Scuola Internazionale Superiore di Studi Avanzati - Trieste

Supersymmetric Solutions in AdS/CFT

CANDIDATE:
Giuseppe Milanesi

SUPERVISOR:
Dott. Martin O'Loughlin

SISSA - Via Beirut 2-4 - 34014 TRIESTE - ITALY

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Chapter 1

Introduction

At present, microscopical physics is well described by Quantum Field Theory. In the Standard Model of particle physics, three of the four fundamental forces of nature are described by the dynamics of quantum fields. The fourth force, gravity, at the subatomic scales is the weakest one and does not fit into the same quantum description. One fundamental feature of Quantum Field Theory is renormalisation: the quantisation of a classical field theory leads to the appearance of divergences; if these divergence are such that they can be reabsorbed into a redefinition of a finite number of parameters of the theory, the theory is called renormalisable. If a Quantum Field Theory has to be a candidate for a fundamental theory, it clearly has to be renormalisable. The naive quantisation of the classical dynamics of gravity, which is best described by General Relativity, unfortunately does not lead to a renormalisable theory. This is one of the reasons why the Standard Model of particle physics is not believed to be a fundamental theory.

One of the key ingredients of Quantum Field Theory is the emergence of particles from the fields. In the absence of gravitational interactions, we may set the theory in a flat Minkowski background. The Hilbert space of a particle carries the smallest possible representation of the smallest symmetry group of the theory: irreducible representations of the Poincare group. Particles emerge naturally out of quantum fields in flat spacetime and viceversa: quantum fields emerge naturally as the the description of the interactions of point particles.

The physical origin of the divergences of quantum field theory resides in the local pointlike interaction between fields. A natural way to avoid this kind of pathologies is to think of point particles as a non fundamental but derived concept. The simplest way of doing this is to consider extended one dimensional objects, strings, as fundamental objects instead of point particles [1, 2]. The quantum relativistic description of a string, a very simple problem at a first sight, leads surprisingly to a huge amount of physical consequences. Although this is not the reason why String Theory was originally introduced, this is the modern perspective we have on it.

The so called bosonic string has only three fundamental ingredients from which an extremely rich variety of consequences can be drawn: strings are fundamental objects, special relativity and quantum mechanics. The dynamics of quantum relativistic strings contains pointlike particles and quantum fields as oscillation modes of the strings but probably its most exciting feature is that it contains also a consistent description of the quantum dynamics of gravity, namely a quantisation of General Relativity. The propagation of bosonic strings is free of quantum anomalies only in 26 spacetime dimensions. However, bosonic strings have several undesired features: most of all they have a tachyonic instability

In order to get a consistent theory of strings a fourth ingredient is extremely helpful: supersymmetry. Supersymmetry is an enlargement of the symmetry group of spacetime obtained by a grading of the Poincare' algebra. In terms of particle and fields is a symmetry relating bosonic and fermionic degrees of freedom.

Supersymmetric String theories are non anomalous only in 10 spacetime dimensions. The four dimensional physics is usually recovered through reduction over a compact six dimensional manifold (compactification).

As we have anticipated, historically, the motivations we have given so far were not the ones which lead to the discovery of string theories. In the 1960's in order to explain the so called Regge trajectory of the lightest hadrons a string model was proposed. Although later it was realised that the fundamental dynamics of strong interactions was better described by a non abelian gauge field theory, QCD, still the connection between strong interactions and String thoeries has remained silent but present troughout the years and found a new impulse of interest in the last ten years or so thanks to the so called AdS/CFT correspondence.

At low energies, QCD becomes strongly coupled and it is not easy to perform calculations. One possible approach is to use numerical simulations on the lattice. It was suggested by 't Hooft [3] that the theory simplifies in the limit of $N = \infty$ and then one could perform a $1/N$ expansion down to $N = 3$. Furthermore, the diagrammatic expansion of the field theory suggests that the large N theory is indeed a string theory. The argument is very general and can be applied for basically any gauge field theory. The resulting string models are four dimensional. As we have already said, strings cannot be quantum mechanically consistent in four dimensions: a scalar field must be introduced in this model. However, since any string theory contains gravitational physics, the introduction of a further field effectively makes the theory five dimensional. As we will see, internal symmetries of the original gauge theory leads us to introduce further dimensions and a correspondence to quantum consistent 10 dimensional strings can be constructed for several supersymmetric gauge theories.

Among the a priori unexpected features of String theory there are the so called D-branes. D-branes are extended dynamical objects of the String theory which can be zero up to nine dimensional. They are solitons of the string theory [4] and in fact their tension is proportional to the inverse of the string coupling g_s.

D-branes are defined in string perturbation theory in a very simple way: they are surfaces where open strings can end. These open strings have some massless modes, which describe the oscillations of the branes, a gauge field living on the brane, and their fermionic partners. If we have N coincident branes the open strings can start and end on different branes, so they carry two indices that run from one to N. This in turn implies that the low energy dynamics is described by a $U(N)$ gauge theory. D-p-branes are charged under $p+1$-form gauge potentials, in the same way that a 0-brane (particle) can be charged under a one-form gauge potential (as in electromagnetism). These $p+1$-form gauge potentials have $p+2$-form field strengths, and they are part of the massless closed string modes, which belong to the supergravity (SUGRA) multiplet containing the massless fields in flat space string theory (before we put in any D-branes). If we now add D-branes they generate a flux of the corresponding field strength, and this flux in turn contributes to the stress energy tensor so the geometry becomes curved. Indeed it is possible to find solutions of the supergravity equations carrying these fluxes. Supergravity is the low-energy limit of string theory, and it is believed that these solutions may be extended to solutions of the full string theory. These solutions are very similar to extremal charged black hole solutions in general relativity, except that in this case they are black branes with p extended spatial dimensions. Like black holes they contain event horizons.

If we consider a set of N coincident D-3-branes the near horizon geometry turns out to be $AdS_5 \times S^5$. On the other hand, the low energy dynamics on their worldvolume is governed by a $U(N)$ gauge theory with $\mathcal{N} = 4$ supersymmetry [5]. These two pictures of D-branes are perturbatively valid for different regimes in the space of possible coupling constants. Perturbative field theory is valid when $g_s N$ is small, while the low-energy gravitational description is perturbatively valid when the radius of curvature is much larger than the string scale, which turns out to imply that $g_s N$ should be very large. As an object is brought closer and closer to the black brane horizon its energy measured by an outside observer is redshifted, due to the large gravitational potential, and the energy seems to be very small. On the other hand low energy excitations on the branes are governed by the Yang-Mills theory. So, it becomes natural to conjecture that Yang-Mills theory at strong coupling is describing the near horizon region of the black brane, whose geometry is $AdS_5 \times S^5$. The first indications that this is the case came from calculations of low energy graviton absorption cross sections [6, 7, 8]. It was noticed there that the calculation done using gravity and the calculation done using super Yang-Mills theory agreed.

$\mathcal{N} = 4$ supersymmetric theories in four dimensions are maximally supersymmetric. Any extension of the supersymmetry algebra would lead to gravitational multiplets. The high level of supersymmetry highly constrains the theory and make it a Conformal Field Theory.

The radius of curvature of Anti-de Sitter space depends on N so that large N corresponds to a large radius of curvature. Thus, by taking N to be large we can make the curvature as small as we want. The theory in AdS includes gravity, since any string theory includes gravity. So in the end we claim that there is an equivalence between a gravitational theory and a field theory. However, the mapping between the gravitational and field theory degrees of freedom is quite non-trivial since the field theory lives in a lower dimension. In some sense the field theory (or at least the set of local observables in the field theory) lives on the boundary of spacetime. One could argue that in general any quantum gravity theory in AdS defines a conformal field theory (CFT) "on the boundary".

Notice that when we say that the theory includes "gravity on AdS" we are considering any finite energy excitation, even black holes in AdS. So this is really a sum over all spacetimes that are asymptotic to AdS at the boundary. This is analogous to the usual flat space discussion of quantum gravity, where asymptotic flatness is required, but the spacetime could have any topology as long as it is asymptotically flat.

The fact that the field theory lives in a lower dimensional space blends in perfectly with some previous speculations about quantum gravity. It was suggested [9, 10] that quantum gravity theories should be holographic, in the sense that physics in some region can be described by a theory at the boundary with no more than one degree of freedom per Planck area. This "holographic" principle comes from thinking about the Bekenstein bound which states that the maximum amount of entropy in some region is given by the area of the region in Planck units [11]. The reason for this bound is that otherwise black hole formation could violate the second law of thermodynamics. We will see that the correspondence between field theories and string theory on AdS space (including gravity) is a concrete realisation of this holographic principle.

Plan of the work

In this Thesis we address the study of several supersymmetric Sectors of the AdS/CFT correspondence. Most of the work presented here is inspired by a beautiful paper [12], in which the correspondence between a class of half BPS operators in the CFT and asymptotically $AdS_5 \times S^5$ geometries was explicitly constructed.

In the next Chapter we introduce in more details the AdS/CFT correspondence. Most of the material presented there is well established (useful reviews are [13, 14, 15]) but some Sections contains original derivations and observations. In Chapter 3 we present the study of a half BPS Sector of the correspondence. The discussion of Section 3.1 has several original points. Section 3.3 contains original yet unpublished material. Chapter 4 basically contains the results published in [16] about regularity of the half BPS geometries that were discussed in the previous Chapter. Chapter 5 contains the extension of the cases studied to less supersymmetric Sectors of the standard AdS/CFT correspondence which were studied in [17] while Chapter 6 contains a construction similar to the one described in Chapter 3 for the less supersymmetric version of the correspondence which relates $\mathcal{N} = 1$ theories to String theory on BPS supergravity solutions [18]. In Chapter 7 we present our conclusive remarks. The Appendix contains a summary of our notations and conventions.

Chapter 2

AdS/CFT correspondence

2.1 Black 3-branes

In this Section we give a description of 3-branes in terms of solutions to the supergravity equations of motion.

2.1.1 Lagrangian and equations of motion

Solutions of type IIB Supergravity which describe p-dimensional extended objects are known. The bosonic part of the action of type IIB Supergravity is[1]

$$S = \frac{1}{2\kappa_{10}^2}\int \sqrt{g}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right) + \\ - \frac{1}{2\kappa_{10}^2}\int\left[\sqrt{g}\left(\frac{1}{2}|F_1|^2 + \frac{1}{2}|\tilde{F}_3|^2 + 4|\tilde{F}_5|^2\right) + 2A_4\wedge H_3\wedge F_3\right] \quad (2.1.1)$$

where the field strengths are defined by

$$\begin{cases} F_1 = \mathrm{d}A_0 \\ H_3 = \mathrm{d}B_2 \\ \tilde{F}_3 = \mathrm{d}A_2 - 4A_0 H_3 \\ \tilde{F}_5 = \mathrm{d}A_4 - \frac{1}{2}A_3\wedge H_2 + \frac{1}{8}B_2\wedge F_3 \end{cases} . \quad (2.1.2)$$

The self-duality condition $\tilde{F}_5 = \star_{10}F_5$ has to be imposed on the equations of motion and not at the level of the Lagrangian. The ten dimensional Newton constant is related to the constant κ_{10} and the string length as

$$16\pi G_{10} = 2\kappa_{10}^2 g_s^2 = (2\pi)^7 l_s^8 g_s^2 . \quad (2.1.3)$$

[1]We use the notation $g = \sqrt{-\det g_{MN}}$ and $F_n = \frac{1}{n!}F_{M_1\cdots M_n}\mathrm{d}x^{M_1}\wedge\cdots\wedge\mathrm{d}x^{M_n}$ with $|F_n|^2 = \frac{1}{n!}F_{M_1\cdots M_n}F^{M_1\cdots M_n}$. See Appendix A

Generic p-branes solutions can be constructed for $p = 1, 3, 5, 7$. A p brane is an electric source for the $p+2$-form while a $6-p$-brane is a magnetic source for the same $p+2$-form. We will focus here on the configuration of N coincident extremal black flat 3-branes.

The only non trivial fields appearing in such a setting are the metric and the RR 5-form. For this reason we have

$$\tilde{F}_5 = F_5 = \mathrm{d}A_4 \,. \tag{2.1.4}$$

The number N of branes is calculated via the Dirac quantisation condition, which, with our normalisations, take the form[2].

$$\int_{\Sigma_5} F_5 = 4\pi^2\alpha'^2 N \tag{2.1.5}$$

where Σ_5 is any five dimensional cycle transverse to the worldvolume of the branes.
The ten dimensional metric in the string frame is given by

$$\mathrm{d}s^2 = H(r)^{-1/2}\mathrm{d}x^\mu \mathrm{d}x_\mu + H(r)^{1/2}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2\right) \tag{2.1.6}$$

where $\mathrm{d}x^\mu\mathrm{d}x_\mu$ is the flat Minkowski four dimensional metric and $\mathrm{d}\Omega_5^2$ is the metric on the unit radius five dimensional sphere. The function $H(r)$ is given by

$$H(r) = 1 + \frac{L^4}{r^4} \tag{2.1.7}$$

while the five form has the expression

$$\begin{aligned} g_s F_5 &= \mathcal{F}_5 + \star_{10}\mathcal{F}_5 \\ \mathcal{F}_5 &= \frac{1}{4}\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}H^{-1} \\ \star_{10}\mathcal{F}_5 &= -\frac{r^5}{4}\frac{\mathrm{d}H}{\mathrm{d}r}\mathrm{d}\Omega_5 \end{aligned} \tag{2.1.8}$$

where $\mathrm{d}\Omega_5$ is the volume form on the unit radius round 5-sphere.
It is not difficult to verify that the equations of motion

$$R_{MN} = \frac{g_s^2}{6} F_{MM_1M_2M_3M_4} F_N{}^{N_1N_2N_3N_4} \tag{2.1.9}$$

are satisfied.
Noticing that

$$r^5\frac{\mathrm{d}H}{\mathrm{d}r} = -4L^4 \tag{2.1.10}$$

we can conclude, from the Dirac quantisation condition (2.1.5) and the expression for $\mathcal{F}_5$:

$$L^4 = 4\pi^2 g_s \alpha'^2 N \tag{2.1.11}$$

[2] The normalisation for F_5 used troughout this thesis differs for a factor of $1/4$ with respect to the most commonly used

Although the metric may look singular at small r, indeed it is not and have a notable regular limit. Let us expand the metric at leading order for $r \to 0$

$$\mathrm{d}s^2 = \frac{r^2}{L^2}\left(\mathrm{d}x^\mu \mathrm{d}x_\mu\right) + \frac{L^2}{r^2}\mathrm{d}r^2 + L^2 \mathrm{d}\Omega_5 \tag{2.1.12}$$

which is the metric of $AdS_5 \times S^5$ where both the factors have radius L. In the same limit, the 5-form has the expression

$$F_5 = \frac{1}{L} Vol_L\left(AdS_5\right) + \frac{1}{L} Vol_L\left(S^5\right) = L^4 Vol\left(AdS_5\right) + L^4 Vol\left(S^5\right) \tag{2.1.13}$$

For later convenience, we will now absorb g_s into the definition of F_5:

$$g_s F_5 \to F_5 \tag{2.1.14}$$

2.1.2 Supersymmetry variation

For a background in which the only non trivial fields are the metric and the five form, the only non vanishing supersymmetry variation is the variation of the gravitino which has the form

$$\begin{aligned}\delta\chi_M &= \nabla_M \psi + \frac{\mathrm{i}}{480} F_{M_1 M_2 M_3 M_4 M_5} \Gamma^{M_1 M_2 M_3 M_4 M_5} \Gamma_M \psi = \\ &= \nabla_M \psi + \frac{\mathrm{i}}{48} F_{M M_2 M_3 M_4 M_5} \Gamma^{M_2 M_3 M_4 M_5} \psi \end{aligned} \tag{2.1.15}$$

where the two 32 component Majorana-Weyl spinor supersymmetry parameters of the IIB theory are grouped into a single complex Weyl spinor ψ obeying the chirality constraint

$$\Gamma_{11}\psi = \psi \tag{2.1.16}$$

The covariant derivative $\nabla_M \psi$ is given by

$$\nabla_M \psi = \partial_M \psi + \frac{1}{4}\omega_{ABM}\Gamma^A \Gamma^B \psi\,, \tag{2.1.17}$$

where the torsionless spin connection 1-form $\omega^A{}_{B\,M}\mathrm{d}x^M$ is defined through the 10-bein $e^A{}_M \mathrm{d}x^M$ by the structure equations

$$\mathrm{d}e^A + \omega^A{}_B \wedge e^B = 0 \tag{2.1.18}$$

and the metricity condition

$$\omega_{AB} = -\omega_{AB}\,. \tag{2.1.19}$$

A supersymmetry parameter which satisfies the equation

$$\delta\chi_M = 0 \tag{2.1.20}$$

is called a Killing spinor. The background describing the black 3-brane has 16 linear independent Killing spinors. This means that it preserves half of the supersymmetries of the theory and thus is denoted as 1/2 BPS.

2.1.3 Supersymmetry and equations of motion

The existence of a Killing spinor ψ constrains the form of the background. In particular, integrability of the equation

$$\mathcal{D}_M\psi = \nabla_M\psi + \frac{\mathrm{i}}{480}F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\Gamma_M\psi = 0 \tag{2.1.21}$$

requires the vanishing of the mixed derivatives

$$\mathcal{R}_{MN}\psi = [\mathcal{D}_M, \mathcal{D}_N]\,\psi = 0 \tag{2.1.22}$$

The commutator of the covariant derivative can be found in [19]. Contraction with three gamma matrices gives

$$\Gamma_P{}^{MN}[\mathcal{D}_M, \mathcal{D}_N]\psi = E_{PM}\Gamma^M\psi + \frac{\mathrm{i}}{3}\Gamma^{M_1M_2M_3}\nabla^{M_4}F_{PM_1M_2M_3M_4}\psi = 0 \tag{2.1.23}$$

where the symmetric tensor E_{MN} gives the equations of motion

$$E_{MN} = R_{MN} - \frac{1}{2}g_{MN}R - \frac{1}{6}F_{MM_1M_2M_3M_4}F_N^{NM_1M_2M_3M_4}\,. \tag{2.1.24}$$

Due to the selfduality of the five form we have

$$\nabla\cdot F_5 = \star_{10}\mathrm{d}\star_{10}F_5 = \star_{10}\mathrm{d}F_5 = 0 \tag{2.1.25}$$

which states that the equations of motion for F_5 coincide with the Bianchi identity. Provided the latter are satisfied, the commutator of the derivatves gives

$$E_{MN}\Gamma^N\psi = 0 \tag{2.1.26}$$

We define an orthogonal frame

$$\mathrm{d}s^2 = 2e^+e^- + \sum_{a=1}^{8}(e^a)^2 \tag{2.1.27}$$

Let $A, B, \ldots$ denote tend dimensional tangent space indices and $a, b, \ldots$ tangent space indices in the positive definite transverse subspace, thus A can take values $+, -, a$. Let E_A be the dual vector fields[3] to e^A defined by $e^A(E_B) = \delta^A_B$.

Following [20], we define the vector $\kappa = \bar{\psi}\Gamma^M\psi\partial_M$. Remembering [21] that, for any type IIB Killing spinor χ the vector κ is null

$$\kappa^2 = 0\,, \tag{2.1.28}$$

[3]Pay attention to the possible confusion between the tensor E_{MN} and the vector fields E_A which are very different objects!

we set

$$\kappa = E_- \quad \Rightarrow \quad e^+ = \kappa_M \mathrm{d}x^M \, . \tag{2.1.29}$$

Going to the tangent frame, the contraction of (2.1.26) with $\bar{\chi}$ on the left gives

$$0 = E_{AB}\kappa^B = E_{A-} \tag{2.1.30}$$

which amounts to 10 of the 55 equations of motion.

On the other hand, if we hit (2.1.26) with $E_{AC}\Gamma^C$ we conclude that

$$E_{AB}E_A^B = 0 \quad \text{no sum on} \, A \, . \tag{2.1.31}$$

Being $E_{A-} = E_A^+ = 0$ this gives

$$0 = E_{AB}E_A^B = E_{Ab}E_A^b \quad \Rightarrow \quad E_{+b} = E_{ab} = 0 \, . \tag{2.1.32}$$

which give other 44 components of the Einstein equation. Thus, the existence of a single Killing spinor and the Bianchi identity for F_5 are sufficient to guarantee that all but one the Einstein equations are satisfied. The only equation of motion which is not implied by the integrability conditions is the E_{++} component.

Assume now that there is a second Killing spinor ψ' such that the bilinear

$$\kappa' = \bar{\psi}'\Gamma^M\psi'\partial_M \tag{2.1.33}$$

is null and not parallel to κ. Then the scalar product is not vanishing

$$\kappa' \cdot \kappa = \gamma(x^M) \neq 0 \tag{2.1.34}$$

and we can choose

$$E_+ = \gamma^{-1}\kappa' \quad \Rightarrow \quad e^- = \gamma^{-1}\kappa'_M \mathrm{d}x^M \, . \tag{2.1.35}$$

We can consider the equation in (2.1.26) with ψ' instead of ψ and contract it with $\bar{\psi}'$ on the left to obtain

$$0 = E_{AB}\bar{\chi}'\Gamma^B\chi' = E_{A+} \tag{2.1.36}$$

which gives in particular the last missing equation

$$E_{++} = 0 \, . \tag{2.1.37}$$

The existence of enough Killing spinors on a specific background is thus sufficient to guarantee that the background satisfies all of the Einstein equations, provided that the Bianchi identities (and thus the equations of motion) for the self dual 5-form are satisfied.

2.2 $\mathcal{N} = 4$ Super Yang - Mills

In this Section we will review some properties of $\mathcal{N} = 4$ $SU(N)$ superconformal gauge field theories which will be useful in the following. We will mainly refer to the well known reviews [14, 15].

2.2.1 Lagrangian and conformal symmetry

The Lagrangian of $\mathcal{N}=4$ $SU(N)$ superconformal gauge field theories is given by

$$\mathcal{L}=\mathrm{Tr}\Big\{\frac{\theta_I}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}-\frac{1}{2g_{YM}^2}\Big(F_{\mu\nu}F^{\mu\nu}+\sum_{i=1}^{6}D_\mu X^i D^\mu X^i+\mathrm{i}\sum_{a=1}^{4}\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda+ \\ -\sum_{a,b,i}C_i^{ab}\lambda_a\left[X^i,\lambda_b\right]-\sum\bar{C}_{iab}\bar{\lambda}^a\left[X^i,\bar{\lambda}^b\right]-\frac{1}{2}\sum_{i,j}\left[X^i,X^j\right]^2\Big)\Big\} \tag{2.2.1}$$

where the constants C_i^{ab} and $\bar{C}_{iab}$ are related to the Clebsch Gordan coefficients in the $SU(4)\cong SO(6)$ R- symmetry group and the reduction of the 10 dimensional Clifford algebra to 4 dimensions. By construction the Lagrangian is invariant under $\mathcal{N}=4$ Poincare supersymmetry, which has 16 generators

$$Q^a_\alpha,\bar{Q}_{a\dot{\alpha}} \qquad a=1,\ldots,4 \qquad \alpha,\dot{\alpha}=\pm 1 \tag{2.2.2}$$

acting as follows [22, 23]:

$$\begin{aligned}\left[Q^a_\alpha,X^i\right]&=C^{iab}\lambda_{\alpha b}\\ \left\{Q^a_\alpha,\lambda^\beta_b\right\}&=F_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta\delta^a_b+\delta^\beta_\alpha[X^i,X^j](C_{ij})^a_b\\ \left\{Q^a_\alpha,\bar{\lambda}_{\dot{\beta}b}\right\}&=\sigma^\mu_{\alpha\dot{\beta}}D_\mu X^i C^{ab}_i\\ \left[Q^a_\alpha,A_\mu\right]&=(\sigma_\mu)_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}}_a\end{aligned} \tag{2.2.3}$$

and their complex conjugate relations.

The transformation of a *conformal field* ϕ under scaling of the coordinates such as

$$x_\mu\to x'_\mu=\lambda^{-1}x_\mu \tag{2.2.4}$$

is specified by an assignment of *conformal dimension* $[\phi]$ as

$$\phi'(x)=\lambda^{[\phi]}\phi(\lambda x)\,. \tag{2.2.5}$$

The Lagrangian (2.2.1) is naively scale invariant assuming that the fields and the parameters appearing in it transform as conformal fields with the following conformal dimensions:

$$[A_\mu]=[X^i]=1 \qquad [\lambda_a]=\frac{3}{2} \qquad [g]=[\theta_I]=0 \tag{2.2.6}$$

The scale symmetry extends to the whole conformal group. Indeed the lagrangian is invariant for any coordinate transformation generated by the infinitesimal transformations

$$\delta x^\mu=x^\mu+v^\mu \tag{2.2.7}$$

with

$$
\begin{aligned}
v_\mu &= a_\mu \\
v_\mu &= \omega_{\mu\nu} x^\nu \quad , \quad \omega_{\mu\nu} = -\omega_{\nu\mu} \\
v_\mu &= \alpha x^\mu \\
v_\mu &= 2 c_\rho x^\rho x_\mu - x^2 c_\mu
\end{aligned}
\tag{2.2.8}
$$

The first two lines represent the generators of the Poincare group, translations, rotations and boosts, the third line represents simple rescaling and the last line shows the so called special conformal transformations.

The action of the conformal algebra on a local conformal field is given by

$$
\begin{aligned}
[P_\mu, \phi(x)] &= \mathrm{i}\partial_\mu \phi(x) \\
[M_{\mu\nu}, \phi(x)] &= \mathrm{i}\left[(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}\right] \phi(x) \\
[D, \phi(x)] &= \mathrm{i}\left(-\Delta + x^\mu \partial_\mu\right) \phi(x) \\
[K_\mu, \phi(x)] &= \left[\mathrm{i}\left(x^2 \partial_\mu - 2 x_\mu x^\nu \partial_\nu + 2 x_\mu \Delta\right) - 2 x^\nu \Sigma_{\mu\nu}\right] \phi(x)
\end{aligned}
\tag{2.2.9}
$$

where $\Sigma_{\mu\nu}$ is the representation of the Lorentz group which acts on the (here suppressed) Lorentz indices of the field $\phi(x)$. One can see that the algebra of the 15 generators above can be reorganised to give the generators J_{IJ} of the isometry group $SO(2,4)$

$$
J_{\mu\nu} = M_{\mu\nu} \qquad J_{\mu 4} = \frac{1}{2}(K_\mu - P_\mu) \qquad J_{\mu 5} = \frac{1}{2}(K_\mu + P_\mu) \qquad J_{45} = D \tag{2.2.10}
$$

which satisfy the commutation relations

$$
[J_{MN}, J_{IJ}] = \mathrm{i}\left(\eta_{NI} J_{MJ} - \eta_{NJ} J_{MI} - \eta_{MI} J_{NJ} + \eta_{MJ} J_{NI}\right) \tag{2.2.11}
$$

where η has signature $(-,+,+,+,+,-)$. Notice in particular that the generators of the maximally compact subgroup $SO(2) \times SO(4)$ of $SO(2,4)$ are given by

$$
J_{05} = \frac{1}{2}(K_0 + P_0) \tag{2.2.12}
$$

and

$$
J_{ij} = M_{ij} \qquad J_{i4} = \frac{1}{2}(K_i - P_i) \qquad i,j = 1,2,3 \tag{2.2.13}
$$

To these bosonic generators one has to add the 16 supersymmetry generators $Q^a_\alpha, \bar{Q}^a_{\dot{\alpha}}$. The inclusion of both conformal invariance and supersymmetry requires the introduction of a further set of symmetry generators $S^a_\alpha, \bar{S}_{a\dot{\alpha}}$ which are known as the superconformal generators. Their origin is due to the fact that the supercharges do not commute with the special conformal generators and the commutator is not a linear combination of the other generators. We give here a schematic description of the commutation rules. More precise expressions can be found in *e.g.* [24, 15, 13]. We first analyse the commutator of the dilatation operator with the other generators:

$$
[D, P] = -\mathrm{i}P \qquad [D, K] = \mathrm{i}K \qquad [D, M] = 0 \tag{2.2.14}
$$

These relations state that that P has $\Delta = 1$ and is a raising operator for the scaling generator D while K has $\Delta = -1$ and is a lowering operator and M commutes with D. This is not hard to understand looking at the action of the symmetry generators on a conformal field as in (2.2.9): P is a derivative operator, while K has two factors of x and one derivative and M has one factor of x and one derivative. Being D real and recalling from standard supersymmetry that $[Q, \bar{Q}]_+ = P$ it is not hard to see that

$$[D, Q] = -\frac{\mathrm{i}}{2} Q \qquad [D, \bar{Q}] = -\frac{\mathrm{i}}{2} \bar{Q} \,. \tag{2.2.15}$$

As we said, the generators S are defined by

$$[K, Q] = S \qquad [K, \bar{Q}] = \bar{S} \tag{2.2.16}$$

which implies

$$[D, S] = \frac{\mathrm{i}}{2} S \qquad [D, \bar{S}] = \frac{\mathrm{i}}{2} \bar{S} \tag{2.2.17}$$

the S generators act as a second set of lowering operators.

In analogy to the relation between supercharges and momentum the anticommutator of the superconformal generators in proportional to the special conformal transformation

$$\left[S, \bar{S}\right] = K \,. \tag{2.2.18}$$

In a Hilbert space which carries a unitary representation of the conformal algebra, cannot exist operators with negative scaling dimension [25, 26]. As such, in any multiplet obtained by acting on a conformal field by elements of the algebra there must be a lowest dimension operator $\mathcal{O}$ which is called a *conformal primary* field. In the conformal case (not superconformal) this is achieved by requiring that

$$[K, \mathcal{O}] = 0 \qquad \text{for}\, x = 0 \,.. \tag{2.2.19}$$

In the superconformal case we we have a further set of lowering operators given by the $S, \bar{S}$ generators. Being the conformal group a subgroup of the superconformal one, in order to have a unitary representation of the superconformal group there must be in the multiplet an operator $\mathcal{O}$ which commutes with the generators S in the origin

$$[S, \mathcal{O}]_\pm = [\bar{S}, \mathcal{O}]_\pm = 0 \,. \tag{2.2.20}$$

where with the index $\pm$ we have included the possibility that such operator is fermionic and thus the grading of the superconformal algebra requires an anticommutator to appear. An operator of this type is called a *superconformal primary* operator. It is clear from the commutation relations $\left\{S, \bar{S}\right\} = K$ and the Jacobi identity, that a superconformal primary operator is also a conformal primary operator.

Another interesting relation is the anticommutator of the Q and S generators

$$[Q, S]_+ = D + M + R \tag{2.2.21}$$

where R represents generators of the R-symmetry group $SO(6)$. A relevant subset of the superconformal primary operators is given by the so called *chiral primary* operators: they commute with some of the supersymmetry generators Q. As such they satisfy

$$[D + M + R, \mathcal{O}] = 0 \tag{2.2.22}$$

which means that their scaling dimension is determined by their R symmetry and Lorentz transformation properties.

2.2.2 Compactification and state operator correspondence

As we will see, one of the key ingredients of the AdS/CFT correspondence is the identification of the isometry group of AdS_5 with the conformal group of $\mathbb{R}^{1,3}$. Therefore it would be useful to review the conformal structure of $\mathbb{R}^{1,3}$ and consider the action of the conformal group under a new perspective.

The metric on the Minkowski space in in 4 dimensions is expressed in polar coordinates as:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2, \tag{2.2.23}$$

where $d\Omega_2$ is the metric on the unit sphere S^2. We can perform a series of coordinate transformation and bring the metric in the following form:

$$\begin{aligned} ds^2 &= -du_+ du_- + \frac{1}{4}(u_+ - u_-)^2 d\Omega_2^2\,, \qquad (u_\pm = t \pm r) \\ &= \frac{1}{\cos^2\tilde{u}_+ \cos^2\tilde{u}_-}\left(-d\tilde{u}_+ d\tilde{u}_- + \frac{1}{4}\sin^2(\tilde{u}_+ - \tilde{u}_-) d\Omega_2^2\right)\,, \qquad (u_\pm = \tan\tilde{u}_\pm) \\ &= \frac{1}{4\cos^2\tilde{u}_+ \cos^2\tilde{u}_-}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_2^2), \qquad (\tilde{u}_\pm = (\tau \pm \theta)/2). \end{aligned} \tag{2.2.24}$$

As shown in figure 2.1, the (t, r) half-plane (for a fixed point on S^2) is mapped into a triangular region in the (τ, θ) plane. The conformally scaled metric

$$ds'^2 = -d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_2^2 \tag{2.2.25}$$

can be analytically continued outside of the triangle, and the maximally extended space with

$$0 \leq \theta \leq \pi, \quad -\infty < \tau < +\infty, \tag{2.2.26}$$

has the geometry of $\mathbb{R} \times S^3$ (Einstein static universe), where $\theta = 0$ and π corresponds to the north and south poles of S^3.

Since

$$\frac{\partial}{\partial\tau} = \frac{1}{2}(1 + u_+^2)\frac{\partial}{\partial u_+} + \frac{1}{2}(1 + u_-^2)\frac{\partial}{\partial u_-}, \tag{2.2.27}$$

the generator H of the global time translation on $\mathbb{R} \times S^3$ is identified with the linear combination

$$H = \frac{1}{2}(P_0 + K_0) = J_{05}, \tag{2.2.28}$$

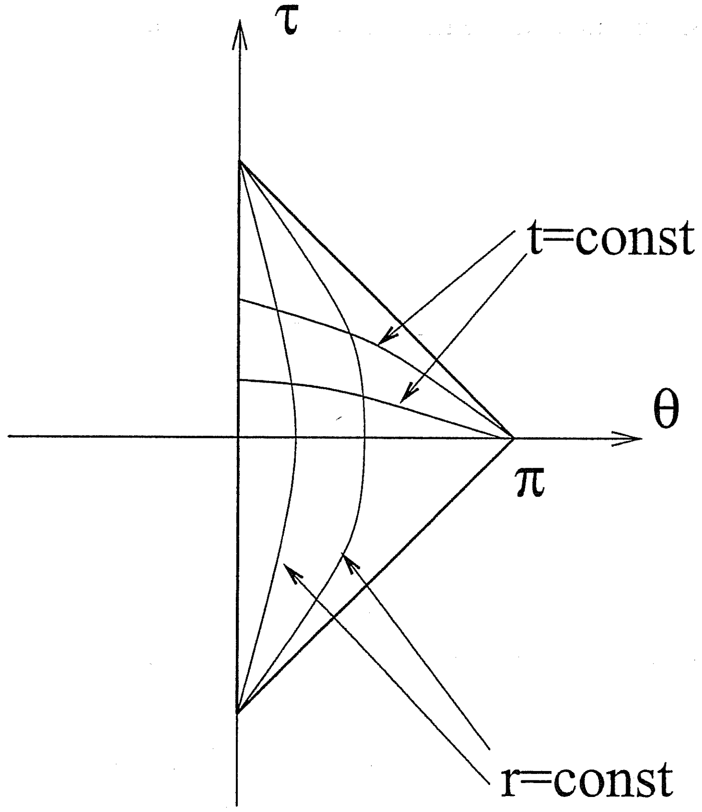


Figure 2.1: The conformal transformation maps the (t, r) half plane into a triangular region in the (τ, θ) plane.

where P_0 and K_0 are given by

$$P_0 : \frac{1}{2}\left(\frac{\partial}{\partial u_+} + \frac{\partial}{\partial u_-}\right), \quad K_0 : \frac{1}{2}\left(u_+^2 \frac{\partial}{\partial u_+} + u_-^2 \frac{\partial}{\partial u_-}\right) \tag{2.2.29}$$

on $\mathbb{R}^{1,3}$ defined in the previous Section. As we have already noticed, the generator $H = J_{05}$ corresponds to the $SO(2)$ part of the maximally compact subgroup $SO(2) \times SO(4)$ of $SO(2,4)$. Thus the subgroup $SO(2) \times SO(4)$ (or to be precise its universal cover) of the conformal group $SO(2,4)$ can be identified with the isometry of the Einstein static universe $\mathbb{R} \times S^3$. The existence of the generator H also guarantees that a correlation function of a CFT on $\mathbb{R}^{1,3}$ can be analytically extended to the entire Einstein static universe $\mathbb{R} \times S^3$.

Notice that it is only in its realisation on $\mathbb{R} \times S^3$ that the full superconformal group (and not just its algebra) has a unitary representation [27, 28]. This is important for the AdS/CFT correspondence in which the conformal group is mapped to the isometry of AdS_5: since AdS_5 is the background of the String theory dual to the CFT, the Hilbert space of the String theory carries a unitary representation of the full conformal group [29];

in order to establish the duality, the Hilbert space of the dual conformal field theory should also carry a unitary representation of the full conformal group.

Another interesting feature of the formulation of the CFT on $\mathbb{R} \times S^3$ is that it doesn't have vacuum degeneracy. From the lagrangian (2.2.1) one can easily derive the potential for the scalars

$$V \propto [X^i, X^j][X_i, X_j]. \tag{2.2.30}$$

Classical vacua are thus specified by the vanishing of the commutators

$$[X^i, X^j] = 0\,. \tag{2.2.31}$$

We can distinguish two classes of solution to this equations

- The *superconformal phase* in which all the X^i vanish. The full gauge algebra and the superconformal are algebra are unbroken.
- The *Coulomb* phase in which some of the X^i don't vanish. The gauge algebra can be broken up to $U(1)^N$ and the superconformal symmetry is also spontaneously broken.

When the theory is formulated on $\mathbb{R} \times S^3$ the six scalars are conformally couple to the curvature of the background and the potential acquire a term

$$V \propto \mathcal{R}\mathrm{Tr}X^2 \tag{2.2.32}$$

where $\mathcal{R}$ is the scalar curvature of the background. This leads to a mass term for the scalars and thus the Coulomb phase is removed.

On a conformal field theory defined on $\mathbb{R} \times S^3$, one can easily apply the standard construction of the *state operator correspondence* in conformal field theories [14]. Such correspondence states that if there is a local conformal field $\phi_\Delta(x)$ with conformal dimension Δ with respect to the dilatation operator D, then there is a corresponding eigenstate $|\Delta\rangle$ of the generator of global time translations (the *Hamiltonian* $H = \frac{1}{2}(P_0 + K_0)$) with eigenvalue $\Delta + E_\Omega$, where E_Ω is the Casimir energy of the vacuum. We will see an explicit realisation of this correspondence pointing out the origin of the Casimir energy in Section 3.1. The state $|\Delta\rangle$ is obtained by acting on the unique vacuum of the theory $|\Omega\rangle$ with the corresponding local operator evaluated on the fixed point of the action of the dilatation D (the *origin*, $x = 0$) $\phi_\Delta(0)$

$$\phi_\Delta(x) \longleftrightarrow |\Delta\rangle = \phi_\Delta(0)|\Omega\rangle \tag{2.2.33}$$

Thanks to this correspondence, we may equivalently consider a representation of the (super)conformal group on the Hilbert space of the theory or on its algebra of operators.

A representation of the superconformal algebra can be obtained by acting on a superconformal primary with different generators of the algebra: operators obtained in this way are called *descendants.* In general a superconformal multiplet will contain several conformal primaries, *i.e.* operators which commute with the special conformal generators K but by the superconformal S. An obvious way of getting such conformal primary is by acting on a superconformal primary with Q. Indeed

$$[K, [Q, \mathcal{O}]_\pm] = -[Q, [\mathcal{O}, K]]_\pm \pm [\mathcal{O}, [K, Q]]_\pm = 0\,. \tag{2.2.34}$$

The complete superalgebra is isomorphic to the superconformal algebra of $SU(2,2|4)$. Unitary representationsions of the latter can be labelled by the quantum numbers of the bosonic subgroup

$$\begin{array}{ccc} SO(1,3) & SO(1,1) & SU(4) \\ (s_+, s_-) & \Delta & [p,q,r] \end{array} \tag{2.2.35, 2.2.36}$$

where $s_\pm$ are positive half integers, representing the value of the Casimirs of $SU(2) \times SU(2) \cong SO(1,3)$, $\Delta \geq 0$ is the dimension of the primary operator in the multiplet and $[p,q,r]$ is the Dynkin label of the representation of $SU(4)$.

2.2.3 Chiral primary operators

As we noticed in Section (2.2.1), a relevant subset of superconformal primary operators is represented by the *chiral primary* operators. They are annihilated by some of the Q charges

$$[Q, \mathcal{O}] = 0 \tag{2.2.37}$$

As a consequence, the multiplet constructed on them are shorter than the standard multiplet because some of the descendants are vanishing. As we noticed, the scaling dimension of a chiral primary is completely determined by its Lorentz and R symmetry quantum numbers. In analogy to what happens in standard extended supersymmetry, where the mass of the short or BPS multiplets is determined by their quantum numbers, chiral primary operators are also called *BPS states.*

It is interesting to have explicit forms for the chiral primary operators in the $\mathcal{N} = 4\, SU(N)$ Super Yang Mills theory. The construction is based on the fact that a superconformal chiral primary is *not* the Q commutator of another operator. Let us recall schematically the value of the Q commutator on the fields of the SYM theory

$$\begin{array}{ll} \{Q, \lambda\} = F + [X, X] & [Q, X] = \lambda \\ \{Q, \bar{\lambda}\} = DX & [Q, F] = D\lambda \end{array} \tag{2.2.38}$$

where D stands for the covariant derivative. The easiest way of constructing chiral primaries is just to avoid the presence of any term which appear in the r.h.s. of the above relations. Gauge invariant superconformal primaries of this type are constructed out of traces of symmetric combination of scalar operators

$$\mathrm{Tr}\left(X^{i_1} \cdots X^{i_p}\right) \tag{2.2.39}$$

where the R-symmetry indices are symmetrised. In general, BPS operators which commute with different Q charges can be obtained by taking products and suitable (anti)symmetrised combinations of such operators [30] which transform under irreducible representations of the $SU(4)$ R symmetry group labelled by integers p, q, r.

Since there is a one-to-one correspondence between chiral primary operators and unitary superconformal multiplets, and so all the state and operator multiplets can be labelled in terms of superconformal chiral primary operators. Recall now that the anticommutator of the Q and S generators is schematically given by

$$[Q, S]_+ = D + M + R \tag{2.2.40}$$

and it vanishes on chiral primaries. As such, the scaling dimension Δ of BPS operators is protected against quantum corrections.

Generic chiral primary operators obtained by sum and products of operators of the form (2.2.39) were classified in [25, 31, 32] and are characterised by the number of Poincare' supersymmetries they preserve which can amount to $\frac{1}{2}$, $\frac{1}{4}$ or $\frac{1}{8}$ of the 16 total supersymmetries. The simplest class is that of $\frac{1}{2}$-BPS operators the elements of which are characterised by the $SO(6)$ R-symmetry representations given by Young tableaux with a single row of length p, i.e. traceless, symmetric $SO(6)$ rank p tensors, or in Dynkin notation, the $[0, p, 0]$ of $SU(4)$. In this case equation (2.2.40) implies that the conformal dimension of the BPS operators is given by

$$\Delta = p \tag{2.2.41}$$

An $SO(6)$ highest weight state of this representation can be obtained by using one of the complex adjoint scalars, $Z \equiv Z_3 \equiv X^5 + iX^6$, which has charge 1 with respect to the $SO(2)$ generator, $J_3 \equiv L_{5,6}$[4] , in $SO(6)$. Out of Z, one can construct multitrace composite $SU(N)$ singlet operators of the form

$$\mathcal{O}_{\{l_i,k_i\}} = \left(\mathrm{Tr}Z^{l_1}\right)^{k_1} \cdots \left(\mathrm{Tr}Z^{l_n}\right)^{k_n} \tag{2.2.42}$$

with

$$\Delta = p = \sum_{i=1}^{n} l_i k_i \,. \tag{2.2.43}$$

We will analyse in detail the structure of gauge invariant 1/2 BPS states in Section 3.1. The lower supersymmetry cases are again best described in terms of $SO(6)$ Young tableaux: the $\frac{1}{4}$ and $\frac{1}{8}$ cases correspond to tableaux with two rows (of lengths $p, q,\ p \geq q$) and three rows (of lengths $p, q, r,\ p \geq q \geq r$) respectively. Again, in discussing highest weight states it is convenient to use the three complex scalars $Z_1 = X^1 + iX^2$, $Z_2 = X^3 + \mathrm{i}X^4$ and Z_3, which have charges (1,0,0), (0,1,0) and (0,0,1) with respect to the three Cartan generators (J_1, J_2, J_3) $SU(4)$ respectively. Highest weight states satisfy the BPS relation $\Delta = J_1 + J_2 = p + q$ and $\Delta = J_1 + J_2 + J_3 = p + q + r$ in the $\frac{1}{4}$ and $\frac{1}{8}$ case, respectively.

This is summarised in the following table:

$p = q = 0,\ r \neq 0$	$p, q \neq 0,\ r = 0$	$p, q, r \neq 0$
$1/2\,BPS$	$1/4\,BPS$	$1/8\,BPS$

Let us consider the $\frac{1}{8}$ case, which is the most generic in this class: given the three complex scalars Z_1, Z_2, Z_3 of the $\mathcal{N} = 4$ $SU(N)$ Super Yang Mills theory one can construct a basis

[4] Here $J_i = L_{2i-1,2i}$ in terms of the standard generators of $SO(6)$

of the ring of gauge invariant, local, composite operators in the $[p, q, r]$ of the R-symmetry group $SU(4)$ as [30]

$$\mathrm{Tr}(Z_1^p)\mathrm{Tr}(Z_2^q)\mathrm{Tr}(Z_3^r) + \cdots \tag{2.2.44}$$

where the dots mean suitable (anti)-symmetrisation and trace structure that projects to the chiral primaries in the (p, q, r) representation of $SU(4)$. We will further analyse the properties of these operators in Section 5.1

2.3 The Maldacena conjecture

In this Section we finally present a formulation of the AdS/CFT correspondence

2.3.1 Low energy and near horizon limit

As we have noticed in Chapter 1, there are are two different descriptions of D-branes in String theory and the dynamics of quantum Strings looks quite different in the two descriptions. Let us specify the features of these two different descriptions in the case of N coincident $D3$-branes.

D-branes can be described as surface where open strings can end. Let us consider the dynamics of strings on a flat background with a flat $\frac{1}{2}$ BPS $D3$-brane. If we consider energies much lower than the string scale l_s^{-1} only massless states can be excited: open strings give the states of 4 dimensional $\mathcal{N} = 4$ $U(N)$ gauge theory and closed strings give a gravity supermultiplet which is described at low energies by the type IIB supergravity Lagrangian. The complete effective action for such massless states include a brane term, a bulk term and an interaction term

$$S = S_{\text{brane}} + S_{\text{bulk}} + S_{\text{int}} \tag{2.3.1}$$

Note that although the action above involves only massless fields it contains contribution from all the massive states that have been integrated out. The brane action is given by the $\mathcal{N} = 4$ $U(N)$ gauge theory plus higher derivatives term which are proportional to powers of α'. We explicit now the way we take the low energy limit of the theory by sending the string length $l_S \to 0$ (which of course implies $\alpha' \to 0$). In this limit gravity becomes free and it decouples from the brane modes. Higher derivatives term in the brane action also vanishes and we are left with just pure $\mathcal{N} = 4$ $U(N)$ SYM and decoupled supergravity on flat space.

Since D-branes source the closed strings modes, we can also describe them as p-branes solution to the supergravity equations. We have already written such solutions

$$\mathrm{d}s^2 = H(r)^{-1/2}\mathrm{d}x^\mu \mathrm{d}x_\mu + H(r)^{1/2}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2\right) \tag{2.3.2}$$

where $\mathrm{d}x^\mu \mathrm{d}x_\mu$ is the flat Minkowski four dimensional metric and $\mathrm{d}\Omega_5^2$ is the metric on the unit radius five dimensional metric. The function $H(r)$ is given by

$$H(r) = 1 + \frac{L^4}{r^4} = 1 + \frac{4\pi g_s N \alpha'^2}{r^4}. \tag{2.3.3}$$

Notice that the energy E_p of an object as measured by any observer at a constant position in r is related to the energy E as measured at infinity by the redshift factor

$$E = H^{-1/4} E_p \,. \tag{2.3.4}$$

We now take again the low energy limit which is specified by sending $\alpha' \rightarrow 0$ and keeping constant r/l_s. There are tow type of low energy excitations (from the point of view of an observer at infinity): massless particles propagating in the bulk with vary large wavelength or any kind of excitations which lives close to $r = 0$. The bulk and the "near horizon" excitations decouple because the absorption cross-section goes like [6] $\sigma \propto R^8/\lambda^3$ where λ is the wavelength. Hence, the low energy theory consists of two decoupled parts, free bulk supergravity and string theory on the near horizon region. The metric in the limit

$$r, l_s \rightarrow 0 \qquad r/l_s = U \,\text{constant} \tag{2.3.5}$$

takes the form

$$\mathrm{d}s^2 = \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}} \left(\mathrm{d}x^\mu \mathrm{d}x_\mu \right) + \sqrt{4\pi g_s N} \frac{\mathrm{d}U^2}{U^2} + \sqrt{4\pi g_s N} \mathrm{d}\Omega_5 \right] \,. \tag{2.3.6}$$

We perform now the further coordinate transformation

$$U = \frac{\sqrt{4\pi g_s N}}{u} \tag{2.3.7}$$

which takes the metric to the form

$$\mathrm{d}s^2 = \alpha' \sqrt{4\pi g_s N} \left[u^2 \left(\mathrm{d}x^\mu \mathrm{d}x_\mu \right) + \frac{\mathrm{d}u^2}{u^2} + \mathrm{d}\Omega_5 \right] \,. \tag{2.3.8}$$

We can now take the string theory non linear sigma model on the background above focusing in particular on the metric part without considering the contribution of the 5-form. Notice that the metric can be written as $g_{MN} = \alpha' \sqrt{4\pi g_s N} \bar{g}_{MN}$ where $\bar{g}_{MN}$ is free of parameters. The sigma model will take the form

$$S_G = \frac{1}{4\pi\alpha'} \int_\Sigma \sqrt{\gamma} \gamma^{\alpha\beta} g_{MN}(X) \partial_\alpha X^M \partial_\beta X^N = \sqrt{\frac{g_s N}{4\pi}} \int_\Sigma \sqrt{\gamma} \gamma^{\alpha\beta} \bar{g}_{MN}(X) \partial_\alpha X^M \partial_\beta X^N \tag{2.3.9}$$

It is clear from this simple derivation that the role that is usually taken in string theory by the parameter $4\pi\alpha'$ is now taken by

$$\sqrt{\frac{4\pi}{\lambda}} \qquad \lambda \equiv g_s N \,. \tag{2.3.10}$$

The low energy effective action of string theory is expressed in terms of a double expansion in α' and g_s. In the context of String Theory on the background of $AdS_5 \times S^5$ it becomes an expansion in λ and g_s. Notice in particular that purely quantum gravitational effects are driven by $g_s^2/\lambda^2 = N^{-2}$.

2.3.2 Formulation of the conjecture

We have just seen that D-branes in String Theory can have two quite different description and in particular that, in a suitable low energy limit, both the descriptions have contain the dynamics of free strings in the bulk. It is thus natural to identify such dynamics on both sides. We are left with $\mathcal{N} = 4$ $U(N)$ SYM with coupling constant $g_{YM} = g_S^{1/2}$ on one side and Type IIB string theory on $AdS_5 \times S^5$ with $L^4 = 4\pi\lambda l_s^4$ on the other side. In a milestone paper Maldacena conjectured [33] that these two theories are actually dual or equivalent: they describe the same dynamics.

The dynamics of a $U(N)$ Yang-Mills theory is determined by a double expansion in the 't Hooft coupling $\lambda = g_{YM}^2 N$ and N. Analysis of loop diagrams can be trusted for small values of λ while the classical gravity description is valid in the incompatible regime of large λ. This makes hard to give a proof of the conjecture.

A notable limit of the gauge theory consist in taking λ fixed and large N. This is known as the 't Hooft limit [3]. This corresponds to a topological expansion of the Feynman digrams of the theory. On the String theory side, keeping finite values of λ while sending $N \to \infty$ implies small g_s and thus corresponds to the limit of weak coupling string perturbation theory, or classical strings.

The AdS/CFT correspondence may be intended in three different formulations of decreasing strength

- The full $\mathcal{N} = 4$ Superconformal $U(N)$ Yang-Mills Theory with arbitrary values of the 't Hooft coupling λ and N is equivalent to String Theory on $AdS_5 \times S^5$ where the mappings of the dimensionless parameters is done as following

$$\frac{L_{AdS}^2}{4\pi\ell_s^2} = \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \qquad g_s = \frac{\lambda}{N}\,. \tag{2.3.11}$$

- The $1/N$ 't Hooft limit expansion of the SYM theory is equivalent to the g_s expansion of Type IIB String Theory on $AdS_5 \times S^5$
- The expansion of $\mathcal{N} = 4$ SYM in powers of $\lambda^{-1/2}$ for large λ and large N is equivalent to the expansion of Classical Type IIB Supergravity on $AdS_5 \times S^5$ at large L/l_s

Notice that of course the strongest formulation implies the two weaker ones but each of the weaker formulations can be valid independently on the validity of the stronger one(s).

A short remark is due here. The SYM theory describing N coincident D-branes is a $U(N) = SU(N) \times U(1)$. The $U(1)$ factor describes collective motion of the branes and decouples from the $SU(N)$. On the gravity side, it correspond to the inclusion of a topological B field on the theory which lives on the boundary and decouples from the dynamics of the remaining degrees of freedom of the theory. One may or may not include the further $U(1)$ factor in the correspondence. In this thesis we will not consider the inclusion of such mode.

2.3.3 Matching of global symmetries

AdS_{p+2} of radius L can be thought as the Lorentzian hyberboloid of radius L in a $p+3$ dimensional flat space with diagonal metric of signature $(- + \cdots + -)$. Introducing coordinates $Y_0, Y_1, \cdots, Y_{p+2}$ it is specified by the equations

$$Y_0^2 + Y_{p+2}^2 - \sum_{i=1}^{p+1} Y_i^2 = L^2 \tag{2.3.12}$$

It is clear that its symmetry group is $SO(2,4)$. Global coordinates on the hyperboloid are given by

$$\begin{aligned} X_0 &= L\cosh\rho\cos\tau \qquad X_{p+2} = L\cosh\rho\sin\tau \\ X_i &= L\Omega_i \sinh\rho \quad \sum_{i=1}^{p+1}\Omega_i^2 = 1 \end{aligned} \tag{2.3.13}$$

The metric in these coordinates is given by

$$ds^2 = L^2\left(-\cosh^2\!\rho\, d\tau^2 + d\rho^2 + \sinh^2\!\rho\, d\Omega_p^2\right) . \tag{2.3.14}$$

The topology of the manifold is $S^1 \times R^{p+1}$ and thus contains closed timelike curves. One can easily avoid such pathology unwrapping the S^1 which amounts to take the universal covering of the hyperboloid. The $SO(2)$ factor in the maximally compact subgroup of the isometry group $SO(2, p+2)$ maps to global time translations. An interesting remark is due here, specifying to the case of our interest: Type IIB String Theory on $AdS_5 \times S^5$. The spectrum of the Kaluza Klein compactification on S^5 of type IIB Supergravity on $AdS_5 \times S_5$ has been classified in [34, 35]. Curiously they found that all the excitations have energies, with respect to the global time translation, which are integer multiple of $(2\pi L)^{-1}$: this amounts to say that supergravity is well defined on the single cover of AdS_5.

The isometry group $SO(2,4)$ of AdS_5 under the duality with the CFT is mapped to the conformal group. The global time translations in AdS_5 are mapped to time translations on $\mathbb{R} \times S^3$ as noted in Section 2.2.2. Moreover in the formulation of the SYM theory on $\mathbb{R} \times S^3$ we have a unitary representation of the full superconformal group, a unique vacuum and thus an unambiguous interpretation of the state operator correspondence in the CFT. For such reasons, when dealing with the AdS/CFT correspondence we will intend the CFT as formulated on $\mathbb{R} \times S^3$

$AdS_5 \times S_5$ is a maximally supersymmetric solution of Type IIB Supergravity and as such it preserves 32 supersymmetries. Half of them are mapped to the 16 supersymmetry generators Q of the CFT and the other half to the 16 superconformal S generators.

2.3.4 Field - operator correspondence

It is straightforward to see that at large values of ρ the metric of global AdS spacetime, after the coordinate transformation

$$z = 2e^{-\rho} \to 0 \tag{2.3.15}$$

takes the form

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left(-\mathrm{d}\tau^2 + \mathrm{d}\Omega_p^2 + \mathrm{d}z^2\right) . \tag{2.3.16}$$

The conformal boundary of AdS_{p+2} is specified by $z = 0$ and is given by $\mathbb{R} \times S^p$. In particular, for AdS_5 the conformal boundary is $\mathbb{R} \times S^3$. Consider the version of the $\mathcal{N} = 4$ CFT on $\mathbb{R} \times S^3$. We will show now that the properties of the conformal boundary of AdS_5 allow us to give a more precise "holographic" formulation of the AdS/CFT correspondence.

For each field propagating in $AdS_5 \times S^5$ we perform a Kaluza Klein expansion over tensor spherical harmonics in S^5.

$$\Phi(x, y) = \sum_i \phi_i(x) Y^i(y) \tag{2.3.17}$$

where x indicates coordinates in AdS_5 and y indicates coordinates in S^5. Lorentz indices have been suppressed. The D'Alambertian operator in AdS_5 for $z \longrightarrow 0$ takes the form

$$\Box_5 = z^2\,\Box_4 + (z\,\partial_z)^2 - 4\,z\,\partial_z\,. \tag{2.3.18}$$

where $\Box_4$ is the D'Alambertian operator in $\mathbb{R} \times S^3$. Near the boundary where z is small, the fields can be approximated by $z^d\,\phi^0(\bar{x})$ where $\bar{x}$ denotes coordinates on $\mathbb{R} \times S^3$. Each Lorentz component of the fields ϕ_i has to be a solution of the Klein-Gordon equation in the anti-de Sitter space, for which the D'Alambertian equals the quadratic Casimir operator $\mathcal{C}_2$ [36]. In terms of the ground state energy E_0 of the anti-de Sitter representation, we have $\mathcal{C}_2 = E_0(E_0 - 4)$, which shows that we have the identification $d = E_0$ or $d = 4 - E_0$. This identification is somewhat remarkable in view of the fact that E_0 is the energy eigenvalue associated with the SO(2) generator of the anti-de Sitter algebra and not with the noncompact scale transformation D which scales z and corresponds to the SO(1, 1) generator.

The quadratic Casimir operator is, by definition, the coefficient of the quadratic term in the action which is usually denoted as m^2. This term in general *does not coincide* with the mass squared M^2 of the field, which can be measured by looking at the wave equation in locally flat coordinates. The relation between the mass and the Casimir operator $\mathcal{C}_2$ depends on the spin of the fields. For example, in generic AdS_D spacetime for the lowest spins it is given by [36]

$$\begin{aligned}
\mathcal{C}_2^{\text{scalar}} &= -\frac{1}{4}D(D-2) + M^2 \\
\mathcal{C}_2^{\text{spinor}} &= -\frac{1}{8}D(D-1) + M^2 \\
\mathcal{C}_2^{\text{vector}} &= M^2 \\
\mathcal{C}_2^{\text{tensor}} &= M^2\,.
\end{aligned} \tag{2.3.19}$$

The first relation in particular gives an elementary interpretation of the well known Breitenlohner-Freedom bound [37] which coincides with the requirement $M^2 \geq 0$.

Generic solutions of the Klein Gordon equations on AdS_5 are thus determined by boundary conditions at $z = 0$ represented by a specification of a field $\phi^0(\bar{x})$ such that

$$\phi(x) \to z^d \phi^0_d(\bar{x}) \tag{2.3.20}$$

The isometry group of AdS_5 acts locally on the field $\phi(x)$, thus the field $\phi^0_d(\bar{x})$ transforms as a conformal field of weight d under scaling of the $\mathbb{R} \times S^3$ coordinates.

Let us focus for simplicity on a scalar field $\phi(x)$. The exponent d can take the two values Δ,$4 - \Delta$ with

$$\Delta = 2 + \sqrt{4 + m^2} \tag{2.3.21}$$

We choose the non normalisable solution determined by the choice $d = 4 - \Delta$. The boundary value $\phi^0_{4-\Delta}$ can be used as a source for an operator $\mathcal{O}$ of dimension Δ in the CFT generating function

$$\exp\left\{\Gamma[\phi^0_\Delta(\bar{x})]\right\} = \left\langle \exp \int_{\mathbb{R}\times S^3} \phi^0_{4-\Delta} \mathcal{O}_\Delta \right\rangle_{CFT} \tag{2.3.22}$$

We can now define the String partition function $Z_S(\phi^0)$ on the KK reduction of $AdS_5 \times S^5$ to AdS_5 with the boundary condition that $\phi(x)$ approaches $z^{4-\Delta}\phi^0_{4-\Delta}(\bar{x})$. For example, in the approximation of classical supergravity, one computes $Z_S(\phi^0)$ by simply solving the equations for $\phi(x)$ with the specified boundary condition and then taking the exponent of the classical supergravity $I_S[\phi]$ action calculated on the solution

$$Z_S\left[\phi^0\right] = \exp\left\{-I_S[\phi]\right\} \tag{2.3.23}$$

The holographic formulation of the AdS/CFT correspondence states that

$$\exp\left\{\Gamma[\phi^0_\Delta(\bar{x})]\right\} = Z_S\left[\phi^0_\Delta\right] \tag{2.3.24}$$

The main problem at this point is to construct the dictionary between conformal operators $\mathcal{O}_\Delta$ in the CFT and fields on the AdS side.

We will describe some steps in this direction it the next Chapters.

Chapter 3

Half-BPS States in AdS/CFT

One of the first steps in understanding the AdS/CFT correspondence is to set up a precise dictionary between the different states and the spectrum of the theories on the two sides of the correspondence. In the previous Chapter we have argued how to identify the conformal dimension of the operators of the CFT with the mass of linear fields propagating on the Supergravity AdS_5 backround.

Type IIB Supergravity is a good approximation of type II B String Theory at low energies compared to the string scale and small string coupling. When the mass and the energy of the linear fields grow, backreaction on the metric should be taken into account.

The identification of the parameters of the $\mathcal{N}=4$ $SU(N)$ SYM, namely $\lambda \equiv g_{YM}^2 N$ and N and the parameters of type IIB String Theory on $AdS_5 \times S^5$, namely L_{AdS}, ℓ_s, g_s, is done as follows

$$\frac{L_{AdS}^2}{4\pi\ell_s^2} = \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \qquad g_s = \frac{\lambda}{N}\,. \tag{3.0.1}$$

We may thus consider solutions to the supergravity equations of motion which are asymptotically $AdS_5 \times S^5$ as good candidates for dual of states in the CFT of generic energy, provided that $N \gg \lambda \gg 1$ and any dimension four curvature invariant of the solutions, $\mathcal{R}_4$, satisfies

$$\mathcal{R}_4 L_{AdS}^4 \ll \lambda \ll N\,. \tag{3.0.2}$$

If one wants to construct the map between the states of the CFT and geometries obtained in this way, supersymmetry is helpful in a twofold way. First, it constrains renormalisation properties of the operators in the CFT and secondly, as any other symmetry assumption, it simplifies the derivation of the solutions to the Einstein equations. In particular, as we have noticed in Section 2.1.3 it reduces the problem of solving the second order differential Einstein equations to the one of solving the first order Killing spinor equations and a few second order equations corresponding to the Bianchi identities for the non vanishing field strength.

One may hope to be able to identify the states of the two theories in the full dual BPS sector. A very beautiful and relatively simple construction of such a dictionary in the half

BPS sector has been performed in [12](LLM). The authors considered geometries dual to half BPS states in the CFT, associated to chiral primary operators which are obtained by taking traces of powers of the operator $Z \equiv Z_3 \equiv X^5 + iX^6$. In LLM, exact half BPS solutions to the supergravity equations of motion are derived by exploiting the bosonic symmetry of the problem.

In this Chapter we will at first give a description of half BPS chiral primaries in terms of free fermions and Young tableaux. We will later summarise the LLM construction and a direct simple generalisation to plane wave geometries.

3.1 Fermion picture of half-BPS CFT chiral primaries

In this section we will give a description of 1/2 BPS chiral primaries in terms of free fermions in a harmonic potential and in terms of Young tableaux. The original construction was developed in [38, 39]. Look also [40].

As we have argued in Section 2.2.3, in every $SO(6)$ multiplet of 1/2 BPS chiral primaries we can identify operators of the form

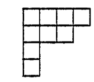
$$\mathcal{O}_{\{l_i,k_i\}} = \left(\mathrm{Tr}Z^{l_1}\right)^{k_1} \cdots \left(\mathrm{Tr}Z^{l_n}\right)^{k_n} \tag{3.1.1}$$

The conformal dimension of such operators is given by

$$\Delta = p = \sum_{i=1}^{n} l_i k_i \,. \tag{3.1.2}$$

from which is clear that (3.1.2) generic half-BPS operators of charge p are characterised by the partitions of p. The partition $\{l_i, k_i\}$ of p can be described by a Young tableau with k_i columns of length l_i putting columns in order f decreasing length. For example

$$\{l_1 = 4, k_1 = 1; l_2 = 2, k_2 = 2; l_3 = 1, k_3 = 1\} \quad \Longrightarrow \tag{3.1.3}$$



Notice that there is no bound on the number of columns but, since $\mathrm{Tr}Z^{N+k}$ with $k > 0$ can be expressed as a linear combination of traces of Z to a lower power, the maximum length of each column is N.

3.1.1 Oscillators basis

We can expand the field $Z(x)$ into spherical harmonics on S^3. The CFT states associated to local operators $\mathcal{O}(x)$ are obtained by acting on the unique conformal vacuum with $\mathcal{O}(0)$ which coincides with the s-wave mode on S^3 of the operator $\mathcal{O}(x)$. Due to this state-operator correspondence we will frequently jump between an operator based and a state based language.

In order to study the CFT states associated to the half-BPS operators, we consider the reduction of the full theory on $\mathbb{R} \times S^3$ to the s-wave modes of the fields Z and $\bar{Z}$: thus, from now on, Z will be a complex matrix. Direct reduction of the lagrangian of the $\mathcal{N}=4$ theory on $\mathbb{R}\times S^3$ gives rise to quadratic as well as quartic terms in $Z, \bar{Z}$:

$$\mathcal{L} = \mu^2 \mathrm{Tr}\left(D_t Z D_t Z^\dagger - Z Z^\dagger\right) + V_D \tag{3.1.4}$$

$$V_D \propto \mathrm{Tr}\left(\left[Z, Z^\dagger\right]\left[Z, Z^\dagger\right]\right) \tag{3.1.5}$$

where

$$Z^\dagger = \bar{Z}^T \qquad \mu^2 = \frac{\Omega_3}{2 g_{YM}^2} \tag{3.1.6}$$

with Ω_3 the area of S^3 and

$$D_t Z = \partial_t Z + \mathrm{i}\left[A_t, Z\right] . \tag{3.1.7}$$

A_t acts as a Lagrange multiplier and can be gauged away. Its equation of motion implies that the generator of gauge symmetries Q vanishes on the states, which is implemented by taking gauge invariant states.

We will further reduce to the abelian sector of the model, assuming that

$$\left[Z, Z^\dagger\right] |\mathcal{O}\rangle = 0 . \tag{3.1.8}$$

for any half-BPS state $|\mathcal{O}\rangle$. In the free theory limit, $V_D = 0$. At one loop, the contribution from the D-term in the scalar potential V_D to the dilatation operator is cancelled by fermion and gauge loops [41, 42]. One can translate this to the dual Hilbert space as,

$$\langle \mathcal{O}' | \mathrm{Tr}\left([Z, Z^\dagger][Z, Z^\dagger]\right) |\mathcal{O}\rangle = 0 . \tag{3.1.9}$$

Since these states are protected by supersymmetry, we expect this to be true independently of the gauge coupling. It is thus natural to guess that the effective dynamics of these states will be described by a normal gauged matrix model with an harmonic oscillator potential [43, 38, 39, 44],

$$\mathcal{L} = \mu^2 \mathrm{Tr}(D_t Z D_t Z^\dagger - Z Z^\dagger) , \quad [Z, \bar{Z}] = 0 . \tag{3.1.10}$$

The momenta conjugated to the fields $Z, \bar{Z}$ are given by

$$\Pi = \mu^2 \partial_t \bar{Z} \tag{3.1.11}$$

$$\bar{\Pi} = \mu^2 \partial_t Z \tag{3.1.12}$$

We define the destruction operators A, B as

$$A = \frac{1}{\sqrt{2}}\left(\mu \bar{Z} + \frac{\mathrm{i}}{\mu}\Pi\right) \qquad B = \frac{1}{\sqrt{2}}\left(\mu Z + \frac{\mathrm{i}}{\mu}\bar{\Pi}\right) \tag{3.1.13}$$

The Hamiltonian is given by

$$\mathcal{H} = \mathrm{Tr}\left(\frac{1}{\mu^2}\Pi^\dagger \Pi + \mu^2 Z^\dagger Z\right) = \mathrm{Tr}\left(A^\dagger A + B^\dagger B + N\right) . \tag{3.1.14}$$

The vacuum energy is thus given by N^2. The $SO(2)$ generator $J = J_3$ of Section 2.2.3 which acts as a phase on Z is given by

$$J = \mathrm{Tr}\left(A^\dagger A - B^\dagger B\right) . \tag{3.1.15}$$

Referring to the discussion of Sections 2.2.2 and 2.2.3, chiral primary states are constructed out of Z and verify

$$H - N^2 = J = \Delta \tag{3.1.16}$$

Since they have to be gauge invariant, they must be obtained acting on the conformal vacuum with traces of the creation operators $A^\dagger$ only[1].

At the beginning of this Section we pointed out that gauge invariant half-BPS states are characterised by partitions of the J charge p. Consider the description of the same states in terms of traces of the operators $A^\dagger$. Generically, they will take the form:

$$\mathcal{O}(A^\dagger)_{\{l_i,k_i\}} = \left(\mathrm{Tr}A^{\dagger\, l_1}\right)^{k_1} \cdots \left(\mathrm{Tr}A^{\dagger\, l_n}\right)^{k_n} \tag{3.1.17}$$

with

$$p = \sum_{i=1}^{n} l_i k_i \, . \tag{3.1.18}$$

Each of these operators is also characterised by a partition of p. Indeed the relation between the description is terms of Z and $A^\dagger$ is very simple. By definition, the operator B annihilates the vacuum $|\Omega\rangle$ and thus we have

$$-\,\mathrm{i}\bar{\Pi}|\Omega\rangle = \mu^2 Z|\Omega\rangle \tag{3.1.19}$$

which implies

$$\left(\mathrm{Tr}A^{\dagger\, l_1}\right)^{k_1} \cdots \left(\mathrm{Tr}A^{\dagger\, l_n}\right)^{k_n} \Omega = \frac{\mu^p}{2^{p/2}} \left(\mathrm{Tr}Z^{l_1}\right)^{k_1} \cdots \left(\mathrm{Tr}Z^{l_n}\right)^{k_n} \tag{3.1.20}$$

3.1.2 Eigenvalues basis

Since we have restricted observables to a single set of oscillators, it should be possible to rewrite the Hamiltonian in terms of a single hermitian matrix of operators. We define

$$Q = \frac{1}{\sqrt{2}}\left(A + A^\dagger\right) \tag{3.1.21}$$

$$P = \frac{1}{\mathrm{i}\sqrt{2}}\left(A - A^\dagger\right) \tag{3.1.22}$$

which obey the commutation relations[2]

$$[Q_{ij}, P_{kl}] = \mathrm{i}\delta_{il}\delta_{jk} \, . \tag{3.1.23}$$

[1] An identical argument can be carried on in the case of half-BPS states defined by $\Delta = -J$, just by exchange the roles of the A and B oscillators.

[2] Notice the non trivial pairing of the indices in the Kronecker symbols.

Neglecting the Casimir energy, we can derive the lagrangian which, in terms of the hermitian matrices $Q, \partial_t Q$ takes the form

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\left[(\partial_t Q)^2 - Q^2\right] \tag{3.1.24}$$

We have thus reduced our model to that one of a hermitian matrix in a harmonic potential. There always exists a unitary matrix U such that[45]

$$Q = U^\dagger \Lambda U \tag{3.1.25}$$

with $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$. We have:

$$\partial_t Q = U^\dagger\left(\dot{\Lambda} + \left[U\dot{U}^\dagger, \Lambda\right]\right)U\,. \tag{3.1.26}$$

We can now perform an $SU(N)$ gauge transformation on the matrix Q. The $SU(N)$ transformation will be in general time dependent, and in order to recover gauge invariance, we have first to restore the gauge potential A_t in the covariant derivative.

$$\partial_t Q \to D_t = \partial_t + \mathrm{i}\left[A_t, Q\right]\,. \tag{3.1.27}$$

So far, we have assumed that $A_t = 0$. We can easily reabsorb the commutator term in (3.1.26) into a gauge transformation and we can reduce the lagrangian of our model to

$$\mathcal{L} = \sum_{i=1}^{N}\frac{1}{2}\left(\dot{\lambda}_i^2 - \lambda_i^2\right) \tag{3.1.28}$$

So the dynamics is governed by the classical motion of the N eigenvalues is that of a harmonic oscillator[3]. However, quantum mechanically there is a change of measure from the matrix basis to the eigenvalue basis. This change of measure is the volume of the gauge orbit of the matrix X, and it is equal to the square of the Van der Monde determinant of the λ_i, namely

$$\mu = \Delta(\lambda)^2 = \prod_{i\neq j}(\lambda_i - \lambda_j) \tag{3.1.29}$$

So that the Hamiltonian in the quantum theory will be given by

$$H\psi = \frac{1}{2}\sum -\mu^{-1}\partial_{\lambda_i}(\mu\partial_{\lambda_i}\psi) + \lambda_i^2\psi \tag{3.1.30}$$

with ψ the wave function of the eigenvalues.

The measure can be absorbed in the wave functions for the λ_i, by attaching a factor of the Van der Monde to the wave function. We define $\psi(\lambda) = \Delta^{-1}(\lambda)\tilde{\psi}(\lambda)$, where $\tilde{\psi}(\lambda)$ is the new wave function in the X variables expressed in terms of the eigenvalues of X

[3]Rigorously, since we are dealing with an $SU(N)$ theory there are only $N-1$ independent eigenvalues. In the large N limit this distinction is irrelevant and we will ignore it here

(these are the λ_i), and the measure for $\tilde{\psi}$ is is just $\prod d\lambda_i$. This can be done for any one matrix model quantum mechanics [46] with a single trace potential. This is a similarity transformation on the space of wave functions, so it affects the form of the Hamilltonian. The new Hamiltonian is

$$\tilde{H} = \frac{1}{2}\sum_i -\partial^2_{\lambda_i} + \lambda_i^2 \tag{3.1.31}$$

so it becomes a Hamiltonian for N free particles in the harmonic oscillator potential well. After this is done the wave functions are completely antisymmetric in the λ_i: the eigenvalues become fermions due to the Van Der Monde determinant. The system is reduced to N free fermions in a given potential, which for us is just $V(x) = x^2/2$. For our setup, an orthogonal basis for the N-particle wave functions is given by Slater determinants of one particle wave functions for the Harmonic oscillator (these are in turn given by Hermite polynomials times a Gaussian factor $H^{n_k}(\lambda)\exp(-\lambda^2/2)$. This basis for the wave functions is given explicitly by

$$\psi(n_1,\ldots,n_N) \sim \det\begin{pmatrix} H^{n_1}(\lambda_1) & H^{n_1}(\lambda_2) & \ldots & H^{n_1}(\lambda_N) \\ H^{n_2}(\lambda_1) & H^{n_2}(\lambda_2) & \ldots & H^{n_2}(\lambda_N) \\ \vdots & \vdots & \ddots & \vdots \\ H^{n_N}(\lambda_1) & H^{n_N}(\lambda_2) & \ldots & H^{n_N}(\lambda_N) \end{pmatrix} \exp(-\sum \lambda^2/2) \tag{3.1.32}$$

In particular, the Fermi statistics imply that all of the n_k are different, and that we can order the n_i so that $n_1 > n_2 > n_3 > \cdots > n_N \geq 0$. The energy of a state is then $\sum_i (n_i + 1/2)$. The ground state of the system is such that the n_i are minimal. This is, $n_{N-k} = k$. From here it follows that the ground state energy of the system is

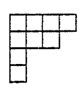
$$\sum_{k=0}^{N-1} \frac{1}{2}(2k+1) = \frac{N^2}{2} \tag{3.1.33}$$

which coincides half of the vacuum energy that we found in (3.1.14). This is due to the fact that we are quantising only one of the two independent harmonic oscillators.

Each excited state can also be parametrised by a set of integers representing the excitation energy of the i-th fermion over its energy in the ground state[39, 47]:

$$r_i = (n_i - n_i^{\Omega}) = n_i - i + 1\,. \tag{3.1.34}$$

The r_i form a non-decreasing set of integers $r_N \geq r_{N-1} \geq \cdots r_i \geq 0$, which can be encoded in a Young diagram T in which the i-th row has length r_i. For example,

$\{4,3,1,1\} \quad \Longrightarrow$  (3.1.35)

Thus, orthogonal excited half-BPS states are in one to one correspondence with Young diagrams. Notice that the maximum length of a column in these type of Young tableaux is N. The energy above the vacuum, and the J charge, of each excited state is equal to the number p of boxes in the Young tableau.

3.1.3 Schur's polynomials basis

We will now describe a second basis of the space of 1/2 BPS chiral primary operators: the Schur polynomials of degree p for the unitary group $SU(N)$. Let us consider a generic $SU(N)$ matrix U. It acts on an N dimensional vector $v \in V \cong \mathbb{C}^n$ of coordinates v^i as

$$(Uv)^i = U^i{}_j v^j \,. \tag{3.1.36}$$

Its action on the symmetrised space $\Omega_p = Sym(V^{\otimes p})$ is given by

$$U(v_1 \otimes \cdots \otimes v_p) = (Uv_1) \otimes \cdots \otimes (Uv_p) \tag{3.1.37}$$

and commutes with the action of the symmetric group S^p on $Sym(V^{\otimes p})$ which simply permutes the vectors v_n, *i.e.* for $\sigma \in S^p$:

$$\sigma(v_1 \otimes \cdots \otimes v_p) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)} \tag{3.1.38}$$

As a consequence, the given representations of the unitary group and of S^p on Ω_p are reducible.
An irreducible representation R of the symmetric group is specified by a Young tableau T_R. To each representation R of the symmetric group is associated a unique irreducible representation of $SU(N)$ which acts on Ω_p as follows

$$R(U)(v_1 \otimes \cdots \otimes v_p)^{i_1 \cdots i^p} = \frac{1}{n!} \sum_{\sigma \in S^p} \chi_R(\sigma) U^{i_1}{}_{\sigma(j_1)} \cdots U^{i_p}{}_{\sigma(j_p)} v_1^{j_1} \cdots v_p^{j_p} \tag{3.1.39}$$

Each representation acts on tensors of a definite symmetry type which is dictated by the corresponding Young tableau. A column of length m in a Young tableau represents complete antisymmetrisation over m indices. Clearly, representations associated to Young tableaux with columns of length greater than N are empty: hence, the maximum length of a column in an allowed Young diagram is N.

Let us consider now the representation of the $su(N)$ algebra which is induced by a representation of the group $SU(N)$. An $SU(N)$ gauge transformation U acts on an element X of the algebra as

$$X \to U^\dagger X U \,. \tag{3.1.40}$$

In each representation R of $SU(N)$, and hence of $su(N)$, we can take the gauge invariant combination

$$\chi_R(X) \equiv \mathrm{Tr}_R X = \frac{1}{p!} \sum_{\sigma \in S^p} \chi_R(\sigma) X^{i_1}{}_{\sigma(i_1)} \cdots X^{i_p}{}_{\sigma(i_p)} \tag{3.1.41}$$

These operators are known as *Schur's polynomials.* With the same definition, Schur's polynomials can be taken for generic complex matrices. They form an orthogonal basis in the space of $SU(N)$ invariant functions of the matrix X, where $SU(N)$ acts on X by conjugation as in (3.1.40).

The dimension of each representation R of $SU(N)$ is easily given by

$$D_R = \chi_R(\mathbb{I}) = \frac{1}{p!}\sum_{\sigma\in S^p} \chi_R(\sigma)\delta^{i_1}{}_{\sigma(i_1)}\cdots\delta^{i_p}{}_{\sigma(i_p)} = \frac{1}{p!}\sum_{\sigma\in S^p}\chi_R(\sigma)N^{C(\sigma)} \tag{3.1.42}$$

where I is the identity matrix and $C(\sigma)$ is the number of cycles of the permutation σ. The generic term in the sum which defines a Schur polynomial for the field Z is proportional to

$$\mathcal{O}_{\{l_i,k_i\}} = \left(\mathrm{Tr}Z^{l_1}\right)^{k_1}\cdots\left(\mathrm{Tr}Z^{l_n}\right)^{k_n} \tag{3.1.43}$$

where the integers l_i, k_i are dictated by the number k_i of cycles of length l_i in the permutation. Since the Schur polynomials are a basis for the half-BPS operators, the relation can be inverted, and any operator of the form $\mathcal{O}_{l_i,k_i}$ can be expressed as a linear combination of Schur polynomials.

Let us consider the simplest example: $p = 2$. We have only 2 permutations

$$I = (1\,2) \tag{3.1.44}$$

$$\sigma = (2\,1)\,. \tag{3.1.45}$$

The identity has two cycles of length one and σ has one cycle of length two. The two possible partitions of $p = 2$ correspond to the Young tableaux

$$T_S = \square\!\square \qquad T_A = \begin{smallmatrix}\square\\ \square\end{smallmatrix}. \tag{3.1.46}$$

We have $\chi_S(1,\sigma) = (1,1)$ and $\chi_A(1,\sigma) = (1,-1)$ and thus

$$D_S = \frac{1}{2}(N^2+N) \qquad D_A = \frac{1}{2}(N^2-N) \tag{3.1.47}$$

The Schur polynomials for the S and A representations are given by

$$\chi_S(X) = \frac{1}{2}\left[\mathrm{Tr}(X)^2 + \mathrm{Tr}(X^2)\right] \tag{3.1.48}$$

$$\chi_A(X) = \frac{1}{2}\left[\mathrm{Tr}(X)^2 - \mathrm{Tr}(X^2)\right] \tag{3.1.49}$$

As a less trivial example, we consider $p = 3$ and the totally symmetric representation corresponding to

$$T_S = \square\!\square\!\square \tag{3.1.50}$$

We have $\chi_S(\sigma) = 1$ for any σ in S^3. The possible permutations and the number of their cycles are

σ	$C(\sigma)$
$(1\,2\,3)$	3
$(1\,3\,2)$	2
$(3\,2\,1)$	2
$(2\,1\,3)$	2
$(2\,3\,1)$	1
$(3\,1\,2)$	1

(3.1.51)

Thus, the dimension of the representation is

$$D_S = \frac{1}{3!}\left(N^3 + 3N^2 + 2N\right) \tag{3.1.52}$$

and the associated Schur polynomial is

$$\chi_S(X) = \frac{1}{3!}\left[\mathrm{Tr}(X)^3 + 3\mathrm{Tr}(X^2)\mathrm{Tr}(X) + 2\mathrm{Tr}(X^3)\right] \tag{3.1.53}$$

3.1.4 Relationships between different bases

So far, we have described three different bases for the half-BPS states in the CFT. The relationship between the oscillators basis and the Schur polynomial basis has been clarified in the previous subsection. Orthogonal elements of both the Schur polynomial basis and the eigenvalues basis are specified by different Young tableaux. We have also associated each Young tableau to orthogonal elements of the eigenvalues basis. Happily an element of the eigenvalues basis associated to a Young diagram T turns out to be exactly the same state as created by the Schur polynomial operator $\chi_T(Z)$ associated to the same Young diagram. This is because the asymptotic behaviour of the Schur polynomial associated to a Young diagram matches the asymptotic behaviour of the Slater determinant [38, 39]. As we have already noticed, the conformal dimension of the operators, Δ, is equal to the number of boxes p appearing in the diagram T.

3.2 Half-BPS geometries in Type II B Supergravity

In this Section we will describe the so called LLM construction, which was developed in [12]. The authors obtained the supergravity duals of the half-BPS states of the CFT that we have studied in the previous Section for large angular momentum $J \backsim N^2$.

When the angular momentum J is small as compared to the rank N of the gauge group, the supergravity duals are given by gravitons and excitations of the RR 5-form propagating in $AdS_5 \times S^5$ [48, 49]. When J is of order N the gravitons effectively become extended object, spherical half-BPS $D3$-branes rotating in S^5 (giant graviton) or propagating in AdS_5 (dual giants) [50, 51, 52, 43] . When the dimension grows to become comparable to N^2 we expect that the backreaction on the geometry is no longer negligible and we get new geometries characterised by a complicated topology. These are the geometries which were constructed in LLM. They represent the geometrical transition between probe giant gravitons or dual giant gravitons and fully backreacted geometries.

3.2.1 The LLM solutions

The construction is based on the assumption that the geometries have to preserve the same amount of bosonic symmetries and supersymmetries that the states in the CFT do. The latter preserve an

$$\mathbb{R}_{BPS} \times SO(4)_R \times SO(4)_s \tag{3.2.1}$$

bosonic symmetry. The first factor is due to invariance under simultaneous dilatation and R-symmetry rotation $D' = D - J$ and is a direct consequence of the chirality of the CFT operators under examination. The second factor is due to invariance under the $SO(4)$ subgroup of the R symmetry which doesn't rotate $X^{5,6}$. The third factor is due to the fact that we are taking the S^3 scalar (s wave) component of the fields. Moreover, on $\mathbb{R} \times S^3$ these states preserve a total amount of 16 supercharges [12].

The problem of finding geometric duals of such CFT states is highly simplified by the presence of supersymmetry. We expect to find solutions of the type IIB Supergravity equations of motion which preserve the same (super)symmetries and have the same charges as the CFT states. Since we are looking at duals of operators which contains only the scalar Z, the only non trivial Supergravity fields will be the metric and the RR 5-form.

In Supergravity, invariance of the background under a supersymmetry transformation is equivalent to the existence of a Killing spinor, as we have outlined in 2.1.2. The LLM procedure amounts to formulate an Ansatz for the metric and the RR 5-form which is invariant under $SO(4) \times SO(4)$ and ask that Killing spinors exist on such a background. Due to the amount of Killing spinors admitted by the LLM solution, following the analysis of 2.1.3, this will be sufficient to guarantee that the background satisfy the full Einstein equations of motion.

The result of the LLM analysis is that the full Supergravity solutions are determined by a single function $z(x_1, x_2, y)$ which is defined on the three dimensional flat halfspace $y > 0$. The complete solutions have the following form:

$$\mathrm{d}s^2 = -h^{-2}(\mathrm{d}t + V_i \mathrm{d}x^i)^2 + h^2(\mathrm{d}y^2 + \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j) + y\mathrm{e}^{G}\mathrm{d}\Omega_3^2 + y\mathrm{e}^{-G}\mathrm{d}\widetilde{\Omega}_3^2 \tag{3.2.2}$$

$$F_{(5)} = F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\Omega + \widetilde{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\widetilde{\Omega} \tag{3.2.3}$$

$$F = \mathrm{e}^{3G} *_4 \widetilde{F} \tag{3.2.4}$$

$$\tag{3.2.5}$$

with $y \geq 0$.

The function $z(x_1, x_2, y)$ determines the entire solution (up to choice of gauge that we will discuss in Chapter 4),

$$z \equiv \frac{1}{2} \tanh G \tag{3.2.6}$$

$$h^{-2} = 2y \cosh G = \frac{y}{\sqrt{(1/2 - z)(1/2 + z)}} \tag{3.2.7}$$

$$\mathrm{d}V = \frac{1}{y} *_3 \mathrm{d}z \tag{3.2.8}$$

$$F = \mathrm{d}(B_t(\mathrm{d}t + V)) + \mathrm{d}\hat{B} \tag{3.2.9}$$

$$\tilde{F} = \mathrm{d}(\tilde{B}_t(\mathrm{d}t + V)) + \mathrm{d}\hat{\tilde{B}} \tag{3.2.10}$$

$$B_t = -\frac{1}{4} y^2 \mathrm{e}^{2G} \qquad \tilde{B}_t = -\frac{1}{4} y^2 \mathrm{e}^{-2G} \tag{3.2.11}$$

$$\mathrm{d}\hat{B} = -\frac{1}{4} y^3 *_3 \mathrm{d}\left(\frac{1/2 + z}{y^2}\right) \qquad \mathrm{d}\hat{\tilde{B}} = \frac{1}{4} y^3 *_3 \mathrm{d}\left(\frac{1/2 - z}{y^2}\right) \tag{3.2.12}$$

where $*_n$ indicate the Hodge dual in n flat dimensions.
For the consistency of (3.2.8) and (3.2.12) we must have

$$(\partial_1^2 + \partial_2^2) z + y \partial_y (\frac{1}{y} \partial_y z) = 0 \tag{3.2.13}$$

The solutions for z and thus the whole supergravity solutions are determined by boundary conditions in the $\{x_1, x_2, y\}$ space. These include the boundary conditions at infinity and on the 2-pane $y = 0$. In particular, with suitable boundary conditions at infinity, if z on only the values $\pm\frac{1}{2}$ on $y = 0$ then the solutions are guaranteed to be singularity free space-times. The value of $\rho = 1/2 - z$ on the boundary plane can be interpreted as a semiclassical fermion density, thus providing direct contact to the CFT dual Yang-Mills theory picture that we have described in the previous Section. We will discuss in details several issues regarding the boundary conditions and regularity of the solutions in Chapter 4.

3.2.2 The LLM construction

In this Subsection we give a sketch of the LLM construction. The complete details can be found in the Appendix of [12]. We will here review some subtle points of their analysis and point out a few observations which will be useful in the following. In Section 3.3 we generalise the LLM construction including the case of generic plane wave geometries.

Statement of the problem and basic assumptions

Our starting point is an Ansatz for the metric and the five-form which exploits the $SO(4)_R \times SO(4)_s$ bosonic symmetry

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{H+G} \mathrm{d}\Omega_3^2 + \mathrm{e}^{H-G} \mathrm{d}\widetilde{\Omega}_3^2 \tag{3.2.14}$$

$$F_{(5)} = F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\Omega + \widetilde{F}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\widetilde{\Omega} \tag{3.2.15}$$

where $d\Omega_3^2$, $d\widetilde{\Omega}_3^2$ denote the metric and $d\Omega$, $d\widetilde{\Omega}$ the volume 3-forms on the two unit radius sphere. The self duality condition for $F_{(5)}$ implies

$$F = e^{3G} *_4 \widetilde{F} \qquad \widetilde{F} = -e^{-3G} *_4 F \qquad F = dB \qquad \widetilde{F} = d\widetilde{B} \tag{3.2.16}$$

the only nontrivial equation for the Killing spinor is

$$\nabla_M \eta + \frac{i}{480} \Gamma_{M_1 M_2 M_3 M_4 M_5} F_{(5)}^{M_1 M_2 M_3 M_4 M_5} \Gamma_M \eta \tag{3.2.17}$$

We choose a basis of tangent space gamma matrices

$$\Gamma_\mu = \gamma_\mu \otimes 1 \otimes 1 \otimes 1 \qquad \Gamma_{\hat{a}} = \gamma_5 \otimes \sigma_1 \otimes \sigma_{\hat{a}} \otimes 1 \qquad \Gamma_{\tilde{a}} = \gamma_5 \otimes \sigma_2 \otimes 1 \otimes \sigma_{\tilde{a}} \tag{3.2.18}$$

where the σ are standard Pauli matrices, $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. The 10 dimensional chirality operator is given by

$$\Gamma_{11} = \Gamma_0 \cdots \Gamma_3 \prod \Gamma_a \prod \Gamma_{\tilde{a}} = \gamma_5 \sigma_3 \tag{3.2.19}$$

and the spinors satisfy the chirality condition

$$\Gamma_{11} \eta = \gamma_5 \sigma_3 \eta = \eta \tag{3.2.20}$$

Killing spinors on the 3-sphere

Let us consider spinors obeying the Killing spinor equation on the round unit radius n-sphere

$$\bar{\nabla}_c \chi = \alpha \frac{i}{2} \sigma_c \chi \tag{3.2.21}$$

being σ_c in the Clifford algebra of $SO(n+1)$ and

$$\bar{\nabla}_c = \Sigma_c + \frac{1}{2} \bar{\Omega}_c{}^{ab} \sigma_{ab} \tag{3.2.22}$$

with $\sigma_{ab} \equiv \frac{1}{2}\sigma_a\sigma_b$ and Σ_c are the vector fields dual to the chosen n dimensional metric connection on the sphere.

Integrability conditions imply $\alpha = \pm 1$. There are exactly $2^{[\frac{n}{2}]}$ linearly independent solutions for each value of α [53, 54].

Specifying this result for $n = 3$ we get 2 solutions for each value of α. We will denote the two doublets as

$$\chi_\pm = \begin{pmatrix} \chi_\pm^1 \\ \chi_\pm^2 \end{pmatrix} \tag{3.2.23}$$

A natural choice for the n-bein of the 3-sphere is given by left invariant $SU(2)$ one forms. The spin connection is then given by (see the Appendix A for a complete derivation)

$$\bar{\Omega}_c{}^{ab} = \epsilon_{abc} \tag{3.2.24}$$

from which we can easily deduce that the spinors in the doublet χ_+ do not depend on the coordinates of the sphere. As we will explain in details in Section 5.5, the doublet χ_+ (resp. χ_-) transforms in the positive (negative) chirality representation of $SO(4)$ under $SO(4)$ isometry transformation on the sphere.

The bilinears

$$L_+^{ij} = \left(\chi_+^i\right)^\dagger \sigma^a \chi_+^j \Sigma_a \qquad L_-^{ij} = \left(\chi_-^i\right)^\dagger \sigma^a \chi_-^j \Sigma_a \tag{3.2.25}$$

are Killing vectors for the S^3 and generate the $SU(2)_+ \times SU(2)_-$ isometry of the metric. We can consider, for each group L_+, L_-, the four real combinations

$$L_\pm^{11}\,,\; L_\pm^{22}\,,\; L_\pm^+ \equiv \frac{1}{\sqrt{2}}\left(L_\pm^{12} + L_\pm^{21}\right)\,,\; L\pm^- \equiv \frac{1}{\mathrm{i}\sqrt{2}}\left(L_\pm^{12} - L_\pm^{21}\right) \tag{3.2.26}$$

Clearly, only three out of the four Killing vectors in each group are linearly independent.

For example, if we consider the positive chirality doublet we may identify the two spinors $\chi_+^{1,2}$ as the two eigenvectors of σ_3

$$\sigma_3 \chi^1 = \chi^1 \tag{3.2.27}$$

$$\sigma_3 \chi^2 = -\chi^2 \tag{3.2.28}$$

If we normalise the spinors such that $\left(\chi_+^i\right)^\dagger \chi_+^j = \delta^{ij}$, we get

$$L_+^{11} = -L^{22} = \Sigma_3 \qquad L_+^+ = \Sigma_1 \qquad L_+^- = -\Sigma_2 \tag{3.2.29}$$

Reduction to a four dimensional problem

The 3-sphere components of the spin connection for the metric (3.2.14) are given by

$$\nabla_{\hat{a}(\bar{a})} = \nabla'_{\hat{a}(\bar{a})} - \frac{1}{4}\Gamma^\mu{}_{\hat{a}(\bar{a})}\partial_\mu (H \pm G) \tag{3.2.30}$$

being $\nabla'_{\hat{a}(\bar{a})} = \partial_{\hat{a}(\bar{a})} + \bar{\Omega}_{\hat{a}(\bar{a})}^{\hat{b}\hat{c}(\bar{b}\bar{c})}\Gamma_{\hat{b}\hat{c}(\bar{b}\bar{c})}$. Let us calculate the contribution from the $F \cdot \Gamma$ contraction

$$\begin{aligned} M\Gamma_M \eta &= \frac{\mathrm{i}}{480}\Gamma_{M_1 \cdots M_5} F_{(5)}^{M_1 \cdots M_5} \Gamma_M \eta = \\ &= \frac{\mathrm{i}}{48}\left(\mathrm{e}^{-\frac{3}{2}(H+G)}\Gamma^{\mu\nu}F_{\mu\nu}\epsilon_{\hat{a}\hat{b}\hat{c}}\Gamma^{\hat{a}\hat{b}\hat{c}} - \mathrm{e}^{-\frac{3}{2}(H-G)}\Gamma^{\mu\nu}\tilde{F}_{\mu\nu}\epsilon_{\bar{a}\bar{b}\bar{c}}\Gamma^{\bar{a}\bar{b}\bar{c}}\right)\Gamma_M \eta = \\ &= -\frac{1}{4}\mathrm{e}^{-\frac{3}{2}(H+G)}\gamma_{\mu\nu}F^{\mu\nu}\gamma_5 \sigma_1 \Gamma_M \eta \end{aligned} \tag{3.2.31}$$

where in the last equality we have used the duality relations coming from (3.2.16), the identity

$$\gamma_5 \gamma_{\mu\nu} = \mathrm{i}\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma} \tag{3.2.32}$$

and the fact that the chirality condition implies

$$\gamma_5 \sigma_2 \eta = \mathrm{i}\sigma_1 \eta \qquad \gamma_5 \sigma_1 \eta = -\mathrm{i}\sigma_2 \eta \tag{3.2.33}$$

The a and $\tilde{a}$ components of (3.2.17) multiplied by respectively $\gamma_5\sigma_1\Gamma_{\hat{a}}$ and $\gamma_5\sigma_2\Gamma_{\tilde{a}}$ take the form

$$\gamma_5\sigma_{1(2)}(\Gamma_{\hat{a}(\tilde{a})}\nabla'_{\hat{a}(\tilde{a})} + \mathrm{e}^{\frac{1}{2}(H\pm G)}\frac{1}{4}\Gamma^\mu\partial_\mu(H\pm G) + \mathrm{e}^{\frac{1}{2}(H\pm G)}M)\eta = 0 \tag{3.2.34}$$

The 4 dimensional components of the spin connection are a function of $g_{\mu\nu}$ only and involve only μ components.

We can write

$$\begin{aligned}\gamma_5\sigma_1\Gamma_{\hat{a}}\nabla'_a &= 1\otimes 1(\sigma_{\hat{a}}\bar{\nabla}_{\hat{a}})\otimes 1\\ \gamma_5\sigma_2\Gamma_{\tilde{a}}\nabla'_{\tilde{a}} &= 1\otimes 1\otimes 1\otimes(\sigma_{\tilde{a}}\bar{\nabla}_{\tilde{a}})\end{aligned} \tag{3.2.35}$$

Thus, the most general solution can be written as

$$\eta = \varepsilon_{a,b}\otimes\chi_a\otimes\tilde{\chi}_b \tag{3.2.36}$$

with χ_a,χ_b obeying (3.2.21) with $\alpha = a, b$ respectively. The ε spinor is acted on by the γ_μ and σ_i matrices and satisfies the equations

$$\left(\mathrm{i}a\mathrm{e}^{-\frac{1}{2}(H+G)}\gamma_5\sigma_1 + \frac{1}{2}\gamma^\mu\partial_\mu(H+G) + 2M\right)\varepsilon = 0 \tag{3.2.37}$$

$$\left(\mathrm{i}b\mathrm{e}^{-\frac{1}{2}(H-G)}\gamma_5\sigma_2 + \frac{1}{2}\gamma^\mu\partial_\mu(H-G) - 2M\right)\varepsilon = 0 \tag{3.2.38}$$

$$\nabla_\mu\varepsilon + M\gamma_\mu\varepsilon = 0 \tag{3.2.39}$$

Spinors bilinears

We define the set of spinor bilinears

$$\begin{aligned}&K_\mu = -\bar{\varepsilon}\gamma_\mu\varepsilon \qquad L_\mu = \bar{\varepsilon}\gamma_5\gamma_\mu\varepsilon \qquad Y_{\mu\nu} = \bar{\varepsilon}\gamma_{\mu\nu}\sigma_1\varepsilon\\ &f_1 = \mathrm{i}\bar{\varepsilon}\sigma_1\varepsilon \qquad f_2 = \mathrm{i}\bar{\varepsilon}\sigma_2\varepsilon\\ &\bar{\varepsilon} = \varepsilon^\dagger\gamma_0\end{aligned} \tag{3.2.40}$$

Using (3.2.39) one can show that

$$\nabla_\mu f_1 = -\mathrm{e}^{-\frac{3}{2}(H-G)}\tilde{F}_{\mu\nu}K^\nu \tag{3.2.41}$$

$$\nabla_\mu f_2 = -\mathrm{e}^{-\frac{3}{2}(H+G)}F_{\mu\nu}K^\nu \tag{3.2.42}$$

$$\nabla_\mu K_\nu = \mathrm{e}^{-\frac{3}{2}(H+G)}F_{\mu\nu}f_2 + \mathrm{e}^{-\frac{3}{2}(H-G)}\tilde{F}_{\mu\nu}f_1 \tag{3.2.43}$$

$$\nabla_\mu L_\nu = \mathrm{e}^{-\frac{3}{2}(H+G)}\left[-\frac{1}{2}g_{\mu\nu}F_{\lambda\rho}Y^{\lambda\rho} - F_\mu{}^\rho Y_{\rho\nu} - F_\nu{}^\rho Y_{\rho\mu}\right] \tag{3.2.44}$$

From (3.2.43) one can see that K_μ is a Killing vector for $g_{\mu\nu}$ and by Fierz rearrangements one can show

$$K\cdot L = 0 \qquad L^2 = -K^2 = f_1^2 + f_2^2 \tag{3.2.45}$$

The standard ten dimensional Killing vector coming from the sandwich of the spinors is given by, in ten dimensional covariant tangent space components

$$\kappa = \left(K(\chi^\dagger\chi)(\tilde\chi^\dagger\tilde\chi), f_2\chi^\dagger\vec\sigma\chi(\tilde\chi^\dagger\tilde\chi), -f_1(\chi^\dagger\chi)\tilde\chi^\dagger\vec\sigma\tilde\chi\right)$$

Here χ stays for any one of the $\chi^i_\pm$. We also have, as expected from general observations, see Section 2.1.3,

$$\kappa^2 = K^2(\chi^\dagger\chi)^2(\tilde\chi^\dagger\tilde\chi)^2 + f_2^2\sum_a(\chi^\dagger\sigma_a\chi)(\chi^\dagger\sigma_a\chi)(\tilde\chi^\dagger\tilde\chi)^2 + f_1^2(\chi^\dagger\chi)^2\sum_a(\tilde\chi^\dagger\sigma_a\tilde\chi)(\tilde\chi^\dagger\sigma_a\tilde\chi) =$$
$$= \left(-f_1^2 - f_2^2 + f_2^2 + f_1^2\right)(\chi^\dagger\chi)^2(\tilde\chi^\dagger\tilde\chi)^2 = 0 \quad (3.2.46)$$

where we have used (3.2.45) and the identity

$$\sum_{a=1}^{3}\sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} = 2\,\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta} \quad (3.2.47)$$

which implies for every two dimensional spinor ζ that

$$\sum_a(\zeta^\dagger\sigma_a\zeta)(\zeta^\dagger\sigma_a\zeta) = (\zeta^\dagger\zeta)^2$$

The Killing spinor equation implies that

$$\partial_{\hat a}(\chi^\dagger\chi) = \partial_{\tilde a}(\tilde\chi^\dagger\tilde\chi) = 0 \quad (3.2.48)$$

and the vectors

$$(\mathbf{0}, e^{\frac{1}{2}(H+G)}\chi^\dagger\vec\sigma\chi, \vec 0) \quad , \quad (\mathbf{0}, \vec 0, e^{\frac{1}{2}(H-G)}\tilde\chi^\dagger\vec\sigma\tilde\chi) \quad (3.2.49)$$

are Killing vectors, corresponding to the $SO(4)\times SO(4)$ symmetries of our Ansatz.

The case $f_1 \neq 0$, $f_2 \neq 0$: LLM geometries

Assume that $f_{1,2} \neq 0$: this is the case that has been considered in [12] and the complete solution for the metric and five form has been given in Subsection 3.2.1. In [12] it is shown that

$$f_{1,2} \propto e^{\frac{1}{2}(H\mp G)} \quad (3.2.50)$$

and thus also the vector

$$(K, \vec 0, \vec 0) = \kappa + J(\tilde\chi^\dagger\tilde\chi) + \tilde J(\chi^\dagger\chi) \quad (3.2.51)$$

obtained as the sum of κ and a Killing vector of the $SO(4)\times SO(4)$ symmetry, is a Killing vector for the full metric. This vector is identified with ∂_t, which is possible since

$$K^2 = -f_1^2 - f_2^2 < 0 \quad (3.2.52)$$

The fact that $f_{1,2} \neq 0$ is thus crucial for all the construction of the 1/2 supersymmetric solutions in [12]. The Killing vector K corresponds to the generator of shifted dilatation operator $D' = D - J$ on the CFT side. The non compact $U(1)$ symmetry of the states which was not made explicit in the Ansatz (3.2.14) is thus correctly recovered as a direct consequence of the supersymmetry of the solutions.

The case $f_1 \neq 0, f_2 = 0$: unphysical solutions

The analysis performed in [12] breaks down in the case one or both of the $f_{1,2}$ functions vanishes. Assuming

$$f_1 \neq 0 \qquad f_2 = 0 \tag{3.2.53}$$

(the argument is analogous when $f_1 = 0, f_2 \neq 0$, simply interchanging the two S^3 spheres) In this case, the sum of (3.2.37),(3.2.38) basically implies

$$\varepsilon = 0 (\text{i.e. no supersymmetry})$$
$$\text{or}$$
$$\mathrm{e}^G = 0$$

If we force the analysis to the case $\mathrm{e}^G = 0$ we get

$$\mathrm{d}s^2 = y\mathrm{e}^{-G}(-\mathrm{d}t^2 + \mathrm{d}\tilde{\Omega}_3^2) + \frac{1}{y}\mathrm{e}^G(\mathrm{d}y^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j) + y\mathrm{e}^G\mathrm{d}\Omega_3^2 \tag{3.2.54}$$

$$\tilde{F} = \mathrm{d}\tilde{B} = -\frac{1}{2}y^2\mathrm{e}^{-2G}\frac{1}{y}\mathrm{d}y \wedge \mathrm{d}t - \frac{1}{2}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \tag{3.2.55}$$

$$F = \mathrm{e}^{3G} *_4 \tilde{F} = -\frac{1}{2}y^2\mathrm{e}^{2G}\frac{1}{y}\mathrm{d}y \wedge \mathrm{d}t - \frac{1}{2}\mathrm{e}^{2G}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \tag{3.2.56}$$

Thus one of the spheres shrinks to zero size and the other blows up to infinity.

3.3 LLM plane waves

An interesting class of solutions to the Killing spinor equations which were not considered in [12] is given by the case

$$f_1 = f_2 = 0 \tag{3.3.1}$$

which implies that κ

$$\kappa = (K(\chi^\dagger\chi)(\tilde{\chi}^\dagger\tilde{\chi}), \vec{0}, \vec{0})$$

is *covariantly constant* and we get immediately

$$\partial H \cdot K = \partial G \cdot K = 0 \tag{3.3.2}$$

We can assume

$$(\chi^\dagger\chi)(\tilde{\chi}^\dagger\tilde{\chi}) = 1 \tag{3.3.3}$$

so there is no distinction between K and κ.

Choice of coordinates

We choose a coordinate v such that $K = \partial_v$. The most general four dimensional metric $g_{\mu\nu}$ will be of the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + A(u, x^1, x^2)\mathrm{d}u^2 + V_i(u, x^1, x^2)\mathrm{d}u\mathrm{d}x^i + h_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

and we can choose the x^i coordinates such that $V_i = 0$ and thus

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + A(u, x^1, x^2)\mathrm{d}u^2 + h_{ij}(u, x^1, x^2)\mathrm{d}x^i\mathrm{d}x^j \tag{3.3.4}$$

The target space index gamma matrices are defined by

$$\gamma_\mu = \gamma_\alpha e^\alpha{}_\mu \tag{3.3.5}$$

where we assume μ, ν... as target space indices and α, β... as tangent space indices and $e^\alpha{}_\mu$ is the vierbein. We thus have

$$\begin{aligned} \gamma_u &= \frac{A}{2}\gamma_- + \gamma_+ \\ \gamma_v &= \gamma_- \\ \gamma_\pm &= \frac{1}{\sqrt{2}}(\gamma_3 \pm \gamma_0) \end{aligned} \tag{3.3.6}$$

First projector on ε

The gauge choice for the v coordinate implies

$$K_u = 1, K_v = K_i = 0 \tag{3.3.7}$$

and thus

$$0 = K_v = -\varepsilon^\dagger\gamma_0\gamma_-\varepsilon = \frac{1}{\sqrt{2}}\varepsilon^\dagger(1 + \gamma_0\gamma_3)\varepsilon\,. \tag{3.3.8}$$

The operator

$$P = \frac{1}{2}(1 + \gamma_0\gamma_3) \tag{3.3.9}$$

is an hermitian projection operator: $P^\dagger = P, P^2 = P$ and thus eq. (3.3.8) implies

$$\gamma_0\gamma_v\varepsilon = 0 \quad \text{i.e.} \quad \gamma_0\gamma_3\varepsilon = -\varepsilon \tag{3.3.10}$$

Let us now study the other consequences of gauge choice we have made. From the first relation in (3.3.7)

$$\begin{aligned} 1 = K_u = -\varepsilon^\dagger\gamma_0(\frac{A}{2}\gamma_- + \gamma_+)\varepsilon = \\ = -\varepsilon^\dagger\gamma_0\gamma_+ = \frac{1}{\sqrt{2}}\varepsilon^\dagger(1 - \gamma_0\gamma_3)\varepsilon = \sqrt{2}\,\varepsilon^\dagger\varepsilon \end{aligned} \tag{3.3.11}$$

i.e. $\varepsilon^\dagger\varepsilon = \frac{1}{\sqrt{2}}$. The other two equations in (3.3.7) are implied by (3.3.10)

$$\begin{gathered} K_i = -\varepsilon^\dagger\gamma_0\gamma_i\varepsilon = \varepsilon^\dagger\gamma_0\gamma_i\gamma_0\gamma_3\varepsilon = \varepsilon^\dagger\gamma_3\gamma_0\gamma_0\gamma_i\varepsilon = \varepsilon^\dagger\gamma_0\gamma_i\varepsilon = -K_i \\ \Rightarrow \\ K_i = 0 \end{gathered} \tag{3.3.12}$$

Constraints on F and its contractions

From (3.2.41),(3.2.42) we get

$$F \cdot K = \tilde{F} \cdot K = 0 \tag{3.3.13}$$
$$\Rightarrow \tag{3.3.14}$$
$$F = F_{ui}\mathrm{d}u \wedge \mathrm{d}x^i \tag{3.3.15}$$

Thus we have

$$M\varepsilon = -\frac{1}{4}\mathrm{e}^{-\frac{3}{2}(H+G)}F_{\mu\nu}\gamma^{\mu\nu}\gamma_5\sigma_1\varepsilon = \frac{1}{4}\mathrm{e}^{-\frac{3}{2}(H+G)}F_{ui}\gamma^i{}_v\varepsilon = 0 \tag{3.3.16}$$

We take the sum of (3.2.37) and (3.2.38), multiplied by σ_1, and its adjoint

$$\sigma_1\partial\!\!\!/ H\varepsilon = (-\mathrm{i}a\mathrm{e}^{-\frac{1}{2}(H+G)}\gamma_5 + b\mathrm{e}^{-\frac{1}{2}(H-G)})\varepsilon \tag{3.3.17}$$
$$\bar{\varepsilon}\sigma_1\partial\!\!\!/ H = -\bar{\varepsilon}(-\mathrm{i}a\mathrm{e}^{-\frac{1}{2}(H+G)}\gamma_5 + b\mathrm{e}^{-\frac{1}{2}(H-G)}) \tag{3.3.18}$$

obtaining

$$\partial_\mu H f_1 = \frac{\mathrm{i}}{2}\bar{\varepsilon}\{\gamma_\mu, \partial\!\!\!/ H\sigma_1\}\varepsilon = -a\mathrm{e}^{-\frac{1}{2}(H+G)}L_\mu \tag{3.3.19}$$

and thus

$$L_\mu = 0 \tag{3.3.20}$$

This equation implies two new constraints on ε

$$\begin{aligned} L_0 &= -\varepsilon^\dagger\gamma_0\gamma_0\gamma_5\varepsilon = \varepsilon^\dagger\gamma_5\varepsilon = 0 \\ L_3 &= \varepsilon^\dagger\gamma_0\gamma_3\gamma_5\varepsilon = -\varepsilon^\dagger\gamma_5\varepsilon = 0 \end{aligned} \tag{3.3.21}$$

which will be taken into account in the following. The other two components of $i = 1, 2$ of (3.3.20) are trivially satisfied

$$\begin{gathered} L_i = -\varepsilon^\dagger\gamma_0\gamma_i\gamma_5\varepsilon = \varepsilon^\dagger\gamma_0\gamma_i\gamma_0\gamma_3\gamma_5\varepsilon = \varepsilon^\dagger\gamma_3\gamma_0\gamma_0\gamma_i\gamma_5\varepsilon = \varepsilon^\dagger\gamma_0\gamma_i\gamma_5\varepsilon = -L_i \\ \Rightarrow \\ L_i = 0\,. \end{gathered} \tag{3.3.22}$$

From (3.2.44) we get

$$\frac{1}{2}g^{\mu\nu}F_{\lambda\rho}Y^{\lambda\rho} + F^\mu{}_\rho Y^{\rho\nu} + F^\nu{}_\rho Y^{\rho\mu} = 0 \tag{3.3.23}$$

As a consequence of (3.3.10) all the components of $Y^{\mu\nu}$ vanish apart from the Y^{vi} which has to satisfy the constraint

$$F_{ui}Y^{vi} = 0 \tag{3.3.24}$$

The latter can be rewritten as

$$(\varepsilon^\dagger \gamma_j \sigma_1 \varepsilon)\, h^{ji}\, F_{ui} = 0 \tag{3.3.25}$$

Solution for G, H and h

We can now write again the sum of (3.2.37),(3.2.38) multiplied by σ_1 and the difference of the two, remembering $M\varepsilon = 0$

$$(\sigma_1 \partial\!\!\!/ H + \mathrm{i}a\mathrm{e}^{-\frac{1}{2}(H+G)}\gamma_5)\varepsilon = b\mathrm{e}^{-\frac{1}{2}(H-G)}\varepsilon \tag{3.3.26}$$

$$(\sigma_1 \partial\!\!\!/ G + \mathrm{i}a\mathrm{e}^{-\frac{1}{2}(H+G)}\gamma_5)\varepsilon = -b\mathrm{e}^{-\frac{1}{2}(H-G)}\varepsilon \tag{3.3.27}$$

Using $\gamma_v\varepsilon = 0$, $\partial H \cdot K = \partial_v H = 0$ we can expand the derivatives into

$$\sigma_1 \partial\!\!\!/ H\varepsilon = (A\gamma_u\partial_u H + \gamma_c\, e^{ci}\partial_i H)\varepsilon \tag{3.3.28}$$

and analogous for G. $e^c{}_i$ is the 2 dimensional vierbein of h_{ij}. Since we have

$$\gamma_5\sigma_3\varepsilon = \varepsilon \tag{3.3.29}$$

$$\gamma_0\gamma_3\varepsilon = -\varepsilon \tag{3.3.30}$$

we need

$$\partial_u H = \partial_u G = 0 \tag{3.3.31}$$

One can note, that in the subspace identified by the two projectors (3.3.29),(3.3.30) above the matrices $\sigma_1\gamma_1, \sigma_1\gamma_2, \gamma_5$ act as the generators of the $SU(2)$ algebra, and we can call them respectively $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3$.

By taking the square (3.3.26),(3.3.27) it is now immediate to get that the functions H and G must satisfy the conditions

$$(\partial H)^2 - \mathrm{e}^{-(H+G)} = \mathrm{e}^{-(H-G)} \tag{3.3.32}$$

$$(\partial G)^2 - \mathrm{e}^{-(H+G)} = \mathrm{e}^{-(H-G)} \tag{3.3.33}$$

We now choose coordinates

$$x^1 = r^1 = \mathrm{e}^{\frac{1}{2}(H+G)} \tag{3.3.34}$$

$$x^2 = r^2 = \mathrm{e}^{\frac{1}{2}(H-G)} \tag{3.3.35}$$

In these coordinates the equations (3.3.32),(3.3.33) can be written

$$\left(\frac{1}{r^1}\right)^2 h^{11} + \left(\frac{1}{r^2}\right)^2 h^{22} + 2\frac{1}{r^1 r^2}h^{12} = \left(\frac{1}{r^1}\right)^2 + \left(\frac{1}{r^2}\right)^2 \tag{3.3.36}$$

and thus

$$h^{11} = h^{22} = 1 \qquad h^{12} = 0\,. \tag{3.3.37}$$

The metric h is thus flat.

Second projector on ε and its consequences

We can now rewrite (3.3.26),(3.3.27) as

$$\left(\frac{1}{r^1}\bar{\sigma}_1 + \frac{1}{r^2}\bar{\sigma}_2 + \mathrm{i}a\frac{1}{r^1}\bar{\sigma}_3\right)\varepsilon = b\frac{1}{r^2}\varepsilon \tag{3.3.38}$$

$$\left(\frac{1}{r^1}\bar{\sigma}_1 - \frac{1}{r^2}\bar{\sigma}_2 + \mathrm{i}a\frac{1}{r^1}\bar{\sigma}_3\right)\varepsilon = -b\frac{1}{r^2}\varepsilon\,; \tag{3.3.39}$$

multiplying by r^2 we get

$$(\mathrm{e}^{-G}\bar{\sigma}_1 + \bar{\sigma}_2 + \mathrm{i}a\mathrm{e}^{-G}\bar{\sigma}_3)\varepsilon = b\varepsilon \tag{3.3.40}$$

$$(\mathrm{e}^{-G}\bar{\sigma}_1 - \bar{\sigma}_2 + \mathrm{i}a\mathrm{e}^{-G}\bar{\sigma}_3)\varepsilon = -b\varepsilon \tag{3.3.41}$$

and taking the sum and the difference of the 2 previous equations we get

$$(\bar{\sigma}_1 + \mathrm{i}a\bar{\sigma}_3)\varepsilon = 0 \tag{3.3.42}$$

$$(\bar{\sigma}_2 - b)\varepsilon = 0\,. \tag{3.3.43}$$

Putting the second equation into the first one we can show that we have non trivial solution if and only if

$$b = -a \tag{3.3.44}$$

$$\bar{\sigma}_2\varepsilon = \sigma_1\gamma_2\varepsilon = b\varepsilon \tag{3.3.45}$$

As a consequence of (3.3.45), we have

$$\varepsilon^\dagger\sigma_1\gamma_i\varepsilon = (0, b\varepsilon^\dagger\varepsilon) \tag{3.3.46}$$

and, using (3.3.25)

$$0 = (\varepsilon^\dagger\gamma_j\sigma_1\varepsilon)\, h^{ji}\, F_{ui} = (\varepsilon^\dagger\gamma_j\sigma_1\varepsilon)\, \delta^{ji}\, F_{ui} = b\varepsilon^\dagger\varepsilon F_{u2}$$

$$\Rightarrow$$

$$F_{u2} = 0 \tag{3.3.47}$$

Moreover, the condition (3.3.21)

$$\varepsilon^\dagger\gamma_5\varepsilon = \varepsilon^\dagger\bar{\sigma}_3\varepsilon = 0 \tag{3.3.48}$$

is satisfied again as consequence of (3.3.45).

Summarising, we have a nonzero solution of (3.2.37),(3.2.38) and the constraints coming from the bilinear equations if

$$\gamma_5\sigma_3\varepsilon = \varepsilon \tag{3.3.49}$$

$$\gamma_0\gamma_3\varepsilon = -\varepsilon \tag{3.3.50}$$

$$\sigma_1\gamma_2\varepsilon = b\varepsilon \tag{3.3.51}$$

$$b = -a \tag{3.3.52}$$

Since the 3 operators appearing on the l.h.s. commute and each of them squares to the identity and is traceless and since each product of one, two or three of them is again traceless, these conditions select a 1 dimensional subspace of ε in the original 8 dimensional one.

Differential equations

We can now go back to the four dimensional part of the Killing spinor equation, that is (3.2.39)

$$\nabla_\mu \varepsilon + M\gamma_\mu \varepsilon = 0 \tag{3.3.53}$$

noting again

$$M = -\frac{1}{4}e^{-\frac{3}{2}(H+G)}\gamma_{\mu\nu}F^{\mu\nu}\gamma_5\sigma_1\gamma_5 \tag{3.3.54}$$

$$\gamma_{\mu\nu} = -i\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma} \tag{3.3.55}$$

$$F^* = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}dx^\mu \wedge dx^\nu = -F_{u1}du \wedge dx^2 \tag{3.3.56}$$

we can write

$$\nabla_\mu \varepsilon - \left(\frac{i}{4}\frac{1}{(r^1)^3}F_{u1}\gamma_v\gamma_2\sigma_1\right)\gamma_\mu\varepsilon = 0 \tag{3.3.57}$$

Calculating the spin connection and using $\gamma_v\varepsilon = 0$ we get

$$\nabla_\mu\varepsilon = \partial_\mu\varepsilon \tag{3.3.58}$$

The $\mu = v, 1, 2$ components of (3.3.57) simply imply

$$\partial_v\varepsilon = \partial_1\varepsilon = \partial_2\varepsilon = 0 \tag{3.3.59}$$

while the u component con be written as

$$\partial_u\varepsilon = \frac{i}{2}\frac{1}{(r^1)^3}F_{u1}\gamma_2\sigma_1\varepsilon = b\frac{i}{2}\frac{1}{(r^1)^3}F_{u1}\,\varepsilon \tag{3.3.60}$$

Solutions of the constraints and final expressions

Consistency with (3.3.59) requires

$$F_{u1} = (r^1)^3\lambda(u) \tag{3.3.61}$$

and thus

$$F_{(5)} = \lambda(u)du \wedge \left((r^1)^3dr_1 \wedge d\Omega + (r^2)^3dr^2 \wedge d\tilde{\Omega}\right) \tag{3.3.62}$$

We can solve the differential equation for ε

$$\varepsilon(u) = e^{b\frac{i}{2}\int_{u0}^{u}\lambda(t)dt}\varepsilon(u_0) \tag{3.3.63}$$

The full ten dimensional metric is now given by

$$ds^2 = 2dudv + A(u, r^1, r^2)du^2 + (dr^1)^2 + (dr^2)^2 + (r^1)^2d\Omega + (r^2)^2d\tilde{\Omega} \tag{3.3.64}$$

which is actually flat in the 8 dimensional transverse space.

Minkowski coordinates

Switching to Minkowski coordinates

$$ds^2 = \mathrm{d}u\mathrm{d}v + A\left(u, \sum_{i=1}^{4}(x^i)^2, \sum_{i=5}^{8}(x^i)^2\right)\mathrm{d}u^2 + \sum_{i=1}^{8}(\mathrm{d}x^i)^2 \tag{3.3.65}$$

$$F_{(5)} = \lambda(u)\mathrm{d}u \wedge \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 + \mathrm{d}x^5 \wedge \mathrm{d}x^6 \wedge \mathrm{d}x^7 \wedge \mathrm{d}x^8\right) \tag{3.3.66}$$

As we will discuss in the next Section, the supersymmetry of the solutions is not sufficient to guarantee that the Einstein equations are satisfied in this case. One equation has still to be imposed:

$$\triangle A = -32\lambda(u)^2 \tag{3.3.67}$$

The complete solutions are the most general half-BPS plane wave which has $SO(4)\times SO(4)$ symmetry. If one of these plane wave solutions could be in the class of solutions studied in [12] it should have 16 extra Killing spinors whose bilinears $f_{1,2}$ do not vanish. This means that the solution must have 32 Killing spinors and it will be the maximally supersymmetric plane wave of [55].

3.4 Einstein equations and counting of supersymmetries

The complex dimension of the subspace of η of the solutions to the Killing spinor equation is given by

$$4 \times 4 \times 8 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{2} = 8 \tag{3.4.1}$$

The first two terms are dimension of the spaces $\chi^i_\pm$, $\tilde{\chi}^i_\pm$ which has to be multiplied by the dimensionality of the ε space. This is reduced by a factor of $\frac{1}{2}$ by 10 dimensional chirality projector, other two factors of $\frac{1}{2}$ come from the two projectors in (3.3.50),(3.3.51) and finally we have to add a factor of $\frac{1}{2}$ because of the condition $a = -b$: we have thus half of the original 16 complex supersymmetries. Looking carefully at the Appendix of [12] one can realise that in the LLM case the counting proceeds in exactly the same manner with the only distinction that the final projector is given by[4] $a = b$.

As we have outlined in Section 2.1.3, having enough supersymmetries can be sufficient to assure that the Einstein equations are satisfied. In particular, in type IIB supergravity, it is necessary and sufficient to have two Killing spinors η, η' whose null vector bilinears

$$\kappa = \eta^\dagger\Gamma^M\eta\,\partial_M \qquad \kappa' = \eta'^\dagger\Gamma^M\kappa,\partial_M \tag{3.4.2}$$

are not proportional and thus have non vanishing scalar product. We have seen in Section 3.2.2 that the bilinear κ for generic η is given by

$$\kappa = \left(K(\chi^\dagger\chi)(\tilde{\chi}^\dagger\tilde{\chi}), f_2\chi^\dagger\vec{\sigma}\chi(\tilde{\chi}^\dagger\tilde{\chi}), -f_1(\chi^\dagger\chi)\tilde{\chi}^\dagger\vec{\sigma}\tilde{\chi}\right)$$

[4]For the dual of states satisfying $\Delta = J$. The dual of states with $\Delta = -J$ satisfy $a = -b$

Each of spinors $\chi, \tilde{\chi}$ can effectively be in a 4 dimensional space and thus carries tho indices

$$\chi = \chi_a^j \qquad \tilde{\chi} = \tilde{\chi}_b^k \tag{3.4.3}$$
$$a, b = \pm \quad i, j = 1, 2$$

The generic κ has thus four indices. However both in the original LLM construction and in the derivation of LLM plane waves the indices a and b are not independent. It must be either $a = b$ or $a = -b$. Hence, there are effectively three indices which can take two values each and give the eight independent Killing spinors that we have counted at the beginning of this Section. Notice that the spinor ε is independent of such indices. Referring to the notation of page 38, let us consider the two independent Killing spinors

$$\eta = \varepsilon \otimes \chi_+^1 \otimes \tilde{\chi}_+^1 \tag{3.4.4}$$
$$\eta' = \varepsilon \otimes \chi_+^1 \otimes \left(\frac{\tilde{\chi}_+^1 + \tilde{\chi}_+^2}{\sqrt{2}} \right) \tag{3.4.5}$$

We choose to normalise the spinors $\chi, \tilde{\chi}$ to unity. In components, the two corresponding Killing vectors have the form

$$\kappa = (K; 0, 0, f_2; 0, 0, -f_1) \tag{3.4.6}$$
$$\kappa' = (K; 0, 0, f_2; -f_1, 0, 0) \tag{3.4.7}$$

The two Killing vector bilinears are precisely of the required form. They are both null, and their scalar product is given by

$$\kappa \cdot \kappa' = K^2 + f_2^2 = -f_1^2 \tag{3.4.8}$$

which is non vanishing in the LLM case.

In the plane wave case, $f_1 = f_2 = 0$, and thus, independently on the choice of spinors $\chi, \tilde{\chi}$, the vector bilinear has the form

$$\kappa = (K; 0, 0, 0; 0, 0, 0) \tag{3.4.9}$$

which means that all the bilinears are proportional to each other, as expected from general observation on plane waves. This is the reason why in the plane wave case we had to impose one of the Einstein equations, the ++ component in the language of Section 2.1.3, at the end of the analysis.

Chapter 4

Singularities and Closed Timelike Curves in LLM Geometries

As we have described, LLM solutions are determined by a single function which obeys an elliptic equation. In this Chapter we consider the most general allowed (on the supergravity side) boundary conditions for the elliptic equation. This means that we study the full set of moduli of this sector of supergravity that consists of metrics asymptotic to $AdS_5 \times S^5$, with an $\mathbb{R} \times SO(4) \times SO(4)$ isometry group and preserving half of the supersymmetry of type IIB string theory. The supergravity solutions in general will be singular. The spacetime singularities appearing are always naked and fall into two distinct classes: null and timelike. The null ones can be considered as seeded by a fermion density between 0 and 1 and have been considered in, for example [56, 57, 58, 59, 60, 47] together with the possible *local* quantum effects responsible for their resolution: the singularity is due to an average over configurations of N fermions in a gas with average density less than one. An individual configuration with the same asymptotics can actually be shown to have as source a collection of N giant gravitons [51, 43] separated one from the other. In the supergravity theory, the resolution of the singularity thus appears as a sort of space-time foam [61, 47] while in the dual CFT one sees that such a configuration corresponds to the Coulomb branch of the theory.

The AdS/CFT correspondence has maybe something more interesting to tell us about the fate of the timelike singularities. The solutions with this kind of singularity are highly "pathological": they have closed timelike curves passing through *any point* of the spacetime and they include unbounded from below negative mass excitations of $AdS_5 \times S^5$.

It has already been conjectured, [62, 63, 64], that geometries with these features should be considered as truly unphysical via *global* considerations in the setting of a full quantum theory of gravity. The AdS/CFT correspondence applied to the space-times of [12] suggests one particular mechanism for the global removal of solutions containing timelike singularities. The deformations of the geometry which produce these singularities apparently correspond to negative dimension operators in the dual field theory. The unitarity of the representations of the superalgebra $SU(2,2|4)$ [25, 26] indicate in particular that unitary operators must have a positive conformal dimension. Our observations indicate that there should actually exist a general proof of the chronology protection conjecture [65] in this sector of supergravity . A first indication of this mechanism linking unitarity to chronology protection can be found in [66] and in the current context in [67, 68].

Extending these works, we prove here that closed timelike curves (CTCs) are unavoidable in any solution with a timelike singularity and that they are excluded in the case of regular and null singular solutions, these being the spacetimes that can be represented in terms of dual fermions, a result anticipated but not proven in [67]. This provides a clear division between these two classes of singular spacetimes which is also reflected in the two different mechanisms responsible for the resolution of their respective spacetime singularities.

In Section 4.1 we review in more details the construction of [12] and we show the most general allowed boundary conditions for a supergravity solution satisfying the symmetry requirements. We clarify the role of the boundary conditions in determining the radius of the asymptotic $AdS_5 \times S^5$ and we show the relation between the boundary conditions and the appearance of spacetime singularities.

In Section 4.2 we exhibit some examples of singular supergravity solutions and we uncover some of their properties such as CTCs and peculiar geometric features. In particular we exhibit unbounded from below (for fixed AdS radius) negative mass excitations of $AdS_5 \times S^5$.

In Section 4.3 we show that most of the interesting features of the examples in Section 4.2, regarding mainly the appearance and the properties of CTCs, are generic for the case of solutions with timelike singularities. Moreover we prove a theorem which clearly relates the appearance of CTCs to the boundary conditions responsible for timelike singularities.

In Section 4.4 we return to a discussion of the meaning of these results, and in particular the possibility of proving the chronology protection conjecture for this class of geometries, by showing that the AdS/CFT correspondence relates naked time machines to non-unitarity in the CFT.

This Chapter is mainly based on [16].

4.1 LLM construction: regular and singular solutions

In the first part of this section we review the construction of [12] in a language adapted to the considerations that follow in the rest of this paper.

In [12] a class of BPS solutions of type IIB supergravity is constructed. This is the most general class of BPS solutions in type IIB supergravity with $SO(4) \times SO(4)$ isometry, one timelike Killing vector and a non-trivial self-dual 5-form field strength $F_{(5)}$. The solutions are given by

$$ds^2 = -h^{-2}(\mathrm{d}t + V_i \mathrm{d}x^i)^2 + h^2(\mathrm{d}y^2 + \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j) + y\mathrm{e}^G \mathrm{d}\Omega_3^2 + y\mathrm{e}^{-G}\mathrm{d}\widetilde{\Omega}_3^2 \tag{4.1.1}$$

$$F_{(5)} = F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\Omega + \widetilde{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\widetilde{\Omega} \tag{4.1.2}$$

$$F = \mathrm{e}^{3G} *_4 \widetilde{F} \tag{4.1.3}$$

$$\tag{4.1.4}$$

with $y \geq 0$.
We can define a function $z = z(x_1, x_2, y)$ which determines the entire solution (up to choice of gauge that we discuss below),

$$z \equiv \frac{1}{2}\tanh G \tag{4.1.5}$$

$$h^{-2} = 2y\cosh G = \frac{y}{\sqrt{(1/2 - z)(1/2 + z)}} \tag{4.1.6}$$

$$\mathrm{d}V = \frac{1}{y} *_3 \mathrm{d}z \tag{4.1.7}$$

$$F = \mathrm{d}(B_t(\mathrm{d}t + V)) + \mathrm{d}\hat{B} \tag{4.1.8}$$

$$\tilde{F} = \mathrm{d}(\tilde{B}_t(\mathrm{d}t + V)) + \mathrm{d}\hat{\tilde{B}} \tag{4.1.9}$$

$$B_t = -\frac{1}{4}y^2\mathrm{e}^{2G} \qquad \tilde{B}_t = -\frac{1}{4}y^2\mathrm{e}^{-2G} \tag{4.1.10}$$

$$\mathrm{d}\hat{B} = -\frac{1}{4}y^3 *_3 \mathrm{d}\left(\frac{1/2 + z}{y^2}\right) \qquad \mathrm{d}\hat{\tilde{B}} = \frac{1}{4}y^3 *_3 \mathrm{d}\left(\frac{1/2 - z}{y^2}\right) \tag{4.1.11}$$

where $*_n$ indicate the Hodge dual in n flat dimensions.
For the consistency of (4.1.7) and (4.1.11) we must have

$$(\partial_1^2 + \partial_2^2)z + y\partial_y(\frac{1}{y}\partial_y z) = 0 \tag{4.1.12}$$

The solutions for z are determined by boundary conditions in the $\{x_1, x_2, y\}$ space as we will now discuss.

4.1.1 Boundary conditions

The solution is well defined for z restricted to the range

$$-1/2 \leq z \leq 1/2 \tag{4.1.13}$$

Equation (4.1.12) implies that z takes its maximum and minimum on the boundary of its domain of definition[1] $\Sigma \subset \mathbb{R}^2 \times \mathbb{R}^+$. A solution of the supergravity equations is thus

[1] The equation (4.1.12) can be rewritten as

$$\left(\partial_1^2 + \partial_2^2 + \partial_y^2 - \frac{1}{y}\partial_y\right) z = 0 \tag{4.1.14}$$

specified by a choice of Σ, and by a function z_0 defined on $\partial\Sigma$ such that

$$\begin{aligned} z &= z_0 \qquad \text{on } \partial\Sigma \\ -1/2 &\leq z_0 \leq 1/2 \end{aligned} \tag{4.1.15}$$

Following [12] one can easily show that if Σ extends to infinity and z goes to either $1/2$ or $-1/2$ for $r^2 = x_1^2 + x_2^2 + y^2 \to \infty$, the solution is asymptotically $AdS_5 \times S^5$. Changing z into $-z$ is a symmetry of the solution and thus we assume for definiteness

$$z \longrightarrow \frac{1}{2} \text{ for } r^2 = x_1^2 + x_2^2 + y^2 \to \infty \tag{4.1.16}$$

We call $\partial\Sigma_0$ the intersection of $\partial\Sigma$ with the $y = 0$ plane, and $\partial\hat{\Sigma} = \partial\Sigma \setminus \partial\Sigma_0$. We note that if $z_0 \neq \pm\frac{1}{2}$ on $\partial\hat{\Sigma}$ then the metric can be analytically continued as far as $y = 0$ or $z = \pm\frac{1}{2}$. In general, after analytically continuing the solution, we have a larger "maximal" domain $\Sigma' \supset \Sigma$ where $-\frac{1}{2} \leq z \leq \frac{1}{2}$.

For convenience we will call again Σ this *maximal* domain of definition. The most general asymptotically $AdS_5 \times S^5$ solution of the supergravity equations is then specified by the domain Σ and a function z_0 on $\partial\Sigma$

$$\begin{cases} -\frac{1}{2} \leq z_0 \leq \frac{1}{2} & \text{on } \partial\Sigma_0 \\ z_0 = \pm\frac{1}{2} & \text{on } \partial\hat{\Sigma} \\ z_0 \to \frac{1}{2} & \text{for } r \to \infty \end{cases} \tag{4.1.17}$$

as illustrated in Figure 1.

We define a new function Φ

$$\Phi \equiv \frac{\frac{1}{2} - z}{y^2} \tag{4.1.18}$$

The equation for z is equivalent to the Laplace equation for Φ on a flat six dimensional space of the form $\mathbb{R}^2 \times \mathbb{R}^4$ where x_1, x_2 are the coordinates on the $\mathbb{R}^2$ and y is the radius for spherical coordinate on the $\mathbb{R}^4$. Since (4.1.12) and the definition of Φ are singular for $y = 0$, Dirichlet boundary conditions for z on $y = 0$ take the role of charge sources for Φ located at $y = 0$. Thus Φ satisfies the equation

$$\begin{cases} (\partial_1^2 + \partial_2^2)\Phi + \frac{1}{y^3}\partial_y(y^3\partial_y\Phi) = *_6\mathrm{d} *_6 \mathrm{d}\Phi = -4\pi^2(\frac{1}{2} - z_0)\delta^{(4)}(y)\chi(\Sigma_0) \\ \Phi = \dfrac{\frac{1}{2} - z_0}{y^2} \text{ on } \partial\hat{\Sigma} \end{cases} \tag{4.1.19}$$

where

$$\chi(\Sigma_0)(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in \Sigma_0 \\ 0 & \text{otherwise} \end{cases}. \tag{4.1.20}$$

Assume that Q is an internal stationary point of z, then clearly $\partial_y z(Q) = 0$. The equation for z implies that $(\partial_1^2 + \partial_2^2 + \partial_y^2)\, z(Q) = 0$, and thus Q cannot be a maximum (nor a minimum).

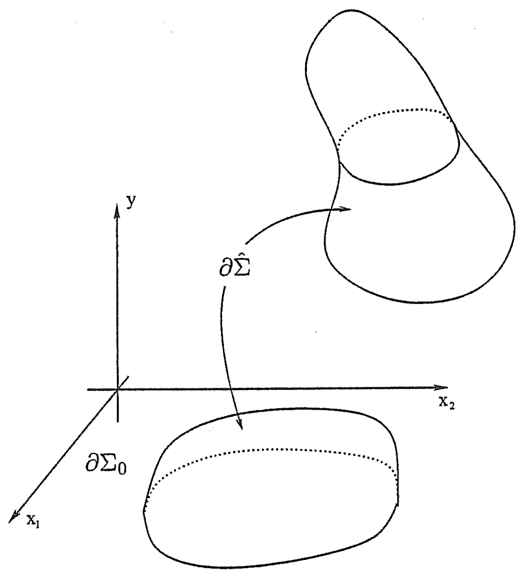


Figure 4.1: The domain of definition Σ

The forms $B, \tilde{B}$ and V are defined up to a gauge transformation. From now on we will use the following convenient gauge for V

$$\mathrm{d} *_3 V = 0 \iff \partial_1 V_1 + \partial_2 V_2 = 0 \tag{4.1.21}$$

4.1.2 Asymptotic behaviour

The boundary conditions (4.1.17) imply that

$$\Phi \longrightarrow \frac{A}{(x_1^2 + x_2^2 + y^2)^2} = \frac{A}{r^4}, \qquad r \to \infty, \qquad A > 0. \tag{4.1.22}$$

Integrating (4.1.19) we obtain

$$A\, 4\pi^3 = \int_{S^5(r)} (- *_6 \mathrm{d}\Phi) = \int_{\hat{\Sigma}^5} (- *_6 \mathrm{d}\Phi) + 4\pi^2 \int_{\partial\Sigma_0} \left(\frac{1}{2} - z_0(x_1, x_2)\right) \mathrm{d}x_1 \mathrm{d}x_2 \tag{4.1.23}$$

where $S^5(r)$ is a 5-sphere of radius r centred on the origin, and $\hat{\Sigma}^5$ is the 5-manifold obtained by the fibration in spherical coordinates of a 3-sphere $S^3(y)$ over $\partial\hat{\Sigma}$.
Going to polar coordinates R, φ in the $\{x_1, x_2\}$ sections and for $R^2 + y^2 \to \infty$, V has the asymptotic behaviour

$$V \approx V_\varphi \mathrm{d}\varphi \qquad V_\varphi \approx -A \frac{R^2}{(R^2 + y^2)^2}. \tag{4.1.24}$$

In [12] it has been shown that the quantity A determines the radius of the asymptotic $AdS_5 \times S^5$

$$R^2_{AdS_5} = R^2_{S^5} = A^{1/2} \tag{4.1.25}$$

In the asymptotic region we can construct a smooth five dimensional manifold $\tilde{\Lambda}_5$ by fibering the three sphere $\tilde{S}^3$ over a surface $\tilde{\Lambda}_2$.
The topology of $\tilde{\Lambda}_5$ is asymptotically S^5. The flux of the five form through this surface is given by

$$N = -\frac{1}{4\pi^4 l_P^4}\int_{\tilde{\Lambda}_5} \mathrm{d}\hat{\tilde{B}} \wedge \mathrm{d}\tilde{\Omega}_3 = -\frac{1}{16\pi^4 l_P^4}\int_{\tilde{\Lambda}_5} *_6 \mathrm{d}\Phi = \frac{1}{4\pi l_P^4} A = \frac{1}{4\pi l_P^4} R^4_{AdS_5} \qquad (4.1.26)$$

which agrees with the standard formula for the relation between the radius of AdS_5 and the flux of $F_{(5)}$.

The mass of the excitation of $AdS_5 \times S^5$ can be computed by looking at subleading terms in the expansion of Φ around $r \to \infty$.

4.1.3 Regular solutions and dual picture

If we choose $\Sigma = \mathbb{R}^2 \times \mathbb{R}^+$ the solution can be written as[2]

$$z = \frac{1}{2} - \Phi\, y^2 = \frac{1}{2} - \frac{y^2}{\pi}\int \frac{\rho(x_1', x_2')\mathrm{d}^2 x'}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2]^2} \qquad (4.1.27)$$

$$V_i = -\frac{1}{\pi}\epsilon_{ij}\int \frac{(x_j - x_j')\rho(x_1', x_2')\mathrm{d}^2 x'}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2]^2} \qquad (4.1.28)$$

with

$$\rho(x_1, x_2) = \frac{1}{2} - z_0(x_1, x_2) \qquad (4.1.29)$$

According to [12], in the dual field theory these excitation of $AdS_5 \times S^5$ are described by N free fermions. The plane $y = 0$ can be identified with the phase space of the dual fermions and the function $\rho(x_1, x_2)$ can be identified with the semiclassical density of these fermions.

It can be shown that the metric is regular if $\Sigma = \mathbb{R}^2 \times \mathbb{R}^+$ and z_0 takes the values $\pm 1/2$ on the $y = 0$ plane [12]. In these cases ρ is non vanishing just inside the "droplets" where $z_0 = -1/2$

$$\rho = \begin{cases} \beta = 1 & \text{inside the droplets} \\ 0 & \text{outside} \end{cases} \qquad (4.1.30)$$

Since we have assumed that $z \to 1/2$ at infinity, we can always find a circle large enough to encircle all "droplets". With these boundary conditions z is given by

$$z = \frac{1}{2} - \frac{y^2}{\pi}\int_{\mathcal{D}} \frac{\mathrm{d}^2 x'}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2]^2} \qquad (4.1.31)$$

[2]Note that

$$\lim_{y\to 0} \frac{y^2}{\pi}\frac{1}{(x_1^2 + x_2^2 + y^2)^2} = \delta^{(2)}(x_1, x_2)$$

$\mathcal{D}$ being the union of the droplets where $z = -1/2$. The V form can be written

$$V_i = -\frac{1}{\pi}\epsilon_{ij}\int_{\mathcal{D}} \frac{(x_j - x_j')\mathrm{d}^2x'}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2]^2} \tag{4.1.32}$$

The determinant of the sections $\{x_1, x_2, y\}$ is given by

$$\tilde{g} = h^4 - V_1^2 - V_2^2 = \frac{1/4 - z^2}{y^2} - V^2 \tag{4.1.33}$$

Note that here and in the following V^2 is formed by contracting indices using the Kronecker delta, i.e. $V^2 \equiv V_1^2 + V_2^2$. Theorem 4.3.1 of Section 4 states that for any[3] $\mathcal{D}$, $\tilde{g} \geq 0$ and the $\{x_1, x_2, y\}$ sections do not contain time-like directions. This guarantees in particular that the original LLM solutions are free of CTCs and are "good" supergravity solutions.

From the analysis of Section 4.1.2, we can deduce that the radius of the asymptotic $AdS_5 \times S^5$ is given by

$$R_{AdS}^4 = \frac{\mathcal{S}}{\pi} = A \tag{4.1.34}$$

where

$$\mathcal{S} = \int_{\mathcal{D}} \mathrm{d}x_1\mathrm{d}x_2 \tag{4.1.35}$$

is the total area of all droplets where $z = -\frac{1}{2}$ $(\rho = 1)$. The quantisation of the flux (4.1.26) gives the quantisation condition on the area of the droplets

$$\mathcal{S} = 4\pi^2 l_p^4 N \tag{4.1.36}$$

If $\mathcal{D}$ consists of one single circular droplet then the spacetime is precisely $AdS_5 \times S^5$. For a generic set of droplets $\mathcal{D}$ the mass (and the angular momentum) of the excitation is given by

$$M = J = \frac{1}{8\pi^2 l_P^8}\left[\frac{1}{2\pi}\int_{\mathcal{D}}(x_1^2 + x_2^2)\mathrm{d}^2x - \left(\frac{1}{2\pi}\int_{\mathcal{D}}\mathrm{d}^2x\right)^2\right] \geq 0 \tag{4.1.37}$$

The origin of the coordinates is chosen such that the dipole vanishes, that is,

$$\int_{\mathcal{D}} x_i \mathrm{d}^2x = 0. \tag{4.1.38}$$

Not surprisingly one can show by a direct calculation that the equality $(M = J = 0)$ holds for a single disk. Given any $\mathcal{D}$ we can build a disk $C_{\mathcal{D}}$ of the same area. The first term is clearly larger for $\mathcal{D}$ than for $C_{\mathcal{D}}$ and thus in general $M > 0$ for the non-singular solutions.

[3] Even extended to infinity, that is relaxing the hypotheses $z \to 1/2$ for $r \to \infty$ and allowing more general asymptotics than $AdS_5 \times S^5$

4.1.4 More general boundary conditions and singularities

In all cases with boundary conditions different from the ones studied in [12] we have spacetime singularities.

It is easy to see that the solutions have a naked time-like singularity when $\partial\hat{\Sigma}$ is non-empty. Consider a surface in the region $y > 0$ on which $z = -1/2$ (the discussion does not change in any substantial way if instead we took $z = 1/2$). Choose a point Q on this surface and define a coordinate ϵ in the $\{x_1, x_2, y\}$ space orthogonal to this surface such that $z = -1/2 + \alpha\epsilon$ for some positive constant α. Complete ϵ to a new orthogonal coordinate system by introducing two coordinates v_i with origin at Q. This is just an orthogonal transformation and translation of the original coordinate system. At Q we can assume that V is finite with a power series expansion away from this point. The subleading terms in this expansion are not important for studying the singularity. We also define a new time coordinate near Q by $T = t + V_i(Q)x_i$. Keeping just the leading divergences and introducing $\rho = (\alpha\epsilon)^{5/4}$ the metric expanded around Q is

$$\mathrm{d}s^2 = \alpha\rho^{-2/5}(-\mathrm{d}T^2 + \mathrm{d}\tilde{\Omega}_3^2) + \frac{16}{25}\mathrm{d}\rho^2 + \rho^{2/5}(\mathrm{d}v_i^2 + \mathrm{d}\Omega_3^2). \tag{4.1.39}$$

A short calculation then shows that the metric is singular with scalar curvature as $\rho \longrightarrow 0$

$$R = -\frac{5}{16\rho^2} \tag{4.1.40}$$

and the singularity is clearly time-like with no horizon.

Singularities are located also on the subset of $\partial\Sigma_0$ where $z \neq \pm\frac{1}{2}$. All these singularities are naked and null.

Indeed assuming that $1/4 - z^2 \longrightarrow \alpha^2$ as $y \longrightarrow 0$ and looking at the $\{t, y\}$ sections we find

$$\mathrm{d}s^2 = -\alpha^{-1}\, y\, \mathrm{d}t^2 + \alpha\, y^{-1}\, \mathrm{d}y. \tag{4.1.41}$$

With the change of variables, $u = \sqrt{y/\alpha}\,\mathrm{e}^{-t/2}$ $v = \sqrt{y/\alpha}\,\mathrm{e}^{t/2}$, the metric becomes simply

$$\mathrm{d}s^2 = \mathrm{d}u\mathrm{d}v \tag{4.1.42}$$

and the singularity is along the curves, $u = 0$ and $v = 0$. The singularity is due to the way in which the radii of the two three spheres, S^3 and $\tilde{S}^3$ go to zero [56].

4.2 Singular solutions: some examples

Interpreting $\rho = 1/2 - z_0$ as the density of the dual fermions, one first natural generalisation of the boundary conditions in [12] is to have density $\rho \neq 1$. We note that for generic $\rho(x_1, x_2)$, the radius of the asymptotic $AdS_5 \times S^5$ is given by

$$R_{AdS}^4 = \frac{1}{\pi}\int \rho(x_1, x_2)\mathrm{d}^2x \tag{4.2.1}$$

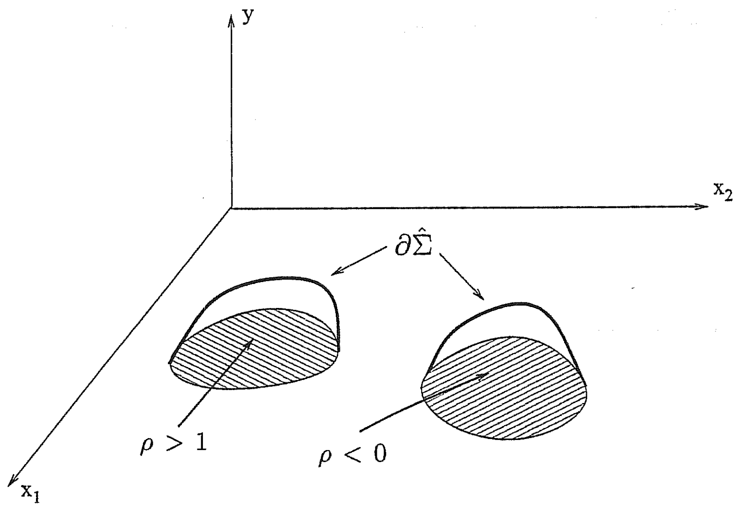


Figure 4.2: Two regions of $y = 0$ plane, one with $\rho > 1$ and the other with $\rho < 0$, leading to a non trivial $\partial\hat{\Sigma}$ attached to the $y = 0$ plane

We have that $0 \leq \rho \leq 1$ (that is $-1/2 \leq z_0 \leq 1/2$), if and only if $\partial\hat{\Sigma} = \emptyset$. In this case all the singularities will be null.

The mass of the excitation is now given by

$$M = \frac{1}{8\pi^2 l_P^8}\left[\frac{1}{2\pi}\int \rho(x_1, x_2)(x_1^2 + x_2^2)\mathrm{d}^2x - \left(\frac{1}{2\pi}\int \rho(x_1, x_2)\mathrm{d}^2x\right)^2\right] \tag{4.2.2}$$

with origin chosen again in such a way that the dipole vanishes

$$\int_{\mathcal{D}} \rho(x_1, x_2)x_i\mathrm{d}^2x = 0. \tag{4.2.3}$$

We note that for fixed value of R_{AdS} there is a lower bound on the mass obtained for $\rho = \pi R_{AdS}^4\delta^{(2)}(x_1, x_2)$,

$$M_{\min} = -\frac{R_{AdS}^8}{32\pi^2 l_P^8} \tag{4.2.4}$$

A priori we can consider also $\rho(x_1, x_2) < 0$ in some domains provided that the integral defining R_{AdS}^4 remains positive. One can easily see that the cases $\rho > 1$ and $\rho < 0$ correspond to choosing a $\partial\hat{\Sigma}$ not empty and attached to the $y = 0$ plane, as in Figure 2. Taking ρ negative in some region we can easily obtain arbitrary large negative value of the mass for fixed R_{AdS}. It's enough to have $\rho < 0$ even in a very small region provided it is located at large $x_1^2 + x_2^2$. In the next subsections we will restrict to the case $\rho \geq 0$, studying some examples with features that will serve as a guide for the general analysis of Section 4.

The appearance of CTCs, which we will show to be unavoidable in Section 4, and unbounded from below negative mass values suggest that one should consider as unphysical the geometries seeded by a density ρ that does not remain between 0 and 1. For the sake of causality and for the stability of the quantum version of the supergravity theory, these solutions should be regarded as unphysical on the basis of some *global* argument. If the singularity was resolved by quantum effects through some local mechanism and "smoothed", then the asymptotics and mass could not change significantly; moreover, we know that the existence of CTCs is a manifestation of global properties of the spacetime. Before discussing the possibility of such a resolution we will study these singular geometries in more detail.

For simplicity, we will first study the case of piecewise constant ρ. Assuming $\rho = \sum_i \beta_i \chi(\mathcal{D}_i)$ the z function can be written as

$$z = 1/2 - \frac{y^2}{\pi} \sum_i \beta_i \int_{\mathcal{D}_i} \frac{\mathrm{d}^2 x'}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2]^2} \tag{4.2.5}$$

The solution seeded by these density distributions can have null singularities or naked time machines. Solutions with null singularities are already discussed in the literature in various places [56, 57, 58, 59, 60] although we have some additional interesting observations to make. These issues will be discussed in section 3.1. In section 3.2 and 3.3 we will discuss the general features of configurations with naked timelike singularities, in particular illustrating a novel geometric mechanism for producing CTCs. In Section 3.4 we study the specific case of the geometry seeded by a circular droplet of density $\beta > 1$.

In Section 4.2.5 we discuss a class of solutions which does not have a density distribution $\rho(x_1, x_2)$ in the $y = 0$ plane as source, but rather appears as a natural continuation of the solutions studied in 4.2.4. These solutions are indeed determined by a $\partial\hat{\Sigma}$ which does not intersect the $y = 0$ plane. They exhibit CTCs and their mass is unbounded from below. Any possible direct connection to the free fermion picture is lost.

4.2.1 $\beta_i \leq 1$ with at least one $\beta_i < 1$

This case was already briefly considered in [12]. These geometries have null singularities located on the $y = 0$ plane inside the droplets. We will show in Section 4.3.3 that also these geometries are free of CTCs.

It is straightforward to show that if $\beta_i \leq 1$ the mass given by (4.2.2) is always nonnegative. These configurations can be viewed as an averaged version of a dilute gas of fermions. In this case one can think that the singularity is resolved by *local* quantum effects by the appearance of a "spacetime foam" [61] and in the dual theory by simply moving to the Coulomb branch of the moduli space.

Geometries corresponding to a single circular droplet of density $\beta < 1$ are precisely the solutions considered in [56].

In the limit that the radius goes to infinity, this describes the $N \to \infty$ limit of the Coulomb branch in the dual gauge theory, as amply discussed in [12]. The corresponding classical geometry is singular but is regularised as above by the dilute Fermi gas, or in geometric language, a dilute gas of giant gravitons, the geometry of which is clearly smooth.

This solution leads one to an interesting relation between a limit of the dual SCFT and the singular homogeneous plane wave metrics that arise generically as the Penrose limit of "reasonable" space-time singularities [69].

For simplicity one can actually consider the boundary condition $\rho = \beta < 1$ for all (x_1, x_2). Consider a null geodesic that ends on the "null" singularity and take the Penrose Limit with respect to this null geodesic.

In such a case it is easy to see that the resulting metric is exactly,

$$ds^2 = 2\mathrm{d}u\mathrm{d}v + (3(x_1^2 + x_2^2) - \sum_{i=1}^{6} w_i^2)\frac{\mathrm{d}u^2}{u^2} + \mathrm{d}x^2 + \mathrm{d}w^2. \tag{4.2.6}$$

In principle this provides a SYM dual description of the singular plane waves as a limit (analogous to the BMN [70] limit of AdS/CFT) of the $N \to \infty$ Coulomb branch in the original dual CFT.

4.2.2 Some $\beta_i > 1$

This boundary condition is equivalent to lifting the surface $z = -1/2$ above the $\{x_1, x_2\}$ plane keeping its boundary fixed at $y = 0$. The continuation of z inside this surface to $y = 0$ will give a non-trivial function everywhere less than $-1/2$. This is the first example of the non-empty $\partial\hat{\Sigma}$ introduced in Section 2.

The emerging geometries have timelike singularities on $\partial\hat{\Sigma}$ and CTCs. They include also negative (but bounded from below) mass excitations of $AdS_5 \times S^5$, as anticipated at the beginning of this section.

In the next subsections we will focus on the $\{x_1, x_2, y\}$ sections. They contain almost all of the interesting features.

4.2.3 Zooming

We consider the leading term of the expansion of z and V for points close to $y = 0$ and the boundary of one droplet of constant density $\beta > 1$. More precisely, with L the typical dimension of the droplet and R the radius of curvature of the boundary, we assume that y and the distance to the boundary are both much smaller than L and R[4]. The leading term can be obtained solving the equations for z with boundary condition

$$\rho(x_1, x_2) = \begin{cases} \beta & , \quad x_2 < 0 \\ 0 & , \quad x_2 > 0 \end{cases} \tag{4.2.7}$$

The case $\beta = 1$ has already been considered in [12] and corresponds to the maximally supersymmetric plane wave [55]. We note here that only in the case of $\beta = 1$ the "zooming"

[4] For a calculation of the subleading terms in such an expansion, see [71].

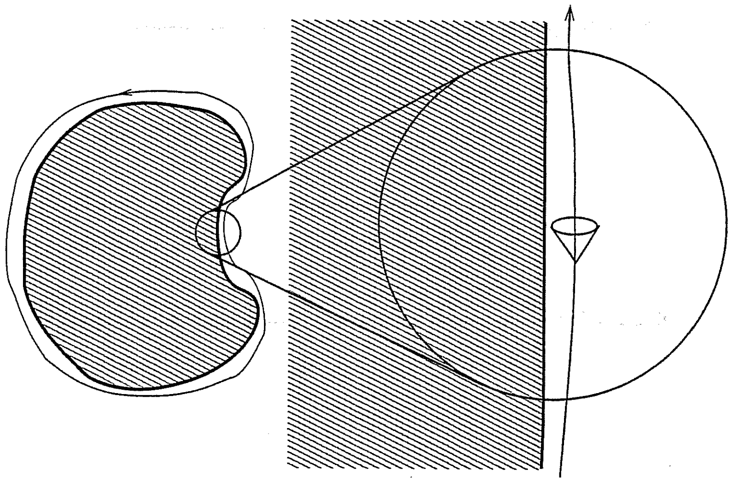


Figure 4.3: Zoom showing light cone near a droplet with $\beta_i > 1$ on the $\{x_1, x_2\}$ plane at $y = 0$.

limit that we are considering here coincides with the Penrose limit. Indeed the BFHP plane wave is *the only* plane wave geometry that can be obtained via the LLM construction and its generalisation with the most general boundary conditions on z considered in previously. All (generalised) LLM metrics, have 16 Killing spinors ψ whose bilinears $\bar{\psi}\Gamma^M\psi$ are null Killing vectors[5] *but not* covariantly constant (c.c.). Any plane wave has 16 Killing spinors with c.c. Killing vector bilinear, and the only one which has 16 extra Killing spinors is the maximally supersymmetric one. The details of the proof can be found in the Appendix.

For generic β we have

$$z = \frac{\beta}{2}\frac{x_2}{\sqrt{x_2^2+y^2}} + \frac{1}{2}(1-\beta) = \frac{\beta}{2}\cos\theta + \frac{1}{2}(1-\beta) \tag{4.2.8}$$

$$V_1 = \frac{\beta}{2\sqrt{x_2^2+y^2}} = \frac{\beta}{2R} \qquad V_2 = 0 \tag{4.2.9}$$

The plane $\cos\theta = \frac{\beta-2}{\beta}$ is $\partial\hat{\Sigma}$ and the domain Σ is defined by

$$1 > \cos\theta > \frac{\beta-2}{\beta} \tag{4.2.10}$$

The vector ∂_{x_1} is a Killing vector and

$$g_{11} = \frac{1}{y\sqrt{\rho(1-\rho)}}\left(\frac{\beta}{2}(1-\beta)(1-\cos\theta)\right) < 0 \tag{4.2.11}$$

[5]As all such bilinears in type IIB solutions [21]

so that it is timelike. The limit $y \to 0, \cos\theta \to 1$ is finite and gives

$$g_{11} \to (1-\beta)\sqrt{\beta}\frac{1}{2x_2} \tag{4.2.12}$$

In the same limit, we have

$$g_{22} = h^2 \to \frac{\sqrt{\beta}}{2x_2} \tag{4.2.13}$$

In a neighbourhood of $\partial\mathcal{D}$, the $\{x_1, x_2\}$ plane is thus a Lorentz submanifold. We note that the opening of the lightcone is given by

$$\tan\phi = \frac{dx_2}{dx_1} = \pm\sqrt{-\frac{g_{11}}{g_{22}}} = \pm\sqrt{(\beta-1)} \tag{4.2.14}$$

From this analysis it is straightforward to conclude that if we have a droplet $\mathcal{D}$ with $\beta_{\mathcal{D}} > 1$ of smooth boundary $\partial\mathcal{D}$, *provided we stay close enough to the* $y=0$ *plane and to* $\partial\mathcal{D}$ *we have CTCs going around* $\mathcal{D}$ (Figure 3). Since these geometries have no horizon a *CTC passes through any point of the spacetime.*

4.2.4 The disk

It is possible to perform a detailed analysis of the geometry seeded by one single circular droplet of constant density $\beta > 1$. The analysis is interesting because it displays some generic features of the timelike singular geometries and it is useful for introducing the more general timelike singularities which we will study in the next section.

We assume that the radius of the droplet is R_0. The radius of the asymptotic $AdS_5 \times S^5$ is thus given by

$$R^4_{AdS} = \frac{1}{\pi}\int \rho = \beta R_0^2 \tag{4.2.15}$$

These geometries have already been studied in [67] where it is shown that they can be viewed as a generalisation of the superstar studied in [56]. The superstar geometries are parameterised by a charge Q and a scale parameter L, which are related to our β and R_0 in the following way.

$$\beta = \frac{1}{1+Q/L^2}\,,\; R_0^2 = L^2(L^2+Q) \;\Rightarrow\; R_{AdS} = L \tag{4.2.16}$$

For fixed value of L we have

$$-L^2 < Q < 0 \Rightarrow \beta > 1 \tag{4.2.17}$$

$Q = -L^2$ corresponds to $\rho = \pi L^4 \delta^{(2)}(x_1, x_2)$. We will discuss the continuation to $Q < -L^2$ in the next section.

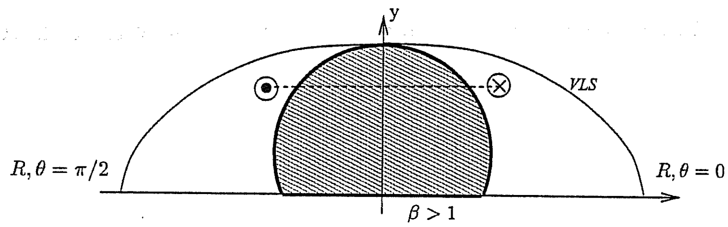


Figure 4.4: Singularity and velocity of light surface for a disk with $\beta > 1$.

Following the analysis in Section 4.2.3 we expect to find CTCs in these geometries. Going to polar coordinates R, φ in the x_1, x_2 plane we have

$$z = \frac{\beta}{2}\frac{R^2 - R_0^2 + y^2}{\sqrt{(R^2 + R_0^2 + y^2)^2 - 4R^2R_0^2}} + \frac{1}{2}(1-\beta) \tag{4.2.18}$$

$$V = V_\varphi \mathrm{d}\varphi \tag{4.2.19}$$

$$V_\varphi = \frac{\beta}{2}\left(1 - \frac{R^2 + R_0^2 + y^2}{\sqrt{(R^2 + R_0^2 + y^2)^2 - 4R^2R_0^2}}\right) \tag{4.2.20}$$

The equation for the $\partial\hat{\Sigma}$ is given by $z = -\frac{1}{2}$

$$R^2 + \left(y - R_0\frac{\beta - 2}{2\sqrt{\beta - 1}}\right)^2 - R_0^2\frac{\beta^2}{4(\beta - 1)} = 0 \tag{4.2.21}$$

Thus the geometry is defined in the $y \geq 0$ halfspace, outside a sphere of radius $\frac{\beta}{2\sqrt{\beta-1}}R_0$ with centre at $R = 0$ and $y = \frac{\beta-2}{2\sqrt{\beta-1}}R_0$. In particular it crosses the y axis at $y = R_0\sqrt{\beta - 1}$.

The square of the Killing vector ∂_φ is

$$g_{\varphi\varphi} = -h^{-2}V_\varphi^2 + R^2h^2 \tag{4.2.22}$$

and we have

$$g_{\varphi\varphi} \geq 0 \iff \frac{y^2}{\beta - 1} + \frac{R^2}{\beta} - R_0^2 \geq 0. \tag{4.2.23}$$

The surface on which $g_{\varphi\varphi} = 0$ is known as the velocity of light surface (VLS).

From this analysis, three main features follow (see Figure 4). We will show in Section 4.3.1 that they are generic for geometries seeded by boundary conditions such that $\partial\hat{\Sigma} \neq \emptyset$.

1. The VLS touches the singularity where[6] $V^2 = 0$. If the VLS did not touch the singularity we would have CTCs which are contractible to a point remaining timelike. At such a point the local orientability of spacetime would be lost - possibly indicating also a change in the signature of spacetime to two time-like directions. The fact that

[6]Note that $V^2 = R^{-2}V_\varphi^2$ and $V_\varphi = O(R^2)$ as $R \to 0$

the VLS touches the singularity at a point, in such a way that there is no loss of time orientability should be guaranteed, but we know of no general theorem that proves this.

2. The opening of the lightcone in the $\{R, \varphi, y\}$ sections inside the ellipsoid is given by

$$\tan\theta = \frac{y^2}{1/4 - z^2} V_\varphi^2 - R^2 \tag{4.2.24}$$

   This means that provided we stay close enough to the singularity at $z = -\frac{1}{2}$ and that we go "around" it in the direction indicated by ∂_φ we have CTCs.

3. All generalised LLM geometries are without horizon and thus a CTC passes through any point of the spacetime.

Following (4.2.2) we can calculate the mass of these excitations over $AdS_5 \times S^5$ as

$$M = \frac{1}{8\pi^2 l_P^8}\left[\frac{1}{\pi}\int \rho\frac{1}{2}(x_1^2 + x_2^2) - \left(\frac{1}{\pi}\int \rho\frac{1}{2}\right)^2\right] = \frac{(\beta R_0^2)^2}{32\pi^2 l_P^8}(\frac{1}{\beta} - 1) \tag{4.2.25}$$

Thus for $\beta > 1$ a circular droplet seeds a negative mass excitation. For fixed value of $R_{AdS}^4 = \beta R_0^2$, the minimum mass is given by $M_{\min} = -\frac{(\beta R_0^2)^2}{32\pi^2 l_P^8} = -\frac{R_{AdS}^8}{32\pi^2 l_P^8}$ and corresponds to $\beta = \infty$, $Q = -L^2$. As expected from the general considerations at the beginning of this Section, this corresponds to $\rho = \pi R_{AdS}^4 \delta^{(2)}(x_1, x_2)$. In this case the surface $z = -1/2$ is a sphere of radius R_{AdS}, tangent to the $\{x_1, x_2\}$ plane and centred on $(R, y) = (0, \frac{1}{2}R_{AdS})$. The VLS is determined by the saturation of the inequality,

$$g_{\varphi\varphi} \geq 0 \iff y^2 + R^2 \geq R_{AdS}^2. \tag{4.2.26}$$

4.2.5 Lifting the sphere

In the previous subsection we have considered geometries seeded by a spherical $\partial\hat{\Sigma}$ intersecting or tangent to the $\{x_1, x_2\}$ plane. One could ask which geometries correspond to a spherical $\partial\hat{\Sigma}$ not touching the $\{x_1, x_2\}$ plane. In this subsection we will answer this question. As in the case of the circle of density $\beta > 1$, these highly symmetric geometries illustrate some features that will be shown to be generic for any solution seeded by a $\partial\hat{\Sigma}$ not attached to the $\{x_1, x_2\}$ plane in Section 4.3.2.

The functions

$$z = \frac{\beta}{2}\frac{R^2 - R_0^2 + y^2}{\sqrt{(R^2 + R_0^2 + y^2)^2 - 4R^2R_0^2}} + \frac{1}{2}(1 - \beta) \tag{4.2.27}$$

$$V_\varphi = \frac{\beta}{2}\left(1 - \frac{R^2 + R_0^2 + y^2}{\sqrt{(R^2 + R_0^2 + y^2)^2 - 4R^2R_0^2}}\right) \tag{4.2.28}$$

determine an asymptotically $AdS_5 \times S^5$ provided $\beta R_0^2 > 0$. Since R_0^2 (and not R_0) appears in these functions we can analytically continue to $\beta < 0$ and $R_0^2 < 0$. Recalling that

$$\beta = \frac{1}{1+Q/L^2} \qquad R_0^2 = L^2(L^2+Q) \tag{4.2.29}$$

this corresponds to $Q < -L^2$.
We define for convenience

$$\tilde{R}_0 \equiv \sqrt{-R_0^2} \tag{4.2.30}$$

and rewrite z and V as

$$z = \frac{\beta}{2}\frac{R^2+\tilde{R}_0^2+y^2}{\sqrt{(R^2+y^2-\tilde{R}_0^2)^2+4R^2\tilde{R}_0^2}} + \frac{1}{2}(1-\beta) \tag{4.2.31}$$

$$V_\varphi = \frac{\beta}{2}\left(1 - \frac{R^2+y^2-\tilde{R}_0^2}{\sqrt{(R^2+y^2-\tilde{R}_0^2)^2+4R^2\tilde{R}_0^2}}\right) \tag{4.2.32}$$

This choice for z corresponds to choosing $\partial\hat{\Sigma}$ to be a sphere of radius $\frac{-\beta}{2\sqrt{1-\beta}}\tilde{R}_0$ with centre at $R=0, y=\frac{2-\beta}{2\sqrt{1-\beta}}\tilde{R}_0$, $\partial\Sigma_0$ coincides with the $\{x_1, x_2\}$ plane and

$$z_0 = \begin{cases} \frac{1}{2} & \text{on } \partial\Sigma_0 \\ -\frac{1}{2} & \text{on } \partial\hat{\Sigma} \end{cases} \tag{4.2.33}$$

The expressions (4.2.31),(4.2.32) are the analytic continuation of the solution for z and V with these constraints. Clearly this continuation cannot be regular everywhere inside the sphere and we expect to find a charge somewhere. Looking at the leading order expansion of Φ for $(R,y) = (R, \tilde{R}_0+\varepsilon) \to (0, \tilde{R}_0)$

$$\Phi = \frac{1/2-z}{y^2} \approx -\frac{\beta}{2\tilde{R}_0}\frac{1}{\sqrt{R^2+\varepsilon^2}} \tag{4.2.34}$$

$$V_\varphi \approx \frac{\beta}{2}\left(1-\frac{\varepsilon}{\sqrt{R^2+\varepsilon^2}}\right) \tag{4.2.35}$$

we can identify the charge and assume that Φ satisfies the equation

$$*_6\, \mathrm{d} *_6 \mathrm{d}\Phi = 4\pi\frac{\beta}{2\tilde{R}_0}\,\delta(y-\tilde{R}_0)\,\delta^{(2)}(R) \tag{4.2.36}$$

We will briefly show in Section 4.3.2 that whenever a subset of $\partial\hat{\Sigma}$ is not attached to the $\{x_1, x_2\}$ plane then we expect Φ to satisfy a similar equation.
Integrating over the five-sphere at infinity we find that

$$A = -\beta\tilde{R}_0^2 = \beta R_0^2 = L^4 \tag{4.2.37}$$

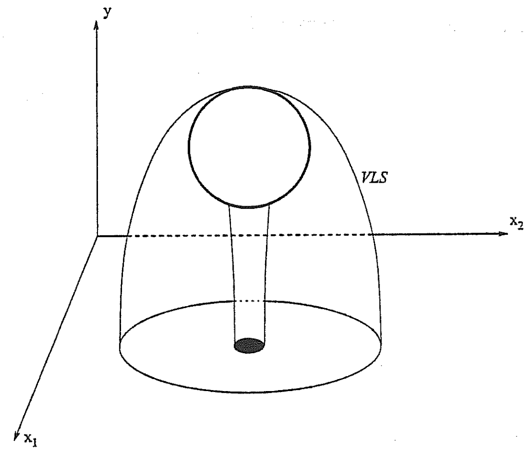


Figure 4.5: "Medusa" diagram: singularity, velocity of light surface and cylinder connecting the singularity ($\partial\hat{\Sigma}$) to the $y = 0$ plane, for the lifted sphere.

and so as expected $R_{AdS} = L$.
We have

$$g_{\varphi\varphi} \geq 0 \iff \frac{y^2}{1-\beta} + \frac{R^2}{-\beta} - \tilde{R}_0^2 \geq 0 \tag{4.2.38}$$

As happened in the case $-L^2 < Q < 0$ also here the velocity of light surface touches the singularity, precisely at $R = 0$ and $y = \tilde{R}_0\sqrt{1-\beta}$. As already mentioned in that case we expect this to be a general feature of geometries with CTCs and we will show this in Section 4.3.2. A more precise way to state this situation is to say that inside the VLS the lightlike direction has a non-trivial π_1.

On the segment of the y axis, between the $y = 0$ plane and the lower intersection with the singularity at $y = \frac{\tilde{R}_0}{\sqrt{1-\beta}}$ we have

$$g_{\varphi\varphi} = -R_{AdS}^2(-\beta)\frac{1}{\sqrt{\tilde{R}_0^2 - (1-\beta)y^2}} \tag{4.2.39}$$

Thus, the segment is actually a cylinder and so again there are no CTCs which are contractible to a point while remaining timelike as shown in Figure 5.

Looking at the next to leading order expansion of the metric for $R^2 + y^2 \to \infty$ we can derive the mass of these excitations of $AdS^5 \times S^5$

$$M = \frac{(\beta R_0^2)^2}{32\pi^2 l_P^8}(\frac{1}{\beta} - 1) \tag{4.2.40}$$

which is clearly negative and, for fixed R_{AdS}, tends to minus infinity for $\beta \to 0^-$.

4.3 Singular solutions: generic properties

In this section we will prove the following

Theorem 4.3.1. *Geometries of the type studied in Section 2 have closed timelike curves if and only if* $\partial\hat{\Sigma} \neq \emptyset$

In particular standard LLM geometries are free of CTCs as well as all geometries seeded by boundary conditions such that $\partial\hat{\Sigma} = \emptyset$ and

$$\frac{1}{2} - z_0(x_1, x_2) = \rho(x_1, x_2) \quad 0 \leq \rho \leq 1 \tag{4.3.1}$$

On the other hand, whenever $\rho > 1$ or $\rho < 0$, (and thus $\partial\hat{\Sigma} \neq \emptyset$), we have CTCs in the spacetime.

We will divide the proof into the 2 subsections 4.1 and 4.3. In subsection 4.2 we will comment on the generic (Lorentz) topology of the solutions and show that some of the interesting features of the examples in Sections 4.2.4 and 4.2.5 are indeed quite general.

4.3.1 Sufficient condition for CTCs

It's easy to show that when $\partial\hat{\Sigma} \neq \emptyset$ we have CTCs.
Looking at the asymptotic expansion for large values of $x_1^2 + x_2^2 + y^2$ in Section 4.1.2, we can see that the vector field

$$\partial_\psi \equiv \frac{1}{\sqrt{V_1^2 + V_2^2}}(V_1\partial_1 + V_2\partial_2) \tag{4.3.2}$$

has closed[7], almost circular orbits at infinity. We can shift V by a constant amount such that $V = 0$ at a point $P \in \partial\hat{\Sigma}$ with $\partial_y z(P) \neq 0$ and the orbits of ∂_ψ are closed around P. Let's assume for definiteness that $z(P) = -\frac{1}{2}$. In a neighbourhood of P we have

$$z(x_1, x_2, y) \approx -\frac{1}{2} + \delta z \tag{4.3.3}$$

$$V_i(x_1, x_2, y) \approx \delta V_i \tag{4.3.4}$$

where δz and δV_i are linear in the co-ordinates $(x_1 - x(P), x_2 - x(P), y - y(P))$. The metric of the sections $\{x_1, x_2, y\}$ is (recalling that $h^4 = \frac{1/4 - z^2}{y^2}$),

$$\begin{aligned}\tilde{g} &= \begin{pmatrix} h^2 - h^{-2}V_1^2 & -h^{-2}V_1V_2 & 0 \\ -h^{-2}V_1V_2 & h^2 - h^{-2}V_2^2 & 0 \\ 0 & 0 & h^2 \end{pmatrix} \approx \\ &\approx \frac{1}{y(P)\sqrt{\delta z}} \begin{pmatrix} \delta z - y(P)^2\delta V_1^2 & y(P)^2\delta V_1\delta V_2 & 0 \\ y(P)^2\delta V_1\delta V_2 & \delta z - y(P)^2\delta V_2^2 & 0 \\ 0 & 0 & \delta z \end{pmatrix}\end{aligned} \tag{4.3.5}$$

[7]This is due to the gauge choice $\partial_1 V_1 + \partial_2 V_2 = 0$

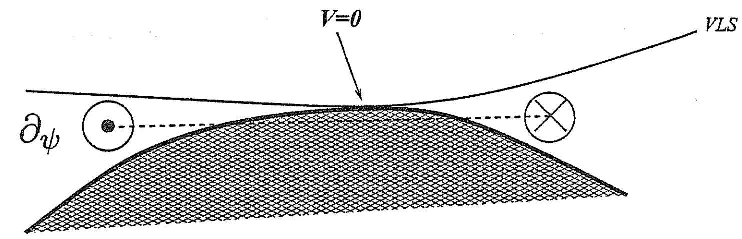


Figure 4.6: $\partial\hat{\Sigma}$ and the VLS touching at $V^2 = 0$.

The vectors

$$\begin{aligned} &\partial_\psi \\ &\partial_\sigma \equiv \frac{1}{\sqrt{V_1^2 + V_2^2}}(-V_2\partial_1 + V_1\partial_2) \\ &\partial_y \end{aligned} \tag{4.3.6}$$

are eigenvectors of $\tilde{g}$ with eigenvalues respectively

$$(h^2 - h^{-2}(V_1^2 + V_2^2)\,,\, h^2\,,\, h^2) \approx \frac{1}{y(P)\sqrt{\delta z}}(\delta z - y(P)^2\delta V^2\,,\, \delta z\,,\, \delta z) \tag{4.3.7}$$

Thus for

$$\delta z - y(P)^2\delta V^2 < 0 \tag{4.3.8}$$

the sections are timelike. This equation also shows that the velocity of light surface always touches the singularity where $V = 0$, as shown in Figure 6.

The opening of the lightcone is given by

$$\tan\theta = h^{-4}(V_1^2 + V_2^2) - 1 \approx \frac{y(P)^2}{\delta z}\delta V^2 - 1 \tag{4.3.9}$$

Thus any closed curve going around P in the sense indicated by ∂_ψ is a CTC provided that we stay close enough to P and $\partial\hat{\Sigma}$ (on which we recall $\delta z = 0$ by definition). Since the CTCs are not hidden by a horizon, also in this general case a CTC passes through any point of the spacetime.

4.3.2 (Lorentz) topology

In the case discussed in Section 4.2.5 we have $z_0 = \frac{1}{2}$ on the entire $\{x_1, x_2\}$ plane and $z = -\frac{1}{2}$ on a sphere centred on the y axis. The appearance of contractible CTCs is excluded by a detailed analysis of the structure of the metric. The Lorentz topology is thus nontrivial, as one could expect in order to preserve the regularity of the local structure of spacetime. The same analysis shows that the topology of the $\{x_1, x_2, y\}$ sections is still $\mathbb{R}^2 \times \mathbb{R}^+$, even if at first sight one would say that a sphere has been removed. This is essentially due to the non vanishing of V_φ along the y axis in the segment between the $y = 0$ plane and the sphere.

Assume we have a connected subset of $\partial\hat{\Sigma}$ which is not attached to the $\{x_1, x_2\}$ plane. We can analytically continue z (and thus Φ) to the $|z| > \frac{1}{2}$ side of $\partial\hat{\Sigma}$. We will necessarily encounter some pole singularity in the equation for Φ, as δ sources centred on some point Q. In a neighbourhood of such a point $(\vec{x}_0, y_0)$ we have to leading order

$$z \approx \sigma y_0^2 \frac{1}{\sqrt{(y-y_0)^2 + R^2}} \tag{4.3.10}$$

$$V \approx V_\varphi \mathrm{d}\varphi \qquad V_\varphi \approx \sigma y_0 \left(1 - \frac{y - y_0}{\sqrt{(y-y_0)^2 + R^2}}\right) \tag{4.3.11}$$

where R, φ are polar coordinates in x_1, x_2 centred on $\vec{x}_0$. By continuity, we can argue that in a neighbourhood of this Q, for $y < y_0$, the vector $V_i\partial_i$ is circulating around a line $\mathcal{L}$ on which it doesn't vanish. Going locally to polar coordinates centred on the intersection of this line with a constant y plane, we have that

$$g_{\varphi\varphi} = -h^{-2}V_\phi^2 + R^2h^2 \tag{4.3.12}$$

is non vanishing at $R = 0$ and thus the line $\mathcal{L}$ is topologically a cylinder. As in section 3.5, the shape of the space-time around such a point Q is similar to the "Medusa" diagram of Figure 5. We expect that several disconnected components of $\partial\hat{\Sigma}$ may give rise to more complicated geometrical structures.

4.3.3 Necessary condition for CTCs

In this Section we will show that if $\partial\hat{\Sigma} = \emptyset$, then there are no CTCs. Looking at the metric (4.1.1) it is clear that if the determinant $\tilde{g}$ of the spatial section $\{x_1, x_2, y\}$ is positive, then there cannot be CTCs. We recall from Section 2 that

$$z(x_1, x_2, y) = \frac{1}{2} - \frac{y^2}{\pi} \int \frac{\rho(x_1', x_2')\mathrm{d}^2x}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2]^2} \tag{4.3.13}$$

$$V_i = -\frac{1}{\pi}\epsilon_{ij} \int \frac{(x_j - x_j')\rho(x_1', x_2')\mathrm{d}^2x'}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2]^2} \tag{4.3.14}$$

$$\tag{4.3.15}$$

and the determinant of the three dimensional sections

$$\begin{aligned}\tilde{g} = h^4 - V^2 = \frac{1/4 - z^2}{y^2} - V^2 = \\ = \frac{1}{\pi} \int \frac{\rho(x_1', x_2')\mathrm{d}^2x}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2]^2} - \frac{y^2}{\pi^2} \left(\int \frac{\rho(x_1', x_2')\mathrm{d}^2x}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2]^2} \right)^2 + \\ - \sum_{i=1,2} \frac{1}{\pi^2} \left(\int \frac{(x_i - x_i')\rho(x_1', x_2')\mathrm{d}^2x'}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2]^2} \right)^2\end{aligned} \tag{4.3.16}$$

Any possible geometry seeded by a function $\rho(x_1, x_2)$ with $0 \leq \rho \leq 1$ can be approximated as well as desired by a piecewise constant $\bar{\rho}$ such that $\bar{\rho} = 0, 1$. So it is enough to prove that the determinant is positive for standard LLM geometries defined by droplets of density $\rho = 1$.

We will prove that, given any possible distribution of droplets $\mathcal{D}$ and any point $P \equiv (x_1(P), x_2(P), y)$, there is a halfplane Π distribution for which $z(P)$ is the same as for the original distribution and $V(P)^2$ is larger. In this way the determinant $\tilde{g}(P)_\Pi$ for the halfplane distribution is smaller than the original determinant $\tilde{g}_\mathcal{D}$. As noted already in [12] a halfplane distribution corresponds to the maximally supersymmetric plane wave and for this metric the determinant always satisfies the relation

$$\tilde{g}_\Pi = \frac{1/4 - z^2}{y^2} - V_\Pi^2 = 0 \tag{4.3.17}$$

So we have $\tilde{g}_\mathcal{D} \geq \tilde{g}_\Pi = 0$.

We first make some assumptions in order to simplify the proof. Given the point $P \equiv (x_1(P), x_2(P), y)$ we move the origin of the $\{x_1, x_2\}$ plane to $(x_1(P), x_2(P))$. We then define a 2-vector $\tilde{V}$ such that $\tilde{V}^2 = V^2$

$$\tilde{V}_i[\mathcal{D}](P) = \frac{1}{\pi} \int_\mathcal{D} \frac{x_i}{(x_1^2 + x_2^2 + y^2)^2} \mathrm{d}^2x \tag{4.3.18}$$

$$\tilde{V}_1^2 + \tilde{V}_2^2 = V_1^2 + V_2^2 \tag{4.3.19}$$

where $\mathcal{D}$ is the union of all the droplets. We also have

$$z_\mathcal{D}(P) = \frac{1}{2} - \Delta_\mathcal{D} z = \frac{1}{2} - \frac{y^2}{\pi} \int_\mathcal{D} \frac{1}{(x_1^2 + x_2^2 + y^2)^2} \mathrm{d}^2x \tag{4.3.20}$$

We identify the direction of $\tilde{V}$ with the x_2 axis. Let us assume that the droplets are all contained in the strip

$$x_{\min} \leq x_2 \leq x_{\max} \tag{4.3.21}$$

where one or even both of $x_{\min}$ and $x_{\max}$ can also be infinite. A distribution corresponding to the (half)plane Π_0 defined by $x_2 \geq x_{\min}$ will give us

$$z_{\Pi_0}(P) \leq z_\mathcal{D}(P) \tag{4.3.22}$$

since $\mathcal{D} \subseteq \Pi_0$

The equality holds just in the case that the original distribution is already a halfplane[8]. In all the other cases, we take a halfplane Π defined by

$$x_2 \geq x \tag{4.3.23}$$

with $x > x_{\min}$ such that

$$z_\Pi(P) = z_\mathcal{D}(P) \tag{4.3.24}$$

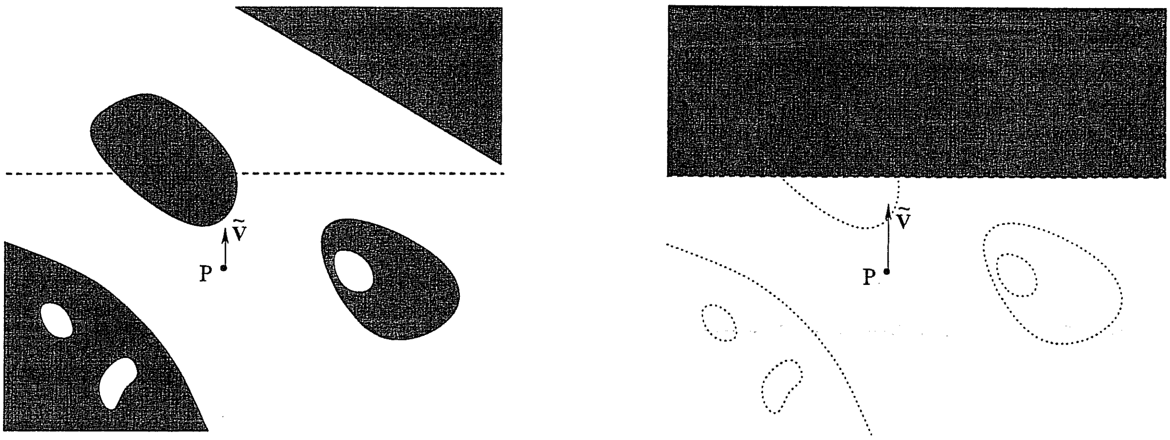


Figure 4.7: Changing $\mathcal{D}$ into Π

We note that for a generic domain $\mathcal{D}$ we have the following relation between $\Delta_{\mathcal{D}} z$ and $\tilde{V}_2[\mathcal{D}](P)$

$$\tilde{V}_2[\mathcal{D}](P) = \frac{\Delta_{\mathcal{D}} z}{y^2} \langle x_2 \rangle_{\mathcal{D}} \tag{4.3.25}$$

with

$$\langle x_2 \rangle_{\mathcal{D}} = \frac{1}{\pi} \int_{\mathcal{D}} \frac{x_2 \mathrm{d}^2 x}{(x_1^2 + x_2^2 + y^2)^2} \left(\frac{1}{\pi} \int_{\mathcal{D}} \frac{\mathrm{d}^2 x}{(x_1^2 + x_2^2 + y^2)^2} \right)^{-1} = \tag{4.3.26}$$

$$= \int x_2 \; \mu_{\mathcal{D}}(x_2) \mathrm{d}x_2$$

$$\mu_{\mathcal{D}}(x_2) = \frac{1}{\pi} \int_{\mathcal{D}} \frac{\mathrm{d}x_1}{(x_1^2 + x_2^2 + y^2)^2} \left(\frac{1}{\pi} \int_{\mathcal{D}} \frac{\mathrm{d}x_1 \mathrm{d}x_2}{(x_1^2 + x_2^2 + y^2)^2} \right)^{-1} \tag{4.3.27}$$

$$\int \mu_{\mathcal{D}}(x_2) \mathrm{d}x_2 = 1 \tag{4.3.28}$$

Thus $\mu_{\mathcal{D}}(x_2)$ acts as a normalised weight function.

From the definition of $\mu_{\mathcal{D}}(x_2)$ and from the fact that, by definition of Π

$$\Delta_{\mathcal{D}} z = \Delta_{\Pi} z \tag{4.3.29}$$

i.e.

$$\frac{1}{\pi} \int_{\mathcal{D}} \frac{1}{(x_1^2 + x_2^2 + y^2)^2} \mathrm{d}^2 x = \frac{1}{\pi} \int_{\Pi} \frac{1}{(x_1^2 + x_2^2 + y^2)^2} \mathrm{d}^2 x \tag{4.3.30}$$

one can easily see that

$$\mu_{\Pi}(x_2) \geq \mu_{\mathcal{D}}(x_2) \quad , \quad x_2 \geq x \tag{4.3.31}$$

$$\mu_{\Pi}(x_2) = 0 \quad , \quad x_2 < x \tag{4.3.32}$$

[8] Or a completely filled plane, which we neglect since it is trivial: the solution is empty Minkowski space

We have

$$\langle x_2\rangle_\Pi = x + \langle (x_2 - x)\rangle_\Pi =$$
$$= x + \int \mu_\Pi(x_2)(x_2 - x)\mathrm{d}x_2 \geq x + \int_{x_2>x} \mu_\mathcal{D}(x_2)(x_2 - x)\mathrm{d}x_2 \; > \; \langle x_2\rangle_\mathcal{D} \quad (4.3.33)$$

The last inequality holds because

$$\langle x_2\rangle_\mathcal{D} = x + \int \mu_\mathcal{D}(x_2)(x_2 - x)\mathrm{d}x_2 =$$
$$= x + \int_{x_2>x} \mu_\mathcal{D}(x_2)(x_2 - x)\mathrm{d}x_2 + \int_{x_2<x} \mu_\mathcal{D}(x_2)(x_2 - x)\mathrm{d}x_2 \quad (4.3.34)$$

and the last term is clearly negative.

Recalling (4.3.25) and (4.3.29) we conclude

$$\tilde{V}_2[\Pi](P) > \tilde{V}_2[\mathcal{D}](P) \quad (4.3.35)$$

and thus we have

$$\tilde{g}_\mathcal{D} = \frac{1/4 - z_\mathcal{D}(P)^2}{y^2} - V_\mathcal{D}^2 > \frac{1/4 - z_\Pi(P)^2}{y^2} - V_\Pi^2 = 0 \quad (4.3.36)$$

The case $y = 0$

In the proof we have implicitly assumed $y > 0$. In the limit $y \to 0$ one can argue, by continuity

$$\tilde{g}_\mathcal{D} \geq 0 \quad (4.3.37)$$

With a bit of effort, we can prove that the equality holds only for the halfplane.

Instead of choosing x in order to fix $z(P)$ we decide to fix

$$\lim_{y\to 0} \frac{1/4 - z^2}{y^2}(P) \quad (4.3.38)$$

which is finite since by hypotheses $z \to \pm\frac{1}{2}$ and is even in y. Recalling that

$$\lim_{y\to 0} \frac{y^2}{\pi} \int_\mathcal{D} \frac{\mathrm{d}^2 x}{(x_1^2 + x_2^2 + y^2)^2} = \begin{cases} 1 & P \in \mathcal{D} \\ 0 & P \in \bar{\mathcal{D}} \end{cases} \quad (4.3.39)$$

we have

$$\frac{1/4 - z^2}{y^2} \to \begin{cases} \frac{1}{y^2}\Delta_\mathcal{D} z \to \frac{1}{\pi}\int_\mathcal{D} \frac{\mathrm{d}^2 x}{(x_1^2+x_2^2)^2} & P \in \bar{\mathcal{D}} \\ -\frac{1}{y^2}\Delta_{\bar{\mathcal{D}}} z \to -\frac{1}{\pi}\int_{\bar{\mathcal{D}}} \frac{\mathrm{d}^2 x}{(x_1^2+x_2^2)^2} & P \in \mathcal{D} \end{cases} \quad (4.3.40)$$

Noting that $V_\mathcal{D} = -V_{\bar{\mathcal{D}}}$, in both cases we can use the same argument as for $y \neq 0$ provided that we change $\mathcal{D}$ into $\bar{\mathcal{D}}$ when $P \in \mathcal{D}$. Thus $\tilde{g} \geq 0$ and again the equality holds only for the halfplane.

4.4 Supergravity singularities and dual field theories

There already exist in the literature on AdS/CFT duality, some indications that geometries with naked time machines are related to non-physical phenomenon in the dual gauge theory. The dual picture should provide a field theory interpretation for the quantum mechanism at work in the resolution of these pathologies, possibly through a careful treatment of unitarity.

In particular, the overrotating solutions of [66] are exactly of this type and as already noted in that paper, and further elucidated in [72, 73], the operator in the corresponding D-brane configuration that takes an underrotating geometry to an overrotating one is non-unitary.

In that case it was first noticed [72] that the overrotating geometries have a VLS that repulses all geodesics that approach from the outside, and thus the region of CTCs is effectively removed from the space-time. It was then noticed in a series of works on the enhancon mechanism that incorporating extra charge sources one can remove the causality violating region[74]. A similar idea is developed for example also in [75]. Our naked time machines do not have a repulsive VLS and as a consequence this method for removing the singularity cannot be applied here.

That some form of chronology protection mechanism should however be present has been conjectured in [67]. In this paper the rotationally symmetric singular configurations that we have studied in Section 4.2.4 are noted to not have a description in terms of the dual free fermion picture as they violate the Pauli exclusion principle.

In general relativity and in supergravity there are of course many geometries that contain CTCs and naked singularities. Is it possible that a similar principle could also rule out those geometries? In particular is it possible that these geometries are in general related to non-unitarity in the dual gauge theories? The violation of the Pauli exclusion principle suggests that our naked time machines may more generally be related to some non-unitary behaviour in the dual gauge theory[9].

The conformal dimension Δ of an operator in the CFT dual to an asymptotically $AdS_5 \times S^5$ geometry is equal to the mass or angular momentum ($M = J$ as a consequence of the BPS condition) of the configuration. For a solution seeded by a density distribution ρ

$$\Delta = M = \frac{1}{8\pi^2 l_P^8}\left[\frac{1}{\pi}\int \rho\frac{1}{2}(x_1^2 + x_2^2) - \left(\frac{1}{\pi}\int \rho\frac{1}{2}\right)^2\right] \tag{4.4.1}$$

As noted in Section 3.4, for a density which is β inside a disk, we have

$$M = \Delta = \frac{\beta R_{AdS}^8}{32\pi^2 l_P^8}\left(\frac{1}{\beta} - 1\right) \tag{4.4.2}$$

From the CFT point of view a configuration with $\beta \gtrsim 1$ can be seen as a "small" deformation of a configuration with $\beta = 1$ and slightly larger radius. Equation (4.4.2) shows that

[9]For a recent and somewhat different perspective on the relationship between unitarity and CTCs, see [76].

this deformation corresponds to an operator with *negative* conformal dimension.

In general we expect, even though we cannot prove it directly, that configurations with ρ not between 0 and 1 correspond to deformations of the CFT by negative conformal dimension operators. As seen in Section 3.5, solutions with more general boundary conditions can still be interpreted as continuous deformations of solutions seeded by density distributions and a similar argument should also relate them to operators of negative conformal dimension.

In a series of papers [25, 26] all unitary irreducible representations of the relevant superconformal algebra, $su(2,2|N)$, are found and in particular unitarity requires that they have positive conformal dimension. The "unphysical" geometries that we have studied in this paper then apparently correspond to deformations by non-unitary operators (with negative conformal dimension) in the dual CFT. This observation together with the observed violation of the Pauli exclusion principle provides strong evidence for the existence of a theorem, for 1/2 BPS configurations in IIB supergravity, relating the chronology protection conjecture to unitarity in the dual CFT.

Chapter 5

1/8 BPS States in AdS/CFT

It is natural to ask how the above very precise correspondence between geometry on the one hand, and features of the quantum mechanical states of the reduced gauge theory on the other, extends to cases with less supersymmetry. There have been various attempts in this direction: for example, in [77] one quarter BPS geometries were found by assuming a non trivial axion-dilaton. This corresponds to putting smeared D7 branes in the background and thus to adding flavour to the gauge theory. A description of one eighth and one quarter BPS geometries in the language of five dimensional gauged supergravity has been given in [78]. The construction of a class of one quarter BPS solutions directly in type IIB appeared in [79, 80] and an interesting further generalisation was presented in [81] Another interesting related work is presented in [82]. This problem was also approached in the probe approximation, where the backreaction on the geometry is neglected: D3 branes can wrap more complex three dimensional surfaces in S^5 and give rise to giant gravitons with fewer supersymmetries [83]. In [84] the authors have been able to count such states. The quantisation of their classical phase space has been performed in [85]. Other works that present interesting connection with ours can be found in [86, 87, 88].

In this Chapter we address the problem of finding BPS supergravity solutions which represent the fully backreacted geometry of a class of 1/8 BPS giant gravitons. Our solutions correspond to gauge theory states associated to linear combinations of composite operators

$$\mathcal{O}(q,r) = \mathrm{Tr}(Z_1^q)\mathrm{Tr}(Z_2^q)\mathrm{Tr}(Z_3^r) + \cdots . \qquad (5.0.1)$$

where Z_1, Z_2 and Z_3 are the three complex scalars of the $\mathcal{N} = 4$ CFT as in Section 2.2.3. The dots signify other terms with suitable (anti)-symmetrisation and trace structures, which have all a total of q Z_1 and Z_2 fields and r Z_3 fields. They are chosen such that $\mathcal{O}(q,r)$ are chiral primary operators which are invariant under the $SU(2)_L$ subgroup of the $SU(2)_L \times SU(2)_R$ acting on Z_1, Z_2. We consider linear combinations of $\mathcal{O}(q,r)$ which have all the same value of q but may have different values of r.

The lowest mode $\mathcal{O}(q,r)$ in the expansion on spherical harmonics on S^3 saturates the BPS bound:

$$\Delta = 2q + r , \qquad (5.0.2)$$

where Δ is the conformal dimension of the operator. The total amount of bosonic symmetry preserved by the corresponding states is thus given by $SO(4)_{KK} \times SU(2)_L \times U(1)_R$. Consequently, we start from an Ansatz for the metric and the self-dual RR 5-form which preserves this amount of symmetry. This implies, as for LLM, that the resulting background will depend non-trivially on three coordinates (an additional symmetry will be associated to the time coordinate, like in LLM). We also require that the background possesses the required amount of supersymmetry by demanding that it possesses a Killing spinor. Applying techniques similar to those in [12, 89, 90, 91, 92, 93] we have been able to express the full solution in terms of four independent functions defined on a three dimensional half-space. As a result of certain Bianchi identities and integrability conditions, these four functions have to satisfy a system of nonlinear, coupled, elliptic differential equations. A unique solution to these equations is obtained once a set of boundary conditions at infinity and on the boundary plane is specified; boundary conditions should be chosen in such a way as to give non-singular geometries with $AdS_5 \times S^5$ asymptotics.

We present here the boundary conditions that give rise to asymptotically non-singular $AdS_5 \times S^5$ geometries. We solve the equations asymptotically up to third order in a large radius expansion. From this analysis we can extract the two dimensionless charges Q and J carried by the solution. These are the charges corresponding to two out of the three $U(1)$ Cartan gauge fields arising from the KK reduction of IIB supergravity on S^5 to five dimensional maximal gauged supergravity. These charges in turn correspond to the q and r charges of the gauge theory side. Moreover, we verify that our solutions saturate the expected BPS bound:

$$M = \frac{\pi L_{AdS}^2}{4G_5}(|J| + 2|Q|)\,. \tag{5.0.3}$$

Unfortunately, a more exhaustive analysis of such boundary conditions is quite difficult due to the complexity (non linearity) of the differential equations. In other words we do not know which of the boundary conditions give rise to globally non-singular backgrounds. We will comment on this issue in the conclusions. This chapter is organised as follows: the conventions and notations are the same used toughout the whole Thesis and reported in the Appendix A; in Section 5.1 we present the gauge theory description of the 1/8 BPS states that we wish to study. In Section 5.2 we show how the 1/8 supersymmetry constrains the components of the metric and 5-form and we reduce these constraints to four differential equations on four scalar functions. The details of this derivations are postponed to Section 5.5. In Section 5.3 we present the large radius asymptotic analysis. Section 5.4 contains a discussion on the results derived in this Chapter. The last two Sections of this Chapter are slightly more technical. As anticipated, Section 5.5 contains the details of the construction of 1/8 BPS geometries. In Section 5.6 we make some observations on the formal which also apply to the original LLM construction. Due to the complexity of the equations involved, the complete analysis has been performed by means of the software Mathematica. All the derivations that are not described in full detail in the text were obtained with the help of such software.

5.1 BPS operators in $\mathcal{N} = 4$ SYM

As outlined is Section 2.2.3, $\frac{1}{8}$ BPS gauge invariant chiral primary operators can be constructed out of the three complex scalars Z_1, Z_2, Z_3 of the $\mathcal{N} = 4$ $SU(N)$ Super Yang Mills theory. Elements of a basis of such operators which transform in the $[p, q, r]$ of the R-symmetry group $SU(4)$ [30] take the form

$$\mathrm{Tr}(Z_1^p)\mathrm{Tr}(Z_2^q)\mathrm{Tr}(Z_3^r) + \cdots . \tag{5.1.1}$$

where the dots mean suitable (anti)-symmetrisation and trace structure that projects to the chiral primaries in the (p, q, r) representation of $SU(4)$.

We are interested in constructing duals of the states corresponding to linear combinations of such operators. However, generic operators of this type break fully the non-abelian $SO(6)$ R-symmetry, up to possible $U(1)$ factors which act by an overall phase on them. However, if

$$p = q . \tag{5.1.2}$$

we can construct operators which are invariant under the $SU(2)_L$ of the $SO(4) = SU(2)_L \times SU(2)_R$ which rotates the four real scalars

$$\begin{pmatrix} X^1 \\ X^2 \\ X^3 \\ X^4 \end{pmatrix} = \begin{pmatrix} \Re Z_1 \\ \Im Z_1 \\ \Re Z_2 \\ \Im Z_2 \end{pmatrix} . \tag{5.1.3}$$

This is best seen by observing that $SU(2)_L$ and $SU(2)_R$ act as left and right multiplication respectively on:

$$\begin{pmatrix} Z_1 & -\bar{Z}_2 \\ Z_2 & \bar{Z}_1 \end{pmatrix}$$

Therefore Z_1 and Z_2 transform as a doublet of $SU(2)_L$, whereas they have the same charge under $J_R^3 = \frac{J_1+J_2}{2}$. The operators with $p = q$ are clearly singlets of $SU(2)_L$, and they acquire an overall phase under J_R^3. They satisfy the relation

$$\Delta = 2q + r . \tag{5.1.4}$$

The bosonic symmetry preserved by these states is:

$$\mathbb{R}_{BPS} \times \left(SU(2)_L \times U(1)_R\right)_{R-\text{charge}} \times SO(4)_{KK} \tag{5.1.5}$$

where the first $\mathbb{R}$ corresponds to the transformations generated by

$$D' \equiv D - 2J_R^3 - J , \tag{5.1.6}$$

with $J = J_3$ acting on Z, D is the dilatation operator and the last $SO(4)$ factor represents the fact that we are considering s-wave modes on S^3 in the reduction of SYM theory on

$\mathbb{R} \times S^3$ [48]. These are the symmetries that will motivate the Ansatz for the metric and five-form on the supergravity side: we will keep a round 3-sphere with the $SO(4)$ isometry corresponding to the $SO(4)$ above. Another S^3 (related to the $SO(4)$ R-symmetry of the $\frac{1}{2}$ BPS case) which is in the S^5 of the $AdS_5 \times S^5$ background, will be squashed with isometry group reduced to $SU(2)_L \times U(1)_R$.

It will be useful for the subsequent analysis of the Killing spinor equation on the supergravity side, to understand the quantum numbers of the preserved supersymmetries. In an $\mathcal{N} = 1$ and $SU(3) \times U(1) \subset SU(4)$ notation, the supersymmetry variations of the complex scalars Z_i are:

$$\delta Z_i = \xi_i \lambda + \xi \psi_i + \epsilon_{ijk} \bar{\xi}^j \bar{\psi}^k, \tag{5.1.7}$$

Here the two-component spinors λ and ψ_i are the gaugino and the chiral matter fermions, while ξ and ξ_i are the supersymmetry parameters. They are in the $\mathbf{1}_{3/2}$ and $\mathbf{3}_{-1/2}$ of $SU(3)_{U(1)}$ respectively. More precisely the Cartan charges of λ, ξ are $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and those of ψ_1, ξ_1 are $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, and similarly for $\psi_{2,3}, \xi_{2,3}$. From (5.1.7) it is clear that the highest weight $\frac{1}{8}$ BPS operators are invariant under the supersymmetry corresponding to $\bar{\xi}$. As for the $SU(2)_L \times SU(2)_R = SO(4) \subset SU(3)$ quantum numbers, the roots of $SU(2)_L$ are $(\pm 1, \mp 1, 0)$ and those of $SU(2)_R$ are $(\pm 1, \pm 1, 0)$. Therefore the preserved supersymmetry parameter $\bar{\xi}$, whose charges are $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, is a singlet of the unbroken $SU(2)_L$ and lowest weight with respect to the broken $SU(2)_R$.

5.2 Generic Solutions

We are looking for supergravity solutions dual to BPS states constructed from linear combinations of the operators

$$\mathcal{O}(q, r) = \mathrm{Tr}(Z_1^q)\mathrm{Tr}(Z_2^q)\mathrm{Tr}(Z_3^r) + \cdots \tag{5.2.1}$$

for constant q, where the meaning of the dots has been explained in the previous two sections. The geometries will thus be invariant under $SU(2)_L \times SO(4)_{KK}$ as defined in the previous section and invariant but charged under the remaining $U(1)_R$. The extra non-compact time-like symmetry ($\mathbb{R}_{BPS}$ of the previous section) is associated to invariance under the transformations generated by D' in the gauge theory and will emerge naturally in our construction[1].

The most generic Ansatz consistent with these symmetries is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \rho_1^2\left[(\sigma^{\hat{1}})^2 + (\sigma^{\hat{2}})^2\right] + \rho_3^2(\sigma^{\hat{3}} - A_\mu \mathrm{d}x^\mu)^2 + \tilde{\rho}^2 \mathrm{d}\tilde{\Omega}_3^2. \tag{5.2.2}$$

where ρ_1, ρ_3, $\tilde{\rho}$, A_μ and $g_{\mu\nu}$ are functions of the four coordinates x^μ. The space is a fibration of a squashed 3-sphere (on which the $SU(2)$ left-invariant 1-forms $\sigma^{\hat{a}}$ are defined) and a round 3-sphere (on which the $SU(2)$ left-invariant 1-forms $\sigma^{\tilde{a}}$ are defined) over a

[1]See Appendix B for details.

four dimensional manifold.
The left invariant 1-forms are given by

$$\begin{aligned}
\sigma^{\hat{1}} &= -\tfrac{1}{2}(\cos\hat{\psi}\, d\hat{\theta} + \sin\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) & \sigma^{\tilde{1}} &= -\tfrac{1}{2}(\cos\tilde{\psi}\, d\tilde{\theta} + \sin\tilde{\psi}\, \sin\tilde{\theta}\, d\tilde{\phi}) \\
\sigma^{\hat{2}} &= -\tfrac{1}{2}(-\sin\hat{\psi}\, d\hat{\theta} + \cos\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) & \sigma^{\tilde{2}} &= -\tfrac{1}{2}(-\sin\tilde{\psi}\, d\tilde{\theta} + \cos\tilde{\psi}\, \sin\tilde{\theta}\, d\tilde{\phi}) \\
\sigma^{\hat{3}} &= -\tfrac{1}{2}(d\hat{\psi} + \cos\hat{\theta}\, d\hat{\phi}) & \sigma^{\tilde{3}} &= -\tfrac{1}{2}(d\tilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi})
\end{aligned} \tag{5.2.3}$$

and satisfy the relations

$$\begin{aligned}
\mathrm{d}\sigma^{\hat{i}} &= \epsilon_{\hat{i}\hat{j}\hat{k}}\sigma^{\hat{j}} \wedge \sigma^{\hat{k}} \\
\mathrm{d}\sigma^{\tilde{i}} &= \epsilon_{\tilde{i}\tilde{j}\tilde{k}}\sigma^{\tilde{j}} \wedge \sigma^{\tilde{k}} .
\end{aligned} \tag{5.2.4}$$

With this normalisation the metric on the unit radius round three sphere is given by

$$d\Omega_3^2 = (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 , \tag{5.2.5}$$

with σ^a being either $\sigma^{\hat{a}}$ or $\sigma^{\tilde{a}}$.
We choose our "d-bein" to be

$$e^m = \varepsilon^m{}_\mu \mathrm{d}x^\mu \tag{5.2.6}$$

$$e^{\hat{a}} = \begin{cases} \rho_1 \sigma^{\hat{a}} & a = 1,2 \\ \rho_3(\sigma^{\hat{3}} - A_\mu \mathrm{d}x^\mu) & a = 3 \end{cases} \tag{5.2.7}$$

$$e^{\tilde{a}} = \tilde{\rho}\sigma^{\tilde{a}} \tag{5.2.8}$$

Since we are looking for the geometric dual to operators which involve only scalar fields in the gauge theory, the only possible non-zero Ramond-Ramond field strength is the five form $F_{(5)}$ and the dilaton is assumed to be constant. The most generic Ansatz for the five form which is invariant under the given symmetries is:

$$\begin{aligned}
F_{(5)} = 2\left(\tilde{G}_{mn} e^m \wedge e^n + \tilde{V}_m e^m \wedge e^{\hat{3}} + \tilde{g} e^{\hat{1}} \wedge e^{\hat{2}}\right) \wedge \tilde{\rho}^3 \mathrm{d}\tilde{\Omega}_3 + \\
2\left(-G_{pq} e^p \wedge e^q \wedge e^{\hat{1}} \wedge e^{\hat{2}} \wedge e^{\hat{3}} + \star_4 \tilde{V} \wedge e^{\hat{1}} \wedge e^{\hat{2}} - \star_4 \tilde{g} \wedge e^{\hat{3}}\right) ,
\end{aligned} \tag{5.2.9}$$

where

$$G_{mn} = \frac{1}{2}\epsilon_{mnpq}\tilde{G}^{mn} \tag{5.2.10}$$

$$\star_4 \tilde{V} = \frac{1}{3!}\epsilon_{mnpq}\tilde{V}^m e^n \wedge e^p \wedge e^q \tag{5.2.11}$$

$$\star_4 \tilde{g} = \tilde{g} e^0 \wedge e^1 \wedge e^2 \wedge e^3 \tag{5.2.12}$$

The Bianchi identity $\mathrm{d}F_{(5)} = 0$ implies:

$$\mathrm{d}\big(\tilde{G}\tilde{\rho}^3 - \tilde{V} \wedge A\rho_3\tilde{\rho}^3\big) = 0 \tag{5.2.13}$$

$$\tilde{V} = \frac{1}{2}\frac{1}{\rho_3\tilde{\rho}^3}\mathrm{d}(\tilde{g}\rho_1^2\tilde{\rho}^3) \tag{5.2.14}$$

$$\mathrm{d}\big(G\rho_1^2\rho_3\big) = 0 \tag{5.2.15}$$

$$\mathrm{d}\big(G\rho_1^2\rho_3 \wedge A + \star_4\tilde{V}\big) - 2\star_4 \tilde{g} = 0 . \tag{5.2.16}$$

Since we are looking for the dual of BPS states, the background should preserve a fraction of the supersymmetry and so there should exist a supersymmetry parameter ψ such that the gravitino variation vanishes:

$$\delta\chi_M = \nabla_M\psi + \frac{\mathrm{i}}{480}F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\Gamma_M\psi = 0\,. \tag{5.2.17}$$

The Bianchi identity and the existence of the spinor ψ are sufficient for our supergravity background to satisfy the full equations of motion of type IIB Supergravity.
The existence of the spinor ψ is also sufficient to express the complete solution in the following form:

$$\begin{aligned}\mathrm{d}s^2 = -h^{-2}(\mathrm{d}t + V_i\mathrm{d}x^i)^2 + h^2\frac{\rho_1^2}{\rho_3^2}(T^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}y^2) + \tilde{\rho}^2\mathrm{d}\tilde{\Omega}_3^2 + \\ + \rho_1^2(\hat{\sigma}_1^2 + \hat{\sigma}_2^2) + \rho_3^2(\hat{\sigma}_3 - A_t\mathrm{d}t - A_i\mathrm{d}x^i)^2\end{aligned} \tag{5.2.18}$$

where the coordinate y is the product of two radii

$$y = \rho_1\tilde{\rho} > 0\,, \tag{5.2.19}$$

and the function h is given by

$$h^{-2} = \tilde{\rho}^2 + \rho_3^2(1 + A_t)^2\,. \tag{5.2.20}$$

The vector ∂_t is the Killing vector which generates the extra non-compact timelike $U(1)$ and thus all the entries of the metric depend only on (x^1, x^2, y), where y is constrained to be positive. They can be expressed in terms of four independent functions:

$$m, n, p, T$$

as follows:

$$\begin{array}{lll} \rho_1^4 = \frac{mp+n^2}{m}y^4 & \rho_3^4 = \frac{p^2}{m(mp+n^2)} & \tilde{\rho}^4 = \frac{m}{mp+n^2} \\ h^4 = \frac{mp^2}{mp+n^2} & A_t = \frac{n-p}{p} & A_i = A_tV_i - \frac{1}{2}\epsilon_{ij}\partial_j\ln T \end{array} \tag{5.2.21}$$

and

$$\mathrm{d}V = -y\star_3[\mathrm{d}n + (nD + 2ym(n-p) + 2n/y)\mathrm{d}y] \tag{5.2.22}$$

$$\partial_y\ln T = D \tag{5.2.23}$$

$$D \equiv 2y(m + n - 1/y^2)\,, \tag{5.2.24}$$

where $\star_3$ indicates the Hodge dual in the three dimensional diagonal metric

$$\mathrm{d}s_3^2 = T^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}y^2\,. \tag{5.2.25}$$

The various four-dimensional forms from which the 5-form field strength is constructed are

$$\tilde{g} = \frac{1}{4\tilde{\rho}}\left[1 - \frac{\rho_3^2}{\rho_1^2}(1 + A_t)\right] \tag{5.2.26}$$

$$\tilde{V} = \frac{1}{2}\frac{1}{\rho_3\tilde{\rho}^3}\mathrm{d}(\tilde{g}\rho_1^2\tilde{\rho}^3) \tag{5.2.27}$$

$$G\rho_1^2\rho_3 = \mathrm{d}B_t \wedge (\mathrm{d}t + V_i\mathrm{d}x^i) + B_t\mathrm{d}V + \mathrm{d}\hat{B} \tag{5.2.28}$$

$$\tilde{G}\tilde{\rho}^3 = \frac{1}{2}g\rho_1^2\tilde{\rho}^3\mathrm{d}A + \mathrm{d}\tilde{B}_t \wedge (\mathrm{d}t + V_i\mathrm{d}x^i) + \tilde{B}_t\mathrm{d}V + \mathrm{d}\hat{\tilde{B}}\,, \tag{5.2.29}$$

where

$$\begin{aligned}
\tilde{B}_t &= -\frac{1}{16}y^2\,\frac{n - 1/y^2}{p}\\
\mathrm{d}\hat{\tilde{B}} &= -\frac{1}{16}y^3 \star_3 [\mathrm{d}m + 2mD\,\mathrm{d}y]\\
B_t &= -\frac{1}{16}y^2\,\frac{n}{m}\\
\mathrm{d}\hat{B} &= \frac{1}{16}y^3 \star_3 [\mathrm{d}p + 4yn(p-n)\mathrm{d}y]\,.
\end{aligned} \tag{5.2.30}$$

Differential equations

The Bianchi identities and the integrability condition for the equation (5.2.22) give

$$\begin{cases} \mathrm{dd}V = 0 \\ \mathrm{dd}\hat{\tilde{B}} = 0 \\ \mathrm{dd}\hat{B} = 0 \end{cases}. \tag{5.2.31}$$

These three conditions together with (5.2.24) give a system of nonlinear coupled elliptic differential equations

$$\begin{aligned}
&y^3(\partial_1^2 + \partial_2^2)n + \partial_y\left(y^3T^2\partial_y n\right) + y^2\partial_y\left[T^2\left(yDn + 2y^2m(n-p)\right)\right] + 4y^2DT^2n = 0\\
&y^3(\partial_1^2 + \partial_2^2)m + \partial_y\left(y^3T^2\partial_y m\right) + \partial_y\left(y^3T^2 2mD\right) = 0\\
&y^3(\partial_1^2 + \partial_2^2)p + \partial_y\left(y^3T^2\partial_y p\right) + \partial_y\left[y^3T^2 4ny(n-p)\right] = 0\\
&\partial_y \ln T = D\,.
\end{aligned} \tag{5.2.32}$$

A solution to these equations is determined by a set of boundary conditions at infinity (large values of y, x^i) and on the plane $y = 0$; they should be chosen in such a way as to give a non-singular geometry asymptotic to $AdS_5 \times S^5$. Due to the non-linearity of the equations the relationship between boundary conditions and non-singular solutions with $AdS_5 \times S^5$ asymptotics is difficult to control. This set of boundary conditions may be regarded as a parametrisation of the space of solutions to our problem.

The LLM limit

The LLM solutions are clearly a subset of ours. They are specified by the additional constraints,

$$n = p = \frac{1}{y^2} - m = \frac{1/2 - z}{y^2} \quad T = 1 \,. \tag{5.2.33}$$

In this case we have

$$D = 0 \quad \rho_1 = \rho_3 = \rho \quad A_t = 0 \quad T = 1 \tag{5.2.34}$$

and the three second order equations collapse to one single linear equation

$$y^3(\partial_1^2 + \partial_2^2)n + \partial_y\left(y^3 T^2 \partial_y n\right) \,. \tag{5.2.35}$$

As this equation is linear it has been possible to completely identify the boundary conditions at $y = 0$ and at infinity that give rise to regular asymptotically $AdS_5 \times S^5$ geometries[12, 16]. This set of boundary conditions can be directly identified with the classical phase space of the dual states in the free fermion picture.

5.3 Asymptotics and Charges

In this section we discuss asymptotic solutions to the differential equations of the previous section which give $AdS_5 \times S^5$ asymptotics [2]. We solve the equations to third order in an expansion for large values of y, x^1, x^2.

We can identify the boundary conditions at infinity by comparing the leading order of this expansion to the same order of LLM, requiring in particular $AdS_5 \times S^5$ asymptotics. The first corrections to the $AdS_5 \times S^5$ geometry capture the global $U(1)$ charges under the gauge fields arising in the Kaluza Klein reduction of IIB supergravity over S^5. We will show that the solutions support non-vanishing fluxes for the the KK gauge fields associated to two of the three Cartan generators of the $SO(6)$ isometry of S^5. In the dual gauge theory picture these generators map to the R-symmetry generators $L_{5,6}$ and $L_{1,2} + L_{3,4}$.

It is not hard to see that the following expressions for our functions

$$\begin{aligned} m &\sim \frac{1}{y^2} - \frac{p_1}{R^4} \\ n &\sim \frac{p_1}{R^4} \\ p &\sim \frac{p_1}{R^4} \\ T &\sim 1 \end{aligned} \tag{5.3.1}$$

satisfy the equations at leading order for large R, with (R, θ, ϕ) polar coordinates in the (x_1, x_2, y) space and p_1 is a constant parameter. We have also, to the same order,

$$V_\phi \sim \frac{p_1 \cos^2\theta}{R^2} \qquad V_r \sim O\left(\frac{1}{R^4}\right) \,, \tag{5.3.2}$$

[2] A study of more general boundary conditions at $y = 0$ will be presented in [94].

with $r^2 = x_1^2 + x_2^2$, $r = R\cos\theta$ and $y = R\sin\theta$.
Defining

$$\begin{aligned} \tilde{R} &= R/\sqrt{p_1} \\ \tilde{\phi} &= \phi - t \end{aligned} \tag{5.3.3}$$

we get

$$\mathrm{d}s^2 = \sqrt{p}_1 \left(-\tilde{R}^2 \mathrm{d}t^2 + \frac{\mathrm{d}\tilde{R}^2}{\tilde{R}^2} + \tilde{R}^2 \mathrm{d}\tilde{\Omega}_3^2 + \mathrm{d}\theta^2 + \cos^2\theta \mathrm{d}\tilde{\phi}^2 + \sin^2\theta \mathrm{d}\hat{\Omega}_3^2 \right) \tag{5.3.4}$$

which is $AdS_5 \times S^5$ in Poincare coordinates. The parameter p_1 and the radius L of AdS_5 are related by

$$L^2 = \sqrt{p_1}\,. \tag{5.3.5}$$

We recall here the expression for the left-invariant one forms

$$\begin{aligned} \sigma^{\hat{1}} &= -\tfrac{1}{2}(\cos\hat{\psi}\, d\hat{\theta} + \sin\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) \\ \sigma^{\hat{2}} &= -\tfrac{1}{2}(-\sin\hat{\psi}\, d\hat{\theta} + \cos\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) \\ \sigma^{\hat{3}} &= -\tfrac{1}{2}(d\hat{\psi} + \cos\hat{\theta}\, d\hat{\phi})\,. \end{aligned} \tag{5.3.6}$$

The metric on the unit radius round three sphere $\mathrm{d}\hat{\Omega}_3$ is

$$\mathrm{d}\hat{\Omega}_3^2 = \left(\sigma^{\hat{1}}\right)^2 + \left(\sigma^{\hat{2}}\right)^2 + \left(\sigma^{\hat{3}}\right)^2 = \frac{1}{4}\left(\mathrm{d}\hat{\theta}^2 + \mathrm{d}\hat{\phi}^2 + \mathrm{d}\hat{\psi}^2 + 2\cos\hat{\theta}\mathrm{d}\hat{\psi}\mathrm{d}\hat{\phi}\right). \tag{5.3.7}$$

We can transform it into the more conventional form

$$\mathrm{d}\hat{\Omega}_3^2 = \mathrm{d}\omega^2 + \cos^2\omega\, \mathrm{d}\phi_1^2 + \sin^2\omega\, \mathrm{d}\phi_2^2 \tag{5.3.8}$$

where

$$\omega = \frac{\hat{\theta}}{2} \qquad \phi_1 = \frac{\hat{\psi} + \hat{\phi}}{2} \qquad \phi_2 = \frac{\hat{\psi} - \hat{\phi}}{2}\,. \tag{5.3.9}$$

We will now consider the next two orders in the asymptotic expansion of our functions and solve the differential equations. For the sake of simplicity we will assume that ∂_ϕ is also a Killing vector of our solutions. Despite this simplifying assumption, in general the solutions will still be charged under the corresponding KK gauge field. From the geometric point of view this means that the solutions are generically stationary. On the gauge theory side, this choice corresponds to looking for duals of linear combinations of states which have all the same L_{56} charge[3] and are thus constructed from linear combinations of $\mathcal{O}(q,r)$ at fixed q and r.

[3] The analog of this choice in the LLM picture would be to consider solutions seeded by rotationally symmetric configurations of bubbles on the $y = 0$ plane.

We thus assume the following expansion of our functions:

$$\begin{aligned} m &\sim \frac{1}{y^2} - \frac{p_1}{R^4} + \frac{m_2(\theta)}{R^6} + \frac{m_3(\theta)}{R^8} \\ n &\sim \frac{p_1}{R^4} + \frac{n_2(\theta)}{R^6} + \frac{n_3(\theta)}{R^8} \\ p &\sim \frac{p_1}{R^4} + \frac{p_2(\theta)}{R^6} + \frac{p_3(\theta)}{R^8} \\ T &\sim 1 + \frac{t_1(\theta)}{R^2} + \frac{t_2(\theta)}{R^4}\,. \end{aligned} \tag{5.3.10}$$

Recalling that $D = 2y(m + n - 1/y^2)$, the equation

$$\partial_y \ln T = D \tag{5.3.11}$$

implies that

$$t_1(\theta) = 0\,. \tag{5.3.12}$$

Moreover we note that

$$V_\phi \sim \frac{p_1 \cos^2\theta}{R^2} + \frac{V_2(\theta)}{R^4} \qquad V_r = 0\,. \tag{5.3.13}$$

With a suitable coordinate transformation

$$\begin{cases} R = \sqrt{p_1}\tilde{R} + \frac{h_1(\tilde{\theta})}{\tilde{R}} \\ \theta = \tilde{\theta} + \frac{g_1(\tilde{\theta})}{\tilde{R}^2} \\ \phi = \tilde{\phi} + t \end{cases} \tag{5.3.14}$$

it should be possible to bring the metric to the following form:

$$\begin{aligned} \mathrm{d}s^2 = \Omega(\tilde{R},\tilde{\theta})L^2 &\left[-(1+\tilde{R}^2 - \frac{\tilde{R}_0^2}{\tilde{R}^2})\mathrm{d}t^2 + \frac{\mathrm{d}\tilde{R}^2}{\tilde{R}^2}(1-\frac{1}{\tilde{R}^2}) + \tilde{R}^2\mathrm{d}\tilde{\Omega}_3^2\right] + \\ &+ L^2\left[g_{\tilde{\theta}\tilde{\theta}}\mathrm{d}\tilde{\theta}^2 + g_{\tilde{\phi}\tilde{\phi}}\cos^2\tilde{\theta}\left(\mathrm{d}\tilde{\phi} + \frac{J}{\tilde{R}^2}\mathrm{d}t\right)^2 + \right. \\ &\left. + g_{\omega\omega}\sin^2\tilde{\theta}\mathrm{d}\omega^2 + g_{\phi\phi}\sin^2\tilde{\theta}\left(\cos^2\omega(\mathrm{d}\phi_1 - \frac{Q}{\tilde{R}^2}\mathrm{d}t)^2 + \sin^2\omega(\mathrm{d}\phi_2 - \frac{Q}{\tilde{R}^2}\mathrm{d}t)^2\right)\right] \end{aligned} \tag{5.3.15}$$

up to subleading corrections. To the leading order the metric components $g_{\tilde{\theta}\tilde{\theta}} = g_{\tilde{\phi}\tilde{\phi}} = g_{\omega\omega} = g_{\phi\phi} = 1$ and reproduce S^5. The constants J and Q are proportional to the total flux of the $U(1)$ gauge fields arising from the KK reduction of the supergravity over S^5. In particular Q is the total charge of the solutions under both the gauge field associated with coordinate transformations generated by $\lambda(\xi)\partial_{\phi_1}$ and $\mu(\xi)\partial_{\phi_2}$ (being ξ coordinates in the AdS_5 factor); these are dual respectively to the $J_1 = L_{1,2}$ and $J_2 = L_{3,4}$ R-symmetry generators. For this reason the expected BPS relation is

$$M = \frac{\pi L^2}{4G_5}(|J| + 2|Q|)\,. \tag{5.3.16}$$

The conformal factor $\Omega(\tilde{R},\tilde{\theta})$ satisfies $\Omega(\tilde{R}=\infty,\tilde{\theta})=1$ and contains terms up to order $\tilde{R}^{-4}$. The mass of the excitations over the AdS_5 vacuum is given by

$$M=\frac{3\pi L^2}{8G_5}\tilde{R}_0 \tag{5.3.17}$$

where G_5 is the five-dimensional Newton constant[4]. We recall now the expression for the metric:

$$\begin{aligned} \mathrm{d}s^2 = &-h^{-2}(\mathrm{d}t^2+V_\phi \mathrm{d}\phi)^2+h^2\frac{\rho_1^2}{\rho_3^2}(T^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j+\mathrm{d}y^2)+\\ &+\tilde{\rho}^2\mathrm{d}\Omega_3^2+\rho_1^2\left[(\sigma^{\hat{1}})^2+(\sigma^{\hat{2}})^2\right]+\rho_3^2(\sigma^{\hat{3}}-A_t\mathrm{d}t-A_\phi\mathrm{d}\phi)^2=\\ &=g_{tt}\mathrm{d}t^2+g_{RR}\mathrm{d}R^2+\tilde{\rho}^2\mathrm{d}\tilde{\Omega}_3^2+2g_{\theta R}\mathrm{d}\theta\mathrm{d}R+\\ &+g_{t\tilde{\phi}}\mathrm{d}t\mathrm{d}\tilde{\phi}+g_{t\hat{3}}\mathrm{d}t\sigma^{\hat{3}}+\\ &+g_{\theta\theta}\mathrm{d}\theta^2+g_{\tilde{\phi}\tilde{\phi}}\mathrm{d}\tilde{\phi}^2+g_{\tilde{\phi}\hat{3}}\mathrm{d}\tilde{\phi}\,\sigma^{\hat{3}}+\rho_1^2\left[(\sigma^{\hat{1}})^2+(\sigma^{\hat{2}})^2\right]+\rho_3^2(\sigma^{\hat{3}})^2 \end{aligned} \tag{5.3.18}$$

with

$$\begin{aligned} g_{tt}&=-h^{-2}(1+V_\phi)^2+h^2\frac{\rho_1^2}{\rho_3^2}r^2T^2+\rho_3^2(A_\phi+A_t)^2\\ g_{RR}&=h^2\frac{\rho_1^2}{\rho_3^2}(\sin^2\theta+T^2\cos^2\theta)\\ g_{\theta R}&=h^2\frac{\rho_1^2}{\rho_3^2}R\sin\theta\cos\theta(1-T^2)\\ g_{t\tilde{\phi}}&=-h^{-2}(1+V_\phi)V_\phi+h^2\frac{\rho_1^2}{\rho_3^2}r^2T^2+\rho_3^2(A_t+A_\phi)A_\phi\\ g_{t\hat{3}}&=-\rho_3^2(A_t+A_\phi)\\ g_{\theta\theta}&=h^2\frac{\rho_1^2}{\rho_3^2}R^2(\cos^2\theta+T^2\sin^2\theta)\\ g_{\tilde{\phi}\tilde{\phi}}&=-h^{-2}V_\phi^2+h^2\frac{\rho_1^2}{\rho_3^2}r^2T^2+\rho_3^2A_\phi^2\\ g_{\tilde{\phi}\hat{3}}&=\rho_3^2A_\phi \end{aligned} \tag{5.3.19}$$

We can now derive the Q charge of our solutions. Using the definition (5.3.6) and the coordinate transformation (5.3.9) we get

$$Q=-\frac{g_{t\hat{3}}}{g_{\hat{3}\hat{3}}}\tilde{R}^2=(A_t+A_\phi)\,. \tag{5.3.20}$$

[4]This approach follows the one in [12]. A more precise and detailed approach can be taken following e.g. the work in [95]

We note that $A_t = (n-p)/p = O(1/R^2)$ and $A_\phi = A_t V_\phi + \frac{1}{2} r\partial_r \ln T = O(1/R^4)$ and thus the leading behaviour of the r.h.s. is determined by A_t and we have

$$Q = \frac{n_2(\theta) - p_2(\theta)}{p_1^2}. \tag{5.3.21}$$

Using these relations we can solve the equations (5.2.32) up to second order in $\frac{1}{R^2}$ and demanding that the solutions are regular we find

$$\begin{cases} p_2(\theta) = d(3\cos^2\theta - 1) \\ n_2(\theta) = p_2(\theta) + p_1^2 Q \\ m_2(\theta) = -p_2(\theta) - 2p_1^2 Q \\ V_2(\theta) = \dfrac{1}{2}\cos^2\theta\left[(Qp_1^2 - d + 3d\cos(2\theta)\right] \end{cases} \tag{5.3.22}$$

where d is a generic real integration constant. The J charge is given by

$$J = \frac{g_{t\tilde{\phi}}}{g_{\tilde{\phi}\tilde{\phi}}} \tilde{R}^2 = \frac{d}{p_1^2} - 1 - Q. \tag{5.3.23}$$

The conserved charges Q and J can be also obtained by evaluating Komar integrals associated with the Killing vectors $\hat{\Sigma}_3$ (the dual vector field to $\hat{\sigma}_3$) and $\frac{\partial}{\partial\phi}$ respectively.

We will now solve the equations to the next order and find the transformation (5.3.14) that brings the metric to the form (5.3.15) enabling us to check that the BPS mass formula

$$M = \frac{\pi L^2}{4G_5}(|J| + 2|Q|) \tag{5.3.24}$$

is satisfied.

We have

$$g_{\tilde{\theta}\tilde{R}} = [h_1'(\tilde{\theta}) - 2\sqrt{p_1}g_1(\tilde{\theta})]\frac{1}{\tilde{R}^3} \tag{5.3.25}$$

which fixes

$$g_1(\tilde{\theta}) = \frac{h_1'(\tilde{\theta})}{2\sqrt{p_1}}. \tag{5.3.26}$$

We are not really interested in the conformal factor $\Omega(\tilde{R},\tilde{\theta})$ and we thus proceed to the calculation of the ratio

$$\frac{g_{\tilde{R}\tilde{R}}}{\tilde{\rho}^2} = \frac{1}{\tilde{R}^4} + \frac{d(3\cos^2\tilde{\theta} - 1) - 6p_1^{3/2}h_1(\tilde{\theta})}{p_1^2} \tag{5.3.27}$$

which should satisfy the equation

$$\frac{g_{\tilde{R}\tilde{R}}}{\tilde{\rho}^2} = \frac{1}{\tilde{R}^4} - \frac{1}{\tilde{R}^6}. \tag{5.3.28}$$

This requirement gives immediately,

$$h_1(\tilde{\theta}) = \frac{p_1^2 + d(3\cos^2\tilde{\theta} - 1)}{6p_1^{3/2}}. \tag{5.3.29}$$

Using this relation we obtain

$$\frac{g_{tt}}{\tilde{\rho}^2} = -1 - \frac{1}{\tilde{R}^2} + \frac{2}{3}\left(\frac{d}{p_1^2} - 1 - 3Q\right)\frac{1}{\tilde{R}^4} \tag{5.3.30}$$

which gives

$$\tilde{R}_0 = \frac{2}{3}(J - 2Q) \tag{5.3.31}$$

and thus

$$M = \frac{3\pi L^2}{8G_5}\tilde{R}_0 = \frac{\pi L^2}{4G_5}(J - 2Q). \tag{5.3.32}$$

This should be compared to

$$M = \frac{\pi L^2}{4G_5}(|J| + 2|Q|), \tag{5.3.33}$$

which apparently requires that $J > 0$ and $Q < 0$. Up to now, J and Q have appeared in the solution to the differential equations as constants of integration. As such, they can take any real value. Constraints on their possible values should come from a global analysis of the solutions [5]. Indeed given the leading behaviour at large R, these subleading corrections should be completely determined by the boundary conditions at $y = 0$. Unfortunately we are not able to express these charges in terms of the data at $y = 0$ plane which could have allowed us to establish the above bounds on J and Q. As a matter of comparison, in the LLM construction only the J charge is present and its value is determined by a set of integrals performed on the $y = 0$ plane. In that case, the bound $J > 0$ is trivially imposed by the specific type of boundary conditions at $y = 0$.

5.4 Discussion and Observations

In this Chapter we have shown how to extend to the 1/8 BPS case the construction of [12]. Due to the reduced amount of symmetry of our background the expressions we find turn out to be rather more complex; in particular the differential equations which determine the background are highly non linear. We performed an asymptotic analysis for large values of R and were able to show that solutions with the desired asymptotics and regularity exist in this limit. Of course, a satisfactory understanding of the boundary conditions at $y = 0$

[5] As in the LLM case, the sign of J is correlated with the relative chirality of the Killing spinor with respect to the two $SO(4)$'s. From the gauge theory side, as follows from the discussion at the end of Section 2, the sign of Q is correlated with the $U(1)_R$ charge of the Killing spinor. As it emerges from the detailed analysis of Appendix B, this charge is captured by the eigenvalue s with respect to a Pauli matrix σ_3. In our analysis we have set for definiteness $s = +1$. Had we chosen $s = -1$, Q would have been positive.

which lead to non-singular solutions is necessary in order to connect the geometry of the supergravity solutions to the phase space of the quantum mechanical system arising from the dual gauge theory on $\mathbb{R} \times S^3$. In particular it would be very interesting to understand the relationship between our construction and the work of [82, 83, 85, 84]. Once the space of solutions is understood from the supergravity point of view one could proceed to its quantisation by a procedure like that presented in [96, 97].

Our solutions have a non empty intersection with the solutions described in [80, 79] and in [78]. A partial discussion of the dictionary between those papers and our work can be found in [81]

Some of the so-called superstar geometries in [56] are also contained in our description. These solutions are known to have singularities and it is possible to identify the boundary conditions at $y = 0$ that are responsible for them. With a more detailed understanding of boundary conditions which give rise to non-singular solutions, and their relation to the CFT, one may better understand the resolution of the singularities in a manner similar to that of [67, 16, 47]. Finally different types of boundary conditions at large R can be studied. Indeed one can find solutions with asymptotics of the form $AdS_5 \times Y^{p,q}$: such geometries correspond to 1/2 BPS operators in the $\mathcal{N} = 1$ superconformal quiver gauge theories and will be the subject of the next Chapter.

5.5 Reduction of the Killing spinor equations

In this Section we present the step by step derivation of the results presented in Section 5.2.

Metric and 5-form Ansatz

The most generic Ansatz for our solutions is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \rho_1^2\big[(\sigma^{\hat{1}})^2 + (\sigma^{\hat{2}})^2\big] + \rho_3^2(\sigma^{\hat{3}} - A_\mu \mathrm{d}x^\mu)^2 + \tilde{\rho}^2 \mathrm{d}\tilde{\Omega}_3^2 \,. \tag{5.5.1}$$

The space is thus made up of a fibration over a four dimensional manifold of a squashed 3-sphere (on which the $SU(2)$ left-invariant 1-forms $\sigma^{\hat{a}}$ are defined) and a round 3-sphere (on which the $SU(2)$ left-invariant 1-forms $\sigma^{\tilde{a}}$ are defined).
The left invariant 1-forms are given by

$$\begin{array}{ll}
\sigma^{\hat{1}} = -\frac{1}{2}(\cos\hat{\psi}\, d\hat{\theta} + \sin\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) & \sigma^{\tilde{1}} = -\frac{1}{2}(\cos\tilde{\psi}\, d\tilde{\theta} + \sin\tilde{\psi}\, \sin\tilde{\theta}\, d\tilde{\phi}) \\
\sigma^{\hat{2}} = -\frac{1}{2}(-\sin\hat{\psi}\, d\hat{\theta} + \cos\hat{\psi}\, \sin\hat{\theta}\, d\hat{\phi}) & \sigma^{\tilde{2}} = -\frac{1}{2}(-\sin\tilde{\psi}\, d\tilde{\theta} + \cos\tilde{\psi}\, \sin\tilde{\theta}\, d\tilde{\phi}) \\
\sigma^{\hat{3}} = -\frac{1}{2}(d\hat{\psi} + \cos\hat{\theta}\, d\hat{\phi}) & \sigma^{\tilde{3}} = -\frac{1}{2}(d\tilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi})
\end{array} \tag{5.5.2}$$

and satisfy the relations

$$\begin{aligned}
\mathrm{d}\sigma^{\hat{i}} &= \epsilon_{\hat{i}\hat{j}\hat{k}}\sigma^{\hat{j}} \wedge \sigma^{\hat{k}} \\
\mathrm{d}\sigma^{\tilde{i}} &= \epsilon_{\tilde{i}\tilde{j}\tilde{k}}\sigma^{\tilde{j}} \wedge \sigma^{\tilde{k}} \,.
\end{aligned} \tag{5.5.3}$$

With this normalisation the metric on the unit radius round three sphere is given by

$$d\Omega_3^2 = (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 \tag{5.5.4}$$

with σ^a being either $\sigma^{\hat{a}}$ or $\sigma^{\tilde{a}}$.
We choose our "d-bein" to be

$$e^m = \varepsilon^m{}_\mu \mathrm{d}x^\mu \tag{5.5.5}$$

$$e^{\hat{a}} = \begin{cases} \rho_1 \sigma^{\hat{a}} & a = 1,2 \\ \rho_3(\sigma^{\hat{3}} - A_\mu \mathrm{d}x^\mu) & a = 3 \end{cases} \tag{5.5.6}$$

$$e^{\tilde{a}} = \tilde{\rho}\sigma^{\tilde{a}} \tag{5.5.7}$$

The only non zero Ramond-Ramond field strength is the five form $F_{(5)}$ and the dilaton is assumed to be constant. The most generic Ansatz for the five form which is invariant under the given symmetries is

$$F_{(5)} = 2\left(\tilde{G}_{mn} e^m \wedge e^n + \tilde{V}_m e^m \wedge e^{\hat{3}} + \tilde{g} e^{\hat{1}} \wedge e^{\hat{2}}\right) \wedge \tilde{\rho}^3 \mathrm{d}\tilde{\Omega}_3 + 2\left(-G_{pq} e^p \wedge e^q \wedge e^{\hat{1}} \wedge e^{\hat{2}} \wedge e^{\hat{3}} + \star_4 \tilde{V} \wedge e^{\hat{1}} \wedge e^{\hat{2}} - \star_4 \tilde{g} \wedge e^{\hat{3}}\right), \tag{5.5.8}$$

where

$$G_{mn} = \frac{1}{2}\epsilon_{mnpq}\tilde{G}^{mn} \tag{5.5.9}$$

$$\star_4 \tilde{V} = \frac{1}{3!}\epsilon_{mnpq}\tilde{V}^m e^n \wedge e^p \wedge e^q \tag{5.5.10}$$

$$\star_4 \tilde{g} = \tilde{g} e^0 \wedge e^1 \wedge e^2 \wedge e^3 \,. \tag{5.5.11}$$

The Bianchi identity $\mathrm{d}F_{(5)} = 0$ gives rise to the set of equations,

$$\mathrm{d}\big(\tilde{G}\tilde{\rho}^3 - \tilde{V} \wedge A \rho_3 \tilde{\rho}^3\big) = 0 \tag{5.5.12}$$

$$\tilde{V} = \frac{1}{2}\frac{1}{\rho_3 \tilde{\rho}^3}\mathrm{d}(\tilde{g}\rho_1^2 \tilde{\rho}^3) \tag{5.5.13}$$

$$\mathrm{d}\big(G\rho_1^2\rho_3\big) = 0 \tag{5.5.14}$$

$$\mathrm{d}\big(G\rho_1^2\rho_3 \wedge A + \star_4 \tilde{V}\big) - 2\star_4 \tilde{g} = 0 \,. \tag{5.5.15}$$

Spin Connection and Covariant Derivative.

The inverse d-bein is

$$E_m = \Xi^\mu{}_m \partial_\mu + A_m \Sigma^{\hat{i}}{}_{\hat{3}} \partial_{\hat{i}} \tag{5.5.16}$$

$$E_{\hat{a}} = \frac{1}{\rho_a}\Sigma^{\hat{i}}{}_{\hat{a}}\partial_{\hat{i}} \tag{5.5.17}$$

$$E_{\tilde{a}} = \frac{1}{\tilde{\rho}}\Sigma^{\tilde{i}}{}_{\tilde{a}}\partial_{\tilde{i}} \,, \tag{5.5.18}$$

where Ξ_m is the inverse vierbein of ε^m and $\Sigma_{\hat{a},\tilde{a}}$ is the inverse of $\sigma^{\hat{a},\tilde{a}}$. We will denote ten-dimensional tangent space indices by A, B, C.... The spin connection is given by

$$\omega_{AB} = -\mathrm{d}e_A \cdot E_B + \mathrm{d}e_B \cdot E_A + \frac{1}{2}\left(e^C \cdot [E_A, E_B]\right) e_C \,. \tag{5.5.19}$$

Using the explicit expressions for E_m we have

$$[E_m, E_n] = [\Xi_m, \Xi_n] + \Sigma_{\hat{3}}\left(\Xi_m(A\cdot\Xi_n) - \Xi_n(A\cdot\Xi_m)\right) \,. \tag{5.5.20}$$

We can thus write, using the relation (A.0.12)

$$\begin{aligned}\omega_{mn} = \tilde{\omega}_{mn} + e^{\hat{3}}\rho_3\frac{1}{2}\left(-A\cdot[\Xi_m,\Xi_n] + \Xi_m(A\cdot\Xi_n) - \Xi_n(A\cdot\Xi_m)\right) = \\ = \tilde{\omega}_{mn} + e^3\rho_3\Xi_m\cdot \mathrm{d}A\cdot\Xi_n \,.\end{aligned} \tag{5.5.21}$$

In order to get the other components of the spin connection we will need the explicit form of the exterior derivative of $e^{\hat{a}} = \rho_{\hat{a}}\sigma^{\hat{a}} - \rho_3\delta^{\hat{a}}_{\hat{3}}A_m e^m$ and of $e^{\tilde{a}} = \tilde{\rho}\sigma^{\tilde{a}}$

$$\mathrm{d}e^{\hat{a}} = \mathrm{d}\rho_{\hat{a}}\wedge\sigma^{\hat{a}} + \rho_{\hat{a}}\mathrm{d}\sigma^{\hat{a}} - \rho_3\delta^{\hat{a}}_{\hat{3}}\mathrm{d}A - \delta^{\hat{a}}_{\hat{3}}\mathrm{d}\rho_3\wedge A_m e^m \tag{5.5.22}$$

$$\mathrm{d}e^{\tilde{a}} = \mathrm{d}\rho_{\tilde{a}}\wedge\sigma^{\tilde{a}} + \rho_{\tilde{a}}\mathrm{d}\sigma^{\tilde{a}} \tag{5.5.23}$$

By definition

$$\begin{aligned}\mathrm{d}\sigma^{\hat{i}} &= \epsilon_{\hat{i}\hat{j}\hat{k}}\sigma^{\hat{j}}\wedge\sigma^{\hat{k}} \\ \mathrm{d}\sigma^{\tilde{i}} &= \epsilon_{\tilde{i}\tilde{j}\tilde{k}}\sigma^{\tilde{j}}\wedge\sigma^{\tilde{k}}\end{aligned} \tag{5.5.24}$$

and thus

$$\begin{aligned}[\Sigma_{\hat{a}}, \Sigma_{\hat{b}}] &= -2\epsilon_{\hat{c}\hat{a}\hat{b}}\Sigma_{\hat{c}} \\ [\Sigma_{\tilde{a}}, \Sigma_{\tilde{b}}] &= -2\epsilon_{\tilde{c}\tilde{a}\tilde{b}}\Sigma_{\tilde{c}} \,,\end{aligned} \tag{5.5.25}$$

so that

$$[E_{\hat{a}}, E_m] = \frac{1}{\rho_a^2}\partial_m\rho_a\Sigma_{\hat{a}} + \frac{1}{\rho_a}A_m[\Sigma_{\hat{a}},\Sigma_{\hat{3}}] = \frac{1}{\rho_a^2}\partial_m\rho_a\Sigma_{\hat{a}} - \frac{2}{\rho_a}A_m\epsilon_{\hat{c}\hat{a}\hat{3}}\Sigma_{\hat{c}} \,. \tag{5.5.26}$$

In the end

$$\begin{aligned}\omega_{\hat{a}m} = -\mathrm{d}e_{\hat{a}}E_m + \frac{1}{2}e^P[E_{\hat{a}}, E_m]e_P = \\ = \partial_m\rho_a\sigma^{\hat{a}} + \delta^{\hat{a}}_{\hat{3}}e^p(\rho_3\frac{1}{2}F_{pm} - A_p\partial_m\rho_3)\end{aligned} \tag{5.5.27}$$

$$\omega_{\tilde{a}m} = -\mathrm{d}e_{\tilde{a}}E_m + \frac{1}{2}e^P[E_{\tilde{a}}, E_m]e_P = \partial_m\tilde{\rho}\,\sigma^{\tilde{a}} \tag{5.5.28}$$

and

$$\omega_{\hat{a}\hat{b}} = -\mathrm{d}e_{\hat{a}} E_{\hat{b}} + \mathrm{d}e_{\hat{b}} E_{\hat{a}} + \frac{1}{2} e^M \cdot [E_{\hat{a}}, E_{\hat{b}}] e_M =$$

$$= \epsilon_{\hat{a}\hat{b}\hat{c}} \left(\frac{\rho_a^2 + \rho_b^2 - \rho_c^2}{\rho_a \rho_b} \right) \sigma^{\hat{c}} + \epsilon_{\hat{a}\hat{b}\hat{3}} \frac{\rho_3^2}{\rho_1^2} A \quad (5.5.29)$$

$$\omega_{\tilde{a}\tilde{b}} = -\mathrm{d}e_{\tilde{a}} E_{\tilde{b}} + \mathrm{d}e_{\tilde{b}} E_{\tilde{a}} + \frac{1}{2} e^M \cdot [E_{\tilde{a}}, E_{\tilde{b}}] e_M = \epsilon_{\tilde{a}\tilde{b}\tilde{c}} \sigma^{\tilde{c}} \,. \quad (5.5.30)$$

The spin connection part of the covariant derivative acting on spinors as presented in Appendix A is

$$\begin{aligned}
&\frac{1}{4}\omega_{MN}\Gamma^M\Gamma^N = \\
&\mathrm{d}x^\mu \Bigg[\frac{1}{4}\tilde{\omega}_{mn}\Gamma^m\Gamma^n - \frac{1}{4}\rho_3 F_{\mu m}\Gamma^m\Gamma^{\hat{3}} + \\
&\qquad - A_\mu \left(-\frac{1}{2}\frac{\rho_3^2}{\rho_1^2}\Gamma^{\hat{1}}\Gamma^{\hat{2}} - \frac{1}{2}\partial_m\rho_3\Gamma^m\Gamma^{\hat{3}} + \frac{1}{8}\rho_3^2 F_{mn}\Gamma^m\Gamma^n \right) \Bigg] + \\
&\sum_{a=1,2} \sigma^{\hat{a}} \left(\frac{1}{2}\frac{\rho_3}{\rho_1}\epsilon_{\hat{a}\hat{b}\hat{3}}\Gamma^{\hat{b}}\Gamma^{\hat{3}} - \frac{1}{2}\partial_m\rho_1\Gamma^m\Gamma^{\hat{a}} \right) + \\
&\quad + \sigma^{\hat{3}} \left(\frac{1}{2}\left(2 - \frac{\rho_3^2}{\rho_1^2}\right)\Gamma^{\hat{1}}\Gamma^{\hat{2}} - \frac{1}{2}\partial_m\rho_3\Gamma^m\Gamma^{\hat{3}} + \rho_3^2\frac{1}{8}F_{mn}\Gamma^m\Gamma^n \right) + \\
&\sum_{a=1,2,3} \sigma^{\tilde{a}} \left(\frac{1}{4}\epsilon_{\tilde{a}\tilde{b}\tilde{c}}\Gamma^{\tilde{b}}\Gamma^{\tilde{c}} - \frac{1}{2}\partial_m\tilde{\rho}\Gamma^m\Gamma^{\tilde{a}} \right)
\end{aligned} \quad (5.5.31)$$

Killing spinor

Conventions and Ansatz

We choose the following ten dimensional gamma matrices

$$\Gamma_m = \gamma_m \otimes 1 \otimes 1 \otimes 1 \quad \Gamma^{\hat{a}} = 1 \otimes \hat{\sigma}_1 \otimes \sigma_{\hat{a}} \otimes 1 \quad \Gamma^{\tilde{a}} = 1 \otimes \hat{\sigma}_2 \otimes 1 \otimes \sigma_{\tilde{a}} \quad (5.5.32)$$

The two 32 component Majorana-Weyl spinor supersymmetry parameters of the IIB theory can be grouped into a single complex Weyl spinor ψ obeying the chirality constraint

$$\Gamma_{11}\psi = \psi \quad (5.5.33)$$

$$\Gamma_{11} = \prod_m \Gamma_m \prod_{\hat{a}} \Gamma_{\hat{a}} \prod_{\tilde{a}} \Gamma_{\tilde{a}} = \gamma_5 \hat{\sigma}_3 \qquad \gamma_5 = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 \,. \quad (5.5.34)$$

The supersymmetry variation of the gravitino χ_M is given by

$$\delta\chi_M = \nabla_M\psi + \frac{\mathrm{i}}{480} F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\Gamma_M\psi \,. \quad (5.5.35)$$

In order to have a supersymmetric background we need to impose that this variation is zero giving rise to the Killing spinor equation on ψ,

$$\nabla_M\psi + \frac{\mathrm{i}}{480}F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\Gamma_M\psi = 0\,. \tag{5.5.36}$$

As a consequence of our symmetry assumptions we look for a ψ of the form

$$\psi = \varepsilon_{(b)} \otimes \hat{\chi} \otimes \tilde{\chi}_{(b)}\,. \tag{5.5.37}$$

Where ψ is an 8 component complex spinor a and $\hat{\chi}, \tilde{\chi}_b$ are 2 components complex spinors defined on the two 3-spheres satisfying

$$\Sigma_{\hat{a}}\hat{\chi} = 0 \qquad \sigma_{\hat{3}}\hat{\chi} = s\chi \tag{5.5.38}$$

$$\nabla'_{\tilde{a}}\tilde{\chi} = b\frac{\mathrm{i}}{2}\sigma_{\tilde{a}}\tilde{\chi}_{(b)} \tag{5.5.39}$$

where ∇' is the covariant derivative on the unit radius three sphere which has spin connection $\omega'_{abc} = \epsilon_{abc}$ and $s, b = \pm 1$. As we are going to show in the following, this choice means that $\hat{\chi}$ is a constant spinor and thus a singlet of the $SU(2)_L$ isometry of the squashed sphere, as required by our analysis of the gauge theory description of supersymmetries in Section 5.1.

Isometries and Spinors

On a unit radius round three sphere there exist two linearly independent solutions to the equation

$$\nabla_a\chi = \beta\frac{\mathrm{i}}{2}\sigma_a\chi \tag{5.5.40}$$

for each choice of $\beta = \pm 1$. The sign of β is correlated with the chirality of the doublet of solutions under the $SO(4) = SU(2)\times SU(2)$ isometry group of S^3. This can be understood as follows.
Given a d-bein $e^a(y)$ and an isometry I we choose a local orthogonal transformation Λ such that

$$\Lambda^a{}_b \mathrm{T}I_*(e^b) = e^a \tag{5.5.41}$$

where $\mathrm{T}I_*$ is the pullback of one forms associated with I. The d-bein is thus invariant under these transformations and it is possible to give meaning to the transformation properties of spinors under the isometries of the metric.
In our case, since $S^3 \approx SU(2)$, we can identify the points y with elements of $SU(2)$. For the round 3-sphere $\tilde{S}^3$ the action of the isometry group $SU(2)_L \times SU(2)_R$ is given by left and right multiplication by generic elements of $SU(2)$. For the squashed three sphere the action of the isometry group $SU(2)_L \times U(1)_R$ is given by left multiplication by generic elements of $SU(2)$ and right multiplication with a $U(1)$ subgroup.
Let's focus on the left isometries L_g. They are defined by

$$L_g(y) = gy\,. \tag{5.5.42}$$

As our 3-bein is built out of left-invariant one forms σ^a, we have by definition

$$\mathrm{T}L_{g*}(\sigma^a) = \sigma^a \tag{5.5.43}$$

which implies that, for such transformations, $\Lambda^a{}_b = \delta^a{}_b$. The action $\mathrm{S}L_g$ on spinors of this isometry is thus very simple

$$\mathrm{S}L_g \chi(gy) = \chi(y)\,. \tag{5.5.44}$$

The action of left multiplications is clearly surjective and thus a spinor χ is invariant under this action if and only if it is a constant spinor. This means that our spinor $\hat{\chi}$ is a singlet under the $SU(2)_L$ isometry of the squashed 3-sphere, while the spinors $\tilde{\chi}_\pm$ transform in the $(0,\frac{1}{2})$ for upper sign and $(\frac{1}{2},0)$ for the lower sign. For a discussion of spinors in squashed 3-spheres see [98, 99].

Equations and bilinears

We turn now to the contribution of the Ramond-Ramond form to the gravitino variation. We define

$$M \equiv \frac{\mathrm{i}}{480} F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\,. \tag{5.5.45}$$

The chirality condition on ψ and the self-duality of $F_{(5)}$ imply that

$$M\Gamma_M\psi = -\left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1\sigma_{\hat{3}} + \mathrm{i}\tilde{g}\sigma_{\hat{3}}\right)\gamma_5\hat{\sigma}_2\Gamma_M\psi\,. \tag{5.5.46}$$

Due to the conditions on the spinor, $\hat{\chi}$ and $\tilde{\chi}_b$ factorise in each component of the gravitino variation equation which then becomes the following system of coupled differential and algebraic equations on ε[6]

$$\left[\tilde{\nabla}_\mu - \frac{1}{4}F_{\mu\nu}\Xi^\nu{}_m\gamma^m\gamma^5\hat{\sigma}_1 s + \mathrm{i}A_\mu s - \left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s + \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\gamma_\mu\right]\varepsilon = 0 \tag{5.5.48}$$

$$\left[\frac{\mathrm{i}}{2}\frac{\rho_3}{\rho_1}\gamma_5\hat{\sigma}_1 + \frac{1}{2}\not{\partial}\rho_1 + \rho_1\left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s - \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\right]\varepsilon = 0 \tag{5.5.49}$$

$$\left[\frac{\mathrm{i}}{2}\left(2 - \frac{\rho_3^2}{\rho_1^2}\right)\gamma_5\hat{\sigma}_1 + \frac{1}{2}\not{\partial}\rho_3 + \frac{1}{8}\rho_3^2\not{F}\gamma_5\hat{\sigma}_1 s + \rho_3\left(\tilde{\not{G}} - \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s + \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\right]\varepsilon = 0 \tag{5.5.50}$$

$$\left[\frac{\mathrm{i}}{2}b\gamma_5\hat{\sigma}_2 + \frac{1}{2}\not{\partial}\tilde{\rho} - \tilde{\rho}\left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s + \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\right]\varepsilon = 0. \tag{5.5.51}$$

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[6] For example the first equation is obtained as follows

$$\begin{aligned}(\nabla_\mu + M\Gamma_\mu)\psi &= \left(\tilde{\nabla}_\mu - \frac{1}{4}\rho_3F_{\mu\nu}\Xi^\nu{}_m\Gamma^m\Gamma^{\hat{3}} + A_\mu\left(\Sigma_{\hat{3}} + \Gamma^{\hat{1}}\Gamma^{\hat{2}}\right) - A_\mu\nabla_{\hat{3}} + M\Gamma_\mu\right)\psi = \\ &= \left(\tilde{\nabla}_\mu - \frac{1}{4}\rho_3F_{\mu\nu}\Xi^\nu{}_m\Gamma^m\Gamma^{\hat{3}} + A_\mu\Gamma^{\hat{1}}\Gamma^{\hat{2}} + M\left(\Gamma_\mu + A_\mu\rho_3\Gamma_{\hat{3}}\right)\right)\psi = \\ &= \left(\tilde{\nabla}_\mu - \frac{1}{4}\rho_3F_{\mu\nu}\Xi^\nu{}_m\gamma^m\sigma^{\hat{3}} + A_\mu\sigma_{\hat{3}} + M\gamma_\mu\right)\psi \end{aligned} \tag{5.5.47}$$

Note that the first equation is a first order differential 4-vector equation for ε while the last three are algebraic 4-scalar equations.
We now define a useful set of bilinears

$$\begin{aligned} &K_\mu = \bar{\varepsilon}\gamma_\mu\varepsilon \qquad L_\mu = \bar{\varepsilon}\gamma_5\gamma_\mu\varepsilon \qquad Y_{\mu\nu} = \bar{\varepsilon}\gamma_{\mu\nu}\sigma_1\varepsilon \\ &f_1 = \mathrm{i}\bar{\varepsilon}\sigma_1\varepsilon \qquad f_2 = \mathrm{i}\bar{\varepsilon}\sigma_2\varepsilon \\ &\bar{\varepsilon} = \varepsilon^\dagger\gamma_0 \end{aligned} \tag{5.5.52}$$

The world indices μ, ν of these bilinears are obtained by contraction of the tangent space indices with the vierbein $\varepsilon^m{}_\mu$. When raising and lowering μ indices we will always use the metric $\tilde{g}_{\mu\nu}$ unless otherwise is specified. By Fierz rearrangements the following relations can be proved

$$K^2 = -L^2 = -f_1^2 - f_2^2 \equiv -h^{-2} \qquad L^\mu K_\mu = 0 \tag{5.5.53}$$

Algebraic relations

By multiplying the algebraic equations (5.5.49),(5.5.50),(5.5.51) with different combinations of gamma matrices and contracting with $\bar{\varepsilon}$ one can obtain the following relations for the spinor bilinears:

$$K^\mu\partial_\mu\rho_1 = 0 \tag{5.5.54}$$

$$K^\mu\partial_\mu\rho_3 = 0 \tag{5.5.55}$$

$$K^\mu\partial_\mu\tilde{\rho} = 0 \tag{5.5.56}$$

$$L_\mu = -\frac{\rho_1}{\rho_3}\frac{f_1}{\tilde{\rho}}\partial_\mu(\rho_1\tilde{\rho}) \tag{5.5.57}$$

$$K^\mu\tilde{V}_\mu = 0 \tag{5.5.58}$$

$$\tilde{g} = \frac{s}{4f_1}\left(b\frac{f_1}{\tilde{\rho}} - \frac{f_2\rho_3}{\rho_1^2}\right) \tag{5.5.59}$$

and also equations for the 2-forms $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\tilde{G}_{\mu\nu}$

$$\begin{aligned} F_{\mu\nu} = -\frac{2}{\rho_3(f_1^2+f_2^2)}\Bigg[&-\left(2-\frac{\rho_3^2}{\rho_1^2}\right)\frac{1}{\rho_3}\epsilon_{\mu\nu\rho\sigma}K^\rho L^\sigma + \frac{b}{\tilde{\rho}}\left(K_\mu L_\nu - K_\nu L_\mu\right) + \\ &- f_1\epsilon_{\mu\nu\rho\sigma}K^\rho\partial^\sigma\ln(\rho_3\tilde{\rho}) - f_2\big(K_\mu\partial_\nu\ln(\rho_3\tilde{\rho}) - K_\nu\partial_\mu\ln(\rho_3\tilde{\rho})\big) + \\ &+ 4f_1\big(K_\mu\tilde{V}_\nu - K_\nu\tilde{V}_\mu\big) + 4f_2\epsilon_{\mu\nu\rho\sigma}K^\sigma\tilde{V}^\rho\Bigg] \end{aligned} \tag{5.5.60}$$

$$\begin{aligned} \tilde{G}_{\mu\nu} = -\frac{1}{2(f_1^2+f_2^2)}\Bigg[&\left(\frac{b}{2\tilde{\rho}} - \tilde{g}s\right)\left(f_1(K_\mu\partial_\nu\ln\tilde{\rho} - K_\nu\partial_\mu\ln\tilde{\rho}) + f_2\epsilon_{\mu\nu\rho\sigma}K^\rho\partial^\sigma\ln\tilde{\rho}\right) + \\ &- f_2\left(K_\mu\tilde{V}_\nu - K_\nu\tilde{V}_\mu\right) + f_1\epsilon_{\mu\nu\rho\sigma}K^\rho\tilde{V}^\sigma\Bigg] \end{aligned} \tag{5.5.61}$$

Differential relations

We can use (5.5.48) to prove the following relations

$$\tilde{\nabla}_\mu K_\nu = 4\left(\tilde{G}_{\mu\nu} f_1 + G_{\mu\nu} f_2\right) - \frac{\rho_3}{2} F_{\mu\nu} f_2 s + 2\epsilon_{\mu\nu\rho\sigma}\tilde{V}^\rho K^\sigma s - 2\tilde{g} Y_{\mu\nu} s \tag{5.5.62}$$

$$\partial_\mu \ln f_1 = \partial_\mu \ln \tilde{\rho} \tag{5.5.63}$$

$$\partial_\mu \left(\frac{f_2}{\rho_3}\right) = F_{\mu\nu} K^\nu s\,. \tag{5.5.64}$$

The first equation says that $K^\mu \partial_\mu$ is a Killing vector for $g_{\mu\nu}$. We make the natural gauge choice

$$K^\mu \partial_\mu = \partial_t\,. \tag{5.5.65}$$

The second equation can be easily integrated to give, with a suitable choice of constant of integration

$$f_1 = \tilde{\rho}\,. \tag{5.5.66}$$

Note further that as a consequence of these equations and of the Bianchi identity

$$\tilde{V} = \frac{1}{2}\frac{1}{\rho_3 \tilde{\rho}^3} \mathrm{d}(\tilde{g} \rho_1^2 \tilde{\rho}^3)\,, \tag{5.5.67}$$

that $F_{\mu\nu}$ is t independent and we can make a gauge choice for A_μ such that $\partial_t A_\mu = 0$. Integrating the equation for f_2 we obtain

$$f_2 = \rho_3 (c + A_t s)\,. \tag{5.5.68}$$

We define the coordinate

$$y \equiv \rho_1 \tilde{\rho} \tag{5.5.69}$$

and thus

$$L_\mu \mathrm{d}x^\mu = -\frac{\rho_1}{\rho_3} \mathrm{d}y\,. \tag{5.5.70}$$

Since $K \cdot L = 0$ there is no cross term g_{ty} in the metric. We can additionally make a coordinate choice such that there are also no g_{yi} cross terms. We have thus reduced our Ansatz for the four dimensional part of the metric to the following

$$\mathrm{d}s^2 = -h^{-2}(\mathrm{d}t + V_1 \mathrm{d}x^1 + V_2 \mathrm{d}x^2)^2 + h^2 \frac{\rho_1^2}{\rho_3^2} \tilde{h}_{ij} \mathrm{d}x^i \mathrm{d}x^j + h^2 \frac{\rho_1^2}{\rho_3^2} \mathrm{d}y^2\,. \tag{5.5.71}$$

Note that

$$h^{-2} = f_1^2 + f_2^2\,. \tag{5.5.72}$$

For convenience we set

$$A_y = 0\,. \tag{5.5.73}$$

All the entries in the metric and in the 5-form are parametrised by a set of functions that we can distinguish on the basis of their transformation properties in the $\{x^1, x^2\}$ plane.

Scalars	Vectors	Symmetric Tensor
$\rho_1, \rho_3, \tilde{\rho}, A_t$	V_i, A_i	$\tilde{h}_{ij}$

Recalling that the scalars are subject to the constraint

$$y = \rho_1 \tilde{\rho} \, . \tag{5.5.74}$$

From now on we will assume for definiteness that $s = 1$.

Specifying the spinor

Due to our gauge choice we have

$$K^0 = e^0{}_t = h^{-1} \quad \Rightarrow \quad \varepsilon^\dagger \varepsilon = h^{-1} \tag{5.5.75}$$

$$L_3 = L_y E^y{}_3 = -\frac{\rho_1}{\rho_3}\frac{\rho_3}{\rho_1} h^{-1} = -h^{-1} \, . \tag{5.5.76}$$

$$\tag{5.5.77}$$

We thus have:

$$\frac{\varepsilon^\dagger \gamma_0 \gamma_5 \gamma_3 \varepsilon}{\varepsilon^\dagger \varepsilon} = -1 \quad \Rightarrow \quad \mathrm{i}\gamma_1\gamma_2 \varepsilon = -\varepsilon \, . \tag{5.5.78}$$

We can now take the sum of equations (5.5.51) and (5.5.49) divided by, respectively, $\tilde{\rho}$ and ρ_1 from which we obtain

$$\left(\sqrt{1 + \mathrm{e}^{-2G}} \gamma_3 \hat{\sigma}_1 + \mathrm{i}\gamma_5 \mathrm{e}^{-G} - 1\right)\varepsilon = 0 \tag{5.5.79}$$

where $\mathrm{e}^{-G} \equiv \frac{f_1}{f_2}$. The solution to this equation is given by

$$\varepsilon = \mathrm{e}^{\mathrm{i}\delta\gamma_5\gamma_3\hat{\sigma}_1} \varepsilon_1 \qquad \gamma_3 \hat{\sigma}_1 \varepsilon_1 = \varepsilon_1 \tag{5.5.80}$$

with $\sinh(2\delta) = \mathrm{e}^{-G}$. The normalisation $h^{-1} = \varepsilon^\dagger \varepsilon$ implies $\varepsilon_1 = f_2^{1/2} \varepsilon_0$ with $\varepsilon_0^\dagger \varepsilon_0 = 1$. These conditions are enough to satisfy all the algebraic equations (5.5.49),(5.5.50),(5.5.51).

Due to the three projectors (5.5.33),(5.5.78),(5.5.80) and the conditions on the $\hat{\chi}, \tilde{\chi}$ spinors, the solution space of the Killing spinor equation is two dimensional and complex.

We will now use the differential equations (5.5.48) and the Bianchi identities (5.2.13)-(5.2.16) to express the unknown vectors and tensors in terms of the scalars.

The spacetime metric and the gauge field A

We define three new bilinears

$$\begin{aligned} \omega_\mu &= \varepsilon^t \gamma_2 \gamma_\mu \varepsilon \\ W^{1,2}_{\mu\nu} &= \varepsilon^t \gamma_2 \gamma_\mu \gamma_\nu \hat{\sigma}_{1,2} \varepsilon \, . \end{aligned} \tag{5.5.81}$$

Using (5.5.48) we can derive

$$\partial_\mu \omega_\nu - \partial_\nu \omega_\mu = -\mathrm{i}\frac{\rho_3}{2} F_\mu{}^\rho W^2_{\nu\rho} - 2\mathrm{i}(A_\mu \omega_\nu - A_\nu \omega_\mu) + 4\epsilon_{\mu\nu\rho\sigma} \tilde{V}^\rho \omega^\sigma - 4\tilde{g} W^1_{\mu\nu} \,. \qquad (5.5.82)$$

We note that

$$\omega_\mu \mathrm{d}x^\mu = -\frac{\rho_1}{\rho_3}(\tilde{e}^1{}_j + \mathrm{i}\tilde{e}^2{}_j)\mathrm{d}x^j \equiv -\frac{\rho_1}{\rho_3}\tilde{e}^z{}_j \mathrm{d}x^j \qquad (5.5.83)$$

where $\tilde{e}^k{}_j$ is a 2-bein for the metric $\tilde{h}_{ij}$. Thus, from (5.5.82) we can get an equation involving $\mathrm{d}\tilde{e}^k$. Singling out the y dependence using the (y, x^i) component of (5.5.82)

$$\partial_y \tilde{e}^z{}_j = -2\frac{h^2}{\rho_3 \rho_1}\left[\frac{\tilde{\rho}}{\rho_3}(\rho_3^3 - \rho_1^2) + \frac{f_2}{\tilde{\rho}}(f_2 \rho_3 - b\rho_1^2)\right] \tilde{e}^z{}_j \equiv D\tilde{e}^z{}_j \,. \qquad (5.5.84)$$

With a further y independent coordinate transformation we can put $\tilde{h}_{ij}$ in diagonal form. We introduce a conformal factor T and set

$$\tilde{e}^i{}_j = T\delta^i{}_j \qquad \partial_y T = DT \qquad (5.5.85)$$

Looking at the (x^1, x^2) component we can establish a relation between the remaining derivatives of T and the connection A_i

$$A_i = (A_t + b - c)V_i - \frac{1}{2}\epsilon_{ij}\partial_j \ln T \,. \qquad (5.5.86)$$

The constant c can be absorbed into a gauge transformation and we will set

$$b = c = 1 \,. \qquad (5.5.87)$$

The right hand side of the $\{t, x^i\}$ component of equation (5.5.82) is proportional to $b - c$ and thus also this equation is consistent with our gauge choice.
We have now an expression for A_i by which we may calculate the components of $F_{\mu\nu}$. This $F_{\mu\nu}$ must be equal to the one obtained in (5.5.60). The contraction with K^μ is trivial. The F_{yi} components give the constraint

$$b = c \qquad (5.5.88)$$

which is solved by our gauge choice. The F_{12} component gives an equation for $(\partial_1^2 + \partial_2^2)T$ that we will discuss later.

We have thus reduced our set of unknowns to five scalars and one 2-vector. Two scalars are constrained by the relations $y = \rho_1 \tilde{\rho}$ and so we have just four independent scalars and one 2-vector.

Scalars	Vector
$\rho_1, \rho_3, \tilde{\rho}, A_t, T$	V_i

We have reduced the four dimensional metric to the form

$$ds^2 = -h^{-2}(\mathrm{d}t + V_i \mathrm{d}x^i)^2 + h^2 \frac{\rho_1^2}{\rho_3^2}(T^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j + \mathrm{d}y^2)\,. \tag{5.5.89}$$

To simplify the final equations we now express the 4 functions $\rho_1, \rho_3, \tilde{\rho}, A_t$ in terms of three independent functions that we will call m, n, p are defined by

$$\begin{array}{ll} \rho_1^4 = y^4 \frac{mp+n^2}{m} & \rho_3^4 = \frac{p^2}{m(mp+n^2)} \\ \tilde{\rho}^4 = \frac{m}{mp+n^2} & A_t = \frac{n-p}{p} \end{array} \tag{5.5.90}$$

With these definitions we have

$$D = 2y(n + m - y^{-2})\,. \tag{5.5.91}$$

With some effort it can be shown that all the equations on the spinor ε are now solved.

Counting of Supersymmetries and Einstein equations

As we noticed in Section 5.5, the space of solutions to the Killing spinor equations is two dimensional and complex. The complex two dimensional space is spanned by the vectors $\tilde{\chi}_+^{1,2}$, in the notation of page 38. As such, there exist two Killing spinors whose null bilinears are not orthogonal and thus are not parallel. They can be constructed precisely in the same way we did for the original LLM case. We repeat here the construction for convenience. Let us consider the two independent Killing spinors

$$\eta = \varepsilon \otimes \chi_+^1 \otimes \tilde{\chi}_+^1 \tag{5.5.92}$$

$$\eta' = \varepsilon \otimes \chi_+^1 \otimes \left(\frac{\tilde{\chi}_+^1 + \tilde{\chi}_+^2}{\sqrt{2}} \right) \tag{5.5.93}$$

We choose to normalise the spinors $\chi, \tilde{\chi}$ to unity. In components, the two corresponding Killing vectors have the form

$$\kappa = (K; 0, 0, f_2; 0, 0, -f_1) \tag{5.5.94}$$

$$\kappa' = (K; 0, 0, f_2; -f_1, 0, 0) \tag{5.5.95}$$

The two Killing vector bilinears are precisely of the required form. They are both null, and their scalar product is given by

$$\kappa \cdot \kappa' = K^2 + f_2^2 = -f_1^2 \tag{5.5.96}$$

which is non vanishing. The existence of the Killing spinors guarantees that the full Einstein equations are satisfied provided that integrability conditions and the Bianchi identities for the Ramond-Ramond 5-form are satisfied. Let us now investigate what the consequence of these final constraints are.

Differential Equations

We will first establish a relation between the vector V_i and the various scalar functions. The equation (5.5.62) is an equation for $\mathrm{d}K$ with

$$K = -h^{-2}(\mathrm{d}t + V_i \mathrm{d}x^i)\,. \tag{5.5.97}$$

We can extract from this equation an expression for $\mathrm{d}V$:

$$\mathrm{d}V = -y \star_3 [\mathrm{d}n + (nD + 2ym(n-p) + 2n/y)\mathrm{d}y] \tag{5.5.98}$$

where by $\star_3$ we mean the Hodge dual in the three dimensional diagonal metric

$$\mathrm{d}s_3^2 = T^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j + \mathrm{d}y^2\,. \tag{5.5.99}$$

Returning to the Bianchi identities

$$\mathrm{d}\big(\tilde{G}\tilde{\rho}^3 - \tilde{V} \wedge A\rho_3\tilde{\rho}^3\big) = 0 \tag{5.5.100}$$

$$\tilde{V} = \frac{1}{2}\frac{1}{\rho_3\tilde{\rho}^3}\mathrm{d}(\tilde{g}\rho_1^2\tilde{\rho}^3) \tag{5.5.101}$$

$$\mathrm{d}\big(G\rho_1^2\rho_3\big) = 0 \tag{5.5.102}$$

$$\mathrm{d}\big(G\rho_1^2\rho_3 \wedge A + \star_4\tilde{V}\big) - 2\star_4\tilde{g} = 0\,. \tag{5.5.103}$$

Substituting in the first equation $\tilde{V}$ as obtained from the second equation we find

$$\mathrm{d}\big(\tilde{G}\tilde{\rho}^3 - \frac{1}{2}\tilde{g}\rho_1^2\tilde{\rho}^3 F\big) = 0\,. \tag{5.5.104}$$

We may thus set locally

$$\begin{aligned}
\mathrm{d}\tilde{B} &= \tilde{G}\tilde{\rho}^3 - \frac{1}{2}\tilde{g}\rho_1^2\tilde{\rho}^3 F \\
\tilde{B} &= \tilde{B}_t(\mathrm{d}t + V) + \hat{\tilde{B}} \\
\mathrm{d}B &= G\rho_1^2\rho_3 \\
B &= B_t(\mathrm{d}t + V) + \hat{B}\,.
\end{aligned} \tag{5.5.105}$$

The algebraic equation (5.5.61) for $\tilde{G}_{\mu\nu}$ and for its dual for $G_{\mu\nu}$ give rise to four new relations

$$\begin{aligned}
\tilde{B}_t &= -\frac{1}{16}y^2\frac{n - 1/y^2}{p} \\
\mathrm{d}\hat{\tilde{B}} &= -\frac{1}{16}y^3 \star_3 [\mathrm{d}m + 2mD] \\
B_t &= -\frac{1}{16}y^2\frac{n}{m} \\
\mathrm{d}\hat{B} &= \frac{1}{16}y^3 \star_3 [\mathrm{d}p + 4yn(p-n)\mathrm{d}y]\,.
\end{aligned} \tag{5.5.106}$$

We need to impose the three equations

$$\begin{cases} \mathrm{d}\mathrm{d}V = 0 \\ \mathrm{d}\mathrm{d}\hat{\hat{B}} = 0 \\ \mathrm{d}\mathrm{d}\hat{B} = 0 \end{cases} . \tag{5.5.107}$$

The last Bianchi identity (5.5.103) is implied by these three. In addition to these equations we have also

$$\partial_y \ln T = D \tag{5.5.108}$$

which together with the previous ones can be used to see that also the consistency equation for F_{12} is satisfied.

We have thus a set of 4 equations for 4 unknowns: m, n, p, T. The equations are defined on the half space

$$(x^1, x^2, y > 0) \tag{5.5.109}$$

and are quite complicated being a set of coupled non-linear second order elliptic differential equations.

$$\begin{aligned} &y^3(\partial_1^2 + \partial_2^2)n + \partial_y\left(y^3T^2\partial_y n\right) + y^2\partial_y\left[T^2\left(yDn + 2y^2m(n-p)\right)\right] + 4y^2DT^2n = 0 \\ &y^3(\partial_1^2 + \partial_2^2)m + \partial_y\left(y^3T^2\partial_y m\right) + \partial_y\left(y^3T^22mD\right) = 0 \\ &y^3(\partial_1^2 + \partial_2^2)p + \partial_y\left(y^3T^2\partial_y p\right) + \partial_y\left[y^3T^24ny(n-p)\right] = 0 \end{aligned} \tag{5.5.110}$$

5.6 Killing vectors and the Kaluza Klein Ansatz

In this appendix we present a geometrical interpretation of the bilinears that we constructed and that we used in Appendix B

Assume we have a fibration of a group manifold over some d dimensional base manifold with metric

$$\mathrm{d}s^2 = \tilde{g}_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu + \beta_{ab}(x)\left(\hat{e}^a(y) - A^a_\mu(x)\mathrm{d}x^\mu\right)\left(\hat{e}^b(y) + A^b_\mu(x)\mathrm{d}x^\mu\right) \tag{5.6.1}$$

where $\hat{e}^a$ is a basis of left-invariant one forms on the group manifold.

We define

$$\kappa = K^\mu\partial_\mu + \alpha(x)^a\hat{E}_a\,. \tag{5.6.2}$$

We recall that given any covariant 2-tensor a and three vector W, V_1, V_2 the Lie derivative of a is given by

$$(\mathcal{L}_W a)(V_1, V_2) = W\left(a(V_1, V_2)\right) - a([W, V_1], V_2) - a(V_1, [W, V_2])\,. \tag{5.6.3}$$

Let us calculate $\mathcal{L}_K g$

$$\begin{aligned} \left(\mathcal{L}_\kappa g\right)(\partial_\mu, \partial_\nu) &= \left(\mathcal{L}_K\tilde{g}\right)(\partial_\mu, \partial_\nu) + K^\rho\partial_\rho\left(\beta_{ab}A^a_\mu A^b_\nu\right) - \partial_\mu\alpha^a\beta_{ab}A^b_\nu - \partial_\nu\alpha^a\beta_{ab}A^b_\mu \\ \left(\mathcal{L}_\kappa g\right)(\hat{E}_a, \hat{E}_b) &= K^\rho\partial_\rho\beta_{ab} + \left(\mathcal{L}_\alpha\hat{g}(x)\right)(\hat{E}_a, \hat{E}_b) \\ \left(\mathcal{L}_\kappa g\right)(\partial_\mu, \hat{E}_a) &= K^\rho\partial_\rho(-\beta_{ab}A^b_\mu) + \beta_{ab}\partial_\mu\alpha^b + \beta_{cd}\alpha^b A^d_\mu f^c_{ba} \end{aligned}$$

where $\hat{g} = \beta_{ab}\hat{e}^a\hat{e}^b$ and f^c_{ba} the structure constants of the group.

When $K = 0, \beta_{ab} = k_{ab}$ with k_{ab} the Killing form of the group and so we obtain the non abelian Kaluza Klein setup.

Assume for the moment that K is a Killing vector of $\tilde{g}$ and $\alpha^a\hat{E}_a$ is a Killing vector of $\hat{g}$, what are the conditions on $\alpha, \beta_{ab}, A^a_\mu$ such that K is a Killing vector for the whole metric? This is easily seen from our previous equations

$$K(\beta_{ab}) = 0 \tag{5.6.4}$$

$$\partial_\mu\alpha^a = K(A^a_\mu) - f^a_{bc}\alpha^b A^c_\mu\,. \tag{5.6.5}$$

We can now specialise to our setting. The group manifold is $SU(2) \times SU(2)$. We can define the ten dimensional vector

$$\begin{aligned}\kappa^M\partial_M = \bar{\psi}\Gamma^M\psi\partial_M = \\ K^\mu\partial_\mu + \left(A_m K^m - \frac{f_2}{\rho_3}s\right)\Sigma^{\hat{i}}{}_{\hat{3}}\partial_{\hat{i}} + \frac{f_1}{\tilde{\rho}}\tilde{\chi}^\dagger\sigma^{\tilde{a}}\tilde{\chi}\Sigma^{\tilde{i}}{}_{\tilde{a}}\partial_{\tilde{i}}\end{aligned} \tag{5.6.6}$$

where we have chosen the normalisation $\chi^\dagger\chi = \tilde{\chi}^\dagger\chi = 1$. κ is a Killing vector and it is null [21]. Since it is null we have

$$K^2 = K_\mu\tilde{g}^{\mu\nu}K^\nu = -f_1^2 - f_2^2 \tag{5.6.7}$$

which was previously seen as consequence of Fierz rearrangements and whereas here we can see its geometrical origin. From the equation on $\nabla_\mu K_\nu$ we know that K is a Killing vector for $\tilde{g}_{\mu\nu}$, and moreover, due to the Killing equation on $\tilde{\chi}$ and the properties of the Ansatz, we have that

$$\Sigma^{\hat{i}}{}_{\hat{3}} \quad , \quad \tilde{\chi}^\dagger\sigma^{\tilde{a}}\tilde{\chi}\Sigma^{\tilde{i}}{}_{\tilde{a}}\partial_{\tilde{i}} \tag{5.6.8}$$

are Killing vector of the group manifolds. We thus conclude

$$K(\rho_1) = K(\rho_3) = K(\tilde{\rho}) = 0 \tag{5.6.9}$$

$$\partial_\mu\left(\frac{f_1}{\tilde{\rho}}\right) = 0 \tag{5.6.10}$$

$$\partial_\mu\left(A_\nu K^\nu - \frac{f_2}{\rho_3}s\right) = K(A_\mu) \tag{5.6.11}$$

The second one can be written in the form we already encountered earlier

$$\partial_\mu\left(\frac{f_2}{\rho_3}\right) = F_{\mu\nu}K^\nu s\,. \tag{5.6.12}$$

We have thus clarified the geometrical origin of the relations between f_1, f_2 and the metric entries.

Chapter 6

Extensions of the Maldacena conjecture

In this Chapter we will briefly introduced some non trivial generalisations of the AdS/CFT correspondence to less supersymmetric cases and then study the half BPS sector of theses extended version of the duality.

6.1 Sasaki-Einstein manifolds and Quiver gauge theories

In the correspondence between string theory on $AdS_5 \times S^5$ and $d = 4$ $\mathcal{N} = 4$ SYM theories, some of the most direct checks, such as protected operator dimensions and the functional form of two- and three-point functions, are determined by properties of the supergroup $SU(2,2|4)$. Many of the normalisations of two- and three-point functions which have been computed explicitly are protected by non-renormalisation theorems. If we want to prove that the correspondence is a fundamental dynamical principle, we have to test it in less (super)symmetric settings. Orbifold theories [100] provide interesting examples; however it has been shown [101, 102] that at large N these theories are a projection of $\mathcal{N} = 4$ super-Yang-Mills theory; in particular many of their Green's functions are dictated by the Green's functions of the $\mathcal{N} = 4$ theory. The projection involved is onto states invariant under the group action that defines the orbifold. Intuitively, this similarity with the $mathcalN = 4$ theory arises because the compact part of the geometry is still (almost everywhere) locally S^5, just with some global identifications. Therefore, to make a more non-trivial test of models with reduced supersymmetry, we are more interested in geometries of the form $AdS_5 \times M_5$ where the compact manifold M_5 is not even locally S^5.

In fact, such compactifications have a long history in the supergravity literature: the direct product geometry $AdS_5 \times M_5$ is known as the Freund-Rubin Ansatz [103]. The curvature of the anti-de Sitter part of the geometry is supported by the five-form of type IIB supergravity. Because this five-form is self-dual, M_5 must also be an Einstein manifold, but with positive cosmological constant: rescaling M_5 if necessary, we can write $\mathcal{R}_{\alpha\beta} = 4g_{\alpha\beta}$.

For simplicity, we are assuming that only the five-form and the metric are involved in the solution.

A trivial but useful observation is that five-dimensional Einstein manifolds with $\mathcal{R}_{\alpha\beta} = 4g_{\alpha\beta}$ are in one-to-one correspondence with Ricci-flat manifolds C_6 whose metric has the conical form

$$ds^2_{C_6} = dr^2 + r^2 ds^2_{M_5} \ . \tag{6.1.1}$$

It can be shown that, given any metric of the form (6.1.1), the ten-dimensional metric

$$ds^2_{10} = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \left(1 + \frac{L^4}{r^4}\right)^{1/2} ds^2_{C_6} \tag{6.1.2}$$

is a solution of the type IIB supergravity equations, provided one puts N units of five-form flux through the manifold M_5, where

$$L^4 = \frac{\sqrt{\pi}}{2} \frac{\kappa_{10} N}{\mathrm{Vol} M_5} = 4\pi g_s N l_s^4 \frac{\pi^3}{\mathrm{Vol} M_5} \tag{6.1.3}$$

Furthermore, it was shown in [104] that the number of supersymmetries preserved by the geometry (6.1.2) is half the number that are preserved by its Ricci-flat $L \longrightarrow 0$ limit. Preservation of supersymmetry therefore amounts to the existence of a Killing spinor on $ds^2_{C_6}$, which would imply that it is a Calabi-Yau metric. In this case the manifold M_5 is called a Sasaki–Einstein manifold. Finally, the $r \ll L$ limit of (6.1.2) is precisely $AdS_5 \times M_5$, and in that limit the number of preserved supersymmetries doubles.

The metric [104] describes a flat 3-brane placed at the tip of the Calabi Yau cone C_6 in a $\mathbb{R}^{3,1} \times C_6$ background. In analogy to the construction we described in Section 2.3, we may equivalently consider the dynamics of such a system as described by open strings ending on a flat D3-brane placed at the tip of the cone.

In this case, the AdS/CFT correspondence states that Type IIB string theory on $AdS_5 \times M_5$ is dual to an $\mathcal{N} = 1$ four–dimensional superconformal field theory.

Until the paper [93], the only Sasaki–Einstein five–manifolds that were known explicitly in the literature were precisely the round metric on S^5 and the homogeneous metric $T^{1,1}$ on $S^2 \times S^3$, or quotients thereof. For the five–sphere the Calabi–Yau cone is simply $\mathbb{C}^3$ and the dual superconformal field theory is the maximally supersymmetric $\mathcal{N} = 4$ $SU(N)$ theory. For $T^{1,1}$ the Calabi–Yau cone is the conifold and the dual $\mathcal{N} = 1$ superconformal field theory was given in [105, 106].

In the paper [93] a countably infinite class of *explicit* Sasaki–Einstein metrics on $S^2 \times S^3$ topology was given explicitly. These were initially found by reduction and T–duality of a class of supersymmetric M–theory solutions discovered in [92]. The family is characterised by two relatively prime positive integers p, q, with $q < p$ and the generic metric is commonly called $Y^{p,q}$. The manifold $Y^{1,0}$ coincides with the already known case of $T^{1,1}$. Each of this metric has an $SU(2) \times U(1) \times U(1)$ local symmetry. In the special case of $T^{1,1}$ this symmetry is enhanced to $SU(2) \times SU(2) \times U(1)$. The topology of each manifold is $S^2 \times S^3$ and thus it has one non homologically non-trivial three cycle. Among the infinite number of homologically equivalent 3-cycles, there are four supersymmetric cycles Σ_i. Here supersymmetric means that the metric cones $C(\Sigma_i)$ are calibrated submanifolds (in fact divisors) in the Calabi–Yau cone. As such D4-branes wrapping these manifolds preserve half of the supersymmetry of the background. They are called *dibaryons* since are the dual of baryonic operators in the corresponding gauge theories.

In the papers [107, 108] the superconformal theories which are dual to String theory on $AdS_5 \times Y^{p,q}$ have been constructed. These are Quiver gauge thoeries with $4p + 2q$ fields transforming in the bifundamental of $2p$ $SU(N)$ gauge groups. Many properties of the theories can be derived from purely geometrical considerations. The field content of the theories is summarised in the following table The $SU(2) \times U(1) \times U(1)$ bosonic symmetry

Field	number	$R-$ charge	$U(1)_B$	$U(1)_F$
Y	$p+q$	$(-4p^2+3q^2+2pq+(2p-q)\sqrt{4p^2-3q^2})/3q^2$	$p-q$	-1
Z	$p-q$	$(-4p^2+3q^2-2pq+(2p+q)\sqrt{4p^2-3q^2})/3q^2$	$p+q$	$+1$
U^α	p	$(2p(2p-\sqrt{4p^2-3q^2}))/3q^2$	$-p$	0
V^α	q	$(3q-2p+\sqrt{4p^2-3q^2})/3q$	q	$+1$

of the $Y^{p,q}$ metric translate into R and flavour symmetry. The fields U^α, V^α transform in the doublet of $SU(2)$ while the fields Y, Z are singlet. In the case of $T^{1,1}$ they combine into a doublet of the second $SU(2)$ factor. The R charges were calculated in [108] and they perfectly agree with the purely geometrical derivation of [93, 107]. Notice that, apart from the R and flavour charges, a further charge is shown in the table. This represents the baryonic charge of each field.

Since all the $2p$ gauge groups have the same rank, N, it is possible to construct simple dibaryonic operators with one type of bifundamental field $A_\alpha^{\ \beta}$:

$$\mathcal{B}[A] = \varepsilon^{\alpha_1 \ldots \alpha_N}\, A_{\alpha_1}^{\ \beta_1} \ldots A_{\alpha_N}^{\ \beta_N}\, \varepsilon_{\beta_1 \ldots \beta_N} \,. \tag{6.1.4}$$

In the $Y^{p,q}$ quivers there are four classes of bifundamental fields, so there are four classes of dibaryonic operators: $\mathcal{B}[Y]$, $\mathcal{B}[Z]$, $\mathcal{B}[U]$ and $\mathcal{B}[V]$. Since the fields U^α and V^α transform in the 2–dimensional representation of the global $SU(2)$, the corresponding baryonic operators transform in the $(N+1)$–dimensional representation, as explained in [109]. Elements of the chiral ring generated by the baryons with conformal dimension of the order N correspond to D3-branes wrapping non trivial supersymmetric three cycles. Since the operators like (6.1.4) are chiral, their scaling dimension is precisely the scaling dimension of the bifundamental A, multiplied by N. These scaling dimensions correspond holographically to the volumes of the corresponding 3–cycles.

In the following Sections we will address the analysis of a class of Supergravity duals of dibaryons with conformal dimension of order N^2. They are given by supersymmetric solutions of the complete supergravity equations of motion.

6.2 Introduction to Quiver half-BPS dual geometries

As we outlined in the previous Sections, the AdS/CFT correspondence extends naturally to the case of String Theory on $AdS^5 Y^{p,q}$ and $\mathcal{N} = 1$ Quiver gauge theories. In particular it would be very interesting to mimic the beautiful LLM construction that we have described in Chapter 3 to the case of the chiral ring of Quiver gauge theories. In the next Sections we address this problem.

Every $Y^{p,q}$ manifold has an $SU(2) \times U(1) \times U(1)$ isometry group and the $AdS_5 \times Y^{p,q}$ solutions preserve 8 of the original 32 supersymmetries of type IIB supergravity. Supersymmetric branes wrapping cycles in $Y^{p,q}$ have been analysed in the probe approximation in [110, 111] and they may be considered as generalisations of giant gravitons. A distinguishing feature of the $Y^{p,q}$ manifolds with respect to the standard S^5 is the presence of a non-trivial 3-cycle. D3-branes can thus wrap such a non trivial cycle and be stable: such branes are dual to baryons in the Gauge theory, the so called dibaryons, which are built out of products of N chiral superfields [112]. Dual giant gravitons were studied in [113, 114]. These are good candidates for the supergravity duals of states in the chiral ring of the $\mathcal{N} = 1$ theories in the regime of conformal dimension of the order of N.

In Chapter 5, we constructed solutions of the type IIB equations of motion with non trivial Ramond Ramond 5-form and $\mathbb{R} \times SO(4) \times SU(2) \times U(1)$ isometry group preserving 4 supercharges. $AdS_5 \times Y^{p,q}$ geometries are clearly contained in this class: the $\mathbb{R} \times SO(4)$ is the non compact version of $U(1) \times SO(4) \subset SO(2,4)$, while the $SU(2) \times U(1) \times U(1)$ isometry group of $Y^{p,q}$ is contained in the generic $SU(2) \times U(1)$ bosonic symmetry.

We will now show in detail how to recover the $AdS_5 \times Y^{p,q}$ geometries from the generic solutions studied in Chapter 5 by requiring that an additional 4 supercharges are preserved. We then study 1/2 BPS excitations of such geometries, namely generic 1/8 BPS solutions of type IIB supergravity with $AdS_5 \times Y^{p,q}$ asymptotics and $\mathbb{R} \times SO(4) \times SU(2) \times U(1)$ isometry: they represent an expansion of the fully backreacted geometries of D3 branes in $AdS_5 \times Y^{p,q}$ and the dual of half-BPS states in the Quiver theories for conformal dimensions of the order of N^2. The description of these geometries is already implicitly contained in the analysis of the last Chapter and is given by the solutions of a system of four coupled non linear differential equations. We will study a class of such solutions which differs from the one studied previously already at the level of the boundary conditions that they satisfy in the asymptotic region. This is due to the fact that we are interested in the dual of states in the Quiver gauge theories and thus the geometries should approach $AdS_5 \times Y^{p,q}$ asymptotically

The brane source of the probe picture is substituted by flux in the same spirit as in the original [12]. The geometries we will obtain carry three net global $U(1)$ charges which are dual to the R-charge, a $U(1)$ flavour charge and the baryonic charge of the gauge theory. In order to specify the asymptotics and charges of the solutions we solve the system of differential equations perturbatively at large AdS_5 radius. The zeroth order fixes the boundary conditions such that the metric and the RR 5-form describe correctly asymptotically $AdS_5 \times Y^{p,q}$ geometries, the first subleading corrections determine the aforementioned global $U(1)$ charges and the second subleading correction is necessary to obtain the value of the mass. Solutions which carry only R-charge have been studied in [115] at the linearised level.

Due to intrinsic complexity of the solutions, we face here a problem which we easily turn around in the previous Chapter. The definition of mass is somewhat subtle in asymptotically AdS spacetimes, [116, 117] but it is even more subtle when one is dealing with states in asymptotically $AdS_5 \times X^5$, with compact non trivial X^5, due to the fact that the subleading terms in the metric, that in principle can be used to determine the mass, mix the AdS_5 and X_5 coordinates. This is due to the fact that it is not possible to reduce the 10 dimensional theory to 5-dimensional gauge Supergravity consistently in the non-Abelian case [115]. We deal with this problem by adopting a 10-dimensional version of the general construction of [118] to find the conserved Hamiltonian and thus the correct definition of the mass. We then determine the mass of our states and check that the BPS condition, relating the mass to the R-charge, is indeed satisfied by our asymptotic solutions.

The paper is organised as follows. In Section 6.3 we give a brief summary of the results of [17]. In Section 6.4 we show how to obtain the $AdS_5 \times Y^{p,q}$ geometries from the general solutions. In Section 6.5 we solve the system of second order equations up to second order in large AdS_5 radius (the detailed form of the second order solutions is given in Section 6.10). In Section 6.6 we show how to obtain the R charge and the $U(1)$ flavour charge of the solutions. In Section 6.7 we discuss subleading corrections to the RR 5-form and derive the baryon charge of the solutions. In Section 6.8 we discuss how to correctly define the mass for a space-time which is asymptotically a product with an AdS_5 factor. Finally, in Section 6.9 we present some discussion.

6.3 Summary of 1/8 BPS geometries

We summarise here for convenience the results of the last Chapter where we constructed generic solutions of type IIB Supergravity preserving 4 of the 32 supersymmetries of the theory and an $\mathbb{R} \times SO(4) \times SU(2) \times U(1)$ bosonic symmetry. The metric takes the form

$$\begin{aligned} ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2\frac{\rho_1^2}{\rho_3^2}(T^2\delta_{ij}dx^i dx^j + dy^2) + \tilde{\rho}^2 d\tilde{\Omega}_3^2 + \\ + \rho_1^2\left((\sigma^{\hat{1}})^2 + (\sigma^{\hat{2}})^2\right) + \rho_3^2(\sigma^{\hat{3}} - A_t dt - A_i dx^i)^2 \quad (6.3.1) \end{aligned}$$

with $i = 1, 2$; the coordinate y is the product of two of the radii,

$$y = \rho_1\tilde{\rho} > 0\,. \tag{6.3.2}$$

and the function h is given by

$$h^{-2} = \tilde{\rho}^2 + \rho_3^2(1 + A_t)^2\,. \tag{6.3.3}$$

The space is a fibration of a squashed 3-sphere (on which the $SU(2)$ left-invariant 1-forms $\sigma^{\hat{a}}$ are defined) and a round 3-sphere $\tilde{\Omega}_3$ (on which the $SU(2)$ left-invariant 1-forms $\sigma^{\tilde{a}}$ are

defined) over a four dimensional manifold.
The left invariant 1-forms are given by:

$$\begin{aligned} \sigma^{\hat 1} &= -\tfrac{1}{2}(\cos\hat\psi\, d\hat\theta + \sin\hat\psi\,\sin\hat\theta\, d\hat\phi) & \sigma^{\tilde 1} &= -\tfrac{1}{2}(\cos\tilde\psi\, d\tilde\theta + \sin\tilde\psi\,\sin\tilde\theta\, d\tilde\phi)\\ \sigma^{\hat 2} &= -\tfrac{1}{2}(-\sin\hat\psi\, d\hat\theta + \cos\hat\psi\,\sin\hat\theta\, d\hat\phi) & \sigma^{\tilde 2} &= -\tfrac{1}{2}(-\sin\tilde\psi\, d\tilde\theta + \cos\tilde\psi\,\sin\tilde\theta\, d\tilde\phi)\\ \sigma^{\hat 3} &= -\tfrac{1}{2}(d\hat\psi + \cos\hat\theta\, d\hat\phi) & \sigma^{\tilde 3} &= -\tfrac{1}{2}(d\tilde\psi + \cos\tilde\theta\, d\tilde\phi) \end{aligned} \tag{6.3.4}$$

and satisfy the relations (with σ^a being either $\sigma^{\hat a}$ or $\sigma^{\tilde a}$)

$$\mathrm{d}\sigma^a = \epsilon_{abc}\sigma^b \wedge \sigma^c\,. \tag{6.3.5}$$

With this normalisation the metric on the unit radius round three sphere is given by

$$d\Omega_3^2 = (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\,. \tag{6.3.6}$$

The only non trivial field strength in our Ansatz is the Ramond-Ramond 5-form: it is more conveniently expressed in terms of the "d-bein"

$$e^0 = h^{-1}(\mathrm{d}t + V_i \mathrm{d}x^i) \tag{6.3.7}$$

$$e^j = h\frac{\rho_1}{\rho_3} T \delta_i^j \mathrm{d}x^i \tag{6.3.8}$$

$$e^3 = h\frac{\rho_1}{\rho_3}\mathrm{d}y \tag{6.3.9}$$

$$e^{\hat a} = \begin{cases} \rho_1\sigma^{\hat a} & \hat a = 1,2 \\ \rho_3(\sigma^{\hat 3} - A_\mu \mathrm{d}x^\mu) & \hat a = 3 \end{cases} \tag{6.3.10}$$

$$e^{\tilde a} = \tilde\rho\sigma^{\tilde a} \tag{6.3.11}$$

as

$$F_{(5)} = 2\left(\tilde G_{mn} e^m \wedge e^n + \tilde V_m e^m \wedge e^{\hat 3} + \tilde g e^{\hat 1}\wedge e^{\hat 2}\right)\wedge \tilde\rho^3 \mathrm{d}\tilde\Omega_3 + \\ 2\left(-G_{pq}e^p \wedge e^q \wedge e^{\hat 1}\wedge e^{\hat 2}\wedge e^{\hat 3} + \star_4 \tilde V \wedge e^{\hat 1}\wedge e^{\hat 2} - \star_4 \tilde g \wedge e^{\hat 3}\right)\,, \tag{6.3.12}$$

where

$$G_{mn} = \frac{1}{2}\epsilon_{mnpq}\tilde G^{pq} \tag{6.3.13}$$

$$\star_4 \tilde V = \frac{1}{3!}\epsilon_{mnpq}\tilde V^m e^n \wedge e^p \wedge e^q \tag{6.3.14}$$

$$\star_4 \tilde g = \tilde g e^0 \wedge e^1 \wedge e^2 \wedge e^3\,. \tag{6.3.15}$$

The complete solution can be expressed in terms of four independent functions m, n, p, T defined on the halfspace (x^1, x^2, y), as follows

$$\begin{aligned} \rho_1^4 &= \tfrac{mp+n^2}{m} y^4 & \rho_3^4 &= \tfrac{p^2}{m(mp+n^2)} & \tilde\rho^4 &= \tfrac{m}{mp+n^2} \\ h^4 &= \tfrac{mp^2}{mp+n^2} & A_t &= \tfrac{n-p}{p} & A_i &= A_t V_i - \tfrac{1}{2}\epsilon_{ij}\partial_j \ln T \end{aligned} \tag{6.3.16}$$

and

$$\mathrm{d}V = -y \star_3 [\mathrm{d}n + (nD + 2ym(n-p) + 2n/y)\mathrm{d}y] \tag{6.3.17}$$

$$\partial_y \ln T = D \tag{6.3.18}$$

$$D \equiv 2y(m + n - 1/y^2)\,, \tag{6.3.19}$$

where $\star_3$ indicates the Hodge dual in the three dimensional diagonal metric

$$\mathrm{d}s_3^2 = T^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j + \mathrm{d}y^2\,. \tag{6.3.20}$$

The various four-dimensional forms from which the 5-form field strength is constructed are given by

$$\tilde{g} = \frac{1}{4\tilde{\rho}}\left[1 - \frac{\rho_3^2}{\rho_1^2}(1 + A_t)\right] \tag{6.3.21}$$

$$\tilde{V} = \frac{1}{2}\frac{1}{\rho_3 \tilde{\rho}^3}\mathrm{d}(\tilde{g}\rho_1^2\tilde{\rho}^3) \tag{6.3.22}$$

$$G\rho_1^2\rho_3 = \mathrm{d}B_t \wedge (\mathrm{d}t + V_i \mathrm{d}x^i) + B_t \mathrm{d}V + \mathrm{d}\hat{B} \tag{6.3.23}$$

$$\tilde{G}\tilde{\rho}^3 = \frac{1}{2}g\rho_1^2\tilde{\rho}^3\mathrm{d}A + \mathrm{d}\tilde{B}_t \wedge (\mathrm{d}t + V_i \mathrm{d}x^i) + \tilde{B}_t \mathrm{d}V + \mathrm{d}\hat{\tilde{B}}\,, \tag{6.3.24}$$

with

$$\begin{aligned} \tilde{B}_t &= -\frac{1}{16}y^2\frac{n - 1/y^2}{p} \\ \mathrm{d}\hat{\tilde{B}} &= -\frac{1}{16}y^3 \star_3 [\mathrm{d}m + 2mD\,\mathrm{d}y] \\ B_t &= -\frac{1}{16}y^2\frac{n}{m} \\ \mathrm{d}\hat{B} &= \frac{1}{16}y^3 \star_3 [\mathrm{d}p + 4yn(p-n)\mathrm{d}y]\,. \end{aligned} \tag{6.3.25}$$

The Bianchi identities on $F_{(5)}$ and the integrability condition for (6.3.17) give three second order differential equations on m, n and p which, together with (6.3.18), give a system of nonlinear coupled elliptic differential equations

$$\begin{aligned} & y^3(\partial_1^2 + \partial_2^2)n + \partial_y\left(y^3T^2\partial_y n\right) + y^2\partial_y\left[T^2\left(yDn + 2y^2m(n-p)\right)\right] + 4y^2DT^2n = 0 \\ & y^3(\partial_1^2 + \partial_2^2)m + \partial_y\left(y^3T^2\partial_y m\right) + \partial_y\left(y^3T^2 2mD\right) = 0 \\ & y^3(\partial_1^2 + \partial_2^2)p + \partial_y\left(y^3T^2\partial_y p\right) + \partial_y\left[y^3T^2 4ny(n-p)\right] = 0 \\ & \partial_y \ln T = D\,. \end{aligned} \tag{6.3.26}$$

6.4 $AdS_5 \times Y^{p,q}$ solutions

Taking solutions for the m, n, p, T function of the previous Chapter which have rotational symmetry in the $\{x^1, x^2\}$ plane, the bosonic symmetry is enhanced to $\mathbb{R} \times SO(4) \times SU(2) \times$

$U(1)\times U(1)$. We will first consider a subset of solutions which preserve 8 supersymmetries (the generic solution preserves only 4 of them as explained in the previous section). The well known $AdS_5\times Y^{p,q}$ [107] are clearly contained in this subset: the round S^3 is a factor in AdS_5, as suggested by the analysis in [17], with $\mathbb{R}\times SO(4)$ the non compact version of $U(1)\times SO(4)\subset SO(2,4)$, while the remaining $SU(2)\times U(1)\times U(1)$ is the isometry group of the generic $Y^{p,q}$ metric.

6.4.1 Constraints for enhanced supersymmetry

Since the solutions described in [17] generically preserve only 4 supersymmetries, the $AdS_5\times Y^{p,q}$ geometries will be specified by a set of constraints on the four functions m, n, p and T. We will now show how these constraints arise.

The supersymmetry parameters that leave invariant our background are the solutions to the Killing spinor equation

$$\delta\chi_M=\nabla_M\psi+\frac{\mathrm{i}}{480}F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\Gamma_M\psi=0\,. \qquad (6.4.1)$$

As a consequence of the symmetry assumptions we look for a solution ψ of the form

$$\psi=\varepsilon\otimes\hat{\chi}\otimes\tilde{\chi}_{(b)}\,. \qquad (6.4.2)$$

Here ε is an 8 component complex spinor and $\hat{\chi},\tilde{\chi}_{(b)}$ are 2 component complex spinors defined on the two 3-spheres satisfying

$$\frac{\partial}{\partial\omega^{\hat{a}}}\hat{\chi}=0 \qquad \sigma_{\hat{3}}\hat{\chi}=s\hat{\chi} \qquad (6.4.3)$$

$$\nabla'_{\tilde{a}}\tilde{\chi}_{(b)}=b\frac{\mathrm{i}}{2}\sigma_{\tilde{a}}\tilde{\chi}_{(b)} \qquad (6.4.4)$$

where $\omega^{\hat{a}},\omega^{\tilde{a}}$ are coordinates on the two spheres. ∇' is the covariant derivative on the unit radius three sphere and $s,b=\pm1$. The spinor $\hat{\chi}$ is a singlet under the $SU(2)_L$ isometry of the squashed 3-sphere, while the spinors $\tilde{\chi}_\pm$ transform as the $(0,\frac{1}{2})$ for upper sign and $(\frac{1}{2},0)$ for the lower sign, of the $SO(4)$ isometry of the round S^3, which is part of AdS_5.

The analysis in [17] fixes $b=s=1$, i.e. $\tilde{\chi}$ has definite chirality in $SO(4)$ and $\hat{\chi}$ is highest weight of the broken $SU(2)_R$. ε is proportional to some ε_0 obeying $\varepsilon_0^\dagger\varepsilon_0=1$. Since we have a doublet of $\tilde{\chi}_{(1)}$, the space of solutions is 2 dimensional and complex giving rise to 4 real preserved supersymmetries. We will show that $AdS_5\times Y^{p,q}$ geometries are obtained by requiring that spinors with $b=-1$, $s=1$ are also solutions of the equations (6.4.1). In this case there are two doublets of $\tilde{\chi}$ and thus 8 real solutions to (6.4.1). This agrees with what one expects from the $\mathcal{N}=1$ SCFT side: there, out of the 4 pairs of Killing spinors $\xi^A_\pm$, $A=1,\ldots,4$ in the **4** of $SU(4)$ of the $\mathcal{N}=4$ theory on $\mathbb{R}\times S^3$, obeying

$$D_\mu\xi^A_\pm=\pm\frac{i}{2}\sigma_\mu\xi^A_\pm \qquad (6.4.5)$$

only the $SU(3)$ singlet $\xi_\pm$ in $SU(3)\times U(1)\subset SU(4)$ survives in the $\mathcal{N}=1$ case. This has $SU(4)$ weights $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and, picking the $SO(4)$ inside $SU(4)$ corresponding for example to the first two entries, we see that it is a singlet of, say, $SU(2)_L$ and highest weight of $SU(2)_R$ in the $SO(4)\subset SU(4)$. Furthermore the two signs in (6.4.5) correspond to the two chiralities of the $SO(4)$ isometry group of S^3. Since this S^3 corresponds to the S^3 inside AdS_5 this checks with the above requirement of $b=\pm 1$.

Due to the conditions on the spinor, $\hat{\chi}$ and $\tilde{\chi}_{(b)}$ factorise in each component of the gravitino variation equation which then becomes equivalent to the following system of coupled differential and algebraic equations on ε[1]

$$\left[\tilde{\nabla}_\mu - \frac{1}{4}F_{\mu\nu}\Xi^\nu{}_m\gamma^m\gamma^5\hat{\sigma}_1 s + \mathrm{i}A_\mu s - \left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s + \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\gamma_\mu\right]\varepsilon = 0 \tag{6.4.7}$$

$$\left[\frac{\mathrm{i}}{2}\frac{\rho_3}{\rho_1}\gamma_5\hat{\sigma}_1 + \frac{1}{2}\not{\partial}\rho_1 + \rho_1\left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s - \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\right]\varepsilon = 0 \tag{6.4.8}$$

$$\left[\frac{\mathrm{i}}{2}\left(2-\frac{\rho_3^2}{\rho_1^2}\right)\gamma_5\hat{\sigma}_1 + \frac{1}{2}\not{\partial}\rho_3 + \frac{1}{8}\rho_3^2\not{F}\gamma_5\hat{\sigma}_1 s + \rho_3\left(\tilde{\not{G}} - \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s + \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\right]\varepsilon = 0 \tag{6.4.9}$$

$$\left[\frac{\mathrm{i}}{2}b\gamma_5\hat{\sigma}_2 + \frac{1}{2}\not{\partial}\tilde{\rho} - \tilde{\rho}\left(\tilde{\not{G}} + \tilde{\not{V}}\gamma_5\hat{\sigma}_1 s + \mathrm{i}\tilde{g}s\right)\gamma_5\hat{\sigma}_2\right]\varepsilon = 0. \tag{6.4.10}$$

Note that the first equation is a first order differential 4-vector equation for ε while the last three are algebraic 4-scalar equations.

We now express all the supergravity fields via the functions m, n, p and T of the previous section. We are thus guaranteed that a solution to the above system with $b=s=1$ exists by the analysis in [17]. We now ask that a second solution to these equations exists for $b=-1$, $s=1$.

We have used Mathematica to solve explicitly the equations. The existence of solutions implies certain constraints on the background, which are more conveniently expressed in terms of the metric entries as

$$\begin{aligned}
&1 + A_t = \rho_1^2/\rho_3^2 \\
&\rho_1^2 - \rho_3^2 = S^4/\rho_1^2 \\
&T^2\partial_y \ln(\rho_1/\rho_3)\left[2\rho_1^2/\rho_3^2 - 2 + y\partial_y\ln(\rho_1/\rho_3)\right] + y\left[\partial_r \ln(\rho_1/\rho_3)\right]^2 = 0
\end{aligned} \tag{6.4.11}$$

where (r,ϕ) are polar coordinates in the (x^1,x^2) plane. Notice that the first two constraints together with the relation $y=\rho_1\tilde{\rho}$ allow us to express the four functions $\rho_1,\rho_3,\tilde{\rho},A_t$ in

[1] For example the first equation is obtained as follows

$$\begin{aligned}
(\nabla_\mu + M\Gamma_\mu)\psi &= \left(\tilde{\nabla}_\mu - \frac{1}{4}\rho_3 F_{\mu\nu}\Xi^\nu{}_m\Gamma^m\Gamma^{\hat{3}} + A_\mu\left(\Sigma_{\hat{3}} + \Gamma^{\hat{1}}\Gamma^{\hat{2}}\right) - A_\mu\nabla_{\hat{3}} + M\Gamma_\mu\right)\psi = \\
&= \left(\tilde{\nabla}_\mu - \frac{1}{4}\rho_3 F_{\mu\nu}\Xi^\nu{}_m\Gamma^m\Gamma^{\hat{3}} + A_\mu\Gamma^{\hat{1}}\Gamma^{\hat{2}} + M\left(\Gamma_\mu + A_\mu\rho_3\Gamma_{\hat{3}}\right)\right)\psi = \\
&= \left(\tilde{\nabla}_\mu - \frac{1}{4}\rho_3 F_{\mu\nu}\Xi^\nu{}_m\gamma^m\sigma^{\hat{3}} + A_\mu\sigma_{\hat{3}} + M\gamma_\mu\right)\psi
\end{aligned} \tag{6.4.6}$$

terms of only one function. The last constraint together with the equation for $\partial_y T$ can be used to eliminate T. Moreover, the three second order differential equations that came from the integrability condition for the 1/8 supersymmetric geometries are reduced to a single equation which is more easily expressed in terms of the function $\tilde{z}$

$$\tilde{z} \equiv \frac{1}{2}\left[1+\tanh\left(\frac{\rho_3(1+A_t)}{\tilde{\rho}}\right)\right] \tag{6.4.12}$$

$$\frac{1}{r}\partial_r\left(r\partial r\tilde{z}\right)+y\partial_y\left\{T^2\frac{1}{y}\left[\partial_y\tilde{z}+4\tilde{z}(1-\tilde{z})\frac{\rho_1^2/\rho_3^2-1}{y}\right]\right\}=0 \tag{6.4.13}$$

where the combination ρ_1^2/ρ_3^2 is given by

$$\frac{\rho_1^2}{\rho_3^2}=\left(\frac{1+\sqrt{1-4S^4\frac{1-\tilde{z}}{y^2\tilde{z}}}}{2}\right)^{-1} \tag{6.4.14}$$

and T can be found by solving the third equation in (6.4.11). The solution is thus specified completely by the single function $\tilde{z}$.

6.4.2 $AdS_5 \times Y^{p,q}$ metrics

In this section, we are going to show how the $AdS_5 \times Y^{p,q}$ geometries arise from the generic description given above. As a first step we present the conditions that should be satisfied by the 1/4 supersymmetric solutions in order that they factorise into,

$$AdS_5 \times X^5\,, \tag{6.4.15}$$

for some supersymmetric five-manifold X^5. These will turn out to be equivalent to a single first order differential equation which implies the second order equation in (6.4.13). The opposite cannot be proven as the generic solution preserving 8 supersymmetries does not appear to be factorisable in general.

As a second step we will prove, by giving the explicit coordinate transformation to the gauge in [107], that the X^5 factor is indeed a generic $Y^{p,q}$ manifold.

First of all we will change to the more convenient set of coordinates $(\tilde{\rho}, \rho_1, \tilde{\phi}, \hat{\psi}')$ defined by

$$\begin{aligned}
y &= \rho_1\tilde{\rho} \\
r &= r(\rho_1,\tilde{\rho}) \\
\phi &= \tilde{\phi}+\tilde{c}\,t \\
\hat{\psi} &= \hat{\psi}'-2\gamma\,t-2\delta\,\phi \equiv \hat{\psi}'-(2\gamma+2\tilde{c}\,\delta)t-2\delta\,\tilde{\phi}
\end{aligned} \tag{6.4.16}$$

Using the constraints in (6.4.11) the solution is completely specified once the explicit form of the function $r(\rho_1,\tilde{\rho})$ is known.

The last shift implies that the left invariant one-form $\sigma^{\hat{3}}$ is shifted to $\sigma^{\hat{3}\,\prime} + (\gamma + \tilde{c}\,\delta)\mathrm{d}t + \delta\,\mathrm{d}\tilde{\phi}$. With a slight abuse of notation we will keep calling this shifted one form $\sigma^{\hat{3}}$. The metric of (6.3.1) is thus

$$\begin{aligned}
\mathrm{d}s^2 = -h^{-2}(\mathrm{d}t^2 + V_\phi \mathrm{d}\phi)^2 + h^2\frac{\rho_1^2}{\rho_3^2}(T^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}y^2) + \\
+ \tilde{\rho}^2\mathrm{d}\Omega_3^2 + \rho_1^2\big[(\sigma^{\hat{1}})^2 + (\sigma^{\hat{2}})^2\big] + \rho_3^2(\sigma^{\hat{3}} - A_t\mathrm{d}t - A_\phi\mathrm{d}\phi)^2 = \\
= g_{tt}\mathrm{d}t^2 + g_{\tilde{\rho}\tilde{\rho}}\mathrm{d}\tilde{\rho}^2 + \tilde{\rho}^2\mathrm{d}\tilde{\Omega}_3^2 + 2g_{t\tilde{\phi}}\mathrm{d}t\mathrm{d}\tilde{\phi} + 2g_{\rho_1\tilde{\rho}}\mathrm{d}\rho_1\mathrm{d}\tilde{\rho} + \\
+ g_{\rho_1\rho_1}\mathrm{d}\rho_1^2 + g_{\tilde{\phi}\tilde{\phi}}\mathrm{d}\tilde{\phi}^2 + \rho_1^2\big[(\sigma^{\hat{1}})^2 + (\sigma^{\hat{2}})^2\big] + \\
+ \rho_3^2\Big[\sigma^{\hat{3}} + \big(\gamma - A_t - \tilde{c}(A_\phi - \delta)\big)\mathrm{d}t - (A_\phi - \delta)\mathrm{d}\tilde{\phi}\Big]^2 ,
\end{aligned} \tag{6.4.17}$$

with

$$\begin{aligned}
g_{tt} &= -h^{-2}(1 + \tilde{c}V_\phi)^2 + \tilde{c}^2h^2\frac{\rho_1^2}{\rho_3^2}T^2r^2 \\
g_{\tilde{\rho}\tilde{\rho}} &= h^2\frac{\rho_1^2}{\rho_3^2}\left[\rho_1^2 + T^2r^2\left(\frac{\partial \ln r}{\partial\tilde{\rho}}\right)^2\right] \\
g_{t\tilde{\phi}} &= -h^{-2}(1 + \tilde{c}V_\phi)V_\phi + \tilde{c}h^2\frac{\rho_1^2}{\rho_3^2}T^2r^2 \\
g_{\rho_1\tilde{\rho}} &= h^2\frac{\rho_1^2}{\rho_3^2}\left[\rho_1\tilde{\rho} + T^2r^2\frac{\partial \ln r}{\partial\tilde{\rho}}\frac{\partial \ln r}{\partial\rho_1}\right] \\
g_{\rho_1\rho_1} &= h^2\frac{\rho_1^2}{\rho_3^2}\left[\tilde{\rho}^2 + T^2r^2\left(\frac{\partial \ln r}{\partial\rho_1}\right)^2\right] \\
g_{\tilde{\phi}\tilde{\phi}} &= -h^{-2}V_\phi^2 + h^2\frac{\rho_1^2}{\rho_3^2}T^2r^2
\end{aligned} \tag{6.4.18}$$

We recall the constraint on the metric components coming from the requirement of 1/4 supersymmetry,

$$\begin{aligned}
1 + A_t &= \frac{\rho_1^2}{\rho_3^2} \\
\rho_1^2 - \rho_3^2 &= \frac{S^4}{\rho_1^2} \\
h^{-2} &= \tilde{\rho}^2 + \rho_1^4/\rho_3^2.
\end{aligned} \tag{6.4.19}$$

In order that the metric factorises we need the $\mathrm{d}t\,\sigma^{\hat{3}}$ term to vanish which requires that

$$A_t + \tilde{c}(A_\phi - \delta) = \gamma\,. \tag{6.4.20}$$

Imposing also $g_{t\tilde{\phi}} = 0$ we obtain

$$\tilde{c}h^2\frac{\rho_1^2}{\rho_3^2}T^2r^2 = h^{-2}(1+\tilde{c}V_\phi)V_\phi\,. \tag{6.4.21}$$

In order to have an AdS_5 factor we should have $-g_{tt} = L^2 + \tilde{\rho}^2$ which gives, using the last relation

$$h^{-2}(1+\tilde{c}V_\phi) = L^2+\tilde{\rho}^2\,. \tag{6.4.22}$$

We also demand that $g_{\tilde{\rho}\tilde{\rho}} = \frac{L^2}{L^2+\tilde{\rho}^2}$ which after a little bit of algebra gives

$$\frac{\partial \ln r}{\partial \tilde{\rho}} = \pm\frac{\tilde{c}\tilde{\rho}}{L^2+\tilde{\rho}^2}. \tag{6.4.23}$$

Requiring that we have a product metric means that we also must impose that $g_{\rho_1\tilde{\rho}} = 0$ which implies

$$\frac{\partial \ln r}{\partial \rho_1} = \mp\tilde{c}\frac{\rho_1^3}{\rho_3^2L^2-\rho_1^4}. \tag{6.4.24}$$

As a result we find immediately that

$$g_{\rho_1\rho_1} = \frac{\rho_1^2L^2}{\rho_3^2L^2-\rho_1^4} \tag{6.4.25}$$

and

$$g_{\tilde{\phi}\tilde{\phi}} = \frac{1}{\tilde{c}^2}\left(L^2-\frac{\rho_1^4}{\rho_3^2}\right)\,. \tag{6.4.26}$$

The generic solution to equation (6.4.23) for the upper sign is

$$r = (L^2+\tilde{\rho}^2)^{\tilde{c}/2}r_0(\rho_1)\rho_1^{\tilde{c}} \tag{6.4.27}$$

where we have extracted the $\rho_1^{\tilde{c}}$ for future convenience. (6.4.24) is an equation for $r_0(\rho_1)$

$$r_0'(\rho_1) = \tilde{c}\frac{L^2(\rho_1^4-S^4)}{\rho_1^7-L^2\rho_1(\rho_1^4-S^4)}r_0(\rho_1) \tag{6.4.28}$$

Using the last constraint in (6.4.11)

$$T^2\partial_y\ln(\rho_1/\rho_3)\left[2\rho_1^2/\rho_3^2-2+y\partial_y\ln(\rho_1/\rho_3)\right]+y\left[\partial_r\ln(\rho_1/\rho_3)\right]^2=0 \tag{6.4.29}$$

we can find T. Note that both the first order differential equation for T

$$\partial_y \ln T = D \tag{6.4.30}$$

and the second order equation in (6.4.13) are satisfied when $r_0(\rho_1)$ satisfies the equation (6.4.28).

6.4.3 Relation to standard $Y^{p,q}$ coordinates

Now we will give the coordinate transformation that takes the metric on X_5 to the standard form of the metric on $Y^{p,q}$ as presented in [107]. We perform the rescaling

$$\tilde{\rho} \longrightarrow L\tilde{\rho}, \qquad \rho_i \longrightarrow L\rho_i, \qquad S \longrightarrow LS \tag{6.4.31}$$

which puts the metric of AdS_5 into the form

$$\mathrm{d}s^2_{AdS_5} = L^2\Big(- (\tilde{\rho}^2+1)\mathrm{d}t^2 + \frac{\mathrm{d}\tilde{\rho}^2}{\tilde{\rho}^2+1} + \tilde{\rho}^2\mathrm{d}\Omega_3^2\Big) \tag{6.4.32}$$

while the metric on the "internal" part is

$$\mathrm{d}s_5^2 = L^2\Bigg[\frac{\rho_1^2}{\rho_3^2-\rho_1^4}\mathrm{d}\rho_1^2 + \frac{1}{\tilde{c}^2}\left(1-\frac{\rho_1^4}{\rho_3^2}\right)\mathrm{d}\tilde{\phi}^2 + \\ + \rho_1^2\big[(\sigma^{\hat{1}})^2+(\sigma^{\hat{2}})^2\big] + \rho_3^2\big(\sigma^{\hat{3}} - (A_\phi-\delta)\mathrm{d}\tilde{\phi}\big)^2\Bigg]. \tag{6.4.33}$$

The standard form for the metric on $Y^{p,q}$ [107] is,

$$\mathrm{d}s^2 = \frac{1-c\hat{y}}{6}(\mathrm{d}\hat{\theta}^2+\sin^2\hat{\theta}\mathrm{d}\hat{\phi}^2) + \frac{1}{w(\hat{y})q(\hat{y})}d\hat{y}^2 + \frac{q(\hat{y})}{9}(\mathrm{d}\hat{\psi}+\cos\hat{\theta}\mathrm{d}\hat{\phi})^2 \\ + w(\hat{y})\left[\mathrm{d}\alpha - \frac{ac-2\hat{y}+\hat{y}^2c}{6(a-\hat{y}^2)}(\mathrm{d}\hat{\psi}+\cos\hat{\theta}\mathrm{d}\hat{\phi})\right]^2 \tag{6.4.34}$$

with[2]

$$w(\hat{y}) = \frac{2(a-\hat{y}^2)}{1-c\hat{y}} \\ q(\hat{y}) = \frac{a-3\hat{y}^2+2c\hat{y}^3}{a-\hat{y}^2}. \tag{6.4.35}$$

Recalling that

$$\sigma^{\hat{1}} = -\tfrac{1}{2}(\cos\hat{\psi}\,d\hat{\theta}+\sin\hat{\psi}\,\sin\hat{\theta}\,d\hat{\phi}) \\ \sigma^{\hat{2}} = -\tfrac{1}{2}(-\sin\hat{\psi}\,d\hat{\theta}+\cos\hat{\psi}\,\sin\hat{\theta}\,d\hat{\phi}) \\ \sigma^{\hat{3}} = -\tfrac{1}{2}(d\hat{\psi}+\cos\hat{\theta}\,d\hat{\phi}) \tag{6.4.36}$$

we immediately get

$$\rho_1^2 = \frac{2}{3}(1-c\hat{y}) \tag{6.4.37}$$

and

$$\frac{1}{4}\rho_3^2 = \frac{2+ac^2-6c\hat{y}+3c^2\hat{y}^2}{18(1-c\hat{y})}. \tag{6.4.38}$$

[2]Notice that the $\hat{\psi}$ of [107] has the opposite sign to that used here.

Recalling that $\rho_1^2 - \rho_3^2 = S^4/\rho_1^2$ we have

$$S^4 = \frac{4}{27}(1 - ac^2)\,. \tag{6.4.39}$$

We also have

$$A_\phi = \frac{1}{\tilde{c}}(\gamma - A_t) + \delta = \frac{1}{\tilde{c}}\left(\gamma + 1 - \frac{\rho_1^2}{\rho_3^2}\right) + \delta\,. \tag{6.4.40}$$

Assuming that $\alpha = \beta\tilde{\phi}$ and equating the $d\tilde{\phi}^2$ component of the metric we get

$$\frac{1}{\tilde{c}^2}\left(1 - \frac{\rho_1^4}{\rho_3^2}\right) + \rho_3^2(A_\phi - \delta)^2 = w(\hat{y})\beta^2 \tag{6.4.41}$$

which implies after some straightforward algebra that

$$\gamma = \frac{1}{2} \quad , \quad \beta = \pm\frac{c}{2\tilde{c}}\,. \tag{6.4.42}$$

The coefficient of the cross term $d\tilde{\phi}\,\sigma^{\hat{3}}$ is

$$\frac{1}{2}\rho_3^2(A_\phi - \delta) = -\beta w(\hat{y})\frac{ac - 2\hat{y} + \hat{y}^2 c}{6(a - \hat{y}^2)} \tag{6.4.43}$$

which implies that we must have $\beta = -\frac{c}{2\tilde{c}}$. We can therefore set

$$\beta = -1 \qquad \tilde{c} = \frac{1}{2}c\,. \tag{6.4.44}$$

Using the expression for r

$$\begin{aligned} r &= (L^2 + \tilde{\rho}^2)^{\tilde{c}/2} r_0(\rho_1)\rho_1^{\tilde{c}} \\ r_0'(\rho_1) &= \tilde{c}\frac{L^2(\rho_1^4 - S^4)}{\rho_1^7 - L^2\rho_1(\rho_1^4 - S^4)} r_0(\rho_1) \end{aligned} \tag{6.4.45}$$

and the definition

$$A_\phi = A_t V_\phi + \frac{1}{2} r\partial_r \ln T \tag{6.4.46}$$

we get

$$A_t + \tilde{c}(A_\phi - \delta) = \frac{1}{2} - \frac{\tilde{c}}{2}(1 + 2\delta) \tag{6.4.47}$$

which gives

$$\delta = -\frac{1}{2} \tag{6.4.48}$$

for $c, \tilde{c} \neq 0$. Finally, the matching of the $d\hat{y}^2$ factor

$$\frac{1}{9}\frac{c^2}{\rho_3^2 - \rho_1^4} = \frac{1}{w(\hat{y})q(\hat{y})} \tag{6.4.49}$$

is identically satisfied. Finally, we observe that the polynomial $q(\hat{y}) = a - 3\hat{y}^2 + 2c\hat{y}^3$, whose zeroes $\hat{y}_1$ and $\hat{y}_2$ (with $\hat{y}_1 < 0$ and $\hat{y}_2$ the smallest between the two other positive zeroes), determine the range of $\hat{y}$, $\hat{y}_1 \leq \hat{y} \leq \hat{y}_2$, can be expressed in terms of ρ_1 as $q = -27(\rho_1^6 - \rho_1^4 + S^4)/4$. Notice also that in the metric (6.4.34) any non zero value of c can be reabsorbed in a rescaling of $\hat{y}$ and α. We may thus set $c = 1$ whenever $c \neq 0$.

c=0 case

Let us now take a look at the singular case

$$c = \tilde{c} = 0 \tag{6.4.50}$$

which corresponds to the Sasaki-Einstein internal manifold $Y^{1,0} \equiv T^{1,1}$. From the equations (6.4.21),(6.4.47) we can immediately obtain

$$\begin{aligned} A_t &= \gamma = \frac{1}{2} \\ V_\phi &= 0 \end{aligned} \tag{6.4.51}$$

and from (6.4.37),(6.4.38),(6.4.39)

$$\begin{aligned} \rho_1^2 &= \rho_3 = \frac{2}{3} \\ S &= \frac{4}{27}\,. \end{aligned} \tag{6.4.52}$$

The equation for ρ_1 implies that

$$y = \sqrt{\frac{2}{3}}L^2\tilde{\rho}\,. \tag{6.4.53}$$

Given these explicit values for ρ_1 and ρ_3, the last constraint in (6.4.11)

$$T^2\partial_y \ln(\rho_1/\rho_3)\left[2\rho_1^2/\rho_3^2 - 2 + y\partial_y \ln(\rho_1/\rho_3)\right] + y\left[\partial_r \ln(\rho_1/\rho_3)\right]^2 = 0 \tag{6.4.54}$$

is automatically satisfied.

The metric on the internal manifold becomes

$$\mathrm{d}s_5^2 = L^2\left[\frac{3}{2}\tau(r)^2(\frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\tilde{\phi}^2) + \frac{2}{3}\left[(\sigma^{\hat{1}})^2 + (\sigma^{\hat{2}})^2\right] + \frac{4}{9}(\sigma^{\hat{3}} - (A_\phi - \delta)\mathrm{d}\tilde{\phi})^2\right] \tag{6.4.55}$$

where

$$T^2 = (\tilde{\rho}^2 + 1)\frac{\tau(r)^2}{r^2} \tag{6.4.56}$$

is such that T solves the equation

$$\partial_y \ln T = D \iff \partial_{\tilde{\rho}} \ln T = \frac{2\tilde{\rho}}{\tilde{\rho}^2 + 1}\,. \tag{6.4.57}$$

We now match this expression with the one in [107]. For $c = 0$, a can be reabsorbed in a coordinate redefinition. We set, for convenience,

$$a = 3 \tag{6.4.58}$$

and obtain,

$$\begin{aligned} ds^2 = \frac{1}{6}(d\hat{\theta}^2 + \sin^2\hat{\theta} d\hat{\phi}^2) + \frac{1}{6(1-\hat{y}^2)}d\hat{y}^2 + \frac{1-\hat{y}^2}{3(3-\hat{y}^2)}(d\hat{\psi} + \cos\hat{\theta} d\hat{\phi})^2 \\ + 2(3-\hat{y}^2)\left[d\alpha + \frac{2\hat{y}}{6(3-\hat{y}^2)}(d\hat{\psi} + \cos\hat{\theta} d\hat{\phi})\right]^2 \end{aligned} \tag{6.4.59}$$

Assuming, as in the generic case, that $\alpha \equiv -\tilde{\phi}$ and equating the $g_{3\alpha}$ and $g_{\alpha\alpha}$ components we get

$$A_\phi - \delta = -3\hat{y} \tag{6.4.60}$$

and

$$\frac{3}{2}\tau^2 + 4\hat{y}^2 = 2(3-\hat{y}^2) \quad \Rightarrow \quad \tau^2 = 4(1-\hat{y}^2)\,. \tag{6.4.61}$$

Assuming $r = r(\hat{y})$ and equating the $d\hat{y}^2$ term gives

$$\frac{\partial \ln r}{\partial \hat{y}} = \pm\frac{1}{6(1-\hat{y}^2)} \quad \Rightarrow \quad r = \lambda\left(\frac{1-\hat{y}}{1+\hat{y}}\right)^{\mp 1/12} \tag{6.4.62}$$

where λ is an arbitrary constant and we fix $\lambda = 1$. We are now able to determine the constant δ through the equation

$$A_\phi = A_t V_\phi + \frac{1}{2} r\partial_r \ln T = -\frac{1}{2} \mp 3\hat{y} \tag{6.4.63}$$

which fixes the upper choice for the sign and

$$\delta = -\frac{1}{2}\,. \tag{6.4.64}$$

In order to bring the metric to the standard $T^{1,1}$ form we set

$$\hat{y} = -\cos\tilde{\theta} \tag{6.4.65}$$

which gives

$$r = \left(\tan\frac{\tilde{\theta}}{2}\right)^{1/6} \quad , \quad \tau = 4\sin^2\tilde{\theta} \tag{6.4.66}$$

and thus

$$\frac{ds_5^2}{L^2} = \frac{1}{6}\left(d\tilde{\theta}^2 + 36\sin^2\tilde{\theta} d\tilde{\phi}^2\right) + \frac{1}{6}\left(d\hat{\theta}^2 + \sin\hat{\theta} d\hat{\phi}^2\right) + \frac{1}{9}\left(d\hat{\psi} + \cos\hat{\theta} d\hat{\phi} + 6\cos\tilde{\theta} d\tilde{\phi}\right)^2 \tag{6.4.67}$$

which is the $T^{1,1}$ metric up to the trivial rescaling

$$\tilde{\phi} \to \frac{1}{6}\tilde{\phi}\,. \tag{6.4.68}$$

6.5 Asymptotic expansion for half BPS states in $AdS_5 \times Y^{p,q}$

In this section we study generic asymptotic perturbations of the $AdS_5 \times Y^{p,q}$ geometries that preserve 1/2 of the bulk supersymmetries. We relax the constraints of (6.4.11) which give back $AdS_5 \times Y^{p,q}$ and solve the differential equations (6.3.26) with the boundary conditions that the solutions approach $AdS_5 \times Y^{p,q}$ at large distances (including also the particular case $c = 0$). We will work in the mixed coordinates $(y, \hat{y})$ or $(y, \tilde{\theta})$ and solve the equation in an expansion for large y, with the simplifying assumption that the solutions are invariant under shifts in $\tilde{\phi}$. We make the following Ansatz for the expansion of our functions,

$$\rho_1 = L\sqrt{\frac{2}{3}(1-c\hat{y})}\left(1+\rho_1^{(1)}(\hat{y})\frac{L^4}{y^2}+\rho_1^{(2)}(\hat{y})\frac{L^8}{y^4}\right) \tag{6.5.1}$$

$$\rho_3 = L\sqrt{\frac{2(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)}{9(1-c\hat{y})}}\left(1+\rho_3^{(1)}(\hat{y})\frac{L^4}{y^2}+\rho_3^{(2)}(\hat{y})\frac{L^8}{y^4}\right) \tag{6.5.2}$$

$$\tilde{\rho} = \frac{y}{\rho_1} \tag{6.5.3}$$

$$A_t = \frac{1-ac^2}{2+ac^2-6c\hat{y}+3c^2\hat{y}^2}\left(1+A_t^{(1)}(\hat{y})\frac{L^4}{y^2}+A_t^{(2)}(\hat{y})\frac{L^8}{y^4}\right) \tag{6.5.4}$$

$$T = \frac{y}{r}\sqrt{\frac{2(a-3\hat{y}^2+2c\hat{y}^3)}{(1-c\hat{y})^3}}\left(1+t^{(1)}(\hat{y})\frac{L^4}{y^2}+t^{(2)}(\hat{y})\frac{L^8}{y^4}\right) \tag{6.5.5}$$

$$V_\phi = \frac{4c(1-c\hat{y})(a-3\hat{y}^2+2c\hat{y}^3)}{3(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)}\frac{L^4}{y^2}+V_\phi^{(2)}(\hat{y})\frac{L^8}{y^4}+V_\phi^{(3)}(\hat{y})\frac{L^{12}}{y^6} \tag{6.5.6}$$

$$r = y^{c/2}r_0(\hat{y})\,. \tag{6.5.7}$$

This expansion reproduces the $c = 0$ limit upon setting $a = 3$, as in the previous section.

In these coordinates, the condition (6.4.28) becomes

$$r_0'(\hat{y}) = \frac{2+ac^2-6c\hat{y}+3c^2\hat{y}^2}{4(1-c\hat{y})(a-3\hat{y}^2+2c\hat{y}^3)}r_0(\hat{y}) \tag{6.5.8}$$

The functions m, n, p are given by

$$m = \frac{1}{\rho_3^2[\tilde{\rho}^2+(1+A_t)^2\rho_3^2]} \tag{6.5.9}$$

$$n = \frac{(1+A_t)\rho_1^2}{y^2[\tilde{\rho}^2+(1+A_t)^2\rho_3^2]} \tag{6.5.10}$$

$$p = \frac{\rho_1^2}{y^2[\tilde{\rho}^2+(1+A_t)^2\rho_3^2]}\,. \tag{6.5.11}$$

The constraints in (6.4.11) and the equations (6.4.12) are satisfied at leading order in y. We rewrite the generic equations (6.3.26) in polar coordinates dividing them by T^2 and exploiting the $U(1)$ symmetry of our solutions

$$\begin{aligned}
&\frac{y^3}{r^2T^2}r\partial_r(r\partial_r n) + \partial_y\left(y^3\partial_y n\right) + y^2\partial_y\left[(yDn + 2y^2m(n-p))\right] \\
&\qquad + 2y^2D(2n + y\partial_y n + yDn) = 0 \\
&\frac{y^3}{r^2T^2}r\partial_r(r\partial_r m) + \partial_y\left(y^3\partial_y m\right) + \partial_y\left(y^3 2mD\right) + 2Dy^3(\partial_y m + 2Dm) = 0 \\
&\frac{y^3}{r^2T^2}r\partial_r(r\partial_r p) + \partial_y\left(y^3\partial_y p\right) + \partial_y\left[y^3 4ny(n-p)\right] + 2Dy^3[\partial_y p + 4ny(n-p)] = 0 \\
&\partial_y \ln T = D
\end{aligned} \tag{6.5.12}$$

where

$$\partial_y f(y,\hat{y}) \equiv \left.\frac{\mathrm{d}f}{\mathrm{d}y}\right|_r = -\frac{c\,r_0(\hat{y})}{2r_0'(\hat{y})}\left.\frac{\mathrm{d}f}{\mathrm{d}\hat{y}}\right|_y + \left.\frac{\mathrm{d}f}{\mathrm{d}y}\right|_{\hat{y}} \tag{6.5.13}$$

$$r\partial_r f(y,\hat{y}) \equiv r\left.\frac{\mathrm{d}f}{\mathrm{d}r}\right|_y = \frac{r_0(\hat{y})}{r_0'(\hat{y})}\left.\frac{\mathrm{d}f}{\mathrm{d}\hat{y}}\right|_y . \tag{6.5.14}$$

The generic asymptotic solutions to these equation are specified, at each order, by 7 integration constants. As in [17], requiring regularity of the solutions implies that not all of them are independent and indeed we have only three independent integration constants.

For the case of $T^{1,1}$ asymptotics, specified by $c = 0$ the first subleading corrections are given by:

$$\rho_1^{(1)}(\tilde{\theta}) = -k + C_1\cos\tilde{\theta} \tag{6.5.15}$$

$$\rho_3^{(1)}(\tilde{\theta}) = \rho_1^{(1)}(\tilde{\theta}) + k^{(1)}(\tilde{\theta}) \tag{6.5.16}$$

$$k^{(1)}(\tilde{\theta}) = k \tag{6.5.17}$$

$$A_t^{(1)}(\tilde{\theta}) = C_2 - 4C_1\cos\tilde{\theta} \tag{6.5.18}$$

$$t^{(1)}(\tilde{\theta}) = \frac{L^2\sqrt{2/3}(1+9k)\sin\tilde{\theta}}{\tan\frac{\tilde{\theta}}{2}} \tag{6.5.19}$$

$$V_\phi^{(2)}(\tilde{\theta}) = -\frac{8}{3}C_2\sin^2\tilde{\theta} \tag{6.5.20}$$

while in the generic case we get:

$$\rho_1^{(1)}(\hat{y}) = \frac{A[2c^2K + 9Ak + 4\hat{y}BC_1]}{6(2 + ac^2 - 6c\hat{y} + 3c^2\hat{y}^2)}$$
$$\rho_3^{(1)}(\hat{y}) = \rho_1^{(1)}(\hat{y}) + k^{(1)}(\hat{y})$$
$$k^{(1)}(\hat{y}) = \frac{A[4c^2LK + 9\big(-2 + 8c\hat{y} - 3c^3\hat{y}^3 - ac^2(4 - c\hat{y})\big)k]}{6(2 + ac^2 - 6c\hat{y} + 3c^2\hat{y}^2)}$$
$$A_t^{(1)}(\hat{y}) = \Big(\frac{-4c^2A^3K}{(2 + ac^2 - 6c\hat{y} + 3c^2\hat{y}^2)^2} + \frac{2A}{L}C_2 - \frac{4}{3}AC_1 +$$
$$- \frac{9c^2\ A[a^2c^2 + \hat{y}^2(12 - 26c\hat{y} + 21c^2\hat{y}^2 - 6c^3\hat{y}^3) + 2a(-2 + 3c\hat{y} - 3c^2\hat{y}^2 + c^3\hat{y}^3)]}{2LB}k$$
$$t^{(1)}(\hat{y}) = \frac{A(4L - 27k)}{6B}$$

where

$$A = 1 - c\hat{y}$$
$$B = 2 + ac^2 - 6c\hat{y}^2 + 2c\hat{y}^3$$
$$K = a - 3\hat{y}^2 + 2c\hat{y}^3$$
$$L = 1 - ac^2$$

The three arbitrary integration constants, C_1, C_2, k will turn out to be related to the supergravity dual of the flavour and baryon charge of the solutions. The second order regular solutions are rather complicated. In general, they will involve new integrations constants together with an inhomogeneous part. The expressions for the inhomogeneous part can be found in the Appendix.

As already noticed, any $c \neq 0$ can be reabsorbed by a redefinition of $\hat{y}$ and so we set $c = 1$.

6.6 U(1) charges

We will now show how the first subleading corrections described in the previous section give rise to the Kaluza-Klein reduction of type IIB supergravity on the $Y^{p,q}$ manifolds respecting the symmetry of our Ansatz. We will calculate the global charges of the solutions under three $U(1)$ massless KK gauge fields living in AdS_5; two of them can be identified with the KK modes of the metric associated to the Killing vectors ∂_α and $\partial_{\hat{\psi}}$ and which are dual to the flavour and R charges of the dual quiver gauge theory (more precisely to linear combinations of the charges) while the third one is associated to the expansion of the RR 4-form potential on the cohomology of $Y^{p,q}$ and it is dual to the baryon charge of the gauge theory. Since the third Betti number of such manifolds is one there is only one baryon charge.

In general the metric on the compact manifold is modified by the metric KK gauge fields

$$ds^2 = g_{\alpha\beta}(d\xi^\alpha + K_I^\alpha A_\mu^I dx^\mu)(d\xi^\beta + K_I^\beta A_\mu^I dx^\mu) \tag{6.6.1}$$

where ξ^α are coordinates in $Y^{p,q}$ and x^μ in AdS_5 and

$$K_I = K_I^\alpha \partial_\alpha \qquad I = 1, \ldots, n \tag{6.6.2}$$

are n Killing vectors of $Y^{p,q}$. In our case only two gauge fields are turned on and they are associated to ∂_α and $\partial_{\hat{\psi}}$. We denote the two global gauge charges respectively as J and Q. The leading order of the corresponding gauge fields A_J, A_Q is thus given by

$$A_J = \frac{J}{\tilde{\rho}^2}dt \qquad A_Q = \frac{Q}{\tilde{\rho}^2}dt\,. \tag{6.6.3}$$

The metric is modified by the shifts

$$d\hat{\psi} \to d\hat{\psi} + \frac{Q}{\tilde{\rho}^2}dt \tag{6.6.4}$$

$$d\alpha \to d\alpha + \frac{J}{\tilde{\rho}^2}dt \tag{6.6.5}$$

to

$$\begin{aligned} ds^2L^{-2} = {} & \frac{1-c\hat{y}}{6}(d\hat{\theta}^2 + \sin^2\hat{\theta}d\hat{\phi}^2) + \frac{1}{w(\hat{y})q(\hat{y})}d\hat{y}^2 + \frac{q(\hat{y})}{9}(d\hat{\psi} + \frac{Q}{\tilde{\rho}^2}dt + \cos\hat{\theta}d\hat{\phi})^2 \\ & + w(\hat{y})\left[d\alpha + \frac{J}{\tilde{\rho}^2}dt - \frac{ac - 2\hat{y} + \hat{y}^2c}{6(a-\hat{y}^2)}(d\hat{\psi} + \frac{Q}{\tilde{\rho}^2}dt + \cos\hat{\theta}d\hat{\phi})\right]^2 . \end{aligned} \tag{6.6.6}$$

Given the expression above for the metric and the solution of the equations of motion up to the first sub-leading order we obtain

$$Q = 3C_2 - 2C_1, \tag{6.6.7}$$

$$J = \frac{1}{2}C_2 - C_1. \tag{6.6.8}$$

Similarly, in the case of $T^{1,1}$ we have

$$\begin{aligned} ds^2L^{-2} = {} & \frac{1}{6}\big(d\tilde{\theta}^2 + 36\sin^2\tilde{\theta}(d\tilde{\phi} + \frac{J}{\tilde{\rho}^2}dt)^2\big) + \frac{1}{6}\big(d\hat{\theta}^2 + \sin\hat{\theta}d\hat{\phi}^2\big) + \\ & + \frac{1}{9}\big(d\hat{\psi} + \frac{Q}{\tilde{\rho}^2}dt + \cos\hat{\theta}d\hat{\phi} + 6\cos\tilde{\theta}(d\tilde{\phi} + \frac{J}{\tilde{\rho}^2}dt)\big)^2 \end{aligned} \tag{6.6.9}$$

with

$$Q = \frac{3}{2}C_2 \tag{6.6.10}$$

$$J = -C_1 \tag{6.6.11}$$

R-charge and Reeb vector

In order to correctly identify the R charge we proceed as in [109, 107, 119]. We define the new coordinates

$$\hat{\psi}' = \hat{\psi} \tag{6.6.12}$$
$$\beta = -6\alpha + c\hat{\psi} \tag{6.6.13}$$

In this coordinate system we can write the metric as a local $U(1)$ fiber over a Kähler-Einstein manifold and $\hat{\psi}'$ is a coordinate on the local $U(1)$ fiber. From (2.17) of [107], we have

$$\mathrm{d}\Omega^2_{Y^{p,q}} = (e^{\hat{\theta}})^2 + (e^{\hat{\phi}})^2 + (e^{\hat{y}})^2 + (e^{\beta})^2 + (e^{\hat{\psi}})^2 \tag{6.6.14}$$

where the one forms on $Y^{p,q}$ are,

$$e^{\hat{\theta}} = \sqrt{\frac{1-c\hat{y}}{6}}d\hat{\theta}\ , \quad e^{\hat{\phi}} = \sqrt{\frac{1-c\hat{y}}{6}}\sin\hat{\theta}\mathrm{d}\hat{\phi}\ , \tag{6.6.15}$$

$$e^{\hat{y}} = \frac{1}{\sqrt{wq}}\mathrm{d}\hat{y}\ , \quad e^{\beta} = \frac{\sqrt{wq}}{6}(\mathrm{d}\beta + c\cos\hat{\theta}\mathrm{d}\hat{\phi})\ , \tag{6.6.16}$$

$$e^{\hat{\psi}'} = \frac{1}{3}(-\mathrm{d}\hat{\psi}' - \cos\hat{\theta}\mathrm{d}\hat{\phi} + \hat{y}(\mathrm{d}\beta + c\cos\hat{\theta}\mathrm{d}\hat{\phi}))\ . \tag{6.6.17}$$

As noted in [109], the R-symmetry is identified with a shift in the angular variable

$$\psi_R = -\frac{1}{2}\hat{\psi}' \tag{6.6.18}$$

at constant β. As a consequence, the $U(1)$ R-symmetry gauge field is given by

$$A_R = -\frac{1}{2}A_Q \tag{6.6.19}$$

and

$$Q_R = -\frac{1}{2}Q\,. \tag{6.6.20}$$

The associated Killing vector is given by

$$K_R = -2\partial_{\hat{\psi}} - \frac{c}{3}\partial_\alpha \tag{6.6.21}$$

which coincides with the Reeb vector of the Sasaki-Einstein manifold. Notice that our Reeb vector differs by a factor of 2/3 from the one in [93],[107].

6.7 The 5-form and baryon charge

The self-dual Ramond-Ramond field strength $F_{(5)}$ can be written as

$$F_{(5)} = \mathcal{F}_5 + \star_{10}\mathcal{F}_5 \,. \tag{6.7.1}$$

With our conventions and normalisations, the leading order for $\mathcal{F}_5$ is given by

$$\mathcal{F}_5^0 = L^4 Vol(Y^{p,q}) \tag{6.7.2}$$

where $Vol(Y^{p,q})$ is the volume form of the unit radius $Y^{p,q}$. The background metric is perturbed by the KK gauge fields as described in the previous chapter: the field strength is also perturbed in order to satisfy the equations of motion. The corrections are known to be of the form [109, 119, 120]

$$\mathcal{F}_5^1 = L^4 \mathrm{d}\left(A_Q \wedge \omega_Q + A_J \wedge \omega_J + A_B \wedge \omega_B\right) \,. \tag{6.7.3}$$

The $Y^{p,q}$ three forms ω_I are defined by

$$\mathrm{d}\omega_I + \iota_{K_I} Vol(Y^{p,q}) = 0 \tag{6.7.4}$$

where K_I, $I = J, Q$ is the Killing vector of $Y^{p,q}$ associated with the A_I gauge field. The 3-forms $\omega_{J,Q}$ are clearly defined up to the addition of a closed form. The 3-form ω_B is the generator of the one dimensional cohomology of the Sasaki-Einstein manifold and A_B is the gauge field dual to the baryon current of the CFT. The arbitrary shift by a closed form of the $\omega_{J,K}$ corresponds to the possibility of shifting the mesonic symmetries of the theory by an arbitrary baryonic one.

In the case of generic $Y^{p,q}$ for $c \neq 0$ we obtain the following form for the subleading corrections to $F_{(5)}$,

$$\begin{aligned}
\mathcal{F}_5^1 = -\frac{2}{\tilde{\rho}^3}\mathrm{d}\tilde{\rho} \wedge \mathrm{d}t \wedge \Bigg\{ \frac{k}{4}\left[\left(\sigma^{\hat{3}} - 3\hat{y}(1-\hat{y})\mathrm{d}\alpha\right) \wedge \sigma^{\hat{1}} \wedge \sigma^{\hat{2}} - \frac{3}{2(1-\hat{y})^2}\sigma^{\hat{3}} \wedge \mathrm{d}\alpha \wedge \mathrm{d}\hat{y}\right] + \\
+\frac{Q}{9}\left[\left(\frac{a-1}{3}\sigma^{\hat{3}} - \frac{a - 2\hat{y}(a-1) - 3\hat{y}^2 + 2\hat{y}^3}{2(1-\hat{y})}\mathrm{d}\alpha\right) \wedge \sigma^{\hat{1}} \wedge \sigma^{\hat{2}} + \frac{2 + a - 6\hat{y} + 3\hat{y}^2}{4(1-\hat{y})^2}\sigma^{\hat{3}} \wedge \mathrm{d}\alpha \wedge \mathrm{d}\hat{y}\right] \\
+\frac{J}{3}\left[\left(\frac{a - 2\hat{y} + \hat{y}^2}{3}\sigma^{\hat{3}} - \frac{a - 2a\hat{y} + \hat{y}^2}{2(1-\hat{y})}\mathrm{d}\alpha\right) \wedge \sigma^{\hat{1}} \wedge \sigma^{\hat{2}} - \frac{a - 2\hat{y} + \hat{y}^2}{2(1-\hat{y})^2}\sigma^{\hat{3}} \wedge \mathrm{d}\alpha \wedge \mathrm{d}\hat{y}\right] \Bigg\} + \\
+\frac{1}{\tilde{\rho}^2}\mathrm{d}t \wedge \left(- Q\frac{2(1-\hat{y})}{9}\mathrm{d}\alpha - J\frac{4(1-\hat{y})}{9}\sigma^{\hat{3}}\right) \wedge \mathrm{d}\hat{y} \wedge \sigma^{\hat{1}} \wedge \sigma^{\hat{2}}
\end{aligned} \tag{6.7.5}$$

while for the case $c = 0$ and going to the natural coordinate $(\tilde{\theta}, \tilde{\phi})$ defined by $(\hat{y}, \alpha) =$

$(-\cos\tilde{\theta}, -\tilde{\phi})$ we get

$$\mathcal{F}_5^1 = -\frac{2}{\tilde{\rho}^3}\mathrm{d}\tilde{\rho}\wedge \mathrm{d}t\Big\{ -\frac{k}{6}\Big[\Big(2\sigma^{\hat{3}} - 6\cos\tilde{\theta}\mathrm{d}\tilde{\phi}\Big)\wedge\sigma^{\hat{1}}\wedge\sigma^{\hat{2}} - 3\sin\tilde{\theta}\sigma^{\hat{3}}\wedge \mathrm{d}\tilde{\theta}\wedge \mathrm{d}\tilde{\phi}\Big]$$
$$+\frac{Q}{9}\Big[\Big(\frac{1}{3}\sigma^{\hat{3}} - \cos\tilde{\theta}\mathrm{d}\tilde{\phi}\Big)\wedge\sigma^{\hat{1}}\wedge\sigma^{\hat{2}} + \frac{1}{2}\sin\tilde{\theta}\sigma^{\hat{3}}\wedge \mathrm{d}\tilde{\theta}\wedge \mathrm{d}\tilde{\phi}\Big]$$
$$+\frac{J}{3}\Big[\Big(-\frac{4}{3}\cos\tilde{\theta}\sigma^{\hat{3}} + \frac{1}{2}(1+7\cos 2\tilde{\theta})\mathrm{d}\tilde{\phi}\Big)\wedge\sigma^{\hat{1}}\wedge\sigma^{\hat{2}} + \frac{1}{2}\sin 2\tilde{\theta}\sigma^{\hat{3}}\wedge \mathrm{d}\tilde{\theta}\wedge \mathrm{d}\tilde{\phi}\Big]\Big\}$$
$$+\frac{1}{\tilde{\rho}^2}\mathrm{d}t\wedge\Big(Q\frac{2}{9}\mathrm{d}\tilde{\phi} + J\frac{4}{9}\sigma^{\hat{3}}\Big)\wedge\sin\tilde{\theta}\mathrm{d}\tilde{\theta}\wedge\sigma^{\hat{1}}\wedge\sigma^{\hat{2}}. \qquad (6.7.6)$$

The volume form on $Y^{p,q}$ is given by[3]

$$\mathrm{Vol}(Y^{p,q}) = -e^{\hat{y}}\wedge e^{\beta}\wedge e^{\hat{\theta}}\wedge e^{\hat{\phi}}\wedge e^{\hat{\psi}'} = \frac{4(1-c\hat{y})}{9}\mathrm{d}\hat{y}\wedge \mathrm{d}\alpha\wedge\sigma^{\hat{1}}\wedge\sigma^{\hat{2}}\wedge\sigma^{\hat{3}}, \qquad (6.7.7)$$

and we define the three forms

$$\omega_\pm \equiv e^{\hat{\psi}'}\wedge(e^{\hat{\theta}}\wedge e^{\hat{\phi}} \pm e^{\hat{y}}\wedge e^{\beta}) =$$
$$= \frac{1}{3}\left(2\sigma^{\hat{3}}(1-c\hat{y}) - 6\hat{y}\mathrm{d}\alpha\right)\wedge\left(\frac{2(1-c\hat{y})}{3}\sigma^{\hat{1}}\wedge\sigma^{\hat{2}} \mp \frac{1}{3}\mathrm{d}\hat{y}\wedge(c\sigma^{\hat{3}} + 3\mathrm{d}\alpha)\right) \qquad (6.7.8)$$

The local Kähler form J_2 is given by

$$J_2 = e^{\hat{\theta}}\wedge e^{\hat{\phi}} - e^{\hat{y}}\wedge e^{\beta} = \frac{1}{2}\mathrm{d}e^{\hat{\psi}'} \qquad (6.7.9)$$

The closed form ω_B is given as in [119] by

$$\omega_B = \frac{9}{8\pi^2(1-c\hat{y})^2}(p^2-q^2)\omega_- \qquad (6.7.10)$$

With this normalisation and assuming that $A_B = \frac{Q_B}{\tilde{\rho}^2}\mathrm{d}t$, the baryon charge Q_B is given by

$$Q_B = \frac{\pi^2}{2(p^2-q^2)}k. \qquad (6.7.11)$$

In the case of $T^{1,1}$ and recalling the change of coordinates $(\hat{y},\alpha) = (-\cos\tilde{\theta}, -\tilde{\phi})$ we get

$$\omega_\pm = \left(\frac{2}{3}\sigma^{\hat{3}} - 2\cos\tilde{\theta}\mathrm{d}\tilde{\phi}\right)\wedge\left(\frac{2}{3}\sigma^{\hat{1}}\wedge\sigma^{\hat{2}} \pm \sin\tilde{\theta}\mathrm{d}\tilde{\theta}\wedge \mathrm{d}\tilde{\phi}\right) \qquad (6.7.12)$$

with

$$\omega_B \equiv \frac{9}{8\pi^2}\omega_- \qquad (6.7.13)$$

[3]The orientation is chosen to satisfy (6.7.2)

and

$$Q_B = \frac{2\pi^2}{3}k\,. \tag{6.7.14}$$

We now rewrite the expansion of $\mathcal{F}_5^1$ as

$$\mathcal{F}_5^1 = L^4 \mathrm{d}\left(A_R \wedge \omega_R + A_\beta \wedge \omega_\beta + A_B \wedge \omega_B\right)\,. \tag{6.7.15}$$

where

$$A_R = -\frac{1}{2}A_Q, \qquad\qquad A_\beta = -6A_J - A_Q \tag{6.7.16}$$

are the gauge fields associated to the Killing vectors

$$K_R = -2\partial_{\hat\psi} - \frac{1}{3}\partial_\alpha, \qquad\qquad \partial_\beta = -\frac{1}{6}\partial_\alpha\,. \tag{6.7.17}$$

The remaining 3-forms are given by

$$\omega_R = -\frac{1}{6}\omega_+ \tag{6.7.18}$$

$$\omega_\beta = -\frac{(a - 2\hat y + \hat y^2)}{18}\sigma^{\hat 3} \wedge \left(\frac{2}{3}\sigma^{\hat 1}\wedge\sigma^{\hat 2} - \frac{1}{2(1-\hat y)^2}\mathrm{d}\alpha\wedge\mathrm{d}\hat y\right) + \tag{6.7.19}$$

$$-\frac{a - 2a\hat y + \hat y^2}{18(1-\hat y)}\mathrm{d}\alpha \wedge \sigma^{\hat 1}\wedge\sigma^{\hat 2}\,. \tag{6.7.20}$$

It can be shown without difficulty that they satisfy the expected relations

$$\mathrm{d}\omega_R + \iota_{(-2\partial_{\hat\psi} - \frac{1}{3}\partial_\alpha)} Vol(Y^{p,q}) = 0\,, \tag{6.7.21}$$

$$\mathrm{d}\omega_J + \iota_{\partial_\beta} Vol(Y^{p,q}) = 0\,. \tag{6.7.22}$$

6.8 Mass in Asymptotically $AdS_5 \times X^5$

In this Section we derive the expression for the mass in the asymptotically $AdS_5 \times Y^{p,q}$ spacetimes under examination. There has been a considerable amount of work over the years on the definition of mass and other conserved charges in general relativity. The issue becomes even subtler in the case of the definition of the mass in asymptotically AdS spaces. For example, the standard expression given in terms of a Komar integral gives a divergent result in this case and the procedure of renormalisation is ambiguous. We will follow the definition of conserved charges given by Wald and collaborators [118, 117], which provides a possible general framework for addressing this issue, and apply it to our case for the computation of the mass. Since our solutions mix, beyond the leading order, AdS and $Y^{p,q}$ coordinates, it is natural to take a ten dimensional approach for the definition of mass as it has the advantage of being relatively simple both from the conceptual and from the technical point of view. A different approach to this problem, more holographic in spirit, which uses a detailed analysis of the KK reduction of the 10 dimensional theory to AdS_5 has been recently followed in [121].

The main result of this section is to prove that, with the adopted definition of mass, the expected BPS relation

$$ML = \frac{3}{2}R, \tag{6.8.1}$$

which follows from the $\mathcal{N} = 1$ superconformal algebra on the field theory side, is satisfied.

6.8.1 Definition of charges in asymptotically $AdS_5 \times X^5$

We are dealing with an asymptotically $AdS_5 \times X^5$ spacetime, where X^5 is a compact manifold.

It is convenient to choose coordinates such that, defining a radial AdS coordinate Ω, $g_{\Omega\Omega} = L^2/\Omega^2$ and $g_{\Omega M} = 0$ for $M \neq \Omega$, M denoting a ten dimensional coordinate. We will also denote the AdS coordinates with $\mu, \nu, \ldots$ and the internal coordinates with $a, b, \ldots$. At leading order for large Ω we have

$$\mathrm{d}s^2 = \frac{L^2}{\Omega^2}\left[\mathrm{d}\Omega^2 - \mathrm{d}t^2 + \mathrm{d}\Omega_3^2\right] + L^2 \mathrm{d}s^2(Y^{p,q}). \tag{6.8.2}$$

We will keep corrections to orders Ω^{2k}, with $k = 0, 1, 2$ for the AdS part, $g_{\mu\nu}$, $k = 1, 2$ for the internal, g_{ab}, and mixed parts, $g_{\mu a}$ respectively. There are of course corrections of higher order in Ω to the background 5-form given by the volume forms on AdS_5 and $Y^{p,q}$ which we will discuss later.

In general the construction of conserved charges proceeds as follows: let us denote for the moment as φ the generic field appearing in a Lagrangian $\mathcal{L}$. The variation of the Lagrangian with respect to φ is given by

$$\delta\mathcal{L} = E(\varphi)\delta\varphi + \mathrm{d}\theta(\varphi, \delta\varphi). \tag{6.8.3}$$

where $E(\varphi)$ denotes the equations of motion. This defines θ, corresponding to the boundary term that arises from integrating by parts in order to remove derivative of $\delta\varphi$. It is a 9-form in spacetime.

We will be interested in the following asymptotic symmetry generator

$$\xi = \frac{\partial}{\partial t} \tag{6.8.4}$$

We want to identify the Hamiltonian generator $\mathcal{H}_\xi$ of this symmetry. The value that it takes on our solution will be the definition of the mass of the metric[4]. $\mathcal{H}_\xi$ is defined via its variation with respect to a generic fluctuation $\delta\phi$, obeying the linearised equations of motion in a given background obeying the full equations of motion [118]:

$$\delta\mathcal{H}_\xi = \int_{\partial\Sigma} (\delta Q_\xi - \xi \cdot \theta) \tag{6.8.5}$$

[4]We are specifying here to a particular symmetry generator since we are interested in the mass, but the same procedure con be applied to the most general asymptotic symmetry generator [118].

where Σ is a 9 dimensional submanifold of the spacetime without boundary, a "slice" corresponding to the vector field ξ. By the integral over $\partial\Sigma$ we mean a limiting process in which the integral is first taken over the boundary ∂K of a compact region inside Σ and then K approaches Σ in a suitable manner. The 8-form Q_ξ is the Noether charge of the asymptotic symmetry ξ. It has a contribution coming from the gravitational lagrangian:

$$Q^{grav}_{\alpha_1\cdots\alpha_8} = -\frac{1}{16\pi G_{10}}\nabla^b\xi^c\epsilon_{bca_1\cdots a_8}\,. \tag{6.8.6}$$

where $\epsilon = \sqrt{-\det g}\ \mathrm{d}^{10}x$ is the volume form. Also, the gravitational contribution to θ is:

$$\theta^{grav}_{a_1\cdots a_9} = \frac{1}{16\pi G_{10}}v^a\epsilon_{aa_1\cdots a_9} \tag{6.8.7}$$

with

$$v^a = \nabla^b\delta g_b{}^a - \nabla^a\delta g_b{}^b \tag{6.8.8}$$

Finally, the RR 5-form contributes both to Q_ξ and θ, giving rise to a single term in the combination $\delta Q_\xi - \xi\cdot\theta$. With our normalisation for the 5-form F_5, the final result for $\delta\mathcal{H}_\xi$ is

$$\delta\mathcal{H}_\xi = \int_{\partial\Sigma}\frac{1}{16\pi G_{10}}\left(-\delta Q^{grav}_\xi - \xi^{a1}\left(v^a\epsilon_{aa_1\cdots a_9} - 128F_{a_1\cdots a_5}\delta A_{a_6\cdots a_9}\right)\right) \tag{6.8.9}$$

where $F^{(5)} = \mathrm{d}A^{(4)}$.

Under mild assumptions [118], a necessary and sufficient condition for the existence of $\mathcal{H}_\xi$ is the integrability of the equation for $\mathcal{H}_\xi$:

$$(\delta_1\delta_2 - \delta_2\delta_1)\mathcal{H}_\xi = 0 \tag{6.8.10}$$

i.e.

$$0 = \xi\cdot[\delta_2\theta(\varphi,\delta_1\varphi) - \delta_1\theta(\varphi,\delta_2\varphi)] \tag{6.8.11}$$

When this condition is satisfied it is guaranteed that there exists an 8-form I_ξ exists whose variation is

$$\delta I_\xi = \delta Q_\xi - \xi\cdot\theta\,. \tag{6.8.12}$$

The value of the global charge associated with the asymptotic isometry generated by ξ is given by a simple "surface" integral, up to an arbitrary constant which can be determined by fixing the value of the charge for a "reference solution",

$$\mathcal{H}_\xi = \int_{\partial\Sigma} I_\xi + \mathcal{H}^0_\xi\,. \tag{6.8.13}$$

Notice that the 8-manifold $\partial\Sigma$ in the present case reduces asymptotically to $S^3\times Y_{pq}$, where S^3 is a 3-sphere of radius L/Ω inside AdS_5. The existence of $\mathcal{H}$ can be explicitly checked for a background with the asymptotic behaviour we have discussed above for the metric. The expression for θ^{grav} in our gauge is proportional to

$$\xi\cdot\theta^{grav}(\delta g) = \left(\Omega^2\delta(g^{tM}\partial_\Omega g_{aM}\sqrt{g}) - g^{MN}\sqrt{g}(\partial_\Omega\delta g_{MN} - \Gamma^P_{\Omega M}\delta g_{PN})\right)\epsilon_{t\Omega M_1\ldots M_8} \tag{6.8.14}$$

One can verify, using the asymptotic expansion for the metric given before, that $\delta_{[1}\theta(\delta_{2]}g) = 0$. The crucial fact for this result to hold is that $\delta\sqrt{g} = \frac{1}{2}g^{MN}\delta g_{MN} = \mathcal{O}(\Omega)$. This is satisfied by our BPS solutions, but can be proven to hold more generally, even for non necessarily BPS solutions of the equations of motion, given an appropriate asymptotic behaviour [94]. One can similarly verify that the contribution of the 5-form to θ is integrable.

Once we have verified the existence of $\mathcal{H}_\xi$, we can define the mass of a generic solution $\mathcal{M}$ to the equations of motion as the value of $\mathcal{H}_\xi$ on such a solution

$$M_{\mathcal{M}} \equiv H_\xi|_{\mathcal{M}} \,. \tag{6.8.15}$$

6.8.2 Calculation of mass and R-charge

We will now proceed to the calculation of the mass and R-charge for the solutions we have described in the previous sections. We are interested in the dependence of the mass M on the integration constants, C_1, C_3 and k. Therefore we will compute, $\frac{\partial M}{\partial C_i}$ and $\frac{\partial M}{\partial k}$, by putting into the formula (6.8.9), the expressions for the background given by our solutions.

Using the expressions for the leading order, first and second subleading orders for the metric and the 5-form given in 6.6 and in the Appendix one can calculate

$$\begin{aligned}
\frac{\partial M}{\partial k} &= \int_{S^3\times Y^{p,q}} \left(\frac{\partial}{\partial k} Q_\xi^{grav} - \xi\cdot\theta^k\right) = 0 \\
\frac{\partial M}{\partial C_1} &= \int_{S^3\times Y^{p,q}} \left(\frac{\partial}{\partial C_1} Q_\xi^{grav} - \xi\cdot\theta^1\right) = 2\frac{\pi L^2}{4G_5} \\
\frac{\partial M}{\partial C_3} &= \int_{S^3\times Y^{p,q}} \left(\frac{\partial}{\partial C_2} Q_\xi^{grav} - \xi\cdot\theta^2\right) = -3\frac{\pi L^2}{4G_5}
\end{aligned} \tag{6.8.16}$$

where G_5 is the 5-dimensional Planck constant $G_5 = G_{10}/Vol(Y^{p,q})$ and

$$\theta^i_{a_1\cdots a_9} = \frac{1}{16\pi G_{10}}\left[\left(\nabla^b\partial_i g_b{}^a - \nabla^a\partial_i g_b{}^b\right)\epsilon_{aa_1\cdots a_9} - 128 F_{a_1\cdots a_5}\partial_i A_{a_6\cdots a_9}\right] \tag{6.8.17}$$

with

$$\partial_i = \frac{\partial}{\partial k}\,,\ \frac{\partial}{\partial C_i}\,. \tag{6.8.18}$$

Putting everything together we conclude that:

$$M = \frac{\pi L^2}{4G_5}(2C_1 - 3C_2) = -\frac{\pi L^2 Q}{4G_5}\,, \tag{6.8.19}$$

where we have set the integration constant to zero. Some comments are in order here. First note that the 8-form to be integrated involves directions orthogonal to t and Ω. The relevant contribution from the 5-form is of the type $F^{(5)}_{t\hat{y}\bar{1}\bar{2}\bar{3}}\partial_i A^{(4)}_{\phi\hat{1}\hat{2}\hat{3}}$, which turns out to be of order Ω^0: $\partial_i A^{(4)}_{\phi\hat{1}\hat{2}\hat{3}}$ goes like Ω^2, and $F^{(5)}_{\Omega\phi\hat{1}\hat{2}\hat{3}} = \partial_\Omega A^{(4)}_{\phi\hat{1}\hat{2}\hat{3}} \sim \Omega$. On the other hand, $F^{(5)}_{t\hat{y}\bar{1}\bar{2}\bar{3}}$, the dual of the latter, goes like Ω^{-2}. Therefore the 5-form term gives a finite contribution to $\partial_i M$. The gravitational contributions to $\partial_i M$ on the other hand contain terms of order $1/\Omega^2$, and are therefore potentially divergent. However the coefficients of these terms turn out to be total derivatives in the internal coordinates: more precisely, the coefficient is proportional to $\frac{\mathrm{d}}{\mathrm{d}\hat{y}}q(\hat{y})$, therefore it gives a vanishing contribution after integrating over $\hat{y}$ between the two zeroes of $q(\hat{y})$, $\hat{y}_1$ and $\hat{y}_2$. Although here we are making reference to our BPS solutions this fact can be proven more generally [94].

Let us now proceed to verify the BPS relation between the mass M and the R-charge R. With our normalisation of the Reeb vector, the BPS relation is given by

$$ML = \frac{3}{2}R \tag{6.8.20}$$

where R is the charge which sources the KK gauge field A_R. The five dimensional equation of motion for its field strength F^R are given by

$$\tau_{RR}\,\mathrm{d}\star_5 F^R = \star_5 J^R\,. \tag{6.8.21}$$

where J^R is the one-form charge current and τ_{RR} comes from the KK reduction. Taking the integral of the equation of motion, the total charge R can be read from the flux at infinity of the field strength

$$R = \lim_{\tilde{\rho}\to\infty} \tau_{RR} \int_{S^3(\tilde{\rho})} F^R \tag{6.8.22}$$

where $S^3(\tilde{\rho})$ is the three dimensional sphere in AdS_5 at constant $t, \tilde{\rho}$. In Section 6.6 we derived

$$A^R \approx -\frac{Q}{2\tilde{\rho}^2}\mathrm{d}t \tag{6.8.23}$$

at leading order in large $\tilde{\rho}$. Following [122] we have

$$\tau_{RR} = \frac{3}{16\pi G_{10}} \int g_{\psi_R\psi_R} vol(Y^{p,q}) = \frac{1}{12\pi G_5} \tag{6.8.24}$$

where we have used $g_{\psi_R\psi_R} = \frac{4}{9}$ as can be seen from (6.6.17) and (6.6.18). We can now explicitly write the value of the total R charge

$$R = -\frac{QL^3}{12\pi G_5} Vol(S^3) = \frac{2}{3}ML \tag{6.8.25}$$

which satisfies the expected relation.

Let us mention that we have also computed M using a 5-dimensional definition involving the intrinsic 5-dimensional Weyl tensor due to Ashtekar and collaborators [116] and rederived in [117],

$$\mathcal{H}_\xi = M = -\frac{1}{8\pi G_5}\int_{S^3} \tilde{E}_{tt} vol(S^3) \tag{6.8.26}$$

where

$$\tilde{E}_{tt} = \frac{1}{2}\Omega^{-2}\tilde{C}_{\Omega t t \Omega}\,. \tag{6.8.27}$$

where $\tilde{C}_{abcd}$ is the Weyl tensor of the unphysical metric $\tilde{g} = \Omega^2 g$. Beyond leading order AdS_5 and $Y^{p,q}$ coordinates mix, so, in general the metric on the deformed AdS_5 depends on the choice of the 5-dimensional slice inside the 10-dimensional manifold. The calculation, done using our explicit form of the perturbed metric and allowing a slice dependence on the internal coordinates, actually reveals that the slice dependence drops out in the Weyl tensor and gives the correct result for the mass, as in the previous 10-dimensional computation. The degree of generality of this result is under investigation [94].

6.9 Observations and discussion

In this paper we have performed an asymptotic, large distance, analysis of 1/2 BPS states in IIB supergravity $AdS_5 \times Y^{p,q}$. The corresponding differential equations are the same as those found in [17], where 1/8 BPS states of IIB supergravity on $AdS_5 \times S^5$ were analysed. The difference resides in the boundary conditions, here we require solutions which are asymptotic to $AdS_5 \times Y^{p,q}$. They carry non trivial charges under the asymptotic isometries which are dual to the R-charge and one $U(1)$ flavour charge of the quiver gauge theories. We have shown that the charges are consistent with the holographic principle which in this case relates $\mathcal{N} = 1$ quiver gauge theories to gravity on $AdS_5 \times Y^{p,q}$. These geometries are therefore the exact analog of those found in [12] for the maximally supersymmetric case.

In the course of the analysis we had to cope with the problem of defining the mass of the states in the asymptotically $AdS_5 \times Y^{p,q}$ spacetime. We adopted a ten dimensional approach, which uses the definition of charges given by Wald and collaborators. It gives a finite (and correct) result. We had indications, however, that at least for our backgrounds an expression due to Ashtekar and Das [116, 117], which involves the intrinsic Weyl tensor in the deformed AdS_5, also gives the correct result. This leads one to ask in the finiteness of the Wald et al. expression can be established in more general terms without relying on a particular form of background. That is, one would like to prove in general, assuming just that the equations of motion hold and with the asymptotic behaviour of the fields implied by AdS/CFT correspondence, that potentially divergent terms are total derivatives in the internal compact manifold. Similarly, it would be interesting to see under which circumstances the ten dimensional approach finally coincides with the 5-dimensional one of Ashtekar et al. This is still an problem at the moment.

6.10 Second order solutions

We give here the complete expression for the second order solutions

$$\rho_1^{(2)}(\hat{y}) = -(L^8(-1+c\hat{y})^2((-4+4ac^2+27k)^2(-80+496c\hat{y}-584\,c^2\hat{y}^2-2696c^3\hat{y}^3+11666c^4\hat{y}^4$$
$$-19494c^5\hat{y}^5+16281c^6\hat{y}^6-6696c^7\hat{y}^7+1080c^8\hat{y}^8+a^3c^6(-65+72c\hat{y}+20c^2\hat{y}^2)+$$
$$+a^2c^4(50+82c\hat{y}+159c^2\hat{y}^2-752c^3\hat{y}^3+380c^4\hat{y}^4)+ac^2(-40-56c\hat{y}+756c^2\hat{y}^2-3572c^3\hat{y}^3+6449c^4\hat{y}^4+$$
$$-4536c^5\hat{y}^5+1080c^6\hat{y}^6))+$$
$$-8(-1+ac^2)(-4+4ac^2+27k)(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2(-20+c\hat{y}(4-84C_1)-264c^3\hat{y}^3(1+4C_1)+$$
$$+120c^4\hat{y}^4(1+4C_1)+c^2\hat{y}^2(133+552C_1)+ac^2(5-60C_1+20c^2\hat{y}^2(1+3C_1)+2c(\hat{y}+54\hat{y}C_1)))+$$
$$+32(-1+ac^2)^2(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2(-10+c\hat{y}(2-84C_1)+10c^4\hat{y}^4(-1+10C_1+27C_1^2)+$$
$$+ac^2(-2c\hat{y}(17+41C_1)+10c^2(\hat{y}+3\hat{y}C_1)^2+5(3+8C_1-6C_2))-2c^3\hat{y}^3(1+143C_1+270C_1^2+30C_2)+$$
$$c^2\hat{y}^2(29+252C_1+180C_1^2+90C_2))))/(4320(-1+ac^2)^2(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^3)$$

$$\rho_3^{(2)}(\hat{y}) = \rho_1^{(2)}(\hat{y}) + k^{(2)}(\hat{y})$$

$$k^{(2)}(\hat{y}) = (L^6(L-cL\hat{y})^2(((1-c\hat{y})(16(-1+ac^2)^2(-1+c\hat{y})(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2$$
$$(11+4ac^2-30c\hat{y}+15c^2\hat{y}^2)-4(-1+ac^2)(-4+4ac^2+27k)(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)$$
$$(-44+284c\hat{y}-576c^2\hat{y}^2+426c^3\hat{y}^3-90c^4\hat{y}^4-9c^5\hat{y}^5+a^2c^4(-14+5c\hat{y})+2ac^2(-34+103c\hat{y}-72c^2\hat{y}^2+$$
$$+12c^3\hat{y}^3))+(-4+4ac^2+27k)^2(a^3c^6(11+c\hat{y})+6a^2c^4(-17+25c\hat{y}-21c^2\hat{y}^2+7c^3\hat{y}^3)+3ac^2(-36+220c\hat{y}+$$
$$-324c^2\hat{y}^2+200c^3\hat{y}^3-51c^4\hat{y}^4+3c^5\hat{y}^5)+2(-22+202c\hat{y}-666c^2\hat{y}^2+894c^3\hat{y}^3-531c^4\hat{y}^4+$$
$$+117c^5\hat{y}^5))))/((1-ac^2)(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2)-12c(4a^2c^3(-4-8C_1+7c(\hat{y}+2\hat{y}C_1)+6C_2)+$$
$$+\hat{y}(27k(2-8c\hat{y}+3c^3\hat{y}^3)(1+2C_1)-4c\hat{y}(-3+2c\hat{y})(-4-8C_1+7c(\hat{y}+2\hat{y}C_1)+6C_2))+$$
$$+ac(4c(-7+27k)\hat{y}(1+2C_1)+56c^4\hat{y}^4(1+2C_1)-4c^3\hat{y}^3(29+58C_1-12C_2)+8(2+4C_1-3C_2)+$$
$$-3c^2\hat{y}^2(-16+9k-32C_1+18kC_1+24C_2)))))/(648(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2)$$

$$t^{(2)}(\hat{y}) = (L^8(-1+c\hat{y})^2(2187k^2(-1+c\hat{y})^2(-2+14c\hat{y}-9c^2\hat{y}^2+ac^2(-7+4c\hat{y}))+$$
$$-216(-1+ac^2)k(-1+c\hat{y})(2+2a^2c^4+27c^2\hat{y}^2-45c^3\hat{y}^3+$$
$$+18c^4\hat{y}^4+ac^2(-13+9c\hat{y}))+16(-1+ac^2)^2(-9(-1+c\hat{y})^2(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)+$$
$$+8(-1+c\hat{y})(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^2C_1+1/1-ac^2(3(-(-1+ac^2)^3C_2-3(-1+ac^2)^2(-1+c\hat{y})^2C_2+$$
$$+27(-1+c\hat{y})^6C[3]-(-1+ac^2)(-1+c\hat{y})^3(-4+4ac^2+(9-9c\hat{y})C_2))))))/(216(1-ac^2)$$
$$(2+ac^2-6c\hat{y}+3c^2\hat{y}^2)^3)$$

Chapter 7

Conclusions and perspectives

The AdS/CFT correspondence is a very promising tool for the understanding of quantum aspects of gravity and dynamical aspects of gauge theories. In this Thesis we mainly focused on gravitational aspects of the correspondence. In particular, in Chapter 4 we have shown how a class of gravitational singularities and pathologies such as closed timelike con be interpreted in the CFT and how unitarity of the CFT provides a mechanism for their resolution.

Later on we have shown how it is possible to study the correspondence in less supersymmetric case. We have first studied a class of $1/8$ BPS asymptotically $AdS_5 \times S^5$ geometries. They are described by a set of four functions which are defined on a half space and obey a set of non linear elliptic differential equations. Solutions are thus specified by boundary conditions. It has been possible to correctly identify the boundary conditions at infinity which give rise to asymptotically regular solutions. Unfortunately, although a discussion of regularity of the geometries on the other boundary, a 2-plane, is not hard, it looses part of its interest since it is not possible to have control on regularity of the full solutions in the bulk. This is mainly due to the complexity of the elliptic equations involved and we do not exclude that it is possible to find a solution to this problem in a reasonably short time. It would be helpful to identify some already known regular solutions which are inside the class we have studied: looking at their structure, one can and try deduce which conditions must be generically satisfied to guarantee regularity. Once this has been achieved, a second step would be to identify singular spacetimes in this class in order to see if the dual CFT picture has something to tell us about the way Strings resolve such singularities, as happened for example in the half-BPS case.

A natural extension of the work in Chapter 5 has lead us to study the half BPS sector of more general formulations of the AdS/CFT correspondence. A certain class of $\mathcal{N} = 1$ quiver gauge theories are conjectured to be dual to Type IIB String theory on $AdS_5 \times Y^{p,q}$ backgrounds. $Y^{p,q}$ are Sasaki Einstein manifolds and the product with AdS_5 gives $1/4$ BPS Type IIB solutions which are inside the class of $1/8$ BPS geometries we have considered. In this setting it has been possible to perform a construction which closely recalls the original

LLM picture: an interesting subset of the half supersymmetric sector of this correspondence has been considered and dual geometries have been constructed. Unfortunately, things are more complicated in this case and a nice picture as in LLM is not possible at the moment. Again, boundary conditions at infinity can be correctly identified and lead to asymptotically regular spacetimes. Some subtleties arise in this case in the calculation the mass, or energy of the excitations. A direct Kaluza Klein reduction over $Y^{p,q}$ to five dimensional gauged Supergravity is not possible because non abelian metric gauge fields source massive excitations of the AdS_5 metric [115]. A full 10 dimensional approach is due here: we applied a general procedure due to Wald and collaborators to our case. In this setting, it would be interesting to ask a series of questions analogous to the ones we posed for the previous case, namely if it is possible to identify regular solutions and, once this is done, if it is possible to identify some mechanism for the resolution of singularity. Moreover if we were able to complete this task at least partially in both the sectors, a comparison of the two would make possible to recognise the generality of the adopted mechanisms for the resolution of singularities. Another interesting point concerning asymptotically $AdS_5 \times Y^{p,q}$ spacetimes is related to the possibility of counting mesonic and baryonic states: this problem has been approached and solved recently in the dual field theories [123], and the correct mapping to the Supergravity side would be highly desirable.

Appendix A

Conventions

We set up our conventions for the wedge product of 1-forms

$$\alpha_1 \wedge \cdots \wedge \alpha_n = \frac{1}{n!} \sum_i \sigma(i) \alpha_{i(1)} \otimes \cdots \otimes \alpha_{i(n)} \tag{A.0.1}$$

where the sum is over the $n!$ permutations i and $\sigma(i)$ is the parity of the permutation.

An n-form α in a d dimensional space ($\alpha \in \Lambda_n$) is given by

$$\alpha = \bar{\alpha}_{\mu_1 \cdots \mu_n} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} = \frac{1}{n!} \alpha_{\mu_1 \cdots \mu_n} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \tag{A.0.2}$$

with $\alpha_{\mu_1 \cdots \mu_n}$ the complete antisymmetrisation of $\bar{\alpha}_{\mu_1 \cdots \mu_n}$.

When a metric is present we can introduce the Hodge dual

$$\star : \Lambda_n \to \Lambda_{d-n} \tag{A.0.3}$$

Given a d-bein of the metric $\{e^m\}_{m=1,\cdots d}$,

$$\star\, e^{m_1} \wedge \cdots \wedge e^{m_n} = \frac{1}{(d-n)!} \epsilon^{m_1, \ldots m_n, m_{n+1}, \cdots, m_d} e_{m_{n+1}} \wedge \cdots \wedge e_{m_d} \tag{A.0.4}$$

where indices are lowered with the tangent space metric. From this definition it follows that

$$\begin{aligned} \star\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} &= \star g^{\mu_1 \mu_1'} \cdots g^{\mu_n \mu_n'} e_{m_1 \mu_1'} \cdots e_{m_n \mu_n'} e^{m_1} \wedge \cdots \wedge e^{m_n} = \\ &= g^{\mu_1 \mu_1'} \cdots g^{\mu_n \mu_n'} e_{m_1 \mu_1'} \cdots e_{m_n \mu_n'} \frac{1}{(d-n)!} \epsilon^{m_1, \cdots m_n, m_{n+1}, \cdots, m_d} e_{m_{n+1}} \wedge \cdots \wedge e_{m_d} = \\ &= \frac{1}{(d-n)!} \sqrt{g} g^{\mu_1 \mu_1'} \cdots g^{\mu_n \mu_n'} \epsilon_{\mu_1', \cdots \mu_n', \mu_{n+1}', \cdots, \mu_d'} \mathrm{d}x^{\mu_{n+1}'} \wedge \cdots \wedge \mathrm{d}x^{\mu_d'} . \end{aligned} \tag{A.0.5}$$

The exterior derivative of a 1-form is defined by

$$\beta = \mathrm{d}\alpha = \partial_\mu \alpha_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2} (\partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{A.0.6}$$

or in terms of components $\beta_{\mu\nu} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu$. The generalisation to any n-form is given by

$$\beta = \mathrm{d}\alpha = \frac{1}{n!}\partial_\mu \alpha_{\nu_1\cdots\nu_n} \mathrm{d}x^\mu \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_n} =$$
$$= \frac{1}{(n+1)!}\partial_{[\mu}\alpha_{\nu_1\cdots\nu_n]}\mathrm{d}x^\mu \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_n} = \frac{1}{(n+1)!}\beta_{\nu_1\cdots\nu_n}\mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_n} \quad \text{(A.0.7)}$$

where now $\beta_{\mu\nu_1\cdots\nu_n} = \partial_{[\mu}\alpha_{\nu_1\cdots\nu_n]}$ and square brackets indicate antisymmetrisation without normalisation.

The torsionless spin connection 1-form is defined by the structure equation:

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0\,. \quad \text{(A.0.8)}$$

Requiring metricity of the connection

$$\omega_{ab} = -\omega_{ba} \quad \text{(A.0.9)}$$

allows us to explicitly express ω_{ab} in terms of the d-bein (E_a are the inverse d-bein vector fields, defined by $e^a \cdot E_b = \delta^a{}_a$),

$$\omega_{ab} = -\mathrm{d}e_a \cdot E_b + \mathrm{d}e_b \cdot E_a + \frac{1}{2}\left(e^c \cdot [E_a, E_b]\right) e_c =$$
$$= \Big[-\frac{1}{2}\left(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}\right) E^\nu{}_b + \frac{1}{2}\left(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}\right) E^\nu{}_a +$$
$$+ \frac{1}{2} e^c{}_\rho \left(E^\nu{}_a \partial_\nu E^\rho{}_b - E^\nu{}_b \partial_\nu E^\rho{}_a\right) e_{c\mu}\Big] \mathrm{d}x^\mu =$$
$$= \Big[-\frac{1}{2}\left(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}\right) E^\nu{}_b + \frac{1}{2}\left(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}\right) E^\nu{}_a +$$
$$- \frac{1}{2} E^\nu{}_a \left(\partial_\nu e^c{}_\rho - \partial_\rho e^c{}_\nu\right) E^\rho{}_b e_{c\mu}\Big] \mathrm{d}x^\mu =$$
$$= -\mathrm{d}e_a \cdot E_b + \mathrm{d}e_b \cdot E_a - \left(E_a \cdot \mathrm{d}e^c \cdot E_b\right) e_c \quad \text{(A.0.10)}$$

where in going from the second to the third line we have used

$$0 = \partial_\mu \eta_{ab} = \partial_\mu \left(e_{a\nu} E^\nu{}_b\right) = \left(\partial_\mu e_{a\nu}\right) E^\nu{}_b + e_{a\nu}\left(\partial_\mu E^\nu{}_b\right)\,. \quad \text{(A.0.11)}$$

This is an explicit realisation of the identity

$$V \cdot \mathrm{d}\alpha \cdot W = \frac{1}{2}\mathrm{d}(\alpha \cdot W) \cdot V - \frac{1}{2}\mathrm{d}(\alpha \cdot V) \cdot W - \frac{1}{2}\alpha \cdot [V, W]\,, \quad \text{(A.0.12)}$$

which holds for any one form α and any pair of vector fields V, W.

The covariant derivative of a spinor is given by

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{4}\omega_{ab\mu}\Gamma^a\Gamma^b\psi\,. \quad \text{(A.0.13)}$$

Group manifolds

Consider a Lie algebra of vector fields on a d-dimensional group manifold. It is a d dimensional vector space of vector fields satisfying

$$[E_a, E_b] = f_{ab}{}^c E_c\,. \tag{A.0.14}$$

The exterior derivative of the dual one forms is given by

$$\mathrm{d}e^c = \frac{1}{2}\alpha_{ab}{}^c e^a \wedge e^b \tag{A.0.15}$$

These are the Maurer Cartan 1-forms. Indeed, we have

$$E_a \cdot \mathrm{d}e^c \cdot E_b = \frac{1}{2}\alpha_{ab}{}^c \tag{A.0.16}$$

and according to (A.0.12)

$$E_a \cdot \mathrm{d}e^c \cdot E_b = -\frac{1}{2}e^c \cdot [E_a, E_b] \tag{A.0.17}$$

which give

$$\alpha_{ab}{}^c = -f_{ab}{}^c\,. \tag{A.0.18}$$

The Lie derivative of a 1-form is defined by

$$(\mathcal{L}_J\omega)\cdot K = \partial_K(\omega\cdot K) - \omega\cdot[J,K] \tag{A.0.19}$$

and thus

$$\mathcal{L}_{E_a} e^c = -f_{ab}{}^c e^b \tag{A.0.20}$$

Taking these e^a as the d-bein, the spin connection on the group manifold is given by

$$\omega_{abc} = \frac{1}{2}(-\alpha_{cba} + \alpha_{cab} + f_{abc}) = \frac{1}{2}(f_{cba} - f_{cab} + f_{abc})\,. \tag{A.0.21}$$

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Bibliography

[1] M. B. Green, J. H. Schwarz, and E. Witten, "Superstring Theory,". Cambridge, Uk: Univ. Pr. (1987) (Cambridge Monographs On Mathematical Physics).

[2] J. Polchinski, "String theory,". Cambridge, UK: Univ. Pr. (1998).

[3] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," *Nucl. Phys.* **B72** (1974) 461.

[4] J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges," *Phys. Rev. Lett.* **75** (1995) 4724–4727, `hep-th/9510017`.

[5] E. Witten, "Bound states of strings and p-branes," *Nucl. Phys.* **B460** (1996) 335–350, `hep-th/9510135`.

[6] I. R. Klebanov, "World-volume approach to absorption by non-dilatonic branes," *Nucl. Phys.* **B496** (1997) 231–242, `hep-th/9702076`.

[7] S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, "String theory and classical absorption by three-branes," *Nucl. Phys.* **B499** (1997) 217–240, `hep-th/9703040`.

[8] S. S. Gubser and I. R. Klebanov, "Absorption by branes and Schwinger terms in the world volume theory," *Phys. Lett.* **B413** (1997) 41–48, `hep-th/9708005`.

[9] G. 't Hooft, "Dimensional reduction in quantum gravity," `gr-qc/9310026`.

[10] L. Susskind, "The World as a hologram," *J. Math. Phys.* **36** (1995) 6377–6396, `hep-th/9409089`.

[11] J. D. Bekenstein, "Entropy bounds and black hole remnants," *Phys. Rev.* **D49** (1994) 1912–1921, `gr-qc/9307035`.

[12] H. Lin, O. Lunin, and J. Maldacena, "Bubbling AdS space and 1/2 BPS geometries," *JHEP* **10** (2004) 025, `hep-th/0409174`.

[13] J. L. Petersen, "Introduction to the Maldacena conjecture on AdS/CFT," *Int. J. Mod. Phys.* **A14** (1999) 3597–3672, `hep-th/9902131`.

[14] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183–386, hep-th/9905111.

[15] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS/CFT correspondence," hep-th/0201253.

[16] G. Milanesi and M. O'Loughlin, "Singularities and closed time-like curves in type IIB 1/2 BPS geometries," *JHEP* **09** (2005) 008, hep-th/0507056.

[17] E. Gava, G. Milanesi, K. S. Narain, and M. O'Loughlin, "1/8 BPS states in AdS/CFT," hep-th/0611065.

[18] E. Gava, K. S. Narain, G. Milanesi, and M. O'Loughlin, "Half BPS states in $AdS_5 \times Y^{p,q}$," arXiv:0709.2856 [hep-th].

[19] G. Papadopoulos and D. Tsimpis, "The holonomy of IIB supercovariant connection," *Class. Quant. Grav.* **20** (2003) L253, hep-th/0307127.

[20] J. P. Gauntlett and S. Pakis, "The geometry of D = 11 Killing spinors. ((T)," *JHEP* **04** (2003) 039, hep-th/0212008.

[21] E. J. Hackett-Jones and D. J. Smith, "Type IIB Killing spinors and calibrations," *JHEP* **11** (2004) 029, hep-th/0405098.

[22] M. F. Sohnius, "Introducing Supersymmetry," *Phys. Rept.* **128** (1985) 39–204.

[23] J. D. Lykken, "Introduction to supersymmetry," hep-th/9612114.

[24] S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, "Superspace, or one thousand and one lessons in supersymmetry," *Front. Phys.* **58** (1983) 1–548, hep-th/0108200.

[25] V. K. Dobrev and V. B. Petkova, "All Positive Energy Unitary Irreducible Representations of Extended Conformal Supersymmetry," *Phys. Lett.* **B162** (1985) 127–132.

[26] V. K. Dobrev and V. B. Petkova, "ON THE GROUP THEORETICAL APPROACH TO EXTENDED CONFORMAL SUPERSYMMETRY: CLASSIFICATION OF MULTIPLETS," *Lett. Math. Phys.* **9** (1985) 287–298.

[27] T. Banks, M. R. Douglas, G. T. Horowitz, and E. J. Martinec, "AdS dynamics from conformal field theory," hep-th/9808016.

[28] M. Luscher and G. Mack, "Global Conformal Invariance in Quantum Field Theory," *Commun. Math. Phys.* **41** (1975) 203–234.

[29] G. T. Horowitz and H. Ooguri, "Spectrum of large N gauge theory from supergravity," *Phys. Rev. Lett.* **80** (1998) 4116–4118, `hep-th/9802116`.

[30] A. V. Ryzhov, "Quarter BPS operators in N = 4 SYM," *JHEP* **11** (2001) 046, `hep-th/0109064`.

[31] L. Andrianopoli and S. Ferrara, "Short and long SU(2,2/4) multiplets in the AdS/CFT correspondence," *Lett. Math. Phys.* **48** (1999) 145–161, `hep-th/9812067`.

[32] L. Andrianopoli, S. Ferrara, E. Sokatchev, and B. Zupnik, "Shortening of primary operators in N-extended SCFT(4) and harmonic-superspace analyticity," *Adv. Theor. Math. Phys.* **4** (2000) 1149–1197, `hep-th/9912007`.

[33] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, `hep-th/9711200`.

[34] M. Gunaydin and N. Marcus, "The Spectrum of the s**5 Compactification of the Chiral N=2, D=10 Supergravity and the Unitary Supermultiplets of U(2, 2/4)," *Class. Quant. Grav.* **2** (1985) L11.

[35] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen, "The Mass Spectrum of Chiral N=2 D=10 Supergravity on S**5," *Phys. Rev.* **D32** (1985) 389.

[36] B. de Wit, "Supergravity," `hep-th/0212245`.

[37] P. Breitenlohner and D. Z. Freedman, "Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity," *Phys. Lett.* **B115** (1982) 197.

[38] S. Corley, A. Jevicki, and S. Ramgoolam, "Exact correlators of giant gravitons from dual N = 4 SYM theory," *Adv. Theor. Math. Phys.* **5** (2002) 809–839, `hep-th/0111222`.

[39] D. Berenstein, "A toy model for the AdS/CFT correspondence," *JHEP* **07** (2004) 018, `hep-th/0403110`.

[40] M. M. Caldarelli and P. J. Silva, "Giant gravitons in AdS/CFT. I: Matrix model and back reaction," *JHEP* **08** (2004) 029, `hep-th/0406096`.

[41] N. R. Constable *et al.*, "PP-wave string interactions from perturbative Yang-Mills theory," *JHEP* **07** (2002) 017, `hep-th/0205089`.

[42] S. E. Vazquez, "BPS condensates, matrix models and emergent string theory," *JHEP* **01** (2007) 101, `hep-th/0607204`.

[43] A. Hashimoto, S. Hirano, and N. Itzhaki, "Large branes in AdS and their field theory dual," *JHEP* **08** (2000) 051, `hep-th/0008016`.

[44] Y. Takayama and A. Tsuchiya, "Complex matrix model and fermion phase space for bubbling AdS geometries," *JHEP* **10** (2005) 004, hep-th/0507070.

[45] I. R. Klebanov, "String theory in two-dimensions," hep-th/9108019.

[46] E. Brezin, C. Itzykson, G. Parisi, and J. B. Zuber, "Planar Diagrams," *Commun. Math. Phys.* **59** (1978) 35.

[47] V. Balasubramanian, J. de Boer, V. Jejjala, and J. Simon, "The library of Babel: On the origin of gravitational thermodynamics," *JHEP* **12** (2005) 006, hep-th/0508023.

[48] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, hep-th/9802150.

[49] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Phys. Lett.* **B428** (1998) 105–114, hep-th/9802109.

[50] N. V. Suryanarayana, "Half-BPS giants, free fermions and microstates of superstars," *JHEP* **01** (2006) 082, hep-th/0411145.

[51] J. McGreevy, L. Susskind, and N. Toumbas, "Invasion of the giant gravitons from anti-de Sitter space," *JHEP* **06** (2000) 008, hep-th/0003075.

[52] M. T. Grisaru, R. C. Myers, and O. Tafjord, "SUSY and Goliath," *JHEP* **08** (2000) 040, hep-th/0008015.

[53] M. Blau, "Killing spinors and SYM on curved spaces," *JHEP* **11** (2000) 023, hep-th/0005098.

[54] C. Bär, "Real Killing spinors and holonomy," *Communications in Mathematical Physics* **154** (1993) 509–521.

[55] M. Blau, J. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos, "A new maximally supersymmetric background of IIB superstring theory," *JHEP* **01** (2002) 047, hep-th/0110242.

[56] R. C. Myers and O. Tafjord, "Superstars and giant gravitons," *JHEP* **11** (2001) 009, hep-th/0109127.

[57] A. Buchel, "Coarse-graining 1/2 BPS geometries of type IIB supergravity," hep-th/0409271.

[58] P. Horava and P. G. Shepard, "Topology changing transitions in bubbling geometries," *JHEP* **02** (2005) 063, hep-th/0502127.

[59] S. S. Gubser and J. J. Heckman, "Thermodynamics of R-charged black holes in AdS(5) from effective strings," *JHEP* **11** (2004) 052, hep-th/0411001.

[60] D. Bak, S. Siwach, and H.-U. Yee, "1/2 BPS geometries of M2 giant gravitons," hep-th/0504098.

[61] V. Balasubramanian, V. Jejjala, and J. Simon, "The library of Babel," hep-th/0505123.

[62] S. S. Gubser, "Curvature singularities: The good, the bad, and the naked," *Adv. Theor. Math. Phys.* 4 (2002) 679–745, hep-th/0002160.

[63] R. C. Myers, "Stress tensors and Casimir energies in the AdS/CFT correspondence," *Phys. Rev.* **D60** (1999) 046002, hep-th/9903203.

[64] G. T. Horowitz and R. C. Myers, "The value of singularities," *Gen. Rel. Grav.* **27** (1995) 915–919, gr-qc/9503062.

[65] S. W. Hawking, "The Chronology protection conjecture," *Phys. Rev.* **D46** (1992) 603–611.

[66] J. C. Breckenridge, R. C. Myers, A. W. Peet, and C. Vafa, "D-branes and spinning black holes," *Phys. Lett.* **B391** (1997) 93–98, hep-th/9602065.

[67] M. M. Caldarelli, D. Klemm, and P. J. Silva, "Chronology protection in anti-de Sitter," hep-th/0411203.

[68] M. Boni and P. J. Silva, "Revisiting the D1/D5 system or bubbling in AdS(3)," hep-th/0506085.

[69] M. Blau, M. Borunda, M. O'Loughlin, and G. Papadopoulos, "The universality of Penrose limits near space-time singularities," *JHEP* **07** (2004) 068, hep-th/0403252.

[70] D. Berenstein, J. M. Maldacena, and H. Nastase, "Strings in flat space and pp waves from N = 4 super Yang Mills," *JHEP* **04** (2002) 013, hep-th/0202021.

[71] Y. Takayama and K. Yoshida, "Bubbling 1/2 BPS geometries and Penrose limits," hep-th/0503057.

[72] G. W. Gibbons and C. A. R. Herdeiro, "Supersymmetric rotating black holes and causality violation," *Class. Quant. Grav.* **16** (1999) 3619–3652, hep-th/9906098.

[73] C. A. R. Herdeiro, "Special properties of five dimensional BPS rotating black holes," *Nucl. Phys.* **B582** (2000) 363–392, hep-th/063.

[74] L. Jarv and C. V. Johnson, "Rotating black holes, closed time-like curves, thermodynamics, and the enhancon mechanism," *Phys. Rev.* **D67** (2003) 066003, hep-th/0211097.

[75] N. Drukker, "Supertube domain-walls and elimination of closed time-like curves in string theory," *Phys. Rev.* **D70** (2004) 084031, hep-th/0404239.

[76] L. Cornalba and M. S. Costa, "Unitarity in the presence of closed timelike curves," hep-th/0506104.

[77] J. T. Liu, D. Vaman, and W. Y. Wen, "Bubbling 1/4 BPS solutions in type IIB and supergravity reductions on S**n x S**n," hep-th/0412043.

[78] Z. W. Chong, H. Lu, and C. N. Pope, "BPS geometries and AdS bubbles," *Phys. Lett.* **B614** (2005) 96–103, hep-th/0412221.

[79] A. Donos, "A description of 1/4 BPS configurations in minimal type IIB SUGRA," hep-th/0606199.

[80] A. Donos, "BPS states in type IIB SUGRA with $SO(4) \times SO(2)_{gauged}$ symmetry," hep-th/0610259.

[81] B. Chen *et al.*, "Bubbling AdS and droplet descriptions of BPS geometries in IIB supergravity," arXiv:0704.2233 [hep-th].

[82] D. Berenstein, "Large N BPS states and emergent quantum gravity," *JHEP* **01** (2006) 125, hep-th/0507203.

[83] A. Mikhailov, "Giant gravitons from holomorphic surfaces," *JHEP* **11** (2000) 027, hep-th/0010206.

[84] G. Mandal and N. V. Suryanarayana, "Counting 1/8-BPS dual-giants," hep-th/0606088.

[85] I. Biswas, D. Gaiotto, S. Lahiri, and S. Minwalla, "Supersymmetric states of N = 4 Yang-Mills from giant gravitons," hep-th/0606087.

[86] N. Kim, "AdS(3) solutions of IIB supergravity from D3-branes," *JHEP* **01** (2006) 094, hep-th/0511029.

[87] J. P. Gauntlett, N. Kim, and D. Waldram, "Supersymmetric AdS(3), AdS(2) and bubble solutions," *JHEP* **04** (2007) 005, hep-th/0612253.

[88] O. Lunin, "On gravitational description of Wilson lines," *JHEP* **06** (2006) 026, hep-th/0604133.

[89] J. P. Gauntlett, D. Martelli, S. Pakis, and D. Waldram, "G-structures and wrapped NS5-branes," *Commun. Math. Phys.* **247** (2004) 421–445, hep-th/0205050.

[90] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis, and H. S. Reall, "All supersymmetric solutions of minimal supergravity in five dimensions," *Class. Quant. Grav.* **20** (2003) 4587–4634, hep-th/0209114.

[91] J. B. Gutowski, D. Martelli, and H. S. Reall, "All supersymmetric solutions of minimal supergravity in six dimensions," *Class. Quant. Grav.* **20** (2003) 5049–5078, hep-th/0306235.

[92] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, "Supersymmetric AdS(5) solutions of M-theory," *Class. Quant. Grav.* **21** (2004) 4335–4366, hep-th/0402153.

[93] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, "Sasaki-Einstein metrics on S(2) x S(3)," *Adv. Theor. Math. Phys.* **8** (2004) 711–734, hep-th/0403002.

[94] E. Gava, G. Milanesi, K. Narain, and M. O'Loughlin, "In preparation,".

[95] K. Skenderis and M. Taylor, "Kaluza-Klein holography," *JHEP* **05** (2006) 057, hep-th/0603016.

[96] L. Grant, L. Maoz, J. Marsano, K. Papadodimas, and V. S. Rychkov, "Minisuperspace quantization of 'bubbling AdS' and free fermion droplets," hep-th/0505079.

[97] L. Maoz and V. S. Rychkov, "Geometry quantization from supergravity: The case of 'bubbling AdS'," *JHEP* **08** (2005) 096, hep-th/0508059.

[98] G. W. Gibbons, "Spectral Asymmetry and Quantum Field Theory in Curved Space-time," *Ann. Phys.* **125** (1980) 98.

[99] N. Hitchin, "Harmonic Spinors," *Adv. Math* **14** (1974) 1.

[100] S. Kachru and E. Silverstein, "4d conformal theories and strings on orbifolds," *Phys. Rev. Lett.* **80** (1998) 4855–4858, hep-th/9802183.

[101] M. Bershadsky, Z. Kakushadze, and C. Vafa, "String expansion as large N expansion of gauge theories," *Nucl. Phys.* **B523** (1998) 59–72, hep-th/9803076.

[102] M. Bershadsky and A. Johansen, "Large N limit of orbifold field theories," *Nucl. Phys.* **B536** (1998) 141–148, hep-th/9803249.

[103] P. G. O. Freund and M. A. Rubin, "Dynamics of Dimensional Reduction," *Phys. Lett.* **B97** (1980) 233–235.

[104] A. Kehagias, "New type IIB vacua and their F-theory interpretation," *Phys. Lett.* **B435** (1998) 337–342, hep-th/9805131.

[105] I. R. Klebanov and E. Witten, "Superconformal field theory on threebranes at a Calabi-Yau singularity," *Nucl. Phys.* **B536** (1998) 199–218, hep-th/9807080.

[106] D. R. Morrison and M. R. Plesser, "Non-spherical horizons. I," *Adv. Theor. Math. Phys.* **3** (1999) 1–81, hep-th/9810201.

[107] D. Martelli and J. Sparks, "Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals," *Commun. Math. Phys.* **262** (2006) 51–89, hep-th/0411238.

[108] S. Benvenuti, S. Franco, A. Hanany, D. Martelli, and J. Sparks, "An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals," *JHEP* **06** (2005) 064, hep-th/0411264.

[109] D. Berenstein, C. P. Herzog, and I. R. Klebanov, "Baryon spectra and AdS/CFT correspondence," *JHEP* **06** (2002) 047, hep-th/0202150.

[110] D. Arean, D. E. Crooks, and A. V. Ramallo, "Supersymmetric probes on the conifold," *JHEP* **11** (2004) 035, hep-th/0408210.

[111] F. Canoura, J. D. Edelstein, L. A. P. Zayas, A. V. Ramallo, and D. Vaman, "Supersymmetric branes on AdS(5) x Y**(p,q) and their field theory duals," *JHEP* **03** (2006) 101, hep-th/0512087.

[112] S. S. Gubser and I. R. Klebanov, "Baryons and domain walls in an N = 1 superconformal gauge theory," *Phys. Rev.* **D58** (1998) 125025, hep-th/9808075.

[113] D. Martelli and J. Sparks, "Dual giant gravitons in Sasaki-Einstein backgrounds," *Nucl. Phys.* **B759** (2006) 292–319, hep-th/0608060.

[114] A. Basu and G. Mandal, "Dual giant gravitons in AdS(m) x Y**n (Sasaki-Einstein)," hep-th/0608093.

[115] A. Buchel and J. T. Liu, "Gauged supergravity from type IIB string theory on Y(p,q) manifolds," *Nucl. Phys.* **B771** (2007) 93–112, hep-th/0608002.

[116] A. Ashtekar and S. Das, "Asymptotically anti-de Sitter space-times: Conserved quantities," *Class. Quant. Grav.* **17** (2000) L17–L30, hep-th/9911230.

[117] S. Hollands, A. Ishibashi, and D. Marolf, "Comparison between various notions of conserved charges in asymptotically AdS-spacetimes," *Class. Quant. Grav.* **22** (2005) 2881–2920, hep-th/0503045.

[118] R. M. Wald and A. Zoupas, "A General Definition of Conserved Quantities in General Relativity and Other Theories of Gravity," *Phys. Rev.* **D61** (2000) 084027, gr-qc/9911095.

[119] C. P. Herzog, Q. J. Ejaz, and I. R. Klebanov, "Cascading RG flows from new Sasaki-Einstein manifolds," *JHEP* **02** (2005) 009, hep-th/0412193.

[120] S. Benvenuti, L. A. Pando Zayas, and Y. Tachikawa, "Triangle anomalies from Einstein manifolds," *Adv. Theor. Math. Phys.* **10** (2006) 395–432, hep-th/0601054.

[121] K. Skenderis and M. Taylor, "Anatomy of bubbling solutions," arXiv:0706.0216 [hep-th].

[122] E. Barnes, E. Gorbatov, K. Intriligator, and J. Wright, "Current correlators and AdS/CFT geometry," *Nucl. Phys.* **B732** (2006) 89–117, hep-th/0507146.

[123] A. Butti, D. Forcella, and A. Zaffaroni, "Counting BPS baryonic operators in CFTs with Sasaki- Einstein duals," *JHEP* **06** (2007) 069, hep-th/0611229.