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Topics in the Stability of Systems of Conservation Laws

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Thesis submitted for the degree of 'Doctor Philosophiae'
Academic Year 1999/2000

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To my Father

Dla mojego Taty

This thesis has been written during my PhD studies that I carried out in Scuola Internazionale Superiore di Studi Avanzati in Trieste, in the period of 1996 - 2000.

The four years spent in Italy were a very intense time of my life, and with large oscillations, I would say. Studying wave patterns of conserved quantities may give one the erroneous impression that in life as in mathematics: solving small interactions only approximately is still fine; or for example that solutions with more than one large wave need a finiteness condition to be stable, even when the waves do not interact. I had the undeserved fortune to keep escaping from these conclusions, and it is impossible to mention all the people who contributed to this work. I would like to embrace them all.

My Italian friends at SISSA, in particular Lorenzo and Duccio, who made my life there nonanonymous. Then – most of all, Marco and his family, always attentive and supportive when I needed help. My friends in Poland, among them Rafał and Maciek, for all they said to me and for how they live.

And finally – someone very special, sharing with me, across the distance between Italy and Poland, every small achievement I made, as well as my failures. This thesis is dedicated to my Father.

Contents

Introduction

In this thesis we gather some recent results concerning the wellposedness of a class of nonlinear PDEs having the following general form:

$$u_t + f(u)_x = 0. \tag{1}$$

Equations (1) are commonly called conservation laws in analogy to the examples of such systems which arise in physics (fluid dynamics, electromagnetism, elasticity, chromatography and others). Here the (unknown) solution vector of conserved quantities $u \in \mathbf{R}^n$ depends on the one dimensional space variable $x \in \mathbf{R}$ and time $t \geq 0$, while $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a smooth flux function, yielding suitable properties - such as hyperbolicity and 'convexity' of characteristic fields, of the associated quasilinear system $u_t + Df(u)u_x = 0$.
It is worth to note that the equation (1), despite its simple form, incorporates basic phenomena of nonlinear wave propagation such as the formation, collision, or cancellation of waves.

In this setting, the recent progress in the field has shown that within the class of initial data $\bar{u} \in \mathrm{L}^1 \cap \mathrm{BV}(\mathbf{R}, \mathbf{R}^n)$ with sufficiently small total variation, the initial-value problem: (1) together with

$$u(0, x) = \bar{u}, \tag{2}$$

is wellposed in $\mathrm{L}^1(\mathbf{R}, \mathbf{R}^n)$. Namely, its admissible solutions constitute a semigroup (called Standard Riemann Semigroup, and abbreviated here to SRS) which is Lipschitz continuous with respect to both time and initial data. The semigroup with these properties is unique and its trajectories are the limits of piecewise constant so-called approximate solutions, generated by the method of wave front tracking. The contraction with respect to a suitably weighted L^1 distance, yielding the Lipschitz continuity of the SRS, is established by defining a Lyapunov functional (called so by comparison with the corresponding idea in ordinary differential equations), decreasing in time and measuring the distance between the time profiles of two arbitrary approximate solutions.

At this point, two natural directions for further research are the following:

- To extend the understanding of the structure of the semigroup solutions to (1) (2), in particular to provide simple, easily checkable conditions sufficient or necessary for a BV solution to be a trajectory of the SRS. In turn, this yields new uniqueness results for the initial-value problem within some general classes of entropy weak solutions.

- To establish the wellposedness of the Cauchy problem (1) (2) for a larger domain of initial data, possibly including functions with large total variation.

In this dissertation we present several results related to these topics.

Chapter 1 contains what was originally published as [BLe]. In the framework of the first point above, we prove that every Lax admissible weak solution of (1) coincides with the corresponding SRS trajectory if and only if it has locally bounded variation along a suitable family of space-like curves. By the uniqueness of SRS, there follows a uniqueness result for (1) (2), within the class of solutions having the mentioned property.

Chapters 2 and 3 are concerned with the L^1 stability of wave patterns containing some non-interacting large shock waves. We study the problem (1) with $\bar{u}$ in (2) being a small $L^1 \cap$ BV perturbation of fixed Riemann data. We a priori assume that the solution of the latter problem is given by a number of (arbitrarily large) Lax compressive and Majda stable shocks of different characteristic families. We formulate the BV and L^1 Stability Conditions that express the expected mutual influence of the large waves. By constructing suitable Glimm and Lyapunov functionals applicable to our setting, we show that the former condition guarantees the existence of a unique, global in time and space, 'admissible' solution to (1) (2); while the latter condition is essential for the stablity of this same solution under (a class of) perturbations of its initial data. This is carried out in Chapter 2, containing the results of [Le1].

In Chapter 3 we present a revised version of the article [Le2], with some new additions. Several authors had investigated the issue of wellposedness of (1) (2) in various contexts, introducing different stability conditions. Some of them require that the eigenvalues of suitable matrices related to wave transmissions - reflections are smaller than 1 in absolute value, other refer to different algebraic properties of the linearised system, such as for example existence of weights with whom the flow of the system becomes a contraction. We explain and compare these conditions, showing that the conditions of Chapter 2 generalize or unify them in appropriate ways.

CHAPTER 1

A uniqueness condition for hyperbolic systems of conservation laws

1. Introduction

Consider a hyperbolic system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0. \tag{1.1}$$

The following standard conditions [L] [Sm] will be assumed throughout. The flux function $f : \Omega \mapsto \mathbf{R}^n$ is smooth and defined on an open set $\Omega \subset \mathbf{R}^n$ containing the origin. The system is strictly hyperbolic, i.e. for each $u \in \Omega$ the Jacobian matrix $A(u) = Df(u)$ has n real distinct eigenvalues: $\lambda_1(u) < \cdots < \lambda_n(u)$. We can thus choose bases of right and left eigenvectors $r_i(u)$, $l_i(u)$, $i = 1, \ldots, n$, normalized so that

$$|r_i| \equiv 1\,, \qquad \langle l_i,\, r_j\rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases} \tag{1.2}$$

for every $i, j \in \{1, \ldots, n\}$ and $u \in \Omega$. For each $i = 1, \ldots, n$, we assume that the i-th field is either linearly degenerate, so that

$$\nabla\lambda_i \cdot r_i(u) \doteq \lim_{h\to 0} \frac{\lambda_i\big(u + hr_i(u)\big) - \lambda_i(u)}{h} = 0 \qquad \text{for every } u \in \Omega,$$

or genuinely nonlinear, so that

$$\nabla\lambda_i \cdot r_i(u) > 0 \qquad \text{for every } u \in \Omega.$$

In this setting, it was proved in [BC2] [BCP] [BLY] that the system (1.1) admits a uniformly Lipschitz continuous semigroup of solutions $S : \mathcal{D} \times [0, \infty) \mapsto \mathcal{D}$, defined on some nontrivial domain $\mathcal{D} \subset \mathrm{L}^1(\mathbf{R}, \mathbf{R}^n)$. More precisely, the following is true (cf. [B3]):

THEOREM 1.1. *There exist a closed domain $\mathcal{D} \subset \mathrm{L}^1$, a continuous mapping $S : [0, \infty) \times \mathcal{D} \longrightarrow \mathcal{D}$, and positive constants δ_0, L depending only on the system (1.1), such that*

- *(i) $\mathcal{D}$ contains all functions $\bar u \in \mathrm{L}^1 \cap \mathrm{BV}(\mathbf{R}, \mathbf{R}^n)$ with $\mathrm{T.V.}(\bar u) < \delta_0$. In particular, if $u \in \mathcal{D}$ then u attains values in Ω.*
- *(ii) $S_0(\bar u) = \bar u$,*
  $S_{t+s}(\bar u) = S_t\left(S_s(\bar u)\right) \quad \forall t, s \geq 0\ \forall \bar u \in \mathcal{D}$.
- *(iii) $\| S_t(\bar u) - S_t(\bar w) \|_{\mathrm{L}^1} \leq L \cdot (|t - s| + \| \bar u - \bar w \|_{\mathrm{L}^1}) \quad \forall t, s \geq 0\ \forall \bar u, \bar w \in \mathcal{D}$.*
- *(iv) If $\bar u \in \mathcal{D}$ is piecewise constant then, for small t, $S_t(\bar u)$ coincide with the solution $u(t, \cdot)$ to (1.1) obtained by piecing together the selfsimilar solutions to Riemann problems arising at the discontinuities of $\bar u$.*

Conditions (ii) (iii) (iv) automatically imply:

(v) *Each trajectory* $t \mapsto S_t(\bar{u})$ *is a solution to (1.1) with a given initial condition*

$$u(0,\cdot) = \bar{u} \in \mathcal{D}. \tag{1.3}$$

Moreover, this solution coincides with the limit of the approximate front tracking solutions of (1.1) (1.3).

Since, given a positively invariant domain $\mathcal{D}$, the semigroup S satisfying conditions (ii)–(iv) above is unique (see [B3]), it is customary to call it Standard Riemann Semigroup (abbreviated in the sequel to SRS). A way to establish the uniqueness of solutions to the Cauchy problem (1.1) (1.3) is thus to prove that every entropy weak solution $u = u(t,x)$ actually coincides with the SRS trajectory:

$$u(t,\cdot) = S_t\bar{u} \tag{1.4}$$

for all $t \geq 0$. Regularity conditions which imply the identity (1.4) were introduced in [BG] [BL]. These conditions provide some control on the oscillation of u in a forward neighborhood of each given point (t,x).

Below we consider an alternative regularity condition, quite simple to state, and prove that it suffices to guarantee uniqueness.

(A3) **(Locally Bounded Variation)** There exists $\delta > 0$ such that, along every space-like curve $t = \gamma(x)$ with $|d\gamma/dx| \leq \delta$ almost everywhere, the total variation of u is locally bounded.

In other words, we require that, whenever $t = \gamma(x)$ is a space-like curve satisfying

$$\big|\gamma(x) - \gamma(x')\big| \leq \delta \cdot |x - x'| \qquad \text{for all } x, x',$$

then the total variation of the composed map $x \mapsto u\big(\gamma(x), x\big)$ is bounded on bounded intervals. For completeness, we restate below our basic assumptions on weak solutions and the Lax entropy conditions.

(A1) **(Conservation Equations)** The function $u = u(t,x)$ is a weak solution of the Cauchy problem (1.1) (1.3), taking values within the domain $\mathcal{D}$ of a Standard Riemann Semigroup S. More precisely, $u : [0,T] \mapsto \mathcal{D}$ is continuous w.r.t. the L^1 distance. The initial condition (1.3) holds, together with

$$\iint \big(u\varphi_t + f(u)\varphi_x\big)\, dxdt = 0 \tag{1.5}$$

for every $\mathcal{C}^1$ function φ with compact support contained inside the open strip $(0,T) \times \mathbf{R}$.

(A2) **(Entropy Condition)** Let u have an approximate jump discontinuity at some point $(\tau,\xi) \in]0,T[\times\mathbf{R}$. More precisely, let there exists states $u^-, u^+ \in \Omega$ and a speed $\lambda \in \mathbf{R}$ such that, calling

$$U(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < \xi + \lambda(t-\tau), \\ u^+ & \text{if} \quad x > \xi + \lambda(t-\tau), \end{cases} \tag{1.6}$$

there holds

$$\lim_{\rho \to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \Big|u(t,x) - U(t,x)\Big|\, dxdt = 0. \tag{1.7}$$

Then, for some $i = 1, \dots, n$, one has the entropy inequality:

$$\lambda_i(u^-) \geq \lambda \geq \lambda_i(u^+). \tag{1.8}$$

With the above assumptions, the main result of this Chapter is the following:

THEOREM 1.2. *Assume that the system (1.1) generates Standard Riemann Semigroup $S : \mathcal{D} \times [0, \infty) \mapsto \mathcal{D}$. Then, for every $\bar{u} \in \mathcal{D}$, $T > 0$, the Cauchy problem (1.1) (1.3) has a unique weak solution $u : [0, T] \mapsto \mathcal{D}$ satisfying the assumptions* **(A1)–(A3)**. *Indeed, these assumptions imply the identity (1.4) for all $t \in [0, T]$.*

A proof of this theorem will be given in Section 4, while in Sections 2 and 3 we collect a number of preliminary estimates.

2. More on SRS

In this Section we recall two useful estimates valid for the trajectories of a Standard Riemann Semigroup S. For their proofs, see [B2].

LEMMA 2.1. *Let $w : [0, T] \mapsto \mathcal{D}$ be Lipschitz continuous. Then for every interval $[a, b] \in \mathbf{R}$ there holds:*

$$\begin{aligned} &\left\| w(T) - S_T w(0) \right\|_{\mathbf{L}^1([a+\lambda^* T, b-\lambda^* T], \mathbf{R}^n)} \\ &= O(1) \cdot \int_0^T \left\{ \liminf_{h \to 0+} \frac{\left\| w(\tau+h) - S_h w(\tau) \right\|_{\mathbf{L}^1([a+\lambda^*(\tau+h), b-\lambda^*(\tau+h)], \mathbf{R}^n)}}{h} \right\} d\tau. \end{aligned} \tag{2.1}$$

Here and in the sequel, with the Landau symbol $O(1)$ we denote a quantity whose absolute value satisfies a uniform bound, depending only on the system (1.1).

Before stating the local integral estimates valid for semigroup trajectories, we need to define two local approximate solutions of (1.1). Let $w \in \mathcal{D}$ and fix a point $\xi \in \mathbf{R}$. Call $\omega = \omega(t, x)$ the unique self-similar entropy solution of the Riemann problem

$$\omega_t + f(\omega)_x = 0, \qquad \omega(0, x) = \begin{cases} w(\xi-) & \text{if } x < 0, \\ w(\xi+) & \text{if } x > 0. \end{cases}$$

For $t \geq 0$, let

$$U^\sharp(t, x) \doteq \begin{cases} \omega(t,\ x - \xi) & \text{if } |x - \xi| \leq \hat{\lambda} t, \\ w(x) & \text{if } |x - \xi| > \hat{\lambda} t. \end{cases}$$

Next, call $\widetilde{A} \doteq Df\big(w(\xi)\big)$ the Jacobian matrix of f computed at $w(\xi)$. For $t \geq 0$, define $U^\flat(t, x)$ to be the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$U^\flat_t + \widetilde{A} U^\flat_x = 0, \qquad U^\flat(0) = w.$$

LEMMA 2.2. *For every function $w \in \mathcal{D}$, every $\xi \in \mathbf{R}$ and $h, \rho > 0$, with the above definitions one has*

$$\begin{aligned} \frac{1}{h} \int_{\xi-\rho+h\hat{\lambda}}^{\xi+\rho-h\hat{\lambda}} \left| (S_h w)(x) - U^\sharp(h,x) \right| \, dx \qquad \\ = O(1) \cdot \text{T.V.}\{w;\ (\xi-\rho,\ \xi) \,\cup\, (\xi,\ \xi+\rho)\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \frac{1}{h} \int_{\xi-\rho+h\hat{\lambda}}^{\xi+\rho-h\hat{\lambda}} \left| (S_h w)(x) - U^\flat(h,x) \right| \, dx \qquad \\ = O(1) \cdot \Big(\text{T.V.}\{w;\ (\xi-\rho,\ \xi+\rho)\} \Big)^2. \end{aligned} \tag{2.3}$$

3. The regularity conditions

In this Section we explain which information on the structure and regularity of solution $u : [0,T] \longrightarrow \mathcal{D}$ to (1.1) is enclosed in the conditions **(A1)**–**(A3)**.

Since $\mathcal{D} \subset \mathrm{L}^1 \cap BV$, for sake of definiteness we shall always work with right-continuous representatives, so that our functions $w \in \mathcal{D}$ will satisfy $w(x) = w(x+)$ for all $x \in \mathbf{R}$. Moreover, given a continuous map $u : [0,T] \mapsto \mathcal{D}$, we will identify it with the corresponding function of two variables $u \in \mathrm{L}^1([0,T] \times \mathbf{R}, \mathbf{R}^n)$, defined in the natural way.

The following two lemmas concern the condition **(A1)**.

LEMMA 3.1. *Let $u : [0,T] \mapsto \mathcal{D}$ satisfy* **(A1)**. *Then u is Lipschitz continuous w.r.t. the L^1 distance.*

LEMMA 3.2. *Let $u : [0,T] \mapsto \mathcal{D}$ satisfy* **(A1)**. *Then $u \in BV(]0,T[\times\mathbf{R};\ \mathbf{R}^n)$. Moreover, there exists a set $\mathcal{N}$ of Lebesgue measure 0, containing the endpoints of the interval $[0,T]$, such that for every $\tau \in [0,T] \setminus \mathcal{N}$ and every $\xi \in \mathbf{R}$ the following holds. Either u is approximately continuous at (τ,ξ), i.e. (1.7) holds with $U(t,x) = u(\tau,\xi-) = u(\tau,\xi+)$, or u has a non-horizontal approximate jump discontinuity at (τ,ξ), so that (1.6) and (1.7) hold. In this latter case one has the additional relations*

$$u^- = u(\tau,\xi-),\ \ u^+ = u(\tau,\xi+),$$
$$\lambda \cdot [u^+ - u^-] = f(u^+) - f(u^-).$$

If u satisfies **(A2)**, *then (1.8) holds for some $i = 1, \dots, n$.*

A proof of Lemma 3.1 can be found in [BG]. The first statement of Lemma 3.2 is a corollary of Lemma 3.1. For the proof of the other statements see [BG] [BL] [EG].

The next two lemmas derive some local properties of u, implied by our assumption **(A3)**.

LEMMA 3.3. *Let $u : [0,T] \mapsto \mathcal{D}$ satisfy* **(A3)**. *Fix $\tau \in [0,T]$ and $\varepsilon > 0$. Then the set*

$$B_{\tau,\varepsilon} = \left\{ \xi \in \mathbf{R};\ \limsup_{t\to\tau+,\ x\to\xi} \left| u(t,x) - u(\tau,\xi) \right| > \varepsilon \right\} \tag{3.1}$$

has no limit points.

PROOF. If the conclusion fails, then there exists a monotone sequence $\{\xi_i\}$ of points in $B_{\tau,\varepsilon}$, converging to some limit point ξ_0. To fix the ideas, let the sequence be decreasing, the other case being entirely similar. For each $i \geq 1$, by the right continuity ot the function $x \mapsto u(\tau, x)$ one can find a point $w_i \in (\xi_i, \xi_{i-1})$ such that $|u(\tau, w_i) - u(\tau, \xi_i)| \leq \varepsilon/2$. Next, let $t_i > \tau$ and $x_i \in]w_{i+1}, w_i[$ satisfy the inequalities

$$|u(t_i, x_i) - u(\tau, \xi_i)| \geq \varepsilon,$$
$$|t_i - \tau| \leq \delta \cdot \max\{|x_i - w_i|,\ |x_i - w_{i+1}|\}. \tag{3.2}$$

Define a space-like curve $t = \gamma(x)$, with $x \in [\xi_0, \xi_1]$, by setting

$$\gamma(x) = \begin{cases} \tau & \text{if } \quad x = \xi_0 \text{ or } x \geq w_1, \\ t_i - (x - x_i) \cdot \frac{t_i - \tau}{w_i - x_i} & \text{if } \quad x \in [x_i, w_i], \\ \tau + (x - w_{i+1}) \cdot \frac{t_i - \tau}{x_i - w_{i+1}} & \text{if } \quad x \in [w_{i+1}, x_i]. \end{cases} \tag{3.3}$$

By (3.2), γ is Lipschitz continuous with Lipschitz constant δ. Since $|u(t_i, x_i) - u(\tau, w_i)| \geq \varepsilon/2$ for all $i \geq 1$, the total variation of the composed map $x \mapsto u(\gamma(x), x)$ on the interval $[\xi_0, \xi_1]$ is infinite. This contradicts the assumption **(A3)**, thus proving Lemma 3.3. □

Throughout the following, we consider a fixed number $\lambda^* \geq 1/\delta$ strictly larger than the absolute values of all propagation speeds λ_i of the system (1.1).

LEMMA 3.4. *Let $u : [0, T] \mapsto \mathcal{D}$ satisfy* **(A3)**. *Then for each $(\tau, \xi) \in (0, T) \times \mathbf{R}$*

$$\lim_{\substack{t \to \tau+,\ x \to \xi\pm \\ |x - \xi| > \lambda^*(t - \tau)}} u(t, x) = u(\tau, \xi\pm).$$

PROOF. Suppose the conclusion of the lemma fails. To fix the ideas, assume that, for some $(\tau, \xi_0) \in (0, T) \times \mathbf{R}$, there exist decreasing sequences $t_j \to \tau+$ and $x_j \to \xi_0+$, such that

$$|x_j - \xi_0| \geq \lambda^* |t_j - \tau|, \qquad \big|u(t_j, x_j) - u(\tau, \xi_0)\big| \geq \varepsilon,$$

for some $\varepsilon > 0$ and every index j. The case $x_j \to \xi_0-$ can be treated in the same way.

Define the sequence of points

$$w_j \doteq x_j + \frac{1}{\delta} \cdot (t_j - \tau)$$

and observe that $w_j \to \xi_0+$ as $j \to \infty$. By possibly taking a subsequence, say $\{(t_i, x_i)\}$, we can assume that the corresponding w_i satisfy

$$x_i \in (w_{i+1}, w_i), \qquad |t_i - \tau| \leq \delta \cdot \max\big\{|x_i - w_i|,\ |x_i - w_{i+1}|\big\} \quad \text{for all } i.$$

Now let γ be the space-like curve defined by (3.3). Since $w_i \to \xi+$, for every i large enough, we have $\big|u(\tau, w_i) - u(\tau, \xi+)\big| \leq \varepsilon/2$, hence $\big|u(t_i, x_i) - u(\tau, w_i)\big| \geq \varepsilon/2$. Therefore, the total variation of the map $x \mapsto u(\gamma(x), x)$ on the interval $[\xi_0, w_1]$ is infinite, in contradiction with **(A3)**. □

We conclude this section by recalling two technical results, that will be needed for proving Theorem 1.2. Their proofs can be found in [BG].

LEMMA 3.5. *Let $w \in \mathrm{L}^1(]a,\ b[, \mathbf{R}^n)$ be such that for some Radon measure μ, one has*

$$\left| \int_{\zeta_1}^{\zeta_2} w(x)\,dx \right| \le \mu([\zeta_1,\ \zeta_2]) \quad \textit{whenever } a < \zeta_1 < \zeta_2 < b.$$

Then

$$\int_a^b \big|w(x)\big|\,dx \le \mu\big((a,\ b)\big).$$

LEMMA 3.6. *Let $u : [0,T] \mapsto \mathcal{D}$ be Lipschitz continuous. At a given point (τ, ξ), let the conditions (1.6) (1.7) hold, for some $u^-, u^+ \in \mathbf{R}^n$, $\lambda \in \mathbf{R}$. Then, for each $\bar\lambda > 0$ one has*

$$\lim_{\rho\to 0+} \sup_{|h|\le\rho} \int_0^{\bar\lambda} \big|u(\tau+h,\ \xi+\lambda h+\rho y) - u^+\big|\,dy = 0,$$

$$\lim_{\rho\to 0+} \sup_{|h|\le\rho} \int_{-\bar\lambda}^0 \big|u(\tau+h,\ \xi+\lambda h+\rho y) - u^-\big|\,dy = 0.$$

4. Proof of the main result

Let u satisfy **(A1)–(A3)**. To deduce (1.4), in view of Lemma 2.1 it suffices to show that for every interval $[a,b] \subset \mathbf{R}$ and a.e. $\tau \in [0,T]$ one has

$$\liminf_{h\to 0+} \frac{\big\|u(\tau+h) - S_h u(\tau)\big\|_{\mathrm{L}^1([a,b],\mathbf{R}^n)}}{h} = 0. \tag{4.1}$$

We will show that (4.1) is valid for every $[a,b] \in \mathbf{R}$, whenever $\tau \in [0,T] \setminus \mathcal{N}$. The proof is divided in 3 steps. The aim of the first two steps is to derive the appropriate estimates on the error

$$\big\|u(\tau+h) - S_h u(\tau)\big\|_{\mathrm{L}^1(I,\mathbf{R}^n)},$$

when $h > 0$ and the interval $I \subset [a,b]$ are small enough. This will be done using the inequalities in Lemma 2.2, namely (2.2) near points where $u(\tau,\cdot)$ has large variation, and (2.3) on intervals where the total variation of $u(\tau,\cdot)$ is suitably small.

In the third step we construct a suitable covering of $[a,b]$ and complete the proof of (4.1) combining the estimates obtained in steps 1 and 2.

STEP 1. Fix $\varepsilon > 0$ and assume $\tau \notin \mathcal{N}$. Then, at every point $\xi \in \mathbf{R}$, the limit (1.7) holds for some u^-, u^+, λ. Observe that $u^+ = u^-$ at a point where u is approximately continuous, while $u^+ \ne u^-$ if u has an approximate jump discontinuity at (τ,ξ). By (1.7), from Lemma 3.6 it follows

$$\begin{aligned}
\lim_{h\to 0+} \frac{1}{h} \int_{\xi-\lambda^* h}^{\xi+\lambda^* h} &\big|u(\tau+h,x) - U(\tau+h,x)\big|\,dx \\
&\le \lim_{h\to 0+} \frac{1}{h} \int_{\xi-\lambda^* h}^{\xi+\lambda^* h} \big|u(\tau+h,x) - u^+\big|\,dx \\
&\quad + \lim_{h\to 0+} \frac{1}{h} \int_{\xi-\lambda^* h}^{\xi+\lambda^* h} \big|u(\tau+h,x) - u^-\big|\,dx = 0.
\end{aligned}$$

Hence
$$\frac{1}{h}\int_{\xi-\lambda^* h}^{\xi+\lambda^* h}\Big|u(\tau+h,x)-U(\tau+h,x)\Big|\,dx\le\varepsilon$$
for all $h>0$ sufficiently small.
By Lemma 3.2, $U(t,x)=U^\sharp(t-\tau,x)$ in a forward neighbourhood of the point (τ,ξ). Hence by (2.1) we get

$$\begin{aligned}\frac{1}{h}\int_{\xi-\lambda^* h}^{\xi+\lambda^* h}&\Big|u(\tau+h,x)-(S_h u(\tau))(x)\Big|\,dx\\ &\le\varepsilon+\frac{1}{h}\int_{\xi-\lambda^* h}^{\xi+\lambda^* h}\Big|(S_h u(\tau))(x)-U(\tau+h,x)\Big|\,dx\\ &=\varepsilon+O(1)\cdot\text{T.V.}\big\{u(\tau);\ (\xi-2\lambda^* h,\,\xi)\ \cup\ (\xi,\,\xi+2\lambda^* h)\big\}\big)\le 2\varepsilon\end{aligned}\tag{4.2}$$

for $h>0$ small enough. Note that here the maximum size of h depends on ξ, τ and ε.

STEP 2. Fix $\varepsilon>0$ and an interval $(c,d)\subset\mathbf{R}$ centered at a point ξ and such that $(c,d)\ \cap\ B_{\tau,\varepsilon}=\emptyset$. Here $B_{\tau,\varepsilon}$ is the set (3.1) of points where the oscillation of u is $>\varepsilon$. For any $h>0$, let Γ_h be a trapezoid defined as
$$\Gamma_h=\Big\{(s,x);\ s\in[\tau,\tau+h],\ x\in\ \big(c+(s-\tau)\lambda^*,\ d-(s-\tau)\lambda^*\big)\Big\}.$$
We first show that for small $h>0$ and every $(s,x)\in\Gamma_h$ one has
$$\big|u(s,x)-u(\tau,\xi)\big|\le 2\varepsilon+\text{T.V.}\big\{u(\tau);\ (c,d)\big\}\tag{4.3}$$
Indeed, by Lemma 3.4 the inequality (4.3) clearly holds for points (s,x) contained in small neighbourhoods of the lower corner points (τ,c) and (τ,d). It thus remains to prove (4.3) in a region of the form $[\tau,\tau+h]\times[c+h',d-h']$, with $h'>0$ given and for some $h>0$ suitably small. Since $[c+h',d-h']\cap B_{\tau,\varepsilon}=\emptyset$, for every $y\in[c+h',d-h']$ we can find $h_y,\rho_y>0$ such that (4.3) holds when $(s,x)\in[\tau,\ \tau+h_y]\times(y-\rho_y,\ y+\rho)$. Covering the compact interval $[c+h',d-h']$ with finitely many open intervals $(y_j-\rho_{y_j},\ y_j+\rho_{y_j})$, $j=1,\dots,N$ and choosing $h\doteq\min h_{y_j}$, we obtain (4.3) for all $(s,x)\in[\tau,\tau+h]\times[c+h',d-h']$. We now show that, for all $h>0$ with $h<(d-c)/2\lambda^*$, the following estimate holds:

$$\begin{aligned}\int_{c+\lambda^* h}^{d-\lambda^* h}&\Big|u(\tau+h,x)-U^\flat(\tau+h,x)\Big|\,dx\\ &=O(1)\cdot\sup_{(s,x)\in\Gamma_h}|u(s,x)-u(\tau,\xi)|\\ &\quad\cdot\int_\tau^{\tau+h}\text{T.V.}\Big\{u(\tau);\ \big[c+\lambda^*(t-\tau),\ d-\lambda^*(t-\tau)\big]\Big\}\,dt.\end{aligned}\tag{4.4}$$

To derive (4.4), we proceed as in [BG]. For each $i=1,\dots,n$ call $\tilde\lambda_i,\tilde l_i,\tilde r_i$ respectively the i-th eigenvalue and the left and right eigenvectors of the matrix $\tilde A=Df(u(\tau,\xi))$, normalized as in (1.2).

Let $\zeta'<\zeta''$ belong to the interval $(c+\lambda^* h,d-\lambda^* h)$. We now need to estimate the quantities
$$E_i\doteq\int_{\zeta_1}^{\zeta_2}\Big[\tilde l_i(u(\tau+h,x)-U^\flat(h,x))\Big]\,dx.$$

Obviously

$$\tilde{l}_i \; U^\flat(h,x) = \tilde{l}_i \; U^\flat(0, x - \tilde{\lambda}_i h) = \tilde{l}_i \; u(\tau, x - \tilde{\lambda}_i h).$$

Integrating (1.1) over the domain

$$\Big\{(s,x);\;\; s \in [\tau,\;\tau+h],\;\; \zeta' + \tilde{\lambda}_i(s-\tau-h) \le x \le \zeta'' + \tilde{\lambda}_i(t-\tau-h)\Big\},$$

we obtain

$$\begin{aligned} E_i &= \int_{\zeta'}^{\zeta''} \tilde{l}_i \; u(\tau+h, x)\,dx - \int_{\zeta'}^{\zeta''} \tilde{l}_i \; u(\tau, x - \tilde{\lambda}_i h)\,dx \\ &= \int_\tau^{\tau+h} \tilde{l}_i \cdot \big(f(u) - \tilde{\lambda}_i u\big)(t,\;\zeta' + (t-\tau-h)\tilde{\lambda}_i)\,dt \\ &\qquad - \int_\tau^{\tau+h} \tilde{l}_i \cdot \big(f(u) - \tilde{\lambda}_i u\big)(t,\;\zeta'' + (t-\tau-h)\tilde{\lambda}_i)\,dt. \end{aligned} \tag{4.5}$$

Consider the states

$$u' \doteq u\big(t,\;\zeta' + (t-\tau-h)\tilde{\lambda}_i\big), \qquad u'' \doteq u\big(t,\;\zeta'' + (t-\tau-h)\tilde{\lambda}_i\big), \qquad \tilde{u} \doteq u(\tau,\xi)$$

and define the averaged matrix

$$A^* \doteq \int_0^1 \Big[Df\big(su'' + (1-s)u'\big) - Df(\tilde{u})\Big]\;ds.$$

One can check that

$$\begin{aligned} &\tilde{l}_i \; \Big(f(u'') - f(u') - \tilde{\lambda}_i(u''-u')\Big) \\ &\qquad = \tilde{l}_i \; \Big(Df(\tilde{u}) \cdot (u''-u') - \tilde{\lambda}_i(u''-u')\Big) + \tilde{l}_i \; A^*(u''-u') \\ &\qquad = \tilde{l}_i \; A^*(u''-u'). \end{aligned}$$

Therefore

$$\begin{aligned} \left|\tilde{l}_i \; \Big(f(u'') - f(u') - \tilde{\lambda}_i(u''-u')\Big)\right| &= O(1) \cdot |u''-u'| \cdot \|A^*\| \\ &= O(1) \cdot |u''-u'| \cdot \big(|u''-\tilde{u}| + |u'-\tilde{u}|\big). \end{aligned}$$

Together with (4.5) this yields:

$$\begin{aligned} |E_i| = O(1) \cdot \int_\tau^{\tau+h} &\Big\{\big|u(t,\zeta' + (t-\tau-h)\tilde{\lambda}_i) - u(t,\zeta'' + (t-\tau-h)\tilde{\lambda}_i)\big| \cdot \\ &\quad \cdot \Big(\big|u(t,\zeta' + (t-\tau-h)\tilde{\lambda}_i) - u(\tau,\xi)\big| \\ &\qquad + \big|u(t,\zeta'' + (t-\tau-h)\tilde{\lambda}_i) - u(\tau,\xi)\big|\Big)\Big\}\;dt \\ = O(1) \cdot &\sup_{(s,x)\in\Gamma_h} |u(s,x) - u(\tau,\xi)| \cdot \\ &\cdot \int_\tau^{\tau+h} \mathrm{T.V.}\Big\{u(t);\;\; [\zeta' + (t-\tau-h)\tilde{\lambda}_i,\;\; \zeta'' + (t-\tau-h)\tilde{\lambda}_i]\Big\}\;dt. \end{aligned}$$

Therefore

$$\left|\int_{\zeta'}^{\zeta''}\Big[u(\tau+h,x)-U^\flat(h,x)\Big]\,dx\right|\leq\sum_{i=1}^{n}|E_i|$$

$$=O(1)\cdot\sup_{(s,x)\in\Gamma_h}|u(s,x)-u(\tau,\xi)|$$

$$\cdot\int_{\tau}^{\tau+h}\left[\sum_{i=1}^{n}\mathrm{T.V.}\Big\{u(t);\ ]\zeta'+(t-\tau-h)\tilde{\lambda}_i,\right.$$

$$\left.\zeta''+(t-\tau-h)\tilde{\lambda}_i]\Big\}\right]dt.$$

In view of Lemma 3.5, this establishes (4.4).

Combining (4.3), (4.4) and (2.3) we obtain

$$\begin{aligned}\int_{c+\lambda^*h}^{d-\lambda^*h}&\Big|u(\tau+h,x)-(S_hu(\tau))(x)\Big|\,dx\\&=O(1)\cdot\Big(2\varepsilon+\mathrm{T.V.}\{u(\tau);\ (c,d)\}\Big)\\&\quad\cdot\int_{\tau}^{\tau+h}\mathrm{T.V.}\Big\{u(t);\ [c+(t-\tau)\lambda^*,\ d-(t-\tau)\lambda^*]\Big\}\,dt\\&\quad+O(1)\cdot h\cdot\Big(\mathrm{T.V.}\{u(\tau);\ (c,d)\}\Big)^2,\end{aligned}\tag{4.6}$$

valid for small $h>0$.

STEP 3. Fix $\varepsilon>0$, a time $\tau\in[0,T]\setminus\mathcal{N}$ and an interval $[a,b]\subset\mathbf{R}$. By Lemma 3.3, the set $B_{\tau,\varepsilon}\cap[a,b]$ contains finitely many points, say $\xi_1<\xi_2<\ldots<\xi_N$. Observe that every point ξ where $u(\tau,\cdot)$ has a jump $>\varepsilon$ is certainly included in the above list.

We can now cover the set $[a,b]\setminus\{\xi_1,\ldots,\xi_N\}$ with open intervals (c_α,d_α), $\alpha=1,\ldots,M$, satisfying the following conditions:

(i) $\{\xi_1,\ldots,\xi_N\}\cap\bigcup_{\alpha=1}^{M}(c_\alpha,d_\alpha)=\emptyset$,

(ii) $\mathrm{T.V.}\{u(\tau);\ (c_\alpha,d_\alpha)\}\leq2\varepsilon$ for every $\alpha=1,\ldots,M$,

(iii) every point of $[a,b]$ is contained in at most two distinct intervals (c_α,d_α).

By steps 1 and 2, for every $h>0$ small enough one has

$$\frac{1}{h}\int_{\xi_i-\lambda^*h}^{\xi_i+\lambda^*h}\Big|u(\tau+h,x)-(S_hu(\tau))(x)\Big|\,dx\leq\frac{\varepsilon}{N},$$

and

$$\int_{c_\alpha+\lambda^* h}^{d_\alpha-\lambda^* h} \Big| u(\tau+h,x) - (S_h u(\tau))(x) \Big| \, dx$$
$$= O(1)\cdot\varepsilon\cdot \int_\tau^{\tau+h} \mathrm{T.V.}\Big\{ u(t);\ \big(c_\alpha + (t-\tau)\lambda^*,\ d_\alpha - (t-\tau)\lambda^*\big) \Big\}\, dt$$
$$+ O(1)\cdot h\varepsilon \cdot \mathrm{T.V.}\big\{ u(\tau);\ (c_\alpha, d_\alpha)\big\}$$

for every $i = 1, \dots, N$ and every $\alpha = 1, \dots, M$. Finally,

$$\begin{aligned}
\frac{1}{h}\int_a^b \Big| u(\tau+h,x) - \big(S_h u(\tau)\big)(x)\Big|\, dx
&\le \sum_{i=1}^N \frac{1}{h}\int_{\xi_i-\lambda^* h}^{\xi_i+\lambda^* h} \Big| u(\tau+h,x) - \big(S_h u(\tau)\big)(x)\Big|\, dx \\
&\qquad + \sum_{\alpha=1}^M \frac{1}{h}\int_{c_\alpha+\lambda^* h}^{d_\alpha-\lambda^* h} \Big| u(\tau+h,x) - \big(S_h u(\tau)\big)(x)\Big|\, dx \\
&\le \varepsilon + O(1)\cdot\frac{\varepsilon}{h}\int_\tau^{\tau+h} \mathrm{T.V.}\big\{u(t);\ \mathbf{R}\big\}\, dt + O(1)\cdot\varepsilon\cdot \mathrm{T.V.}\big\{u(\tau);\ \mathbf{R}\big\} \\
&= O(1)\cdot\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain (4.1). We have thus shown that if u satisfies **(A1)**–**(A3)**, then it must coincide with the corresponding semigroup trajectory $t \mapsto S_t\bar{u}$. On the other hand, one can easily check that the assumptions **(A1)**–**(A3)** are satisfied by all semigroup trajectories, because these are obtained as limits of wave-front tracking approximations. The proof of Theorem 1.2 is thus completed. □

CHAPTER 2

L^1 stability of patterns of non-interacting large shock waves

1. Introduction

In the previous Chapter we dealt with systems of conservation laws with initial data that are small in L^1 and BV. We recalled the basic result (Theorem 1.1, Chapter 1) saying that the entropy solutions to such Cauchy problems constitute a semigroup which is Lipschitz continuous with respect to time and initial data. A major question which remains open is whether the uniqueness of solutions also holds for arbitrarily large initial data. Observe that, because of the finite propagation speed, this is essentially a local problem, which can thus be reduced to proving the wellposedness of the Cauchy problem with the initial data $\bar{u}$ being a small perturbation of a fixed Riemann problem. The solution of the latter consists in general of M (large) waves of different characteristic families, but we shall restrict ourselves to the case where all these waves are (large) shocks.

More precisely, in the n-dimensional state space $M+1$ ($M \in \{2\dots n\}$) distinct states $\{u_0^q\}_{q=0}^M$ are fixed, with their corresponding open disjoint neighbourhoods $\{\Omega^q\}_{q=0}^M$ such that:

- The flux function $f : \Omega \longrightarrow \mathbf{R}^n$ is smooth and defined on $\Omega = \bigcup_{q=0}^{M} \Omega^q \subset \mathbf{R}^n$.
- f is strictly hyperbolic in Ω, that is: at every point $u \in \Omega$ the matrix $Df(u)$ has n real and simple eigenvalues $\lambda_1(u) < \ldots < \lambda_n(u)$. Note that consequently one has:
$$|\lambda_k(u) - \lambda_s(v)| \geq c \qquad \forall k \neq s \ \forall q : 0 \dots M \ \forall u, v \in \Omega^q \tag{1.1}$$
with some positive constant c, if only the neighbourhoods Ω^q are sufficiently small.
- Each characteristic field of the system:
$$u_t + f(u)_x = 0 \tag{1.2}$$
is either linearly degenerate or genuinely nonlinear, that is: with a basis $\{r_k(u)\}_{k=1}^n$ of corresponding right eigenvectors of $Df(u)$; $Df(u)r_k(u) = \lambda_k(u)r_k(u)$, each of the n directional derivatives $r_k \nabla \lambda_k$ vanishes either identically or nowhere.

We assume that the Riemann problem: (1.2) with:
$$u(0, \cdot) = \bar{u}, \tag{1.3}$$

$$\bar{u}(x) = \begin{cases} u_0^0 & x < 0 \\ u_0^M & x > 0 \end{cases} \tag{1.4}$$

has an M-shock solution:

$$u(t,x) = \begin{cases} u_0^0 & x < \Lambda^1 t \\ u_0^q & \Lambda^q t < x < \Lambda^{q+1} t, \quad q : 1 \dots M-1 \\ u_0^M & x > \Lambda^M t, \end{cases} \tag{1.5}$$

in which the states u_0^q are joined by M (large) shocks (u_0^{q-1}, u_0^q), $q : 1 \dots M$, travelling with respective speeds Λ^q.

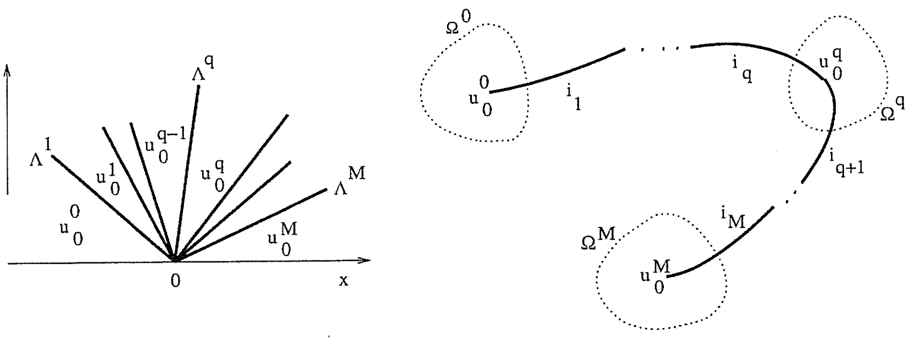

FIGURE 1.1

The following standard conditions on the nature of the large shocks are assumed. For (1.5) to be a distributional solution of (1.2) (1.3) (1.4), we need that for every $q : 1 \dots M$ the Rankine-Hugoniot conditions are satisfied:

$$f(u_0^{q-1}) - f(u_0^q) = \Lambda^q (u_0^{q-1} - u_0^q). \tag{1.6}$$

Moreover, the shocks (u_0^{q-1}, u_0^q) are said to belong to the corresponding i_q-characteristic families $(1 \leq i_1 < i_2 < \dots < i_M \leq n)$ and assumed to be compressive in the sense of Lax [L]:

$$\lambda_{i_q}(u_0^{q-1}) > \Lambda^q > \lambda_{i_q}(u_0^q). \tag{1.7}$$

Note that (1.7) yields in particular that the shocks of characteristic families carrying bigger indices travel with the faster speed: $\Lambda^1 < \dots < \Lambda^M$, as in Figure 1.1.

Finally, we require that all large shocks are stable in the sense of Majda [M], that is:

> The n vectors
> $$r_1(u_0^{q-1}), \dots, r_{i_q - 1}(u_0^{q-1}), u_0^q - u_0^{q-1}, r_{i_q + 1}(u_0^q), \dots, r_n(u_0^q) \tag{1.8}$$
> are linearly independent.

for every $q : 1 \dots M$. This condition, satisfied automatically when the shock (u_0^{q-1}, u_0^q) has small amplitude, will be explained and discussed in detail in the next Chapter.

The following questions arise naturally:

I. Do we have the (global in time and space) existence of a solution u to (1.2) (1.3) when $\bar{u}$ stays 'close' to the Riemann data (1.4)?

II. In case the answer to **I** is positive, is the solution u stable under small perturbations of its initial data?

Differently from the case of small initial data, the assumptions introduced so far are not sufficient to ensure the positive answer to any of the above questions. The more, even the solvability of Riemann problems (u^-, u^+) with $u^-, u^+ \in \Omega$ is not just a simple consequence of the existence of the solution (1.5) but requires an additional hypothesis. This and other stability conditions implying positive answers to questions **I** and **II** will be discussed in Chapter 3.
The purpose of this Chapter is to prove the wellposedness of (1.2) (1.3) under the following assumptions:

WEIGHTED BV STABILITY CONDITION

There exist a constant $\theta \in (0,1)$ and positive weights $w_1^q, \ldots, w_n^q$ (for every $q : 0 \ldots M$) such that the following holds. Consider a small wave of a family $k \leq i_q$, hitting from the right the large initial i_q-shock (u_0^{q-1}, u_0^q), as in Figure 1.2. Condition (1.8) guarantees that the Riemann problem (u_0^{q-1}, u_0^q) can be uniquely solved (see Section 2 in Chapter 3). Then

$$\sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| < \theta \tag{1.9}$$

at $\epsilon_1^{in} = \ldots = \epsilon_k^{in} = \ldots = \epsilon_n^{in} = 0$.

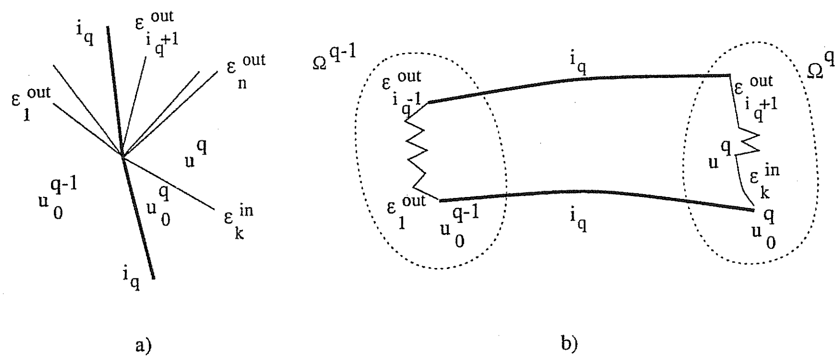

FIGURE 1.2

Symetrically, in case when a small k-wave with $k \geq i_q$ hits the shock (u_0^{q-1}, u_0^q) from the left (as in Figure 1.3), there holds:

$$\sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| < \theta \tag{1.10}$$

at $\epsilon_1^{in} = \ldots = \epsilon_k^{in} = \ldots = \epsilon_n^{in} = 0$.

Regarding w_s^q as the weight given to an s-wave located in the region between the $q-1$ and the q-th large shock, conditions (1.9) (1.10) simply say that, every time

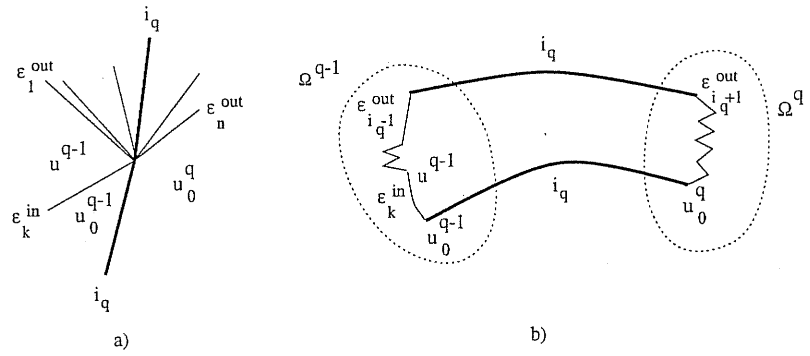

FIGURE 1.3

a small wave hits a large shock, the total weighted strength of the outgoing small waves is smaller than the weighted strength of the incoming wave.

WEIGHTED L^1 STABILITY CONDITION

In the setting of Figure 1.2:

$$
\begin{aligned}
&\sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right) \right| \\
&\qquad + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right) \right| < \theta
\end{aligned}
\tag{1.11}
$$

at $\epsilon_1^{in} = \ldots = \epsilon_k^{in} = \ldots = \epsilon_n^{in} = 0$, while in the setting of Figure 1.3:

$$
\begin{aligned}
&\sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right) \right| \\
&\qquad + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right) \right| < \theta
\end{aligned}
\tag{1.12}
$$

at $\epsilon_1^{in} = \ldots = \epsilon_k^{in} = \ldots = \epsilon_n^{in} = 0$.

Observe that it is always possible to define the weights $\{w_k^0\}$ and $\{w_k^M\}$ such that (1.9) - (1.12) are satisfied, provided that the suitable weights $\{w_k^q\}$, $q \notin \{0, M\}$ exist.

Also, as follows from Theorem 5.2 in Chapter 3, the Weighted L^1 Stability Condition is indeed stronger than the Weighted BV Stability Condition.

Now we turn to the main point of this Chapter. Define the domain $\widetilde{\mathcal{D}}_{\delta_0}$ by:

$$\widetilde{\mathcal{D}}_{\delta_0} = \mathrm{cl}\left\{ u : \mathbf{R} \longrightarrow \mathbf{R}^n;\ \text{there exist points } x^1 < x^2 < \ldots < x^M \text{ in } \mathbf{R} \right.$$

$$\text{such that calling } \widetilde{u}(x) = \begin{cases} u_0^0 & x < x^1 \\ u_0^q & x^q < x < x^{q+1},\ q: 1 \ldots M-1 \\ u_0^M & x > x^M \end{cases} \tag{1.13}$$

$$\left. \text{we have: } u - \widetilde{u} \in \mathrm{L}^1(\mathbf{R}, \mathbf{R}^n) \text{ and T.V.}(u - \widetilde{u}) \leq \delta_0 \right\},$$

with the closure taken in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^n)$.

Our main results are the following:

THEOREM 1.1. *If the Weighted* BV *Stability Condition (1.9) (1.10) is satisfied, then there exists $\delta_0 > 0$ such that for every $\bar{u} \in \widetilde{\mathcal{D}}_{\delta_0}$ (1.2) (1.3) has a solution (defined for all times $t \geq 0$).*

THEOREM 1.2. *If the Weighted* L^1 *Stability Condition (1.11) (1.12) is satisfied, then there exists $\delta_0 > 0$, $L > 0$, a closed domain $\mathcal{D}_{\delta_0} \subset \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^n)$ containing $\widetilde{\mathcal{D}}_{\delta_0}$, and a continuous semigroup $S : [0, \infty) \times \mathcal{D}_{\delta_0} \longrightarrow \mathcal{D}_{\delta_0}$ such that:*

(i) $S_0(\bar{u}) = \bar{u}$,
$S_{t+s}(\bar{u}) = S_t(S_s(\bar{u})) \quad \forall t, s \geq 0\ \forall \bar{u} \in \mathcal{D}_{\delta_0}$.

(ii) $\| S_t(\bar{u}) - S_s(\bar{w}) \|_{\mathrm{L}^1} \leq L \cdot (|t - s| + \| \bar{u} - \bar{w} \|_{\mathrm{L}^1}) \quad \forall t, s \geq 0\ \forall \bar{u}, \bar{w} \in \mathcal{D}_{\delta_0}$.

(iii) Each trajectory $t \mapsto S_t(\bar{u})$ is a solution of (1.2) (1.3).

The rest of this Chapter is organised as follows. Toward the proof of Theorem 1.1, in Section 2 we describe the wave front tracking algorithm working in the presence of large shocks and generating the approximate solutions of the system (1.2). The main features of the wave front tracking approximate solutions are listed in Theorem 3.4. In Section 3 we define the Glimm potentials, measuring the total strength of all small waves in these approximate solutions, and the amount of interaction between themselves or against the large shocks. The existence of the Glimm potential implies the validity of Theorem 1.1 in the standard way.
Section 4 contains the definition of the Lyapunov functional and the basic L^1 stability estimates for the wave front tracking algorithm. Our functional is motivated by the similar one in [BLY]; the difference is that it now contains some extra terms accounting for the interactions and coupling of the small waves against the large shocks. In Section 5 we prove the stated stability estimates, concluding the proof of Theorem 1.2. Finally, in Section 6 we comment on the relation of these results to other papers.

2. Wave front tracking approximations

Given a Cauchy problem (1.2) (1.3), one of the main strategies [B3] [D] to obtain the existence of its global in time solution is the following:

(i) Approximate the initial data $\bar{u}$ by piecewise constant data $\bar{u}_\epsilon$.

(ii) Give a recipe for construction an approximate solution u_ϵ to (1.2) with $u_\epsilon(0, \cdot) = \bar{u}_\epsilon$. The function u_ϵ should have relatively simple structure, e.g.

be piecewise constant, with finitely many jumps occuring along straight discontinuity lines.

(iii) Show that for some parameter sequence $\epsilon_n \to 0$, the sequence u_{ϵ_n} has a limit in $\mathrm{L}^1_{\mathrm{loc}}$, and that this limit is a solution to (1.2) (1.3).

In this Section our goal is to realize (ii) by means of the wave front tracking algorithm [BJ] [R], that we adjust to work in the presence of large shocks.

The 'fundamental block' for building the approximate solutions u_ϵ is provided by the suitable piecewise constant approximation of the self-similar solution to an arbitrary Riemann problem (u^-, u^+). If both the states u^-, u^+ belong to the same set Ω^q, then this solution is given by the already standard Accurate or Simplified Riemann Solvers [BJ]. Their construction depend on two positive parameters: δ which bounds the strength of the wave fronts in every rarefaction fan approximating centered rarefaction wave in the real solution, and $\hat{\lambda}$ (strictly larger than all characteristic speeds of (1.2)) that is the speed of non-physical waves, generated whenever the simplified method is used.

Below we present the corresponding solvers for the large amplitude Riemann problems $(u^-, u^+) \in \Omega^{q-1} \times \Omega^q$. Having it done, the wave front tracking algorithm works as follows.

> Assume we are given a piecewise constant function $u(0,\cdot) \in \widetilde{\mathcal{D}}_{\delta_0}$, for some small $\delta_0 > 0$ (such functions are dense in the domain $\widetilde{\mathcal{D}}_{\delta_0}$).
> Fix a treshold parameter $\rho > 0$. At each discontinuity point x of $u(0,\cdot)$, its corresponding Riemann problem $(u(0,x-), u(0,x+))$ is solved approximately on a forward neighbourhood of $(0,x)$ using Accurate Riemann Solver. The approximate solution $u(\cdot,\cdot)$ is thus defined until a time t_1, when the first interaction(s) between two or more wave fronts takes place. Decreasing slightly the speed of the slowest front, we may without loss of generality assume that at the first interaction time t_1 exactly two fronts interact. This interaction creates a new Riemann problem, that is to be solved according to the following rules.
>
> First, define the strength of any large wave connecting two states in Ω^{q-1} and Ω^q to be equal to some fixed number $B \leq 1$, bigger than all strengths of small waves.
>
> Now if both incoming fronts (small or large) are physical, and their strengths ϵ, ϵ' satisfy $|\epsilon\epsilon'| > \rho$, then the Accurate Riemann Solver is used. On the other hand, if $|\epsilon\epsilon'| \leq \rho$ or if one of the incoming fronts is a non-physical wave (thus with speed $\hat{\lambda}$), we then use the Simplified Riemann Solver. (Note that whenever a big wave is involved in the interaction then we use one of the new Solvers, described below.)
> This way a piecewise constant approximate solution $u(\cdot,\cdot)$ is generated up to the time t_2 when the second interaction takes place. We use the decribed procedure again to solve the new Riemann problem and thus prolong the function u up to the time t_3, and so on ...

This completes the definition of the algorithm. Next we introduce the Riemann Solvers, assuming that the problem under consideration is $(u^-, u^+) \in \Omega^{q-1} \times \Omega^q$.

Accurate Riemann Solver

Acurate Riemann Solver is the self-similar admissible solution of the Riemann problem (u^-, u^+), described in Theorem 2.1 in Chapter 3, with every rarefaction wave $(w, \mathcal{R}_k(w)(\epsilon))$ replaced by a piecewise constant rarefaction fan:

$$u(t,x) = \mathcal{R}_k(w)(s\tilde{\epsilon}) \quad \text{for } \frac{x}{t} \in \Big(\lambda_k(\mathcal{R}_k(w)(s\tilde{\epsilon})),\ \lambda_k(\mathcal{R}_k(w)((s+1)\tilde{\epsilon}))\Big) \qquad \forall s: 0 \dots N-1$$

where $N = [\epsilon/\delta] + 1$, $\tilde{\epsilon} = \epsilon/N$.

Simplified Riemann Solver

Case 1. Let $k > i_q$ be the family of a small (physical) wave of strength ϵ_k^{in}, interacting from the left with a large i_q shock, as in Figure 2.1.

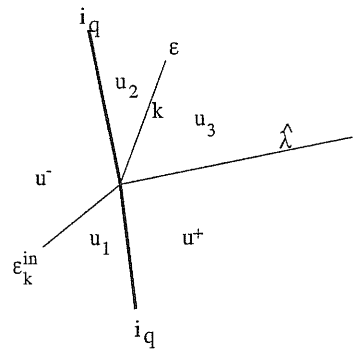

Figure 2.1

We solve the Riemann problem (u^-, u^+) in the following way:

$$\begin{cases} u^- & \text{for } x/t < \Lambda^{i_q}(u^-, u_2) \\ u_2 & \text{for } x/t \in (\Lambda^{i_q}(u^-, u_2), \lambda_k(u_2, u_3)) \\ u_3 = \Psi_k(u_2, \epsilon) & \text{for } x/t \in (\lambda_k(u_2, u_3), \hat{\lambda}) \\ u^+ & \text{for } x/t > \hat{\lambda}. \end{cases}$$

Here $\Psi_k(u_2, \cdot)$ is the k-th wave curve through its left state u_2 ([L]). The speed

$$\Lambda^{i_q}(u^-, u_2) = \frac{\langle f(u^-) - f(u_2), u^- - u_2 \rangle}{|u^- - u_2|^2}$$

is the Rankine-Hugoniot speed of the outgoing large shock. The outgoing strength ϵ is equal the strength ϵ_k^{out} in the solution of the problem (u^-, u^+) given in Theorem 2.1 in Chapter 3. The corresponding speed:

$$\lambda_k(u_2, u_3) = \begin{cases} \lambda_k(u_2) & \text{if } \epsilon > 0 \text{ and the } k\text{-field is genuinely nonlinear} \\ \dfrac{\langle f(u_2) - f(u_3), u_2 - u_3 \rangle}{|u_2 - u_3|^2} & \text{otherwise.} \end{cases}$$

The middle state u_2 is taken such that $|u_2 - u^-| = |u_1 - u^+|$ and that u_2 can be connected to u^- by a large i_q shock. The existence and uniqueness of u_2 is guaranteed by the analysis in Section 2 in Chapter 3.

If the small k-wave hits the large i_q-shock from the right ($k < i_q$), we construct the Solver in the analogous way; letting the k-wave pass through the shock with its strength changed by an appropriate factor, and create a new non-physical wave travelling with speed $\hat{\lambda}$.

CASE 2. A large i_q-shock is hit by a small wave of the same family or by a non-physical discontinuity. This case is treated exactly as in [BJ].

3. Glimm's potentials

Once the steps of the algorithm have been defined, it remains to show that it generates a piecewise constant function u_ϵ, defined for all times $t \geq 0$, and that the sequence $\{u_\epsilon\}$ converges to a solution of (1.2) (1.3) as ϵ tends to 0, for a suitable choice of the algorithm parameters ρ and δ (depending on ϵ). Since the large shocks of different families do not interact, the standard analysis (as in the case of only weak shocks present [B1] [BJ] [R]) applies, provided one finds suitable a priori bounds on the Total Variation of the profiles $u_\epsilon(t, \cdot)$ and bounds on the global number of wave fronts and the total strength of all non-physical waves. To this end, we shall define the so-called Glimm's potentials. Their properties are stated in Proposition 3.3.

Let $u(t, x)$ be a piecewise constant approximate solution, generated by the wave front tracking algorithm. At a fixed time $t > 0$, the function $u(t, \cdot)$ is piecewise constant, with jumps located at the wave front positions. There are precisely M large jumps, while the others are small, their left and right states belonging to the same set Ω^q.

DEFINITION 3.1. *(Approaching waves)*

(i) *We say that two small (possibly non-physical) fronts α and β, located at points $x_\alpha < x_\beta$ and belonging to the characteristic families $k_\alpha, k_\beta \in \{1 \dots n+1\}$ respectively, approach each other iff the following two conditions hold simultaneously:*
- *x_α and x_β lay together in one of the $M+1$ intervals (M of them unbounded) into which $\mathbf{R}$ is partitioned by the locations of large shocks. In other words: the states joined by the fronts under consideration both belong to the same set Ω^q.*
- *Either $k_\alpha < k_\beta$ or else $k_\alpha = k_\beta$ and at least one of the waves is a genuinely nonlinear shock.*

In this case we write: $(\alpha, \beta) \in \mathcal{A}$.

(ii) *We say that a small wave α of the characteristic family $k_\alpha \in \{1 \dots n+1\}$, located at x_α, approaches a large shock of family $k_\beta = i_q$ for some $q : 1 \dots M$, located at a point x_β iff one of the following conditions hold:*
- *The states u^-, u^+ joined by the small wave under consideration both belong to Ω^{q-1} and $k_\alpha \geq i_q$.*
- *The states u^-, u^+ belong to Ω^q and $k_\alpha \leq i_k$.*

We then write: $\alpha \in \mathcal{A}_{i_q}$.

We adopt the following notation. For a small wave of family k and strength ϵ_k, connecting two states u^- and u^+, we define its weighted strength as

$$b_k = w_k^q \cdot \epsilon_k \qquad \text{if } u^-, u^+ \in \Omega^q, \tag{3.1}$$

w_k^q as in the Weighted BV Stability Condition.

DEFINITION 3.2. *For a fixed $t > 0$ we define the following (weighted) total variation and interaction potentials:*

$$V(t) = \sum \{|b_\alpha|;\ \alpha \text{ - the small waves of all families}\},$$
$$Q_{\mathcal{A}}(t) = \sum_{(\alpha,\beta)\in\mathcal{A}} |b_\alpha b_\beta|,$$
$$Q_{i_q}(t) = \sum_{\alpha\in\mathcal{A}_{i_q}} |b_\alpha|, \quad q : 1 \dots M.$$

The Glimm's potentials:

$$Q(t) = \kappa Q_{\mathcal{A}}(t) + \sum_{q=1}^{M} Q_{i_q}(t),$$
$$\Gamma(t) = V(t) + \tilde{\kappa} Q(t) + \sum_{q=1}^{M-1} |u_q^*(t) - u_0^q|,$$

where $\kappa, \tilde{\kappa} > 0$ are constants to be specified later. The vectors $u_q^*(t)$ are the right states of the i_q-th large shock in $u(t,\cdot)$, respectively.

The following result is implied by the assumed Weighted BV Stability Condition (1.9) (1.10).

PROPOSITION 3.3. *Assume that the Weighted* BV *Stability Condition holds. Then for some constants $c, \kappa, \tilde{\kappa}, \delta_0 > 0$ the following is satisfied. If $u(0,\cdot)$ is piecewise constant and belongs to $\widetilde{\mathcal{D}}_{\delta_0}$, then for any $t > 0$ when two wave fronts of families α and β interact we have:*

(i)

$$\Delta Q(t) = Q(t+) - Q(t-)$$
$$\leq \begin{cases} -c|b_\alpha \cdot b_\beta| & \text{if both waves are small} \\ -c|b_\alpha| & \text{if } \alpha \text{ is a small wave and } \beta \text{ is a large shock.} \end{cases}$$

(ii) The same estimate as in (i) holds for $\Delta\Gamma(t) = \Gamma(t+) - \Gamma(t-)$.

The proof of Proposition 3.3 is standard, the details can be found in [LT].

Following [BLY] below we gather the main properties of the wave front tracking approximate solutions.

THEOREM 3.4. *Assume that the Weighted* BV *Stability Condition is satisfied and let $u(0,\cdot) \in \widetilde{\mathcal{D}}_{\delta_0}$ be a piecewise constant function, for δ_0 suitably small. Fix $\epsilon > 0$. Then for some parameters $\rho, \delta > 0$ (depending on ϵ) the wave front tracking algorithm generates the function $u_\epsilon : [0,\infty) \times \mathbf{R} \longrightarrow \mathbf{R}^n$ with the properties:*

(i) u_ϵ *is piecewise constant, with discontinuities occuring along finitely many lines in the* (t,x) *plane. Only finitely many interactions take place, each involving exactly two incoming fronts. Jumps can be of four types: small shocks (or contact discontinuities), rarefactions, non-physical waves and large shocks, denoted as* $\mathcal{J} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP} \cup \mathcal{LS}$.

(ii) *Along each shock (or contact discontinuity)* $x = x_\alpha$, $\alpha \in \mathcal{S}$, *its left and right states satisfy:*

$$u_\epsilon(t, x_\alpha+) = \Psi_{k_\alpha}(u_\epsilon(t, x_\alpha-), \epsilon_\alpha)$$

for some $k_\alpha : 1 \dots n$ *and the corresponding wave strength* ϵ_α. *Here* $\Psi_{k_\alpha}(u^-, \cdot)$ *stands for the wave curve through a fixed left state* u^- *(see* [L]*). If the* k_α *characteristic family is genuinely nonlinear, then* $\epsilon_\alpha < 0$. *Moreover, the speed of the shock satisfies:*

$$|\dot{x}_\alpha - \lambda_{k_\alpha}(u_\epsilon(t, x_\alpha-), u_\epsilon(t, x_\alpha+))| \leq \epsilon. \tag{3.2}$$

(iii) *Along each rarefaction front* $x = x_\alpha$, $\alpha \in \mathcal{R}$, *one has:*

$$u_\epsilon(t, x_\alpha+) = \Psi_{k_\alpha}(u_\epsilon(t, x_\alpha-), \epsilon_\alpha)$$

for some genuinely nonlinear family k_α *and the corresponding wave strength* $\epsilon_\alpha \in (0, \epsilon)$. *Moreover:*

$$|\dot{x}_\alpha - \lambda_{k_\alpha}(u_\epsilon(t, x_\alpha+)| \leq \epsilon. \tag{3.3}$$

(iv) *Every non-physical front* $x = x_\alpha$, $\alpha \in \mathcal{NP}$, *has the same speed* $\dot{x}_\alpha = \hat{\lambda}$, *where* $\hat{\lambda}$ *is a fixed constant strictly greater than all characterictic speeds. The total strength of all non-physical waves in* $u(t, \cdot)$ *remains uniformly small:*

$$\sum_{\alpha \in \mathcal{NP}} |u_\epsilon(t, x_\alpha-) - u_\epsilon(t, x_\alpha+)| \leq \epsilon \quad \forall t \geq 0. \tag{3.4}$$

(v) *Each of the* M *large shocks* $x = x_\alpha$, $\alpha \in \mathcal{LS}$ *is admissible in the sense of Theorem 2.1 in Chapter 3, and travels with the exact speed* $\dot{x}_\alpha = \Lambda^{k_\alpha}(u_\epsilon(t, x_\alpha-), u_\epsilon(t, x_\alpha+))$.

The function u_ϵ *as above will be called an* ϵ*-approximate solution of (1.2).*

4. The Lyapunov functional

This Section serves to define the Lyapunov functional Φ [LY1] [LY2] [LY3] [BLY], measuring the L^1 distance between the time profiles of two arbitrary ϵ-approximate solutions $u, v : [0, \infty) \times \mathbf{R} \longrightarrow \mathbf{R}^n$ constructed by wave front tracking algorithm (see Theorem 3.4). The two crucial features of Φ will be the following:

$$\frac{1}{C} \cdot \| u(t, \cdot) - v(t, \cdot) \|_{L^1} \leq \Phi(u(t, \cdot), v(t, \cdot)) \leq C \cdot \| u(t, \cdot) - v(t, \cdot) \|_{L^1}, \tag{4.1}$$

$$\Phi(u(t, \cdot), v(t, \cdot)) \leq \Phi(u(s, \cdot), v(s, \cdot)) + O(1) \cdot \epsilon \cdot (t - s) \quad \forall t > s \geq 0. \tag{4.2}$$

Fix a time $t > 0$ and consider a space point $x \in \mathbf{R}$ which is not a discontinuity point of the functions $u = u(t, \cdot)$, $v = v(t, \cdot)$. Let $u(x) \in \Omega^i$, $v(x) \in \Omega^j$, for some $i, j : 0 \dots M$. We define the scalar quantities $\{b_k(x)\}_{k=1}^n$ as the weighted strengths of the corresponding shock waves in the jump $(u(x), v(x))$. More precisely, we consider the Riemann data:

$$(w^-, w^+) = \begin{cases} (u(x), v(x)) & \text{if } i \leq j \\ (v(x), u(x)) & \text{if } i > j. \end{cases} \tag{4.3}$$

By a slight modification of Corollary 5.3 in Chapter 3 one can see that the Riemann problem (1.2) (4.3) has a unique self-similar solution, whose all small waves are shocks (possibly nonadmissible). The weighted strengths of the waves in this solution wii be called $b_k(x)$. In particular, if for example $u(x), v(x) \in \Omega^0$, then for every $k : 1 \dots n$ we have $b_k(x) = w_k^0 \cdot \epsilon_k(x)$ where the strengths $\{\epsilon_k(x)\}_{k=1}^n$ are implicitely defined by:

$$v(x) = \mathcal{S}_n(\dots, \mathcal{S}_1(u(x), \epsilon_1(x)), \dots, \epsilon_n(x)).$$

By $\lambda_k(x)$ we denote the corresponding speed of the k-th wave $\epsilon_k(x)$.

We define the functional:

$$\Phi(u, v) := \sum_{k=1}^{n} \int_{-\infty}^{\infty} |b_k(x)| W_k(x) dx,$$

where the weights W_k are given by:

$$W_k(x) := 1 + \kappa_1 A_k(x) + \kappa_2 [Q(u) + Q(v)]. \tag{4.4}$$

The constants κ_1, κ_2 in (4.4) are to be defined later. Q is the Glimm interaction potential, introduced in Definition 3.2. The amount $A_k(x)$ of waves in u and v, approaching the wave $\epsilon_k(x)$ is defined in the following way:

$$A_k(x) = \begin{cases} B_k(x) + C_k(x) & \text{if } k\text{-wave } b_k(x) \text{ is small, joining} \\ & \qquad \text{the states in } \Omega^s,\ s : 0 \dots M \\ D_k(x) + F_k(x) & \text{if } k\text{-wave } b_k(x) = B \text{ is large,} \\ & \qquad k = i_s \text{ for some } s : 1 \dots M \end{cases} \tag{4.5}$$
$$+ \begin{cases} G_k(x) & \text{if } k\text{-field is genuinely nonlinear and } k\text{-wave } b_k(x) \\ & \qquad \text{is small, joining the states } \Omega^s,\ s : 0 \dots M \\ 0 & \text{otherwise} \end{cases}$$

The summands in (4.5) are defined as follows:

$$B_k(x) = \left[\sum_{\substack{\alpha \in \mathcal{LS},\ k_\alpha \in \{i_s, i_{s+1}\} \\ x_\alpha < x,\ k_\alpha > k}} + \sum_{\substack{\alpha \in \mathcal{LS},\ k_\alpha \in \{i_s, i_{s+1}\} \\ x_\alpha > x,\ k_\alpha < k}} \right] |\epsilon_\alpha|$$
$$+ \begin{cases} \displaystyle\sum_{\substack{\alpha \in \mathcal{LS} \\ x_\alpha < x,\ k_\alpha = i_s}} |\epsilon_\alpha| & \text{if } k = i_s \\ \displaystyle\sum_{\substack{\alpha \in \mathcal{LS} \\ x_\alpha > x,\ k_\alpha = i_{s+1}}} |\epsilon_\alpha| & \text{if } k = i_{s+1}, \end{cases}$$

$$C_k(x) = \left[\sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{LS},\ x_\alpha<x,\ k<k_\alpha\le n, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} + \sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{LS},\ x_\alpha>x,\ 1\le k_\alpha<k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} \right] |\epsilon_\alpha|,$$

$$D_k(x) = \left[\sum_{\substack{\alpha\in\mathcal{LS}, \\ x_\alpha>x,\ k_\alpha=i_{s-1}}} + \sum_{\substack{\alpha\in\mathcal{LS}, \\ x_\alpha<x,\ k_\alpha=i_{s+1}}} \right] |\epsilon_\alpha|,$$

$$F_k(x) = \left[\sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{LS},\ x_\alpha<x,\ k<k_\alpha\le n, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^{s-1} \text{ or in } \Omega^s}} + \sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{LS},\ x_\alpha>x,\ 1\le k_\alpha<k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^{s-1} \text{ or in } \Omega^s}} \right] |\epsilon_\alpha|$$
$$+ \left[\sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{LS},\ x_\alpha<x,\ k_\alpha=k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^{s-1}}} + \sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{LS},\ x_\alpha>x,\ k_\alpha=k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} \right] |\epsilon_\alpha|,$$

$$G_k(x) = \begin{cases} \left[\sum_{\substack{\alpha\in\mathcal{J}(u)\setminus\mathcal{LS},\ x_\alpha<x,\ k_\alpha=k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} + \sum_{\substack{\alpha\in\mathcal{J}(v)\setminus\mathcal{LS},\ x_\alpha>x,\ k_\alpha=k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} \right] |\epsilon_\alpha| & \text{if } b_k(x) < 0 \\ \left[\sum_{\substack{\alpha\in\mathcal{J}(v)\setminus\mathcal{LS},\ x_\alpha<x,\ k_\alpha=k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} + \sum_{\substack{\alpha\in\mathcal{J}(u)\setminus\mathcal{LS},\ x_\alpha>x,\ k_\alpha=k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} \right] |\epsilon_\alpha| & \text{if } b_k(x) > 0. \end{cases}$$

Here ϵ_α stands for the (nonweighted) strength of the wave $\alpha \in \mathcal{J}$, located at point x_α and belonging to the characteristic family k_α. $\mathcal{J} = \mathcal{J}(u) \cup \mathcal{J}(v)$ is the set of all waves in u and v, by $\mathcal{LS}, \mathcal{R}, \mathcal{S}, \mathcal{C}$ we denote respectively: the large shocks, rarefactions, (weak) shocks and non-physical waves in u and v.

We assume the convention that in the above definitions we sum only the terms whose indices lie in their admissible ranges; for example if $s = M$, then obviously there are no large waves with the index i_{s+1} and thus we do not treat the corresponding terms calling the strengths of these waves.

We comment briefly on the formula (4.5). The summands $B_k(x)$ and $D_k(x)$ account for the large waves approaching the k-wave under consideration. However, only these large waves are considered, whose right or left state belongs to the set Ω^s containing at least one of the states joined by the k-wave.
$C_k(x)$ and $G_k(x)$ are the usual summands, identical with the ones in the corresponding definition of $A_k(x)$ in [BLY]. Their presence says that a small k-wave is approached by any wave of a faster family, located to the left and any wave of a slower family, located to the right. Only small physical waves, 'living' in the same set Ω^s as the k-wave, are involved.
The summand $F_k(x)$ contains the strengths of the small physical waves approaching a large k-wave under consideration, according to their locations and speeds. The convention as in the definition of $B_k(x)$ is valid. The presence of the second term in $F_k(x)$ is due to the assumed Lax stability of large shocks.

Let α be a wave in u (or v), located at a point x_α, with speed $\dot{x}_\alpha$. Following [BLY] define:

$$E_{\alpha,k} = |b_k(x_\alpha+)|W_k(x_\alpha+)(\lambda_k(x_\alpha+) - \dot{x}_\alpha) - |b_k(x_\alpha-)|W_k(x_\alpha-)(\lambda_k(x_\alpha-) - \dot{x}_\alpha).$$

The standard argument [BLY] [LT] shows that (4.1) (4.2) are implied by:

$$\sum_{k=1}^{n} E_{\alpha,k} = O(1) \cdot |\epsilon_\alpha| \qquad \forall \alpha \in \mathcal{C} \tag{4.6}$$

$$\sum_{\alpha \in \mathcal{J} \setminus \mathcal{C}} \sum_{k=1}^{n} E_{\alpha,k} = O(1) \cdot \epsilon \tag{4.7}$$

If t is an interaction time of two fronts in u or v then all weights $W_k(x)$ decrease at time t. (4.8)

The statements (4.6) and (4.8) are proved as in [LT], using Definition 3.2. In the remaining part of the article we will focus on (4.7). As usual, if no ambiguity created, we abbreviate the notation and for a particular wave α under consideration write: b_k^+ instead of $b_k(x_\alpha+)$, W_k^- instead of $W_k(x_\alpha-)$, etc.

Keeping in mind a possible 'representative' configuration of wave locations in u and v, as in Figure 4.1 we formulate the following condition:

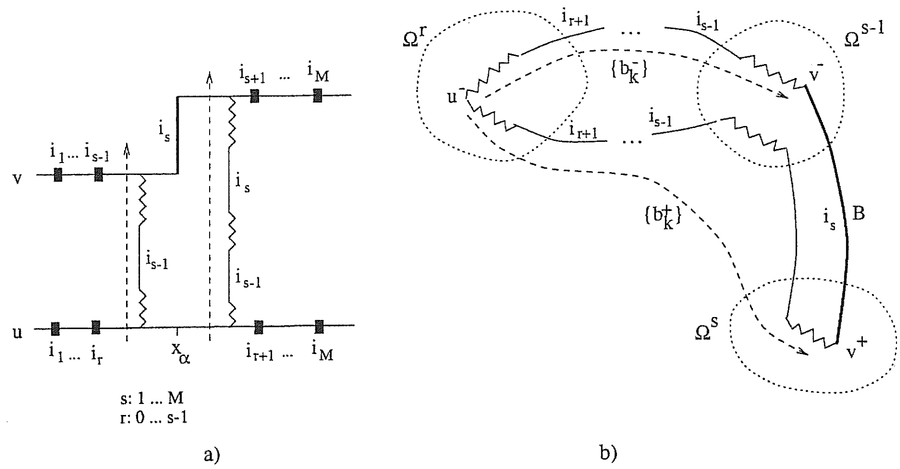

FIGURE 4.1

At least for one wave $\alpha \in \mathcal{LS}$ (of the family i_s) both wave vectors $\{b_k^-\}_{k=1}^n$ and $\{b_k^+\}_{k=1}^n$ contain a large wave of the same family i_k. (4.9)

The proof of (4.7) will be performed according to whether (4.9) holds or is violated.

CASE 1. – (4.9) holds.
Note that one may always take $i_k \in \{i_{s-1}, i_{s+1}\}$ so that, by (4.5):

$$E_{i_k} = B \cdot \left[(W_{i_k}^+ - W_{i_k}^-)(\lambda_{i_k}^\pm - \dot{x}_\alpha) + W_{i_k}^\mp(\lambda_{i_k}^+ - \lambda_{i_k}^-)\right] \leq -\kappa_1 B^2 c, \tag{4.10}$$

where $c > 0$ is a small uniform constant, bounded away from zero. The inequality in (4.10) follows from the fact that $\lambda^+_{i_k} - \lambda^-_{i_k}$ in there is of the order of the sum of all small waves in $\{\epsilon^-_k\}^n_{k=1}$ and $\{\epsilon^+_k\}^n_{k=1}$.

We thus see that E_{i_k} provides a big negative term that eventually overwhelms all the other terms E_k, because:

$$\begin{aligned}
&E_k = B \cdot W^\pm_k(\lambda^+_k - \lambda^-_k) \quad \text{if } b^+_k = b^-_k = B \text{ and } k \notin \{i_{s-1}, i_{s+1}\},\\
&E_k = |b^+_k| \cdot W^+_k(\lambda^+_k - \dot{x}_\alpha) - |b^-_k| \cdot W^-_k(\lambda^-_k - \dot{x}_\alpha) \quad \text{if both } b^+_k \text{ and } b^-_k \text{ are small},\\
&E_{i_s} = B \cdot W^\pm_{i_s}(\lambda^\pm_{i_s} - \dot{x}_\alpha) - |b^\mp_{i_s}| \cdot W^\mp_{i_s}|\lambda^\mp_{i_s} - \dot{x}_\alpha| \leq B \cdot W^\pm_{i_s}(\lambda^\pm_{i_s} - \dot{x}_\alpha).
\end{aligned}$$

In all the above cases:

$$E_k = O(1) \cdot \left[\sum_{\substack{k=1 \\ \alpha_k \notin \mathcal{LS}}}^{n} |\epsilon^-_k| + \sum_{\substack{k=1 \\ \alpha_k \notin \mathcal{LS}}}^{n} |\epsilon^+_k| \right]. \tag{4.11}$$

Similar analysis works for E^β_k with $\beta \in \mathcal{LS}$, $\beta \neq \alpha$. In case $\alpha \in \mathcal{S} \cup \mathcal{R}$, the following estimate will be shown in Section 4:

$$\sum_{k=1}^{n} E_k = O(1) \cdot |\epsilon_\alpha|. \tag{4.12}$$

Observing that by Proposition 3.3 the quantity

$$\sum_{\alpha \in \mathcal{J} \setminus \mathcal{LS}} |\epsilon_\alpha|$$

is bounded (uniformly in time), one sees that (4.10) - (4.12) imply (4.7) if only κ_1 is big and δ_0 in (1.13) is small enough.

CASE 2. – (4.9) is violated.

The above is possible if and only if no large wave can be found between the locations of any pair of the large shocks of the same family (occuring in u and v). In other words: one of the immediate large successors or predecessors of any large wave in u or v, must be of the same characteristic family as this wave – see Figure 4.2.

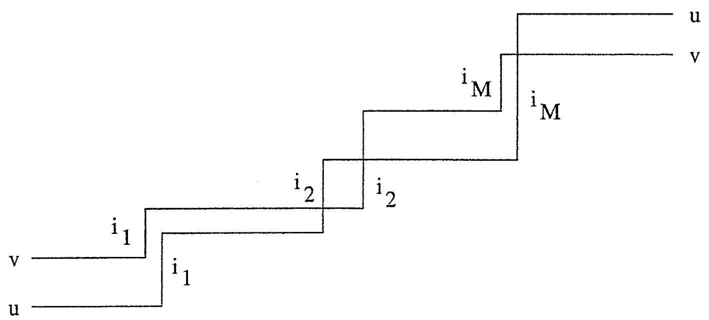

FIGURE 4.2

For a fixed $s : 1 \dots M$, denote by α the wave in v of the family i_s, and by β the large jump in u of the same family. Then, as shown in Section 5:

$$\sum_{k=1}^{n} E_{\alpha,k} + \sum_{k=1}^{n} E_{\beta,k} \leq 0, \tag{4.13}$$

and

$$\sum_{k=1}^{n} E_{\alpha,k} = O(1) \cdot \epsilon \cdot |\epsilon_\alpha| \quad \forall \alpha \in \mathcal{S} \cup \mathcal{R}. \tag{4.14}$$

Certainly (4.13) and (4.14) imply (4.7).

5. Proofs of the stability estimates

CASE OF LARGE SHOCKS
– THE ESTIMATE (4.13)

We assume that the waves location pattern looks as in Figure 5.1, all the other possible configurations can be treated in entirely the same way.

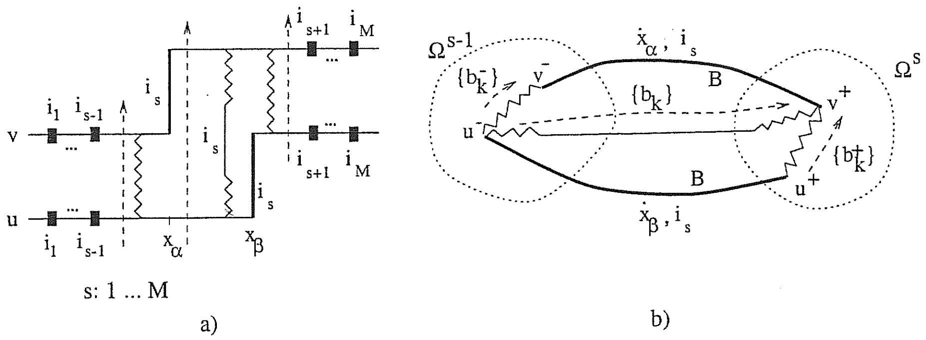

FIGURE 5.1

Using notation of Figure 5.1, we will show that:

$$\begin{aligned}\sum_{k=1}^{n} E_{\alpha,k} + \sum_{k=1}^{n} E_{\beta,k} = \sum_{k=1}^{n} &\left[|b_k| \cdot W_k(\lambda_k - \dot{x}_\alpha) - |b_k^-| \cdot W_k^-(\lambda_k^- - \dot{x}_\alpha)\right] \\ + \sum_{k=1}^{n} &\left[|b_k^+| \cdot W_k^+(\lambda_k^+ - \dot{x}_\beta) - |b_k| \cdot W_k(\lambda_k - \dot{x}_\beta)\right] \leq 0.\end{aligned} \tag{5.1}$$

First, we estimate $\sum_{k=1}^{n} E_{\alpha,k}$. By Lemma 5.1 in [LT] and definitions (4.5) we get:

$$E_{\alpha,i_s} = B \cdot O(1) \cdot \sum_{k \geq i_s} |b_k^-| - |b_{i_s}^-| \cdot (O(1) + 2\kappa_1 B) \cdot |\lambda_{i_s}^- - \dot{x}_\alpha|, \tag{5.2}$$

$$
\begin{aligned}
\sum_{k\le i_{s-1}} E_{\alpha,k} &= \sum_{k\le i_{s-1}} \Big[|b_k|\cdot(\lambda_k-\dot{x}_\alpha)(W_k-W_k^-) \\
&\qquad\qquad + W_k^-\big(|b_k|(\lambda_k-\dot{x}_\alpha)-|b_k^-|(\lambda_k^- -\dot{x}_\alpha)\big)\Big] \\
&\le \sum_{k\le i_{s-1}} \Big[-|b_k|\cdot|\lambda_k-\dot{x}_\alpha|\kappa_1 B \\
&\qquad\qquad + 2\kappa_1 B\big(|b_k^-||\lambda_k^- -\dot{x}_\alpha|-|b_k||\lambda_k-\dot{x}_\alpha|\big)\Big] \\
&\quad + O(1)\cdot\left[\sum_{k<i_s}|b_k|+\sum_{k\ge i_s}|b_k^-|\right] \\
&= -3\kappa_1 B\cdot\sum_{k\le i_{s-1}}|b_k||\lambda_k-\dot{x}_\alpha|+2\kappa_1 B\cdot\sum_{k\le i_{s-1}}|b_k^-||\lambda_k^- -\dot{x}_\alpha| \\
&\quad + O(1)\cdot\left[\sum_{k<i_s}|b_k|+\sum_{k\ge i_s}|b_k^-|\right],
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
\sum_{i_{s-1}<k<i_s} E_{\alpha,k} &= \sum_{i_{s-1}<k<i_s} \Big[|b_k|\cdot(\lambda_k-\dot{x}_\alpha)(W_k-W_k^-) \\
&\qquad\qquad + W_k^-\big(|b_k|(\lambda_k-\dot{x}_\alpha)-|b_k^-|(\lambda_k^- -\dot{x}_\alpha)\big)\Big] \\
&\le \sum_{i_{s-1}<k<i_s} \big[-|b_k|\cdot|\lambda_k-\dot{x}_\alpha|\kappa_1 B\big] \\
&\quad + O(1)\cdot\left[\sum_{k<i_s}|b_k|+\sum_{k\ge i_s}|b_k^-|\right],
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
\sum_{i_s<k<i_{s+1}} E_{\alpha,k} &= \sum_{i_s<k<i_{s+1}} \Big[|b_k^-|\cdot(\lambda_k^- -\dot{x}_\alpha)(W_k-W_k^-) \\
&\qquad\qquad + W_k\big(|b_k|(\lambda_k-\dot{x}_\alpha)-|b_k^-|(\lambda_k^- -\dot{x}_\alpha)\big)\Big] \\
&\le \sum_{i_s<k<i_{s+1}} \Big[-|b_k^-|\cdot|\lambda_k^- -\dot{x}_\alpha|\kappa_1 B \\
&\qquad\qquad + \kappa_1 B\big(|b_k||\lambda_k-\dot{x}_\alpha|-|b_k^-||\lambda_k^- -\dot{x}_\alpha|\big)\Big] \\
&\quad + O(1)\cdot\sum_{k\ge i_s}|b_k^-| \\
&= -2\kappa_1 B\cdot\sum_{i_s<k<i_{s+1}}|b_k^-||\lambda_k^- -\dot{x}_\alpha| \\
&\quad + \kappa_1 B\cdot\sum_{i_s<k<i_{s+1}}|b_k||\lambda_k-\dot{x}_\alpha|+O(1)\cdot\sum_{k\ge i_s}|b_k^-|,
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
\sum_{k\geq i_{s+1}} E_{\alpha,k} = & \sum_{k\geq i_{s+1}} \Big[|b_k|\cdot(\lambda_k-\dot{x}_\alpha)(W_k-W_k^-) \\
& \qquad + W_k^-\left(|b_k|(\lambda_k-\dot{x}_\alpha)-|b_k^-|(\lambda_k^- -\dot{x}_\alpha)\right)\Big] \\
\leq & \sum_{k\geq i_{s+1}} \Big[|b_k|\cdot|\lambda_k-\dot{x}_\alpha|\kappa_1 B \\
& \qquad + 2\kappa_1 B\left(|b_k||\lambda_k-\dot{x}_\alpha|-|b_k^-||\lambda_k^- -\dot{x}_\alpha|\right)\Big] \\
& + O(1)\cdot\sum_{k\geq i_s}|b_k^-| \\
= & -2\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k^-||\lambda_k^- -\dot{x}_\alpha| + 3\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k||\lambda_k-\dot{x}_\alpha| \\
& + O(1)\cdot\sum_{k\geq i_s}|b_k^-|.
\end{aligned}
\tag{5.6}
$$

As usual, we do not take into account these terms in the above formulae, that contain the 'nonexisting' indices i_{s-1} or i_{s+1}.

Summing the inequalities (5.2) - (5.6) we obtain:

$$
\begin{aligned}
\sum_{k=1}^{n} E_{\alpha,k} \leq & -3\kappa_1 B\cdot\sum_{k\leq i_{s-1}}|b_k||\lambda_k-\dot{x}_\alpha| - \kappa_1 B\cdot\sum_{i_{s-1}<k<i_s}|b_k||\lambda_k-\dot{x}_\alpha| \\
& + \kappa_1 B\cdot\sum_{i_s<k<i_{s+1}}|b_k||\lambda_k-\dot{x}_\alpha| + 3\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k||\lambda_k-\dot{x}_\alpha| \\
& + 2\kappa_1 B\cdot\sum_{k\leq i_{s-1}}|b_k^-||\lambda_k^- -\dot{x}_\alpha| - 2\kappa_1 B\cdot\sum_{k\geq i_s}|b_k^-||\lambda_k^- -\dot{x}_\alpha| \\
& + O(1)\cdot\left[\sum_{k<i_s}|b_k| + \sum_{k\geq i_s}|b_k^-|\right].
\end{aligned}
\tag{5.7}
$$

Now we estimate the terms in $\sum_{k=1}^n E_{\beta,k}$:

$$
E_{\beta,i_s} = B\cdot O(1)\cdot\sum_{k\leq i_s}|b_k^+| - |b_{i_s}^+|\cdot(O(1)+2\kappa_1 B)\cdot|\lambda_{i_s}^+ -\dot{x}_\beta|, \tag{5.8}
$$

$$
\begin{aligned}
\sum_{k\le i_{s-1}} E_{\beta,k} &= \sum_{k\le i_{s-1}} \Big[|b_k|\cdot(\lambda_k-\dot{x}_\beta)(W_k-W_k^-) \\
&\qquad\qquad + W_k^+\left(|b_k^+|(\lambda_k^+-\dot{x}_\beta)-|b_k|(\lambda_k-\dot{x}_\beta)\right)\Big] \\
&\le \sum_{k\le i_{s-1}} \Big[|b_k|\cdot|\lambda_k-\dot{x}_\beta|\kappa_1 B \\
&\qquad\qquad + 2\kappa_1 B\left(|b_k||\lambda_k-\dot{x}_\beta|-|b_k^+||\lambda_k^+-\dot{x}_\beta|\right)\Big] \\
&\quad + O(1)\cdot\left[\sum_{k\le i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right] \\
&= 3\kappa_1 B\cdot\sum_{k\le i_{s-1}}|b_k||\lambda_k-\dot{x}_\beta| - 2\kappa_1 B\cdot\sum_{k\le i_{s-1}}|b_k^+||\lambda_k^+-\dot{x}_\beta| \\
&\quad + O(1)\cdot\left[\sum_{k\le i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right],
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
\sum_{i_{s-1}<k<i_s} E_{\beta,k} &= \sum_{i_{s-1}<k<i_s} \Big[|b_k^+|\cdot(\lambda_k^+-\dot{x}_\beta)(W_k-W_k^-) \\
&\qquad\qquad + W_k\left(|b_k^+|(\lambda_k^+-\dot{x}_\beta)-|b_k|(\lambda_k-\dot{x}_\beta)\right)\Big] \\
&\le \sum_{i_{s-1}<k<i_s} \Big[-|b_k^+|\cdot|\lambda_k^+-\dot{x}_\alpha|\kappa_1 B\Big] \\
&\qquad\qquad + \kappa_1 B\left(|b_k||\lambda_k-\dot{x}_\beta|-|b_k^+||\lambda_k^+-\dot{x}_\beta|\right)\Big] \\
&\quad + O(1)\cdot\left[\sum_{k\le i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right] \\
&= -2\kappa_1 B\cdot\sum_{i_{s-1}<k<i_s}|b_k^+||\lambda_k^+-\dot{x}_\beta| \\
&\quad + \kappa_1 B\cdot\sum_{i_{s-1}<k<i_s}|b_k||\lambda_k-\dot{x}_\beta| \\
&\quad + O(1)\cdot\left[\sum_{k\le i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right],
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
\sum_{i_s<k<i_{s+1}} E_{\beta,k} &= \sum_{i_s<k<i_{s+1}} \Big[|b_k|\cdot(\lambda_k-\dot{x}_\beta)(W_k-W_k^-) \\
&\qquad\qquad + W_k^+\left(|b_k^+|(\lambda_k^+-\dot{x}_\beta)-|b_k|(\lambda_k-\dot{x}_\beta)\right)\Big] \\
&\le \sum_{i_s<k<i_{s+1}} \Big[-|b_k|\cdot|\lambda_k-\dot{x}_\alpha|\kappa_1 B\Big] \\
&\quad + O(1)\cdot\left[\sum_{k\le i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right],
\end{aligned}
\tag{5.11}
$$

$$\begin{aligned}
\sum_{k\geq i_{s+1}} E_{\beta,k} = & \sum_{k\geq i_{s+1}} \Big[|b_k|\cdot(\lambda_k-\dot{x}_\beta)(W_k-W_k^-) \\
& \qquad\qquad + W_k^+\left(|b_k^+|(\lambda_k^+-\dot{x}_\beta)-|b_k|(\lambda_k-\dot{x}_\beta)\right)\Big] \\
\leq & \sum_{k\geq i_{s+1}} \Big[-|b_k|\cdot|\lambda_k-\dot{x}_\beta|\kappa_1 B \\
& \qquad\qquad + 2\kappa_1 B\left(|b_k^+||\lambda_k^+-\dot{x}_\beta|-|b_k||\lambda_k-\dot{x}_\beta|\right)\Big] \\
& + O(1)\cdot\left[\sum_{k\leq i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right] \\
= & -3\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k||\lambda_k-\dot{x}_\beta|+2\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k^+||\lambda_k^+-\dot{x}_\beta| \\
& + O(1)\cdot\left[\sum_{k\leq i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right].
\end{aligned} \tag{5.12}$$

Thus, in view of (5.8) - (5.12), we get:

$$\begin{aligned}
\sum_{k=1}^{n} E_{\beta,k} \leq & 3\kappa_1 B\cdot\sum_{k\leq i_{s-1}}|b_k||\lambda_k-\dot{x}_\beta|+\kappa_1 B\cdot\sum_{i_{s-1}<k<i_s}|b_k||\lambda_k-\dot{x}_\beta| \\
& -\kappa_1 B\cdot\sum_{i_s<k<i_{s+1}}|b_k||\lambda_k-\dot{x}_\beta|-3\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k||\lambda_k-\dot{x}_\beta| \\
& -2\kappa_1 B\cdot\sum_{k\leq i_s}|b_k^+||\lambda_k^+-\dot{x}_\beta|+2\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k^+||\lambda_k^+-\dot{x}_\beta| \\
& +O(1)\cdot\left[\sum_{k\leq i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right].
\end{aligned} \tag{5.13}$$

Summing (5.7) with (5.13) and recalling that

$$|\dot{x}_\alpha-\dot{x}_\beta| = O(1)\cdot\left[\sum_{k\geq i_s}|b_k^-|+\sum_{k\leq i_s}|b_k^+|\right],$$

we get:

$$\begin{aligned}
& \sum_{k=1}^{n} E_{\alpha,k}+\sum_{k=1}^{n} E_{\beta,k} \\
& \quad \leq 2\kappa_1 B\cdot\Big(\sum_{k\leq i_{s-1}}|b_k^-||\lambda_k^--\dot{x}_\alpha|-\sum_{k\geq i_s}|b_k^-||\lambda_k^--\dot{x}_\alpha| \\
& \qquad\qquad -\sum_{k\leq i_s}|b_k^+||\lambda_k^+-\dot{x}_\beta|+\sum_{k\geq i_{s+1}}|b_k^+||\lambda_k^+-\dot{x}_\beta|\Big) \\
& \quad +O(1)\cdot\left[\sum_{k\geq i_s}|b_k^-|+\sum_{k\neq i_s}|b_k|+\sum_{k\leq i_s}|b_k^+|\right].
\end{aligned} \tag{5.14}$$

Since

$$\sum_{k\neq i_s}|b_k| = O(1)\cdot\left[\sum_{k\geq i_s}|b_k^-| + \sum_{k\leq i_s}|b_k^+|\right],$$

we see that once we fix $\gamma\in(0,1)$, without loss of generality the following estimate holds:

$$\begin{aligned}\sum_{k=1}^{n}E_{\alpha,k} + \sum_{k=1}^{n}E_{\beta,k}&\\ \leq 2\kappa_1 B\cdot\Big[\sum_{k\leq i_{s-1}}|b_k^-||\lambda_k^- - \dot{x}_\beta| &+ \sum_{k\geq i_{s+1}}|b_k^+||\lambda_k^+ - \dot{x}_\beta|\\ &-\gamma\cdot\Big(\sum_{k\geq i_s}|b_k^-||\lambda_k^- - \dot{x}_\beta| + \sum_{k\leq i_s}|b_k^+||\lambda_k^+ - \dot{x}_\beta|\Big)\Big].\end{aligned}\tag{5.15}$$

Note that if $\gamma > \theta$ then the right hand side of (5.15) is nonpositive by the following two estimates:

$$\begin{aligned}\sum_{k\leq i_{s-1}}|b_k(\lambda_k - \Lambda^s) - b_k^-(\lambda_k^- - \Lambda^s)| + \sum_{k\geq i_{s+1}}|b_k||\lambda_k - \Lambda^s|&\\ \leq\gamma\cdot\sum_{k\geq i_s}|b_k^-||\lambda_k^- - \Lambda^s|,&\end{aligned}\tag{5.16}$$

$$\begin{aligned}\sum_{k\geq i_{s+1}}|b_k(\lambda_k - \Lambda^s) - b_k^+(\lambda_k^+ - \Lambda^s)| + \sum_{k\leq i_{s-1}}|b_k||\lambda_k - \Lambda^s|&\\ \leq\gamma\cdot\sum_{k\leq i_s}|b_k^+||\lambda_k^+ - \Lambda^s|,&\end{aligned}\tag{5.17}$$

that are the consequences of the Stability Condition (1.11) (1.12) and can be proved as in Lemma 5.5. in [LT]. Indeed, summing (5.16) with (5.17), we have:

$$\begin{aligned}\gamma\cdot\Big(\sum_{k\geq i_s}|b_k^-||\lambda_k^- - \Lambda^s| + |b_k^+||\lambda_k^+ - \Lambda^s|\Big)&\\ \geq\Big[\sum_{k\leq i_{s-1}}|b_k^-||\lambda_k^- - \Lambda^s| - \sum_{k\leq i_{s-1}}|b_k||\lambda_k - \Lambda^s| + \sum_{k\geq i_{s+1}}|b_k||\lambda_k - \Lambda^s|\Big]&\\ +\Big[\sum_{k\geq i_{s+1}}|b_k^+||\lambda_k^+ - \Lambda^s| - \sum_{k\geq i_{s+1}}|b_k||\lambda_k - \Lambda^s| + \sum_{k\leq i_{s-1}}|b_k||\lambda_k - \Lambda^s|\Big]&\\ = \sum_{k\leq i_{s-1}}|b_k^-||\lambda_k^- - \Lambda^s| + \sum_{k\geq i_{s+1}}|b_k^+||\lambda_k^+ - \Lambda^s|\Big],&\end{aligned}$$

that implies (4.13) in view of (5.15).

CASE OF SMALL PHYSICAL WAVES – THE ESTIMATES (4.12) AND (4.14)

Denote by $\dot{y}_\alpha$ the 'real' speed of the α wave under consideration, that is: $\dot{y}_\alpha = \lambda_{k_\alpha}(v^-, v^+)$ in case $\alpha\in\mathcal{S}$ or $\dot{y}_\alpha = \lambda_{k_\alpha}(v^+)$ in case $\alpha\in\mathcal{R}$. For $k : 1\ldots n$ let us estimate the difference between E_k and a similar expression where $\dot{y}_\alpha$ replaces $\dot{x}_\alpha$:

$$\begin{aligned} E_k - \left[|b_k^+|W_k^+(\lambda_k^+ - \dot{y}_\alpha) - |b_k^-|W_k^-(\lambda_k^- - \dot{y}_\alpha)\right] \\ = (\dot{y}_\alpha - \dot{x}_\alpha)\left[|b_k^+|W_k^+ - |b_k^-|W_k^-\right] = O(1)\cdot\epsilon\cdot|\epsilon_\alpha|, \end{aligned} \tag{5.18}$$

because $|\dot{y}_\alpha - \dot{x}_\alpha| \le \epsilon$.

Below, we will assume that $\dot{y}_\alpha = \dot{x}_\alpha$ and prove that under this hypothesis, (4.12) holds in Case 1, while

$$\sum_{k=1}^{n} E_k \le 0 \tag{5.19}$$

holds in Case 2. These together with (5.18) will yield, respectively, (4.12) and (4.14).

We assume that α – the wave under consideration – is located in Ω^s, for some $s : 0, \dots, M$. In other words, both states joined by α belong to Ω^s.

Case A. Assume first that both wave vectors $\{b_k^-\}_{k=1}^n$ and $\{b_k^+\}_{k=1}^n$ contain a large wave of the family $i_k \in \{i_s, i_{s+1}\}$. We treat here the case $i_k = i_s$ with wave configuration as in Figure 5.2, the other cases being similar. We have:

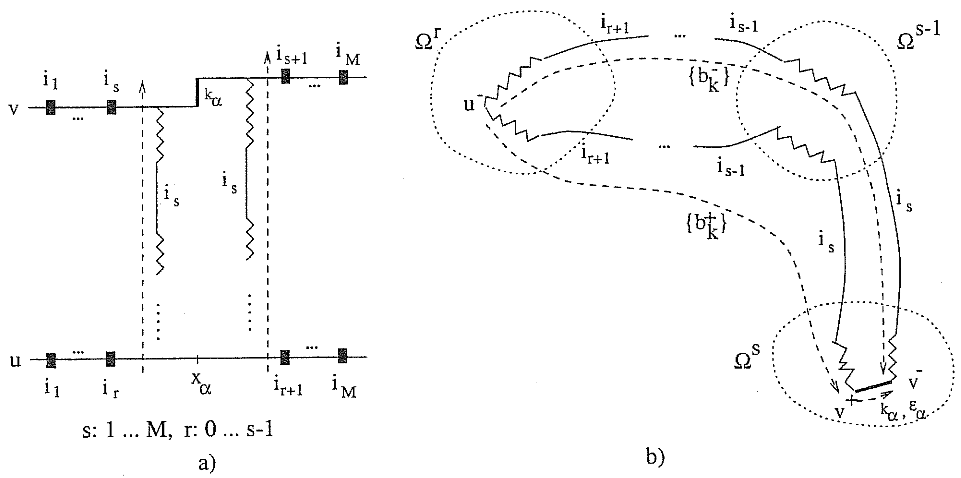

FIGURE 5.2

$$\begin{aligned} E_{i_s} &= B\cdot\left[(W_{i_s}^+ - W_{i_s}^-)(\lambda_{i_s}^\pm - \dot{x}_\alpha| + W_{i_s}^\mp(\lambda_{i_s}^\pm - \lambda_{i_s}^\mp)\right] \\ &\le -Bc\kappa_1\cdot|\epsilon_\alpha| + O(1)\cdot B\cdot|\epsilon_\alpha| \end{aligned} \tag{5.20}$$

(the choice of the upper or lower superindices depends on the family number k_α).

For indices k such that b_k^+ and b_k^- are small, as in [LT] we obtain:

$$\begin{aligned} E_k = |b_k^\pm|\cdot(W_k^+ - W_k^-)\cdot(\lambda_k^\pm - \dot{x}_\alpha) \\ + W_k^\mp\left(|b_k^+|(\lambda_k^+ - \dot{x}_\alpha) - |b_k^-|(\lambda_k^- - \dot{x}_\alpha)\right) \\ \le (O(1) + 4\kappa_1 B)\cdot\left(O(1)\cdot|b_k^+ - b_k^-| + O(1)\cdot|b_k^-||\epsilon_\alpha|\right) + O(1)\cdot|\epsilon_\alpha|. \end{aligned} \tag{5.21}$$

If the k wave is large $b_k^+ = b_k^- = B$, but $k \neq i_s$, then $W_k^+ = W_k^-$ and

$$E_k = B \cdot W_k^+(\lambda_k^+ - \lambda_k^-)) = O(1) \cdot (O(1) + 4\kappa_1 B) \cdot |\epsilon_\alpha|. \tag{5.22}$$

Now, summing (5.20) with (5.21) we receive:

$$\begin{aligned} E_{i_s} + \sum_{\substack{k:1\ldots n \\ b_k^\pm \neq B}} E_k \leq &- B\kappa_1 c \cdot |\epsilon_\alpha| + O(1) \cdot |\epsilon_\alpha| \\ &+ O(1) \cdot 4\kappa_1 B \cdot \sum_{\substack{k:1\ldots n \\ b_k^\pm \neq B}} \Big[|b_k^+ - b_k^-| + |b_k^-||\epsilon_\alpha| \Big] \leq 0, \end{aligned} \tag{5.23}$$

if only κ_1 is big enough and all the weights are w_k^s small.

Note that in Case 2:

$$\sum_{k=1}^{n} E_k = E_{i_s} + \sum_{\substack{k:1\ldots n \\ b_k^\pm \neq B}} E_k,$$

so (5.19) follows from (5.23).

In Case 1 some terms of the form (5.22) may be added to (5.23), thus we can hope only for the weaker inequality (4.12), which indeed follows from (5.20) (5.21) (5.22).

CASE B. – Figure 5.3

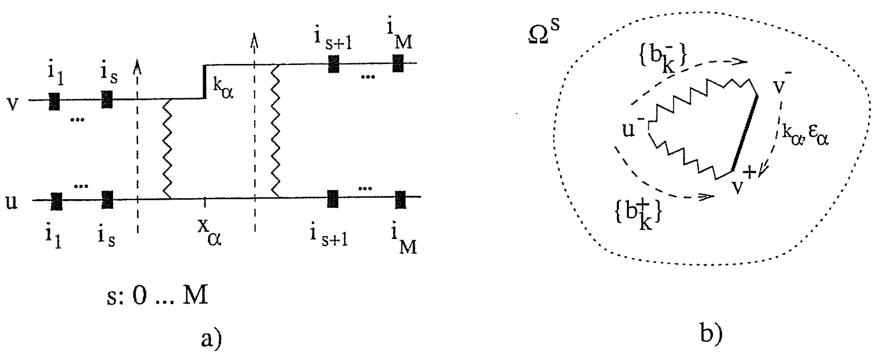

FIGURE 5.3

This case has been treated in [BLY]. If the constant B is small enough and κ_1 is big (with respect to the uniform constants $O(1)$ in all the formulae), we get (5.19) as in [BLY].

6. Relations to previous works

In this short Section we comment on the relations of the results presented in this Chapter to other works.

In [S], Schochet was the first to introduce a BV stability condition, giving positive answer to question **I**. The global solution to the Cauchy problem under consideration is constructed there by the method of Glimm's scheme.

In [BC1], Bressan and Colombo consider the general Riemann problem for systems of two equations and assuming the corresponding stability condition, answer question **II** positively.
More recently, the paper [LT] proves Theorems 1.1 and 1.2 (for systems of $n \geq 2$ equations) in the presence of two large shocks. Substantial differences between $M = 2$ and $M > 2$ occur in particular in the proof of Theorem 1.2. Namely, the straightforward generalization of the Lyapunov functional introduced in [LT] does not provide a functional decreasing along the wave front tracking solutions, when $M > 2$. On the other hand, our functional defined in Section 4, reduces when $M = 2$ to a Lyapunov functional that can be seen as a simplification of the one from [LT]. Also, instead of Majda's criterion (1.8), the paper [LT], following [BC1], used a differently stated assumption. We will show the equivalence of the two conditions in Chapter 3.
Other connections of our results to existing papers (in particular [Ch] [LiY] [W]) will be pointed out in the next Chapter.

CHAPTER 3

Stability conditions for patterns of non-interacting large shock waves

1. Introduction

In the study of local existence and stability of solutions to the Cauchy problem for a $n \times n$ system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0 \tag{1.1}$$

$$u(0, \cdot) = \bar{u}, \tag{1.2}$$

due to the finite speed of propagation, one is led to consider the special case where the initial data $\bar{u}$ is a small perturbation of a Riemann data:

$$\bar{u}(x) = \begin{cases} u^- & x < 0 \\ u^+ & x > 0. \end{cases} \tag{1.3}$$

In the previous Chapter we have shown that existence and stability of solutions can be obtained under suitable linearized stability conditions for the solutions of (1.1) (1.2) (1.3). The main purpose of this Chapter is to compare the various assumptions of this kind, appearing in the literature, and prove their equivalence. As before, we shall restrict ourselves to the case where the solution of (1.1) (1.2) (1.3) consists of $M+1$ constant states, $M \in \{2, \ldots, n\}$, separated by (possibly large) admissible shocks, say in the characteristic families $i_1 < \ldots < i_M$. Calling $u_0^0 = u^-, u_0^1, u_0^2, \ldots, u_0^M = u^+$ these intermediate states, and Λ^q the speed of the i_q shock, the linearized system has the form:

$$v_t + Df(u_0^q) \cdot v_x = 0, \qquad x/t \in (\Lambda^q, \Lambda^{q+1}). \tag{1.4}$$

Along shock lines we have the boundary conditions obtained by linearizing the Rankine-Hugoniot equations, that yield the linear dependence of the strengths of the outgoing waves on the components of the incoming wave vector interacting with the i_q large shock under consideration:

$$\epsilon_s^{out} = \sum_{\substack{k:1\ldots n \\ incoming}} a_{sk}^q \cdot \epsilon_k^{in} \tag{1.5}$$

(see Figure 1.1).

This Chapter is organized as follows. In Section 2 we focus on the admissibilty and stability of a single large shock in the reference solution (1.1) (1.2) (1.3). In particular the Majda [M] and Chern [Ch] stability conditions are studied.

In Section 3 we formulate a so-called Finiteness Condition, which guarantees the stability of a multiple-shocks pattern within the class of Riemann data.

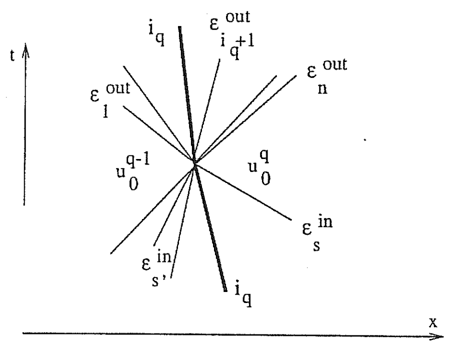

FIGURE 1.1

Section 4 gathers some preliminary facts on matrix analysis. Sections 5, 6, and 7 discuss different BV and L[1] Stability Conditins. In Chapter 2 (as in [Le1]), these conditions are formulated in terms of the existence of a suitable family of weights such that the corresponding BV or L[1] norm of any solution of the linearized system (1.4) is non-increasing in time. In Section 5, we give various equivalent formulations of these conditions, requiring that the eigenvalues of corresponding matrices, related to wave reflections-transmissions, are smaller than 1 in absolute value. This enables us to compare our conditions with the Finiteness Condition (Theorem 5.2), as well as to prove the equivalence of our conditions with other assumptions of this kind that can be found in [S] (Theorem 6.1) and [LiY] (Remark 5.4).

In the last Section we treat the case of systems of $n = 2$ equations, with the presence of $M = 2$ large shocks and deal with the corresponding conditions introduced in [BC1] [W] [LT].

We end this Section recalling the setting of the Cauchy problem (1.1) (1.2) from Chapter 2. In the n-dimensional state space $M+1$ distinct states $\{u_0^q\}_{q=0}^M$ are fixed, with their corresponding open disjoint neighbourhoods $\{\Omega^q\}_{q=0}^M$ such that:

- $f : \Omega \longrightarrow \mathbf{R}^n$ is smooth and defined on $\Omega = \bigcup_{q=0}^{M} \Omega^q \subset \mathbf{R}^n$.
- f is strictly hyperbolic in Ω, that is: at each point $u \in \Omega$, the matrix $Df(u)$ has n real and simple eigenvalues $\lambda_1(u) < \ldots < \lambda_n(u)$.
- Each characteristic field of (1.1) is either linearly degenerate or genuinely nonlinear, that is: with a basis $\{r_k(u)\}_{k=1}^n$ of corresponding right eigenvectors of $Df(u)$, $Df(u)r_k(u) = \lambda_k(u)r_k(u)$, each of the n directional derivatives $r_k \nabla \lambda_k$ vanishes either identically or nowhere.

The solution to (1.1) (1.2) with the initial data

$$\bar{u}(x) = \begin{cases} u_0^0 & x < 0 \\ u_0^M & x > 0 \end{cases} \tag{1.6}$$

is given by M shocks (u_0^{q-1}, u_0^q), $q : 1 \dots M$, belonging to respective characteristic families i_q and travelling with respective speeds Λ^q:

$$u(t,x) = \begin{cases} u_0^0 & x < \Lambda^1 t \\ u_0^q & \Lambda^q t < x < \Lambda^{q+1} t, \quad q : 1 \dots M-1 \\ u_0^M & x > \Lambda^M t, \end{cases} \tag{1.7}$$

as in Figure 1.1 in Chapter 2.

2. Stability of a single large shock

In this Section we discuss conditions yielding the stability of a single (possibly large) shock in the solution of (1.1) (1.2). We assume that the shock connects the states $u^- \neq u^+$, belonging to their respective disjoint open neighbourhoods Ω^- and Ω^+, and travels with the speed Λ. In the notation of the previous Section this reads $u_0^0 = u^-, u_0^1 = u^+, \Omega^0 = \Omega^-, \Omega^1 = \Omega^M = \Omega^+, \Lambda^1 = \Lambda$.

The shock satisfies the Rankine-Hugoniot conditions:

$$f(u^-) - f(u^+) = \Lambda(u^- - u^+), \tag{2.1}$$

belongs to the i-th characteristic family, and is compressive in the sense of Lax [L], that is:

$$\lambda_i(u^-) > \Lambda > \lambda_i(u^+). \tag{2.2}$$

If we drop the convention $\Omega^- \cap \Omega^+ = \emptyset$ and assume instead that $\Omega = \Omega^- \cup \Omega^+$ is convex, then for any pair $u, u' \in \Omega$ the following averaged matrix can be defined:

$$A(u,u') = \int_0^1 Df(\theta u + (1-\theta)u')d\theta.$$

It is then easily seen that (2.1) holds if and only if Λ is an eigenvalue of $A(u^-, u^+)$ with the corresponding eigenvector $u^- - u^+$. In particular, if u^- is close to u^+, then in view of (2.2), the vector equation (2.1) is equivalent to the following one

$$\langle l_k(u^-, u^+), u^- - u^+ \rangle = 0 \quad \forall k \neq i, \tag{2.3}$$

where by $\{l_k(u,u')\}_{k=1}^n$ we denote the base of left eigenvectors of the strictly hyperbolic matrix $A(u,u')$.

For $n = 2$ and $i = 1$ these $n-1$ equations reduce to the scalar condition:

$$\Psi(u^-, u^+) = 0,$$

where

$$\Psi(u,u') = \langle l_2(u,u'), u - u' \rangle. \tag{2.4}$$

Note that if the strength of the shock (u^-, u^+) is large, then in general one does not expect the matrix $A(u^-, u^+)$ to be strictly hyperbolic. In the spirit of this reasoning, the following condition has been assumed in [BC1] (for $n = 2$ equations in (1.1)):

(i) $A(u,u')$ is defined and strictly hyperbolic for every pair $u,u' \in \Omega$.
(ii) The 1-shock (u^-,u^+) is stable, that is

$$\left\langle \frac{\partial}{\partial u}\Psi(u^-,u^+), r_2(u^+) \right\rangle \neq 0,$$

with Ψ as in (2.4).

In the general case, a further extension of (2.3) was given in [LT]:

There exists a smooth function $\Psi : \Omega^- \times \Omega^+ \longrightarrow \mathbf{R}^{n-1}$ such that:
(i) $\Psi(u^0,u^1) = 0$ iff the states u^0 and u^1 can be connected by a (large) shock of the i-th characteristic family, with the speed $\Lambda(u^0,u^1)$. The Rankine-Hugoniot condition holds: $f(u^0) - f(u^1) = \Lambda(u^0,u^1)(u^0 - u^1)$. In particular $\Psi(u^-,u^+) = 0$ and $\Lambda(u^-,u^+) = \Lambda$.
(ii) The $n-1$ vectors:

$$\left\{ \frac{\partial\Psi}{\partial u^0}(u^-,u^+)\cdot r_k(u^-) \right\}_{k=1}^{i-1} \cup \left\{ \frac{\partial\Psi}{\partial u^1}(u^-,u^+)\cdot r_k(u^+) \right\}_{k=i+1}^{n} \tag{2.5}$$

are linearly independent.

Under this assumption we are going to show that if only the sets Ω^-,Ω^+ are small enough, then any Riemann problem with the states in Ω^-,Ω^+ has a unique self-similar solution. Note that in the proof of this fact, the condition (2.5) is actually equivalent to the hypothesis of the implicit function theorem.

THEOREM 2.1. *Assume (2.5) and (2.2). Then the following is true.*
(i) Every Riemann problem $(u^0,u^1) \in \Omega^- \times \Omega^+$ *for (1.1) has a unique self-similar solution, composed of* n *shock or rarefaction waves, connecting the states* $u^0 = u_0, u_1, \dots, u_{i-1} \in \Omega^-$ *and* $u_i, u_{i+1}, \dots, u_n = u^1 \in \Omega^+$, *as in Figure 2.1.*
(ii) The admissibility of this solution is understood in the following sense. Call $\Psi_k(u,\cdot)$ *the* k*-th wave curve passing through a left state* u *(see* [L]*). Then for every* $k \neq i$ *one has* $u_k = \Psi_k(u_{k-1},\epsilon_k)$, *for some parameter* ϵ_k. *Moreover, the* i*-th wave* (u_{i-1},u_i) *is a (large) compressive Lax shock, that is:*

$$\Psi(u_{i-1},u_i) = 0,$$
$$\lambda_i(u_{i-1}) > \Lambda(u_{i-1},u_i) > \lambda_i(u_i).$$

PROOF. Define the C^2 function $F : \Omega^- \times \Omega^+ \times I^{n-1} \to \mathbf{R}^{n-1}$ (I denotes a small interval containing $0 \in \mathbf{R}$):

$$\begin{aligned} &F(u^0,u^1,\epsilon_1,\dots,\epsilon_{i-1},\epsilon_i,\dots,\epsilon_n) = \\ &\Psi\Big(\Psi_{i-1}(\dots\Psi_2(\Psi_1(u^0,\epsilon_1),\epsilon_2)\dots\epsilon_{i-1}), \\ &\qquad \tilde\Psi_{i+1}(\dots\tilde\Psi_{n-1}(\tilde\Psi_n(u^1,-\epsilon_n),-\epsilon_{n-1})\dots-\epsilon_{i+1}\Big). \end{aligned} \tag{2.6}$$

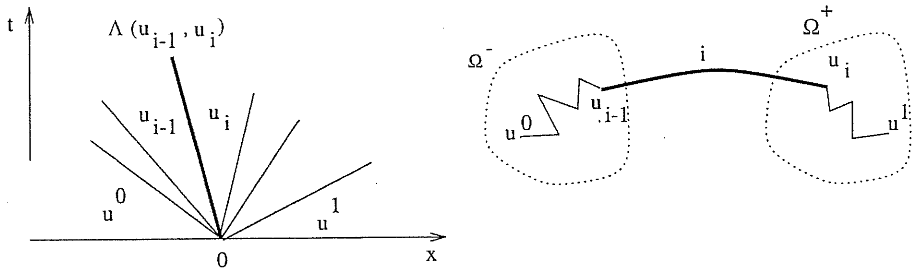

FIGURE 2.1

Here the fuctions $\tilde{\Psi}_k(\cdot,\cdot)$ are such that $\tilde{\Psi}_k(\Psi_k(u,\epsilon),-\epsilon) = u$. Note that

$$F(u^-,u^+,0) = \Psi(u^-,u^+) = 0$$

and the derivative

$$\frac{\partial}{\partial(\epsilon_1,\dots,\epsilon_{i-1},\epsilon_i,\dots,\epsilon_n)} F(u^-,u^+,0)$$

is a $(n-1)\times(n-1)$ matrix, having the $n-1$ vectors in (2.5)(ii) as its columns.

Therefore, by (2.5)(ii) and the implicit function theorem, for any given pair of states $(u^0,u^1) \in \Omega^- \times \Omega^+$ there exists exactly one $(n-1)$-dimensional wave vector $(\epsilon_1,\dots,\epsilon_{i-1},\epsilon_{i+1},\dots,\epsilon_n)$ (depending in a C^2 way on (u^0,u^1)) such that

$$F(u^0,u^1,\epsilon_1,\dots,\epsilon_{i-1},\epsilon_{i+1},\dots,\epsilon_n) = 0.$$

The states $\{u_k\}_{k=0}^n$ are then defined as follows:

$$\begin{cases} u_0 = u^0 \\ u_k = \Psi_k(\dots\Psi_2(\Psi_1(u^0,\epsilon_1),\epsilon_2)\dots,\epsilon_k) & \text{for } k = 1\dots i-1 \\ u_k = \tilde{\Psi}_{k+1}(\dots\tilde{\Psi}_{n-1}(\tilde{\Psi}_n(u^1,-\epsilon_n),-\epsilon_{n-1})\dots,-\epsilon_{k+1}) & k = i\dots n-1 \\ u_n = u^1. \end{cases}$$

□

In Chapter 2, the stability of the large shock (u^-,u^+) satisfying (2.1) (2.2) is understood in the classical sense of Majda [M]:

The n vectors
$$r_1(u^-),\dots,r_{i-1}(u^-),u^- - u^+,r_{i+1}(u^+),\dots,r_n(u^+) \tag{2.7}$$
are linearly independent.

Again, for weak shocks the condition (2.7) is always satisfied. The main result of this Section is the following.

THEOREM 2.2. *Let (u^-,u^+) be a Rankine-Hugoniot shock, such that its speed Λ in (2.1) is not an eigenvalue of $Df(u^-)$ neither of $Df(u^+)$. Then the conditions (2.5) and (2.7) are equivalent.*

The proof of Theorem 2.2 relies on the construction of a particular function Ψ_0, whose zero level set consists of those pairs of states $(u^0, u^1) \in \Omega^- \times \Omega^+$ that can be connected by an admissible i-shock as in (2.5)(i).
We define Ψ_0 as follows:

$$\Psi_0(u^0, u^1) = \left\{ \left\langle f(u^1) - f(u^0),\ V_k(u^1 - u^0) \right\rangle \right\}_{k=1}^{n-1}, \tag{2.8}$$

where V_k are any smooth functions defined on a neighbourhood of the vector $u_0 = u^+ - u^- \neq 0$ with values in $\mathbf{R}^n$, and such that for every u the space

$$\operatorname{span}\{V_1(u), \dots, V_{n-1}(u)\}$$

is the orthogonal complement of the vector u.

LEMMA 2.3. *$\{V_k\}_{k=1}^{n-1}$ can be taken so that:*

$$V_k(u_0) = -[DV_k(u_0)]^T \cdot u_0 \quad \forall k : 1 \dots n-1. \tag{2.9}$$

PROOF. By $e_1, \dots, e_n$ we denote the standard Euclidean base of $\mathbf{R}^n$.
For u close to e_n define the vectors $\{\widetilde{V}_k(u)\}_{k=1}^{n-1}$ applying the Gramm-Schmidt ortogonalization process to n linearly independent vectors: $u, e_1, \dots, e_{n-1}$. Namely, set:

$$\begin{aligned} &\widetilde{V}_1(u) = e_1 - \langle e_1, u\rangle \cdot \tfrac{u}{|u|^2} \\ &\widetilde{V}_k(u) = e_k - \left[\langle e_k, u\rangle \cdot \tfrac{u}{|u|^2} + \textstyle\sum_{s=1}^{k-1} \langle e_k, \widetilde{V}_s(u)\rangle \cdot \widetilde{V}_s(u)\right] \quad \forall k : 2 \dots n-1. \end{aligned} \tag{2.10}$$

Note that:

$$\widetilde{V}_k(e_n) = e_n \quad \forall k : 1 \dots n-1 \tag{2.11}$$

and:

- $\left\langle \widetilde{V}_k(u), u \right\rangle = 0 \quad \forall k : 1 \dots n-1$
- $\{\widetilde{V}_k\}_{k=1}^{n-1}$ are smooth functions of u.

Thus, $\operatorname{span}\{\widetilde{V}_1(u), \dots, \widetilde{V}_{n-1}(u)\}$ always complements orthogonally the vector u.
Moreover, using (2.11) and the fact that $\widetilde{V}_k \in \operatorname{span}(e_1, \dots, e_k, u)$, by the explicit formulas (2.10) one proves inductively that:

$$D\widetilde{V}_k(e_n) = [d_{sl}]_{s,l:1\dots n}, \quad d_{sl} = \begin{cases} -1 & \text{for } (s,l) = (n,k) \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

Now for u close to u_0 define:

$$V_k(u) = A^{-1} \cdot \widetilde{V}_k(Au), \tag{2.13}$$

where A is an orthogonal transformation composed with an appropriate dilatation such that $Au_0 = e_n$. Consequently

$$A^{-1} = |u_0|^2 A^T. \tag{2.14}$$

Obviously $\{V_k\}_{k=1}^{n-1}$ are smooth functions, and by the corresponding property of $\{\widetilde{V}_k\}_{k=1}^{n-1}$ they span the orthogonal complement of its argument vector.

By (2.13) (2.14) (2.12) and (2.11) we get:

$$\begin{aligned}[DV_k(u_0)]^T \cdot u_0 =& A^T \cdot [D\widetilde{V}_k(e_n)]^T \cdot (A^T)^{-1} \cdot u_0 = A^{-1} \cdot [D\widetilde{V}_k(e_n)]^T \cdot Au_0 \\ =& - A^{-1}e_k = -A^{-1} \cdot \widetilde{V}_k(Au_0) = -V_k(u_0),\end{aligned}$$

which proves (2.9). □

Using the above lemma one finds a convenient formula for the derivatives of Ψ_0:

$$\frac{\partial \Psi_0}{\partial u^0}(u^-, u^+) = -V \cdot [Df(u^-) - \Lambda Id], \tag{2.15}$$

$$\frac{\partial \Psi_0}{\partial u^1}(u^-, u^+) = V \cdot [Df(u^+) - \Lambda Id]. \tag{2.16}$$

Here V is the $(n-1) \times n$ matrix, whose rows are the vectors $V_1(u_0), \ldots, V_{n-1}(u_0)$. Note that since rank $V = n-1$, then Λ not an eigenvalue of neither $Df(u^-)$ nor $Df(u^+)$, this in view of (2.15) (2.16) implies:

$$\mathrm{rank}\frac{\partial \Psi_0}{\partial u^0}(u^-, u^+) = \mathrm{rank}\frac{\partial \Psi_0}{\partial u^1}(u^-, u^+) = n-1. \tag{2.17}$$

PROOF OF THEOREM 2.2.
STEP 1. By (2.15) (2.16) we get:

$$\frac{\partial \Psi_0}{\partial u^0}(u^-, u^+) \cdot r_k(u^-) = -(\lambda_k(u^-) - \Lambda) \cdot V \cdot r_k(u^-) \quad \forall k : 1 \ldots i-1,$$
$$\frac{\partial \Psi_0}{\partial u^1}(u^-, u^+) \cdot r_k(u^+) = (\lambda_k(u^+) - \Lambda) \cdot V \cdot r_k(u^+) \quad \forall k : i+1 \ldots n.$$

Since $\Lambda \notin \{\lambda_k(u^-)\}_{k=1}^{i-1} \cup \{\lambda_k(u^+)\}_{k=i+1}^{n}$ we see that the condition (2.5)(ii) for our function Ψ_0 is satisfied iff the vectors $\{V \cdot r_k(u^-)\}_{k=1}^{i-1} \cup \{V \cdot r_k(u^+)\}_{k=i+1}^{n}$ are linearly independent, which is in turn equivalent to Majda's condition (2.7), as ker $V = \mathrm{span}(u_0)$. We have thus shown that (2.7) is equivalent to (2.5)(ii) for the function Ψ_0. One sees this way that (2.7) implies (2.5).

STEP 2. Now we turn towards proving the converse implication. Let Ψ be any function satisfying (2.5). In particular, by (2.5)(ii), rank $D\Psi(u^-, u^+)$ is maximal and equal to $n-1$. The same is true for $D\Psi_0(u^-, u^+)$, by (2.17), so:

$$\mathrm{rank}\ D\Psi(u^-, u^+) = \mathrm{rank}\ D\Psi_0(u^-, u^+). \tag{2.18}$$

Another important remark is that

$$\ker\ D\Psi(u^-, u^+) = \ker\ D\Psi_0(u^-, u^+). \tag{2.19}$$

The spaces in (2.19) both coincide with the tangent space of the manifold $(\Psi_0)^{-1}(0)$ at point (u^-, u^+).

The following simple fact of linear algebra will be used in the sequel:

LEMMA 2.4. *Let $A, B : \mathbf{R}^n \longrightarrow \mathbf{R}^s$ be two linear operators, $s < n$. Assume that* rank A = rank $B = s$ *and* ker A = ker B. *Then for any s vectors $v_1, \ldots, v_s \in \mathbf{R}^n$ there holds: the vectors $\{Av_k\}_{k=1}^{s}$ are linearly independent iff $\{Bv_k\}_{k=1}^{s}$ are linearly independent.*

In view of (2.18) (2.19), we can apply Lemma 2.4 to the linear operators

$$D\Psi(u^-,u^+),\ D\Psi_0(u^-,u^+) : \mathbf{R}^{2n} \longrightarrow \mathbf{R}^{n-1}$$

and the following set of $n-1$ test vectors in $\mathbf{R}^{2n}$:

$$\{[r_k(u^-)^T, 0\dots 0]^T\}_{k=1}^{i-1} \cup \{[0\dots 0, r_k(u^+)^T]^T\}_{k=i+1}^{n}.$$

By (2.5)(ii) we receive that the same condition is satisfied by our function Ψ_0. This in turn, is equivalent to (2.7), as shown in step 1. □

REMARK 2.5. The proof of Theorem 2.2 shows that if a function Ψ as in (2.5) exists, then it can be replaced by the function Ψ_0, in this case necessarily enjoying the properties (2.5)(i) (2.5)(ii) and (2.17). This last property is crucial for the construction of the wave front tracking approximate solutions to (1.1) in Section 2, Chapter 2.

In the remaining part of this Section we recall the so-called Chern stability condition [Ch] for a single Lax shock (u^-,u^+) if the i-th characteristic family:

> Every Riemann problem $(u^0,u^1) \in \Omega^- \times \Omega^+$ for (1.1) can be solved uniquely by a composite of n elementary waves. The states separating these waves lie in $\Omega^- \cup \Omega^+$ and depend on (u^0,u^1) smoothly. The i-th wave in this solution is a Lax admissible shock. (2.20)

Condition (2.20) has been introduced in [Ch] to study the stability and truncation error of the Glimm scheme – it has been shown, in particular, that under this condition the Glimm approximate solutions converge to a global solution of (1.1) (1.2) with initial data $\bar{u}$ containing only one strong discontinuity (u^0,u^1). Note that the existence and stability of this solution follows immediately from our analysis in Chapter 2 – indeed, for $M=1$ the Weighted BV and L^1 Stability Conditions are always satisfied. The only apparently missing point is the Majda stability of the single large shock in the reference solution.

However, by Theorems 2.1 and 2.2 one easily gets: :

COROLLARY 2.6. *The three conditions: (2.20), (2.5) and (2.7) are equivalent.*

3. The Riemann problem

The analysis in Section 2 has shown that for a single i-shock (u^-,u^+), the Majda stability condition (2.7) implies the assertions (i) and (ii) of Theorem 2.1. Below we study the similar problem for a general pattern of M (large) shock waves (1.7). It appears that the solvability of Riemann problems close to (u_0^0, u_0^M) is not just a simple consequence of the admissibility of each single shock in (1.7), but requires an additional hypothesis about the mutual influence of the large shocks. To introduce this 'finiteness' condition we will need some special notation.

First of all, assume that every large shock in the reference solution (1.7) is Majda stable and Lax compressive (that is, conditions (1.6) (1.7) (1.8) in Chapter 2 hold).

Then, for every $q : 1\dots M$ define the $(n-1)\times i_q$ matrix G_q^r, expressing the strengths of the small outgoing waves in terms of the strengths of waves interacting from the right with the large i_q-shock. More precisely, using the notation as in Figure 1.2 in Chapter 2, we have:

$$G_q^r = [a_{sk}^q]_{\substack{s:1\ldots n,\ s\neq i_q,\\ k:1\ldots i_q}} \quad a_{sk}^q = \frac{\partial \epsilon_s^{out}}{\partial \epsilon_k^{in}}_{|\epsilon_k^{in}=0} . \tag{3.1}$$

Analogously, in case when the interacting k-waves approach the large shock from the left (see Figure 1.3 in Chapter 2), we have the $(n-1)\times(n-i_q+1)$ matrix G_q^l:

$$G_q^l = [a_{sk}^q]_{\substack{s:1\ldots n,\ s\neq i_q,\\ k:i_q\ldots n}} \quad a_{sk}^q = \frac{\partial \epsilon_s^{out}}{\partial \epsilon_k^{in}}_{|\epsilon_k^{in}=0} . \tag{3.2}$$

Define now the square $M\cdot(n-1)$ dimensional matrix $\mathbf{G}$, called in the sequel the 'finiteness matrix':

$$\mathbf{G} = \begin{bmatrix} [\Theta] & G_1^r & & & \\ G_2^l & [\Theta] & G_2^r & & \\ & G_3^l & [\Theta] & G_3^r & \\ & & \ddots & \ddots & \\ & & & G_M^l & [\Theta] \end{bmatrix} \tag{3.3}$$

(here $[\Theta]$ stands for the $(n-1)\times(n-1)$ zero matrix).

We are ready to state our new stability condition for the pattern (1.7).

FINITENESS CONDITION : 1 is not an eigenvalue of the matrix $\mathbf{G}$. (3.4)

The next theorem can be seen as a generalization of Theorem 2.1.

THEOREM 3.1. *In the above setting, let the Finiteness Condition (3.4) hold. Then any Riemann problem* $(u^-,u^+)\in\Omega^0\times\Omega^M$ *for (1.1) has a unique self-similar solution, attaining* $n+1$ *states, consecutively connected by:*

- *weak waves of the corresponding families (if both left and right states of a pair under consideration belong to the same set* Ω^q*),*
- M *admissible large shocks, joining the states belonging to different sets* Ω^q*,*

as in Figure 3.1.

PROOF. By Theorem 2.2, for every shock (u_0^{q-1},u_0^q) there exists its constitutive function Ψ^q, as in the condition (2.5), whose zeros are the pairs of nearby states $(u^{q-1},u^q)\in\Omega^{q-1}\times\Omega^q$ that can be joined by an admissible large i_q shock. The function (compare (2.6))

$$F:\left(\Omega^0\times\Omega^1\times\ldots\times\Omega^M\right)\times I^{i_1-1}\times I^{i_2-i_1-1}\times I^{i_3-i_2-1}\times\ldots\times I^{i_M-i_{M-1}-1}\times I^{n-i_M}\longrightarrow\mathbf{R}^{M\cdot(n-1)}$$

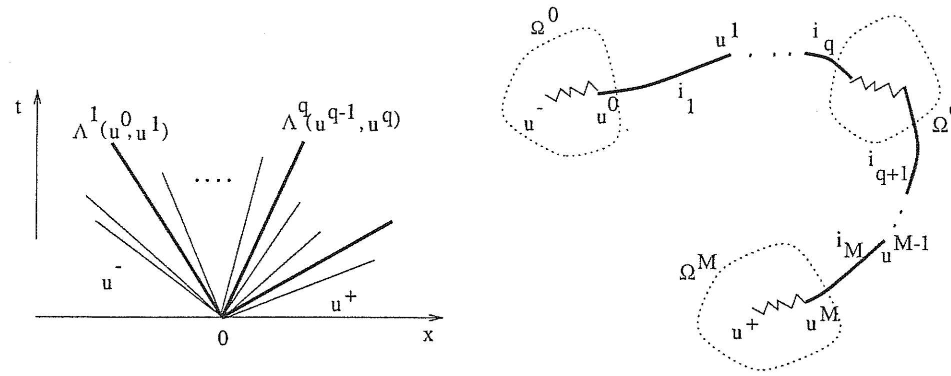

FIGURE 3.1

has now the following form:

$$
\begin{aligned}
&F\Big((u^-, u^1, u^2, \dots u^{M-1}, u^+),\\
&\qquad\qquad (\epsilon_1, \epsilon_2, \dots, \epsilon_{i_1-1}), (\epsilon_{i_1+1}, \dots, \epsilon_{i_2-1}), \dots, (\epsilon_{i_M+1}, \dots, \epsilon_n)\Big)\\
&\quad = \Psi^1\Big(\Psi_{i_1-1}(\epsilon_{i_1-1}, \dots \Psi_1(\epsilon_1, u^-)\dots),\ u^1\Big),\\
&\qquad \Psi^2\Big(\Psi_{i_2-1}(\epsilon_{i_2-1}, \dots \Psi_{i_1+1}(\epsilon_{i_1+1}, u^1)\dots),\ u^2\Big),\\
&\qquad \dots\\
&\qquad \Psi^M\Big(\Psi_{i_M-1}(\epsilon_{i_M-1}, \dots \Psi_{i_{M-1}+1}(\epsilon_{i_{M-1}+1}, u^{M-1})\dots),\\
&\qquad\qquad\qquad\qquad \tilde{\Psi}_{i_M+1}(-\epsilon_{i_M+1}, \dots \tilde{\Psi}_n(-\epsilon_n, u^+)\dots)\Big).
\end{aligned}
$$

The functions $\tilde{\Psi}_k$ and Ψ_k are as in the proof of Theorem 2.1.

Call A the $M \cdot (n-1)$ dimensional square matrix that is the derivarive of F with respect to the variables $(u^1, \dots, u^{M-1}), (\epsilon_1, \dots, \epsilon_n)$ at the point

$$
((u_0^0, \dots, u_0^M), (0, \dots, 0)) .
$$

We will show that A is invertible if and only if the Finiteness Condition (3.4) holds; by implicit function theorem the proof will be then complete.

Note first, that the invertibility of A is equivalent to the invertibility of the following matrix (which without loss of generality we also call A), of the same dimension:

$$
A = \begin{bmatrix}
A_1 & B_1^r & & & & \\
 & B_1^l & A_2 & B_2^r & & \\
 & & & B_2^l & & \\
 & & & \ddots & \ddots & \\
 & & & & A_M & \widetilde{A}_M
\end{bmatrix} . \tag{3.5}
$$

Here

$$A_q = \begin{cases} \dfrac{\partial \Psi^1}{\partial u^0}\,(u_0^0, u_0^1) \cdot \left[r_1(u_0^0) \dots r_{i_1-1}(u_0^0)\right] & \text{for } q = 1 \\[2ex] \dfrac{\partial \Psi^q}{\partial u^{q-1}}\,(u_0^{q-1}, u_0^q) \cdot \left[r_{i_{q-1}+1}(u_0^{q-1}) \dots r_{i_q-1}(u_0^{q-1})\right] & \text{for } q : 2 \dots M \end{cases}$$

and

$$\begin{aligned} \widetilde{A}_M &= \frac{\partial \Psi^M}{\partial u^M}\,(u_0^{M-1}, u_0^M) \cdot \left[r_{i_M+1}(u_0^M) \dots r_n(u_0^M)\right], \\ B_q^l &= \frac{\partial \Psi^q}{\partial u^{q-1}}\,(u_0^{q-1}, u_0^q) \cdot \left[r_1(u_0^{q-1}) \dots r_n(u_0^{q-1})\right], \\ B_q^r &= \frac{\partial \Psi^q}{\partial u^q}\,(u_0^{q-1}, u_0^q) \cdot \left[r_1(u_0^{q-1}) \dots r_n(u_0^{q-1})\right]. \end{aligned}$$

For every $q : 1 \dots M$ define a square $(n-1)$ dimensional matrix:

$$C_q = \Bigg[-\frac{\partial \Psi^q}{\partial u^{q-1}}\,(u_0^{q-1}, u_0^q) \cdot \left[r_1(u_0^{q-1}) \dots r_{i_q-1}(u_0^{q-1})\right], \\ \frac{\partial \Psi^q}{\partial u^q}\,(u_0^{q-1}, u_0^q) \cdot \left[r_{i_q+1}(u_0^q) \dots r_n(u_0^q)\right] \Bigg].$$

In view of Theorem 2.1 one can find (using again the implicit function theorem) the following formulae, expressing some of the minors of our matrix A in terms of the blocks constituting the matrix $\mathbf{G}$ in the Finiteness Condition (3.4):

$$\begin{aligned} \frac{\partial \Psi^q}{\partial u^{q-1}}\,(u_0^{q-1}, u_0^q) \cdot \left[r_{i_q}(u_0^{q-1}) \dots r_n(u_0^{q-1})\right] &= -\, C_q \cdot G_q^l, \\ \frac{\partial \Psi^q}{\partial u^q}\,(u_0^{q-1}, u_0^q) \cdot \left[r_1(u_0^q) \dots r_{i_q}(u_0^q)\right] &= C_q \cdot G_q^r, \end{aligned} \tag{3.6}$$

for every $q : 1 \dots M$.

Introducing (3.6) in (3.5), and permuting the columns of A in the appropriate way, we receive that A is invertible if and only if the following matrix (which we again denote by A) is invertible:

$$A = \begin{bmatrix} -C_1 & C_1 \cdot G_1^r & & & \\ C_2 \cdot G_2^l & -C_2 & C_2 \cdot G_2^r & & \\ & & \ddots & \ddots & \\ & & & C_M \cdot G_M^l & -C_M \end{bmatrix}. \tag{3.7}$$

Note that by (ii) in the stability condition (2.5), each martix C_q is invertible. We can thus multiply A by the square block matrix:

$$\begin{bmatrix} C_1^{-1} & & & \\ & C_2^{-1} & & \\ & & \ddots & \\ & & & C_M^{-1} \end{bmatrix},$$

and conclude that the invertibility of A in (3.7) is equivalent to the invertibility of the matrix $\mathbf{G} - Id$, which is exactly the Finiteness Condition (3.4). □

4. Three lemmas on matrix analysis

In this Section we present three abstract lemmas on matrix analysis, that will be used in the sequel.

LEMMA 4.1. *Let $Q = [q_{sk}]_{s,k:1\dots n}$ be a $n \times n$ matrix with nonnegative entries: $q_{sk} \geq 0$. The following conditions are equivalent:*

(*i*) $\mathrm{spRad}(Q) < 1$.

(*ii*) *There exists a diagonal matrix $W = \mathrm{diag}(w_1, \dots, w_n)$ with positive diagonal entries $w_s > 0$, such that $\| WQW^{-1} \|_1 < 1$.*
Here the norm of a $n \times n$ matrix $P = [p_{sk}]_{s,k:1\dots n}$ is defined by:

$$\| P \|_1 = \max_{k:1\dots n} \sum_{s=1}^{n} |p_{sk}|.$$

PROOF. (*ii*) $\Rightarrow$ (*i*). Since Spec $(WQW^{-1}) =$ Spec Q, without loss of generality we may assume that $W = Id$.

The matrix Q^T is nonnegative, so (see [G]) there exists a real nonnegative (maximal) eigenvalue λ and a corresponding eigenvector $r = [r_1, \dots, r_n]^T$ with nonnegative components, such that for any other eigenvalue ρ there must be: $|\rho| \leq \lambda$. Let $k \in \{1, \dots, n\}$ be such that:

$$r_k = \max_{s:1\dots n} r_s > 0.$$

We have:

$$\lambda r_k = \sum_{s=1}^{n} q_{sk} r_s \leq \Big(\sum_{s=1}^{n} q_{sk}\Big) r_k \leq \| Q \|_1 \, r_k < r_k.$$

Since λ is also a maximal eigenvalue of Q, this part of the proof is done.

(*i*) $\Rightarrow$ (*ii*). We prove that (*i*) implies the existence of a diagonal matrix $W = \mathrm{diag}(w_1, \dots, w_n)$ with each $w_k > 0$ and such that $\| (WQW^{-1})^T \|_1 < 1$. This is enough since $(W^{-1}QW)^T = WQ^TW^{-1}$ and Spec $Q =$ Spec Q^T.

We proceed by induction on n. For $n = 1$ the assertion is trivially satisfied. For $n > 1$, let λ be a maximal eigenvalue of Q, with a corresponding nonnegative eigenvector $r = [r_1, \dots, r_n]^T \neq 0$. By (*i*) there must be $\lambda < 1$.

Let $\mathcal{I} = \{k : 1 \dots n;\ r_k \neq 0\}$. Define:

$$w_k = r_k \quad \text{for } k \in \mathcal{I}. \tag{4.1}$$

In case the set $\{1 \dots n\} \setminus \mathcal{I}$ is nonempty, consider the square nonnegative matrix $\widetilde{Q}$, composed of the rows and columns of Q carrying the indices not belonging to $\mathcal{I}$. Since for $k \notin \mathcal{I}$ one has:

$$\sum_{s=1}^{n} q_{ks} r_s = 0,$$

there must be:

$$q_{ks} = 0 \quad \forall s \in \mathcal{I}, k \notin \mathcal{I}.$$

Thus the maximal eigenvalue $\widetilde{\lambda}$ of $\widetilde{Q}$ belongs to Spec Q, so $\widetilde{\lambda} < 1$. By inductive assumption, there exists a diagonal matrix $\widetilde{W} = \operatorname{diag}(\{\widetilde{w}_k;\ k \notin \mathcal{I}\})$ with positive diagonal entries such that $\| (\widetilde{W}\widetilde{Q}\widetilde{W}^{-1})^T \|_1 < 1$. Fix $\epsilon > 0$ and define:

$$w_k = \epsilon \widetilde{w}_k \quad \text{for } k \notin \mathcal{I}. \tag{4.2}$$

By (4.1) (4.2) we have:

$$\begin{aligned}
\forall k \notin \mathcal{I} \quad \sum_{s=1}^{n}(W^{-1}QW)_{ks} &= w_k^{-1}(\sum_{s=1}^{n} q_{ks}w_s) = w_k^{-1}(\sum_{s\notin\mathcal{I}} q_{ks}w_s)\\
&= \widetilde{w}_k^{-1}(\sum_{s\notin\mathcal{I}} q_{ks}\widetilde{w}_s) \leq \| (\widetilde{W}\widetilde{Q}\widetilde{W}^{-1})^T \|_1 < 1,\\
\forall k \in \mathcal{I} \quad \sum_{s=1}^{n}(W^{-1}QW)_{ks} &= w_k^{-1}(\sum_{s=1}^{n} q_{ks}w_s) = r_k^{-1}(\sum_{s\in\mathcal{I}} q_{ks}r_s \ + \ \sum_{s\notin\mathcal{I}} q_{ks}r_s)\\
&= r_k^{-1}(\sum_{s=1}^{n} q_{ks}r_s) \ + \ r_k^{-1}(\sum_{s\notin\mathcal{I}} q_{ks}\epsilon\widetilde{w}_s)\\
&= r_k^{-1}\lambda r_k \ + \ \epsilon r_k^{-1}(\sum_{s\notin\mathcal{I}} \widetilde{w}_s) < 1,
\end{aligned}$$

if only ϵ in (4.2) is small enough. Thus we get $\| (W^{-1}QW)^T \|_1 < 1$. □

LEMMA 4.2. *Let A, B be two $n \times n$ matrices with nonnegative entries:*

$$A = [a_{sk}]_{s,k:1\ldots n}, \qquad B = [b_{sk}]_{s,k:1\ldots n}.$$

Assume that there exist two sets of indices: $col, ver \subset \{1 \ldots n\}$ with the properties:

- $col \cap ver = \emptyset$,
- $\forall k \notin col \ \forall s : 1 \ldots n \quad a_{sk} = b_{ks} = 0$,
- $\forall s \notin ver \ \forall k : 1 \ldots n \quad a_{sk} = b_{ks} = 0$.

Then the following two statements are equivalent:

(*i*) *There exists $W = \operatorname{diag}(w_1, \ldots, w_n)$ with all $w_k > 0$ such that $\| WAW^{-1} \|_1 < 1$ and $\| WBW^{-1} \|_1 < 1$.*
(*ii*) *There exists $W = \operatorname{diag}(w_1, \ldots, w_n)$ with all $w_k > 0$ such that $\| WABW^{-1} \|_1 < 1$.*

The matrix norm $\| \cdot \|_1$ is defined as in Lemma 4.1.

PROOF. (*i*) $\Rightarrow$ (*ii*). For every $k : 1 \ldots n$ we have:

$$\begin{aligned}
\sum_{s=1}^{n}(WABW^{-1})_{sk} &= \sum_{s=1}^{n}(WAW^{-1} \cdot \ WBW^{-1})_{sk}\\
&= \sum_{s=1}^{n} \Big[(WBW^{-1})_{sk} \ \cdot \ \sum_{l=1}^{n}(WAW^{-1})_{ls}\Big]\\
&\leq \| WAW^{-1} \|_1 \cdot \| WBW^{-1} \|_1 < 1,
\end{aligned}$$

which implies:

$$\| WABW^{-1} \|_1 < 1.$$

$(ii) \Rightarrow (i)$. Since $WABW^{-1} = (WAW^{-1})(WBW^{-1})$, we may without loss of generality assume that $\| AB \|_1 < 1$ and prove the existence of a diagonal matrix W satisfying (i).

By (ii) we have:

$$\sum_{s\in col} \left[b_{sk} \cdot \sum_{r\in ver} a_{rs} \right] < 1 \quad \forall k \in ver.$$

For a fixed $\epsilon > 0$ define:

$$w_k = \begin{cases} \sum_{s\in ver} a_{sk} + \epsilon & \text{for } k \in col, \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{s\in ver} w_s a_{sk} = \sum_{s\in ver} a_{sk} < w_k \quad \forall k \in col,$$

$$\sum_{s\in col} w_s b_{sk} = \sum_{s\in col} \Big(\sum_{r\in ver} a_{rs} \Big) b_{sk} + \sum_{s\in col} \epsilon b_{sk} < 1 = w_k \quad \forall k \in ver,$$

provided that ϵ is small enough.
We have thus proved that $\| WAW^{-1} \|_1 < 1$ and $\| WBW^{-1} \|_1 < 1$. □

LEMMA 4.3. *Let A, B be two $n \times n$ matrices with nonnegative entries and such that $\| A + B \|_1 < 1$. Then $\| B \cdot (Id - A)^{-1} \|_1 < 1$.*

PROOF. Note first that since $\| A \|_1 < 1$, then the matrix $Id - A$ is invertible and its inverse

$$(Id - A)^{-1} = Id + A + A^2 + \dots$$

has nonnegative entries. From the assumption it follows moreover that:

$$\forall k : 1 \dots n \quad \sum_{i=1}^{n} [B]_{ik} < 1 - \sum_{i=1}^{n} [A]_{ik} = \sum_{i=1}^{n} [Id - A]_{ik},$$

and thus

$$\begin{aligned} \forall k : 1 \dots n \quad \sum_{i=1}^{n} [B \cdot (Id - A)^{-1}]_{ik} &= \sum_{s=1}^{n} \Big(\sum_{i=1}^{n} [B]_{is} \Big) \cdot [(Id - A)^{-1}]_{sk} \\ &< \sum_{s=1}^{n} \Big(\sum_{i=1}^{n} [Id - A]_{is} \Big) \cdot [(Id - A)^{-1}]_{sk} \\ &= \sum_{i=1}^{n} [(Id - A) \cdot (Id - A)^{-1}]_{ik} = 1, \end{aligned}$$

which proves our lemma. □

5. Stability conditions compared

In this and the next Sections we discuss different BV and L^1 stability conditions, appearing in the literature, and compare them to the conditions (1.9) - (1.12) in Chapter 2. Recall that these last conditions guarantee the wellposedness of the problem (1.1) (1.2) and the existence of the Lipschitz continuous semigroup of solutions, whose domain contains all the small $L^1 \cap BV$ perturbations of the initial data $\bar{u}$ in (1.6) (Theorems 1.1 and 1.2 in Chapter 2).

First, we reformulate the conditions of Chapter 2, using the language of matrix analysis. Define the square $M \cdot (n-1)$ dimensional matrix $\mathbf{H}$, in the same manner as the finiteness martix $\mathbf{G}$ in (3.3), but with the submatrices G_q^x replaced with the corresponding H_q^x. The matrices H_q^x ($x \in \{l, r\}$) have the form as in (3.1) and (3.2), the only difference is that now their elements enclose also the shift ratios in the wave interaction patterns:

$$a_{sk}^q = \frac{\partial}{\partial \epsilon_k^{in}}_{|\epsilon_k^{in}=0} \left(\epsilon_s^{out} \cdot \frac{\lambda_s^{out} - \Lambda^q}{\lambda_k^{in} - \Lambda^q} \right).$$

For a given matrix Q, by $|Q|$ we denote the nonnegative matrix that consists of the absolute values of the elements of Q. By specRad Q we mean the spectral radius of Q.

We are now ready to reformulate and compare our stability conditions:

$$\text{BV STABILITY CONDITION}: \qquad \text{specRad}\, |\mathbf{G}| < 1, \tag{5.1}$$

$$\text{L}^1 \text{ STABILITY CONDITION}: \qquad \text{specRad}\, |\mathbf{H}| < 1. \tag{5.2}$$

THEOREM 5.1. *The* BV *Stability Condition (5.1) is equivalent to the Weighted* BV *Stability Condition (1.9) (1.10) in Chapter 2. The* L^1 *Stability Condition (5.2) is equivalent to the Weighted* L^1 *Stability Condition (1.11) (1.12) in Chapter 2.*

The proof of Theorem 5.1 is a direct application of Lemma 4.1.

THEOREM 5.2. *The Finiteness Condition (3.4) is weaker than the* BV *Stability Condition (5.1), which is in turn implied by the* L^1 *Stability Condition (5.2).*

PROOF. To prove the first assertion, note that if (5.1) holds, then by Lemma 4.1 without loss of generality we have $\| \, |\mathbf{G}^T| \, \|_1 < 1$. Assume that $\mathbf{G} \cdot r = r$, for some nonzero vector r of the appropriate dimension. Let s be the index such that:

$$|r_s| = \max_k |r_k|.$$

Then:

$$|r_s| = \left| \sum_k \mathbf{G}_{ks} \cdot r_k \right| \leq \sum_k |\mathbf{G}_{ks}| \cdot |r_k| \leq \left(\sum_k |\mathbf{G}|_{ks} \right) \cdot |r_s| < |r_s|$$

which is a contradiction.

To prove the second implication, we use Theorem 5.1. Assume that the Weighted L^1 Stability Condition (1.11) (1.12) in Chapter 2 is satisfied. For $q : 1 \ldots M-1$ and $s : 1 \ldots n$ define

$$\widetilde{w}_s^q = \left| \lambda_s(u_0^q) - \Lambda^{q+1} \right| \cdot w_s^q,$$

while $\widetilde{w}_s^q$ are set to be: small for $s : i_M + 1 \dots n$, and big for $s : 1 \dots i_M$. We will show that the inequalities (1.9) (1.10) in Chapter 2 hold for all $q : 1 \dots M$, with the new weights $\{\widetilde{w}_s^q\}$.

Indeed, to prove (1.9), compute:

$$\begin{aligned}\sum_{s=1}^{i_{q-1}} \frac{\widetilde{w}_s^{q-1}}{\widetilde{w}_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| + \sum_{s=i_q+1}^{n} \frac{\widetilde{w}_s^q}{\widetilde{w}_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| \\ = \sum_{s=1}^{i_{q-1}} \frac{w_s^{q-1}}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| \cdot \frac{|\lambda_s(u_0^{q-1}) - \Lambda^q|}{|\lambda_k(u_0^q) - \Lambda^{q+1}|} \\ + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| \cdot \frac{|\lambda_s(u_0^q) - \Lambda^{q+1}|}{|\lambda_k(u_0^q) - \Lambda^{q+1}|} < 1,\end{aligned}$$

by the assumption and the following easily received inequalities:

$$\begin{aligned}|\lambda_k(u_0^q) - \Lambda^{q+1}| &> |\lambda_k(u_0^q) - \Lambda^q| \quad \forall k \leq i_q, \\ |\lambda_k(u_0^q) - \Lambda^{q+1}| &< |\lambda_k(u_0^q) - \Lambda^q| \quad \forall k \geq i_{q+1}.\end{aligned}$$

The other estimate (1.10) is justified in a similar way. □

From Theorems 5.2 and 3.1 it follows:

COROLLARY 5.3. *Let the Weighted* BV *Stability Condition hold. With any Riemann data* (u^-, u^+), $u^- \in \Omega^i, u^+ \in \Omega^j, 0 \leq i \leq j \leq M$, *(1.1) has a unique self-similar solution, attaining* $n + 1$ *states, consecutively connected by:*

- *weak waves of the corresponding families (if both left and right states of a pair under consideration belong to the same set* Ω^q, $i \leq q \leq j$*),*
- $j - i$ *large admissible shocks, joining the states belonging to different sets* Ω^q.

This result has been used in Chapter 2, to define the weighted L^1 distance between two approximate solutions of (1.1) (1.2).

REMARK 5.4. It has recently been brought to our attention that conditions similar to our conditions (5.1) and (5.2) can be found in the book [LiY].

The authors investigate on the (short time) existence and regularity of classical solutions to the so-called typical boundary value problems on fan-shaped domains, for quasilinear hyperbolic systems with smooth coefficients. In particular, they show the existence of a unique C^1 solution to this problem, provided that the so-called minimal characterizing number of the characterizing matrix for the typical boundary value problem, is smaller than 1 (Theorem 1.1 in Chapter 4). If the same holds for the second characterizing matrix (see Paragraph 4 in Chapter 7), then the corresponding solution is C^2 regular (Theorem 1.1 in Chapter 7).

These results can well be applied to the quasilinear system (1.4) with the boundary conditions (1.5) along the boundaries of the angular domains given by the large shocks in the solution of (1.1) (1.2) (1.3). The boundary conditions (1.5) appear already in the solvable form (see Lemma 5.10 in Chapter 2) that is, some of the components of u at the vertex $x = 0, t = 0$ (namely, the components corresponding to the outgoing modes) are explicitely expressed as functions of the others (corresponding to the incoming modes). It is not hard to notice, that the characterizing matrix

of this problem is made up of the quantities $\left\{\frac{\partial}{\partial\epsilon_k^{in}}\,\epsilon_s^{out}\right\}$ in such a way that its minimal characterizing number is smaller than 1 if and only if our BV Stability Condition holds. In a similar manner, the mentioned solvability condition for the second characterizing matrix, containing the numbers $\left\{\frac{\partial}{\partial\epsilon_k^{in}}\left(\frac{\epsilon_s^{out}\cdot(\lambda_s^{out}-\Lambda^q)}{(\lambda_k^{in}-\Lambda^q)}\right)\right\}$, is equivalent to our L^1 Stability Condition. It is worth noting that for nonnegative matrices the minimal characterizing number is nothing else but the spectral radius (see Appendix 1). In particular, our Lemma 4.1, which came up independently in the investigations leading to [Le2], can be seen as a corollary of this result.
The analysis in [LiY] implies thus the local in time existence of the piecewise C^1 (respectively, C^2) solution to the problem (1.1) (1.2) with $\bar{u}$ smooth except at the point $x = 0$, where it induces the Riemann problem 'close' to (u^-, u^+). We expect, that this solution actually coincides with the solution given as the limit of the wave front tracking approximations (see Chapter 2 in this thesis).

We end this Section with a technical lemma (to be used in Section 6), showing some other possible reformulations of the Weighted BV Stability Condition. For every $q : 1 \dots M$ define four nonnegative matrices:

$$Q_q^{rl} = [|a_{sk}^q|]_{\substack{s:1\dots i_{q-1},\\ k:1\dots i_q}}, \qquad Q_q^{rr} = [|a_{sk}^q|]_{\substack{s:i_{q+1}\dots n,\\ k:1\dots i_q}},$$

whose elements are absolute values of these defined in (3.1), and

$$Q_q^{ll} = [|a_{sk}^q|]_{\substack{s:1\dots i_{q-1},\\ k:i_q\dots n}}, \qquad Q_q^{lr} = [|a_{sk}^q|]_{\substack{s:i_{q+1}\dots n,\\ k:i_q\dots n}},$$

with a_{sk}^q as in (3.2).

Note that in the above, the range of s (indexing the outgoing small waves) depends on the neighbouring large shock (of the family i_{q-1} or i_{q+1}). Indeed, it is relevant to keep track of only these new born waves that in the future may possibly interact with large shocks, thus changing the global wave pattern.
Keeping in mind the above comment, we also remark that the notation for the matrices $Q_1^{rl}, Q_1^{lr}, Q_1^{ll}, Q_M^{rr}, Q_M^{lr}, Q_M^{rl}$ is ambiguous, however in view of what we have said the precise form of these matrices is irrelevant in the following analysis.

Finally, for every matrix Q_q^{xy}, $x, y \in \{l, r\}$ define the corresponding square $n \times n$ matrix $\widetilde{Q}_q^{xy}$, by completing all the 'missing' entries with zeros. For example:

$$\widetilde{Q}_1^{rr} = [|\widetilde{a}_{sk}|]_{s,k:1\dots n} \qquad \widetilde{a}_{sk} = \begin{cases} a_{sk} & \text{for } s : i_2 \dots n,\ k : 1 \dots i_1, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 5.5. *The following conditions are equivalent to the Weighted* BV *Stability Condition (1.9) (1.10) in Section 2:*

(*i*) *There exist* $M-1$ *diagonal matrices* $\{W^q\}_{q=1}^{M-1}$, *with positive diagonal entries such that:*

$$\| W^1\widetilde{Q}_1^{rr}(W^1)^{-1} \|_1 < 1, \tag{5.3}$$

$$\begin{array}{l} \| W^{q-1}\widetilde{Q}_q^{ll}(W^{q-1})^{-1} + W^q\widetilde{Q}_q^{lr}(W^{q-1})^{-1} \|_1 < 1 \\ \| W^q\widetilde{Q}_q^{rr}(W^q)^{-1} + W^{q-1}\widetilde{Q}_q^{rl}(W^q)^{-1} \|_1 < 1 \end{array} \quad \forall q : 2 \dots M-1, \tag{5.4}$$

$$\| W^{M-1}\widetilde{Q}_M^{ll}(W^{M-1})^{-1} \|_1 < 1. \tag{5.5}$$

(*ii*) *Define two block square matrices of the dimension* $(M-1)\cdot n$*:*

$$Odd_M = \begin{bmatrix} \widetilde{Q}_1^{rr} & 0 & \dots & \dots & 0 \\ 0 & \widetilde{Q}_3^{ll} & \widetilde{Q}_3^{rl} & 0 & \vdots \\ \vdots & \widetilde{Q}_3^{lr} & \widetilde{Q}_3^{rr} & 0 & \\ \vdots & 0 & 0 & \widetilde{Q}_5^{ll} & \\ 0 & \dots & & & \ddots \end{bmatrix},$$

$$Even_M = \begin{bmatrix} \widetilde{Q}_2^{ll} & \widetilde{Q}_2^{rl} & 0 & \dots & 0 \\ \widetilde{Q}_2^{lr} & \widetilde{Q}_2^{rr} & 0 & \dots & \\ 0 & 0 & \widetilde{Q}_4^{ll} & \widetilde{Q}_4^{rl} & \\ \vdots & \vdots & \widetilde{Q}_4^{lr} & \widetilde{Q}_4^{rr} & \\ 0 & & & & \ddots \end{bmatrix}.$$

Then

$$\mathrm{spRad}(Odd_M \cdot Even_M) < 1. \tag{5.6}$$

Proof. The condition (i) is obviously equivalent to the Weighted BV Stability Condition, if we set $W^q = \mathrm{diag}(w_1^q, \dots, w_n^q)$ for all $q : 1 \dots M-1$.

Note that (5.3) (5.4) (5.5) are equivalent to

$$\| W \cdot Odd_M \cdot W^{-1} \| < 1, \qquad \| W \cdot Even_M \cdot W^{-1} \| < 1, \tag{5.7}$$

where W is the block diagonal matrix of the dimension $(M-1)\cdot n$, given by:

$$W = \mathrm{diag}(W^1, \dots, W^{M-1}).$$

By Lemma 4.1 and 4.2, (5.7) is in turn equivalent to (5.6), what proves (ii). □

6. The Schochet BV Stability Condition

Below we state the Schochet BV Stability Condition from [S] and prove its equivalence to our BV Stability Condition. Once Theorem 5.1 has been proved, in what follows with the term 'BV Stability Condition' we shall refer to (1.9) (1.10) in Chapter 2, or (5.1), as necessary.

Consider the first pair of large shocks: (u_0^0, u_0^1) and (u_0^1, u_0^2) and a tuple $\gamma = [\gamma_k]_{k:i_2\dots n}$ of small waves travelling in the region between these shocks, and approaching the second one. By interaction of γ with (u_0^1, u_0^2), then interaction of the new born 'reflected' waves with (u_0^0, u_0^1), and so on, further waves travelling in the region between the two shocks under consideration are produced. Call

$$R^1 = Q_1^{rr}. \tag{6.1}$$

The total strength of such waves, belonging to the characteristic families $k \geq i_2$ is then seen to be:

$$\left[Id + R^1 Q_2^{ll} + \left(R^1 Q_2^{ll}\right)^2 + \dots\right] |\gamma| = \left(Id - R^1 Q_2^{ll}\right)^{-1} |\gamma| \doteq P^{1-2}|\gamma|,$$

(where $|\gamma| = [|\gamma_k|]_{k:i_2\dots n}$), provided that the first stability requirement:

$$\text{all eigenvalues of } R^1 \cdot Q_2^{ll} \text{ are } < 1 \text{ in absolute value} \tag{6.2}$$

is satisfied.

Now, view the pair of the first two large shocks as a single entity. The reflexion matrix R^{1-2}, expressing the strengths of the outgoing small waves of families $k \geq i_3$, exiting the region between the first and the second large waves to the right of the latter one, in terms of the incoming waves of the families $k \leq i_2$, possibly interacting with the $(i_1 - i_2)$ couple of large shocks from the right, has the form:

$$R^{1-2} = Q_2^{rr} + Q_2^{lr} P^{1-2} R^1 Q_2^{rl}.$$

Thus, the natural stability requirement for the triple $(i_1 - i_2 - i_3)$ of large shocks, analogous to (6.2) is:

$$\text{all eigenvalues of } R^{1-2} \cdot Q_3^{ll} \text{ are } < 1 \text{ in absolute value.}$$

Proceeding in the same manner and viewing any fixed combination $(i_1 - \dots - i_q)$ of consequtive large shocks as a single entity, influencing its succeeding large wave i_{q+1}, we obtain the following $(M-1)$ assertions, that constitute:

THE SCHOCHET BV STABILITY CONDITION [S]

$$\begin{aligned} &\mathrm{spRad}(F^{1-2}) < 1, \\ &\mathrm{spRad}(F^{1-2-3}) < 1, \\ &\quad\vdots \\ &\mathrm{spRad}(F^{1-\dots-M}) < 1. \end{aligned} \tag{6.3}$$

The stability matrices F are defined inductively together with the corresponding reflection and production matrices R, P, by recalling (6.1) and setting:

$$F^{1-\dots-q} \doteq R^{1-\dots-(q-1)} \cdot Q_q^{ll} \qquad \text{for } q: 2\dots M \tag{6.4}$$

$$P^{1-\dots-q} \doteq \left(Id - F^{1-\dots-q}\right)^{-1} \qquad \text{for } q: 2\dots M \tag{6.5}$$

$$R^{1-\dots-q} \doteq Q_q^{rr} + Q_q^{lr} P^{1-\dots-q} R^{1-\dots-(q-1)} Q_q^{rl} \qquad \text{for } q: 2\dots M-1. \tag{6.6}$$

The main result of this Section is:

THEOREM 6.1. *The* BV *Stability Condition (5.1) is equivalent to the Schochet* BV *Stability Condition (6.3).*

PROOF. STEP 1. (5.1) $\Rightarrow$ (6.3). We use the equivalent form of the BV Stability Condition (5.1) given in Lemma 5.5 (i).

We first show that

$$\forall q : 1 \dots M-1 \quad \| W^q \cdot \widetilde{R}^{1-\dots-q} \cdot (W^q)^{-1} \|_1 < 1. \tag{6.7}$$

We proceed by induction on q. For $q = 1$, (6.7) is equivalent to (5.3) in view of (6.1). For $q : 2 \dots M-1$, by (6.6) we have:

$$\begin{aligned} W^q \cdot \widetilde{R}^{1-\dots-q} \cdot (W^q)^{-1} &= W^q \widetilde{Q}_q^{rr} (W^q)^{-1} \\ &+ \left[W^q \widetilde{Q}_q^{lr} \widetilde{P}^{1-\dots-q} \widetilde{R}^{1-\dots-(q-1)} (W^{q-1})^{-1} \right] \cdot \left[W^{q-1} \widetilde{Q}_q^{rl} (W^q)^{-1} \right]. \end{aligned}$$

The desired conclusion (6.7) will thus follow from the second inequality in (5.4) provided that

$$\| W^q \widetilde{Q}_q^{lr} \widetilde{P}^{1-\dots-q} \widetilde{R}^{1-\dots-(q-1)} (W^{q-1})^{-1} \|_1 < 1. \tag{6.8}$$

Note that:

$$\begin{aligned} &W^q \widetilde{Q}_q^{lr} \widetilde{P}^{1-\dots-q} \widetilde{R}^{1-\dots-(q-1)} (W^{q-1})^{-1} \\ &= W^q \widetilde{Q}_q^{lr} \cdot \left(Id - \widetilde{R}^{1-\dots-(q-1)} \widetilde{Q}_q^{ll} \right)^{-1} \cdot \widetilde{R}^{1-\dots-(q-1)} (W^{q-1})^{-1} \\ &= \left[W^q \widetilde{Q}_q^{lr} (W^{q-1})^{-1} \right] \\ &\quad \cdot \left\{ Id - \left[W^{q-1} \widetilde{R}^{1-\dots-(q-1)} (W^{q-1})^{-1} \right] \cdot \left[W^{q-1} \widetilde{Q}_q^{ll} (W^{q-1})^{-1} \right] \right\}^{-1} \\ &\quad \cdot \left[W^{q-1} \widetilde{R}^{1-\dots-(q-1)} (W^{q-1})^{-1} \right]. \end{aligned} \tag{6.9}$$

Setting

$$A = W^{q-1} \widetilde{Q}_q^{ll} (W^{q-1})^{-1}, \qquad B = W^q \widetilde{Q}_q^{lr} (W^{q-1})^{-1}$$

and combining Lemma 4.3 with the inductive assumption:

$$\| W^{q-1} \cdot \widetilde{R}^{1-\dots-(q-1)} \cdot (W^{q-1})^{-1} \|_1 < 1,$$

we get (6.8) by (6.9) and thus complete the proof of (6.7).

We now prove inductively that the BV Stability Condition (5.1) implies (6.3). For $M = 2$, the conditions (5.3) and (5.5) are by Lemma 6.1 and Lemma 4.1 equivalent to:

$$\text{all eigenvalues of } \widetilde{Q}_1^{rr} \cdot \widetilde{Q}_2^{ll} \text{ are } < 1 \text{ in absolute value.} \tag{6.10}$$

But

$$\text{Spec } Q_1^{rr} Q_2^{ll} \subset \text{Spec } \widetilde{Q}_1^{rr} \widetilde{Q}_2^{ll} \subset \left(\text{Spec } Q_1^{rr} Q_2^{ll} \right) \cup \{0\},$$

so (6.10) is equivalent to

$$\text{spRad}(F^{1-2}) < 1,$$

that is in turn precisely the condition (6.3).
Note that above we proved even more than we need at this point - we proved the equivalence of (5.1) and (6.3) in case $M = 2$ of only two large shocks present.

Let now $M > 2$. Since (5.4) for $q = M - 1$ implies

$$\| W^{q-2} \widetilde{Q}_{q-1}^{ll} (W^{q-1})^{-1} \|_1 < 1,$$

by the inductive assumption we get:

$$\operatorname{spRad}(F^{1-\dots-q}) < 1 \quad \forall q : 2 \dots M-1.$$

But, by (5.5) and (6.7) for $q = M-1$, in view of Lemma 4.2 and definition (6.4)

$$\| W^{M-1} \widetilde{F}^{1-\dots-M} (W^{M-1})^{-1} \|_1 < 1,$$

which by Lemma 4.1 implies finally

$$\operatorname{spRad}(F^{1-\dots-M}) < 1.$$

This finishes the proof of (5.1) $\Rightarrow$ (6.3). □

STEP 2. (6.3) $\Rightarrow$ (5.1). We use the equivalent form of the BV Stability Condition (5.1) given in Lemma 5.5 (ii).

We proceed by induction on M. For $M = 2$ the assertion has already been established in step 1. Let $M > 2$ and fix $\lambda \geq 1$. We will show that

$$\det(Odd_M \cdot Even_M - \lambda Id) \neq 0, \tag{6.11}$$

that by the property of nonnegative matrices mentioned in the proof of Lemma 4.1, will prove the theorem.

Assume first that M is an odd number. By known formulae on the determinant of block matrices (see [G]), and few easy computations one gets:

$$\begin{aligned} &\det(Odd_M \cdot Even_M - \lambda Id) \\ &\quad = \det(Odd_{M-1} \cdot Even_{M-1} - \lambda Id) \\ &\qquad \cdot \det\Big(\widetilde{Q}^{ll}_M \widetilde{Q}^{rr}_{M-1} + \widetilde{Q}^{ll}_M \cdot A_M \cdot (\lambda Id - Odd_{M-1} \cdot Even_{M-1})^{-1} \\ &\qquad\qquad \cdot B_m \cdot \widetilde{Q}^{rl}_{M-1} - \lambda Id\Big), \end{aligned} \tag{6.12}$$

where A_M is a $n \times ((M-2) \cdot n)$ block matrix of the form:

$$A_M = \begin{bmatrix} 0 & \dots & \dots & 0 & \widetilde{Q}^{lr}_{M-1} \end{bmatrix},$$

and B_M is a $((M-2) \cdot n) \times n$ block matrix:

$$B_M = \begin{bmatrix} 0 & \dots & 0 & \widetilde{Q}^{rl}_{M-2} & \widetilde{Q}^{rr}_{M-2} \end{bmatrix}^T,$$

while Odd_{M-1} and $Even_{M-1}$, are defined analogously to Odd_M and $Even_M$ as in Lemma 5.5 (ii).

Note that the Schochet condition (6.3) implies (by the inductive assumption):

$$\det(Odd_{M-1} \cdot Even_{M-1} - \lambda Id) \neq 0, \tag{6.13}$$

$$\operatorname{spRad}(F^{1-\dots-M}) < 1. \tag{6.14}$$

By the definitions (6.4) – (6.6):

$$F^{1-\dots-M} = Q^{ll}_M \cdot \left[Q^{rr}_{M-1} + Q^{lr}_{M-1} \big(Id - F^{1-\dots-(M-1)}\big)^{-1} \cdot R^{1-\dots-(M-2)} Q^{rl}_{M-1} \right].$$

Thus, in view of (6.13) and (6.14), the needed (6.11) will follow from (6.12) provided that:

$$\begin{aligned} A_M \cdot (Id - Odd_{M-1} \cdot Even_{M-1})^{-1} \cdot B_M \\ = \widetilde{Q}^{lr}_{M-1} \cdot \left(Id - \widetilde{F}^{1-\ldots-(M-1)}\right)^{-1} \cdot \widetilde{R}^{1-\ldots-(M-2)}. \end{aligned} \tag{6.15}$$

By the same kind of reasoning it is possible to prove that for M even, (6.11) is a consequence of the formula:

$$\begin{aligned} C_M \cdot (Id - Odd_{M-1} \cdot Even_{M-1})^{-1} \cdot D_M \\ = \left(Id - \widetilde{F}^{1-\ldots-(M-1)}\right)^{-1} \cdot \widetilde{R}^{1-\ldots-(M-2)} \cdot \widetilde{Q}^{rl}_{M-1}, \end{aligned} \tag{6.16}$$

where C_M is a $n \times ((M-2) \cdot n)$ block matrix of the form:

$$C_M = \begin{bmatrix} 0 & \ldots & 0 & \widetilde{Q}^{lr}_{M-2} & \widetilde{Q}^{rr}_{M-2} \end{bmatrix},$$

and D_M is a $((M-2) \cdot n) \times n$ block matrix:

$$D_M = \begin{bmatrix} 0 & \ldots & \ldots & 0 & \widetilde{Q}^{rl}_{M-1} \end{bmatrix}^T.$$

In the remaining part of the proof we will concentrate on showing that (6.15) holds for every odd number M. The proof of (6.16) is entirely the same. We are going to prove (6.15) by induction on odd numbers M. For $M = 3$, the left hand side of (6.15) reduces to:

$$\widetilde{Q}^{lr}_2 \cdot (Id - \widetilde{Q}^{rr}_1 \cdot \widetilde{Q}^{ll}_2)^{-1} \cdot \widetilde{Q}^{rr}_1,$$

which is precisely equal to $\widetilde{Q}^{lr}_2 \cdot \left(Id - \widetilde{F}^{1-2}\right)^{-1} \cdot \widetilde{R}^1$, by (6.1) and (6.4).

For $M > 3$ and odd, computing $(Id - Odd_{M-1} \cdot Even_{M-1})^{-1}$ in terms of the matrices $Odd_{M-3}, Even_{M-3}$, and the basic block-interaction matrices Q^{xy}_q, we get the equivalent form of the left hand side of the formula (6.15):

$$\begin{aligned} & A_M \cdot (Id - Odd_{M-1} \cdot Even_{M-1})^{-1} \cdot B_M \\ & = \begin{bmatrix} 0 & \widetilde{Q}^{lr}_{M-1} \end{bmatrix} \\ & \quad \cdot \left\{ Id - \begin{bmatrix} \widetilde{Q}^{ll}_{M-2} & \widetilde{Q}^{rl}_{M-2} \\ \widetilde{Q}^{lr}_{M-2} & \widetilde{Q}^{rr}_{M-2} \end{bmatrix} \cdot \begin{bmatrix} \widetilde{Q}^{rr}_{M-3} & 0 \\ 0 & \widetilde{Q}^{ll}_{M-1} \end{bmatrix} \right. \\ & \qquad - \begin{bmatrix} \widetilde{Q}^{ll}_{M-2} \\ \widetilde{Q}^{lr}_{M-2} \end{bmatrix} \cdot A_{M-2} \cdot (Id - Odd_{M-3} \cdot Even_{M-3})^{-1} \\ & \qquad \left. \cdot B_{M-2} \cdot \begin{bmatrix} \widetilde{Q}^{rl}_{M-3} & 0 \end{bmatrix} \right\}^{-1} \cdot \begin{bmatrix} \widetilde{Q}^{rl}_{M-2} \\ \widetilde{Q}^{rr}_{M-2} \end{bmatrix}. \end{aligned} \tag{6.17}$$

Using the inductive assumption and the definition (6.6) we reformulate the right hand side of (6.17):

$$
\begin{aligned}
&A_M \cdot (Id - Odd_{M-1} \cdot Even_{M-1})^{-1} \cdot B_M \\
&= \begin{bmatrix} 0 & \widetilde{Q}^{lr}_{M-1} \end{bmatrix} \\
&\quad \cdot \Bigg\{ Id - \begin{bmatrix} \widetilde{Q}^{ll}_{M-2} & \widetilde{Q}^{rl}_{M-2} \\ \widetilde{Q}^{lr}_{M-2} & \widetilde{Q}^{rr}_{M-2} \end{bmatrix} \cdot \begin{bmatrix} \widetilde{Q}^{rr}_{M-3} & 0 \\ 0 & \widetilde{Q}^{ll}_{M-1} \end{bmatrix} \\
&\qquad - \begin{bmatrix} \widetilde{Q}^{ll}_{M-2} \\ \widetilde{Q}^{lr}_{M-2} \end{bmatrix} \cdot \widetilde{Q}^{lr}_{M-3} \cdot (Id - \widetilde{F}^{1-\ldots-(M-3)})^{-1} \\
&\qquad \cdot \widetilde{R}^{1-\ldots-(M-4)} \begin{bmatrix} \widetilde{Q}^{rl}_{M-3} & 0 \end{bmatrix} \Bigg\}^{-1} \cdot \begin{bmatrix} \widetilde{Q}^{rl}_{M-2} \\ \widetilde{Q}^{rr}_{M-2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & \widetilde{Q}^{lr}_{M-1} \end{bmatrix} \cdot \Bigg\{ Id - \begin{bmatrix} \widetilde{Q}^{ll}_{M-2} & \widetilde{Q}^{rl}_{M-2} \\ \widetilde{Q}^{lr}_{M-2} & \widetilde{Q}^{rr}_{M-2} \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} \begin{array}{l} \widetilde{Q}^{rr}_{M-3} + \\ \quad \widetilde{Q}^{lr}_{M-3}(Id - \widetilde{F}^{1-\ldots-(M-3)})^{-1} \cdot \\ \qquad \cdot \widetilde{R}^{1-\ldots-(M-4)} \widetilde{Q}^{rl}_{M-3} \end{array} & 0 \\ 0 & \widetilde{Q}^{ll}_{M-1} \end{bmatrix} \Bigg\}^{-1} \cdot \begin{bmatrix} \widetilde{Q}^{rl}_{M-2} \\ \widetilde{Q}^{rr}_{M-2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & \widetilde{Q}^{lr}_{M-1} \end{bmatrix} \cdot \Bigg\{ Id - \begin{bmatrix} \widetilde{Q}^{ll}_{M-2} & \widetilde{Q}^{rl}_{M-2} \\ \widetilde{Q}^{lr}_{M-2} & \widetilde{Q}^{rr}_{M-2} \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} \widetilde{R}^{1-\ldots-(M-3)} & 0 \\ 0 & \widetilde{Q}^{ll}_{M-1} \end{bmatrix} \Bigg\}^{-1} \cdot \begin{bmatrix} \widetilde{Q}^{rl}_{M-2} \\ \widetilde{Q}^{rr}_{M-2} \end{bmatrix} .
\end{aligned}
\tag{6.18}
$$

Calling

$$
\begin{aligned}
X &= Id - \widetilde{Q}^{ll}_{M-2} \widetilde{R}^{1-\ldots-(M-3)}, \\
Y &= -\widetilde{Q}^{rl}_{M-2} \widetilde{Q}^{ll}_{M-1}, \\
Z &= -\widetilde{Q}^{lr}_{M-2} \widetilde{R}^{1-\ldots-(M-3)}, \\
W &= Id - \widetilde{Q}^{lr}_{M-2} \widetilde{Q}^{ll}_{M-1},
\end{aligned}
$$

we rewrite the right hand side of (6.18):

$$
\begin{aligned}
&\begin{bmatrix} 0 & \widetilde{Q}^{lr}_{M-1} \end{bmatrix} \cdot \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}^{-1} \cdot \begin{bmatrix} \widetilde{Q}^{rl}_{M-2} \\ \widetilde{Q}^{rr}_{M-2} \end{bmatrix} \\
&\quad = \widetilde{Q}^{lr}_{M-1} \cdot \Big(-(W - ZX^{-1}Y)^{-1} ZX^{-1} \cdot \widetilde{Q}^{rl}_{M-2} \\
&\qquad\qquad + (W - ZX^{-1}Y)^{-1} \cdot \widetilde{Q}^{rr}_{M-2} \Big) \\
&\quad = \widetilde{Q}^{lr}_{M-1} \cdot (W - ZX^{-1}Y)^{-1} \cdot (\widetilde{Q}^{rr}_{M-2} - ZX^{-1} \cdot \widetilde{Q}^{rl}_{M-2}) \\
&\quad = \widetilde{Q}^{lr}_{M-1} \cdot (Id - \widetilde{R}^{1-\ldots-(M-2)} \widetilde{Q}^{ll}_{M-1})^{-1} \cdot \widetilde{R}^{1-\ldots-(M-2)},
\end{aligned}
\tag{6.19}
$$

because, by definitions (6.4) – (6.6):

$$W - ZX^{-1}Y = Id - \widetilde{R}^{1-\dots-(M-2)}\widetilde{Q}^{ll}_{M-1},$$
$$\widetilde{Q}^{rr}_{M-2} - ZX^{-1}\cdot\widetilde{Q}^{rl}_{M-2} = \widetilde{R}^{1-\dots-(M-2)}.$$

The equality (6.19) together with (6.18) prove (6.15). The proof of step 2 and thus also the proof of Theorem 6.1 is complete. □ □

7. Systems of two equations

In the particular case $n = M = 2$, $i_1 = 1$, $i_2 = 2$, the matrices Q_1^{rr} and Q_2^{ll} reduce to single numbers, and the BV Stability Condition (5.1) appears in a simple form:

$$\left|\frac{\partial\epsilon_2^{out}}{\partial\epsilon_1^{in}}\,|\epsilon_1^{in}=0\right|\cdot\left|\frac{\partial\epsilon_1^{out}}{\partial\epsilon_2^{in}}\,|\epsilon_2^{in}=0\right| < 1. \tag{7.1}$$

Similarly, the L^1 Stability Condition (5.2) is equivalent to:

$$\left|\frac{\partial\epsilon_2^{out}}{\partial\epsilon_1^{in}}\,|\epsilon_1^{in}=0\right|\cdot\left|\frac{\partial\epsilon_1^{out}}{\partial\epsilon_2^{in}}\,|\epsilon_2^{in}=0\right|\cdot\frac{\lambda_1(u_0^1)-\Lambda^2}{\lambda_1(u_0^1)-\Lambda^1}\cdot\frac{\lambda_2(u_0^1)-\Lambda^1}{\lambda_2(u_0^1)-\Lambda^2} < 1. \tag{7.2}$$

In both (7.1) and (7.2) the first derivative corresponds to the right interaction with the large shock of the first family, while the second derivative corresponds to the left interaction with the large shock of the second characteristic family.

In this Section, we show that (7.1) and (7.2) are equivalent, respectively, to the appropriate conditions providing stability results in [BC1] and [W].

Indeed, in the setting of [BC1]:

$$\kappa_1 = \frac{\partial\epsilon_2^{out}}{\partial\epsilon_1^{in}}\,|\epsilon_1^{in}=0 = -\frac{\left\langle\frac{\partial\Psi^2(u_0^0,u_0^1)}{\partial u^1}, r_1(u_0^1)\right\rangle}{\left\langle\frac{\partial\Psi^2(u_0^0,u_0^1)}{\partial u^1}, r_2(u_0^1)\right\rangle}$$

and

$$\kappa_2 = \frac{\partial\epsilon_1^{out}}{\partial\epsilon_2^{in}}\,|\epsilon_2^{in}=0 = -\frac{\left\langle\frac{\partial\Psi^1(u_0^1,u_0^2)}{\partial u^1}, r_2(u_0^1)\right\rangle}{\left\langle\frac{\partial\Psi^1(u_0^1,u_0^2)}{\partial u^1}, r_1(u_0^1)\right\rangle},$$

where

$$\Psi^1(u^1,u^2) = \langle l_1(u^1,u^2), u^1-u^2\rangle,$$
$$\Psi^2(u^0,u^1) = \langle l_2(u^0,u^1), u^0-u^1\rangle,$$

l_1 and l_2 being the left eigenvectors of the averaged flux gradient matrix between the reference points u (compare (2.4)).

One sees that the Bressan-Colombo Stability Condition

$$\left|\kappa_1\cdot\frac{\lambda_1(u_0^1)-\Lambda^2}{\lambda_1(u_0^1)-\Lambda^1}\right|\cdot\left|\kappa_2\cdot\frac{\lambda_2(u_0^1)-\Lambda^1}{\lambda_2(u_0^1)-\Lambda^2}\right| < 1$$

is precisely (7.2).

In [W], (1.1) (1.7) is assumed to satisfy the following Stability Condition.

Let

$$\begin{aligned}(\Lambda^1 Id - Df(u_0^1))^{-1}(u_0^1 - u_0^0) &= \alpha r_1(u_0^1) + \beta r_2(u_0^1),\\ (Df(u_0^1) - \Lambda^2 Id)^{-1}(u_0^2 - u_0^1) &= \gamma r_1(u_0^1) + \delta r_2(u_0^1).\end{aligned} \tag{7.3}$$

Then:

$$|\beta\gamma| < |\alpha\delta|. \tag{7.4}$$

This condition is a reduction of a multidimensional BV stability condition (to be found in [Me]) to the case of one space dimension.

THEOREM 7.1. *Assume that both shocks in the reference solution (1.7) (recall that $M = 2$) are Majda stable and Lax admissible. Then the condition (7.4) is equivalent to the* BV *Stability Condition (7.1).*

PROOF. It is enough to show that in the context of (7.3) (7.4) (7.1), there hold:

$$\left|\frac{\beta}{\alpha}\right| = \left|\frac{\partial\epsilon_2^{out}}{\partial\epsilon_1^{in}}_{|\epsilon_1^{in}=0}\right|, \tag{7.5}$$

$$\left|\frac{\gamma}{\delta}\right| = \left|\frac{\partial\epsilon_1^{out}}{\partial\epsilon_2^{in}}_{|\epsilon_2^{in}=0}\right|. \tag{7.6}$$

We focus on (7.5), thus the case when the large shock (u_0^0, u_0^1) is hit from the right by a small wave of the first characteristic family and strength ϵ_1^{in}. The proof of (7.6) is entirely similar, so we omit it.

Let $F : \Omega^0 \times \Omega^1 \times I \longrightarrow \mathbf{R}$ be defined as follows:

$$F(u^-, u^+, \epsilon) = \Psi_0(u^-, \widetilde{\Psi}_2(u^+, -\epsilon)),$$

where Ψ_0 is the constitutive function of the large 1-shock, as in (2.8) (compare Theorem 2.2 and Remark 2.5), and the functions $\Psi_k, \widetilde{\Psi}_k$ are as in the proof of Theorem 2.1.

The fundamental equation relating the strengths ϵ_1^{in} and ϵ_2^{out} in (7.5) has the form:

$$F(u_0^0, \Psi_1(u_0^1, \epsilon_1^{in}), \epsilon_2^{out}) = 0. \tag{7.7}$$

Differentiating (7.7) with respect to ϵ_1^{in} at $\epsilon_1^{in} = 0$ and using (2.16), we receive:

$$\begin{aligned}0 &= \frac{\partial\Phi_0}{\partial u^1}(u_0^0, u_0^1)\cdot r_1(u_0^1) + \frac{\partial\Phi_0}{\partial u^1}(u_0^0, u_0^1)\cdot r_2(u_0^1)\cdot \frac{\partial\epsilon_2^{out}}{\partial\epsilon_1^{in}}_{|\epsilon_1^{in}=0}\\ &= V_1(u_0^1 - u_0^0)^T \cdot \left[Df(u_0^1) - \Lambda^1 Id\right]\cdot\left(r_1(u_0^1) + r_2(u_0^1)\cdot\frac{\partial\epsilon_2^{out}}{\partial\epsilon_1^{in}}_{|\epsilon_1^{in}=0}\right).\end{aligned} \tag{7.8}$$

Since $V_1(u_0^1 - u_0^0)$ is orthogonal to $u_0^1 - u_0^0$, (7.8) is equivalent to:

$$\left[Df(u_0^1) - \Lambda^1 Id\right]\cdot\left(r_1(u_0^1) + r_2(u_0^1)\cdot\frac{\partial\epsilon_2^{out}}{\partial\epsilon_1^{in}}_{|\epsilon_1^{in}=0}\right) = s\cdot(u_0^1 - u_0^0), \tag{7.9}$$

with some $s \neq 0$, as Λ^1 is not an eigenvalue of $Df(u_0^1)$. The first formula in (7.3) is equivalent to:

$$\left[Df(u_0^1) - \Lambda^1 Id\right]\cdot(-\alpha r_1(u_0^1) - \beta r_2(u_0^1)) = (u_0^1 - u_0^0),$$

and thus by (7.9) we get (7.5). □

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