) \le \widetilde{L}(\delta u_m^N, \sigma_m^N) \le \widetilde{L}(v, \sigma_m^N) \quad \text{for all } v \in V_+,\ \tau \in Q \cap \mathcal{K}. \tag{4.73}$$

As $\sigma_m^N \in Q \cap \mathcal{K}$, we have that $\sigma_m^N \in \mathcal{K}$ and $[\sigma_m^N \nu] = F$ on Γ_1. Moreover, by (4.73),

$$\int_\Omega (v - \delta u_m^N) \cdot \operatorname{div} \sigma_m^N \, dx \le - \int_\Omega f_m^N \cdot (v - \delta u_m^N)\, dx,$$

which is in fact an equality, valid for all $v \in V_+$. Hence,

$$\operatorname{div} \sigma_m^N = -f_m^N \in L^n. \tag{4.74}$$

The first inequality in (4.73) yields

$$\begin{aligned}
&\int_\Omega \Big[\varepsilon(\delta w_m^N) : \sigma_m^N + (\delta w_m^N - \delta u_m^N) \cdot \operatorname{div} \sigma_m^N - g^*(\sigma_m^N) + \sigma_m^N : e_{m-1}^N\Big]\, dx \ge\\
&\ge \int_\Omega \Big[\varepsilon(\delta w_m^N) : \tau + (\delta w_m^N - \delta u_m^N) \cdot \operatorname{div} \tau - g^*(\tau) + \tau : e_{m-1}^N\Big]\, dx
\end{aligned} \tag{4.75}$$

On the other hand, by the integration by parts formula (see [KT83, Theorem 3.2]) for $\delta u_m^N \in BD(\Omega)$ and $\sigma_m^N \in \Sigma$ with $-\operatorname{div} \sigma_m^N = f_m^N$ and $[\sigma_m^N \nu] = F_m^N$ on Γ_1:

$$\begin{aligned}
&\int_\Omega (\delta w_m^N - \delta u_m^N) \operatorname{div} \sigma_m^N \, dx = -\langle \varepsilon^D(\delta w_m^N - \delta u_m^N), \sigma_m^{ND}\rangle -\\
&- \frac{1}{n} \int_\Omega \operatorname{div} (\delta w_m^N - \delta u_m^N) \operatorname{tr} \sigma_m^N \, dx + \int_{\partial\Omega} [\sigma_m^N \nu] \cdot (\delta w_m^N - \delta u_m^N).
\end{aligned} \tag{4.76}$$

We note that, strictly speaking, in the boundary term the integrand is just a distribution, an element of $(C^1(\partial\Omega))'$. However, as $(\delta w_m^N - \delta u_m^N) \cdot \nu = 0$ on Γ_0 and $[\sigma_m^N \nu] = F_m^N \in L^\infty(\Gamma_1)$ by

[KT83, Proposition 3.4] one has

$$\int_{\partial\Omega} [\sigma_m^N \nu] \cdot (\delta w_m^N - \delta u_m^N) =$$
$$\int_{\Gamma_1} (\delta w_m^N - \delta u_m^N) \cdot F_m^N \, d\mathcal{H}^{n-1} + \int_{\Gamma_0} (\delta w_m^N - \delta u_m^N)_\tau \cdot [\sigma_m^N \nu]_\tau \, d\mathcal{H}^{n-1}.$$

This relation together with (4.75) and (4.76) implies

$$\langle \varepsilon^D(\delta u_m^N), \tau^D - \sigma_m^{ND} \rangle - \int_\Omega \frac{1}{2}(\mathbb{A}\tau, \tau)\, dx +$$
$$+ \int_\Omega \frac{1}{2}(\mathbb{A}\sigma_m^N, \sigma_m^N)\, dx + \frac{1}{n}\int_\Omega \operatorname{div} \delta u_m^N \operatorname{tr}(\tau - \sigma_m^N)\, dx +$$
$$+ \int_\Omega (\tau - \sigma_m^N) : e_{m-1}^N \, dx + \int_{\Gamma_0} (\delta w_m^N - \delta u_m^N)_\tau \cdot [\tau - \sigma_m^N]_\tau \, d\mathcal{H}^{n-1} \leq 0,$$

and hence

$$\langle \varepsilon^D(\delta u_m^N), \tau^D - \sigma_m^{ND} \rangle + \int_\Omega (\tau - \sigma_m^N) : e_{m-1}^N \, dx - \int_\Omega \mathbb{A}\sigma_m^N : (\tau - \sigma_m^N)\, dx +$$
$$+ \frac{1}{n}\int_\Omega \operatorname{div} \delta u_m^N \operatorname{tr}(\tau - \sigma_m^N)\, dx +$$
$$+ \int_{\Gamma_0} (\delta w_{m-1}^N - \delta u_m^N)_\tau \cdot [\tau - \sigma_m^N]_\tau \, d\mathcal{H}^{n-1} - \int_\Omega \frac{1}{2}\mathbb{A}(\sigma_m^N - \tau) : (\sigma_m^N - \tau)\, dx \leq 0.$$

Now, taking $\widetilde{\tau} = \sigma_m^N + \alpha(\tau - \sigma_m^N) \in \mathcal{K}$ and letting $\alpha \to 0$ one gets

$$\langle \varepsilon^D(\delta u_m^N), \tau^D - \sigma_m^{ND} + \rangle + \int_\Omega (\tau - \sigma_m^N) : e_{m-1}^N \, dx + \frac{1}{n}\int_\Omega \operatorname{div} \delta u_m^N \operatorname{tr}(\tau - \sigma_m^N)\, dx -$$

$$- \int_\Omega \mathbb{A}\sigma_m^N : (\tau - \sigma_m^N)\, dx + \int_{\Gamma_0} (\delta w_m^N - \delta u_m^N)_\tau \cdot [\tau - \sigma_m^N]_\tau \, d\mathcal{H}^{n-1} =$$
$$= \langle \varepsilon^D(\delta u_m^N), \tau^D - \sigma_m^{ND} \rangle + \int_\Omega (\tau^D - \sigma_m^{ND}) : (e_{m-1D}^N - e_m^{ND})\, dx +$$
$$+ \frac{1}{n}\int_\Omega (\operatorname{div} \delta u_m^N - \operatorname{tr} \delta e_m^N) \operatorname{tr}(\tau - \sigma_m^N)\, dx + \int_{\Gamma_0} (\delta w_m^N - \delta u_m^N)_\tau \cdot [\tau - \sigma_m^N]_\tau \, d\mathcal{H}^{n-1} \leq 0.$$

for all $\tau \in Q \cup \mathcal{K}$. Taking $\tau \in C_c^\infty(\Omega; \mathbb{M}_{sym}^{n\times n})$ with $\tau^D = 0$ we conclude that

$$\operatorname{tr}(\varepsilon(\delta u_m^N) - \delta e_m^N) = \operatorname{div} \delta u_m^N - \operatorname{tr} \delta e_m^N = 0 \quad \text{a.e. in } \Omega,$$

and the induction hypothesis $\operatorname{tr}(\varepsilon(u_{m-1}^N) - e_{m-1}^N) = 0$ a.e. in Ω implies that $\operatorname{tr}(\varepsilon(u_m^N) - e_m^N) = 0$ a.e. in Ω, and thus

$$p_m^N \in M_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n\times n}).$$

Therefore we have the following inequality

$$\langle p_m^N - p_{m-1}^N, \tau - \sigma \rangle \leq 0$$

for all $\tau \in Q \cap \mathcal{K}$.

The last relation, in its turn, implies that

$$\mathcal{H}(p_m^N - p_{m-1}^N) = \langle p_m^N - p_{m-1}^N, \sigma\rangle,$$

which yields the following

$$\begin{aligned}\mathcal{H}(\varepsilon q + p_m^N - p_{m-1}^N) - \mathcal{H}(p_m^N - p_N) - \langle \varepsilon q, \sigma\rangle \geq \\ \geq \langle \varepsilon q + p_m^N - p_{m-1}^N, \sigma\rangle - \langle p_m^N - p_{m-1}^N, \sigma\rangle - \langle \varepsilon q, \sigma\rangle \geq 0,\end{aligned}$$

for every triple $(v, \eta, q) \in A(0)$.

The latter inequality and (4.74) imply that $(u_m^N, e_m^N, p_m^N) \in A(w_m^N)$ is a solution to problem (4.49). □

4.6 Approximations

In this section we will show that some solutions of the relaxed minimax problem (4.67) possess an important property of being approximated by more regular functions in a way that allows us to get the higher regularity of stresses for our evolutionary problem.

Now we consider a family of regularized problems and show that their solutions converge to a saddle point of (4.67) in a suitable weak sense.

4.6.1 Regularized problems

We consider a family of variational problems depending on a parameter $\alpha \in (0, 1)$

$$\begin{cases}\text{find } u_m^\alpha \in V_* \\ I_\alpha(u_m^\alpha) = \inf\{I_\alpha(v) : v \in \delta w_m^N + V_*\},\end{cases} \tag{4.77}$$

where

$$\begin{gathered}V_* = V_0 \cap W^{1,2}(\Omega; \mathbb{R}^n), \\ I_\alpha(v) = \frac{\alpha}{2}\int_\Omega |\varepsilon^D(v) + e_{m-1}^{ND}|^2\,dx + I(v) = \\ \frac{\alpha}{2}\int_\Omega |\varepsilon^D(v) + e_{m-1}^{ND}|^2\,dx + \int_\Omega g(\varepsilon(v) + e_{m-1}^N)\,dx - \int_\Omega f_m^N \cdot v\,dx - \int_{\Gamma_1} F_m^N \cdot v\,d\mathcal{H}^{n-1}.\end{gathered}$$

As it is easy to see, for each $\alpha > 0$, the coercivity estimate (4.60) and Korn inequality guarantee that the functional I_α is coercive on V_*, so that problem (4.77) has a unique minimizer $u_m^\alpha \in V_*$ which satisfies a nonlinear system of PDE's of elliptic type:

$$\int_\Omega \sigma_m^\alpha : \varepsilon(v)\,dx = \mathcal{F}_m^N(v) \equiv \int_\Omega f_m^N \cdot v\,dx + \int_{\Gamma_1} F_m^N \cdot v\,d\mathcal{H}^{n-1} \quad \text{for all } v \in V_*, \tag{4.78}$$

where

$$\begin{aligned}\sigma_m^\alpha = \alpha(\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}) + \frac{\partial g}{\partial \varkappa}(\varepsilon(u_m^\alpha) + e_{m-1}^N) = \\ = \alpha(\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}) + K_0(\operatorname{div} u_m^\alpha + \operatorname{tr} e_{m-1}^N)\mathbf{1} + \\ + g_0'(|\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|)\frac{\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}}{|\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|}.\end{aligned} \tag{4.79}$$

Therefore,

$$\operatorname{div}\sigma_m^\alpha + f_m^N = 0 \quad \text{in } \Omega. \tag{4.80}$$

Remark 4.6.1. We note, that the functional $I_\alpha(v)$ is of the form $I(v) + \frac{\alpha}{2}\|\varepsilon^D(v)\|^2_{L^2}$, where the second summand is added to make it coercive in $W^{1,2}$.

Lemma 4.6.2. *For any $\alpha \in (0,1)$ the following estimate is true*

$$\sqrt{\alpha}\|\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}\|_{L^2(\Omega)} + \|\operatorname{div} u_m^\alpha\|_{L^2(\Omega)} + \|\varepsilon^D(u_m^\alpha)\|_{L^1(\Omega)} + \|u_m^\alpha\|_{L^{\frac{n}{n-1}}(\Omega)} \le C, \tag{4.81}$$

where $C = C(\|f_m^N\|_{L^n(\Omega)},\ \|F_m^N\|_{L^\infty}(\Gamma_1),\ \|\delta w_m^N\|_{W^{1,2}(\Omega;\mathbb{R}^n)}, \|e_{m-1}^N\|_{L^2})$.

Moreover for a subsequence the following hold:

$$u_m^\alpha \rightharpoonup \delta u_m^N \quad \text{in } L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n), \tag{4.82}$$

$$u_m^\alpha \to \delta u_m^N \quad \text{in } L^r(\Omega;\mathbb{R}^n) \quad \text{for } r \in [1, n/(n-1)), \tag{4.83}$$

$$\int_\Omega \tau : \varepsilon(u_m^\alpha)\,dx \to \int_\Omega \tau : \varepsilon(\delta u_m^N)\,dx \quad \text{for every } \tau \in C_c^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}), \tag{4.84}$$

$$\operatorname{div} u_m^\alpha \rightharpoonup \operatorname{div}\delta u_m^N \quad \text{in } L^2(\Omega;\mathbb{R}^n), \tag{4.85}$$

$$\alpha\int_\Omega |\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|^2\,dx \to 0, \tag{4.86}$$

$$\sigma_m^\alpha \rightharpoonup \sigma_m^N \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \tag{4.87}$$

$$\sigma_m^{\alpha D} - \alpha(\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}) \overset{*}{\rightharpoonup} \sigma_m^{ND} \quad \text{in } L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym}), \tag{4.88}$$

where δu_m^N is a solution to problem (4.69) *and σ_m^N is the unique solution to problem* (4.59).

Proof: From the coercivity estimate (4.60) one immediately obtains (4.81).

It follows from (4.79) that the sequences $\{\sigma_m^\alpha\}$ and $\{\sigma_m^{\alpha D} - \alpha(\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND})\}$ are bounded in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$ respectively.

So we get the convergences (4.82)-(4.85), (4.87) and (4.88). It remains to show that u and σ are solutions of (4.69) and (4.59) and that (4.86) holds.

As $\varkappa^\alpha := \sigma_m^\alpha - \alpha(\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}) \in \mathcal{K}$, and since $\mathcal{K}$ is weakly closed in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, it follows that $\sigma \in \mathcal{K}$. Now passing to the limit in (4.78) and using the condition (4.6) we can extend (4.78) to V_0 and thus $\sigma_m^N \in Q_{f_m^N}$.

On the other hand, the duality relations imply that

$$\varkappa^\alpha : (\varepsilon(u_m^\alpha) + e_{m-1}^N) - g(\varepsilon(u_m^\alpha) + e_{m-1}^N) - g^*(\tau^\alpha) = 0 \quad \text{a.e. in } \Omega.$$

But then, by (4.78) and (4.79) one gets

$$I_\alpha(u_m^\alpha) = \frac{\alpha}{2}\int_\Omega |\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|^2\,dx +$$
$$+\int_\Omega \Big[\tau^\alpha : (\varepsilon(u_m^\alpha) + e_{m-1}^N) - g^*(\varkappa^\alpha)\Big]\,dx - \mathcal{F}_m^N(u_m^\alpha) =$$
$$= -\frac{\alpha}{2}\int_\Omega |\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|^2\,dx + \int_\Omega \Big[\sigma_m^\alpha : (\varepsilon(u_m^\alpha) + e_{m-1}^N) - g^*(\varkappa^\alpha)\Big]\,dx - \mathcal{F}_m^N(u_m^\alpha).$$

By Theorem 4.4.1, applied to problems (4.58) and (4.59), we get

$$\begin{aligned} &\sup\{R(\tau)\ :\ \tau \in Q_{f_m^N}\cap\mathcal{K}\} = \inf\{I(v)\ :\ v\in \delta w_m^N + V_0\} \le I(u_m^\alpha) \le I_\alpha(u_m^\alpha) = \\ &= -\frac{\alpha}{2}\int_\Omega |\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|^2\,dx - \int_\Omega g^*(\varkappa^\alpha)\,dx + \int_\Omega \sigma_m^\alpha : (\varepsilon(u_m^\alpha) + e_{m-1}^N)\,dx - \\ &-\mathcal{F}_m^N(u_m^\alpha) = -\frac{\alpha}{2}\int_\Omega |\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|^2\,dx - \int_\Omega g^*(\varkappa^\alpha)\,dx + \\ &+\int_\Omega \sigma_m^\alpha : (\varepsilon(\delta w_m^N) + e_{m-1}^N)\,dx - \mathcal{F}_m^N(\delta w_m^N), \end{aligned} \tag{4.89}$$

where the Euler equation (4.78) was used.

Since

$$-\int_\Omega g^*(\sigma_m^N)\,dx + \int_\Omega \sigma : (\varepsilon(\delta w_m^N) + e_{m-1}^N)\,dx - \mathcal{F}_m^N(\delta w_m^N) = R(\sigma_m^N),$$

by exploiting the convergences (4.87) and (4.88) we obtain from (4.89):

$$\lim_{\alpha\to 0} I_\alpha(u_m^\alpha) \le R(\sigma_m^N) - \limsup_{\alpha\to 0}\ \frac{\alpha}{2}\int_\Omega |\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|^2\,dx.$$

Thus, proceeding with (4.89) we obtain

$$\begin{aligned} R(\sigma_m^N) &\le \sup\{R(\tau)\ :\ \tau\in Q_{f_m^N}\cap\mathcal{K}\} = \inf\{I(v)\ :\ v\in u_0 + V_0\} \le \\ &\le \liminf_{\alpha\to 0} I(u_m^\alpha) \le \limsup_{\alpha\to 0} I(u_m^\alpha) \le \lim_{\alpha\to 0} I_\alpha(u_m^\alpha) \le \\ &\le R(\sigma_m^N) - \limsup_{\alpha\to 0}\ \frac{\alpha}{2}\int_\Omega |\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|^2\,dx \le R(\sigma_m^N), \end{aligned}$$

which gives (4.86) and ensures that σ_m^N is a solution to problem (4.59).

Moreover one has the identity

$$\lim_{\alpha\to 0} I(u_m^\alpha) = \inf\{I(v)\ :\ v\in \delta w_m^N + V_0\}, \tag{4.90}$$

which implies that u_m^α is a minimizing sequence for the problem (4.58), and therefore it converges weakly in $L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ to a solution of problem (4.69). □

4.6.2 Convergence of variations

Let us show, that the approximating sequence enjoys better convergence properties, than those stated in Lemma 4.6.2. Namely, the following result holds.

Lemma 4.6.3. *We have*

$$|\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}| \overset{*}{\rightharpoonup} |\varepsilon^D(\delta u_m^N) + e_{m-1}^{ND}| \quad \text{in } M_b(\Omega)\,. \tag{4.91}$$

Proof: By Lemma 4.6.2, Theorem 4.5.4 and (4.90)

$$\lim_{\alpha\to 0} \Phi(u_m^\alpha) = \lim_{\alpha\to 0} I(u_m^\alpha) = \inf_{v\in\delta w_m^N + V_0} I(v) = \inf_{v\in V_+} \Phi(v) = \Phi(\delta u_m^N),$$

that is,

$$\lim_{\alpha\to 0} \int_\Omega g(\varepsilon(u_m^\alpha) + e_{m-1}^N)\,dx = \int_\Omega g(\varepsilon(\delta u_m^N) + e_{m-1}^N). \tag{4.92}$$

Properties (4.84) and (4.85) imply that

$$\lim_{\alpha\to 0} \int_\Omega g_0(|\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|)\,dx \geq \int_\Omega g_0(|\varepsilon^D(\delta u_m^N) + e_{m-1}^{ND}|),$$

and

$$\lim_{\alpha\to 0} \int_\Omega |\operatorname{div} u_m^\alpha + \operatorname{tr} e_{m-1}^{ND}|^2\,dx \geq \int_\Omega |\operatorname{div} \delta u_m^N + \operatorname{tr} e_{m-1}^{ND}|^2\,dx.$$

Combined with (4.92), the latter inequalities give

$$\lim_{\alpha\to 0} \int_\Omega g_0(|\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|) = \int_\Omega g_0(|\varepsilon^D(\delta u_m^N) + e_{m-1}^{ND}|). \tag{4.93}$$

To show the validity of (4.91) we argue in the following way: the sequence $|\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|$ is bounded in $M_b(\Omega)$, hence there exists a nonnegative measure $\lambda \in M_b(\Omega)$, such that

$$|\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}| \overset{*}{\rightharpoonup} \lambda \quad \text{in } M_b(\Omega),\ \text{as } \alpha \to 0\,. \tag{4.94}$$

Thus, $\lambda \geq |\varepsilon^D(\delta u_m^N) + e_{m-1}^{ND}|$ in $M_b(\Omega)$, and, in particular, the inequality holds also for $\mathcal{L}^n$-absolutely continuous and singular parts:

$$\begin{aligned} \lambda^a &\geq |\varepsilon^D(\delta u_m^N) + e_{m-1}^{ND}|^a, \\ \lambda^s &\geq |\varepsilon^D(\delta u_m^N) + e_{m-1}^{ND}|^s. \end{aligned} \tag{4.95}$$

By the weak* lower-semicontinuity of convex functionals of measures and since for g_0 as in (4.55) the recession function of g is of the form $g_0^\infty(t) = k_*\sqrt{2}\,t$,

$$\lim_{\alpha\to 0} \int_\Omega g_0(|\varepsilon^D(u^\alpha) + e_{m-1}^{ND}|)\,dx \geq \int_\Omega g_0(\lambda) = \int_\Omega g_0(\lambda^a)\,dx + k_*\sqrt{2}\,\lambda^s(\Omega). \tag{4.96}$$

We have

$$\begin{aligned} &\int_\Omega g_0(|\varepsilon^D(\delta u_m^N) + e_{m-1}^{ND}|) = \\ &= \int_\Omega g_0(|\varepsilon^D(\delta u_m^N) + e_{m-1}^{ND}|^a)\,dx + k_*\sqrt{2}\,|\varepsilon^D(\delta u_m^N) + e_{m-1}^{ND}|^s(\Omega). \end{aligned} \tag{4.97}$$

As the function g_0 is strictly monotone increasing, from (4.93)-(4.97) it follows that

$$\lambda = |\varepsilon^D(\delta u_m^N) + e_{m-1}^{ND}|.$$

Now (4.91) is a consequence of (4.94). □

4.6.3 Technical lemmas

Note, that the definition (4.79) of σ^α implies, that

$$\sigma^\alpha_m = \alpha(\varepsilon^D(u^\alpha_m) + e^{ND}_{m-1}) + K_0(\operatorname{div} u^\alpha_m + \operatorname{tr} e^N_{m-1})\mathbf{1}+ \\ + \begin{cases} 2\mu(\varepsilon^D(u^\alpha_m) + e^{ND}_{m-1}), & \text{if } |\varepsilon^D(u^\alpha_m) + e^{ND}_{m-1}| \le \frac{k_*}{\sqrt{2}\mu} \\ k_*\sqrt{2}\frac{\varepsilon^D(u^\alpha_m)+e^{ND}_{m-1}}{|\varepsilon^D(u^\alpha_m)+e^{ND}_{m-1}|}, & \text{if } |\varepsilon^D(u^\alpha_m) + e^{ND}_{m-1}| > \frac{k_*}{\sqrt{2}\mu}. \end{cases} \tag{4.98}$$

According to the chain rule of [MT03] we have the following expression for the derivatives of σ^α_m:

$$\sigma^\alpha_{m,k} = \alpha(\varepsilon^D(u^\alpha_{m,k}) + e^{ND}_{m-1,k}) + \frac{\partial^2 g}{\partial \varkappa^2}(\varepsilon(u^\alpha_m) + e^N_{m-1})(\varepsilon^D(u^\alpha_{m,k}) + e^{ND}_{m-1,k}). \tag{4.99}$$

Here and henceforth the subscript $_{,k}$ denotes the partial derivative with respect to x_k.

In the sequel we will use the notation

$$\tau^\alpha_m := \varepsilon(u^\alpha_m) + e^N_{m-1}. \tag{4.100}$$

Put $\overline{g}_0(\varkappa) = g_0(|\varkappa^D|)$. Let us introduce two bilinear forms, that depend on α and implicitly on the point $x \in \Omega$:

$$E^\alpha_1(\varepsilon, \varkappa) = \left(\frac{\partial^2 \overline{g}_0}{\partial \tau^2}(\tau^\alpha_m)\,\varepsilon\right) : \varkappa = \\ = \frac{g_0'(|\tau^{\alpha D}_m|)}{|\tau^{\alpha D}_m|}\,\varepsilon : \varkappa + \left(g_0''(|\tau^{\alpha D}_m|) - \frac{g_0'(|\tau^{\alpha D}_m|)}{|\tau^{\alpha D}_m|}\right)\frac{\tau^{\alpha D}_m : \varepsilon}{|\tau^{\alpha D}_m|}\,\frac{\tau^{\alpha D}_m : \varkappa}{|\tau^{\alpha D}_m|} \tag{4.101}$$

and

$$E^\alpha_2(\varepsilon, \varkappa) = \alpha\,\varepsilon^D : \varkappa^D + K_0 \operatorname{tr}\varepsilon \operatorname{tr}\varkappa + E^\alpha_1(\varepsilon^D, \varkappa^D). \tag{4.102}$$

Below we list some relations which will be extensively used in the remaining part of the paper.

Lemma 4.6.4. *The following relations hold true:*

$$\sigma^\alpha_{m,k} : \varkappa = E^\alpha_2(\tau^\alpha_{m,k}, \varkappa), \tag{4.103}$$

$$E^\alpha_2(\varkappa^D, \varkappa^D) \le \alpha|\varkappa^D|^2 + \begin{cases} 2\mu\,|\varkappa^D|^2, & \text{if } |\tau^{\alpha D}_m| \le \frac{k_*}{\sqrt{2}\mu} \\ k_*\sqrt{2}\,\frac{|\varkappa^D|^2}{|\tau^{\alpha D}_m|}, & \text{if } |\tau^{\alpha D}_m| > \frac{k_*}{\sqrt{2}\mu} \end{cases} \tag{4.104}$$

for any $\varkappa \in \mathbb{M}^{n\times n}_{sym}$. *Moreover*

$$E^\alpha_2(\mathbb{A}\sigma^\alpha_{m,k}, \mathbb{A}\sigma^\alpha_{m,k}) \le \tfrac{1}{n}\operatorname{tr}\mathbb{A}\sigma^\alpha_{m,k}\operatorname{tr}\sigma^\alpha_{m,k} + \tfrac{\alpha}{2\mu}\mathbb{A}\sigma^{\alpha D}_{m,k} : \sigma^{\alpha D}_{m,k} + \\ + \begin{cases} \mathbb{A}\sigma^{\alpha D}_{m,k} : \sigma^{\alpha D}_{m,k}, & \text{if } |\tau^{\alpha D}_m| \le \frac{k_*}{\sqrt{2}\mu} \\ \frac{k_*}{\sqrt{2}\mu|\tau^{\alpha D}|}\mathbb{A}\sigma^{\alpha D}_{m,k} : \sigma^{\alpha D}_{m,k}, & \text{if } |\tau^{\alpha D}_m| > \frac{k_*}{\sqrt{2}\mu} \end{cases} \tag{4.105}$$

PROOF: Identity (4.103) and inequality (4.104) follows from (4.98), (4.99), the definitions (4.101), (4.102) and the expression for g_0, as in (4.55).

To prove (4.105) we first use the definition (4.102) of E_2^α and (4.1) of $\mathbb{A}$, obtaining

$$E_2^\alpha(\mathbb{A}\sigma_{m,k}^\alpha, \mathbb{A}\sigma_{m,k}^\alpha) = \tfrac{\alpha}{2\mu}\,\mathbb{A}\sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D} + \frac{(\operatorname{tr}\sigma_{m,k}^\alpha)^2}{n^2 K_0} + \tfrac{1}{4\mu^2} E_1^\alpha(\sigma_{m,k}^{\alpha D}, \sigma_{m,k}^{\alpha D}). \tag{4.106}$$

Now let us study the last summand. Taking (4.104) into account we have the following

$$\frac{1}{4\mu^2} E_1^\alpha(\sigma_{m,k}^{\alpha D}, \sigma_{m,k}^{\alpha D}) \le \frac{1}{4\mu^2}\begin{cases} 2\mu\,\sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D}, & \text{if } |\tau_m^{\alpha D}| \le \frac{k_*}{\sqrt{2}\mu} \\ \frac{\sqrt{2}k_*}{|\tau^{\alpha D}|}\sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D}, & \text{if } |\tau_m^{\alpha D}| > \frac{k_*}{\sqrt{2}\mu} \end{cases} \le$$

$$\le \frac{1}{2\mu}\begin{cases} \sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D}, & \text{if } |\tau_m^{\alpha D}| \le \frac{k_*}{\sqrt{2}\mu} \\ \frac{k_*}{\sqrt{2}\mu|\tau^{\alpha D}|}\sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D}, & \text{if } |\tau_m^{\alpha D}| > \frac{k_*}{\sqrt{2}\mu} \end{cases} = \begin{cases} \mathbb{A}\sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D}, & \text{if } |\tau_m^{\alpha D}| \le \frac{k_*}{\sqrt{2}\mu} \\ \frac{k_*}{\sqrt{2}\mu|\tau^{\alpha D}|}\mathbb{A}\sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D}, & \text{if } |\tau_m^{\alpha D}| > \frac{k_*}{\sqrt{2}\mu} \end{cases}$$

Thus, (4.105) follows now from (4.106). □

Corollary 4.6.5. *As a consequence of* (4.105)*, we have the following estimate:*

$$E_2^\alpha(\mathbb{A}\sigma_{m,k}^\alpha, \mathbb{A}\sigma_{m,k}^\alpha) \le \left(1 + \tfrac{\alpha}{2\mu}\right)\mathbb{A}\sigma_{m,k}^\alpha : \sigma_{m,k}^\alpha. \tag{4.107}$$

Lemma 4.6.6. *We have*

$$-E_2^\alpha(\tau_{m,k}^{\alpha D}; \tau_{m,k}^{\alpha D}) = -\sigma_{m,k}^{\alpha D} : \tau_{m,k}^{\alpha D} \le$$

$$\le \begin{cases} -\mathbb{A}\sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D} + \frac{\alpha}{2\mu}\,E_2^\alpha[\tau_{m,k}^{\alpha D}; \tau_{m,k}^{\alpha D}], & \textit{if } |\tau_m^{\alpha D}| \le \frac{k_*}{\sqrt{2}\mu} \\ -\frac{\sqrt{2}\mu}{k_*}|\tau^{\alpha D}|\,\mathbb{A}\sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D} + \frac{\alpha}{k_*\sqrt{2}}\,|\tau_m^{\alpha D}|\,E_2^\alpha[\tau_{m,k}^{\alpha D}; \tau_{m,k}^{\alpha D}], & \textit{if } |\tau_m^{\alpha D}| > \frac{k_*}{\sqrt{2}\mu}\ . \end{cases} \tag{4.108}$$

PROOF: Suppose, $|\tau_m^{\alpha D}| \le \frac{k_*}{\sqrt{2}\mu}$. Then $\sigma_m^{\alpha D} = \alpha\tau_m^{\alpha D} + 2\mu\tau_m^{\alpha D}$, and

$$-\sigma_{m,k}^{\alpha D} : \tau_{m,k}^{\alpha D} = -\sigma_{m,k}^{\alpha D} : \left(\tau_{m,k}^{\alpha D} + \tfrac{1}{2\mu}\alpha\,\tau_{m,k}^{\alpha D}\right) + \tfrac{1}{2\mu}\alpha\,\sigma_{m,k}^{\alpha D} : \tau_{m,k}^{\alpha D} =$$
$$= -\mathbb{A}\sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D} + \tfrac{1}{2\mu}\alpha\,E_2^\alpha[\tau_{m,k}^{\alpha D}; \tau_{m,k}^{\alpha D}].$$

Now let $|\tau_m^{\alpha D}| > \frac{k_*}{\sqrt{2}\mu}$. In this case $\sigma_m^{\alpha D} = \alpha\tau_m^{\alpha D} + k_*\sqrt{2}\frac{\tau_m^{\alpha D}}{|\tau_m^{\alpha D}|}$, that is

$$\sigma_{m,k}^{\alpha D} = \alpha\tau_{m,k}^{\alpha D} + k_*\sqrt{2}\left[\frac{\tau_m^{\alpha D}}{|\tau_m^{\alpha D}|}\right]_{,k}, \tag{4.109}$$

thus

$$\tau_{m,k}^{\alpha D} = \frac{|\tau_m^{\alpha D}|}{k_*\sqrt{2}}\left(\sigma_{m,k}^{\alpha D} - \alpha\tau_{m,k}^{\alpha D}\right) + \tau_m^{\alpha D}\,\frac{\tau_m^{\alpha D} : \tau_{m,k}^{\alpha D}}{|\tau_m^{\alpha D}|^2},$$

and

$$-\sigma^{\alpha D}_{m,k} : \tau^{\alpha D}_{m,k} = -\tfrac{\sqrt{2}\mu}{k_*}|\tau^{\alpha D}_m|\mathbb{A}\sigma^{\alpha D}_{m,k} : \sigma^{\alpha D}_{m,k} + \tfrac{\alpha}{k_*\sqrt{2}}\,|\tau^{\alpha D}_m|\,\sigma^{\alpha D}_{m,k} : \tau^{\alpha D}_{m,k} +$$
$$-\frac{\tau^{\alpha D}_m : \tau^{\alpha D}_{m,k}}{|\tau^{\alpha D}_m|^2}\sigma^{\alpha D}_{m,k} : \tau^{\alpha D}_m \le$$
$$-\frac{\sqrt{2}\mu}{k_*}|\tau^{\alpha D}_m|\mathbb{A}\sigma^{\alpha D}_{m,k} : \sigma^{\alpha D}_{m,k} + \tfrac{\alpha}{k_*\sqrt{2}}\,|\tau^{\alpha D}_m|\,\sigma^{\alpha D}_{m,k} : \tau^{\alpha D}_{m,k},$$

where we have used (4.109) and the orthogonality of $\big[\frac{\tau^{\alpha D}_m}{|\tau^{\alpha D}_m|}\big]_{,k}$ and $\tau^{\alpha D}_m$ to show that

$$-\frac{\tau^{\alpha D}_m : \tau^{\alpha D}_{m,k}}{|\tau^{\alpha D}_m|^2}\sigma^{\alpha D}_{m,k} : \tau^{\alpha D}_m = -\alpha\left(\frac{\tau^{\alpha D}_m : \tau^{\alpha D}_{m,k}}{|\tau^{\alpha D}_m|}\right)^2 \le 0.$$

The claim is proved. □

4.7 $W^{1,2}_{loc}$-estimates of stresses in the incremental problems

In this section we deduce the iterative estimate of L^2-norms of gradients of functions σ^α, defined by means of (4.79) via the solutions of regularized problems (4.77), and we show, that for every given m and N we have $\sigma^N_m \in W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$. We note, however, that in this section we are concerned only with the problem of regularity of each σ^N_m, that is, we do not care about the uniformity of estimates with respect to m and N. Having these estimates in hand, we conclude that the convergence of approximate solutions σ^α to σ^N_m, which was known to take place in the weak topology of $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ (see (4.87)), is actually better, and is determined by the critical exponent of the Sobolev embedding.

We note that, to underline the dependence of σ^α and u^α on m, we sometimes write them as σ^α_m and u^α_m. Remark, that in what follows C_m will denote a constant independent of α, which may change from line to line. This constant might depend on m, N, and, in case of local estimates, on a domain $\Omega' \subset\subset \Omega$. We will use the notation C only when this constant does not depend on m and N.

Thus, our objective now is the following estimate

$$\int_{\Omega'} \mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\,dx \le C(m,N,\Omega'), \tag{4.110}$$

valid for any $\Omega' \subset\subset \Omega$.

Suppose, by induction, that we have already proved that $\sigma^N_{m-1} \in W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$. To simplify the notation, in this section we will omit writing the index N for the solutions of the incremental problem (4.49). Let us turn to the regularized problem (4.77). Since u^α_m is a solution of the nonlinear elliptic system (4.80) with $f_m \in L^n(\Omega;\mathbb{R}^n)$ and $e^N_{m-1} \in W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$, one can show, by considering the difference quotients, that

$$u^\alpha_m \in W^{2,2}_{loc}(\Omega;\mathbb{R}^n), \qquad \varepsilon(u^\alpha_m),\ \sigma^\alpha_m \in W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym}). \tag{4.111}$$

In the rest of the paper we adopt the summation convention over repeated indices (excluding m, k and α). As

$$(\sigma^\alpha_m)_{ij,j} = -(f_m)_i \quad \text{a.e. in } \Omega, \tag{4.112}$$

one has

$$\int_\Omega \sigma^\alpha_{m,k} : \varepsilon(v)\, dx = -\int_\Omega f_m \cdot v_{,k}\, dx \quad \text{for all } v \in C_0^\infty(\Omega;\mathbb{R}^n),\ k = 1,\dots,n. \tag{4.113}$$

By using formula (4.103), estimate (4.107) and the definition (4.100) of τ^α_m we obtain:

$$\begin{aligned} \mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k} = E^\alpha_2(\tau^\alpha_{m,k}, \mathbb{A}\sigma^\alpha_{m,k}) \le \\ \le \left[E^\alpha_2(\tau^\alpha_{m,k},\tau^\alpha_{m,k})\right]^{1/2}\left[E^\alpha_2(\mathbb{A}\sigma^\alpha_{m,k},\mathbb{A}\sigma^\alpha_{m,k})\right]^{1/2} \le \\ \le \tfrac{1}{2}E^\alpha_2(\tau^\alpha_{m,k},\tau^\alpha_{m,k}) + \tfrac{1}{2}E^\alpha_2(\mathbb{A}\sigma^\alpha_{m,k},\mathbb{A}\sigma^\alpha_{m,k}) \le \\ \le \tfrac{1}{2}\sigma^\alpha_{m,k} : \tau^\alpha_{m,k} + \left(\tfrac{1}{2} + \tfrac{\alpha}{4\mu}\right)\mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k} \le \\ \le \tfrac{1}{2}\sigma^\alpha_{m,k} : \varepsilon(u^\alpha_{m,k}) + \tfrac{1}{2}\sigma^\alpha_{m,k} : \mathbb{A}\,\sigma^N_{m-1,k} + \left(\tfrac{1}{2} + \tfrac{\alpha}{4\mu}\right)\mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k} \end{aligned} \tag{4.114}$$

By applying Cauchy inequality to $\mathbb{A}\,\sigma^\alpha_{m,k} : \sigma^N_{m-1,k}$, we get

$$\left(1 - \tfrac{\alpha}{\mu}\right)\mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k} \le \mathbb{A}\,\sigma^N_{m-1,k} : \sigma^N_{m-1,k} + 2\sigma^\alpha_{m,k} : \varepsilon(u^\alpha_{m,k}). \tag{4.115}$$

Thus, it remains to prove the boundedness in $L^1_{loc}(\Omega)$ of the second summand of (4.115).
Let us introduce the notation

$$\sigma^\alpha := \sigma^\alpha_m, \quad f := f_m, \quad u^\alpha := u^\alpha_m,$$

omitting also the index m for further convenience. Let $\varphi \in C^3_0(\Omega)$ be a arbitrary cut-off function, such that $\varphi \equiv 1$ on Ω' and $\operatorname{supp}\varphi \subset \Omega'' \subset\subset \Omega$. By (4.111) we can put the function

$$v = \varphi^6 u^\alpha_{,k}$$

into the identity (4.113).

To avoid parentheses, we agree that expressions like $\varphi^6_{,i}$ always mean $(\varphi^6)_{,i}$. We start by

$$\begin{aligned} \int_\Omega \sigma^\alpha_{,k} : \varepsilon(\varphi^6 u^\alpha_{,k})\, dx = -\int_\Omega f \cdot (\varphi^6 u^\alpha_{,k})_{,k}\, dx = \\ = -\int_\Omega f \cdot \varphi^6 \Delta u^\alpha\, dx - \int_\Omega f \cdot \varphi^6_{,k} u^\alpha_{,k}\, dx. \end{aligned} \tag{4.116}$$

As

$$\frac{1}{2}\Delta u^\alpha = \operatorname{div}\varepsilon(u^\alpha) - \frac{1}{2}\nabla \operatorname{div} u^\alpha,$$

we go on

$$\begin{aligned}
-\int_\Omega f\cdot\varphi^6\Delta u^\alpha\,dx-\int_\Omega f\cdot\varphi^6_{,k}u^\alpha_{,k}\,dx=-2\int_\Omega f\cdot\varphi^6\operatorname{div}\varepsilon(u^\alpha)\,dx+\\
+\int_\Omega f\cdot\varphi^6\nabla\operatorname{div}u^\alpha\,dx-\int_\Omega f\cdot\varphi^6_{,k}u^\alpha_{,k}\,dx=2\int_\Omega\varepsilon(f):\varphi^6\varepsilon(u^\alpha)\,dx+\\
+2\int_\Omega(f\odot\nabla\varphi^6):\varepsilon(u^\alpha)\,dx+\int_\Omega\varphi^6\,f\cdot\nabla\operatorname{div}u^\alpha\,dx-\int_\Omega f\cdot\varphi^6_{,k}u^\alpha_{,k}\,dx=\\
=2\int_\Omega\varphi^6\varepsilon(f):\varepsilon(u^\alpha)\,dx+\int_\Omega\varphi^6\,f\cdot\nabla\operatorname{div}u^\alpha\,dx+\int_\Omega f_i\,\varphi^6_{,j}u^\alpha_{j,i}\,dx=\\
=2\int_\Omega\varphi^6\varepsilon(f):\varepsilon(u^\alpha)\,dx+\\
+\int_\Omega\varphi^6\,f\cdot\nabla\operatorname{div}u^\alpha\,dx-\int_\Omega\nabla\varphi^6\cdot u^\alpha\operatorname{div}f\,dx-\int_\Omega(f\odot u^\alpha):\nabla^2\varphi^6\,dx.
\end{aligned}\tag{4.117}$$

Thus (4.116) and (4.117) yield

$$J^\alpha_m:=\int_\Omega\varphi^6\sigma^\alpha_{,k}:\varepsilon(u^\alpha_{,k})\,dx=J_1+J_2+J_3,\tag{4.118}$$

where

$$J_1:=-2\int_\Omega\sigma^\alpha_{ij,k}\varphi^6_{,i}\varepsilon_{kj}(u^\alpha)\,dx,\tag{4.119}$$

$$J_2:=\int_\Omega\sigma^\alpha_{ij,k}\varphi^6_{,i}u^\alpha_{k,j}\,dx,\tag{4.120}$$

$$\begin{aligned}J_3:=\int_\Omega\Big[2\varphi^6\varepsilon(f):\varepsilon(u^\alpha)+\varphi^6\,f\cdot\nabla\operatorname{div}u^\alpha\,dx-\\
\nabla\varphi^6\cdot u^\alpha\operatorname{div}f-(f\odot u^\alpha):\nabla^2\varphi^6\Big]\,dx.\end{aligned}\tag{4.121}$$

Now, by using the orthogonal decomposition of $\mathbb{M}^{n\times n}_{sym}=\mathbb{M}^D+\mathbb{R}\,\mathbf{1}$:

$$\varepsilon(u^\alpha)=\varepsilon^D(u^\alpha)+\frac{1}{n}\operatorname{div}u^\alpha\mathbf{1},\qquad\sigma^\alpha=\sigma^{\alpha D}+\frac{1}{n}\operatorname{tr}\sigma^\alpha\mathbf{1}$$

and the Euler equation (4.112), one gets

$$\begin{aligned}
J_1 = -2\int_\Omega \sigma^\alpha_{ij,k}\varphi^6_{,i}\varepsilon^D_{jk}(u^\alpha)\,dx - \frac{2}{n}\int_\Omega \sigma^\alpha_{ij,j}\varphi^6_{,i}\mathrm{div}\,u^\alpha\,dx = \\
-\frac{2}{n}\int_\Omega \mathrm{tr}\,\sigma^\alpha_{,k}\varphi^6_{,i}\varepsilon^D_{ik}(u^\alpha)\,dx - 2\int_\Omega \sigma^{\alpha D}_{ij,k}\varphi^6_{,i}\varepsilon^D_{jk}(u^\alpha)\,dx+ \\
+\frac{2}{n}\int_\Omega f\cdot\nabla\varphi^6\mathrm{div}\,u^\alpha\,dx = 2\int_\Omega (f_k + \sigma^{\alpha D}_{ks,s})\varphi^6_{,i}\varepsilon^D_{ik}(u^\alpha)\,dx- \\
-2\int_\Omega \sigma^{\alpha D}_{ij,k}\varphi^6_{,i}\varepsilon^D_{jk}(u^\alpha)\,dx + \frac{2}{n}\int_\Omega f\cdot\nabla\varphi^6\mathrm{div}\,u^\alpha\,dx = \\
= 2\int_\Omega (f\odot\nabla\varphi^6):\varepsilon^D(u^\alpha)\,dx + \frac{2}{n}\int_\Omega f\cdot\nabla\varphi^6\mathrm{div}\,u^\alpha\,dx+ \\
+12\int_\Omega \varphi^5\sigma^{\alpha D}_{ij,k}\Big(-\varphi_{,i}\varepsilon^D_{jk}(u^\alpha) + \delta_{ik}\varphi_{,s}\varepsilon^D_{js}(u^\alpha)\Big)\,dx = \\
= 2\int_\Omega (f\odot\nabla\varphi^6):\varepsilon^D(u^\alpha)\,dx + 12\int_\Omega \varphi^5\sigma^{\alpha D}_{,k}:S^{(k)}\,dx + \frac{2}{n}\int_\Omega f\cdot\nabla\varphi^6\mathrm{div}\,u^\alpha\,dx.
\end{aligned} \tag{4.122}$$

where the matrices $S^{(k)}$ are defined by

$$S^{(k)}_{ij} := \Big(\delta_{ik}\varphi_{,s}\varepsilon^D_{js}(u^\alpha) - \varphi_{,i}\varepsilon^D_{jk}(u^\alpha)\Big). \tag{4.123}$$

It follows immediately from the definition that

$$\mathrm{tr}\,(S^{(k)}) = \delta_{ik}\varphi_{,s}\varepsilon^D_{is}(u^\alpha) - \varphi_{,i}\varepsilon^D_{ik}(u^\alpha) = 0. \tag{4.124}$$

Now let us turn to J_2. Integrating by parts and using (4.112) we get

$$\begin{aligned}
J_2 = -\int_\Omega \Big[\sigma^\alpha_{ij}\varphi^6_{,ik}u^\alpha_{k,j} + \sigma^\alpha_{ij}\varphi^6_{,i}\mathrm{div}\,u^\alpha_{,j}\Big] = \\
= \int_\Omega \sigma^\alpha_{ij,j}\varphi^6_{,ik}u^\alpha_k\,dx + \int_\Omega \sigma^\alpha_{ij}\varphi^6_{,ijk}u^\alpha_k\,dx + \int_\Omega \sigma^\alpha_{ij,j}\varphi^6_{,i}\mathrm{div}\,u^\alpha\,dx+ \\
+\int_\Omega \sigma^\alpha_{ij}\varphi^6_{,ij}\mathrm{div}\,u^\alpha\,dx = -\int_\Omega (f\odot u^\alpha):\nabla^2\varphi^6\,dx- \\
-\int_\Omega f\cdot\nabla\varphi^6\mathrm{div}\,u^\alpha\,dx + \int_\Omega \sigma^\alpha_{ij}\varphi^6_{,ijk}u^\alpha_k\,dx + \int_\Omega \sigma^\alpha:\nabla^2\varphi^6\mathrm{div}\,u^\alpha\,dx.
\end{aligned}$$

The latter equality and (4.118)-(4.122) give the estimate:

$$J^\alpha_m \le I^\alpha_0 + I^\alpha_1 + I^\alpha_2 + I^\alpha_3 \tag{4.125}$$

with

$$\begin{aligned} I_0^\alpha := 2\int_\Omega \Big[&(f\odot\nabla\varphi^6) : \varepsilon^D(u^\alpha) + \varphi^6\varepsilon(f) : \varepsilon(u^\alpha) - \\ &-(f\odot u^\alpha) : \nabla^2\varphi^6\Big]\,dx + \left(\tfrac{2}{n}-1\right)\int_\Omega f\cdot\nabla\varphi^6 \operatorname{div} u^\alpha\,dx \\ &-\int_\Omega \Big(\varphi^6 \operatorname{div} f \operatorname{div} u^\alpha + \nabla\varphi^6\cdot u^\alpha \operatorname{div} f\Big)\,dx, \end{aligned} \tag{4.126}$$

$$I_1^\alpha := 12\int_\Omega \varphi^5\sigma_{,k}^{\alpha D} : S^{(k)}\,dx, \quad I_2^\alpha := \int_\Omega \sigma_{ij}^\alpha \varphi_{,ijk}^6 u_k^\alpha\,dx,$$

$$I_3^\alpha := \int_\Omega \sigma^\alpha : \nabla^2\varphi^6 \operatorname{div} u^\alpha\,dx.$$

Estimate of I_0^α. By using the convergence $u_m^\alpha \overset{*}{\rightharpoonup} \delta u_m^N$ in $BD(\Omega)$ (Theorem 4.5.5 and Lemma 4.6.2), I_0^α can be estimated as

$$|I_0^\alpha| \le C(\|f_m\|_{C^1(\Omega')},\ \|\varphi\|_{C^1(\Omega)}) \sup_\alpha \|u_m^\alpha\|_{BD} \le C(m, N, \Omega'). \tag{4.127}$$

Estimate of I_1^α. As $\sigma_{m-1}^N \in \mathbb{K}$, its deviator σ_{m-1}^{ND} is bounded, and so is e_{m-1}^{ND}. We conclude, that

$$\begin{aligned} &\int_{|\varepsilon^D(u_m^\alpha)+e_{m-1}^{ND}|\le \frac{k_*}{\sqrt{2}\mu}} |\varepsilon^D(u_m^\alpha)|^2\,dx + \\ &+\int_{|\varepsilon^D(u_m^\alpha)+e_{m-1}^{ND}|> \frac{k_*}{\sqrt{2}\mu}} \frac{|\varepsilon^D(u_m^\alpha)|^2}{|\varepsilon^D(u_m^\alpha)+e_{m-1}^{ND}|}\,dx \le C_m\|\varepsilon^D(u_m^\alpha)\|_1 + C_m, \end{aligned} \tag{4.128}$$

which is uniformly bounded with respect to α by Lemma 4.6.2.

Using (4.124), (4.103), (4.100), (4.86), (4.104) and (4.128) we obtain

$$\begin{aligned} |I_1^\alpha| &\le C_m \int_\Omega \varphi^5 E_2^\alpha(\tau_{,k}^\alpha, S^{(k)})\,dx \le \\ &\le \tfrac{1}{100}\int_\Omega \varphi^6 E_2^\alpha(\tau_{m,k}^\alpha, \tau_{m,k}^\alpha)\,dx + C_m\int_\Omega \varphi^4 E_2^\alpha(S^{(k)}, S^{(k)})\,dx \le \\ &\le \tfrac{1}{100}\int_\Omega \varphi^6\,\sigma_{m,k}^\alpha : \tau_{m,k}^\alpha\,dx + C_m\,\alpha\int_\Omega \varphi^4|S^{(k)}|^2\,dx + \\ &+C_m\int_{|\tau_m^{\alpha D}|\le\frac{k_*}{\sqrt{2}\mu}} \varphi^4|S^{(k)}|^2\,dx + C_m\int_{|\tau_m^{\alpha D}|>\frac{k_*}{\sqrt{2}\mu}} \frac{\varphi^4|S^{(k)}|^2}{|\tau_m^{\alpha D}|}\,dx \le \\ &\le \tfrac{1}{100}\Big(J_m^\alpha + \int_\Omega \varphi^6\mathbb{A}\sigma_{m,k}^\alpha : \sigma_{m,k}^\alpha\,dx + \int_\Omega \varphi^6\mathbb{A}\sigma_{m-1,k} : \sigma_{m-1,k}\,dx\Big) + \\ &+C_m\int_{|\tau_m^{\alpha D}|\le\frac{k_*}{\sqrt{2}\mu}} |\varepsilon^D(u_m^\alpha)|^2\,dx + C_m\int_{|\tau_m^{\alpha D}|>\frac{k_*}{\sqrt{2}\mu}} \frac{|\varepsilon^D(u_m^\alpha)|^2}{|\tau_m^{\alpha D}|}\,dx + C_m\,\alpha\|\varepsilon^D(u_m^\alpha)\|_2^2 \le \\ &\le \tfrac{1}{100}\Big(J_m^\alpha + \int_\Omega \varphi^6\mathbb{A}\sigma_{m,k}^\alpha : \sigma_{m,k}^\alpha\,dx + \int_\Omega \varphi^6\mathbb{A}\sigma_{m-1,k} : \sigma_{m-1,k}\,dx\Big) + \\ &+C_m\int_\Omega |\varepsilon^D(u_m^\alpha)|\,dx + C_m. \end{aligned} \tag{4.129}$$

Estimate of I_2^α. As to the second summand, Lemma 4.6.2 and the embedding $W_0^{1,2}(\Omega;\mathbb{M}^{n\times n}_{sym}) \hookrightarrow L^n(\Omega;\mathbb{M}^{n\times n}_{sym})$ for $n=2,3$ allows one to make the following estimates

$$\begin{aligned}|I_2^\alpha| &\le C_m\|\varphi^3\sigma_m^\alpha\|_{n;\Omega}\,\|u_m^\alpha\|_{\frac{n}{n-1};\Omega} \le C_m\|\nabla(\varphi^3\,\sigma_m^\alpha)\|_2\|u_m^\alpha\|_{\frac{n}{n-1}} \le \\ &\le C_m\left[\int_\Omega \varphi^6\mathbb{A}\,\sigma_{m,l}^\alpha : \sigma_{m,l}^\alpha\,dx + \int_\Omega \varphi^4|\nabla\varphi|^2|\sigma_m^\alpha|^2\,dx\right]^{1/2} \le \\ &\le \tfrac{1}{100}\int_\Omega \varphi^6\mathbb{A}\,\sigma_{m,l}^\alpha : \sigma_{m,l}^\alpha\,dx + C_m.\end{aligned} \tag{4.130}$$

Estimate of I_3^α. As

$$\operatorname{div} u_m^\alpha = \frac{1}{nK_0}\operatorname{tr}\sigma_m^\alpha - \operatorname{tr} e_{m-1}^N,$$

we can bound I_3^α as

$$|I_3^\alpha| \le C_m(\|\sigma_m^\alpha\|_2^2 + \|\sigma_{m-1}^N\|_2^2) \tag{4.131}$$

So, (4.118), (4.125), (4.127)-(4.131) and the regularity of σ_{m-1}^N, proved at the previous step, imply that

$$J_m^\alpha \le C_m + \tfrac{1}{100}J_m^\alpha + \tfrac{2}{100}\int_\Omega \varphi^6\mathbb{A}\,\sigma_{m,l}^\alpha : \sigma_{m,l}^\alpha\,dx.$$

Therefore (4.115) allow us to conclude that (4.110) holds, and thus

$$\limsup_{\alpha\to 0}\|\nabla\sigma_m^\alpha\|_{L^2(\Omega')} \le C(m,N,\Omega'), \tag{4.132}$$

where this constant depends on the domain Ω', the step m, and the data of the problem.

Remark 4.7.1. Inequality (4.132) and the convergence $\sigma_m^\alpha \rightharpoonup \sigma_m^N$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, see (4.87), imply that

$$\begin{aligned}&\sigma_m^N \in W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym}),\\ &\sigma_m^\alpha \rightharpoonup \sigma_m^N \quad \text{in } W^{1,2}_{loc}(\Omega),\\ \text{and}\quad &\sigma^\alpha \to \sigma_m^N \quad \text{in } L^n_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym}),\end{aligned} \tag{4.133}$$

where the strong convergence in $L^n_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$ is guaranteed by the Sobolev embedding for $n = 2,3$.

4.8 Auxiliary estimates

In this section we prove a fine convergence estimate for the approximate solutions of regularized problems (Lemma 4.8.1) and get some analytical estimates, which are the core of the proof of the uniform boundedness of σ_N^m in $W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym})$ (Lemmas 4.8.2, 4.8.3 and Corollary 4.8.4).

In these estimates it is crucial that constants C does not depend on m and N, although they may depend on φ.

In the rest of the paper $\omega_m(\alpha)$ will denote a generic function converging to 0 as $\alpha\to 0$, which may change from line to line and depend upon m and N.

4.8.1 Fine properties of approximating sequence

Lemma 4.8.1. *For any non-negative function $\varphi \in C_0(\Omega)$ we have*

$$\int_{|\tau_m^{\alpha D}|>(1+\frac{1}{N})\frac{k_*}{\sqrt{2}\mu}} \varphi^4\Big(|\varepsilon^D(u_m^\alpha)+e_{m-1}^{ND}|-|e_{m-1}^{ND}|\Big)\,dx \le \frac{C}{N}+\omega_m(\alpha). \tag{4.134}$$

Proof: As $|e_{m-1}^{ND}| = |\mathbb{A}\sigma_{m-1}^{ND}| = \frac{1}{2\mu}|\sigma_{m-1}^{ND}| \le \frac{k_*}{\sqrt{2}\mu}$, we have

$$\begin{aligned}
&\int_{(1+\frac{1}{N})\frac{k_*}{\sqrt{2}\mu}\le|\tau_m^{\alpha D}|} \varphi^4\Big(|\varepsilon^D(u_m^\alpha)+e_{m-1}^{ND}|-|e_{m-1}^{ND}|\Big)\,dx \le\\
&\le \int_{\frac{k_*}{\sqrt{2}\mu}\le|\tau_m^{\alpha D}|} \varphi^4\Big(|\varepsilon^D(u_m^\alpha)+e_{m-1}^{ND}|-|e_{m-1}^{ND}|\Big)\,dx =\\
&= \int_\Omega \varphi^4\Big(|\varepsilon^D(u_m^\alpha)+e_{m-1}^{ND}|-|e_{m-1}^{ND}|\Big)\,dx-\\
&- \int_{|\tau_m^{\alpha D}|<\frac{k_*}{\sqrt{2}\mu}} \varphi^4\Big(|\varepsilon^D(u_m^\alpha)+e_{m-1}^{ND}|-|e_{m-1}^{ND}|\Big)\,dx.
\end{aligned} \tag{4.135}$$

On the set $\{|\tau_m^{\alpha D}| < \frac{k_*}{\sqrt{2}\mu}\}$ one has $\sigma_m^{\alpha D} = \alpha\tau_m^{\alpha D} + 2\mu\tau_m^{\alpha D}$ by (4.98). From (4.133) and (4.15) it follows, that

$$\begin{aligned}
&- \int_{|\tau_m^{\alpha D}|<\frac{k_*}{\sqrt{2}\mu}} \varphi^4\Big(|\varepsilon^D(u_m^\alpha)+e_{m-1}^{ND}|-|e_{m-1}^{ND}|\Big)\,dx \le\\
&\le \frac{1}{2\mu}\int_{|\tau_m^{\alpha D}|<\frac{k_*}{\sqrt{2}\mu}} \varphi^4\big(|\sigma_m^{\alpha D}-\sigma_{m-1}^{ND}|+\alpha|\tau_m^{\alpha D}|\big)\,dx \le\\
&\le \tfrac{1}{2\mu}\int_\Omega \varphi^4|\sigma_m^{\alpha D}-\sigma_{m-1}^{ND}|\,dx + C\alpha \le\\
&\le \tfrac{1}{2\mu}\int_\Omega \varphi^4|\sigma_m^{ND}-\sigma_{m-1}^{ND}|\,dx + \tfrac{1}{2\mu}\int_\Omega \varphi^4|\sigma_m^N-\sigma_m^\alpha|\,dx + C\alpha \le \frac{C}{N}+\omega_m(\alpha).
\end{aligned}$$

On the other hand, by (4.91) and (4.15)

$$\begin{aligned}
&\lim_{\alpha\to 0}\int_\Omega \varphi^4\Big(|\varepsilon^D(u_m^\alpha)+e_{m-1}^{ND}|-|e_{m-1}^{ND}|\Big)\,dx =\\
&= \langle\varphi^4, |\varepsilon^D(\delta u_m^N)+e_{m-1}^{ND}|-|e_{m-1}^{ND}|\cdot\mathcal{L}^n\rangle \le |\varepsilon^D(\delta u_m^N)|(\Omega) \le \frac{C}{N}.
\end{aligned}$$

The result now follows from (4.135) and the last two inequalities. □

4.8.2 Analytic estimates

Lemma 4.8.2. *The following inequality holds for J_m^α defined in (4.118):*

$$J_m^\alpha \le \frac{C}{N} + \frac{1}{N}\int_\Omega \varphi^6 \mathbb{A}\sigma^N_{m,l} : \sigma^N_{m,l}\,dx + 12\int_\Omega \varphi^5 \sigma_m^{\alpha D} : S^{(k)}\,dx + \omega_m(\alpha) \tag{4.136}$$

PROOF: Recalling (4.125) we have $J_m^\alpha \le I_0^\alpha + I_1^\alpha + I_2^\alpha + I_3^\alpha$ with I_i^α, $i = 0, \dots, 3$ defined in (4.126). We will show, that the sum $I_0^\alpha + I_2^\alpha + I_3^\alpha$ is of order $\frac{1}{N}$ when $\alpha \to 0$.

Estimates of I_0^α: Since $f_m \in C^1_{loc}(\Omega;\mathbb{R}^n)$, one can employ (4.82)-(4.85) to pass to the limit in (4.126), and use estimates (4.15) of $\|\delta u^m\|_{BD;\Omega}$ to get

$$|I_0^\alpha| \le C(\|f\|_{L^\infty([0,T];C^1(\Omega';\mathbb{R}^n))},\ \|\varphi\|_{C^2(\Omega)})\,\frac{1}{N} + \omega_m(\alpha), \tag{4.137}$$

where $\mathrm{supp}\varphi \subset \Omega'' \subset\subset \Omega$.

Estimates of I_2^α: To pass to the limit in I_2^α, we exploit (4.82) and (4.133):

$$\lim_{\alpha\to 0} I_2^\alpha = \lim_{\alpha\to 0}\int_\Omega \sigma^\alpha_{ij}\varphi^6_{,ijk}u^\alpha_k\,dx = \int_\Omega (\sigma^N_m)_{ij}\,\varphi^6_{,ijk}(\delta u^N_m)_k\,dx =: I_2.$$

Now let us use the embedding $W^{1,2}_0(\Omega;\mathbb{M}^{n\times n}_{sym}) \hookrightarrow L^n(\Omega;\mathbb{M}^{n\times n}_{sym})$ for $n = 2,3$, (4.11) and (4.15)

$$\begin{aligned} |I_2| &\le C\|\varphi^3\sigma^N_m\|_{n;\Omega}\,\|\delta u^N_m\|_{\frac{n}{n-1};\Omega} \le \\ &\le C\biggl(\int_\Omega \varphi^6\sigma^N_{m,l} : \sigma^N_{m,l}\,dx + \|\sigma^N_m\|^2_{2;\Omega}\biggr)^{1/2}\|\delta u^N_m\|_{\frac{n}{n-1};\Omega} \le \\ &\le \frac{1}{N}\int_\Omega \varphi^6\mathbb{A}\sigma^N_{m,l} : \sigma^N_{m,l}\,dx + \frac{C}{N}. \end{aligned} \tag{4.138}$$

Estimates of I_3^α: The relations (4.85) and (4.133) allow one to pass to the limit in I_3^α:

$$\lim_{\alpha\to 0} I_3^\alpha = \lim_{\alpha\to 0}\int_\Omega \sigma^\alpha_m : \nabla^2\varphi^6 \mathrm{div}\,u^\alpha_m\,dx = \int_\Omega \sigma^N_m : \nabla^2\varphi^6\mathrm{div}\,\delta u^N_m\,dx =: I_3,$$

so in view of the equality $\mathrm{div}\,\delta u^N_m = \mathrm{tr}\,\delta e^N_m$, by (4.11) and (4.15) we conclude that

$$|I_3| \le C\|\sigma^N_m\|_{2;\Omega}\,\|\mathrm{tr}\,\delta e^N_m\|_{2;\Omega} \le \frac{C}{N}. \tag{4.139}$$

Thus, (4.137)-(4.139) imply (4.136). □

Lemma 4.8.3. *The following "iterative" estimate holds true:*

$$\begin{aligned} \int_\Omega \varphi^6 E_2^\alpha(\tau^\alpha_{m,k},\tau^\alpha_{m,k})\,dx &\le \tfrac{100}{99}\int_\Omega \varphi^6\mathbb{A}\sigma^\alpha_{m,k} : \sigma^N_{m-1,k}\,dx + \\ &+ \sum_{s=1}^{9N-1}\frac{1}{s+10}\int_{F_s}\varphi^6\mathbb{A}\sigma^{\alpha D}_{m,k} : \sigma^{\alpha D}_{m,k}\,dx - \\ -\tfrac{1}{99}\int_{|\tau^{\alpha D}|\le 10\frac{k_*}{\sqrt{2\mu}}} \varphi^6\,E_2^\alpha(\tau^{\alpha D}_{m,k},\tau^{\alpha D}_{m,k})\,dx &- \tfrac{1}{99}\int_\Omega \varphi^6\,\tfrac{1}{n}\,\mathrm{tr}\,\mathbb{A}\sigma^\alpha_{m,k}\,\mathrm{tr}\,\sigma^\alpha_{m,k}\,dx + \\ +\frac{C}{N}\int_\Omega \varphi^6\,\mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\,dx &+ \frac{C}{N}\int_\Omega \varphi^6\,\mathbb{A}\sigma^N_{m,l} : \sigma^N_{m,l}\,dx + \frac{C}{N} + \omega_m(\alpha), \end{aligned} \tag{4.140}$$

where F_s, $s = 1,\dots,9N-1$ is defined by $F_s = \{\frac{k_*}{\sqrt{2\mu}}(1+\frac{9}{s+1}) < |\tau^{\alpha D}| \le \frac{k_*}{\sqrt{2\mu}}(1+\frac{9}{s})\}$

Proof: By (4.103), (4.100), (4.118) and (4.136)

$$
\begin{aligned}
\int_\Omega \varphi^6 E_2^\alpha(\tau^\alpha_{m,k},\tau^\alpha_{m,k})\,dx \le 12\int_\Omega \varphi^5\sigma^\alpha_{m,k} : S^{(k)}\,dx+ \\
+\int_\Omega \varphi^6\mathbb{A}\sigma^\alpha_{m,k} : \sigma^N_{m-1,k} + \tfrac{1}{N}\int_\Omega \varphi^6\mathbb{A}\sigma^N_{m,l} : \sigma^N_{m,l}\,dx + \frac{C}{N} + \omega_m(\alpha) = \\
= B_1^\alpha + B_2^\alpha + B_3^\alpha + B_4^\alpha + \int_\Omega \varphi^6\mathbb{A}\sigma^\alpha_{m,k} : \sigma^N_{m-1,k}+ \\
+\tfrac{1}{N}\int_\Omega \varphi^6\mathbb{A}\sigma^N_{m,l} : \sigma^N_{m,l}\,dx + \frac{C}{N} + \omega_m(\alpha),
\end{aligned}
\tag{4.141}
$$

where

$$
B_i^\alpha := \int_{\Omega_i} \varphi^5\,\sigma^\alpha_{m,k} : S^{(k)}\,dx, \quad i = 1,\dots,4,
$$

and

$$
\begin{aligned}
\Omega_1 = \left\{|\tau^{\alpha D}_m| \le \tfrac{k_*}{\sqrt{2}\mu}\right\}, \qquad \Omega_2 = \left\{\tfrac{k_*}{\sqrt{2}\mu} < |\tau^{\alpha D}_m| \le (1+\tfrac{1}{N})\tfrac{k_*}{\sqrt{2}\mu}\right\}, \\
\Omega_3 = \left\{(1+\tfrac{1}{N})\tfrac{k_*}{\sqrt{2}\mu} < |\tau^{\alpha D}_m| \le 10\tfrac{k_*}{\sqrt{2}\mu}\right\}, \qquad \Omega_4 = \left\{10\tfrac{k_*}{\sqrt{2}\mu} < |\tau^{\alpha D}_m|\right\}.
\end{aligned}
\tag{4.142}
$$

Estimate of B_1^α: According to (4.98) and (4.100), in the region Ω_1 the following identity holds:

$$
2\mu\,\varepsilon^D(u^\alpha_m) = \sigma^{\alpha D}_m - \sigma^{ND}_{m-1} - \alpha\tau^{\alpha D}_m.
$$

Hence, by (4.123)

$$
|S^{(k)}|^2 \le C|\varepsilon^D(u^\alpha_m)|^2 \le C\big(|\sigma^{\alpha D}_m - \sigma^{ND}_{m-1}|^2 + \alpha^2|\tau^{\alpha D}_m|^2\big),
$$

and we have

$$
\int_{\Omega_1} \varphi^4|S^{(k)}|^2\,dx \le C\,\alpha^2 + C\|\sigma^{\alpha D}_m - \sigma^{ND}_{m-1}\|^2_{2,\Omega''}.
$$

Thus, the convergence (4.133) and the increment estimate (4.15), it follows

$$
\begin{aligned}
B_1^\alpha \le \frac{1}{N}\int_{\Omega_1} \varphi^6\mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\,dx+ \\
+CN\int_{\Omega_1} \varphi^4|S^{(k)}|^2\,dx \le \\
\le \frac{1}{N}\int_{\Omega_1} \varphi^6\mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\,dx + \frac{C}{N} + \omega_m(\alpha).
\end{aligned}
\tag{4.143}
$$

Estimate of B_2^α: Let us note that (4.98) and (4.100) yield that for $|\tau^{\alpha D}_m| \ge \frac{k_*}{\sqrt{2}\mu}$ one has

$$
k_*\sqrt{2}\varepsilon^D(u^\alpha_m) = \sigma^{\alpha D}_m\big(|\tau^{\alpha D}_m| - \tfrac{k_*}{\sqrt{2}\mu}\big) - \tfrac{k_*}{\sqrt{2}\mu}(\sigma^{ND}_{m-1} - \sigma^{\alpha D}_m) - \alpha\tau^{\alpha D}_m\,|\tau^{\alpha D}_m|,
\tag{4.144}
$$

so that in the region Ω_2 we have

$$
|\varepsilon^D(u^\alpha_m)|^2 \le \frac{C}{N^2}|\sigma^{\alpha D}_m|^2 + C|\sigma^{\alpha D}_m - \sigma^{ND}_{m-1}|^2 + \alpha^2|\tau^{\alpha D}_m|^4.
$$

From the inequality $|S^{(k)}| \le C|\varepsilon^D(u_m^\alpha)|$

$$\varphi^5 \sigma_{m,k}^\alpha : S^{(k)} \le \frac{1}{N}\varphi^6\, \mathbb{A}\sigma_{m,k}^\alpha : \sigma_{m,k}^\alpha + CN\,\varphi^4|\varepsilon^D(u_m^\alpha)|^2,$$

so that by the former estimate, the boundedness of $\tau_m^{\alpha D}$ and $\sigma_m^{\alpha D}$ on Ω_2 (see (4.98)), (4.133) and (4.15) we have

$$B_2^\alpha \le \frac{1}{N}\int_{\Omega_2} \varphi^6 \mathbb{A}\sigma_{m,k}^\alpha : \sigma_{m,k}^\alpha \, dx + \frac{C}{N} + \omega_m(\alpha). \tag{4.145}$$

Estimate of B_3^α: Using the notation $F_s = \{\frac{k_*}{\sqrt{2}\mu}(1+\frac{9}{s+1}) < |\tau_m^{\alpha D}| \le \frac{k_*}{\sqrt{2}\mu}(1+\frac{9}{s})\}$ we write

$$\begin{gathered} B_3^\alpha = 12\sum_{s=1}^{9N-1}\int_{F_s}\varphi^5\sigma_{m,k}^\alpha : S^{(k)}\,dx \le \\ \le \sum_{s=1}^{9N-1}\left[\frac{1}{2(s+10)}\int_{F_s}\varphi^6\mathbb{A}\sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D}\,dx + C(s+10)\int_{F_s}\varphi^4|\varepsilon^D(u_m^\alpha)|^2\,dx\right]. \end{gathered} \tag{4.146}$$

Now we show, that the last sum can be bounded by $\frac{C}{N} + \omega_m(\alpha)$.

Thanks to (4.144) on F_s we have

$$|\varepsilon^D(u_m^\alpha)|^2 \le \frac{9}{s}\,\frac{1}{2\,k_*^2}\,|\sigma_m^{\alpha D}|^2\left(|\tau_m^{\alpha D}| - \tfrac{k_*}{\sqrt{2}\mu}\right) + C|\sigma_m^{\alpha D} - \sigma_{m-1}^{ND}|^2 + \alpha^2|\tau_m^{\alpha D}|^4,$$

so that by (4.133), (4.11), (4.15) and the boundedness of $\sigma_m^{\alpha D}$ and $\tau_m^{\alpha D}$ on F_s (see (4.98)) we have

$$\begin{gathered} \sum_{s=1}^{9N-1}(s+10)\int_{F_s}\varphi^4|\varepsilon^D(u_m^\alpha)|^2\,dx \le \\ \sum_{s=1}^{9N-1}\int_{F_s}\varphi^4\left(\tfrac{9\,(s+10)}{s}\,\tfrac{1}{2\,k_*^2}|\sigma_m^{\alpha D}|^2\left(|\tau_m^{\alpha D}| - \tfrac{k_*}{\sqrt{2}\mu}\right) + CN|\sigma_m^{\alpha D} - \sigma_{m-1}^{ND}|^2 + CN\alpha^2\right)dx \\ \le C\int_{\Omega_3}|\sigma_m^{\alpha D}|\left(|\tau_m^{\alpha D}| - \tfrac{k_*}{\sqrt{2}\mu}\right)\,dx + \\ + CN\|\sigma_m^{\alpha D} - \sigma_{m-1}^{ND}\|_{2;\Omega''}^2 + CN\,\alpha^2 \end{gathered} \tag{4.147}$$

Since by (4.98) and (4.100)

$$|\sigma_m^{\alpha D}||\tau_m^{\alpha D}| \le \alpha|\tau_m^{\alpha D}|^2 + k_*\sqrt{2}|\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|,$$

and as the triangle inequality $|\sigma_m^{\alpha D}| \ge |\sigma_{m-1}^{ND}| - |\sigma_{m-1}^{ND} - \sigma_m^{\alpha D}|$ implies

$$-|\sigma_m^{\alpha D}|\tfrac{k_*}{\sqrt{2}\mu} \le -k_*\sqrt{2}|e_{m-1}^{ND}| + \tfrac{k_*}{\sqrt{2}\mu}|\sigma_{m-1}^{ND} - \sigma_m^{\alpha D}|.$$

Therefore we can bound the right-hand side of (4.147) by

$$C\int_{\Omega_3}\varphi^4\Big(|\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}| - |e_{m-1}^{ND}|\Big)\,dx + \frac{C}{N} + \omega_m(\alpha).$$

Thus, by (4.134) we conclude, that

$$B_3^\alpha \le \sum_{s=1}^{9N-1} \frac{1}{2(s+10)} \int_{F_s} \varphi^6 \mathbb{A}\sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D}\, dx + \frac{C}{N} + o_\alpha(N). \tag{4.148}$$

Estimate of B_4^α: By applying the Cauchy inequality

$$\varphi^5\, E_2^\alpha[\tau_{m,k}^{\alpha D}; S^{(k)}] \le \tfrac{1}{100}\varphi^6\, E_2^\alpha[\tau_{m,k}^{\alpha D}; \tau_{m,k}^{\alpha D}] + C\,\varphi^4\, E_2^\alpha[S^{(k)}; S^{(k)}],$$

and using (4.103), (4.104), (4.123) and (4.86) we conlcude that

$$\begin{aligned} B_4^\alpha = \int_{\Omega_4} \varphi^5 E_\alpha^2[\tau_{m,k}^{\alpha D}; S^{(k)}]\, dx \le \\ \le \frac{1}{100}\int_{\Omega_4} \varphi^6 E_2^\alpha[\tau_{m,k}^{\alpha D}; \tau_{m,k}^{\alpha D}]\, dx + C\int_{\Omega_4} \varphi^4 \frac{|\varepsilon^D(u_m^\alpha)|^2}{|\tau_m^{\alpha D}|} + o_\alpha(1). \end{aligned} \tag{4.149}$$

To show that the last summand is of order $\frac{1}{N}$, we first note that on the set Ω_4 the inequality

$$\frac{|\varepsilon^D(u_m^\alpha)|^2}{|\tau_m^{\alpha D}|} < 10(|\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}| - |e_{m-1}^{ND}|) \tag{4.150}$$

is valid. To see this, we multiply both sides by $|\tau_m^{\alpha D}| = |\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|$. Using the inequality $|e_{m-1}^{ND}| \le \frac{k_*}{\sqrt{2}\mu}$, which follows from $\sigma_{m-1}^N \in \mathbb{K}$, the right-hand side of (4.150) can be estimated as

$$\begin{aligned} 10\big(|\varepsilon^D(u^\alpha)_m|^2 + 2\varepsilon^D(u_m^\alpha) : e_{m-1}^{ND} + |e_{m-1}^{ND}|^2 - |e_{m-1}^{ND}| \cdot |\varepsilon^D(u_m^\alpha) + e_{m-1}^{ND}|\big) \ge \\ \ge 10\big(|\varepsilon^D(u_m^\alpha)|^2 - 3|e_{m-1}^{ND}| \cdot |\varepsilon^D(u_m^\alpha)|\big) \ge 10|\varepsilon^D(u_m^\alpha)|^2 - 30\,\tfrac{k_*}{\sqrt{2}\mu}|\varepsilon^D(u_m^\alpha)|. \end{aligned}$$

Using again $|e_{m-1}^{ND}| \le \frac{k_*}{\sqrt{2}\mu}$, in the region Ω_4 we obtain $|\varepsilon^D(u_m^\alpha)| > 9\frac{k_*}{\sqrt{2}\mu}$, which yields that

$$10|\varepsilon^D(u^\alpha)|^2 - 30\tfrac{k_*}{\sqrt{2}\mu}|\varepsilon^D(u^\alpha)| > |\varepsilon^D(u_m^\alpha)|^2 + 51\tfrac{k_*}{\sqrt{2}\mu}|\varepsilon^D(u^\alpha)| > |\varepsilon^D(u_m^\alpha)|^2,$$

and (4.150) is proved.

From (4.149), (4.150) and (4.134) we have

$$B_4^\alpha \le \frac{1}{100}\int_{\Omega_4} \varphi^6 E_2^\alpha[\tau_{m,k}^{\alpha D}; \tau_{m,k}^{\alpha D}]\, dx + \frac{C}{N} + \omega_m(\alpha). \tag{4.151}$$

Collecting (4.141)-(4.145), (4.148) and (4.151) we obtain

$$\begin{aligned} \int_\Omega \varphi^6 E_2^\alpha(\tau_{m,k}^\alpha, \tau_{m,k}^\alpha)\, dx \le \int_\Omega \varphi^6 \mathbb{A}\sigma_{m,k}^\alpha : \sigma_{m-1,k}^N\, dx + \\ + \sum_{s=1}^{9N-1} \frac{1}{2(s+10)} \int_{F_s} \varphi^6 \mathbb{A}\sigma_{m,k}^{\alpha D} : \sigma_{m,k}^{\alpha D}\, dx + \frac{1}{100}\int_{\Omega_4} \varphi^6 E_2^\alpha[\tau_{m,k}^{\alpha D}; \tau_{m,k}^{\alpha D}]\, dx + \\ + \tfrac{C}{N} + \tfrac{C}{N}\int_\Omega \varphi^6 \mathbb{A}\sigma_{m,k}^\alpha : \sigma_{m,k}^\alpha\, dx + \tfrac{1}{N}\int_\Omega \varphi^6 \mathbb{A}\sigma_{m,l}^N : \sigma_{m,l}^N\, dx + \omega_m(\alpha), \end{aligned}$$

or, by obvious transformations, using definition (4.102) of the quadratic form E_2^α

$$\tfrac{99}{100}\int_\Omega \varphi^6 E_2^\alpha(\tau_{m,k}^\alpha,\tau_{m,k}^\alpha)\,dx \le \int_\Omega \varphi^6\,\mathbb{A}\sigma_{m,k}^\alpha : \sigma_{m-1,k}^N\,dx+$$
$$+\sum_{s=1}^{9N-1}\frac{1}{2(s+10)}\int_{F_s}\varphi^6\mathbb{A}\sigma_{m,k}^{\alpha D}:\sigma_{m,k}^{\alpha D}\,dx+$$
$$+\tfrac{C}{N}\int_\Omega \varphi^6\,\mathbb{A}\sigma_{m,k}^\alpha:\sigma_{m,k}^\alpha\,dx+\tfrac{C}{N}\int_\Omega\varphi^6\,\mathbb{A}\sigma_{m,l}^N:\sigma_{m,l}^N\,dx-$$
$$-\tfrac{1}{100}\int_{\Omega_1\cup\Omega_2\cup\Omega_3}E_2^\alpha(\tau_{m,k}^{\alpha D},\tau_{m,k}^{\alpha D})\,dx-\tfrac{1}{100}\int_\Omega\varphi^6\tfrac{1}{n}\operatorname{tr}\mathbb{A}\sigma_{m,k}^\alpha\operatorname{tr}\sigma_{m,k}^\alpha\,dx+\frac{C}{N}+\omega_m(\alpha),$$

so the claim (4.140) follows by multiplying the latter inequality by $\frac{100}{99}$. □

By using Lemmas 4.6.4 and 4.6.6 we can express (4.140) in a different form, which is more suitable for our uniform estimates of $\sigma_{m,k}^\alpha$.

Corollary 4.8.4. *The following estimate is valid*

$$\tfrac12\int_\Omega\varphi^6E_2^\alpha(\tau_{m,k}^\alpha,\tau_{m,k}^\alpha)\,dx+\tfrac12\int_\Omega\varphi^6E_2^\alpha(\mathbb{A}\sigma_{m,k}^\alpha,\mathbb{A}\sigma_{m,k}^\alpha)\,dx\le$$
$$\le\big(\tfrac14\cdot\tfrac{296}{99}+\tfrac{C}{N}+\omega_m(\alpha)\big)\int_\Omega\varphi^6\mathbb{A}\sigma_{m,k}^\alpha:\sigma_{m,k}^\alpha\,dx+ \tag{4.152}$$
$$+\tfrac14\cdot\tfrac{100}{99}\int_\Omega\varphi^6\mathbb{A}\sigma_{m-1,k}^N:\sigma_{m-1,k}^N\,dx+\tfrac{C}{N}\int_\Omega\varphi^6\mathbb{A}\sigma_{m,l}^N:\sigma_{m,l}^N\,dx+\frac{C}{N}+\omega_m(\alpha).$$

Proof: We consider the situation in each of the following domains Ω_i, defined in (4.142). First, remark that from (4.108) we obtain

$$-E_2^\alpha(\tau_{m,k}^{\alpha D},\tau_{m,k}^{\alpha D})\le-(1+\omega_m(\alpha))\,\mathbb{A}\sigma_{m,k}^{\alpha D}:\sigma_{m,k}^{\alpha D} \tag{4.153}$$

on $\Omega_1\cup\Omega_2\cup\Omega_3$. We will apply (4.140), and divide the integral over Ω in three integrals over the domains just defined.

$$\tfrac12\int_\Omega\varphi^6E_2^\alpha(\tau_{m,k}^\alpha,\tau_{m,k}^\alpha)\,dx+\tfrac12\int_\Omega\varphi^6E_2^\alpha(\mathbb{A}\sigma_{m,k}^\alpha,\mathbb{A}\sigma_{m,k}^\alpha)\,dx\le$$
$$\le\tfrac12\int_\Omega\varphi^6E_2^\alpha(\mathbb{A}\sigma_{m,k}^\alpha,\mathbb{A}\sigma_{m,k}^\alpha)\,dx+\tfrac14\cdot\tfrac{100}{99}\int_\Omega\varphi^6\mathbb{A}\sigma_{m,k}^\alpha:\sigma_{m,k}^\alpha\,dx-$$
$$-\tfrac12\cdot\tfrac{1}{99}\int_\Omega\varphi^6\frac{\operatorname{tr}\mathbb{A}\sigma_{m,k}^\alpha\cdot\operatorname{tr}\sigma_{m,k}^\alpha}{n}\,dx-\tfrac12\cdot\tfrac{1}{99}\int_{\Omega_1\cup\Omega_2\cup\Omega_3}\varphi^6E_2^\alpha(\tau_{m,k}^{\alpha D},\tau_{m,k}^{\alpha D})\,dx+ \tag{4.154}$$
$$+\tfrac12\sum_{s=1}^{9N-1}\frac{1}{s+10}\int_{F_s}\varphi^6\mathbb{A}\sigma_{m,k}^{\alpha D}:\sigma_{m,k}^{\alpha D}\,dx+\tfrac14\cdot\tfrac{100}{99}\int_\Omega\varphi^6\mathbb{A}\sigma_{m-1,k}^N:\sigma_{m-1,k}^N\,dx+$$
$$+\tfrac{C}{N}\int_\Omega\varphi^6\mathbb{A}\sigma_{m,k}^\alpha:\sigma_{m,k}^\alpha\,dx+\tfrac{C}{N}\int_\Omega\varphi^6\mathbb{A}\sigma_{m,l}^N:\sigma_{m,l}^N\,dx+\tfrac{C}{N}+\omega_m(\alpha)$$

Estimates over $\Omega_1 \cup \Omega_2$**:** By (4.105) and (4.153) the sum of the integrals on $\Omega_1 \cup \Omega_2$ corresponding to the first four terms in (4.154) is bounded by

$$\begin{aligned}\left(\tfrac{1}{4}\cdot\tfrac{100}{99}+\tfrac{1}{2}-\tfrac{1}{2}\cdot\tfrac{1}{99}+\omega_m(\alpha)\right)\int_{\Omega_1\cup\Omega_2}\varphi^6\mathbb{A}\sigma^\alpha_{m,k}:\sigma^\alpha_{m,k}\,dx \leq \\ \leq \left(\tfrac{1}{4}\cdot\tfrac{296}{99}+\omega_m(\alpha)\right)\int_{\Omega_1\cup\Omega_2}\varphi^6\mathbb{A}\sigma^\alpha_{m,k}:\sigma^\alpha_{m,k}\,dx\end{aligned} \tag{4.155}$$

Estimates over Ω_3**:** The integral on Ω_3 is estimated by considering the integrals on the sets F_s, defined in (4.140). Using (4.105), (4.153) and the bounds

$$\tfrac{k_*}{\sqrt{2}\mu}\cdot\tfrac{s+10}{s+1} < |\tau^{\alpha D}_m| \leq \tfrac{k_*}{\sqrt{2}\mu}\cdot\tfrac{s+9}{s}$$

on each of F_s, the sum of the integrals on F_s corresponding to the first five terms in (4.154) is bounded by

$$\begin{aligned}\sum_{s=1}^{9N-1}\Bigg[\Big(\tfrac{1}{4}\cdot\tfrac{100}{99}+\tfrac{1}{2}\cdot\tfrac{s+1}{s+10}+\tfrac{1}{2}\cdot\tfrac{1}{s+10}-\tfrac{1}{2}\cdot\tfrac{1}{99}\cdot\tfrac{s+10}{s+1}+\omega_m(\alpha)\Big)\cdot \\ \cdot\int_{F_s}\varphi^6\mathbb{A}\sigma^{\alpha D}_{m,k}:\sigma^{\alpha D}_{m,k}\,dx+ \\ +\left(\tfrac{1}{4}\cdot\tfrac{100}{99}+\tfrac{1}{2}-\tfrac{1}{2}\cdot\tfrac{1}{99}\right)\int_{F_s}\varphi^6\frac{\operatorname{tr}\mathbb{A}\sigma^\alpha_{m,k}\cdot\operatorname{tr}\sigma^\alpha_{m,k}}{n}\,dx\Bigg] \leq \\ \leq \left(\tfrac{1}{4}\cdot\tfrac{296}{99}+\omega_m(\alpha)\right)\int_{\Omega_3}\varphi^6\mathbb{A}\sigma^\alpha_{m,k}:\sigma^\alpha_{m,k}\,dx.\end{aligned} \tag{4.156}$$

Estimates over Ω_4**:** By (4.105) and the lower bound $|\tau^{\alpha D}_m| > 10\frac{k_*}{\sqrt{2}\mu}$, the sum of the integrals on $\Omega_1 \cup \Omega_2$ corresponding to the first three terms in (4.154) is bounded by

$$\begin{aligned}\left(\tfrac{1}{4}\cdot\tfrac{100}{99}+\tfrac{1}{20}+\omega_m(\alpha)\right)\int_{\Omega_4}\varphi^6\mathbb{A}\sigma^{\alpha D}_{m,k}:\sigma^{\alpha D}_{m,k}\,dx+ \\ +\left(\tfrac{1}{4}\cdot\tfrac{100}{99}+\tfrac{1}{2}-\tfrac{1}{2}\cdot\tfrac{1}{99}\right)\int_{\Omega_4}\varphi^6\frac{\operatorname{tr}\mathbb{A}\sigma^\alpha_{m,k}\cdot\operatorname{tr}\sigma^\alpha_{m,k}}{n}\,dx+ \\ \leq \left(\tfrac{1}{4}\cdot\tfrac{296}{99}+\omega_m(\alpha)\right)\int_{\Omega_4}\varphi^6\mathbb{A}\sigma^\alpha_{m,k}:\sigma^\alpha_{m,k}\,dx.\end{aligned} \tag{4.157}$$

The claim now follows from (4.154)-(4.157). □

4.9 Uniform $W^{1,2}_{loc}$-estimates of stresses

To carry out the proof of the uniform boundedness of $\|\sigma_N\|_{L^\infty((0,T);W^{1,2}_{loc}(\Omega;\mathbb{M}^{n\times n}_{sym}))}$ we will make use of the refined version of iterative estimate (4.115), deduced in the previous section, which results in a discrete analogue of Gronwall inequality. To this aim, we need to have the estimate of the last term of (4.115). To make the estimates uniform, we will use the convergence of u^α_m to δu^m as in (4.82)-(4.86), and the convergence of σ^α_m to σ^m as in (4.133).

So, the goal of this section is to prove the following inequality first

$$\big(1-\tfrac{C}{N}\big)\int_\Omega \varphi^6 \mathbb{A}\sigma^N_{m,l} : \sigma^N_{m,l}\,dx \le \big(1+\tfrac{C}{N}\big)\int_\Omega \varphi^6 \mathbb{A}\sigma^N_{m-1,l} : \sigma^N_{m-1,l}\,dx + \frac{C}{N}, \tag{4.158}$$

with C independent of N, and then to deduce Theorem 4.2.1.

We begin as in (4.114), using (4.103) and (4.100):

$$\int_\Omega \varphi^6 \mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\,dx \le \tfrac12 \int_\Omega \varphi^6 E^\alpha_2(\tau^\alpha_{m,k},\tau^\alpha_{m,k})\,dx + \tfrac12\int_\Omega \varphi^6 E^\alpha_2(\mathbb{A}\sigma^\alpha_{m,k},\mathbb{A}\sigma^\alpha_{m,k})\,dx.$$

Thus, (4.152) yields

$$\begin{aligned}\big(\tfrac14\cdot\tfrac{100}{96}+\omega_m(\alpha)\big)&\int_\Omega \varphi^6 \mathbb{A}\sigma^\alpha_{m,k} : \sigma^\alpha_{m,k}\,dx \le \\ &\le \tfrac14\cdot\tfrac{100}{96}\int_\Omega \varphi^6 \mathbb{A}\sigma^N_{m-1,k} : \sigma^N_{m-1,k}\,dx + \\ &+\tfrac{C}{N}\int_\Omega \varphi^6 \mathbb{A}\sigma^N_{m,l} : \sigma^N_{m,l}\,dx + \tfrac{C}{N} + \omega_m(\alpha).\end{aligned} \tag{4.159}$$

Now, to deduce (4.158) it remains to pass to the limit with respect to α in (4.159), to use (4.133) and the lower semicontinuity of the norm, and to sum the resulting expressions with respect to k.

Proof of Theorem 4.2.1:
Iterating (4.158) we get the following for every $m = 1,\dots,N$

$$\begin{aligned}\int_\Omega \varphi^6 \mathbb{A}\sigma^N_{m,l} : \sigma^N_{m,l}\,dx &\le \frac{\big(1+\frac{C}{N}\big)^N}{\big(1-\frac{C}{N}\big)^N}\int_\Omega \varphi^6 \mathbb{A}\sigma_{0,l} : \sigma_{0,l}\,dx + \tfrac{2C}{N}\sum_{i=1}^N \frac{\big(1+\frac{C}{N}\big)^{i-1}}{\big(1-\frac{C}{N}\big)^i} \le \\ &\le e^{2C}\int_\Omega \varphi^6 \mathbb{A}\sigma_{0,l} : \sigma_{0,l}\,dx + 2C\,e^{2C}.\end{aligned} \tag{4.160}$$

Thus, we obtain

$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]} \|\sigma_N(t)\|_{W^{1,2}(\Omega';\mathbb{M}^{n\times n}_{sym})} \le C(\Omega'),$$

and the conclusion follows from convergence of $\sigma_N(t) \rightharpoonup \sigma(t)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ for every $t\in[0,T]$.
□

4.10 Examples

We conclude the paper by presenting two examples which show that we cannot expect any kind of spatial regularity for $u(t)$ and $p(t)$. Classical example of quasistatic evolutions, where $u(t)$ develops a jump, or $p(t)$ is singular, can be found in [Ser85, Suq88].

However, these examples are based on non-uniqueness, and besides the irregular solution $(u(t),e(t),p(t))$ there exists also a regular solution $(\tilde u(t),\tilde e(t),\tilde p(t))$ with $\tilde e(t)\equiv e(t)$.

The main feature of the examples presented here is that the data is smooth, the solution $(u(t),e(t),p(t))$ corresponding to this data is unique, but $u(t)$ and $p(t)$ are irregular.

We consider two particular cases of the periodic problem in dimension two, where not only the stress tensor $\sigma(t)$ is unique, but so are the displacement $u(t)$ and a plastic part of the strain $p(t)$. In this case the problem is reduced to a one-dimensional one.

In the first example we show that there are smooth data for which the displacement $u(t,\cdot)$, corresponding to the unique solution $(u(t), e(t), p(t))$, develops a space discontinuity along a hyperplane at a given time t^*.

In the second example we determine a wide class of (possibly singluar) measures, such that for every μ in this class there exist smooth data for which the quasistatic evolution $(u(t), e(t), p(t))$ is unique and $p(t^*) = \mu$ at a prescribed time t^*.

We consider the case of simple shear for the Dirichlet-periodic problem in dimension $n = 2$. Similar examples can be easily constructed also in higher dimensions. We consider the unit cube $\Omega = (0,1) \times (0,1)$ and x_1-periodic solutions with boundary data of the form

$$\begin{aligned} u(t, x_1, 0) &= (0,0), \\ u(t, x_1, 1) &= (\sqrt{2}\varphi(t), 0), \\ u(t, 0, x_2) &= u(t, 1, x_2). \end{aligned} \tag{4.161}$$

Let us introduce a linear isometry $M : \mathbb{R} \to \mathbb{M}^{2\times 2}_{sym}$ as

$$M(\alpha) = \begin{pmatrix} 0 & \frac{\alpha}{\sqrt{2}} \\ \frac{\alpha}{\sqrt{2}} & 0 \end{pmatrix}. \tag{4.162}$$

Assume, that the volume force has the form

$$f(t,x) = \frac{1}{\sqrt{2}}(f^R(t,x_2), 0), \tag{4.163}$$

where we require the safe-load assumption to hold, and the initial conditions $(u_0, e_0, 0)$ are

$$u_0(x_1,x_2) = \begin{pmatrix} \sqrt{2}u_0^R(x_2) \\ 0 \end{pmatrix} \quad \text{and} \quad e_0(x_1,x_2) = M(e_0^R(x_2)), \tag{4.164}$$

for some functions u_0^R, e_0^R.

First, we will show, that in this particular situation all solutions of the quasistatic problem can be obtained from the solutions of a suitable one-dimensional problem. The definition of quasistatic evolution in dimension one can be obtained from Definitions 4.3.1 and 4.3.6 by replacing the spaces $\mathbb{M}^{n\times n}_{sym}$ and $\mathbb{M}^{n\times n}_D$ by $\mathbb{R}$, the compliance tensor $\mathbb{A}$ by $\frac{1}{2\mu}$, and the set $\mathbb{K}$ by $\mathbb{K}^R = [-\sqrt{2}k_*, \sqrt{2}k_*]$.

Let $W \subset \mathbb{M}^{2\times 2}_{sym}$ be defined as follows

$$W = \left\{ \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} : a \in \mathbb{R} \right\}.$$

Given $p^R \in M_b([0,1])$, the measure $M(p^R) \in M_b([0,1]\times[0,1]; \mathbb{M}^{2\times 2}_{sym})$ is defined by

$$\begin{gathered} M(p^R)(A \times B) = M(p^R(B))\,\mathcal{L}^1(A) \\ \text{for every pair of Borel sets } A, B \subset [0,1], \text{ that is} \\ \langle M(p^R), \psi\rangle = \sqrt{2}\int_0^1 \langle p^R, \psi_{12}(x_1,\cdot)\rangle\, dx_1 \\ \text{for every } \psi \in C^0([0,1]\times[0,1]; \mathbb{M}^{2\times 2}_{sym}). \end{gathered} \tag{4.165}$$

Theorem 4.10.1. *Suppose that the boundary condition is given as in* (4.161) *and the load as in* (4.163) *with* $f^R \in AC([0,T]; L^2(0,1))$. *Suppose, that the triple* $(u_0, e_0, 0)$ *is kinematically admissible and satisfies the stability condition of Theorem* (4.3.7). *Then* (u,e,p) *is a solution of the quasistatic problem with initial conditions* (4.164) *if and only if it has the form:*

$$u(t,x_1,x_2) = \begin{pmatrix} \sqrt{2}u^R(t,x_2) \\ 0 \end{pmatrix}, \quad e(t,x_1,x_2) = M(e^R(t,x_2)), \quad p(t) = M(p^R(t)) \tag{4.166}$$

with $M(p^R(t))$ *defined in* (4.165), *where* $(u^R(t), e^R(t), p^R(t))$ *is a solution of the one-dimensional quasistatic problem on* $\Omega^R = (0,1)$, *with initial data* $(u_0^R, e_0^R, 0)$, *Dirichlet boundary conditions* $u^R(t,0) = 0$, $u^R(t,1) = \varphi(t)$, *and load* $f^R(t,y)$.

Proof: Consider the quasistatic problem with initial data $(u_0^R, e_0^R, 0)$ in dimension one with domain $\Omega^R = (0,1)$, the compliance tensor $\mathbb{A}^R = \frac{1}{2\mu}$, volume force $f^R(t,y)$ and the Dirichlet boundary data $u^R(t,0) = 0$, $u^R(t,1) = \varphi(t)$. Let $(u^R(t,y), e^R(t,y), p^R(t,y))$ be a solution to this problem.

Now we show that the function (u,e,p) defined as follows

$$u(t,x_1,x_2) = (\sqrt{2}u^R(t,x_2), 0), \quad e(t,x_1,x_2) = M(e^R(t,x_2)), \quad p(t) = M(p^R(t)), \tag{4.167}$$

with M defined in (4.162) and (4.165), is a quasistatic evolution in dimension two.

To this aim, let us check conditions (qs1) and (qs2) of Definition 4.3.6 with $w(t,x_1,x_2) = (\sqrt{2}\varphi(t)x_2, 0)$.

(qs1): The kinematic admissibility condition for (u,e,p) in dimension two (see Definition 4.3.1) follows from the corresponding condition for (u^R, e^R, p^R) in dimension one. As the minimality condition in (qs1) is equivalent to $-\operatorname{div}\sigma = f$ and $\sigma \in \mathcal{K}$, and these properties follow form the properties of σ^R.

(qs2): Since M is an isometry, the energy balance for (u,e,p) follows from the analogous property of (u^R, e^R, p^R).

Thus the function $(u(t), e(t), p(t))$ defined in (4.167) is a quasistatic evolution in dimension two.

By the uniqueness of the strain σ we know, that for any quasistatic evolution in dimension two, the stress field $\sigma(t)$ is given by

$$\sigma(t,x_1,x_2) = \begin{pmatrix} 0 & \sigma_{12}(t,x_2) \\ \sigma_{12}(t,x_2) & 0 \end{pmatrix}. \tag{4.168}$$

By the pointwise formulation of the flow rule, proved in [DDM06, Theorem 6.4], and taking into account the fact, that $\sigma(t)$ is continuous, we have that for a.e. $t \in [0,T]$

$$g(x) := \frac{d\dot{p}(t)}{d|\dot{p}(t)|} \in W \quad \text{for } |\dot{p}(t)|\text{-a.e. } x \in [0,1]\times[0,1].$$

As $\dot{p}(t) = g(x)\,|\dot{p}(t)|$ for a.e. $t \in [0,T]$, it follows that

$$\dot{p}(t) \in M_b([0,1]\times[0,1]; W).$$

Thus, as

$$\langle p(t), \varphi \rangle = \int_0^t \langle \dot{p}(s), \varphi \rangle \, ds,$$

for every $\varphi \in C([0,T] \times [0,T]; \mathbb{M}^{n\times n}_{sym})$, we conclude that $p(t) \in M_b([0,1] \times [0,1]; W)$ for a.e. $t \in [0,T]$.

So, from (4.168) and the last relation we deduce by the additive decomposition, that

$$\varepsilon(u) \in M_b(\Omega; W).$$

In particular, it implies, that

$$u_{1,1}(t,x) = 0, \quad \text{and} \quad u_{2,2}(t,x) = 0,$$

that is,

$$u_1(t,x) = u_1(t,x_2) \quad \text{and} \quad u_2(t,x) = u_2(t,x_1).$$

However, from the relaxed form of boundary conditions (4.161), which take the form

$$u_2(t,x_1,x_2) = u_2(t,x_1,0) = (-u(t,x_1,0) \odot \nu(x_1))_{22} = 0,$$

we have that $u_2(t,x_1,x_2) \equiv 0$.

Thus, $u(t,x_1,x_2) = (u_1(t,x_2),0)$. Defining $u^R(t,x_2) = \frac{1}{\sqrt{2}} u_1(t,x_2)$, we see that (4.166) holds for suitable $e^R(t)$ and $p^R(t)$. It is then easy to see, that the triple $(u^R(t), e^R(t), p^R(t))$ is a solutuion to the one-dimensional quasistatic problem. □

4.10.1 Example 1

In this situation the data of the problem are the following: the domain Ω^R is $(0,1)$, the time interval is $[0,T] = [0, \frac{3}{2}]$, the constraint set $\mathbb{K}^R = [-1,1]$ and the elasticity tensor $\mathbb{A}^R$ is the identity. Taking the initial data to be $(u_0, e_0, p_0) = (0,0,0)$ we show that there exists a unique quasistatic evolution in dimension one, and that the displacement u of the solution has a jump at a point $x = \frac{1}{2}$ after time $t^* = 1$.

We choose a function $G \in C_0^\infty(0,1)$ such that

$$\int_0^1 G(y)\, dy = 0,$$
$$G(1/2) = 1, \quad G(y) < 1 \quad \text{for } y \neq \tfrac{1}{2},$$
$$G(y) > -\frac{1}{10} \quad \text{for } y \in [0,1],$$

and denote its derivative by $g(y)$.

So, we consider the one-dimensional quasistatic problem with the following C^∞ data:

$$\begin{aligned} (u_0^R, e_0^R, p_0^R) &= (0,0,0), \\ u^R(t,0) &= u^R(t,1) = 0, \\ f^R(t,y) &= -t\, g(y). \end{aligned} \tag{4.169}$$

According to Theorem 4.10.1 all solutions of the corresponding two-dimensional Dirichlet-periodic problem are generated by the solutions of one-dimensional problem (4.169).

Consider the functions $(u^R(t), e^R(t), p^R(t))$ as follows:

$$u^R(t,y) = \begin{cases} t\displaystyle\int_0^y G(z)\,dz, & \text{for } t \le 1; \\ (1-t)y + (t-1)\chi_{(\frac{1}{2},1)}(y) + t\displaystyle\int_0^y G(z)\,dz, & \text{for } 1 < t \le \frac{3}{2}; \end{cases}$$

$$e^R(t,y) = \sigma^R(t,y) = \begin{cases} tG(y), & \text{for } t \le 1; \\ tG(y) + 1 - t, & \text{for } 1 < t \le \frac{3}{2}; \end{cases}$$

$$p^R(t) = \begin{cases} 0, & \text{for } t \le 1; \\ (t-1)\delta_{1/2}, & \text{for } 1 < t \le \frac{3}{2}. \end{cases}$$

It is easy to see, that this triple satisfies

$$(u^R(t), e^R(t), p^R(t)) \in A^R(0) \quad \text{for all } t \in [0,T].$$

In view of Remark 4.3.8 the global minimality condition is ensured by the fact that $|\sigma^R(t,y)| \le 1$ and $\sigma^R_y(t,y) = tg(y) = -f^R(t,y)$.

By [DDM06, Theorem 6.4] the energy balance is equivalent to the pointwise formulation of the flow rule. Since $\dot p^R(t) = \delta_{\frac{1}{2}}$ for $t > 1$ and as $\sigma^R(t,\frac{1}{2}) = 1$, and $|\sigma^R(t,y)| < 1$ for $y \ne \frac{1}{2}$ or $t < 1$ we have that

$$1 = \frac{d\dot p^R(t)}{d|\dot p^R(t)|}(y) \in N_{\mathbb{K}^R}(\sigma^R(t,y)) \quad \text{for } |\dot p^R(t)|\text{-a.e. } y \in [-1,1],$$

which is precisely the pointwise expression of the flow rule. Thus, (u^R, e^R, p^R) constructed above is a quasistatic evolution in dimension one.

Now we show that the solution constructed is the unique one. Let us suppose, that there exists another quasistatic evolution $(v^R(t), \eta^R(t), q^R(t))$. By the uniqueness of the stress, $\eta^R \equiv e^R$. Now let us show that $q^R \equiv p^R$. As the energy balance (qs2) is satisfied for (v^R, η^R, q^R), the pointwise formulation of the flow rule yields

$$\frac{d\dot q^R(t)}{d|\dot q^R(t)|} \in N_{\mathbb{K}^R}(\sigma^R(t,y)) \quad \text{for } |\dot q^R(t)|\text{-a.e. } y \in [-1,1].$$

By the properties of $\sigma^R(t)$ it follows, that $\operatorname{supp} \dot q^R(t) \subset \{\frac{1}{2}\}$ for a.e. $t \in [1,\frac{3}{2}]$, while $\dot q^R(t) = 0$ for a.e. $t \in [0,1]$.

Thus the formula

$$\langle q^R(t), \varphi\rangle_{M_b;C_0} = \int_0^t \langle \dot q^R(s), \varphi\rangle_{M_b;C_0}\,ds \quad \text{for any } \varphi \in C_0(0,1),$$

yields that $q^R(t) = \beta(t)\delta_{\frac{1}{2}}$ with $\beta \ge 0$, and the boundary conditions (4.169) imply that $\beta(t) = t-1$, that is $q^R \equiv p^R$. This yields also that $v^R(t) = u^R(t)$, and we obtain the uniqueness of $u^R(t)$.

4.10.2 Example 2

We are given the domain $\Omega^R = (0,1)$, the time interval $[0,1]$, the constraint set $\mathbb{K}^R = [-1,1]$ and the elasticity tensor $\mathbb{A}^R = 1$. Let $\mu^R \in M_b^+(0,1)$ be a diffuse measure, that is $\mu^R(\{x\}) = 0$. Suppose, that $\mu^R([0,1]) = 1$.

We will choose the C^∞ data of the problem so that the quasistatic evolution is unique and satisfies $p^R(t,\cdot) = \mu^R$ for $t = 1$.

Let us take the continuous nondecreasing function $\Phi(s) = \mu^R([0,s])$. We consider the left-continuous inverse

$$X(t) := \sup\{s : \Phi(s) < t\},$$

so that $\Phi(X(t)) \equiv t$. Let us take the set $\{(t, X(t)) : t \in [0,1]\}$ and denote its closure by E:

$$E := \mathrm{cl}\,\{(t, X(t)) : t \in [0,1]\} = \bigcup_{t\in[0,1]} \{(t, X(t)),\ (t, X(t+0))\}.$$

Then there exists a function $\phi(t,y)$, such that

$$\begin{gathered} \phi \in C_0^\infty(\mathbb{R}^2),\ 0 < \phi \le 1, \\ \phi^{-1}(\{1\}) = E. \end{gathered}$$

The data of the one-dimensional problem we would like to solve are the following:

$$\begin{gathered} u_0^R(y) = \int_0^y \phi(0,z)\,dz, \quad e_0^R(y) = \phi(0,y), \quad p_0^R = 0, \\ u^R(t,0) = 0, \quad u^R(t,1) = \int_0^1 \phi(t,y)\,dy + t, \\ f^R(t,y) = -\phi_y(t,y). \end{gathered} \tag{4.170}$$

Now consider a function $\mu^R : [0,T] \to M_b^+([0,1])$, defined as

$$\mu^R(t)(B) = \mu^R(B \cap [0, X(t)])$$

for every Borel set $B \subset [0,1]$. The estimate

$$\|\mu^R(t) - \mu^R(s)\|_1 = \mu^R((x(s), x(t)]) = \Phi(X(t)) - \Phi(X(s)) = t - s,$$

shows that $\mu^R \in AC([0,T]; M_b^+([0,1]))$. Moreover, the very definition of $X(t)$ yields

$$\dot\mu^R(t) = \delta_{X(t)}.$$

Consider the following functions:

$$\begin{gathered} u^R(t,y) = \int_0^y \phi(t,z)\,dz + \mu^R(t)(0,y), \\ e^R(t,y) = \sigma^R(t,y) = \phi(t,y), \\ p^R(t) = \mu^R(t), \end{gathered} \tag{4.171}$$

and let us show that (u^R, e^R, p^R) defined in this way is the unique solution of the quasistatic problem (4.170).

First of all, it is obvious that the initial conditions are satisfied and the triple (u_0^R, e_0^R, p_0^R) satisfies the stability condition. Let us check the conditions (qs1) and (qs2) as in the Definition 4.3.6.
(qs1): As

$$(\mu^R(t)(0,y))_y = \mu^R(t) \quad \text{in } \mathcal{D}'(0,1),$$

the kinematic admissibility condition in $(0,1)$ is trivially satisfied by (4.171). As the boundary conditions hold in the strong sense and $p = 0$ on $\partial\Omega$, we have that the triple $(u(t), e(t), p(t))$ is kinematically admissible for its boundary data.

What about the global stability, it follows from the equivalent condition (see Remark 4.3.8)

$$-\sigma_y^R(t,y) = f^R(t,y) \quad \text{and} \quad |\sigma^R(t,y)| \leq 1.$$

(qs2): As $\sigma^R(t, X(t)) = \sigma^R(t, X(t+0)) = 1$ and $|\sigma^R(t,y)| < 1$ otherwise, the pointwise formulation of the flow rule, which is equivalent to the energy balance, is satisfied:

$$\dot{p}^R(t) = \delta_{X(t-0)}, \quad 1 = \frac{d\dot{p}^R(t)}{d|\dot{p}^R(t)|}(y) \in N_{\mathbb{K}^R}(\sigma^R(t,y)), \quad \text{for } |\dot{p}^R(t)|\text{-a.e. } y \in (0,1).$$

So, (4.171) is a solution to (4.170).

Now let us take any solution $(v^R(t), \eta^R(t), q^R(t))$ to quasistatic problem (4.170). As the stress is unique, we have $\eta^R(t) \equiv e^R(t)$. Now, the pointwise formulation of the flow rule implies

$$\frac{d\dot{q}^R(t)}{d|\dot{q}^R(t)|}(y) \in N_{\mathbb{K}^R}(\sigma^R(t,y)), \quad \text{for } |\dot{q}^R(t)|\text{-a.e. } y \in (0,1),$$

that is $\operatorname{supp} \dot{q}^R(t) \subset \{X(t), X(t+0)\}$. As $X(t)$ is a monotone function, it has at most countable number of discontinuities, that is for a.e. $t \in [0,1]$ we have

$$\operatorname{supp} \dot{q}^R(t) \subset \{X(t)\}. \tag{4.172}$$

The boundary conditions for $v^R(t)$ yield:

$$v^R(t,1) = \int_0^1 \phi(t,y)\,dy + q^R(t)(0,1) = \int_0^1 \phi(t,y)\,dy + t,$$

which in its turn implies that $q^R(t)(0,1) = t$, and from (4.172) it follows, that

$$\dot{q}^R(t) = \delta_{X(t)}.$$

That is, $q^R(t) \equiv p^R(t)$ and (4.171) is the unique solution corresponding to the data (4.170).

Chapter 5

Regularity of bending moments for perfect elasto-plastic plates

5.1 Introduction

In this paper we study the regularity of the bending moments of the quasistatic evolution of clamped perfectly elasto-plastic plates under the action of a time-dependent transversal body force. Before introducing the regularity result, we describe the mechanical model. The reference configuration is a bounded open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary and the elastic domain $\mathbb{K}$ is a bounded closed convex subset of $\mathbb{M}^{2\times 2}_{sym}$ (the space of symmetric 2×2 matrices) with nonempty interior, whose boundary $\partial\mathbb{K}$ plays the role of the yield surface.

Given a scalar valued function $f(t,x)$ defined for $t \in [0,T]$ and $x \in \Omega$, which represents the transversal body force, the strong formulation of the evolution problem consists in finding a scalar valued function $u(t,x)$ (the vertical displacement) and three matrix-valued functions $e(t,x)$, $p(t,x)$ and $M(t,x)$ (the elastic and plastic curvatures and the bending moments) such that for every $t \in [0,T]$, for every $x \in \Omega$ the following conditions hold:

1. kinematic admissibility: $D^2u(t,x) = e(t,x) + p(t,x)$ in Ω,
   $u(t,x) = 0$, $\frac{\partial u}{\partial \nu}(t,x) = 0$ on $\partial\Omega$
2. constitutive equation: $M(t,x) = \mathbb{C}\, e(t,x)$,
3. equilibrium: $\operatorname{div}\operatorname{div} M(t,x) = f(t,x)$ in Ω,
4. moment constraint: $M(t,x) \in \mathbb{K}$,
5. associative flow rule: $\dot{p}(t,x) \in N_{\mathbb{K}}(M(t,x))$,

where $\nu(x)$ is the outer unit normal to $\partial\Omega$ and $\mathbb{C}$ is the rigidity tensor. The symbol $N_{\mathbb{K}}(\xi)$ denotes the normal cone to the set $\mathbb{K}$ at the point ξ in the sense of convex analysis. The problem is supplemented by initial conditions at time $t = 0$.

The boundary conditions $u = 0$ and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ reflect the mechanical assumption the plate is clamped.

For the regularity *we restrict ourselves* to the *isotropic case* where $\mathbb{K}$ *is a ball, centered at the origin,* and $\mathbb{C}$ *is a multiple of identity tensor* $\mathbb{I}$, which can be reduced to considering

$$\mathbb{K} = B_1(0), \quad \mathbb{C} = \mathbb{I}.$$

Existence of weak solutions to problems in perfect plasticity has been extensively studied during last decades (see, for example, [Anz84, DDD07, DDM06, Dem89, FS00, Tem85]). Since the variational formulation of the problem used in the definition of weak solutions involves an integral with linear growth in D^2u, the natural functional spaces for the problem are $BH(\Omega)$ of functions with bounded Hessian for the vertical displacements u, and $L^2(\Omega; \mathbb{M}^{2\times2}_{sym})$ for the bending moments M.

However, in a similar problem for Prandtl-Reuss perfect plasticity it was shown in [BF96, Dem08b] that the stress (which is the counterpart of the bending moments) belongs to $W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times2}_{sym})$ (see also [FS00, Ser87, Ser96, Ser94, Ser94, Ser93b, Ser94] for similar results for some static models).

In the present paper we study the spatial regularity of the bending moments $M(t,\cdot)$ for the quasistatic problem for prefect elastoplastic plates. The main result obtained (see Theorem 5.2.2 below) for the model under consideration is

$$M \in L^\infty([0,T]; W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times2}_{sym})). \tag{5.1}$$

As in [Dem08b], our strategy for an evolutionary quasistatic problem relies on a regularity result for an analogous static problem, obtained in [Ser94], where it was shown that in a static situation the bending moments enjoy the following differentiability condition:

$$M \in W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times2}_{sym}).$$

We discretize time by points $(t^N_r)^N_{r=1}$

$$0 = t^N_0 < t^N_1 < \cdots < T^N_N = T$$

and we approximate the original quasistatic problem by a sequence of incremental "static" problems, finding for each $r = 1, \ldots, N$ the updated values of $(u^N_r, e^N_r, p^N_r, M^N_r)$, provided that $(u^N_{r-1}, e^N_{r-1}, p^N_{r-1}, M^N_{r-1})$ is already found. Shortly, the main idea is to generalize the estimates of [Ser94] in order to take into account the influence of the previous steps.

To be more precise, following [Dem09], we apply the standard method of constructing piecewise constant approximations

$$(u_N(t), e_N(t), p_N(t), M_N(t)) = (u^N_r, e^N_r, p^N_r, M^N_r) \quad \text{for} \quad t^N_r \le t < t^N_{r+1},$$

with $0 \le r < N$, of the continuous-time energy formulation of rate-independent processes (see [Mie02] for the survey of this approach). Our aim is to get a uniform estimate of the form

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \|M_N(t)\|_{W^{1,2}_{loc}(\Omega;\mathbb{M}^{2\times2}_{sym})} \le C, \tag{5.2}$$

which clearly implies (5.1).

To get (5.2) we consider the updated values of $(u^N_r, e^N_r, p^N_r, M^N_r)$ as saddle points of some minimax problem, similar to the one considered in [FS00, Ser87, Ser94] for static cases in perfect plasticity. The main difference from the purely static problem is the presence of a term which takes into account the outcome of the preceding step. Approximating each incremental problem with a sequence of regularized problems, depending on a real parameter $\alpha \in (0,1)$, we obtain that their solutions M_r^α converge to M_r^N, a solution to the corresponding incremental problem, weakly in $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$, as $\alpha \to 0$. Then one can show, that for every incremental problem the bound

$$\sup_{\alpha>0} \|M_r^\alpha\|_{W^{1,2}(\Omega';\mathbb{M}^{2\times 2}_{sym})} \leq C_r$$

holds for any domain $\Omega' \subset\subset \Omega$, where the constant C_r depends on the discretization step and on Ω'. Thus, M_r^N is itself in $W^{1,2}_{loc}(\Omega;\mathbb{M}^{2\times 2}_{sym})$, and the compactness of Sobolev embedding improves the convergence of M_r^α to M_r^N. Then we do some analytical work to make the last estimate uniform in r and N, and we obtain (5.2).

Notice that all the arguments used below are purely local, and cannot be used for studying the behavior of bending moments up to the boundary $\partial\Omega$ (see [Ser99] for the discussion of the global regularity issues in an analogous case of Hencky perfect plasticity). As far as we know, the only global regularity result in perfect plasticity was obtained in [Kne06] for Hencky perfect plasticity.

The paper is organized as follows: in Section 5.2 we introduce the notation and state the main result. We present a weak formulation of the problem and prove a time-continuity result in Section 5.3. A minimax formulation of incremental problems in spirit of [FS00, Ser87, Ser94] is presented in Section 5.4. In Section 5.5 we introduce some regularized problems, depending on a real parameter $\alpha \in (0,1)$, whose solutions are smooth and "approximate", as $\alpha \to 0$, a solution $(u_r^N, e_r^N, p_r^N, M_r^N)$ of the corresponding incremental problem. We obtain $W^{1,2}_{loc}$ estimates of the solutions of regularized problems in Section 5.6, and conclude that

$$\sup_{t\in[0,T]} \|M_N(t)\|_{W^{1,2}(\Omega';\mathbb{M}^{2\times 2}_{sym})} \leq C(N,\Omega')$$

for each $\Omega' \subset\subset \Omega$ and $N \in \mathbb{N}$. Section 5.7 contains some analytical estimates, that will be used for making $W^{1,2}_{loc}$ estimates uniform with respect to N. Finally, in Section 5.8 we apply the results of Section 5.7 to obtain the uniform estimates of Sobolev norms and to conclude the proof of Theorem 5.2.2.

5.2 Preliminaries

5.2.1 Notation and definitions

We adopt the following notation:

$\mathbb{R}^n$ denotes the n-dimensional Euclidian space,

$\mathbb{M}^{2\times 2}_{sym}$ denotes the space of all 2×2 symmetric matrices, equipped with the Hilbert-Schmidt scalar product $\sigma : \xi = \sigma_{ij}\xi_{ij}$,

$a \odot b$ stands for the symmetrized tensor product of two vectors $a, b \in \mathbb{R}^n$, given by the formula $(a\odot b)_{ij} = \frac{1}{2}(a_i b_j + a_j b_i)$,

$L^p(\Omega;\mathbb{R}^m)$ is the Lebesgue space of functions from Ω into $\mathbb{R}^m$, having finite norm $(\int_\Omega |f|^p\,dx)^{1/p}$,

$W^{l,p}(\Omega;\mathbb{R}^m)$ is the Sobolev space of all functions from Ω into $\mathbb{R}^m$ with the norm

$$\|f\|_{l,p;\Omega} := \Big(\int_\Omega \sum_{\alpha=0}^{l} |\nabla^\alpha f|^p\Big)^{1/p},$$

$\mathcal{L}^2$ stands for the Lebesgue measure on $\mathbb{R}^2$,
$\mathcal{H}^1$ is the one-dimensional Hausdorf measure,
$M_b(\overline{\Omega};\mathbb{R}^m)$ is the space of all bounded Radon measures on $\overline{\Omega}$ with values in $\mathbb{R}^m$,
For $\mu \in M_b(\overline{\Omega};\mathbb{R}^m)$, we denote its total variation by $|\mu|$, which is an element of $M_b(\overline{\Omega})$, with $\|\mu\|_{1;\overline{\Omega}} = |\mu|(\overline{\Omega})$, while by μ^a and μ^s we denote its absolutely continuous and singular part with respect to $\mathcal{L}^n$,
$BH(\Omega)$ is the space of all functions in $L^1(\Omega)$ such that $Du \in BV(\Omega;\mathbb{R}^2)$, with norm $\|u\|_{2,1;\Omega} := \|u\|_{1,1;\Omega} + \|D^2u\|_{1;\Omega}$,
$\langle\cdot|\cdot\rangle$ denotes a duality product depending upon the context.

Remark 5.2.1. We refer to [Tem85, Chapter III] for the main properties of $BH(\Omega)$ and for the definition of weak* convergence in $BH(\Omega)$. Remark, that for $u \in BH(\Omega)$ we have the following embedding:

$$u \in C(\overline{\Omega}), \quad \nabla u \in L^2(\Omega;\mathbb{R}^2).$$

Let us introduce the notation

$$S(\Omega) = \Big\{ M \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) : \operatorname{div}\operatorname{div} M \in L^2(\Omega) \Big\},$$

$$\mathcal{K}(\Omega) = \Big\{ m \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) : m(x) \in \mathbb{K} \quad \text{for a.e. } x \in \Omega \Big\}.$$

5.2.2 The main result

We impose the following assumption on the data of the problem:

$$f \in AC([0,T];L^2(\Omega)) \cap L^\infty([0,T];W^{1,2}_{loc}(\Omega)). \tag{5.3}$$

We also assume the so-called uniform safe-load condition:

$$\begin{aligned}&\text{there exists a function } m^1 \in AC([0,T];L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})), \text{ such that}\\ &\operatorname{div}\operatorname{div} m(t) = f(t) \text{ in } \Omega \text{ for every } t \in [0,T], \text{ and}\\ &|m^1(t,x)| \le (1-\lambda) \text{ for some } 0<\lambda<1, \text{ a.e. } x \in \Omega, \text{ for every } t \in [0,T].\end{aligned} \tag{5.4}$$

Here and in the rest of the paper div always denotes the divergence with respect to space variables.
The main result of the paper is the following regularity theorem.

Theorem 5.2.2. *Suppose that the set $\mathbb{K}$ is a ball, centered at the origin, and $\mathbb{C}$ is a multiple of the identity, and that assumptions* (5.3), (5.4) *are satisfied. Then for the solution (u,e,p) of the quasistatic problem, see Definition 5.3.4 below, we have*

$$M \in L^\infty([0,T];W^{1,2}_{loc}(\Omega;\mathbb{M}^{2\times 2}_{sym})),$$

with $M(t,x) = \mathbb{C}e(t,x)$.

Remark 5.2.3. As already mentioned, we consider the case $\mathbb{K} = B_1(0)$ and $\mathbb{C} = \mathbb{I}$. It means, that $M \equiv e$, and we will be using both notations M and e for the same object.

5.3 Weak formulation of the quasistatic problem

Below we give the possible definition of weak solution to the quasistatic problem. The formulation we use (see [Dem09]) is expressed in terms of energy balance and energy dissipation.

5.3.1 Weak formulation: quasistatic evolution

Now we give the definition of a kinematically admissible triple. The first condition describes the additive decomposition, the second one gives the boundary conditions for u, while the third one reflects the boundary conditions for Du in a relaxed form, which is typical in the variational theory of functionals with linear growth.

Definition 5.3.1. A triple $(u, e, p) \in BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ is called kinematically admissible, if the following conditions hold

$$D^2 u = e + p \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

$$p = -\nabla u \odot \nu \, \mathcal{H}^1 \quad \text{on } \partial\Omega.$$

Definition 5.3.2. For a kinematically admissible triple (u, e, p) and $M \in S(\Omega)$ we define a measure $[M : p] \in M_b(\overline{\Omega})$ by putting

$$\begin{aligned} [M : p] &= M : p^a + [M : D^2 u]^s = [M : D^2 u] - M : e \quad \text{in } \Omega, \\ [M : p] &= -\frac{\partial u}{\partial \nu} M_{ij} \nu_i \nu_j \, d\mathcal{H}^1 \quad \text{on } \partial\Omega, \end{aligned} \tag{5.5}$$

where the measure $[M : D^2 u]$ is defined in [Dem89].

Thus, a duality pairing between $S(\Omega)$ and $\Pi(\Omega)$ is defined by

$$\langle M | p \rangle := [M : p](\overline{\Omega}). \tag{5.6}$$

One can prove the following integration by parts formula (see [Dem09, Proposition 2.8]).

Proposition 5.3.3. *For a kinematically admissible triple (u, e, p) and $M \in S(\Omega)$ with* $\operatorname{div}\operatorname{div} M = f \in L^2(\Omega)$ *we have*

$$[M : p](\overline{\Omega}) = \int_\Omega u \, f \, dx - \int_\Omega (M : e) \, dx. \tag{5.7}$$

Let us define the functionals which appear in the energy formulation of the problem. The quadratic form $\mathcal{Q} : L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \to \mathbb{R}$, corresponding to the stored elastic energy, is defined by

$$\mathcal{Q}(e) = \frac{1}{2} \int_\Omega e : e \, dx.$$

Since in our case the function H considered in [Dem09, Section 2.1] reduces to the norm in $\mathbb{M}^{2\times 2}_{sym}$, the dissipation in any time interval $[s,t] \subset [0,T]$ is defined by

$$\mathcal{D}(p; s, t) = \sup \left\{ \sum_{j=1}^{M} \|p(t_j) - p(t_{j-1})\|_{1;\overline{\Omega}} \; : \; s = t_0 \leq \cdots \leq t_M = t, \; M \in \mathbb{N} \right\}.$$

Now we are in a position to give a variational formulation of the quasistatic problem. In the following definition $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\Omega)$.

Definition 5.3.4. A quasistatic evolution is a function $t \mapsto (u(t), e(t), p(t))$ from $[0,T]$ into $BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ which satisfies the following conditions:

(qs1) for every $t \in [0,T]$ the triple $(u(t), e(t), p(t))$ is kinematically admissible and

$$\mathcal{Q}(e(t)) - \langle f(t), u(t) \rangle \leq \mathcal{Q}(\eta) + \|q - p(t)\|_{1;\overline{\Omega}} - \langle f(t), v \rangle \tag{5.8}$$

for every kinematically admissible triple (v, η, q);

(qs2) the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\Omega; \mathbb{M}^{2\times 2}_{sym})$ has bounded variation and for every $t \in [0,T]$

$$\begin{aligned} &\mathcal{Q}(e(t)) + \mathcal{D}(p; 0, t) - \langle f(t), u(t) \rangle = \\ &= \mathcal{Q}(e(0)) - \langle f(0), u(0) \rangle - \int_0^t \langle \dot{f}(s), u(s) \rangle \, ds. \end{aligned} \tag{5.9}$$

5.3.2 Existence result and time-discretization

The following theorem establishes the existence of a solution to the quasistatic problem in perfect plasticity.

Theorem 5.3.5. *Let a kinematically admissible triple (u_0, e_0, p_0) satisfy the stability condition*

$$Q(e_0) - \langle f(0), u_0 \rangle \leq Q(\eta) + \|q - p_0\|_{1;\overline{\Omega}} - \langle f(0), v \rangle,$$

for every kinematically admissible triple (v, η, q). Then there exists a quasistatic evolution

$$(u(t), e(t), p(t)),$$

such that

$$u(0) = u_0,\ e(0) = e_0,\ p(0) = p_0.$$

Moreover, the elastic part $t \mapsto e(t)$ *of* $D^2u(t)$ *is unique and a quasistatic evolution* (u, e, p) *as a function from* $[0, T]$ *to* $BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ *is absolutely continuous in time.*

In [Dem09] this theorem is proved by a discretization of time. We divide the interval $[0, T]$ into N equal parts of length T/N by points $(t^N_r)_{r=0,\dots,N}$. For $r = 0, \dots, N$ we set

$$f^r_N = f(t^N_r) \quad \text{and } (m^1)^N_r = m^1(t^N_r). \tag{5.10}$$

For every N we define u^N_r, e^N_r and p^N_r by induction. We set

$$(u^N_0, e^N_0, p^N_0) = (u_0, e_0, p_0),$$

while for every $r = 1, \dots, N$ we define (u^N_r, e^N_r, p^N_r) as a solution to the incremental problem

$$\min_{(u,e,p)} \left\{ \mathcal{Q}(e) + \|p - p^N_{r-1}\|_{1;\overline{\Omega}} - \int_\Omega f^N_r \, u \, dx \right\}, \tag{5.11}$$

where the minimisation is carried out over all kinematically admissible triples (see Definition 5.3.1).

Remark 5.3.6. We note, that (u, e, p) is a solution to (5.11) if and only if one of the following conditions holds:

1. for every kinematically admissible triple (v, η, q) one has

$$-\|q\|_{1;\overline{\Omega}} \le \langle e, \eta \rangle - \langle f^N_r, v \rangle \le \|q\|_{1;\overline{\Omega}}$$

2. $e \in S(\Omega) \cap \mathcal{K}$ with $\operatorname{div}\operatorname{div} e = -f^N_r$.

For $r = 0, \dots, N$ we set $M^N_r = e^N_r$ and for every $t \in [0, T]$ we define piecewise constant interpolations

$$u_N(t) = u^N_r, \quad e_N(t) = e^N_r, \quad p_N(t) = p^N_r, \quad M_N(t) = M^N_r,$$
$$f_N(t) = f^N_r, \quad m^1_N(t) = (m^1)^N_r,$$

where r is the largest integer such that $t^N_r \le t$. By definition $(u_N(t), e_N(t), p_N(t))$ is kinematically admissible for every $t \in [0, T]$.

In the proof of the existence, it was shown that for approximate solutions one has the estimate

$$\sup_{t\in[0,T]} \|e_N(t)\|_{2;\Omega} + \operatorname{Var}(p_N; 0, T) + \sup_{t\in[0,T]} \|u_N\|_{2,1;\Omega} \le C, \tag{5.12}$$

which is uniform with respect to N, and it was established that these functions converge pointwise (with respect to t) to a solution of the quasistatic evolution problem.

5.3.3 Continuity estimates of solutions of the incremental problems

In [Dem09] it was established that the quasistatic evolution is absolutely continuous in time. However, as we will deal precisely with the solutions of the time-discretized problems, we would need the continuity estimates of solutions at the level of incremental problems.

The following notation will be often used below: given a function $h : [0,T] \to X$,

$$\delta h_r^N := h(t_r^N) - h(t_{r-1}^N). \tag{5.13}$$

We also consider the increment of the data of the problem, defined by

$$D_r^N := \|\delta (m^1)_r^N\|_{2;\Omega} + \|\delta f_r^N\|_{2;\Omega}. \tag{5.14}$$

By (5.3) and (5.4), after time reparametrization, we may assume that

$$f \in \mathrm{Lip}([0,T]; L^2(\Omega)) \cap L^\infty([0,T]; W^{1,2}_{loc}(\Omega)),$$

and

$$m^1 \in \mathrm{Lip}([0,T]; L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})).$$

Indeed, every absolutely continuous function can be made Lipschitz just by time reparametrization, and this leads to a corresponding reparametrization of the solutions, the problem being rate-independent. In other words, we may suppose, that

$$D_r^N \leq \frac{C}{N}. \tag{5.15}$$

Theorem 5.3.7. *For solutions of the incremental problems* (u_r^N, e_r^N, p_r^N) *the following inequality holds:*

$$\|\delta e_r^N\|_{2;\Omega} + \|\delta p_r^N\|_{1;\overline{\Omega}} + \|\delta u_r^N\|_{2,1;\Omega} \leq D_r^N, \tag{5.16}$$

where δh_r^N *in understood as in* (5.13) *and* D_r^N *denotes the increment of the data of the problem, defined by* (5.14).

Proof: As the triple

$$(u_{r-1}^N, e_{r-1}^N, p_{r-1}^N)$$

is kinematically admissible, the minimality condition (5.11) and the integration by parts formula (5.7) imply

$$\mathcal{Q}(e_r^N) - \int_\Omega (m^1)_r^N : e_r^N \, dx + \|p_r^N - p_{r-1}^N\|_{1;\overline{\Omega}} - \langle (m^1)_r^N, p_r^N - p_{r-1}^N \rangle \leq$$
$$\leq \mathcal{Q}(e_{r-1}^N) - \int_\Omega (m^1)_r^N : e_{r-1}^N \, dx$$

Developing the quadratic form in the right-hand side we arrive at:

$$\begin{aligned} &\frac{1}{2}\int_\Omega M_r^N : e_r^N \, dx - \frac{1}{2}\int_\Omega M_{r-1}^N : e_{r-1}^N \, dx + \mathcal{H}(p_r^N - p_{r-1}^N) \leq \\ &+ \langle (m^1)_r^N, p_m^N - p_{m-1}^N \rangle - \int_\Omega (m^1)_r^N : e_{r-1}^N \, dx + \int_\Omega (m^1)_r^N : e_r^N \, dx. \end{aligned} \tag{5.17}$$

Now consider the functions

$$v = u_r^N - u_{r-1}^N,\ \eta = e_r^N - e_{r-1}^N,$$
$$q = p_r^N - p_{r-1}^N.$$

Since (v, η, q) is kinematically admissible and $(u_{r-1}^N, e_{r-1}^N, p_{r-1}^N)$ is a solution of the corresponding minimum problem at the previous step, we obtain, by means of Remark 5.3.6 and the integration by parts formula (5.7)

$$\begin{gathered} -\int_\Omega M_{r-1}^N : (e_r^N - e_{r-1}^N)\,dx + \int_\Omega (m^1)_{r-1}^N : (e_r^N - e_{r-1}^N)\,dx + \\ \langle (m^1)_{r-1}^N, p_r^N - p_{r-1}^N\rangle \le \mathcal{H}(p_r^N - p_{r-1}^N). \end{gathered} \tag{5.18}$$

By combining (5.17) and (5.18) we get the following

$$\begin{gathered} \mathcal{Q}(e_r^N - e_{r-1}^N) = \frac{1}{2}\int_\Omega M_r^N : e_r^N\,dx - \frac{1}{2}\int_\Omega M_{r-1}^N : e_{r-1}^N\,dx - \\ -\int_\Omega M_{r-1}^N : (e_r^N - e_{r-1}^N)\,dx \le \langle (m^1)_r^N, p_r^N - p_{r-1}^N\rangle - \\ -\int_\Omega (m^1)_r^N : e_{r-1}^N\,dx + \int_\Omega (m^1)_r^N : e_r^N\,dx - \\ -\int_\Omega (m^1)_{r-1}^N : (e_r^N - e_{r-1}^N)\,dx - \langle (m^1)_{r-1}^N, p_r^N - p_{r-1}^N\rangle, \end{gathered} \tag{5.19}$$

where $\langle \cdot, \cdot \rangle$ is the duality defined in (5.6).

Let us apply the integration by parts formula (5.7) to compute $\langle (m^1)_m^N, p_m^N - p_{m-1}^N\rangle$:

$$\begin{gathered} \langle (m^1)_r^N, p_r^N - p_{r-1}^N\rangle = \\ -\int_\Omega (m^1)_r^N : (e_r^N - e_{r-1}^N)\,dx + \int_\Omega f_r^N\,(u_r^N - u_{r-1}^N)\,dx. \end{gathered} \tag{5.20}$$

A similar formula holds for $\langle (m^1)_{r-1}^N, p_r^N - p_{r-1}^N\rangle$.

Putting (5.20) into (5.19) we end up with the estimate

$$\begin{gathered} \mathcal{Q}(e_r^N - e_{r-1}^N) \le \int_\Omega (f_r^N - f_{r-1}^N)\cdot(u_r^N - u_{r-1}^N)\,dx + \\ +\|f_r^N - f_{r-1}^N\|_{2;\Omega}\,\|u_r^N - u_{r-1}^N\|_{2,1;\Omega}. \end{gathered} \tag{5.21}$$

Now let us estimate $\|p_r^N - p_{r-1}^N\|_{1;\overline{\Omega}}$ in terms of the data of the problem. First of all, the safe load condition yields

$$\lambda\|p_r^N - p_{r-1}^N\|_{1;\overline{\Omega}} \le \mathcal{H}(p_r^N - p_{r-1}^N) - \langle (m^1)_r^N, p_r^N - p_{r-1}^N\rangle.$$

Now, the relation (5.17) and the boundedness of $\|\varrho_r^N\|_{\infty;\Omega}$, $\|e_r^N\|_{2;\Omega}$ and $\|p_r^N\|_{1;\overline{\Omega}}$ imply

$$\|p_r^N - p_{r-1}^N\|_{1;\overline{\Omega}} \le C(\|e_r^N - e_{r-1}^N\|_{2;\Omega} + D_r^N) \tag{5.22}$$

Taking into account the inequality

$$\|u_r^N - u_{r-1}^N\|_{2,1;\Omega} \le C(\|e_r^N - e_{r-1}^N\|_{2;\Omega} + \|p_r^N - p_{r-1}^N\|_{1;\overline{\Omega}},$$

proved in [Dem09, estimate (3.9)], the estimate

$$\|p_r^N - p_{r-1}^N\|_{1;\overline{\Omega}} + \|e_r^N - e_{r-1}^N\|_{2;\Omega} \le CD_r^N \tag{5.23}$$

follows now from (5.21), (5.22) and the application of the Cauchy inequality.

To prove

$$\|D^2u_r^N - D^2u_{r-1}^N\|_{1;\Omega} \le CD_r^N, \tag{5.24}$$

we recall the additive decomposition $D^2u = e + p$ and make use of (5.23).

Finally to show the validity of (5.16), it remains to estimate $\|u_r^N - u_{r-1}^N\|_{1;\Omega}$. By the Poincare inequality for BH the result follows from (5.15), (5.23), (5.24) and the latter inequality. □

5.4 Minimax problem

In this section we briefly discuss the minimax formulation of the incremental problem. We follow the general scheme, described in [FS00], which was applied in [Ser94] for studying the regularity of solutions of static problems in the theory of perfect elastoplastic plates.

We refer to [FS00, Chapter 1] for the complete exposition of an abstract theory and to [Dem08b, Section 4] for its short presentation. The following calculations follow closely [Dem08b, Section 5], making use of constructions developed in [Ser94].

Recall that the time-discretization procedure, that provides us a way of constructing approximate solutions to the quasistatic problem for perfect elasto-plastic plates, leads one to solving a sequence of the following incremental problems:

$$\min_{(u,M,p)} \{\mathcal{Q}(M) + \|p - p_{r-1}^N\|_{1;\overline{\Omega}} - \langle f_r^N, u\rangle\}, \tag{5.25}$$

where the minimum is taken over all kinematically admissible triples (see Definition 5.3.1), with p_{r-1}^N be a solution of the corresponding incremental problem, obtained at the previous step.

5.4.1 Functional setting of the problem

We set

$$V_0 = W_0^{2,1}(\Omega), \quad U = W_0^{1,2}(\Omega),$$

U^* is the dual space of U. If $1 < p \le +\infty$, the space $L^p(\Omega)$, is embedded in U^* by the usual identification

$$\langle f, u\rangle = \int_\Omega f\,u\,dx \quad \text{for any } u \in W_0^{1,2}(\Omega).$$

Put

$$P = L^1(\Omega;\mathbb{M}^{2\times 2}_{sym}) \quad P^* = L^\infty(\Omega;\mathbb{M}^{2\times 2}_{sym}).$$

We have the following

$$\begin{aligned}&\text{the embedding of } V_0 \text{ into } U \text{ is continuous}\\ &V_0 \text{ is dense in } U.\end{aligned} \tag{5.26}$$

Let us introduce the functionals $G : P \to \overline{\mathbb{R}}$ and $\mathcal{L} : U \to \overline{\mathbb{R}}$ by

$$\begin{aligned} G(m) &= \int_\Omega g(m + e^N_{r-1})\, dx, \quad m \in P, \\ \mathcal{L}(v) &= -\int_\Omega f^N_r \cdot v\, dx, \quad v \in U. \end{aligned} \tag{5.27}$$

Thus, G and $\mathcal{L}$ are continuous and it is easy to see that the Legendre transform of G is

$$G^*(M) = \int_\Omega \Big(g^*(M) - M : e^N_{r-1} \Big)\, dx, \quad \text{for } M \in P^*. \tag{5.28}$$

Here $g : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{R}$ has the form

$$g(m) \equiv g_0(|m|) := \begin{cases} \frac{1}{2}|m|^2, & \text{if } |m| \le 1; \\ |m| - \frac{1}{2}, & \text{if } |m| > 1, \end{cases} \tag{5.29}$$

while its Legendre transform $g^* : \mathbb{M}^{2\times 2}_{sym} \to \overline{\mathbb{R}}$ is given by the formula

$$g^*(M) = \begin{cases} \frac{1}{2}|M|^2, & \text{if } |M| \le 1; \\ +\infty, & \text{otherwise.} \end{cases}$$

5.4.2 Saddle-point problem in its strong formulation

Introduce the continuous linear operator $A : V_0 \to P$ as

$$Av := D^2 v, \quad v \in V_0.$$

We define the Lagrangian $\ell : V \times \mathcal{K}(\Omega) \to \overline{\mathbb{R}}$ by

$$\ell(v, m) = \int_\Omega \Big(D^2 v : m + m : e^{r-1}_N \Big)\, dx - \int_\Omega g^*(m)\, dx + \mathcal{L}(v),$$

and consider the following minimax problem

$$\begin{cases} \text{find a pair } (u, M) \in V_0 \times \mathcal{K}(\Omega) \text{ such that} \\ \ell(u, m) \le \ell(u, M) \le \ell(v, M), \text{ for all } v \in V_0,\ m \in \mathcal{K}(\Omega). \end{cases} \tag{5.30}$$

The minimax problem (5.30) generates a pair of dual problems, the primal one

$$\begin{cases} \text{find } \delta u^N_r \in V_0 \text{ such that} \\ I(\delta u^N_r) = \inf\{ I(v) : v \in V_0 \}, \end{cases} \tag{5.31}$$

where the functional I is given by

$$I(v) = G(Av) + \mathcal{L}(v) = \int_\Omega g(D^2 v + e^{r-1}_N)\, dx - \int_\Omega f^N_r \cdot v\, dx,$$

and the dual one

$$\begin{cases} \text{find } M_r^N \in Q_{f_r^N} \cap \mathcal{K}(\Omega) \text{ such that} \\ R(M_r^N) = \sup\{R(m) : m \in Q_{f_r^N} \cap \mathcal{K}(\Omega)\}, \end{cases} \tag{5.32}$$

where

$$R(m) = \begin{cases} \ell(0,m) = \int_\Omega \Big(m : e_{r-1}^N - g^*(m) \Big)\, dx, & m \in Q_{f_r^N} \cap \mathcal{K}(\Omega); \\ -\infty, & m \notin Q_{f_r^N} \cap \mathcal{K}(\Omega), \end{cases} \quad \text{for } m \in \mathcal{K}(\Omega),$$

with $Q_{f_r^N}$ being defined as

$$Q_{f_r^N} = \Big\{ m \in S(\Omega) : \operatorname{div}\operatorname{div} m = f_r^N \Big\}.$$

The following theorem (see [FS00, Chapter 1]) shows that under very mild assumptions the dual problem (5.32) has a solution and one can exchange inf and sup signs.

Theorem 5.4.1. *Suppose that the following two conditions hold*

$$C := \inf\{I(v) : v \in V_0\} \in \mathbb{R}, \tag{5.33}$$

$$\begin{gathered} \textit{there exists } u_1 \in V \textit{ such that } G(Au_1) < +\infty,\ \mathcal{L}(u_1) < +\infty \\ \textit{and the function } p \mapsto G(Au_1 + p) \textit{ is continuous at zero.} \end{gathered} \tag{5.34}$$

Then problem (5.32) *has at least one solution and the identity*

$$C = \sup\{R(m) : m \in P^*\}$$

is valid.

Condition (5.34) is obviously satisfied. It is easy to see, that the safe load condition (5.4) yields condition (5.33) and the coercivity of the functional I with respect to the norm of V_0. However, as the space V_0 is not reflexive, one needs to construct a suitable relaxation of the variational problem (5.30), (5.31).

5.4.3 The relaxed problem

We construct a variational extension of the problem. To this aim we construct a relaxation of problem (5.30). We will make use of an auxiliary space D, defined in the following way: a function m belongs to D if and only if there exists $u^* \in U^*$ such that

$$\int_\Omega u^*\, v\, dx = \int_\Omega m : D^2 v\, dx \quad \text{for all } v \in V_0 .$$

Thus,

$$D = \Big\{ M \in P^* : \operatorname{div}\operatorname{div} M \in U^* \Big\},$$

According to the general procedure (see [FS00, Chapter 1]) we define an extension V_+ of the space V as

$$V_+ = \Big\{ v \in U : \sup_{\|M\|_{\infty;\Omega}\le 1,\ M\in D} \int_\Omega v \operatorname{div}\operatorname{div} M\, dx < +\infty \Big\}.$$

In particular, taking the test fields $M \in C_0^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym})$ we conclude that $v \in BH(\Omega)$.

Introduce the relaxed Lagrangian

$$L(v,m) = \int_\Omega (\operatorname{div}\operatorname{div} m - f_N^r)\, v\, dx + \int_\Omega m : e_N^{r-1}\, dx - \int_\Omega g^*(m)\, dx$$

for $v \in V_+$ and $m \in \mathcal{K}(\Omega) \cap D$. Consider the minimax problem for this relaxed Lagrangian L:

$$\begin{cases} \text{find a pair } (u,M) \in V_+ \times (\mathcal{K}(\Omega)\cap D) \text{ such that} \\ L(u,m) \le L(u,M) \le L(v,M), \text{ for all } v \in V_+,\ m \in \mathcal{K}(\Omega)\cap D. \end{cases} \tag{5.35}$$

Arguing as in [FS00, Chapter 1] and [Dem08b, Section 5] we conclude that the following result holds (see Theorem 5.4.2 below): there exists a saddle point $u \in V_+$ and $M \in \mathcal{K}(\Omega)\cap D$ of the Lagrangian L on the set $V_+ \times (\mathcal{K}(\Omega)\cap D)$. In this case the tensor M is the unique solution of the problem (5.32) and $w \in V_+$ is a solution of the problem

$$\begin{cases} \text{find } w \in V_+ \text{ such that} \\ \Phi(w) = \inf\{\Phi(v) : v \in V_+\}, \end{cases} \tag{5.36}$$

where

$$\Phi(v) := \sup\{L(v,m) : m \in \mathcal{K}(\Omega)\cap D\}.$$

The precise result is expressed in the following Theorem, which is a a consequence of [FS00, Theorem 1.2.2] (see also [Ser94, assertions (2.7)-(2.9)] for a similar construction in the static problem).

Theorem 5.4.2. *Suppose that $f_r^N \in L^2(\Omega)$ and condition (5.4) holds. Then there exists at least one solution $(\delta u_r^N, M_r^N)$ to the minimax problem (5.35) in $V_+ \times (\mathcal{K}\cap D)$. Moreover, M_r^N is the unique solution to the dual variational problem (5.32) and δu_r^N is a solution to (5.36). The identity*

$$\Phi(\delta u_r^N) = R(M_r^N)$$

holds. Finally, every minimizing sequence of the problem (5.31) contains a subsequence which converges to some solution of (5.36) weakly in U and strongly in $W_0^{1,p}(\Omega;\mathbb{R}^2)$ for $1 \le p < 2$.

We have the following representation result.

Lemma 5.4.3. *For $u \in V_+$ we have*

$$\Phi(u) = \int_\Omega g(D^2 u + e_{r-1}^N) - \int_\Omega f_r^N\, u\, dx,$$

where the corresponding integral is understood as functional of measure (see [Tem85, Chapter II]).

Proof: We refer to [Dem08b, Lemma 5.3] for the proof. □

5.4.4 Saddle points generate solutions of the incremental problems

Let us show, that if we interpret a saddle point $(\delta u_r^N, M_r^N)$ of (5.35) as the increment of u and the updated value of M, then we get a solution to the incremental problem (5.25).

Theorem 5.4.4. *Let $(\delta u_r^N, M_r^N) \in V_+ \times (D \cap \mathcal{K}(\Omega))$ be a saddle point of the relaxed Lagrangian L. Then the triple (u_r^N, e_r^N, p_r^N), constructed as*

$$\begin{aligned} u_r^N &= u_{r-1}^N + \delta u_r^N, \\ e_r^N &= M_r^N, \\ p_r^N &= D^2 u_r^N - e_r^N \quad \text{in } \Omega, \\ p_r^N &= -\nabla u_r^N \odot \nu \, \mathcal{H}^1 \quad \text{on } \partial\Omega \end{aligned}$$

is kinematically admissible and is a solution to the incremental problem (5.25).

Proof: First of all, kinematic admissibility of the triple (u_r^N, e_r^N, p_r^N) is obvious by its construction. Let us prove that it solves (5.25).

As $(\delta u_r^N, M_r^N) \in V_+ \times (D \cap \mathcal{K}(\Omega))$ is a saddle point of L, we have

$$L(\delta u_r^N, m) \le L(\delta u_r^N, M_r^N) \le L(v, M_r^N), \quad \text{for all } v \in V_+ \text{ and } m \in (D \cap \mathcal{K}(\Omega)). \tag{5.37}$$

Since $M_r^N \in D \cap \mathcal{K}(\Omega)$, we already know that $M_r^N \in \mathcal{K}(\Omega)$, while the second part of (5.37) implies

$$\operatorname{div}\operatorname{div} M_r^N = f_r^N \in L^2(\Omega). \tag{5.38}$$

The first part of inequality (5.37) yields

$$\begin{aligned} \int_\Omega \Big[\operatorname{div}\operatorname{div} M_r^N \cdot \delta u_r^N - g^*(M_r^N) + M_r^N : M_{r-1}^N \Big] \, dx \ge \\ \ge \int_\Omega \Big[\operatorname{div}\operatorname{div} m \cdot \delta u_r^N - g^*(m) + m : M_{r-1}^N \Big] \, dx \end{aligned} \tag{5.39}$$

for every $m \in \mathcal{K}(\Omega) \cap D$.

For $\delta u_r^N \in BH(\Omega)$ with $\delta u_r^N = 0$ on $\partial\Omega$ and $m \in S(\Omega)$, the integration by parts formula [Dem89, Proposition 2.3] takes the form

$$\int_\Omega \operatorname{div}\operatorname{div} m \cdot \delta u_r^N \, dx = [D^2(\delta u_r^N) : m](\Omega) - \int_{\partial\Omega} \frac{\partial(\delta u_r^N)}{\partial \nu} m_{ij} \nu_i \nu_j \, d\mathcal{H}^1.$$

Thus, from (5.39) we deduce

$$\begin{aligned} \langle D^2(\delta u_r^N), m - M_r^N \rangle - \frac{1}{2} \int_\Omega \Big(|m|^2 - |M_r^N|^2 \Big) \, dx + \int_\Omega (m - M_r^N) : M_{r-1}^N \, dx - \\ - \int_{\partial\Omega} \frac{\partial(\delta u_r^N)}{\partial \nu} (m_{ij} - M_{ij}) \nu_i \nu_j \, d\mathcal{H}^1 \le 0. \end{aligned}$$

By taking $\widetilde{m} = M_r^N + \alpha(m - M_r^N) \in \mathcal{K} \cap D$ and letting $\alpha \to 0$ one obtains

$$\langle D^2(\delta u_r^N), m - M_r^N \rangle - \int_\Omega (m - M_r^N) : \delta e_r^N \, dx - \int_{\partial\Omega} \frac{\partial(\delta u_r^N)}{\partial \nu} (m_{ij} - M_{ij}) \nu_i \nu_j \, d\mathcal{H}^1 \leq 0,$$

that is

$$\langle \delta p_r^N, m - M_r^N \rangle \leq 0$$

for all $m \in D \cap \mathcal{K}(\Omega)$. Hence, by [Dem09, Proposition 2.3]

$$\|\delta p_r^N\|_{1;\overline{\Omega}} = \langle \delta p_r^N, M_r^N \rangle,$$

and we have

$$\|q + \delta p_r^N\|_{1;\overline{\Omega}} - \|\delta p_r^N\|_{1;\overline{\Omega}} - \langle q, M_r^N \rangle \geq 0$$

for every kinematically admissible triple (v, η, q). The latter inequality and (5.38) imply that (u_r^N, e_r^N, p_r^N) is a solution to problem (5.25). ☐

5.5 Approximations

In this section we show that some solutions of the relaxed minimax problem (5.35) can be approximated by more regular functions in a way that allows us to get higher regularity of bending moments.

We also prove some technical lemmas to be used in the rest of the paper.

5.5.1 Regularized problems

As in [Ser94, Section 3] and [Dem08b, Section 6] we consider the family of variational problems, depending on a positive parameter $\alpha \in (0, 1]$:

$$\begin{cases} \text{find } u_r^\alpha \in W_0^{2,2}(\Omega) \text{ such that} \\ I_\alpha(u_r^\alpha) = \inf\{I_\alpha(v) : v \in W_0^{2,2}(\Omega)\}, \end{cases} \tag{5.40}$$

where

$$\begin{aligned} I_\alpha(v) &= \frac{\alpha}{2} \int_\Omega |D^2 v + M_{r-1}^N|^2 \, dx + I(v) = \\ &= \frac{\alpha}{2} \int_\Omega |D^2 v + M_{r-1}^N|^2 \, dx + \int_\Omega g(D^2 v + M_{r-1}^N) \, dx - \int_\Omega f_r^N \, v \, dx. \end{aligned} \tag{5.41}$$

It is easy to see that problem (5.40) has a unique solution $u_r^\alpha \in W_0^{2,2}(\Omega)$, which satisfies a nonlinear system of PDEs:

$$\int_\Omega M_r^\alpha : D^2 v \, dx = \int_\Omega f_r^N \, v \, dx \quad \text{for all } v \in C_0^\infty(\Omega), \tag{5.42}$$

that is

$$\operatorname{div}\operatorname{div} M_r^\alpha = f_r^N, \tag{5.43}$$

where

$$M_r^\alpha = \alpha(D^2 u_r^\alpha + M_{r-1}^N) + \frac{\partial g}{\partial \tau}(D^2 u_r^\alpha + M_{r-1}^N). \tag{5.44}$$

Lemma 5.5.1. *Under conditions* (5.3), (5.4) *and* (5.29) *the following estimates hold*

$$\sqrt{\alpha}\|u_r^\alpha\|_{2,2;\Omega} + \|u_r^\alpha\|_{2,1;\Omega} + \|u_r^\alpha\|_{1,2;\Omega} + \|u_r^\alpha\|_{\infty;\Omega} \le C,$$

where the constant $C = C(\|f_r^N\|_{2;\Omega}, \|M_{r-1}^N\|_{2;\Omega;\mathbb{M}^{2\times 2}_{sym}})$ *does not depend on the parameter* α.

Proof: The safe-load condition (5.4) implies

$$\int_\Omega f_N^r\, u_r^\alpha\, dx = \int_\Omega m^1 : D^2 u_r^\alpha\, dx,$$

and using definition (5.41) of I_α we deduce the estimate

$$I_1(0) \ge I_\alpha(u_r^\alpha) \ge \int_\Omega \Big\{ \frac{\alpha}{2}|D^2u_r^\alpha + M_{r-1}^N|^2 + g(D^2u_r^\alpha + M_{r-1}^N) - c|D^2u_r^\alpha| \Big\}\, dx.$$

The claim now follows from the embedding theorems. □

Lemma 5.5.2. *Under the conditions of Lemma 5.5.1 we can find subsequences, denoted by* u_r^α *and* M_r^α, *such that as* $\alpha \to 0$ *we have*

$$M_r^\alpha \rightharpoonup M_r^N \quad \textit{weakly in } L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}), \tag{5.45}$$

$$u_r^\alpha \to \delta u_r^N \quad \textit{strongly in } W_0^{1,p}(\Omega),\ \textit{for } 1 \le p < 2, \tag{5.46}$$

$$u_r^\alpha \rightharpoonup \delta u_r^N \quad \textit{weakly in } W_0^{1,2}(\Omega), \tag{5.47}$$

$$\nabla u_r^\alpha \overset{*}{\rightharpoonup} \nabla \delta u_r^N \quad \textit{weakly}^* \textit{ in } BV(\Omega;\mathbb{R}^2), \tag{5.48}$$

$$\alpha \int_\Omega |D^2u_r^\alpha + M_{r-1}^N|^2\, dx \to 0, \tag{5.49}$$

$$M_0^\alpha := \frac{\partial g}{\partial \tau}(D^2u_r^\alpha + M_{r-1}^N) \overset{*}{\rightharpoonup} M_r^N \quad \textit{weakly}^* \textit{ in } L^\infty(\Omega;\mathbb{M}^{2\times 2}_{sym}), \tag{5.50}$$

where the pair $(\delta u_r^N, M_r^N)$ *is a solution to problem* (5.35).

Proof: Assertions (5.45)-(5.48) and (5.50) follow from (5.44), Lemma 5.5.1 and embedding theorems.

Thus, it remains to prove (5.49) and that the pair $(\delta u_r^N, M_r^N)$ is a solution to problem (5.35).

As $M_0^\alpha \in \mathcal{K}(\Omega)$, the convergence (5.50) yields that $M_r^N \in \mathcal{K}(\Omega)$. Hence, from the Euler equation (5.42) and the convergence (5.45) we conclude that $M_r^N \in Q_{f_r^N} \cap \mathcal{K}(\Omega)$.

The duality relations imply that

$$M_0^\alpha : (D^2u_r^\alpha + M_{r-1}^N) = g(D^2u_r^\alpha + M_{r-1}^N) + g^*(M_0^\alpha) \quad \text{a.e. in } \Omega.$$

Therefore, by using the Euler equation (5.42) we can rewrite the functional I_α as

$$I_\alpha(u_r^\alpha) = \int_\Omega \Big[M_r^\alpha - \alpha(D^2u^\alpha + M_{r-1}^N)\Big] : (D^2u_r^\alpha + M_{r-1}^N)\,dx -$$
$$- \int_\Omega g^*(M_0^\alpha)\,dx - \int_\Omega f\,u_r^\alpha\,dx + \frac{\alpha}{2}\int_\Omega |D^2u_r^\alpha + M_{r-1}^N|^2\,dx =$$
$$= -\frac{\alpha}{2}\int_\Omega |D^2u_r^\alpha + M_{r-1}^N|^2\,dx - \int_\Omega g^*(M_0^\alpha) + \int_\Omega M_r^\alpha : M_{r-1}^N\,dx.$$

By Theorem 5.4.1 applied to problems (5.31) and (5.32), we get

$$\begin{aligned}\sup\{R(m) : m \in Q_{f_r^N} \cap \mathcal{K}\} = \inf\{I(v) : v \in V_0\} \le I(u_r^\alpha) \le I_\alpha(u_r^\alpha) = \\ = -\frac{\alpha}{2}\int_\Omega |D^2u_r^\alpha + M_{r-1}^N|^2\,dx - \int_\Omega g^*(M_0^\alpha) + \int_\Omega M_r^\alpha : M_{r-1}^N\,dx.\end{aligned} \tag{5.51}$$

As

$$-\int_\Omega g^*(M_r^N) + \int_\Omega M_r^N : M_{r-1}^N = R(M_r^N),$$

by making use of convergence (5.45) and (5.50) it follows, that

$$\lim_{\alpha\to 0} I_\alpha(u_r^\alpha) \le R(M_r^N) - \limsup_{\alpha\to 0}\frac{\alpha}{2}\int_\Omega |D^2u_r^\alpha + M_{r-1}^N|^2\,dx.$$

According to (5.51) we have

$$\sup\{R(m) : m \in Q_{f_r^N} \cap \mathcal{K}\} = \inf\{I(v) : v \in V_0\} \le \liminf_{\alpha\to 0} I(u_r^\alpha) \le$$
$$\le \lim_{\alpha\to 0} I_\alpha(u_r^\alpha) \le R(M_r^N) - \limsup_{\alpha\to 0}\frac{\alpha}{2}\int_\Omega |D^2u_r^\alpha + M_{r-1}^N|^2\,dx \le R(M_r^N),$$

which implies the relation (5.49) and ensures that M_r^N is a solution to problem (5.32).

Moreover, the identity

$$\lim_{\alpha\to 0} I(u_r^\alpha) = \inf\{I(v) : v \in V_0\} \tag{5.52}$$

yields that u_r^α is a minimizing sequence for problem (5.31), and therefore it converges to a solution of problem (5.36) as in Theorem 5.4.2. □

5.5.2 Convergence of variations

Now we show, that the approximating sequence enjoys better convergence properties, than those stated in Lemma 5.5.2.

Lemma 5.5.3. *We have*

$$|D^2u_r^\alpha + M_{r-1}^N| \overset{*}{\rightharpoonup} |D^2(\delta u_r^N) + M_{r-1}^N| \quad \text{in } M_b(\Omega). \tag{5.53}$$

Proof: By Lemma 5.5.2, Theorem 5.4.2 and (5.52) we have

$$\lim_{\alpha\to 0}\Phi(u_r^\alpha) = \lim_{\alpha\to 0} I(u_r^\alpha) = \inf_{v\in V_0} I(v) = \inf_{V_+}\Phi(v) = \Phi(\delta u_r^N),$$

so that in view of Lemma 5.4.3.

$$\int_\Omega g_0(|D^2u_r^\alpha + M_{r-1}^N|)\,dx \to \int_\Omega g_0(|D^2(\delta u_r^N) + M_{r-1}^N|). \tag{5.54}$$

The sequence $|D^2u_r^\alpha + M_{r-1}^N|$ is bounded in $M_b(\Omega)$, therefore there exists a nonnegative measure $\lambda \in M_b(\Omega)$, such that

$$|D^2u_r^\alpha + M_{r-1}^N| \overset{*}{\rightharpoonup} \lambda \quad \text{weakly}^* \text{ in } M_b(\Omega), \quad \text{as } \alpha \to 0. \tag{5.55}$$

Thus, $\lambda \geq |D^2(\delta u_r^N) + M_{r-1}^N|$ in $M_b(\Omega)$, and the inequality holds true also for $\mathcal{L}^n$-absolutely continuous and singular parts:

$$\begin{aligned} \lambda^a &\geq |D^2(\delta u_r^N) + M_{r-1}^N|^a, \\ \lambda^s &\geq |D^2(\delta u_r^N) + M_{r-1}^N|^s. \end{aligned} \tag{5.56}$$

By the weak* lower-semicontinuity of convex functionals of measures, and using the explicit form of the recession function of g_0, which is $g_0^\infty(t) = t$, we obtain

$$\lim_{\alpha\to 0}\int_\Omega g_0(|D^2u_r^\alpha + M_{r-1}^N|)\,dx \geq \int_\Omega g_0(\lambda) = \int_\Omega g_0(\lambda^a)\,dx + \lambda^s(\Omega). \tag{5.57}$$

On the other hand we have

$$\begin{aligned} &\lim_{\alpha\to 0}\int_\Omega g_0(|D^2u_r^\alpha + M_{r-1}^N|)\,dx = \\ &= \int_\Omega g_0(|D^2(\delta u_r^N) + M_{r-1}^N|^a)\,dx + |D^2(\delta u_r^N) + M_{r-1}^N|^s(\Omega). \end{aligned} \tag{5.58}$$

As the function g_0 is strictly monotone increasing, from (5.54)-(5.58) we conclude that

$$\lambda = |D^2(\delta u_r^N) + M_{r-1}^N|.$$

Now the result follows from (5.55). □

5.5.3 Technical estimates

By the definition (5.44) of M_r^α we have

$$M_r^\alpha = \alpha(D^2u_r^\alpha + M_{r-1}^N) + \begin{cases} D^2u_r^\alpha + M_{r-1}^N, & \text{if } |D^2u_r^\alpha + M_{r-1}^N| \leq 1; \\ \frac{D^2u_r^\alpha + M_{r-1}^N}{|D^2u_r^\alpha + M_{r-1}^N|}, & \text{if } |D^2u_r^\alpha + M_{r-1}^N| > 1. \end{cases} \tag{5.59}$$

According to the chain rule of [MT03] the following expression for the derivatives of M_r^α is valid

$$M_{r,k}^\alpha = \alpha(D^2u_{r,k}^\alpha + M_{r-1,k}^N) + \frac{\partial^2 g}{\partial\kappa^2}(D^2u_r^\alpha + M_{r-1}^N)(D^2u_{r,k}^\alpha + M_{r-1,k}^N). \tag{5.60}$$

Here and henceforth the subscript $_{,k}$ denotes the partial derivative with respect to x_k.

In what follows we adopt the notation

$$\tau_r^\alpha := D^2 u_r^\alpha + M_{r-1}^N. \tag{5.61}$$

Let us introduce two bilinear forms, that depend on α and implicitly on the point $x \in \Omega$:

$$\begin{aligned} E_1^\alpha(\varepsilon, \varkappa) &= \left(\frac{\partial^2 g}{\partial \tau^2}(\tau_r^\alpha)\,\varepsilon\right) : \varkappa = \\ &= \frac{g_0'(|\tau_r^\alpha|)}{|\tau_r^\alpha|}\,\varepsilon : \varkappa + \left(g_0''(|\tau_r^\alpha|) - \frac{g_0'(|\tau_r^\alpha|)}{|\tau_r^\alpha|}\right)\,\frac{\tau_r^\alpha : \varepsilon}{|\tau_r^\alpha|}\,\frac{\tau_r^\alpha : \varkappa}{|\tau_r^\alpha|} \end{aligned} \tag{5.62}$$

and

$$E_2^\alpha(\varepsilon, \varkappa) = \alpha\varepsilon : \varkappa + E_1^\alpha(\varepsilon, \varkappa). \tag{5.63}$$

Below we establish some technical inequalities to be used in the remaining sections.

Lemma 5.5.4. *The following relations hold true:*

$$M_{r,k}^\alpha : \varkappa = E_2^\alpha(\tau_{r,k}^\alpha, \varkappa), \tag{5.64}$$

$$E_2^\alpha(\varkappa, \varkappa) \le \alpha|\varkappa|^2 + \begin{cases} |\varkappa|^2, & \text{if } |\tau_r^\alpha| \le 1 \\ \frac{|\varkappa|^2}{|\tau_r^\alpha|}, & \text{if } |\tau_r^\alpha| > 1 \end{cases} \tag{5.65}$$

for any $\varkappa \in \mathbb{M}^{2\times 2}_{sym}$.

Proof: Identity (5.64) and inequality (5.65) follow from (5.59)-(5.63) and the expression of g_0 as in (5.29). □

Corollary 5.5.5. *The following estimates are valid*

$$E_2^\alpha(M_{r,k}^\alpha, M_{r,k}^\alpha) \le \alpha M_{r,k}^\alpha : M_{r,k}^\alpha + \begin{cases} M_{r,k}^\alpha : M_{r,k}^\alpha, & \text{if } |\tau_r^\alpha| \le 1 \\ \frac{1}{|\tau_r^\alpha|} M_{r,k}^\alpha : M_{r,k}^\alpha, & \text{if } |\tau_r^\alpha| > 1. \end{cases} \tag{5.66}$$

In particular, we have

$$E_2^\alpha(M_{r,k}^\alpha, M_{r,k}^\alpha) \le (1+\alpha)\, M_{r,k}^\alpha : M_{r,k}^\alpha. \tag{5.67}$$

Lemma 5.5.6.

$$\begin{aligned} -E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha) &= -M_{r,k}^\alpha : \tau_{r,k}^\alpha \le \\ &\le \begin{cases} -M_{r,k}^\alpha : M_{r,k}^\alpha + \alpha\, E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha), & \text{if } |\tau_r^\alpha| \le 1; \\ -|\tau_r^\alpha| M_{r,k}^\alpha : M_{r,k}^\alpha + \alpha\, |\tau_r^\alpha|\, E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha), & \text{if } |\tau_r^\alpha| > 1. \end{cases} \end{aligned} \tag{5.68}$$

PROOF: Suppose, $|\tau_r^\alpha| \le 1$. Then $M_r^\alpha = \alpha\,\tau_r^\alpha + \tau_r^\alpha$, and thus

$$\begin{aligned} -M_{r,k}^\alpha : \tau_{r,k}^\alpha = -M_{r,k}^\alpha : \Big(\alpha\,\tau_{r,k}^\alpha + \tau_{r,k}^\alpha\Big) + \alpha\, M_{r,k}^\alpha : \tau_{r,k}^\alpha = \\ = -M_{r,k}^\alpha : M_{r,k}^\alpha + \alpha\, E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha). \end{aligned} \tag{5.69}$$

Now let $|\tau_r^\alpha| > 1$. Then $M_r^\alpha = \alpha\,\tau_r^\alpha + \frac{\tau_r^\alpha}{|\tau_r^\alpha|}$, and hence

$$M_{r,k}^\alpha = \alpha\,\tau_{r,k}^\alpha + \left[\frac{\tau_r^\alpha}{|\tau_r^\alpha|}\right]_{,k}. \tag{5.70}$$

Expressing $\tau_{r,k}^\alpha$ from the latter relation we get

$$\tau_{r,k}^\alpha = |\tau_{r,k}^\alpha|\left(M_{r,k}^\alpha - \alpha\,\tau_{r,k}^\alpha\right) + \tau_r^\alpha\,\frac{\tau_r^\alpha : \tau_{r,k}^\alpha}{|\tau_r^\alpha|},$$

which yields

$$\begin{aligned} -M_{r,k}^\alpha : \tau_{r,k}^\alpha = -|\tau_r^\alpha|\, M_{r,k}^\alpha : M_{r,k}^\alpha + \alpha\,|\tau_r^\alpha|\, M_{r,k}^\alpha : \tau_{r,k}^\alpha - \\ -\frac{\tau_r^\alpha : \tau_{r,k}^\alpha}{|\tau_r^\alpha|^2}\, M_{r,k}^\alpha : \tau_r^\alpha \le -|\tau_r^\alpha|\, M_{r,k}^\alpha : M_{r,k}^\alpha + \alpha\,|\tau_r^\alpha|\, M_{r,k}^\alpha : \tau_{r,k}^\alpha, \end{aligned} \tag{5.71}$$

where (5.70) and the orthogonality of $\left[\frac{\tau_r^\alpha}{|\tau_r^\alpha|}\right]_{,k}$ and τ_r^α was used:

$$-\frac{\tau_r^\alpha : \tau_{r,k}^\alpha}{|\tau_r^\alpha|^2}\, M_{r,k}^\alpha : \tau_r^\alpha = -\alpha\left(\frac{\tau_r^\alpha : \tau_{r,k}^\alpha}{|\tau_r^\alpha|}\right)^2 \le 0.$$

The claim now follows from (5.69) and (5.71). □

5.6 $W_{loc}^{1,2}$ estimates of bending moments in the incremental problems

In this section we deduce some iterative estimates for the L^2 norms of the gradients of the functions M_r^α, defined by means of (5.44), and we show that for every given r and N we have $M_r^N \in W_{loc}^{1,2}(\Omega; \mathbb{M}_{sym}^{2\times 2})$. We note that for the moment we are concerned only with the problem of regularity of each M_r^N, that is, we don't care about the uniformity of estimates with respect to r and N. Having obtained the L^2 bounds, we conclude that the approximate solutions M_r^α, whih were known to converge to M_r^N weakly in $L^2(\Omega; \mathbb{M}_{sym}^{2\times 2})$, actually converge strongly.

Remark that in what follows C_r will denote a constant independent of α, which may change from line to line. This constant may depend on r, N, and, in case of local estimates, on a domain $\Omega' \subset\subset \Omega$. We will use the notation C only when this constant does not depend on r and N.

For the moment, our objective is the following estimate:

$$\int_{\Omega'} M_{r,k}^\alpha : M_{r,k}^\alpha\, dx \le C(r, N, \Omega'), \tag{5.72}$$

valid for any $\Omega' \subset\subset \Omega$.

Suppose, by induction, that we have already proved that $M^N_{r-1} \in W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym})$. To simplify the notation, in this section we sometimes omit writing the index N for the solutions of the incremental problem (5.25). Let us examine the regularized problem (5.40). Since u^α_r is a solution of the nonlinear elliptic system (5.43) with $f^N_r \in L^2(\Omega)$ and $e^N_{r-1} \in W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym})$, one can show, by working with difference quotients, that

$$\begin{gathered} u^\alpha_m \in W^{3,2}_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym}), \\ M^\alpha_m,\ D^2 u^\alpha_m \in W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym}). \end{gathered} \tag{5.73}$$

By using formula (5.64) , estimate (5.67) and the definition (5.61) of τ^α_r we obtain

$$\begin{gathered} M^\alpha_{r,k} : M^\alpha_{r,k} = E^\alpha_2(\tau^\alpha_{r,k},\ M^\alpha_{r,k}) \le \\ \le \Big[E^\alpha_2(\tau^\alpha_{r,k}, \tau^\alpha_{r,k})\Big]^{1/2} \Big[E^\alpha_2(M^\alpha_{r,k}, M^\alpha_{r,k})\Big]^{1/2} \le \\ \le \tfrac{1}{2} E^\alpha_2(\tau^\alpha_{r,k}, \tau^\alpha_{r,k}) + \tfrac{1}{2} E^\alpha_2(M^\alpha_{r,k}, M^\alpha_{r,k}) \le \\ \le \tfrac{1}{2} M^\alpha_{r,k} : \tau^\alpha_{r,k} + \left(\tfrac{1}{2} + \tfrac{\alpha}{2}\right) M^\alpha_{r,k} : M^\alpha_{r,k} \le \\ \le \tfrac{1}{2} M^\alpha_{r,k} : D^2 u^\alpha_{r,k} + \tfrac{1}{2} M^\alpha_{r,k} : M^N_{r-1,k} + \left(\tfrac{1}{2} + \tfrac{\alpha}{2}\right) M^\alpha_{r,k} : M^\alpha_{r,k}. \end{gathered} \tag{5.74}$$

By applying the Cauchy inequality to $M^\alpha_{r,k} : M^N_{r-1,k}$ we get

$$(1-\alpha) M^\alpha_{r,k} : M^\alpha_{r,k} \le M^N_{r-1,k} : M^N_{r-1,k} + 2 M^\alpha_{r,k} : D^2 u^\alpha_{r,k}. \tag{5.75}$$

Thus, it remains to prove the boundedness in $L^1_{loc}(\Omega)$ of the second summand of

Let us introduce the notation

$$M^\alpha := M^\alpha_r, \quad f := f^N_r, \quad u^\alpha := u^\alpha_r,$$

omitting index m for further convenience. Let $\varphi \in C^3_0(\Omega)$ be an arbitrary cut-off function, such that $\varphi \equiv 1$ on Ω', and $\operatorname{supp}\varphi \subset \Omega'' \subset\subset \Omega$. By (5.73) we can put the function

$$v = \varphi^4\, u^\alpha_{,k}$$

into the Euler equation (5.42).

We start by

$$\int_\Omega M^\alpha_{,k} : D^2(\varphi^4\, u^\alpha_{,k})\, dx = \int_\Omega \varphi^4\, \nabla f \cdot \nabla u^\alpha\, dx.$$

This equality can be expressed in the following way

$$\begin{gathered} J^\alpha_r := \int_\Omega \varphi^4 M^\alpha_{,k} : D^2 u^\alpha_{,k}\, dx = \int_\Omega \varphi^4\, f_{,k}\, u^\alpha_{,k}\, dx - \\ -2 \int_\Omega M^\alpha_{ij,k}\, \varphi^4_{,j}\, u^\alpha_{,ki}\, dx - \int_\Omega M^\alpha_{ij,k}\, \varphi^4_{,ij}\, u^\alpha_{,k}\, dx. \end{gathered} \tag{5.76}$$

Thus, we have

$$J^\alpha_r \le I^\alpha_1 + I^\alpha_2 + I^\alpha_3, \tag{5.77}$$

with

$$
\begin{gathered}
I_1^\alpha := \int_\Omega \varphi^4\, f_{,k}\, u^\alpha_{,k}\, dx, \quad I_2^\alpha := -2\int_\Omega M^\alpha_{ij,k}\, \varphi^4_{,j}\, u^\alpha_{,ki}\, dx \\
I_3^\alpha := -\int_\Omega M^\alpha_{ij,k}\, \varphi^4_{,ij}\, u^\alpha_{,k}\, dx.
\end{gathered} \tag{5.78}
$$

Estimate of I_1^α.

$$
|I_1^\alpha| \le \|f\|_{1,2;\Omega''}\, \|u^\alpha\|_{1,2;\Omega} \le C_r. \tag{5.79}
$$

Estimate of I_2^α. Let us introduce the matrices $S^{(k)} = (S^{(k)}_{ij})$ defined by

$$
S^{(k)}_{ij} := -\varphi_{,j}\, u^\alpha_{,ki}. \tag{5.80}
$$

Then by using (5.64), (5.66), (5.61) and the fact that $\|M^N_{r-1}\|_{\infty;\Omega} \le 1$ we obtain

$$
\begin{gathered}
I_2^\alpha = -2\int_\Omega M^\alpha_{ij,k}\, \varphi^4_{,j}\, u^\alpha_{,ki}\, dx = 8\int_\Omega \varphi^3 E_2^\alpha(\tau^\alpha_{r,k}, S^{(i)})\, dx \le \\
\le \tfrac{1}{100}\int_\Omega \varphi^4 E_2^\alpha(\tau^\alpha_{r,k}, \tau^\alpha_{r,k})\, dx + C_r\int_\Omega \varphi^2 E_2^\alpha(S^{(k)}, S^{(k)})\, dx \le \\
\le \tfrac{1}{100}\int_\Omega \varphi^4\, M^\alpha_{r,k} : \tau^\alpha_{r,k} + \alpha\, C_r\int_\Omega \varphi^2 |S^{(k)}|^2\, dx + \\
+ C_r\int_{|\tau^\alpha_r|\le 1} \varphi^2 |S^{(k)}|^2\, dx + C_r\int_{|\tau^\alpha_r|>1} \frac{\varphi^2|S^{(k)}|^2}{|\tau^\alpha_r|}\, dx \le \\
\le \tfrac{1}{100}\left(J^\alpha_r + \int_\Omega \varphi^4\, M^\alpha_{r,k} : M^\alpha_{r,k}\, dx + \int_\Omega \varphi^4\, M^N_{r-1,k} : M^N_{r-1,k}\, dx\right) + \\
+ C_r\int_{|\tau^\alpha_r|\le 1} \varphi^2 |D^2 u^\alpha_r|^2\, dx + C_r\int_{|\tau^\alpha_r|>1} \frac{\varphi^2 |D^2 u^\alpha_r|^2}{|\tau^\alpha_r|}\, dx + \alpha\, C_r \|D^2 u^\alpha_r\|^2_{2;\Omega} \le \\
\le \tfrac{1}{100}\left(J^\alpha_r + \int_\Omega \varphi^4\, M^\alpha_{r,k} : M^\alpha_{r,k}\, dx + \int_\Omega \varphi^4\, M^N_{r-1,k} : M^N_{r-1,k}\, dx\right) + \\
+ C_r\int_\Omega |D^2 u^\alpha_r|\, dx + C_r.
\end{gathered} \tag{5.81}
$$

Estimate of I_3^α. Using (5.64) and Lemma 5.5.1

$$
\begin{gathered}
I_3^\alpha = -\int_\Omega M^\alpha_{ij,k}\, \varphi^4_{,ij}\, u^\alpha_{,k}\, dx = \\
= -4\int_\Omega \varphi^3\, u^\alpha_{,k}\, E_2^\alpha(\tau^\alpha_{r,k}, \nabla^2\varphi)\, dx - 12\int_\Omega \varphi^2\, u^\alpha_{,k}\, E_2^\alpha(\tau^\alpha_{r,k}, \nabla\varphi\otimes\nabla\varphi)\, dx \le \\
\le \tfrac{1}{100}\int_\Omega \varphi^4 E_2^\alpha(\tau^\alpha_{r,k}, \tau^\alpha_{r,k})\, dx + \\
+ C_r\int_\Omega |\nabla u^\alpha|^2 \left(\varphi^2\, E_2^\alpha(\nabla^2\varphi, \nabla^2\varphi) + E_2^\alpha(\nabla\varphi\otimes\nabla\varphi, \nabla\varphi\otimes\nabla\varphi)\right)\, dx \le \\
\le \tfrac{1}{100}\left(J^\alpha_r + \int_\Omega \varphi^4\, M^\alpha_{r,k} : M^\alpha_{r,k}\, dx + \int_\Omega \varphi^4\, M^N_{r-1,k} : M^N_{r-1,k}\, dx\right) + C_r.
\end{gathered} \tag{5.82}
$$

So, (5.76), (5.77), (5.79)-(5.82), and the regularity of M_{r-1}^N proved at the previous step, imply that

$$J_r^\alpha \leq C_r + \tfrac{2}{100} J_r^\alpha + \tfrac{2}{100} \int_\Omega \varphi^4 \, M_{r,k}^\alpha : M_{r,k}^\alpha \, dx.$$

Therefore, (5.75) allows us to conclude that (5.72) holds for every $k = 1, 2$, and thus

$$\limsup_{\alpha \to 0} \|\nabla M_r^\alpha\|_{2;\Omega'} \leq C(r, N, \Omega'). \tag{5.83}$$

Remark 5.6.1. Inequality (5.83) and the convergence $M_r^\alpha \rightharpoonup M_r^N$ in $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$, see (5.45), imply that

$$\begin{gathered} M_r^N \in W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym}), \\ M_r^\alpha \rightharpoonup M_r^N \quad \text{in } W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym}), \\ \text{and} \quad M_r^\alpha \to M_r^N \quad \text{in } L^2_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym}), \end{gathered} \tag{5.84}$$

where the strong convergence in $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ is guaranteed by Sobolev embedding.

5.7 Auxiliary estimates

In this section we prove a fine convergence estimate for the approximate solutions of regularized problems (Lemmas 5.7.1 and 5.7.3) and get analytic estimates, which are the core of the proof of the uniform boundedness of M_r^N in $W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym})$ (Lemmas 5.7.4, 5.7.5 and Corollary 5.7.6).

In these estimates it is crucial that the constants C does not depend on r and N, although they might depend on φ.

In the rest of the paper $\omega_r(\alpha)$ will denote a generic function, converging to 0 as $\alpha \to 0$, which may change from line to line and may depend on r and N.

5.7.1 Fine properties of approximating sequence

Lemma 5.7.1. *For any function $\psi \in C_0(\Omega)$ with $0 \leq \psi \leq 1$, we have*

$$\int_{|\tau_r^\alpha| > 1} \psi \left(|D^2 u_r^\alpha + M_{r-1}^N| - |M_{r-1}^N| \right) dx \leq \frac{C}{N} + \omega_r(\alpha), \tag{5.85}$$

where the constant C and the quantity $\omega_r(\alpha)$ may depend on the properties of ψ.

Proof: As $|M_{r-1}^N| \leq 1$, we have

$$\begin{aligned} \int_{|\tau_r^\alpha| > 1} & \psi \left(|D^2 u_r^\alpha + M_{r-1}^N| - |M_{r-1}^N| \right) dx = \\ &= \int_\Omega \psi \left(|D^2 u_r^\alpha + M_{r-1}^N| - |M_{r-1}^N| \right) dx - \\ &- \int_{|\tau_r^\alpha| \leq 1} \psi \left(|D^2 u_r^\alpha + M_{r-1}^N| - |M_{r-1}^N| \right) dx. \end{aligned} \tag{5.86}$$

Equality (5.59) implies that on the set $\{|\tau_r^\alpha| \le 1\}$ one has $M_r^\alpha = \alpha\,\tau_r^\alpha + \tau_r^\alpha$. Thus, by (5.59), (5.61), Lemma 5.5.1, (5.84) and (5.16) we obtain

$$-\int_{|\tau_r^\alpha|\le 1} \psi \left(|D^2u_r^\alpha + M_{r-1}^N| - |M_{r-1}^N|\right)\,dx \le$$
$$\le \int_{|\tau_r^\alpha|\le 1} \psi \left(|M_r^\alpha - M_{r-1}^N| + \alpha\,|\tau_r^\alpha|\right)\,dx \le$$
$$\le \int_\Omega \psi|M_r^\alpha - M_{r-1}^N|\,dx + C\,\alpha \le$$
$$\le \int_\Omega \psi|M_r^N - M_{r-1}^N|\,dx + \int_\Omega \psi|M_r^N - M_r^\alpha|\,dx + C\,\alpha \le \frac{C}{N} + \omega_m(\alpha).$$

On the other hand, by (5.53) and (5.16)

$$\lim_{\alpha\to 0}\int_\Omega \psi \left(|D^2u_r^\alpha + M_{r-1}^N| - |M_{r-1}^N|\right)\,dx =$$
$$\langle \psi, |D^2(\delta u_r^N) + M_{r-1}^N| - |M_{r-1}^N|\cdot\mathcal{L}^n\rangle \le |D^2(\delta u_r^N)|(\Omega) \le \frac{C}{N}.$$

The estimate (5.85) follows from last two estimates and (5.86). □

As a corollary, we prove a local estimate for $|D^2u_r^\alpha|$.

Corollary 5.7.2. *We have*

$$\int_\Omega \psi|D^2u_r^\alpha|\,dx \le \frac{C}{N} + \omega_r(\alpha) \tag{5.87}$$

for every function $\psi \in C_0(\Omega)$ with $0 \le \psi \le 1$, where the constant C and the quantity $\omega_r(\alpha)$ may depend on ψ.

Proof: Introduce the notation

$$\widetilde{\Omega}_1 = \{|\tau_r^\alpha| \le 1\}, \quad \widetilde{\Omega}_2 = \{1 < |\tau_r^\alpha|\}$$

Now we divide the integral over Ω into two integrals over $\widetilde{\Omega}_i$, $i = 1, 2$, and estimate each one of them, as in (5.87).

Estimate over $\widetilde{\Omega}_1$. According to (5.59) and (5.61) in the region $\widetilde{\Omega}_1$ we have

$$D^2u_r^\alpha = M_r^\alpha - M_{r-1}^N + \alpha\tau_r^\alpha,$$

and hence, by (5.84) and (5.16) we obtain

$$\int_{\widetilde{\Omega}_1} \psi\,|D^2u_r^\alpha|\,dx \le \int_{\widetilde{\Omega}_1} \psi\,|M_r^\alpha - M_{r-1}^N|\,dx + \alpha\int_{\widetilde{\Omega}_1} \psi\,dx \le \frac{C}{N} + \omega_r(\alpha). \tag{5.88}$$

Estimate over $\widetilde{\Omega}_2$. By (5.59) and (5.61) in $\widetilde{\Omega}_2$ one has

$$D^2u_r^\alpha = M_r^\alpha\,(|\tau_r^\alpha| - 1) - (M_{r-1}^N - M_r^\alpha) - \alpha\,\tau_r^\alpha\,|\tau_r^\alpha|. \tag{5.89}$$

Again, by (5.59) and (5.61) we get

$$|M_r^\alpha|\,|\tau_r^\alpha| \le \alpha\,|\tau_r^\alpha|^2 + |D^2 u_r^\alpha + M_{r-1}^N|,$$

and the triangle inequality $|M_r^\alpha| \ge |M_{r-1}^N| - |M_{r-1}^N - M_r^\alpha|$ yields

$$-|M_r^\alpha| \le -|M_{r-1}^N| + |M_{r-1}^N - M_r^\alpha|.$$

By the last two estimates, the relation (5.89) becomes

$$\int_{\widetilde{\Omega}_2} \psi|D^2 u_r^\alpha|\,dx \le \int_{\widetilde{\Omega}_2} \psi\left(|D^2 u_r^\alpha + M_{r-1}^N| - |M_{r-1}^N|\right)\,dx + \\ +2\int_{\widetilde{\Omega}_2} \psi|M_{r-1}^N - M_r^\alpha|\,dx + 2\alpha\int_{\widetilde{\Omega}_2} |\tau_r^\alpha|^2\,dx$$

Using (5.85), (5.49), the convergence (5.84) and (5.16), by the last estimate we conclude, that

$$\int_{\widetilde{\Omega}_2} \psi|D^2 u_r^\alpha|\,dx \le \frac{C}{N} + \omega_r(\alpha) \tag{5.90}$$

Now the claim (5.87) follows from (5.88) and (5.90). □

Lemma 5.7.3. *The following estimate holds:*

$$\int_\Omega \psi^2|\nabla u_r^\alpha|^2\,dx \le \frac{C}{N^2} + \omega_r(\alpha), \tag{5.91}$$

for any function $\psi \in C_0^2(\Omega)$ with $0 \le \psi \le 1$. Remark, that the constant C and the quantity $\omega_r(\alpha)$ depend upon $\|\psi\|_{2,\infty;\Omega}$.

Proof: We begin by defining the functions $v_r^\alpha := u_r^\alpha\,\psi \in W_0^{2,2}(\Omega)$, which satisfy the following equalities:

$$\begin{aligned} \nabla v_r^\alpha &= \psi\,\nabla u_r^\alpha + u_r^\alpha\,\nabla\psi, \\ D^2 v_r^\alpha &= \psi\,D^2 u_r^\alpha + 2\,\nabla\psi \odot \nabla u_r^\alpha + u_r^\alpha\,\nabla^2\psi. \end{aligned} \tag{5.92}$$

Then, by using (5.92), the Sobolev embeddings $W^{2,1}(\Omega) \hookrightarrow W^{1,2}(\Omega)$ and $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$, and the Poincare inequality for $W_0^{2,1}(\Omega)$ we can estimate the integral considered as follows

$$\begin{aligned} &\int_\Omega |\psi\,\nabla u_r^\alpha|^2\,dx \le 2\int_\Omega |\psi\,\nabla u_r^\alpha + u_r^\alpha\,\nabla\psi|^2\,dx + 2\int_\Omega |u_r^\alpha\,\nabla\psi|^2\,dx \le \\ &\le C\int_\Omega |\nabla v_r^\alpha|^2\,dx + C\int_\Omega |u_r^\alpha|^2\,dx \le C\,\|v_r^\alpha\|_{2,1;\Omega}^2 + C\,\|u_r^\alpha\|_{1,1;\Omega}^2 \le \\ &\le C\left(\int_\Omega |D^2 v_r^\alpha|\,dx\right)^2 + C\,\|u_r^\alpha\|_{1,1;\Omega}^2 \le \\ &\le C\left(\int_\Omega \psi\,|D^2 u_r^\alpha|\,dx + \int_\Omega |\nabla u_r^\alpha|\,dx + \int_\Omega |u_r^\alpha|\,dx\right)^2 + C\|u_r^\alpha\|_{1,1;\Omega}^2\,. \end{aligned} \tag{5.93}$$

Now we use (5.93), the estimate (5.87), the convergence $u_r^\alpha \to \delta u_r^N$ in $W^{1,1}(\Omega)$, as in (5.46), the embedding $BH(\Omega) \hookrightarrow W^{1,1}(\Omega)$, and (5.16) to obtain

$$\int_\Omega \psi^2 |\nabla u_r^\alpha|^2\, dx \le C \left(\int_\Omega \psi\, |D^2 u_r^\alpha|\, dx \right)^2 + C \|u_r^\alpha\|^2_{1,1;\Omega} \le$$
$$\le \frac{C}{N^2} + C\, \|\delta u_r^N\|^2_{BH(\Omega)} + \omega_r(\alpha) \le \frac{C}{N^2} + \omega_r(\alpha).$$

The claim is proved. □

5.7.2 Analytic estimates

Lemma 5.7.4. *The following inequality holds for J_r^α defined in* (5.76)*:*

$$J_r^\alpha \le -2 \int_\Omega M^\alpha_{ij,k}\, \varphi^4_{,j}\, u^\alpha_{,ki}\, dx + \tfrac{1}{N} \int_\Omega \varphi^4\, E_2^\alpha(\tau^\alpha_{r,k}, \tau^\alpha_{r,k}) + \frac{C}{N} + \omega_r(\alpha). \tag{5.94}$$

PROOF: Recalling (5.77), we have $J_r^\alpha \le I_1^\alpha + I_2^\alpha + I_3^\alpha$ with I_i^α, $i = 1, \ldots, 3$ defined in (5.78). We show, that I_1^α and I_3^α are of order $\frac{1}{N}$ when $\alpha \to 0$.

Estimate of I_1^α. Since $f_r^N \in W^{1,2}_{loc}(\Omega)$, one can employ the convergence (5.47) to pass to the limit in I_1^α, and use the estimates (5.16) of $\|\delta u_r^N\|_{BH(\Omega)}$ to obtain

$$|I_1^\alpha| \le C(\|f\|_{L^\infty([0,T];W^{1,2}(\Omega''))})\, \frac{1}{N} + \omega_r(\alpha). \tag{5.95}$$

Estimate of I_3^α. First of all, remark that the function

$$\varphi \left(E_2^\alpha(\nabla^2\varphi, \nabla^2\varphi) + E_2^\alpha(\nabla\varphi \otimes \nabla\varphi, \nabla\varphi \otimes \nabla\varphi) \right)$$

is bounded and has a compact support, which is a subset of $\operatorname{supp}\varphi$. Let us choose a function $\psi \in C_0^\infty(\Omega)$, such that

$$\psi \equiv 1 \text{ on } \operatorname{supp}\varphi \quad \text{and} \quad \operatorname{supp}\psi \subset \Omega''.$$

$$\begin{aligned} I_3^\alpha &= -\int_\Omega M^\alpha_{ij,k}\, \varphi^4_{,ij}\, u^\alpha_{,k}\, dx = \\ &= -4 \int_\Omega \varphi^3\, u^\alpha_{r,k}\, E_2^\alpha(\tau^\alpha_{r,k}, \nabla^2\varphi)\, dx - 12 \int_\Omega \varphi^2\, u^\alpha_{r,k}\, E_2^\alpha(\tau^\alpha_{r,k}, \varphi \otimes \varphi)\, dx \le \\ &\le \tfrac{1}{N} \int_\Omega \varphi^4\, E_2^\alpha(\tau^\alpha_{r,k}, \tau^\alpha_{r,k})\, dx + CN \int_\Omega \psi^2 |\nabla u_r^\alpha|^2\, dx, \end{aligned} \tag{5.96}$$

with ψ chosen above, using the fact that

$$\varphi \left(E_2^\alpha(\nabla^2\varphi, \nabla^2\varphi) + E_2^\alpha(\nabla\varphi \otimes \nabla\varphi, \nabla\varphi \otimes \nabla\varphi) \right) \le C\, \psi^2.$$

Thus, by (5.77), (5.95), (5.96) and (5.91) we obtain (5.94). □

Lemma 5.7.5. *The following "iterative" estimate holds true:*

$$
\begin{aligned}
\big(1-\tfrac{2}{N}\big)\int_\Omega \varphi^4 E_2^\alpha(\tau^\alpha_{r,k},\tau^\alpha_{r,k})\,dx \le \tfrac{100}{99}\int_\Omega M^\alpha_{r,k} : M^N_{r-1,k}\,dx+\\
+\sum_{s=1}^{9N-1}\frac{1}{s+10}\int_{F_s}\varphi^4 M^\alpha_{r,k} : M^\alpha_{r,k}\,dx - \tfrac{1}{99}\int_{|\tau^\alpha_r|\le 10}\varphi^4 E_2^\alpha(\tau^\alpha_{r,k},\tau^\alpha_{r,k})\,dx+\\
+\tfrac{C}{N}\int_\Omega \varphi^4 M^\alpha_{r,k} : M^\alpha_{r,k}\,dx + \tfrac{C}{N}\int_\Omega \varphi^4 M^N_{r-1,k} : M^N_{r-1,k}\,dx + \frac{C}{N} + \omega_r(\alpha),
\end{aligned}
\tag{5.97}
$$

where $F_s,\ s=1,\dots,9N-1$ *is defined by* $F_s = \{1+\frac{9}{s+1} < |\tau^\alpha_r| \le 1+\frac{9}{s}\}$.

Proof: By (5.64), (5.61), (5.76) and (5.94)

$$
\begin{aligned}
\big(1-\tfrac{1}{N}\big)\int_\Omega \varphi^4 E_2^\alpha(\tau^\alpha_{r,k},\tau^\alpha_{r,k})\,dx \le -2\int_\Omega M^\alpha_{ij,k}\,\varphi^4_{,j}\,u^\alpha_{,ki}\,dx+\\
+\int_\Omega \varphi^4 M^\alpha_{r,k} : M^N_{r-1,k}\,dx + \frac{C}{N} + \omega_r(\alpha) =\\
= B^\alpha_1 + B^\alpha_2 + B^\alpha_3 + B^\alpha_4 + \int_\Omega \varphi^4 M^\alpha_{r,k} : M^N_{r-1,k}\,dx + \frac{C}{N} + \omega_r(\alpha),
\end{aligned}
\tag{5.98}
$$

where

$$
B^\alpha_i := 8\int_{\Omega_i}\varphi^3\, M^\alpha_{r,k} : S^{(k)}\,dx, \quad i=1,\dots,4,
$$

with $S^{(k)}$ defined in (5.80) and

$$
\begin{aligned}
\Omega_1 = \{|\tau^\alpha_r|\le 1\}, \qquad \Omega_2 = \big\{1 < |\tau^\alpha_r| \le 1+\tfrac{1}{N}\big\},\\
\Omega_3 = \big\{1+\tfrac{1}{N} < |\tau^\alpha_r| \le 10\big\}, \qquad \Omega_4 = \{10 < |\tau^\alpha_r|\}.
\end{aligned}
\tag{5.99}
$$

Estimate of B^α_1: According to (5.59) and (5.61), in the region Ω_1 the following identity holds:

$$
D^2 u^\alpha_r = M^\alpha_r - M^N_{r-1} - \alpha\,\tau^\alpha_r .
$$

Hence, by (5.80)

$$
|S^{(k)}|^2 \le C\,|D^2u^\alpha_r|^2 \le C\left(|M^\alpha_r - M^N_{r-1}|^2 + \alpha^2\,|\tau^\alpha_r|^2\right),
$$

and we have

$$
\int_{\Omega_1}\varphi^2|S^{(k)}|^2\,dx \le C\,\alpha^2 + C\,\|M^\alpha_r - M^N_{r-1}\|^2_{2;\Omega''}.
$$

Thus, from the convergence (5.84) and the increment estimate (5.16), it follows that

$$
\begin{aligned}
B^\alpha_1 \le \tfrac{1}{N}\int_{\Omega_1}\varphi^4\, M^\alpha_{r,k} : M^\alpha_{r,k}\,dx + CN\int_{\Omega_1}\varphi^2\,|S^{(k)}|^2\,dx \le\\
\le \tfrac{1}{N}\int_{\Omega_1}\varphi^4\, M^\alpha_{r,k} : M^\alpha_{r,k}\,dx + \tfrac{C}{N} + \omega_r(\alpha).
\end{aligned}
\tag{5.100}
$$

Estimate of B^α_2: We remark, that (5.59) and (5.61) yield that for $|\tau^\alpha_r|\ge 1$ one has

$$
D^2u^\alpha_r = M^\alpha_r\,(|\tau^\alpha_r|-1) - \big(M^N_{r-1} - M^\alpha_r\big) - \alpha\,\tau^\alpha_r\,|\tau^\alpha_r|,
\tag{5.101}
$$

so that in the region Ω_2 we have

$$|D^2u_r^\alpha|^2 \le \tfrac{C}{N^2}\,|M_r^\alpha|^2 + C\,|M_r^\alpha - M_{r-1}^N|^2 + C\,\alpha^2\,|\tau_r^\alpha|^4.$$

By the inequality $|S^{(k)}| \le C|D^2u_r^\alpha|$, see (5.80),

$$8\,\varphi^3\,M_{r,k}^\alpha : S^{(k)} \le \tfrac{1}{N}\varphi^4\,M_{r,k}^\alpha : M_{r,k}^\alpha + C\,N\,\varphi^2\,|D^2u_r^\alpha|^2,$$

so that by the former estimate, the boundedness of τ_r^α and M_r^α on Ω_2 (see (5.59)), (5.84) and (5.16) we have

$$B_2^\alpha \le \tfrac{1}{N}\int_{\Omega_2}\varphi^4\,M_{r,k}^\alpha : M_{r,k}^\alpha\,dx + \frac{C}{N} + \omega_r(\alpha). \tag{5.102}$$

Estimate of B_3^α: Using the notation $F_s = \{1 + \frac{9}{s+1} < |\tau_r^\alpha| \le 1 + \frac{9}{s}\}$ for $s = 1,\dots,9N-1$, we write

$$\begin{aligned} B_3^\alpha = 8\sum_{s=1}^{9N-1}\int_{F_s}\varphi^3\,M_{r,k}^\alpha : S^{(k)}\,dx \le \\ \le \sum_{s=1}^{9N-1}\left[\tfrac{1}{2(s+10)}\int_{F_s}\varphi^4\,M_{r,k}^\alpha : M_{r,k}^\alpha\,dx + C\,(s+10)\int_{F_s}\varphi^2|D^2u_r^\alpha|^2\,dx\right]. \end{aligned} \tag{5.103}$$

Now we show, that the last summand can be bounded by $\frac{C}{N} + \omega_r(\alpha)$.

Thanks to (5.101) on F_s we have

$$|D^2u_r^\alpha|^2 \le \tfrac{9}{s}\,|M_r^\alpha|^2\,(|\tau_r^\alpha| - 1) + C|M_r^\alpha - M_{r-1}^N|^2 + \alpha^2\,|\tau_r^\alpha|^4,$$

so that by (5.84), (5.12), (5.16), and the boundedness of M_r^α and τ_r^α on F_s (see (5.59)) we have

$$\begin{aligned} \sum_{s=1}^{9N-1}(s+10)\int_{F_s}\varphi^2\,|D^2u_r^\alpha|^2\,dx \le \\ \le \sum_{s=1}^{9N-1}\int_{F_s}\varphi^2\left(\tfrac{9(s+10)}{s}\,|M_r^\alpha|^2\,(|\tau_r^\alpha|-1) + CN\,|M_r^\alpha - M_{r-1}^N|^2 + CN\alpha^2\right)dx \le \\ \le C\int_{\Omega_3}\varphi^2|M_r^\alpha|\,(|\tau_r^\alpha| - 1)\;dx + CN\|M_r^\alpha - M_{r-1}^N\|_{2;\Omega''}^2 + CN\alpha^2. \end{aligned} \tag{5.104}$$

By (5.59) and (5.61) we have $|M_r^\alpha|\,|\tau_r^\alpha| \le \alpha\,|\tau_r^\alpha|^2 + |D^2u_r^\alpha + M_{r-1}^N|$, and by the triangle inequality $|M_r^\alpha| \ge |M_{r-1}^N| - |M_{r-1}^N - M_r^\alpha|$ we have also $-|M_r^\alpha| \le -|M_{r-1}^N| + |M_{r-1}^N - M_r^\alpha|$. Therefore using (5.84) and (5.16) we can bound the right-hand side of (5.104) by

$$C\int_{\Omega_3}\varphi^2\left(|D^2u_r^\alpha + M_{r-1}^N| - |M_{r-1}^N|\right) + \frac{C}{N} + \omega_r(\alpha).$$

Thus, by (5.85) we conclude that

$$B_3^\alpha \le \sum_{s=1}^{9N-1}\tfrac{1}{2(s+10)}\int_{F_s}\varphi^4\,M_{r,k}^\alpha : M_{r,k}^\alpha\,dx + \frac{C}{N} + \omega_r(\alpha). \tag{5.105}$$

□

Estimate of B_4^α: Applying the Cauchy inequality

$$\varphi^3\, E_2^\alpha(\tau_{r,k}^\alpha, S^{(k)}) \le \tfrac{1}{100}\varphi^4\, E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha) + C\varphi^2\, E_2^\alpha(S^{(k)}, S^{(k)}),$$

and using (5.64), (5.65), (5.80) and (5.49) we obtain

$$\begin{aligned} B_4^\alpha = \int_{\Omega_4} \varphi^4\, E_2^\alpha(\tau_{r,k}^\alpha, S^{(k)})\, dx \le \\ \le \tfrac{1}{100} \int_{\Omega_4} \varphi^4\, E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha)\, dx + C \int_{\Omega_4} \varphi^2\, \frac{|D^2 u_r^\alpha|^2}{|\tau_r^\alpha|}\, dx + \omega_r(\alpha). \end{aligned} \tag{5.106}$$

To show that the last summand is of order $\frac{1}{N}$, we first note that on the set Ω_4 the inequality

$$\frac{|D^2 u_r^\alpha|^2}{|\tau_r^\alpha|} < 10\left(|D^2 u_r^\alpha + M_{r-1}^N| - |M_{r-1}^N|\right) \tag{5.107}$$

holds. To prove it, we multiply both sides by $|\tau_r^\alpha| = |D^2 u_r^\alpha + M_{r-1}^N|$. Using the inequality $|M_{r-1}^N| \le 1$, which follows from $M_{r-1}^N \in \mathbb{K}$, the right-hand side of (5.107) can be bounded from below by

$$\begin{aligned} 10\left(|D^2 u_r^\alpha|^2 + 2\, D^2 u_r^\alpha : M_{r-1}^N + |M_{r-1}^N|^2 - |M_{r-1}^N| \cdot |D^2 u_r^\alpha + M_{r-1}^N|\right) \ge \\ \ge 10\left(|D^2 u_r^\alpha|^2 - 3\, |M_{r-1}^N| \cdot |D^2 u_r^\alpha|\right) \ge 10\, |D^2 u_r^\alpha|^2 - 30\, |D^2 u_r^\alpha|. \end{aligned}$$

Using again $|M_{r-1}^N| \le 1$, in the region Ω_4 we have that $|D^2 u_r^\alpha| > 9$, which yields that

$$10\, |D^2 u_r^\alpha|^2 - 30\, |D^2 u_r^\alpha| \ge |D^2 u_r^\alpha|^2 + 51\, |D^2 u_r^\alpha| > |D^2 u_r^\alpha|^2,$$

and (5.107) is proved.

From (5.106), (5.107), and (5.85) we have

$$B_4^\alpha \le \tfrac{1}{100} \int_{\Omega_4} \varphi^4\, E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha)\, dx + \frac{C}{N} + \omega_r(\alpha). \tag{5.108}$$

Collecting (5.98), (5.100), (5.102), (5.105), and (5.108) we obtain

$$\begin{aligned} \left(1 - \tfrac{1}{N}\right) \int_\Omega \varphi^4\, E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha)\, dx \le \int_\Omega \varphi^4\, M_{r,k}^\alpha : M_{r-1,k}^N\, dx + \\ + \sum_{s=1}^{9N-1} \tfrac{1}{2(s+10)} \int_{F_s} \varphi^4\, M_{r,k}^\alpha : M_{r,k}^\alpha\, dx + \tfrac{1}{100} \int_{\Omega_4} \varphi^4\, E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha)\, dx + \\ + \tfrac{1}{N} \int_\Omega \varphi^4\, M_{r,k}^\alpha : M_{r,k}^\alpha\, dx + \frac{C}{N} + \omega_r(\alpha), \end{aligned}$$

or, by easy transformations,

$$\begin{aligned} \left(\tfrac{99}{100} - \tfrac{1}{N}\right) \int_\Omega \varphi^4\, E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha)\, dx \le \int_\Omega \varphi^4\, M_{r,k}^\alpha : M_{r-1,k}^N\, dx + \\ + \sum_{s=1}^{9N-1} \tfrac{1}{2(s+10)} \int_{F_s} \varphi^4\, M_{r,k}^\alpha : M_{r,k}^\alpha\, dx - \tfrac{1}{100} \int_{\Omega_1 \cup \Omega_2 \cup \Omega_3} \varphi^4 E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha)\, dx + \\ + \tfrac{C}{N} \int_\Omega \varphi^4\, M_{r,k}^\alpha : M_{r,k}^\alpha\, dx + \frac{C}{N} + \omega_r(\alpha). \end{aligned}$$

The claim (5.97) now follows by multiplying the last inequality by $\frac{100}{99}$. □

By using Lemmas 5.5.4 and 5.5.6 we can express (5.97) in a different form, which is more suitable for our uniform estimates of $M^{\alpha}_{r,k}$.

Corollary 5.7.6. *The following estimate holds*

$$\begin{aligned}\tfrac{1}{2}\int_\Omega \varphi^4\, E_2^\alpha(\tau^\alpha_{r,k},\tau^\alpha_{r,k})\,dx + \tfrac{1}{2}\int_\Omega \varphi^4\, E_2^\alpha(M^\alpha_{r,k},M^\alpha_{r,k})\,dx \le\\ \le \left(\tfrac{1}{4}\cdot\tfrac{296}{99} + \tfrac{C}{N} + \omega_r(\alpha)\right)\int_\Omega \varphi^4\, M^\alpha_{r,k} : M^\alpha_{r,k}\,dx + \\ + \left(\tfrac{1}{4}\cdot\tfrac{100}{99} + \tfrac{C}{N}\right)\int_\Omega \varphi^4\, M^N_{r-1,k} : M^N_{r-1,k}\,dx + \frac{C}{N} + \omega_r(\alpha).\end{aligned} \tag{5.109}$$

Proof: We consider each of the domains $\Omega_i,\ i=1,\dots,4$, defined in (5.99). First, remark, that (5.68) yields

$$-E_2^\alpha(\tau^\alpha_{r,k},\tau^\alpha_{r,k}) \le -(1+\omega_r(\alpha))\, M^\alpha_{r,k} : M^\alpha_{r,k} \tag{5.110}$$

on $\Omega_1\cup\Omega_2$. We apply (5.97), dividing the integral over Ω into three integrals over the domains just defined.

$$\begin{aligned}\left(\tfrac{1}{2}-\tfrac{1}{N}\right)\int_\Omega \varphi^4\, E_2^\alpha(\tau^\alpha_{r,k},\tau^\alpha_{r,k})\,dx + \tfrac{1}{2}\int_\Omega \varphi^4\, E_2^\alpha(M^\alpha_{r,k},M^\alpha_{r,k})\,dx \le\\ \le \tfrac{1}{2}\int_\Omega \varphi^4\, E_2^\alpha(M^\alpha_{r,k},M^\alpha_{r,k})\,dx + \tfrac{1}{4}\cdot\left(\tfrac{100}{99}+\tfrac{C}{N}\right)\int_\Omega \varphi^4\, M^\alpha_{r,k} : M^\alpha_{r,k}\,dx - \\ -\tfrac{1}{2}\cdot\tfrac{1}{99}\int_{\Omega_1\cup\Omega_2\cup\Omega_3} \varphi^4\, E_2^\alpha(\tau^\alpha_{r,k},\tau^\alpha_{r,k})\,dx + \tfrac{1}{2}\sum_{s=1}^{9N-1}\frac{1}{s+10}\int_{F_s}\varphi^4 M^\alpha_{r,k} : M^\alpha_{r,k}\,dx + \\ +\tfrac{1}{4}\cdot\left(\tfrac{100}{99}+\tfrac{C}{N}\right)\int_\Omega \varphi^4\, M^N_{r-1,k} : M^N_{r-1,k}\,dx + \frac{C}{N} + \omega_r(\alpha).\end{aligned} \tag{5.111}$$

Estimates over $\Omega_1\cup\Omega_2$: By (5.67) and (5.110) the sum of the integrals over $\Omega_1\cup\Omega_2$ corresponding to the first three terms in (5.111) is bounded by

$$\begin{aligned}\left(\tfrac{1}{4}\cdot\tfrac{100}{99}+\tfrac{1}{2}-\tfrac{1}{2}\cdot\tfrac{1}{99}+\tfrac{C}{N}+\omega_r(\alpha)\right)\int_{\Omega_1\cup\Omega_2}\varphi^4\, M^\alpha_{r,k} : M^\alpha_{r,k}\,dx \le\\ \le \left(\tfrac{1}{4}\cdot\tfrac{296}{99}+\omega_r(\alpha)+\tfrac{C}{N}\right)\int_{\Omega_1\cup\Omega_2}\varphi^4\, M^\alpha_{r,k} : M^\alpha_{r,k}\,dx.\end{aligned} \tag{5.112}$$

Estimates over Ω_3: The integral over Ω_3 is estimated by considering the integrals over the sets F_s, defined in (5.97). Using (5.65), (5.68), the bounds

$$\tfrac{s+10}{s+1} < |\tau^\alpha_r| \le \tfrac{s+9}{s},$$

on each F_s, and the inequality

$$\tfrac{1}{2}\cdot\tfrac{s+1}{s+10} - \tfrac{1}{2}\cdot\tfrac{1}{99}\cdot\tfrac{s+10}{s+1} + \tfrac{1}{2}\cdot\tfrac{1}{s+10} < \tfrac{1}{2}\cdot\tfrac{98}{99}$$

valid for $s \in \mathbb{N}$, the sum of the integrals over F_s corresponding to the first four terms in (5.111) is bounded by

$$\begin{aligned} \sum_{s=1}^{9N-1} \Big[\Big(\tfrac{1}{4} \cdot \tfrac{100}{99} + \tfrac{1}{2} \cdot \tfrac{s+1}{s+10} - \tfrac{1}{2} \cdot \tfrac{1}{99} \cdot \tfrac{s+10}{s+1} + \tfrac{1}{2} \cdot \tfrac{1}{s+10} + \omega_r(\alpha) + \tfrac{C}{N} \Big) \Big] \cdot \\ \cdot \int_{F_s} \varphi^4 \, M^\alpha_{r,k} : M^\alpha_{r,k} \, dx \le \\ \le \big(\tfrac{1}{4} \cdot \tfrac{296}{99} + \tfrac{C}{N} + \omega_r(\alpha) \big) \int_{\Omega_3} \varphi^4 \, M^\alpha_{r,k} : M^\alpha_{r,k} \, dx. \end{aligned} \tag{5.113}$$

Estimates over Ω_4: By (5.65) and the lower bound $|\tau^\alpha_r| > 10$, the sum of the integrals over Ω_4 corresponding to the first three terms in (5.111) is bounded by

$$\begin{aligned} \big(\tfrac{1}{4} \cdot \tfrac{100}{99} + \tfrac{1}{20} + \omega_r(\alpha) + \tfrac{C}{N} \big) \int_{\Omega_4} \varphi^4 \, M^\alpha_{r,k} : M^\alpha_{r,k} \, dx \le \\ \le \big(\tfrac{1}{4} \cdot \tfrac{296}{99} + \omega_r(\alpha) + \tfrac{C}{N} \big) \int_{\Omega_4} \varphi^4 \, M^\alpha_{r,k} : M^\alpha_{r,k} \, dx. \end{aligned} \tag{5.114}$$

The claim now follows from (5.112)-(5.114). □

5.8 Uniform $W^{1,2}_{loc}$ estimates of approximate solutions

To carry out the proof of the uniform boundedness of $\|M_N\|_{L^\infty((0,T);W^{1,2}_{loc}(\Omega;\mathbb{M}^{2\times 2}_{sym}))}$ we will make use of the refined version of iterative estimate (5.75), deduced in the previous section, which results in a discrete analogue of Gronwall inequality. To this aim, we need to estimate the last term of (5.75). To make the estimates uniform, we will use the convergence of u^α_r to δu^N_r as in (5.46)-(5.48), and the convergence of M^α_r to M^N_r as in (5.84).

So, the goal of this section is to prove the following inequality first

$$\big(1 - \tfrac{C}{N}\big) \int_\Omega \varphi^4 M^N_{r,l} : M^N_{r,l} \, dx \le \big(1 + \tfrac{C}{N}\big) \int_\Omega \varphi^4 M^N_{r-1,l} : M^N_{r-1,l} \, dx + \frac{C}{N}, \tag{5.115}$$

with C independent of N, and then to deduce Theorem 5.2.2.

We begin as in (5.74), using (5.61) and (5.64):

$$\int_\Omega \varphi^4 M^\alpha_{r,k} : M^\alpha_{r,k} \, dx \le \tfrac{1}{2} \int_\Omega \varphi^4 E^\alpha_2(\tau^\alpha_{r,k}, \tau^\alpha_{r,k}) \, dx + \tfrac{1}{2} \int_\Omega \varphi^4 E^\alpha_2(M^\alpha_{r,k}, M^\alpha_{r,k}) \, dx.$$

Thus, (5.109) yields

$$\begin{aligned} \big(\tfrac{1}{4} \cdot \tfrac{100}{96} - \tfrac{C}{N} + \omega_r(\alpha) \big) \int_\Omega M^\alpha_{r,k} : M^\alpha_{r,k} \, dx \le \\ \le \tfrac{1}{4} \cdot \tfrac{100}{96} \int_\Omega \varphi^4 M^N_{r-1,k} : M^N_{r-1,k} \, dx + \\ + \tfrac{C}{N} \int_\Omega \varphi^4 M^N_{r,k} : M^N_{r,k} \, dx + \tfrac{C}{N} + \omega_r(\alpha). \end{aligned} \tag{5.116}$$

Now, to deduce (5.115) it remains to pass to the limit with respect to α in (5.116), to use (5.84) and the lower semicontinuity of the norm, and to sum the resulting expressions with respect to k.

Proof of Theorem 5.2.2: Iterating (5.115) we get the following for every $r = 1, \ldots, N$

$$\begin{aligned}\int_\Omega \varphi^4 M^N_{r,l} : M^N_{r,l}\, dx &\le \frac{\left(1+\frac{C}{N}\right)^N}{\left(1-\frac{C}{N}\right)^N} \int_\Omega \varphi^4 M_{0,l} : \sigma_{0,l}\, dx + \tfrac{2C}{N} \sum_{i=1}^N \frac{\left(1+\frac{C}{N}\right)^{i-1}}{\left(1-\frac{C}{N}\right)^i} \le \\ &\le e^{2C} \int_\Omega \varphi^4 M_{0,l} : M_{0,l}\, dx + 2C\, e^{2C}.\end{aligned} \tag{5.117}$$

Thus, we obtain

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \|M_N(t)\|_{1,2;\Omega'} \le C(\Omega'),$$

and the conclusion follows from convergence of $M_N(t) \rightharpoonup M(t)$ in $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ for every $t \in [0,T]$.
□

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