Dimensional Reduction and Approximation of Measures and Weakly Differentiable Homeomorphisms

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Ph.D. Thesis

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November 2007 - 27th October 2011

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Introduction

This thesis is devoted to the study of two different problems: the properties of the disintegration of the Lebesgue measure on the faces of a convex function and the existence of smooth approximations of bi-Lipschitz orientation-preserving homeomorphisms in the plane.

The first subject is analyzed in Part I, Chapters 1-3, while the second subject is treated in Part II, Chapters 4-5. Our contribution to the material contained in this thesis is mainly contained in a joint work with L. Caravenna [**19**] and in two papers obtained in collaboration with A. Pratelli [**23**], [**24**].

In **Part I** we deal with the explicit disintegration of the n-dimensional Hausdorff measure on the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ w.r.t. the partition given by its faces. By faces of a convex function we mean the convex sets obtained by the intersection of the graph with its tangent hyperplanes.

As the graph of a convex function naturally supports the n-dimensional Hausdorff measure, its faces, being convex, have a well defined linear dimension, and then they naturally support a proper dimensional Hausdorff measure.
Our main result is that the conditional measures induced by the disintegration are equivalent to the Hausdorff measure on the faces on which they are concentrated.

THEOREM 0.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\mathscr{H}^n \llcorner \operatorname{graph} f$ be the Hausdorff measure on its graph. Consider the partition of the graph of f into the relative interiors of the faces $\{\mathsf{F}_\alpha\}_{\alpha\in\mathsf{A}}$.*
Then, the Lebesgue measure on the graph of the convex function admits a unique disintegration

$$\mathscr{H}^n \llcorner \operatorname{graph} f = \int_{\mathsf{A}} \lambda_\alpha \, dm(\alpha) \tag{0.1}$$

w.r.t. this partition and the conditional measure λ_α which is concentrated on the relative interior of the face F_α is equivalent to $\mathscr{H}^k \llcorner \mathsf{F}_\alpha$, where k is the linear dimension of F_α.

In particular, as yield by formula (0.1), we recover the $\mathscr{H}^n$-negligibility of the set of relative boundary points of the faces whose dimension is greater or equal than 1, which was first obtained with other methods in [**40**].

The absolute continuity of the conditional probabilities w.r.t. the proper dimensional Hausdorff measures of the sets on which they are concentrated, despite seeming an intuitive and natural fact, does not always hold for Borel partitions of $\mathbb{R}^n$ into locally affine sets, i.e. relatively open subsets of affine planes of $\mathbb{R}^n$. The only cases in which the result is trivial are the partitions into 0-dimensional sets (i.e., single points) and n-dimensional

sets. Indeed, on one hand existence and uniqueness of a disintegration can be obtained by classical theorems relying on weak measurability conditions on the quotient maps defining the partition. On the other hand, as soon as $n \geq 3$, there exist examples of Borel collections of disjoint segments such that the conditional measures induced by the disintegration of the Lebesgue measure are Dirac deltas (see the counterexamples in [**40**], [**3**]).
Hence, for this kind of result further regularity properties are needed. For partitions defined by a Lipschitz quotient map between two Euclidean spaces, the absolute continuity of the conditional probabilities is guaranteed by the Coarea Formula (see e.g. [**5**]), and also for local Lipschitz conditions one can reduce to the same tools (in applications to optimal mass transport problem, see for example [**56**], [**17**], [**4**], [**7**]). In particular, when $n = 2$ the directions of a family of disjoint segments satisfy the local Lipschitz property, up to an $\mathscr{L}^2$-negligible set, and then Theorem 0.1 holds (see [**6**]).

In our case, up to our knowledge, the directions of the faces of a convex function do not have any weak differentiability property which can be used to prove Theorem 0.1 by standard Coarea Formulas. Therefore, our result, other than answering a quite natural question, enriches the regularity properties of the faces of a convex function, which have been intensively studied for example in [**27**], [**40**], [**41**], [**6**], [**49**].

In absence of any Lipschitz regularity for the directions of the faces, we look for another kind of "regularity". In particular, we show that the directions of the faces of a convex function can be approximated, in a suitable sense, by vector fields of partitions belonging to a class for which the absolute continuity property holds and, moreover, passes to the limit in the approximation process (the *cone vector fields* defined in 1.8). While the construction of the approximating vector fields and their convergence heavily depends on the specific problem, namely the fact that the directions we are approximating lie in the faces of a convex function, the definition of this suitable approximation property and the fact that, when satisfied, it guarantees the absolute continuity of the conditional probabilities, can be extended to arbitrary Borel partitions into locally affine sets –namely, sets which have a well defined linear dimension. Due to the structure of the approximating vector fields, this property will be called *cone approximation property* and, in this thesis, it will be studied for general Borel locally affine partitions. For families of disjoint segments, this approximation property was first introduced in [**13**] in order to solve a variational problem and it has been successfully applied to show the existence of optimal transport maps for strictly convex norms in [**18**].

Our result is the first dealing with the absolute continuity problem for locally affine partitions into sets of arbitrary dimension (i.e., possibly greater than 1). Actually, the approximation technique developed in the thesis can be also interesting for possible applications to other fields. Indeed, the disintegration theorem is an effective tool in dimensional reduction arguments, where it may be essential to have an explicit expression for the conditional measures. In particular, in the optimal transportation framework, the problem of the absolute continuity of the conditional measures on a partition given by locally affine sets was addressed by V.N. Sudakov in [**55**]. While trying to solve the Monge problem for general convex norms in $\mathbb{R}^n$ and absolutely continuous initial measures, which is straightforward when $n = 1$ due to monotonic rearrangement, he had the idea of reducing the transport problem, via disintegration of the initial and final measures, on the partition into locally affine sets where the potential of the dual problem is an affine function and the norm is linear. In particular, for strictly convex norms, these sets form a collection of disjoint segments. Then, the absolute continuity of the conditional mesures of the first marginal would permit to solve the so obtained family of independent 1-dimensional Monge problems and finally, gluing the so obtained optimal transport maps, to find a solution of the original Monge problem on $\mathbb{R}^n$. Unfortunately, the proof was based on a faulty lemma, which claimed the absolutely continuity property to be true for all Borel partitions into locally affine sets. Several years later, the counterexample in [**3**] opened the problem of filling the gap in Sudakov's proof. For uniformly smooth and convex norms and absolutely continuous initial mesures, thanks to a kind of Lipschitz regularity satisfied by the segments of the partition in this case, it was solved by L. Ambrosio in [**4**] (see also [**7**] for the extension to non-compactly supported measures, and [**17**] and [**56**] for the case in which both the initial and final measures are compactly supported and absolutely continuous). As mentioned before, for general strictly convex norms the problems was solved by L. Caravenna [**18**], proving that the segments where the potential is affine satisfy the approximation property introduced in [**13**]. The general convex case has been recently settled in a joint work with S. Bianchini [**12**].

Just to give an idea of how this technique works, focus on a collection of 1-dimensional faces $\mathscr{C}$ which are transversal to a fixed hyperplane $H_0 = \{x \in \mathbb{R}^n : x \cdot \mathrm{e} = 0\}$ and such that the projection of each face on the line spanned by the fixed vector e contains the interval $[h^-, h^+]$, with $h^- < 0 < h^+$. Indeed, we will obtain the disintegration of the Lebesgue measure on the k-dimensional faces, with $k > 1$, from a reduction argument to this case. First, we slice $\mathscr{C}$ with the family of affine hyperplanes $H_t = \{x \cdot \mathrm{e} = t\}$, where $t \in [h^-, h^+]$, which are parallel to H_0. In this way, by Fubini-Tonelli Theorem, the Lebesgue measure $\mathscr{L}^n$ of $\mathscr{C}$ can be recovered by integrating the $(n-1)$-dimensional Hausdorff measures of the sections of $\mathscr{C} \cap H_t$ over the segment $[h^-, h^+]$ which parametrizes the parallel hyperplanes. Then, as the faces in $\mathscr{C}$ are transversal to H_0, one can see each point in $\mathscr{C} \cap H_t$ as the image of a map σ^t defined on $\mathscr{C} \cap H_0$ which couples the points lying on the same face. Suppose that the $(n-1)$-dimensional Hausdorff measure $\mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_t)$ is absolutely continuous w.r.t. the pushforward measure $\sigma^t_\#(\mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_0))$ with Radon-Nikodym derivative α^t. Then we can reduce each integral over the section $\mathscr{C} \cap H_t$ to an integral over the section $\mathscr{C} \cap H_0$:

$$\int_{\mathscr{C}} d\mathscr{L}^n = \int_{[h^-,h^+]} \mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_t)\, dt = \int_{[h^-,h^+]} \int_{\mathscr{C} \cap H_0} \alpha^t(\sigma^t(z))\, d\mathscr{H}^{n-1}(z)\, dt.$$

Exchanging the order of the last iterated integrals, we obtain the following:

$$\int_{\mathscr{C}} d\mathscr{L}^n = \int_{\mathscr{C} \cap H_0} \int_{[h^-,h^+]} \alpha^t(\sigma^t(z))\, dt\, d\mathscr{H}^{n-1}(z).$$

Since the sets $\{\sigma^{[h^-,h^+]}(z)\}_{z \in \mathscr{C} \cap H_0}$ are exactly the elements of our partition, the last equality provides the explicit disintegration we are looking for: in particular, the conditional measure concentrated on $\sigma^{[h^-,h^+]}(z)$ is absolutely continuous w.r.t. $\mathscr{H}^{n-1} \llcorner \sigma^{[h^-,h^+]}(z)$.

The core of the proof is then to show that

$$\mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_t) \ll \sigma^t_\sharp(\mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_0)).$$

We prove this fact as a consequence of the following quantitative estimate: for all $0 \leq t \leq h^+$ and $S \subset \mathscr{C} \cap H_0$

$$\mathscr{H}^{n-1}(\sigma^t(S)) \leq \left(\frac{t-h^-}{-h^-}\right)^{n-1} \mathscr{H}^{n-1}(S). \tag{0.2}$$

This fundamental estimate, as in [**13**], [**18**], is proved approximating the 1-dimensional faces with a sequence of finitely many cones with vertex in $\mathscr{C} \cap H_{h^-}$ and basis in $\mathscr{C} \cap H_t$. At this step of the technique, the construction of such approximating sequence heavily depends on the nature of the partition one has to deal with. In this case, our main task is to find the suitable cones relying on the fact that we are approximating the faces of a convex function.
One can also derive an estimate symmetric to the above one, showing that $\sigma^t_\#(\mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_0))$ is absolutely continuous w.r.t. $\mathscr{H}^{n-1} \llcorner (\mathscr{C} \cap H_t)$: as a consequence, α^t is strictly positive and therefore the conditional measures are not only absolutely continuous w.r.t. the proper Hausdorff measure, but equivalent to it.

The fundamental estimate (0.2) implies moreover a Lipschitz continuity and BV regularity of $\alpha^t(z)$ w.r.t t: this yields an improvement of the regularity of the partition that now we are going to describe. In Chapter 3 we present these results for general locally affine Borel partitions satisfying the cone approximation property, while here for simplicity we consider the partition given by the faces of a convex function.
Consider a vector field v which at each point x is parallel to the face through that point x. If we restrict the vector field to an open Lipschitz set Ω which does not contain points in the relative boundaries of the faces, then we prove that its distributional divergence is the sum of two terms: an absolutely continuous measure, and a $(n-1)$-rectifiable measure representing the flux of v through the boundary of Ω. The density $(\operatorname{div} \mathrm{v})_{\mathrm{a.c.}}$ of the absolutely continuous part is related to the density of the conditional measures defined by the disintegration above.
In the case of the set $\mathscr{C}$ previously considered, if the vector field is such that $\mathrm{v} \cdot \mathrm{e} = 1$, the expression of the density of the absolutely continuous part of the divergence is

$$\partial_t \alpha^t = (\operatorname{div} \mathrm{v})_{\mathrm{a.c.}} \alpha^t.$$

Up to our knowledge, no piecewise BV regularity of the vector field v of faces directions is known. Therefore, it is a remarkable fact that a divergence formula holds.

The divergence of the whole vector field v is the limit, in the sense of distributions, of the sequence of measures which are the divergence of truncations of v on the elements $\{\mathscr{K}_\ell\}_{\ell \in \mathbb{N}}$ of a suitable partition of $\mathbb{R}^n$. However, in general, it fails to be a measure.

In the last part of Chapter 3, we change point of view: instead of looking at vector fields constrained to the faces of the convex function, we describe the faces as an $(n+1)$-uple of currents, the k-th one corresponding to the family of k-dimensional faces, for $k = 0, \dots, n$. The regularity results obtained for the vector fields can be rewritten as regularity results for these currents. More precisely, we prove that they are locally flat chains. When truncated on a set Ω as above, they are locally normal, and we give an explicit formula for their border; the $(n+1)$-uple of currents is the limit, in the flat norm, of the truncations on the elements of a partition.

An application of this kind of further regularity is presented in Section 8 of [**13**]. Given a vector field v constrained to live on the faces of f, the divergence formula we obtain allows to reduce the transport equation

$$\operatorname{div} \rho \mathrm{v} = g$$

to a PDE on the faces of the convex function. We do not pursue this issue in the thesis.

In **Part II** we deal with approximations of bi-Lipschitz orientation-preserving homeomorphisms $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \Delta \subseteq \mathbb{R}^2$, where Ω and $\Delta = u(\Omega)$ are two open bounded subsets of $\mathbb{R}^2$. In particular, we show that both u and its inverse can be approximated in the $W^{1,p}$-norm ($p \in [1, +\infty)$) by piecewise affine or smooth orientation-preserving homeomorphisms. Our main theorem is the following.

THEOREM 0.2. *Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set, and let $u : \Omega \longrightarrow \Delta$ be a bi-Lipschitz orientation-preserving homeomorphism. Then, for any $\bar{\varepsilon} > 0$ and any $1 \leq p < \infty$, there exists a bi-Lipschitz orientation-preserving homeomorphism $v : \Omega \longrightarrow \Delta$ such that $u = v$ on $\partial\Omega$,*

$$\|u - v\|_{L^\infty(\Omega)} + \|u^{-1} - v^{-1}\|_{L^\infty(\Delta)} + \|Du - Dv\|_{L^p(\Omega)} + \|Du^{-1} - Dv^{-1}\|_{L^p(\Delta)} \leq \bar{\varepsilon}\,, \quad (0.3)$$

and v is either countably piecewise affine or smooth. More precisely, there exist two geometric constants C_1 and C_2 such that, if u is L bi-Lipschitz, then the countably piecewise affine approximation can be chosen to be $C_1 L^4$ bi-Lipschitz, while the smooth approximation can be chosen to be $C_2 L^{28/3}$ bi-Lipschitz.

Thanks to a result by C. Mora-Corral and A. Pratelli [**46**] (see Theorem 4.1 below), the problem of finding smooth approximations can be actually reduced to find countably piecewise affine ones –i.e. affine on the elements of a locally finite triangulation of Ω, see Definition 4.4.

The fact that v might not be (finitely) piecewise affine but countably piecewise affine is due to the fact that we require $u = v$ on $\partial\Omega$ and it may happen that either Ω is not a polygon or that u is not piecewise affine on the boundary. In fact, we also prove the following

THEOREM 0.3. *If under the assumptions of Theorem 0.2 one has also that Ω is polygonal and u is piecewise affine on $\partial\Omega$, then there exists a (finitely) piecewise affine approximation $v : \Omega \longrightarrow \Delta$ as in Theorem 0.2 which is $C_1 C'(\Omega) L^4$ bi-Lipschitz.*

About the dependence of $C'(\Omega)$ in Theorem 0.3 on the domain Ω, see Remark 4.22.

The first naive idea coming to one's mind in order to construct a piecewise affine approximation of u could be the following: first, to select an arbitrary locally affine triangulation of Ω with triangles of sufficiently small diameter; then, to define v as the function which, on every triangle, is the affine interpolation of the values of u on its vertices. Unfortunately, if on one hand the functions defined in this way provide an approximation of u in L^∞, on the other hand they may fail to be homeomorphisms. The problem is due to the fact that, taking arbitrary nondegenerate triangles in Ω –no matter how small– then the affine interpolation of u on the vertices of the triangles can be orientation-preserving on some triangles and orientation-reversing on the others (see Figure 1). This prevents the affine interpolation to be injective since an homeomorphism on a connected domain in $\mathbb{R}^2$ must be either orientation-preserving or orientation-reversing on every subdomain. An explicit example of a function with such a bad behaviour can be found in [**53**].

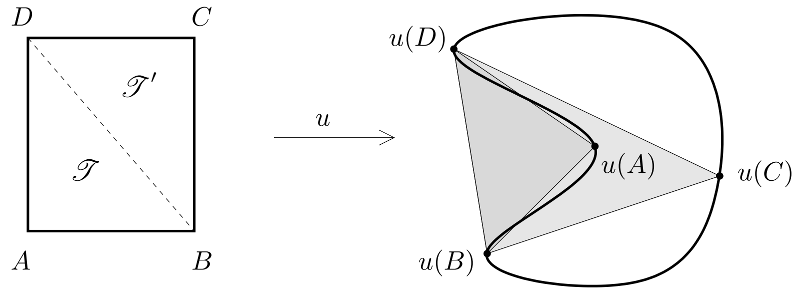

FIGURE 1. The square $ABCD$ is divided in the triangles $\mathscr{T}$ and $\mathscr{T}'$. The affine interpolation v of u on $ABCD$ is not injective, since $v(\mathscr{T}) \subseteq v(\mathscr{T}')$ ($v(\mathscr{T})$ and $v(\mathscr{T}')$ are shaded). Moreover, u is orientation-preserving in the square while v is orientation-reversing on $\mathscr{T}$.

The general problem of finding suitable approximations of homeomorphisms $u : \mathbb{R}^d \supseteq \Omega \longrightarrow u(\Omega) \subseteq \mathbb{R}^d$ with piecewise affine homeomorphisms has a long history. As far as we know, in the simplest non-trivial setting (i.e. $d = 2$, approximations in the L^∞-norm) the problem was solved by Radó [**50**]. Due to its fundamental importance in geometric topology, the problem of finding piecewise affine homeomorphic approximations in the L^∞-norm and dimensions $d > 2$ was deeply investigated in the 50s and 60s. In particular, it was solved by Moise [**43**] and Bing [**14**] in the case $d = 3$ (see also the survey book [**44**]), while for contractible spaces of dimension $d \geq 5$ the result follows from theorems of Connell [**20**], Bing [**15**], Kirby [**38**] and Kirby, Siebenmann and Wall [**39**] (for a proof see, e.g., Rushing [**52**] or Luukkainen [**42**]). Finally, twenty years later, while studying a class of quasi-conformal varietes, Donaldson and Sullivan [**25**] proved that the result is false in dimension 4.

Let us consider now homeomorphisms u which are bi-Sobolev, i.e. $u \in W^{1,p}$ and $u^{-1} \in W^{1,p}$ for some $p \in [1, +\infty]$. As pointed out by Ball (see [**9**], see also Evans [**26**]), the problem of proving the existence of piecewise affine approximations of bi-Sobolev homeomorphisms as in (0.3) arises naturally when one wants to approximate with finite elements the solutions of minimization problems in nonlinear elasticity (e.g. the minima of neo-hookean functionals, see also [**8**], [**10**], [**21**], [**54**]). In that context, the function u represents the physical deformation of a material with no interpenetration of matter (in particular,

$d = 2$ as in the present paper, or $d = 3$). The reason why one requires that the approximation in the Sobolev norm holds also for the inverse of u is that the functionals of nonlinear elasticity usually depend on functions of the Jacobian of u which explode both at 0 and $+\infty$. The physical meaning of choosing such functionals is that too high compressions or strecthings require high energy.

The additional difficulty in this case is to keep under control the derivatives of the approximating piecewise affine functions. In particular, one has to prevent the angles and the sides of the triangulations defining v from becoming comparatively too big/small.

The results available in the literature provide, under increasingly weaker hypotheses on the derivatives of u, piecewise affine or smooth approximations of u and its derivatives, but not of its inverse u^{-1}. The first results were obtained by Mora-Corral [**45**] (for planar bi-Sobolev mappings that are smooth outside a finite set) and by Bellido and Mora-Corral [**16**], in which they prove that if $u \in C^{0,\alpha}$ for some $\alpha \in (0, 1]$, then one can find piecewise affine approximations v in $C^{0,\beta}$, where $\beta \in (0, \alpha)$ depends only on α.

Recently, Iwaniec, Kovalev and Onninen [**37**] almost completely solved the approximation problem of planar Sobolev homeomorphisms, proving that whenever u belongs to $W^{1,p}$ for some $1 < p < +\infty$, then it can be approximated by smooth homeomorphisms v in the $W^{1,p}$-norm (improving the previous result for homeomorphisms in $W^{1,2}$ [**36**]).

However, as mentioned also by the authors of [**37**] themselves, the original problem posed by Ball and Evans of finding approximations of bi-Sobolev homeomorphisms together with their inverses still remained a completely open problem.

Our construction is the first one to take care also of the distance of the inverse maps, leading to a partial result towards the solution of the general problem: in fact, we are able to deal with homeomorphisms which are bi-Sobolev for $p = +\infty$. The techniques adopted in [**16**] and [**37**] are completely different with respect to the ones which will be used throughout this paper. While the proof in [**16**] is based on a refinement of the supremum norm approximation of Moise [**43**] (which, as pointed out by the authors themselves, cannot be extended to deal with the Sobolev case) and the approach of [**37**] makes use of the identification $\mathbb{R}^2 \simeq \mathbb{C}$ and involves coordinate-wise p-harmonic functions, our proof is constructive, thus lomg, but it does not make use of more sophisticated tools than the Lebesgue differentiation Theorem for L^1-maps in $\mathbb{R}^d$ and the Jordan curve Theorem.

Here we give a very short and rough scheme of the construction, simply aiming at introducing the main chapters and sections of the second part of the thesis. First, exploiting the nicer properties of u around the Lebesgue points of Du, we show that we can cover an arbitrarily large (in the Lebesgue sense) part of Ω with a family of uniform squares on which we can take v equal to the piecewise affine interpolation of the values of u on the vertices of the squares. Indeed, we prove that, in a sufficiently small neighborhood of a Lebesgue point of Du, the phenomenon depicted in Figure 1 cannot happen. Then, we cover the remaining part of Ω with a countable "tiling" made of right squares and we construct the piecewise affine approximation of u on the sides of these squares. Finally, to complete the construction, we have to define v in their interiors.

In order to do so, we use a planar bi-Lipschitz extension theorem for homeomorphic images of squares obtained in a joint work with A. Pratelli [**24**].

The proof of this bi-Lipschitz extension theorem will be the subject of Chapter 5. In particular, we prove that, given a planar bi-Lipschitz homeomorphism $\tilde{u}$ defined on the boundary of the unit square, it is possible to extend it to a function $\tilde{v}$ of the whole square, in such a way that $\tilde{v}$ is still bi-Lipschitz. Denoting by L and $\widetilde{L}$ the bi-Lipschitz constants of $\tilde{u}$ and $\tilde{v}$, with our construction one has $\widetilde{L} \leq CL^4$ (being C an explicit geometrical constant). The existence of a bi-Lipschitz extension of a planar bi-Lipschitz homeomorphism defined on the boundary of the unit square had already been proved in 1980 by Tukia (see [**57**]), using a completely different argument, but without any estimate on the constant $\widetilde{L}$. Moreover, we show that if $\tilde{u}$ is a piecewise affine function, then $\tilde{v}$ can be taken piecewise affine too. Hence, taking $\tilde{u}$ equal to the restriction of v to the boundary of any square of the countable "tiling" and defining $v = \tilde{v}$ in the interior, we complete the construction of the countably piecewise affine approximation of u. The explicit bound for the Lipschitz constants of our bi-Lipschitz extensions in terms of the Lipscitz constant of u will be actually essential in order to choose the approximating v as close as we want to u in the sense of (0.3).

More precisely, let $\tilde{u} : \partial\mathcal{D} \to u(\partial\mathcal{D}) \subseteq \mathbb{R}^2$ be a bi-Lipschitz orientation-preserving homeomorphism on the boundary of the unit square $\mathcal{D} = \mathcal{D}(0,1)$. By the Jordan curve Theorem, its image $\tilde{u}(\partial\mathcal{D})$ is the boundary $\partial\Gamma$ of a bounded closed Lipschitz domain $\Gamma \subseteq \mathbb{R}^2$.

Our main result is the construction of a piecewise affine bi-Lipschitz extension when $\tilde{u}$ is a piecewise affine function, hence Γ is a closed polygon.

THEOREM 0.4. *Let $\tilde{u} : \partial\mathcal{D} \to \partial\Gamma$ be an L bi-Lipschitz orientation-preserving piecewise affine map. Then there exists a piecewise affine extension $\tilde{v} : \mathcal{D} \to \Gamma$ which is CL^4 bi-Lipschitz, being C a purely geometric constant. Moreover, there exists also a smooth extension $\tilde{v}' : \mathcal{D} \to \Gamma$, which is $C'L^{28/3}$ bi-Lipschitz.*

Moreover, thanks to the geometric Lemmas 4.19 and 4.20 and Theorem 0.2 of Chapter 4, we also prove the existence of a (countably piecewise affine) bi-Lipschitz extension for any bi-Lipschitz map.

THEOREM 0.5. *Let $\tilde{u} : \partial\mathcal{D} \to \partial\Gamma \subseteq \mathbb{R}^2$ be an L bi-Lipschitz orientation-preserving map. Then there exists an extension $\tilde{v} : \mathcal{D} \to \Gamma \subseteq \mathbb{R}^2$ which is $C''L^4$ bi-Lipschitz, being C'' a purely geometric constant. Moreover, $\tilde{v}$ can also be taken countably piecewise affine with constant $\tilde{C}L^4$ or smooth with constant $\tilde{C}'L^{28/3}$.*

Hence, Theorem 0.4 is needed in the proof of Theorem 0.2, while Theorem 0.3 is needed in the proof of Theorem 0.5.

Our proof of Theorem 0.4 is constructive, thus quite intricate. However, the overall idea is simple and we try to make it as clear as possible in Chapter 5.

We conclude observing that, in the proof of Theorems 0.2, 0.3, 0.4 and 0.5, we can give an explicit –though rough– bound on the values of the costants C_1, C_2 and C_3 (while the constant $C'(\Omega)$ depends on the set Ω, see Remark 4.22)

$$C_1 = 72^4 C_3\,, \qquad C_2 = 70 C_1^{7/3}\,, \qquad C_3 = 636000$$

$$C = 636000\,, \qquad C' = 70C^{7/3}\,, \qquad C'' = 236C\,.$$

Plan of the thesis

In Chapter 1 we deal with general Borel partitions of $\mathbb{R}^n$ into locally affine sets (see Definition 1.12).
In Section 1.1 we recall the definition of (strongly consistent) disintegration of a measure over a partition and a general abstract theorem which guarantees the existence and uniqueness of a disintegration in the cases analyzed in the thesis.
The main issue of this chapter is to show that, whenever collections of segments obtained by slicing the sets of the partition with transversal affine planes satisfy a suitable "cone approximation property", then the conditional probabilities of the disintegration of the Lebesgue measure on the sets of the partition are absolutely continuous w.r.t. the proper dimensional Hausdorff measures of the sets on which they are concentrated. These collections of segments are the 1-*dimensional model sets*, introduced in Section 1.2.
The aim of Section 1.2 is to define the "cone approximation property" for 1-*dimensional model sets* and show that it implies the absolute continuity w.r.t. $\mathscr{H}^1$ of the conditional probabilities.
In Section 1.3 we show how to reduce the problem of absolute continuity of the disintegration on Borel locally affine partitions to testing the approximation property on the 1-*dimensional model sets* obtained with suitable slicings of the sets of the partition.

In Chapter 2 we deal with the locally affine partition given by the relative interiors of the faces of a convex function $f : \mathbb{R}^n \to \mathbb{R}$. After giving the main notation and preliminary definitions (Section 2.1), in Section 2.2 we prove that this partition satisfies the required measurability properties in order to support a unique and strongly consistent disintegration. In Lemma 2.3 of Section 2.3 we prove the "cone approximation property" for the 1-*dimensional slices* of the faces. Finally, in Section 2.4 we show that the "cone approximation property" and the fact that the faces are convex sets imply the Lebesgue-negligibility of the relative boundary points of the faces. Thus, we can conclude the proof of our main Theorem 0.1.

Chapter 3 deals with the divergence of the directions of the sets of locally affine partitions satisfying the "approximation property" as in Chapter 1. In particular, the results we obtain apply to the directions of the faces of a convex function. Section 5.2 contains a study of the regularity properties of the density function of the conditional probabilities of the disintegration w.r.t. the Hausdorff measures on the sets of the partition. Sections 3.2 and 3.3 describe, with two different approaches, how the regularity properties of the density function reflect on the regularity of the divergence of the directions of the sets of the partition. In Section 3.2 we consider the divergence of any vector field which at each point $x \in \mathbb{R}^n$ is parallel to the face of f through x. In Section 3.3 we consider the boundaries of the $(n+1)$-uple of currents associated to the faces of f, the k-th one acting on k-forms on $\mathbb{R}^n$.

In Chapter 4 we prove that any planar bi-Lipschitz orientation-preserving homeomorphism u can be approximated by countably piecewise affine homeomorphisms v as in Theorem 0.2. In Section 4.1 we give an idea of the proof. The construction is based on a suitable subdivision of the domain Ω into a tiling made of "Lebesgue squares" and into a countable tiling of "non-Lebesgue squares", up to the boundary of Ω. In Section 4.3 we define the "Lebesgue squares" and construct a piecewise affine approximation of u on these sets. In Section 4.4 we complete the construction defining v out of the "Lebesgue squares". Finally, in Section 4.5 we prove the existence of (finitely) piecewise affine approximations of u under the assumptions of Theorem 0.3.

Chapter 5 is devoted to the proof of the piecewise affine bi-Lipschitz extension Theorem 0.4 for piecewise affine bi-Lipschitz maps defined on the boundary of the squares. In Section 5.1 we introduce the main notation and Section 5.2 contains a brief scheme of the construction. Each of the Sections 5.3-5.10 corresponds to a step of the proof. Finally, in Section 5.11 we show how to get smooth bi-Lipschitz extensions, also for general bi-Lipschitz maps as in Theorem 0.5.

Acknowledgments. I would like to thank S. Bianchini for sharing with me his enthusiasm, ideas and trust, and for the interesting and beautiful problems he proposed to me during my Ph.D.. I am also very grateful to A. Pratelli for his constant support and guidance during our collaboration, and for the friendly and scientifically stimulating atmosphere during the last year spent in Pavia.

I thank all my colleagues and friends in Trieste for these four beautiful years, planting with me a piece of root in its magic land; I thank Carla for her support and our unique friendship. Moreover, I thank all my dearest friends spread all over the world for never fearing the distance.

My special thanks go to my family and Annibale, for always being so close though physically so far, and for feeling and sharing at my side every moment of this important experience.

Part I. The disintegration of the Lebesgue measure on the faces of a convex function

CHAPTER 1

Disintegration on locally affine partitions of $\mathbb{R}^n$

In this chapter we prove that, whenever suitable subpartitions into 1-dimensional sets of a locally affine partition of $\mathbb{R}^n$ –called 1-*dimensional slices* (Definition 1.23)– satisfy a regularity property called *cone approximation property* (Definition 1.11), then the conditional probabilities of the disintegration of Lebesgue measure on these sets are equivalent to the proper-dimensional Hausdorff measure of the sets on which they are concentrated. Our main result is the following

THEOREM 1.1. *Let* $\{X^k_\alpha\}_{\substack{\alpha\in\mathsf{A}_k\\ k=0,\dots,n}} \subset \mathbb{R}^n$ *be a Borel partition into locally affine sets whose* 1-dimensional slices *satisfy the* cone approximation property*. Then,*

$$\mathscr{L}^n \llcorner \bigcup_{k,\alpha} X^k_\alpha = \int \mu^k_\alpha \, dm(k,\alpha),$$

where the conditional probability μ^k_α *which is concentrated on the* k*-dimensional set* X^k_α *is equivalent to* $\mathscr{H}^k \llcorner X^k_\alpha$*, for* m*-a.e.* (k,α).

The definition of the regularity property for general locally affine partitions needed in Theorem 1.1 is not straightforward. However, the core of the technical role of this property at the aim of proving the absolute continuity of the conditional probabilities is already clear in the particular case in which the locally affine sets are 1-dimensional sets (i.e., segments) whose projections on a fixed direction of $\mathbb{R}^d$ are given by a fixed segment. Collections of disjoint segments of this kind will be called 1-*dimensional model sets* (Definition 1.7). The aim of Section 1.2 is to define the cone approximation property for 1-dimensional model sets (Definition 1.11). Instead of defining it at the beginning of the section, we introduce it as the final ingredient of a disintegration technique which permits to show the absolute continuity property. The content of Section 1.2 was first presented in [**13**] for a partition into segments coming from a variational problem, though not stated within this general framework.

In Section 1.3 we first deal with partitions into higher dimensional sets, called k-*dimensional model sets*. All the sets of such partitions have the same dimension (equal to some $k \in \{1,\dots,n-1\}$) and their projection on a fixed k-dimensional subspace of $\mathbb{R}^n$ is given by a fixed k-dimensional parallelogram. In particular, when $k = 1$ the definition is consistent with the one of 1-dimensional model set given in Section 1.2. By a Fubini-Tonelli argument, we will see that whenever all the 1-dimensional model sets obtained by slicing a k-dimensional model set with transversal affine planes (i.e., the 1-*dimensional sices* of the model set) satisfy the cone approximation property defined in Section 1.2, then the result of Theorem 1.1 holds. Finally, in Subsection 1.3.1 we deal with general locally affine partitions. After showing how to reduce via a countable covering argument from general locally affine partitions to k-dimensional model sets called *k-dimensional $\mathcal{D}$-cylinders* (as in [**19**] for the faces of a convex function), we complete the proof of Theorem 1.1, which is implied by the validity of the cone approximation property defined in Section 1.2 for the 1-dimensional slices of the so obtained k-dimensional $\mathcal{D}$-cylinders.

1.1. An abstract disintegration theorem

A disintegration of a measure over a partition of the space on which it is defined is a way to write that measure as a "weighted sum" of probability measures which are possibly concentrated on the elements of the partition.

Let (X, Σ, μ) be a measure space (which will be called the *ambient space* of the disintegration), i.e. Σ is a σ-algebra of subsets of X and μ is a measure with finite total variation on Σ and let $\{X_\alpha\}_{\alpha\in\mathsf{A}} \subset X$ be a partition of X. After defining the following equivalence relation on X

$$x \sim y \quad \Leftrightarrow \quad \exists\, \alpha \in \mathsf{A} : \; x, y \in X_\alpha,$$

we make the identification $\mathsf{A} = X/_\sim$ and we denote by p the quotient map $p : x \in X \mapsto [x] \in \mathsf{A}$.

Moreover, we endow the quotient space A with the measure space structure given by the largest σ-algebra that makes p measurable, i.e.

$$\mathscr{A} = \{F \subset \mathsf{A} : \; p^{-1}(F) \in \Sigma\},$$

and by the measure $\nu = p_\# \mu$.

Definition 1.2 (Disintegration). *A* disintegration *of μ consistent with the partition $\{X_\alpha\}_{\alpha\in\mathsf{A}}$ is a family $\{\mu_\alpha\}_{\alpha\in\mathsf{A}}$ of probability measures on X such that*

1. $\forall\, E \in \Sigma, \quad \alpha \mapsto \mu_\alpha(E)$ *is ν-measurable*;

2. $\mu = \int \mu_\alpha \, d\nu$, *i.e.*

$$\mu(E \cap p^{-1}(F)) = \int_F \mu_\alpha(E)\, d\nu(\alpha), \quad \forall\, E \in \Sigma,\; F \in \mathscr{A}. \tag{1.1}$$

The disintegration is unique *if the measures μ_α are uniquely determined for ν-a.e. $\alpha \in \mathsf{A}$.*
The disintegration is strongly consistent with p *if $\mu_\alpha(X \setminus X_\alpha) = 0$ for ν-a.e. $\alpha \in \mathsf{A}$.*
The measures μ_α are also called conditional probabilities of μ w.r.t. ν.

Remark 1.3. *When a disintegration exists, formula* (1.1) *can be extended by Beppo Levi theorem to measurable functions $f : X \to \mathbb{R}$ as*

$$\int f \, d\mu = \int \Big(\int f \, d\mu_\alpha \Big)\, d\nu(\alpha).$$

The existence and uniqueness of a disintegration can be obtained under very weak assumptions which concern only the ambient space. Nevertheless, in order to have the strong consistency of the conditional probabilities w.r.t. the quotient map we have to make structural assumptions also on the quotient measure algebra, otherwise in general $\mu_\alpha(X_\alpha) \neq 1$ (i.e. the disintegration is consistent but not strongly consistent). The more general result of existence of a disintegration which is consistent with a given partition is contained in [**48**], while a weak sufficient condition in order that a consistent disintegration is also strongly consistent is given in [**35**].

In the following we recall an abstract disintegration theorem, in the form presented in [**11**]. It guarantees, under suitable assumptions on the ambient and on the quotient measure spaces, the existence, uniqueness and strong consistency of a disintegration. Before stating it, we recall that a measure space (X, Σ) is countably-generated if Σ coincides with the σ-algebra generated by a sequence of measurable sets $\{B_n\}_{n\in\mathbb{N}} \subset \Sigma$.

THEOREM 1.4. *Let (X, Σ) be a countably-generated measure space and let μ be a measure on X with finite total variation. Then, given a partition $\{X_\alpha\}_{\alpha\in\mathsf{A}}$ of X, there exists a unique consistent disintegration $\{\mu_\alpha\}_{\alpha\in\mathsf{A}}$. Moreover, if there exists an injective measurable map from $(\mathsf{A}, \mathscr{A})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, where $\mathscr{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$, the disintegration is strongly consistent with p.*

REMARK 1.5. *If the total variation of μ is not finite, a disintegration of μ consistent with a given partition as defined in* (1.1) *in general does not exists, even under the assumptions on the ambient and on the quotient space made in Theorem 1.4 (take for example $X = \mathbb{R}^n$, $\Sigma = \mathscr{B}(\mathbb{R}^n)$, $\mu = \mathscr{L}^n$ and $X_\alpha = \{x : x \cdot z = \alpha\}$, where z is a fxed vector in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$).*

Nevertheless, if μ is σ-finite and (X, Σ), $(\mathsf{A}, \mathscr{A})$ satisfy the hypothesis of Theorem 1.4, as soon as we replace the possibly infinite-valued measure $\nu = p_\#\mu$ with an equivalent σ-finite measure m on $(\mathsf{A}, \mathscr{A})$, we can find a family of σ-finite measures $\{\tilde{\mu}_\alpha\}_{\alpha\in\mathsf{A}}$ on X such that

$$\mu = \int \tilde{\mu}_\alpha \, dm(\alpha) \tag{1.2}$$

and

$$\tilde{\mu}_\alpha(X \backslash X_\alpha) = 0 \quad \text{for } m\text{-a.e. } \alpha \in \mathsf{A}. \tag{1.3}$$

For example, we can take $m = p_\#\theta$, where θ is a finite measure equivalent to μ.

We recall that two measures μ_1 and μ_2 are equivalent *if and only if*

$$\mu_1 \ll \mu_2 \quad \text{and} \quad \mu_2 \ll \mu_1. \tag{1.4}$$

Moreover, if λ and $\{\tilde{\lambda}_\alpha\}_{\alpha\in\mathsf{A}}$ satisfy (1.2) *and* (1.3) *as well as m and $\{\tilde{\mu}_\alpha\}_{\alpha\in\mathsf{A}}$, then λ is equivalent to m and*

$$\tilde{\lambda}_\alpha = \frac{dm}{d\lambda}(\alpha)\tilde{\mu}_\alpha,$$

where $\frac{dm}{d\lambda}$ is the Radon-Nikodym derivative of m w.r.t. λ.

Whenever μ is a σ-finite measure with infinite total variation, by disintegration of μ strongly consistent with a given partition we will mean any family of σ-finite measures $\{\tilde{\mu}_\alpha\}_{\alpha\in\mathsf{A}}$ which satisfy the above properties; in fact, whenever μ has finite total variation we will keep the definition of disintegration given in (1.1).

Finally, we recall that any disintegration of a σ-finite measure μ can be recovered by the disintegrations of the finite measures $\{\mu \llcorner K_n\}_{n\in\mathbb{N}}$, where $\{K_n\}_{n\in\mathbb{N}} \subset X$ is a partition of X into sets of finite μ-measure.

Throughout this part of the thesis it will be convenient to denominate in a different way the partitions of $\mathbb{R}^n$ satisfying the assumptions of Theorem 1.4 with $\Sigma = \mathscr{B}(\mathbb{R}^n)$.

DEFINITION 1.6. *We say that a partition of $\mathbb{R}^n$ into sets $\{X_\alpha\}_{\alpha\in\mathsf{A}} \subset \mathbb{R}^n$ is a* Borel partition *if the quotient map $p : \bigcup_{\alpha\in\mathsf{A}} X_\alpha \to \mathsf{A}$ is $(\mathscr{B}(\mathbb{R}^n), \mathscr{A})$-measurable and there exists an injective measurable map from $(\mathsf{A}, \mathscr{A})$ to $(\mathbb{R}^m, \mathscr{B}(\mathbb{R}^m))$ for some $m \in \mathbb{N}$.*

The unique strongly consistent disintegration of a finite measure μ over a Borel partition $\{X_\alpha\}_{\alpha\in\mathsf{A}}$ will be denoted as

$$\mu = \int \mu_\alpha \, dm(\alpha), \quad \mu_\alpha(X_\alpha) = 1 \quad \text{for } m\text{-a.e. } \alpha \in \mathsf{A}. \tag{1.5}$$

1.2. Disintegration on 1-dimensional model sets

In this section we consider partitions of $\mathbb{R}^d$ into segments as in the following

DEFINITION 1.7 (1-dimensional model set). *A* 1-dimensional model set *is a σ-compact subset of $\mathbb{R}^d$ of the form*

$$\mathscr{C}^1 = \bigcup_{r\in\mathsf{R}} \ell_r,$$

where $\{\ell_r\}_{r\in\mathsf{R}} \subset \mathbb{R}^d$ is a collection of disjoint segments for which there exist a unit vector $\mathrm{e} \in \mathbb{S}^{d-1}$ and two real numbers $h^- < h^+$ such that

$$\{x \cdot \mathrm{e} : x \in \ell_r\} = [h^-, h^+], \quad \forall\, r \in \mathsf{R}.$$

Moreover, let us assume that for all $t \in [h^-, h^+]$, the map

$$\sigma^t : \ell_r \ni z \mapsto \ell_r \cap \{x \cdot \mathrm{e} = t\}, \quad \forall\, r \in \mathsf{R}$$

has σ-compact graph and that $\mathscr{L}^d(\mathscr{C}^1) < +\infty$.

For an example of 1-dimensional model set see Figure 1.

Notice that, by the abstract disintegration Theorem 1.4 there exists a unique strongly consistent disintegration

$$\mathscr{L}^d \llcorner \mathscr{C}^1 = \int \mu^1_r \, dm(r), \quad \mu^1_r(\ell_r) = 1 \quad \text{for } m\text{-a.e. } r \in \mathsf{R}. \tag{1.6}$$

Our aim is to show that, if the segments of a 1-dimensional model set satisfy an additional "regularity property" called *cone approximation property* (Definition 1.7), then we get the equivalence w.r.t. $\mathscr{H}^1$ of the conditional probabilities μ^1_r. In particular, this property will be presented as a final tool which permits to complete a disintegration technique for showing absolute continuity, first developed by S. Bianchini and M. Gloyer in [**13**].

Given a 1-dimensional model set $\mathscr{C}^1$, we always fix a vector e and two real numbers h^-, h^+ as in Definition 1.7. We also define the *transversal sections* $Z_t = \mathscr{C}^1 \cap \{x : x\cdot\mathrm{e} = t\}$, for all $t \in [h^-, h^+]$.

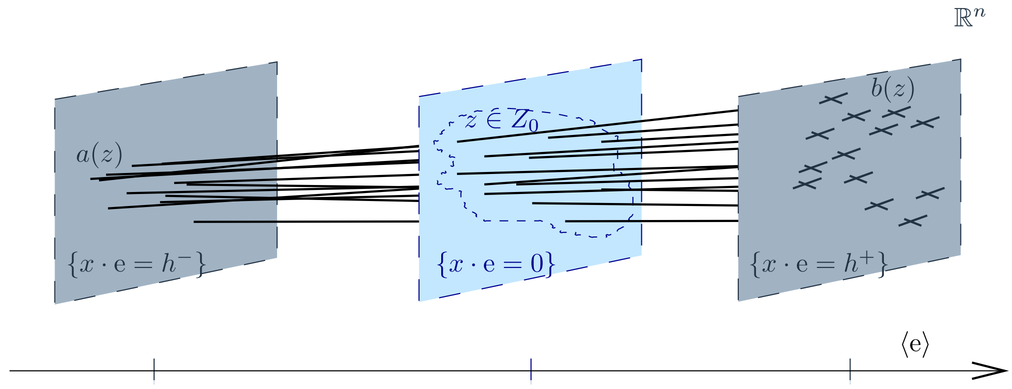

FIGURE 1. A 1-dimensional model set. Given a subset Z_0 of the hyperplane $\{x \cdot \mathrm{e} = 0\}$, the above model set is made of disjoint segments passing through some $z \in Z_0$, truncated between $\{x \cdot \mathrm{e} = h^-\}$ and $\{x \cdot \mathrm{e} = h^+\}$.

1.2.1. A Fubini-Tonelli technique and absolute continuity on transversal hyperplanes. In this short section we show that, by a Fubini-Tonelli argument, we can revert the problem of absolute continuity w.r.t. $\mathscr{H}^1$ of the conditional probabilities $\{\mu^1_r\}_{r \in \mathsf{R}}$ to the absolute continuity w.r.t. $\mathscr{H}^{d-1}$ of the push forward by the flow induced by the directions of the segments of the $\mathscr{H}^{d-1}$-measure on transversal sections.

First of all, we cut the set $\mathscr{C}^1$ with the affine hyperplanes which are perpendicular to the segment $[h^-\mathrm{e}, h^+\mathrm{e}]$, we apply Fubini-Tonelli theorem and we get

$$\int_{\mathscr{C}^1} \varphi(x)\, d\mathscr{L}^d(x) = \int_{h^-}^{h^+} \int_{Z_t} \varphi \, d\mathscr{H}^{d-1}\, dt, \quad \forall\, \varphi \in C^0_c(\mathbb{R}^d). \tag{1.7}$$

Then we observe the following: for every $s, t \in [h^-, h^+]$, the points of Z_s are in bijective correspondence with the points of the section Z_t and a bijection is obtained by pairing the points that belong to the same segment ℓ_r, for some $r \in \mathsf{R}$. In particular, this bijection is given by the map $\sigma^t_{|Z_s} : Z_s \to Z_t$.

Therefore, as soon as we fix a transversal section of $\mathscr{C}^1$, say for e.g. $Z_0 = \{x \cdot \mathrm{e} = 0\} \cap \mathscr{C}$ assuming that $0 \in (h^-, h^+)$, we can try to rewrite the inner integral in the r.h.s. of (1.7) as an integral of the function $\varphi \circ \sigma^t_{|Z_0}$ w.r.t. to the $\mathscr{H}^{d-1}$ measure of the fixed section Z_0. Setting for simplicity of notation $\sigma^t = \sigma^t_{|Z_0}$ and $Z = Z_0$, this can be done if

$$(\sigma^t)^{-1}_{\#}(\mathscr{H}^{d-1} \llcorner \sigma^t(Z)) \ll \mathscr{H}^{d-1} \llcorner Z. \tag{1.8}$$

Indeed,

$$\int_{\sigma^t(Z)} \varphi(y)\, d\mathscr{H}^{d-1}(y) = \int_Z \varphi(\sigma^t(z))\, d(\sigma^t)^{-1}_{\#}(\mathscr{H}^{d-1} \llcorner \sigma^t(Z))(z)$$

and if (1.8) is satisfied at least for $\mathscr{L}^1$-a.e. $t \in [h^-, h^+]$, then

$$(1.7) = \int_{h^-}^{h^+} \int_Z \varphi(\sigma^t(z))\, \alpha(t,z)\, d\mathscr{H}^{d-1}(z)\, dt,$$

where $\alpha(t,z)$ is the Radon-Nikodym derivative of $(\sigma^t)^{-1}_{\#}(\mathscr{H}^{d-1} \llcorner \sigma^t(Z))$ w.r.t. $\mathscr{H}^{d-1} \llcorner Z$.

Having turned the r.h.s. of (1.7) into an iterated integral over a product space isomorphic to $Z + [h^-\mathrm{e}, h^+\mathrm{e}]$, the final step consists in applying Fubini-Tonelli theorem again so as to exchange the order of the integrals and get

$$\int_{\mathscr{C}^1} \varphi(x)\, d\mathscr{L}^d(x) = \int_Z \int_{h^-}^{h^+} \varphi(\sigma^t(z))\, \alpha(t,z)\, dt\, d\mathscr{H}^{d-1}(z).$$

This final step can be done if α is Borel-measurable and locally integrable in (t,z).

By the uniqueness of the disintegration stated in Theorem 1.4 we have that, calling $r(z)$ the index $r \in \mathsf{R}$ such that $z \in Z \cap \ell_r$,

$$d\mu^1_{r(z)}(t) = \frac{\alpha(t\,,z) \cdot d\mathscr{H}^1 \llcorner \ell_r(t)}{\int_{h^-}^{h^+} \alpha(s,z)\, ds}, \quad \text{for } \mathscr{H}^{d-1}\text{-a.e. } z \in Z. \tag{1.9}$$

Moreover, we observe that if also the reverse absolute continuity estimate

$$\sigma^t_{\#}(\mathscr{H}^{d-1} \llcorner Z) \ll \mathscr{H}^{d-1} \llcorner \sigma^t(Z) \tag{1.10}$$

holds, then $\alpha(t,\cdot) > 0$ $\mathscr{H}^{d-1}$-a.e. on Z and the conditional probabilities (1.9) are equivalent to the 1-dimensional Hausdorff measure on the segments.

1.2.2. An approximation property and absolute continuity estimates. In this section we want to find additional conditions on the segments of a 1-dimensional model set so that (1.8) and (1.10) hold for a.e. $t \in [h^-, h^+]$. In this way, by the reasoning made in Section 1.2.1, we get the equivalence to $\mathscr{H}^1$ of the conditional probabilities (1.6). These conditions will be expressed by the *cone approximation property* defined in Definition 1.11.

First of all, given a 1-dimensional model set $\mathscr{C}^1$ we denote by $v_{\mathrm{e}} : \mathscr{C}^1 \to \mathbb{S}^{d-1}$ the unit vector field such that $v_{\mathrm{e}} \cdot \mathrm{e} > 0$ and $\langle v_{\mathrm{e}}(z)\rangle = \langle \ell_{r(z)} - z\rangle$ and we call it *direction vector field* of $\mathscr{C}^1$. Let us observe that v_{e} is constant on each segment ℓ_r, thus a 1-dimensional model set is completely identified by e, h^-, h^+ and v_{e}. Moreover, notice that, for all $z \in Z$, $s \in [h^-, h^+]$

$$\sigma^s(z) = z + s\frac{v_{\mathrm{e}}(z)}{v_{\mathrm{e}}(z) \cdot \mathrm{e}}. \tag{1.11}$$

We also set $v^s_{\mathrm{e}} = v_{\mathrm{e}|Z_s}$.

Now we fix $u \in (h^-, h^+)$ and we introduce a class of collections of segments which will be shown to satisfy (1.8) or (1.10).

Definition 1.8 ((Finite union of) Cones). *A* cone *with basis $Z^+_u \subseteq \{x \cdot \mathrm{e} = u\}$ ($Z^-_u \subseteq \{x \cdot \mathrm{e} = u\}$) and vertex in Z_{h^+} (Z_{h^-}) is a set of the form*

$$\bigcup_{z \in Z^+_u} [z, y^+]$$

(resp. $\bigcup_{z \in Z^-_u} [z, y^-]$) for some $y^+ \in Z_{h^+}$ ($y^- \in Z_{h^-}$). The point y^+ (y^-) is called vertex *of the cone. A* finite union of cones *is a subset of $\mathbb{R}^d$ given by a finite collection of cones which intersect at most in a $\mathscr{H}^{d-1}$-negligible subset of $\{x \cdot \mathrm{e} = u\}$.*

We notice that, for all $s \in [u, h^+)$ the intersection of a finite union of cones $\mathscr{C}^1_{u,+}$ with basis Z^+_u and vertices in Z_{h^+} with the set $\{x \in \mathbb{R}^d : x \cdot \mathrm{e} \in [u, s]\}$ is a 1-dimensional model set. The same holds by simmetry for finite union of cones with basis Z^-_u and vertices in Z_{h^-}, taking $s \in (h^-, u]$. Therefore, on $\mathscr{C}^1_{u,+} \cap \{x \in \mathbb{R}^d : x \cdot \mathrm{e} \in [u, h^+)\}$ we can define a direction vector field $v^u_{\mathrm{e},+}$ as done for the 1-dimensional model set $\mathscr{C}^1$.On $\mathscr{C}^1_{u,-} \cap \{x \in \mathbb{R}^d : x \cdot \mathrm{e} \in (h^-, u]\}$ we define the direction field $v^u_{\mathrm{e},-}$ in such a way that $v_{\mathrm{e},-} \cdot \mathrm{e} < 0$. Moreover, we define

$$\sigma^{u,s}_+(z) = z + (s-u)\frac{v^u_{\mathrm{e},+}(z)}{v^u_{\mathrm{e},+}(z) \cdot \mathrm{e}}, \quad \forall\, s \in [u, h^+],\, z \in Z^+_u$$

$$\sigma^{u,s}_-(z) = z + (s-u)\frac{v^u_{\mathrm{e},-}(z)}{v^u_{\mathrm{e},-}(z) \cdot \mathrm{e}}, \quad \forall\, s \in [h^-, u],\, z \in Z^-_u.$$

The importance of the finite unions of cones lies in the fact that they satisfy the following quantitative estimates on the push-forward of the $\mathscr{H}^{d-1}$-measure on transversal sections.

LEMMA 1.9. *Let $\mathscr{C}^1_{u,+}$ be a finite union of cones with basis Z^+_u and vertices in Z_{h^+}. Then, for all $s \in [u, h^+)$ and for all $A \subseteq Z^+_u$*

$$\mathscr{H}^{d-1}\big(\sigma^{u,s}_+(A)\big) \geq \mathscr{H}^{d-1}(A)\left(\frac{h^+ - s}{h^+ - u}\right)^{d-1}. \tag{1.12}$$

Let $\mathscr{C}^1_{u,-}$ be a finite union of cones with basis Z^-_u and vertices in Z_{h^-}. Then, for all $s \in (h^-, u]$ and for all $A \subseteq Z^-_u$

$$\mathscr{H}^{d-1}\big(\sigma^{u,s}_-(A)\big) \geq \mathscr{H}^{d-1}(A)\left(\frac{h^- - s}{h^- - u}\right)^{d-1}. \tag{1.13}$$

Moreover, (1.12) *(resp.* (1.13)*) holds also for finite union of cones with basis in $\{x \cdot \mathrm{e} = h^-\}$ (resp. $\{x \cdot \mathrm{e} = h^+\}$).*

The proof of Lemma 1.9 is a straightforward consequence of the similitude criteria for triangles.

Now, let us see how the estimates (1.12),(1.13) imply (1.8) and (1.10). If $t > 0$, taking $u = 0$ and $s = t$ in (1.12) we get (1.10), while choosing $u = t$ and $s = 0$ in (1.13) we obtain (1.8). If instead $t < 0$, then (1.8) (resp. (1.10)) is obtained taking $u = t$ and $s = 0$ in (1.12) (resp. $u = 0$ and $s = t$ in (1.13)).

We are then ready to state and prove the main result of this section. We prove that, if the restrictions to transversal sections of the direction vector field of a 1-dimensional model set can be pointwise approximated with finite unions of cones with vertices in $Z_{h^\pm}$, then the $(d-1)$-dimensional Hausdorff measures of the transversal sections satisfy the estimates (1.12) and (1.13), thus yielding the equivalence w.r.t. $\mathscr{H}^1$ of the conditional probabilities.

LEMMA 1.10. *Let $\mathscr{C}^1$ be a 1-dimensional model set and let $v_\mathrm{e} : \mathscr{C}^1 \to \mathbb{S}^{d-1}$ be its direction vector field. Let us assume that, for all $u \in (h^-, h^+)$, there exist two sequence of* *direction vector fields of finite union of cones* $\{v^u_{\mathrm{e},+,j}\}_{j\in\mathbb{N}}$ *and* $\{v^u_{\mathrm{e},-,j}\}_{j\in\mathbb{N}}$ *with bases in* Z_u *and vertices respectively in* Z_{h^+} *and* Z_{h^-} *such that*

$$\mathscr{H}^{d-1}(Z_u \setminus \operatorname{dom} v^u_{\mathrm{e},\pm,j}) = 0, \qquad \forall\, j \in \mathbb{N} \tag{1.14}$$

$$v^u_{\mathrm{e},\pm,j} \longrightarrow \pm v^u_{\mathrm{e}}, \qquad \mathscr{H}^{d-1}\text{-a.e. on } Z_u. \tag{1.15}$$

Then, for all $h^- < s \le t < h^+$ *and for all* $A \subseteq Z$

$$\left(\frac{h^+ - s}{h^+ - t}\right)^{d-1} \mathscr{H}^{d-1}\big(\sigma^s(A)\big) \le \mathscr{H}^{d-1}\big(\sigma^t(A)\big) \le \mathscr{H}^{d-1}\big(\sigma^s(A)\big)\left(\frac{h^- - t}{h^- - s}\right)^{d-1}. \tag{1.16}$$

Moreover, the right estimate in (1.16) *holds also for* $t = h^+$ *in case there exists an approximating sequence* $\{v^{h^+}_{\mathrm{e},-,j}\}_{j\in\mathbb{N}}$ *and the left estimate holds also for* $s = h^-$ *if there exists an approximating sequence* $\{v^{h^-}_{\mathrm{e},+,j}\}_{j\in\mathbb{N}}$ *as above.*

Before proving Lemma 1.10, we give the following

Definition 1.11 (Cone approximation property). *We say that a 1-dimensional model set* $\mathscr{C}^1 \subseteq \mathbb{R}^d$ *satisfies the* cone approximation property *if it satisfies the assumptions of Lemma 1.10.*

Proof of Lemma 1.10: First of all we observe that, by symmetry, it is sufficient to fix $t = 0 \in (h^-, h^+]$ and prove that, if $\exists\, \{v^0_{\mathrm{e},-,j}\}_{j\in\mathbb{N}}$ converging to $-v^0_{\mathrm{e}}$ as in (1.14) and (1.15), then the r.h.s. of (1.16) holds for all $s \in (h^-, 0]$.

The estimate (1.13) of Lemma 1.9 with $u = 0$ tells us that, for all $j \in \mathbb{N}$, $s \in (h^-, 0]$ and $A \subseteq Z$

$$\mathscr{H}^{d-1}\big(\sigma^{0,s}_{-,j}(A)\big) \ge \mathscr{H}^{d-1}(A)\left(\frac{h^- - s}{h^-}\right)^{d-1}. \tag{1.17}$$

Moreover, (1.17) is stable under pointwise limit. Indeed, take any compact set $A_\varepsilon \subseteq A$ such that: $\mathscr{H}^{d-1}(A \setminus A_\varepsilon) < \varepsilon$, and v^0_{e} and $v^0_{\mathrm{e},-,j}$ are continuous on A_ε for all $j \in \mathbb{N}$. In particular, $\{\sigma^{0,s}_{-,j}(A_\varepsilon)\}_{j\in\mathbb{N}}$ is a sequence of compact sets that, due to (1.14), converge w.r.t. the Hasudorff distance to the compact set $\sigma^s(A_\varepsilon) \subseteq \sigma^s(A)$. By the upper-semicontinuity of the $(d-1)$-dimensional Hausdorff measure on compact sets of $\{x \cdot \mathrm{e} = s\}$ converging w.r.t. the Hausdorff distance, we have

$$\begin{aligned}\mathscr{H}^{d-1}\big(\sigma^s(A)\big) &\ge \mathscr{H}^{d-1}\big(\sigma^s(A_\varepsilon)\big) \ge \limsup_{j\to\infty} \mathscr{H}^{d-1}\big(\sigma^{0,s}_{-,j}(A_\varepsilon)\big) \\ &\overset{(1.17)}{\ge} \mathscr{H}^{d-1}(A_\varepsilon)\left(\frac{h^- - s}{h^-}\right)^{d-1} \ge (\mathscr{H}^{d-1}(A) - \varepsilon)\left(\frac{h^- - s}{h^-}\right)^{d-1}.\end{aligned}$$

Letting ε tend to 0 we get exactly the r.h.s. of (1.16) for $t = 0$.

□

1.3. Disintegration on locally affine partitions

The aim of this section is to define the 1-*dimensional slices* for Borel locally affine partitions in $\mathbb{R}^n$ and to give a proof of Theorem 1.1.

First of all we give a rigorous definition of locally affine set.

Definition 1.12. *A nonempty set $E \subseteq \mathbb{R}^n$ is* locally affine *if it consists of a single point or if there exist $k \in \{1, \dots, n\}$ and an affine k-dimensional plane $V \subseteq \mathbb{R}^n$ such that $E \subseteq V$ and E is relatively open in V. By relatively open in V we mean open in the relative topology induced by $\mathbb{R}^n$ on V.*

Given a set $E' \subseteq \mathbb{R}^n$, we denote by $\mathrm{aff}(E')$ the *affine hull* of E', namely the minimal (w.r.t. set inclusion) affine subspace of $\mathbb{R}^n$ containing E'. We denote by $\mathrm{ri}(E')$ the *relative interior* of E', which is the interior of E' in the relative topology induced by $\mathbb{R}^n$ on $\mathrm{aff}(E')$ and by $\mathrm{rb}(E')$ its *relative boundary.* If $\emptyset \neq \mathrm{ri}(E') \subseteq E' \subseteq \mathrm{clos}\,(ri(E'))$, the *dimension* of E' is defined as $\dim(E') := \dim(\mathrm{aff}(E'))$. Whenever $\dim(E') = k$, we say that E' is a k-*dimensional* set and single points are equivalently called 0-*dimensional* sets. An example of sets whose linear dimension is well defined are the convex sets.

Concerning the disintegration on 0-dimensional and n-dimensional sets we make the following

Remark 1.13. *The result of Theorem 1.1 for $k = 0, n$ is trivial. Indeed, for all $\alpha \in \mathsf{A}_0$ we must put $\mu^0_\alpha = \delta_{\{X^0_\alpha\}}$, where δ_{x_0} is the Dirac mass supported in x_0, whereas if $\alpha \in \mathsf{A}_n$ we have that $\mu^n_\alpha = \frac{\mathscr{L}^n \llcorner X^n_\alpha}{|\mathscr{L}^n \llcorner X^n_\alpha|}$.*

Hence, from now onwards we will care only about the disintegration on the sets of dimension $k \in \{1, \dots, n-1\}$.

1.3.1. Disintegration on k-dimensional model sets. In this subsection we define a class of locally affine partitions into sets of a fixed dimension $k \in \{1, \dots, n-1\}$ called k-*dimensional model sets*, which generalize the concept of 1-dimensional model set introduced in Definition 1.7. Then, we define the 1-*dimensional slices* for these kinds of partitions and prove Theorem 1.1 in this special case.

We fix $k \in \{1, \dots, n-1\}$ and a plane $V \in \mathscr{G}(k, n)$, where $\mathscr{G}(k, n)$ is the Grassmanian of k-dimensional planes of $\mathbb{R}^n$ passing through the origin. We denote by $\mathbb{S}^{n-1} \cap V$ the $(k-1)$-dimensional unit sphere of V w.r.t. the Euclidean norm and $\pi_V : \mathbb{R}^n \to V$ the projection map on the k-plane V. We consider an orthonormal set $\{\mathrm{e}_1, \dots, \mathrm{e}_k\}$ in $\mathbb{R}^n$ such that $\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle = V$ and two k-uple of points $\mathrm{l} = (l_1, \dots, l_k)$, $\mathrm{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$ with $\mathrm{l}_j < \mathrm{m}_j$ for all $j = 1, \dots, k$. Then we define the k-dimensional

$$C^k(\mathrm{l}, \mathrm{m}) := \prod_{j=1}^{k} [l_j\, \mathrm{e}_j, m_j\, \mathrm{e}_j].$$

Now we are ready to give the definition of k-*dimensional model sets*

Definition 1.14. *A* k-dimensional model set *is a σ-compact subset of $\mathbb{R}^n$ of the form*

$$\mathscr{C}^k = \bigcup_{r \in \mathsf{R}_k} \mathscr{C}^k_r, \tag{1.18}$$

where $\{\mathscr{C}^k_r\}_{r \in \mathsf{R}_k}$ is a collection of disjoint closed k-dimensional sets such that

$$\pi_V(\mathscr{C}^k_r) = C^k(\mathrm{l}, \mathrm{m}).$$

Moreover we assume that, for all $\mathrm{w} \in C^k(\mathrm{l},\mathrm{m})$ *the map*

$$\mathscr{C}_r^k \ni z \mapsto \mathscr{C}_r^k \cap \pi_V^{-1}(\mathrm{w})$$

has σ*-compact graph.*

Again by the measurability assumptions made in Definition 1.14, Theorem 1.4 applies and then there exists a unique strongly consistent disintegration

$$\mathscr{L}^n \llcorner \mathscr{C}^k = \int \mu_r^k \, dm(k,r), \quad \mu_r^k(\mathscr{C}_r^k) = 1. \tag{1.19}$$

Given a k-dimensional model set $\mathscr{C}^k$, we always fix a k-plane V and a k-dimensional rectangle $C^k(\mathrm{l},\mathrm{m})$ as in Definition 1.14. We also define the *transversal sections* $Z_{\mathrm{w}}^k = \mathscr{C}^k \cap \pi_V^{-1}(\mathrm{w})$, for all $\mathrm{w} \in C^k(\mathrm{l},\mathrm{m})$

As in Section 1.2, the problem of the absolute continuity (or equivalence) of the conditional probabilities w.r.t. $\mathscr{H}^k$ can be reduced to the problem of absolute continuity (equivalence) w.r.t. $\mathscr{H}^{n-k}$ of the push-forward of the $(n-k)$-dimensional Hausdorff measure on transversal sections.

As soon as we fix a transversal section $Z^k = Z_{\mathrm{w}}^k$, the set $\mathscr{C}^k$ can be parametrized with the maps

$$\sigma^{\mathrm{w}+t\mathrm{e}} : Z^k \to Z^k_{\mathrm{w}+t\mathrm{e}}, \quad \sigma^{\mathrm{w}+t\mathrm{e}}(z) = z + t\frac{v_{\mathrm{e}}(z)}{|\pi_V(v_{\mathrm{e}}(z))|}, \tag{1.20}$$

where e is a unit vector in $\mathbb{S}^{n-1} \cap V$, $t \in \mathbb{R}$ satisfies $\mathrm{w} + t\mathrm{e} \in C^k(\mathrm{l},\mathrm{m})$ and $v_{\mathrm{e}}(z)$ is the unit direction contained in the set $\mathscr{C}_r^k$ passing through z which is such that $\frac{\pi_V(v_{\mathrm{e}}(z))}{|\pi_V(v_{\mathrm{e}}(z))|} = \mathrm{e}$. We observe that, according to our notation,

$$(\sigma^{\mathrm{w}+t\mathrm{e}})^{-1} = \sigma^{(\mathrm{w}+t\mathrm{e})-t\mathrm{e}}.$$

For all $\mathrm{w} \in C^k(\mathrm{l},\mathrm{m})$ we also define the real numbers

$$h^+(\mathrm{w},\mathrm{e}) = \sup\{t : \mathrm{w} + t\mathrm{e} \in C^k(\mathrm{l},\mathrm{m})\}, \quad h^-(\mathrm{w},\mathrm{e}) = \inf\{t : \mathrm{w} + t\mathrm{e} \in C^k(\mathrm{l},\mathrm{m})\}$$

The first step of the proof of Theorem 1.1 for the partition (1.18) consists in cutting the set $\mathscr{C}^k$ with affine hyperplanes which are perpendicular to e_i for $i = 1, \dots, k$ and apply k-times the Fubini-Tonelli theorem. Then, the main point is again to show that, for every e and t as above,

$$(\sigma^{t\mathrm{e}})^{-1}_{\#}(\mathscr{H}^{n-k} \llcorner Z^k_{\mathrm{w}+t\mathrm{e}}) \text{ is equivalent to } \mathscr{H}^{n-k} \llcorner Z^k \tag{1.21}$$

and, after this, that the Radon-Nikodym derivative between the above measures satisfies proper measurability and integrability conditions.

The main observation is now that the sets of the form

$$\bigcup_{z \in Z^k} \left\{ \sigma^{\mathrm{w}+t\mathrm{e}}(z) : t \in [h^-(\mathrm{w},\mathrm{e}), h^+(\mathrm{w},\mathrm{e})] \right\} \tag{1.22}$$

are 1-dimensional model sets as in Definition 1.7, living in the $d = (n-k+1)$-dimensional space $\pi_V^{-1}\big([h^-(\mathrm{w},\mathrm{e}), h^+(\mathrm{w},\mathrm{e})]\mathrm{e}\big)$. Hence, if the 1-dimensional model sets (1.22) satisfy the approximation property of Definition 1.11, in the same way as shown in the previous section one gets that (1.21) holds.

The role of the 1-dimensional model sets (1.22) in proving Theorem 1.1 leads us to give the following definition

DEFINITION 1.15. *We call* 1-dimensional slice *of a k-dimensional model set $\mathscr{C}^k$ any* 1*-dimensional model set of the form* (1.22)$= \mathscr{C}^k \cap \pi_V^{-1}\big([h^-(\mathrm{w},\mathrm{e}), h^+(\mathrm{w},\mathrm{e})]\mathrm{e}\big)$.

As we have just observed, Lemma 1.10 of Section 1.2 can be rewritten adapting to this context in the following way.

LEMMA 1.16. *Let $\mathscr{C}^k \cap \pi_V^{-1}\big([h^-(\mathrm{w},\mathrm{e}), h^+(\mathrm{w},\mathrm{e})]\mathrm{e}\big)$ be a* 1*-dimensional slice of a k-dimensional model set $\mathscr{C}^k$ which satisfies the cone approximation property. Then, for all $h^-(\mathrm{w},\mathrm{e}) < s \le t < h^+(\mathrm{w},\mathrm{e})$ and for all $A \subseteq Z^k_{\mathrm{w}}$*

$$\left(\frac{h^+(\mathrm{w},\mathrm{e}) - s}{h^+(\mathrm{w},\mathrm{e}) - t}\right)^{n-k} \mathscr{H}^{n-k}\big(\sigma^{\mathrm{w}+s\mathrm{e}}(A)\big) \le \mathscr{H}^{n-k}\big(\sigma^{\mathrm{w}+t\mathrm{e}}(A)\big)$$
$$\le \mathscr{H}^{n-k}\big(\sigma^{\mathrm{w}+s\mathrm{e}}(A)\big)\left(\frac{h^-(\mathrm{w},\mathrm{e}) - t}{h^-(\mathrm{w},\mathrm{e}) - s}\right)^{n-k}. \tag{1.23}$$

Moreover, the right estimate in (1.23) *holds also for $t = h^+(\mathrm{w},\mathrm{e})$ and the left estimate holds also for $s = h^-(\mathrm{w},\mathrm{e})$.*

Indeed, Lemma 1.10 implies the following

COROLLARY 1.17. *Let $\mathscr{C}^k$ be a k-dimensional model set whose* 1*-dimensional slices satisfy the cone approximation property and let $\sigma^{\mathrm{w}+s\mathrm{e}}(Z^k)$, $\sigma^{\mathrm{w}+t\mathrm{e}}(Z^k)$ be two sections of $\mathscr{C}^k$ with s and t as in Lemma 1.16.. Then, if we put $\mathrm{s} = \mathrm{w} + s\mathrm{e}$ and $\mathrm{t} = \mathrm{w} + t\mathrm{e}$, we have that*

$$\sigma_{\#}^{\mathrm{t}-|\mathrm{s}-\mathrm{t}|\mathrm{e}}\big(\mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{t}}(Z^k)\big) \ll \mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{s}}(Z^k) \tag{1.24}$$

and by the Radon-Nikodym theorem there exists a function $\alpha(\mathrm{t},\mathrm{s},\cdot)$ which is $\mathscr{H}^{n-k}$-a.e. defined on $\sigma^{\mathrm{s}}(Z^k)$ and is such that

$$\sigma_{\#}^{\mathrm{t}-|\mathrm{s}-\mathrm{t}|\mathrm{e}}\big(\mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{t}}(Z^k)\big) = \alpha(\mathrm{t},\mathrm{s},\cdot)\cdot \mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{s}}(Z^k). \tag{1.25}$$

PROOF. Without loss of generality we can assume that $s = 0$. If $\mathscr{H}^{n-k}(A) = 0$ for some $A \subset Z^k$, by definition of push forward of a measure we have that

$$(\sigma^{\mathrm{w}+t\mathrm{e}})^{-1}_{\#}\big(\mathscr{H}^{n-k} \llcorner\, \sigma^{\mathrm{w}+t\mathrm{e}}(Z^k)\big)(A) = \mathscr{H}^{n-k}(\sigma^{\mathrm{w}+t\mathrm{e}}(A)) \tag{1.26}$$

and taking $s = 0$ in (1.23) we find that $\mathscr{H}^{n-k}(A) = 0$ implies that $\mathscr{H}^{n-k}(\sigma^{\mathrm{w}+t\mathrm{e}}(A)) = 0$. □

In the proof of Theorem 1.1 we will also need the following

REMARK 1.18. *The fuction $\alpha = \alpha(\mathrm{t},\mathrm{s},y)$ defined in* (1.25) *is measurable w.r.t. y and, for $\mathscr{H}^{n-k}$-a.e. $y' \in \sigma^{\mathrm{w}+t\mathrm{e}}(Z^k)$, we have that*

$$\alpha(\mathrm{s},\mathrm{t},y') = \alpha(\mathrm{t},\mathrm{s},\sigma^{\mathrm{t}-|\mathrm{s}-\mathrm{t}|\mathrm{e}}(y'))^{-1}. \tag{1.27}$$

Moreover, since

$$(\sigma^{\mathrm{w}+s\mathrm{e}})^{-1} = (\sigma^{\mathrm{w}+t\mathrm{e}})^{-1} \circ \sigma^{\mathrm{s}+|\mathrm{s}-\mathrm{t}|\mathrm{e}}$$

we have that

$$\begin{aligned}
\alpha(\mathrm{s},\mathrm{w},z)\mathscr{H}^{n-k} \llcorner Z^k &= (\sigma^{\mathrm{w}+s\mathrm{e}})^{-1}_{\#}(\mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{s}}(Z^k)) \\
&= (\sigma^{\mathrm{w}+t\mathrm{e}})^{-1}_{\#}\big(\sigma^{\mathrm{s}+|\mathrm{s}-\mathrm{t}|\mathrm{e}}_{\#}\mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{s}}(Z^k)\big) \\
&= (\sigma^{\mathrm{w}+t\mathrm{e}})^{-1}_{\#}\Big(\alpha(\mathrm{s},\mathrm{t},y)\cdot \mathscr{H}^{n-k} \llcorner \sigma^{\mathrm{t}}(Z^k)\Big) \\
&= \alpha(\mathrm{t},\mathrm{w},z)\cdot\alpha(\mathrm{s},\mathrm{t},\sigma^{\mathrm{t}}(z))\cdot \mathscr{H}^{n-k} \llcorner Z. \qquad (1.28)
\end{aligned}$$

From (1.23) *we immediately get the uniform bounds:*

$$\begin{aligned}
&\left(\frac{h^+(\mathrm{t},\mathrm{e})-u}{h^+(\mathrm{t},\mathrm{e})}\right)^{n-k} \le \alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},\cdot) \le \left(\frac{u-h^-(\mathrm{t},\mathrm{e})}{-h^-(\mathrm{t},\mathrm{e})}\right)^{n-k} \quad \textit{if } u\in[0,h^+(\mathrm{t},\mathrm{e})],\\
&\left(\frac{u-h^-(\mathrm{t},\mathrm{e})}{-h^-(\mathrm{t},\mathrm{e})}\right)^{n-k} \le \alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},\cdot) \le \left(\frac{h^+(\mathrm{t},\mathrm{e})-u}{h^+(\mathrm{t},\mathrm{e})}\right)^{n-k} \quad \textit{if } u\in[h^-(\mathrm{t},\mathrm{e}),0]. \qquad (1.29)
\end{aligned}$$

In particular, $\alpha(\cdot,\mathrm{w},z)>0$ *on* $C^k(\mathrm{l},\mathrm{m})$ *for m-a.e.* $z\in Z^k$.

Now we can finally give the proof of Theorem 1.1 for k-dimensional model sets.

Proof of Theorem 1.1 for k-dimensional model sets: To end the proof of the theorem we only have to make rigorous the Fubini-Tonelli Tonelli argument which leads to check 1.21 on the 1-dimensional slices. For simplicity of notation, let us fix $\mathrm{w}=0\in C^k(\mathrm{l},\mathrm{m})$ and let us set $Z^k=Z^k_{\mathrm{w}}$, $h_i^{\pm}=h^{\pm}(\mathrm{w},\mathrm{e}_i)$, for all $i=1,\dots,k$ and $\sigma^{(t_1\mathrm{e}_1+\cdots+t_k\mathrm{e}_k)}=\sigma^{\mathrm{w}+(t_1\mathrm{e}_1+\cdots+t_k\mathrm{e}_k)}$.

Our aim is then to show that, for all $\varphi\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\int_{\mathscr{C}^k}\varphi\,d\mathscr{L}^n = \int_{Z^k}\int_{h_k^-}^{h_k^+}\!\!\dots\int_{h_1^-}^{h_1^+}\alpha(t_1\mathrm{e}_1+\cdots+t_k\mathrm{e}_k,0,z)\,\varphi(\sigma^{(t_1\mathrm{e}_1+\cdots+t_k\mathrm{e}_k)}(z))\,dt_1\dots dt_k\,d\mathscr{H}^{n-k}(z), \qquad (1.30)$$

where α is the function defined in (1.25).

We proceed using the disintegration technique which was presented in Section 1.2.

$$\begin{aligned}
\int_{\mathscr{C}^k}\varphi(x)\,d\mathscr{L}^n(x) &= \int_{h_k^-}^{h_k^+}\dots\int_{h_1^-}^{h_1^+}\int_{\mathscr{C}^k\cap\{x\cdot\mathrm{e}_k=t_k\}\cap\dots\cap\{x\cdot\mathrm{e}_1=t_1\}}\varphi\,d\mathscr{H}^{n-k}\\
&\underset{(1.24)}{=} \int_{h_k^-}^{h_k^+}\dots\int_{h_1^-}^{h_1^+}\int_{Z^k}\alpha(t_k\mathrm{e}_k,0,z)\dots\alpha(t_1\mathrm{e}_1+\cdots+t_k\mathrm{e}_k,t_2\mathrm{e}_2+\cdots+t_k\mathrm{e}_k,\sigma^{(t_2\mathrm{e}_2+\cdots+t_k\mathrm{e}_k)}(z))\\
&\qquad\cdot\varphi(\sigma^{(t_1\mathrm{e}_1+\cdots+t_k\mathrm{e}_k)}(z))\,d\mathscr{H}^{n-k}(z)\,dt_1\dots dt_k\\
&\underset{(1.28)}{=} \int_{h_k^-}^{h_k^+}\dots\int_{h_1^-}^{h_1^+}\int_{Z^k}\alpha(t_1\mathrm{e}_1+\cdots+t_k\mathrm{e}_k,0,z)\,\varphi(\sigma^{(t_1\mathrm{e}_1+\cdots+t_k\mathrm{e}_k)}(z))\,d\mathscr{H}^{n-k}(z)\,dt_1\dots dt_k\\
&\underset{\substack{(1.29)\\ \text{Rem } 1.18}}{=} \int_{Z^k}\int_{h_k^-}^{h_k^+}\dots\int_{h_1^-}^{h_1^+}\alpha(t_1\mathrm{e}_1+\cdots+t_k\mathrm{e}_k,0,z)\,\varphi(\sigma^{(t_1\mathrm{e}_1+\cdots+t_k\mathrm{e}_k)}(z))\,dt_1\dots dt_k\,d\mathscr{H}^{n-k}(z).
\end{aligned}$$

By the uniqueness and strong consistency of the disintegration guaranteed by Theorem 1.4, analogously to (1.9) we have that

$$\mu^k_{r(z)}(dt_1 \ldots dt_k) = \frac{\alpha(t_1 \mathrm{e}_1 + \cdots + t_k \mathrm{e}_k, 0, z) \, \mathscr{H}^k \llcorner \mathscr{C}^k_{r(z)}(dt_1 \ldots dt_k)}{\int_{h_k^-}^{h_k^+} \ldots \int_{h_1^-}^{h_1^+} \alpha(s_1 \mathrm{e}_1 + \cdots + s_k \mathrm{e}_k, 0, z) \, ds_1 \ldots ds_k}, \tag{1.31}$$

□

1.3.2. Reduction to k-dimensional model sets and global disintegration. In this section we finally consider a general Borel locally affine partition into sets $\{X^k_\alpha\}_{\substack{\alpha \in \mathsf{A}_k \\ k=0,\ldots,n}} \subset \mathbb{R}^n$. Neglecting by Remark 1.13 the 0-dimensional sets, we show that the set

$$\mathcal{T} := \bigcup_{k \in \{1,\ldots,n\}} \bigcup_{\alpha \in \mathsf{A}_k} X^k_\alpha \tag{1.32}$$

can be partitioned, up to an $\mathscr{L}^n$-negligible set, into a countable family of k-dimensional model sets called *k-dimensional $\mathcal{D}$-cylinders*. Finally, we complete the proof of Theorem 1.1.

In order to find a countable partition of $\mathcal{T}$ into model sets like the set $\mathscr{C}^k$ which was defined in Section 1.3.1, w.l.o.g. we start assuming that $\mathcal{T}$ is a σ-compact set and that the quotient map p of Definition 1.6 has σ-compact graph. In fact, it is sufficient to remove an $\mathscr{L}^n$-negligible set. We also define

$$X^k = \bigcup_{\alpha \in \mathsf{A}_k} X^k_\alpha, \quad \forall\, k = 0, \ldots, n \tag{1.33}$$

and the (multivalued) equivalence map $\mathcal{R} : \underset{k,\alpha}{\cup} X^k_\alpha \to \underset{k,\alpha}{\cup} X^k_\alpha$

$$X^k_\alpha \ni x \mapsto \mathcal{R}(x) = X^k_\alpha, \quad \text{for all } k = 0, \ldots, n,\ \alpha \in \mathsf{A}_k. \tag{1.34}$$

Moreover, we define the multivalued *direction map*

$$\mathcal{D} : \mathcal{T} \to \mathbb{S}^{n-1}, \quad \mathcal{D}(x) := \left\{ \frac{y - x}{\|y - x\|} : y \in \mathcal{R}(x) \setminus \{x\} \right\}. \tag{1.35}$$

By the assumptions on the quotient map p, it is easy to check that the maps $\mathcal{R}$ and $\mathcal{D}$ have σ-compact graphs.

Now we can start to build the partition of X^k into k-dimensional model sets.

Definition 1.19. *For all $k = 1, \ldots, n$, we call* sheaf set *a σ-compact subset of X^k of the form*

$$\mathscr{Z}^k = \underset{z \in Z^k}{\cup} \mathcal{R}(z), \tag{1.36}$$

where Z^k is a σ-compact subset of X^k which is contained in an affine $(n-k)$-plane in $\mathbb{R}^n$ and is such that

$$\mathcal{R}(z) \cap Z^k = \{z\}, \quad \forall\, z \in Z^k.$$

We call sections *of $\mathscr{Z}^k$ all the sets Y^k that satisfy the same properties of Z^k in the Definition 1.19.*

A subsheaf *of a sheaf set* $\mathscr{Z}^k$ *is a sheaf set* $\mathscr{W}^k$ *of the form*

$$\mathscr{W}^k = \underset{w \in W^k}{\cup} \mathcal{R}(w),$$

where W^k *is a* σ*-compact subset of a section of the sheaf set* $\mathscr{Z}^k$*.*

Now we prove that the set X^k can be covered with countably many disjoint sets of the form (1.36).
First of all, let us take a dense sequence $\{V_i\}_{i\in\mathbb{N}} \subset \mathscr{G}(k,n)$ and fix, $\forall\, i \in \mathbb{N}$, an orthonormal set $\{\mathrm{e}_{i_1}, \dots, \mathrm{e}_{i_k}\}$ in $\mathbb{R}^n$ such that

$$V_i = \langle \mathrm{e}_{i_1}, \dots, \mathrm{e}_{i_k} \rangle. \tag{1.37}$$

Recalling the notation set at the beginning of Section 1.3.1, we denote by $\pi_i = \pi_{V_i} : \mathbb{R}^n \to V_i$ the projection map on the k-plane V_i. For every fixed $0 < \varepsilon < 1$ the following sets form a disjoint covering of the k-dimensional unit spheres in $\mathbb{R}^n$:

$$\mathbb{S}_i^{k-1} = \Big\{ \mathbb{S}^{n-1} \cap V : V \in \mathscr{G}(k,n), \inf_{x \in \mathbb{S}^{n-1}\cap V} \|\pi_V(x)\| \geq 1 - \varepsilon \Big\} \setminus \bigcup_{j=1}^{i-1} \mathbb{S}_j^{k-1}, \qquad i = 1, \dots, I,$$

where $I \in \mathbb{N}$ depends on the ε we have chosen.

In order to determine a countable partition of X^k into sheaf sets we consider the k-dimensional rectangles in the k-planes (1.37) whose boundary points have dyadic coordinates. For all

$$\mathrm{l} = (l_1, \dots, l_k),\ \mathrm{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k \quad \text{with } l_j < m_j \ \ \forall\, j = 1, \dots, k$$

and for all $i = 1, \dots, I,\ p \in \mathbb{N}$, let $C^k_{ip\mathrm{lm}}$ be the rectangle

$$C^k_{ip\mathrm{lm}} = 2^{-p} \prod_{j=1}^{k} [l_j\, \mathrm{e}_{i_j}, m_j\, \mathrm{e}_{i_j}]. \tag{1.38}$$

LEMMA 1.20. *The following sets are sheaf sets covering* X^k*: for* $i = 1, \dots, I$*,* $p \in \mathbb{N}$*, and* $S \subset \mathbb{Z}^k$ *take*

$$\mathscr{Z}^k_{ipS} = \Big\{ x \in X^k : \mathcal{D}(x) \subseteq \mathbb{S}_i^{k-1} \text{ and } S \subseteq \mathbb{Z}^k \text{ is the maximal set such that}$$

$$\underset{\mathrm{l}\in S}{\cup} C^k_{ip\mathrm{l}(\mathrm{l}+1)} \subseteq \pi_i(\mathcal{R}(x)) \Big\}. \tag{1.39}$$

Moreover, a disjoint family of sheaf sets that cover X^k *is obtained in the following way: in case* $p = 1$ *we consider all the sets* $\mathscr{Z}^k_{ipS}$ *as above, whereas for all* $p > 1$ *we take a set* $\mathscr{Z}^k_{ipS}$ *if and only if the set* $\underset{\mathrm{l}\in S}{\cup} C^k_{ip\mathrm{l}(\mathrm{l}+1)}$ *does not contain any rectangle of the form* $C^k_{ip'\mathrm{l}(\mathrm{l}+1)}$ *for every* $p' < p$*.*

As soon as a nonempty sheaf set $\mathscr{Z}^k_{ipS}$ *belongs to this partition, it will be denoted by* $\bar{\mathscr{Z}}^k_{ipS}$*.*

For the proof of this lemma we refer to the analogous Lemma 2.6 in [**18**].

Then, we can refine the partition into sheaf sets by cutting them with sections which are perpendicular to fixed k-planes.

Definition 1.21. *(See Figure 2) A* k-dimensional $\mathcal{D}$-cylinder *of* $\{X^k_\alpha\}_{\substack{\alpha\in\mathsf{A}_k\\ k=1,\dots,n}}$ *is a k-dimensional model set of the form*

$$\mathscr{C}^k = \mathscr{Z}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(C^k), \tag{1.40}$$

where $\mathscr{Z}^k$ is a k-dimensional sheaf set, $\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle$ is any fixed k-dimensional subspace which is perpendicular to a section of $\mathscr{Z}^k$ and C^k is a rectangle in $\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle$ of the form

$$C^k = \prod_{i=1}^k [t_i^- \mathrm{e}_i, t_i^+ \mathrm{e}_i],$$

with $-\infty < t_i^- < t_i^+ < +\infty$ for all $i = 1,\dots,k$, such that

$$C^k \subseteq \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(\mathcal{R}(z)) \quad \forall\, z \in \mathscr{Z}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(C^k).$$

We set $\mathscr{C}^k = \mathscr{C}^k(\mathscr{Z}^k, C^k)$ when we want to refer explicitily to a sheaf set $\mathscr{Z}^k$ and to a rectangle C^k that can be taken in the definition of $\mathscr{C}^k$.
The k-plane $\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle$ is called the axis *of the k-dimensional model set and every set Z^k of the form*

$$\mathscr{C}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(\mathrm{w}), \quad \textit{for some } \mathrm{w} \in \mathrm{ri}(C^k)$$

is called a section *of the $\mathcal{D}$-cylinder.*

We also define the border of $\mathscr{C}^k$ transversal to $\mathcal{D}$ and its outer unit normal as

$$\begin{aligned} \mathfrak{d}\mathscr{C}^k &= \mathscr{C}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(\mathrm{rb}(C^k)), \\ \hat{n}_{|_{\mathfrak{d}\mathscr{C}^k}}(x) &= \text{ outer unit normal to } \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(C^k) \text{ at } x, \quad \text{for all } x \in \mathfrak{d}\mathscr{C}^k. \end{aligned} \tag{1.41}$$

Lemma 1.22. *The set $\mathcal{T}$ can be covered by the $\mathcal{D}$-cylinders*

$$\mathscr{C}^k(\mathscr{Z}^k_{ipS}, C^k_{ip\mathrm{l}(\mathrm{l}+1)}), \tag{1.42}$$

where $S \subseteq \mathbb{Z}^k$, $\mathrm{l} \in S$ and $\mathscr{Z}^k_{ipS}$, $C^k_{ip\mathrm{l}(\mathrm{l}+1)}$ are the sets defined in (1.39),(1.38).

Moreover, there exists a countable covering of $\mathcal{T}$ with $\mathcal{D}$-cylinders of the form (1.42) *such that*

$$\pi_i\Big[\mathscr{C}^k(\mathscr{Z}^k_{ipS}, C^k_{ip\mathrm{l}(\mathrm{l}+1)}) \cap \mathscr{C}^k(\mathscr{Z}^k_{ip'S'}, C^k_{ip'\mathrm{l}'(\mathrm{l}'+1)})\Big] \subset \mathrm{rb}[C^k_{ip\mathrm{l}(\mathrm{l}+1)}] \cap \mathrm{rb}[C^k_{ip'\mathrm{l}'(\mathrm{l}'+1)}] \tag{1.43}$$

for any couple of $\mathcal{D}$-cylinders which belong to this countable family (if $i \neq i'$, it follows from the definition of sheaf set that $\mathscr{C}^k(\mathscr{Z}^k_{ipS}, C^k_{ip\mathrm{l}(\mathrm{l}+1)}) \cap \mathscr{C}^k(\mathscr{Z}^k_{i'p'S'}, C^k_{i'p'\mathrm{l}'(\mathrm{l}'+1)})$ must be empty).

Proof. The fact that the $\mathcal{D}$-cylinders defined in (1.42) cover $\mathcal{T}$ follows directly from Definitions 1.19 and 1.21 as in [**18**].

Our aim is then to construct a countable covering of $\mathcal{T}$ with $\mathcal{D}$-cylinders wich satisfy property (1.43).
First of all, let us fix a nonempty sheaf set $\bar{\mathscr{Z}}^k_{ipS}$ which belongs to the countable partition of $\mathcal{T}$ given in Lemma 1.20.

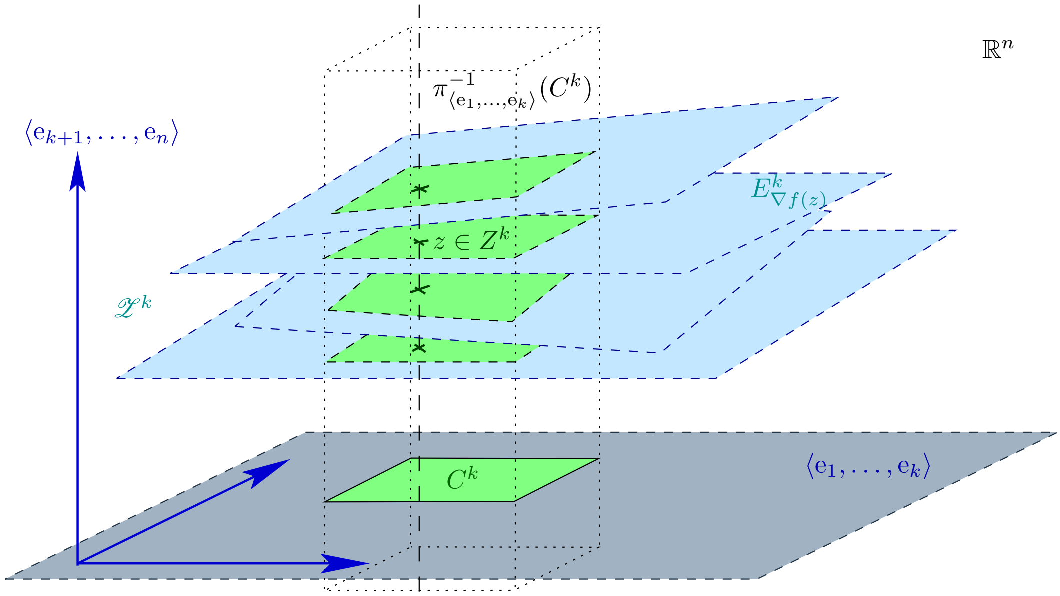

FIGURE 2. Sheaf sets and $\mathcal{D}$-cylinders (Definitions 1.19, 1.21). Roughly, a sheaf set $\mathscr{Z}^k$ is a collection of k-dimjensional sets, which intersect exactly at one point some set Z^k contained in a $(n-k)$-dimensional plane. A $\mathcal{D}$-cylinder $\mathscr{C}^k$ is the intersection of a sheaf set with $\pi^{-1}_{\langle e_1,\dots,e_k\rangle}(C^k)$, for some rectangle $C^k = \mathrm{conv}(\{t_i^- e_i, t_i^+ e_i\}_{i=1,\dots,k})$, where $\{e_1,\dots,e_n\}$ are an orthonormal basis of $\mathbb{R}^n$. Such sections Z^k are called basis, while the k-plane $\langle e_1,\dots,e_k\rangle$ is an axis.

In the following we will determine the $\mathcal{D}$-cylinders of the countable covering which are contained in $\bar{\mathscr{Z}}^k_{ipS}$; the others can be selected in the same way starting from a different sheaf set of the partition given in Lemma 1.20.
Then, the $\mathcal{D}$-cylinders that we are going to choose are of the form

$$\mathscr{C}^k\big(\mathscr{Z}^k_{i\hat{p}\hat{S}}, C^k_{i\hat{p}\hat{\mathrm{l}}(\hat{\mathrm{l}}+1)}\big),$$

where $\mathscr{Z}^k_{i\hat{p}\hat{S}}$ is a subsheaf of the sheaf set $\bar{\mathscr{Z}}^k_{ipS}$.

The construction is done by induction on the natural number $\hat{p}$ which determines the diameter of the squares $C^k_{i\hat{p}\hat{\mathrm{l}}(\hat{\mathrm{l}}+1)}$ obtained projecting the $\mathcal{D}$-cylinders contained in $\bar{\mathscr{Z}}^k_{ipS}$ on the axis $\langle e_{i_1},\dots,e_{i_k}\rangle$. Then, as the induction step increases, the diameter of the k-dimensional rectangles associated to the $\mathcal{D}$-cylinders that we are going to add to our countable partition will be smaller and smaller (see Figure 3).

By definition (1.39) and by the fact that $\bar{\mathscr{Z}}^k_{ipS}$ is a nonempty element of the partition defined in Lemma 1.20, the smallest natural number $\hat{p}$ such that there exists a k-dimensional rectangle of the form $C^k_{i\hat{p}\hat{\mathrm{l}}(\hat{\mathrm{l}}+1)}$ which is contained in $\pi_i(\bar{\mathscr{Z}}^k_{ipS})$ is exactly p; then, w.l.o.g., we can assume in our induction argument that $p = 1$.

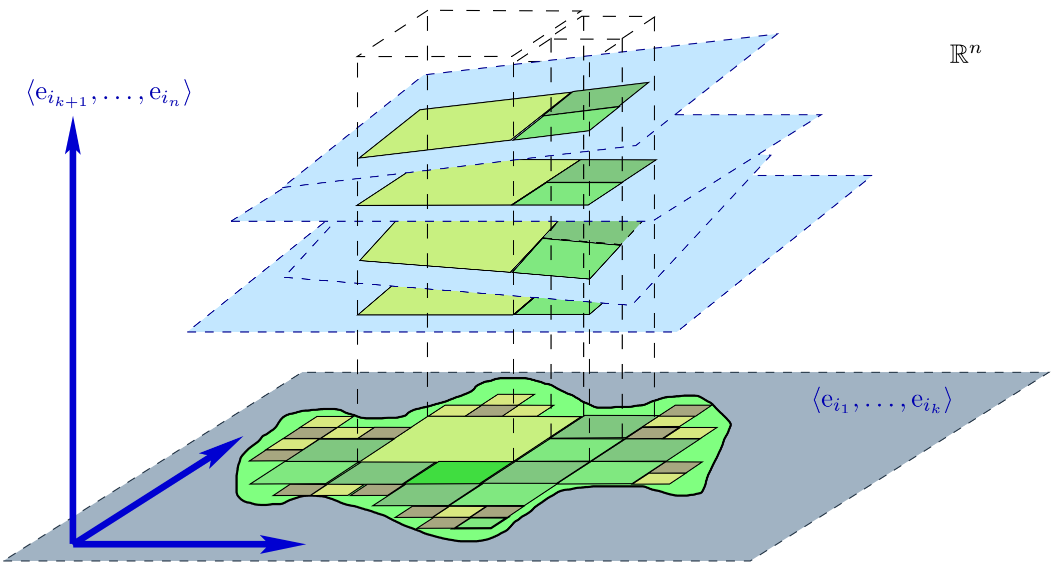

FIGURE 3. Partition of E^k into $\mathcal{D}$-cylinders (Lemma 1.22).

For all $\hat{p} \in \mathbb{N}$, we call $Cyl_{\hat{p}}$ the collection of the $\mathcal{D}$-cylinders which have been chosen up to step $\hat{p}$.
When $\hat{p} = 1$ we set

$$Cyl_1 = \{\mathscr{C}^k(\bar{\mathscr{Z}}^k_{i1S}, C^k_{i1\mathrm{l}(\mathrm{l}+1)}) : \mathrm{l} \in S\}.$$

Now, let us suppose to have determined the collection of $\mathcal{D}$-cylinders $Cyl_{\hat{p}}$ for some $\hat{p} \in \mathbb{N}$.

Then, we define

$$Cyl_{\hat{p}+1} = Cyl_{\hat{p}} \bigcup \Big\{ \mathscr{C}^k = \mathscr{C}^k(\mathscr{Z}^k_{i(\hat{p}+1)\tilde{S}}, C^k_{i(\hat{p}+1)\tilde{\mathrm{l}}(\tilde{\mathrm{l}}+1)}) : \mathscr{Z}^k_{i(\hat{p}+1)\tilde{S}} \text{ is a subsheaf of } \bar{\mathscr{Z}}^k_{ipS} \text{ and}$$
$$\mathscr{C}^k \nsubseteq \mathscr{C}^k(\mathscr{Z}^k_{ip'S'}, C^k_{ip'\mathrm{l}'(\mathrm{l}'+1)}) \text{ for all } \mathscr{C}^k(\mathscr{Z}^k_{ip'S'}, C^k_{i'p'\mathrm{l}'(\mathrm{l}'+1)}) \in Cyl_{\hat{p}} \Big\}.$$

□

Before completing the proof of Theorem 1.1, we give the following

DEFINITION 1.23. *We call* 1-dimensional slice *of a locally affine partition any 1-dimensional slice of the k-dimensional sets given by its k-dimensional $\mathcal{D}$-cylinders as in Definition 1.15, for all $k = 1, \dots, n-1$.*

Indeed, by Remark 1.13, the disintegration on n-dimensional $\mathcal{D}$-cylinders is simply a consequence the σ-additivity of the Lebesgue measure

PROOF OF THEOREM 1.1. As we observed in Remark 1.13, it is sufficient to prove the theorem for the disintegration of the Lebegue measure on the set X^k when $k \in \{1, \dots, n-1\}$.

Moreover, by (1.43), for all $k = 1, \dots, n-1$ there exists a $\mathscr{L}^n$-negligible set N^k such that

$$X^k \setminus N^k = \bigcup_{j \in \mathbb{N}} \mathscr{C}_j^k \setminus \mathfrak{d}\mathscr{C}_j^k,$$

where $\{\mathscr{C}_j^k\}_{j\in\mathbb{N}}$ is the countable collection of k-dimensional $\mathcal{D}$-cylinders covering X^k which was constructed in Lemma 1.22, so that the sets $\hat{\mathscr{C}}_j^k = \mathscr{C}_j^k \setminus \mathfrak{d}\mathscr{C}_j^k$ are disjoint.

The fundamental observation is the following:

$$\bigcup_{j\in\mathbb{N}} \hat{\mathscr{C}}_j^k = \bigcup_{j\in\mathbb{N}} \bigcup_{\alpha\in\mathsf{A}_k} X_{\alpha,j}^k = \bigcup_{\alpha\in\mathsf{A}_k} \bigcup_{j\in\mathbb{N}} X_{\alpha,j}^k = \bigcup_{\alpha\in\mathsf{A}_k} X_\alpha^k \setminus N^k,$$

where $X_{\alpha,j}^k = X_\alpha^k \cap \hat{\mathscr{C}}_j^k$.

For all $j \in \mathbb{N}$, we set

$$\mathsf{A}_{k,j} = \{\alpha \in \mathsf{A}_k : \, X_{\alpha,j}^k \neq \emptyset\},$$

we denote by $p_{k,j} : \hat{\mathscr{C}}_j^k \to \mathsf{A}_{k,j}$ the quotient map corresponding to the partition

$$\hat{\mathscr{C}}_j^k = \bigcup_{\alpha\in\mathsf{A}_k} X_{\alpha,j}^k$$

and we set $\nu_{k,j} = p_{k,j\#}\mathscr{L}^n \llcorner \hat{\mathscr{C}}_j^k$.

Since the quotient space $(\mathsf{A}_{k,j}, \mathscr{B}(\mathsf{A}_{k,j}))$ is isomorphic to $(Z_j^k, \mathscr{B}(Z_j^k))$, where Z_j^k is a section of $\mathscr{C}_j^k$, and since by assumption the 1-dimensional slices defined in Definition 1.23 satisfy the cone approximation property, by the proof of Theorem 1.1 for k-dimensional model sets given in Section 1.3.1 we have that

$$\mathscr{L}^n \llcorner \mathscr{C}_j^k(E_j \cap p_j^{-1}(F_j)) = \int_{F_j} \mu_{\alpha,j}^k(E_j)\, d\nu_{k,j}(y), \quad \forall E_j \in \mathscr{B}(\mathscr{C}_j^k),\ F_j \in \mathscr{B}(\mathsf{A}_{k,j}), \tag{1.44}$$

where $\mu_{\alpha,j}^k$ is equivalent to $\mathscr{H}^k \llcorner X_{\alpha,j}^k$ for $\nu_{k,j}$-a.e. $\alpha \in \mathsf{A}_{k,j}$.

Moreover, for every $E \in \mathscr{B}(\mathbb{R}^n) \cap X^k$ there exist sets $E_j \in \mathscr{B}(\mathscr{C}_j^k)$ such that

$$E = \bigcup_{j\in\mathbb{N}} E_j$$

and for all $F \in \mathscr{B}(\mathsf{A}_k)$, where $\mathsf{A}_k = \bigcup_{j\in\mathbb{N}} \mathsf{A}_{k,j}$, there exist sets $F_j \in \mathscr{B}(\mathsf{A}_{k,j})$ such that

$$F = \bigcup_{j\in\mathbb{N}} F_j \quad \text{and} \quad p^{-1}(F) = \bigcup_{j\in\mathbb{N}} p_j^{-1}(F_j).$$

Then,

$$\begin{aligned}
\mathscr{L}^n \llcorner K(E \cap p^{-1}(F)) &= \sum_{j=1}^{+\infty} \mathscr{L}^n \llcorner \mathscr{C}_j^k(E_j \cap p_j^{-1}(F_j)) \\
&\underset{(1.44)}{=} \sum_{j=1}^{+\infty} \int_{F_j} \mu_{\alpha,j}^k(E_j)\, d\nu_{k,j}(\alpha) \\
&= \sum_{j=1}^{+\infty} \int_{\mathsf{A}_{k,j}} \mathbf{1}_{F_j}(\alpha) \mu_{\alpha,j}^k(E_j)\, d\nu_{k,j}(\alpha)
\end{aligned}$$

$$= \sum_{j=1}^{+\infty} \int_{\mathsf{A}_k} \mathbf{1}_{F_j}(\alpha) \mu^k_{\alpha,j}(E_j) f_j(\alpha)\, d\nu_k(\alpha), \tag{1.45}$$

where f_j is the Radon-Nikodym derivative of $\nu_{k,j}$ w.r.t. the measure ν_k on A_k given by $p_\# \mathscr{L}^n \llcorner (K \cap X^k)$.

Since, by Theorem 1.4, there exists a unique disintegration $\{\mu^k_\alpha\}_{\substack{\alpha \in \mathsf{A}_k \\ k=0,\dots,n}}$ such that

$$\mathscr{L}^n \llcorner K(E \cap \nabla f^{-1}(F)) = \int_F \mu^k_\alpha(E)\, dp_\# \mathscr{L}^n \llcorner K(k, \alpha) \quad \text{for all} \quad E \in \mathscr{B}(\mathbb{R}^n),\ F \in \mathscr{B}(\mathsf{A}),$$

we conclude that the last term in (1.45) converges and

$$\mu^k_\alpha = \sum_{j=1}^{+\infty} f_j(\alpha)\, \mu^k_{\alpha,j} \quad \text{for } \nu_k\text{-a.e. } \alpha \in \mathsf{A}_k,$$

so that the Theorem is proved. □

CHAPTER 2

Disintegration on the faces of a convex function

In this chapter we deal with the Disintegration Theorem 0.1. For the proof we will show that the partition given by the relative interiors of the faces of a convex function satisfies the approximation property of Theorem 1.1 (Lemma 2.3) and that the points in the relative boundaries of the faces of dimension greater or equal than 1 are a negligible set (Lemma 2.4).

For notational convenience, instead of considering the disintegration of the n-dimensional Hausdorff measure on the faces of the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$, we consider the disitegration of the n-dimensional Lebegue measure on the sets given by the projections of the faces on the first n-coordinates (2.4). This and other preliminary questions and definitions will be discussed in Section 2.1. Moreover, in Section 2.1 we will state Lemma 2.3 and 2.4 and prove that the validity of Lemma 2.3 and Lemma 2.4 immediately yields Theorem 0.1 (see the proof at Page 37). In Section 2.2 we prove that the partition into "projected faces" satisfies the measurability assumptions of Definition 1.6, which guarantee the existence and uniqueness of a disintegration (see Theorem 1.4). Section 2.3 contains the proof of the approximation Lemma 2.3. In Lemma 2.4 of Section 2.4 we finally prove the negligibility of the relative boundary points of the faces.

2.1. Preliminaries and main results

In this section we set some notation and basic definitions which enter into the statement and the proof of our main Theorem 0.1.

First of all, let us consider the **ambient space** (see Section 1.1)

$$(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), \mathscr{L}^n \llcorner K),$$

where $\mathscr{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$, $\mathscr{B}(\mathbb{R}^n)$ is the Borel σ-algebra, K is any set of finite Lebesgue measure and $\mathscr{L}^n \llcorner K$ is the restriction of the Lebesgue measure to the set K. Indeed, by Remark 1.5 in Section 1.1, the disintegration of the Lebesgue measure w.r.t. a given partition is determined by the disintegrations of the Lebesgue measure restricted to finite measure sets.

Then, let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function.
We recall that the subdifferential of f at a point $x \in \mathbb{R}^n$ is the set $\partial^- f(x)$ of all $r \in \mathbb{R}^n$ such that

$$f(w) - f(x) \geq r \cdot (w - x), \quad \forall\, w \in \mathbb{R}^n. \tag{2.1}$$

From the basic theory of convex functions, as f is real-valued and is defined on all $\mathbb{R}^n$, $\partial^- f(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$, it is a closed convex set and it consists of a single point if and

only if f is differentiable at x. Moreover, in that case, $\partial^- f(x) = \{\nabla f(x)\}$, where $\nabla f(x)$ is the differential of f at the point x.
We denote by $\operatorname{dom}\nabla f$ a σ-compact set where f is differentiable and such that $\mathbb{R}^n \setminus \operatorname{dom}\nabla f$ is Lebesgue negligible. $\nabla f : \operatorname{dom}\nabla f \to \mathbb{R}$ denotes the differential map and $\operatorname{Im}\nabla f$ the image of $\operatorname{dom}\nabla f$ with the differential map. By the convexity of f, we can moreover assume w.l.o.g. that the intersection of $\nabla f^{-1}(y)$ with $\operatorname{dom}\nabla f$ is convex, for all $y \in \operatorname{Im}\nabla f$.

Now we give the formal definition of **face of a convex function** and relate this object to the sets $\nabla f^{-1}(y)$.

Definition 2.1. *A* tangent hyperplane *to the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is a subset of $\mathbb{R}^{n+1}$ of the form*

$$H_y = \{(z, h_y(z)) : z \in \mathbb{R}^n, \text{ and } h_y(z) = f(x) + y \cdot (z - x)\}, \tag{2.2}$$

where $x \in \nabla f^{-1}(y)$.

We note that, by convexity, the above definition is independent of $x \in \nabla f^{-1}(y)$.

Definition 2.2. *A* face of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ *is a set of the form*

$$H_y \cap \operatorname{graph} f_{|\operatorname{dom}\nabla f}. \tag{2.3}$$

Definition 2.2 corresponds to the notion of *exposed face* of the epigraph of f given in [**51**]. The faces of a convex function are a family of disjoint convex sets whose union covers $\operatorname{graph} f_{|\operatorname{dom}\nabla f}$.
It is easy to check that, $\forall\, y \in \operatorname{Im}\nabla f$ and $\forall\, z$ such that $(z, f(z)) \in H_y \cap \operatorname{graph} f_{|\operatorname{dom}\nabla f}$, we have that $y = \nabla f(z)$.

If we denote by $\pi_{\mathbb{R}^n} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection map on the first n coordinates, one can see that, for all $y \in \operatorname{Im}\nabla f$,

$$\nabla f^{-1}(y) = \pi_{\mathbb{R}^n}(H_y \cap \operatorname{graph} f_{|\operatorname{dom}\nabla f}). \tag{2.4}$$

For this reason, the sets $\{\nabla f^{-1}(y)\}_{y\in\operatorname{Im}\nabla f}\} \subset \mathbb{R}^n$ will be called **projected faces** of f. Notice that the projected faces are a partition in $\mathbb{R}^n$ into convex sets whose union covers $\operatorname{dom}\nabla f$. For notational convenience, the set $\nabla f^{-1}(y)$ will be denoted as F_y. We also write F_y^k instead of F_y whenever we want to emphasize the fact that the latter has dimension k, for $k = 0, \dots, n$ (where the dimension of a convex set C, denoted by $\dim(C)$, is the dimension of its affine hull $\operatorname{aff}(C)$). We also denote by $\operatorname{ri}(F_y)$ the relative interior of a face and by $\operatorname{rb}(F_y)$ its relative boundary (see Definition 1.12).

Due to the fact that the map $\operatorname{graph} f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$ is bi-Lipschitz and that its inverse $\pi_{\mathbb{R}^n}$ preserves the convexity of the faces, their disjointness and their linear dimension, we have the disintegration Theorem 0.1 is implied by the following two lemmas.

Lemma 2.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\{\operatorname{ri}(F_y)\}_{y\in\operatorname{Im}\nabla f} \subset \mathbb{R}^n$ be the partition given by the relative interiors of the projected faces of f. Then, the* 1-dimensional slices *of $\{\operatorname{ri}(F_y)\}_{y\in\operatorname{Im}\nabla f} \subset \mathbb{R}^n$ (see Definition 1.23) satisfy the* cone approximation property *(see Definition 1.11).*

LEMMA 2.4. *The set*

$$\mathscr{N} = \bigcup_{y:\, \dim F_y \geq 1} \mathrm{rb}(F_y) \tag{2.5}$$

is $\mathscr{L}^n$-negligible.

Indeed, we have the following

PROOF OF THEOREM 0.1: Let us assume that Lemma 2.3 and Lemma 2.4 hold. Then, by Theorem 1.1, for all $K \subset \mathbb{R}^n$ such that $\mathscr{L}^n(K) < +\infty$

$$\mathscr{L}^n \llcorner K = \int \mu_y^k \, d\nabla f_\#(\mathscr{L}^n \llcorner K)(y), \tag{2.6}$$

where μ_y^k is equivalent to $\mathscr{H}^k \llcorner F_y^k$ for $\nabla f_\#(\mathscr{L}^n \llcorner K)$-a.e. $y \in \mathrm{Im}\, \nabla f$.

Finally, by the properties of the map graph $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$ mentioned above, this immediately yields the Disintegration Theorem 0.1.

□

2.2. Measurability of the faces directions

The aim of this subsection is to show that the set of the relative interiors of the projected faces of a convex function f (2.4) can be parametrized by a Borel measurable (and multivalued) map as in Definition 1.6. This will give the existence of a unique strongly consistent disintegration and will allow us to decompose $\mathrm{dom}\, \nabla f$ into a countable family of k-dimensional model sets as in Section 1.3.2.

Since ∇f is a Borel map, we can assume that the quotient map p of Definition 1.2 is given by the restriction of ∇f to the set $\cup_{y \in \mathrm{Im}\, \nabla f} \mathrm{ri}(F_y)$ and that the quotient space is given by $(\mathrm{Im}\, \nabla f, \mathscr{B}(\mathrm{Im}\, \nabla f))$, which is measurably included in $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$.

Therefore, we are left to prove the set $\cup_{y \in \mathrm{Im}\, \nabla f} \mathrm{ri}(F_y)$ is Borel.

To this aim, let us set some more notation and define sets and (multivalued) maps relative to the projected faces, which will be used also in the rest of the chapter.

First of all, we define the equivalence maps

$$\mathrm{dom}\, \nabla f \ni x \mapsto \mathcal{P}(x) := \big\{ z \in \mathrm{dom}\, \nabla f : \exists\, y \in \mathrm{Im}\, \nabla f \text{ s.t. } x, z \in F_y \big\} \tag{2.7}$$

and

$$\mathrm{dom}\, \nabla f \ni x \mapsto \mathcal{R}(x) := \big\{ z \in \mathrm{dom}\, \nabla f : \exists\, y \in \mathrm{Im}\, \nabla f \text{ s.t. } x \in F_y \text{ and } z \in \mathrm{ri}(F_y) \big\}. \tag{2.8}$$

We notice that $\mathcal{P}(x)$ is the projected face of f to which x belongs, while $\mathcal{R}(x)$ is its relative interior. Moreover,

$$\bigcup_{y \in \mathrm{Im}\, \nabla f} \mathrm{ri}(F_y) = \pi_2(\mathrm{graph}\, \mathcal{R}), \tag{2.9}$$

where $\pi_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the projection onto the second factor.

Since the disintegration over the 0-dimensional faces is trivial, we will restrict our attention to the set (see (1.32))

$$\mathcal{T} = \{ x \in \pi_2(\mathrm{graph}\, \mathcal{R}) : \mathcal{R}(x) \neq \{x\} \}. \tag{2.10}$$

On $\mathcal{T}$ we can also define the *direction map* (see (1.35))

$$\mathcal{T} \ni x \mapsto \mathcal{D}(x) := \left\{ \frac{y-x}{\|y-x\|} \,:\, y \in \mathcal{R}(x) \right\}. \tag{2.11}$$

Our aim is then prove the following lemma. We recall that a multivalued map is defined to be Borel measurable if the counterimage of any open set is Borel.

LEMMA 2.5. *The graphs of the multivalued maps $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{D}$ are a σ-compact sets.*

As a consequence of Lemma 2.5 and equations (2.9)-(2.10), the domain of our partition $\bigcup_{y\in\operatorname{Im}\nabla f} \operatorname{ri}(F_y)$ and $\mathcal{T}$ are σ-compact, thus Borel.

PROOF OF LEMMA 2.5: From the continuity of f and from the upper-semicontinuity of its subdifferential (2.1), we have that the graph of the multivalued map $\tilde{\mathcal{P}} : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$\tilde{\mathcal{P}}(x) := \left\{ z \in \mathbb{R}^n \,:\, \exists\, y \in \partial^- f(x) \text{ s.t. } f(z) = f(x) + y \cdot (z-x) \right\}$$

is closed in $\mathbb{R}^n \times \mathbb{R}^n$.

Hence, since $\operatorname{dom} \nabla f$ is σ-compact (see Section 2.1), the sets

$$\operatorname{graph} \mathcal{P} = \operatorname{graph} \tilde{\mathcal{P}} \cap \operatorname{dom} \nabla f \times \mathbb{R}^n, \quad \operatorname{graph} \mathcal{R} = \operatorname{graph} \tilde{\mathcal{P}} \cap \operatorname{dom} \nabla f \times \operatorname{dom} \nabla f.$$

The fact that the graph of $\mathcal{D}$ is σ-compact follows from the continuity of the map $\mathbb{R}^n \times \mathbb{R}^n \ni (x,z) \mapsto \frac{z-x}{|z-x|}$ out of the diagonal. $\square$

2.3. Validity of the cone approximation property

This section is devoted to the proof of Lemma 2.3. Using the notation of Chapter 1, with $X^k_\alpha = \operatorname{ri}(F^k_y)$ for some $y \in \operatorname{Im} \nabla f$, we can restrict to prove the cone approximation property for a 1-dimensional $\mathcal{D}$-cylinder $\mathscr{C}$ with axis $\langle \mathrm{e} \rangle$ generated by a unit vector $\mathrm{e} \in \mathbb{S}^{n-1}$, direction vector field v_e and parameterizing map $\sigma^{\mathrm{w}+t\mathrm{e}}$ (see Definition 1.21). Moreover, w.l.o.g., we can assume that $\mathrm{w} = 0$ and we set, for simplicity of notation, $\sigma^t := \sigma^{\mathrm{w}+t\mathrm{e}}$, $h^\pm = h^\pm(\mathrm{w},\mathrm{e})$ and $Z_t = Z_{\mathrm{w}+t\mathrm{e}}$. By symmetry, it is sufficient to prove the existence of approximating finite union of cones $\{v^t_{\mathrm{e},-,j}\}_{j\in\mathbb{N}}$ for $-v^t_\mathrm{e}$ as in Lemma 1.10 with bases in Z_t for some fixed $t \in (h^-,h^+]$ and vertices in Z_{h^-}. We also introduce the following notation

$$H_t := \{x \in \mathbb{R}^n : x \cdot \mathrm{e} = t\}, \quad \text{where} \quad t \in [h^-,h^+] \quad \text{and} \quad h^-,h^+ \in \mathbb{R} : h^- < 0 < h^+;$$

$$\mathcal{B}^{n-1}_R(x) = \{z \in H_{\{x\cdot\mathrm{e}\}} : \|z-x\| \leq R\};$$

$$\mathscr{C}_t = \bigcup_{s\in[h^-,t]} H_s \cap \mathscr{C};$$

$$l_t(x) = \mathcal{R}(x) \cap \mathscr{C}_t, \quad \forall\, x \in \mathscr{C}_t;$$

$$\forall\, x \in \mathbb{R}^n, \quad \tilde{x} := (x, f(x)) \in \mathbb{R}^{n+1} \quad \text{and} \quad \forall\, A \subset \mathbb{R}^n, \quad \tilde{A} := \operatorname{graph} f_{|A};$$

$$v^t_j = v^t_{\mathrm{e},-,j}.$$

Moreover, we recall the following definitions:

DEFINITION 2.6. *The* convex envelope *of a set of points $X \subset \mathbb{R}^n$ is the smaller convex set* conv(X) *that contains X. The following characterization holds:*

$$\operatorname{conv}(X) = \Big\{ \sum_{j=1}^{J} \lambda_j\, x_j \,:\, x_j \in X,\, 0 \leq \lambda_j \leq 1,\, \sum_{j=1}^{J} \lambda_j = 1,\, J \in \mathbb{N} \Big\}. \tag{2.12}$$

DEFINITION 2.7. *The* graph of a compact convex set $C \subset \mathbb{R}^{n+1}$, *that we denote by* graph(C), *is the graph of the function $g : \pi_{\mathbb{R}^n}(C) \to \mathbb{R}$ which is defined by*

$$g(x) = \min\{t \in \mathbb{R} : (x, t) \in C\}.$$

DEFINITION 2.8. *A* supporting hyperplane *to the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is an affine hyperplane in $\mathbb{R}^{n+1}$ of the form*

$$H = \{w \in \mathbb{R}^{n+1} : w \cdot b = \beta\},$$

where $b \neq 0$, $w \cdot b \leq \beta$ for all $w \in \operatorname{epi} f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t \geq f(x)\}$ and $w \cdot b = \beta$ for at least one $w \in \operatorname{epi} f$. As f is defined and real-valued on all $\mathbb{R}^n$, every supporting hyperplane is of the form

$$H_y = \{(z, h_y(z)) \,:\, z \in \mathbb{R}^n, h_y(z) = f(x) + y \cdot (z - x)\},$$

for some $y \in \partial^- f(x)$. Whenever $y \in \operatorname{Im} \nabla f$, H_y is a tangent hyperplane to the graph of f according to Definition 2.1.

DEFINITION 2.9. *A* supporting k-plane *to the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is an affine k-dimensional subspace of a supporting hyperplane to the graph of f (see Definition 2.8) whose intersection with* graph f *is nonempty.*

DEFINITION 2.10. *An R-face of a convex set $C \subset \mathbb{R}^d$ is a convex subset C' of C such that every closed segment in C with a relative interior point in C' has both endpoints in C'. The zero-dimensional R-faces of a convex set are also called* extreme points *and the set of all extreme points in a convex set C will be denoted by* ext(C).

The definition of R-face corresponds to the definition of *extremal face* of a convex set in [**51**].
We also recall the following propositions, for which we refer to Section 18 of [**51**].

PROPOSITION 2.11. *Let $C = \operatorname{conv}(D)$, where D is a set of points in $\mathbb{R}^d$, and let C' be a nonempty R-face of C. Then $C' = \operatorname{conv}(D')$, where D' consists of the points in D which belong to C'.*

PROPOSITION 2.12. *Let C be a bounded closed convex set. Then $C = \operatorname{conv}(\operatorname{ext}(C))$.*

In our construction we first approximate the 1-dimensional faces that lie on the graph of f restricted to the given $\mathcal{D}$-cylinder and then we get the approximating vector fields $\{v_j^t\}_{j \in \mathbb{N}}$ simply projecting the directions of those approximations on the first n coordinates.

PROOF OF LEMMA 2.3: **Step 1** *Preliminary considerations*
Eventually partitioning $\mathscr{C}$ into a countable collection of sets, we can assume that $\sigma^t(Z)$ and $\sigma^{h^-}(Z)$ are bounded, with $\sigma^t(Z) \subset \mathcal{B}^{n-1}_{R_1}(x_1) \subset H_t$ and $\sigma^{h^-}(Z) \subset \mathcal{B}^{n-1}_{R_2}(x_2) \subset H_{h^-}$. Then,

if we call K_t the convex envelope of $\mathcal{B}^{n-1}_{R_1}(x_1) \cup \mathcal{B}^{n-1}_{R_2}(x_2)$, the function $f_{|K_t}$ is uniformly Lipschitz with a certain Lipschitz constant L_f.

Step 2 *Construction of approximating functions* (see Figure 1)
Now we define a sequence of functions $\{f_j\}_{j\in\mathbb{N}}$ whose 1-dimensional faces approximate,

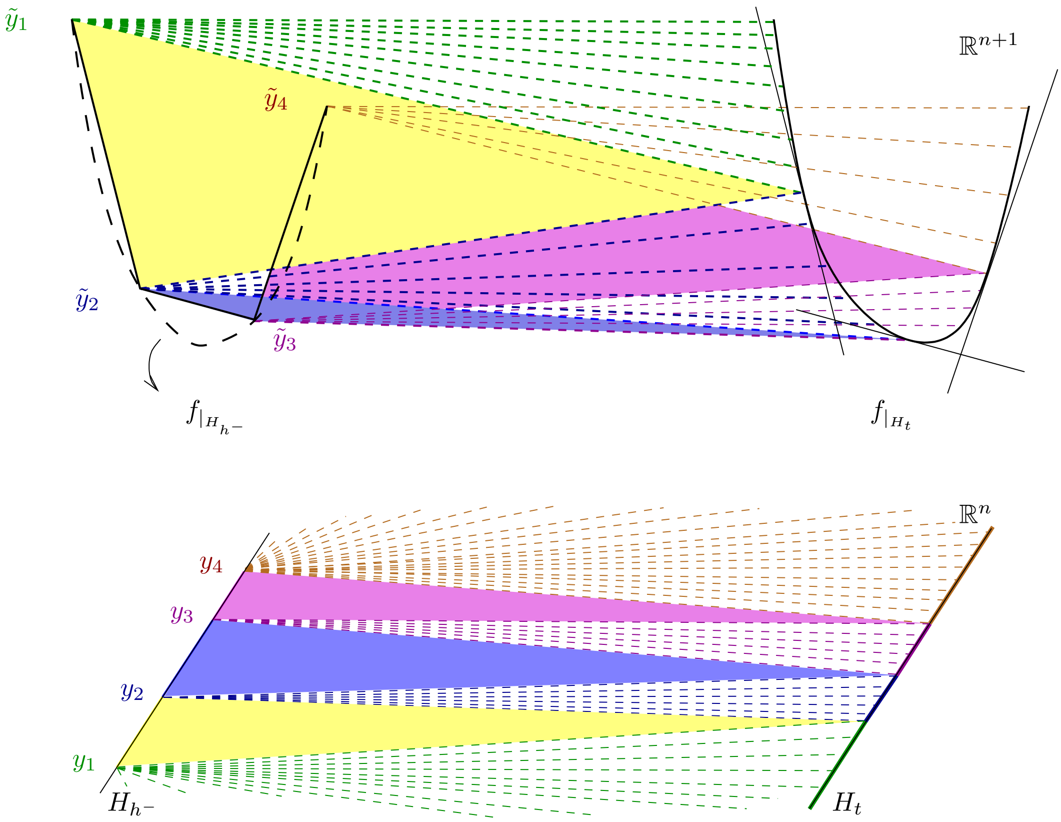

FIGURE 1. Illustration of a vector field approximating the one dimensional faces of f (Lemma 2.3). One can see in the picture the graph of f_4, which is the convex envelope of $\{\tilde{y}_i\}_{i=1,\dots,4}$ and $f_{|H_t}$. The faces of f_j connect $\mathscr{H}^{n-1}$-a.e. point of H_t to a single point among the $\{\tilde{y}_i\}_i$, while the remaining points of H_t correspond to some convex envelope $\mathrm{conv}(\{\tilde{y}_{i_\ell}\}_\ell)$ — here represented by the segments $[\tilde{y}_i, \tilde{y}_{i+1}]$. The region where the vector field v^t_4, giving the directions of the faces of f_j, is multivalued corresponds to the 'planar' faces of f_4. The affine span of these planar faces, restricted to suitable planes contained in H_t, provides a supporting hyperplane for the restriction of f to these latter planes — in the picture they are depicted as tangent lines. The intersection of $\sigma^t(Z) \subset H_t$ with any supporting plane to the graph of $f_{|H_t}$ must contain just one point, otherwise $\mathcal{D}$ would be multivalued at some point of $\sigma^t(Z)$.

in a certain sense, the pieces of the 1-dimensional faces of f which are contained in $\mathscr{C}_t$.

The directions of a properly chosen subcollection of the 1-dimensional faces of f_j will give, when projected on the first n coordinates, the approximate vector field v_j^t.

First of all, take a sequence $\{\tilde{y}_i\}_{i\in\mathbb{N}} \subset \tilde{\sigma}^{h^-}(Z)$ such that the collection of segments $\{\tilde{l}_t(y_i)\}_{i\in\mathbb{N}}$ is dense in $\bigcup_{y\in\sigma^{h^-}(Z)} \tilde{l}_t(y)$.

For all $j \in \mathbb{N}$, let C_j be the convex envelope of the set

$$\{\tilde{y}_i\}_{i=1}^{j} \quad \cup \quad \operatorname{graph} f_{|_{\mathbf{B}_{R_1}^{n-1}(x_1)}}$$

and call $f_j : \pi_{\mathbb{R}^n}(C_j) \to \mathbb{R}$ the function whose graph is the graph of the convex set C_j. We note that $\pi_{\mathbb{R}^n}(C_j) \cap H_{h^-} = \operatorname{conv}(\{y_i\}_{i=1}^{j})$ and $\operatorname{graph} f_{j|_{\operatorname{conv}(\{y_i\}_{i=1}^{j})}} = \operatorname{graph}(\operatorname{conv}(\{\tilde{y}_i\}_{i=1}^{j}))$.

We claim that the graph of f_j is made of segments that connect the points of $\operatorname{graph}(\operatorname{conv}(\{\tilde{y}_i\}_{i=1}^{j}))$ to the graph of $f_{|_{\mathbf{B}_{R_1}^{n-1}(x_1)}}$ (indeed, by convexity and by the fact that $\tilde{y}_i = (y_i, f(y_i))$, $f_j = f$ on $\mathcal{B}_{R_1}^{n-1}(x_1)$).

In order to prove this, we first observe that, by definition, all segments of this kind are contained in the set C_j. On the other hand, by (2.12), all the points in C_j are of the form

$$w = \sum_{i=1}^{J} \lambda_i \, w_i,$$

where $\sum_{i=1}^{J} \lambda_i = 1$, $0 \leq \lambda_i \leq 1$ and $w_i \in \{\tilde{y}_i\}_{i=1}^{j} \cup \operatorname{graph} f_{|_{\mathcal{B}_{R_1}^{n-1}(x_1)}}$. In particular, we can write

$$w = \alpha z + (1-\alpha) r, \quad \text{where } 0 \leq \alpha \leq 1, \quad z \in \operatorname{conv}\big(\{\tilde{y}_i\}_{i=1}^{j}\big) \quad \text{and } r \in \operatorname{epi} f_{|_{\mathcal{B}_{R_1}^{n-1}(x_1)}}. \tag{2.13}$$

Moreover, if we take two points $z' \in \operatorname{graph}(\operatorname{conv}(\{\tilde{y}_i\}_{i=1}^{j}))$, $r' \in \operatorname{graph} f_{|_{\mathcal{B}_{R_1}^{n-1}(x_1)}}$ such that $\pi_{\mathbb{R}^n}(z') = \pi_{\mathbb{R}^n}(z)$ and $\pi_{\mathbb{R}^n}(r') = \pi_{\mathbb{R}^n}(r)$, we have that the point

$$w' = \alpha z' + (1-\alpha) r'$$

belongs to C_j, lies on a segment which connects $\operatorname{graph}(\operatorname{conv}(\{\tilde{y}_i\}_{i=1}^{j}))$ to $\operatorname{graph} f_{|_{\mathcal{B}_{R_1}^{n-1}(x_1)}}$ and its $(n+1)$ coordinate is less than the $(n+1)$ coordinate of w.

The graph of f_j contains also all the pieces of 1-dimensional faces $\{\tilde{l}_t(y_i)\}_{i=1}^{j}$, since by construction it contains their endpoints and it lies over the graph of $f_{|_{\pi_{\mathbb{R}^n}(C_j)}}$.

Step 3 *Construction of approximating vector fields* (see Figure 1)

Among all the segments in the graph of f_j that connect the points of $\operatorname{graph}(\operatorname{conv}(\{\tilde{y}_i\}_{i=1}^{j}))$ to the graph of $f_{|_{\mathcal{B}_{R_1}^{n-1}(x_1)}}$, we select those of the form $[\tilde{x}, \tilde{y}_k]$, where $x \in \sigma^t(Z)$, $y_k \in \{y_i\}_{i=1}^{j}$, and we show that for $\mathscr{H}^{n-1}$-a.e. $x \in \sigma^t(Z)$ there exists only one segment within this class which passes through $\tilde{x}$. The approximating vector field will be given by the projection on the first n coordinates of the directions of these segments.

First of all, we claim that for all $x \in \mathcal{B}^{n-1}_{R_1}(x_1)$ the graph of f_j contains at least a segment of the form $[\tilde{x}, \tilde{y}_i]$ for some $i \in \{1, \dots, j\}$.
Indeed, we show that if $\tilde{x}$ is the endpoint of a segment of the form $[\tilde{x}, (y, f_j(y))]$ where $y \in \mathrm{conv}\big(\{y_i\}_{i=1}^j\big)$ but $(y, f_j(y)) \notin \mathrm{ext}\big(\mathrm{conv}\big(\{y_i\}_{i=1}^j\big)\big)$, then there are at least two segments of the form $[\tilde{x}, \tilde{y}_k]$ with $\tilde{y}_k \in \mathrm{ext}(\mathrm{conv}(\{\tilde{y}_i\}_{i=1}^j)) \subset \{\tilde{y}_i\}_{i=1}^j$ (here we assume that $j \geq 2$).
In order to prove this, take a point $(z, f_j(z))$ in the open segment $(\tilde{x}, (y, f_j(y)))$ and a supporting hyperplane $H(z)$ to the graph of f_j that contains that point. By definition, $H(z)$ contains the whole segment $[\tilde{x}, (y, f_j(y))]$ and the set $H(z) \cap \big(H_{h^-} \times \mathbb{R}\big)$ is a supporting hyperplane to the set $\mathrm{graph}(\mathrm{conv}(\{\tilde{y}_i\}_{i=1}^j))$ that contains the point $(y, f_j(y))$.
Now, take the smallest R-face C of $\mathrm{conv}\big(\{\tilde{y}_i\}_{i=1}^j\big)$ which is contained in $\mathrm{graph}(\mathrm{conv}\big(\{\tilde{y}_i\}_{i=1}^j\big))$ and contains the point $(y, f_j(y))$, that is given by the intersection of all R-faces which contain $(y, f_j(y))$.
By Propositions 2.11 and 2.12, $C = \mathrm{conv}\big[\mathrm{ext}\big(\mathrm{conv}\big(\{\tilde{y}_i\}_{i=1}^j\big)\big) \cap C\big]$ and as $(y, f_j(y)) \notin \mathrm{ext}\big(\mathrm{conv}\big(\{\tilde{y}_i\}_{i=1}^j\big)\big)$, $\dim(C) \geq 1$ and the set $\mathrm{ext}\big(\mathrm{conv}\big(\{\tilde{y}_i\}_{i=1}^j\big)\big) \cap C$ contains at least two points $\tilde{y}_k$, $\tilde{y}_l$.
In particular, since both C and $\tilde{x}$ belong to $H(z) \cap \mathrm{graph}(f_j)$, by definition of supporting hyperplane we have that the graph of f_j contains the segments $[\tilde{x}, \tilde{y}_k]$, $[\tilde{x}, \tilde{y}_l]$ and our claim is proved.

Now, for each $j \in \mathbb{N}$, we define the (possibly multivalued) map $\mathcal{D}^t_j : \mathcal{B}^{n-1}_{R_1}(x_1) \to \mathbb{R}^n$ as follows:

$$\mathcal{D}^t_j :\ x \mapsto \left\{ \frac{y_i - x}{|y_i - x|} \ :\ [\tilde{x}, \tilde{y}_i] \subset \mathrm{graph}(f_j) \right\}$$

and we prove that the set

$$B_j := \sigma^t(Z) \cap \{x \in \mathcal{B}^{n-1}_{R_1}(x_1) : \mathcal{D}^t_j(x) \text{ is multivalued } \} \tag{2.14}$$

is $\mathscr{H}^{n-1}-$negligible, $\forall\, j \in \mathbb{N}$.
Thus, if we neglect the set $B = \bigcup_{j \in \mathbb{N}} B_j$, we can define our approximating vector field as

$$v^t_j(x) = \{\mathcal{D}^t_j(x)\}, \quad \forall\, x \in \sigma^t(Z) \backslash B, \quad \forall\, j \in \mathbb{N}. \tag{2.15}$$

In order to show that $\mathscr{H}^{n-1}(B_j) = 0$ we first prove that, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathcal{B}^{n-1}_{R_1}(x_1)$, whenever $\mathcal{D}^t_j(x)$ contains the directions of two segments, f_j must be linear on their convex envelope.
Indeed, suppose that the graph of f_j contains two segments $[\tilde{x}, \tilde{y}_{i_k}]$, where $i_k \in \{1, \dots, j\}$ and $k = 1, 2$, and consider two points $(z_k, f_j(z_k)) \subset [\tilde{x}, \tilde{y}_{i_k}]$ such that

$$z_1 = x + s\mathrm{e} + a_1 v_1, \quad s \in [h^- - t, 0),\ v_1 \in H_0;$$
$$z_2 = x + s\mathrm{e} + a_2 v_2, \quad s \in [h^- - t, 0),\ v_2 \in H_0.$$

As f_j is linear on $[x, y_{i_k}]$, we have that

$$f_j(z_k) = f_j(x) + r_k \cdot (s\mathrm{e} + a_k v_k), \tag{2.16}$$

where $r_k \in \partial^- f_j(x), \quad k = 1, 2.$

Moreover, since

$$\pi_{H_0}(\partial^- f_j(x)) = \partial^- f_{|_{\mathcal{B}^{n-1}_{R_1}(x_1)}}(x)$$

and the set where $\partial^- f_{|_{\mathcal{B}^{n-1}_{R_1}(x_1)}}$ is multivalued is $\mathscr{H}^{n-2}$-rectifiable (see for e.g. [**58, 1**]), we have that, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathcal{B}^{n-1}_{R_1}(x_1)$

$$r \cdot v = \nabla(f_{|_{\mathcal{B}^{n-1}_{R_1}(x_1)}})(x) \cdot v, \quad \forall\, r \in \partial^- f_j(x),\ \forall\, v \in H_0.$$

Then, if we put $w = \nabla(f_{|_{\mathcal{B}^{n-1}_{R_1}(x_1)}})(x)$, (2.16) becomes

$$f_j(z_k) = f_j(x) + r_k \cdot s\mathrm{e} + w \cdot a_k v_k. \tag{2.17}$$

If $\bar{z} = (1-\lambda)z_1 + \lambda z_2$, we have that

$$\begin{aligned} f_j(\bar{z}) &\le (1-\lambda) f_j(z_1) + \lambda f_j(z_2) \\ &\overset{(2.17)}{=} f_j(x) + s((1-\lambda)r_1 + \lambda r_2)\cdot \mathrm{e} + w \cdot ((1-\lambda)a_1 v_1 + \lambda a_2 v_2). \end{aligned} \tag{2.18}$$

As $((1-\lambda)r_1 + \lambda r_2) \in \partial^- f_j(x)$, we also obtain that

$$\begin{aligned} f_j(\bar{z}) \ge\ & f_j(x) + s((1-\lambda)r_1 + \lambda r_2)\cdot \mathrm{e} \\ & + ((1-\lambda)r_1 + \lambda r_2)\cdot((1-\lambda)a_1 v_1 + \lambda a_2 v_2) = \\ & = f_j(x) + s((1-\lambda)r_1 + \lambda r_2)\cdot \mathrm{e} + w\cdot((1-\lambda)a_1 v_1 + \lambda a_2 v_2) = \\ & \overset{(2.17)}{=} (1-\lambda) f_j(z_1) + \lambda f_j(z_2). \end{aligned}$$

Thus, we have that $f_j((1-\lambda)z_1 + \lambda z_2) = (1-\lambda) f_j(z_1) + \lambda f_j(z_2)$ and our claim is proved.

In particular, there exists a supporting hyperplane to the graph of f_j which contains the affine hull of the convex envelope of $\{[\tilde{x}, \tilde{y}_{i_k}]\}_{k=1,2}$ and then this affine hull must intersect $H_t \times \mathbb{R}$ into a supporting line to the graph of $f_{|_{\mathcal{B}^{n-1}_{R_1}(x_1)}}$ which is parallel to the segment $[\tilde{y}_{i_1}, \tilde{y}_{i_2}]$.

Thus, if all the supporting lines to the graph of $f_{|_{\mathcal{B}^{n-1}_{R_1}(x_1)}}$ which are parallel to a segment $[\tilde{y}_k, \tilde{y}_m]$ (with $k, m \in \{1, \dots, j\}$, $k \neq m$) are parametrized as

$$l_{k,m} + w,$$

where $l_{k,m}$ is the linear subspace of $\mathbb{R}^{n+1}$ which is parallel to $[\tilde{y}_k, \tilde{y}_m]$ and $w \in W_{k,m} \subset H_t \times \mathbb{R}$ is perpendicular to $l_{k,m}$, we have that

$$B_j = \sigma^t(Z) \cap \Bigg[\bigcup_{\substack{k,m\in\{1,\dots,j\} \\ k<m}} \ \bigcup_{w \in W_{k,m}} \pi_{\mathbb{R}^n}(l_{k,m} + w) \Bigg].$$

By this characterization of the set B_j and by Fubini theorem on H_t w.r.t. the partition given by the lines which are parallel to $\pi_{\mathbb{R}^n}(l_{k,m})$ for every k and m, in order to show that $\mathscr{H}^{n-1}(B_j) = 0$ it is sufficient to prove that, $\forall\, w \in W_{k,m}$,

$$\mathscr{H}^{n-1}(\sigma^t(Z) \cap \pi_{\mathbb{R}^n}(l_{k,m} + w)) = 0. \tag{2.19}$$

Finally, (2.19) follows from the fact that a supporting line to the graph of $f|_{\mathcal{B}^{n-1}_{R_1}(x_1)}$ cannot contain two distinct points of $\tilde{\sigma}^t(Z)$, because otherwise they would be contained in a higher dimensional face of the grapf of f contraddicting the definition of $\tilde{\sigma}^t(Z)$.

Then, the vector field defined in (2.15) is defined $\mathscr{H}^{n-1}$-a.e..

Step 4 *Convergence of the approximating vector fields*

Here we prove the convergence property of the vector field defined in (2.15) as stated in Lemma 1.10.
This result is obtained as a consequence of the uniform convergence of the approximating functions f_j to the function $\hat{f}$ which is the graph of the set

$$\hat{C} = \mathrm{conv}\big(\{\tilde{l}_t(y_i)\}_{i\in\mathbb{N}}\big).$$

First of all we observe that, since $C_j \nearrow \hat{C}$,

$$\mathrm{dom}\, f_j = \pi_{\mathbb{R}^n}(C_j) \nearrow \mathrm{dom}\, \hat{f} = \pi_{\mathbb{R}^n}(\hat{C}) \quad \text{and} \quad f_j(x) \searrow \hat{f}(x) \quad \forall\, x \in \mathrm{ri}(\pi_{\mathbb{R}^n}(\hat{C})),$$

where $f_j(x)$ is defined $\forall\, j \geq j_0$ such that $x \in \pi_{\mathbb{R}^n}(C_{j_0})$.

In order to prove that $f_j(x) \searrow \hat{f}(x)$ uniformly, we show that the functions f_j are uniformly Lipschitz on their domain, with uniformly bounded Lipschitz constants.
We recall that the graph of f_j is made of segments that connect the points of graph $f|_{\mathcal{B}^{n-1}_{R_1}(x_1)}$ to the points of graph(conv$\big(\{\tilde{y}_i\}_{i=1}^j\big)$).

In order to find and upper bound for the incremental ratios between points $z, w \in \mathrm{dom}\, f_j$, we distinguish two cases.

<u>Case 1</u>: $[z,w] \subset [x, y_k]$, where $x \in \mathcal{B}^{n-1}_{R_1}(x_1)$, $y_k \in \{y_i\}_{i=1}^j$ and $[\tilde{x}, \tilde{y}_k] \subset \mathrm{graph}(f_j)$.

In this case we have that

$$\frac{|f_j(z) - f_j(w)|}{|z-w|} = \frac{|f_j(x) - f_j(y_k)|}{|x-y_k|} = \frac{|f(x) - f(y_k)|}{|x-y_k|} \leq L_f,$$

where L_f is the Lipschitz constant of f on K_t.

<u>Case 2</u>: Otherwise we observe that, since f_j is convex,

$$|f_j(z) - f_j(w)| \leq \sup_{r\in\partial^- f_j(z)\cup\partial^- f_j(w)} |r\cdot(z-w)|. \tag{2.20}$$

Let then $r \in \partial^- f_j(z) \cup \partial^- f_j(w)$ be a maximizer of the r.h.s. of (2.20) and let us suppose, without loss of generality, that $r \in \partial^- f_j(z)$. If $x \in \mathcal{B}^{n-1}_{R_1}(x_1)$ is such that $(z, f_j(z)) \subset [(y, f_j(y)), \tilde{x}] \subset \mathrm{graph}(f_j)$ for some $y \in \mathrm{conv}(\{y_i\}_{i=1}^j)$, we have the following unique decomposition

$$w - z = \beta_j(z,w)\left(\frac{x-z}{|x-z|}\right) + \gamma_j(z,w)q,$$

where $q \in \mathbb{S}^{n-1} \cap H_0$ and $\beta_j(z,w)$, $\gamma_j(z,w) \in \mathbb{R}$.

Then,

$$r\cdot(w-z) = \beta_j(z,w)\left(r\cdot\frac{x-z}{|x-z|}\right) + \gamma_j(z,w)(r\cdot q). \tag{2.21}$$

The first scalar product in (2.21) can be estimated as in Case 1.

As for the second term, we note that the supporting hyperplane to the graph of f_j given by the graph of the affine function $h(p) = f_j(z) + r \cdot (p - z)$ contains the segment $[(z, f_j(z)), \tilde{x}]$ and its intersection with the hyperplane $H_t \times \mathbb{R}$ is given by a supporting hyperplane to the graph of $f_{|_{\mathcal{B}^{n-1}_{R_1}(x_1)}}$ which contains the point $\tilde{x}$.

Moreover, as $q \in H_0$, we have that

$$r \cdot q = \pi_{H_0}(r) \cdot q,$$

and we know that $\pi_{H_0}(r) \in \partial^- f_{|_{\mathbf{B}^{n-1}_{R_1}(x_1)}}(x)$.
By definition of subdifferential, for all $s \in \partial^- f_{|_{\mathcal{B}^{n-1}_{R_1}(x_1)}}(x)$ and for all $\lambda > 0$ such that $x + \lambda q,\ x - \lambda q \in \mathcal{B}^{n-1}_{R_1}(x_1)$,

$$\frac{f(x) - f(x - \lambda q)}{\lambda} \le s \cdot q \le \frac{f(x + \lambda q) - f(x)}{\lambda} \tag{2.22}$$

and so the term $|r \cdot q|$ is bounded from above by the Lipschitz constant of f.
As the scalar products $\beta_j(z, w)$, $\gamma_j(z, w)$ are uniforlmly bounded w.r.t. j on $\operatorname{dom} f_j \subset \operatorname{dom} \hat{f}$, we conclude that the functions $\{f_j\}_{j \in \mathbb{N}}$ are uniformly Lipschitz on the sets $\{\operatorname{dom} f_j\}_{j \in \mathbb{N}}$ and their Lipschitz constants are uniformly bounded by some positive constant $\hat{L}$.
If we call $\hat{f}_j$ a Lipschitz extension of f_j to the set $\operatorname{dom} \hat{f}$ which has the same Lipschitz constant (Mac Shane lemma), by Ascoli-Arzelá theorem we have that

$$\hat{f}_j \to \hat{f} \quad \text{uniformly on } \operatorname{dom} \hat{f}.$$

Now we prove that, for $\mathscr{H}^{n-1}$-a.e. $x \in \sigma^t(Z) \backslash B$, $v^t_j(x) \to v_{\mathrm{e}}(x)$.
Given a point $x \in \sigma^t(Z) \backslash B$, we call $\tilde{y}_{j(x)}$, where $j \in \mathbb{N}$, the unique point $\tilde{y}_k \in \{\tilde{y}_i\}_{i=1}^{j}$ such that

$$v^t_j(x) = \frac{y_k - x}{|y_k - x|}.$$

By compactness of graph(conv($\{\tilde{y}_i\}_{i \in \mathbb{N}}$)) , there is a subsequence $\{j_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ such that

$$\tilde{y}_{j_n(x)} \to \hat{y} \in \operatorname{graph} f,$$

hence

$$v^t_{j_n}(x) \to \hat{v} = \frac{\hat{y} - x}{|\hat{y} - x|}.$$

As the functions f_j converge to $\hat{f}$ uniformly, the point $\hat{y}$ and the whole segment $[\tilde{x}, \hat{y}]$ belong to the graph of $\hat{f}$.
So, there are two segments $\tilde{l}_t(x)$ and $[\tilde{x}, \hat{y}]$ which belong to the graph of $\hat{f}$ and pass through the point $\tilde{x}$.

Since $\hat{f}_{|_{\mathcal{B}^{n-1}_{R_1}(x_1)}} = f_{|_{\mathcal{B}^{n-1}_{R_1}(x_1)}}$, we can apply the same reasoning we made in order to prove that the set (2.14) was $\mathscr{H}^{n-1}$-negligible to conclude that the set

$$\sigma^t(Z) \cap \Big\{ x \in \mathcal{B}^{n-1}_{R_1}(x_1) : \exists \text{ more than two segments in the graph of } \hat{f} \\ \text{that connect } \tilde{x} \text{ to a point of graph}(\mathrm{conv}(\{\tilde{y}_i\}_{i\in\mathbb{N}})) \Big\}$$

has zero $\mathscr{H}^{n-1}$-measure.

Then, $[\tilde{x}, \hat{y}] = \tilde{l}_t(x)$ and $\hat{v} = -v^t_{\mathrm{e}}(x)$ for $\mathscr{H}^{n-1}$-a.e. $x \in \sigma^t(Z)$, so that the lemma is proved.

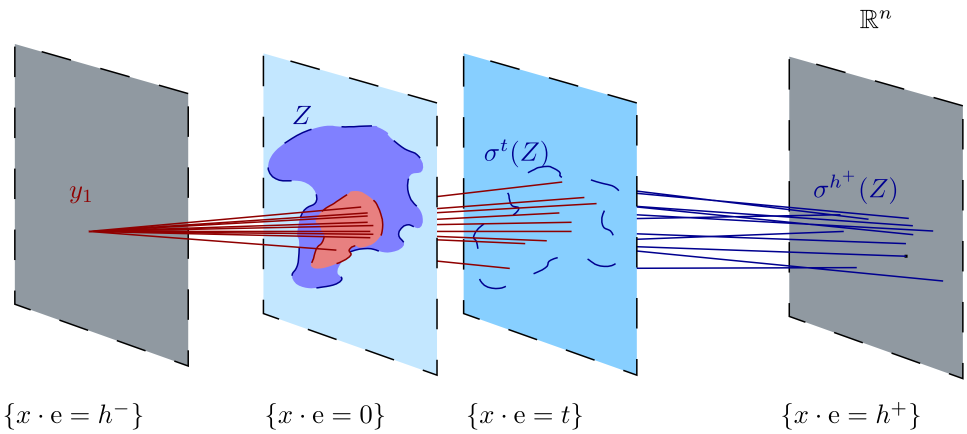

FIGURE 2. The vector field v_{e} is approximated by directions of approximating cones, in the picture one can see the first one. At the same time, Z is approximated by the push forward of $\sigma^t(Z)$ with the approximating vector field: compare the blue area with the red one.

□

For the proof of Lemma 2.4 in the next section, we will use the following remark, which extends the cone approximation property also to the 1-dimensional model sets containing relative boundary points of the faces.

REMARK 2.13. *In Lemma 2.4 we proved the cone approximation property for 1-dimensional $\mathcal{D}$-cylinders, i.e. 1-dimensional model sets given by the intersection of* $\bigcup_{y\in\nabla f(\tilde{F}^k)} \mathrm{ri}(F^k_y)$ *with some $(n-k+1)$-dimensional affine plane, for some $\tilde{F}^k \subseteq F^k$ and $k \in \{1,\dots,n-1\}$. Now we observe that, since the sets F^k_y are convex, then also every point* $x \in \bigcup_{y\in\nabla f(\tilde{F}^k)} \mathrm{rb}(F^k_y)$ *belongs to a 1-dimensional model set obtained intersecting* $\bigcup_{y\in\tilde{F}^k} F^k_y$ *with some $(n-k+1)$-dimensional affine plane. In fact, for all $x \in \mathrm{rb}(F^k_y)$ there exists some $\mathrm{e}(x) \in \mathbb{S}^{n-1}$ and either some $t^+ > 0$ such that $(x, x+t^+\mathrm{e}(x)] \subseteq \mathrm{ri}(F^k_y)$ or some $t^- < 0$*

such that $[x + t^{-}\mathrm{e}(x), x) \subseteq \mathrm{ri}(F^k_y)$. *Repeating exactly the same reasoning made in the proof of Lemma 2.4, one can prove that for the points in* $\cup_y \mathrm{rb}(F_y)$ *the cone approximation property of Lemma 1.10 holds in the direction* $\pm\mathrm{e}$ *of the corresponding* 1*-dimensional model set pointing towards the relative interior of the faces. Hence, for such points, one of the two absolute continuity estimates for the endpoints of the segments of a* 1*-dimensional set as in* (1.23) *holds.*

2.4. Negligibility of relative boundary points

In this section we prove Lemma 2.4.

We observe that the union of the borders of the n-dimensional faces has zero Lebesgue measure by convexity and by the fact that the n-dimensional faces of f are at most countable.

For faces of dimension k, with $1 < k < n$, the proof is by contradiction: one considers a Lebesgue point of suitable subsets of $\cup_y \mathrm{rb}(F^k_y)$ and applies one of the two fundamental estimates of (1.23) (see Remark 2.13) in order to show that the complementary is too big.

Equation (2.5) was first proved using a different technique in [**40**] –where it was shown that the union of the relative boundaries of the R-faces (see Definition 2.10) of an n-dimensional convex body C which have dimension at least 1 has zero $\mathscr{H}^{n-1}$-measure.

Proof of Lemma 2.4: Consider any n-dimensional face F^n_y. Being convex, it has nonempty interior. As a consequence, since two different faces cannot intersect, there are at most countably many n-dimensional faces $\{F^n_{y_i}\}_{i\in\mathbb{N}}$; moreover, by convexity, each $F^n_{y_i}$ has an $\mathscr{L}^n$-negligible boundary. Thus

$$\mathscr{L}^n\Big(\bigcup_i \mathrm{rb}\left(F^n_{y_i}\right)\Big) = 0.$$

Since $\mathscr{N} \subset \bigcup_{k=1}^{n} F^k \cup \Sigma^1(f)$, where $\Sigma^1(f)$ is the $\mathscr{L}^n$-negligible set of non-differentiability points of f, the thesis is reduced to showing that, for $1 \leq k < n$,

$$\mathscr{L}^n\Big(F^k \setminus E^k\Big) = 0, \tag{2.23}$$

with $E^k = \cup_y \mathrm{ri}(F^k_y)$.

Given a k dimensional subspace $V \in \mathscr{G}(k,n)$, a unit direction $\mathrm{e} \in \mathbb{S}^{n-1} \cap V$, and $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, define the set $\mathcal{A}^{p,\mathrm{e},V}$ of those $x \in \mathscr{N} \cap F^k_{\nabla f(x)}$ which satisfy the two relations

$$\inf_{d\in\mathcal{D}(x)} \|\pi_V(d)\| \geq 1/\sqrt{2} \tag{2.24}$$

$$\pi_V\big(F^k_{\nabla f(x)}\big) \supset \mathrm{conv}\Big(\big\{\pi_V(x)\big\} \cup \pi_V(x) + 2^{-p+1}\mathrm{e} + 2^{-p}\big(\mathbb{S}^{n-1} \cap V\big)\Big). \tag{2.25}$$

Choosing (p, e, V) in a sequence $\{(p_i, \mathrm{e}_i, V_i)\}_{i\in\mathbb{N}}$ which is dense in $\mathbb{N}_0 \times \big(S^{n-1} \cap V\big) \times \mathscr{G}(k,n)$, the family $\{\mathcal{A}^{p_i,\mathrm{e}_i,V_i}\}_{i\in\mathbb{N}}$ provides a countable covering of $F^k \setminus E^k$ with measurable sets. The measurability of each $\mathcal{A}^{p,\mathrm{e},V}$ can be deduced as follows. The set defined by (2.24) is exactly

$$\mathcal{D}^{-1} \circ \pi_V^{-1}\Big(V \setminus \mathrm{ri}\Big(\frac{1}{\sqrt{2}}\mathcal{B}^n\Big)\Big).$$

Moreover, (2.25) is equivalent to

$$\pi_V\big(\mathcal{P}(x) - x\big) \supset \mathrm{conv}(2^{-p+1}\mathrm{e} + 2^{-p}\big(\mathbb{S}^{n-1} \cap V\big)).$$

Since $\mathcal{P}$ and $\mathcal{D}$ are measurable (Lemma 2.5), then the measurability of $\mathcal{A}^{p,\mathrm{e},V}$ follows.

In particular, if by absurd (2.23) does not hold, then there exists a subset $\mathcal{A}^{p,\mathrm{e},V}$ of $F^k \backslash E^k$ with positive Lebesgue measure. Up to rescaling, one can assume w.l.o.g. that $p = 0$, $V = \langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle$, where $\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$ is an orthonormal basis of $\mathbb{R}^n$, and $\mathrm{e} = \mathrm{e}_1$. Moreover, we will denote $\mathcal{A}^{p,\mathrm{e},V}$ simply with $\mathcal{A}$.

Before reaching the contradiction $\mathscr{L}^n(\mathcal{A}) = 0$, we need the following remarks.
First of all we notice that, thanks to Remark 2.13, for $0 \leq h \leq 3$ and $\mathrm{t} \in \pi_V(\mathcal{A})$, one can prove the fundamental estimate

$$\mathscr{H}^{n-k}\big(\sigma^{\mathrm{t}+h\mathrm{e}}(S)\big) \geq \Big(\frac{3-h}{3}\Big)^{n-k} \mathscr{H}^{n-k}\big(S\big) \qquad \forall S \subset \mathcal{A} \cap \pi_V^{-1}(\mathrm{t}) \tag{2.26}$$

exactly as in Lemma 1.10, with the approximating vector field given in Step 3, Page 41. Indeed, the $(n-k+1)$-plane $\pi_V^{-1}(\mathbb{R}\mathrm{e})$ cuts the face of each $z \in \mathcal{A} \cap \pi_V^{-1}(\mathrm{t})$ into exactly one line l; this line has projection on V containing at least $[\mathrm{t}, \mathrm{t} + 3\mathrm{e}]$.
Notice moreover that, by (2.25), each point $x \in \mathrm{l}$, with $\pi_V(x) \in \mathrm{ri}\,\big([\mathrm{t}, \mathrm{t} + 3\mathrm{e}]\big)$, is a point in the relative interior of the face. In particular, it does not belong to $\mathcal{A}$.

Let us now prove the claim, assuming by contradiction that $\mathscr{L}^n(\mathcal{A}) > 0$ (see also Figure 3). Fix any $\varepsilon > 0$ small enough. w.l.o.g. one can suppose the origin to be a Lebesgue point of $\mathcal{A}$. Therefore, for every $0 < r < \bar{r}(\varepsilon) < 1$, there exists $T \subset \prod_{i=1}^k [0, r\mathrm{e}_i]$, with $\mathscr{H}^k(T) > (1-\varepsilon)r^k$, such that

$$\mathscr{H}^{n-k}\big(\mathcal{A} \cap \pi_V^{-1}(\mathrm{t}) \cap [0, r]^n\big) \geq (1-\varepsilon) r^{n-k} \qquad \text{for all } \mathrm{t} \in T. \tag{2.27}$$

Moreover, there is a set $Q \subset [0, r\mathrm{e}]$, with $\mathscr{H}^1(Q) > (1-2\varepsilon)r$, such that

$$\mathscr{H}^{k-1}\big(T \cap \pi_{\langle \mathrm{e} \rangle}^{-1}(\mathrm{q})\big) > (1-\varepsilon) r^{k-1} \qquad \text{for } \mathrm{q} \in Q. \tag{2.28}$$

Consider two points $\mathrm{q}, \mathrm{s} := \mathrm{q} + 2\varepsilon r\mathrm{e} \in Q$, and take $\mathrm{t} \in T \cap \pi_{\langle \mathrm{e} \rangle}^{-1}(\mathrm{q})$. By the fundamental estimate (2.26), one has

$$\mathscr{H}^{n-k}\big(\sigma^{\mathrm{t}+2\varepsilon r\mathrm{e}}\big(S_{\mathrm{t},r}\big)\big) \geq (1-\varepsilon)^{n-k} \mathscr{H}^{n-k}\big(S_{\mathrm{t},r}\big) \qquad \text{where } S_{\mathrm{t},r} := \mathcal{A} \cap \pi_V^{-1}(\mathrm{t}) \cap [0, r]^n.$$

Furthermore, condition (2.23) implies that $\|x + 2\varepsilon r\mathrm{e} - \sigma^{\mathrm{t}+2\varepsilon r\mathrm{e}}(x)\| \leq 2\varepsilon r$ for each $x \in A \cap \pi_V^{-1}(\mathrm{t})$. Moving points within $\pi_V^{-1}(\mathrm{t}) \cap [0, r]^n$ by means of the map $\sigma^{\mathrm{t}+2\varepsilon r\mathrm{e}}$, they can therefore reach only the square $\pi_V^{-1}(\mathrm{s}) \cap [-2\varepsilon r, (1+2\varepsilon)r]^n$. Notice that for ε small, since our proof is needed for $n \geq 3$ and $k \geq 1$,

$$\mathscr{H}^{n-k}([-2\varepsilon r, (1+2\varepsilon)r]^n \backslash [0, r]^n) = (1+4\varepsilon)^{n-k} r^{n-k} - r^{n-k} \leq 4(n-k)\varepsilon r^{n-k} + o(\varepsilon) < n2^n \varepsilon r^{n-k}.$$

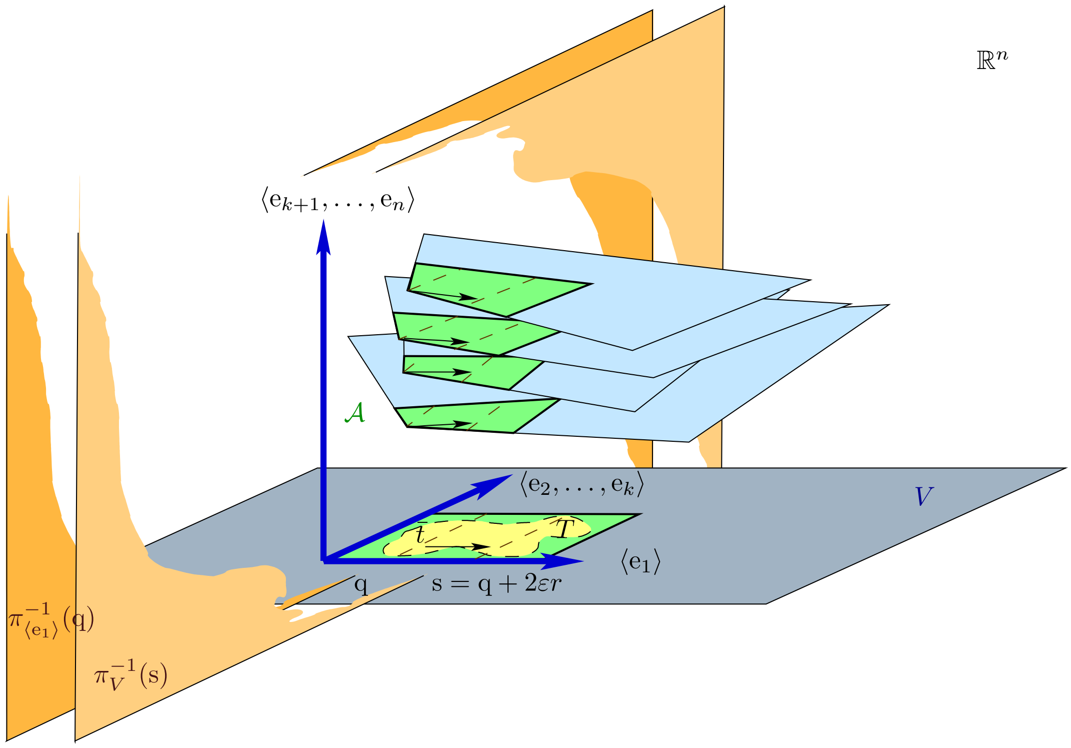

FIGURE 3. Illustration of the construction in the proof of Lemma 2.4. $\mathcal{A}$ is the set of points on the border of k-faces of f, projected on $\mathbb{R}^n$, having directions close to $V = \langle e_1, dots, e_k \rangle$ and such that, for each point $x \in \mathcal{A}$, $\pi_V(F^k_{\nabla f(x)})$ contains a fixed half k-cone centered at x with direction e_1. T is a subset of the square $\prod_{i=1}^k [0, re_i]$ such that, for every $t \in T$, $\pi_V^{-1}(t) \cap \mathcal{A}$ is 'big'. Finally, q, $s = q + 2\varepsilon r e_1$ are points on $[0, re_1]$ such that the intersection of T with the affine hyperplanes $\pi^{-1}_{\langle e_1 \rangle}(q)$, $\pi^{-1}_{\langle e_1 \rangle}(s)$ is 'big'. The absurd arises from the following. Due to the fundamental estimate, translating by $2\varepsilon r e_1$ the points $T \cap \pi^{-1}_{\langle e_1 \rangle}(q)$, one finds points in the complementary of T. Since $T \cap \pi^{-1}_{\langle e_1 \rangle}(q)$ was 'big', then $T \setminus \pi^{-1}_{\langle e_1 \rangle}(s)$ should be big, contradicting the fact that $T \cap \pi^{-1}_{\langle e_1 \rangle}(s)$ is 'big'.

As a consequence, the portion which exceeds $\pi_V^{-1}(s) \cap [0, r]^n$ can be estimated as follows:

$$\mathscr{H}^{n-k}\left(\sigma^{t+2\varepsilon re}\left(S_{t,r}\right) \cap [0, r]^n\right) \geq \mathscr{H}^{n-k}\left(\sigma^{t+2\varepsilon re}\left(S_{t,r}\right)\right) - n 2^n \varepsilon r^{n-k}.$$

As notice before, condition (2.25) implies that the points $\sigma^{t+2\varepsilon re}\left(S_{t,r}\right) \cap [0, r]^n$ belong to the complementary of $\mathcal{A}$. By the above inequalities we obtain then

$$\mathscr{H}^{n-k}\left(\mathcal{A}^c \cap \pi_V^{-1}(t + 2\varepsilon r e) \cap [0, r]^n\right) \quad \geq \quad \mathscr{H}^{n-k}\left(\sigma^{t+2\varepsilon re}\left(S_{t,r}\right) \cap [0, r]^n\right)$$

$$\geq (1-\varepsilon)^{n-k}\mathscr{H}^{n-k}\big(S_{\mathrm{t},r}\big) - n2^n\varepsilon r^{n-k}$$
$$\overset{(2.27)}{\geq} (1-\varepsilon)^{n-k+1}r^{n-k} - n2^n\varepsilon r^{n-k}$$
$$\geq \frac{1}{2}r^{n-k}.$$

The last estimate shows that, for each $\mathrm{t} \in T \cap \pi_{\langle \mathrm{e}\rangle}^{-1}(\mathrm{q})$, the point $\mathrm{s} = \mathrm{t} + 2\varepsilon r\mathrm{e}$ does not satisfy the inequality in (2.27): thus $\big(T \cap \pi_{\langle \mathrm{e}\rangle}^{-1}(\mathrm{q})\big) + 2\varepsilon r\mathrm{e}$ lies in the complementary of T. In particular

$$\mathscr{H}^{k-1}\big(T \cap \pi_{\langle \mathrm{e}\rangle}^{-1}(\mathrm{s})\big) < r^{k-1} - \mathscr{H}^{k-1}\big(T \cap \pi_{\langle \mathrm{e}\rangle}^{-1}(\mathrm{q})\big).$$

However, by construction both t and s belong to Q. This yields the contradiction, by definition of Q:

$$\frac{1}{2}r^{k-1} \overset{(2.28)}{<} \mathscr{H}^{k-1}\big(T \cap \pi_{\langle \mathrm{e}\rangle}^{-1}(\mathrm{s})\big) < r^{k-1} - \mathscr{H}^{k-1}\big(T \cap \pi_{\langle \mathrm{e}\rangle}^{-1}(\mathrm{t})\big) \overset{(2.28)}{<} \frac{1}{2}r^{k-1}. \qquad \square$$

CHAPTER 3

A divergence formula

The disintegration technique developed in Chapter 1 for locally affine partitions satisfying the assumptions of Theorem 1.1 led to define of a function α, on any $\mathcal{D}$-cylinder $\mathscr{C}^k = \mathscr{C}^k(\mathscr{Z}^k, C^k)$, as the Radon-Nikodym derivative in (1.25).
In the present section we find that on $\mathscr{C}^k$ the function α satisfies the system of ODEs

$$\partial_{t_\ell}\alpha\bigg(\mathrm{t} = \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i \mathrm{v}_i(x)\bigg) = (\mathrm{div}\, \mathrm{v}_\ell)_{\mathrm{a.c.}}(x)\alpha\bigg(\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i \mathrm{v}_i(x)\bigg)$$

for $\ell = 1, \dots, k$, where we assume w.l.o.g. that $0 \in C^k$, $\langle \mathrm{e}_1, \dots, \mathrm{e}_k\rangle$ is an axis of $\mathscr{C}$, $\mathrm{v}_i(x)$ is the vector field

$$x \mapsto \mathbf{1}_{\mathscr{C}^k}(x)(\langle \mathcal{D}(x)\rangle \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(\mathrm{e}_i))$$

and $(\mathrm{div}\, \mathrm{v}_i)_{\mathrm{a.c.}}(x)$ is the density of the absolutely continuous part of the divergence of v_i, that we prove to be a measure.
This is a consequence of the Disintegration Theorem 1.1 and of the regularity estimates on α which will be proved in Proposition 3.1.
Notice that even the fact that the divergence of v_i is a measure is not trivial, since the vector field is not weakly differentiable.

Heuristically, the ODEs above can be formally derived as follows.
In Section 1.2.1 we saw that $\mathscr{C}^k$ is the image of the product space $C^k + Z^k$, where $Z^k = \mathscr{C}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(0)$ is a section of $\mathscr{C}^k$, with the change of variable

$$\Phi(\mathrm{t}+\mathrm{z}) = \mathrm{z} + \sum_{i=1}^k t_i \mathrm{v}_i(\mathrm{z}) = \sigma^{\mathrm{t}}(\mathrm{z}) \qquad \text{for all } \mathrm{t} = \sum_{i=1}^k t_i \mathrm{e}_i \in C^k,\ \mathrm{z} \in Z^k. \tag{3.1}$$

In Theorem 1.1 we found that the weak Jacobian of this change of variable is defined, and given by

$$|\mathfrak{J}(\mathrm{t}+\mathrm{z})| = \alpha(\mathrm{t}, 0, \mathrm{z}).$$

From (3.1) one finds that, if v_i was smooth instead of only Borel, this Jacobian would be

$$\mathfrak{J}(\mathrm{t}+\mathrm{z}) = \det\left(\left[[\mathrm{v}_j\cdot \mathrm{e}_i]_{\substack{i=1,\dots,n\\ j=1,\dots,k}} \middle| \left[\sum_{\ell=1}^k t_\ell \partial_{z_j}\langle \mathrm{v}_\ell(\mathrm{z})\cdot \mathrm{e}_i\rangle + \delta_{i,j}\right]_{\substack{i=1,\dots,n\\ j=k+1,\dots,n}} \right]\right);$$

by direct computations with Cramer rule and the multilinearity of the determinant, moreover, from the last two equations above one would prove the relation

$$\partial_{t_\ell}\mathfrak{J}(\mathrm{t}+\mathrm{z}) = \mathrm{trace}\Big(J\mathrm{v}_\ell(\mathrm{z})\,(J\Phi(\mathrm{t}+\mathrm{z}))^{-1}\Big)\mathfrak{J}(\mathrm{t}+\mathrm{z}),$$

where Jg denotes the Jacobian matrix of a function g.
By the Lipschitz regularity of α w.r.t. the $\{t_i\}_{i=1}^k$ variables given in Proposition 3.1, one could then expect that

$$\partial_{t_\ell}\alpha(\mathrm{t},0,\mathrm{z}) = \Big(\sum_{j=1}^n \partial_{x_j}(\mathrm{v}_i(\Phi^{-1}(x))\cdot \mathrm{e}_j)|_{x=\Phi(\mathrm{t}+\mathrm{z})}\Big)\alpha(\mathrm{t},0,\mathrm{z}). \tag{3.2}$$

Notice that $\sum_j \partial_{x_j}(\mathrm{v}_i(\Phi^{-1}(x))\cdot \mathrm{e}_j)|_{x=\Phi(\mathrm{t}+\mathrm{z})}$ is the pointwise divergence of the vector field $\mathrm{v}_i(\Phi^{-1}(x))$ evaluated at $x=\Phi(\mathrm{t}+\mathrm{z})$. In this article, we denote it with $(\mathrm{div}(\mathrm{v}_i\circ\Phi^{-1}))_{\mathrm{a.c.}}$. Finally, given a regular domain $\Omega\subset\mathbb{R}^n$, by the Green-Gauss-Stokes formula one should have

$$\int_\Omega (\mathrm{div}(\mathrm{v}_i\circ\Phi^{-1}))_{\mathrm{a.c.}}\,d\mathscr{L}^n(x) = \int_{\partial\Omega}\mathrm{v}_i(\Phi^{-1}(x))\cdot\hat{n}\,d\mathscr{H}^{n-1}(x), \tag{3.3}$$

where $\hat{n}$ is the outer normal to the boundary of Ω.

The analogue of Formulas (3.2) and (3.3) is the additional regularity we prove in this section, in a weak context, for direction vector fields parallel to the sets of the partition and for the current of the k-dimensional sets. We give now the idea of the proof, in the case of partitions into 1-dimensional sets.

Fix the attention on a 1-dimensional $\mathcal{D}$-cylinder $\mathscr{C}$ with axis e and basis $Z=\mathscr{C}\cap\pi_{\langle\mathrm{e}\rangle}^{-1}(0)$. Consider the distributional divergence of the vector field v giving pointwise on $\mathscr{C}$ the direction of projected faces, normalized with $\mathrm{v}\cdot\mathrm{e}=1$, and vanishing elsewhere (i.e., the *direction vector field* of the 1-dimensional slice of $\mathscr{C}$ in the direction e defined in Section 1.2.2). The Disintegration Theorem 1.1 (see the proof at page 26) decomposes integrals on $\mathscr{C}$ to integrals first on the sets $\{\mathcal{R}(z)\}_{z\in Z}$, with the additional density factor α, then on Z. By means of it, one then reduces the integral $\int_{\mathscr{C}}\nabla\varphi\cdot\mathrm{v}$, defining the distributional divergence, to the following integrals on the sets of the partition:

$$-\int_{[h^-\mathrm{e},h^+\mathrm{e}]}\nabla\varphi(x)|_{x=z+t_1\mathrm{v}(z)}\cdot\mathrm{v}(z)\alpha(\mathrm{t},0,z)\,d\mathscr{H}^1(\mathrm{t})\qquad \text{where } z \text{ varies in } Z.$$

Since α is Lipschitz in t and $\nabla\varphi|_{x=\sigma^{\mathrm{w}+t_1\mathrm{e}}(z)}\cdot\mathrm{v}=\partial_{t_1}(\varphi\circ\sigma^{\mathrm{w}+\mathrm{t}}(z))$, by integrating by parts one arrives to

$$\int_{[h^-\mathrm{e},h^+\mathrm{e}]}\varphi\circ\sigma^{\mathrm{w}+\mathrm{t}}(z)\partial_{t_1}\alpha(\mathrm{t},0,\mathrm{z})\,d\mathscr{H}^1(\mathrm{t}) - [\varphi\circ\sigma^{\mathrm{w}+\mathrm{t}}(z)\alpha(\mathrm{t},0,z)]\Big|_{\mathrm{t}=h^-\mathrm{e}}^{\mathrm{t}=h^+\mathrm{e}}.$$

Applying again the disintegration theorem in the other direction, by the invertibility of α, one comes back to integrals on the $\mathcal{D}$-cylinder, where in the first addend φ is now integrated with the factor $\partial_{t_1}\alpha/\alpha$.

An argument of this kind yields an explicit representation of the distributional divergence of the truncation of a vector field v, parallel at each point x to the set $\mathcal{R}(x)$ through x, to $\mathscr{C}^k$. This divergence is a Radon measure, the absolutely continuous part is basically given by (3.2) and, as in (3.3), there is moreover a singular term representing the flux through the border of $\mathscr{C}^k$ transversal to $\mathcal{D}$, already defined as

$$\mathfrak{d}\mathscr{C}^k = \mathscr{C}^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(\mathrm{rb}(C^k)), \qquad \hat{n}_{|_{\mathfrak{d}\mathscr{C}^k}} \text{ outer unit normal to } \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(C^k). \tag{3.4}$$

As $\mathscr{C}^k$ are not regular sets, but just σ-compact, there is a loss of regularity for the divergence of v in the whole $\mathbb{R}^n$. In general, the distributional divergence will just be a series of measures.

3.1. Regularity of the density function

In this section, we show that the quantitative estimates of Lemma 1.16 allow not only to derive the absolute continuity of the push forward with $\sigma^{\mathrm{w}+t\mathrm{e}}$ and prove Theorem 1.1, but also to find regularity estimates on the density function defined in 1.25. This regularity properties will be used in the rest of this chapter.

Proposition 3.1. *Let $\mathscr{C}^k(\mathscr{Z}^k, C^k)$ be a k-dimensional $\mathcal{D}$-cylinder parametrized as in Section 1.3.2 and assume without loss of generality that $\mathrm{w} = \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(Z^k) = 0$. Then, the function $\alpha(\mathrm{t},0,z)$ defined in (1.25) is locally Lipschitz in $\mathrm{t} \in \mathrm{ri}(C^k)$ (and so jointly measurable in (t,z)). Moreover, for $\mathscr{H}^{n-k}$-a.e. $y \in \sigma^{\mathrm{t}}(Z)$ the following estimates hold:*

1. *Derivative estimate*

$$-\left(\frac{n-k}{h^+(\mathrm{t},\mathrm{e})-u}\right)\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y) \leq \frac{d}{du}\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y) \leq \left(\frac{n-k}{u-h^-(\mathrm{t},\mathrm{e})}\right)\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y); \tag{3.5}$$

2. *Integral estimate*

$$\left(\frac{|h^+(\mathrm{t},\mathrm{e})-u|}{|h^+(\mathrm{t},\mathrm{e})|}\right)^{n-k}(-1)^{\mathbf{1}_{\{u<0\}}} \leq \alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y)\,(-1)^{\mathbf{1}_{\{u<0\}}} \leq \left(\frac{|h^-(\mathrm{t},\mathrm{e})-u|}{|h^-(\mathrm{t},\mathrm{e})|}\right)^{n-k}(-1)^{\mathbf{1}_{\{u<0\}}}; \tag{3.6}$$

3. *Total variation estimate*

$$\int_{h^-(\mathrm{t},\mathrm{e})}^{h^+(\mathrm{t},\mathrm{e})}\left|\frac{d}{du}\alpha(\mathrm{t}+u\mathrm{e},0,z)\right|du \leq 2\alpha(\mathrm{t},0,z)\left[\frac{|h^+-h^-|^{n-k}}{|h^+|^{n-k}}+\frac{|h^+-h^-|^{n-k}}{|h^-|^{n-k}}-1\right], \tag{3.7}$$

where h^+, h^- stand for $h^+(\mathrm{t},\mathrm{e}), h^-(\mathrm{t},\mathrm{e})$.

Proof. *Lipschitz regularity estimate* First we prove the local Lipschitz regularity of $\alpha(\mathrm{t},0,z)$ w.r.t. $\mathrm{t} \in \mathrm{ri}(C^k)$. We repeat the reasoning made in Remark 1.18.
Given $\mathrm{s},\mathrm{t} \in C^k$, we set $\mathrm{e} = \frac{\mathrm{s}-\mathrm{t}}{|\mathrm{s}-\mathrm{t}|}$.

As

$$\sigma^{\mathrm{s}-|\mathrm{s}|\frac{\mathrm{s}}{|\mathrm{s}|}} = \sigma^{\mathrm{t}-|\mathrm{t}|\frac{\mathrm{t}}{|\mathrm{t}|}} \circ \sigma^{\mathrm{s}-|\mathrm{s}-\mathrm{t}|\mathrm{e}},$$

then

$$\begin{aligned}\sigma_{\#}^{\mathrm{s}-|\mathrm{s}|\frac{\mathrm{s}}{|\mathrm{s}|}}(\mathscr{H}^{n-k}\llcorner\,\sigma^{\mathrm{s}}(Z)) &= \sigma_{\#}^{\mathrm{t}-|\mathrm{t}|\frac{\mathrm{t}}{|\mathrm{t}|}}\big(\sigma_{\#}^{\mathrm{s}-|\mathrm{s}-\mathrm{t}|\mathrm{e}}\mathscr{H}^{n-k}\llcorner\,\sigma^{\mathrm{s}}(Z)\big)\\ &= \sigma_{\#}^{\mathrm{t}-|\mathrm{t}|\frac{\mathrm{t}}{|\mathrm{t}|}}\Big(\alpha(\mathrm{s},\mathrm{t},y)\cdot\mathscr{H}^{n-k}\llcorner\,\sigma^{\mathrm{t}}(Z)\Big)\\ &= \alpha(\mathrm{t},0,z)\cdot\alpha(\mathrm{s},\mathrm{t},\sigma^{\mathrm{t}}(z))\cdot\mathscr{H}^{n-k}_{|Z}. \end{aligned} \tag{3.8}$$

By definition of α it follows that

$$\alpha(\mathrm{s},0,z)-\alpha(\mathrm{t},0,z)=\alpha(\mathrm{t},0,z)[\alpha(\mathrm{s},\mathrm{t},\sigma^{\mathrm{t}}(z))-1]. \tag{3.9}$$

Now we want to estimate the term $[\alpha(\mathrm{s},\mathrm{t},\sigma^{\mathrm{t}}(z))-1]$ with the lenght $|\mathrm{s}-\mathrm{t}|$ times a constant which is locally bounded w.r.t. t. In order to do this, we proceed as in the Corollary 2.19 of [**18**] using the estimate

$$\begin{aligned}\left(\frac{h^+(\mathrm{t},\mathrm{e})-u_2}{h^+(\mathrm{t},\mathrm{e})-u_1}\right)^{n-k}\mathscr{H}^{n-k}(\sigma^{\mathrm{t}+u_1\mathrm{e}}(S)) &\le \mathscr{H}^{n-k}(\sigma^{\mathrm{t}+u_2\mathrm{e}}(S))\\ &\le \left(\frac{u_2-h^-(\mathrm{t},\mathrm{e})}{u_1-h^-(\mathrm{t},\mathrm{e})}\right)^{n-k}\mathscr{H}^{n-k}(\sigma^{\mathrm{t}+u_1\mathrm{e}}(S)),\end{aligned} \tag{3.10}$$

which holds $\forall\ h^-(\mathrm{t},\mathrm{e})<u_1\le u_2<h^+(\mathrm{t},\mathrm{e})$ and $\forall\, S\subset\sigma^{\mathrm{t}}(Z)$.

Indeed, (3.10) can be rewritten in the following way:

$$\begin{aligned}\left(\frac{h^+(\mathrm{t},\mathrm{e})-u_2}{h^+(\mathrm{t},\mathrm{e})-u_1}\right)^{n-k}\int_S\alpha(\mathrm{t}+u_1\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y) &\le \int_S\alpha(\mathrm{t}+u_2\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y)\\ &\le \left(\frac{u_2-h^-(\mathrm{t},\mathrm{e})}{u_1-h^-(\mathrm{t},\mathrm{e})}\right)^{n-k}\int_S\alpha(\mathrm{t}+u_1\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y).\end{aligned} \tag{3.11}$$

Therefore, there is a dense sequence $\{u_i\}_{i\in\mathbb{N}}$ in $(h^-(\mathrm{t},\mathrm{e}),h^+(\mathrm{t},\mathrm{e}))$ such that for $\mathscr{H}^{n-k}$-a.e. $y\in S$ and for all $u_i\le u_j$, $i,j,\in\mathbb{N}$ the following inequalities hold

$$\begin{aligned}\left[\left(\frac{h^+(\mathrm{t},\mathrm{e})-u_j}{h^+(\mathrm{t},\mathrm{e})-u_i}\right)^{n-k}-1\right]\alpha(\mathrm{t}+u_i\mathrm{e},\mathrm{t},y) &\le \alpha(\mathrm{t}+u_j\mathrm{e},\mathrm{t},y)-\alpha(\mathrm{t}+u_i\mathrm{e},\mathrm{t},y)\\ &\le \left[\left(\frac{u_j-h^-(\mathrm{t},\mathrm{e})}{u_i-h^-(\mathrm{t},\mathrm{e})}\right)^{n-k}-1\right]\alpha(\mathrm{t}+u_i\mathrm{e},\mathrm{t},y).\end{aligned} \tag{3.12}$$

Thanks to the uniform bounds (1.29), for all $y\in\sigma^{\mathrm{t}}(Z)$ such that (3.12) holds, the function $\alpha(\mathrm{t}+\cdot\mathrm{e},\mathrm{t},y)$ is locally Lipschitz on $\{u_i\}_{i\in\mathbb{N}}$ and for every $[a,b]\subset(h^-(\mathrm{t},\mathrm{e}),h^+(\mathrm{t},\mathrm{e}))$ the Lipschitz constants of α on $\{u_i\}_{i\in\mathbb{N}}\cap[a,b]$ are uniformly bounded w.r.t. y.
Then, on every compact interval $[a,b]\subset(h^-(\mathrm{t},\mathrm{e}),h^+(\mathrm{t},\mathrm{e}))$ there exists a Lipschitz extension $\tilde{\alpha}(\mathrm{t}+\cdot\mathrm{e},\mathrm{t},y)$ of $\alpha(\mathrm{t}+\cdot\mathrm{e},\mathrm{t},y)$ which has the same Lipschitz constant.

By the dominated convergence theorem, whenever $\{u_{j_n}\}_{n\in\mathbb{N}} \subset \{u_j\}_{j\in\mathbb{N}}$ converges to some $u \in [a,b]$ we have

$$\int_S \alpha(\mathrm{t}+u_{j_n}\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y) \quad \longrightarrow \quad \int_S \tilde{\alpha}(\mathrm{t}+u\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y), \quad \forall\, S \subset \sigma^{\mathrm{t}}(Z).$$

However, the integral estimate (3.11) implies that

$$\int_S \alpha(\mathrm{t}+u_{j_n}\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y) \quad \longrightarrow \quad \int_S \alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y)\,d\mathscr{H}^{n-k}(y),$$

so that the Lipschitz extension $\tilde{\alpha}$ is an $L^1(\mathscr{H}^{n-k})$ representative of the original density α for all $u \in [a,b]$. Repeating the same reasoning for an increasing sequence of compact intervals $\{[a_n,b_n]\}_{n\in\mathbb{N}}$ that converge to $(h^-(\mathrm{t},\mathrm{e}),h^+(\mathrm{t},\mathrm{e}))$, we can assume that the density function $\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y)$ is locally Lipschitz in u with a Lipschitz constant that depends continuously on t and on e.
Then, by (3.9), the local Lipschitz regularity in t of the function $\alpha(\mathrm{t},0,z)$ is proved.

Derivative estimate If we derive w.r.t. u the pointwise estimate (3.12) (which holds for all $u \in (h^-(\mathrm{t},\mathrm{e}),h^+(\mathrm{t},\mathrm{e}))$ by the first part of the proof) we obtain the derivative estimate (3.5).

Integral estimate (3.5) implies the monotonicity of the following quantities:

$$\frac{d}{du}\left(\frac{\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y)}{(h^+(\mathrm{t},\mathrm{e})-u)^{n-k}}\right) \geq 0, \quad \frac{d}{du}\left(\frac{\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},y)}{(u-h^-(\mathrm{t},\mathrm{e}))^{n-k}}\right) \leq 0.$$

Integrating the above inequalities from $u \in (h^-(\mathrm{t},\mathrm{e}),h^+(\mathrm{t},\mathrm{e}))$ to 0 we obtain (3.6).

Total variation estimate In order to prove (3.7) we proceed as in Corollary 2.19 of [**18**].

$$\begin{aligned}
\int_{h^-(\mathrm{t},\mathrm{e})}^0 \left|\frac{d}{du}\alpha(\mathrm{t}+u\mathrm{e},0,z)\right| du &\leq \int_{\{\frac{d}{du}\alpha(\mathrm{t}+u\mathrm{e},0,z)>0\}\cap\{u\in(h^-(\mathrm{t},\mathrm{e}),0)\}} \frac{d}{du}\alpha(\mathrm{t}+u\mathrm{e},0,z)\,du \\
&\quad + \int_{h^-(\mathrm{t},\mathrm{e})}^0 \frac{(n-k)\alpha(\mathrm{t}+u\mathrm{e},0,z)}{|h^+(\mathrm{t},\mathrm{e})-u|}\,du \\
&\leq \int_{h^-(\mathrm{t},\mathrm{e})}^0 \frac{d}{du}\alpha(\mathrm{t}+u\mathrm{e},0,z)\,du+ \\
&\quad + 2\int_{h^-(\mathrm{t},\mathrm{e})}^0 \frac{(n-k)\alpha(\mathrm{t}+u\mathrm{e},0,z)}{|h^+(\mathrm{t},\mathrm{e})-u|}\,du \\
&\leq \alpha(\mathrm{t},0,z) + 2\int_{h^-(\mathrm{t},\mathrm{e})}^0 \frac{(n-k)\alpha(\mathrm{t}+u\mathrm{e},0,z)}{|h^+(\mathrm{t},\mathrm{e})-u|}\,du. \qquad (3.13)
\end{aligned}$$

From (3.9) we know that $\alpha(\mathrm{t}+u\mathrm{e},0,z) = \alpha(\mathrm{t},0,z)\,\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},\sigma^{\mathrm{t}}(z))$.
Moreover, since $u < 0$

$$\alpha(\mathrm{t}+u\mathrm{e},\mathrm{t},\sigma^{\mathrm{t}}(z)) \underset{(3.6)}{\leq} \left(\frac{|h^+(\mathrm{t},\mathrm{e})-u|}{|h^+(\mathrm{t},\mathrm{e})|}\right)^{n-k}.$$

If we substitute this inequality in (3.13) we find that

$$(3.13) \leq \alpha(\mathrm{t}, 0, z) + 2\alpha(\mathrm{t}, 0, z) \int_{h^-(\mathrm{t},\mathrm{e})}^{0} \frac{(n-k)|h^+(\mathrm{t}, \mathrm{e}) - u|^{n-k-1}}{|h^+(\mathrm{t}, \mathrm{e})|^{n-k}} \, du$$
$$= -\alpha(\mathrm{t}, 0, z) + 2\alpha(\mathrm{t}, 0, z) \frac{|h^+(\mathrm{t}, \mathrm{e}) - h^-(\mathrm{t}, \mathrm{e})|^{n-k}}{|h^+(\mathrm{t}, \mathrm{e})|^{n-k}}. \tag{3.14}$$

Adding the symmetric estimate on $(0, h^+(\mathrm{t}, \mathrm{e}))$ we obtain (3.7). □

3.2. Divergence of direction vector fields

In the present subsection, we study the regularity of a vector field parallel, at each point, to the corresponding k-dimensional set through that point.

3.2.1. Study on $\mathcal{D}$-cylinders. As a preliminary step, fix the attention on the $\mathcal{D}$-cylinder

$$\mathscr{C}^k = \mathscr{C}^k(\mathscr{Z}^k, C^k).$$

One can assume w.l.o.g. that the axis of $\mathscr{C}^k$ is identified by vectors $\{\mathrm{e}_1, \dots, \mathrm{e}_k\}$ which are the first k coordinate vectors of $\mathbb{R}^n$ and that C^k is the square

$$C^k = \prod_{i=1}^{k} [-\mathrm{e}_i, \mathrm{e}_i].$$

Denote with Z^k the section $\mathscr{Z}^k \cap \pi^{-1}_{\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle}(0)$.

Definition 3.2 (Coordinate vector fields). *We define on* $\mathbb{R}^n$ k-coordinate vector fields *for* $\mathscr{C}^k$ *as follows:*

$$\mathrm{v}_i(x) = \begin{cases} 0 & \text{if } x \notin \mathscr{C}^k \\ \mathrm{v} \in \langle \mathcal{D}(x) \rangle \quad \text{such that } \pi_{\langle \mathrm{e}_1, \dots, \mathrm{e}_k \rangle} \mathrm{v} = \mathrm{e}_i & \text{if } x \in \mathscr{C}^k. \end{cases}$$

The k-coordinate vector fields are a basis for the module on the algebra of measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ constituted by the vector fields with values in $\langle \mathcal{D}(x) \rangle$ at each point $x \in \mathscr{C}^k$, and vanishing elsewhere.

Consider the distributional divergence of v_i, denoted by $\operatorname{div} \mathrm{v}_i$. As a consequence of the absolute continuity of the push forward with σ, and by the regularity of the density α, one gains more regularity of the divergence.

Let us fix a notation. Given any vector field $\mathrm{v} : \mathbb{R}^n \to \mathbb{R}^n$ whose distributional divergence is a Radon measure, we will denote with $(\operatorname{div} \mathrm{v})_{\mathrm{a.c.}}$ the density of the absolutely continuous part of the measure $\operatorname{div} \mathrm{v}$.

LEMMA 3.3. *The distribution* $\operatorname{div} \mathrm{v}_i$ *is a Radon measure. Its absolutely continuous part has density*

$$(\operatorname{div} \mathrm{v}_i)_{\mathrm{a.c.}}(x) = \frac{\partial_{t_i}\alpha\big(\mathrm{t} = \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i \mathrm{v}_i(x)\big)}{\alpha\big(\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i \mathrm{v}_i(x)\big)} \mathbf{1}_{\mathscr{C}^k}(x). \tag{3.15}$$

Its singular part is $\mathscr{H}^{n-1} \llcorner \big(\mathscr{C}^k \cap \{x\cdot \mathrm{e}_i = -1\}\big) - \mathscr{H}^{n-1} \llcorner \big(\mathscr{C}^k \cap \{x\cdot \mathrm{e}_i = 1\}\big)$.

PROOF. Consider any test function $\varphi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and apply the Disintegration Theorem 1.1, Page 26:

$$\langle \operatorname{div} \mathrm{v}_i, \varphi\rangle := -\int_{\mathscr{C}^k} \nabla\varphi(x)\cdot \mathrm{v}_i(x)\, d\mathscr{L}^n(x) = -\int_{Z^k}\int_{C^k} \alpha(\mathrm{t},0,z)\,\nabla\varphi(\sigma^{\mathrm{t}}(z))\cdot \mathrm{v}_i(z)\, d\mathscr{H}^k(\mathrm{t})\, d\mathscr{H}^{n-k}(z),$$

where we used that v_i is constant on the sets of the partition, i.e. $\mathrm{v}_i(z) = \mathrm{v}_i(\sigma^{\mathrm{t}}(z))$. Being $\sigma^{\mathrm{t}}(z) = z + \sum_{i=1}^k t_i \mathrm{v}_i(z)$, one has

$$\nabla_x\varphi(x = \sigma^{\mathrm{t}}(z))\cdot \mathrm{v}_i(z) = \nabla_x\varphi(x=\sigma^{\mathrm{t}}(z))\cdot \partial_{t_i}(\sigma^{\mathrm{t}}(z)) = \partial_{t_i}(\varphi(\sigma^{\mathrm{t}}(z))).$$

The inner integral is thus

$$\int_{C^k} \nabla\varphi(\sigma^{\mathrm{t}}(z))\cdot \mathrm{v}_i(z)\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t}) = \int_{C^k} \partial_{t_i}(\varphi(\sigma^{\mathrm{t}} z))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t}).$$

Since Proposition 3.1 ensures that α is Lipschitz in t, for $\mathrm{t} \in C^k$, one can integrate by parts:

$$\begin{aligned}\int_{C^k} \partial_{t_i}(\varphi(\sigma^{\mathrm{t}}(z)))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t}) = &-\int_{C^k} \varphi(\sigma^{\mathrm{t}}(z))\partial_{t_i}\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t})\\ &+\int_{C^k\cap\{t_i=1\}} \varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^{k-1}(\mathrm{t})\\ &-\int_{C^k\cap\{t_i=-1\}} \varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^{k-1}(\mathrm{t}).\end{aligned}$$

Substitute in the first expression. Recall moreover the definition of α in (1.25), as a Radon-Nikodym derivative of a push-forward measure, and its invertibility and Lipschitz estimates (Remark 1.18, Proposition 3.1), among with in particular the L^1 estimate on the function $\partial_{t_i}\alpha/\alpha$. Then, pushing the measure from $\mathrm{t} = 0$ to a generic t, one comes back to the integral on the $\mathcal{D}$-cylinder

$$\begin{aligned}\langle \operatorname{div} \mathrm{v}_i, \varphi\rangle = &\int_{Z^k}\int_{C^k} \varphi(\sigma^{\mathrm{t}}(z))\partial_{t_i}\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t})\, d\mathscr{H}^{n-k}(z)\\ &-\int_{Z^k}\int_{C^k\cap\{t_i=1\}} \varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^{k-1}(\mathrm{t})\, d\mathscr{H}^{n-k}(z)\\ &+\int_{Z^k}\int_{C^k\cap\{t_i=-1\}} \varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^{k-1}(\mathrm{t})\, d\mathscr{H}^{n-k}(z)\\ = &\int_{\mathscr{C}^k} \varphi(x)(\operatorname{div} \mathrm{v}_i)_{\mathrm{a.c.}}(x)\, d\mathscr{L}^n(x) - \int_{\mathscr{C}^k\cap\{x\cdot \mathrm{e}_i=1\}} \varphi(x)\, d\mathscr{H}^{n-1}(x) + \int_{\mathscr{C}^k\cap\{x\cdot \mathrm{e}_i=-1\}} \varphi(x)\, d\mathscr{H}^{n-1}(x).\end{aligned}$$

where $(\operatorname{div} \mathrm{v}_i)_{\mathrm{a.c.}}$ is the function $\frac{\partial_{t_i}\alpha}{\alpha}$ precisely written in the statement. Thus we have just proved the thesis, consisting in the last formula. □

Remark 3.4. *Consider a function* $\lambda \in L^1(\mathscr{C}^k;\mathbb{R})$ *constant on each set of the partition, meaning that* $\lambda(\sigma^{\mathrm{t}}(z)) = \lambda(z)$ *for* $\mathrm{t} \in C^k$ *and* $z \in Z^k$. *One can regard this* λ *as a function of* $\nabla f(x)$. *Then the same statement of Lemma 3.3 applies to the vector field* $\lambda \mathrm{v}_i$, *but the divergence is clearly* $\operatorname{div}(\lambda \mathrm{v}_i) = \lambda \operatorname{div} \mathrm{v}_i$. *The proof is the same, observing that*

$$
\begin{aligned}
\langle \operatorname{div}(\lambda \mathrm{v}_i), \varphi \rangle &:= -\int_{\mathscr{C}^k} \nabla\varphi(x) \cdot \lambda(x)\mathrm{v}_i(x)\, d\mathscr{L}^n(x) \\
&\overset{(1.30)}{=} -\int_{Z^k}\int_{C^k} \lambda(z)\nabla\varphi(\sigma^{\mathrm{t}}(z)) \cdot \mathrm{v}_i(z)\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t})\, d\mathscr{H}^{n-k}(z) \\
&= -\int_{Z^k}\int_{C^k} \lambda(z)\partial_{t_i}(\varphi(\sigma^{\mathrm{t}}(z)))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t})\, d\mathscr{H}^{n-k}(z) \\
&= \int_{Z^k}\int_{C^k} \lambda(z)\varphi(\sigma^{\mathrm{t}}(z))\partial_{t_i}\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t})\, d\mathscr{H}^{n-k}(z) \qquad (3.16)\\
&\qquad - \int_{Z^k}\int_{C^k\cap\{t_i=1\}} \lambda(z)\varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^{k-1}(\mathrm{t})\, d\mathscr{H}^{n-k}(z) \\
&\qquad + \int_{Z^k}\int_{C^k\cap\{t_i=-1\}} \lambda(z)\varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^{k-1}(\mathrm{t})\, d\mathscr{H}^{n-k}(z) \\
&\overset{(1.30)}{=} \int_{\mathscr{C}^k} \varphi(x)\lambda(x)(\operatorname{div}\mathrm{v}_i)_{\mathrm{a.c.}}(x)\, d\mathscr{L}^n(x) - \int_{\mathscr{C}^k\cap\{x\cdot \mathrm{e}_i=1\}} \varphi(x)\lambda(x)\, d\mathscr{H}^{n-1}(x) \\
&\qquad + \int_{\mathscr{C}^k\cap\{x\cdot \mathrm{e}_i=-1\}} \varphi(x)\lambda(x)\, d\mathscr{H}^{n-1}(x).
\end{aligned}
$$

Suitably adapting the integration by parts in the above equality (3.16) *with*

$$
\begin{aligned}
&\int_{C^k} \lambda(\sigma^{\mathrm{t}}(z))\partial_{t_i}(\varphi(\sigma^{\mathrm{t}}(z)))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t}) = \\
&-\int_{C^k} \lambda(\sigma^{\mathrm{t}}(z))\varphi(\sigma^{\mathrm{t}}(z))\partial_{t_i}\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t}) - \int_{C^k} \partial_{t_i}\lambda(\sigma^{\mathrm{t}}(z))\varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^k(\mathrm{t}) \\
&+\int_{C^k\cap\{t_i=1\}} \lambda(\sigma^{\mathrm{t}}(z))\varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^{k-1}(\mathrm{t}) - \int_{C^k\cap\{t_i=-1\}} \lambda(\sigma^{\mathrm{t}}(z))\varphi(\sigma^{\mathrm{t}}(z))\alpha(\mathrm{t},0,z)\, d\mathscr{H}^{k-1}(\mathrm{t})
\end{aligned}
$$

one finds moreover that for all $\lambda \in L^1(\mathbb{R}^n;\mathbb{R})$ *continuously differentiable along* v_i *with integrable directional derivative* $\partial_{\mathrm{v}_i}\lambda$, *the following relation holds:*

$$
\operatorname{div}(\lambda \mathrm{v}_i) = \lambda \operatorname{div} \mathrm{v}_i + \partial_{\mathrm{v}_i}\lambda\, d\mathscr{L}^n \qquad (3.17)
$$

Notice that in (3.17) *there is the addend* $\lambda\mathscr{H}^{n-1} \llcorner (\mathscr{C}^k \cap \{x \cdot \mathrm{e}_i = 1\})$, *which would make no sense for a general* $\lambda \in L^1(\mathbb{R}^n;\mathbb{R})$. *Now we prove that the restriction to* $\mathscr{C}^k \cap \{x \cdot \mathrm{e}_i = 1\}$ *of each representative of* λ *which is* $\mathcal{C}^1(\mathcal{R}(z) \cap \mathscr{C}^k)$, *for* $\mathscr{H}^{n-k}$*-a.e.* $z \in Z^k$, *identifies the same function in* $L^1(\mathscr{C}^k \cap \{x \cdot \mathrm{e}_i = 1\})$.
Indeed, any two representatives $\tilde{\lambda}$, $\hat{\lambda}$ *of the* L^1*-class of* λ *can differ only on a* $\mathscr{L}^n$*-negligible set* N. *By the Disintegration Theorem 1.1, and using moreover Fubini theorem for reducing the integral on* C^k *to integrals on lines parallel to* e_i, *one has that the intersection of* N *with each of the* 1*-dimensional slices whose projection on* $\langle \mathrm{e}_1, \dots, \mathrm{e}_k\rangle$ *is parallel to* e_i *is almost always negligible:*

$$
\mathscr{H}^1\big(N \cap \{q + \langle \mathrm{v}_i(q)\rangle\}\big) = 0 \qquad \text{for } q \in \mathscr{C}^k \cap \{x \cdot \mathrm{e}_i = 0\} \setminus M, \text{ with } \mathscr{H}^{n-1}(M)=0.
$$

Being continuously differentiable along v_i*, one can redefine* $\tilde{\lambda}$*,* $\hat{\lambda}$ *in such a way that* $N \cap \{q + \langle \mathrm{v}_i(q)\rangle\} = \emptyset$ *for all* $q \in \mathscr{C}^k \cap \{x \cdot \mathrm{e}_i = 0\} \setminus M$*. As a consequence* $N \cap \{x \cdot \mathrm{e}_i = t\}$ *is a subset of* $\tau^{t\mathrm{e}_i}(M)$*, where* $\tau^{t\mathrm{e}_i}$ *is the map moving along each set of the partition with* $t\mathrm{v}_i$*:*

$$\mathscr{C}^k \cap \{x \cdot \mathrm{e}_i = 0\} \ni q \mapsto \tau^{t\mathrm{e}_i}(q) := q + t\mathrm{v}_i = \sigma^{(\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(q)) + t\mathrm{e}_1}(q).$$

By the push forward formula (1.25)*, denoting* $\mathrm{w}_q := \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(q)$ *and* $z_q := \pi_{\langle \mathrm{e}_{k+1},\dots,\mathrm{e}_n\rangle}(q)$

$$\mathscr{H}^{n-1} \llcorner (\tau^{t\mathrm{e}_i}(S)) = \alpha(\mathrm{w}_q, \mathrm{w}_q + t\mathrm{e}_i, z_q)\tau^{t\mathrm{e}_i}_\sharp(\mathscr{H}^{n-1}(q) \llcorner S)) \qquad \text{for } S \subset \mathscr{C}^k \cap \{x \cdot \mathrm{e}_1 = 0\}.$$

Therefore, as $\mathscr{H}^{n-1}(M) = 0$*, one has that* $\tilde{\lambda}$ *and* $\hat{\lambda}$ *identify the same integrable function on each section of* $\mathscr{C}^k$ *perpendicular to* e_i*, showing that the measure* $\lambda\mathscr{H}^{n-1} \llcorner (\{x \cdot \mathrm{e}_i = 1\})$ *is well defined.*

Actually, the same argument as above should be used in (3.16) *in order to show that* $\lambda(z)$ *is integrable on* Z^k*, so that one can separate the three integrals as we did. Indeed, being constant on set of the partition by assumption, the restriction of* λ *to a section is trivially well defined as associating to a point the value of* λ *corresponding to the set of that point, but the integrability w.r.t.* $\mathscr{H}^{n-1}$ *on each section is a consequence of the push forward estimate.*

As a direct consequence of (3.17), by linearity, one gets a divergence formula for any sufficiently regular vector field which, at each point of $\mathscr{C}^k$, is parallel to the corresponding set of the partition.

COROLLARY 3.5. *Consider any vector field* $\mathrm{v} = \sum_{i=1}^k \lambda_i \mathrm{v}_i$ *with* $\lambda_i \in L^1(\mathscr{C}^k;\mathbb{R})$ *continuously differentiable along* v_i*, with directional derivative* $\partial_{\mathrm{v}_i}\lambda_i$ *integrable on* $\mathscr{C}^k$*. Then the divergence of* v *is a Radon measure and for every* $\varphi \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R}^n)$

$$\langle \operatorname{div} \mathrm{v}, \varphi\rangle = \int_{\mathscr{C}^k} \varphi(x)(\operatorname{div} \mathrm{v})_{\mathrm{a.c.}}(x)\, d\mathscr{L}^n(x) - \int_{\mathfrak{d}\mathscr{C}^k} \varphi(x)\, \mathrm{v}(x) \cdot \hat{n}(x)\, d\mathscr{H}^{n-1}(x),$$

where $\mathfrak{d}\mathscr{C}^k$*, the border of* $\mathscr{C}^k$ *transversal to* $\mathcal{D}$*, and* $\hat{n}$*, the outer unit normal, are define in Formula* (3.4)*. Moreover, for* $x \in \mathscr{C}^k$

$$(\operatorname{div} \mathrm{v})_{\mathrm{a.c.}}(x) = \sum_{i=1}^k \lambda_i(x) \frac{\partial_{t_i}\alpha(\mathrm{t} = \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x \cdot \mathrm{e}_i \mathrm{v}_i(x))}{\alpha(\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x \cdot \mathrm{e}_i \mathrm{v}_i(x))} + \sum_{i=1}^k \partial_{\mathrm{v}_i}\lambda_i(x). \tag{3.18}$$

REMARK 3.6. *The result is essentially based on the application of the integration by parts formula when the integral on* $\mathscr{C}^k$ *is reduced, by the Disintegration Theorem 1.1, to integrals on* C^k*: this is why we assume the* C^1 *regularity of the* λ_i*, w.r.t. the directions of the k-dimensional set passing through each point of* $\mathscr{C}^k$*. Such regularity could be further weakened, however we do not pursue this issue here. As a consequence, one can easily extend the statement of the previous corollary to sets of the form* $\mathscr{C}^k_\Omega = X^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_n\rangle}(\overline{\Omega})$*, for an open set* $\Omega \subset \langle \mathrm{e}_1, \dots, \mathrm{e}_k\rangle$ *with piecewise Lipschitz boundary, defining* $\mathfrak{d}\mathscr{C}^k_\Omega := X^k \cap \pi^{-1}_{\langle \mathrm{e}_1,\dots,\mathrm{e}_n\rangle}(\mathrm{rb}(\Omega))$*.*

3.2.2. Global Version. We study now the distributional divergence of an integrable vector field v on $\mathcal{T}$, as we did in Subsection 3.2.1 for such a vector field truncated on $\mathcal{D}$-cylinders.

Corollary 3.7. *Consider a vector field* $\mathrm{v} \in L^1(\mathcal{T};\mathbb{R}^n)$ *such that* $\mathrm{v}(x) \in \langle \mathcal{D}(x)\rangle$ *for* $x \in \mathbb{R}^n$, *where we define* $\mathcal{D}(x) = 0$ *for* $x \notin \mathcal{T}$. *Suppose moreover that the restriction to every set* X^k_α, *for* $\alpha \in \mathsf{A}_k$, *is continuously differentiable with integrable derivatives.*
Then, for every $\varphi \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R}^n)$ *one can write*

$$\langle \operatorname{div} \mathrm{v}, \varphi\rangle = \lim_{\ell\to\infty} \sum_{i=1}^{\ell} \left\{ \int_{\mathscr{C}_i} \varphi(x)(\operatorname{div}(\mathbf{1}_{\mathscr{C}_i}\mathrm{v}))_{\mathrm{a.c.}}(x)\, d\mathscr{L}^n(x) - \int_{\mathfrak{d}\mathscr{C}_i} \varphi(x)\,\mathrm{v}(x)\cdot \hat{n}_i(x)\, d\mathscr{H}^{n-1}(x)\right\}. \tag{3.19}$$

where $\{\mathscr{C}_\ell\}_{\ell\in\mathbb{N}}$ *is the countable partition of* $\mathcal{T}$ *in* $\mathcal{D}$*-cylinders given in Lemma 1.22, while* $(\operatorname{div}(\mathbf{1}_{\mathscr{C}_i}\mathrm{v}))_{\mathrm{a.c.}}$ *is the one of Corollary 3.5 and* $\mathfrak{d}\mathscr{C}_i$, $\hat{n}_i$ *are defined in Formula* (3.4).

Remark 3.8. *By construction of the partition, each of the second integrals in the r.h.s. of* (3.19) *appears two times in the series, with opposite sign. Intuitively, the finite sum of these border terms is the integral on a perimeter which tends to the singular set.*

Remark 3.9. *Suppose that* $\operatorname{div}\mathrm{v}$ *is a Radon measure. Then Corollary 3.7 implies that*

$$\mathbf{1}_{\mathscr{C}^k}(\operatorname{div}\mathrm{v})_{\mathrm{a.c.}} \equiv (\operatorname{div}(\mathbf{1}_{\mathscr{C}^k}\mathrm{v}))_{\mathrm{a.c.}}.$$

Proof of Corollary 3.7. The partition of $\mathcal{T}$ into such sets $\{\mathscr{C}_\ell\}_{\ell\in\mathbb{N}}$ is given exactly by Lemma 1.22. Therefore, by dominated convergence theorem one finds that

$$\langle \operatorname{div}\mathrm{v},\varphi\rangle = -\int_{\mathcal{T}} \mathrm{v}(x)\cdot\nabla\varphi(x)\, d\mathscr{L}^n(x) = -\lim_{\ell\to\infty}\sum_{i=1}^{\ell}\int_{\mathscr{C}_i} \mathrm{v}(x)\cdot\nabla\varphi(x)\, d\mathscr{L}^n(x).$$

The addends in the r.h.s. are, by definition, the distributional divergence of the vector fields $\mathrm{v}\mathbf{1}_{\mathscr{C}_i}$ applied to φ. In particular, by Corollary 3.5, they are equal to

$$-\int_{\mathscr{C}_i} \mathrm{v}(x)\cdot\nabla\varphi(x)\, d\mathscr{L}^n(x) = \int_{\mathscr{C}_i}\varphi(x)\big(\operatorname{div}\mathrm{v}\big)_{a.c.}(x)\, d\mathscr{L}^n(x) + \int_{\mathfrak{d}\mathscr{C}^k_i}\varphi(x)\,\mathrm{v}(x)\cdot\hat{n}_i(x)\, d\mathscr{H}^{n-1}(x),$$

proving the thesis. □

3.3. Divergence of the currents of k-dimensional sets

3.3.1. The currents of k-sets. In the present subsection, we change point of view. Instead of looking at vector fields constrained to the sets of the partition, we regard the k-dimensional sets as a k-dimensional current. We establish that this current is a locally flat chain, providing a sequence of normal currents converging to it in the mass norm. The border of these normal currents has the same representation one would have in a smooth setting.

Before proving it, we devote Subsection 3.3.1.1 to recalls on this argument, in order to fix the notations. They are taken mainly from Chapter 4 of [**47**] and Sections 1.5.1, 4.1 of [**28**].

3.3.1.1. *Recalls.* Let $\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$ be a basis of $\mathbb{R}^n$. The **wedge product** between vectors is multilinear and alternating, i.e.:

$$\Big(\sum_{i=1}^n \lambda_i \mathrm{e}_i\Big) \wedge u_1 \wedge \cdots \wedge u_m = \sum_{i=1}^n \lambda_i (\mathrm{e}_i \wedge u_1 \wedge \cdots \wedge u_m) \qquad m \in \mathbb{N},\ \lambda_1, \dots, \lambda_n \in \mathbb{R}$$

$$u_0 \wedge \cdots \wedge u_i \wedge \cdots \wedge u_m = (-1)^i u_i \wedge u_0 \wedge \cdots \wedge \widehat{u_i} \wedge \cdots \wedge u_m \quad 0 < i \le m,\ u_0, \dots, u_m \in \mathbb{R}^n,$$

where the element under the hat is missing. The space of all linear combinations of

$$\Big\{\mathrm{e}_{i_1 \dots i_m} := \mathrm{e}_{i_1} \wedge \cdots \wedge \mathrm{e}_{i_m} :\ i_1 < \cdots < i_m \text{ in } \{1, \dots, n\}\Big\}$$

is the space of m**-vectors**, denoted by $\Lambda_m \mathbb{R}^n$. The space $\Lambda_0 \mathbb{R}$ is just $\mathbb{R}$. $\Lambda_m \mathbb{R}^n$ has the **inner product** given by

$$\mathrm{e}_{i_1 \dots i_m} \cdot \mathrm{e}_{j_1 \dots j_m} = \prod_{k=1}^m \delta_{i_k j_k} \qquad \text{where } \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

The induced **norm** is denoted by $\|\cdot\|$. An m**-vector field** is a map $\xi : \mathbb{R}^n \to \Lambda_m \mathbb{R}^n$.

The dual Hilbert space to $\Lambda_m \mathbb{R}^n$, denoted by $\Lambda^m \mathbb{R}^n$, is the space of m**-covectors**. The element dual to $\mathrm{e}_{i_1 \dots i_m}$ is denoted by $d\mathrm{e}_{i_1 \dots i_m}$. A **differential** m**-form** is a map $\omega : \mathbb{R}^n \to \Lambda^m \mathbb{R}^n$.

We denote with $\langle \cdot, \cdot \rangle$ the duality pairing between m-vectors and m-covectors. Moreover, the same symbol denotes in this paper the bilinear **pairing**, which is a map $\Lambda^p \mathbb{R}^n \times \Lambda_q \mathbb{R}^n \to \Lambda^{p-q} \mathbb{R}^n$ for $p > q$ and $\Lambda^p \mathbb{R}^n \times \Lambda_q \mathbb{R}^n \to \Lambda_{q-p} \mathbb{R}^n$ for $q > p$ whose non-vanishing images on a basis are

$$d\mathrm{e}_{i_1 \dots i_\ell} = \big\langle d\mathrm{e}_{i_1 \dots i_\ell} \wedge d\mathrm{e}_{i_{\ell+1} \dots i_{\ell+m}},\ \mathrm{e}_{i_{\ell+1} \dots i_{\ell+m}} \big\rangle \qquad \text{if } p = \ell + m > m = q$$

$$\mathrm{e}_{i_{\ell+1} \dots i_{\ell+m}} = \big\langle d\mathrm{e}_{i_1 \dots i_\ell},\ \mathrm{e}_{i_1 \dots i_\ell} \wedge \mathrm{e}_{i_{\ell+1} \dots i_{\ell+m}} \big\rangle \qquad \text{if } p = \ell < \ell + m = q.$$

Consider any differential m-form

$$\omega = \sum_{i_1 \dots i_m} \omega_{i_1 \dots i_m} \, d\mathrm{e}_{i_1 \dots i_m}$$

which is differentiable. The **exterior derivative** $d\omega$ of ω is the differential $(m+1)$-form

$$d\omega = \sum_{i_1 \dots i_m} \sum_{j=1}^n \frac{\partial \omega_{i_1 \dots i_m}}{\partial x_j} \, d\mathrm{e}_j \wedge d\mathrm{e}_{i_1 \dots i_m}.$$

If $\omega \in \mathcal{C}^i(\mathbb{R}^n; \Lambda^m \mathbb{R}^n)$, the i-th exterior derivative is denoted with $d^i \omega$.
Consider any m-vector field

$$\xi = \sum \xi_{i_1 \dots i_m} \, \mathrm{e}_{i_1 \dots i_m}$$

which is differentiable. The **pointwise divergence** $(\operatorname{div} \xi)_{\text{a.c.}}$ of ξ is the $(m-1)$-vector field

$$(\operatorname{div} \xi)_{\text{a.c.}} = \sum_{i_1 \dots i_m} \sum_{j=1}^n \frac{\partial \xi_{i_1 \dots i_m}}{\partial x_j} \big\langle d\mathrm{e}_j,\ \mathrm{e}_{i_1 \dots i_m} \big\rangle.$$

Consider the space $\mathscr{D}^m$ of $\mathcal{C}^\infty$-differential m-form with compact support. The topology is generated by the seminorms

$$\nu_K^i(\phi) = \sup_{x\in K,\, 0\le j\le i} \|d^j\phi(x)\| \qquad \text{with } K \text{ compact subset of } \mathbb{R}^n,\ i\in\mathbb{N}.$$

The dual space to $\mathscr{D}^m$, endowed with the weak topology, is called the space of m-**dimensional currents** and it is denoted by $\mathscr{D}_m$. The support of a current $T \in \mathscr{D}_m$ is the smallest close set $K \subset \mathbb{R}^n$ such that $T(\omega) = 0$ whenever $\omega \in \mathscr{D}^m$ vanishes out of K. The **mass** of a current $T \in \mathscr{D}_m$ is defined as

$$\mathbf{M}(T) = \sup\Big\{T(\omega):\ \omega\in\mathscr{D}^m,\ \sup_{x\in\mathbb{R}^n}\|\omega(x)\|\le 1\Big\}.$$

The **flat norm** of a current $T \in \mathscr{D}_m$ is defined as

$$\mathbf{F}(T) = \sup\Big\{T(\omega):\ \omega\in\mathscr{D}^m,\ \sup_{x\in\mathbb{R}^n}\|\omega(x)\|\le 1,\ \sup_{x\in\mathbb{R}^n}\|d\omega(x)\|\le 1\Big\}.$$

An m-dimensional current $T \in \mathscr{D}_m$ is **representable by integration**, and we denote it by $T = \mu\wedge\xi$, if there exists a Radon measure μ over $\mathbb{R}^n$ and a μ-locally integrable m-vector field ξ such that

$$T(\omega) = \int_{\mathbb{R}^n}\big\langle\omega,\,\xi\big\rangle\,d\mu \qquad \forall\omega\in\mathscr{D}^m.$$

If $m \ge 1$, the **boundary** of an m-dimensional current T is defined as

$$\partial T\in\mathscr{D}_{m-1}, \qquad \big(\partial T\big)(\omega) = T(d\omega) \text{ whenever } \omega\in\mathscr{D}^{m-1}.$$

If either $m = 0$, or both T and ∂T are representable by integration, then we will call T **locally normal**. If T is locally normal and compactly supported, then T is called **normal**. The $\mathbf{F}$-closure, in $\mathscr{D}_m$, of the normal currents is the space of **locally flat chains**. Its subspace of currents with finite mass is the $\mathbf{M}$-closure, in $\mathscr{D}_m$, of the normal currents.

To each $\mathscr{L}^n$-measurable m-vector field ξ such that $\|\xi\|$ is locally integrable there corresponds the current $\mathscr{L}^n\wedge\xi\in\mathscr{D}_m(\mathbb{R}^n)$. If ξ is of class $\mathcal{C}^1$, then this current is locally normal and the divergence of ξ is related to the boundary of the corresponding current by

$$-\partial\big(\mathscr{L}^n\wedge\xi\big) = \mathscr{L}^n\wedge(\operatorname{div}\xi)_{\text{a.c.}},$$

Moreover, if Ω is an open set with $\mathcal{C}^1$ boundary, $\hat{n}$ is its outer unit normal and $d\hat{n}$ the dual of $\hat{n}$, then

$$\partial\big(\mathscr{L}^n\wedge(\mathbf{1}_\Omega\xi)\big) = -(\mathscr{L}^n\llcorner\Omega)\wedge(\operatorname{div}\xi)_{\text{a.c.}} + (\mathscr{H}^{n-1}\llcorner\partial\Omega)\wedge\big\langle d\hat{n},\,\xi\big\rangle. \tag{3.20}$$

In the next subsection, we are going to find the analogue of the Green-Gauss Formula (3.20) for the k-dimensional current associated to k-faces, restricted to $\mathcal{D}$-cylinders. In order to do this, we will re-define the function $(\operatorname{div}\xi)_{\text{a.c.}}$ for a less regular k-vector field and this definition will be an extension of the above one.

3.3.1.2. *Divergence of the Current of k-dimensional sets on $\mathcal{D}$-cylinders.* As a preliminary study, restrict again the attention to a $\mathcal{D}$-cylinder as in Subsection 3.2.1, and keep the notation we had there.

The k-dimensional sets, restricted to $\mathscr{C}^k$, define a k-vector field

$$\xi(x) = \mathbf{1}_{\mathscr{C}^k} \mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_k.$$

In general, this vector field does not enjoy much regularity. Nevertheless, as a consequence of the study of Section 3.2, one can find a representation of $\partial\big(\mathscr{L}^n \wedge \xi\big)$ like the one in a regular setting, (3.20). This involves the density α of the push-forward with σ which was studied before, see (1.25).

LEMMA 3.10. *Consider a function λ such that it is continuously differentiable on each set of the partition and assume $\mathscr{C}^k$ bounded.*
Then, the k-dimensional current $\big(\mathscr{L}^n \wedge \lambda\xi\big)$ is normal and the following formula holds

$$\partial\big(\mathscr{L}^n \wedge \lambda\xi\big) = -\mathscr{L}^n \wedge (\operatorname{div} \lambda\xi)_{\text{a.c.}} + \big(\mathscr{H}^{n-1} \llcorner \mathfrak{d}\mathscr{C}^k\big) \wedge \big\langle d\hat{n},\, \lambda\xi\big\rangle,$$

where $\mathfrak{d}\mathscr{C}^k$, $\hat{n}$ are defined in (3.4)*, $d\hat{n}$ is the differential* 1*-form at each point dual to the vector field $\hat{n}$, and $(\operatorname{div} \lambda\xi)_{\text{a.c.}}$ is defined here as*

$$(\operatorname{div} \lambda\xi)_{\text{a.c.}} := \sum_{i=1}^{k} (-1)^{i+1} (\operatorname{div} \lambda \mathrm{v}_i)_{\text{a.c.}}\, \mathrm{v}_1 \wedge \cdots \wedge \widehat{\mathrm{v}}_i \wedge \cdots \wedge \mathrm{v}_k$$

with the functions $(\operatorname{div} \mathrm{v}_i)_{\text{a.c.}}$ of (3.15)*:*

$$(\operatorname{div} \lambda \mathrm{v}_i)_{\text{a.c.}}(x) = \left(\lambda(x) \frac{\partial_{t_i} \alpha\big(\mathrm{t} = \pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i \mathrm{v}_i(x)\big)}{\alpha\big(\pi_{\langle \mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x), 0, x - \sum_{i=1}^k x\cdot \mathrm{e}_i \mathrm{v}_i(x)\big)} + \partial_{\mathrm{v}_i}\lambda(x)\right) \mathbf{1}_{\mathscr{C}^k}(x).$$

PROOF. Actually, this is consequence of Corollary 3.5 in Subsection 3.2.1, reducing to computations in coordinates. One has to verify the equality of the two currents on a basis.

For simplicity, consider first

$$\omega = \phi\, d\mathrm{e}_2 \wedge \cdots \wedge d\mathrm{e}_k.$$

with $\phi \in \mathcal{C}^1(\mathbb{R}^n)$. Then

$$d\omega = \partial_{x_1}\phi\, d\mathrm{e}_1 \wedge \cdots \wedge d\mathrm{e}_k + \sum_{i=k+1}^{n} \partial_{x_i}\phi\, d\mathrm{e}_i \wedge \cdots \wedge d\mathrm{e}_n,$$

$$\langle d\omega, \xi\rangle = \nabla\phi \cdot \mathrm{v}_1 \qquad \big\langle \omega,\, (\operatorname{div} \lambda\xi)_{\text{a.c.}}\big\rangle = (\operatorname{div} \lambda \mathrm{v}_1)_{\text{a.c.}} \phi \qquad \big\langle \omega,\, \big\langle d\hat{n},\, \xi\big\rangle\big\rangle = \phi\, \hat{n}\cdot \mathrm{e}_1$$

and the thesis reduces exactly to Lemma 3.3, and Remark 3.4:

$$\partial\big(\mathscr{L}^n \wedge \lambda\xi\big)(\omega) := \int_{\mathscr{C}^k} \langle d\omega, \lambda\xi\rangle\, d\mathscr{L}^n \overset{3.3}{=} -\int_{\mathscr{C}^k} \big\langle \omega,\, (\operatorname{div} \lambda\xi)_{\text{a.c.}}\big\rangle\, d\mathscr{L}^n + \int_{\mathfrak{d}\mathscr{C}^k} \big\langle \omega,\, \big\langle d\hat{n},\, \lambda\xi\big\rangle\big\rangle\, d\mathscr{H}^{n-1}$$
$$=: -\mathscr{L}^n \wedge (\operatorname{div} \lambda\xi)_{\text{a.c.}} + (\mathscr{H}^{n-1} \llcorner \mathfrak{d}\mathscr{C}^k) \wedge (\hat{n} \wedge \lambda\xi).$$

The same lemma applies with $(-1)^{i+1}\mathrm{v}_i$ instead of v_1 if

$$\omega = \phi\, d\mathrm{e}_1 \wedge \cdots \wedge \widehat{d\mathrm{e}_i} \wedge \cdots \wedge d\mathrm{e}_k,$$

since the following formulas hold:

$$\langle d\omega, \xi\rangle = (-1)^{i+1}\nabla\phi{\cdot}\mathrm{v}_i \qquad \big\langle\omega,\, (\operatorname{div}\lambda\xi)_{\text{a.c.}}\big\rangle = (-1)^{i+1}(\operatorname{div}\lambda\mathrm{v}_i)_{\text{a.c.}}\phi \qquad \big\langle\omega,\, \big\langle d\hat{n},\, \xi\big\rangle\big\rangle = (-1)^{i+1}\phi\,\hat{n}{\cdot}\mathrm{e}_i.$$

Let us show the equality more in general. By a direct computation, one can verify that

$$\mathrm{v}_1\wedge\cdots\wedge\widehat{\mathrm{v}}_i\wedge\cdots\wedge\mathrm{v}_k = \sum_{h=0}^{k-1}\sum_{\substack{k<i_{h+1}<\dots\\ \dots<i_{k-1}\le n}}\sum_{\substack{\sigma\in S(1\dots\hat{\imath}\dots k-1)\\ \sigma(1)<\dots<\sigma(h)}}\operatorname{sgn}\sigma\,\mathrm{v}_{\sigma(h+1)}^{i_{h+1}}\cdots\mathrm{v}_{\sigma(k-1)}^{i_{k-1}}\,\mathrm{e}_{\sigma(1)\dots\sigma(h)i_{h+1}\dots i_{k-1}},$$

where v_i^j is the j-th component of v_i, $S(1\dots\hat{\imath}\dots k)$ denotes the group of permutation of the integers $\{1,\dots,\hat{\imath},\dots,k\}$, with i is missing, and, if $\sigma\in S(1\dots\hat{\imath}\dots k)$, $\operatorname{sgn}\sigma$ is 1 if the permutation is even, -1 otherwise.
On the other hand, consider now a $(k-1)$ form $\omega = \phi\,d\mathrm{e}_{i_1\dots i_h}\wedge d\mathrm{e}_{i_{h+1}\dots i_{k-1}}$, where $1\le i_1<\dots<i_h\le k$, and $k<i_{h+1}<\dots<i_{k-1}\le n$. Then, again by direct computation,

$$\big\langle d\omega,\,\xi\big\rangle = \sum_{\substack{\sigma\in S(1\dots k)\\ \sigma(2)=i_1,\dots,\sigma(h+1)=i_h}}(\nabla\phi\cdot\mathrm{v}_{\sigma(1)})\operatorname{sgn}\sigma\,\mathrm{v}_{\sigma(h+2)}^{i_{h+1}}\cdots\mathrm{v}_{\sigma(k)}^{i_{k-1}},$$

$$\begin{aligned}\big\langle\omega,\,(\operatorname{div}\lambda\xi)_{\text{a.c.}}\big\rangle &= \phi\sum_{i=1}^{k}(-1)^{i+1}(\operatorname{div}\lambda\mathrm{v}_i)_{\text{a.c.}}\sum_{\substack{\sigma\in S(1\dots\hat{\imath}\dots k-1)\\ \sigma(1)=i_1,\dots,\sigma(h)=i_h}}\operatorname{sgn}\sigma\,\mathrm{v}_{\sigma(h+1)}^{i_{h+1}}\cdots\mathrm{v}_{\sigma(k-1)}^{i_{k-1}}\\ &= \sum_{\substack{\sigma\in S(1\dots k)\\ \sigma(2)=i_1,\dots,\sigma(h+1)=i_h}}(\phi\cdot(\operatorname{div}\lambda\mathrm{v}_{\sigma(1)})_{\text{a.c.}})\operatorname{sgn}\sigma\,\mathrm{v}_{\sigma(h+2)}^{i_{h+1}}\cdots\mathrm{v}_{\sigma(k)}^{i_{k-1}},\end{aligned}$$

and finally

$$\begin{aligned}\big\langle\omega,\,\big\langle d\hat{n},\,\xi\big\rangle\big\rangle &= \sum_{i=1}^{k}(-1)^{i+1}(\hat{n}\cdot\mathrm{e}_i)\,\big\langle\omega,\,\mathrm{v}_1\wedge\cdots\wedge\widehat{\mathrm{v}}_i\wedge\cdots\wedge\mathrm{v}_k\big\rangle\\ &= \sum_{\substack{\sigma\in S(1\dots k)\\ \sigma(2)=i_1,\dots,\sigma(h+1)=i_h}}(\phi\,\hat{n}\cdot\mathrm{v}_{\sigma(1)})\operatorname{sgn}\sigma\,\mathrm{v}_{\sigma(h+2)}^{i_{h+1}}\cdots\mathrm{v}_{\sigma(k)}^{i_{k-1}}\end{aligned}$$

Therefore the thesis reduces to Corollary 3.5, being each v_j^i constant on each face. □

3.3.1.3. *Divergence of the current of k-dimensional sets in the whole space.* In the previous section, we considered a k-dimensional current $\big(\mathscr{L}^n\llcorner\mathscr{C}^k\big)\wedge\xi$ identified by the restriction to a $\mathcal{D}$-cylinder $\mathscr{C}^k$ of the k-dimensional sets of the partition, projected on $\mathbb{R}^n$. We established the formula analogous to (3.20) for the border of this current, which is representable by integration w.r.t. the measures $\mathscr{L}^n\llcorner\mathscr{C}^k$ and $\mathscr{H}^{n-1}\llcorner\mathfrak{d}\mathscr{C}^k$. In particular, when $\mathscr{C}^k$ is bounded it is a normal current.
Moreover, we have related the density of the absolutely continuous part to the function α by

$$(\operatorname{div}\xi)_{\text{a.c.}} = \sum_{i=1}^{k}(-1)^{i+1}\frac{\partial_{t_i}\alpha\big(\mathrm{t}=\pi_{\langle\mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x),0,x-\sum_{i=1}^{k}x\cdot\mathrm{e}_i\mathrm{v}_i(x)\big)}{\alpha\big(\pi_{\langle\mathrm{e}_1,\dots,\mathrm{e}_k\rangle}(x),0,\sum_{i=1}^{k}x-x\cdot\mathrm{e}_i\mathrm{v}_i(x)\big)}\mathbf{1}_{\mathscr{C}^k}(x)\,\mathrm{v}_1\wedge\cdots\wedge\widehat{\mathrm{v}}_i\wedge\cdots\wedge\mathrm{v}_k.$$

We observe now that the partition of $\mathbb{R}^n$ into the sets $\{X^k\}_{k=1}^n$, and the remaining set that we call now $\widetilde{X}^0$, define a $(n+1)$-uple of currents. The elements of this $(n+1)$-uple are described by the following statement, which is basically Corollary 3.7 when rephrased in this setting.

COROLLARY 3.11. *Let $\{\mathscr{C}_\ell^k\}_{\ell\in\mathbb{N}}$ be a countable partition of X^k in $\mathcal{D}$-cylinders as in Lemma 1.22 and, up to a refinement of the partition, assume moreover that the $\mathcal{D}$-cylinders are bounded.*

Consider a k-vector field $\xi_k \in L^1(\mathbb{R}^n;\Lambda_k\mathbb{R}^n)$ corresponding, at each point $x\in X^k$, to the k-plane $\langle\mathcal{D}(x)\rangle$, and vanishing elsewhere. Assume moreover that it is continuously differentiable if restricted to any set X^k_α, with locally integrable derivatives, meaning more precisely that $\xi_k\circ\sigma^{\mathrm{w}_\ell+\mathrm{t}}(z)$ belongs to $L^1_{\mathscr{H}^{n-k}(z)}(Z^k_\ell;\mathcal{C}^1_{\mathrm{t}}(C^k;\Lambda_k\mathbb{R}^n))$ for each ℓ.

Then, the k-dimensional current $\mathscr{L}^n\wedge\xi_k$ is a locally flat chain, since it is the limit in the flat norm of normal currents: indeed, for $k>0$ one has

$$\partial\big(\mathscr{L}^n\wedge\xi_k\big) = \mathbf{F}\text{-}\lim_{\ell}\sum_{i=1}^{\ell}\Big\{-\mathscr{L}^n\wedge(\operatorname{div}(\mathbf{1}_{\mathscr{C}_i^k}\xi_k))_{\mathrm{a.c.}} + \big(\mathscr{H}^{n-1}\llcorner\mathfrak{d}\mathscr{C}_i^k\big)\wedge\big\langle d\hat{n}_i,\,\xi_k\big\rangle\Big\}, \qquad (3.21)$$

where $(\operatorname{div}\mathbf{1}_{\mathscr{C}_i^k}\xi_k)_{\mathrm{a.c.}}$ is the one of Lemma 3.10, $\mathfrak{d}\mathscr{C}_i^k$, the border of $\mathscr{C}_i^k$ transversal to $\mathcal{D}$, and $\hat{n}_i$, the outer unit normal, are defined in Formula (3.4), and $d\hat{n}_i$ is the dual to $\hat{n}_i$.

Notice finally that the current $\mathscr{L}^n\wedge\xi_k$ is itself locally normal if restricted to the interior of X^k. However, in general X^k can have empty interior. If $\partial\big(\mathscr{L}^n\wedge\xi_k\big)$ is representable by integration, then the density of its absolutely continuous part w.r.t. $\mathscr{L}^n$, at any point $x\in\mathscr{C}_\ell^k$, is given by $\operatorname{div}(\mathbf{1}_{\mathscr{C}_\ell^k}\xi_k)_{\mathrm{a.c.}}(x)$.

The following table collects some of the notations used in this part of the thesis.

$\mathscr{B}(\mathbb{R}^n)$	Borel sets in $\mathbb{R}^n$
$\mathscr{L}^n$	n-dimensional Lebesgue measure
$\mathscr{H}^k$	k-dimensional Hausdorff outer measure
(X, Σ, μ)	$\Sigma =$ σ-algebra of subsets of X and $\mu =$ measure on Σ, i.e. $\mu : \Sigma \to [0, +\infty]$, $\mu(\emptyset) = 0$ and μ is countably additive on disjoint sets of Σ
$L^1_{(\mathrm{loc})}(\mu)$	(locally) integrable functions (w.r.t. μ)
$L^\infty_{(\mathrm{loc})}$	(locally) essentially bounded functions
$\mathcal{C}^k_{(\mathrm{c})}$	k-times continuously differentiable functions (with compact support)
$\mathbf{1}_A$	$\mathbf{1}_A(x) = 1$ if $x \in A$, $\mathbf{1}_A(x) = 0$ otherwise
$\mu \llcorner A$	restriction of a measure μ to a set A
$\mu = \int \mu_\alpha \, d\nu$	disintegration of μ, see Definition 1.2
$\mu \ll \nu$	$\mu(A) = 0$ whenever $\nu(A) = 0$ (absolute continuity of a measure μ w.r.t. ν)
equivalent	μ is eqivalent to ν if $\mu \ll \nu$ and $\nu \ll \mu$ (1.4)
separated	two sets A and B sets are separated if each is disjoint from the other's closure
perpendicular	A set A is perpendicular to an affine plane H of $\mathbb{R}^d$ if $\exists\, w \in H$ s.t. $\pi_H(A) = w$
$v \cdot w$	Euclidean scalar product in $\mathbb{R}^n$
$\lVert\cdot\rVert$	Euclidean norm in $\mathbb{R}^n$
$\mathbb{S}^{n-1}, \mathcal{B}^n$	$\{x \in \mathbb{R}^n : \lVert x\rVert = 1\}$, $\{x \in \mathbb{R}^n : \lVert x\rVert \leq 1\}$
$\mathscr{G}(k, n)$	Grassmaniann of k-dimensional vector spaces in $\mathbb{R}^n$
π_L	orthogonal projection from $\mathbb{R}^n$ to the affine plane $L \subseteq \mathbb{R}^n$
$\langle\cdot,\cdot\rangle$	pairing, see Subsection 3.3.1.1
$\langle \mathrm{v}_1, \ldots, \mathrm{v}_k \rangle$	linear span of vectors $\{\mathrm{v}_1, \ldots, \mathrm{v}_k\}$ in $\mathbb{R}^n$
$\mathrm{aff}(A)$	affine hull of A, the smallest affine plane containing A
$\mathrm{conv}(A)$	convex envelope of A, the smallest convex set containing A
$\dim(A)$	linear dimension of $\mathrm{aff}(A)$
$\mathrm{ri}(C)$	relative interior of C, the interior of C w.r.t. the topology of $\mathrm{aff}(C)$
$\mathrm{rb}(C)$	relative boundary of C, the boundary of C w.r.t. the topology of $\mathrm{aff}(C)$
R-face	see Definition 2.10
extreme points	zero-dimensional R-faces
$\mathrm{ext}(C)$	extreme points of a convex set C
$\mathrm{dom}\, g$	the domain of a function g
$\mathrm{graph}\, g$	$\{(x, g(x)) : x \in \mathrm{dom}\, g\}$ (graph)
$\mathrm{epi}\, g$	$\{(x, t) : x \in \mathrm{dom}\, g,\ t \geq g(x)\}$ (epigraph)

∇g	gradient of g
$\partial^- g$	subdifferential of g, see Page 35
$g\|_a$	evaluation of g at the point a
$g\|_b^a$	the difference $g(b) - g(a)$
$g_{\|A}$	the restriction of g to a subset A of $\operatorname{dom} g$
f	a fixed convex function $\mathbb{R}^n \to \mathbb{R}$
$\operatorname{dom} \nabla f$	a fixed σ-compact set where f is differentiable, see Section 2.1
$\operatorname{Im} \nabla f$	$\{\nabla f(x) : x \in \operatorname{dom} \nabla f\}$, see Section 1.3
face of f	intersection of graph $f_{\|\operatorname{dom} \nabla f}$ with a tangent hyperplane
k-face of f	k-dimensional face of f
F_y	$\nabla f^{-1}(y) = \{x \in \operatorname{dom} \nabla f : \nabla f(x) = y\}$
F_y^k	F_y when $\dim(F_y) = k, \quad k = 0, \dots, n$
F^k	the set $\underset{y}{\cup} F_y^k$
$\mathcal{P}(x)$	see Formula (2.7)
$\mathcal{R}(x)$	see (1.34) and (2.8) for the faces of a convex function
$\mathcal{T}$	$\{x \in \operatorname{dom} \nabla f \cap \pi_2(\operatorname{graph} \mathcal{R}) : \mathcal{R}(x) \neq \{x\}\}$
$\mathcal{D}$	multivalued map of set directions, see Formula (1.35) and (2.11) for the faces of a convex function
Z^k	section of a sheaf set, see Definition 1.19
$\mathscr{Z}^k$	sheaf set, see Definition 1.19
$[\mathrm{v}, \mathrm{w}]$	segment that connects v to w, i.e. $\{(1-\lambda)\mathrm{v} + \lambda \mathrm{w} : \lambda \in [0,1]\}$
$\Pi_{i=1}^k [\mathrm{v}_i, \mathrm{w}_i]$	k-dimensional rectangle in $\mathbb{R}^n$ with sides parallel to $\{[\mathrm{v}_i, \mathrm{w}_i]\}_{i=1}^k$, equal to the convex envelope of $\{\mathrm{v}_i, \mathrm{w}_i\}_{i=1}^k$
$\mathscr{C}^k(\mathscr{Z}^k, C^k)$	k-dimensional $\mathcal{D}$-cylinder $\mathscr{C}^k$, see Definition 1.21
$\mathfrak{d}\mathscr{C}^k$, $\hat{n}_{\|\mathfrak{d}\mathscr{C}^k}$	border of $\mathscr{C}^k$ transversal to $\mathcal{D}$ and outer unit normal, see Formula (3.4)

$\sigma^{\mathrm{w}+t\mathrm{e}}$	a map which parametrizes a $\mathcal{D}$-cylinder $\mathscr{C}^k(\mathscr{Z}^k, C^k)$, see Formula (1.20)
$\sigma^{t\mathrm{e}}$	$\sigma^{t\mathrm{e}} = \sigma^{0+t\mathrm{e}}$, where $\mathrm{e} \in \mathbb{S}^{n-1}$, $t \in \mathbb{R}$
σ^{t}	if we write $\mathrm{t} = t\mathrm{e}$ with e a unit direction, then $\sigma^{\mathrm{t}} = \sigma^{0+t\mathrm{e}}$
$\alpha(t, s, x)$	see Formula (1.25)
$\operatorname{div} \mathrm{v}$	if $\mathrm{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$, its divergence is the distribution $\mathcal{C}^1_{\mathrm{c}}(\mathbb{R}^n) \ni \varphi \mapsto -\int \mathrm{v} \cdot \nabla \varphi$
$(\operatorname{div} \mathrm{v})_{\mathrm{a.c.}}$	see Notation 3.2.1, Formula (3.18)
v_i	see Definition 3.2
$(\operatorname{div} \mathrm{v}_i)_{\mathrm{a.c.}}$	see Formula (3.15)

Part II. Approximation of orientation-preserving homeomorphisms

CHAPTER 4

Smooth approximation of planar bi-Lipschitz orientation-preserving homeomorphisms

4.1. Scheme of the proof

The aim of this section is to present a short scheme of the construction of a countably piecewise affine approximation of u as in Theorem 0.2. Indeed, as mentioned in the introduction, the smooth extension readily follows by the following recent result from [**46**].

THEOREM 4.1. *Let $v : \Omega \longrightarrow \mathbb{R}^2$ be a (countably) piecewise affine homeomorphism, bi-Lipschitz with constant L. Then there exist a smooth diffeomorphism $\hat{v} : \Omega \longrightarrow v(\Omega)$ such that $\hat{v} \equiv v$ on $\partial\Omega$, $\hat{v}$ is bi-Lipschitz with constant at most $70L^{7/3}$, and*

$$\|\hat{v} - v\|_{L^\infty(\Omega)} + \|D\hat{v} - Dv\|_{L^p(\Omega)} + \|\hat{v}^{-1} - v^{-1}\|_{L^\infty(v(\Omega))} + \|D\hat{v}^{-1} - Dv^{-1}\|_{L^p(v(\Omega))} \leq \varepsilon\,.$$

Approximation of u on Lebesgue squares. The first idea is to use the fact that, in a sufficiently small neighborhood of each Lebesgue point z for the differential Du, the map u is arbitrarily close, both in $W^{1,p}$ and in L^∞, to an affine L bi-Lipschitz map (given by its linearization around the point z). The $W^{1,p}$ estimate is simply a restatement of the definition of Lebesgue point of Du, while the L^∞ estimate is proven in Lemma 4.13. Indeed we prove that, given a square $\tilde{\mathcal{D}} \subseteq \Omega$ (e.g. a neighborhood of z), the more Du is close in $L^p(\tilde{\mathcal{D}})$ to an L bi-Lipschitz matrix M (given e.g. by $Du(z)$), the more u is close in $L^\infty(\tilde{\mathcal{D}})$ to an L bi-Lipschitz affine map u_M with $Du_M = M$. Moreover, since u is bi-Lipschitz, we have that also the inverse of u is close both in $W^{1,p}(u(\tilde{\mathcal{D}}))$ and in $L^\infty(u(\tilde{\mathcal{D}}))$ to the inverse of u_M.

The main implication of these estimates towards the construction of a piecewise affine bi-Lipschitz map approximating u is the following. Let us take a square $\tilde{\mathcal{D}} \subseteq \Omega$ as above and let us consider the piecewise affine function v which coincides with u on the vertices of $\tilde{\mathcal{D}}$ and is affine on each of the two triangles obtained dividing $\tilde{\mathcal{D}}$ with a diagonal. If $\|Du - M\|_{L^p(\tilde{\mathcal{D}})}$ is sufficiently small, then the L^∞ estimate implies that $u(\partial\tilde{\mathcal{D}})$ is uniformly relatively close to the parallelogram of side lengths at least $\mathrm{side}(\tilde{\mathcal{D}})/L$ given by $u_M(\partial\tilde{\mathcal{D}})$. Hence, since $v = u$ on the vertices of the square and is affine on each side of $\partial\tilde{\mathcal{D}}$, the same uniform estimate holds also for v. In particular, the map v is orientation-preserving, injective, and approximates u and its inverse as desired.

Finally, thanks to the fact that the Lebesgue points of Du have full measure in Ω, we fix two orthonormal vectors $\mathrm{e}_1,\, \mathrm{e}_2 \in \mathbb{R}^2$ and, $\forall\, \varepsilon > 0$, we find a set $\Omega_\varepsilon \subset\subset \Omega$ with $\mathscr{L}(\Omega \setminus \Omega_\varepsilon) \leq \varepsilon$ which is made by a uniform "tiling" $\{\mathcal{D}_\alpha\}_\alpha$ of squares with sides parallel to $\mathrm{e}_1,\, \mathrm{e}_2$ with the following property. On each square $\mathcal{D}_\alpha$ of the tiling, Du is sufficiently close to an L bi-Lipschitz matrix M (in particular, M will be equal to $Du(z)$ for some Lebesgue point $z \in \mathcal{D}_\alpha$). Then, putting together the previous remarks, one can show that the piecewise affine function v obtained interpolating between the values of u on the vertices of the squares is injective and satisfies (0.3) on Ω_ε. Moreover, v is $L+\varepsilon$ bi-Lipschitz. The squares of the tiling covering Ω_ε will be called *Lebesgue squares*, and the set Ω_ε *right polygon*, due to its shape –see Figure 1.

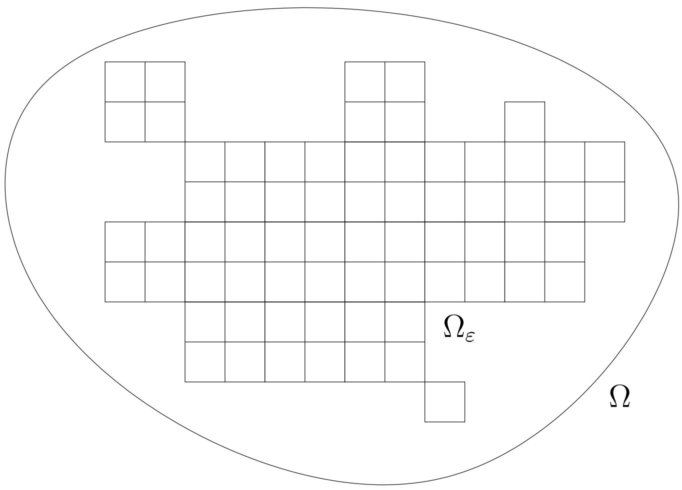

FIGURE 1. An open set Ω and a right polygon $\Omega_\varepsilon \subset\subset \Omega$

Thus, the first idea of the proof leads to define a piecewise affine approximation of u on a set whose Lebesgue measure is as close as we want to $\mathscr{L}(\Omega)$. In order to complete the construction, we have to define v in the interior of the set $\Omega \setminus \Omega_\varepsilon$.

Countably piecewise affine bi-Lipschitz extension. The second idea of our proof is to reduce to the following model case: $\Omega \setminus \Omega_\varepsilon$ is a square of Lebesgue measure at most ε and $u_{|\partial(\Omega\setminus\Omega_\varepsilon)}$ is a piecewise affine function. In particular, by the previous construction, $v = u$ on $\partial\Omega_\varepsilon$.

In this case, an approximating function v is provided by the planar bi-Lipschitz extension Theorem 0.4 proved in [**24**]. Indeed, it is sufficient to take $\tilde{u} = v_{|\partial\Omega_\varepsilon}$ and $v = \tilde{v}$. In particular, as mentioned at the end of the Introduction, in [**24**] it is shown that one can take the geometric constant $C_3 = C = 636000$.

It is then easy to verify that, provided ε is chosen sufficiently small at the beginning, such an extension of $u_{|\partial(\Omega\setminus\Omega_\varepsilon)}$ together with the already defined piecewise affine interpolation of u on the Lebesgue squares, satisfies the assumptions of Theorem 0.2. Indeed, by definition, v is injective on the whole Ω. Moreover, we know by the previous construction that it satisfies (0.3) on Ω_ε. On the other hand, on $\Omega \setminus \Omega_\varepsilon$, $|Du|$ and $|Dv|$ are bounded by the two Lipschitz constants L and C_3L^4 (together with their inverses) on a set of small area and then the $W^{1,p}$ estimates in (0.3) follow. Finally, since $\Omega \setminus \Omega_\varepsilon$ and $u(\Omega \setminus \Omega_\varepsilon)$ have small Lebesgue measure, v and v^{-1} are also close to u and u^{-1} in L^∞.

In order to reduce to this model case, we perform the following steps:

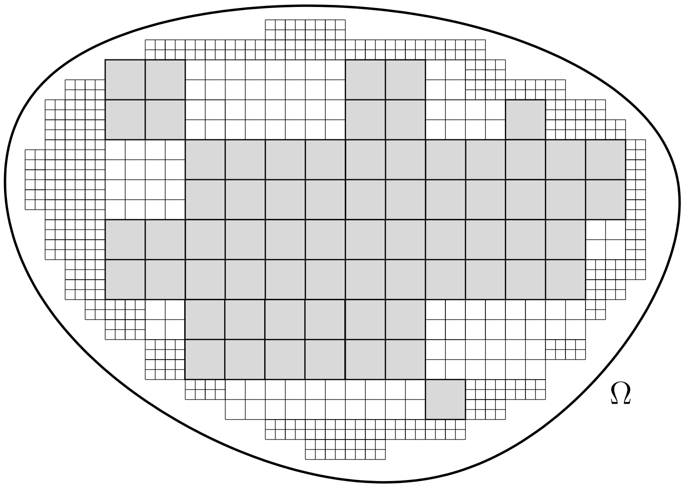

FIGURE 2. The countable tiling of $\Omega \setminus \Omega_\varepsilon$ (the shaded region is Ω_ε)

1. We cover $\Omega \setminus \Omega_\varepsilon$ with a countable (locally finite in Ω) "tiling" of small squares whose sides are parallel to e_1 and e_2 (see Figure 2).

2. On the 1-dimensional grid $\mathcal{Q}$ given by the boundaries of the squares of the tiling we define the piecewise affine approximation v in such a way that $v(\mathcal{Q}) \subseteq \Delta$ and v is $72L$ bi-Lipschitz.

3. We "fill" the squares of the tiling extending $v_{|Q}$ by means of Theorem 0.4, thus getting a globally $C_3(72L)^4$ bi-Lipschitz function on $\Omega \setminus \Omega_\varepsilon$.

The fact that the Lipschitz constant of v on Ω_ε depends only on the Lipschitz constant of u will tell us that, as in the model case, the $W^{1,p}$ and L^∞ norms of $u - v$ and $u^{-1} - v^{-1}$ can be made as small as we want –provided we choose ε sufficiently small at the beginning. Thus we end the proof of Theorem 0.2.

Let us also give a very rough idea of how the proofs of Steps **1**, **2** and **3** works. While Step **1** is a simple geometric construction, Step **2** essentially consists in approximating u on the grid $\mathcal{Q}$ with a piecewise affine function. This will be possible thanks to Lemma 4.19, which tells that it is possible to approximate u on the segments, and Lemma 4.20, which takes care of the "crosses". Finally, Proposition 4.15 in Section 4.4 concludes the argument of Steps **2** and **3**. The essential idea there is that, since on the "non-Lebesgue squares" the behaviour of u is wilder, one cannot simply take v equal to the affine interpolation of u on the vertices. Indeed, as already pointed out in the introduction, this could easily give a non-injective function. However, since the total area of the non-Lebesgue squares is small, *any* approximation of u on them is ok, and this is why we use the extension of $v_{|Q}$ given by Theorem 0.4 in Step **3**.

4.2. Notation and Preliminaries

In this section we give some preliminary definitions and fix some useful notation which will be used in this part of the thesis.

We denote by Ω a bounded open subset of $\mathbb{R}^2$ and by $\operatorname{clos}\Omega$ its closure.

First we recall the following

Definition 4.2 (L bi-Lipschitz map). *We say that a function $u : \Omega \to u(\Omega)$ is L bi-Lipschitz for some $L > 0$ if*

$$\frac{1}{L}\,|y-x| \le |u(y)-u(x)| \le L|y-x|\,, \quad \forall\, x,y \in \Omega. \tag{4.1}$$

In particular, $L \ge 1$.

Then we recall the definition of orientation-preserving (resp. reversing) homeomorphism.

Definition 4.3 (Orientation-preserving (reversing) homeomorphism). *We say that an homeomorphism $u : \Omega \longrightarrow u(\Omega) \subseteq \mathbb{R}^2$ is orientation-preserving (reversing) if whenever a simple closed curve $[0,1] \ni t \mapsto \gamma(t) \in \Omega$ is parameterized clockwise, then $[0,1] \ni t \mapsto u(\gamma(t)) \in u(\Omega)$ is parameterized clockwise (resp. anti-clockwise).*

It is well known that if Ω is connected, then any homeomorphism $u : \Omega \longrightarrow u(\Omega) \subseteq \mathbb{R}^2$ is either orientation-preserving or orientation-reversing. Moreover, if u is a diffeomorphism being orientation-preserving (reversing) is equivalent to satisfy $\det Du \ge 0$ ($\det Du \le 0$) pointwise on Ω.

Next, we define the class of (countably) piecewise affine functions, in which we look for approximations of orientation-preserving homeomorphisms. To this aim we recall the definitions of *(finite) triangulation* of a polygon and of *locally finite triangulation* of Ω. A *polygon* is an open connected subset of $\mathbb{R}^2$ whose boundary is given by a finite union of segments intersecting only at their endpoints.

Definition 4.4 ((Finite) triangulation). *A* (finite) triangulation *of a polygon $\Omega' \subseteq \mathbb{R}^2$ is a finite collection of closed triangles $\{T_i\}_{i=1}^N$ whose union is equal to* $\operatorname{clos}\Omega'$ *and for all $i \neq j$*

$$T_i \cap T_j \quad \text{is either empty, or a common vertex, or a common side of } T_i \text{ and } T_j\,. \tag{4.2}$$

Definition 4.5 (Locally finite triangulation). *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set. A* locally finite triangulation *of Ω is a locally finite collection of closed triangles $\{T_i\}_{i\in\mathbb{N}}$ such that $\Omega \subseteq \bigcup_{i\in\mathbb{N}} T_i \subseteq \operatorname{clos}\Omega$ and* (4.2) *holds.*

We notice that, unless Ω is a polygon, the number of elements of a triangulation cannot be finite.

Definition 4.6 (Piecewise affine and countably piecewise affine function). *A function $v : \Omega \longrightarrow \mathbb{R}^2$ is* countably piecewise affine *if $v_{|T}$ is affine on every triangle T of a suitable locally finite triangulation of Ω. If Ω is a polygon and the triangulation is finite, then we say that v is* (finitely) piecewise affine.

In order to build the triangulation on which the function v of Theorem 0.2 is countably piecewise affine, we will use, on a subset of Ω of Lebesgue measure as close as we want to $\mathscr{L}(\Omega)$ (i.e., the Lebesgue measure of Ω), uniform triangulations into right triangles. The union of such triangles will be called a right polygon, according to the following definition. From now on, $\mathrm{e}_1, \mathrm{e}_2$ will be two fixed orthonormal vectors in $\mathbb{R}^2$.

Definition 4.7 (Right polygon). *An open bounded set $\Omega' \subset \mathbb{R}^2$ is called a* right polygon of side-length r *(or simply an r-*polygon*) if it is a finite union of closed polygons whose sides are all parallel to e_1, e_2, and have lengths which are integer multiples of $r > 0$.*

Points in Ω will be denoted by $z \in \mathbb{R}^2$ or by $(x, y) \in \mathbb{R} \times \mathbb{R}$, with $z = x\mathrm{e}_1 + y\mathrm{e}_2$. We denote with $\mathcal{B}(z, r)$ the ball of center z and radius r and with $\mathcal{D}(z, r)$ the square of center z, side length r and sides parallel to e_1, e_2. Moreover, the generic square of a collection of squares $\{\mathcal{D}(z_\alpha, r_\alpha)\}_{\alpha\in\mathbb{N}}$ will be also sometimes denoted simply by $\mathcal{D}_\alpha$. Instead of working directly with triangulations, it will be convenient, in order to apply our method, to subdivide Ω into a countable and locally finite family of squares called tiling.

Definition 4.8 (Tiling). *Given an open, bounded set Ω, a* tiling of Ω *is a locally finite (in Ω) collection of closed squares $\{\mathcal{D}_\alpha(z_\alpha, r_\alpha)\}_{\alpha\in\mathbb{N}}$ whose union is contained between Ω and* $\operatorname{clos}\Omega$ *and such that, $\forall \alpha \neq \beta \in \mathbb{N}$, $\mathcal{D}_\alpha \cap \mathcal{D}_\beta$ is either empty, or a common vertex of $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$, or a side of one of the two. Two squares of a tiling are said to be* adjacent *if their intersection is nonempty.*

Notice that a tiling of Ω can be either finite or countable, and in particular it is surely countable if Ω is not a right polygon.

It will be often useful to regard a given tiling of Ω as the union of the finite tiling corresponding to a right polygon $\Omega' \subset\subset \Omega$ and a countable tiling of $\Omega \setminus \Omega'$, locally finite in Ω. Since these kinds of "sub-tilings" will be frequently used in the paper, we define them separately.

Definition 4.9 (r-Tiling of a right polygon and tiling of (Ω, Ω')). *Given an r-polygon Ω', the* r-tiling of Ω' *is the (unique) finite collection of closed squares $\{\mathcal{D}(z_\alpha, r)\}_{\alpha\in\mathscr{I}(r)}$ whose union is equal to* $\operatorname{clos}\Omega'$ *and, $\forall\, \alpha \neq \beta \in \mathscr{I}(r)$, $\mathcal{D}_\alpha \cap \mathcal{D}_\beta$ is either empty, or a common vertex, or a common side of $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$. Given a bounded, open set Ω and an r-polygon $\Omega' \subset\subset \Omega$, a* tiling of (Ω, Ω') *is a tiling of Ω whose restriction to Ω' is the r-tiling of Ω'.*

The 1-dimensional skeleton of a tiling will be called grid, according to the following definition.

Definition 4.10 (Grid). *Let $\{\mathcal{D}_\alpha\}_{\alpha\in\mathbb{N}}$ be a tiling of Ω. We call* grid of the tiling *the 1-dimensional set given by the union of the boundaries of the squares of the tiling. Each side (resp. vertex) of the squares of a tiling will be called* side *(resp.* vertex*) of the grid.*

A possible definition of a piecewise affine approximation of u on a given r-polygon, which will be used in Section 4.3, is the following.

Definition 4.11 ((Ω', r)-interpolation of u). *Let $\Omega' \subset\subset \Omega$ and $\{\mathcal{D}_\alpha\}_{\alpha\in\mathscr{I}(r)}$ be an r-right polygon and its r-tiling. We call* (Ω', r)-interpolation of u *the piecewise affine function $v : \Omega' \longrightarrow v(\Omega') \subseteq \mathbb{R}^2$ which coincides with u on the vertices of the r-tiling and,* *for each* $\alpha \in \mathscr{I}(r)$*, is affine on the two right triangles forming* $\mathcal{D}_\alpha$ *and having as common hypothenuse the north-east/south-west diagonal of* $\mathcal{D}_\alpha$*.*

We conclude this section with a table collecting the main notation.

$\mathbb{S}^1$	unit sphere of $\mathbb{R}^2$,
$\Omega \subseteq \mathbb{R}^2$	a given open bounded set ,
$u : \Omega \longrightarrow \Delta$	a given L bi-Lipschitz function ,
$\mathscr{M}(2 \times 2)$	two by two real matrices ,
$\lvert M \rvert$	$\sup\left\{\lvert Mv \rvert : \lvert v \rvert = 1\right\}$,
$\mathscr{M}(2 \times 2; L)$	$\left\{M \in \mathscr{M}(2 \times 2) : \mathrm{Det}\, M > 0, \lvert M \rvert \leq L, \lvert M^{-1} \rvert \leq L\right\}$,
e_1, e_2	two fixed positively oriented orthonormal vectors in $\mathbb{R}^2$,
$\mathcal{B}(z, r)$	ball with center z and radius r ,
$\mathcal{D}(z, r)$	square with center z, side length r and sides parallel to e_1, e_2 ,
$\mathcal{D}$	$\mathcal{D}(0, 1)$,
$\mathscr{L}$	Lebesgue measure on $\mathbb{R}^2$,
$\mathscr{H}^1$	1-dimensional Hausdorff measure ,
$\mathrm{int}\, A$	interior of a set $A \subseteq \mathbb{R}^2$,
$\mathrm{clos}\, A$	closure of A ,
∂A	boundary of $A \subseteq \mathbb{R}^2$,
$\Omega' \subset\subset \Omega$	$\mathrm{clos}\, \Omega' \subseteq \Omega$,
$d(A, B)$	$\inf\{\lvert z - w \rvert : z \in A, w \in B\}$.

4.3. Approximation on "Lebesgue squares"

The aim of this section is to prove the following

PROPOSITION 4.12. *For every* $\varepsilon > 0$ *there exists a right polygon* $\Omega_\varepsilon \subset\subset \Omega$ *of side length* r *such that the* (Ω_ε, r)*-interpolation* $v : \Omega_\varepsilon \longrightarrow v(\Omega_\varepsilon) \subseteq \mathbb{R}^2$ *is* $L + \varepsilon$ *bi-Lipschitz and satisfies*

$$\Delta_\varepsilon := v(\Omega_\varepsilon) \subset\subset \Delta\,, \tag{4.3}$$

$$\|v - u\|_{L^\infty(\Omega_\varepsilon)} + \|v^{-1} - u^{-1}\|_{L^\infty(\Delta_\varepsilon)} + \|Du - Dv\|_{L^p(\Omega_\varepsilon)} + \|Du^{-1} - Dv^{-1}\|_{L^p(\Delta_\varepsilon)} \leq \varepsilon\,, \tag{4.4}$$

$$\mathscr{L}(\Omega \setminus \Omega_\varepsilon) \leq \varepsilon\,, \quad \mathscr{L}(\Delta \setminus \Delta_\varepsilon) \leq \varepsilon\,, \quad d(\Omega_\varepsilon, \mathbb{R}^2 \setminus \Omega) \geq 2r\,, \tag{4.5}$$

$$\|v - u\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{\sqrt{2}r}{6L^3}\,. \tag{4.6}$$

The reason why the piecewise affine interpolation of u will be injective on Ω_ε is that, for each square $\mathcal{D}_\alpha$ of the r-tiling of Ω_ε, the function u will be uniformly close to an affine L bi-Lipschitz function on the nine squares around $\mathcal{D}_\alpha$. The linear part of each of these affine functions will be the differential of u at some Lebesgue point for Du inside $\mathcal{D}_\alpha$. For this reason, the squares of such r-tiling will be called "Lebesgue squares".

The plan of this section is the following. Section 4.3.0.4 contains Lemma 4.13, which is the main ingredient in the proof of Proposition 4.12. Indeed, Lemma 4.13 says that, when on a square Du is close in average to an L bi-Lipschitz matrix M, then u is close in L^∞ to an affine function u_M with $Du_M = M$. Then, in Section 4.3.0.5, we will determine Ω_ε as a suitable union of squares of an r-tiling on which Lemma 4.13 holds and provides a sufficiently strong L^∞ estimate. Finally, in Section 4.3.0.6 we show that the (Ω_ε, r)-interpolation of u satisfies the required properties.

4.3.0.4. *An L^∞ Lemma.* We are now ready to begin the proof of Proposition 4.12, starting from the following fundamental lemma. Here and in the following, by $\mathscr{M}(2\times 2; L)$ we denote the set of the two by two invertible matrices M such that the affine map $z \mapsto M(z)$ is L bi-Lipschitz. Moreover, Ω and u will always be a set and a function as in the assumptions of Theorem 0.2.

LEMMA 4.13. *For any $\eta > 0$ there exists $\delta = \delta(\eta) > 0$ such that, if $\bar z \in \Omega$, $M \in \mathscr{M}(2\times 2; L)$ and $\rho > 0$ are so that $\mathcal{D}(\bar z, \rho) \subset\subset \Omega$ and*

$$⨍_{\mathcal{D}(\bar z,\rho)} |Du(z) - M|\, dz \le \delta, \tag{4.7}$$

then there exists an affine function $u_M : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $Du_M = M$ and so that

$$|u(z) - u_M(z)| \le \eta\rho \qquad\qquad \forall\, z \in \mathcal{D}(\bar z, \rho)\,. \tag{4.8}$$

PROOF. Up to a translation, we are allowed to assume for simplicity that $\bar z = u(\bar z) = (0,0) \in \mathbb{R}^2$. Let us then call, for a big constant R to be specified later,

$$B^1 := \left\{x \in \big[-\rho/2, \rho/2\big] : \int_{-\rho/2}^{\rho/2} |Du(x,t) - M|\, dt \le \rho R\delta\right\},$$
$$B^2 := \left\{y \in \big[-\rho/2, \rho/2\big] : \int_{-\rho/2}^{\rho/2} |Du(t,y) - M|\, dt \le \rho R\delta\right\}.$$

Notice that, since u is bi-Lipschitz on Ω, then so is its restriction to the horizontal and vertical segments of the square $\mathcal{D}(0,\rho)$. Hence, the above integrals make sense for every x and y. By (4.7) and Fubini–Tonelli Theorem, we readily obtain

$$\mathscr{H}^1\Big(\big[-\rho/2,\rho/2\big] \setminus B^1\Big) \le \frac{\rho}{R}, \qquad\qquad \mathscr{H}^1\Big(\big[-\rho/2,\rho/2\big] \setminus B^2\Big) \le \frac{\rho}{R}. \tag{4.9}$$

Define now $u_M(z) = Mz$, and $\varphi(z) = u(z) - u_M(z)$. For any $x_1, x_2 \in B^1$ and $y_1, y_2 \in B^2$ we immediately get

$$\begin{aligned}\Big|\varphi(x_1,y_1) - \varphi(x_2,y_2)\Big| &\le \Big|\varphi(x_1,y_1) - \varphi(x_2,y_1)\Big| + \Big|\varphi(x_2,y_1) - \varphi(x_2,y_2)\Big| \\ &\le \int_{x_1}^{x_2} \Big|Du(t,y_1) - M\Big|\, dt + \int_{y_1}^{y_2} \Big|Du(x_2,t) - M\Big|\, dt \le 2\rho R\delta\,.\end{aligned} \tag{4.10}$$

Let now $(x,y) \in \mathcal{D}(\bar z,\rho)$ be a generic point. By (4.9), there exist $x_1 \in B^1$ and $y_1 \in B^2$ so that

$$\Big|x - x_1\Big| \le \frac{\rho}{R}, \qquad\qquad \Big|y - y_1\Big| \le \frac{\rho}{R},$$

and since u and u_M are L bi-Lipschitz, thus φ is $2L$-Lipschitz, we get

$$\Big|\varphi(x,y) - \varphi(x_1,y_1)\Big| \le \frac{2\sqrt{2}\rho L}{R}\,. \tag{4.11}$$

Let finally (x,y) and $(\tilde x,\tilde y)$ be two generic points in $\mathcal{D}(\bar z, r)$. Putting together (4.10) and (4.11) we immediately get

$$\Big|\varphi(x,y) - \varphi(\tilde x,\tilde y)\Big| \le \frac{4\sqrt{2}\rho L}{R} + 2\rho R\delta \le \eta\rho\,,$$

where the last inequality is true up to take R big enough and then δ small enough. Since $\varphi(0,0) = 0$, this concludes the proof. □

4.3.0.5. *A large right polygon made of Lebesgue squares.* In this section we show that, for any $\eta > 0$, it is possible to construct a right polygon $\Omega_\eta \subset\subset \Omega$ of side length r_η such that $\mathscr{L}(\Omega \setminus \Omega_\eta) \leq \eta$ and such that, for any square $\mathcal{D}(z, r_\eta)$ of the r_η-tiling of Ω_η, the assumption (4.7) of Lemma 4.13 holds on the bigger square $\mathcal{D}(z, 3r_\eta)$. As we will show in Section 4.3.0.6, if we choose η and then r_η small enough, the corresponding (Ω_η, r_η)-interpolation of u satisfies the requirement of Proposition 4.12. Then, Ω_η will turn out to be the right polygon of Lebesgue squares we are looking for. The goal of this section is to show the following estimate.

Lemma 4.14. *For every $\eta > 0$ there exists a constant $r = r(\eta) > 0$ and an r-polygon $\Omega_\eta \subset\subset \Omega$ such that $\mathscr{L}\big(\Omega \setminus \Omega_\eta\big) \leq \eta$ and each square of the r-tiling $\{\mathcal{D}(z_\alpha, r)\}_{\alpha \in \mathscr{I}(r)}$ satisfies the following properties,*

$$\mathcal{D}(z_\alpha, 3r) \subset\subset \Omega \quad \forall\, \alpha \in \mathscr{I}(r), \tag{4.12}$$

$$\fint_{\mathcal{D}(z_\alpha,3r)} |Du(z) - M|\, dz \leq \delta(\eta) \text{ for some } M = M(\alpha) \in \mathscr{M}(2 \times 2; L). \tag{4.13}$$

Proof. We start selecting some $r_0 = r_0(\eta) > 0$ and an r_0-polygon $\Omega_0 \subset\subset \Omega$ such that $\mathscr{L}\big(\Omega \setminus \Omega_0\big) \leq \eta/2$ and each square of the r_0-tiling of Ω_0 satisfies (4.12). Then, for every r such that $r_0 \in r\mathbb{N}$, we can regard Ω_0 also as an r-polygon, and consequently call $\{\mathcal{D}(z_\alpha, r)\}_{\alpha \in \mathscr{I}_0(r)}$ its r-tiling. We define the set

$$\mathscr{I}(r) := \Big\{\alpha \in \mathscr{I}_0(r) : \fint_{\mathcal{D}(z_\alpha,3r)} |Du - M| \leq \delta \text{ for some } M = M(\alpha) \in \mathscr{M}(2 \times 2; L)\Big\},$$

where $\delta = \delta(\eta)$ is given by Lemma 4.13, and we let

$$\Omega_\eta := \bigcup_{\alpha \in \mathscr{I}(r)} \mathcal{D}(z_\alpha, r)\,.$$

Since property (4.13) is true by construction, to conclude the proof it is enough to select a suitable $r = r(\eta)$ in such a way that $\mathscr{L}\big(\Omega_0 \setminus \Omega_\eta\big) \leq \eta/2$.

To do so, we apply the Lebesgue Differentiation Theorem to the map Du finding that, for $\mathscr{L}$-a.e. $z \in \Omega_0$, there exists $r(z) > 0$ such that $\mathcal{D}(z, 4r(z)) \subseteq \Omega_0$ and

$$\fint_{\mathcal{D}(z,\rho)} \Big|Du(w) - Du(z)\Big|\, dw \leq \frac{\delta}{2} \qquad\qquad \forall\, 0 < \rho \leq 4r(z)\,.$$

We can then choose $r = r(\eta)$ so small that the set $A(r) := \big\{z \in \Omega_0 : r(z) \leq r\big\}$ satisfies

$$\mathscr{L}\big(A(r)\big) \leq \eta/2\,. \tag{4.14}$$

We now claim that, for each $\alpha \in \mathscr{I}_0(r)$,

$$\mathcal{D}(z_\alpha, r) \not\subseteq A(r) \qquad \Longrightarrow \qquad \alpha \in \mathscr{I}(r). \tag{4.15}$$

Indeed, letting $M = Du(z)$ for some $z \in \mathcal{D}(z_\alpha, r) \setminus A(r)$, by definition of $A(r)$ and $r(z)$ we get

$$\fint_{\mathcal{D}(z_\alpha,3r)} |Du - M| = \frac{1}{9r^2}\int_{\mathcal{D}(z_\alpha,3r)} |Du - M| \leq \frac{1}{9r^2}\int_{\mathcal{D}(z,4r)} |Du - M| = \frac{16}{9}\fint_{\mathcal{D}(z,4r)} |Du - M| \leq \frac{8}{9}\,\delta\,,$$

thus (4.15) is obtained. As a consequence, by (4.14) we have that

$$\mathscr{L}\left(\Omega_0 \setminus \Omega_\eta\right) = \mathscr{L}\left(\bigcup_{\alpha\in\mathscr{I}_0(r)\setminus\mathscr{I}(r)} \mathcal{D}(z_\alpha, r)\right) \leq \mathscr{L}\big(A(r)\big) \leq \frac{\eta}{2}$$

and, as we noticed above, this concludes the proof. □

4.3.0.6. *Affine approximation of u on Lebesgue squares.* In this section we complete the proof of Proposition 4.12. At this point, the proof reduces to show that, provided we choose η small enough, the (Ω_η, r)-interpolation of u on the right polygon Ω_η as in Lemma 4.14 satisfies the properties of Proposition 4.12.

PROOF OF PROPOSITION 4.12: Let $\varepsilon > 0$ be a given constant. Then, let $\eta = \eta(\varepsilon)$ be a sufficiently small constant, whose value will be precised later. Define now $\delta = \delta\big(\eta(\varepsilon)\big)$ as in Lemma 4.13, and define also $r = r\big(\eta(\varepsilon)\big)$ and $\Omega_\varepsilon = \Omega_{\eta(\varepsilon)}$ according to Lemma 4.14. We will show that the right polygon Ω_ε fulfills all the requirements of the proposition as soon as $\eta(\varepsilon)$ is small enough. To this aim we call, as in the statement, $v : \Omega_\varepsilon \longrightarrow \Delta_\varepsilon$ the (Ω_ε, r)-interpolation of u (see Definition 4.11) on the right polygon Ω_ε.

Let us briefly fix some notation which will be used through the proof. For any $\alpha \in \mathscr{I}(r)$, we define $M_\alpha \in \mathscr{M}(2\times 2; L)$ so that (4.13) holds. Applying Lemma 4.13 with $\rho = 3r$, we get an affine function $u_\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $Du_\alpha = M_\alpha$ and

$$\Big|u - u_\alpha\Big| \leq 3\eta r \qquad \text{on } \mathcal{D}(z_\alpha, 3r)\,. \tag{4.16}$$

Figure 3 depicts the functions u, v and u_α.

We can then start the proof, which will be divided in some steps for clarity.

<u>Step I.</u> *For any $\alpha \in \mathscr{I}(r)$, $v\big(\mathcal{D}(z_\alpha, r)\big) \subseteq u\big(\mathcal{D}(z_\alpha, 3r)\big)$.*

Take $\alpha \in \mathscr{I}(r)$. Keeping in mind (4.16) and recalling the definition of v, we get that

$$v\big(\mathcal{D}(z_\alpha, r)\big) \subseteq \mathcal{B}\Big(u_\alpha\big(\mathcal{D}(z_\alpha, r)\big), 3\eta r\Big)\,. \tag{4.17}$$

Similarly, we get that

$$u\big(\mathcal{D}(z_\alpha, 3r)\big) \supseteq \Big\{x : \mathcal{B}(x, 3\eta r) \subseteq u_\alpha\big(\mathcal{D}(z_\alpha, 3r)\big)\Big\}\,. \tag{4.18}$$

Hence, the step is concluded if

$$\mathcal{B}\Big(u_\alpha\big(\mathcal{D}(z_\alpha, r)\big), 6\eta r\Big) \subseteq u_\alpha\big(\mathcal{D}(z_\alpha, 3r)\big)\,,$$

which in turn, recalling that $Du_\alpha \equiv M_\alpha \in \mathscr{M}(2\times 2; L)$, is true as soon as $\eta < (6L)^{-1}$.

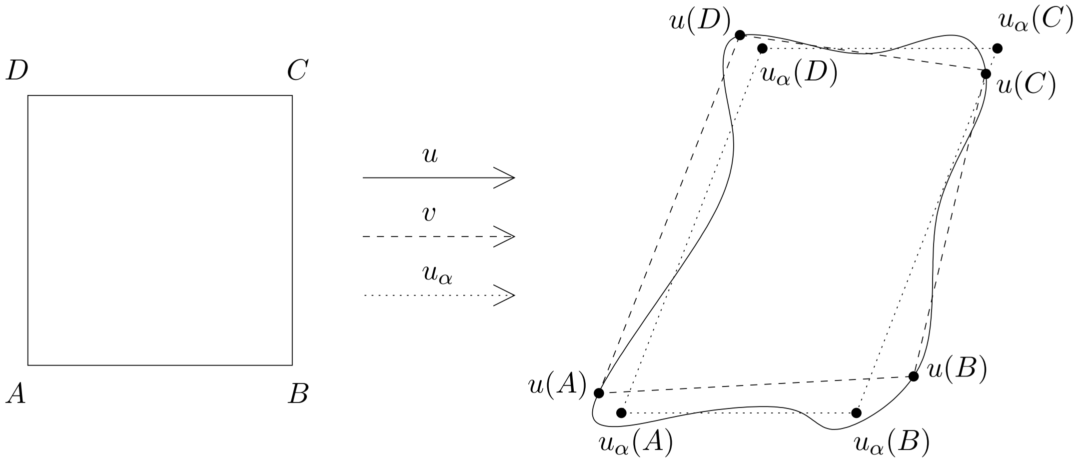

FIGURE 3. The functions u, v and u_α on a square.

Observe that, as an immediate consequence of this step and (4.12), we have $\Delta_\varepsilon \subset\subset \Delta$, that is, (4.3) holds.

Step II. Injectivity of v.

Take $\alpha \in \mathscr{I}(r)$. Applying again (4.16) as in Step I, we deduce that v is injective on $\mathcal{D}(z_\alpha, 3r) \cap \Omega_\varepsilon$ as soon as $\eta < (6L)^{-1}$. To conclude that v is injective, then, we have to show that $v\big(\mathcal{D}(z_\alpha, r)\big) \cap v\big(\mathcal{D}(z_\beta, r)\big) = \emptyset$ if $\mathcal{D}(z_\alpha, r)$ and $\mathcal{D}(z_\beta, r)$ are two non-adjacent squares of the tiling of Ω_ε. But in fact, if $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ are non-adjacent, then

$$\mathcal{D}(z_\alpha, 3r) \cap \mathcal{D}(z_\beta, r) = \mathcal{D}(z_\beta, 3r) \cap \mathcal{D}(z_\alpha, r) = \emptyset\,,$$

thus the fact that $v\big(\mathcal{D}(z_\alpha, r)\big) \cap v\big(\mathcal{D}(z_\beta, r)\big) = \emptyset$ follows as an immediate consequence of (4.17) and (4.18) applied to α and β.

Step III. Estimate on $\|v - u\|_{L^\infty(\Omega_\varepsilon)}$ and on $\|v^{-1} - u^{-1}\|_{L^\infty(\Delta_\varepsilon)}$.

Fix a generic square $\mathcal{D}_\alpha$ of the r-tiling of Ω_ε, and observe that $\|u_\alpha - u\|_{L^\infty(\mathcal{D}_\alpha)} \leq 3\eta r$ by (4.16). Moreover, v and u_α are both affine on each of the two right triangles on which $\mathcal{D}_\alpha$ is divided, and since on the vertices of these triangles v equals u, again by (4.16) we deduce also $\|v - u_\alpha\|_{L^\infty(\mathcal{D}_\alpha)} \leq 3\eta r$. Thanks to these two estimates, we deduce

$$\begin{aligned}\|v - u\|_{L^\infty(\Omega_\varepsilon)} &= \sup_{\alpha \in \mathscr{I}(r)} \|v - u\|_{L^\infty(\mathcal{D}_\alpha)} \leq \sup_{\alpha \in \mathscr{I}(r)} \|v - u_\alpha\|_{L^\infty(\mathcal{D}_\alpha)} + \|u_\alpha - u\|_{L^\infty(\mathcal{D}_\alpha)} \\ &\leq 6\eta r \leq \frac{\varepsilon}{4L}\,,\end{aligned} \tag{4.19}$$

where the last inequality is true as soon as η, hence also r, is small enough.

Since we have already proven that v is injective, the L^∞ estimate for the inverse maps is now a simple consequence. Indeed, taking a generic point $w = v(z) \in \Delta_\varepsilon$, with $x \in \Omega_\varepsilon$, by (4.19) we have

$$\Big|u^{-1}(w) - v^{-1}(w)\Big| = \Big|u^{-1}(v(z)) - u^{-1}(u(z))\Big| \leq L\Big|v(z) - u(z)\Big| \leq \frac{\varepsilon}{4}\,,$$

so that

$$\|u^{-1} - v^{-1}\|_{L^\infty(\Delta_\varepsilon)} \leq \frac{\varepsilon}{4}\,. \tag{4.20}$$

Step IV. Estimate on $\|Dv - Du\|_{L^p(\Omega_\varepsilon)}$.

Let us start observing that, since by construction $|Du| \leq L$ and $|Dv| \leq \sqrt{2}L$, one has

$$\begin{aligned}
\|Dv - Du\|^p_{L^p(\Omega_\varepsilon)} &= \sum_{\alpha\in\mathscr{I}(r)} \|Dv - Du\|^p_{L^p(\mathcal{D}_\alpha)} \\
&\leq \big(3L\big)^{p-1} \sum_{\alpha\in\mathscr{I}(r)} \|Dv - Du\|_{L^1(\mathcal{D}_\alpha)} \\
&\leq \big(3L\big)^{p-1} \sum_{\alpha\in\mathscr{I}(r)} \|Dv - Du_\alpha\|_{L^1(\mathcal{D}_\alpha)} + \|Du_\alpha - Du\|_{L^1(\mathcal{D}_\alpha)}\,.
\end{aligned} \tag{4.21}$$

By (4.13), we already know that for each $\alpha \in \mathscr{I}(r)$ it is

$$\|Du-Du_\alpha\|_{L^1(\mathcal{D}_\alpha)} = \int_{\mathcal{D}(z_\alpha,r)} \big|Du-Du_\alpha\big| \leq 9r^2 \fint_{\mathcal{D}(z_\alpha,3r)} \big|Du-M_\alpha\big| \leq 9\delta r^2 = 9\delta\big|\mathcal{D}_\alpha\big|\,. \tag{4.22}$$

Let us then concentrate on $\|Dv - Du_\alpha\|_{L^1(\mathcal{D}_\alpha)}$. Consider the triangle $T = z_1z_2z_3$, being

$$z_1 \equiv z_\alpha + \big(-r/2, -r/2\big)\,, \qquad z_2 \equiv z_\alpha + \big(r/2, -r/2\big)\,, \qquad z_3 \equiv z_\alpha + \big(r/2, r/2\big)\,.$$

Since both v and u_α are affine on T, then in particular $Dv - Du_\alpha$ is a constant linear function on T. Recalling again (4.16), let us then calculate

$$\begin{aligned}
\Big|\big(Dv_{|T} - Du_\alpha\big)(r\mathrm{e}_1)\Big| &= \Big|\big(v(z_2) - v(z_1)\big) - \big(u_\alpha(z_2) - u_\alpha(z_1)\big)\Big| \\
&= \Big|\big(u(z_2) - u(z_1)\big) - \big(u_\alpha(z_2) - u_\alpha(z_1)\big)\Big| \leq 6\eta r\,,
\end{aligned}$$

and similarly

$$\Big|\big(Dv_{|T} - Du_\alpha\big)(r\mathrm{e}_2)\Big| = \Big|\big(v(z_3) - v(z_2)\big) - \big(u_\alpha(z_3) - u_\alpha(z_2)\Big| \leq 6\eta r\,.$$

We deduce that $\|Dv - Du_\alpha\|_{L^\infty(T)} \leq 6\sqrt{2}\eta$. We can argue in the very same way for all the different triangles in which $\mathcal{D}(z_\alpha, 3r) \cap \Omega_\varepsilon$ is divided, thus we get

$$\Big\|Dv - Du_\alpha\Big\|_{L^\infty(\mathcal{D}(z_\alpha,3r)\cap\Omega_\varepsilon)} \leq 6\sqrt{2}\eta \leq 9\eta\,. \tag{4.23}$$

Inserting this estimate and (4.22) into (4.21), we get

$$\|Dv - Du\|^p_{L^p(\Omega_\varepsilon)} \leq \big(3L\big)^{p-1} 9\big(\delta+\eta\big) \sum_{\alpha\in\mathscr{I}(r)} \big|\mathcal{D}_\alpha\big| = \big(3L\big)^{p-1} 9\big(\eta+\delta\big)\big|\Omega_\varepsilon\big| \leq \left(\frac{\varepsilon}{4}\right)^p, \tag{4.24}$$

where again the last inequality holds true as soon as η, hence also δ, is small enough.

Step V. Bi-Lipschitz estimate for v.

Take a point $z \in \mathcal{D}(z_\alpha, 3r) \cap \Omega_\varepsilon$. Recalling that u_α is L bi-Lipschitz and (4.23), we get

$$\frac{1}{L} - 9\eta \leq \big|Dv(z)\big| \leq L + 9\eta\,. \tag{4.25}$$

Let then z, $z' \in \Omega_\varepsilon$ be two generic points, and assume that $z \in \mathcal{D}(z_\alpha, r)$. If one has $z' \in \mathcal{D}(z_\alpha, 3r)$, then an immediate geometric argument using the definition of v and (4.25) yields

$$\left(\frac{1}{L} - 9\eta\right)\left|z - z'\right| \leq \left|v(z) - v(z')\right| \leq \big(L + 9\eta\big)\left|z - z'\right|. \tag{4.26}$$

On the other hand, assume that $z' \notin \mathcal{D}(z_\alpha, 3r)$, so that $|z - z'| \geq r$. In this case, the L^∞ estimate (4.19) gives

$$\left|v(z) - v(z')\right| \leq \left|u(z) - u(z')\right| + \left|v(z) - u(z)\right| + \left|v(z') - u(z')\right| \leq \big(L + 12\eta\big)\left|z - z'\right|, \tag{4.27}$$

and similarly

$$\left|v(z) - v(z')\right| \geq \left|u(z) - u(z')\right| - \left|v(z) - u(z)\right| - \left|v(z') - u(z')\right| \geq \left(\frac{1}{L} - 12\eta\right)\left|z - z'\right|. \tag{4.28}$$

Putting together (4.26), (4.27) and (4.28), provided that η is small enough we conclude that v is $L + \varepsilon$ bi-Lipschitz.

Step VI. Estimate on $\|Dv^{-1} - Du^{-1}\|_{L^p(\Delta_\varepsilon)}$.

Fix a generic $\alpha \in \mathscr{I}(r)$, and recall the elementary fact that, given two invertible matrices A and B, one always has $\left|B^{-1} - A^{-1}\right| \leq \left|A^{-1}\right|\left|B^{-1}\right|\left|B - A\right|$. Since by construction u and u_α are L bi-Lipschitz, and Du_α is constant on $\mathcal{D}_\alpha$, then Step I, (4.13) and (4.22) ensure that

$$\begin{aligned}
\|Du^{-1} - Du_\alpha^{-1}\|_{L^1(v(\mathcal{D}_\alpha))} &= \int_{v(\mathcal{D}(z_\alpha, r))} \left|Du^{-1}(z) - Du_\alpha^{-1}(z)\right| dz \\
&\leq L^2 \int_{u(\mathcal{D}(z_\alpha, 3r))} \left|Du\big(u^{-1}(z)\big) - M_\alpha\right| dz \leq L^4 \int_{\mathcal{D}(z_\alpha, 3r)} \left|Du(w) - M_\alpha\right| dw \\
&= 9r^2 L^4 \fint_{\mathcal{D}(z_\alpha, 3r)} \left|Du - M_\alpha\right| \leq 9r^2 L^4 \delta = 9L^4 \delta \left|\mathcal{D}_\alpha\right|.
\end{aligned}$$

On the other hand, again using $\left|B^{-1} - A^{-1}\right| \leq \left|A^{-1}\right|\left|B^{-1}\right|\left|B - A\right|$, the fact that Du_α is constant on $\mathcal{D}_\alpha$, the fact that u_α is L bi-Lipschitz by definition while v is $(L+\varepsilon)$ bi-Lipschitz by Step V, and (4.23), we readily obtain

$$\left\|Dv^{-1} - Du_\alpha^{-1}\right\|_{L^\infty(v(\mathcal{D}_\alpha))} \leq L(L + \varepsilon) 9\eta \leq 18L^2\eta.$$

We can then repeat the same argument as in (4.21) to get

$$\begin{aligned}
\|Dv^{-1} - Du^{-1}\|^p_{L^p(\Delta_\varepsilon)} &\leq \big(3L\big)^{p-1} \sum_{\alpha \in \mathscr{I}(r)} \|Dv^{-1} - Du_\alpha^{-1}\|_{L^1(v(\mathcal{D}_\alpha))} + \|Du_\alpha^{-1} - Du^{-1}\|_{L^1(v(\mathcal{D}_\alpha))} \\
&\leq \big(3L\big)^{p-1}\Big(18L^2\eta\left|\Delta_\varepsilon\right| + 9L^4\delta\left|\Omega_\varepsilon\right|\Big) \leq \left(\frac{\varepsilon}{4}\right)^p,
\end{aligned} \tag{4.29}$$

where as usual the last estimate holds up to possibly further decrease η and then also δ.

Step VII. Conclusion.

Let us now conclude the proof of Proposition 4.12 by checking that Ω_ε fulfills all the requirements of the statement. The fact that v is $L + \varepsilon$ bi-Lipschitz is given by Step V.

The validity of (4.3) has been observed in Step I. The estimate (4.4) just follows by adding (4.19), (4.20), (4.24) and (4.29). Concerning (4.5), the facts that $\Omega\setminus\Omega_\varepsilon$ is small and that $d(\Omega_\varepsilon,\mathbb{R}^2\setminus\Omega)\geq 2r$ are given by Lemma 4.14, while the fact that also $\Delta\setminus\Delta_\varepsilon$ is small is immediate by the bi-Lipschitz property of u and the L^∞ estimate (4.8) of Lemma 4.13. Finally, (4.6) is an immediate up to choose $\eta\leq\sqrt{2}/(36L^3)$, since (4.19) ensures that $\|v-u\|_{L^\infty(\Omega_\varepsilon)}\leq 6\eta r$. □

4.4. Approximation out of “Lebesgue squares”

In this section we complete the proof of Theorem 0.2, defining the countably piecewise affine approximation of u out of the large right polygon Ω_ε of “Lebesgue squares” constructed in Proposition 4.12. Following the scheme outlined in Section 4.1, the construction is carried out in three steps: the covering of $\Omega\setminus\Omega_\varepsilon$ with a suitable (locally finite) tiling, the definition of a bi-Lipschitz piecewise affine approximation of u on the grid of the tiling and, finally, the extension of the approximating function to the interior of the grid by means of Theorem 0.4. The main result of this section is the following.

Proposition 4.15. *Let $v_\varepsilon:\Omega_\varepsilon\longrightarrow\Delta_\varepsilon$ be a piecewise affine bi-Lipschitz function as in Proposition 4.12. Then, there exists a C_1L^4 bi-Lipschitz countably piecewise affine function $\tilde v_\varepsilon:\Omega\setminus\Omega_\varepsilon\longrightarrow\Delta\setminus\Delta_\varepsilon$, where C_1 is a geometric constant, such that $\tilde v_\varepsilon=u$ on $\partial\Omega$ and $\tilde v_\varepsilon=v_\varepsilon$ on $\partial\Omega_\varepsilon$.*

We can immediately notice that Theorem 0.2 will follow as an easy consequence of Propositions 4.12 and 4.15.

Proof of Theorem 0.2: Take $\varepsilon>0$, and apply Proposition 4.12 to get an r-polygon $\Omega_\varepsilon\subset\subset\Omega$ and a piecewise affine bi-Lipschitz function $v_\varepsilon:\Omega_\varepsilon\longrightarrow\Delta_\varepsilon$. By Proposition 4.15, we have a C_1L^4 bi-Lipschitz function $\tilde v_\varepsilon:\Omega\setminus\Omega_\varepsilon\longrightarrow\Delta\setminus\Delta_\varepsilon$, so we can define the function $v:\Omega\longrightarrow\Delta$ as $v\equiv v_\varepsilon$ on Ω_ε and $v\equiv\tilde v_\varepsilon$ on $\Omega\setminus\Omega_\varepsilon$. Since v_ε (resp., $\tilde v_\varepsilon$) is bi-Lipschitz with constant $L+\varepsilon$ (resp., C_1L^4), and $\tilde v_\varepsilon=v_\varepsilon$ on $\partial\Omega_\varepsilon$, we have that v is a bi-Lipschitz homeomorphism with constant C_1L^4. Moreover, by construction v is countably piecewise affine, and it is orientation-preserving since so is u and $v\equiv u$ on $\partial\Omega$. We are then left with showing that v satisfies (0.3), and by (4.4) this basically reduces to consider what happens in $\Omega\setminus\Omega_\varepsilon$. Since $\tilde v_\varepsilon$ is bi-Lipschitz with constant C_1L^4, by (4.5) we clearly have

$$\|Dv-Du\|_{L^p(\Omega\setminus\Omega_\varepsilon)}\leq\|Dv-Du\|_{L^\infty(\Omega\setminus\Omega_\varepsilon)}\Big|\Omega\setminus\Omega_\varepsilon\Big|^{1/p}\leq\big(L+C_1L^4\big)\varepsilon^{1/p}\,,\tag{4.30}$$

and similarly

$$\|Dv^{-1}-Du^{-1}\|_{L^p(\Delta\setminus\Delta_\varepsilon)}\leq\big(L+C_1L^4\big)\varepsilon^{1/p}\,.\tag{4.31}$$

Concerning the L^∞ estimates, since $\Big|\Omega\setminus\Omega_\varepsilon\Big|\leq\varepsilon$ then for every $z\in\Omega\setminus\Omega_\varepsilon$ there exist $z'\in\Omega_\varepsilon$ such that $|z-z'|\leq\sqrt{\varepsilon/\pi}$, thus by (4.4) we find

$$\begin{aligned}|v(z)-u(z)|&\leq|v(z)-v(z')|+|v(z')-u(z')|+|u(z')-u(z)|\\&\leq\big(L+C_1L^4\big)\sqrt{\frac{\varepsilon}{\pi}}+\|v_\varepsilon-u\|_{L^\infty(\Omega_\varepsilon)}\leq\big(L+C_1L^4\big)\sqrt{\frac{\varepsilon}{\pi}}+\varepsilon\,.\end{aligned}$$

Arguing in the very same way to bound $|v^{-1}(w)-u^{-1}(w)|$ for a generic $w\in\Delta\setminus\Delta_\varepsilon$ yields

$$\|v-u\|_{L^\infty(\Omega\setminus\Omega_\varepsilon)}\leq\big(L+C_1L^4\big)\sqrt{\frac{\varepsilon}{\pi}}+\varepsilon\,,\quad \|v^{-1}-u^{-1}\|_{L^\infty(\Delta\setminus\Delta_\varepsilon)}\leq\big(L+C_1L^4\big)\sqrt{\frac{\varepsilon}{\pi}}+\varepsilon\,. \tag{4.32}$$

Putting together (4.30), (4.31) and (4.32), we find that v satisfies (0.3) as soon as ε is chosen small enough, depending on $\bar\varepsilon$. Hence, we have found the countably piecewise affine approximation as required. Concerning the smooth approximation, its existence directly follows applying Theorem 4.1, thus we have in particular $C_2=70C_1^{7/3}$. ☐

We have now to prove Proposition 4.15. To do so, let us fix some notation. Recall that Ω_ε is an r-polygon for some $r=r(\varepsilon)$. We will then start by selecting a suitable tiling $\{\mathcal{D}_j=\mathcal{D}(z_j,r_j)\}_{j\in\mathbb{N}}$ of $(\Omega,\Omega_\varepsilon)$, according with Definition 4.9. This means that $\{\mathcal{D}_j\}$ is a tiling of Ω whose restriction to Ω_ε coincides with the r-tiling of Ω_ε. The only requirements that we ask to $\{\mathcal{D}_j\}$ are the following,

$$r_j=r\quad\forall\,j:\ \operatorname{clos}\mathcal{D}_j\cap\partial\Omega_\varepsilon\neq\emptyset\,, \tag{4.33}$$

$$\mathcal{D}_j\subset\subset\Omega\quad\forall\,j\in\mathbb{N}\,. \tag{4.34}$$

Notice that (4.33) is possible thanks to (4.5), while (4.34) basically means that the tiling has to be countable instead of finite, and the squares have to become smaller and smaller when approaching the boundary of Ω. Of course, in the particular case when Ω itself is a right polygon, instead of (4.34) one could have asked the tiling to be finite (we will discuss this possibility more in detail in Remark 4.21).

Since it is of course possible to find a tiling of $(\Omega,\Omega_\varepsilon)$ which satisfies (4.33) and (4.34), from now on we fix such a tiling, and we denote by $\mathcal{Q}$ its associated 1-dimensional grid according with Definition 4.10. Moreover, we set $\mathcal{Q}'=\mathcal{Q}\cap(\Omega\setminus\operatorname{clos}\Omega_\varepsilon)$, which is the part of the grid on which we really need to work. Notice that $\mathcal{Q}'$ is a 1-dimensional set, made by all the sides of the grid $\mathcal{Q}$ which lie in $\Omega\setminus\operatorname{clos}\Omega_\varepsilon$.

Let us call now w_α the generic vertex of $\mathcal{Q}'$, hence, the generic vertex of the grid $\mathcal{Q}$ which does not belong to Ω_ε (but it may belong to $\partial\Omega_\varepsilon$). Each vertex w_α is of the form $w_\alpha=z_j+(\pm r_j/2,\pm r_j/2)$ for some j, and it is one extreme of either three, or four sides of $\mathcal{Q}$. To shorten the notation, we will denote the other extremes of these sides by w_α^i with $1\leq i\leq\bar\imath(\alpha)$, being then $\bar\imath(\alpha)\in\{3,4\}$. Finally, we will denote by ℓ_α the minimum of the lengths of the sides $w_\alpha w_\alpha^i$. Observe that if $w_\alpha\notin\partial\Omega_\varepsilon$, then w_α is one extreme of either three or four sides of $\mathcal{Q}'\subseteq\mathcal{Q}$. On the other hand, if $w_\alpha\in\partial\Omega_\varepsilon$, then by (4.33) it is one extreme of four sides of $\mathcal{Q}$, either one or two of these four sides lies in $\mathcal{Q}'$, and $\ell_\alpha=r$.

Thanks to Theorem 0.4, to obtain the piecewise affine function $\tilde v_\varepsilon$ of Proposition 4.15 we essntially have to define it, in a suitable way, on the 1-dimensional grid $\mathcal{Q}'$. To do so, our main ingredients are the following two lemmas. The first one, Lemma 4.19, states that, on any given segment of Ω, u can be approximated as well as desired in L^∞ with suitable piecewise affine $4L$ bi-Lipschitz functions. This is of course of primary importance to define the piecewise affine approximation $\tilde v_\varepsilon$ of u on the sides of the grid $\mathcal{Q}'$, but it is still not enough. In fact, we have to take some additional care to treat the "crosses" of $\mathcal{Q}'$ (that is, the regions around the vertices), in order to be sure that our affine $\tilde{v}_\varepsilon$ on $\mathcal{Q}'$ remains injective. This will be obtained thanks to the second Lemma 4.20.

To state the next two lemmas, it will be useful to introduce some piece of notation.

Definition 4.16 (Interpolation of u). *Given a segment $pq \subset\subset \Omega$, let $\{z_i z_{i+1}\}_{0\leq i<N}$ be N essentially disjoint segments whose union is pq, with $z_0 = p$ and $z_N = q$. For any such subdivision of the segment, will call* interpolation of u *the finitely piecewise affine function $u_{pq} : pq \longrightarrow \mathbb{R}^2$ such that, for any $0 \leq i \leq N-1$ and any $0 \leq t \leq 1$,*

$$u_{pq}\Big(z_i + t\big(z_{i+1} - z_i\big)\Big) = u(z_i) + t\big(u(z_{i+1}) - u(z_i)\big)\,.$$

Definition 4.17 (Adjusted function and crosses). *Let $\{\xi_\alpha\}_{\alpha\in\mathbb{N}}$ be a sequence such that for any α one has $3L\xi_\alpha \leq \ell_\alpha$. For any $\alpha \in \mathbb{N}$ and any $1 \leq i \leq \bar{i}(\alpha)$, we define ξ_α^i as the biggest number such that*

$$\left\{\begin{array}{ll} \Big|u(w_\alpha) - u\big(w_\alpha + \xi_\alpha^i(w_\alpha^i - w_\alpha)\big)\Big| \leq \xi_\alpha & \text{if } w_\alpha w_\alpha^i \subset \mathcal{Q}'\,,\\ \Big|u(w_\alpha) - v_\varepsilon\big(w_\alpha + \xi_\alpha^i(w_\alpha^i - w_\alpha)\big)\Big| \leq \xi_\alpha & \text{if } w_\alpha w_\alpha^i \subset \mathcal{Q}\setminus\mathcal{Q}'\,. \end{array}\right.$$

We will call adjusted function *the function $u_{\rm adj} : \mathcal{Q} \longrightarrow \mathbb{R}^2$ defined as follows. First of all, we set $u_{\rm adj} = v_\varepsilon$ on $\mathcal{Q}\setminus\mathcal{Q}'$. Then, let $w_\alpha w_\beta$ be a side of $\mathcal{Q}'$, thus being $w_\beta = w_\alpha^i$ and $w_\alpha = w_\beta^j$ for two suitable i and j. We define*

$$u_{\rm adj}\big(w_\alpha + t(w_\beta - w_\alpha)\big) := \left\{\begin{array}{ll} u(w_\alpha) + \frac{t}{\xi_\alpha^i}\Big(u\big(w_\alpha + \xi_\alpha^i(w_\beta - w_\alpha)\big) - u(w_\alpha)\Big) & \text{in } (0, \xi_\alpha^i)\,,\\ u\big(w_\alpha + t(w_\beta - w_\alpha)\big) & \text{in } (\xi_\alpha^i, 1 - \xi_\beta^j)\,,\\ u(w_\beta) + \frac{(1-t)}{\xi_\beta^j}\Big(u\big(w_\beta + \xi_\beta^j(w_\alpha - w_\beta)\big) - u(w_\beta)\Big) & \text{in } (1 - \xi_\beta^j, 1). \end{array}\right.$$

In words, for any side in $\mathcal{Q}'$, $u_{\rm adj}$ coincides with u in the internal part of the side, while the two parts closest to the vertices w_α and w_β are replaced with segments. Moreover, for any vertex w_α of $\mathcal{Q}'$ we will define its associated cross *as*

$$Z_\alpha = \bigcup_{i=1}^{\bar{i}(\alpha)} \Big\{w_\alpha + t(w_\alpha^i - w_\alpha) :\ 0 \leq t \leq \xi_\alpha^i\Big\}\,.$$

Remark 4.18. *Some remarks are now in order. First of all, since u is L bi-Lipschitz on Ω, and also v_ε is L bi-Lipschitz on any segment $w_\alpha w_\alpha^i \subseteq \mathcal{Q}\setminus\mathcal{Q}'$, by the choice $3L\xi_\alpha \leq \ell_\alpha$ we directly deduce that $0 < \xi_\alpha^i \leq 1/3$ for any α and any i. Thus, two different crosses have always empty intersection. For the same reason, each of the $\bar{i}(\alpha)$ extremes of the cross Z_α has a distance at least ξ_α/L from w_α. Finally, for all different α and β one has $\mathcal{B}(u(w_\alpha), \xi_\alpha) \cap \mathcal{B}(u(w_\beta), \xi_\beta) = \emptyset$. Indeed, assuming without loss of generality that $\ell_\alpha \geq \ell_\beta$, we have $\Big|u(w_\beta) - u(w_\alpha)\Big| \geq \ell_\alpha/L$. And as a consequence, $\xi_\alpha + \xi_\beta \leq \ell_\alpha/(3L) + \ell_\beta/(3L) \leq 2\ell_\alpha/(3L) < \Big|u(w_\beta) - u(w_\alpha)\Big|$.*

LEMMA 4.19. *For every segment $pq \subset\subset \Omega$ and every $\delta > 0$, there exist a function $u^\delta_{pq} : pq \longrightarrow \Delta$ which is a $4L$ bi-Lipschitz interpolation of u with the property that $\|u^\delta_{pq} - u\|_{L^\infty(pq)} \leq \delta$.*

LEMMA 4.20. *There exists a sequence $\{\xi_\alpha\}_{\alpha\in\mathbb{N}}$ such that the associated adjusted function $u_{\rm adj} : \mathcal{Q} \longrightarrow \mathbb{R}^2$ is $18L$ bi-Lipschitz and $u_{\rm adj}(\mathcal{Q}) \subseteq \Delta$.*

Before giving the proof of Lemmas 4.19 and 4.20, we show how they enter into the proof of Proposition 4.15.

PROOF OF PROPOSITION 4.15: To define the searched function $\tilde v_\varepsilon : \Omega\setminus\Omega_\varepsilon \longrightarrow \Delta\setminus\Delta_\varepsilon$, let first $u_{\rm adj} : \mathcal{Q} \longrightarrow \mathbb{R}^2$ be an adjusted function according with Lemma 4.20, corresponding to the sequence $\{\xi_\alpha\}$. Our strategy will be first to define a suitable piecewise affine and injective function $u'_{\rm adj} : \mathcal{Q} \longrightarrow \Delta$, coinciding with $u_{\rm adj}$ near the vertices w_α, and then to obtain $\tilde v_\varepsilon$ extending $u'_{\rm adj}$ in the interior of each square making use of Theorem 0.4. We divide the proof in some steps.

Step I. Definition of $u'_{\rm adj} : \mathcal{Q} \longrightarrow \Delta$.

First of all, we define $u'_{\rm adj} = u_{\rm adj} = v_\varepsilon$ on $\mathcal{Q} \setminus \mathcal{Q}'$. Then, consider a generic side $w_\alpha w_\beta \subseteq \mathcal{Q}'$, and define pq the *internal segment* of the side $w_\alpha w_\beta$, that is, p and q are the extremes of the segment $w_\alpha w_\beta \setminus \big(Z_\alpha \cup Z_\beta\big)$. Taken now a small constant $\delta = \delta(\alpha, \beta)$, to be precised later, we set $u'_{\rm adj} = u_{\rm adj}$ on $w_\alpha w_\beta \cap \big(Z_\alpha \cup Z_\beta\big)$, and $u'_{\rm adj} = u^\delta_{pq}$ on pq, where u^δ_{pq} is given by Lemma 4.19.

By definition, it is clear that $u'_{\rm adj}$ is a continuous, countably piecewise affine function on $\mathcal{Q}$. Moreover, since the different constants $\delta(\alpha, \beta)$ are independent and any internal segment pq is compactly supported in Ω, one clearly has that $u'_{\rm adj}(\mathcal{Q}) \subseteq \Delta$ up to choose the constants small enough. In addition, since $u'_{\rm adj}$ is obtained glueing the $4L$ bi-Lipschitz functions u^δ_{pq} and the $18L$ bi-Lipschitz function $u_{\rm adj}$, we clearly have that $u'_{\rm adj}$ is $18\sqrt{2}L$-Lipschitz (but, *a priori*, not bi-Lipschitz!). To conclude the proof, we will first show that in fact $u'_{\rm adj}$ is bi-Lipschitz, thus in particular injective, then eventually we will extend $u'_{\rm adj}$ to the interior of the squares of the tiling, hence to the whole $\Omega \setminus \Omega_\varepsilon$.

Let us then fix two points $z, z' \in \mathcal{Q}$. In the next Steps II–IV we will show that

$$\left|u'_{\rm adj}(z) - u'_{\rm adj}(z')\right| \geq \frac{1}{72L}|z - z'|\,, \tag{4.35}$$

considering separately the different possible positions of z and z'.

Step II. The case when $z \in pq \subseteq w_\alpha w_\beta$, $z' \notin w_\alpha w_\beta$.

The first case is when z belongs to an internal segment pq which is contained in the side $w_\alpha w_\beta \subseteq \mathcal{Q}'$, and z' does not belong to the side $w_\alpha w_\beta$. In this case, as observed in Remark 4.18, we know that $|z - z'| \geq \xi_\alpha/L$. Thus, there are two subcases. If z' does not belong to any internal segment (hence, either z' belongs to some cross, or $z' \in \Omega_\varepsilon$), then $u'_{\rm adj}(z') = u_{\rm adj}(z')$ and then by Lemma 4.20, provided that we choose

$$\delta(\alpha, \beta) \leq \frac{\min\{\xi_\alpha, \xi_\beta\}}{36L^2}\,, \tag{4.36}$$

we have

$$\begin{aligned}\left|u'_{\rm adj}(z)-u'_{\rm adj}(z')\right| &= \left|u^\delta_{pq}(z)-u_{\rm adj}(z')\right| \geq \left|u_{\rm adj}(z)-u_{\rm adj}(z')\right| - \left|u^\delta_{pq}(z)-u_{\rm adj}(z)\right|\\ &= \left|u_{\rm adj}(z)-u_{\rm adj}(z')\right| - \left|u^\delta_{pq}(z)-u(z)\right| \geq \frac{1}{18L}\left|z-z'\right| - \delta(\alpha,\,\beta)\\ &\geq \frac{1}{18L}\left|z-z'\right| - \frac{\xi_\alpha}{36L^2} \geq \frac{1}{36L}\left|z-z'\right|,\end{aligned}$$

so that (4.35) is proved.

Consider now the other subcase, namely, when z' belongs to some other internal segment $p'q' \subseteq w_{\alpha'}w_{\beta'}$. In that case, since by construction and (4.36) it is $|z-z'| \geq 36L\delta(\alpha,\,\beta)$ and $|z-z'| \geq 36L\delta(\alpha',\,\beta')$, one directly has

$$\begin{aligned}\left|u'_{\rm adj}(z)-u'_{\rm adj}(z')\right| &= \left|u^\delta_{pq}(z)-u^{\delta'}_{p'q'}(z')\right| \geq \left|u(z)-u(z')\right| - \left|u(z)-u^\delta_{pq}(z)\right| - \left|u(z')-u^{\delta'}_{p'q'}(z')\right|\\ &\geq \frac{1}{L}\left|z-z'\right| - \delta(\alpha,\,\beta) - \delta(\alpha',\,\beta') \geq \frac{17}{18L}|z-z'|,\end{aligned}$$

hence again (4.35) is established.

<u>Step III.</u> *The case when $z \in pq \subseteq w_\alpha w_\beta$, $z' \in w_\alpha w_\beta$.*

The second case is when z still belongs to an internal segment pq contained in the side $w_\alpha w_\beta \subseteq \mathcal{Q}'$, and also z' belongs to the side $w_\alpha w_\beta$. In particular, if also z' is in the internal segment pq then we already know the validity of (4.35) because $u'_{\rm adj}(z) = u^\delta_{pq}(z)$ and $u'_{\rm adj}(z') = u^\delta_{pq}(z')$, while u^δ_{pq} is $4L$ bi-Lipschitz. Therefore, we can directly assume that $z' \in w_\alpha p$, being the case $z' \in q w_\beta$ clearly the same.

By Definition 4.17 we know that $u'_{\rm adj}(z') = u_{\rm adj}(z')$ lies in the segment $u(w_\alpha)u(p)$, which is a radius of the ball $\mathcal{B}\big(u(w_\alpha),\xi_\alpha\big)$. Hence, for any point s outside the same ball, a trivial geometric argument tells us that

$$\left|s-u_{\rm adj}(z')\right| \geq \frac{\left|s-u(p)\right| + \left|u(p)-u_{\rm adj}(z')\right|}{3}\,. \tag{4.37}$$

Notice now that it is not true, in general, that $u'_{\rm adj}(z) = u^\delta_{pq}(z)$ lies outside the ball $\mathcal{B}\big(u(w_\alpha),\xi_\alpha\big)$. However, recalling that u^δ_{pq} is an interpolation of u, by Definition 4.16 we know that $u^\delta_{pq}(z)$ is in a segment whose both extremes are out of the ball. Thus, if $\left|u^\delta_{pq}(z)-u(w_\alpha)\right| < \xi_\alpha$, it is anyway surely true that

$$\xi_\alpha - \left|u^\delta_{pq}(z)-u(w_\alpha)\right| \ll \left|u^\delta_{pq}(z)-u(p)\right|,$$

up to possibly decreasing $\delta(\alpha,\,\beta)$. Putting this observation together with (4.37) we readily obtain that

$$\begin{aligned}\left|u^\delta_{pq}(z)-u_{\rm adj}(z')\right| &\geq \frac{\left|u^\delta_{pq}(z)-u(p)\right| + \left|u(p)-u_{\rm adj}(z')\right|}{4}\\ &= \frac{\left|u^\delta_{pq}(z)-u^\delta_{pq}(p)\right| + \left|u_{\rm adj}(p)-u_{\rm adj}(z')\right|}{4}\,,\end{aligned}$$

recalling that $u_{\rm adj}(p) = u^\delta_{pq}(p) = u(p)$ (of course, by selecting $\delta(\alpha, \beta)$ small enough, we could have used any number greater than 3, instead of 4, in the above estimate). Therefore, since u^δ_{pq} is $4L$ bi-Lipschitz while $u_{\rm adj}$ is $18L$ bi-Lipschitz, we readily obtain

$$\left|u'_{\rm adj}(z) - u'_{\rm adj}(z')\right| = \left|u^\delta_{pq}(z) - u_{\rm adj}(z')\right| \geq \frac{\left|u^\delta_{pq}(z) - u^\delta_{pq}(p)\right|}{4} + \frac{\left|u_{\rm adj}(p) - u_{\rm adj}(z')\right|}{4}$$
$$\geq \frac{|z-p|}{16L} + \frac{|p-z'|}{72L} \geq \frac{|z-z'|}{72L},$$

recalling that z, p and z' are aligned. Hence, (4.35) is checked once again also in this case.

Step IV. The case when neither z nor z' are in some internal segment.

Thanks to Step II and Step III, and by the symmetry of the inequality (4.35), we are left to consider only the situation where no one between z and z' is inside some internal segment. In other words, both z and z' must be either in $\mathcal{Q} \setminus \mathcal{Q}'$ or in some cross. By the definition of $u'_{\rm adj}$, this means that $u'_{\rm adj}(z) = u_{\rm adj}(z)$ and $u'_{\rm adj}(z') = u_{\rm adj}(z')$. And thus, since $u_{\rm adj}$ is $18L$ bi-Lipschitz thanks to Lemma 4.20, the validity of (4.35) is already known. Summarizing, we have shown the validity of (4.35) in any possible case, and this means that the function $u'_{\rm adj} : \mathcal{Q} \longrightarrow \Delta$ is injective and $72L$ bi-Lipschitz.

Step V. Conclusion.

We have now to define the piecewise affine and bi-Lipschitz function $\tilde v_\varepsilon : \Omega \setminus \Omega_\varepsilon \longrightarrow \Delta \setminus \Delta_\varepsilon$, matching u on $\partial\Omega$ and matching v_ε on $\partial\Omega_\varepsilon$. To do so, consider each square $\mathcal{D}_j$ of the tiling contained in $\Omega \setminus \Omega_\varepsilon$. The function $u'_{\rm adj}$ is $72L$ bi-Lipschitz from $\partial\mathcal{D}_j$ to a subset of Δ, then by Theorem 0.4 it can be continuously extended to a piecewise affine bi-Lipschitz function of the whole square $\mathcal{D}_j$, with bi-Lipschitz constant $C_3 72^4 L^4$. Define $\tilde v_\varepsilon$ as the countably piecewise affine function on $\Omega \setminus \Omega_\varepsilon$ which gathers all these extensions on all the squares $\mathcal{D}_j \subseteq \Omega \setminus \Omega_\varepsilon$ of the tiling.

For each square $\mathcal{D}_j \subseteq \Omega \setminus \Omega_\varepsilon$, we clearly have that $\partial\big(\tilde v_\varepsilon(\mathcal{D}_j)\big) = u'_{\rm adj}(\partial\mathcal{D}_j)$. This yields that $\tilde v_\varepsilon$ is injective. Moreover, by continuity it is clear that $\tilde v_\varepsilon = u$ on $\partial\Omega$, and by construction $\tilde v_\varepsilon = v_\varepsilon$ on $\partial\Omega_\varepsilon$. As a consequence, $\tilde v_\varepsilon : \Omega \setminus \Omega_\varepsilon \longrightarrow \Delta \setminus \Delta_\varepsilon$ fulfills all our requirements. In particular, one has $C_1 = 72^4 C_3$. □

Let us now make a simple observation, which will be useful in the sequel.

REMARK 4.21. *Assume that Ω is a right polygon of side-length $\bar r$ and that u is piecewise affine on $\partial\Omega$. Then, consider the right polygon Ω_ε of side-length r given by Proposition 4.12. By the construction of Section 4.3, it is not restrictive to assume that $\bar r \in r\mathbb{N}$, and that Ω_ε is a subset of the r-tiling of Ω. Therefore, we can repeat* verbatim *the construction of Proposition 4.15 using, as tiling, the r-tiling of Ω. Notice that in this case assumption (4.34) is not valid –see the remark right after (4.34)– but in fact if Ω is a polygon, and u is affine on its sides, there is no need for the tiling to use smaller and smaller squares at the boundary. As a consequence, the bi-Lipschitz approximation provided by Proposition 4.15 is (finitely) piecewise affine instead of countably piecewise affine. Observe that the assumption that u*

is piecewise affine of $\partial\Omega$ is essential, because otherwise the approximation $\tilde v_\varepsilon$ would not coincide with u on $\partial\Omega$.

To conclude the proof of Theorem 0.2, we then only need to give the proofs of Lemma 4.19 and of Lemma 4.20.

Proof of Lemma 4.19: Let $\rho > 0$ be a small number, to be fixed later. Define then $t_0 = 0$, $z_0 = p$ and then recursively

$$t_{i+1} := \max\Big\{t > t_i : \Big|u(z_i) - u\big(p + t(q-p)\big)\Big| \leq \rho\Big\}\,, \qquad z_{i+1} := p + t_{i+1}(q-p)\,.$$

In this way, we have selected a finite sequence of points $z_0 = p$, $z_1, \dots z_N = q$ in the segment pq, where $N = N(p,q,\rho)$. We can then already define the function u^δ_{pq} by setting, for any $0 \leq i \leq N-1$ and any $0 \leq t \leq 1$,

$$u^\delta_{pq}\Big(z_i + t\big(z_{i+1} - z_i\big)\Big) = u(z_i) + t\big(u(z_{i+1}) - u(z_i)\big)\,.$$

Hence, u^δ_{pq} is the interpolation of u associated with the points $\{z_i\}$, according with Definition 4.16. The function u^δ_{pq} is by construction finitely piecewise affine and L-Lipschitz, and by the uniform continuity of u in pq it is also clear that the bound $\|u - u^\delta_{pq}\|_{L^\infty(pq)} \leq \delta$ holds true as soon as ρ is small enough. To conclude, we have thus only to check that

$$\Big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\Big| \geq \frac{1}{4L}|z - z'| \tag{4.38}$$

for all z, z' in pq. If both z and z' belong to a same segment z_iz_{i+1}, then the estimate is true because u^δ_{pq} is affine on that segment and u is L bi-Lipschitz.

Assume then that $z \in z_iz_{i+1}$ and $z' \in z_jz_{j+1}$ with $j > i$. If $j = i+1$, thus z and z' belong to two consecutive segments, then by the definition of the points z_i the angle $u^\delta_{pq}(z)\widehat{u^\delta_{pq}(z_{i+1})}u^\delta_{pq}(z')$ is at least $\pi/3$, hence

$$\begin{aligned}\Big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\Big| &\geq \frac{\Big|u^\delta_{pq}(z) - u^\delta_{pq}(z_{i+1})\Big|}{2} + \frac{\Big|u^\delta_{pq}(z_{i+1}) - u^\delta_{pq}(z')\Big|}{2}\\ &= |z - z_{i+1}|\,\frac{\big|u(z_i) - u(z_{i+1})\big|}{2|z_i - z_{i+1}|} + |z_{i+1} - z'|\,\frac{\big|u(z_{i+1}) - u(z_{i+2})\big|}{2|z_{i+1} - z_{i+2}|} \geq \frac{|z - z'|}{2L}\,,\end{aligned}$$

so that (4.38) is checked.

Instead, consider what happens if $j > i+1$. In this case, since $u^\delta_{pq}(z) \in u(z_i)u(z_{i+1})$ and for all $l > i+1$ one has $u(z_l) \notin \mathcal{B}(u(z_i),\rho) \cup \mathcal{B}(u(z_{i+1}),\rho)$, an immediate geometric argument ensures that $|u^\delta_{pq}(z) - u^\delta_{pq}(z')| \geq \sqrt{3}\rho/2$. As a consequence, we have

$$\Big|u(z_i) - u(z_{j+1})\Big| \leq \Big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\Big| + 2\rho \leq \bigg(1 + \frac{4}{3}\sqrt{3}\bigg)\Big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\Big| \leq 4\Big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\Big|\,,$$

which yields

$$\Big|u^\delta_{pq}(z) - u^\delta_{pq}(z')\Big| \geq \frac{\Big|u(z_i) - u(z_{j+1})\Big|}{4} \geq \frac{\Big|z_i - z_{j+1}\Big|}{4L} \geq \frac{\Big|z - z'\Big|}{4L}\,,$$

hence (4.38) holds true also in this case and we conclude the proof. □

PROOF OF LEMMA 4.20: Let us take a vertex w_α of the grid $\mathcal{Q}'$. Take then a constant $\xi_\alpha \leq \ell_\alpha/(3L)$, with $\xi_\alpha = \ell_\alpha/(3L) = r/(3L)$ if $w_\alpha \in \partial\Omega_\varepsilon$, while if $w_\alpha \notin \partial\Omega_\varepsilon$ the inequality can be strict. In particular, it is admissible to ask that for any α one has

$$\xi_\alpha < \frac{r}{2L}\,. \tag{4.39}$$

Define now ξ^i_α as in Definition 4.17 and, for any $1 \leq i \leq \bar{i}(\alpha)$, let $p_i = w_\alpha + \xi^i_\alpha\big(w^i_\alpha - w_\alpha\big)$. If $w_\alpha \in \Omega \setminus \partial\Omega_\varepsilon$, then we have

$$u(w_\alpha)u(p_i) \subset\subset \Delta \qquad \forall\, 1 \leq i \leq \bar{i}(\alpha)\,, \tag{4.40}$$

up to possibly decrease the value of ξ_α. Instead, if $w_\alpha \in \partial\Omega_\varepsilon$, then (4.40) is already ensured by (4.6) and (4.5) in Proposition 4.12, without any need of changing ξ_α.

We introduce then the adjusted function $u_{\rm adj}$ of Definition 4.17: to obtain the thesis, we need to check that it fulfills the requirements of Lemma 4.20. Thanks to (4.40), we already know that $u_{\rm adj} : \mathcal{Q} \longrightarrow \Delta$. Hence, all we have to do is to check that

$$\frac{|z - z'|}{18L} \leq \Big|u_{\rm adj}(z) - u_{\rm adj}(z')\Big| \leq 18L|z - z'|\,. \tag{4.41}$$

for all $z,\, z' \in \mathcal{Q}$. We will do it in some steps.

<u>*Step I.*</u> *For all α, $u^{-1}_{\rm adj}\Big(\operatorname{clos}\mathcal{B}\big(u(w_\alpha), \xi_\alpha\big)\Big) = Z_\alpha$.*

We start observing an important property, that is, for any α and for any $z \in \mathcal{Q}$ we have that $\Big|u_{\rm adj}(z) - u(w_\alpha)\Big| \leq \xi_\alpha$ if and only if $z \in Z_\alpha$. In fact, if $z \in Z_\alpha$ then $z \in w_\alpha p_i$ for some $1 \leq i \leq \bar{i}(\alpha)$, and since $u_{\rm adj}$ is affine in the segment $w_\alpha p_i$, while $\Big|u_{\rm adj}(p_i) - u(w_\alpha)\Big| = \xi_\alpha$, then of course $\Big|u_{\rm adj}(z) - u(w_\alpha)\Big| \leq \xi_\alpha$.

On the other hand, assume that $z \notin Z_\alpha$: we have to show that $\Big|u_{\rm adj}(z) - u(w_\alpha)\Big| > \xi_\alpha$. If $z \in w_\alpha w^i_\alpha$ for some $1 \leq i \leq \bar{i}(\alpha)$, then there are three possibilities. First, if $w_\alpha w^i_\alpha \subset \mathcal{Q} \setminus \mathcal{Q}'$, then $u_{\rm adj} = v_\varepsilon$ is affine on the side $w_\alpha w^i_\alpha$, so the claim is trivial. Second, if $w_\alpha w^i_\alpha \subset \mathcal{Q}'$ and z belongs to the cross Z_β associated to the vertex $w_\beta = w^i_\alpha$, then again the claim is immediate since $u_{\rm adj}(z)$ belongs to the ball $\mathcal{B}\big(u(w_\beta), \xi_\beta\big)$, which does not intersect $\mathcal{B}\big(u(w_\alpha), \xi_\alpha\big)$ by Remark 4.18. Lastly, if $w_\alpha w^i_\alpha \subset \mathcal{Q}'$ and $z \notin Z_\beta$, then $u_{\rm adj}(z) = u(z)$, thus the claim is again obvious by the definition of ξ^i_α.

To conclude the step, we have to consider a point $z \notin Z_\alpha$ which does not belong to any side of $\mathcal{Q}$ starting at w_α. We have again to distinguish some possible cases. If z belongs to the cross Z_β for some β, then again the claim follows by the fact that $\mathcal{B}\big(u(w_\alpha), \xi_\alpha\big) \cap \mathcal{B}\big(u(w_\beta), \xi_\beta\big) = \emptyset$. If z does not belong to any cross and $z \in \mathcal{Q}'$, then $u_{\rm adj}(z) = u(z)$ so the claim follows because, using the bi-Lipschitz property of u and the fact that $\xi_\alpha \leq \ell_\alpha/(3L)$, we have

$$u(z) \in \mathcal{B}\big(u(w_\alpha), \xi_\alpha\big) \quad \Longrightarrow \quad \Big|z - w_\alpha\Big| \leq \frac{\ell_\alpha}{3}\,,$$

which is impossible because $|z - w_\alpha| > \ell_\alpha$. Finally, consider the case when $z \in \mathcal{Q} \setminus \mathcal{Q}'$. In this case, we surely have $|z - w_\alpha| \geq r$ by construction, thus by (4.6) and (4.39) we get

$$\begin{aligned}\Big|u_{\rm adj}(z) - u(w_\alpha)\Big| &= \Big|v_\varepsilon(z) - u(w_\alpha)\Big| \geq \Big|u(z) - u(w_\alpha)\Big| - \Big|u(z) - v_\varepsilon(z)\Big|\\ &\geq \frac{\Big|z - w_\alpha\Big|}{L} - \frac{\sqrt{2}r}{6L^3} \geq \frac{r}{2L} > \xi_\alpha\,,\end{aligned}$$

thus the first step is concluded.

Now, taken two points $z,\, z' \in \mathcal{Q}$, we have to show the validity of (4.41).

<u>*Step II.*</u> *Validity of (4.41) if* $z,\, z' \in Z_\alpha$.

Let us first suppose that both z and z' belong to a same cross Z_α. By construction, $u_{\rm adj}$ is L bi-Lipschitz on each segment $w_\alpha p_i$, hence to show (4.41) we can assume without loss of generality that $z \in w_\alpha p_1$ and $z' \in w_\alpha p_2$. Therefore, on one side we have

$$\begin{aligned}\Big|u_{\rm adj}(z) - u_{\rm adj}(z')\Big| &\leq \Big|u_{\rm adj}(z) - u_{\rm adj}(w_\alpha)\Big| + \Big|u_{\rm adj}(w_\alpha) - u_{\rm adj}(z')\Big| \leq L\big(|z - w_\alpha| + |w_\alpha - z'|\big)\\ &\leq \sqrt{2}\,L\,|z - z'|\,.\end{aligned}$$

On the other side, to estimate $\Big|u_{\rm adj}(z) - u_{\rm adj}(z')\Big|$ from below, assume without loss of generality that $\Big|u_{\rm adj}(w_\alpha) - u_{\rm adj}(z)\Big| \leq \Big|u_{\rm adj}(w_\alpha) - u_{\rm adj}(z')\Big|$, and define $z'' \in w_\alpha z'$ so that

$$\Big|u_{\rm adj}(w_\alpha) - u_{\rm adj}(z)\Big| = \Big|u_{\rm adj}(w_\alpha) - u_{\rm adj}(z'')\Big|\,.$$

Since the triangle $u_{\rm adj}(w_\alpha)u_{\rm adj}(z)u_{\rm adj}(z'')$ is isosceles, then

$$u_{\rm adj}(z)\widehat{u_{\rm adj}(z'')}u_{\rm adj}(z') \geq \frac{\pi}{2}\,. \tag{4.42}$$

Moreover, we claim that

$$\frac{\Big|u_{\rm adj}(z) - u_{\rm adj}(z'')\Big|}{\Big|z - z''\Big|} \geq \frac{1}{2L}\,. \tag{4.43}$$

Indeed, if both $w_\alpha w_\alpha^1$ and $w_\alpha w_\alpha^2$ belong to $\mathcal{Q}'$, then by definition

$$\frac{\Big|u_{\rm adj}(z) - u_{\rm adj}(z'')\Big|}{\Big|z - z''\Big|} = \frac{\Big|u_{\rm adj}(p_1) - u_{\rm adj}(p_2)\Big|}{\Big|p_1 - p_2\Big|} = \frac{\Big|u(p_1) - u(p_2)\Big|}{\Big|p_1 - p_2\Big|} \geq \frac{1}{L}\,,$$

so (4.43) holds true. Conversely, if both $w_\alpha w_\alpha^1$ and $w_\alpha w_\alpha^2$ belong to $\mathcal{Q} \setminus \mathcal{Q}'$, then since v_ε is $L + \varepsilon$ bi-Lipschitz we have

$$\frac{\Big|u_{\rm adj}(z) - u_{\rm adj}(z'')\Big|}{\Big|z - z''\Big|} = \frac{\Big|u_{\rm adj}(p_1) - u_{\rm adj}(p_2)\Big|}{\Big|p_1 - p_2\Big|} = \frac{\Big|v_\varepsilon(p_1) - v_\varepsilon(p_2)\Big|}{\Big|p_1 - p_2\Big|} \geq \frac{1}{L + \varepsilon}\,,$$

so again (4.43) holds true. Finally, assume that $w_\alpha w_\alpha^1 \subseteq \mathcal{Q}'$ while $w_\alpha w_\alpha^2 \subseteq \mathcal{Q} \setminus \mathcal{Q}'$ (the case of $w_\alpha w_\alpha^1 \subseteq \mathcal{Q} \setminus \mathcal{Q}'$ and $w_\alpha w_\alpha^2 \subseteq \mathcal{Q}'$ being completely equivalent). In this case, it must clearly be $w_\alpha \in \partial\Omega_\varepsilon$, hence by Remark 4.18 we know that $|p_1 - w_\alpha|$ and $|p_2 - w_\alpha|$ are both at least $\xi_\alpha/L = r/(3L^2)$, thus $|p_1 - p_2| \geq \sqrt{2}r/(3L^2)$. Therefore, recalling again (4.6), we have

$$\frac{\big|u_{\rm adj}(z) - u_{\rm adj}(z'')\big|}{\big|z - z''\big|} = \frac{\big|u_{\rm adj}(p_1) - u_{\rm adj}(p_2)\big|}{\big|p_1 - p_2\big|} = \frac{\big|u(p_1) - v_\varepsilon(p_2)\big|}{\big|p_1 - p_2\big|}$$
$$\geq \frac{\big|u(p_1) - u(p_2)\big|}{\big|p_1 - p_2\big|} - \frac{\big|u(p_2) - v_\varepsilon(p_2)\big|}{\big|p_1 - p_2\big|} \geq \frac{1}{L} - \frac{\sqrt{2}r/(6L^3)}{\sqrt{2}r/(3L^2)} = \frac{1}{2L}\,,$$

thus (4.43) has been finally checked in all the possible cases. This inequality, together with (4.42) and again with the fact that $u_{\rm adj}$ is L bi-Lipschitz on the segment $z'z'' \subseteq w_\alpha p_2$, yields

$$\big|u_{\rm adj}(z) - u_{\rm adj}(z')\big| \geq \frac{\sqrt{2}}{2}\Big(\big|u_{\rm adj}(z) - u_{\rm adj}(z'')\big| + \big|u_{\rm adj}(z'') - u_{\rm adj}(z')\big|\Big)$$
$$\geq \frac{\sqrt{2}}{2}\bigg(\frac{|z - z''|}{2L} + \frac{|z'' - z'|}{L}\bigg) \geq \frac{\sqrt{2}}{4L}|z - z'|\,.$$

Summarizing, under the assumptions of this step

$$\frac{\sqrt{2}}{4L}|z - z'| \leq \big|u_{\rm adj}(z) - u_{\rm adj}(z')\big| \leq \sqrt{2}L|z - z'|\,. \tag{4.44}$$

Therefore, (4.41) is shown and this step is concluded.

Step III. Validity of (4.41) if for all α one has $z,\, z' \notin {\rm int}\, Z_\alpha$.

Consider now the situation when neither z nor z' belong to the interior of any cross. In this case, we have that $u_{\rm adj}(z) = u(z)$ if $z \in \mathcal{Q}'$, while $u_{\rm adj}(z) = v_\varepsilon(z)$ if $z \in \mathcal{Q} \setminus \mathcal{Q}'$, and the same holds for z'. Since u is L bi-Lipschitz while v_ε is $L + \varepsilon$ bi-Lipschitz, the validity of (4.41) is obvious if both $z,\, z' \in \mathcal{Q}'$, as well as if both $z,\, z' \in \mathcal{Q} \setminus \mathcal{Q}'$. Therefore, we can just concentrate on the case when $z \in \mathcal{Q}'$, $z' \in \mathcal{Q} \setminus \mathcal{Q}'$.

In this case, the main observation is that $|z-z'| \geq \sqrt{2}r/(3L^2)$, since both z and z' must be at distance at least $r/(3L^2)$ from any vertex $w_\alpha \in \partial\Omega_\varepsilon$, because they do not belong to any cross Z_α. As a consequence, again by (4.6) we get

$$\big|u_{\rm adj}(z) - u_{\rm adj}(z')\big| = \big|u(z) - v_\varepsilon(z')\big| \geq \big|u(z) - u(z')\big| - \big|u(z') - v_\varepsilon(z')\big|$$
$$\geq \frac{|z - z'|}{L} - \frac{\sqrt{2}r}{6L^3} \geq \frac{|z - z'|}{2L}\,,$$

while

$$\big|u_{\rm adj}(z) - u_{\rm adj}(z')\big| = \big|u(z) - v_\varepsilon(z')\big| \leq \big|u(z) - u(z')\big| + \big|u(z') - v_\varepsilon(z')\big|$$
$$\leq L\,|z - z'| + \frac{\sqrt{2}r}{6L^3} \leq \bigg(L + \frac{1}{2L}\bigg)|z - z'|\,,$$

thus also in this case (4.41) is proven (keep in mind that, since u is a L bi-Lipschitz map, then of course $L \geq 1$!). In particular, under the assumptions of this step one has

$$\frac{|z-z'|}{2L} \leq \Big|u_{\rm adj}(z) - u_{\rm adj}(z')\Big| \leq \frac{3}{2}\, L\, |z-z'|\,. \tag{4.45}$$

Step IV. Validity of (4.41) if $z \in Z_\alpha$ and for all β one has $z' \notin \operatorname{int} Z_\beta$.

We pass now to consider the case when z belongs to some cross Z_α, while z' does not belong to the interior of any cross. In particular, we can assume that $z \in w_\alpha p_1$. To get the above estimate in (4.41), it is enough to make a trivial geometric observation, namely, that there exists $1 \leq i \leq \bar\imath(\alpha)$ such that

$$|z-z'| \geq \frac{\sqrt{2}}{2}\Big(|z-p_i| + |p_i - z'|\Big)\,,$$

not necessarily with $i=1$. As a consequence, we can use the estimate (4.44) of Step II for the points z and p_i –which both belong to Z_α– and the estimate (4.45) of Step III for the points p_i and z' –none of which belongs to the interior of some Z_β– to get

$$\begin{aligned}\Big|u_{\rm adj}(z) - u_{\rm adj}(z')\Big| &\leq \Big|u_{\rm adj}(z) - u_{\rm adj}(p_i)\Big| + \Big|u_{\rm adj}(p_i) - u_{\rm adj}(z')\Big| \leq \sqrt{2}L|z-p_i| + \frac{3}{2}\, L|p_i - z'| \\ &\leq \frac{3}{2}\sqrt{2}\, L|z-z'|\,.\end{aligned}$$

On the other hand, to get the below estimate in (4.41), let us recall that by Step I we have

$$u_{\rm adj}(z) \in \operatorname{clos}\mathcal{B}\big(u(w_\alpha),\xi_\alpha\big)\,, \qquad\qquad u_{\rm adj}(z') \notin \mathcal{B}\big(u(w_\alpha),\xi_\alpha\big)\,. \tag{4.46}$$

Since $u_{\rm adj}(z)$ belongs to the radius $u(w_\alpha)u_{\rm adj}(p_1)$, then an immediate geometric argument from (4.46) implies, as already observed in (4.37), that

$$\Big|u_{\rm adj}(z) - u_{\rm adj}(z')\Big| \geq \frac{\Big|u_{\rm adj}(z) - u_{\rm adj}(p_1)\Big| + \Big|u_{\rm adj}(p_1) - u_{\rm adj}(z')\Big|}{3}\,. \tag{4.47}$$

Thus, using the L bi-Lipschitz property of $u_{\rm adj}$ in the segment $w_\alpha p_1$, and the estimate (4.45) of Step III for p_1 and z', we get

$$\Big|u_{\rm adj}(z) - u_{\rm adj}(z')\Big| \geq \frac{|z-p_1|}{3L} + \frac{|p_1 - z'|}{6L} \geq \frac{|z-z'|}{6L}\,.$$

Summarizing, under the assumptions of this step we have

$$\frac{|z-z'|}{6L} \leq \Big|u_{\rm adj}(z) - u_{\rm adj}(z')\Big| \leq \frac{3}{2}\sqrt{2}\, L|z-z'|\,, \tag{4.48}$$

hence in particular (4.41) is again checked.

Step V. Validity of (4.41) if $z \in Z_\alpha$ and $z' \in Z_\beta$.

The last situation which is left to consider, is when z and z' belong to two different crosses. This situation will be very similar to that of Step IV. Indeed, for the above estimate in (4.41) we can again start observing that for some $1 \leq i \leq \bar{i}(\alpha)$ it must be

$$|z - z'| \geq \frac{\sqrt{2}}{2}\Big(|z - p_i| + |p_i - z'|\Big)\,.$$

Then, we use the estimate (4.44) of Step II for the points z, $p_i \in Z_\alpha$, and the estimate (4.48) of Step IV for the points $z' \in Z_\beta$ and p_i –which does not belong to the interior of any cross– getting

$$\begin{aligned}\Big|u_{\rm adj}(z) - u_{\rm adj}(z')\Big| &\leq \Big|u_{\rm adj}(z) - u_{\rm adj}(p_i)\Big| + \Big|u_{\rm adj}(p_i) - u_{\rm adj}(z')\Big|\\ &\leq \sqrt{2}L|z - p_i| + \frac{3}{2}\sqrt{2}\,L|p_i - z'| \leq 3L|z - z'|\,.\end{aligned}$$

Finally, to find the below estimate in (4.41) we notice again that (4.47) is in force, and we use the L bi-Lipschitz property of $u_{\rm adj}$ in $w_\alpha p_1$ and the estimate (4.48) of Step IV for p_1 and z', obtaining

$$\Big|u_{\rm adj}(z) - u_{\rm adj}(z')\Big| \geq \frac{|z - p_1|}{3L} + \frac{|p_1 - z'|}{18L} \geq \frac{|z - z'|}{18L}\,.$$

Thus, we have finally checked (4.41) in all the possible cases, so that the proof is concluded. □

4.5. Finitely piecewise affine approximation on polygonal domains

In this last short section we give a proof of Theorem 0.3. In fact, the proof is quite short, since it is just a simple adaptation of the arguments of Section 4.4.

Proof of Theorem 0.3: First of all, assume that Ω is an $\bar{r}$-right polygon and that u is piecewise affine on $\partial\Omega$. Then, as already underlined in Remark 4.21, we can slightly modify the proofs of Proposition 4.12 and Proposition 4.15 to get what follows. First of all, there exist some r such that $\bar{r} \in r\mathbb{N}$, an r-right polygon $\Omega_\varepsilon \subset\subset \Omega$, which is part of the r-tiling of Ω, and an $L + \varepsilon$ bi-Lipschitz and piecewise affine function $v_\varepsilon : \Omega_\varepsilon \longrightarrow \mathbb{R}^2$ for which (4.3), (4.4), (4.5) and (4.6) hold. Moreover, there exists also a finitely piecewise affine map $\tilde{v}_\varepsilon : \Omega \setminus \Omega_\varepsilon \longrightarrow \Delta \setminus \Delta_\varepsilon$ which is C_1L^4 bi-Lipschitz and which coincides with u on $\partial\Omega$ and with v_ε on $\partial\Omega_\varepsilon$. Therefore, glueing v_ε and $\tilde{v}_\varepsilon$ exactly as in the proof of Theorem 0.2, we immediately get the required C_1L^4 bi-Lipschitz and (finitely) piecewise affine approximation of u.

Consider now the general situation of a polygon Ω with a map u which is piecewise affine on $\partial\Omega$. Of course, there exists a right polygon $\widehat{\Omega}$ and a bi-Lipschitz map $\Phi : \Omega \longrightarrow \widehat{\Omega}$, having bi-Lipschitz constant $C(\Omega)$. The map $u \circ \Phi^{-1}$ is a $C(\Omega)L$ bi-Lipschitz map from the right polygon $\widehat{\Omega}$ to Δ, which is piecewise affine on the boundary. Then, we can apply the first part of the proof to get an approximation $v : \widehat{\Omega} \longrightarrow \Delta$ which is finitely piecewise affine and $C_1C(\Omega)^4L^4$ bi-Lipschitz. Finally, $v \circ \Phi : \Omega \longrightarrow \Delta$ is a $C_1C(\Omega)^5L^4$ bi-Lipschitz approximation of u as desired. Thus, the proof is concluded by setting $C'(\Omega) = C(\Omega)^5$. □

Remark 4.22. *Observe that the (best) constant* $C'(\Omega)$ *depends on the geometric features of* Ω*, such as the minimum and the maximum angles of its boundary. However, by the construction above one has that* $C'(\Omega) = 1$ *whenever* Ω *is a right polygon.*

CHAPTER 5

A planar bi-Lipschitz extension theorem

In this chapter we let $\tilde{u} : \partial\mathcal{D} \to \tilde{u}(\partial\mathcal{D}) \subseteq \mathbb{R}^2$ be a bi-Lipschitz orientation-preserving homeomorphism on the boundary of the unit square $\mathcal{D} = \mathcal{D}(0,1)$. By the Jordan curve Theorem, its image $\tilde{u}(\partial\mathcal{D})$ is the boundary $\partial\Gamma$ of a bounded closed Lipschitz domain $\Gamma \subseteq \mathbb{R}^2$. As mentioned in the Introduction, our aim is to find a bi-Lipschitz extension of $\tilde{u}$ to the whole square $\mathcal{D}$, with a Lipschitz constant which can be explicitely estimated in terms of the Lipschitz constant of $\tilde{u}$ times a geometric constant.

Our main result is Theorem 0.4, in which we construct a piecewise affine bi-Lipschitz extension when $\tilde{u}$ is a piecewise affine function, hence Γ is a closed polygon. In particular, as observed in the Introduction and in the previous Chapter, Theorem 0.4 permits to end the proof of Theorem 0.2.

Moreover, by an approximation argument exploiting also Theorem 0.2 of the previous chapter, in Section 5.11 we prove Theorem 0.5, yielding the existence of a (countably piecewise affine) bi-Lipschitz extension for any bi-Lipschitz map $\tilde{u}$.

5.1. Notation

In this short section, we briefly fix some notation that will be used throughout the chapter. We list here only the notation which is common to all the different steps of the proof of Theorem 0.4: some steps, in fact, use some additional specific notation which will be specified only when needed.

We recall that $\mathcal{D} = [-1/2, 1/2]^2$ is the unit square in $\mathbb{R}^2$ with center at $O = (0,0)$. The points of $\mathcal{D}$ will be always denoted by capital letters, such as A, B, P, Q and so on. On the other hand, points of Γ will be always denoted by bold capital letters, such as $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ and similar. To shorten the notation and help the reader, whenever we use the same letter for a point in $\partial\mathcal{D}$ and (in bold) for a point in $\partial\Gamma$, say $P \in \partial\mathcal{D}$ and $\boldsymbol{P} \in \partial\Gamma$, this always means that $\tilde{u}(P) = \boldsymbol{P}$. Similarly, whenever the same letter refers to a point P in $\mathcal{D}$ and (in bold) to a point $\boldsymbol{P}$ in Γ, this always means that the extension $\tilde{v}$ that we are constructing is done in such a way that $\tilde{v}(P) = \boldsymbol{P}$.

For any two points P, $Q \in \mathcal{D}$, we call PQ and $\ell(PQ)$ the segment connecting P and Q and its length. In the same way, for any $\boldsymbol{P}$, $\boldsymbol{Q} \in \Gamma$, by $\boldsymbol{PQ}$ and by $\ell\big(\boldsymbol{PQ}\big)$ we will denote the segment joining $\boldsymbol{P}$ and $\boldsymbol{Q}$ and its length. Since Γ is not, in general, a convex set, we will use the notation $\boldsymbol{PQ}$ only if the segment $\boldsymbol{PQ}$ is contained in Γ.

Moreover, for any two points P, $Q \in \partial\mathcal{D}$, we call $\widehat{PQ}$ the shortest path inside $\partial\mathcal{D}$ connecting P and Q, and by $\ell(\widehat{PQ}) \in [0,2]$ its lenght. Notice that $\widehat{PQ}$ is well-defined unless P and Q are opposite points of $\partial\mathcal{D}$. In that case, the length $\ell(\widehat{PQ})$ is still well-defined, being 2, while the notation $\widehat{PQ}$ may refer to any of the two minimizing paths (and we write $\widehat{PQ}$ only after having specified which one). Accordingly, given two points $\boldsymbol{P}$ and $\boldsymbol{Q}$ on $\partial\Gamma$, we write $\widehat{\boldsymbol{PQ}}$ to denote the path $\tilde{u}(\widehat{PQ})$. Observe that, if $\tilde{u}$ is piecewise affine on $\partial\mathcal{D}$, then $\widehat{\boldsymbol{PQ}}$ is a piecewise affine path for any $\boldsymbol{P}$ and $\boldsymbol{Q}$ in $\partial\Gamma$.

5.2. An overview of the construction

In this section we give an overview of the proof of Theorem 0.4, which will be the object of Sections 5.3-5.10.

First of all (Section 5.3) we determine a "central ball" $\widehat{\boldsymbol{\mathcal{B}}}$, which is a suitable ball contained in Γ and whose boundary touches the boundary of Γ in some points $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_N$, being $N \geq 2$. The image through $\tilde{v}$ of the central part of the square $\mathcal{D}$ will eventually be contained inside this central ball.

For any two consecutive points $\boldsymbol{A}_i$, $\boldsymbol{A}_{i+1}$ among those just described, we consider the part of Γ which is "beyond" the segment $\boldsymbol{A}_i\boldsymbol{A}_{i+1}$ (by construction, the interior of this segment lies in the interior of Γ). We call "primary sectors" these regions, and we give their formal definition and study their main properties in Section 5.4. It is to be observed that Γ is the essentially disjoint union of these primary sectors and of the "internal polygon" having the points $\boldsymbol{A}_i$ as vertices (see Figure 2 for an example).

We start then to consider a given sector, with the aim of defining an extension of $\tilde{u}$ which is bi-Lipschitz between a suitable subset of the square $\mathcal{D}$ and this sector. In order to do so, we first give a method (Section 5.5) to partition a sector in triangles. We will call vertex of a sector any point $\boldsymbol{P} \in \partial\Gamma$ which is a vertex of one of these triangles. Then, using this partition, for any point $\boldsymbol{P}$ we define a suitable piecewise affine path γ, which starts from $\boldsymbol{P}$ and ends on a point $\boldsymbol{P'}$ on the segment $\boldsymbol{A}_i\boldsymbol{A}_{i+1}$ (Section 5.6). These paths will be called "paths inside a primary sector". We also need a bound on the lengths of these paths, found in Section 5.7.

Then we can define our extension. Basically, the idea is the following. Take any point $P \in \partial\mathcal{D}$ such that $\boldsymbol{P}$ is a vertex of $\partial\Gamma$ inside our given sector. A temptative method to define the extension of $\tilde{u}$ is the following. Denoting by O the center of the square $\mathcal{D}$, we send the first part of the segment PO of the square (i.e., a suitable segment $PP' \subseteq PO$) onto the path γ found in Section 5.6, while the last part $P'O$ of PO is sent onto the segment connecting $\boldsymbol{P'}$ with a special point $\boldsymbol{O}$ of the central ball $\widehat{\boldsymbol{\mathcal{B}}}$ (in most cases $\boldsymbol{O}$ will be the center of $\widehat{\boldsymbol{\mathcal{B}}}$). Unfortunately, this method does not work if we simply send PP' onto γ at constant speed; instead, we have to carefully define speed functions for all the different vertices $\boldsymbol{P}$ of the sector, and the speed function of any point will affect the speed functions of the other points. This will be done in Section 5.8.

At this stage, we have already defined the extension $\tilde{v}$ of $\tilde{u}$ on many segments (paths inside a primary sector) of the square, thus it is easy to extend $\tilde{v}$ so as to cover the whole primary sectors. To define formally this map, and in particular to check that it is CL^4 bi-Lipschitz, is the goal of Section 5.9. Finally, in Section 5.10, we put together all the maps done for the different primary sectors and fill also the "internal polygon", while keeping the bi-Lipschitz property. The whole construction is done in such a way that the resulting extending map $\tilde{v}$ is piecewise affine. In Section 5.11 we conclude the proof of Theorem 0.4, showing the existence of a smooth extension $\tilde{v}$. This will obtained from the piecewise affine map thanks to a recent result by Mora-Corral and the second author in [**46**], see Theorem 5.32. Moreover, thanks to the results obtained in Chapter 2, we give the proof of Theorem 0.5.

5.3. Choice of a "central ball"

Our first step consists in determining a suitable ball, that will be called "central ball", whose interior is contained in the interior of Γ, and whose boundary touches the boundary of $\partial\Gamma$. Before starting, let us briefly explain why we do so. Consider a very simple situation, i.e. when Γ is convex. In this case, the easiest way to build an extension $\tilde{u}$ as required by Theorem 0.4 is first to select a point $\boldsymbol{O} = \tilde{v}(O)$ having distance of order at least $1/L$ from $\partial\Gamma$, and then to define the obvious piecewise affine extension of $\tilde{u}$, that is, for any two consecutive vertices P, $Q \in \partial\mathcal{D}$ we send the triangle OPQ onto the triangle $\boldsymbol{OPQ}$ in the affine way. This very coarse idea does not suit the general case, because in general Γ can be very complicated and a priori there is no reason why the triangle $\boldsymbol{OPQ}$ should be contained in Γ. Nevertheless, our construction is somehow reminiscent of this idea. In fact, we will select a suitable point $\boldsymbol{O} = u(O) \in \Gamma$ in such "central ball" and we will build the image of a triangle like OPQ as a "triangular shape", suitably defining the "sides" $\boldsymbol{OP}$ and $\boldsymbol{OQ}$ which will be, in general, piecewise affine curves instead of straight lines. Since our "central ball" will be sufficiently big, in a neighborhood of $\boldsymbol{O}$ of order at least $1/L$ the construction will be carried out as in the convex case.

The goal of this step is only to determine a suitable "central ball" $\widehat{\boldsymbol{\mathcal{B}}}$. The point $\boldsymbol{O}$ will be chosen only in Section 5.10, and it will be in the interior of this ball –in fact, in most cases $\boldsymbol{O}$ will be the center of $\widehat{\boldsymbol{B}}$.

LEMMA 5.1. *There exists an open ball* $\widehat{\boldsymbol{\mathcal{B}}} \subseteq \Gamma$ *such that the intersection* $\partial\widehat{\boldsymbol{\mathcal{B}}} \cap \partial\Gamma$ *consists of* $N \geq 2$ *points* $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\dots \boldsymbol{A}_N$, *taken in the anti-clockwise order on the circle* $\partial\widehat{\boldsymbol{\mathcal{B}}}$, *and with the property that* $\partial\mathcal{D}$ *is the union of the paths* $\widehat{A_iA_{i+1}}$, *with the usual convention* $N + 1 \equiv 1$.

REMARK 5.2. *Before giving the proof of our lemma, some remarks are in order. First of all, since the ball* $\widehat{\boldsymbol{\mathcal{B}}}$ *is contained in* Γ, *then* $\partial\Gamma \cap \widehat{\boldsymbol{\mathcal{B}}} = \emptyset$. *As a consequence, the path* $\partial\Gamma$ *meets all the points* $\boldsymbol{A}_i$ *in the same order as* $\partial\widehat{\boldsymbol{\mathcal{B}}}$, *hence also the points* $A_i \in \partial\mathcal{D}$ *are in the anti-clockwise order (since* $\tilde{u}$ *is orientation preserving). Hence, the thesis is equivalent to say that for each* i, *among the two injective paths connecting* A_i *and* A_{i+1} *on* $\partial\mathcal{D}$, *the anti-clockwise one is shorter than the other.*

In addition, notice that from the thesis one has two possibilities. If $N = 2$, *then necessarily* $\ell(A_1A_2) = 2$, *so that the two paths* $\widehat{A_1A_2}$ *and* $\widehat{A_2A_1}$ *have the same length. On the other hand, if* $N \geq 3$, *then it is immediate to observe that there must be two points* A_i *and* A_j, *not necessarily consecutive, such that* $\ell\big(\widehat{A_iA_j}\big) \geq 4/3$. *In any case, by the* *bi-Lipschitz property of* $\tilde{u}$*, this ensures that the radius of* $\widehat{\boldsymbol{\mathcal{B}}}$ *is at least* $\frac{2}{3L}$*, since the circle* $\partial\widehat{\boldsymbol{\mathcal{B}}}$ *contains two points having distance at least* $\frac{4}{3L}$*.*

Finally notice that, given a ball $\mathcal{B}$ *contained in* Γ *and such that* $\partial\Gamma \cap \partial\mathcal{B}$ *contains at least two points, there is a simple method to check whether* $\widehat{\mathcal{B}} = \mathcal{B}$ *satisfies the requirements of the lemma. Indeed,* $\mathcal{B}$ *is a central ball unless there is an arc of length* 2 *in* $\partial\mathcal{D}$ *whose image does not contain any point of* $\partial\Gamma \cap \partial\mathcal{B}$*.*

Proof of Lemma 5.1. First of all, we define the set

$$S = \big\{(\boldsymbol{A}, \boldsymbol{P}) \subseteq \partial\Gamma \times \partial\Gamma : \boldsymbol{A} \neq \boldsymbol{P} \text{ and } \exists\, \mathcal{B} \subseteq \Gamma \text{ s.t. } \{\boldsymbol{A}, \boldsymbol{P}\} \subseteq \partial\mathcal{B} \cap \partial\Gamma\big\}.$$

We notice that S is symmetric and nonempty. Indeed, since Γ is a polygon, for almost all $\boldsymbol{A} \in \partial\Gamma$ there exists an *inward normal at* $\boldsymbol{A}$, i.e. a direction $\nu \in \mathbb{S}^1$ such that for some $\varepsilon > 0$

$$\mathcal{B}\big(\boldsymbol{A} + \varepsilon\nu, \varepsilon\big) \subseteq \Gamma.$$

Of course, if ν is an inward normal at $\boldsymbol{A}$, then the above inclusion holds true for all $0 < \varepsilon \leq \bar{\varepsilon} = \bar{\varepsilon}(\boldsymbol{A}, \nu)$ and then any point $\boldsymbol{P} \in \partial\Gamma \cap \overline{\mathcal{B}}(\boldsymbol{A} + \bar{\varepsilon}\nu, \bar{\varepsilon})$ is such that $(\boldsymbol{A}, \boldsymbol{P}) \in S$. In the following, we will denote by $N(\boldsymbol{A}) \subseteq \mathbb{S}^1$ the set of inward normals at $\boldsymbol{A}$. Notice that, for any $\boldsymbol{A} \in \partial\Gamma$, the set $N(\boldsymbol{A})$ is either empty (at internal corners), or a single point, or a proper closed interval of $\mathbb{S}^1$ (at external corners). Then, we will say that $\nu < \nu'$ if ν' follows ν in the anti-clockwise order on $N(\boldsymbol{A})$. In the same way, we will say that $\boldsymbol{A} < \boldsymbol{P}$ on $\partial\Gamma$ if $\boldsymbol{A}$ is the first point of $\widehat{\boldsymbol{AP}}$ according to the anti-clockwise order. Moreover, we denote by $\mathcal{B}(\boldsymbol{A}, \boldsymbol{P})$ a ball contained in Γ such that $\{\boldsymbol{A}, \boldsymbol{P}\} \subseteq \partial\mathcal{B}(\boldsymbol{A}, \boldsymbol{P}) \cap \partial\Gamma$.

Then, there are two cases. The first case is when there exists $(\boldsymbol{A}, \boldsymbol{P}) \in S$ with $\ell(\widehat{AP}) = 2$. If this happens, then by Remark 5.2 any ball $\mathcal{B}(\boldsymbol{A}, \boldsymbol{P})$ is a central ball.

Now, let us consider the case in which $\ell(\widehat{AP}) < 2$ for all $(\boldsymbol{A}, \boldsymbol{P}) \in S$. By a compactness and continuity argument we take $(\boldsymbol{A}, \boldsymbol{P}) \in S$ such that $\ell(\widehat{AP})$ is maximal in S. Indeed it is easy to see that, for all $n \in \mathbb{N}$, the set

$$\big\{(\boldsymbol{A}, \boldsymbol{P}) \in S : \ell(\widehat{AP}) \geq 1/n\big\}$$

is compact and that the length of the minimal arcs is a continuous function. Moreover, supposing e.g. that $\boldsymbol{A} < \boldsymbol{P}$, among the balls contained in Γ and containing $\boldsymbol{A}$ and $\boldsymbol{P}$ on their boundary, we take $\mathcal{B}(\boldsymbol{A}, \boldsymbol{P}) = \mathcal{B}\big(\boldsymbol{A} + \bar{\varepsilon}\nu, \bar{\varepsilon}\big)$ such that ν is maximal in $N(\boldsymbol{A})$.

The proof of the lemma is then concluded once we have shown the following

<u>Claim:</u> *There is some point* $\boldsymbol{R} \in \partial\mathcal{B}(\boldsymbol{A}, \boldsymbol{P}) \cap \partial\Gamma \setminus \widehat{\boldsymbol{AP}}$*.*

In fact, let $\boldsymbol{R}$ be a point in $\partial\mathcal{B}(\boldsymbol{A}, \boldsymbol{P}) \cap \partial\Gamma \setminus \widehat{\boldsymbol{AP}}$, which exists thanks to the claim, and consider the three points A, P and R on $\partial\mathcal{D}$ and the corresponding shorter paths $\widehat{AP}$, $\widehat{AR}$ and $\widehat{PR}$. Since $R \notin \widehat{AP}$ by construction, by the maximality of $\ell(\widehat{AP})$ we derive that $\widehat{AR}$ does not contain P and $\widehat{PR}$ does not contain A. Thus, $\partial\mathcal{D}$ is the essentially disjoint union of the three paths $\widehat{AP}$, $\widehat{AR}$ and $\widehat{PR}$. But then, if we take any path of length 2 in $\partial\mathcal{D}$, this intersects at least one between A, P and R. Thanks to the last observation of Remark 5.2, this gives a central ball $\widehat{\mathcal{B}} = \mathcal{B}(\boldsymbol{A}, \boldsymbol{P})$.

Let us now prove the claim. We show that, if such $\boldsymbol{R}$ does not exist, then the pair $(\boldsymbol{A}, \boldsymbol{P})$ is not maximal, thus contradicting our assumptions. Since $\ell(\widehat{AP}) < 2$, we can find two points $\boldsymbol{A'}, \boldsymbol{P'} \in \partial\Gamma$ such that $\boldsymbol{A'} < \boldsymbol{A}$, $\boldsymbol{P} < \boldsymbol{P'}$, $\widehat{A'P'} \supseteq \widehat{AP}$ and $\ell(\widehat{A'P'}) < 2$ (see Figure 1). Moreover, since we have assumed that $\partial\mathcal{B}(\boldsymbol{A}, \boldsymbol{P}) \cap \partial\Gamma \setminus \widehat{\boldsymbol{AP}} = \emptyset$, there exists $\delta > 0$ such that $\operatorname{dist}(\partial\Gamma \setminus \widehat{\boldsymbol{A'P'}}, \partial\mathcal{B}(\boldsymbol{A}, \boldsymbol{P})) \geq \delta$. Then, by a simple continuity argument, we claim that there exist two points $\boldsymbol{A'} \leq \boldsymbol{A''} \leq \boldsymbol{A}$ and $\boldsymbol{P} \leq \boldsymbol{P''} \leq \boldsymbol{P'}$ such that $(\boldsymbol{A''}, \boldsymbol{P''}) \in S$ and the center of the ball $\mathcal{B}(\boldsymbol{A''}, \boldsymbol{P''})$ lies out of the region $\boldsymbol{\Theta}$ shaded in Figure 1, i.e. the closed subset of Γ enclosed by $\widehat{\boldsymbol{AP}}$ and by the two radii of $\mathcal{B}(\boldsymbol{A}, \boldsymbol{P})$ passing through $\boldsymbol{A}$ and $\boldsymbol{P}$. By the maximality of ν, at least one between $\boldsymbol{A''}$ and $\boldsymbol{P''}$ must be different from $\boldsymbol{A}$, resp. $\boldsymbol{P}$. Then, since $\boldsymbol{A''} \geq \boldsymbol{A'}$ and $\boldsymbol{P''} \leq \boldsymbol{P'}$, we have that $\widehat{A''P''} \supseteq \widehat{AP}$ and $\ell(\widehat{A''P''}) > \ell(\widehat{AP})$, thus contradicting the maximality of $(\boldsymbol{A}, \boldsymbol{P})$ in S.

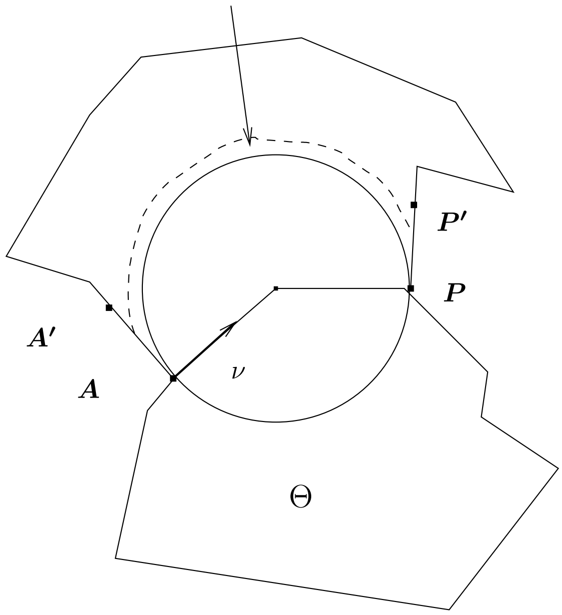

FIGURE 1. Argument for the proof of the Claim of Lemma 5.1

□

5.4. Primary sectors

In this step, we will give the definition of "sectors" of Γ, we will study their main properties, and we will call some of them "primary sectors". Let us start with some notation.

DEFINITION 5.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two points in $\partial\Gamma$ such that $\operatorname{int}\boldsymbol{AB} \subseteq \operatorname{int}\Gamma$. Let moreover $\widehat{\boldsymbol{AB}}$ be, as usual, the image under u of the shortest path connecting A and B on*

$\partial\mathcal{D}$ (or of a given one of the two injective paths, if A and B are opposite). We will call sector between $\boldsymbol{A}$ and $\boldsymbol{B}$*, and denote it as $\mathcal{S}(\boldsymbol{AB})$, the subset of Γ enclosed by the closed path $\boldsymbol{AB}\cup\widehat{\boldsymbol{AB}}$.*

Remark 5.4. *It is useful to notice what follows. If $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}\in\partial\Gamma$, and $\boldsymbol{C}$, $\boldsymbol{D}\in\widehat{\boldsymbol{AB}}$, then $\widehat{\boldsymbol{CD}}\subseteq\widehat{\boldsymbol{AB}}$. Moreover, if both $\operatorname{int}\boldsymbol{AB}$ and $\operatorname{int}\boldsymbol{CD}$ lie in the interior of Γ, then one also has*

$$\mathcal{S}(\boldsymbol{CD})\subseteq\mathcal{S}(\boldsymbol{AB})\,.$$

We observe now a very simple property, which will play a crucial role in our future construction, namely that the length of a shortest path in $\partial\mathcal{D}$ can be bounded by the length of the corresponding segment in Γ.

Lemma 5.5. *Let $\boldsymbol{P}$, $\boldsymbol{Q}$ be two points in $\partial\Gamma$ such that the segment $\boldsymbol{PQ}$ is contained in Γ. Then one has*

$$\ell\big(\widehat{PQ}\big)\leq\sqrt{2}L\,\ell(\boldsymbol{PQ})\,. \tag{5.1}$$

Proof. The inequality simply comes from the Lipschitz property of u, and from the fact that $\mathcal{D}$ is a square. Indeed,

$$\ell\big(\widehat{PQ}\big)\leq\sqrt{2}\,\ell(PQ)\leq\sqrt{2}L\,\ell(\boldsymbol{PQ})\,.$$

□

Remark 5.6. *We observe that, of course, the estimate (5.1) holds true because $\widehat{PQ}$ is the shortest path between P and Q in $\partial\mathcal{D}$ (however, this does not necessarily imply that $\widehat{\boldsymbol{PQ}}$ is the shortest path between $\boldsymbol{P}$ and $\boldsymbol{Q}$ in $\partial\Gamma$). We will see that the estimate* (5.1) *is the reason why we had to perform the complicated construction of Lemma 5.1 so as to find points A_j on $\partial\Gamma$ such that each path $\widehat{A_iA_{i+1}}$ does not pass through the other points A_j.*

We now fix a central ball $\hat{\mathcal{B}}$ as in Lemma 5.1 and define the "primary sectors", which are the sectors between consecutive points $\boldsymbol{A}_i$.

Definition 5.7. *We call* primary sector *each of the sectors $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$, being $\boldsymbol{A}_j$ the points of $\hat{\mathcal{B}}$ obtained by Lemma 5.1.*

Notice that the above definition makes sense, because the points $\boldsymbol{A}_i$ are all on the boundary of the central ball $\widehat{\boldsymbol{\mathcal{B}}}$ and $\widehat{\boldsymbol{\mathcal{B}}}$ does not intersect $\partial\Gamma$, thus the open segments $\operatorname{int}\boldsymbol{A}_i\boldsymbol{A}_{i+1}$ are entirely contained in the interior of Γ. Moreover, by the claim of Lemma 5.1 it follows that the sectors $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$ are essentially pairwise disjoint. The set Γ is the essentially disjoint union of the sectors $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$ and of the polygon $S=\boldsymbol{A}_1\boldsymbol{A}_2\dots\boldsymbol{A}_N$, as Figure 2 illustrates.

5.5. Triangulation of primary sectors

In view of the preceding sections, we aim to extend the function $\tilde{u}$ in order to cover a whole given primary sector. This extension of the function $\tilde{u}$, which is the main part of the proof, will be quite delicate and long, being the scope of Sections 5.5-5.9. In this

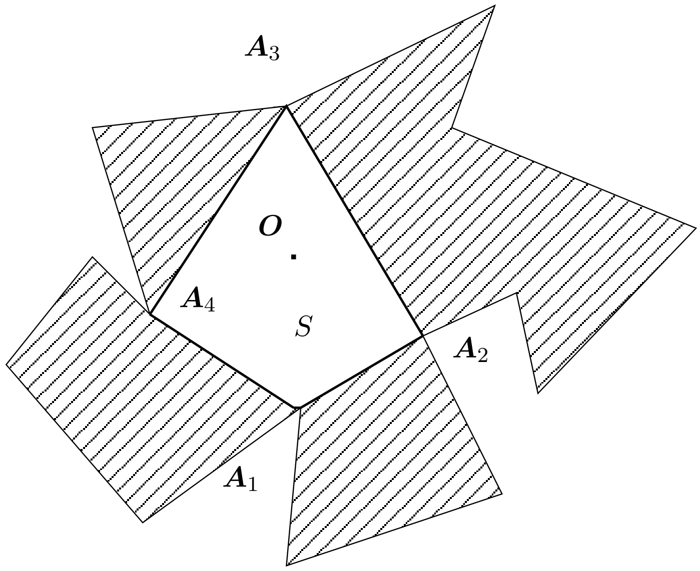

FIGURE 2. A set Γ with four primary sectors

section, we describe a method to partition a given sector in triangles. Let us then start with a technical definition.

DEFINITION 5.8. *Let $\mathcal{S}(\boldsymbol{AB})$ be a sector, and let $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ be three points in $\widehat{\boldsymbol{AB}}$ such that the triangle $\boldsymbol{PQR}$ is not degenerate and is contained in Γ. We say that $\boldsymbol{PQR}$ is an* admissible triangle *if each of its open sides is entirely contained either in $\partial\Gamma$, or in $\Gamma\setminus\partial\Gamma$. If $\boldsymbol{PQR}$ is an admissible triangle, we say that $\boldsymbol{PR}$ is its* exit side *if $\widehat{\boldsymbol{PR}} = \widehat{\boldsymbol{PQ}} \cup \widehat{\boldsymbol{QR}}$.*

Figure 3 shows a sector $\mathcal{S}(\boldsymbol{AB})$, drawn in black, with five numbered triangles, having dotted sides. Triangles 1 and 3 are not admissible because they contain a side which is neither all contained in $\partial\Gamma$, nor all in $\Gamma \setminus \partial\Gamma$, in particular triangle 1 has a side which is half in $\partial\Gamma$ and half in $\Gamma \setminus \partial\Gamma$, while triangle 3 has a side which is all contained in $\Gamma \setminus \partial\Gamma$ except for a point. On the other hand, triangles 2, 4 and 5 are admissible, and an arrow indicates the exit side for each of them.

REMARK 5.9. *It is important to observe that each admissible triangle has exactly one exit side. As the figure shows, an admissible triangle can have all the three sides in the interior of Γ, as triangle 2, or two, as triangle 5, or just one, as triangle 4. In any case, the exit side is always in the interior of Γ.*

It is also useful to understand the reason for the choice of the name. Consider a point $\boldsymbol{T} \in \widehat{\boldsymbol{PR}}$, being $\boldsymbol{PR}$ the exit side of the admissible triangle $\boldsymbol{PQR}$, and consider the segment TO which connects $T = \tilde{u}^{-1}(\boldsymbol{T})$ to the center O of the square $\mathcal{D}$. If $\tilde{v} : \mathcal{D} \to \Gamma$ is an extension as required by Theorem 0.4, then the image of the segment TO under $\tilde{v}$ must be a path inside Γ which connects $\boldsymbol{T}$ to $\boldsymbol{O}$. This path must clearly exit from the triangle $\boldsymbol{PQR}$ through the exit side $\boldsymbol{PR}$.

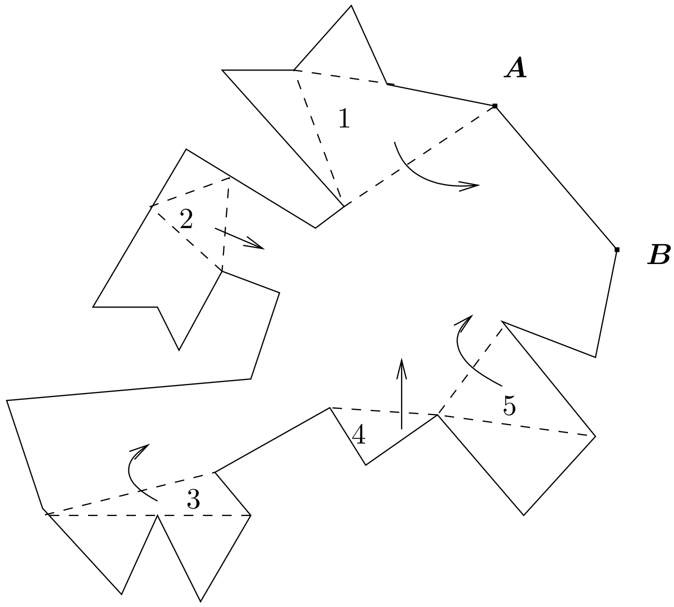

FIGURE 3. Some (admissible or not) triangles in a sector

Before stating and proving the main result of this step we fix some further notation. Recall that Γ is a polygon obtained as the image of $\partial\mathcal{D}$ under $\tilde{u}$. Hence, $\partial\mathcal{D}$ is divided in a finite number of segments and $\tilde{u}$ is affine on each of these segments. We will then call *vertex* on $\partial\mathcal{D}$ each extreme point of any of these segments. Therefore, the four corners of $\partial\mathcal{D}$ are of course vertices, but there are usually much more vertices. Correspondingly, we call *vertex* on $\partial\Gamma$ the image of each vertex on $\partial\mathcal{D}$. Thus, all the points of $\partial\Gamma$ which are "vertices" in the usual sense of the polygon (i.e. corners), are clearly also vertices in our notation. However, there may be also other vertices which are not corners, hence which are in the interior of some segment contained in $\partial\Gamma$. We will also call *side* in $\partial\Gamma$ any segment connecting two consecutive vertices on $\partial\Gamma$. Hence, some of the segments which are sides of $\partial\Gamma$ in the sense of polygons are in fact sides according to our notation, but there might be also some segments contained in $\partial\Gamma$ which are not sides, but finite union of sides.

Finally, notice that it is admissible to add (finitely many!) new vertices to $\partial\mathcal{D}$ and then correspondingly to $\partial\Gamma$. This means that we will possibly split some side in two or more parts, which is possible since of course $\tilde{u}$ will be affine on each of those parts.

REMARK 5.10. *As an immediate application of this possibility of adding a finite set of new vertices, we will assume without loss of generality that for any two consecutive vertices* P *and* Q *in* $\mathcal{D}$*, one always has* $\widehat{POQ} \leq 1/50L$.

We can finally state and prove the main result of this step.

LEMMA 5.11. *Let* $\mathcal{S}(\boldsymbol{AB})$ *be a sector. There exists a partition of* $\mathcal{S}(\boldsymbol{AB})$ *in a finite number of admissible triangles such that:*

a) *each vertex in* $\mathcal{S}(\boldsymbol{AB})$ *is vertex of some triangle of the partition,*

b) *for each triangle $\boldsymbol{PQR}$ of the partition, whose exit side is $\boldsymbol{PR}$, the orthogonal projection of $\boldsymbol{Q}$ on the straight line through $\boldsymbol{PR}$ lies in the closed segment $\boldsymbol{PR}$ (equivalently, the angles $\widehat{\boldsymbol{PRQ}}$ and $\widehat{\boldsymbol{RPQ}}$ are at most $\pi/2$).*

To show this result, it will be convenient to associate to any possible sector a number, which we will call "weight".

DEFINITION 5.12. *Let $\mathcal{S}(\boldsymbol{AB})$ be a sector, and for any point $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$ (different from $\boldsymbol{A}$ and $\boldsymbol{B}$) let us call $\boldsymbol{P}_\perp$ the orthogonal projection of $\boldsymbol{P}$ onto the straight line through $\boldsymbol{AB}$. We will say that $\boldsymbol{AB}$* "sees" *$\boldsymbol{P}$ if $\boldsymbol{P}_\perp$ belongs to the closed segment $\boldsymbol{AB}$ and the open segment $\boldsymbol{PP}_\perp$ is entirely contained in the interior of Γ. Let now ω be the number of sides of the path $\widehat{\boldsymbol{AB}}$. We will say that the* weight of the sector $\mathcal{S}(\boldsymbol{AB})$ *is ω if $\boldsymbol{AB}$ sees at least a* vertex *$\boldsymbol{P}$ in $\widehat{\boldsymbol{AB}}$. Otherwise, we will say that weight of $\mathcal{S}(\boldsymbol{AB})$ is $\omega + \frac{1}{2}$.*

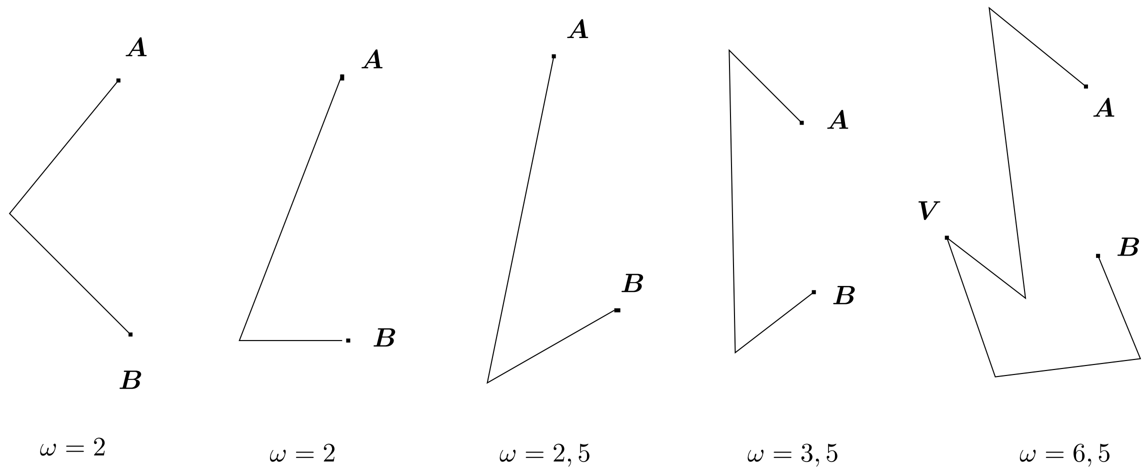

FIGURE 4. Some simple sectors and their weights

In other words, the weight of any sector is an half-integer corresponding to the number of sides of the sector, augmented of a "penalty" $1/2$ in case that the segment $\boldsymbol{AB}$ does not see any vertex of $\widehat{\boldsymbol{AB}}$. For instance, Figure 4 shows some simple sectors and the corresponding weights. Notice that the last sector has non-integer weight because $\boldsymbol{AB}$ does *not* see the vertex $\boldsymbol{V}$, since the segment $\boldsymbol{VV}_\perp$ lies also out of Γ. We now show a simple technical lemma, and then we pass to the proof of Lemma 5.11.

LEMMA 5.13. *If the sector $\mathcal{S}(\boldsymbol{AB})$ has a non-integer weight, then there exists a side $\boldsymbol{A}^+\boldsymbol{B}^-$ in $\widehat{\boldsymbol{AB}}$ such that $\boldsymbol{AB}$ sees only points of the side $\boldsymbol{A}^+\boldsymbol{B}^-$.*

PROOF. First, notice that the property that we are going to show appears evident from the last three examples of Figure 4.

Let us now pass to the proof. For any point $\boldsymbol{D}$ which belongs to the segment $\boldsymbol{AB}$, there exists exactly a point $\boldsymbol{C} \in \widehat{\boldsymbol{AB}}$ such that $\boldsymbol{AB}$ sees $\boldsymbol{C}$ and $\boldsymbol{C}_\perp = \boldsymbol{D}$. This point is simply obtained by taking the half-line orthogonal to $\boldsymbol{AB}$, starting from $\boldsymbol{D}$ and going inside the sector: $\boldsymbol{C}$ is the first point of this half-line which belongs to $\partial\Gamma$, and in particular it belongs to $\widehat{\boldsymbol{AB}}$ by construction.

The proof is then concluded once we show that all such points $\boldsymbol{C}$s are on the same side. Indeed, if it was not so, there would be some such $\boldsymbol{C}$ which is a vertex, contradicting the fact that the sector has non-integer weight. ☐

Proof of Lemma 5.11. We will show the result by induction on the (half-integer) weight of the sector.

If $\mathcal{S}(\boldsymbol{AB})$ has weight 2, which is the least possible weight, then the two sides of the sector must be $\boldsymbol{AC}$ and $\boldsymbol{CB}$ for a vertex $\boldsymbol{C}$. Moreover, $\boldsymbol{AB}$ sees $\boldsymbol{C}$, because otherwise the weight would be 2.5. Hence, the sector coincides with the triangle $\boldsymbol{ABC}$, which is a (trivial) partition as required.

Let us now consider a sector of weight $\omega > 2$, and assume by induction that we already know the validity of our claim for all the sectors of weight less than ω. In the proof, we distinguish three cases.

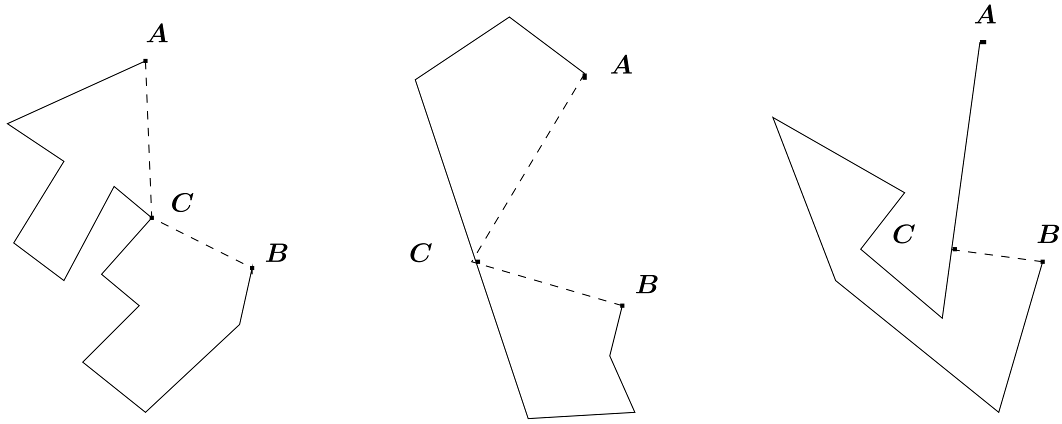

Figure 5. The three possible cases in Lemma 5.11

Case 1. $\omega \in \mathbb{N}$.

In this case, there are by definition some vertices which are seen by $\boldsymbol{AB}$. Among these vertices, let us call $\boldsymbol{C}$ the one which is closest to the segment $\boldsymbol{AB}$. Let us momentarily assume that neither $\boldsymbol{AC}$ nor $\boldsymbol{BC}$ is entirely contained in $\partial\Gamma$. Then, by the minimality property of $\boldsymbol{C}$, the open segments $\boldsymbol{AC}$ and $\boldsymbol{BC}$ lie entirely in the interior of Γ, as depicted in Figure 5 (left). Hence, one can consider the sectors $\mathcal{S}(\boldsymbol{AC})$ and $\mathcal{S}(\boldsymbol{BC})$, as ensured by Remark 5.4. Moreover, of course the weights of both $\mathcal{S}(\boldsymbol{AC})$ and $\mathcal{S}(\boldsymbol{BC})$ are strictly less than ω, so by inductive assumption we know that it is possible to find a suitable partition in triangles for both the sectors $\mathcal{S}(\boldsymbol{AC})$ and $\mathcal{S}(\boldsymbol{BC})$. Finally, since by construction the sectors $\mathcal{S}(\boldsymbol{AC})$ and $\mathcal{S}(\boldsymbol{BC})$ are essentially disjoint, and the union of them with the triangle

$\boldsymbol{ABC}$ is the whole sector $\mathcal{S}(\boldsymbol{AB})$, putting together the two decompositions and the triangle $\boldsymbol{ABC}$ we get the desired partition of $\mathcal{S}(\boldsymbol{AB})$.

Let us now consider the possibility that $\boldsymbol{AC} \subseteq \partial\Gamma$ (if, instead, $\boldsymbol{BC} \subseteq \partial\Gamma$, then the completely symmetric argument clearly works). If it is so, we can anyway repeat almost exactly the same argument as before. In fact, $\boldsymbol{BC}$ is entirely contained in the interior of Γ, again by the minimality property of $\boldsymbol{C}$ and by the fact that $\omega > 2$. Moreover, the sector $\mathcal{S}(\boldsymbol{BC})$ has weight strictly less than ω, so by induction we can find a good partition of $\mathcal{S}(\boldsymbol{BC})$, and adding the triangle $\boldsymbol{ABC}$ we get the desired partition of $\mathcal{S}(\boldsymbol{AB})$.

<u>Case 2.</u> *$\omega \notin \mathbb{N}$, $\boldsymbol{A}^+ \not\equiv \boldsymbol{A}$, $\boldsymbol{B}^- \not\equiv \boldsymbol{B}$.*

In this case, we can use the same idea of Case 1 with a slight modification. In fact, define $\boldsymbol{C} \in \boldsymbol{A}^+\boldsymbol{B}^-$ the point such that $\boldsymbol{C}_\perp$ is the middle point of the segment $\boldsymbol{AB}$ (this point is well-defined as shown in the proof of Lemma 5.13). Again, by definition and by Lemma 5.13 we have that the open segments $\boldsymbol{AC}$ and $\boldsymbol{BC}$ are in the interior of Γ, see Figure 5 (center).

Let us then *decide* that the point $\boldsymbol{C}$ is a new vertex of $\partial\Gamma$. This means that from now on we consider the point $\boldsymbol{C}$ as a vertex, and consequently we stop considering $\boldsymbol{A}^+\boldsymbol{B}^-$ as a side of $\partial\Gamma$, instead, we think of it as the union of the two sides $\boldsymbol{A}^+\boldsymbol{C}$ and $\boldsymbol{CB}^-$. However, notice carefully that this choice modifies the weight of $\mathcal{S}(\boldsymbol{AB})$! In fact, the number of sides of $\mathcal{S}(\boldsymbol{AB})$ is increased by 1, and since $\boldsymbol{AB}$ sees $\boldsymbol{C}$ by construction, then the new weight of $\mathcal{S}(\boldsymbol{AB})$ is $\omega + \frac{1}{2}$.

We can now argue as in Case 1. In fact, again the sector $\mathcal{S}(\boldsymbol{AB})$ is the union of the triangle $\boldsymbol{ABC}$ with the two sectors $\mathcal{S}(\boldsymbol{AC})$ and $\mathcal{S}(\boldsymbol{BC})$, so it is enough to put together the triangle $\boldsymbol{ABC}$ and the two partitions given by the inductive assumption applied on the sectors $\mathcal{S}(\boldsymbol{AC})$ and $\mathcal{S}(\boldsymbol{BC})$. To do so, we have of course to be sure that the weight of both sectors is strictly less than the original weight of $\mathcal{S}(\boldsymbol{AB})$, that is, ω (and not $\omega + \frac{1}{2}$!). This is clear by the assumption that $\boldsymbol{A}^+ \not\equiv \boldsymbol{A}$ and $\boldsymbol{B}^- \not\equiv \boldsymbol{B}$, since the side $\boldsymbol{A}^+\boldsymbol{B}^-$ is neither the first nor the last of the path $\widehat{\boldsymbol{AB}}$, thus the weight of both sectors is at most $\omega - 1$.

<u>Case 3.</u> *$\omega \notin \mathbb{N}$ and $\boldsymbol{A}^+ \equiv \boldsymbol{A}$ or $\boldsymbol{B}^- \equiv \boldsymbol{B}$.*

By symmetry, let us assume that $\boldsymbol{A}^+ \equiv \boldsymbol{A}$. In this case, we cannot argue exactly as in Case 2, because if we did so the sector $\mathcal{S}(\boldsymbol{BC})$ may have weight either ω or $\omega - \frac{1}{2}$, and in the first possibility we could not use the inductive hypothesis.

Anyway, it is enough to make a slight variation of the argument of Case 2. Define $\boldsymbol{C}$, as in Figure 5 (right), the point of $\boldsymbol{AB}^-$ such that $\boldsymbol{BC}$ is orthogonal to $\boldsymbol{AB}^-$. This time, it is clear that the open segment $\boldsymbol{BC}$ lies in the interior of Γ. Let us now *decide*, exactly as in Case 2, that the point $\boldsymbol{C}$ is from now on a vertex, thus changing the weight of $\mathcal{S}(\boldsymbol{AB})$ from ω to $\omega + \frac{1}{2}$.

By construction, the segment $\boldsymbol{AB}$ sees the point $\boldsymbol{C}$, and the sector $\mathcal{S}(\boldsymbol{AB})$ is the union of the sector $\mathcal{S}(\boldsymbol{BC})$ and of the triangle $\boldsymbol{ABC}$. Hence, we conclude exactly as in the other cases if we can use the inductive assumption on the sector $\mathcal{S}(\boldsymbol{BC})$. In this case, notice that the number of sides of $\mathcal{S}(\boldsymbol{BC})$ equals exactly the original number of sides of $\mathcal{S}(\boldsymbol{AB})$, that is, $\omega - \frac{1}{2}$. Hence, in principle, the weight of $\mathcal{S}(\boldsymbol{BC})$ could be either $\omega - \frac{1}{2}$ or ω, as observed before. But in fact, by our definition of $\boldsymbol{C}$, we have that the segment $\boldsymbol{BC}$ sees the vertex $\boldsymbol{B}^-$, so that the actual weight of $\mathcal{S}(\boldsymbol{BC})$ is $\omega - \frac{1}{2}$, hence strictly less than ω, and then we can use the inductive assumption. □

To give some examples, let us briefly consider the three cases drawn in Figure 5. In the left case, the weight of $\mathcal{S}(\boldsymbol{AB})$ was $\omega = 12$, and the weights of the sectors $\mathcal{S}(\boldsymbol{AC})$ and $\mathcal{S}(\boldsymbol{BC})$ are both 6. In the central case, the weight of $\mathcal{S}(\boldsymbol{AB})$ was $\omega = 6.5$, then it becomes 7 because we add the new vertex $\boldsymbol{C}$, and the weights of the sectors $\mathcal{S}(\boldsymbol{AC})$ and $\mathcal{S}(\boldsymbol{BC})$ are respectively 3 and 4.5. Finally, in the right case, the weight of $\mathcal{S}(\boldsymbol{AB})$ was $\omega = 7.5$, it becomes 8 as we add $\boldsymbol{C}$, and the weight of the sector $\mathcal{S}(\boldsymbol{BC})$ is 7.

An explicit example of a sector with a partition in triangles done according with the construction of Lemma 5.11 can be seen in Figure 6.

We conclude this step by noticing a natural partial order on the triangles of the partition given by Lemma 5.11 and by adding some remarks.

DEFINITION 5.14. *Let $\mathcal{S}(\boldsymbol{AB})$ be a sector, and consider a partition satifying the properties of Lemma 5.11. We define the partial order $\leq$ between the triangles of the partition as $\boldsymbol{PQR} \leq \boldsymbol{STU}$ if the exit side of $\boldsymbol{PQR}$ is one of the sides of $\boldsymbol{STU}$. Equivalently, let $\boldsymbol{PQR}$ and $\boldsymbol{STU}$ be two triangles of the partition, being $\boldsymbol{SU}$ the exit side of the latter. One has $\boldsymbol{PQR} \leq \boldsymbol{STU}$ if and only if all the points $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ belong to the path $\widehat{\boldsymbol{SU}}$.*

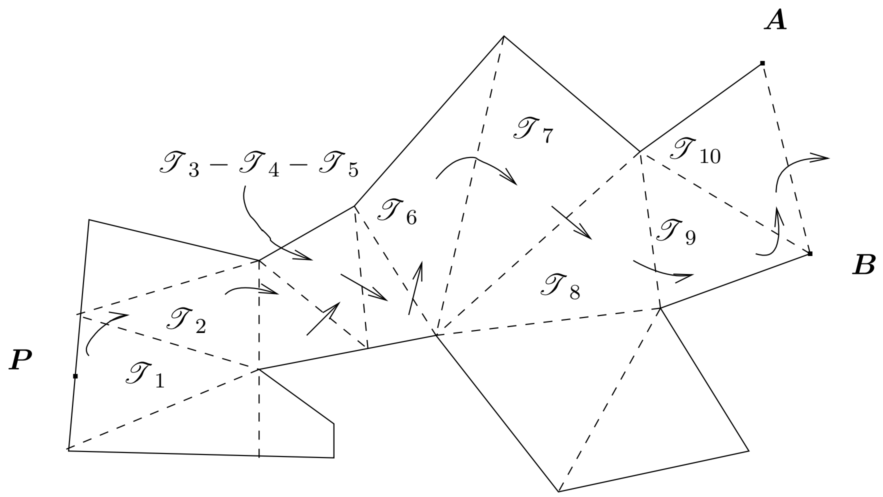

FIGURE 6. Partition of a sector in triangles, and natural sequence of triangles related to some $\boldsymbol{P}$

REMARK 5.15. *Notice that the relation defined above admits as greatest element the triangle having $\boldsymbol{AB}$ as its (exit) side. Moreover, each triangle except the maximizer has a unique successor.*

We remark also that, since the triangles are a finite number, in all the following constructions we will always be allowed to consider a single triangle of the partition and to *assume that the costruction has been done in all the triangles which are smaller in the sense of the order.*

Definition 5.16. *Let* $\mathcal{S}(\boldsymbol{AB})$ *be a sector subdivided in triangles according to Lemma 5.11, and consider a point* $\boldsymbol{P} \in \widehat{\boldsymbol{AB}} = \partial\Gamma \cap \mathcal{S}(\boldsymbol{AB})$. *We will call* natural sequence of triangles related to $\boldsymbol{P}$ *the sequence* $(\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_N)$ *of triangles of the partition satisfying the following requirements,*

- $\mathscr{T}_1$ *is the minimal triangle containing* $\boldsymbol{P}$ *in the sense of the order of Definition 5.14,*
- $\mathscr{T}_N$ *is the triangle having* $\boldsymbol{AB}$ *as its (exit) side,*
- $\mathscr{T}_{i+1}$ *is the successor of* $\mathscr{T}_i$ *for all* $1 \leq i \leq N-1$.

It is immediate, thanks to the above remark, to observe that this sequence is univoquely determined. Figure 6 shows a sector subdivided in triangles and a point $\boldsymbol{P}$ with the related natural sequence of triangles $(\mathscr{T}_1, \ldots, \mathscr{T}_{10})$, with the arrows on the exit sides.

5.6. Paths inside primary sectors

In this step we define non-intersecting piecewise affine paths starting from any point $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$, where $\mathcal{S}(\boldsymbol{AB})$ is a given sector. This is the most important and delicate point of our construction. The goal of this step is to provide the "first part" of the piecewise affine path from a vertex $\boldsymbol{P}$ to the center $\boldsymbol{O}$, that is, the part which is inside the primary sector $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$ to which $\boldsymbol{P}$ belongs. Of course, to eventually obtain the bi-Lipschitz property for the function $\tilde{v}$, we have to take care that all the paths starting from different points $\boldsymbol{P} \neq \boldsymbol{Q}$ do not become neither too far nor too close to each other. We can now give a simple definition and then state and prove the main result of this step.

Definition 5.17. *Let* $\mathcal{S}(\boldsymbol{AB})$ *be a sector, and let* $\boldsymbol{P} \in \partial\Gamma \cap \mathcal{S}(\boldsymbol{AB})$. *Let moreover* $(\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_N)$ *be the natural sequence of triangles related to* $\boldsymbol{P}$, *according to Definition 5.16. We will call* good path corresponding to $\boldsymbol{P}$ *any piecewise affine path* $\boldsymbol{P}\boldsymbol{P}_1\boldsymbol{P}_2\cdots\boldsymbol{P}_N$ *such that each* $\boldsymbol{P}_i$ *belongs to the exit side of the triangle* $\mathscr{T}_i$ *(then* $\boldsymbol{P}_N \in \boldsymbol{AB}$*). Notice that* N *depends on* $\boldsymbol{P}$.

Figure 7 shows a sector $\mathcal{S}(\boldsymbol{AB})$ subdivided in triangles as in Lemma 5.11 and shows two good paths corresponding to the points $\boldsymbol{P}$ and $\boldsymbol{Q}$.

Lemma 5.18. *Let* $\mathcal{S}(\boldsymbol{AB})$ *be a sector. Then there exist good paths* $\boldsymbol{P}\boldsymbol{P}_1\boldsymbol{P}_2\cdots\boldsymbol{P}_N$ *corresponding to each vertex* $\boldsymbol{P}$ *of* $\partial\Gamma \cap \mathcal{S}(\boldsymbol{AB})$, *with* $N = N(\boldsymbol{P})$, *satisfying the following properties:*

(i) *for any* $\boldsymbol{P}$ *and for any* $1 \leq i \leq N(\boldsymbol{P})$, *the segment* $\boldsymbol{P}_{i-1}\boldsymbol{P}_i$ *makes an angle of at least* $\arcsin\left(\frac{1}{6L^2}\right)$ *with the side of* $\mathscr{T}_i$ *to which* $\boldsymbol{P}_{i-1}$ *belongs, and an angle of at least* $\pi/12 = 15^\circ$ *with the exit side of* $\mathscr{T}_i$;

(ii) *for any* $\boldsymbol{P}$, $\ell\big(\widehat{\boldsymbol{P}\boldsymbol{P}_N}\big) = \ell(\boldsymbol{P}\boldsymbol{P}_1) + \ell(\boldsymbol{P}_1\boldsymbol{P}_2) + \cdots + \ell(\boldsymbol{P}_{N-1}\boldsymbol{P}_N) \leq 4\,\ell\big(\widehat{\boldsymbol{AB}}\big)$;

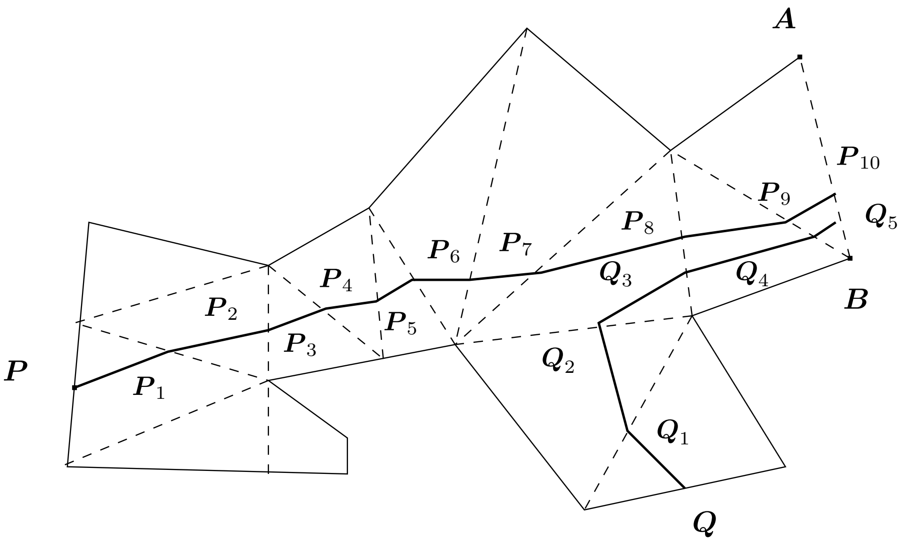

FIGURE 7. A sector with two good paths corresponding to $\boldsymbol{P}$ and $\boldsymbol{Q}$

(iii) *for any $\boldsymbol{P}$, $\boldsymbol{Q}$, if for some $1 \leq i \leq N(\boldsymbol{P})$ and $1 \leq j \leq N(\boldsymbol{Q})$ one has that $\boldsymbol{P}_i$ and $\boldsymbol{Q}_j$ belong to the same exit side of some triangle, then*

$$\frac{\ell(\widehat{PQ})}{7L} \leq \ell(\boldsymbol{P}_i\boldsymbol{Q}_j) \leq \ell(\widehat{\boldsymbol{PQ}}),$$

and moreover, if $i < N(\boldsymbol{P})$ then

$$\ell(\boldsymbol{P}_{i+1}\boldsymbol{Q}_{j+1}) \leq \ell(\boldsymbol{P}_i\boldsymbol{Q}_j);$$

(iv) *the piecewise affine paths $\boldsymbol{P}\boldsymbol{P}_1\boldsymbol{P}_2\cdots\boldsymbol{P}_N$ are pairwise disjoint.*

For the sake of clarity, let us briefly discuss the meaning of the requirements of Lemma 5.18, having in mind the example of Figure 7. Condition (i), considered for the point $\boldsymbol{P}$ and with $i = 3$ (so that $\mathscr{T}_i = \boldsymbol{CDE}$) means that

$$\sin\Big(\boldsymbol{P}_3\widehat{\boldsymbol{P}_2}\boldsymbol{D}\Big) \geq \frac{1}{6L^2}, \quad \sin\Big(\boldsymbol{P}_3\widehat{\boldsymbol{P}_2}\boldsymbol{E}\Big) \geq \frac{1}{6L^2}, \quad \boldsymbol{P}_2\widehat{\boldsymbol{P}_3}\boldsymbol{C} \geq \frac{\pi}{12}, \quad \boldsymbol{P}_2\widehat{\boldsymbol{P}_3}\boldsymbol{E} \geq \frac{\pi}{12}.$$

Condition (ii) just means that $\ell(\widehat{\boldsymbol{PP}_7}) \leq 4\,\ell(\widehat{\boldsymbol{AB}})$, where $\widehat{\boldsymbol{PP}_7}$ denotes the piecewise affine path $\boldsymbol{P}\boldsymbol{P}_1\boldsymbol{P}_2\cdots\boldsymbol{P}_7$. Similarly, $\ell(\widehat{\boldsymbol{QQ}_3}) \leq 4\,\ell(\widehat{\boldsymbol{AB}})$.

Condition (iii) ensures that

$$\frac{\ell(\widehat{PQ})}{7L} \leq \ell(\boldsymbol{P}_7\boldsymbol{Q}_3) \leq \ell(\boldsymbol{P}_6\boldsymbol{Q}_2) \leq \ell(\widehat{\boldsymbol{PQ}}).$$

In particular, concerning the second half of (iii), notice that by construction if $\boldsymbol{P}_i$ and $\boldsymbol{Q}_j$ belong to the same exit side of a triangle, then also the points $\boldsymbol{P}_{i+1}$ and $\boldsymbol{Q}_{j+1}$ belong to the same exit side of a triangle and so on. Hence, the second half of (iii) is saying that the function $l \mapsto \ell\big(\boldsymbol{P}_{i+l}\boldsymbol{Q}_{j+l}\big)$ is a decreasing function of l for $0 \le l \le N(\boldsymbol{P}) - i = N(\boldsymbol{Q}) - j$.

Finally, condition (iv) illustrates the whole idea of the construction of this step, that is, the piecewise affine paths starting from the external part $\partial\Gamma \cap \mathcal{S}(\boldsymbol{AB})$ and arriving to the segment $\boldsymbol{AB}$ do not intersect to each other, as in Figure 7.

Proof of Lemma 5.18. We will show the thesis arguing by induction on the weight of the structure $\mathcal{S}(\boldsymbol{AB})$, as in Lemma 5.11. In fact, instead of proving that the thesis is true for structures of weight 2 (recall that this is the minimal possible weight) and then giving an inductive argument, we will prove everything at once. In other words, we take a structure $\mathcal{S}(\boldsymbol{AB})$ and we assume that *either* $\mathcal{S}(\boldsymbol{AB})$ has weight 2, *or* the result has been already shown for all the structures of weight less than the weight of $\mathcal{S}(\boldsymbol{AB})$.

Let us call $\boldsymbol{C} \in \partial\Gamma \cap \mathcal{S}(\boldsymbol{AB})$ the point such that $\boldsymbol{ABC}$ is the greatest triangle of the partition of $\mathcal{S}(\boldsymbol{AB})$ with the order of Definition 5.14.

Consider now the segment $\boldsymbol{AC}$, which lies entirely either in the interior of Γ or on $\partial\Gamma$. In the first case, $\mathcal{S}(\boldsymbol{AC})$ is a sector of weight strictly less than that of $\mathcal{S}(\boldsymbol{AB})$. Then, by inductive assumption, there are piecewise affine paths $\boldsymbol{PP}_1 \cdots \boldsymbol{P}_N$ for each vertex $\boldsymbol{P} \in \widehat{\boldsymbol{AC}}$, with $\boldsymbol{P}_N \in \boldsymbol{AC}$, satisfying conditions (i)–(iv) with $\mathcal{S}(\boldsymbol{AC})$ in place of $\mathcal{S}(\boldsymbol{AB})$. We have then to connect the points $\boldsymbol{P}_N$ on $\boldsymbol{AC}$ with the segment $\boldsymbol{AB}$. In the second case, i.e. if $\boldsymbol{AC} \subseteq \partial\Gamma$, then $\widehat{\boldsymbol{AC}} = \boldsymbol{AC}$, thus we have to connect all the vertices contained in $\boldsymbol{AC}$ (which are not necessarily only $\boldsymbol{A}$ and $\boldsymbol{C}$!) with the segment $\boldsymbol{AB}$. The same considerations hold for $\boldsymbol{BC}$ in place of $\boldsymbol{AC}$.

The construction of the segments between $\boldsymbol{AC} \cup \boldsymbol{BC}$ and $\boldsymbol{AB}$ will be divided, for clarity, in several parts.

Part 1. *Definition of* $\boldsymbol{C}_1$.

By definition, $\boldsymbol{C}$ is a vertex of $\partial\Gamma$. Hence, the first thing to do is to define the good path corresponding to $\boldsymbol{C}$, that is a suitable segment $\boldsymbol{CC}_1$ with $\boldsymbol{C}_1 \in \boldsymbol{AB}$. Let us first define two points $\boldsymbol{C}^+$ and $\boldsymbol{C}^-$, on the straight line containing $\boldsymbol{AB}$, as in Figure 8. These two points are defined by

$$\ell(\boldsymbol{BC}^+) = \ell(\boldsymbol{BC})\,, \qquad\qquad \ell(\boldsymbol{AC}^-) = \ell(\boldsymbol{AC})\,.$$

In Figure 8, $\boldsymbol{C}^\pm$ both belong to the segment $\boldsymbol{AB}$, but of course it may even happen that $\boldsymbol{C}^+$ stays above $\boldsymbol{A}$, and/or that $\boldsymbol{C}^-$ stays below $\boldsymbol{B}$. Let us now give a temptative definition of $\boldsymbol{C}_1$ by letting $\widetilde{\boldsymbol{C}}_1$ be the point of $\boldsymbol{AB}$ such that

$$\frac{\ell\big(\widehat{AC}\big)}{\ell\big(\widehat{AB}\big)} = \frac{\ell(\boldsymbol{A}\widetilde{\boldsymbol{C}}_1)}{\ell\big(\boldsymbol{AB}\big)}\,. \tag{5.2}$$

Taking $\boldsymbol{C}_1 = \widetilde{\boldsymbol{C}}_1$ would be a good choice from many points of view, but unfortunately one would eventually obtain estimates weaker than (i)–(iv).

Instead, we give our next definition. We define $\boldsymbol{C}_1$ to be the point of the segment $\boldsymbol{C}^-\boldsymbol{C}^+$ which is closest to $\widetilde{\boldsymbol{C}}_1$. In other words, we can say that we set $\boldsymbol{C}_1 = \widetilde{\boldsymbol{C}}_1$ if $\widetilde{\boldsymbol{C}}_1$

belongs to $\boldsymbol{C}^+\boldsymbol{C}^-$, while otherwise we set $\boldsymbol{C}_1 = \boldsymbol{C}^+$ (resp. $\boldsymbol{C}_1 = \boldsymbol{C}^-$) if $\widetilde{\boldsymbol{C}}_1$ is above $\boldsymbol{C}^+$ (resp. below $\boldsymbol{C}^-$).

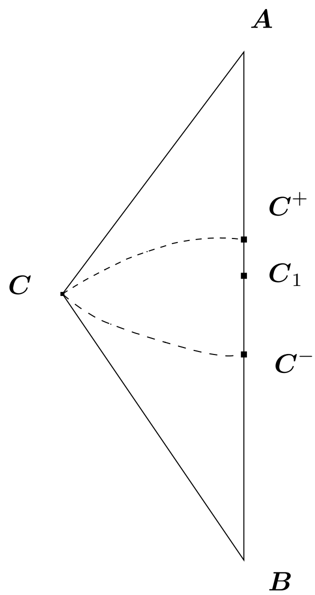

FIGURE 8. The triangle $\boldsymbol{ABC}$ with the points $\boldsymbol{C}^+$, $\boldsymbol{C}^-$ and $\boldsymbol{C}_1$

Notice that $\boldsymbol{C}_1$ belongs to $\boldsymbol{AB}$, since so does $\widetilde{\boldsymbol{C}}_1$ thanks to (5.2). It is also important to underline that

$$\ell\big(\widehat{AC}\big) \leq \sqrt{2}L\,\ell\big(\boldsymbol{AC}_1\big)\,, \qquad\qquad \ell\big(\widehat{BC}\big) \leq \sqrt{2}L\,\ell\big(\boldsymbol{BC}_1\big)\,. \tag{5.3}$$

By symmetry, let us only show the first inequality. Recall that by (5.1) we know

$$\ell\big(\widehat{AC}\big) \leq \sqrt{2}L\,\ell\big(\boldsymbol{AC}\big)\,, \qquad\qquad \ell\big(\widehat{AB}\big) \leq \sqrt{2}L\,\ell\big(\boldsymbol{AB}\big)\,.$$

As a consequence, either $\boldsymbol{C}_1 = \boldsymbol{C}^-$, and then

$$\ell\big(\boldsymbol{AC}_1\big) = \ell\big(\boldsymbol{AC}^-\big) = \ell\big(\boldsymbol{AC}\big) \geq \frac{\ell\big(\widehat{AC}\big)}{\sqrt{2}L}\,,$$

or $\ell\big(\boldsymbol{AC}_1\big) \geq \ell\big(\boldsymbol{A}\widetilde{\boldsymbol{C}}_1\big)$, and then by (5.2)

$$\ell\big(\boldsymbol{AC}_1\big) \geq \ell\big(\boldsymbol{A}\widetilde{\boldsymbol{C}}_1\big) = \ell\big(\widehat{AC}\big)\,\frac{\ell\big(\boldsymbol{AB}\big)}{\ell\big(\widehat{AB}\big)} \geq \frac{\ell\big(\widehat{AC}\big)}{\sqrt{2}L}\,.$$

Recall now that, to show the thesis, all we have to do is to take each vertex $\boldsymbol{D} \in \boldsymbol{AC} \cup \boldsymbol{BC}$ and to find a suitable corresponding point $\boldsymbol{D'} \in \boldsymbol{AB}$, in such a way that the requirements (i)–(v) are satisfied. Having defined $\boldsymbol{C}_1$, we have then to send the points $\boldsymbol{P}_N$ of $\boldsymbol{AC}$ in $\boldsymbol{AC}_1$ and those of $\boldsymbol{BC}$ in $\boldsymbol{BC}_1$.

We claim that the two segments can be considered independently, that is, we can limit ourselves to describe how to send $\boldsymbol{BC}$ on $\boldsymbol{BC}_1$ and check that the properties (i)–(iv) hold for points of $\widehat{\boldsymbol{BC}}$. Indeed, if we do so, by symmetry the same definitions can be repeated for $\boldsymbol{AC}$, and the properties (i)–(iv) hold separately for points of $\widehat{\boldsymbol{BC}}$ and $\widehat{\boldsymbol{AC}}$. The only thing which would be missing, then, would be to check the validity of (iii) for two points $\boldsymbol{P} \in \widehat{\boldsymbol{AC}}$ and $\boldsymbol{Q} \in \widehat{\boldsymbol{BC}}$. Moreover, this will be trivially true, because since $\boldsymbol{C}$ belongs to both the segments $\boldsymbol{AC}$ and $\boldsymbol{BC}$, then it is enough to use (iii) once with $\boldsymbol{P}$ and $\boldsymbol{C}$, and once with $\boldsymbol{C}$ and $\boldsymbol{Q}$, recalling that clearly

$$\ell\big(\widehat{PQ}\big) = \ell\big(\widehat{PC}\big) + \ell\big(\widehat{CQ}\big)\,, \qquad\qquad \ell\big(\boldsymbol{P}_i\boldsymbol{Q}_j\big) = \ell\big(\boldsymbol{P}_i\boldsymbol{C}_1\big) + \ell\big(\boldsymbol{C}_1\boldsymbol{Q}_j\big)\,.$$

For this reason, from now on we will concentrate ourselves only on the segment $\boldsymbol{BC}$. We will call $\boldsymbol{D}$ the generic point of $\boldsymbol{BC}$, which clearly corresponds to $\boldsymbol{P}_{N-1}$ for some $\boldsymbol{P} \in \widehat{\boldsymbol{BC}}$, as discussed at the beginning of the proof.

<u>Part 2.</u> *Construction for the case* $\boldsymbol{C}_1 = \boldsymbol{C}^+$.

In this case, for any $\boldsymbol{D} \in \boldsymbol{BC}$ we set its image as the point $\boldsymbol{D'} \in \boldsymbol{BC}_1$ for which $\ell(\boldsymbol{BD}) = \ell(\boldsymbol{BD'})$. Then in particular all the segments $\boldsymbol{DD'}$ are parallel to $\boldsymbol{CC}_1$. Let us now check the validity of (i)–(iii), since (iv) is trivially true.

Let us start with (i). Given $\boldsymbol{D} \in \boldsymbol{BC}$, and $\boldsymbol{D'}$ its image, call $\beta = \boldsymbol{A}\widehat{\boldsymbol{B}}\boldsymbol{C} \in (0, \pi/2]$. Then one has

$$\boldsymbol{D}\widehat{\boldsymbol{D'}}\boldsymbol{B} = \boldsymbol{D'}\widehat{\boldsymbol{D}}\boldsymbol{B} = \frac{\pi-\beta}{2}\,, \qquad\qquad \boldsymbol{D}\widehat{\boldsymbol{D'}}\boldsymbol{A} = \boldsymbol{D'}\widehat{\boldsymbol{D}}\boldsymbol{C} = \frac{\pi+\beta}{2}\,,$$

thus (i) holds true.

Let us now consider (ii). Given a point $\boldsymbol{D} \in \boldsymbol{BC}$, by construction one has

$$\ell\big(\boldsymbol{DD'}\big) \le \ell\big(\boldsymbol{CC}_1\big) \le \ell\big(\boldsymbol{AC}\big) \le \ell\big(\widehat{\boldsymbol{AC}}\big)\,. \tag{5.4}$$

We can then consider separately two cases. If $\boldsymbol{BC} \subseteq \partial\Gamma$, then one simply has $\boldsymbol{P} \equiv \boldsymbol{D}$ and $\boldsymbol{P}_N \equiv \boldsymbol{P}_1 \equiv \boldsymbol{D'}$, so clearly

$$\ell\big(\widehat{\boldsymbol{PP}_N}\big) = \ell\big(\boldsymbol{DD'}\big) \le \ell\big(\widehat{\boldsymbol{AC}}\big) \le \ell\big(\widehat{\boldsymbol{AB}}\big)\,.$$

On the other hand, if the open segment $\boldsymbol{BC}$ lies in the interior of Γ, then one has

$$\ell\big(\widehat{\boldsymbol{PP}_{N-1}}\big) \le 4\ell\big(\widehat{\boldsymbol{BC}}\big) \tag{5.5}$$

by inductive assumption, thus (5.4) and (5.5) give

$$\ell\big(\widehat{\boldsymbol{PP}_N}\big) = \ell\big(\widehat{\boldsymbol{PP}_{N-1}}\big) + \ell\big(\boldsymbol{DD'}\big) \le 4\ell\big(\widehat{\boldsymbol{BC}}\big) + \ell\big(\widehat{\boldsymbol{AC}}\big) \le 4\ell\big(\widehat{\boldsymbol{AB}}\big)\,,$$

hence also (ii) is done.

It remains now to consider (iii). Thus we take two points $\boldsymbol{D} \equiv \boldsymbol{P}_{N-1}$ and $\boldsymbol{E} \equiv \boldsymbol{Q}_{\widetilde{N}-1}$ on $\boldsymbol{BC}$, denoting for brevity $N = N(\boldsymbol{P})$ and $\widetilde{N} = N(\boldsymbol{Q})$. We have to consider separately the two cases arising if $\boldsymbol{BC}$ lies in the boundary or in the interior of Γ. In the first case, $\boldsymbol{P} \equiv \boldsymbol{D}$ and $\boldsymbol{Q} \equiv \boldsymbol{E}$, thus by the Lipschitz property of u we have

$$\frac{\ell\big(\widehat{PQ}\big)}{L} \leq \ell\big(\widehat{\boldsymbol{PQ}}\big) = \ell\big(\boldsymbol{DE}\big) = \ell\big(\boldsymbol{D'E'}\big)\,,$$

so that (iii) is trivially true. In the second case, $\ell\big(\boldsymbol{D'E'}\big) = \ell\big(\boldsymbol{DE}\big)$, so (iii) is true by inductive assumption.

To conclude the proof, we now have to see what happens when $\boldsymbol{C}_1 \neq \boldsymbol{C}^+$. We will further subdivide this last case depending on whether $\beta > \pi/12 = 15°$ or not, being $\beta = \boldsymbol{A}\widehat{\boldsymbol{B}}\boldsymbol{C}$.

<u>Part 3.</u> *Construction for the case $\boldsymbol{C}_1 \neq \boldsymbol{C}^+$, $\beta \geq \pi/12$.*

In this case, for any $\boldsymbol{D} \in \boldsymbol{BC}$ we define $\boldsymbol{D'} \in \boldsymbol{BC}_1$ as the point satisfying

$$\ell\big(\boldsymbol{BD'}\big) = \min\left\{\ell\big(\boldsymbol{BD}\big)\,,\,\ell\big(\boldsymbol{BC}_1\big) - \frac{\ell\big(\widehat{PC}\big)}{7L}\right\}, \tag{5.6}$$

being as usual $\boldsymbol{P} \in \widehat{\boldsymbol{BC}}$ the point such that $\boldsymbol{D} = \boldsymbol{P}_{N-1}$. Observe that this definition makes sense since, also using (5.3), one has that the minimum in (5.6) is between 0 and $\ell\big(\boldsymbol{BC}_1\big)$ for each $\boldsymbol{D} \in \boldsymbol{BC}$. In particular, the minimum is strictly increasing between 0 and $\ell\big(\boldsymbol{BC}_1\big)$ as soon as $\boldsymbol{D}$ moves from $\boldsymbol{B}$ to $\boldsymbol{C}$, so (iv) is already checked. Let us then check the validity of (i)–(iii).

We first concentrate on (i). Just for a moment, let us call $\boldsymbol{D^*} \in \boldsymbol{BC}^+$ the point for which $\ell\big(\boldsymbol{BD}\big) = \ell\big(\boldsymbol{BD^*}\big)$, so that the triangle $\boldsymbol{BDD^*}$ is isosceles. Therefore, one immediately has

$$\boldsymbol{D}\widehat{\boldsymbol{D'}}\boldsymbol{B} \geq \boldsymbol{D}\widehat{\boldsymbol{D^*}}\boldsymbol{B} = \frac{\pi-\beta}{2} \geq \frac{\pi}{4}\,, \qquad \boldsymbol{D'}\widehat{\boldsymbol{D}}\boldsymbol{C} \geq \boldsymbol{D^*}\widehat{\boldsymbol{D}}\boldsymbol{C} = \frac{\pi+\beta}{2} \geq \frac{\pi}{2}\,. \tag{5.7}$$

Moreover, by construction it is clear that

$$\boldsymbol{D}\widehat{\boldsymbol{D'}}\boldsymbol{A} \geq \boldsymbol{D}\widehat{\boldsymbol{B}}\boldsymbol{A} = \beta \geq \frac{\pi}{12}\,. \tag{5.8}$$

To conclude, we have to estimate $\boldsymbol{D'}\widehat{\boldsymbol{D}}\boldsymbol{B}$, and we start claiming the bound

$$\ell\big(\boldsymbol{BD'}\big) \geq \frac{\ell\big(\boldsymbol{BD}\big)}{\sqrt{2}L^2}\,. \tag{5.9}$$

In fact, recalling (5.6), either $\ell\big(\boldsymbol{BD'}\big) = \ell\big(\boldsymbol{BD}\big)$, and then (5.9) clearly holds, or otherwise by (5.3) and the Lipschitz property of u

$$\begin{aligned}\ell\big(\boldsymbol{BD'}\big) = \ell\big(\boldsymbol{BC}_1\big) - \frac{\ell\big(\widehat{PC}\big)}{7L} \geq \frac{\ell\big(\widehat{BC}\big)}{\sqrt{2}L} - \frac{\ell\big(\widehat{PC}\big)}{7L} \geq \frac{\ell\big(\widehat{BC}\big) - \ell\big(\widehat{PC}\big)}{\sqrt{2}L} = \frac{\ell\big(\widehat{BP}\big)}{\sqrt{2}L} \geq \frac{\ell\big(\widehat{\boldsymbol{BP}}\big)}{\sqrt{2}L^2}\\ \geq \frac{\ell\big(\boldsymbol{BD}\big)}{\sqrt{2}L^2}\,,\end{aligned}$$

thus again (5.9) is checked. Concerning the last inequality, namely $\ell\big(\widehat{\boldsymbol{BP}}\big) \geq \ell\big(\boldsymbol{BD}\big)$, this is an equality if the segment $\boldsymbol{BC}$ belongs to $\partial\Gamma$, while otherwise it is true by inductive assumption on the sector $\mathcal{S}(\widehat{\boldsymbol{BC}})$, applying (iii) to the points $\boldsymbol{P}$ and $\boldsymbol{Q} \equiv \boldsymbol{B}$. Consider now the triangle $\boldsymbol{DBD'}$: immediate trigonometric arguments tell us that

$$\ell\big(\boldsymbol{DD'}\big)\sin\big(\boldsymbol{D'}\widehat{\boldsymbol{D}}\boldsymbol{B}\big) = \ell\big(\boldsymbol{BD'}\big)\sin\beta\,, \quad \ell\big(\boldsymbol{BD}\big)\sin\beta = \ell\big(\boldsymbol{DD'}\big)\sin\Big(\boldsymbol{D'}\widehat{\boldsymbol{D}}\boldsymbol{B} + \beta\Big)\,,$$

from which we get, using also (5.9),

$$\sin\big(\boldsymbol{D'}\widehat{\boldsymbol{D}}\boldsymbol{B}\big) = \frac{\ell\big(\boldsymbol{BD'}\big)}{\ell\big(\boldsymbol{BD}\big)}\sin\Big(\boldsymbol{D'}\widehat{\boldsymbol{D}}\boldsymbol{B} + \beta\Big) \geq \frac{\sin(\pi/12)}{\sqrt{2}L^2} \geq \frac{1}{6L^2}\,. \tag{5.10}$$

Putting together (5.7), (5.8) and (5.10), we conclude the inspection of (i).

Concerning (ii), it is enough to observe that

$$\frac{\ell\big(\boldsymbol{DD'}\big)}{\ell\big(\widehat{\boldsymbol{AC}}\big)} \leq \frac{\ell\big(\boldsymbol{DD'}\big)}{\ell\big(\boldsymbol{AC}\big)} = \frac{\sin\big(\boldsymbol{C}\widehat{\boldsymbol{A}}\boldsymbol{B}\big)}{\sin\big(\boldsymbol{D}\widehat{\boldsymbol{D'}}\boldsymbol{A}\big)} \leq \frac{1}{\sin(15^\circ)} \leq 4\,. \tag{5.11}$$

Therefore, as in Part 2, either $\boldsymbol{BC} \subseteq \partial\Gamma$, and then

$$\ell\big(\widehat{\boldsymbol{PP}_N}\big) = \ell\big(\boldsymbol{DD'}\big) \leq 4\ell\big(\widehat{\boldsymbol{AC}}\big) \leq 4\ell\big(\widehat{\boldsymbol{AB}}\big)\,,$$

or thanks to the inductive assumption one has

$$\ell\big(\widehat{\boldsymbol{PP}_N}\big) = \ell\big(\widehat{\boldsymbol{PP}_{N-1}}\big) + \ell\big(\boldsymbol{DD'}\big) \leq 4\ell\big(\widehat{\boldsymbol{BC}}\big) + 4\ell\big(\widehat{\boldsymbol{AC}}\big) = 4\ell\big(\widehat{\boldsymbol{AB}}\big)\,,$$

so (ii) is again easily checked.

Let us now consider (iii). As in Part 2, we take on $\boldsymbol{BC}$ two points $\boldsymbol{D} \equiv \boldsymbol{P}_{N-1}$ and $\boldsymbol{E} \equiv \boldsymbol{Q}_{\widetilde{N}-1}$ with $N = N(\boldsymbol{P})$ and $\widetilde{N} = N(\boldsymbol{Q})$, and we assume by symmetry that $\ell\big(\boldsymbol{BD}\big) \leq \ell\big(\boldsymbol{BE}\big)$. Since it is surely $\ell\big(\boldsymbol{DE}\big) \leq \ell\big(\widehat{\boldsymbol{PQ}}\big)$, either as a trivial equality if $\boldsymbol{BC} \subseteq \partial\Gamma$, or by inductive assumption otherwise, showing (iii) consists in proving that

$$\frac{\ell\big(\widehat{PQ}\big)}{7L} \leq \ell\big(\boldsymbol{D'E'}\big) \leq \ell\big(\boldsymbol{DE}\big)\,. \tag{5.12}$$

We start with the right inequality. Recalling the definition (5.6), if $\ell\big(\boldsymbol{BD'}\big) = \ell\big(\boldsymbol{BD}\big)$ then, since $\ell\big(\boldsymbol{BE'}\big) \leq \ell\big(\boldsymbol{BE}\big)$, one has

$$\ell\big(\boldsymbol{D'E'}\big) = \ell\big(\boldsymbol{BE'}\big) - \ell\big(\boldsymbol{BD'}\big) \leq \ell\big(\boldsymbol{BE}\big) - \ell\big(\boldsymbol{BD}\big) = \ell\big(\boldsymbol{DE}\big)\,.$$

On the other hand, if

$$\ell\big(\boldsymbol{BD'}\big) = \ell\big(\boldsymbol{BC}_1\big) - \frac{\ell\big(\widehat{PC}\big)}{7L}\,,$$

then we get

$$\ell\big(\boldsymbol{D'E'}\big) = \ell\big(\boldsymbol{BE'}\big) - \ell\big(\boldsymbol{BD'}\big) \leq \left(\ell\big(\boldsymbol{BC}_1\big) - \frac{\ell\big(\widehat{QC}\big)}{7L}\right) - \left(\ell\big(\boldsymbol{BC}_1\big) - \frac{\ell\big(\widehat{PC}\big)}{7L}\right)$$
$$= \frac{\ell\big(\widehat{PQ}\big)}{7L} \leq \ell\big(\boldsymbol{DE}\big)\,,$$

where again the last inequality is true either by the Lipschitz property of u if $\boldsymbol{PQ} = \boldsymbol{DE}$, or by inductive assumption otherwise. Thus, the right inequality in (5.12) is established, and we pass to consider the left one.

Still recalling (5.6), if $\ell\big(\boldsymbol{BE'}\big) = \ell\big(\boldsymbol{BE}\big)$ then

$$\ell\big(\boldsymbol{D'E'}\big) = \ell\big(\boldsymbol{BE'}\big) - \ell\big(\boldsymbol{BD'}\big) \geq \ell\big(\boldsymbol{BE}\big) - \ell\big(\boldsymbol{BD}\big) = \ell\big(\boldsymbol{DE}\big) \geq \frac{\ell\big(\widehat{PQ}\big)}{7L}\,,$$

being again the last equality true either by the Lipschitz property of u or by inductive assumption. Finally, if

$$\ell\big(\boldsymbol{BE'}\big) = \ell\big(\boldsymbol{BC}_1\big) - \frac{\ell\big(\widehat{QC}\big)}{7L}\,,$$

then again we get

$$\ell\big(\boldsymbol{D'E'}\big) = \ell\big(\boldsymbol{BE'}\big) - \ell\big(\boldsymbol{BD'}\big) \geq \left(\ell\big(\boldsymbol{BC}_1\big) - \frac{\ell\big(\widehat{QC}\big)}{7L}\right) - \left(\ell\big(\boldsymbol{BC}_1\big) - \frac{\ell\big(\widehat{PC}\big)}{7L}\right) = \frac{\ell\big(\widehat{PQ}\big)}{7L}\,,$$

so the estimate (5.12) is completely shown and then this part is concluded.

<u>Part 4.</u> *Construction for the case $\boldsymbol{C}_1 \neq \boldsymbol{C}^+$, $\beta < \pi/12$.*

We are now ready to consider the last –and hardest– possible situation, namely when $\boldsymbol{C}_1 \neq \boldsymbol{C}^+$ and the angle β is small. Roughly speaking, the fact that $\boldsymbol{C}_1$ is below $\boldsymbol{C}^+$ tells us that the segment $\boldsymbol{BC}$ has to shrink, in order to fit into $\boldsymbol{BC}_1$. On the other hand, the fact that β can be very small makes it hard to obtain simultaneously the estimate (iii) on the lengths and the (i) on the angles. As in Figure 9, we call $\boldsymbol{H}$ the orthogonal projection of $\boldsymbol{C}$ on $\boldsymbol{AB}$.

Since $\beta < \pi/12$, the point $\boldsymbol{C}^-$ belongs to the segment $\boldsymbol{AB}$, and then we obtain, by a trivial geometrical argument, that

$$\ell\big(\boldsymbol{BC}_1\big) \geq \ell\big(\boldsymbol{BC}^-\big) \geq \ell\big(\boldsymbol{BH}\big) - \ell\big(\boldsymbol{CH}\big) = \ell\big(\boldsymbol{BC}\big)\Big(\cos\beta - \sin\beta\Big) \geq \frac{\sqrt{2}}{2}\,\ell\big(\boldsymbol{BC}\big)\,. \tag{5.13}$$

Let us immediately go into our definition of $\boldsymbol{P}_N$ for every vertex $\boldsymbol{P} \in \widehat{\boldsymbol{BC}}$. First of all, since we need to work with consecutive vertices, let us enumerate all the vertices of $\widehat{\boldsymbol{BC}}$ as $\boldsymbol{P}^0 = \boldsymbol{B}$, $\boldsymbol{P}^1$, $\boldsymbol{P}^2$, ..., $\boldsymbol{P}^M = \boldsymbol{C}$. The simplest idea to define the points $\boldsymbol{P}^i_N$ would be to shrink all the segment $\boldsymbol{BC}$ so to fit into $\boldsymbol{BC}_1$, thus getting, for any pair $\boldsymbol{P}^i$, $\boldsymbol{P}^{i+1}$ of consecutive vertices,

$$\ell\big(\boldsymbol{P}^i_N\boldsymbol{P}^{i+1}_N\big) = \frac{\ell\big(\boldsymbol{BC}_1\big)}{\ell\big(\boldsymbol{BC}\big)}\,\ell\big(\boldsymbol{P}^i_{N-1}\boldsymbol{P}^{i+1}_{N-1}\big)\,.$$

Unfortunately, this does not work, since from the inductive assumption

$$\ell\big(\boldsymbol{P}^i_{N-1}\boldsymbol{P}^{i+1}_{N-1}\big) \geq \frac{1}{7L}\,\ell\big(\widehat{P^iP^{i+1}}\big)$$

one would be led to deduce

$$\ell\big(\boldsymbol{P}^i_N\boldsymbol{P}^{i+1}_N\big) \geq \frac{\ell\big(\boldsymbol{BC}_1\big)}{\ell\big(\boldsymbol{BC}\big)}\,\frac{1}{7L}\,\ell\big(\widehat{P^iP^{i+1}}\big) \geq \frac{\sqrt{2}}{14L}\,\ell\big(\widehat{P^iP^{i+1}}\big)\,,$$

by (5.13), so the induction would not work.

However, our idea to overcome the problem is very simple, that is, among all the pairs $\boldsymbol{P}^i$, $\boldsymbol{P}^{i+1}$ of consecutive vertices we will shrink only those which are still "shrinkable", that is, for which the ratio

$$\varrho_i := \frac{\ell\big(\boldsymbol{P}^i_{N-1}\boldsymbol{P}^{i+1}_{N-1}\big)}{\ell\big(\widehat{P^iP^{i+1}}\big)} \tag{5.14}$$

is not already too small, more precisely, not smaller than $1/(3L)$. Let us make this formally. Define

$$\delta := \sum\left\{\ell\big(\boldsymbol{P}^i_{N-1}\boldsymbol{P}^{i+1}_{N-1}\big)\,:\;\varrho_i \leq \frac{1}{3L}\right\}, \tag{5.15}$$

and notice that

$$\ell\big(\widehat{BC}\big) \geq \sum\left\{\ell\big(\widehat{P^iP^{i+1}}\big)\,:\;\varrho_i \leq \frac{1}{3L}\right\} \geq 3L\delta\,,$$

then by (5.1)

$$\delta \leq \frac{\ell\big(\widehat{BC}\big)}{3L} \leq \frac{\sqrt{2}}{3}\,\ell\big(\boldsymbol{BC}\big)\,. \tag{5.16}$$

Finally, we define the points $\boldsymbol{P}^i_N$ in such a way that any segment $\boldsymbol{P}^i_N\boldsymbol{P}^{i+1}_N$ has the same length as $\boldsymbol{P}^i_{N-1}\boldsymbol{P}^{i+1}_{N-1}$ if ϱ_i is small, and otherwise it is rescaled by a factor $\lambda < 1$ (constant through all $\boldsymbol{BC}$). In other words, setting the increasing sequence δ_i as

$$\delta_i := \sum\left\{\ell\big(\boldsymbol{P}^j_{N-1}\boldsymbol{P}^{j+1}_{N-1}\big)\,:\;j<i,\;\varrho_j \leq \frac{1}{3L}\right\}, \tag{5.17}$$

so that comparing with (5.15) one has $\delta_0 = 0$ and $\delta_M = \delta$, we define $\boldsymbol{P}^i_N$ to be the point of $\boldsymbol{BC}_1$ such that

$$\ell\big(\boldsymbol{BP}^i_N\big) = \delta_i + \lambda\Big(\ell\big(\boldsymbol{BP}^i_{N-1}\big) - \delta_i\Big)\,. \tag{5.18}$$

The constant λ is easily estimated by the constraint that $\boldsymbol{P}^M_N = \boldsymbol{C}_1$ and by (5.13) and (5.16), getting

$$1 > \lambda = \frac{\ell\big(\boldsymbol{BC}_1\big) - \delta}{\ell\big(\boldsymbol{BC}\big) - \delta} \geq \frac{\frac{\sqrt{2}}{2}\ell\big(\boldsymbol{BC}\big) - \delta}{\ell\big(\boldsymbol{BC}\big) - \delta} \geq \frac{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{3}}{1 - \frac{\sqrt{2}}{3}} > \frac{3}{7}\,. \tag{5.19}$$

For future reference, it is also useful to notice here another estimate of λ which depends on β, obtained exactly as the one above from (5.13) and (5.16), that is,

$$\lambda = \frac{\ell(\boldsymbol{BC}_1) - \delta}{\ell(\boldsymbol{BC}) - \delta} \geq \frac{\ell(\boldsymbol{BC})(\cos\beta - \sin\beta) - \delta}{\ell(\boldsymbol{BC}) - \delta} \geq \frac{\cos\beta - \sin\beta - \frac{\sqrt{2}}{3}}{1 - \frac{\sqrt{2}}{3}} . \tag{5.20}$$

Notice that by (5.17) and (5.18) one readily gets

$$\ell(\boldsymbol{P}^i_N \boldsymbol{P}^{i+1}_N) = \begin{cases} \ell(\boldsymbol{P}^i_{N-1} \boldsymbol{P}^{i+1}_{N-1}) & \text{if } \varrho_i \leq \dfrac{1}{3L} , \\ \lambda\, \ell(\boldsymbol{P}^i_{N-1} \boldsymbol{P}^{i+1}_{N-1}) & \text{otherwise} . \end{cases} \tag{5.21}$$

Now that we have given the definition of the points $\boldsymbol{P}^i_N$, we only have to check the validity of (i)–(iii), since again (iv) is trivial by definition.

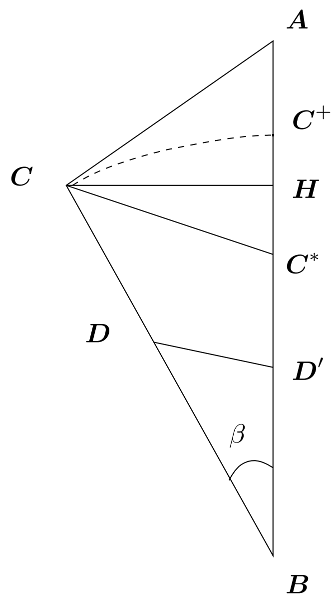

FIGURE 9. The triangle $\boldsymbol{ABC}$ in Part 4

Let us start with (i). Take $0 \leq i \leq M$ and call, as before, $\boldsymbol{D} = \boldsymbol{P}^i_{N-1}$ and $\boldsymbol{D}' = \boldsymbol{P}^i_N$. Since by construction $\ell(\boldsymbol{BD}') \leq \ell(\boldsymbol{BD})$, then one immediately gets $\boldsymbol{D}\widehat{\boldsymbol{D}'}\boldsymbol{B} \geq \boldsymbol{D}'\widehat{\boldsymbol{D}}\boldsymbol{B}$, from which one directly gets

$$\boldsymbol{D}\widehat{\boldsymbol{D}'}\boldsymbol{B} \geq \frac{\pi - \beta}{2} \geq \frac{11}{24}\,\pi , \qquad \boldsymbol{D}'\widehat{\boldsymbol{D}}\boldsymbol{C} = \pi - \boldsymbol{D}'\widehat{\boldsymbol{D}}\boldsymbol{B} \geq \frac{\pi + \beta}{2} \geq \frac{\pi}{2} , \tag{5.22}$$

so that the first two angles are checked and we need to estimate $\boldsymbol{D}'\widehat{\boldsymbol{D}}\boldsymbol{B}$ and $\boldsymbol{D}\widehat{\boldsymbol{D}'}\boldsymbol{A}$. To do so, let us call $\boldsymbol{C}^* \in \boldsymbol{AB}$ the point such that $\ell(\boldsymbol{BC}^*) = \lambda\, \ell(\boldsymbol{BC})$, so that by construction

$$\boldsymbol{D}'\widehat{\boldsymbol{D}}\boldsymbol{B} \geq \boldsymbol{C}^*\widehat{\boldsymbol{C}}\boldsymbol{B} , \qquad \boldsymbol{D}\widehat{\boldsymbol{D}'}\boldsymbol{A} \geq \boldsymbol{C}\widehat{\boldsymbol{C}^*}\boldsymbol{A} . \tag{5.23}$$

The point $\boldsymbol{C}^*$ must lie either between $\boldsymbol{H}$ and $\boldsymbol{C}^+$ or between $\boldsymbol{B}$ and $\boldsymbol{H}$. In the first case also the other two angles are immediately estimated, since then by (5.23) one has

$$\boldsymbol{D'}\widehat{\boldsymbol{D}}\boldsymbol{B} \geq \boldsymbol{C}^*\widehat{\boldsymbol{C}}\boldsymbol{B} \geq \boldsymbol{H}\widehat{\boldsymbol{C}}\boldsymbol{B} = \frac{\pi}{2} - \beta \geq \frac{5}{12}\pi\,, \qquad \boldsymbol{D}\widehat{\boldsymbol{D'}}\boldsymbol{A} \geq \boldsymbol{C}\widehat{\boldsymbol{C}^*}\boldsymbol{A} \geq \frac{\pi}{2}\,. \tag{5.24}$$

Assume then that, as in Figure 9, $\boldsymbol{C}^*$ is between $\boldsymbol{B}$ and $\boldsymbol{H}$. Then we can estimate, also recalling (5.20),

$$\begin{aligned}\ell\big(\boldsymbol{C}^*\boldsymbol{H}\big) &= \ell\big(\boldsymbol{B}\boldsymbol{H}\big) - \ell\big(\boldsymbol{B}\boldsymbol{C}^*\big) = \ell\big(\boldsymbol{B}\boldsymbol{C}\big)\Big(\cos\beta - \lambda\Big)\\ &\leq \ell\big(\boldsymbol{B}\boldsymbol{C}\big)\left(\cos\beta - \frac{\cos\beta - \sin\beta - \frac{\sqrt{2}}{3}}{1-\frac{\sqrt{2}}{3}}\right) = \ell\big(\boldsymbol{B}\boldsymbol{C}\big)\frac{\frac{\sqrt{2}}{3}\,\frac{\sin\beta}{1+\cos\beta}+1}{1-\frac{\sqrt{2}}{3}}\,\sin\beta\,.\end{aligned}$$

As a consequence, we have

$$\begin{aligned}\boldsymbol{H}\widehat{\boldsymbol{C}}\boldsymbol{C}^* &= \arctan\left(\frac{\ell\big(\boldsymbol{C}^*\boldsymbol{H}\big)}{\ell\big(\boldsymbol{C}\boldsymbol{H}\big)}\right) \leq \arctan\left(\frac{\frac{\sqrt{2}}{3}\,\frac{\sin\beta}{1+\cos\beta}+1}{1-\frac{\sqrt{2}}{3}}\right)\\ &\leq \arctan\left(\frac{\frac{\sqrt{2}}{3}\,\frac{\sin 15^\circ}{1+\cos 15^\circ}+1}{1-\frac{\sqrt{2}}{3}}\right) \leq 0.36\,\pi \leq 65^\circ\,.\end{aligned}$$

Finally, from this estimate and (5.23) we get

$$\begin{aligned}&\boldsymbol{D'}\widehat{\boldsymbol{D}}\boldsymbol{B} \geq \boldsymbol{C}^*\widehat{\boldsymbol{C}}\boldsymbol{B} = \frac{\pi}{2} - \beta - \boldsymbol{H}\widehat{\boldsymbol{C}}\boldsymbol{C}^* > \frac{\pi}{18}\,,\\ &\boldsymbol{D}\widehat{\boldsymbol{D'}}\boldsymbol{A} \geq \boldsymbol{C}\widehat{\boldsymbol{C}^*}\boldsymbol{A} = \frac{\pi}{2} - \boldsymbol{H}\widehat{\boldsymbol{C}}\boldsymbol{C}^* \geq 25^\circ\,.\end{aligned} \tag{5.25}$$

Putting together the first two estimates from (5.22), and the last two estimates either from (5.24) or from (5.25), we conclude the proof of (i).

Let us now check (ii). Repeating the argument of Part 3, we have that (ii) follows at once as soon as one shows (5.11), that is, $\ell\big(\boldsymbol{D}\boldsymbol{D'}\big) \leq 4\,\ell\big(\widehat{\boldsymbol{A}\boldsymbol{C}}\big)$. But in fact, using (5.25), we immediately get

$$\ell\big(\boldsymbol{D}\boldsymbol{D'}\big) \leq \frac{\ell\big(\boldsymbol{C}\boldsymbol{H}\big)}{\sin\big(\boldsymbol{D}\widehat{\boldsymbol{D'}}\boldsymbol{A}\big)} \leq \frac{\ell\big(\boldsymbol{A}\boldsymbol{C}\big)}{\sin\big(\boldsymbol{D}\widehat{\boldsymbol{D'}}\boldsymbol{A}\big)} \leq \frac{\ell\big(\widehat{\boldsymbol{A}\boldsymbol{C}}\big)}{\sin\big(25^\circ\big)} < 2.5\,\ell\big(\widehat{\boldsymbol{A}\boldsymbol{C}}\big) \leq 4\,\ell\big(\widehat{\boldsymbol{A}\boldsymbol{C}}\big)\,.$$

Let us then consider (iii). It is of course sufficient to check the validity of the inequality only when $\boldsymbol{P}$ and $\boldsymbol{Q}$ are two consecutive vertices of $\widehat{\boldsymbol{B}\boldsymbol{C}}$. Let us then take $0 \leq i < M$ and recall that we have to show

$$\frac{\ell\big(\widehat{P^iP^{i+1}}\big)}{7L} \leq \ell\big(\boldsymbol{P}^i_N\boldsymbol{P}^{i+1}_N\big) \leq \ell\big(\boldsymbol{P}^i_{N-1}\boldsymbol{P}^{i+1}_{N-1}\big) \tag{5.26}$$

knowing, again either by inductive assumption or by the Lipschitz property,

$$\frac{\ell\big(\widehat{P^iP^{i+1}}\big)}{7L} \leq \ell\big(\boldsymbol{P}^i_{N-1}\boldsymbol{P}^{i+1}_{N-1}\big) \leq \ell\big(\widehat{\boldsymbol{P}^i\boldsymbol{P}^{i+1}}\big)\,. \tag{5.27}$$

The right inequality in (5.26) is an immediate consequence of (5.21), being $\lambda < 1$. Concerning the left inequality, it is also quick to check, distinguishing whether ϱ_i is small or not. In fact, if $\varrho_i \leq 1/(3L)$, then by (5.21) also the left inequality in (5.26) derives from the analogous inequality in (5.27). Otherwise, if $\varrho_i > 1/(3L)$, then one directly has by (5.21), (5.14) and (5.19) that

$$\ell\big(\boldsymbol{P}^i_N \boldsymbol{P}^{i+1}_N\big) = \lambda\, \ell\big(\boldsymbol{P}^i_{N-1} \boldsymbol{P}^{i+1}_{N-1}\big) = \lambda \varrho_i\, \ell\big(\widehat{P^i P^{i+1}}\big) > \frac{1}{3L}\, \lambda\, \ell\big(\widehat{P^i P^{i+1}}\big) > \frac{1}{7L}\, \ell\big(\widehat{P^i P^{i+1}}\big)\,,$$

thus concluding the proof. □

5.7. Lenght of paths inside a sector

In Section 5.6, we have described how to get a piecewise affine path $\boldsymbol{P}\boldsymbol{P}_1\boldsymbol{P}_2\cdots\boldsymbol{P}_N$ which starts from a given point $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$ and ends on the segment $\boldsymbol{AB}$, being $\mathcal{S}(\boldsymbol{AB})$ a given sector. In this step, we want to improve the estimate from above of the length of this path. This is important because this path will be (up to a small correction in the future) part of the image of the segment $PO \subseteq \mathcal{D}$ under the extension $\tilde{v}$ of $\tilde{u}$ that we are building, and then its length gives a lower bound to the Lipschitz constant of the map $\tilde{v}$. After a short definition, we will state the main result of this step.

Definition 5.19. *Let $\mathcal{S}(\boldsymbol{AB})$ be a given sector, $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$ and let $\boldsymbol{P}\boldsymbol{P}_1\boldsymbol{P}_2\cdots\boldsymbol{P}_N$ be the piecewise affine path given by Lemma 5.18. We will then denote this piecewise affine path as $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$. More in general, for any $1 \leq i < j \leq N$, we will denote by $\widehat{\boldsymbol{P}_i\boldsymbol{P}_j}$ the piecewise affine path $\boldsymbol{P}_i\boldsymbol{P}_{i+1}\cdots\boldsymbol{P}_j$.*

Lemma 5.20. *Let $\mathcal{S}(\boldsymbol{AB})$ be a sector. Then, for any $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$ one has*

$$\ell\big(\widehat{\boldsymbol{P}\boldsymbol{P}_N}\big) \leq 113\, \min\Big\{\ell\big(\widehat{\boldsymbol{AP}}\big),\, \ell\big(\widehat{\boldsymbol{PB}}\big)\Big\}\,.$$

Let us fix a generic point $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$. The proof of the lemma will require a detailed analysis of the different triangles of the natural sequence of triangles related to $\boldsymbol{P}$. Recall that the natural sequence of triangles, according with Definition 5.16, is the sequence $\big(\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_N\big)$ such that every $\boldsymbol{P}_i$ of the path $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$ belongs to the exit side of $\mathscr{T}_i$. Let us start by calling for simplicity $\boldsymbol{A}_i\boldsymbol{B}_i$ the exit side of the triangle $\mathscr{T}_i$, being $\boldsymbol{A}_i \in \widehat{\boldsymbol{AP}}$ and $\boldsymbol{B}_i \in \widehat{\boldsymbol{PB}}$, so that in particular $\boldsymbol{A}_N = \boldsymbol{A}$ and $\boldsymbol{B}_N = \boldsymbol{B}$. Moreover, we call $\boldsymbol{A}_0\boldsymbol{B}_0$ the side of $\mathscr{T}_1$ which contains $\boldsymbol{P} = \boldsymbol{P}_0$. Notice that, by the construction of the triangles done in Section 5.5, for any i the exit side of the triangle $\mathscr{T}_i$ is a side of the triangle $\mathscr{T}_{i+1}$, thus the exit sides of $\mathscr{T}_i$ and $\mathscr{T}_{i+1}$ have exactly one point in common. In other words, either $\boldsymbol{A}_{i+1} = \boldsymbol{A}_i$, or $\boldsymbol{B}_{i+1} = \boldsymbol{B}_i$.

We give now an idea of how our estimate works. Let us assume, by simmetry, that e.g. $\ell(\widehat{\boldsymbol{PB}}) \leq \ell(\widehat{\boldsymbol{AP}})$. Since

$$\ell(\widehat{\boldsymbol{PB}}) = \ell(\widehat{\boldsymbol{PB}_N}) \geq \ell(\boldsymbol{P}_0\boldsymbol{B}_0) + \sum_{i=0}^{N-1} \ell(\boldsymbol{B}_i\boldsymbol{B}_{i+1}) = \ell(\boldsymbol{P}_0\boldsymbol{B}_0) + \ell(\widehat{\boldsymbol{B}_0\boldsymbol{B}_N}),$$

where $\widehat{\boldsymbol{B}_0\boldsymbol{B}_N} = \boldsymbol{B}_0\boldsymbol{B}_1 + \cdots + \boldsymbol{B}_{N-1}\boldsymbol{B}_N$, we will prove that

$$\ell(\widehat{\boldsymbol{P}\boldsymbol{P}_N}) \leq 113\big((\boldsymbol{P}_0\boldsymbol{B}_0) + \ell(\widehat{\boldsymbol{B}_0\boldsymbol{B}_N})\big) = 113\big(\ell(\widehat{\boldsymbol{P}_0\boldsymbol{B}_N})\big). \tag{5.28}$$

On one hand, if $\boldsymbol{B}_{i+1} \neq \boldsymbol{B}_i$ for some triangle $\mathscr{T}_{i+1}$, by property (i) of Lemma 5.18

$$\ell(\boldsymbol{P}_i\boldsymbol{P}_{i+1}) \leq 4\ell(\boldsymbol{B}_i\boldsymbol{B}_{i+1}).$$

Indeed, this is a consequence of the fact that $\boldsymbol{P}_i\widehat{\boldsymbol{P}_{i+1}}\boldsymbol{A}_{i+1} \geq \pi/12$. If instead $\boldsymbol{B}_i = \boldsymbol{B}_{i+1}$, the length of the segment $\boldsymbol{P}_i\boldsymbol{P}_{i+1}$ does not apparently contribute to the increase of the path $\ell(\widehat{\boldsymbol{P}_0\boldsymbol{B}_N})$. However, due to the fact that by construction

$$\ell(\boldsymbol{P}_{i+1}\boldsymbol{B}_{i+1}) \leq \ell(\boldsymbol{P}_i\boldsymbol{B}_i) \quad \text{if} \quad \boldsymbol{B}_i = \boldsymbol{B}_{i+1},$$

the sum of the angles of the form $\boldsymbol{A}_i\widehat{\boldsymbol{B}_i}\boldsymbol{A}_{i+1}$ cannot increase too much unless triangles $\mathscr{T}_i$ for which $\boldsymbol{A}_i\widehat{\boldsymbol{B}_i}\boldsymbol{A}_{i+1} > 0$ (or, equivalently, $\boldsymbol{B}_i = \boldsymbol{B}_{i+1}$) alternate sufficiently often with triangles with $\boldsymbol{B}_i\widehat{\boldsymbol{A}_i}\boldsymbol{B}_{i+1} > 0$ (and then $\boldsymbol{B}_i \neq \boldsymbol{B}_{i+1}$). In other words, the worst case is when $\mathcal{S}(\boldsymbol{AB})$ has a spiral/snake shape. In order to overcome this problem, we will first subdivide the natural sequence of triangles $\big(\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_N\big)$ into sequences of consecutive triangles $\mathscr{U} = \big(\mathscr{T}_i, \mathscr{T}_{i+1}, \ldots, \mathscr{T}_{i+j}\big)$ called "units", then we will group consecutive sequences of "units" into "systems of units" $\mathscr{S} = \big(\mathscr{U}_i, \mathscr{U}_{i+1}, \ldots, \mathscr{U}_{i+j}\big)$ and consecutive sequences of "systems of units" into "blocks of systems" $\mathscr{B} = \big(\mathscr{S}_i, \mathscr{S}_{i+1}, \ldots, \mathscr{S}_{i+j}\big)$. Passing from one category of objects to the one containing it, the spiraling effect increases: however, we will see that at every step this effect can be controlled, leading to the estimate (5.28).

We can now introduce the first category.

DEFINITION 5.21. *Let $0 \leq i \leq j \leq N$ be such that $\{i, i+1, \cdots, j-1, j\}$ is a maximal sequence with the property that $\boldsymbol{B}_l$ is the same point for all $i \leq l \leq j$ (by "maximal" we mean that either $i = 0$ or $\boldsymbol{B}_{i-1} \neq \boldsymbol{B}_i$, as well as either $j = N$ or $\boldsymbol{B}_j \neq \boldsymbol{B}_{j+1}$). We will then say that $\mathscr{U} = \big(\mathscr{T}_{i+1}, \mathscr{T}_2, \ldots, \mathscr{T}_{j+1}\big)$ is a* unit of triangles*, where $i+1$ is substituted by i if $i = 0$, and $j+1$ is substituted by j if $j = N$, and then no unit is defined if $i = j = N > 1$. To any unit we associate two angles, that is,*

$$\theta^+ := \boldsymbol{P}_i\widehat{\boldsymbol{B}_i}\boldsymbol{A}_j\,, \qquad\qquad \theta^- := \boldsymbol{B}_j\widehat{\boldsymbol{A}_j}\boldsymbol{B}_{j+1}\,,$$

while $\theta^+ := \boldsymbol{P}_0\widehat{\boldsymbol{B}_0}\boldsymbol{A}_j$ if $i = 0$, where $P_0 = P$.

The reason for this strange definition with $i+1$ and $j+1$ will be clear later. The meaning of the definition is quite simple: the first unit starts with $\mathscr{T}_1$ and ends with $\mathscr{T}_j$, where j is the smaller index such that $\boldsymbol{B}_j \neq \boldsymbol{B}_1$. The second unit starts with $\mathscr{T}_{j+1}$ and ends with $\mathscr{T}_{j'}$, where j' is the smaller index, possibly $j+1$ itself, for which $\boldsymbol{B}_j \neq \boldsymbol{B}_{j'}$. And so on, until one reaches $\mathscr{T}_N$, and then one stops regardless whether or not $\boldsymbol{B}_N$ is different from $\boldsymbol{B}_{N-1}$. It is immediate from the definition to observe that the sequence of triangles $\big(\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_N\big)$ is the concatenation of units of triangles. To understand how the units work, it can be useful to check the example of Figure 10, where $N = 10$ and the units of triangles are $\big(\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3, \mathscr{T}_4,\big)$, $\big(\mathscr{T}_5\big)$, $\big(\mathscr{T}_6, \mathscr{T}_7, \mathscr{T}_8\big)$, $\big(\mathscr{T}_9\big)$ and $\big(\mathscr{T}_{10}\big)$. Notice also

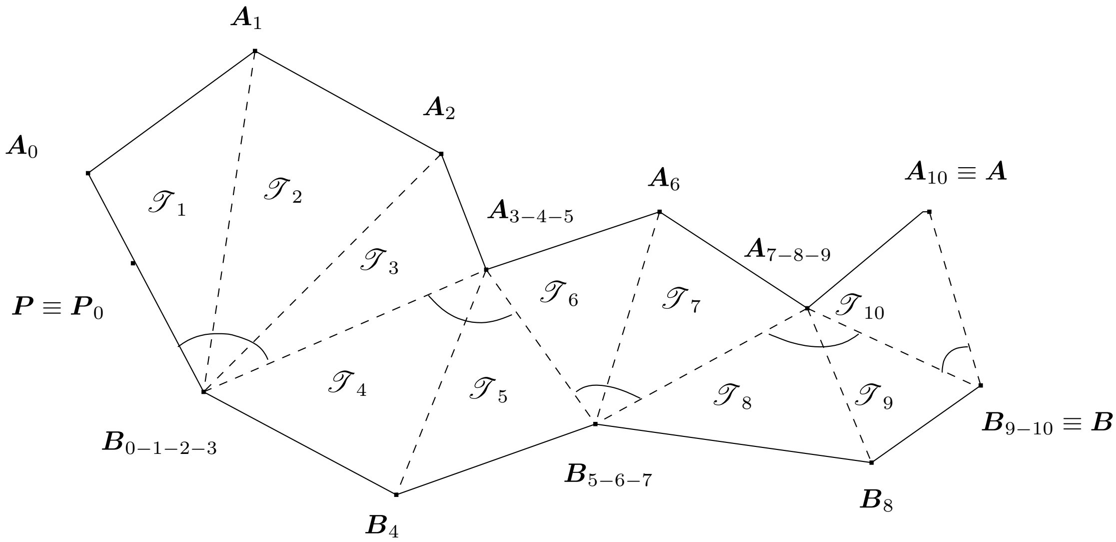

FIGURE 10. A natural sequence of triangles $\mathscr{T}_i$ with the points $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ and the angles $\theta^\pm$

that for any unit of triangles one has $\theta^+ > 0$, unless the unit is made by a single triangle, as $\big(\mathscr{T}_5\big)$ in the figure. Similarly, one has that $\theta^- > 0$, unless $j = N$ and $\boldsymbol{B}_j = \boldsymbol{B}_{j-1}$, as $\big(\mathscr{T}_{10}\big)$ in the figure.

The role of the units is contained in the following result.

LEMMA 5.22. *Let* $\mathscr{U} = \big(\mathscr{T}_i, \mathscr{T}_{i+1}, \dots, \mathscr{T}_j\big)$ *be a unit of triangles. Then one has*

$$\ell\big(\widehat{\boldsymbol{P}_{i-1}\boldsymbol{P}_j}\big) \leq \big(1+\theta^+\big)\,\ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big) - \ell\big(\boldsymbol{P}_j\boldsymbol{B}_j\big) + 5\,\ell\big(\boldsymbol{B}_{i-1}\boldsymbol{B}_j\big)\,, \tag{5.29}$$

$$\ell\big(\boldsymbol{B}_{i-1}\boldsymbol{B}_j\big) \geq \frac{\theta^-}{\pi}\,\ell\big(\boldsymbol{P}_j\boldsymbol{B}_j\big)\,, \tag{5.30}$$

$$\ell\big(\boldsymbol{P}_j\boldsymbol{B}_j\big) \leq \ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big) + \ell\big(\boldsymbol{B}_{i-1}\boldsymbol{B}_j\big)\,. \tag{5.31}$$

PROOF. The proof will follow from simple geometric considerations thanks to Lemma 5.18. To help the reader, the situation is depicted in Figure 11. First of all, one has by definition

$$\ell\big(\widehat{\boldsymbol{P}_{i-1}\boldsymbol{P}_j}\big) = \ell\big(\widehat{\boldsymbol{P}_{i-1}\boldsymbol{P}_{j-1}}\big) + \ell\big(\boldsymbol{P}_{j-1}\boldsymbol{P}_j\big)\,. \tag{5.32}$$

We claim that

$$\ell\big(\widehat{\boldsymbol{P}_{i-1}\boldsymbol{P}_{j-1}}\big) \leq \big(1+\theta^+\big)\,\ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big) - \ell\big(\boldsymbol{P}_{j-1}\boldsymbol{B}_{i-1}\big)\,. \tag{5.33}$$

In fact, if $i = j$ then $\ell\big(\widehat{\boldsymbol{P}_{i-1}\boldsymbol{P}_{j-1}}\big) = 0$ and then (5.33) is trivially true. Otherwise, let us consider the triangle $\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\boldsymbol{P}_i$. Thanks to property (iii) in Lemma 5.18, one has

$$\ell\big(\boldsymbol{P}_i\boldsymbol{B}_{i-1}\big) \leq \ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big)\,,$$

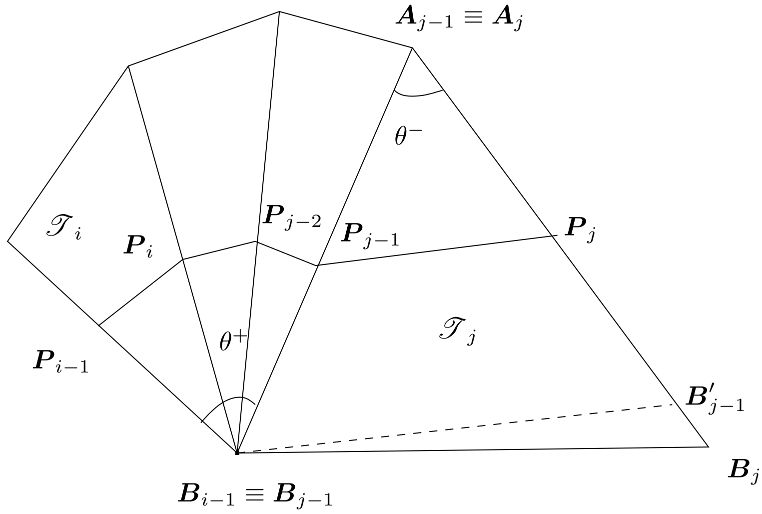

FIGURE 11. Situation in Lemma 5.22

and then an immediate trigonometric argument tells us that

$$\begin{aligned}\ell\big(\boldsymbol{P}_{i-1}\boldsymbol{P}_i\big) &\leq 2\ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big)\sin\left(\frac{\boldsymbol{P}_{i-1}\widehat{\boldsymbol{B}_{i-1}}\boldsymbol{P}_i}{2}\right) + \ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big) - \ell\big(\boldsymbol{P}_i\boldsymbol{B}_{i-1}\big) \\ &\leq \ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big)\cdot \boldsymbol{P}_{i-1}\widehat{\boldsymbol{B}_{i-1}}\boldsymbol{P}_i \; + \ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big) - \ell\big(\boldsymbol{P}_i\boldsymbol{B}_{i-1}\big)\,.\end{aligned}$$

We can repeat the same argument more in general. In fact, for any $i \leq l \leq j-1$ one has from Lemma 5.18 that

$$\ell\big(\boldsymbol{P}_l\boldsymbol{B}_{i-1}\big) \leq \ell\big(\boldsymbol{P}_{l-1}\boldsymbol{B}_{i-1}\big) \leq \cdots \leq \ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big)\,, \tag{5.34}$$

hence the previous trigonometric argument implies

$$\ell\big(\boldsymbol{P}_{l-1}\boldsymbol{P}_l\big) \leq \ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big)\cdot \boldsymbol{P}_{l-1}\widehat{\boldsymbol{B}_{i-1}}\boldsymbol{P}_l \; + \ell\big(\boldsymbol{P}_{l-1}\boldsymbol{B}_{i-1}\big) - \ell\big(\boldsymbol{P}_l\boldsymbol{B}_{i-1}\big)\,.$$

Adding this inequality for all $i \leq l \leq j-1$ one gets

$$\begin{aligned}\ell\big(\widehat{\boldsymbol{P}_{i-1}\boldsymbol{P}_{j-1}}\big) &= \sum_{l=i}^{j-1}\ell\big(\boldsymbol{P}_{l-1}\boldsymbol{P}_l\big) \\ &\leq \sum_{l=i}^{j-1}\ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big)\cdot \boldsymbol{P}_{l-1}\widehat{\boldsymbol{B}_{i-1}}\boldsymbol{P}_l \; + \ell\big(\boldsymbol{P}_{l-1}\boldsymbol{B}_{i-1}\big) - \ell\big(\boldsymbol{P}_l\boldsymbol{B}_{i-1}\big) \\ &= \theta^+\ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big) + \ell\big(\boldsymbol{P}_{i-1}\boldsymbol{B}_{i-1}\big) - \ell\big(\boldsymbol{P}_{j-1}\boldsymbol{B}_{i-1}\big)\,,\end{aligned}$$

which is (5.33).

Let us now point our attention to the triangle $\mathscr{T}_j$. First of all, let us call $\boldsymbol{H}$ (resp. $\boldsymbol{B}_\perp$) the orthogonal projection of $\boldsymbol{P}_{j-1}$ (resp. $\boldsymbol{B}_{i-1}$) on the straight line passing through $\boldsymbol{A}_j\boldsymbol{B}_j$ (these two points are not indicated in the figure, for the sake of clarity). Since by (i) of Lemma 5.18 we have $\boldsymbol{P}_{j-1}\widehat{\boldsymbol{P}_j}\boldsymbol{H} \geq 15°$, it is

$$\ell\big(\boldsymbol{P}_{j-1}\boldsymbol{P}_j\big) = \frac{\ell\big(\boldsymbol{P}_{j-1}\boldsymbol{H}\big)}{\sin\Big(\boldsymbol{P}_{j-1}\widehat{\boldsymbol{P}_j}\boldsymbol{H}\Big)} \leq \frac{1}{\sin 15°}\,\ell\big(\boldsymbol{P}_{j-1}\boldsymbol{H}\big) \leq 4\,\ell\big(\boldsymbol{P}_{j-1}\boldsymbol{H}\big)\,, \tag{5.35}$$

and similarly

$$\begin{aligned}\ell\big(\boldsymbol{B}_{i-1}\boldsymbol{B}_j\big) &\geq \ell\big(\boldsymbol{B}_{i-1}\boldsymbol{B}_\perp\big) = \ell\big(\boldsymbol{A}_{j-1}\boldsymbol{B}_{i-1}\big)\sin\theta^- \geq \ell\big(\boldsymbol{P}_{j-1}\boldsymbol{B}_{i-1}\big)\sin\theta^- \\ &\geq \frac{2\theta^-}{\pi}\,\ell\big(\boldsymbol{P}_{j-1}\boldsymbol{B}_{i-1}\big)\,,\end{aligned} \tag{5.36}$$

recalling the by definition of the triangles of the sectors one has $\theta^- \leq \pi/2$. Moreover, since $\boldsymbol{P}_{j-1} \in \boldsymbol{A}_{j-1}\boldsymbol{B}_{i-1}$, then clearly $\ell\big(\boldsymbol{P}_{j-1}\boldsymbol{H}\big) \leq \ell\big(\boldsymbol{B}_{i-1}\boldsymbol{B}_\perp\big)$, so (5.35) and (5.36) imply

$$\ell\big(\boldsymbol{P}_{j-1}\boldsymbol{P}_j\big) \leq 4\,\ell\big(\boldsymbol{B}_{i-1}\boldsymbol{B}_j\big)\,. \tag{5.37}$$

Let us now call, as in the figure, $\boldsymbol{B}'_{j-1}$ the first point of the piecewise affine path which starts from $\boldsymbol{B}_{j-1}$ and arrives to $\boldsymbol{AB}$ according to Lemma 5.18 –with the notation of Lemma 5.18 we should have called that point $(\boldsymbol{B}_{j-1})_1$. Applying twice condition (iii) of Lemma 5.18 we get

$$\ell\big(\boldsymbol{P}_j\boldsymbol{B}_j\big) = \ell\big(\boldsymbol{P}_j\boldsymbol{B}'_{j-1}\big) + \ell\big(\boldsymbol{B}'_{j-1}\boldsymbol{B}_j\big) \leq \ell\big(\boldsymbol{P}_{j-1}\boldsymbol{B}_{i-1}\big) + \ell\big(\boldsymbol{B}_{i-1}\boldsymbol{B}_j\big)\,.$$

This inequality allows us to conclude. Indeed, together with (5.32), (5.33) and (5.37) it concludes the proof of (5.29). Then, together with (5.34), it yields (5.31). And finally, together with (5.36), it gives (5.30) since

$$\begin{aligned}2\ell\big(\boldsymbol{B}_{i-1}\boldsymbol{B}_j\big) &\geq \frac{2\theta^-}{\pi}\,\ell\big(\boldsymbol{B}_{i-1}\boldsymbol{B}_j\big) + \ell\big(\boldsymbol{B}_{i-1}\boldsymbol{B}_j\big) \\ &\geq \frac{2\theta^-}{\pi}\Big(\ell\big(\boldsymbol{P}_j\boldsymbol{B}_j\big) - \ell\big(\boldsymbol{P}_{j-1}\boldsymbol{B}_{j-1}\big)\Big) + \frac{2\theta^-}{\pi}\,\ell\big(\boldsymbol{P}_{j-1}\boldsymbol{B}_{j-1}\big) = \frac{2\theta^-}{\pi}\,\ell\big(\boldsymbol{P}_j\boldsymbol{B}_j\big)\,.\end{aligned}$$

□

After this result, we can stop thinking about triangles, and we can start working only with units. In fact, notice that any unit of triangles, say $\mathscr{U} = \big(\mathscr{T}_i, \mathscr{T}_{i+1}, \ldots, \mathscr{T}_j\big)$, starts with the exit side of $\mathscr{T}_{i-1}$ and finishes with the exit side of $\mathscr{T}_j$ and that the estimates (5.29), (5.30) and (5.31) are already written only in terms of points of those sides. Let us then number the units as $\mathscr{U}_1, \mathscr{U}_2, \ldots, \mathscr{U}_M$, with $M \leq N$, and let us define i_l and j_l, for $1 \leq l \leq M$, in such a way that $\mathscr{U}_l = \big(\mathscr{T}_{i_l}, \mathscr{T}_{i_l+1}, \ldots, \mathscr{T}_{j_l}\big)$. Notice that $i_1 = 1$, $j_M = N$, and $j_l + 1 = i_{l+1}$ for each $1 \leq l < M$. Let us give the following definitions,

$$\boldsymbol{Q}_l := \boldsymbol{P}_{j_l}\,,\quad \boldsymbol{C}_l := \boldsymbol{A}_{j_l}\,,\quad \boldsymbol{D}_l := \boldsymbol{B}_{j_l}\,,\quad \boldsymbol{Q}_0 := \boldsymbol{P}_0 = \boldsymbol{P}\,,\quad \boldsymbol{D}_0 := \boldsymbol{B}_0 = \boldsymbol{B}_1\,, \tag{5.38}$$

where the last two definitions are done to be consistent. Call also $\theta_l^\pm$ the angles $\theta^\pm$ related to the unit $\mathscr{U}_l$. Hence, the claim of Lemma 5.22 can be rewritten as

$$\ell\big(\widehat{\boldsymbol{Q}_{l-1}\boldsymbol{Q}_l}\big) \leq \big(1+\theta_l^+\big)\,\ell\big(\boldsymbol{Q}_{l-1}\boldsymbol{D}_{l-1}\big) - \ell\big(\boldsymbol{Q}_l\boldsymbol{D}_l\big) + 5\,\ell\big(\boldsymbol{D}_{l-1}\boldsymbol{D}_l\big)\,, \tag{5.29'}$$

$$\ell\big(\boldsymbol{D}_{l-1}\boldsymbol{D}_l\big) \geq \frac{\theta_l^-}{\pi}\,\ell\big(\boldsymbol{Q}_l\boldsymbol{D}_l\big)\,, \tag{5.30'}$$

$$\ell\big(\boldsymbol{Q}_l\boldsymbol{D}_l\big) \leq \ell\big(\boldsymbol{Q}_{l-1}\boldsymbol{D}_{l-1}\big) + \ell\big(\boldsymbol{D}_{l-1}\boldsymbol{D}_l\big)\,. \tag{5.31'}$$

Before passing to the definition of "systems" of units, and in order to help understanding its meaning, it can be useful to give a proof of Lemma 5.20 in a very peculiar case.

LEMMA 5.23. *The claim of Lemma 5.20 holds true if*

$$\ell\big(\widehat{\boldsymbol{D}_0\boldsymbol{D}_{M-1}}\big) \leq \frac{\ell\big(\boldsymbol{Q}_0\boldsymbol{D}_0\big)}{4}\,, \tag{5.39}$$

$$\ell\big(\boldsymbol{Q}_l\boldsymbol{D}_l\big) \geq \frac{\ell\big(\boldsymbol{Q}_0\boldsymbol{D}_0\big)}{2} \qquad \forall\, 1 \leq l \leq M-1\,. \tag{5.40}$$

PROOF. First of all notice that, by the two assumptions and an easy geometrical argument (recalling that all the triangles $\mathscr{T}_i$ are disjoint, hence in particular the segments $\boldsymbol{Q}_l\boldsymbol{D}_l$ cannot intersect), one finds that

$$\sum_{l=1}^{M}\theta_l^+ - \sum_{l=1}^{M-1}\theta_l^- \leq \frac{13}{6}\,\pi\,. \tag{5.41}$$

Moreover, by (5.31') and (5.39), one gets

$$\ell\big(\boldsymbol{Q}_l\boldsymbol{D}_l\big) \leq \frac{5}{4}\,\ell\big(\boldsymbol{Q}_0\boldsymbol{D}_0\big) \qquad \forall\, 0 \leq l \leq M-1\,. \tag{5.42}$$

We can now evaluate, using (5.29'), (5.42), (5.41), (5.40) and (5.30'),

$$\begin{aligned}
\ell\big(\widehat{\boldsymbol{Q}_0\boldsymbol{Q}_M}\big) &= \sum_{l=1}^{M}\ell\big(\widehat{\boldsymbol{Q}_{l-1}\boldsymbol{Q}_l}\big) \leq \sum_{l=1}^{M}\big(1+\theta_l^+\big)\,\ell\big(\boldsymbol{Q}_{l-1}\boldsymbol{D}_{l-1}\big) - \ell\big(\boldsymbol{Q}_l\boldsymbol{D}_l\big) + 5\,\ell\big(\boldsymbol{D}_{l-1}\boldsymbol{D}_l\big)\\
&\leq \frac{5}{4}\,\ell\big(\boldsymbol{Q}_0\boldsymbol{D}_0\big)\sum_{l=1}^{M}\theta_l^+ + \ell\big(\boldsymbol{Q}_0\boldsymbol{D}_0\big) - \ell\big(\boldsymbol{Q}_M\boldsymbol{D}_M\big) + 5\ell\big(\widehat{\boldsymbol{D}_0\boldsymbol{D}_M}\big)\\
&\leq \ell\big(\boldsymbol{Q}_0\boldsymbol{D}_0\big)\bigg(1+\frac{65}{24}\,\pi\bigg) + \frac{5}{4}\,\ell\big(\boldsymbol{Q}_0\boldsymbol{D}_0\big)\sum_{l=1}^{M-1}\theta_l^- + 5\ell\big(\widehat{\boldsymbol{D}_0\boldsymbol{D}_M}\big)\\
&\leq 10\,\ell\big(\boldsymbol{Q}_0\boldsymbol{D}_0\big) + \frac{5}{2}\sum_{l=1}^{M-1}\theta_l^-\,\ell\big(\boldsymbol{Q}_l\boldsymbol{D}_l\big) + 5\ell\big(\widehat{\boldsymbol{D}_0\boldsymbol{D}_M}\big)\\
&\leq 10\,\ell\big(\boldsymbol{Q}_0\boldsymbol{D}_0\big) + \frac{5}{2}\,\pi\sum_{l=1}^{M-1}\ell\big(\boldsymbol{D}_{l-1}\boldsymbol{D}_l\big) + 5\ell\big(\widehat{\boldsymbol{D}_0\boldsymbol{D}_M}\big)\\
&\leq 10\,\ell\big(\boldsymbol{Q}_0\boldsymbol{D}_0\big) + \big(5+\frac{5}{2}\,\pi\big)\ell\big(\widehat{\boldsymbol{D}_0\boldsymbol{D}_M}\big) \leq 10\,\ell\big(\boldsymbol{Q}_0\boldsymbol{D}_0\big) + 13\ell\big(\widehat{\boldsymbol{D}_0\boldsymbol{D}_M}\big)\,.
\end{aligned} \tag{5.43}$$

Finally, recall that

$$\ell\big(\widehat{\boldsymbol{P}\boldsymbol{B}}\big) = \ell\big(\widehat{\boldsymbol{P}\boldsymbol{B}_0}\big) + \ell\big(\widehat{\boldsymbol{B}_0\boldsymbol{B}}\big) \geq \ell\big(\boldsymbol{P}\boldsymbol{B}_0\big) + \ell\big(\widehat{\boldsymbol{B}_0\boldsymbol{B}_N}\big) = \ell\big(\boldsymbol{Q}_0\boldsymbol{D}_0\big) + \ell\big(\widehat{\boldsymbol{D}_0\boldsymbol{D}_M}\big)\,,$$

hence from (5.43) we directly get $\ell\big(\widehat{\boldsymbol{P}\boldsymbol{P}_N}\big) = \ell\big(\widehat{\boldsymbol{Q}_0\boldsymbol{Q}_M}\big) \leq 13\ell\big(\widehat{\boldsymbol{P}\boldsymbol{B}}\big)$. Since it is admissible to assume, by symmetry, that $\ell\big(\widehat{\boldsymbol{P}\boldsymbol{B}}\big) \leq \ell\big(\widehat{\boldsymbol{A}\boldsymbol{P}}\big)$, we conclude the proof of Lemma 5.20 under the assumptions (5.39) and (5.40). □

It is to be noticed carefully that the key point in the above proof is the validity of (5.41), which is a simple consequence of (5.39) and (5.40), but which one cannot hope to have in general. Basically, (5.41) fails whenever the sector $\mathcal{S}(\boldsymbol{AB})$ has a spiral shape, and in fact (5.39) and (5.40) precisely prevent the sector to be an enlarging and a shrinking spiral respectively.

Since the assumptions (5.39) and (5.40) do not hold, in general, through all the units, we will group the units in "systems" in which they are valid.

DEFINITION 5.24. *Let $k_0 = 0$. We define recursively the increasing finite sequence $\{k_1, \cdots, k_W\}$ as follows. For each $j \geq 0$, if $k_j = M$ then we conclude the construction (and thus $W = j$), while otherwise we define $k_j < k_{j+1} \leq M$ to be the biggest number such that*

$$\ell\big(\widehat{\boldsymbol{D}_{k_j}\boldsymbol{D}_{k_{j+1}-1}}\big) \leq \frac{\ell\big(\boldsymbol{Q}_{k_j}\boldsymbol{D}_{k_j}\big)}{4}\,, \tag{5.39'}$$

$$\ell\big(\boldsymbol{Q}_l\boldsymbol{D}_l\big) \geq \frac{\ell\big(\boldsymbol{Q}_{k_j}\boldsymbol{D}_{k_j}\big)}{2} \qquad \forall\, k_j < l < k_{j+1}\,. \tag{5.40'}$$

Notice that the sequence is well-defined, since if $k_j < M$ then the assumptions (5.39') and (5.40') trivially hold with $k_{j+1} = k_j + 1$. Hence, $W \leq M \leq N$. We define then system of units *each collection of units of the form $\mathscr{S}_j = \big(\mathscr{U}_{k_{j-1}+1}, \mathscr{U}_{k_{j-1}+2}, \ldots, \mathscr{U}_{k_j}\big)$, for $1 \leq j \leq W$.*

Thanks to this definition, we can rephrase the claim of Lemma 5.23 as follows: "the claim of Lemma 5.20 holds true if there is only one system of units". But in fact, the argument of Lemma 5.23 still gives some useful information for each different system, as we will see in a moment with Lemma 5.25. Before doing so, in order to avoid too many indices, it is convenient to introduce some new notation in order to work only with systems instead of with units. Hence, in analogy with (5.38), we set

$$\boldsymbol{R}_j := \boldsymbol{Q}_{k_j}\,, \quad \boldsymbol{E}_j := \boldsymbol{C}_{k_j}\,, \quad \boldsymbol{F}_j := \boldsymbol{D}_{k_j}\,, \quad \boldsymbol{R}_0 := \boldsymbol{Q}_0 = \boldsymbol{P}\,, \quad \boldsymbol{F}_0 := \boldsymbol{D}_0 = \boldsymbol{B}_1\,. \tag{5.44}$$

We can now observe an estimate for the systems which comes directly from the argument of Lemma 5.23.

LEMMA 5.25. *Let $\mathscr{S}_j$ be a system of units. Then one has*

$$\ell\big(\widehat{\boldsymbol{R}_{j-1}\boldsymbol{R}_j}\big) \leq 13\,\ell\big(\widehat{\boldsymbol{F}_{j-1}\boldsymbol{F}_j}\big) + 10\,\ell\big(\boldsymbol{R}_{j-1}\boldsymbol{F}_{j-1}\big)\,. \tag{5.45}$$

and moreover

$$\ell\big(\boldsymbol{R}_j\boldsymbol{F}_j\big) \leq \ell\big(\boldsymbol{R}_{j-1}\boldsymbol{F}_{j-1}\big) + \ell\big(\widehat{\boldsymbol{F}_{j-1}\boldsymbol{F}_j}\big)\,. \tag{5.46}$$

PROOF. First of all, repeat *verbatim*, substituting 0 with k_{j-1} and M with k_j, the proof of Lemma 5.23 until the estimate (5.43), which then reads as

$$\ell\big(\widehat{\boldsymbol{Q}_{k_{j-1}}\boldsymbol{Q}_{k_j}}\big) \leq 10\,\ell\big(\boldsymbol{Q}_{k_{j-1}}\boldsymbol{D}_{k_{j-1}}\big) + 13\,\ell\big(\widehat{\boldsymbol{D}_{k_{j-1}}\boldsymbol{D}_{k_j}}\big)\,.$$

This estimate is exactly (5.45), rewritten with the new notations (5.44). On the other hand, concerning (5.46), it is enough to add the inequality (5.31') with all $k_{j-1}+1 \leq l \leq k_j$, thus obtaining

$$\sum_{l=k_{j-1}+1}^{k_j} \ell\big(\boldsymbol{Q}_l\boldsymbol{D}_l\big) \leq \sum_{l=k_{j-1}+1}^{k_j} \ell\big(\boldsymbol{Q}_{l-1}\boldsymbol{D}_{l-1}\big) + \sum_{l=k_{j-1}+1}^{k_j} \ell\big(\boldsymbol{D}_{l-1}\boldsymbol{D}_l\big)\,,$$

which is equivalent to

$$\ell\big(\boldsymbol{Q}_{k_j}\boldsymbol{D}_{k_j}\big) \leq \ell\big(\boldsymbol{Q}_{k_{j-1}}\boldsymbol{D}_{k_{j-1}}\big) + \ell\big(\widehat{\boldsymbol{D}_{k_{j-1}}\boldsymbol{D}_{k_j}}\big)\,.$$

This estimate corresponds to (5.46) when using the new notations. □

Notice that, by adding (5.45) for all $1 \leq j \leq W$, one obtains

$$\ell\big(\widehat{\boldsymbol{P}\boldsymbol{P}_N}\big) = \ell\big(\widehat{\boldsymbol{Q}_0\boldsymbol{Q}_M}\big) = \ell\big(\widehat{\boldsymbol{R}_0\boldsymbol{R}_W}\big) \leq 13\,\ell\big(\widehat{\boldsymbol{F}_0\boldsymbol{F}_W}\big) + 10\sum_{j=0}^{W-1} \ell\big(\boldsymbol{R}_j\boldsymbol{F}_j\big)\,,$$

and since $\widehat{\boldsymbol{F}_0\boldsymbol{F}_W} = \widehat{\boldsymbol{B}_1\boldsymbol{B}_N} \subseteq \widehat{\boldsymbol{P}\boldsymbol{B}}$, to conclude Lemma 5.20 one needs to estimate the last sum.

Having done this remark, we can now introduce our last category, namely the "blocks" of systems. To do so, notice that by Definition 5.24 of systems of units and using the new notations (5.44), for any $1 \leq j < W$ one must have, by maximality of k_j,

$$\text{either} \qquad \ell\big(\widehat{\boldsymbol{F}_{j-1}\boldsymbol{F}_j}\big) > \frac{\ell\big(\boldsymbol{R}_{j-1}\boldsymbol{F}_{j-1}\big)}{4}\,, \qquad \text{or} \qquad \ell\big(\boldsymbol{R}_j\boldsymbol{F}_j\big) < \frac{\ell\big(\boldsymbol{R}_{j-1}\boldsymbol{F}_{j-1}\big)}{2}\,. \tag{5.47}$$

We can then give our definition.

Definition 5.26. *Let $p_0 = 0$. We define recursively the increasing sequence $\{p_1, \cdots, p_H\}$ as follows. For each $i \geq 0$, if $p_i = W$ then we conclude the construction (and thus $H = i$), while otherwise we define $p_i < p_{i+1} \leq W$ to be the biggest number such that*

$$\ell\big(\boldsymbol{R}_j\boldsymbol{F}_j\big) < \frac{\ell\big(\boldsymbol{R}_{j-1}\boldsymbol{F}_{j-1}\big)}{2} \qquad \forall\, p_i < j < p_{i+1}\,.$$

Notice again that this strictly increasing sequence is well-defined since the inequality is emptily true for $p_{i+1} = p_i + 1$. We then define block of systems *each collection $\mathscr{B}_i = \big(\mathscr{S}_{p_{i-1}+1}, \mathscr{S}_{p_{i-1}+2}, \ldots, \mathscr{S}_{p_i}\big)$, for $1 \leq i \leq H$.*

We can now show the important properties of the blocks of systems.

Lemma 5.27. *For any $0 \leq i < H$, the following estimate concerning the block $\mathscr{B}_i$ holds true,*

$$\ell\big(\widehat{\boldsymbol{R}_{p_i}\boldsymbol{R}_{p_{i+1}}}\big) \leq 13\,\ell\big(\widehat{\boldsymbol{F}_{p_i}\boldsymbol{F}_{p_{i+1}}}\big) + 20\,\ell\big(\boldsymbol{R}_{p_i}\boldsymbol{F}_{p_i}\big)\,. \tag{5.48}$$

Moreover, for any $0 \leq i < H-1$, one also has

$$\ell\big(\boldsymbol{R}_{p_{i+1}}\boldsymbol{F}_{p_{i+1}}\big) \leq 5\,\ell\big(\widehat{\boldsymbol{F}_{p_i}\boldsymbol{F}_{p_{i+1}}}\big)\,. \tag{5.49}$$

Proof. It is enough to add (5.45) for $p_i+1\le j\le p_{i+1}$ to obtain

$$\begin{aligned}\ell\big(\widehat{\boldsymbol{R}_{p_i}\boldsymbol{R}_{p_{i+1}}}\big)&=\sum_{j=p_i+1}^{p_{i+1}}\ell\big(\widehat{\boldsymbol{R}_{j-1}\boldsymbol{R}_j}\big)\le 13\sum_{j=p_i+1}^{p_{i+1}}\ell\big(\widehat{\boldsymbol{F}_{j-1}\boldsymbol{F}_j}\big)+10\sum_{j=p_i+1}^{p_{i+1}}\ell\big(\boldsymbol{R}_{j-1}\boldsymbol{F}_{j-1}\big)\\&=13\,\ell\big(\widehat{\boldsymbol{F}_{p_i}\boldsymbol{F}_{p_{i+1}}}\big)+10\sum_{j=p_i}^{p_{i+1}-1}\ell\big(\boldsymbol{R}_j\boldsymbol{F}_j\big)<13\,\ell\big(\widehat{\boldsymbol{F}_{p_i}\boldsymbol{F}_{p_{i+1}}}\big)+20\,\ell\big(\boldsymbol{R}_{p_i}\boldsymbol{F}_{p_i}\big)\,,\end{aligned}$$

thus (5.48) is already obtained.

Consider now (5.49). Recalling the definition of the blocks, the maximality of p_{i+1} tells us that either $p_{i+1}=W$ (and this is excluded by $i<H-1$) or

$$\ell\big(\boldsymbol{R}_{p_{i+1}}\boldsymbol{F}_{p_{i+1}}\big)\ge\frac{\ell\big(\boldsymbol{R}_{p_{i+1}-1}\boldsymbol{F}_{p_{i+1}-1}\big)}{2}\,.$$

Hence, keeping in mind (5.47) with $j=p_{i+1}$, we also have that

$$\ell\big(\widehat{\boldsymbol{F}_{p_{i+1}-1}\boldsymbol{F}_{p_{i+1}}}\big)>\frac{\ell\big(\boldsymbol{R}_{p_{i+1}-1}\boldsymbol{F}_{p_{i+1}-1}\big)}{4}\,.$$

Let us apply now (5.46) with $j=p_{i+1}$, to get

$$\ell\big(\boldsymbol{R}_{p_{i+1}}\boldsymbol{F}_{p_{i+1}}\big)\le\ell\big(\boldsymbol{R}_{p_{i+1}-1}\boldsymbol{F}_{p_{i+1}-1}\big)+\ell\big(\widehat{\boldsymbol{F}_{p_{i+1}-1}\boldsymbol{F}_{p_{i+1}}}\big)\le 5\,\ell\big(\widehat{\boldsymbol{F}_{p_{i+1}-1}\boldsymbol{F}_{p_{i+1}}}\big)\le 5\,\ell\big(\widehat{\boldsymbol{F}_{p_i}\boldsymbol{F}_{p_{i+1}}}\big)\,,$$

and so also (5.49) is proved. □

We finally end this step with the proof of Lemma 5.20.

Proof of Lemma 5.20. By symmetry, we can assume that $\min\Big\{\ell\big(\widehat{\boldsymbol{AP}}\big),\,\ell\big(\widehat{\boldsymbol{PB}}\big)\Big\}=\ell\big(\widehat{\boldsymbol{PB}}\big)$. Using (5.48) and (5.49), we then estimate

$$\begin{aligned}\ell\big(\widehat{\boldsymbol{P}_0\boldsymbol{P}_N}\big)&=\ell\big(\widehat{\boldsymbol{Q}_0\boldsymbol{Q}_M}\big)=\ell\big(\widehat{\boldsymbol{R}_0\boldsymbol{R}_W}\big)=\sum_{i=0}^{H-1}\ell\big(\widehat{\boldsymbol{R}_{p_i}\boldsymbol{R}_{p_{i+1}}}\big)\\&\le\sum_{i=0}^{H-1}13\,\ell\big(\widehat{\boldsymbol{F}_{p_i}\boldsymbol{F}_{p_{i+1}}}\big)+\sum_{i=0}^{H-1}20\,\ell\big(\boldsymbol{R}_{p_i}\boldsymbol{F}_{p_i}\big)\\&=13\sum_{i=0}^{H-1}\ell\big(\widehat{\boldsymbol{F}_{p_i}\boldsymbol{F}_{p_{i+1}}}\big)+20\,\ell\big(\boldsymbol{R}_0\boldsymbol{F}_0\big)+20\sum_{i=0}^{H-2}\ell\big(\boldsymbol{R}_{p_{i+1}}\boldsymbol{F}_{p_{i+1}}\big)\\&\le 13\sum_{i=0}^{H-1}\ell\big(\widehat{\boldsymbol{F}_{p_i}\boldsymbol{F}_{p_{i+1}}}\big)+20\,\ell\big(\boldsymbol{R}_0\boldsymbol{F}_0\big)+100\sum_{i=0}^{H-2}\ell\big(\widehat{\boldsymbol{F}_{p_i}\boldsymbol{F}_{p_{i+1}}}\big)\\&\le 113\sum_{i=0}^{H-1}\ell\big(\widehat{\boldsymbol{F}_{p_i}\boldsymbol{F}_{p_{i+1}}}\big)+20\,\ell\big(\boldsymbol{R}_0\boldsymbol{F}_0\big)=113\,\ell\big(\widehat{\boldsymbol{F}_0\boldsymbol{F}_W}\big)+20\,\ell\big(\boldsymbol{R}_0\boldsymbol{F}_0\big)\\&=113\,\ell\big(\widehat{\boldsymbol{B}_1\boldsymbol{B}_N}\big)+20\,\ell\big(\boldsymbol{P}_0\boldsymbol{B}_1\big)\le 113\,\ell\big(\widehat{\boldsymbol{P}_0\boldsymbol{B}_N}\big)=113\,\ell\big(\widehat{\boldsymbol{PB}}\big)\,.\end{aligned}$$

□

5.8. Speed of paths inside a sector

A temptative choice to complete the path from $\boldsymbol{P}$ to $\boldsymbol{O}$ could be the piecewise affine path $\Gamma_P = \widehat{\boldsymbol{P}\boldsymbol{P}_N} \cup \boldsymbol{P}_N\boldsymbol{O}$, which consists of the path $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$ defined in Section 5.6 followed by the segment $\boldsymbol{P}_N\boldsymbol{O}$. However, we can easily see that sending the segment PO to the path Γ_P at constant speed is not the right choice. Basically, the reason is the following: if two points $\boldsymbol{P}$ and $\boldsymbol{Q}$ in $\widehat{\boldsymbol{A}\boldsymbol{B}}$ have distance $\varepsilon > 0$, the lengths of $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$ and of $\widehat{\boldsymbol{Q}\boldsymbol{Q}_N}$ may differ of $K\varepsilon$ for any big constant K (e.g. spiral shape of Section 5.7), thus if we use the constant speed in the definition of $\tilde{v}$ we end up with a piecewise affine function making use of triangles with arbitrarily small and big angles, thus with an arbitrarily large bi-Lipschitz constant. For this reason, we parameterize the paths $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$ with a non constant speed. Choosing the correct speed is precisely the aim of this step.

Let us start with the definition of a *possible speed function*.

DEFINITION 5.28. *Let $\mathcal{S}(\boldsymbol{A}\boldsymbol{B})$ be a sector, and let Σ be the union of the paths $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$ for all the vertices $\boldsymbol{P}$ of $\widehat{\boldsymbol{A}\boldsymbol{B}}$ (such union is disjoint by Lemma 5.18). We say that $\tau : \Sigma \to \mathbb{R}^+$ is a* possible speed function *if for any vertex $\boldsymbol{P} \in \widehat{\boldsymbol{A}\boldsymbol{B}}$ one has*

- $\tau(\boldsymbol{P}) = 0$ *,*
- *for each vertex $\boldsymbol{P} \in \widehat{\boldsymbol{A}\boldsymbol{B}}$ and each $0 \le i < N(\boldsymbol{P})$, the restriction of τ to the closed segment $\boldsymbol{P}_i\boldsymbol{P}_{i+1}$ is affine.*

Moreover, for any $\boldsymbol{S}$ belonging to the open segment $\boldsymbol{P}_i\boldsymbol{P}_{i+1}$, we shall write

$$\tau'(\boldsymbol{S}) := \frac{\tau(\boldsymbol{P}_{i+1}) - \tau(\boldsymbol{P}_i)}{\ell\big(\boldsymbol{P}_i\boldsymbol{P}_{i+1}\big)} . \tag{5.50}$$

To avoid misunderstandings in the following result it is useful to underline that, if one considers $\tau(\boldsymbol{S})$ as the time at which the curve $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$ passes through $\boldsymbol{S}$, then in fact $\tau'(\boldsymbol{S})$ corresponds to the *inverse* of the speed of the curve. Let us then state and prove the main result of this step.

LEMMA 5.29. *There exists a possible speed function τ such that*

$$\frac{1}{60L} \le \tau'(\boldsymbol{S}) \le 1 \qquad \forall\, \boldsymbol{S} \in \Sigma\,, \tag{5.51}$$

if $\boldsymbol{P}_i$ and $\boldsymbol{Q}_j$ belong to the same exit side of a triangle, then
$$|\tau(\boldsymbol{P}_i) - \tau(\boldsymbol{Q}_j)| \le 170L\,\ell\big(\widehat{\boldsymbol{P}\boldsymbol{Q}}\big)\,. \tag{5.52}$$

PROOF. We start noticing that, in order to define τ, it is enough to fix τ' within the whole path $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$ for any vertex $\boldsymbol{P} \in \widehat{\boldsymbol{A}\boldsymbol{B}}$. We argue again by induction on the weight of the sector.

<u>Case I.</u> *The weigth of $\mathcal{S}(\boldsymbol{A}\boldsymbol{B})$ is 2.*

In this case, the sector is a triangle $\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}$, and we directly set $\tau' \equiv 1$ within all Σ, so that (5.51) is clearly true. Consider now (5.52). Since there is only a single triangle, then necessarily $i = j = 1$ and $\boldsymbol{P}_1$ and $\boldsymbol{Q}_1$ belong to $\boldsymbol{AB}$, so that

$$\tau(\boldsymbol{P}_1) = \ell\big(\boldsymbol{PP}_1\big)\,, \qquad\qquad \tau(\boldsymbol{Q}_1) = \ell\big(\boldsymbol{QQ}_1\big)\,,$$

by the choice $\tau' \equiv 1$. It is then enough to recall Lemma 5.18 (iii) and to use the triangular inequality to get

$$|\tau(\boldsymbol{P}_1) - \tau(\boldsymbol{Q}_1)| = \Big|\ell\big(\boldsymbol{PP}_1\big) - \ell\big(\boldsymbol{QQ}_1\big)\Big| \leq \ell\big(\boldsymbol{PQ}\big) + \ell\big(\boldsymbol{P}_1\boldsymbol{Q}_1\big) \leq 2\ell\big(\boldsymbol{PQ}\big)\,,$$

so that (5.52) holds true.

<u>Case II.</u> *The weigth of* $\mathcal{S}(\boldsymbol{AB})$ *is at least* 3.

In this case, let us consider the maximal triangle $\boldsymbol{ABC}$. Then, we can assume that τ has been already defined in the sectors $\mathcal{S}(\boldsymbol{AC})$ and $\mathcal{S}(\boldsymbol{BC})$, emptily if the segment $\boldsymbol{AC}$ (resp. $\boldsymbol{BC}$) belongs to $\partial\Gamma$, and by inductive assumption otherwise, and with the properties that

$$\Big|\tau(\boldsymbol{P}_{N-1}) - \tau(\boldsymbol{Q}_{N-1})\Big| \leq 170L\,\ell\big(\boldsymbol{PQ}\big)\,, \tag{5.53}$$

for every $\boldsymbol{P}, \boldsymbol{Q} \in \widehat{\boldsymbol{AB}}$, and that $1/60L \leq \tau'(\boldsymbol{S}) \leq 1$ for every $\boldsymbol{S} \in \mathcal{S}(\boldsymbol{AC}) \cup \mathcal{S}(\boldsymbol{BC})$. Thus, we only have to define τ in the triangle $\boldsymbol{ABC}$ and by definition of possible speed function it is enough to set τ on the segment $\boldsymbol{AB}$ or, equivalently, to set τ' on the triangle $\boldsymbol{ABC}$.

Let us begin with a temptative definition, namely, we define $\tilde\tau$ by putting $\tilde\tau' \equiv 1/60L$ in $\boldsymbol{ABC}$, and we will define τ as a modification –if necessary– of $\tilde\tau$. Notice that, for any $\boldsymbol{P}_{N-1} \in \boldsymbol{AC} \cup \boldsymbol{BC}$, our definition consists in setting

$$\tilde\tau(\boldsymbol{P}_N) = \tau(\boldsymbol{P}_{N-1}) + \frac{1}{60L}\,\ell\big(\boldsymbol{P}_{N-1}\boldsymbol{P}_N\big)\,. \tag{5.54}$$

Of course the function $\tilde\tau$ satisfies (5.51), but in general it is not true that (5.52) holds.

We can now define the function τ by setting

$$\tau(\boldsymbol{P}_N) := \tilde\tau(\boldsymbol{P}_N) \vee \max\Big\{\tilde\tau(\boldsymbol{Q}_N) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big) :\, \boldsymbol{Q} \in \widehat{\boldsymbol{AB}}\Big\}\,, \tag{5.55}$$

for any vertex $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$. Since by definition $\tau \geq \tilde\tau$, it is clear that $\tau' \geq \tilde\tau' = 1/60L$ in the triangle $\boldsymbol{ABC}$, so the first inequality in (5.51) holds true also for τ.

It is also easy to check (5.52). Indeed, take $\boldsymbol{P}$ and $\boldsymbol{Q}$ in $\widehat{\boldsymbol{AB}}$, and consider two possibilities. If $\tau(\boldsymbol{Q}_N) = \tilde\tau(\boldsymbol{Q}_N)$, then

$$\tau(\boldsymbol{P}_N) \geq \tilde\tau(\boldsymbol{Q}_N) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big) = \tau(\boldsymbol{Q}_N) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big)\,. \tag{5.56}$$

On the other hand, if $\tau(\boldsymbol{Q}_N) = \tilde\tau(\boldsymbol{R}_N) - 170L\,\ell\big(\widehat{\boldsymbol{QR}}\big)$ for some $\boldsymbol{R} \in \widehat{\boldsymbol{AB}}$, then

$$\begin{aligned}\tau(\boldsymbol{P}_N) &\geq \tilde\tau(\boldsymbol{R}_N) - 170L\,\ell\big(\widehat{\boldsymbol{PR}}\big) \geq \tilde\tau(\boldsymbol{R}_N) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big) - 170L\,\ell\big(\widehat{\boldsymbol{QR}}\big)\\ &= \tau(\boldsymbol{Q}_N) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big)\,,\end{aligned}$$

so that (5.56) is true in both cases. Exchanging the roles of $\boldsymbol{P}$ and $\boldsymbol{Q}$ immediately yields (5.52). Summarizing, to conclude the thesis we only have to check that $\tau' \leq 1$

on $\boldsymbol{ABC}$, which by induction amounts to check that for any $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$ one has

$$\tau(\boldsymbol{P}_N) - \tau(\boldsymbol{P}_{N-1}) \leq \ell\big(\boldsymbol{P}_{N-1}\boldsymbol{P}_N\big)\,.$$

Let us then assume the existence of some vertex $\boldsymbol{P} \in \widehat{\boldsymbol{AB}}$ such that

$$\tau(\boldsymbol{P}_N) - \tau(\boldsymbol{P}_{N-1}) > \ell\big(\boldsymbol{P}_{N-1}\boldsymbol{P}_N\big)\,, \tag{5.57}$$

and the searched inequality will follow once we find some contradiction. By symmetry, we assume that $\boldsymbol{P}_{N-1} \in \boldsymbol{AC}$. Of course, if $\tau(\boldsymbol{P}_N) = \tilde\tau(\boldsymbol{P}_N)$ then (5.54) already prevents the validity of (5.57). Therefore, keeping in mind (5.55), we obtain the existence of some vertex $\boldsymbol{Q} \in \widehat{\boldsymbol{AB}}$ such that

$$\tau(\boldsymbol{P}_N) = \tilde\tau(\boldsymbol{Q}_N) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big)\,, \tag{5.58}$$

which gives

$$\tau(\boldsymbol{P}_N) = \tau(\boldsymbol{Q}_{N-1}) + \frac{1}{60L}\,\ell\big(\boldsymbol{Q}_{N-1}\boldsymbol{Q}_N\big) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big)\,.$$

Recalling (5.53) and (5.57), and eventually applying the triangle inequality if $\boldsymbol{Q}_{N-1} \in \boldsymbol{BC}$, we deduce

$$\begin{aligned}\tau(\boldsymbol{P}_{N-1}) &\geq \tau(\boldsymbol{Q}_{N-1}) - 170L\,\ell\big(\widehat{\boldsymbol{PQ}}\big) = \tau(\boldsymbol{P}_N) - \frac{1}{60L}\,\ell\big(\boldsymbol{Q}_{N-1}\boldsymbol{Q}_N\big)\\ &> \tau(\boldsymbol{P}_{N-1}) + \ell\big(\boldsymbol{P}_{N-1}\boldsymbol{P}_N\big) - \frac{1}{60L}\,\ell\big(\boldsymbol{Q}_{N-1}\boldsymbol{Q}_N\big)\,,\end{aligned}$$

so that

$$\ell\big(\boldsymbol{Q}_{N-1}\boldsymbol{Q}_N\big) > 60L\,\ell\big(\boldsymbol{P}_{N-1}\boldsymbol{P}_N\big)\,. \tag{5.59}$$

Call now, as in Figure 12, $\boldsymbol{P}_\perp$ and $\boldsymbol{Q}_\perp$ the orthogonal projections of $\boldsymbol{P}_{N-1}$ and $\boldsymbol{Q}_{N-1}$ on the segment $\boldsymbol{AB}$, and note that by a trivial geometrical argument –recalling that $\boldsymbol{P}_{N-1} \in \boldsymbol{AC}$– one has

$$\frac{\ell\big(\boldsymbol{P}_{N-1}\boldsymbol{P}_\perp\big)}{\ell\big(\boldsymbol{Q}_{N-1}\boldsymbol{Q}_\perp\big)} \geq \frac{\ell\big(\boldsymbol{AP}_{N-1}\big)}{\ell\big(\boldsymbol{AQ}_{N-1}\big)}\,,$$

where the inequality is an equality if $\boldsymbol{Q}_{N-1} \in \boldsymbol{AC}$, while it is strict if $\boldsymbol{Q}_{N-1} \in \boldsymbol{BC}$. Then, recalling Lemma 5.18 (i) and (5.59), one has

$$\begin{aligned}\ell\big(\boldsymbol{P}_{N-1}\boldsymbol{P}_N\big) &\geq \ell\big(\boldsymbol{P}_{N-1}\boldsymbol{P}_\perp\big) = \ell\big(\boldsymbol{Q}_{N-1}\boldsymbol{Q}_\perp\big)\,\frac{\ell\big(\boldsymbol{AP}_{N-1}\big)}{\ell\big(\boldsymbol{AQ}_{N-1}\big)}\\ &= \ell\big(\boldsymbol{Q}_{N-1}\boldsymbol{Q}_N\big)\,\sin\big(\boldsymbol{Q}_{N-1}\widehat{\boldsymbol{Q}_N}\boldsymbol{A}\big)\,\frac{\ell\big(\boldsymbol{AP}_{N-1}\big)}{\ell\big(\boldsymbol{AQ}_{N-1}\big)} \geq \frac{1}{4}\,\ell\big(\boldsymbol{Q}_{N-1}\boldsymbol{Q}_N\big)\,\frac{\ell\big(\boldsymbol{AP}_{N-1}\big)}{\ell\big(\boldsymbol{AQ}_{N-1}\big)}\\ &> 15L\,\ell\big(\boldsymbol{P}_{N-1}\boldsymbol{P}_N\big)\,\frac{\ell\big(\boldsymbol{AP}_{N-1}\big)}{\ell\big(\boldsymbol{AQ}_{N-1}\big)}\,,\end{aligned}$$

which means

$$\ell\big(\boldsymbol{AQ}_{N-1}\big) \geq 15L\,\ell\big(\boldsymbol{AP}_{N-1}\big)\,.$$

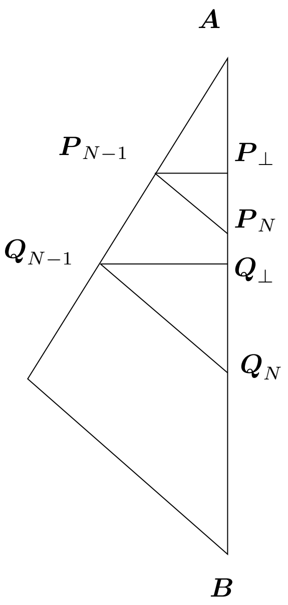

FIGURE 12. The triangle $\boldsymbol{ABC}$ with the points $\boldsymbol{P}_{N-1}$, $\boldsymbol{P}_N$, $\boldsymbol{P}_\perp$ and $\boldsymbol{Q}_{N-1}$, $\boldsymbol{Q}_N$, $\boldsymbol{Q}_\perp$

Making again use of Lemma 5.18 (iii), we then have

$$\begin{aligned}\ell\big(\widehat{\boldsymbol{PQ}}\big) &\geq \ell\big(\boldsymbol{P}_{N-1}\boldsymbol{Q}_{N-1}\big) = \ell\big(\boldsymbol{AQ}_{N-1}\big) - \ell\big(\boldsymbol{AP}_{N-1}\big) \geq 14L\,\ell\big(\boldsymbol{AP}_{N-1}\big) \geq 2\,\ell\big(\widehat{AP}\big)\\ &\geq \frac{2}{L}\,\ell\big(\widehat{\boldsymbol{AP}}\big)\,,\end{aligned}$$

so that

$$3\,\ell\big(\widehat{\boldsymbol{PQ}}\big) \geq \left(1+\frac{2}{L}\right)\ell\big(\widehat{\boldsymbol{PQ}}\big) \geq \frac{2}{L}\left(\ell\big(\widehat{\boldsymbol{AP}}\big)+\ell\big(\widehat{\boldsymbol{PQ}}\big)\right) \geq \frac{2}{L}\,\ell\big(\widehat{\boldsymbol{AQ}}\big)\,.$$

Hence, by (5.58) and the Lipschitz property of u,

$$\tilde{\tau}(\boldsymbol{Q}_N) \geq 170L\,\ell\big(\boldsymbol{PQ}\big) \geq \frac{340}{3}\,\ell\big(\widehat{\boldsymbol{AQ}}\big)\,. \tag{5.60}$$

On the other hand, by definition and inductive assumption,

$$\tilde{\tau}(\boldsymbol{Q}_N) = \tau(\boldsymbol{Q}_{N-1}) + \frac{1}{60L}\,\ell\big(\boldsymbol{Q}_{N-1}\boldsymbol{Q}_N\big) \leq \ell\big(\widehat{\boldsymbol{QQ}_{N-1}}\big) + \frac{1}{60L}\,\ell\big(\boldsymbol{Q}_{N-1}\boldsymbol{Q}_N\big) \leq \ell\big(\widehat{\boldsymbol{QQ}_N}\big)\,,$$

which recalling Lemma 5.20 of Section 5.7 gives

$$\tilde{\tau}(\boldsymbol{Q}_N) \leq 113\,\ell\big(\widehat{\boldsymbol{AQ}}\big) < \frac{340}{3}\,\ell\big(\widehat{\boldsymbol{AQ}}\big)\,.$$

Finally, this gives a contradiction with (5.60) and the proof of the lemma is concluded. □

5.9. Extension in the primary sectors

We are finally ready to define the extension of $\tilde{u}$ inside a primary sector. The goal of this step is to take a primary sector $\mathcal{S}(\boldsymbol{AB})$, being $\boldsymbol{A} = \tilde{u}(A)$ and $\boldsymbol{B} = \tilde{u}(B)$, with $A,\, B \in \partial\mathcal{D}$ as usual, and to define a piecewise affine bi-Lipschitz extension $\tilde{u}_{AB}$ of $\tilde{u}$ which sends a suitable subset $\mathcal{D}_{AB}$ of the square $\mathcal{D}$ onto $\mathcal{S}(\boldsymbol{AB})$ (see Figure 13). First we observe a simple trigonometric estimate for the bi-Lipschitz constant of an affine map between two triangles and then we state and prove the main result of this step.

Lemma 5.30. *Let $\mathscr{T}$ and $\mathscr{T}'$ be two triangles in $\mathbb{R}^2$, and let ϕ be a bijective affine map sending $\mathscr{T}$ onto $\mathscr{T}'$. Call a, b and α the lengths of two sides of $\mathscr{T}$ and the angle between them, and let a', b' and α' be the correponding lengths and angle in $\mathscr{T}'$. Then, the Lipschitz constant of the map ϕ can be bounded as*

$$\operatorname{Lip}(\phi) \leq \frac{a'}{a} + \frac{b'\sin\alpha'}{b\sin\alpha} + \left|\frac{b'\cos\alpha'}{b\sin\alpha} - \frac{a'\cos\alpha}{a\sin\alpha}\right| \leq \frac{a'}{a} + \frac{2b'}{b\sin\alpha} + \frac{a'}{a\sin\alpha}\,. \tag{5.61}$$

Proof. Let us take an orthonormal basis $\{\mathrm{e}_1,\, \mathrm{e}_2\}$ of $\mathbb{R}^2$. Up to an isometry of the plane, we can assume that the two sides of lengths a and a' are both on the line $\{\mathrm{e}_2 = 0\}$, that the two triangles $\mathscr{T}$ and $\mathscr{T}'$ both lie in the half-space $\{\mathrm{e}_2 \geq 0\}$ and that the vertices whose angles are given by α, α' coincide with the point $(0,0)$. Hence, one has that $\phi(x) = M\,x + \omega$, for some vector $\omega \in \mathbb{R}^2$ and a 2×2 matrix M. We have then

$$\operatorname{Lip}(\phi) = |M| = \sup_{\nu\neq 0} \frac{|M\nu|}{|\nu|}\,.$$

With our choice of coordinates, we have clearly

$$M\big(a,0\big) = \big(a',0\big)\,, \qquad\qquad M\big(b\cos\alpha, b\sin\alpha\big) = \big(b'\cos\alpha', b'\sin\alpha'\big)\,,$$

which immediately gives

$$M = \begin{pmatrix} \dfrac{a'}{a} & \dfrac{b'\cos\alpha'}{b\sin\alpha} - \dfrac{a'\cos\alpha}{a\sin\alpha} \\ 0 & \dfrac{b'\sin\alpha'}{b\sin\alpha} \end{pmatrix},$$

from which the estimate (5.61) immediately follows. □

Lemma 5.31. *Let $\mathcal{S}(\boldsymbol{AB})$ be a primary sector. Then there exists a polygonal subset $\mathcal{D}_{AB}$ of $\mathcal{D}$, and a piecewise affine map $\tilde{u}_{AB} : \mathcal{D}_{AB} \to \mathcal{S}(\boldsymbol{AB})$ such that:*

(i) *for any $P \in \partial\mathcal{D}$, one has $\mathcal{D}_{AB} \cap OP = \emptyset$ if $P \notin \widehat{AB}$, $\mathcal{D}_{AB} \cap OP = \{P\}$ if $P \in \{A,\, B\}$, and $\mathcal{D}_{AB} \cap OP = PP_N$ with $P_N = tO + (1-t)P$ and $0 < t = t(P) < 4/5$ if $P \in \widehat{AB} \setminus \{A,\, B\}$.*

(ii) *$\tilde{u}_{AB} = \tilde{u}$ on $\widehat{AB} = \partial D \cap \mathcal{D}_{AB}$.*

(iii) *$\tilde{u}_{AB}$ is bi-Lipschitz with constant $212000L^4$.*

(iv) *For any two consecutive vertices $P,\, Q \in \widehat{AB}$, one has $P_{N(P)}\widehat{Q_{N(Q)}}O \geq \dfrac{1}{87L}$.*

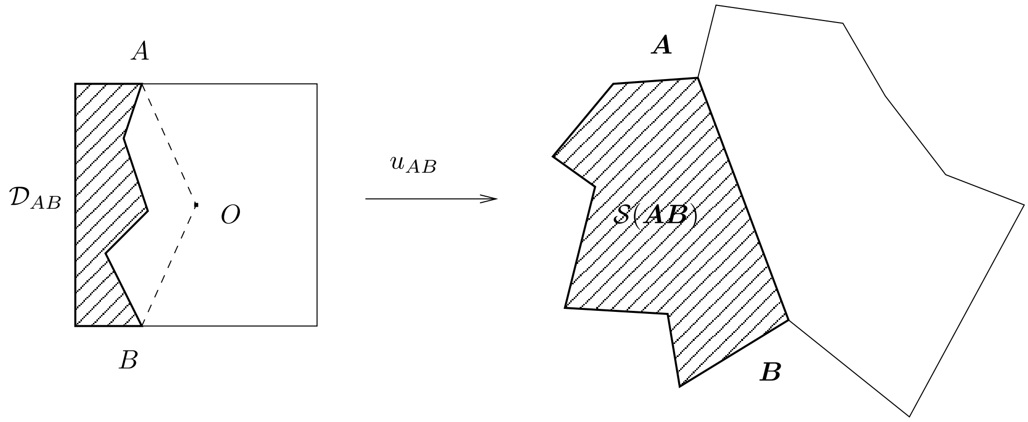

FIGURE 13. The function $\tilde{u}_{AB} : \mathcal{D}_{AB} \to \mathcal{S}(\boldsymbol{AB})$

PROOF. We will divide the proof in three parts.

Part 1. Definition of $\bar{\gamma}$, $\bar{\boldsymbol{\gamma}}$, $\tilde{u}_{AB} : \partial\bar{\gamma} \to \partial\bar{\boldsymbol{\gamma}}$, and validity of (i) and (ii).

First of all, we take a vertex $P \in \widehat{AB}$ and, for any $1 \le i \le N = N(\boldsymbol{P})$, we set

$$P_i = t_{P,i}\, O + (1 - t_{P,i})P\,, \qquad \text{with} \qquad t_{P,i} = \frac{\tau(\boldsymbol{P}_i)}{10L}\,, \tag{5.62}$$

where τ is the function of Lemma 5.29. Then, we define $\tilde{u}_{AB}$ on the segment PP_N as the piecewise affine function such that for all i one has $\tilde{u}_{AB}(P_i) = \boldsymbol{P}_i$. It is important to observe that

$$0 \le t_{P,i} \le \frac{4}{5}\,, \qquad \forall P \in \widehat{AB},\ 1 \le i \le N = N(\boldsymbol{P})\,. \tag{5.63}$$

Indeed, using (5.51) in Lemma 5.29, (ii) in Lemma 5.18, and the Lipschitz property of $\tilde{u}$, one has that

$$\tau(\boldsymbol{P}_i) \le \tau(\boldsymbol{P}_N) \le \sum_{j=1}^{N} \ell\big(\boldsymbol{P}_{j-1}\boldsymbol{P}_j\big) = \ell\big(\widehat{\boldsymbol{P}\boldsymbol{P}_N}\big) \le 4\,\ell\big(\widehat{\boldsymbol{AB}}\big) \le 4L\,\ell\big(\widehat{AB}\big) \le 8L\,,$$

so by (5.62) we get (5.63).

We are now ready to define the set $\mathcal{D}_{AB}$. Let us enumerate, just for one moment, the vertices of $\widehat{AB}$ as $P^0 \equiv A$, P^1, P^2 ... , P^{W-1}, $P^W \equiv B$, following the order of $\widehat{AB}$. The set $\mathcal{D}_{AB}$ is then defined as the polygon whose boundary is the union of $\widehat{AB}$ with the path $AP^1_{N(1)}P^2_{N(2)} \cdots P^{W-1}_{N(W-1)}B$, as in Figure 13, where for each $0 < i < W$ we have written $N(i) = N(\boldsymbol{P}^i)$. Hence, property (i) is true by construction and by (5.63).

Then we take two generic consecutive vertices $P, Q \in \widehat{AB}$, and we call $\bar{\gamma} \subseteq \mathcal{D}_{AB}$ the quadrilater PP_NQ_MQ, and $\bar{\boldsymbol{\gamma}} \subseteq \mathcal{S}(\boldsymbol{AB})$ the polygon whose boundary is $\boldsymbol{PQ} \cup \widehat{\boldsymbol{QQ}_M} \cup \boldsymbol{Q}_M\boldsymbol{P}_N \cup \widehat{\boldsymbol{P}_N\boldsymbol{P}}$, where we have set $N = N(\boldsymbol{P})$ and $M = N(\boldsymbol{Q})$. Notice that, varying the consecutive vertices P and Q, $\mathcal{D}_{AB}$ is the union of the different polygons $\bar{\gamma}$, while $\mathcal{S}(\boldsymbol{AB})$ is the union of the polygons $\bar{\boldsymbol{\gamma}}$. We will then define the function $\tilde{u}_{AB}$ so that $\tilde{u}_{AB}(\bar{\gamma}) = \bar{\boldsymbol{\gamma}}$.

Let us start with the definition of $\tilde{u}_{AB}$ from $\partial\bar{\gamma}$ to $\partial\bar{\boldsymbol{\gamma}}$. The function $\tilde{u}_{AB}$ has been already defined from the segment PP_N to the path $\widehat{\boldsymbol{P}\boldsymbol{P}_N}$ and from the segment QQ_M to the path $\widehat{\boldsymbol{Q}\boldsymbol{Q}_{\boldsymbol{M}}}$. Hence we conclude defining $\tilde{u}_{AB}$ to be affine from the segment PQ to the segment $\boldsymbol{PQ}$, and from P_NQ_M to $\boldsymbol{P}_N\boldsymbol{Q}_M$. Notice that, as a consequence, also property (ii) is true by construction.

Now we see how to extend $\tilde{u}_{AB}$ from the interior of $\bar{\gamma}$ to the interior of $\bar{\boldsymbol{\gamma}}$ satisfying properties (iii) and (iv).

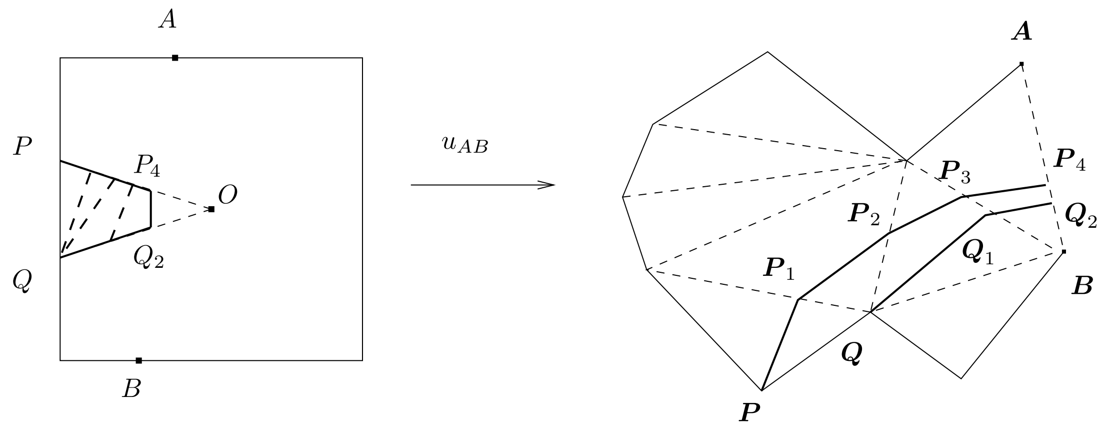

FIGURE 14. The sets $\bar{\gamma}$ and $\bar{\boldsymbol{\gamma}}$

Recalling the partition of $\mathcal{S}(\boldsymbol{AB})$ in triangles done in Section 5.5, $\boldsymbol{PQ}$ is a side of some triangle $\boldsymbol{PQR}$, and since $\boldsymbol{PQ} \subseteq \partial\bar{\boldsymbol{\gamma}}$ it cannot be the exit side. Let us then assume, without loss of generality, that the exit side is $\boldsymbol{QR}$. Hence, it follows that $N > M$. Moreover, if $\big(\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_N\big)$ is the natural sequence of triangles related to $\boldsymbol{P}$, as in Definition 5.16, then it is immediate to observe that $\boldsymbol{Q}$ belong to the exit side of $\mathscr{T}_i$ for all $1 \leq i \leq N-M$. Figure 14 shows an example in which $N = 4$ and $M = 2$. In the following two parts, we will define $\tilde{u}_{AB}$ separately on the triangle $PP_{N-M}Q$ and on the quadrilateral $P_{N-M}P_NQ_MQ$, whose union is $\bar{\gamma}$.

Part 2. Definition of $\tilde{u}_{AB}$ in the triangle $PP_{N-M}Q$, and validity of (iii) and (iv).

In this second part we define $\tilde{u}_{AB}$ from the triangle $PP_{N-M}Q$ to the polygon in $\bar{\boldsymbol{\gamma}}$ whose boundary is $\widehat{\boldsymbol{P}\boldsymbol{P}_{N-M}} \cup \boldsymbol{P}_{N-M}\boldsymbol{Q} \cup \boldsymbol{QP}$. The definition is very simple, namely, for any $0 \leq i < N - M$ we let $\tilde{u}_{AB}$ be the affine function sending the triangle $P_iP_{i+1}Q$ onto the triangle $\boldsymbol{P}_i\boldsymbol{P}_{i+1}\boldsymbol{Q}$, as shown in Figure 15. We now have to check the validity of (iii) and (iv) in the triangle $PP_{N-M}Q$. Keeping in mind Lemma 5.30, to show (iii) it is enough to compare the lengths of P_iP_{i+1} and $\boldsymbol{P}_i\boldsymbol{P}_{i+1}$, those of $P_{i+1}Q$ and $\boldsymbol{P}_{i+1}\boldsymbol{Q}$, and the angles $P_i\widehat{P_{i+1}}Q$ and $\boldsymbol{P}_i\widehat{\boldsymbol{P}_{i+1}}\boldsymbol{Q}$.

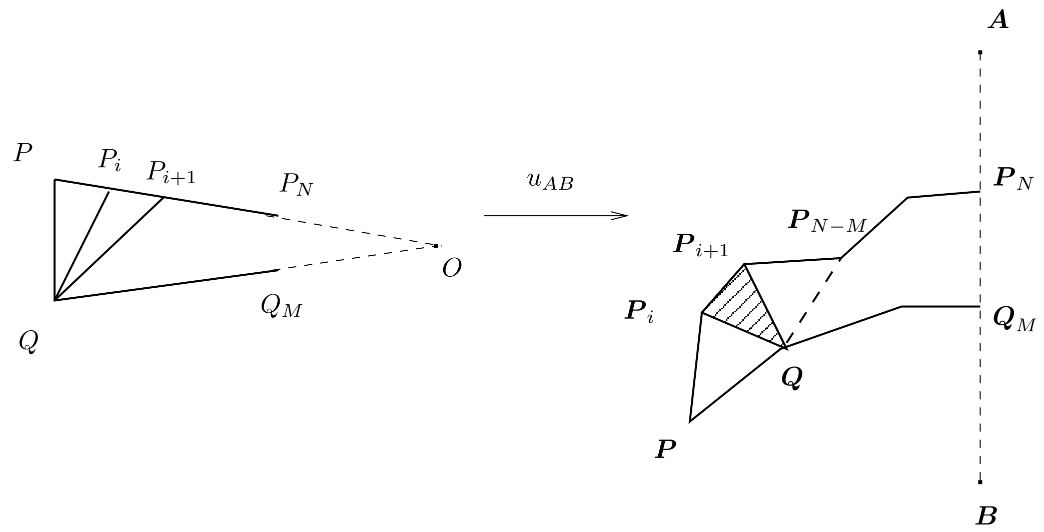

FIGURE 15. The situation in Part 2

We start recalling that (iii) in Lemma 5.18, together with the Lipschitz property of $\tilde{u}$, ensures

$$\frac{\ell(PQ)}{7L} \leq \ell(\boldsymbol{P}_{i+1}\boldsymbol{Q}) \leq \ell(\boldsymbol{PQ}) \leq L\ell(PQ) \tag{5.64}$$

(keep in mind that, since P and Q are consecutive vertices, then $PQ = \widehat{PQ}$ and $\boldsymbol{PQ} = \widehat{\boldsymbol{PQ}}$). Recalling now (5.52) of Lemma 5.29 and (5.62), we get

$$t_{P,i+1} = t_{P,i+1} - t_{Q,0} = \frac{\tau(\boldsymbol{P}_{i+1}) - \tau(\boldsymbol{Q}_0)}{10L} \leq 17\,\ell(\boldsymbol{PQ}) \leq 17L\,\ell(PQ)\,. \tag{5.65}$$

We want now to estimate $\ell(P_{i+1}Q)$. To do so, let us assume, as in Figure 15 and without loss of generality, that P and Q belong to the left side of the square $\mathcal{D}$ and that P is above Q. Call also $V \equiv (-\frac{1}{2}, -\frac{1}{2})$ the southwest corner of $\mathcal{D}$, and let δ_x and δ_y be the horizontal and vertical components of the vector $P_{i+1} - Q$, so that

$$\ell(P_{i+1}Q) = \sqrt{\delta_x^2 + \delta_y^2}\,.$$

By construction one clearly has $\delta_x = t_{P,i+1}/2$. We claim that

$$\frac{\sqrt{2}}{2}\,\ell(PQ) \leq \ell(P_{i+1}Q) \leq 17\,\frac{\sqrt{2}}{2}\,L\,\ell(PQ)\,. \tag{5.66}$$

In fact, since P_{i+1} belongs to the segment PO, then one surely has

$$\ell(P_{i+1}Q) \geq \ell(PQ)\sin\big(O\widehat{P}V\big) \geq \frac{\sqrt{2}}{2}\,\ell(PQ)\,,$$

so that the left inequality in (5.66) holds. To show the right inequality in (5.66) we consider two cases, depending on whether P_{i+1} is above or below Q or, in other words, whether $P_{i+1}\widehat{Q}V$ is bigger or smaller than $\pi/2$. If P_{i+1} is above Q, then $\delta_y \le \ell\big(PQ\big)$, so that thanks to (5.65) we get

$$\begin{aligned}\ell\big(P_{i+1}Q\big) &= \sqrt{\delta_x^2+\delta_y^2} \le \sqrt{\left(\frac{t_{P,i+1}}{2}\right)^2 + \ell\big(PQ\big)^2} \le \sqrt{\left(\frac{17}{2}\,L\,\ell\big(PQ\big)\right)^2 + \ell\big(PQ\big)^2} \\ &= \frac{\sqrt{293}}{2}\,L\,\ell\big(PQ\big) \le 17\,\frac{\sqrt{2}}{2}\,L\,\ell\big(PQ\big)\,.\end{aligned} \tag{5.67}$$

On the other hand, if P_{i+1} is below Q, then

$$\frac{\pi}{4} \le O\widehat{Q}V \le P_{i+1}\widehat{Q}V \le \frac{\pi}{2}\,,$$

hence

$$\ell\big(P_{i+1}Q\big) = \frac{\delta_x}{\sin\big(P_{i+1}\widehat{Q}V\big)} \le \sqrt{2}\delta_x = \frac{\sqrt{2}}{2}\,t_{P,i+1} \le 17\,\frac{\sqrt{2}}{2}\,L\,\ell\big(PQ\big)\,,$$

which in both cases yields (5.66).

Keeping in mind (5.64), from (5.66) we obtain

$$\frac{\sqrt{2}}{2L} \le \frac{\ell\big(P_{i+1}Q\big)}{\ell\big(\boldsymbol{P}_{i+1}\boldsymbol{Q}\big)} \le 17\cdot 7\,\frac{\sqrt{2}}{2}\,L^2 \le 85L^2\,. \tag{5.68}$$

It is much easier to compare $\ell\big(P_iP_{i+1}\big)$ and $\ell\big(\boldsymbol{P}_i\boldsymbol{P}_{i+1}\big)$. Indeed, by immediate geometrical argument, recalling (5.62), (5.50) and condition (5.51) of Lemma 5.29, and letting $\boldsymbol{S}$ be any point in the interior of $\boldsymbol{P}_i\boldsymbol{P}_{i+1}$, one has

$$\begin{aligned}\ell\big(P_iP_{i+1}\big) &\le \frac{\sqrt{2}}{2}\,\big(t_{P,i+1}-t_{P,i}\big) = \frac{\sqrt{2}}{20L}\,\big(\tau(\boldsymbol{P}_{i+1})-\tau(\boldsymbol{P}_i)\big) = \frac{\sqrt{2}}{20L}\,\tau'(\boldsymbol{S})\,\ell\big(\boldsymbol{P}_i\boldsymbol{P}_{i+1}\big) \\ &\le \frac{\sqrt{2}}{20L}\,\ell\big(\boldsymbol{P}_i\boldsymbol{P}_{i+1}\big)\,,\end{aligned}$$

and analogously

$$\ell\big(P_iP_{i+1}\big) \ge \frac{t_{P,i+1}-t_{P,i}}{2} = \frac{\tau(\boldsymbol{P}_{i+1})-\tau(\boldsymbol{P}_i)}{20L} = \frac{\tau'(\boldsymbol{S})}{20L}\,\ell\big(\boldsymbol{P}_i\boldsymbol{P}_{i+1}\big) \ge \frac{1}{1200L^2}\,\ell\big(\boldsymbol{P}_i\boldsymbol{P}_{i+1}\big)\,.$$

Thus, we have

$$\frac{1}{1200L^2} \le \frac{\ell\big(P_iP_{i+1}\big)}{\ell\big(\boldsymbol{P}_i\boldsymbol{P}_{i+1}\big)} \le \frac{\sqrt{2}}{20L}\,. \tag{5.69}$$

Let us finally compare the angles $P_i\widehat{P_{i+1}}Q$ and $\boldsymbol{P}_i\widehat{\boldsymbol{P}_{i+1}}\boldsymbol{Q}$. Concerning $\boldsymbol{P}_i\widehat{\boldsymbol{P}_{i+1}}\boldsymbol{Q}$, it is enough to recall (i) of Lemma 5.18 to obtain

$$15^\circ \le \boldsymbol{P}_i\widehat{\boldsymbol{P}_{i+1}}\boldsymbol{Q} \le 165^\circ\,. \tag{5.70}$$

On the other hand, concerning $P_i\widehat{P_{i+1}}Q$, we start observing

$$P_i\widehat{P_{i+1}}Q = P\widehat{P_{i+1}}Q \le \pi - O\widehat{P}Q \le \frac{3}{4}\,\pi\,. \tag{5.71}$$

To obtain an estimate from below to $P_i\widehat{P_{i+1}}Q$, instead, we call for brevity $\alpha := P_i\widehat{P_{i+1}}Q = P\widehat{P_{i+1}}Q$ and $\theta := O\widehat{P}V - \frac{\pi}{2} \in \big[-\pi/4, \pi/4\big)$, so that an immediate trigonometric argument gives

$$\ell\big(PQ\big) = \frac{t_{P,i+1}}{2}\Big(\tan(\theta+\alpha) - \tan\theta\Big)\,. \tag{5.72}$$

We aim then to show that

$$\alpha \geq \frac{1}{19L}\,. \tag{5.73}$$

In fact, if

$$\theta + \alpha \geq \frac{\pi}{4} + \frac{1}{19}\,,$$

then since $\theta \leq \pi/4$ we immediately deduce the validity of (5.73). On the contrary, if

$$\theta + \alpha < \frac{\pi}{4} + \frac{1}{19}\,,$$

then recalling (5.72), the fact that $\theta \geq -\pi/4$, (5.65) and the Lipschitz property of u we get

$$\ell\big(PQ\big) = \frac{t_{P,i+1}}{2}\Big(\tan(\theta+\alpha) - \tan\theta\Big) \leq \frac{t_{P,i+1}}{2}\,\frac{\alpha}{\cos^2\Big(\frac{\pi}{4}+\frac{1}{19}\Big)} \leq \frac{17}{2}\,L\,\ell\big(PQ\big)\,\frac{\alpha}{\cos^2\Big(\frac{\pi}{4}+\frac{1}{19}\Big)}\,,$$

from which we get

$$\alpha \geq \frac{2\cos^2\Big(\frac{\pi}{4}+\frac{1}{19}\Big)}{17L} \geq \frac{1}{19L}\,,$$

so that (5.73) is concluded. Putting it together with (5.71), we deduce

$$\frac{1}{19L} \leq P_i\widehat{P_{i+1}}Q \leq \frac{3}{4}\,\pi\,. \tag{5.74}$$

Finally we show the validity of (iii), simply applying (5.61) of Lemma 5.30. Indeed, let us call ϕ the affine map which sends the triangle $P_iP_{i+1}Q$ onto $\boldsymbol{P}_i\boldsymbol{P}_{i+1}\boldsymbol{Q}$ and, for brevity and according with the notation of Lemma 5.30, let us write

$$\begin{aligned} a &= \ell\big(P_{i+1}Q\big)\,, & b &= \ell\big(P_iP_{i+1}\big)\,, & \alpha &= P_i\widehat{P_{i+1}}Q\,,\\ a' &= \ell\big(\boldsymbol{P}_{i+1}\boldsymbol{Q}\big)\,, & b' &= \ell\big(\boldsymbol{P}_i\boldsymbol{P}_{i+1}\big)\,, & \alpha' &= \boldsymbol{P}_i\widehat{\boldsymbol{P}_{i+1}}\boldsymbol{Q}\,. \end{aligned}$$

Then, the estimates (5.68), (5.69), (5.70) and (5.74) can be rewritten as

$$\frac{\sqrt{2}}{2L} \leq \frac{a}{a'} \leq 85L^2\,, \qquad \frac{1}{1200L^2} \leq \frac{b}{b'} \leq \frac{\sqrt{2}}{20L}\,, \qquad \sin\alpha' \geq \frac{1}{4}\,, \qquad \sin\alpha \geq \frac{1}{20L}\,, \tag{5.75}$$

where for the last estimate we used that

$$\sin\alpha \geq \sin\left(\frac{1}{19L}\right) = \frac{1}{19L}\left(19L\sin\left(\frac{1}{19L}\right)\right) \geq \frac{1}{19L}\left(19\sin\left(\frac{1}{19}\right)\right) \geq \frac{1}{20L}\,. \tag{5.76}$$

Therefore, (5.61) and (5.75) give us

$$\mathrm{Lip}(\phi) \leq \frac{a'}{a} + \frac{2b'}{b\sin\alpha} + \frac{a'}{a\sin\alpha} \leq \sqrt{2}\,L + 48000L^3 + 20\sqrt{2}\,L^2\,.$$

On the other hand, exchanging the roles of the triangles, we get

$$\mathrm{Lip}(\phi^{-1}) \leq \frac{a}{a'} + \frac{2b}{b'\sin\alpha'} + \frac{a}{a'\sin\alpha'} \leq 85L^2 + \frac{2\sqrt{2}}{5L} + 340L^2 \,.$$

To conclude this part, we want to check (iv) for the pairs of consecutive vertices P, Q such that the side P_NQ_M is in the triangle $PP_{N-M}Q$. Notice that this happens only when $M = 0$, or in other words, if $Q \equiv A$ or $Q \equiv B$. Let us then assume that Q is either A or B, and let us show that (iv) holds, that is,

$$Q\widehat{P_N}O \geq \frac{1}{87L}\,, \qquad\qquad P_N\widehat{Q}O \geq \frac{1}{87L}\,. \tag{5.77}$$

Taking $i = N-1$ and applying the second inequality in (5.74), we immediately find

$$Q\widehat{P_N}O = \pi - P_{N-1}\widehat{P_N}Q \geq \frac{\pi}{4} > \frac{1}{87L}\,.$$

In the same way, applying the first inequality in (5.74) and recalling Remark 5.10, one has

$$P_N\widehat{Q}O = \pi - Q\widehat{P_N}O - Q\widehat{O}P_N = P_{N-1}\widehat{P_N}Q - P\widehat{O}Q \geq \frac{1}{19L} - \frac{1}{50L} > \frac{1}{87L}\,.$$

Hence, (5.77) is checked.

<u>Part 3.</u> *Definition of u_{AB} in the quadrilateral $P_{N-M}P_NQ_MQ$, and validity of (iii) and (iv).* The definition is again trivial: we take any $N-M \leq i < N$ and, setting $j = i - N + M \in [0, M)$, we have to send the quadrilateral $P_iP_{i+1}Q_{j+1}Q_j$ on the quadrilateral $\boldsymbol{P}_i\boldsymbol{P}_{i+1}\boldsymbol{Q}_{j+1}\boldsymbol{Q}_j$. To do so, we send the triangle $P_iP_{i+1}Q_{j+1}$ (resp. $Q_{j+1}Q_jP_i$) onto the triangle $\boldsymbol{P}_i\boldsymbol{P}_{i+1}\boldsymbol{Q}_{j+1}$ (resp. $\boldsymbol{Q}_{j+1}\boldsymbol{Q}_j\boldsymbol{P}_i$) in the bijective affine way, as depicted in Figure 16. Then, we have to check the validity of (iii) and (iv). As in Part 2, checking (iii) basically relies, thanks to Lemma 5.30, on a comparison between the lengths of the corresponding sides and between the corresponding angles. The argument will be very similar to that already used in Part II, but for the sake of clarity we are going to underline all the changes in the proof.

First of all, the argument leading to (5.69) can be *verbatim* repeated for both the segments P_iP_{i+1} and Q_jQ_{j+1}, leading to

$$\frac{1}{1200L^2} \leq \frac{\ell\big(P_iP_{i+1}\big)}{\ell\big(\boldsymbol{P}_i\boldsymbol{P}_{i+1}\big)} \leq \frac{\sqrt{2}}{20L}\,, \qquad \frac{1}{1200L^2} \leq \frac{\ell\big(Q_jQ_{j+1}\big)}{\ell\big(\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\big)} \leq \frac{\sqrt{2}}{20L}\,. \tag{5.78}$$

The argument that we used in Part 2 to bound the length of the segment $P_{i+1}Q$ works, with minor modifications, to estimate the lengths of P_iQ_j and $P_{i+1}Q_{j+1}$. Let us do it in detail for P_iQ_j, being the case of $P_{i+1}Q_{j+1}$ exactly the same. First of all, assuming without loss of generality that P and Q lie in the left side of $\mathcal{D}$, and that P is above Q, let us call $x_j \in (-1/2, -1/5)$ the first coordinate of Q_j, set $V_j \equiv (x_j, -1/2)$, $V \equiv (-1/2, -1/2)$, and define $P^\perp$ the point of the segment OP having first coordinate equal to x_j. We also assume w.l.o.g. that $\tau(P_i) > \tau(P^\perp)$.

As in (5.65), then, we obtain

$$t_{P,i} - t_{Q,j} \leq 17L\,\ell\big(PQ\big)\,, \qquad\qquad t_{P,i+1} - t_{Q,j+1} \leq 17L\,\ell\big(PQ\big)\,. \tag{5.79}$$

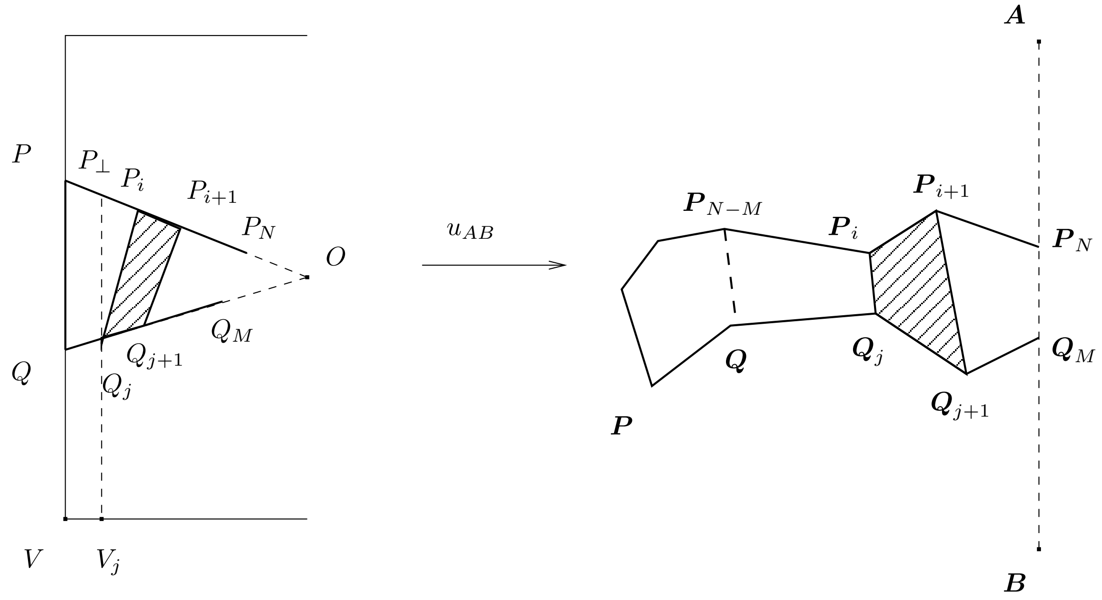

FIGURE 16. The situation in Part 3

We claim that

$$\frac{\sqrt{2}}{10}\,\ell\big(PQ\big) \leq \ell\big(P_iQ_j\big) \leq 17\,\frac{\sqrt{2}}{2}\,L\,\ell\big(PQ\big)\,. \tag{5.80}$$

–notice the presence of $\sqrt{2}/10$ in the left hand side, while there was $\sqrt{2}/2$ in the corresponding term in (5.66). To show the left inequality in (5.80) we start observing that, being P_i in OP, one has

$$\ell\big(P_iQ_j\big) \geq \ell\big(P^\perp Q_j\big)\sin\big(O\widehat{P^\perp}Q_j\big) = \ell\big(P^\perp Q_j\big)\sin\big(O\widehat{P}V\big) \geq \frac{\sqrt{2}}{2}\,\ell\big(P^\perp Q_j\big)\,.$$

Moreover, the segment $P^\perp Q_j$ is parallel to PQ, then (5.63) immediately gives $\ell\big(P^\perp Q_j\big) \geq \ell\big(PQ\big)/5$. Hence, we get $\ell\big(P_iQ_j\big) \geq \frac{\sqrt{2}}{10}\ell\big(PQ\big)$, that is the left inequality of (5.80).

Let us now pass to the right inequality. To do so we call again δ_x and δ_y the horizontal and vertical components of P_iQ_j, so that $\ell(P_iQ_j) = \sqrt{\delta_x^2+\delta_y^2}$. Notice that by construction $\delta_x = |t_{P,i} - t_{Q,j}|/2$. If P_i is above Q_j, as in Figure 16, then $\delta_y \leq \ell\big(PQ\big)$, so that exactly as in (5.67) we get, using (5.79) and the Lipschitz property of u,

$$\begin{aligned}\ell\big(P_iQ_j\big) = \sqrt{\delta_x^2+\delta_y^2} &\leq \sqrt{\bigg(\frac{t_{P,i}-t_{Q,j}}{2}\bigg)^2 + \ell\big(PQ\big)^2} \leq \sqrt{\bigg(\frac{17}{2}\,L\,\ell\big(PQ\big)\bigg)^2 + \ell\big(PQ\big)^2}\\ &= \frac{\sqrt{293}}{2}\,L\,\ell\big(PQ\big) \leq 17\,\frac{\sqrt{2}}{2}\,L\,\ell\big(PQ\big)\,.\end{aligned}$$

On the other hand, if P_i is below Q_j, then surely

$$\frac{\pi}{4} \leq O\widehat{Q_j}V_j \leq P_i\widehat{Q_j}V_j \leq \frac{\pi}{2}\,,$$

thus

$$\ell\big(P_iQ_j\big) = \frac{\delta_x}{\sin\big(P_i\widehat{Q_j}V_j\big)} \leq \sqrt{2}\delta_x = \frac{\sqrt{2}}{2}\left|t_{P,i} - t_{Q,j}\right| \leq 17\,\frac{\sqrt{2}}{2}\,L\,\ell\big(PQ\big)\,,$$

and so the validity of the right inequality in (5.80) is established in both cases. Since (iii) of Lemma 5.18 gives

$$\frac{\ell\big(PQ\big)}{7L} \leq \ell\big(\boldsymbol{P}_i\boldsymbol{Q}_j\big) \leq \ell\big(\boldsymbol{P}\boldsymbol{Q}\big) \leq L\,\ell\big(PQ\big)\,,$$

from (5.80) we immediately obtain

$$\frac{\sqrt{2}}{10L} \leq \frac{\ell\big(P_iQ_j\big)}{\ell\big(\boldsymbol{P}_i\boldsymbol{Q}_j\big)} \leq 85L^2\,. \tag{5.81}$$

The same argument, exchanging i and j with $i+1$ and $j+1$ respectively, gives also

$$\frac{\sqrt{2}}{10L} \leq \frac{\ell\big(P_{i+1}Q_{j+1}\big)}{\ell\big(\boldsymbol{P}_{i+1}\boldsymbol{Q}_{j+1}\big)} \leq 85L^2\,. \tag{5.82}$$

We now have to consider the angles $P_i\widehat{P_{i+1}}Q_{j+1}$, $Q_{j+1}\widehat{Q_j}P_i$ and their correspondent ones in $\bar{\gamma}$. By Lemma 5.18 (i), we already know that

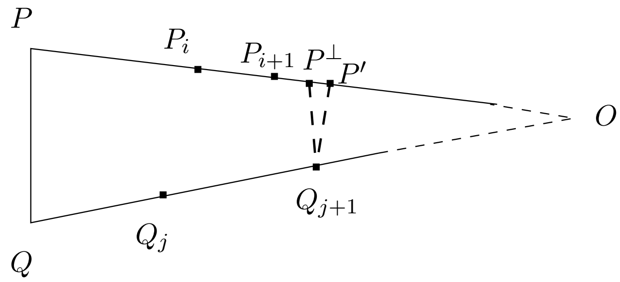

FIGURE 17. Position of the points P_i, P_{i+1}, Q_j, Q_{j+1}, $P^\perp$ and P'

$$15^\circ \leq \boldsymbol{P}_i\widehat{\boldsymbol{P}_{i+1}}\boldsymbol{Q}_{j+1} \leq 165^\circ\,, \qquad\qquad \sin\Big(\boldsymbol{Q}_{j+1}\widehat{\boldsymbol{Q}_j}\boldsymbol{P}_i\Big) \geq \frac{1}{6L^2}\,. \tag{5.83}$$

As in Figure 17, let us then call P' the orthogonal projection of Q_{j+1} on the segment OP, and $P^\perp$ the point of the segment OP with the same first coordinate as Q_{j+1}. Assume for a moment that, as in the figure, P' does not belong to PP_{i+1}. By (5.79) and by (5.63) we have

$$\ell\big(P_{i+1}P^\perp\big)=\frac{|t_{P,i+1}-t_{Q,j+1}|}{2\sin\big(O\widehat{P}Q\big)}\leq\frac{\sqrt{2}}{2}\,17L\,\ell\big(PQ\big)\,,\quad \ell\big(P^\perp Q_{j+1}\big)\geq\frac{\ell\big(PQ\big)}{5}\,,$$
$$\ell\big(P^\perp P'\big)=\ell\big(P^\perp Q_{j+1}\big)\cos\big(O\widehat{P}Q\big)\,,\qquad \ell\big(Q_{j+1}P'\big)=\ell\big(P^\perp Q_{j+1}\big)\sin\big(O\widehat{P}Q\big)\,. \tag{5.84}$$

Therefore, we can evaluate

$$\begin{aligned}\tan\big(P'\widehat{P_{i+1}}Q_{j+1}\big)&=\frac{\ell\big(Q_{j+1}P'\big)}{\ell\big(P_{i+1}P'\big)}\geq\frac{\ell\big(Q_{j+1}P'\big)}{\ell\big(P^\perp P'\big)+\ell\big(P_{i+1}P^\perp\big)}\geq\frac{\frac{\sqrt{2}}{2}\ell\big(P^\perp Q_{j+1}\big)}{\frac{\sqrt{2}}{2}\ell\big(P^\perp Q_{j+1}\big)+\frac{\sqrt{2}}{2}17L\,\ell\big(PQ\big)}\\&=\frac{\ell\big(P^\perp Q_{j+1}\big)}{\ell\big(P^\perp Q_{j+1}\big)+17L\,\ell\big(PQ\big)}\geq\frac{1}{86L}\,,\end{aligned}$$

which immediately gives

$$P_i\widehat{P_{i+1}}Q_{j+1}=\pi-P'\widehat{P_{i+1}}Q_{j+1}\leq\pi-\arctan\left(\frac{1}{86L}\right). \tag{5.85}$$

Notice that, if P' belongs to PP_{i+1}, then $P_i\widehat{P_{i+1}}Q_{j+1}\leq\pi/2$, so (5.85) holds *a fortiori* true.

We claim that one also has

$$P_i\widehat{P_{i+1}}Q_{j+1}\geq\frac{1}{87L}\,. \tag{5.86}$$

To show this, we are going to argue in a very similar way to what already done in Part 2. In fact, if $t_{P,i+1}\leq t_{Q,j+1}$ then (5.86) trivially holds true. Assuming, on the contrary, that $t_{P,i+1}>t_{Q,j+1}$, we call for brevity $\alpha:=P_i\widehat{P_{i+1}}Q_{j+1}$ and $\theta:=O\widehat{P^\perp}Q_{j+1}-\frac{\pi}{2}\in\big[-\frac{\pi}{4},\frac{\pi}{4}\big)$, and we notice that an immediate trigonometric argument gives

$$\ell\big(P^\perp Q_{j+1}\big)=\frac{t_{P,i+1}-t_{Q,j+1}}{2}\Big(\tan(\theta+\alpha)-\tan\theta\Big). \tag{5.87}$$

We can assume that

$$\theta+\alpha\leq\frac{\pi}{4}+\frac{1}{87}\,,$$

since otherwise (5.86) is already established. Hence, recalling (5.84), (5.87), the fact that $\theta\geq-\pi/4$, (5.79) and the Lipschitz property of u we get

$$\ell\big(PQ\big)\leq5\,\ell\big(P^\perp Q_{j+1}\big)=\frac{5}{2}\left(t_{P,i+1}-t_{Q,j+1}\right)\Big(\tan(\theta+\alpha)-\tan\theta\Big)\leq\frac{85L\,\ell\big(PQ\big)}{2}\,\frac{\alpha}{\cos^2\left(\frac{\pi}{4}+\frac{1}{87}\right)}\,,$$

which implies

$$\alpha\geq\frac{2\cos^2\left(\frac{\pi}{4}+\frac{1}{87}\right)}{85L}\geq\frac{1}{87L}\,.$$

Thus, (5.86) is now established. If we repeat exactly the same argument that we used to obtain (5.85) and (5.86) in the symmetric way, that is, substituting P_i, P_{i+1} and Q_{j+1} with Q_{j+1}, Q_j and P_i respectively, then we get

$$Q_{j+1}\widehat{Q_j}P_i \leq \pi - \arctan\left(\frac{1}{86L}\right), \qquad Q_{j+1}\widehat{Q_j}P_i \geq \frac{1}{87L}. \tag{5.88}$$

Finally, we can check the validity of (iii) by making use of (5.61) of Lemma 5.30. Indeed, let us call ϕ (resp. $\tilde{\phi}$) the affine maps which send $P_iP_{i+1}Q_{j+1}$ on $\boldsymbol{P}_i\boldsymbol{P}_{i+1}\boldsymbol{Q}_{j+1}$ (resp. $Q_{j+1}Q_jP_i$ on $\boldsymbol{Q}_{j+1}\boldsymbol{Q}_j\boldsymbol{P}_i$). According with the notation of Lemma 5.30, let us write

$$\begin{aligned}
a &= \ell\big(P_{i+1}Q_{j+1}\big), & b &= \ell\big(P_iP_{i+1}\big), & \alpha &= P_i\widehat{P_{i+1}}Q_{j+1}, \\
a' &= \ell\big(\boldsymbol{P}_{i+1}\boldsymbol{Q}_{j+1}\big), & b' &= \ell\big(\boldsymbol{P}_i\boldsymbol{P}_{i+1}\big), & \alpha' &= \boldsymbol{P}_i\widehat{\boldsymbol{P}_{i+1}}\boldsymbol{Q}_{j+1}, \\
\tilde{a} &= \ell\big(P_iQ_j\big), & \tilde{b} &= \ell\big(Q_jQ_{j+1}\big), & \tilde{\alpha} &= Q_{j+1}\widehat{Q_j}P_i, \\
\tilde{a}' &= \ell\big(\boldsymbol{P}_i\boldsymbol{Q}_j\big), & \tilde{b}' &= \ell\big(\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\big), & \tilde{\alpha}' &= \boldsymbol{Q}_{j+1}\widehat{\boldsymbol{Q}_j}\boldsymbol{P}_i.
\end{aligned}$$

The estimates (5.78), (5.81) and (5.82) for the sides, and (5.83), (5.85), (5.86) and (5.88) for the angles, give us

$$\frac{\sqrt{2}}{10L} \leq \frac{a}{a'} \leq 85L^2, \qquad \frac{1}{1200L^2} \leq \frac{b}{b'} \leq \frac{\sqrt{2}}{20L}, \qquad \sin\alpha' \geq \frac{1}{4}, \qquad \sin\alpha \geq \frac{1}{88L}, \tag{5.89}$$

$$\frac{\sqrt{2}}{10L} \leq \frac{\tilde{a}}{\tilde{a}'} \leq 85L^2, \qquad \frac{1}{1200L^2} \leq \frac{\tilde{b}}{\tilde{b}'} \leq \frac{\sqrt{2}}{20L}, \qquad \sin\tilde{\alpha}' \geq \frac{1}{6L^2}, \qquad \sin\tilde{\alpha} \geq \frac{1}{88L}, \tag{5.90}$$

where the estimates for α and $\tilde{\alpha}$ can be obtained in the very same way as (5.76). As in Part 2, then, we can apply (5.61) together with (5.89) and (5.90) to obtain

$$\begin{aligned}
\mathrm{Lip}(\phi) &\leq \frac{a'}{a} + \frac{2b'}{b\sin\alpha} + \frac{a'}{a\sin\alpha} \leq 5\sqrt{2}\,L + 211200L^3 + 440\sqrt{2}\,L^2 \leq 212000L^3, \\
\mathrm{Lip}(\phi^{-1}) &\leq \frac{a}{a'} + \frac{2b}{b'\sin\alpha'} + \frac{a}{a'\sin\alpha'} \leq 85L^2 + \frac{2\sqrt{2}}{5L} + 340\,L^2, \\
\mathrm{Lip}(\tilde{\phi}) &\leq \frac{\tilde{a}'}{\tilde{a}} + \frac{2\tilde{b}'}{\tilde{b}\sin\tilde{\alpha}} + \frac{\tilde{a}'}{\tilde{a}\sin\tilde{\alpha}} \leq 5\sqrt{2}\,L + 211200L^3 + 440\sqrt{2}\,L^2 \leq 212000L^3, \\
\mathrm{Lip}(\tilde{\phi}^{-1}) &\leq \frac{\tilde{a}}{\tilde{a}'} + \frac{2\tilde{b}}{\tilde{b}'\sin\tilde{\alpha}'} + \frac{\tilde{a}}{\tilde{a}'\sin\tilde{\alpha}'} \leq 85L^2 + \frac{3\sqrt{2}L}{5} + 500\,L^4 \leq 212000L^4.
\end{aligned}$$

Thus, we have checked the validity of (iii).

Concerning (iv), we have to show that

$$P_N\widehat{Q_M}O \geq \frac{1}{87L}, \qquad Q_M\widehat{P_N}O \geq \frac{1}{87L}. \tag{5.91}$$

In fact, applying (5.85) with $i = N-1$ and then $j = M-1$, we have that

$$Q_M\widehat{P_N}O = \pi - P_{N-1}\widehat{P_N}Q_M \geq \arctan\left(\frac{1}{86L}\right) \geq \frac{1}{87L},$$

and the same argument, exchanging the roles of P_N and Q_M ensures the validity of (5.91). Thus, (iv) is established and the proof is concluded. □

5.10. Extension in the whole square

We finally come to the explicit definition of the piecewise affine map $\tilde{v}$. It is important to recall now Lemma 5.1 of Section 5.3. It provides us with a central ball $\widehat{\boldsymbol{\mathcal{B}}} \subseteq \Gamma$ which is such that the intersection of its boundary with $\partial\Gamma$ consists of N points $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_N$, with $N \geq 2$. Moreover, for each $1 \leq i \leq N$ one has that the path $\widehat{A_iA_{i+1}}$ does not contain other points A_j with $j \neq i, i+1$. Or, in other words, that for each $1 \leq i \leq N$ the anticlockwise path connecting A_i and A_{i+1} on $\partial\mathcal{D}$ has length at most 2 (keep in mind Remark 5.2). Notice that this implies, in the case $N = 2$, that the points A_1 and A_2 are opposite points of $\partial\mathcal{D}$. The set Γ is then subdivided in N primary sectors $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$, plus the remaining polygon Π (see e.g. Figure 18, where Π is a (coloured) quadrilateral).

Moreover, thanks to Section 5.9, we have N disjoint polygonal subsets $\mathcal{D}_i$ as in the Figure, and N extensions $\tilde{u}_i : \mathcal{D}_i \to \mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$. It is then easy to guess a possible definition of $\tilde{v}$, that is setting $\tilde{v} \equiv \tilde{u}_i$ on each $\mathcal{D}_i$ and then sending in the obvious piecewise affine way the set $\mathcal{D} \setminus \cup_i\mathcal{D}_i$ (coloured in the figure) into the polygon Π, defining $\tilde{u}(O)$ as the center of $\widehat{\boldsymbol{\mathcal{B}}}$. Unfortunately, this strategy does not always work. For instance, if $N = 2$, then Π is a degenerate empty polygon, thus it cannot be the bi-Lipschitz image of the non-empty region $\mathcal{D} \setminus \cup_i\mathcal{D}_i$. And also for $N \geq 3$, it may happen that the polygon Π does not contain the center of $\widehat{\boldsymbol{\mathcal{B}}}$, which is instead inside some sector $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$. And in that case, obviously, the center of $\widehat{\boldsymbol{\mathcal{B}}}$ can not be the point $\tilde{u}(O)$. Having these possibilities in mind, we are now ready to give the proof of the first part of Theorem 0.4, that is, the existence of the piecewise affine extension $\tilde{v}$ of $\tilde{u}$.

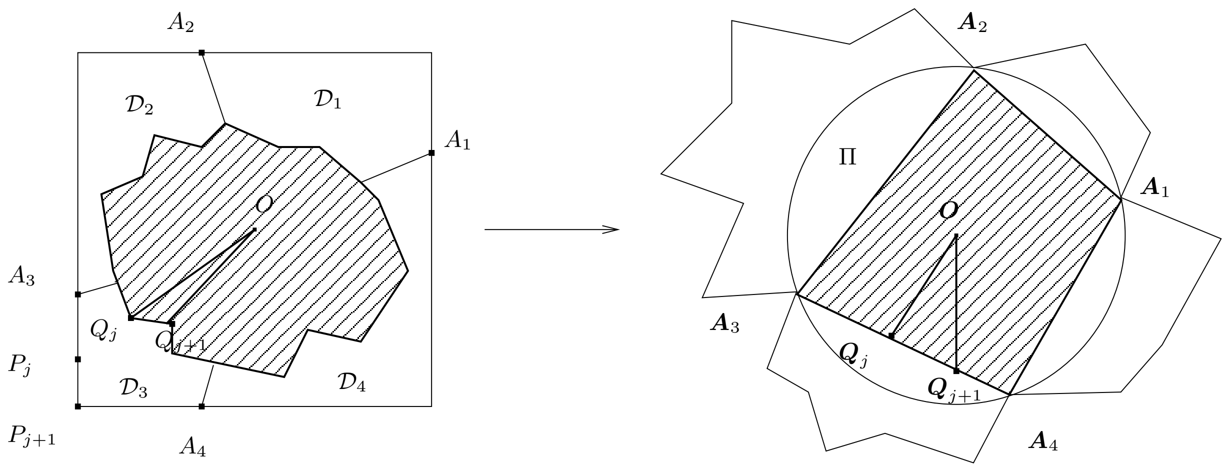

FIGURE 18. The sets $\mathcal{D}_i$ in $\mathcal{D}$ and the set Π in Γ

PROOF OF THEOREM 0.4 (PIECEWISE AFFINE EXTENSION). We need to consider three possible situations. To distinguish between them, let us start with a definition. For any $1 \leq i \leq N$, we call d_i the signed distance between the segment $\boldsymbol{A}_i\boldsymbol{A}_{i+1}$ and the center of $\widehat{\boldsymbol{\mathcal{B}}}$, where the sign is positive if the center does not belong to $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$, and negative otherwise –for instance, in the situation of Figure 18 all the distances d_i are positive. Let us also call r the radius of $\widehat{\boldsymbol{\mathcal{B}}}$, and observe that

$$\frac{2}{3L} \leq r \leq \frac{2L}{\pi}\,. \tag{5.92}$$

The first inequality has been already pointed out in Remark 5.2. Concerning the second one, it immediately follows by observing that the perimeter of Γ is at least $2\pi r$ by geometric reasons, and on the other hand it is less than $4L$ since it is the $L-$Lipschitz image of the square $\mathcal{D}$ which has perimeter 4. We can then give our proof in the different cases.

Case A. *For each $1 \leq i \leq N$, one has $d_i \geq r/4$.*

This is the simplest of the three cases, and the situation is already shown in Figure 18. We start by calling $\boldsymbol{O}$ the center of $\widehat{\boldsymbol{\mathcal{B}}}$. Then, for all $1 \leq i \leq N$, let us define $\tilde{v} \equiv \tilde{u}_i$ on $\mathcal{D}_i$. We have now to send $\mathcal{D} \setminus \cup_i \mathcal{D}_i$ into Π. In order to do so, consider all the vertices P_j of $\partial\mathcal{D}$. For each vertex P_j, which belongs to some set $\mathcal{D}_i$ for a suitable $i = i(j)$, there exists a point Q_j, which is the last point of the segment P_jO which belongs to $\partial\mathcal{D}_i$. Indeed, the segment P_jO intersects $\partial\mathcal{D}_i$ only at P_j and at Q_j, and the two points are the same if and only if $P_j \equiv A_i$ or $P_j \equiv A_{i+1}$). By the construction of Step VII, we know that $\tilde{v}(Q_j) = (\boldsymbol{P}_j)_{N(P_j)}$, and we will write for brevity $\boldsymbol{Q}_j := (\boldsymbol{P}_j)_{N(P_j)}$. Now notice that $\mathcal{D} \setminus \cup_i \mathcal{D}_i$ is the union of the triangles $Q_jQ_{j+1}O$, and on the other hand Π is the union of the triangles $\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\boldsymbol{O}$. We then conclude our definition of $\tilde{v}$ by imposing that $\tilde{v}$ sends in the affine way each triangle $Q_jQ_{j+1}O$ into the triangle $\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\boldsymbol{O}$. Hence, it is clear that $\tilde{v}$ is a piecewise affine homeomorphism between $\mathcal{D}$ and Γ, which extends the original function $\tilde{u}$. Thus, to finish the proof we only have to check that $\tilde{v}$ is bi-Lipschitz with the right constant. Since this is already ensured by Lemma 5.31 on each primary sector, it is enough to consider a single triangle $Q_jQ_{j+1}O$. Using again Lemma 5.30 from Section 5.9 to estimate the bi-Lipschitz constant of the affine map on the triangle, we have to give upper and lower bounds for the quantities

$$\begin{aligned} a &= \ell\big(Q_jQ_{j+1}\big)\,, & b &= \ell\big(Q_jO\big)\,, & \alpha &= O\widehat{Q_j}Q_{j+1}\,,\\ a' &= \ell\big(\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\big)\,, & b' &= \ell\big(\boldsymbol{Q}_j\boldsymbol{O}\big)\,, & \alpha' &= \boldsymbol{O}\widehat{\boldsymbol{Q}_j}\boldsymbol{Q}_{j+1}\,. \end{aligned}$$

Let us then collect all the needed estimates: first of all, notice that the ratio a/a' has already been evaluated in Lemma 5.31, either in Part 2 or in Part 3. Thus, recalling (5.75) and (5.89), we already know that

$$\frac{\sqrt{2}}{10L} \leq \frac{a}{a'} \leq 85L^2\,. \tag{5.93}$$

Concerning the ratio b/b', notice that by geometric reasons and recalling (5.63), we have

$$\frac{1}{10} \leq b \leq \frac{\sqrt{2}}{2}\,, \tag{5.94}$$

while by (5.92) and the assumption of this case

$$\frac{1}{6L} \leq \frac{r}{4} \leq b' \leq r \leq \frac{2L}{\pi}\,. \tag{5.95}$$

Thus,

$$\frac{\pi}{20L} \leq \frac{b}{b'} \leq 3\sqrt{2}L\,. \tag{5.96}$$

Let us finally consider the angles α and α'. Concerning α, property (iv) of Lemma 5.31 tells us that

$$\frac{1}{87L} \leq \alpha \leq \pi - \frac{1}{87L}\,. \tag{5.97}$$

On the other hand, by the assumption of this case we clearly have

$$\arcsin\frac{1}{4} \leq \alpha' \leq \pi - \arcsin\frac{1}{4}\,,$$

and then

$$\frac{1}{\sin\alpha} \leq 88L\,, \qquad\qquad \frac{1}{\sin\alpha'} \leq 4\,. \tag{5.98}$$

We can then apply (5.61) making use of (5.93), (5.96) and (5.98) to get

$$\begin{aligned}
\mathrm{Lip}(\phi) &\leq \frac{a'}{a} + \frac{2b'}{b\sin\alpha} + \frac{a'}{a\sin\alpha} \leq 5\sqrt{2}L + \frac{3520}{\pi}\,L^2 + 440\sqrt{2}L^2\,,\\
\mathrm{Lip}(\phi^{-1}) &\leq \frac{a}{a'} + \frac{2b}{b'\sin\alpha'} + \frac{a}{a'\sin\alpha'} \leq 85L^2 + 24\sqrt{2}L + 340L^2\,,
\end{aligned}$$

thus the claim of the theorem is obtained in this first case.

Case B. *There exists some $1 \leq i \leq N$ such that $-r/2 \leq d_i < r/4$.*

Also in this case, we set $\tilde v(O) = \boldsymbol{O}$ to be the center of $\widehat{\boldsymbol{\mathcal{B}}}$. Let us write now $\mathcal{D} = \cup_i \mathcal{A}_i$, where each $\mathcal{A}_i$ is the subset of $\mathcal{D}$ whose boundary is $A_iO \cup A_{i+1}O \cup \widehat{A_iA_{i+1}}$. Notice that for each i, one has $\mathcal{D}_i \subseteq \mathcal{A}_i$, and in particular we set $\mathcal{I}_i = \mathcal{A}_i \setminus \mathcal{D}_i$, the "internal part" of $\mathcal{A}_i$. Our definition of $\tilde v$ will be done in such a way that, for each $1 \leq i \leq N$, $v(\mathcal{A}_i)$ will be the union of the sector $\boldsymbol{\mathcal{S}}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$ and the triangle $\boldsymbol{A}_i\boldsymbol{A}_{i+1}\boldsymbol{O}$. Observe that, in the Case A, we had defined $\tilde v$ so that for each i one had $\tilde v(\mathcal{D}_i) = \boldsymbol{\mathcal{S}}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$ and $\tilde v(\mathcal{I}_i) = \boldsymbol{A}_i\boldsymbol{A}_{i+1}\boldsymbol{O}$.

Let us fix a given $1 \leq i \leq N$, and notice that either $d_i \geq r/4$, or $-r/2 \leq d_i < r/4$. In fact, since we assume the existence of some i for which $-r/2 \leq d_i < r/4$, then it is not possible that there exist some other i with $d_i < -r/2$.

If $d_i \geq r/4$, then we define $\tilde v$ exactly as in Case A, that is, we set $\tilde v \equiv \tilde u_i$ on $\mathcal{D}_i$, and for any two consecutive vertices P_j, $P_{j+1} \in \widehat{A_iA_{i+1}}$ we let $\tilde v$ be the affine function transporting the triangle $Q_jQ_{j+1}O$ of $\mathcal{D}$ onto the triangle $\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\boldsymbol{O}$ of Γ, where $\boldsymbol{Q}_k = \boldsymbol{P}_{N(P_k)}$. In this case, $\tilde v$ is bi-Lipschitz on $\mathcal{A}_i$ with constant at most $5\sqrt{2}L + 3520L^2/\pi + 440\sqrt{2}L^2$, as we already showed in Case A.

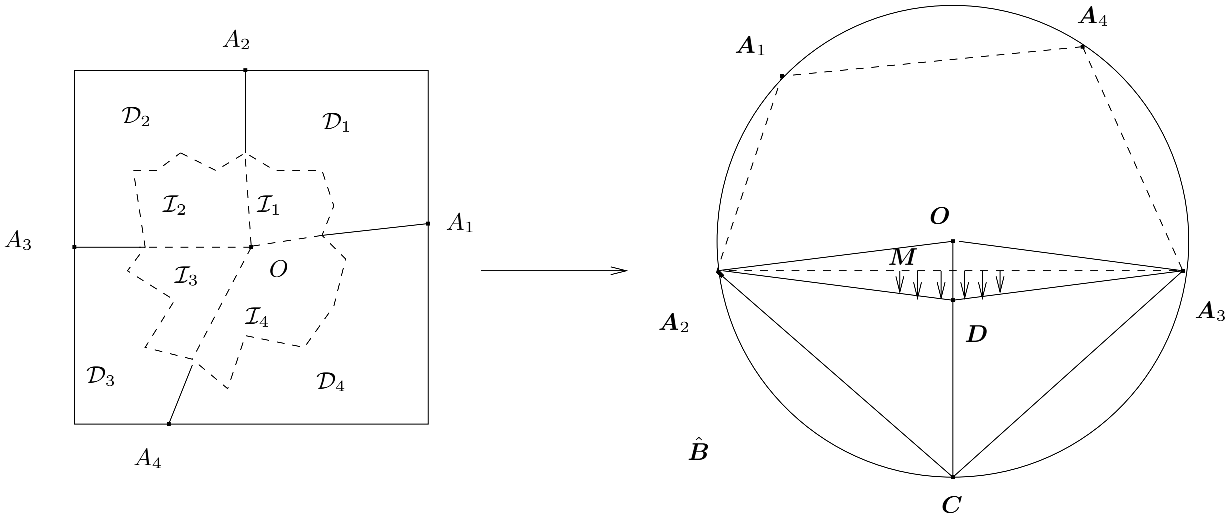

FIGURE 19. The situation for Case B, with the sets $\mathcal{A}_i$ and the points $\boldsymbol{M}$, $\boldsymbol{D}$ and $\boldsymbol{C}$

Consider then the case of an index i such that $-r/4 \leq d_i \leq r/4$, as it happens for $i = 2$ in Figure 19 (where d_2 is positive but smaller than $r/4$). As in the figure, let us call $\boldsymbol{C} \in \partial\widehat{\boldsymbol{\mathcal{B}}}$ the point belonging to the axis of the segment $\boldsymbol{A}_i\boldsymbol{A}_{i+1}$ and to the sector $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$, and let also $\boldsymbol{D} \in \boldsymbol{OC}$ be the point such that $\ell\big(\boldsymbol{OD}\big) = r/4$. We now introduce a bi-Lipschitz and piecewise affine function $\Phi : \boldsymbol{A}_i\boldsymbol{A}_{i+1}\boldsymbol{C} \to \boldsymbol{A}_i\boldsymbol{D}\boldsymbol{A}_{i+1}\boldsymbol{C}$. If we call $\boldsymbol{M}$ the mid-point of $\boldsymbol{A}_i\boldsymbol{A}_{i+1}$, the function Φ is simply given by the affine map between the triangle $\boldsymbol{A}_i\boldsymbol{MC}$ and $\boldsymbol{A}_i\boldsymbol{DC}$, and by the affine map between $\boldsymbol{A}_{i+1}\boldsymbol{MC}$ and $\boldsymbol{A}_{i+1}\boldsymbol{DC}$. The fact that Φ is piecewise affine is clear, being Φ defined gluing two affine maps. Moreover, by the fact that $-r/2 \leq d_i < r/4$, Φ is $2-$Lipschitz and Φ^{-1} is $3-$Lipschitz. We will extend $\Phi : \mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1}) \to \mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1})$, whitout need of changing the name, as the identity out of the triangle $\boldsymbol{A}_i\boldsymbol{A}_{i+1}\boldsymbol{C}$. Of course also the extended Φ is $2-$Lipschitz and its inverse is $3-$Lipschitz.

We are now ready to define $\tilde{v}$ in $\mathcal{A}_i$. First of all, we set $\tilde{v} \equiv \Phi \circ \tilde{u}_i$ on $\mathcal{D}_i$. Thanks to Lemma 5.31 and the properties of Lipschitz functions, we have that $\tilde{v}$ is piecewise affine and bi-Lipschitz with constant $3 \cdot 212000L^4 = 636000L^4$ on its image, which is $\mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1}) \setminus \boldsymbol{A}_i\boldsymbol{A}_{i+1}\boldsymbol{D}$. To conclude, we need to send $\mathcal{I}_i$ onto the quadrilater $\boldsymbol{A}_i\boldsymbol{O}\boldsymbol{A}_{i+1}\boldsymbol{D}$. To do so, consider all the vertices $P_j \in \widehat{A_iA_j}$, and define $Q_j \in \partial\mathcal{D}_i$ as in Case A. This time, we will not set $\boldsymbol{Q}_j = \tilde{u}_i(Q_j)$: instead, $\boldsymbol{Q}_j$ will be defined as $\boldsymbol{Q}_j := \Phi\big(\tilde{u}_i(Q_j)\big)$, so that $\tilde{v}(Q_j) = \boldsymbol{Q}_j$ as usual. Notice that, again, $\mathcal{I}_i$ is the union of the triangles $Q_jQ_{j+1}O$, while the quadrilateral $\boldsymbol{A}_i\boldsymbol{O}\boldsymbol{A}_{i+1}\boldsymbol{D}$ is the union of the triangles $\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\boldsymbol{O}$ (up to the possible addition of a new vertex corresponding to $\boldsymbol{D}$). The map $\tilde{v}$ on $\mathcal{I}_i$ will be then the map which sends each triangle $Q_jQ_{j+1}O$ onto $\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\boldsymbol{O}$ in the affine way. Clearly the map v is then a piecewise affine homeomorphism, so that again we only have to check its bi-Lipschitz constant (Figure 20 may help the reader to follow the construction). As usual, we will apply (5.61) of Lemma 5.30, so we set the quantities

$$a = \ell\big(Q_jQ_{j+1}\big)\,, \qquad b = \ell\big(Q_jO\big)\,, \qquad \alpha = O\widehat{Q_j}Q_{j+1}\,,$$
$$a' = \ell\big(\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\big)\,, \qquad b' = \ell\big(\boldsymbol{Q}_j\boldsymbol{O}\big)\,, \qquad \alpha' = \boldsymbol{O}\widehat{\boldsymbol{Q}_j}\boldsymbol{Q}_{j+1}\,.$$

Recall that, studying Case A, we have already found in (5.93) that for each vertex $P_j \in$

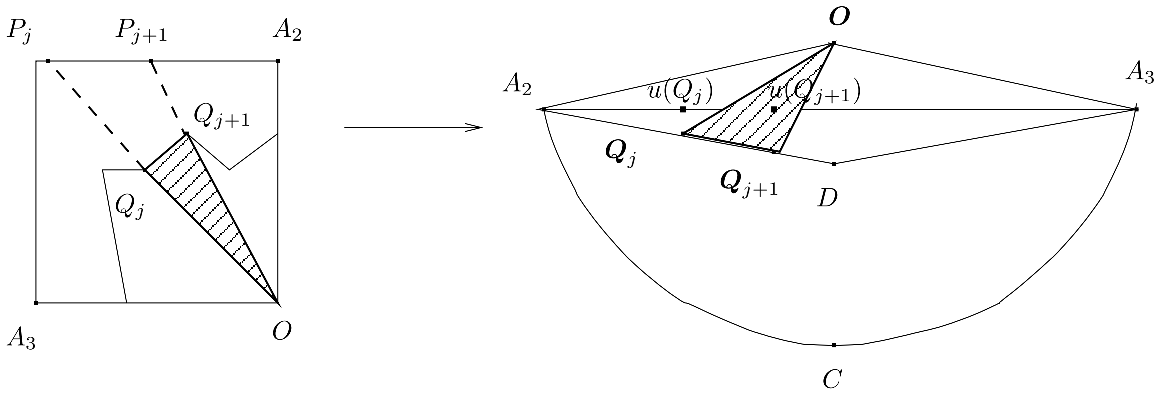

FIGURE 20. A zoom for Case B, with Q_j, Q_{j+1}, $\tilde u_2(Q_j)$, $\tilde u_2(Q_{j+1})$, $\boldsymbol{Q}_j$ and $\boldsymbol{Q}_{j+1}$

$\widehat{A_iA_{i+1}}$ one has

$$\frac{\sqrt{2}}{10L} \leq \frac{\ell\big(Q_jQ_{j+1}\big)}{\ell\Big(\tilde u_i(Q_j)\tilde u_i(Q_{j+1})\Big)} \leq 85L^2\,. \tag{5.99}$$

Notice also that now we have $\ell\big(Q_jQ_{j+1}\big) = a$, exactly as in Case A, but it is no more true that $\ell\Big(\tilde u_i(Q_j)\tilde u_i(Q_{j+1})\Big) = a'$. However, since Φ is 2−Lipschitz and Φ^{-1} is 3−Lipschitz, we have

$$a' = \ell\big(\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\big) = \ell\Big(\Phi\big(\tilde u_i(Q_j)\big)\Phi\big(\tilde u_i(Q_{j+1})\big)\Big) \leq 2\,\ell\Big(\tilde u_i(Q_j)\tilde u_i(Q_{j+1})\Big)\,,$$
$$a' = \ell\big(\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\big) = \ell\Big(\Phi\big(\tilde u_i(Q_j)\big)\Phi\big(\tilde u_i(Q_{j+1})\big)\Big) \geq \frac{\ell\Big(\tilde u_i(Q_j)\tilde u_i(Q_{j+1})\Big)}{3}\,,$$

which by (5.99) ensures

$$\frac{\sqrt{2}}{20L} \leq \frac{a}{a'} \leq 255L^2\,. \tag{5.100}$$

To bound the ratio b/b', we have to estimate both b and b'. Concerning b, we already know by (5.94) that

$$\frac{1}{10} \leq b \leq \frac{\sqrt{2}}{2}\,.$$

On the other hand, let us study b'. The estimate from above, exactly as in (5.95), is simply obtained by (5.92) as

$$b' \leq r \leq \frac{2L}{\pi}\,.$$

Instead, to get the estimate from below, it is enough to recall that $\boldsymbol{Q}_j$ belongs to the segment $\boldsymbol{A}_i\boldsymbol{D}$ (or $\boldsymbol{A}_{i+1}\boldsymbol{D}$). Thus, being $d_i \leq r/4$, an immediate geometric argument and again (5.92) give

$$b' \geq \frac{1}{2\sqrt{7}}\,r \geq \frac{1}{3\sqrt{7}L}$$

Collecting the inequalities that we just found, we get

$$\frac{\pi}{20L} \leq \frac{b}{b'} \leq \frac{3}{2}\sqrt{14}L\,. \tag{5.101}$$

Concerning the angles, (5.97) already tells us that

$$\frac{1}{87L} \leq \alpha \leq \pi - \frac{1}{87L}\,.$$

Moreover, an immediate geometric argument ensures that $\sin\alpha'$ is minimal if $\alpha' = \boldsymbol{O}\widehat{\boldsymbol{A}_i}\boldsymbol{D}$, and in turn this last angle depends only on d_i and it is minimal when $d_i = -r/2$. A simple calculation ensures that, in this extremal case, one has

$$\alpha' = \arctan\frac{1}{\sqrt{3}/2} - \arctan\frac{1/2}{\sqrt{3}/2} > 15^\circ\,,$$

thus we have

$$\frac{1}{\sin\alpha} \leq 88L\,, \qquad\qquad \frac{1}{\sin\alpha'} \leq 4\,. \tag{5.102}$$

Therefore, by applying (5.61) having (5.100), (5.101) and (5.102) at hand, we get

$$\begin{aligned}
\mathrm{Lip}(\phi) &\leq \frac{a'}{a} + \frac{2b'}{b\sin\alpha} + \frac{a'}{a\sin\alpha} \leq 10\sqrt{2}L + \frac{3520}{\pi}\,L^2 + 880\sqrt{2}L^2\,,\\
\mathrm{Lip}(\phi^{-1}) &\leq \frac{a}{a'} + \frac{2b}{b'\sin\alpha'} + \frac{a}{a'\sin\alpha'} \leq 255L^2 + 12\sqrt{14}L + 1020L^2\,.
\end{aligned}$$

<u>Case C.</u> *There exists some $1 \leq i \leq N$ such that $d_i < -r/2$.*

Notice that this i is necessarily unique, since $d_1 < -r/2$ implies that for all $i \neq 1$ one has $d_i > r/2$. In this case, differently from the preceding ones, we will *not* set $\boldsymbol{O}$ to be the center of $\widehat{\boldsymbol{\mathcal{B}}}$. Instead, as in Figure 21, let us call $\boldsymbol{M}$ the midpoint of $\boldsymbol{A}_1\boldsymbol{A}_2$, $\boldsymbol{C} \in \widehat{\boldsymbol{\mathcal{B}}}$ the point such that the triangle $\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{C}$ is equilateral, and $\boldsymbol{D}$ and $\boldsymbol{O}$ the two points which divide the segment $\boldsymbol{CM}$ in three equal parts. We aim at defining the extension $\tilde{v}$ in such a way that $\tilde{v}(O) = \boldsymbol{O}$.

Before starting, we need to underline a basic estimate, that is,

$$\frac{4}{3L} \leq \ell\big(\boldsymbol{A}_1\boldsymbol{A}_2\big) \leq \frac{2\sqrt{3}}{\pi}\,L\,. \tag{5.103}$$

The right estimate is an immediate consequence of the assumption $d_1 < -r/2$ and of (5.92). Concerning the left estimate, recall that, as noticed in Remark 5.2, there must be two points $\boldsymbol{A}_i\boldsymbol{A}_j \in \partial\widehat{\boldsymbol{\mathcal{B}}}$ such that $\ell(\boldsymbol{A}_i\boldsymbol{A}_j) \geq 4/3L$. Thus the left estimate follows simply by observing that the distance $\ell(\boldsymbol{A}_i\boldsymbol{A}_j)$ is maximal, under the assumption of this Case C, for $i = 1$ and $j = 2$.

We can now start our construction. Exactly as in Case B, call $\Phi : \mathcal{S}(\boldsymbol{A}_1\boldsymbol{A}_2) \to \mathcal{S}(\boldsymbol{A}_1\boldsymbol{A}_2)$ the piecewise affine function which equals the identity out of $\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{C}$, and which sends in the affine way the triangle $\boldsymbol{A}_1\boldsymbol{M}\boldsymbol{C}$ (resp. $\boldsymbol{A}_2\boldsymbol{M}\boldsymbol{C}$) onto the triangle $\boldsymbol{A}_1\boldsymbol{D}\boldsymbol{C}$ (resp. $\boldsymbol{A}_2\boldsymbol{D}\boldsymbol{C}$). Also in this case, one easily finds that Φ is $2-$Lipschitz, while Φ^{-1} is $3-$Lipschitz. We are now ready to define the function $\tilde{v}$. As in Case B, for any i our definition will be so that $\tilde{v}(\mathcal{A}_i) = \mathcal{S}(\boldsymbol{A}_i\boldsymbol{A}_{i+1}) \cup \boldsymbol{A}_i\boldsymbol{A}_{i+1}\boldsymbol{O}$.

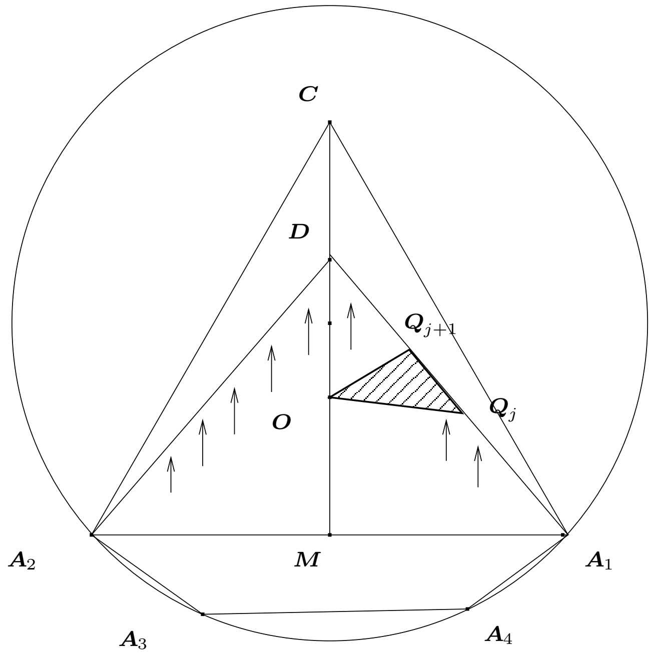

FIGURE 21. Situation in case C, with $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{M}$ and $\boldsymbol{O}$

Let us start with $i = 1$. First of all, we define $\tilde{v} : \mathcal{D}_1 \to \Gamma$ as $\tilde{v} = \Phi \circ \tilde{u}_1$, which is, exactly as in Case B, a $636000L^4$ bi-Lipschitz piecewise affine homeomorphism between $\mathcal{D}_1$ and $\mathcal{S}(\boldsymbol{A}_1\boldsymbol{A}_2) \setminus \boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{D}$. Moreover, defining Q_j and $\boldsymbol{Q}_j$ as in Case B, the internal part $\mathcal{I}_1$ is the union of the triangles $Q_jQ_{j+1}O$, while $\boldsymbol{A}_1\boldsymbol{O}\boldsymbol{A}_2\boldsymbol{D}$ is the union of the triangles $\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\boldsymbol{O}$ (again, possibly adding a vertex corresponding to $\boldsymbol{D}$). We will then define again $\tilde{v} : \mathcal{I}_1 \to \mathcal{D}$ by sending in the affine way each triangle in its corresponding one, and since $\tilde{v}$ is again a piecewise affine homeomorphism by definition we have to check its bi-Lipschitz constant. To do so, we define as in Case B the constants

$$a = \ell(Q_jQ_{j+1})\,, \qquad b = \ell(Q_jO)\,, \qquad \alpha = O\widehat{Q_j}Q_{j+1}\,,$$

$$a' = \ell\big(\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\big)\,, \qquad b' = \ell\big(\boldsymbol{Q}_j\boldsymbol{O}\big)\,, \qquad \alpha' = \boldsymbol{O}\widehat{\boldsymbol{Q}_j}\boldsymbol{Q}_{j+1}\,.$$

The very same arguments which lead to (5.100) and (5.97) give again

$$\frac{\sqrt{2}}{20L} \leq \frac{a}{a'} \leq 255L^2\,, \qquad \frac{1}{\sin\alpha} \leq 88L\,. \tag{5.104}$$

Since (5.94) is still true, to estimate b/b' we again need to bound b' from above and from below. By easy geometric arguments, since $\boldsymbol{Q}_j$ belongs to $\boldsymbol{A}_1\boldsymbol{D}$ or to $\boldsymbol{A}_2\boldsymbol{D}$, we find

$$\frac{\sqrt{7}}{14}\,\ell\big(\boldsymbol{A}_1\boldsymbol{A}_2\big) \leq b' \leq \ell\big(\boldsymbol{A}_1\boldsymbol{O}\big) = \frac{\sqrt{3}}{3}\,\ell\big(\boldsymbol{A}_1\boldsymbol{A}_2\big)\,.$$

(recall that Figure 21 depicts the situation and the position of the points). Thanks to (5.103), then, we deduce

$$\frac{2\sqrt{7}}{21L} \leq b' \leq \frac{2}{\pi}\,L\,,$$

which by (5.94) yields

$$\frac{\pi}{20L} \leq \frac{b}{b'} \leq \frac{3}{4}\sqrt{14}\,L\,. \tag{5.105}$$

Finally, we have to estimate $\sin\alpha'$. As is clear from Figure 21, $\sin\alpha'$ is minimal if $\boldsymbol{Q}_j \equiv \boldsymbol{A}_1$, thus if $\alpha' = \boldsymbol{O}\widehat{\boldsymbol{A}_1}\boldsymbol{D}$. Since in this extremal case one has

$$\alpha' = \arctan\frac{2\sqrt{3}}{3} - \arctan\frac{\sqrt{3}}{3} > 15^\circ\,,$$

we obtain

$$\sin\alpha' \geq \frac{1}{4}\,. \tag{5.106}$$

Applying then once more (5.61), thanks to (5.104), (5.105) and (5.106) we get

$$\begin{aligned}
\mathrm{Lip}(\phi) &\leq \frac{a'}{a} + \frac{2b'}{b\sin\alpha} + \frac{a'}{a\sin\alpha} \leq 10\sqrt{2}L + \frac{3520}{\pi}\,L^2 + 880\sqrt{2}L^2\,,\\
\mathrm{Lip}(\phi^{-1}) &\leq \frac{a}{a'} + \frac{2b}{b'\sin\alpha'} + \frac{a}{a'\sin\alpha'} \leq 255L^2 + 6\sqrt{14}L + 1020L^2\,.
\end{aligned}$$

To conclude, we have now to consider that case $i \neq 1$. Notice that now we cannot simply rely on the calculations done in Case A as we did in Case B, because this time $\boldsymbol{O}$ is not the center of $\widehat{\boldsymbol{\mathcal{B}}}$. Nevertheless, we still define $\tilde{v} \equiv \tilde{u}_i$ on $\mathcal{D}_i$, which is $212000L^4$ bi-Lipschitz by Section 5.10, and again, to conclude, we have to send $\mathcal{I}_i$ onto $\boldsymbol{A}_i\boldsymbol{A}_{i+1}\boldsymbol{O}$. Since the first set is the union of the triangles $Q_jQ_{j+1}O$, while the latter is the union of the triangles $\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\boldsymbol{O}$, we define $\tilde{v}$ on $\mathcal{I}_i$ as the piecewise affine map which sends each triangle onto its correspondent one, and we only have to check the bi-Lipschitz constant of $\tilde{v}$ on $\mathcal{I}_1$. As usual, we set

$$\begin{aligned}
a &= \ell\big(Q_jQ_{j+1}\big)\,, & b &= \ell\big(Q_jO\big)\,, & \alpha &= O\widehat{Q_j}Q_{j+1}\,,\\
a' &= \ell\big(\boldsymbol{Q}_j\boldsymbol{Q}_{j+1}\big)\,, & b' &= \ell\big(\boldsymbol{Q}_j\boldsymbol{O}\big)\,, & \alpha' &= \boldsymbol{O}\widehat{\boldsymbol{Q}_j}\boldsymbol{Q}_{j+1}\,.
\end{aligned}$$

Let us now make the following observation. Even though the situation is not the same as in Case A, as we pointed out above, the only difference is in fact that now $\boldsymbol{O}$ is not the center of $\widehat{\boldsymbol{\mathcal{B}}}$. And this difference clearly affects only b' and α', thus (5.93), (5.94) and (5.97) already tell us

$$\frac{\sqrt{2}}{10L} \leq \frac{a}{a'} \leq 85L^2\,, \qquad \frac{1}{10} \leq b \leq \frac{\sqrt{2}}{2}\,, \qquad \frac{1}{87L} \leq \alpha \leq \pi - \frac{1}{87L}\,.$$

Concerning b', since any point $\boldsymbol{Q}_j$ is below $\boldsymbol{A}_1\boldsymbol{A}_2$ by construction (recall Figure 21), we immediately deduce that

$$b' \geq \ell\big(\boldsymbol{MO}\big) = \frac{\sqrt{3}}{6}\,\ell\big(\boldsymbol{A}_1\boldsymbol{A}_2\big) \geq \frac{2\sqrt{3}}{9L}\,,$$

also using (5.103). On the other hand, by the assumption $d_1 < -r/2$ and by construction it immediately follows that $\boldsymbol{O}$ is below the center of $\widehat{\boldsymbol{\mathcal{B}}}$, then keeping in mind (5.92) we have

$$b' \leq r \leq \frac{2L}{\pi}\,.$$

Finally, concerning α', it is clear by construction that both α' and $\pi - \alpha'$ are strictly bigger than $\boldsymbol{A}_1\widehat{\boldsymbol{A}_2}\boldsymbol{O}$, thus

$$\sin\alpha' \geq \sin \boldsymbol{A}_1\widehat{\boldsymbol{A}_2}\boldsymbol{O} = \sin\left(\arctan\frac{\sqrt{3}}{3}\right) = \frac{1}{2}\,.$$

Summarizing, we have

$$\frac{\sqrt{2}}{10L} \leq \frac{a}{a'} \leq 85L^2\,, \qquad \frac{\pi}{20L} \leq \frac{b}{b'} \leq \frac{3\sqrt{6}L}{4}\,, \qquad \sin\alpha \geq \frac{1}{88L}\,, \qquad \sin\alpha' \geq \frac{1}{2}\,.$$

Now, it is enough to use (5.61) for a last time to obtain

$$\begin{aligned}
\mathrm{Lip}(\phi) &\leq \frac{a'}{a} + \frac{2b'}{b\sin\alpha} + \frac{a'}{a\sin\alpha} \leq 5\sqrt{2}L + \frac{3520}{\pi}\,L^3 + 440\sqrt{2}L^2\,,\\
\mathrm{Lip}(\phi^{-1}) &\leq \frac{a}{a'} + \frac{2b}{b'\sin\alpha'} + \frac{a}{a'\sin\alpha'} \leq 85L^2 + 3\sqrt{6}L + 170L^2
\end{aligned}$$

and then the proof of the first part of Theorem 0.4 is finally concluded. □

5.11. Smooth extension and Proof of Theorem 0.5

In this last section, we show the existence of the smooth extension $\tilde{v}'$ of $\tilde{u}$, thus concluding the proof of Theorem 0.4, and we prove the existence of bi-Lipschitz extensions for a general bi-Lipschitz function $\tilde{u}$ (i.e., not necessarily piecewise affine) as in Theorem 0.5.

The proof of the last statement of Theorem 0.4 is an immediate corollary of the following recent result by C.Mora-Corral and A.Pratelli (see [**46**, Theorem A]; in fact, we prefer to claim here only the part of that result that we need in this paper).

THEOREM 5.32. *Let $v : \Omega \to \mathbb{R}^2$ be a (countably) piecewise affine homeomorphism, bi-Lipschitz with constant L. Then there exists a smooth diffeomorfism $\hat{v} : \Omega \to v(\Omega)$ such that $\hat{v} \equiv v$ on $\partial\Omega$, $\hat{v}$ is bi-Lipschitz with constant at most $70L^{7/3}$, and*

$$\|\hat{v} - v\|_{L^\infty(\Omega)} + \|D\hat{v} - Dv\|_{L^p(\Omega)} + \|\hat{v}^{-1} - v^{-1}\|_{L^\infty(v(\Omega))} + \|D\hat{v}^{-1} - Dv^{-1}\|_{L^p(v(\Omega))} \leq \varepsilon\,.$$

PROOF OF THEOREM 0.4 (SMOOTH EXTENSION). Let $\tilde{v}$ be an affine extension of $\tilde{u}$ having bi-Lipschitz constant at most CL^4, which exists thanks to the proof of the first part of the Theorem 0.4, Section 5.10. By Theorem 5.32, there exists a map $\tilde{v}'$ which is smooth, concides with $\tilde{v}$ on $\partial\mathcal{D}$, and has bi-Lipschitz constant at most $70C^{7/3}L^{28/3}$. This map $\tilde{v}'$ is a smooth extension of $\tilde{u}$ as required. □

We now give the proof of Theorem 0.5, which will be obtained from Theorem 0.4 by a quick extension argument. We will use the following geometric result, which is a simple adaptation of Lemmas 4.19 and 4.20 of Chapter 4 to define piecewise affine approximations on the boundary of a square.

LEMMA 5.33. *Let $\varphi : \partial\mathcal{D} \to \mathbb{R}^2$ be an L bi-Lipschitz map. Then, for any $\varepsilon > 0$, there exists a piecewise affine map $\varphi_\varepsilon : \partial\mathcal{D} \to \mathbb{R}^2$ which is $4L$ bi-Lipschitz and such that*

$$|\varphi(P) - \varphi_\varepsilon(P)| \leq \varepsilon \qquad \forall P \in \partial\mathcal{D}\,.$$

We can now show our Theorem 0.5.

PROOF OF THEOREM 0.5. Let $\tilde{u} : \partial\mathcal{D} \to \mathbb{R}^2$ be an L bi-Lipschitz map. Fix $\varepsilon > 0$ and apply Lemma 5.33, obtaining a $4L$ bi-Lipschitz and piecewise affine map $\tilde{u}_\varepsilon : \partial\mathcal{D} \to \mathbb{R}^2$, with $\|\tilde{u}_\varepsilon - \tilde{u}\|_{L^\infty(\partial\mathcal{D})} \leq \varepsilon$. Theorem 0.4, applied to $\tilde{u}_\varepsilon$, gives then an extension $\tilde{v}_\varepsilon : \mathcal{D} \to \mathbb{R}^2$ which is $236CL^4$ bi-Lipschitz and satisfies $\tilde{v}_\varepsilon = \tilde{u}_\varepsilon$ on

Then, applying Theorem 0.2 to $\tilde{v}$, we obtain respectively a countable piecewise affine function $\bar{v}$ which is very close to $\tilde{v}$, coincides with $\tilde{v}$ on $\partial\mathcal{D}$, and is $C_1\big(C''L^4\big)^4$ bi-Lipschitz, and with a smooth function $\bar{v}'$, again very close to $\tilde{v}$, coinciding with $\tilde{v}$ on $\partial\mathcal{D}$ and $C_2(C''L^4)^{28/3}$ bi-Lipschitz. These two function $\bar{v}$ and $\bar{v}'$ are the searched extensions of $\tilde{u}$ in the secon claim of the theorem. □

We conclude the chapter with a last observation.

REMARK 5.34. *One could be not satisfied to pass from the first to the secon claim of Theorem 0.5 passing from L^4 to L^{16} (resp. $L^{112/3}$). In fact, it is possible to modify the construction of Theorem 0.4 so as to directly obtain, in the case of a general L bi-Lipschitz function $\tilde{u} : \partial\mathcal{D} \to \mathbb{R}^2$, a countably piecewise affine extension $\tilde{v}$ of $\tilde{u}$ which is $\widetilde{C}L^4$ bi-Lipschitz. And then, thanks to Theorem 5.32, one would also get a smooth extension $\tilde{v}$ which is $70\widetilde{C}^{7/3}L^{28/3}$ bi-Lipschitz.*

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