Scuola Internazionale Superiore di Studi Avanzati - Trieste

Asymptotic problems and approximation results in variational models of quasistatic crack growth

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Thesis submitted for the degree of *Doctor Philosophiae*
Academic Year 2003-2004

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Contents

Introduction

This thesis deals with variational problems arising from a model for quasistatic crack growth in linearly elastic bodies proposed by G.A. Francfort and J.-J. Marigo in 1998 [54] and inspired by the classical Griffith's criterion. The main feature of the model is that the path of the crack is not prescribed a priori, but the evolution of the crack is determined through a competition between volume and surface energies.

In order to make the ideas precise, let $\Omega \subseteq \mathbb{R}^3$ be a linearly elastic, isotropic and homogeneous body in the reference configuration, $\partial_D\Omega$ a part of its boundary, and let $g : \partial_D\Omega \to \mathbb{R}^3$ be the displacement at the points of $\partial_D\Omega$. Following [54], the total energy associated to a configuration (u, K) given by a crack $K \subseteq \overline{\Omega}$ and a displacement $u : \Omega \setminus K \to \mathbb{R}^3$ with $u = g$ on $\partial_D\Omega \setminus K$ is given by

$$\mathcal{E}(u,K) := \int_\Omega \mu|Eu|^2 + \frac{\lambda}{2}|trEu|^2\,dx + k\mathcal{H}^2(K). \tag{1}$$

Here Eu denotes the symmetric part of the gradient of u, tr denotes the trace of the matrix, and $\mathcal{H}^2$ denotes the surface measure (two dimensional Hausdorff measure). The coefficients μ, λ (*Lamé coefficients*) and k depend on the material. The boundary condition is required only on $\partial_D\Omega \setminus K$ because the displacement in a fractured region is supposed to be not transmitted.

The total energy contains a volume part (*bulk energy*) given by

$$\int_\Omega \mu|Eu|^2 + \frac{\lambda}{2}|trEu|^2\,dx$$

and a surface part (*surface energy*) given by

$$k\mathcal{H}^2(K).$$

Suppose that the boundary displacement g varies with the time $t \in [0,T]$. The quasistatic evolution $t \to (u(t), K(t))$ proposed in [54] requires that:

(*a*) $K(t)$ is increasing in time, i.e., $K(t_1) \subseteq K(t_2)$ for all $0 \le t_1 \le t_2 \le T$;

(*b*) $\mathcal{E}(u(t), K(t)) \le \mathcal{E}(u, H)$ for all cracks H such that $\cup_{s<t} K(s) \subseteq H$ and all displacements $v : \Omega \setminus H \to \mathbb{R}^3$ with $v = g$ on $\partial_D\Omega \setminus H$;

(*c*) the total energy $\mathcal{E}(u(t), K(t))$ is absolutely continuous in time, and its derivative is equal to the power of external forces.

Condition (*a*) stands for the *irreversibility* of the evolution: the crack can only increase in time, i.e., no *healing processes* are admitted.

Condition (*b*) can be interpreted as a *static equilibrium* at each time t. It requires that the configuration $(u(t), K(t))$ minimizes the total energy $\mathcal{E}$ among all admissible configurations (v, H) with H containing the cracks $K(s)$ at all previous times $s < t$. As a consequence of (*b*), the displacement $u(t)$ satisfies the usual equilibrium equation in $\Omega \setminus K(t)$ and the crack $K(t)$ is *traction* *free*, i.e., setting $Su(t) := 2\mu Eu(t) + \lambda \mathrm{tr} Eu(t) I$ (*stress* of the configuration)

$$\begin{cases} \mathrm{Div}(Su(t)) = 0 & \text{in } \Omega \setminus K(t), \\ u(t) = g(t) & \text{on } \partial_D\Omega \setminus K(t), \\ Su(t) \cdot n = 0 & \text{on } \partial_N\Omega \cup K(t). \end{cases} \tag{2}$$

Condition (c) stands for the *nondissipativity* of the process in the following sense: in a regular context, i.e., if Γ is a submanifold and u is regular enough, we can perform integration by part getting

$$\frac{d}{dt}\mathcal{E}(u(t), K(t)) = \int_{\partial_D\Omega\setminus K(t)} Su(t)n \cdot \dot{g}(t)\, dx,$$

where n is the normal vector to $\partial_D\Omega \setminus K(t)$, so that the variation of the total energy is due to the power introduced in the system by the variation of the boundary displacement $g(t)$.

In view of conditions (b) and (c), this model of quasistatic crack propagation can be interpreted in the general framework of the theory of rate independent processes proposed by A. Mielke and coauthors: in this direction, we refer to [70] and references therein.

In the same paper [54], Francfort and Marigo suggest that a quasistatic crack growth relative to the boundary displacement $g(t)$ can be obtained as limit of a discrete in time evolution obtained through a step by step minimization process. To be precise, considering a subdivision I_δ of the time interval $[0, T]$ with step less than δ, i.e.,

$$I_\delta := \{0 = t_0^\delta < t_1^\delta < \cdots < t_{N_\delta}^\delta = T\}$$

with $\max_i(t_{i+1}^\delta - t_i^\delta) < \delta$, let (u_0^δ, K_0^δ) be a solution of the problem

$$\min_{u,K}\{\mathcal{E}(u, K) : u = g(0) \text{ on } \partial_D\Omega \setminus K\}, \tag{3}$$

and let (u_i^δ, K_i^δ) be a solution of the problem

$$\min_{u,K}\{\mathcal{E}(u, K) \ :\ K_{i-1}^\delta \subseteq K, u = g(t_i^\delta) \text{ on } \partial_D\Omega \setminus K\}. \tag{4}$$

Let us consider the piecewise constant in time interpolation

$$u^\delta(t) := u_i^\delta \qquad K^\delta(t) := K_i^\delta \qquad \text{for every } t \in [t_i^\delta, t_{i+1}^\delta[. \tag{5}$$

Letting $\delta \to 0$ along a suitable sequence $(\delta_n)_{n\in\mathbb{N}}$, and supposing that

$$u^{\delta_n}(t) \to u(t) \qquad \text{and} \qquad K^{\delta_n}(t) \to K(t), \tag{6}$$

it is expected that $t \to (u(t), K(t))$ is a quasistatic crack growth relative to the boundary displacement $g(t)$. In fact problems (3) and (4) provides K_i^δ satisfying irreversibility ($K_i^\delta \subseteq K_j^\delta$ for $i \leq j$) and static equilibrium at every time t_i^δ. Moreover also a weak form of nondissipativity holds, namely

$$\mathcal{E}(u_i^\delta, K_i^\delta) \leq \mathcal{E}(u_0^\delta, K_0^\delta) + \int_0^{t_i^\delta} \int_\Omega Su^\delta(t) E\dot{g}(t)\, dx\, dt + e(\delta), \tag{7}$$

where $e(\delta) \to 0$ as $\delta \to 0$. Assuming g sufficiently regular, in the limit one expects that irreverisbility, static equilibrium and half of nondissipativity hold. The missing inequality for nondissipativity could be derived indeed by irreversibility and static equilibrium of the limit $K(t)$. In fact it is not hard to prove that (see for example [59])

$$\mathcal{E}(g(t_i^\delta), K(t_i^\delta)) \geq \mathcal{E}(g(0), K(0)) + \int_0^{t_i^\delta} \int_\Omega S\tilde{u}^\delta(t) E\dot{g}(t)\, dx\, dt + e(\delta),$$

where $\tilde{u}^\delta(t) := u(t^\delta_{i+1})$ for every $t \in]t^\delta_i, t^\delta_{i+1}]$ and $e(\delta) \to 0$ for $\delta \to 0$. If the limit u enjoys some continuity property, the missing inequality is readily obtained (for a similar problem in the theory of rate independent processes, see [71]).

From a mathematical point of view, in order to put the variational theory of Francfort and Marigo into sound bases, one needs to find a functional space for the displacements and a class of admissible cracks so that problems (3) and (4) admit solutions, the convergences (6) are well defined, and the passage to the limit in (7) is possible. Moreover irreversibility and static equilibrium should be preserved in the limit.

This program has been carried out basically in the papers by G. Dal Maso and R. Toader [45], by G.A. Francfort and C.J. Larsen [53], and by G. Dal Maso, G.A. Francfort and R. Toader [44].

In [45] Dal Maso and Toader consider the case of *antiplanar shear*. Since everything can be referred to a cross section of the body, the total energy of a configuration (u, K) becomes (normalizing all the constants)

$$\mathcal{E}(u,K) := \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^1(K). \tag{8}$$

The admissible cracks are given by the family $\mathcal{K}^f_m(\overline{\Omega})$ of compact sets in $\overline{\Omega}$ with finite $\mathcal{H}^1$-measure and with m connected components. The class of admissible displacements for a crack K is given by the Deny-Lions space

$$L^{1,2}(\Omega \setminus K) := \{u \in L^2_{loc}(\Omega \setminus K), \nabla u \in L^2(\Omega \setminus K)\}.$$

The space $L^{1,2}(\Omega \setminus K)$ contains the Sobolev space $H^1(\Omega \setminus K)$, but it could be strictly larger since $\Omega \setminus K$ is a priori not sufficiently regular. A natural convergence for the cracks (under a bound on the surface energy) is given by Hausdorff convergence of compact sets (see Section 1.3). As for the displacements, a natural convergence is given by the weak convergence in $L^2(\Omega; \mathbb{R}^2)$ of the gradients (with the convention of considering $\nabla u = 0$ on K).

Problem (4) becomes in this setting

$$\min\{\int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^1(K) \,:\, K^\delta_{i-1} \subseteq K, u \in L^{1,2}(\Omega \setminus K), u = g(t^\delta_i) \text{ on } \partial_D\Omega \setminus K\}. \tag{9}$$

The restriction on the number of connected components of the cracks permits to solve the problem using the direct method of the Calculus of Variations, i.e., looking for the behavior of a minimizing sequence $(u_n, K_n)_{n\in\mathbb{N}}$. In fact up to a subsequence we can assume that

$$K_n \to K \qquad \text{in the Hausdorff metric,}$$

and that

$$\nabla u_n \rightharpoonup \nabla u \qquad \text{weakly in } L^2(\Omega, \mathbb{R}^2)$$

with $u \in L^{1,2}(\Omega \setminus K)$. Since

$$\int_\Omega |\nabla u|^2 \, dx \le \liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^2 \, dx,$$

the pair (u, K) is a minimizer of problem (9) provided that the $\mathcal{H}^1$-measure is lower semicontinuous along $(K_n)_{n\in\mathbb{N}}$, i.e.,

$$\mathcal{H}^1(K) \le \liminf_{n\to+\infty} \mathcal{H}^1(K_n). \tag{10}$$

The lower semicontinuity (10) is not true in general if we do not impose some restrictions on $(K_n)_{n\in\mathbb{N}}$. The celebrated Gołąb's theorem (see for example [50]) asserts that the $\mathcal{H}^1$-measure is lower semicontinuous if the K_n's are connected, and an extension to the case in which K_n has a uniformly bounded number of connected components is straightforward, see [45].

The static equilibrium for the pair $(u(t), K(t))$ obtained as a limit of the discretized evolution $(u^{\delta_n}(t), K^{\delta_n}(t))$ is recovered by Dal Maso and Toader through an approximation argument: for all $K \in \mathcal{K}_m^f(\overline{\Omega})$ containing $K(t)$ they construct $K_n \in \mathcal{K}_m^f(\overline{\Omega})$ with $K^{\delta_n}(t) \subseteq K_n$, $K_n \to K$ in the Hausdorff metric and

$$\limsup_{n \to +\infty} \mathcal{H}^1(K_n \setminus K^{\delta_n}(t)) \leq \mathcal{H}^1(K \setminus K(t)).$$

Moreover they prove that if $K_n \in \mathcal{K}_m^f(\overline{\Omega})$ and $K_n \to K$ in the Hausdorff metric, then for all $v \in L^{1,2}(\Omega \setminus K)$ there exists $v_n \in L^{1,2}(\Omega \setminus K_n)$ with $v_n = g^{\delta_n}(t)$ on $\partial_D \Omega \setminus K_n$ and such that (setting $\nabla v_n = 0$ and $\nabla v = 0$ on K_n and K respectively)

$$\nabla v_n \to \nabla v \qquad \text{strongly in } L^2(\Omega; \mathbb{R}^2). \tag{11}$$

The static equilibrium of the pair $(u(t), K(t))$ is thus readily implied by static equilibrium of $(u^{\delta_n}(t), K^{\delta_n}(t))$.

In Chapter 2, which contains the results of [55], we extend the lower semicontinuity (10) to the case of surface energies of the form

$$\int_K \varphi(x, \nu_x)\, d\mathcal{H}^1(x),$$

where ν_x is the *unit normal vector* at x to K and $\varphi : \overline{\Omega} \times \mathbb{R}^2 \to [0, \infty[$ is a continuous function, positively 1-homogeneous, even and convex in the second variable. Our proof is based on Reshetnyak's lower semicontinuity theorem for measures [79], and on a blow-up argument. As noticed by L. Ambrosio, the same result can also be obtained from Gołąb's theorem in abstract metric space (see [9]) considering on $\mathbb{R}^2$ the Finsler metric induced by φ.

This extension of Gołąb's theorem is employed in order to obtain an existence result for quasistatic crack growth in the setting of Dal Maso and Toader for linearly elastic, anisotropic and inhomogeneous bodies, i.e., for total energies of the form

$$\int_\Omega A(x) \nabla u \nabla u \, dx + \int_K \varphi(x, \nu_x)\, d\mathcal{H}^1(x),$$

where $A \in L^\infty(\Omega; M^{2\times 2})$ with

$$c_1|\xi|^2 \leq A\xi\xi \leq c_2|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^2, \qquad c_1, c_2 > 0.$$

In Chapter 3, which contains the results of [56], we try to extend the approximation (11) to higher dimensions. The context in which the problem is treated is that of *stability of Neumann problems*, a topic which has its own interest besides the applications to fracture mechanics (see for example [65], [67], [35], [74], [82],[83], [48], [34], [24], [39] and [66])

In fact an approximation like (11) under the assumption $K_n \to K$ in the Hausdorff metric turns out to be equivalent to the convergence of solutions of elliptic equations with Neumann conditions on $\partial\Omega \cup K_n$. More precisely if $u_n \in H^1(\Omega \setminus K_n)$ is the solution to the problem

$$\begin{cases} -\Delta u_n + u_n = f & \text{in } \Omega \setminus K_n \\ \frac{\partial u_n}{\partial \nu} = 0 & \text{on } \partial\Omega \cup K_n \end{cases} \tag{12}$$

with $f \in L^2(\Omega)$, and $u \in H^1(\Omega \setminus K)$ is the solution of

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \setminus K \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \cup K, \end{cases} \tag{13}$$

then the approximation result (11) is equivalent to the convergence

$$u_n \to u \qquad \text{strongly in } L^2(\Omega)$$

and

$$\nabla u_n \to \nabla u \qquad \text{strongly in } L^2(\Omega, \mathbb{R}^N),$$

where we extend $u_n, \nabla u_n$ and $u, \nabla u$ to zero on K_n and K respectively (in particular ∇u is the distributional gradient of u only in $\Omega \setminus K$).

In dimension $N \geq 3$ the connectedness of K_n is no longer sufficient to guarantee stability of Neumann problems as shown in the counterexample of the Neumann sieve, where a transmission condition on K (different from $\frac{\partial u}{\partial \nu} = 0$) appears in the limit (see for example [74]). A typical cause of nonstability is the presence on K_n of small holes that induces some homogenization effects in the limit.

In order to preserve stability of Neumann problems in dimension $N \geq 3$, we assume that K_n satisfies a suitable cone condition, which implies that the sets K_n's are locally sufficiently regular subsets of $(N-1)$-dimensional Lipschitz submanifolds of $\mathbb{R}^N$ in such a way that homogenization effects due to the possible presence of holes cannot occur.

Let C be a fixed $(N-1)$-dimensional cone. The sequence $(K_n)_{n \in \mathbb{N}}$ satisfies the C-condition if there exist constants $\delta, L_1, L_2 > 0$ such that, for all n and for all $x \in K_n$, there exists $\Phi_x : B_\delta(x) \to \mathbb{R}^N$ with

(a) for all $z_1, z_2 \in B_\delta(x)$:

$$L_1|z_1 - z_2| \leq |\Phi_x(z_1) - \Phi_x(z_2)| \leq L_2|z_1 - z_2|;$$

(b) $\Phi_x(x) = 0$ and $\Phi_x(B_\delta(x) \cap K_n) \subseteq \{x_N = 0\}$;

(c) for all $y \in B_{\frac{\delta}{2}}(x) \cap K_n$,

$$\Phi_x(y) \in C_y \subseteq \Phi_x(B_\delta(x) \cap K_n)$$

for some finite closed cone C_y congruent (up to a rototranslation) to C. Conditions (a), (b) imply that, near x, K_n is a subset of a $(N-1)$-dimensional Lipschitz submanifold $M_{n,x}$ of $\mathbb{R}^N$ and condition (c) implies that K_n is sufficiently regular in $M_{n,x}$, essentially a finite union of Lipschitz subsets. A particular class of cracks which satisfy the C-condition is given for example by $(\Psi_n(\overline{A}))_{n \in \mathbb{N}}$, where A is a Lipschitz bounded open subset of $\{x_N = 0\}$ and (Ψ_n) is a sequence of bi-Lipschitz maps from $\mathbb{R}^N$ into itself with constants L_1 and L_2; another example is given by $(\Psi_n(\tilde{K}_n))_{n \in \mathbb{N}}$, where $(\tilde{K}_n)_{n \in \mathbb{N}}$ is a sequence of compact subsets of $\{x_N = 0\}$ satisfying the cone condition with respect to a finite close cone C.

The main result of Chapter 3 is that if $(K_n)_{n \in \mathbb{N}}$ is a sequence of compact sets in $\mathbb{R}^N$ satisfying the C-condition with respect to a $(N-1)$-dimensional cone C, and K_n converges to K in the Hausdorff metric, then the Neumann problems (12) are stable.

Francfort and Larsen proposed in [53] a *weak* approach to the problem of quasistatic crack evolution in which they got rid of the restriction on the number of connected components of the cracks. They employed the space SBV of functions of bounded variation which was introduced by E. De Giorgi and L. Ambrosio [46] to deal with free discontinuity problems, and which has turned out to be very useful in problems where a competition between volume and surface energies is involved (image segmentation, fracture mechanics, plasticity, liquid crystals, see [8] for more details).

Francfort and Larsen treat the case of *generalized antiplanar shear*, i.e., they consider the antiplanar setting with a N-dimensional base $\Omega \subseteq \mathbb{R}^N$.

The class of admissible cracks is given by the family of rectifiable sets in $\overline{\Omega}$, while the class of admissible displacements is given by the space $SBV(\Omega)$. The boundary datum is given by (traces of) functions in the Sobolev space $H^1(\Omega)$: for technical reasons, g is assumed to be also in

$L^\infty(\Omega)$. In order to treat the boundary condition in a simpler way, they consider (using extension operators) boundary data $g \in H^1(\Omega')$, where Ω' is an open set containing $\overline{\Omega}$, and they extended the displacement u to Ω' setting $u = g$ on $\Omega' \setminus \Omega$. The total energy considered in [53] is

$$\mathcal{E}(u, K) := \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(K \setminus \partial_N \Omega), \tag{14}$$

where u is an admissible displacement for K and g, i.e., $u = g$ on $\Omega' \setminus \Omega$, and $S(u) \subseteq K$. In this setting problem (4) takes the form

$$\min \left\{ \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{N-1}\Big(S(u) \setminus (K_{i-1}^\delta \cup \partial_N \Omega)\Big) \ : \ u \in SBV(\Omega'), u = g(t_i^\delta) \text{ on } \Omega' \setminus \Omega \right\}, \tag{15}$$

where $K_{i-1}^\delta = \bigcup_{j=1}^{i-1} S(u_j^\delta)$. Problem (15) admits solutions as a consequence of Ambrosio's lower semicontinuity theorems (see for example [4]).

The convergence for the displacements is given by weak convergence for SBV functions (see subsection 1.1.1), and the cracks in the limit are reconstructed looking for the jumps of the displacements. More precisely they fix a countable and dense subset D of $[0, T]$, and for all $t \in D$ the displacement $u(t)$ in the limit is given by the weak limit for $\delta \to 0$ of $u^\delta(t)$ defined in (5), while $K(t)$ is given by the union of jumps at previous times, i.e., $K(t) = \bigcup_{s \leq t, s \in D} S(u(s))$. The missing times are treated in the following way: they set

$$K(t) := \bigcup_{s \in D, s < t} K(s), \tag{16}$$

and they consider $u(t)$ as a minimum energy displacement associated to $K(t)$ and $g(t)$.

The static equilibrium property for $(u(t), K(t))$ is derived from that of $(u^\delta(t), K^\delta(t))$ by means of a geometric construction which Francfort and Larsen called the Transfer of Jump Sets. In its basic form, the Transfer of Jump Sets can be formulated in the following way. If $g \in H^1(\Omega')$ is a boundary displacement, $(u_n)_{n \in \mathbb{N}}$ is a sequence in $SBV(\Omega')$ with $u_n = g$ on $\Omega' \setminus \Omega$ and $u_n \rightharpoonup u$ weakly in $SBV(\Omega')$, then for all $v \in SBV(\Omega')$ with $v = g$ on $\Omega' \setminus \Omega$ there exists $(v_n)_{n \in \mathbb{N}}$ sequence in $SBV(\Omega')$ with $v_n = g$ on $\Omega' \setminus \Omega$ such that

$$\nabla v_n \to \nabla v \qquad \text{strongly in } L^2(\Omega'; \mathbb{R}^N)$$

and

$$\limsup_{n \to +\infty} \mathcal{H}^{N-1}\Big(S(v_n) \setminus (S(u_n) \cup \partial_N \Omega)\Big) \leq \mathcal{H}^{N-1}\Big(S(v) \setminus (S(u) \cup \partial_N \Omega)\Big).$$

This result readily implies that if $(u_n)_{n \in \mathbb{N}}$ enjoys the minimality property

$$\int_\Omega |\nabla u_n|^2 \, dx \leq \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1}\Big(S(v) \setminus (S(u_n) \cup \partial_N \Omega)\Big) \tag{17}$$

for all $v \in SBV(\Omega')$ with $v = g$ on $\Omega' \setminus \Omega$, then the same holds for the weak limit u. The stability of static equilibrium in the problem of quasistatic crack evolution is treated by a more complex version of this approximation argument (see Theorem 1.4.3).

In Chapter 4, which contains the result of [61] in collaboration with M. Ponsiglione, we prove that the stability of static equilibrium has a variational character. In fact, referring for example to the basic problem (17), we have that u_n is a (absolute) minimizer of the functional

$$\mathcal{E}_n(v) := \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1}\Big(S(v) \setminus (S(u_n) \cup \partial_N \Omega)\Big).$$

The problem of stability is then dealt with the tool of Γ-convergence (see Section 1.2).

The Γ-convergence of functionals introduced by E. De Giorgi and T. Franzoni [47] turns out to be very useful in Calculus of Variations basically because of the following property: if a sequence of functionals $(F_n)_{n\in\mathbb{N}}$ defined on a topological space X Γ-converges to a functional F, and if x_n is a minimum point for F_n with $x_n \to x$, then x is a minimum point for F.

In our case we get that the weak limit u of $(u_n)_{n\in\mathbb{N}}$ is a minimum for the Γ-limit $\mathcal{E}$ of the functionals $\mathcal{E}_n$: it turns out that the qualitative properties of $\mathcal{E}$ permit to recover the minimality property (17) for u in the limit.

The same approach is used to treat general unilateral minimality properties of the form

$$\int_\Omega f_n(x, \nabla u_n)\, dx \le \int_\Omega f_n(x, \nabla v)\, dx + \int_{S(v)\setminus K_n} g_n(x, \nu), \tag{18}$$

where $v \in SBV(\Omega)$, K_n is rectifiable in $\overline{\Omega}$, and f_n, g_n are densities of volume and surface energies satisfying standard structure assumptions. The Γ-convergence approach provides effective volume and surface energy densities f, g and a limit crack K containing $S(u)$ so that the limit u of $(u_n)_{n\in\mathbb{N}}$ satisfies the minimality property (18) with respect to f, g, and K.

The problem of stability in the case of fixed energy densities f and g has been treated by Dal Maso, Francfort and Toader [44] by means of a variational notion of convergence for rectifiable sets which they called σ^p-convergence (to recover K in the limit), and of a suitable Transfer of Jump Sets for the sets K_n (to recover minimality). The crack K constructed by our Γ-convergence approach contains the σ^p-limit of the cracks $(K_n)_{n\in\mathbb{N}}$, so that the minimality property in the limit turns out to be improved. Moreover the Γ-convergence approach provides a Transfer of Jump Sets adapted to the case of varying energies, which seems difficult to be derived directly.

The stability result for minimality properties of the form (18) is used to deal with the study of quasistatic crack evolutions in composite materials. More precisely we study the asymptotic behavior of a quasistatic evolution $t \to (u_n(t), K_n(t))$ relative to the bulk energy f_n and the surface energy g_n and we prove that it converges to a quasistatic evolution $t \to (u(t), K(t))$ relative to the effective bulk and surface energy densities f and g, with convergence for bulk and surface energies at all times. This analysis applies to the case of composite materials, i.e., materials obtained through a fine mixture of different phases. The model case is that of periodic homogenization, i.e., materials with total energy given by

$$\mathcal{E}_\varepsilon(u, K) := \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u(x)\right)\, dx + \int_K g\left(\frac{x}{\varepsilon}, \nu\right)\, d\mathcal{H}^{N-1}(x),$$

where ε is a small parameter giving the size of the inhomogeneities in the mixture, and f, g are periodic in x. Our result implies that a quasisistatic crack evolution $t \to (u_\varepsilon(t), K_\varepsilon(t))$ for ε small is very near to a quasistatic evolution for the homogeneous material having bulk and surface energy densities f_{hom} and g_{hom}, which are obtained from f and g through periodic homogenization formulas available in the literature (see for example [21] and [20]). This result provides a mathematical justification of considering homogeneous bulk and surface energy densities in the macroscopic description of materials whose toughness properties depend on x at a microscopic level.

In Chapter 5, which contains the results of [58], we study size effects in quasistatic crack growth. More precisely we prove that the model of quasistatic crack evolution proposed by Francfort and Marigo [54] can take into account the fact that *ductility* is influenced by the size of the structure, and in particular that the structure tends to become brittle if its size increases (see for example [32], and references therein). With this aim, in the framework of generalized antiplanar shear, we consider surface energies of the form (Barenblatt's type [13])

$$\int_K \varphi(|[u]|(x))\, d\mathcal{H}^{N-1}(x), \tag{19}$$

where $[u](x) := u^+(x) - u^-(x)$ is the difference of the traces of u on both sides of K, and $\varphi : [0, +\infty[\to [0, +\infty[$ (which depends on the material) is such that $\varphi(0) = 0$. Setting $\sigma := \varphi'$, $\sigma(|[u]|(x))$ can be interpreted as the surface density of forces in x that act between the two lips of the crack K whose displacements are $u^+(x)$ and $u^-(x)$ respectively. Typically σ is decreasing, and $\sigma(s) = 0$ for $s \geq \bar{s}$, i.e., the interaction between the two lips of the crack decreases as the opening increases, and it disappears when the opening is greater than a critical length $\bar{s}$. We assume that $a = \varphi'(0) < +\infty$, and $\lim_{s\to+\infty} \varphi(s) = 1$. The first hypothesis is given by mechanical considerations, since $\varphi'(0)$ is the maximal tensile strain which the material can sustain before breaking.

We consider a total energy of the form

$$\mathcal{E}(u, K) := \int_\Omega |\nabla u|^2 + \int_{K\setminus\partial_N\Omega} \varphi(|[u]|)\, d\mathcal{H}^{N-1}. \tag{20}$$

Problem (3) takes in this setting the form (we use Ω' to treat the boundary condition as explained before)

$$\min\left\{\int_\Omega |\nabla u|^2\, dx + \int_{S(u)\setminus\partial_N\Omega} \varphi(|[u]|)\, d\mathcal{H}^{N-1}, u \in SBV(\Omega), u = g(0) \text{ on } \Omega' \setminus \Omega\right\}. \tag{21}$$

We set $K_0^\delta := S(u_0^\delta)$, and $\psi_0^\delta := |[u_0^\delta]|$. Supposing to have constructed $(u_{i-1}^\delta, K_{i-1}^\delta, \psi_{i-1}^\delta)$, problem (4) takes the form

$$\min\left\{\int_\Omega |\nabla u|^2 + \int_{(S(u)\cup K_{i-1}^\delta)\setminus\partial_N\Omega} \varphi(|[u]| \vee \psi_{i-1}^\delta)\, d\mathcal{H}^{N-1}, u \in SBV(\Omega), u = g(t_i^\delta) \text{ on } \Omega' \setminus \Omega\right\}, \tag{22}$$

where $|[u]| \vee \psi_{i-1}^\delta := \max\{|[u]|, \psi_{i-1}^\delta\}$. We set $K_i^\delta := K_{i-1}^\delta \cup S(u_i^\delta)$ and $\psi_i^\delta := \psi_{i-1}^\delta \vee |[u_i^\delta]|$.

Problem (22) takes into account an *irreversibility condition* in the growth of the crack. Indeed, while on $S(u) \setminus K_{i-1}^\delta$ the surface energy which comes in the minimization of (22) is exactly as in (19), on $S(u) \cap K_{i-1}^\delta$ the surface energy involved takes into account the previous work made on K_i^δ. The surface energy is of the form of (19) only if $|[u]| > \psi_{i-1}^\delta$, that is only if the opening is increased. If $|[u]| \leq \psi_i^\delta$ no energy is spent, that is displacements of this form along the crack do not contribute to the surface energy. Notice finally that the irreversibility condition involves only the modulus of $[u]$: this is an assumption which is reasonable since we are considering only antiplanar displacements. Clearly more complex irreversibility conditions can be formulated, involving for example a partial release of energy when the opening decreases: the one we study is the first straightforward extension of the irreversibility condition given in [54] adapted to the energy (14).

Since $a = \varphi'(0) < +\infty$, problems (21) and (22) con not be solved directly in the space $SBV(\Omega)$ using the direct method of the Calculus of Variations: we solve them in the space $BV(\Omega)$ of functions of bounded variation in the relaxed form

$$\int_\Omega f(\nabla u)\, dx + \int_{K\setminus\partial_N\Omega} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} + a|D^c u|(\Omega),$$

where $a = \varphi'(0)$, f is a suitable modification of $|\xi|^2$ determined by a (see (5.9)), and $D^c u$ indicates the Cantorian part of the derivative of u.

The analysis for $\delta \to 0$ with h fixed presents several difficulties, the main one being the stability of the minimality property of the discrete in time evolutions. The main purpose of the chapter is to prove that these difficulties disappear as the size of the reference configuration increases, thank to the fact that the body response tends to become more and more brittle in spite of the presence of cohesive forces on the cracks.

In order to point out the size effects, we consider discrete in time evolutions in $\Omega_h := h\Omega$ with h large under suitable boundary displacements, and we study the asymptotic behavior of the displacements and of the cracks suitably rescaled to the fixed configuration Ω. The boundary displacements on $\partial_D\Omega_h := h\partial_D\Omega$ are taken of the form

$$g_h(t, x) := h^\alpha g\left(t, \frac{x}{h}\right),$$

where $\alpha > 0$. Let us denote by $(u^{\delta,h}(t), K^{\delta,h}(t), \psi^{\delta,h}(t))$ the piecewise constant interpolation of the discrete in time evolution of the crack in Ω_h relative to the boundary displacement g_h.

The size effects come out in the asymptotic behavior of $(u^{\delta,h}(t), K^{\delta,h}(t), \psi^{\delta,h}(t))$ for $\delta \to 0$ and $h \to +\infty$. It turns out that the behavior depends on α: in the case $\alpha = \frac{1}{2}$ the discrete in time evolutions suitably rescaled to Ω converge to a quasistatic evolution for the total energy (14) in the sense of Francfort and Larsen, so that brittle effects occur as the size of the body increases. We investigate also the cases $\alpha \in (0, \frac{1}{2})$ and $\alpha > \frac{1}{2}$ which lead to an elastic problem and to a rupture problem in Ω respectively.

In Chapters 6, 7 and 8 we provide some approximation results for quasistatic crack evolutions which were suggested by numerics (see Bourdin, Francfort and Marigo [18]). The basic idea is to replace the total energy $\mathcal{E}$ given in (14) by suitable approximations $\mathcal{E}_\varepsilon$ in the sense of Γ-convergence, to reformulate problems (3) and (4) in term of $\mathcal{E}_\varepsilon$, and to study the asymptotics as $\delta \to 0$ and $\varepsilon \to 0$ (δ being the time step discretization).

We remark that in order to recover the irreversibility condition in the limit, a suitable notion of irreversibility at the level of the functionals $\mathcal{E}_\varepsilon$ has to be imposed (for instance in problem (4)), and this makes the problem of approximating quasistatic crack evolutions different from a simple collection of approximations of minimization problems to which Γ-convergence can be directly applied.

In Chapter 6, which contains the results of [57], the total energy of the system is approximated by the Ambrosio-Tortorelli functional given by

$$F_\varepsilon(u,v) = \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2\,dx + \frac{\varepsilon}{2}\int_\Omega |\nabla v|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1-v)^2\,dx \tag{23}$$

where $(u,v) \in H^1(\Omega) \times H^1(\Omega)$, $0 \le v \le 1$, and $0 < \eta_\varepsilon << \varepsilon$. F_ε contains an *elliptic part*

$$\int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2\,dx \tag{24}$$

and a *surface part*

$$MM_\varepsilon(v) := \frac{\varepsilon}{2}\int_\Omega |\nabla v|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1-v)^2\,dx \tag{25}$$

which is a term of Modica-Mortola type (see [72]).

L. Ambrosio and V.M. Tortorelli [10], [11] proved that the functional (23) provides an approximation in the sense of Γ-convergence of the total energy

$$\mathcal{E}(u) := \int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{N-1}(S(u))$$

defined on $SBV(\Omega)$.

The passage from the H^1 context to the SBV one in the Γ-limit process can be explained in the following way. If a sequence $(u_\varepsilon, v_\varepsilon)$ is such that $F_\varepsilon(u_\varepsilon, v_\varepsilon) + ||u_\varepsilon||_\infty \le C$, then $v_\varepsilon \to 1$ strongly in $L^2(\Omega)$, and it turns out that, up to a subsequence, $u_\varepsilon \to u$ in measure for some $u \in SBV(\Omega)$; some jumps in the limit may appear because the gradient of u_ε can become larger and larger in the thin regions in which v_ε approaches zero.

As for the applications to crack propagation, the function u_ε has to be considered as a regularization of the displacement u, while the function v_ε has to be intended as a function which tends to 0 in the region where the crack $S(u)$ will appear in the limit, and which tends to 1 elsewhere. Moreover (24) and (25) have to be interpreted as regularizations of the bulk and surface energy of the system.

In relation with quasistatic crack evolutions, we define through a variational argument the following notion of quasistatic evolution for the functional F_ε: for every $\varepsilon > 0$ we find a map $t \to (u_\varepsilon(t), v_\varepsilon(t))$ from $[0,T]$ to $H^1(\Omega) \times H^1(\Omega)$, $0 \le v_\varepsilon(t) \le 1$, $u_\varepsilon(t) = g(t)$, $v_\varepsilon(t) = 1$ on $\partial_D\Omega$ such that:

(a) for all $0 \leq s < t \leq T$: $v_\varepsilon(t) \leq v_\varepsilon(s)$;

(b) for all $(u, v) \in H^1(\Omega) \times H^1(\Omega)$ with $u = g(t)$, $v = 1$ on $\partial_D\Omega$, $0 \leq v \leq v_\varepsilon(t)$:

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \leq F_\varepsilon(u, v);$$

(c) the energy $\mathcal{E}_\varepsilon(t) := F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))$ is absolutely continuous and for all $t \in [0, T]$

$$\mathcal{E}_\varepsilon(t) = \mathcal{E}_\varepsilon(0) + 2\int_0^t \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau))\nabla u_\varepsilon(\tau)\nabla \dot{g}(\tau)\, dx\, d\tau;$$

(d) there exists a constant C depending only on g such that $\mathcal{E}_\varepsilon(t) \leq C$ for all $t \in [0, T]$.

Condition (a) permits to recover in this regular context the fact that the crack is increasing in time: in fact, as $v_\varepsilon(t)$ determines the crack in the regions where it is near zero, the condition $v_\varepsilon(t) \leq v_\varepsilon(s)$ ensures that existing cracks are preserved at subsequent times. Condition (b) reproduces the static equilibrium condition at each time, while condition (c) reproduces the nondissipavity condition in this context. Condition (d) gives the necessary compactness in order to let $\varepsilon \to 0$. The requirement $v_\varepsilon(t) = 1$ on $\partial_D\Omega$ for all $t \in [0, 1]$ is made in such a way that, letting $\varepsilon \to 0$, the surface energy of the crack in the limit is the usual one also for the part touching the boundary $\partial_D\Omega$.

The main result of the chapter is that, as $\varepsilon \to 0$, the quasistatic evolution $t \to (u_\varepsilon(t), v_\varepsilon(t))$ for the Ambrosio-Tortorelli functional converges to a quasistatic evolution for brittle fracture in the sense of Francfort and Larsen [53], providing an approximation of the total energy at any time. More precisely, there exists a quasistatic evolution $t \to (u(t), K(t))$, $u(t) \in SBV(\Omega)$, relative to the boundary data g and a sequence $\varepsilon_n \to 0$ such that for all $t \in [0, 1]$ which are not discontinuity points of $\mathcal{H}^{N-1}(K(\cdot))$ we have

$$v_{\varepsilon_n}(t)\nabla u_{\varepsilon_n}(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega, \mathbb{R}^N),$$

$$\int_\Omega (\eta_{\varepsilon_n} + v_{\varepsilon_n}(t)^2)|\nabla u_{\varepsilon_n}(t)|^2\, dx \to \int_\Omega |\nabla u(t)|^2\, dx,$$

and

$$MM_{\varepsilon_n}(v_{\varepsilon_n}(t)) \to \mathcal{H}^{N-1}(K(t)).$$

We thus obtain an approximation of the total energy at any time, and an approximation of the strain, of the bulk and the surface energy at all time up to a countable set.

The fact that nothing can be said for the times belonging to the countable set of discontinuity points of $\mathcal{H}^{N-1}(K(\cdot))$ is due to possible non uniqueness of quasistatic crack growth relative to the boundary displacement g. Following Francfort and Larsen we considered evolutions which are *left continuous* outside a countable set (see equation (16)), but the Ambrosio-Tortorelli approximation could as well individuate a suitable crack $\tilde{K}(t)$ contained between the left and right envelope of $K(\cdot)$. As a consequence approximation of bulk and surface parts of the energy could not hold at these times.

In Chapter 7, which contains the results of [59] in collaboration with M. Ponsiglione, we provide a finite element approximation of quasistatic crack evolutions. We employ the Γ-convergence results by M. Negri (see [75], [76]) concerning suitable discontinuous finite element approximations for the total energy (14).

Restricting our analysis to a two dimensional polygonal reference configuration, the discretization of the domain Ω is carried out considering two parameters $\varepsilon > 0$ and $a \in]0, \frac{1}{2}[$. We consider a regular triangulation $\mathbf{R}_\varepsilon$ of size ε of Ω, i.e., we assume that there exist two constants c_1 and c_2 so that every triangle $T \in \mathbf{R}_\varepsilon$ contains a ball of radius $c_1\varepsilon$ and is contained in a ball of radius $c_2\varepsilon$. On each edge $[x, y]$ of $\mathbf{R}_\varepsilon$ we consider a point z such that $z = tx + (1 - t)y$ with $t \in [a, 1 - a]$. These points are called *adaptive vertices.* Connecting together the adaptive vertices, we divide every $T \in \mathbf{R}_\varepsilon$ into four triangles. We take the new triangulation $\mathbf{T}$ obtained after this division as the discretization of Ω. The family of all such triangulations is denoted by $\mathcal{T}_{\varepsilon,a}(\Omega)$.

The class of discretized admissible displacements is the family of functions u which are affine on the triangles of some triangulation $\mathbf{T}(u) \in \mathcal{T}_{\varepsilon,a}(\Omega)$ and are allowed to jump across the edges of $\mathbf{T}(u)$. The discretized energy is obtained simply restricting the total energy (14) to admissible the displacements.

We reformulate problems (15) in this discretized context obtaining a discrete in time evolution $(u^{\delta,i}_{\varepsilon,a}, K^{\delta,i}_{\varepsilon,a})$. Interpolating in time as usual, we obtain a discrete evolution $t \to (u^\delta_{\varepsilon,a}(t), K^\delta_{\varepsilon,a}(t))$ depending on ε, a and δ.

The main result of the chapter is that as $\delta \to 0$, $\varepsilon \to 0$ and $a \to 0$ (along suitable sequences), the evolution $t \to (u^\delta_{\varepsilon,a}(t), K^\delta_{\varepsilon,a}(t))$ converges to a quasistatic crack growth $t \to (u(t), K(t))$ in the sense of Francfort and Larsen. More precisely we have convergence of total energies for all times $t \in [0,T]$, and for all times which are not discontinuity points for $\mathcal{H}^1(K(\cdot))$ we have convergence of the stress

$$\nabla u^\delta_{\varepsilon,a}(t) \to \nabla u(t) \qquad \text{strongly in } L^2(\Omega;\mathbb{R}^2), \tag{26}$$

and convergence of the surface energy

$$\mathcal{H}^1(K^\delta_{\varepsilon,a}(t)) \to \mathcal{H}^1(K(t)). \tag{27}$$

As in the case of the Ambrosio-Tortorelli approximation, the restriction to times which are continuity points for $\mathcal{H}^1(K(\cdot))$ in order to get approximation of both bulk and surface energies is due to the possible nonuniqueness of quasistatic crack growth relative to a time dependent boundary displacement.

In Chapter 8, which contains the results of [59] in collaboration with M. Ponsiglione, we employ the discontinuous finite element approximation used in Chapter 7 in order to approximate a general model of quasistatic crack growth in nonlinear elasticity proposed by Dal Maso, Francfort and Toader in [44]. This model takes into account possible body and traction forces, so that the total energy considered has the form

$$\mathcal{E}(t)(u,K) := \int_\Omega W(x,\nabla u)\,dx - \int_\Omega F(t,x,u)\,dx - \int_{\partial_s\Omega} G(t,x,u)\,dx + \int_{K\setminus\partial_N\Omega} k(x,\nu)\,d\mathcal{H}^{N-1}. \tag{28}$$

Here W is the hyperelastic potential with p-growth estimates in the gradient ($p > 1$), F and G are the time dependent potentials of body and traction forces respectively, $\partial_s\Omega$ is the part of the boundary of Ω on which the traction forces are applied, while k is the density of the surface energy of the crack K. In this model the bulk energy of the deformation u is given by

$$\mathcal{E}^b(t)(u) := \int_\Omega W(x,\nabla u)\,dx - \int_\Omega F(t,x,u)\,dx - \int_{\partial_s\Omega} G(t,x,u)\,dx, \tag{29}$$

while the surface energy of the crack K is given by

$$\mathcal{E}^s(K) := \int_K k(x,\nu)\,d\mathcal{H}^{N-1}. \tag{30}$$

The admissible deformations are vector valued functions in the space of *generalized functions of bounded variation* $GSBV(\Omega;\mathbb{R}^m)$, while the admissible cracks are the rectifiable sets in $\overline{\Omega}$.

Considering the case of $\Omega \subseteq \mathbb{R}^2$ polygonal and $m = 2$, we construct a discrete evolution $t \to (u^\delta_{\varepsilon,a}(t), K^\delta_{\varepsilon,a}(t))$ in a similar way to Chapter 7. Then we prove that there exist a quasistatic evolution $t \to (u(t), K(t))$ in the sense of Dal Maso, Francfort and Toader and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, such that setting

$$u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t), \qquad K_n(t) := K^{\delta_n}_{\varepsilon_n,a_n}(t),$$

for all $t \in [0,T]$ the following facts hold:

(a) $(u_n(t))_{n\in\mathbb{N}}$ is strongly precompact in $L^1(\Omega;\mathbb{R}^2)$, and every accumulation point $\tilde{u}(t)$ is admissible for $g(t)$ and $K(t)$, and $(\tilde{u}(t), K(t))$ satisfy the static equilibrium property with respect to the total energy $\mathcal{E}(t)(\cdot,\cdot)$; moreover there exists a subsequence $(\delta_{n_k}, \varepsilon_{n_k}, a_{n_k})_{k\in\mathbb{N}}$ of $(\delta_n, \varepsilon_n, a_n)_{n\in\mathbb{N}}$ (depending on t) such that

$$u_{n_k}(t) \to u(t) \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^2);$$

and

$$\nabla u_{n_k}(t) \to \nabla u(t) \qquad \text{weakly in } L^p(\Omega; M^{2\times 2}),$$

where $M^{2\times 2}$ denotes the space of 2×2 matrices;

(b) convergence of the total energy holds, and more precisely elastic and surface energies converge separately, that is

$$\mathcal{E}^b(t)(u_n(t)) \to \mathcal{E}^b(t)(u(t)), \qquad \mathcal{E}^s(K_n(t)) \to \mathcal{E}^s(K(t)).$$

By point (a), the approximation of the deformation $u(t)$ is available only up to a subsequence depending on t: this is due to the possible non uniqueness of the minimum energy deformation associated to $K(t)$. In the case $\mathcal{E}^b(t)(u)$ is strictly convex, it turns out that the deformation $u(t)$ is uniquely determined, and we prove that

$$\nabla u_n(t) \to \nabla u(t) \qquad \text{strongly in } L^p(\Omega; M^{2\times 2})$$

and

$$u_n(t) \to u(t) \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^2).$$

By point (b) the approximation of bulk and surface energies is available at every time $t \in [0,T]$, and this is an improvement with respect to the approximation results of Chapter 6 and Chapter 7. This improvement is obtained adapting to our context the notion of σ^p-convergence for rectifiable sets formulated by Dal Maso, Francfort and Toader in [44] in order to determine the crack $K(t)$ which is approximated by $K^{\delta}_{\varepsilon,a}(t)$ at all times t.

Chapter 1

Preliminaries

In this chapter we recall some basic notions about functions of bounded variations which will be used throughout the thesis. Moreover we recall the definition of Γ-convergence for functionals defined on metric spaces, and the definition of Hausdorff convergence of compact sets. Finally we state the mathematical results obtained by Dal Maso and Toader [45] and Francfort and Larsen [53] relative to the model of quasistatic crack evolutions of Francfort and Marigo [54].

1.1 Functions of bounded variation

For a comprehensive treatment of the theory of functions of bounded variation, we refer to [8]. Let A be an open set in $\mathbb{R}^N$, and let $m \geq 1$. We say that $u \in BV(A, \mathbb{R}^m)$ if $u \in L^1(A, \mathbb{R}^m)$, and its distributional derivative Du is a bounded vector-valued Radon measure on A. In this case it turns out that the set $S(u)$ of points $x \in A$ which are not Lebesgue points of u is rectifiable, that is there exists a sequence of C^1 manifolds $(M_i)_{i \in \mathbb{N}}$ such that $S(u) \subseteq \cup_i M_i$ up to a set of $\mathcal{H}^{N-1}$-measure zero. As a consequence $S(u)$ admits a normal $\nu_u(x)$ at $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$. Moreover for $\mathcal{H}^{N-1}$ a.e. $x \in S(u)$, there exist $u^+(x), u^-(x) \in \mathbb{R}^m$ such that

$$\lim_{r \to 0} \frac{1}{|B_r^{\pm}(x)|} \int_{B_r^{\pm}(x)} |u(y) - u^{\pm}(x)| \, dy = 0,$$

where $B_r^{\pm}(x) := \{y \in B_r(x) \,:\, (y - x) \cdot \nu_u(x) \gtrless 0\}$, and $B_r(x)$ is the ball with center x and radius r. It turns out that Du can be represented as

$$Du(A) = \int_A \nabla u(x) \, dx + \int_{A \cap S(u)} (u^+(x) - u^-(x)) \otimes \nu_u(x) \, d\mathcal{H}^{N-1}(x) + D^c u(A),$$

where ∇u denotes the approximate gradient of u and $D^c u$ is the Cantor part of Du. $BV(A, \mathbb{R}^m)$ is a Banach space with respect to the norm $\|u\|_{BV(A,\mathbb{R}^m)} := \|u\|_{L^1(A,\mathbb{R}^m)} + |Du|(A)$.

We will often use the following result: if A is bounded and Lipschitz, and if $(u_k)_{k \in \mathbb{N}}$ is a bounded sequence in $BV(A, \mathbb{R}^m)$, then there exists a subsequence $(u_{k_h})_{h \in \mathbb{N}}$ and $u \in BV(A, \mathbb{R}^m)$ such that

$$\begin{aligned} u_{k_h} &\to u \qquad \text{strongly in } L^1(A, \mathbb{R}^m), \\ Du_{k_h} &\overset{*}{\rightharpoonup} Du \qquad \text{weakly}^* \text{ in the sense of measures.} \end{aligned} \tag{1.1}$$

We say that $u_k \overset{*}{\rightharpoonup} u$ weakly* in $BV(A, \mathbb{R}^m)$ if (1.1) holds.

Finally in the context of fracture problems we will use the following notation: if A is Lipschitz, and if $\partial_D A \subseteq \partial A$, then for all $u, g \in BV(A)$ we set

$$S^g(u) := S(u) \cup \{x \in \partial_D A \,:\, u(x) \neq g(x)\}, \tag{1.2}$$

where the inequality on $\partial_D A$ is intended in the sense of traces. Moreover, we set for all $x \in S(u)$

$$[u](x) := u^+(x) - u^-(x),$$

and for all $x \in \partial_D A$ we set $[u](x) := u(x) - g(x)$, where the traces of u and g on ∂A are used.

1.1.1 The space SBV of special function of bounded variation

We say that $u \in SBV(A, \mathbb{R}^m)$ if $u \in BV(A, \mathbb{R}^m)$ and $D^c u = 0$. The space $SBV(A, \mathbb{R}^m)$ is called the space of *special functions of bounded variation* with values in $\mathbb{R}^m$. Note that if $u \in SBV(A, \mathbb{R}^m)$, then the singular part of Du is concentrated on $S(u)$.

The space SBV is very useful when dealing with variational problems involving volume and surface energies because of the following compactness and lower semicontinuity result due to Ambrosio ([3], [5]).

Theorem 1.1.1. *Let A be open and bounded in $\mathbb{R}^N$, let $m \geq 1$ and let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $SBV(A, \mathbb{R}^m)$. Assume that there exists $q > 1$ and $C \in [0; +\infty[$ such that*

$$\int_A |\nabla u_k|^q \, dx + \mathcal{H}^{N-1}(S(u_k)) + ||u_k||_{L^\infty(A,\mathbb{R}^m)} \leq C$$

for every $k \in \mathbb{N}$. Then there exists a subsequence $(u_{k_h})_{h\in\mathbb{N}}$ and a function $u \in SBV(A, \mathbb{R}^m)$ such that

$$\begin{aligned} u_{k_h} &\to u \quad \textit{strongly in } L^1(A, \mathbb{R}^m), \\ \nabla u_{k_h} &\rightharpoonup \nabla u \quad \textit{weakly in } L^1(A; \mathbb{R}^{m\times N}), \\ \mathcal{H}^{N-1}(S(u)) &\leq \liminf_{h\to+\infty} \mathcal{H}^{N-1}(S(u_{k_h})). \end{aligned} \tag{1.3}$$

Moreover if $\mathcal{H}^{N-1} \llcorner S(u_k) \stackrel{*}{\rightharpoonup} \mu$ weakly in the sense of measures, then $\mathcal{H}^{N-1} \llcorner S(u) \leq \mu$ as measures.*

We will often use the following notion of convergence in $SBV(A, \mathbb{R}^m)$

Definition 1.1.2. *Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $SBV(A, \mathbb{R}^m)$. We say that $u_k \rightharpoonup u$ weakly in $SBV(A, \mathbb{R}^m)$ if u_k and u satisfy* (1.3).

1.1.2 The space $GSBV$ of generalized special function of bounded variation

The space $GSBV(A, \mathbb{R}^m)$ is defined as the set of functions $u : A \to \mathbb{R}^m$ such that $\varphi(u) \in SBV_{loc}(A)$ for every $\varphi \in C^1(\mathbb{R}^m)$ such that the support of $\nabla\varphi$ has compact closure in $\mathbb{R}^m$. If $p \in]1, +\infty[$, we set

$$GSBV^p(A, \mathbb{R}^m) := \{u \in GSBV(A, \mathbb{R}^m) \, : \, \nabla u \in L^p(A, \mathcal{M}^{m\times n}), \, \mathcal{H}^{n-1}(S(u)) < +\infty\}.$$

By [44, Proposition 2.2] the space $GSBV^p(A, \mathbb{R}^m)$ coincide with $(GSBV^p(A, \mathbb{R}))^m$, that is $u := (u_1, \ldots, u_m) \in GSBV^p(A, \mathbb{R}^m)$ if and only if $u_i \in GSBV^p(A, \mathbb{R})$ for every $i = 1, \ldots, m$.

The following compactness and lower semicontinuity result will be used in the following sections. For a proof, we refer to [4].

Theorem 1.1.3. *Let A be an open and bounded subset of $\mathbb{R}^N$. Let $g(x, u) : A \times \mathbb{R}^m \to [0, \infty]$ be a Borel function, lower semicontinuous in u and satisfying the condition*

$$\lim_{|u|\to\infty} g(x, u) = +\infty \textit{ for a.e. } x \in A.$$

Let $(u_k)_{k\in\mathbb{N}}$ *be a sequence in* $GSBV^p(A;\mathbb{R}^m)$ *such that*

$$\sup_k \int_A |\nabla u_k(x)|^p\,dx + \mathcal{H}^{N-1}(S(u_k)) + \int_A g(x,u_k(x))\,dx < +\infty.$$

Then there exists a subsequence $(u_{k_h})_{h\in\mathbb{N}}$ *and a function* $u\in GSBV^p(A;\mathbb{R}^m)$ *such that*

$$\begin{aligned} u_{k_h} &\to u && \text{in measure,} \\ \nabla u_{k_h} &\rightharpoonup \nabla u && \text{weakly in } L^p(A;M^{m\times N}). \end{aligned} \tag{1.4}$$

Moreover we have that

$$\mathcal{H}^{N-1}(S(u)) \le \liminf_h \mathcal{H}^{N-1}(S(u_{k_h})).$$

1.2 Γ-convergence

Let us recall the definition and some basic properties of De Giorgi's Γ-*convergence* in metric spaces. We refer the reader to [43] for an exhaustive treatment of this subject. Let (X,d) be a metric space. We say that a sequence $F_h: X\to[-\infty,+\infty]$ Γ-converges to $F: X\to[-\infty,+\infty]$ (as $h\to+\infty$) if for all $u\in X$ we have

(i) (Γ-*liminf inequality*) for every sequence $(u_h)_{h\in\mathbb{N}}$ converging to u in X,

$$\liminf_{h\to+\infty} F_h(u_h) \ge F(u);$$

(ii) (Γ-*limsup inequality*) there exists a sequence $(u_h)_{h\in\mathbb{N}}$ converging to u in X, such that

$$\limsup_{h\to+\infty} F_h(u_h) \le F(u).$$

The function F is called the Γ-limit of $(F_h)_{h\in\mathbb{N}}$ (with respect to d). Γ-convergence is a convergence of variational type as explained in the following proposition.

Proposition 1.2.1. *Assume that the sequence* $(F_h)_{h\in\mathbb{N}}$ Γ*-converges to* F *and that there exists a compact set* $K\subseteq X$ *such that for all* $h\in\mathbb{N}$

$$\inf_{u\in K} F_h(u) = \inf_{u\in X} F_h(u).$$

Then F *admits a minimum on* X, $\inf_X F_h \to \min_X F$, *and any limit point of any sequence* $(u_h)_{h\in\mathbb{N}}$ *such that*

$$\lim_{h\to+\infty}\Big(F_h(u_h) - \inf_{u\in X} F_h(u)\Big) = 0,$$

is a minimizer of F.

Moreover the following compactness result holds.

Proposition 1.2.2. *If* (X,d) *is separable, and* $(F_h)_{h\in\mathbb{N}}$ *is a sequence of functionals on* X*, then there exists a subsequence* $(F_{h_k})_{k\in\mathbb{N}}$ *and a function* $F: X\to[-\infty;+\infty]$ *such that* $(F_{h_k})_{k\in\mathbb{N}}$ Γ*-converges to* F.

1.3 Hausdorff convergence of compact sets

Let Ω be an open bounded set in $\mathbb{R}^N$. We indicate the set of all compact subsets of $\overline{\Omega}$ by $\mathcal{K}(\overline{\Omega})$, $\mathcal{K}(\overline{\Omega})$ can be endowed by the Hausdorff metric d_H defined by

$$d_H(K_1,K_2) := \max\left\{\sup_{x\in K_1}\operatorname{dist}(x,K_2),\ \sup_{y\in K_2}\operatorname{dist}(y,K_1)\right\}$$

with the conventions $\operatorname{dist}(x,\emptyset)=\operatorname{diam}(\Omega)$ and $\sup\emptyset = 0$, so that $d_H(\emptyset,K)=0$ if $K=\emptyset$ and $d_H(\emptyset,K)=\operatorname{diam}(\Omega)$ if $K\neq\emptyset$. It turns out that $\mathcal{K}(\overline{\Omega})$ endowed with the Hausdorff metric is a compact space (see e.g. [81]).

1.4 The model by Francfort and Marigo for quasistatic evolution in brittle fracture: the mathematical results

For a brief description of the variational model for quasistatic crack evolution proposed by Francfort and Marigo in [54], we refer to the Introduction.

In the rest of the section we state precisely the mathematical formulations of the problem given by Dal Maso and Toader in [45], and by Francfort and Larsen in [53].

1.4.1 The result by Dal Maso and Toader

Let $\Omega \subseteq \mathbb{R}^2$ be open and bounded. Dal Maso and Toader treat the case of *antiplanar shear*, i.e. they consider an elastic body of the form $\Omega \times \mathbb{R}$ and assume that the cracks are of the form $K \times \mathbb{R}$ with $K \subseteq \overline{\Omega}$, and that the displacements $u : \Omega \times \mathbb{R} \to \mathbb{R}^3$ depend only on x_1, x_2, and are of the form $u(x_1, x_2) = (0, 0, u_3(x_1, x_2))$.

Referring to the cross section Ω, they consider as family of admissible cracks the set

$$\mathcal{K}^f_m(\overline{\Omega}) := \{K \subseteq \overline{\Omega},\ K \text{ has at most } m \text{ connected components}, \mathcal{H}^1(K) < +\infty\}.$$

The family of admissible displacement associated to a crack K is given by

$$L^{1,2}(\Omega \setminus K) := \{u \in H^1_{loc}(\Omega \setminus K), \nabla u \in L^2(\Omega \setminus K)\}.$$

In the model case, Dal Maso and Toader consider a total energy of the form

$$\int_\Omega |\nabla u(x)|^2\, dx + \mathcal{H}^1(K).$$

Notice that if $u \in L^{1,2}(\Omega \setminus K)$ with $K \in \mathcal{K}^f_m(\overline{\Omega})$, then ∇u is defined almost everywhere in Ω, so that we can see ∇u as an element of $L^2(\Omega)$ and compute its norm. The quantity $\|\nabla u\|^2$ is referred to as *bulk energy* of u. The quantity $\mathcal{H}^1(K)$ is referred to as *surface energy* of K.

The admissible boundary displacements on $\partial_D\Omega \subseteq \partial\Omega$ open in the relative topology and with a finite number of connected components, are given by the traces of the functions in the Sobolev space $H^1(\Omega)$. The displacement $u_{g,K}$ of the elastic body relative to g and the crack K is obtained minimizing $\|\nabla u\|^2$ under the condition $u = g$ on $\partial_D\Omega \setminus K$ (which is intended in the sense of traces). It turns out that u is determined up to a constant in the connected components of $\Omega \setminus K$ which do not touch $\partial_D\Omega$. The *total energy* associated to (g, K) is given by

$$\mathcal{E}(g, K) := \|\nabla u_{g,K}\|^2 + \mathcal{H}^1(K).$$

Let $g \in AC([0,T]; H^1(\Omega))$, i.e. let g be absolutely continuous from $[0,T]$ to $H^1(\Omega)$ (see [22] for a definition). The main result of [45] is the following theorem.

Theorem 1.4.1. *There exists a map $t \to K(t)$ from $[0,T]$ to $\mathcal{K}^f_m(\overline{\Omega})$ such that the following facts hold*

(a) $K(t_1) \subseteq K(t_2)$ *for* $0 \le t_1 \le t_2 \le T$;

(b) $K(0)$ *is such that for every* $K \in \mathcal{K}^f_m(\overline{\Omega})$

$$\mathcal{E}(g(0), K(0)) \le \mathcal{E}(g(0), K),$$

while for every $t \in]0,T]$ *we have that for every* $K \in \mathcal{K}^f_m(\overline{\Omega})$ *with* $K(t) \subseteq K$

$$\mathcal{E}(g(t), K(t)) \le \mathcal{E}(g(t), K).$$

(c) $t \to \mathcal{E}(g(t), K(t))$ *is absolutely continuous and*

$$\frac{d}{dt}\mathcal{E}(g(t), K(t)) = \int_\Omega \nabla u(t) \nabla \dot{g}(t)\, dx,$$

where $\dot{g}$ denotes the time derivative of g.

1.4.2 The result by Francfort and Larsen

Francfort and Larsen [53] consider the case of generalized antiplanar shear, that is they consider $\Omega \subseteq \mathbb{R}^N$. The family of admissible cracks is given by the rectifiable sets in $\overline{\Omega}$, while the family of admissible displacements is given by the functions in $SBV(\Omega)$.

If $\partial_D\Omega \subseteq \partial\Omega$, the class of admissible prescribed displacements on $\partial_D\Omega$ is given by the traces of functions in $H^1(\Omega)$. Given a boundary displacement g and a crack Γ, a displacement u is associated to g and Γ if

$$S(u) \cup \{x \in \partial_D\Omega \,:\, u(x) \neq g(x)\} \,\tilde{\subseteq}\, \Gamma,$$

where $A \,\tilde{\subseteq}\, B$ means that $A \subseteq B$ up to a set of $\mathcal{H}^{N-1}$-measure zero, and the inequality on $\partial_D\Omega$ is intended in the sense of traces.

Let $g : [0,1] \to H^1(\Omega)$ be absolutely continuous. The main result of [53] is the following theorem.

Theorem 1.4.2. *There exists a crack $\Gamma(t) \subseteq \overline{\Omega}$ and a field $u(t) \in SBV(\Omega)$ such that*

(a) $\Gamma(t)$ increases with t;

(b) $u(0)$ minimizes

$$\int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}(S(v) \cup \{x \in \partial_D\Omega : v(x) \neq g(0)(x)\})$$

among all $v \in SBV(\Omega)$ (inequalities on $\partial_D\Omega$ are intended for the traces of v and g);

(c) for $t > 0$, $u(t)$ minimizes

$$\int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{N-1}\left([S(v) \cup \{x \in \partial_D\Omega : v(x) \neq g(t)(x)\}] \setminus \Gamma(t)\right)$$

among all $v \in SBV(\Omega)$;

(d) $S(u(t)) \cup \{x \in \partial_D\Omega \,:\, u(t)(x) \neq g(t)(x)\} \subseteq \Gamma(t)$, up to a set of $\mathcal{H}^{N-1}$–measure 0.

Furthermore, the total energy

$$\mathcal{E}(t) := \int_\Omega |\nabla u(t)|^2\, dx + \mathcal{H}^{N-1}(\Gamma(t))$$

is absolutely continuous and is given by

$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau)\nabla \dot{g}(\tau)\, dx\, d\tau.$$

Finally, for any countable, dense set $I \subseteq [0,1]$, the crack $\Gamma(t)$ and the field $u(t)$ can be chosen such that

$$\Gamma(t) = \bigcup_{\tau \in I, \tau \le t} (S(u(\tau)) \cup \{x \in \partial_D\Omega \,:\, u(\tau)(x) \neq g(\tau)(x)\})$$

The main tool in the proof of Theorem 1.4.2 is the Transfer of Jump Sets [53, Theorem 2.1].

Theorem 1.4.3 (Transfer of Jump Sets). *Let $\overline{\Omega} \subseteq \Omega'$, with $\partial\Omega$ Lipschitz, and let for $r = 1, \dots, i$ (u_n^r) be a sequence in $SBV(\Omega')$ such that*

(1) $S(u_n^r) \subseteq \overline{\Omega}$;

(2) $|\nabla u_n^r|$ weakly converges in $L^1(\Omega')$; and

(3) $u_n^r \to u^r$ strongly in $L^1(\Omega')$,

where $u^r \in BV(\Omega')$ *with* $\mathcal{H}^{N-1}(S(u^r)) < \infty$. *Then for every* $\phi \in SBV(\Omega')$ *with* $\mathcal{H}^{N-1}(S(\phi)) < \infty$ *and* $\nabla\phi \in L^q(\Omega; \mathbb{R}^N)$ *for some* $q \in [1, +\infty[$, *there exists a sequence* (ϕ_n) *in* $SBV(\Omega')$ *with* $\phi_n \equiv \phi$ *on* $\Omega' \setminus \overline{\Omega}$ *such that*

(a) $\phi_n \to \phi$ *strongly in* $L^1(\Omega')$;

(b) $\nabla\phi_n \to \nabla\phi$ *strongly in* $L^q(\Omega')$; *and*

(c) $\mathcal{H}^{N-1}\left([S(\phi_n) \setminus \bigcup_{r=1}^{i} S(u_n^r)] \setminus [S(\phi) \setminus \bigcup_{r=1}^{i} S(u^r)]\right) \to 0.$

In particular

$$\limsup_n \mathcal{H}^{N-1}\left(S(\phi_n) \setminus \bigcup_{r=1}^{i} S(u_n^r)\right) \leq \mathcal{H}^{N-1}\left(S(\phi) \setminus \bigcup_{r=1}^{i} S(u^r)\right). \tag{1.5}$$

Chapter 2

A generalization of Gołąb's theorem and applications to fracture mechanics

In this chapter [1] we extend the result of Dal Maso and Toader [45] to *anisotropic linearly elastic inhomogeneous bodies* subjected to *anti-planar* or *planar shear*. *Anisotropy* will be considered both in the *bulk* and in the *surface energy*.

The class of admissible cracks is given by compact sets with at most m connected components and finite length. The *surface energy* of a crack K is given by

$$\int_K \varphi(x,\nu_x)\,d\mathcal{H}^1(x),$$

where ν_x is the *unit normal vector* at x to K and $\varphi : \overline{\Omega}\times\mathbb{R}^2 \to [0,\infty[$ is a continuous function, positively 1-homogeneous, even and convex in the second variable. Notice that the integral is well defined: in fact, even if K is not in general the union of m regular curves, it turns out that it is possible to define at $\mathcal{H}^1$-a.e. point $x\in K$ an *approximate unit normal vector* ν_x completely determined up to the sign. In the case K is regular, ν_x coincides with the usual normal vector.

In order to deal with an *anisotropic* and *inhomogeneous* surface energy, the main step is to prove a lower semicontinuity theorem for the functional

$$\mathcal{F}(K) := \int_K \varphi(x,\nu_x)\,d\mathcal{H}^1$$

along Hausdorff converging sequences. Notice that for $\varphi \equiv 1$, the semicontinuity result reduces to the celebrated Gołąb's theorem on the lower semicontinuity of $\mathcal{H}^1$ measure under Hausdorff convergence.

The chapter is organized as follows. After recalling some preliminary results, we prove the lower semicontinuity theorem in Section 2.3. In Sections 2.4 and 2.5, we deal with the study of quasi-static growth of brittle cracks in the anti-planar and planar cases. Using shape continuity results proved in [25] and [33], we can treat inhomogeneous bulk energies.

2.1 Preliminary results

In what follows, $\Omega \subseteq \mathbb{R}^2$ is a bounded open set with Lipschitz boundary, $\partial_D\Omega$ is a subset of $\partial\Omega$ open in the relative topology and with a finite number of connected components.

[1] The results of this chapter are contained in the paper:
Giacomini A.: A generalization of Gołab theorem and applications to fracture mechanics. *Math. Models Methods Appl. Sci.* **12** (2002) 1245-1267.

Sets with finite perimeter. We indicate the perimeter of E in Ω by $\mathrm{P}(E,\Omega)$. Let E be a set of finite perimeter in Ω; the reduced boundary ∂^*E and the approximate inner normal ν at points of ∂^*E are defined such that the following identity holds:

$$\forall g \in C_c(\Omega,\mathbb{R}^2) \quad -\int_E \operatorname{div} g \, d\mathcal{L}^2 = \int_{\partial^*E} g\cdot\nu \, d\mathcal{H}^1.$$

Set $\mu_E = \nu\mathcal{H}^1 \llcorner \partial^*E$. For all $x \in \partial^*E$, indicated the map $\xi \to \frac{1}{\lambda}(\xi - x)$ by D_λ, the following *blow up property* holds: for $\lambda \to 0^+$

$$\mu_{D_\lambda(E)} \overset{*}{\rightharpoonup}_{loc} \mathcal{H}^1 \llcorner T_\nu,$$

locally weakly star in the sense of measures, where T_ν is the subspace orthogonal to ν.

We say that a sequence (E_h) of subset of Ω converges in $L^1_{\mathrm{loc}}(\Omega)$ to E, if the corresponding characteristic functions χ_{E_h} converge in $L^1_{\mathrm{loc}}(\Omega)$ to χ_E. If there exists $C \geq 0$ such that $\mathrm{P}(E_h,\Omega) \leq C$ for all h and $E_h \to E$ in $L^1_{\mathrm{loc}}(\Omega)$, then E has finite perimeter in Ω and $\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E$ in the weak star topology of $\mathcal{M}_b(\Omega,\mathbb{R}^2)$. For further details on sets of finite perimeter, the reader is referred to [8].

Structure of compact connected sets with finite $\mathcal{H}^1$ measure. It can be proved (see e.g. [50]) that if K is compact and connected in $\overline{\Omega}$ for a.e. $x \in K$ there exists an approximate unit normal vector ν_x which is characterized by

$$\mu_{D_\lambda(E)} \overset{*}{\rightharpoonup}_{loc} \mathcal{H}^1 \llcorner T_{\nu_x} \quad \text{for } \lambda \to 0^+ \tag{2.1}$$

locally weakly star in the sense of measures, where T_{ν_x} is the subspace of $\mathbb{R}^2$ orthogonal to ν_x. Moreover the map $x \to \nu_x$ is Borel measurable, so that for every continuous function $\varphi : \overline{\Omega}\times\mathbb{R}^2 \to [0,\infty[$ even in the second variable the integral

$$\int_K \varphi(x,\nu)\, d\mathcal{H}^1$$

is well defined. Clearly the functional is well defined also for $K \in \mathcal{K}^f_m(\overline{\Omega})$ with $m \geq 1$.

In section 2.3 we will be concerned in the problem of the lower semicontinuity of the function $K \to \int_K \varphi(x,\nu)\, d\mathcal{H}^1$ under the Hausdorff convergence.

We will use the fact that a connected set C with finite $\mathcal{H}^1$ measure is arcwise connected and moreover $\mathcal{H}^1(\overline{C}) = \mathcal{H}^1(C)$: see e.g. [45].

Reshetnyak's theorems on measures. The following theorem gives a lower semicontinuity result for functionals defined on measures; for a proof, the reader is referred to [8]. If μ is a measure, let $|\mu|$ be its total variation and let $\frac{d\mu}{d|\mu|}$ be the Radon-Nicodym derivative of μ with respect to $|\mu|$.

Theorem 2.1.1. *Let Ω be an open subset of $\mathbb{R}^n$ and μ, μ_k be $\mathbb{R}^m$-valued finite Radon measures in Ω; if $\mu_h \to \mu$ weakly star in $\mathcal{M}_b(\Omega,\mathbb{R}^m)$ then*

$$\int_\Omega f\left(x, \frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x) \leq \liminf_{h\to\infty} \int_\Omega f\left(x, \frac{d\mu_h}{d|\mu_h|}(x)\right) d|\mu_h|(x)$$

for every lower semicontinuous function $f : \Omega \times \mathbb{R}^m \to [0,+\infty]$, positively 1-homogeneous and convex in the second variable.

We say that μ_n converges strictly to μ in $\mathcal{M}_b(\Omega,\mathbb{R}^m)$ if $\mu_n \to \mu$ weakly star and $|\mu_n|(\Omega) \to |\mu|(\Omega)$. The following theorem gives a continuity result for functional defined on measures: for a proof see [8].

Theorem 2.1.2. *Let Ω be an open subset of $\mathbb{R}^n$ and μ, μ_k be $\mathbb{R}^m$-valued finite Radon measures in Ω; if $\mu_h \to \mu$ strictly in $\mathcal{M}_b(\Omega, \mathbb{R}^m)$ then*

$$\lim_{h\to\infty} \int_\Omega f\left(x, \frac{d\mu_h}{d|\mu_h|}(x)\right) d|\mu_h|(x) = \int_\Omega f\left(x, \frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x)$$

for every continuous and bounded function $f : \Omega \times S^{m-1} \to \mathbb{R}$.

Deny-Lions spaces. If A is an open subset of $\mathbb{R}^2$, the Deny-Lions space $L^{1,2}(A)$ is defined as

$$L^{1,2}(A) := \left\{u \in W^{1,2}_{\mathrm{loc}}(A) \, : \, \nabla u \in L^2(A, \mathbb{R}^2)\right\}. \tag{2.2}$$

In the case in which A is regular $L^{1,2}(A)$ coincides with the usual Sobolev space while if it is irregular, it can be strictly larger. In what follows, given $K \subseteq \overline{\Omega}$ compact and $u \in L^{1,2}(\Omega \setminus K)$, we extend ∇u to 0 on K, so that $\nabla u \in L^2(\Omega, \mathbb{R}^2)$ although ∇u is the distributional derivative of u only on $\Omega \setminus K$. The following theorem proved in [25] will be used in Section 2.4.

Theorem 2.1.3. *Let $m \geq 1$, K_n a sequence of compact sets in $\overline{\Omega}$ with at most m connected components converging to K in the Hausdorff metric and such that $\mathcal{L}^2(\Omega \setminus K_n) \to \mathcal{L}^2(\Omega \setminus K)$. Then for every $u \in L^{1,2}(\Omega \setminus K)$, there exists $u_n \in L^{1,2}(\Omega \setminus K_n)$ such that $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^2)$.*

Consider now for A open subset of $\mathbb{R}^2$

$$LD^{1,2}(A) := \left\{u \in W^{1,2}_{\mathrm{loc}}(A; \mathbb{R}^2) \, : \, E(u) \in L^2(A, \mathrm{M}^{n\times n}_{\mathrm{sym}})\right\}, \tag{2.3}$$

where $Eu := \frac{1}{2}(\nabla u + (\nabla u)^t)$ is the symmetric part of the gradient of u and $\mathrm{M}^{n\times n}_{\mathrm{sym}}$ is the space of 2×2 symmetric matrices endowed with the standard scalar product $B_1{:}B_2 := tr(B_1^t B_2)$ and the corresponding norm $|B| := (B{:}B)^{\frac{1}{2}}$.

In what follows, given $K \subseteq \overline{\Omega}$ compact and $u \in LD^{1,2}(\Omega \setminus K)$, we extend Eu to 0 on K although it coincides with the symmetric part of the distributional gradient of u only on $\Omega \setminus K$. The following result, which can be obtained combining the density result proved in [33] and Theorem 2.1.3, will be used in Section 2.5.

Theorem 2.1.4. *Let $m \geq 1$, K_n a sequence of compact sets in $\overline{\Omega}$ with at most m connected components converging to K in the Hausdorff metric and such that $\mathcal{L}^2(\Omega \setminus K_n) \to \mathcal{L}^2(\Omega \setminus K)$. Then for every $u \in LD^{1,2}(\Omega \setminus K)$, there exists a sequence $u_n \in LD^{1,2}(\Omega \setminus K_n)$ such that $Eu_n \to Eu$ strongly in $L^2(\Omega; \mathrm{M}^{n\times n}_{\mathrm{sym}})$.*

2.2 The main theorems

Let $\Omega \subseteq \mathbb{R}^2$ be open, bounded and with Lipschitz boundary. For every $m \geq 1$ let $\mathcal{K}_m(\overline{\Omega})$ be the family of compact subsets of $\overline{\Omega}$ which have less than m connected components. Let

$$\mathcal{K}^f_m(\overline{\Omega}) := \{K \in \mathcal{K}_m(\overline{\Omega}) \, : \, \mathcal{H}^1(K) < +\infty\}.$$

Let $\varphi : \overline{\Omega} \times \mathbb{R}^2 \to [0, +\infty[$ a continuous function, positively 1-homogeneous, even and convex in the second variable such that for $c_1, c_2 > 0$

$$\forall (x, \nu) \in \overline{\Omega} \times \mathbb{R}^2 \, : \, c_1|\nu| \leq \varphi(x, \nu) \leq c_2|\nu|. \tag{2.4}$$

The main result of the chapter is the following lower semicontinuity theorem.

Theorem 2.2.1. *The functional*

$$\begin{array}{rcl} \mathcal{F} : \mathcal{K}_m^f(\overline{\Omega}) & \longrightarrow & [0,\infty[\\ K & \mapsto & \int_K \varphi(x,\nu)\, d\mathcal{H}^1 \end{array}$$

is lower semicontinuous if $\mathcal{K}_m^f(\overline{\Omega})$ is endowed with the Hausdorff metric.

The previous theorem will be used to deal with the problem of evolution of brittle cracks in linearly elastic bodies.

Let $a \in L^\infty(\Omega, \mathrm{M}^{n\times n}_{\mathrm{sym}})$ such that for $\alpha_1, \alpha_2 > 0$

$$\forall x \in \Omega\,, \forall \xi \in \mathbb{R}^2 \,:\, \alpha_1|\xi|^2 \le a(x)\xi\cdot\xi \le \alpha_2|\xi|^2. \tag{2.5}$$

Let $(\cdot,\cdot)_a$ denote the associated scalar product on $L^2(\Omega,\mathbb{R}^2)$ defined as

$$(v,w)_a = \int_\Omega \sum_{i,j=1}^2 a(x) v_i(x) w_j(x)\, d\mathcal{L}^2(x)$$

and let $||\cdot||_a$ be the relative norm.

For every $g \in H^1(\Omega)$ and $K \in \mathcal{K}_m^f(\overline{\Omega})$, we set

$$\mathcal{E}(g,K) := \min_{v\in\Gamma(g,K)} \left\{ \int_{\Omega\setminus K} a(x)\nabla v\cdot\nabla v\, d\mathcal{L}^2 + \int_K \varphi(x,\nu)\, d\mathcal{H}^1 \right\}, \tag{2.6}$$

where

$$\Gamma(g,K) := \left\{ u \in L^{1,2}(\Omega\setminus K)\,,\, u = g \,\mathrm{on}\, \partial_D\Omega \setminus K \right\}. \tag{2.7}$$

The following theorem states the existence of a quasi-static evolution for brittle cracks in linear elastic bodies under anti-planar displacement: note that both the bulk and the surface energy depend in a possibly inhomogeneous way on the anisotropy of the body.

Theorem 2.2.2. *Let $m \ge 1$, $g \in AC([0,1], H^1(\Omega))$, $K_0 \in \mathcal{K}_m^f(\overline{\Omega})$. Then there exists a function $K : [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ such that, letting $u(t)$ be a solution of the minimum problem (2.6) which defines $\mathcal{E}(g(t), K(t))$ for all $t \in [0,1]$,*

(a) $$K_0 \subseteq K(s) \subseteq K(t) \text{ for } 0 \le s \le t \le 1;$$

(b) $$\mathcal{E}(g(0),K(0)) \le \mathcal{E}(g(0),K) \quad \forall K \in \mathcal{K}_m^f(\overline{\Omega}),\ K_0 \subseteq K;$$

(c) $$\forall t \in]0,1] : \mathcal{E}(g(t),K(t)) \le \mathcal{E}(g(t),K) \quad \forall K \in \mathcal{K}_m^f(\overline{\Omega}),\ \cup_{s<t} K(s) \subseteq K;$$

(d) $$t \mapsto \mathcal{E}(g(t),K(t)) \text{ is absolutely continuous on } [0,1];$$

(e) $$\frac{d}{dt}\mathcal{E}(g(t),K(t)) = 2(\nabla u(t), \nabla\dot{g}(t))_a \quad \text{for a.e. } t \in [0,1],$$

(f) $$\frac{d}{ds}\mathcal{E}(g(t),K(s))|_{s=t} = 0 \quad \text{for a.e. } t \in [0,1].$$

Let $\mathcal{L}(\mathrm{M}^{n\times n}_{\mathrm{sym}})$ be the space of automorphism of $\mathrm{M}^{n\times n}_{\mathrm{sym}}$ and let $A \in L^\infty(\Omega, \mathcal{L}(\mathrm{M}^{n\times n}_{\mathrm{sym}}))$ such that there exist $\alpha_1, \alpha_2 > 0$ with

$$\forall x \in \Omega \,:\, \alpha_1|M|^2 \le A(x)M{:}M \le \alpha_2|M|^2.$$

Let us pose $(Eu,Ev)_A := \int_{\Omega\setminus K} A(x)Eu{:}Ev\, d\mathcal{L}^2$ and $||Eu||_A := (Eu,Eu)_A^{\frac{1}{2}}$.

For every $g \in H^1(\Omega;\mathbb{R}^2)$ and $K \in \mathcal{K}^f_m(\overline{\Omega})$, set

$$\mathcal{G}(g,K) = \min_{v \in \mathcal{V}(g,K)} \left\{ \int_{\Omega \setminus K} A(x)Eu{:}Eu\, d\mathcal{L}^2 + \int_K \varphi(x,\nu)\, d\mathcal{H}^1 \right\}, \tag{2.8}$$

where

$$\mathcal{V}(g,K) = \{u \in LD^{1,2}(\Omega \setminus K)\,,\, u = g \,\mathrm{on}\, \partial_D \Omega \setminus K\}. \tag{2.9}$$

The following theorem states the existence of a quasi-static evolution for brittle cracks in inhomogeneous anisotropic linearly elastic bodies under planar displacement.

Theorem 2.2.3. *Let $m \geq 1$, $g \in AC([0,1], H^1(\Omega;\mathbb{R}^2))$, $K_0 \in \mathcal{K}^f_m(\overline{\Omega})$. Then there exists a function $K : [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ such that, letting $u(t)$ be a solution of the minimum problem (2.8) which defines $\mathcal{G}(g(t),K(t))$ for all $t \in [0,1]$,*

(a) $$K_0 \subseteq K(s) \subseteq K(t) \text{ for } 0 \leq s \leq t \leq 1;$$

(b) $$\mathcal{G}(g(0),K(0)) \leq \mathcal{G}(g(0),K) \quad \forall K \in \mathcal{K}^f_m(\overline{\Omega}),\, K_0 \subseteq K;$$

(c) $$\forall t \in]0,1] : \mathcal{G}(g(t),K(t)) \leq \mathcal{G}(g(t),K) \quad \forall K \in \mathcal{K}^f_m(\overline{\Omega}),\, \cup_{s<t} K(s) \subseteq K;$$

(d) $$t \mapsto \mathcal{G}(g(t),K(t)) \text{ is absolutely continuous on } [0,1];$$

(e) $$\frac{d}{dt}\mathcal{G}(g(t),K(t)) = 2(Eu(t), E\dot{g}(t))_A \quad \text{for a.e. } t \in [0,1],$$

(f) $$\frac{d}{ds}\mathcal{G}(g(t),K(s))|_{s=t} = 0 \quad \text{for a.e. } t \in [0,1].$$

Remark 2.2.4. It turns out that for every function $K : [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ which satisfies (a)-(d) of Theorem 2.2.2, then conditions (e) and (f) are equivalent. Similarly, for every function $K : [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ which satisfies (a)-(d) of Theorem 2.2.3, conditions (e) and (f) are equivalent.

We will prove theorem 2.2.1 in section 2.3 using a comparison of measures which involves a blow-up technique; theorems 2.2.2 and 2.2.3 will be proved in sections 2.4 and 2.5 respectively: a discretization in time procedure will be employed and, in the particular case in which $g(0) = 0$, we prove that this method gives an approximation of the total energy of the solution.

2.3 A generalization of Gołąb theorem

Throughout this section we employ the notations introduced in Section 2.2. Let $\mathcal{C}$ be the subset of $L^1(\Omega)$ composed by characteristic functions of sets with finite perimeter in Ω.

Theorem 2.3.1. *Consider the functional $\mathcal{G} : \mathcal{C} \to [0,\infty[$ defined by*

$$\mathcal{G}(E) = \int_{\partial^* E} \varphi(x,\nu)\, d\mathcal{H}^1$$

where ν denotes the inner normal of E. Then $\mathcal{G}$ is lower semicontinuous with respect to the L^1 topology.

Proof. Let (E_h) be a sequence of sets with finite perimeter in Ω with $E_h \to E$ in $L^1(\Omega)$: it is sufficient to consider the case $\mathrm{P}(E_h,\Omega) \leq C$ for some $C \geq 0$ independent of h. As noted in Section 2.1, $\mu_{E_h} \stackrel{*}{\rightharpoonup} \mu_E$ in the weak star topology of $\mathcal{M}_b(\Omega,\mathbb{R}^2)$. Since the inner normal to E_h (resp. E) is given by $\frac{d\mu_{E_h}}{d\mathcal{H}^1}$ (resp. by $\frac{d\mu_E}{d\mathcal{H}^1}$), we can use Reshetnyak lower semicontinuity theorem (see Section 2.1) to get the conclusion. $\square$

Theorem 2.3.2. *Let U be an open subset of $\mathbb{R}^2$. The functional*

$$\mathcal{F} : \mathcal{K}_m^f(\overline{\Omega}) \longrightarrow [0,\infty[$$
$$K \mapsto \int_{K\cap U} \varphi(x,\nu)\, d\mathcal{H}^1$$

is lower semicontinuous if $\mathcal{K}_m^f(\overline{\Omega})$ is endowed with the Hausdorff metric.

Proof. We consider preliminarily the case $m = 1$.

Let $K_n, K \in \mathcal{K}_1^f(\overline{\Omega})$ with $K_n \to K$ in the Hausdorff metric: our aim is to verify that

$$\int_{K\cap U} \varphi(x,\nu)\, d\mathcal{H}^1 \le \liminf_n \int_{K_n\cap U} \varphi(x,\nu)\, d\mathcal{H}^1.$$

Without loss of generality we may consider sequences (K_n) such that

$$\sup_n \int_{K_n\cap U} \varphi(x,\nu)\, d\mathcal{H}^1 < +\infty.$$

Let us consider the positive measures μ_n, μ in $\mathcal{M}_b(U)$

$$\mu_n(B) = \int_{K_n\cap B} \varphi(x,\nu)\, d\mathcal{H}^1,$$

$$\mu(B) = \int_{K\cap B} \varphi(x,\nu)\, d\mathcal{H}^1.$$

By (2.4), (μ_n) is bounded in $\mathcal{M}_b(U)$ and so up to a subsequence it converges in the weak-star topology of $\mathcal{M}_b(U)$ to a measure μ_0 whose support is contained in $K\cap U$. By weak convergence we have

$$\mu_0(U) \le \liminf_n \mu_n(U),$$

and so it is sufficient to prove that

$$\mu(U) \le \mu_0(U). \tag{2.10}$$

We prove instead that $\mu \le \mu_0$ using a density argument which requires a blow-up technique: we obtain (2.10) as a consequence.

Firstly consider $K_n \in \mathcal{K}_1^f(\overline{B}_1(0))$, $\mathcal{H}^1(K_n) \le C$ for some $C \ge 0$ and $K_n \to K$ in the Hausdorff metric where K is the diameter connecting the points $e_1 := (1,0)$ and $-e_1$. Note that for every strip $S_\eta = \{x \in \mathbb{R}^2 : -\eta \le x_2 \le \eta\}$ with $\eta > 0$, $K_n \subseteq S_\eta$ and $K_n \cap \partial S_\eta = \emptyset$ for n large enough. Given $\varepsilon > 0$, let $V^\varepsilon := \{x \in \mathbb{R}^2 : -1+\varepsilon \le x_1 \le 1-\varepsilon\}$, $\partial^\pm V^\varepsilon$ the connected components of ∂V^ε containing the points $(1-\varepsilon)e_1$ and $-(1-\varepsilon)e_1$ respectively. For n large enough, since K_n is connected, there exist points $x_n^\pm \in \partial^\pm V^\varepsilon \cap K_n$ such that $x_n^\pm \to \pm(1-\varepsilon)e_1$. Let L_n be the union of the segments connecting x_n^- to $-(1-\varepsilon)e_1$ and $-(1-\varepsilon)e_1$ to $-e_1$, x_n^+ to $(1-\varepsilon)e_1$ and $(1-\varepsilon)e_1$ to e_1. Note that $H_n := K_n \cup L_n$ is connected and that

$$\mathcal{H}^1(L_n) \le 3\varepsilon$$

for n large enough.

Let E_n be the connected component of $B_1(0) \setminus H_n$ containing $\frac{1}{2}e_2$, where $e_2 := (0,1)$. As $\pm e_1 \in H_n$ and H_n converges to K in the Hausdorff metric, it is easy to see that E_n converges in L^1 to $B_1^+(0) := \{x \in B_1(0) : x_2 > 0\}$. E_n has finite perimeter because $\partial E_n \subseteq H_n$ and these sets have finite $\mathcal{H}^1$ measure (see Proposition 3.62 of [8]). By Theorem 2.3.1 we have

$$\begin{aligned}\int_K \varphi(x,\nu)\,d\mathcal{H}^1 = \int_{\partial^* B_1^+(0)} \varphi(x,\nu)\,d\mathcal{H}^1 &\leq \liminf_n \int_{\partial^* E_n} \varphi(x,\nu)\,d\mathcal{H}^1 \leq \\ &\leq \liminf_n \int_{H_n} \varphi(x,\nu)\,d\mathcal{H}^1 \leq \\ &\leq \liminf_n \int_{K_n} \varphi(x,\nu)\,d\mathcal{H}^1 + c_2 \limsup_n \mathcal{H}^1(L_n) \leq \\ &\leq \liminf_n \int_{K_n} \varphi(x,\nu)\,d\mathcal{H}^1 + 3c_2\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$, we have

$$\int_K \varphi(x,\nu)\,d\mathcal{H}^1 \leq \liminf_n \int_{K_n} \varphi(x,\nu)\,d\mathcal{H}^1. \tag{2.11}$$

To obtain the thesis, we need to prove that for $\mathcal{H}^1$-almost all points x_0 of $K \cap U$

$$\limsup_{\rho \to 0^+} \frac{\mu(B_\rho(x_0))}{2\rho} \geq \varphi(x_0, \nu_{x_0}) \tag{2.12}$$

where ν_{x_0} indicates the normal to K at x_0: this is sufficient in order to compare μ_0 and μ (see Theorem 2.56 of [8]).

Up to a rotation we may assume that $\nu_{x_0} = e_2$. Let $\rho_k \to 0^+$ and let T_k be the map defined by

$$T_k(\xi) = \frac{1}{\rho_k}(\xi - x_0)$$

which brings the ball $B_{\rho_k}(x_0)$ to the unit ball of the plane. By our choice of x_0, $\mathcal{H}^1 \llcorner T_k(K)$ converges locally weakly star in the sense of measures to $\mathcal{H}^1 \llcorner H$ where H denotes the horizontal axis of the plane.

Note that for $k \to \infty$

$$T_k(K) \cap \overline{B}_1(0) \to H \cap \overline{B}_1(0) \tag{2.13}$$

in the Hausdorff metric. In fact, up to a subsequence, $T_k(K) \cap \overline{B}_1(0) \to \widetilde{K}$ by compactness of the Hausdorff metric. Clearly $H \cap \overline{B}_1(0) \subseteq \widetilde{K}$ because if $y \in (H \cap \overline{B}_1(0)) \setminus \widetilde{K}$, there exists $\rho > 0$ such that $T_k(K) \cap B_\rho(y) = \emptyset$ definitively and so

$$\mathcal{H}^1(H \cap \overline{B}_1(0) \cap B_\rho(y)) \leq \liminf_k \mathcal{H}^1(T_k(K) \cap B_\rho(y)) = 0$$

which is absurd. Conversely, $\widetilde{K} \subseteq H \cap \overline{B}_1(0)$ because if $y \in \widetilde{K} \setminus (H \cap \overline{B}_1(0))$, there exists $\rho > 0$ such that $H \cap \overline{B}_1(0) \cap \overline{B}_\rho(y) = \emptyset$ and by the inequality

$$\limsup_k \mathcal{H}^1(\overline{B}_\rho(y) \cap T_k(K)) \leq \mathcal{H}^1(\overline{B}_\rho(y) \cap H \cap \overline{B}_1(0))$$

we deduce

$$\limsup_k \mathcal{H}^1(\overline{B}_\rho(y) \cap T_k(K)) = 0. \tag{2.14}$$

But we proved that $H \cap \overline{B}_1(0) \subseteq \widetilde{K}$ and so the points of $H \cap \overline{B}_1(0)$ are limit of points of $T_k(K)$: since every $T_k(K)$ is arcwise connected (they are connected and have finite $\mathcal{H}^1$ measure), we have that $\mathcal{H}^1(\overline{B}_\rho(y) \cap T_k(K)) \geq \rho$ definitively and this contradicts (2.14).

We may suppose that ρ_k's are chosen in such a way that

$$\mu(\partial B_{\rho_k}(x_0)) = 0 \qquad \lim_n \mu_n(\overline{B}_{\rho_k}(x_0)) = \mu(B_{\rho_k}(x_0)). \tag{2.15}$$

Since $T_k(K_n) \to T_k(K)$ in the Hausdorff metric for $n \to +\infty$ by (2.13) and (2.15) there exists a subsequence n_k such that

$$T_k(K_{n_k}) \cap \overline{B}_1(0) \to H \cap \overline{B}_1(0) \tag{2.16}$$

in the Hausdorff metric for $k \to +\infty$ and

$$\mu_{n_k}(\overline{B}_{\rho_k}(x_0)) \leq \mu(B_{\rho_k}(x_0)) + \rho_k^2.$$

We now want to use the device of the first part of the proof: we employ the notations introduced before. Let $\varepsilon > 0$, $\eta > 0$, $R^\varepsilon_\eta := S_\eta \cap V^\varepsilon$, $\partial^\pm R^\varepsilon_\eta := R^\varepsilon_\eta \cap \partial^\pm V^\varepsilon$; for k large enough $T_k(K_{n_k}) \cap V^\varepsilon \subseteq R^\varepsilon_\eta$ and if $C^\pm_k$ is the connected component of $(T_k(K_{n_k}) \cap \overline{B}_1(0)) \cup \partial^- R^\varepsilon_\eta \cup \partial^+ R^\varepsilon_\eta$ containing $\partial^\pm R^\varepsilon_\eta$, we have $(T_k(K_{n_k}) \cap \overline{B}_1(0)) \cup \partial^- R^\varepsilon_\eta \cup \partial^+ R^\varepsilon_\eta = C^-_k \cup C^+_k$. In fact if $\xi \notin C^-_k \cup C^+_k$ and C^ξ_k be the connected component of $(T_k(K_{n_k}) \cap \overline{B}_1(0)) \cup \partial^- R^\varepsilon_\eta \cup \partial^+ R^\varepsilon_\eta$ containing ξ, by (2.16), $C^\xi_k \cap \partial R^\varepsilon_\eta = \emptyset$ for k large enough and so C^ξ_k would be connected against the connectedness of $T_k(K_{n_k}) \cup \partial^- R^\varepsilon_\eta \cup \partial^+ R^\varepsilon_\eta$.

By (2.16), we deduce easily that it is possible to join a point of C^+_k and a point of C^-_k through a line $l_k \subseteq \overline{B}_1(0)$ such that $\mathcal{H}^1(l_k) \leq \varepsilon$ for k large enough.

Considering $H_k := (T_k(K_{n_k}) \cap \overline{B}_1(0)) \cup L_k \cup l_k$, H_k is connected in $\overline{B}_1(0)$ and converges to $H \cap \overline{B}_1(0)$ in the Hausdorff metric. Applying (2.11) with $\varphi = \varphi(x_0, \cdot)$, and since $\sup\{|\varphi(x_0 + \rho_k(\cdot), \nu) - \varphi(x_0, \nu)|\} \to 0$ in $\overline{B}_1(0) \times S^1$ uniformly by the continuity of φ, we get

$$2\varphi(x_0, e_2) \leq \liminf_k \int_{H_k} \varphi(x_0, \nu)\, d\mathcal{H}^1 \leq \liminf_k \int_{T_k(K_{n_k}) \cap \overline{B}_1(0)} \varphi(x_0 + \rho_k x, \nu)\, d\mathcal{H}^1 + 3c_2\varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$2\varphi(x_0, e_2) \leq \liminf_k \int_{T_k(K_{n_k}) \cap \overline{B}_1(0)} \varphi(x_0 + \rho_k x, \nu)\, d\mathcal{H}^1. \tag{2.17}$$

Now we are ready to conclude: in fact

$$\begin{aligned}
\limsup_{\rho \to 0} \frac{\mu(B_\rho(x_0))}{2\rho} &\geq \limsup_k \frac{\mu(B_{\rho_k}(x_0))}{2\rho_k} \geq \\
&\geq \liminf_k \frac{\mu_{n_k}(\overline{B}_{\rho_k}(x_0))}{2\rho_k} = \\
&= \frac{1}{2} \liminf_k \int_{T_k(K_{n_k}) \cap \overline{B}_1(0)} \varphi(x_0 + \rho_k x, \nu)\, d\mathcal{H}^1 \geq \varphi(x_0, e_2),
\end{aligned}$$

the last inequality coming from (2.17).

Let's now turn to the case $m \geq 2$. Let $(K_n) \in \mathcal{K}^f_m(\overline{\Omega})$ converges to K; up to a subsequence, we may suppose that there exists $m' \leq m$ such that each K_n has exactly m' connected components $\widehat{K}^1_n, \cdots, \widehat{K}^{m'}_n$. We may suppose moreover that for all i, $\widehat{K}^i_n \to \widehat{K}^i$ in the Hausdorff metric: it is readily seen that $K = \cup_{i=1}^{m'} \widehat{K}^i$ so that, using the lower semicontinuity for the case $m = 1$, we obtain

$$\begin{aligned}
\int_{K \cap U} \varphi(x, \nu)\, d\mathcal{H}^1 &\leq \sum_{i=1}^{m'} \int_{\widehat{K}^i \cap U} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \\
&\leq \liminf_n \sum_{i=1}^{m'} \int_{\widehat{K}^i_n \cap U} \varphi(x, \nu)\, d\mathcal{H}^1 = \\
&= \liminf_n \int_{K_n \cap U} \varphi(x, \nu)\, d\mathcal{H}^1.
\end{aligned}$$

□

Theorem 2.2.1 is now proved: it is sufficient to apply Theorem 2.3.2 with $U = B_R(0)$, $\overline{\Omega} \subseteq B_R(0)$.

Corollary 2.3.3. *Let (H_n) be a sequence in $\mathcal{K}(\overline{\Omega})$ which converges to H in the Hausdorff metric. Let $m \geq 1$ and let (K_n) be a sequence in $\mathcal{K}_m^f(\overline{\Omega})$ which converges to K in the Hausdorff metric. Then*

$$\int_{K \setminus H} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \liminf_n \int_{K_n \setminus H_n} \varphi(x, \nu)\, d\mathcal{H}^1.$$

Proof. Let $\varepsilon > 0$ and let $H^\varepsilon = \{x \in \overline{\Omega} : \operatorname{dist}(x, H) \leq \varepsilon\}$. Definitively $H_n \subseteq H^\varepsilon$ so that $K_n \setminus H^\varepsilon \subseteq K_n \setminus H_n$. Applying Theorem 2.3.2 with $U = \mathbb{R}^2 \setminus H^\varepsilon$, we have

$$\int_{K \setminus H^\varepsilon} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \liminf_n \int_{K_n \setminus H^\varepsilon} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \liminf_n \int_{K_n \setminus H_n} \varphi(x, \nu)\, d\mathcal{H}^1.$$

Letting ε go to zero, we obtain the thesis. □

The following result will be useful in sections 2.4 and 2.5.

Theorem 2.3.4. *Given $m \geq 1$, let (H_n) be a sequence in $\mathcal{K}_m^f(\overline{\Omega})$ which converges to H in the Hausdorff metric, and let $K \in \mathcal{K}_m^f(\overline{\Omega})$ with $H \subseteq K$. Then there exists a sequence (K_n) in $\mathcal{K}_m^f(\overline{\Omega})$ which converges to K in the Hausdorff metric and such that $H_n \subseteq K_n$ and*

$$\int_{K \setminus H} \varphi(x, \nu)\, d\mathcal{H}^1 = \lim_n \int_{K_n \setminus H_n} \varphi(x, \nu)\, d\mathcal{H}^1. \tag{2.18}$$

Proof. Following Lemma 3.8 of [45], the connected components C_i of $K \setminus H$ are at least countable and satisfy $\mathcal{H}^1(\overline{C_i}) = \mathcal{H}^1(C_i)$. Since $H_n \to H$ in the Hausdorff metric and Ω has Lipschitz boundary, we can find arcs Z_n^i in $\overline{\Omega}$ joining H_n and C_i such that $\mathcal{H}^1(Z_n^i) \to 0$ as $n \to \infty$. Given h, consider $K_n^h := \cup_{i=1}^h Z_n^i \cup \cup_{i=1}^h \overline{C}_i$; we have $K_n^h \in \mathcal{K}_h^f(\overline{\Omega})$, $K_n^h \to K^h := \cup_{i=1}^h \overline{C}_i$ in the Hausdorff metric. Note that $\nu\mathcal{H}^1 \llcorner K_n^h \to \nu\mathcal{H}^1 \llcorner K^h$ strictly for $n \to \infty$. By Theorem 2.1.2, since $\mathcal{H}^1(\overline{C_i}) = \mathcal{H}^1(C_i)$, we have

$$\begin{aligned}
\lim_n \int_{K_n^h} \varphi(x, \nu)\, d\mathcal{H}^1 &= \int_{K^h} \varphi(x, \nu)\, d\mathcal{H}^1 = \\
&= \int_{\cup_{i=1}^h C_i} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \int_{K \setminus H} \varphi(x, \nu)\, d\mathcal{H}^1.
\end{aligned}$$

Choose $h_n \to +\infty$ such that

$$\limsup_n \int_{K_n^{h_n}} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \int_{K \setminus H} \varphi(x, \nu)\, d\mathcal{H}^1,$$

so that

$$\lim_n \sum_{i=1}^{h_n} \mathcal{H}^1(Z_n^i) = 0.$$

If we pose $K_n := H_n \cup K_n^{h_n}$, we have $K_n \in \mathcal{K}_m^f(\overline{\Omega})$, $K_n \to K$ in the Hausdorff metric and

$$\limsup_n \int_{K_n \setminus H_n} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \limsup_n \int_{K_n^{h_n}} \varphi(x, \nu)\, d\mathcal{H}^1 \leq \int_{K \setminus H} \varphi(x, \nu)\, d\mathcal{H}^1.$$

The converse inequality comes from Corollary 2.3.3. □

2.4 The anti-planar anisotropic case

In this section we deal with quasi-static growth of brittle cracks in inhomogeneous anisotropic linearly elastic bodies under anti-planar displacements. We employ the notations of Section 2.2. For every $\lambda \geq 0$ let

$$\mathcal{K}_m^\lambda(\overline{\Omega}) := \{K \in \mathcal{K}_m(\overline{\Omega}) \,:\, \mathcal{H}^1(K) < \lambda\}.$$

We begin with the following lemma which extends Theorem 2.1.3 considering boundary data. The idea is due to A. Chambolle.

Lemma 2.4.1. *Let $m \geq 1$, K_n a sequence in $\mathcal{K}_m(\overline{\Omega})$ which converges to K in the Hausdorff metric and such that $\mathcal{L}^2(\Omega \setminus K_n) \to \mathcal{L}^2(\Omega \setminus K)$. Let $g_n \to g$ strongly in $H^1(\Omega)$ and let $\Gamma(g_n, K_n)$ and $\Gamma(g, K)$ be the sets introduced in (2.7). Then for every $u \in \Gamma(g, K)$, there exists $u_n \in \Gamma(g_n, K_n)$ such that $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^2)$.*

Proof. Consider Ω' a regular open set containing $\overline{\Omega}$ and pose $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. Since $\partial\Omega$ is regular, we may extend g_n and g to $H^1(\Omega')$ and suppose $g_n \to g$ strongly in $H^1(\Omega')$. Note that if $H_n := K_n \cup \overline{\partial_N\Omega}$ and $H = K \cup \overline{\partial_N\Omega}$, $H_n, H \in \mathcal{K}_{m'}(\overline{\Omega'})$, $H_n \to H$ in the Hausdorff metric and $\mathcal{L}^2(\Omega' \setminus H_n) \to \mathcal{L}^2(\Omega' \setminus H)$. Consider

$$v := \begin{cases} u & \text{in } \Omega \\ g & \text{in } \Omega' \setminus \Omega \end{cases}$$

Clearly $v \in L^{1,2}(\Omega' \setminus H)$; we may apply Theorem 2.1.3 and deduce that there exists $v_n \in L^{1,2}(\Omega' \setminus H_n)$ such that $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega', \mathbb{R}^2)$. Note that we may assume $(v_n - v)$ has null average on $\Omega' \setminus \overline{\Omega}$, because we are allowed to add constants to v_n; since $\Omega' \setminus \overline{\Omega}$ is regular, by Poincaré inequality we obtain $v_n \to v$ strongly in $H^1(\Omega' \setminus \overline{\Omega})$. Let E_Ω be a linear extension operator from $H^1(\Omega' \setminus \overline{\Omega})$ to $H^1(\Omega')$. If $w_n := (v_n - v)|_{\Omega' \setminus \overline{\Omega}}$, we can choose

$$u_n := v_n - E_\Omega w_n + (g_n - g)$$

restricted to Ω. It is readily seen that $u_n \in \Gamma(g_n, K_n)$ and $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^2)$. $\square$

By standard arguments, it can be proved that the minimum of problem (2.6) is attained. Moreover, it can be shown that, since $\Omega \setminus K$ is not guaranteed to be regular, this minimum is in general not attained in $H^1(\Omega \setminus K)$ when the boundary data g is not bounded: the reader is referred to [68]. The following proposition deals with the behavior of minima when the compact set K varies.

Proposition 2.4.2. *Let $m \geq 1$, $\lambda \geq 0$, (K_n) a sequence in $\mathcal{K}_m^\lambda(\overline{\Omega})$ which converges to K in the Hausdorff metric, (g_n) a sequence in $H^1(\Omega)$ which converges to g strongly in $H^1(\Omega)$. Let u_n be a solution of the minimum problem*

$$\min_{v \in \Gamma(g_n, K_n)} ||\nabla u||_a^2 \tag{2.19}$$

and let u be a solution of the minimum problem

$$\min_{v \in \Gamma(g, K)} ||\nabla u||_a^2, \tag{2.20}$$

where $\Gamma(g_n, K_n)$ and $\Gamma(g, K)$ are defined as in (2.7).

Then $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^2)$.

Proof. Using g_n as test function, we obtain

$$||\nabla u_n||_a \leq ||\nabla g_n||_a \leq c < +\infty.$$

By (2.5), there exists $\psi \in L^2(\Omega, \mathbb{R}^2)$ such that, up to a subsequence, $\nabla u_n \rightharpoonup \psi$ weakly in $L^2(\Omega, \mathbb{R}^2)$. It is not difficult to prove that there exists $u \in L^2_{loc}(\Omega)$ such that $\nabla u = \psi$ in $\Omega \setminus K$. Moreover by means of Poincaré inequality, we deduce that $u = g$ on $\partial_D\Omega \setminus K$. According to Lemma 2.4.1, let $v_n \in \Gamma(g_n, K_n)$ with $\nabla v_n \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^2)$; since $||u_n||_a \leq ||v_n||_a$ by minimality of u_n, we obtain $\limsup_n ||u_n||_a \leq ||u||_a$. This proves $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega, \mathbb{R}^2)$. $\square$

We now turn to the proof of Theorem 2.2.2. We use a discretization in time. Given $\delta > 0$, let N_δ be the largest integer such that $\delta N_\delta \leq 1$; for $i \geq 0$ we pose $t_i^\delta = i\delta$ and for $0 \leq i \leq N_\delta$ we pose $g_i^\delta = g(t_i^\delta)$. Define K_i^δ as a solution of the minimum problem

$$\min_K \left\{ \mathcal{E}(g_i^\delta, K) \,:\, K \in \mathcal{K}_m^f(\overline{\Omega}),\, K_{i-1}^\delta \subseteq K \right\}, \tag{2.21}$$

where $K_{-1}^\delta = K_0$.

Lemma 2.4.3. *The minimum problem (2.21) admits a solution.*

Proof. We proceed by induction. Suppose K_{i-1}^δ is constructed and that $\lambda > \mathcal{E}(g_i^\delta, K_{i-1}^\delta)$. Let (K_n) be a minimizing sequence of problem (2.21) and let u_n be a solution of the minimum problem (2.6) which defines $\mathcal{E}(g_i^\delta, K_n)$. Up to a subsequence, $K_n \to K$ in the Hausdorff metric and $K_{i-1}^\delta \subseteq K$. Since

$$||\nabla u_n||_a^2 + \int_{K_n} \varphi(x,\nu)\, d\mathcal{H}^1 \leq \lambda$$

for n large, we have that

$$\int_{K_n} \varphi(x,\nu)\, d\mathcal{H}^1 \leq \lambda.$$

We have $K_n \in \mathcal{K}_m^{\alpha_1^{-1}\lambda}(\overline{\Omega})$ and applying Proposition 2.4.2, we have $||u_n||_a \to ||u||_a$ where u is a solution of problem (2.6) which defines $\mathcal{E}(g_i^\delta, K)$; moreover by Theorem 2.3.2, we get

$$\int_K \varphi(x,\nu)\, d\mathcal{H}^1 \leq \liminf_n \int_{K_n} \varphi(x,\nu)\, d\mathcal{H}^1 \leq \lambda.$$

Thus $K \in \mathcal{K}_m^f(\overline{\Omega})$ and $\mathcal{E}(g_i^\delta, K) \leq \liminf_n \mathcal{E}(g_i^\delta, K_n)$. We conclude that K is a solution of the minimum problem (2.21). $\square$

Now, consider the following piecewise constant interpolation: put $g^\delta(t) = g_i^\delta$, $K^\delta(t) = K_i^\delta$, $u^\delta(t) = u_i^\delta$ for $t_i^\delta \leq t < t_{i+1}^\delta$, where u_i^δ is a solution of problem (2.6) which defines $\mathcal{E}(g_i^\delta, K_i^\delta)$.

Lemma 2.4.4. *There exists a positive function $\rho(\delta)$, converging to zero as $\delta \to 0$, such that for all $s < t$ in $[0,1]$,*

$$\begin{aligned} ||\nabla u^\delta(t)||_a^2 + \int_{K^\delta(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \quad \leq \quad & ||\nabla u^\delta(s)||_a^2 + \int_{K^\delta(s)} \varphi(x,\nu)\, d\mathcal{H}^1 + \\ & +2\int_{t_i^\delta}^{t_j^\delta} (\nabla u^\delta(t), \nabla \dot{g}(t))_a\, dt + \rho(\delta) \end{aligned} \tag{2.22}$$

where $t_i^\delta \leq s < t_{i+1}^\delta$ and $t_j^\delta \leq t < t_{j+1}^\delta$.

Proof. Inequality (2.22) is precisely

$$||\nabla u_j^\delta||_a^2 + \int_{K_j^\delta} \varphi(x,\nu)\, d\mathcal{H}^1 \leq ||\nabla u_i^\delta||_a^2 + \int_{K_i^\delta} \varphi(x,\nu)\, d\mathcal{H}^1 + 2\int_{t_i^\delta}^{t_j^\delta} (\nabla u^\delta(t), \nabla \dot{g}(t))_a\, dt + \rho(\delta).$$

To obtain this one, it is sufficient to adapt the proof of Lemma 7.3 in [45]. $\square$

Lemma 2.4.5. *There exists a constant C, depending only on g and K_0, such that*

$$||\nabla u^\delta(t)||_a \leq C \qquad \int_{K^\delta(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \leq C$$

for every $\delta > 0$ and $t \in [0,1]$. In particular, there exists $\lambda > 0$ such that for all $t \in [0,1]$, $K^\delta(t) \in \mathcal{K}_m^\lambda(\overline{\Omega})$.

Proof. Put $\eta = \max_t\{||\nabla g(t)||_a, ||\nabla \dot{g}(t)||_a\}$. Clearly $||\nabla u^\delta(t)||_a \leq ||\nabla g^\delta(t)||_a \leq \eta$ since $g^\delta(t)$ is an admissible displacement for $K^\delta(t)$. Clearly from inequality (2.22) with $s=0$, we obtain

$$\begin{aligned} ||\nabla u^\delta(t)||_a^2 + \int_{K^\delta(t)} \varphi(x,\nu)\, d\mathcal{H}^1 &\leq ||\nabla u^\delta(0)||_a^2 + \int_{K^\delta(0)} \varphi(x,\nu)\, d\mathcal{H}^1 + \\ &+2\int_0^{t_j^\delta} (\nabla u^\delta(t), \nabla \dot{g}(t))_a\, dt + \rho(\delta) \leq \\ &\leq ||\nabla u_0^\delta||_a^2 + \int_{K_0} \varphi(x,\nu)\, d\mathcal{H}^1 + 2\eta^2 + \rho(\delta). \end{aligned}$$

The last term depends only on g and K_0 and so we obtain the first part of the thesis. The second one comes from (2.4). □

Lemma 2.4.6. *Let C be the constant of Lemma 2.4.5. There exists an increasing function $K : [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ (that is $K(s) \subseteq K(t)$ for every $0 \leq s \leq t \leq 1$), such that, for every $t \in [0,1]$, $K^\delta(t)$ converges to $K(t)$ in the Hausdorff metric as $\delta \to 0$ along a suitable sequence independent of t. Moreover if $u(t)$ is a solution of the minimum problem (2.6) which defines $\mathcal{E}(g(t), K(t))$, for every $t \in [0,1]$ we have $\nabla u^\delta(t) \to \nabla u(t)$ strongly in $L^2(\Omega, \mathbb{R}^2)$.*

Proof. The first part is a variant of Helly's theorem for monotone function: for a proof see Lemma 7.5 of [45]; the second part comes directly from Lemma 2.4.5 and Proposition 2.4.2. □

Fix now the sequence (δ_n) and the increasing map $t \to K(t)$ given by Lemma 2.4.6. We indicate $K^{\delta_n}(t)$ by $K_n(t)$ and $u^{\delta_n}(t)$ by $u_n(t)$.

The following property of the pair $(g(t), K(t))$ is important for subsequent results.

Lemma 2.4.7. *For every $t \in [0,1]$ we have*

$$\mathcal{E}(g(t), K(t)) \leq \mathcal{E}(g(t), K) \quad \forall K \in \mathcal{K}_m^f(\overline{\Omega}),\, K(t) \subseteq K. \tag{2.23}$$

Moreover

$$\mathcal{E}(g(0), K(0)) \leq \mathcal{E}(g(0), K) \quad \forall K \in \mathcal{K}_m^f(\overline{\Omega}),\, K_0 \subseteq K. \tag{2.24}$$

Proof. Let $t \in [0,1]$ and $K \in \mathcal{K}_m^f(\overline{\Omega})$ with $K(t) \subseteq K$. Since $K_n(t) \to K(t)$ in the Hausdorff metric as $\delta_n \to 0$, by Theorem 2.3.4 there exists a sequence (K_n) in $\mathcal{K}_m^f(\overline{\Omega})$ converging to K in the Hausdorff metric, such that $K_n(t) \subseteq K_n$ and

$$\int_{K_n \setminus K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \to \int_{K\setminus K(t)} \varphi(x,\nu)\, d\mathcal{H}^1. \tag{2.25}$$

By Lemma 2.4.5, there exists $\lambda > 0$ such that $K_n(t) \in \mathcal{K}_m^\lambda(\overline{\Omega})$ for all n. By (2.25), we deduce that there exists $\lambda' > \lambda$ with $K_n \in \mathcal{K}_m^{\lambda'}(\overline{\Omega})$ for all n.

Let v_n and v solutions of problems (2.6) which define $\mathcal{E}(g_n(t), K_n)$ and $\mathcal{E}(g(t), K)$. By minimality of $K_n(t)$ we have $\mathcal{E}(g_n(t), K_n(t)) \leq \mathcal{E}(g_n(t), K_n)$ and so

$$||\nabla u_n(t)||_a^2 \leq ||\nabla v_n||_a^2 + \int_{K_n \setminus K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1; \tag{2.26}$$

as $\delta_n \to 0$, $\nabla u_n(t) \to \nabla u(t)$ and $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega, \mathbb{R}^2)$ by Proposition 2.4.2: passing to the limit in (2.26) and adding to both sides $\int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1$, by (2.25) we have the thesis.

A similar proof holds for (2.24). □

Lemma 2.4.8. *The function $t \to \mathcal{E}(g(t), K(t))$ is absolutely continuous and*

$$\frac{d}{dt}\mathcal{E}(g(t), K(t)) = 2(\nabla u(t), \nabla \dot{g}(t))_a \quad \textit{for a.e } t \in [0,1]$$

where $u(t)$ is a solution of the minimum problem (2.6) which defines $\mathcal{E}(g(t), K(t))$.

Proof. We rewrite (2.22) in the following form

$$||\nabla u_n(t)||_a^2 + \int_{K_n(t)\setminus K_n(s)} \varphi(x,\nu)\, d\mathcal{H}^1 \le ||\nabla u_n(s)||_a^2 + 2\int_{t_i^{\delta_n}}^{t_j^{\delta_n}} (\nabla u_n(t), \nabla \dot{g}(t))_a\, dt + \rho(\delta_n)$$

for $s \le t$ and $t_i^{\delta_n} \le s < t_{i+1}^{\delta_n}$ and $t_j^{\delta_n} \le t < t_{j+1}^{\delta_n}$. Passing to the limit for $\delta_n \to 0$, using Corollary 2.3.3 we obtain

$$||\nabla u(t)||_a^2 + \int_{K(t)\setminus K(s)} \varphi(x,\nu)\, d\mathcal{H}^1 \le ||\nabla u(s)||_a^2 + 2\int_s^t (\nabla u(\tau), \nabla \dot{g}(\tau))_a\, d\tau,$$

so that

$$\begin{aligned} ||\nabla u(t)||_a^2 + \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \quad &\le \quad ||\nabla u(s)||_a^2 + \int_{K(s)} \varphi(x,\nu)\, d\mathcal{H}^1 + \\ &\quad +2\int_s^t (\nabla u(\tau), \nabla \dot{g}(\tau))_a\, d\tau. \end{aligned}$$

Following Lemma 6.5 of [45], we can prove that the function $F(g) := \mathcal{E}(g, K(t))$ is differentiable on $H^1(\Omega)$ and its differential is given by $dF(g)h = 2(\nabla u(t), \nabla h)_a$ where $u(t)$ is a solution of problem (2.6) which defines $\mathcal{E}(g, K(t))$. By Lemma 2.4.7, we obtain

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(s)) \ge \mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(t)) = 2\int_s^t (\nabla u(\tau,t), \nabla \dot{g}(\tau))_a d\tau$$

where $u(\tau,t)$ is a solution of the minimum problem (2.6) which defines $\mathcal{E}(g(\tau), K(t))$. We can conclude that $t \to \mathcal{E}(g(t), K(t))$ is absolutely continuous since $||\nabla u(t)||_a$ and $||\nabla u(\tau,t)||_a$ are bounded by Lemma 2.4.5. Moreover, dividing the previous inequalities by $t-s$ and letting $s \to t$, since $\nabla u(\tau,t) \to \nabla u(t)$ strongly in $L^2(\Omega, \mathbb{R}^2)$ for $\tau \to t$, we obtain

$$\frac{d}{dt}\mathcal{E}(g(t), K(t)) = 2(\nabla u(t), \nabla \dot{g}(t))_a \quad \textit{for a.e } t \in [0,1].$$

□

We now turn to the proof of Theorem 2.2.2. Points (a) and (b) are proved in lemmas 2.4.6 and 2.4.7 while points (d) and (e) are proved in Lemma 2.4.8. Point (f) and its equivalence to point (e) stated in Remark 2.2.4 are proved adapting Lemma 6.4 of [45]. To prove point (c), we need the following lemma.

Lemma 2.4.9. *Let $K : [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ be a map which satisfies lemmas 2.4.7 and 2.4.8. Then for every $t \in]0,1]$,*

$$\mathcal{E}(g(t), K(t)) \le \mathcal{E}(g(t), K) \quad \forall K \in \mathcal{K}_m^f(\overline{\Omega}) : \cup_{s<t} K(s) \subseteq K.$$

Proof. Consider $t \in]0,1]$ and $K \in \mathcal{K}_m^f(\overline{\Omega})$ such that $\cup_{s<t} K(s) \subseteq K$. For $0 \le s < t$ we have $K(s) \subseteq K$ and so by Lemma 2.4.7, $\mathcal{E}(g(s), K(s)) \le \mathcal{E}(g(s), K)$. By Lemma 2.4.8, these expressions continuously depend on s and so passing to the limit for $s \to t$, we obtain the thesis. □

Consider now the particular case in which $g(0) = 0$: there exists a solution $K(t)$ to the problem of evolution such that $K(0) = K_0$ because in the time discretization method employed, we can choose $K^\delta(0) = K_0$. Under this assumption, we prove that this method gives an approximation of the energy of the solution.

We pose

$$\mathcal{E}_n(t) = ||\nabla u_n(t)||_a^2 + \int_{K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1$$

and

$$\mathcal{E}(t) = \mathcal{E}(g(t), K(t)) = ||\nabla u(t)||_a^2 + \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1.$$

The following convergence result holds.

Proposition 2.4.10. *For all $t \in [0,1]$ the following facts hold:*

(a) $K_n(t) \to K(t)$ *in the Hausdorff metric;*

(b) $\nabla u_n(t) \to \nabla u(t)$ *strongly in* $L^2(\Omega, \mathbb{R}^2)$*;*

(c) $\int_{K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \to \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1$.

In particular $\mathcal{E}_n(t) \to \mathcal{E}(t)$ *for all* $t \in [0,1]$.

Proof. We have already proved points (a) and (b) in Lemma 2.4.6. Since the functions $t \to \int_{K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1$ are increasing and bounded, we may suppose that, by Helly's theorem, they converge pointwise to a bounded increasing function $h : [0,1] \to [0,\infty[$ i.e. for all $t \in [0,1]$

$$\lim_n \int_{K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1 = h(t).$$

Moreover by Theorem 2.3.2 we have that $\int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \le h(t)$ for all $t \in [0,1]$ and by construction $\lambda(0) = \int_{K(0)} \varphi(x,\nu)\, d\mathcal{H}^1$; in particular we have for all $t \in [0,1]$

$$\mathcal{E}(t) \le ||\nabla u(t)||_a^2 + h(t)$$

and $\mathcal{E}(0) = ||\nabla u(0)||_a^2 + h(0)$. Passing to the limit in (2.22), by (b) we obtain

$$||\nabla u(t)||_a^2 + h(t) \le ||\nabla u(s)||_a^2 + h(s) + 2\int_s^t (\nabla u(t), \nabla \dot{g}(t))_a\, dt. \tag{2.27}$$

Since by condition (e) of Theorem 2.2.2

$$\mathcal{E}(t) - \mathcal{E}(0) = 2\int_0^t (\nabla u(\tau), \nabla \dot{g}(\tau))_a\, d\tau,$$

we have

$$\begin{aligned} ||\nabla u(t)||_a^2 + h(t) \quad &- \quad \mathcal{E}(t) = \\ &\le \quad 2\int_0^t (\nabla u(\tau), \nabla \dot{g}(\tau))_a\, d\tau - 2\int_0^t (\nabla u(\tau), \nabla \dot{g}(\tau))_a\, d\tau = 0. \end{aligned}$$

We conclude that $h(t) = \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1$ for all $t \in [0,1]$. This proves point (c) and the thesis is obtained. □

2.5 The planar anisotropic case

In this section we briefly sketch the modifications of the arguments used in the previous section in order to deal with the evolution of cracks in inhomogeneous anisotropic linearly elastic bodies under planar displacements. We employ the notations of Section 2.2. For $\lambda \ge 0$ let

$$\mathcal{K}_m^\lambda(\overline{\Omega}) := \{K \in \mathcal{K}_m(\overline{\Omega}) \,:\, \mathcal{H}^1(K) < \lambda\}.$$

The following lemma can be obtained with arguments similar to those of Lemma 2.4.1.

Lemma 2.5.1. *Let $m \ge 1$, K_n a sequence in $\mathcal{K}_m(\overline{\Omega})$ which converges to K in the Hausdorff metric and such that $\mathcal{L}^2(\Omega \setminus K_n) \to \mathcal{L}^2(\Omega \setminus K)$. Let $g_n \to g$ strongly in $H^1(\Omega, \mathbb{R}^2)$ and let $\mathcal{V}(g_n, K_n)$ and $\mathcal{V}(g, K)$ be the sets introduced in (2.9). Then for every $u \in \mathcal{V}(g,K)$, there exists $u_n \in \mathcal{V}(g_n, K_n)$ such that $Eu_n \to Eu$ strongly in $L^2(\Omega, \mathrm{M}^{n\times n}_{\mathrm{sym}})$.*

By standard techniques, it can be proved that the minimum in problem (2.8) is attained. The following result is similar to Proposition 2.4.2 and deals with the behavior of these minima as K varies.

Proposition 2.5.2. *Let $m \geq 1$ and $\lambda \geq 0$, let K_n be a sequence in $\mathcal{K}_m^\lambda(\overline{\Omega})$ which converges to K in the Hausdorff metric, and let g_n be a sequence in $H^1(\Omega)$ which converges to g strongly in $H^1(\Omega)$. Let u_n be a solution of the minimum problem*

$$\min_{v \in \mathcal{V}(g_n, K_n)} ||Ev||_A^2, \tag{2.28}$$

and let u be a solution of the minimum problem

$$\min_{v \in \mathcal{V}(g,K)} ||Ev||_A^2 \tag{2.29}$$

where $\mathcal{V}(g_n, K_n)$ and $\mathcal{V}(g, K)$ are defined as in (2.9).
Then $Eu_n \to Eu$ strongly in $L^2(\Omega, \mathbb{M}^{n\times n}_{\mathrm{sym}})$.

Proof. Using g_n as test function we obtain $||Eu_n||_A \leq ||Eg_n||_A \leq c < +\infty$. By assumption on A, there exists $\sigma \in L^2(\Omega, \mathbb{M}^{n\times n}_{\mathrm{sym}})$ such that up to a subsequence $Eu_n \rightharpoonup \sigma$ weakly in $L^2(\Omega, \mathbb{M}^{n\times n}_{\mathrm{sym}})$. It is not difficult to prove that there exists $u \in L^2_{loc}(\Omega, \mathbb{R}^2)$ such that $Eu = \sigma$ in $\Omega \setminus K$. Moreover by means of Korn-Poincaré inequality, we deduce that $u = g$ on $\partial_D\Omega \setminus K$. According to Lemma 2.5.1, let $v_n \in \mathcal{V}(g_n, K_n)$ with $Ev_n \to Eu$ strongly in $L^2(\Omega, \mathbb{M}^{n\times n}_{\mathrm{sym}})$; since $||Eu_n||_A \leq ||Ev_n||_A$ by minimality of u_n, we obtain

$$\limsup_n ||Eu_n||_A \leq \limsup_n ||Ev_n||_A = ||Eu||_A.$$

This proves $Eu_n \to Eu$ strongly in $L^2(\Omega, \mathbb{M}^{n\times n}_{\mathrm{sym}})$. □

We employ again a time discretization process. As before given $\delta > 0$, let N_δ be the largest integer such that $\delta N_\delta \leq 1$; for $i \geq 0$ we pose $t_i^\delta = i\delta$ and for $0 \leq i \leq N_\delta$ we pose $g_i^\delta = g(t_i^\delta)$. Define K_i^δ as a solution of the minimum problem

$$\min_K \left\{\mathcal{G}(g_i^\delta, K) \,:\, K \in \mathcal{K}_m^f(\overline{\Omega}),\, K_{i-1}^\delta \subseteq K\right\}, \tag{2.30}$$

where $K_{-1}^\delta = K_0$.

Lemma 2.5.3. *The minimum problem (2.30) admits a solution.*

Proof. We proceed by induction. Suppose K_{i-1}^δ is constructed and that $\lambda > \mathcal{G}(g_i^\delta, K_{i-1}^\delta)$. Let (K_n) be a minimizing sequence of problem (2.30) and let u_n be a solution of the minimum problem (2.8) which defines $\mathcal{G}(g_i^\delta, K_n)$. Up to a subsequence $K_n \to K$ in the Hausdorff metric and $K_{i-1}^\delta \subseteq K$. Since

$$||Eu_n||_A^2 + \int_{K_n} \varphi(x,\nu)\, d\mathcal{H}^1 \leq \lambda$$

for n large enough, we have that

$$\int_{K_n} \varphi(x,\nu)\, d\mathcal{H}^1 \leq \lambda;$$

We have $K_n \in \mathcal{K}_m^{\alpha_1^{-1}\lambda}(\overline{\Omega})$ and applying Proposition 2.5.2, we have $||Eu_n||_A \to ||Eu||_A$ where u is a solution of problem (2.8) which defines $\mathcal{E}(g_i^\delta, K)$; by Theorem 2.3.2, we get

$$\int_K \varphi(x,\nu)\, d\mathcal{H}^1 \leq \liminf_n \int_{K_n} \varphi(x,\nu)\, d\mathcal{H}^1 \leq \lambda.$$

Thus $K \in \mathcal{K}_m^f(\overline{\Omega})$ and $\mathcal{G}(g_i^\delta, K) \leq \liminf_n \mathcal{G}(g_i^\delta, K_n)$. We conclude that K is a solution of the minimum problem (2.30). □

Consider as before the piecewise constant interpolation obtained putting $g^\delta(t) = g_i^\delta$, $K^\delta(t) = K_i^\delta$, $u^\delta(t) = u_i^\delta$ for $t_i^\delta \le t < t_{i+1}^\delta$, where u_i^δ is a solution of problem (2.8) which defines $\mathcal{G}(g_i^\delta, K_i^\delta)$.

Lemma 2.5.4. *There exists a positive function $\rho(\delta)$, converging to zero as $\delta \to 0$, such that for all $s < t$ in $[0,1]$*

$$\begin{aligned} \|Eu^\delta(t)\|_A^2 + \int_{K^\delta(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \quad &\le \quad \|Eu^\delta(s)\|_A^2 + \int_{K^\delta(s)} \varphi(x,\nu)\, d\mathcal{H}^1 + \\ &\quad + 2\int_{t_i^\delta}^{t_j^\delta} (Eu^\delta(t), E\dot{g}(t))_A \, dt + \rho(\delta) \end{aligned}$$

where $t_i^\delta \le s < t_{i+1}^\delta$ and $t_j^\delta \le t < t_{j+1}^\delta$. In particular there exists $C > 0$ depending only on g and K_0 such that for all $t \in [0,1]$

$$\|Eu^\delta(t)\|_A \le C \qquad \int_{K^\delta(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \le C.$$

Proof. It is sufficient to adapt lemmas 2.4.4 and 2.4.5. □

Using Proposition 2.5.2 and the previous lemma we obtain

Lemma 2.5.5. *There exists an increasing function $K : [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ (that is $K(s) \subseteq K(t)$ for every $0 \le s \le t \le 1$), such that, for every $t \in [0,1]$, $K^\delta(t)$ converges to $K(t)$ in the Hausdorff metric as $\delta \to 0$ along a suitable sequence independent of t. Moreover if $u(t)$ is a solution of the minimum problem (2.8) which defines $\mathcal{G}(g(t), K(t))$, for every $t \in [0,1]$ we have $Eu^\delta(t) \to Eu(t)$ strongly in $L^2(\Omega, \mathrm{M}^{n\times n}_{\mathrm{sym}})$.*

The proof of Theorem 2.2.3 can now be obtained using arguments similar to those of lemmas 2.4.7, 2.4.8 and 2.4.9 of Section 2.4.

Consider now the particular case in which $g(0) = 0$: there exists a solution $K(t)$ to the problem of evolution such that $K(0) = K_0$ because in the time discretization method employed we can choose $K^\delta(0) = K_0$. Under this assumption, as in the anti-planar case, the discretization method gives an approximation of the energy of the solution.

In fact, if we pose $K_n(t) := K^{\delta_n}(t)$ and

$$\mathcal{G}_n(t) := \|Eu_n(t)\|_A^2 + \int_{K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1,$$

$$\mathcal{G}(t) := \mathcal{G}(g(t), K(t)) = \|Eu(t)\|_A^2 + \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1,$$

the following approximation result holds.

Proposition 2.5.6. *As $\delta_n \to 0$ for all $t \in [0,1]$ the following facts hold:*

(a) $K_n(t) \to K(t)$ in the Hausdorff metric;

(b) $Eu_n(t) \to Eu(t)$ strongly in $L^2(\Omega, \mathrm{M}^{n\times n}_{\mathrm{sym}})$;

(c) $\int_{K_n(t)} \varphi(x,\nu)\, d\mathcal{H}^1 \to \int_{K(t)} \varphi(x,\nu)\, d\mathcal{H}^1$.

In particular $\mathcal{G}_n(t) \to \mathcal{G}(t)$ for all $t \in [0,1]$.

Proof. It is sufficient to adapt Proposition 2.4.10. □

Chapter 3

A stability result for Neumann problems in dimension $N \geq 3$

In this chapter [1] we deal with stability of Neumann problems in N-dimensional domains ($N \geq 3$) containing cracks.

Given Ω open and bounded in $\mathbb{R}^N$, (K_n) a sequence of compact sets in $\mathbb{R}^N$, consider the following Neumann problems

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \setminus K_n \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \cup (\partial K_n \cap \Omega) \end{cases} \tag{3.1}$$

with $f \in L^2(\Omega)$: we intend (3.1) satisfied in the usual weak sense of Sobolev spaces, that is $u \in H^1(\Omega \setminus K_n)$ and

$$\int_{\Omega\setminus K_n} \nabla u \nabla \varphi + \int_{\Omega\setminus K_n} u\varphi = \int_{\Omega\setminus K_n} f\varphi$$

for all $\varphi \in H^1(\Omega \setminus K_n)$. If (K_n) converges to a compact set K in the Hausdorff metric , we look for conditions on the sequence (K_n) such that, considered the problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \setminus K \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \cup (\partial K \cap \Omega), \end{cases} \tag{3.2}$$

the solutions u_n of (3.1) (extended to 0 on $K_n \cap \Omega$) converge to the solution u of (3.2) (extended to 0 on $K \cap \Omega$). If this is the case, we say that the Neumann problems (3.1) are stable.

The problem of stability for elliptic problems under Neumann boundary conditions has been widely investigated. Usually, since in general the domains $\Omega \setminus K_n$ are not regular, it is not possible to deal with the problem using extension operators (see for example [65], [67]).

In dimension $N = 2$, Chambolle and Doveri [34] in 1997 proved a stability result under a uniform limitation of $\mathcal{H}^1(K_n)$ and of the number of the connected components of K_n; Bucur and Varchon [24] in 2000 proved that if K_n has at most m connected components ($m \in \mathbb{N}$), the stability of the problems is equivalent to the condition $\mathcal{L}^2(\Omega \setminus K_n) \to \mathcal{L}^2(\Omega \setminus K)$.

In dimension $N \geq 3$, the bound on the number of the connected components of K_n is not a relevant feature and a condition similar to that of Bucur and Varchon doesn't hold: in fact, problems (3.1) could be not stable even if the sets K_n are connected. In 1997, Cortesani [39] proved that in general, if K is contained in a C^1 submanifold of $\mathbb{R}^N$, the limit of solutions of (3.1) satisfies a transmission condition on K. Several results on this transmission condition are known under additional assumptions on (K_n). In the case in which K_n is contained in a hyperplane M and is the complement in M of a periodic grid of $(N-1)$ dimensional balls, the problem is treated

[1] The results of this chapter are contained in the paper:
Giacomini A.: A stability result for Neumann problems in dimension $N \geq 3$ *J. Convex Anal.* 11 (2004) 41-58.

in [78]. In [29], a continuity result is obtained in the case $K_n \subseteq M$ and K_n satisfies appropriate capacity conditions on the boundary. In Murat [74] and Del Vecchio [48] (see also [82],[83]), the case of a sieve (Neumann sieve) is considered: the transmission conditions that occur in the limit are determined in relation to capacity properties of the holes of the sieve.

In this chapter, we suppose that the sets K_n, locally, are sufficiently regular subsets of $(N-1)$-dimensional Lipschitz submanifolds of $\mathbb{R}^N$ in such a way that homogenization effects due to the possible holes cannot occur.

Let π be the hyperplane $x_N = 0$ in $\mathbb{R}^N$ and let C be an $(N-1)$-dimensional finite closed cone with nonempty relative interior (for a precise definition see Section 3.2). We say that the sequence (K_n) satisfies the C-condition if there exist constants $\delta, L_1, L_2 > 0$ such that, for all n and for all $x \in K_n$, there exists $\Phi_x : B_\delta(x) \to \mathbb{R}^N$ with

(a) for all $z_1, z_2 \in B_\delta(x)$:

$$L_1|z_1 - z_2| \le |\Phi_x(z_1) - \Phi_x(z_2)| \le L_2|z_1 - z_2|;$$

(b) $\Phi_x(x) = 0$ and $\Phi_x(B_\delta(x) \cap K_n) \subseteq \pi$;

(c) for all $y \in B_{\frac{\delta}{2}}(x) \cap K_n$,

$$\Phi_x(y) \in C_y \subseteq \Phi_x(B_\delta(x) \cap K_n)$$

for some finite closed cone C_y in π congruent (up to a rototranslation) to C. Conditions (a), (b) imply that, near x, K_n is a subset of an $(N-1)$-dimensional Lipschitz submanifold $M_{n,x}$ of $\mathbb{R}^N$ and condition (c) implies that K_n is sufficiently regular in $M_{n,x}$, essentially a finite union of Lipschitz subsets. A particular class of cracks which satisfy the C-condition is given for example by $(\Psi_n(\overline{A}))$, where A is a Lipschitz bounded open subset of π and (Ψ_n) is a sequence of bi-Lipschitz maps from $\mathbb{R}^N$ into itself with constants L_1 and L_2; another example is given by $(\Psi_n(\tilde{K}_n))$, where $(\tilde{K}_n)$ is a sequence of compact subsets of π satisfying the cone condition with respect to a finite close cone C (see Definition 3.2.1).

The main result of the chapter is that, if the sequence (K_n) satisfies the C-condition and $K_n \to K$ in the Hausdorff metric, then the spaces $W^{1,p}(\Omega \setminus K_n)$ converge in the sense of Mosco (see Section 3.1) to the space $W^{1,p}(\Omega \setminus K)$ for $1 < p \le 2$. As a consequence for the case $p = 2$, the problems (3.1) are stable, that is transmission conditions in the limit are avoided.

The hypotheses above are not sufficient to cover the case $p > 2$; moreover, point (b) in C-condition cannot be omitted: in fact a sort of "curvilinear" cone condition given only by points (a) and (c) does not provide the Mosco convergence. We will see these facts through explicit examples.

Finally we mention [35] and [66] in which compactness properties for domains satisfying appropriate uniform cone and segment conditions are used to deal with shape optimization problems: the present work is in spirit close to these papers, the main difference being that the moving boundary can be inside Ω, so that Ω can lie on both side of the boundary.

The chapter is organized as follows: in Section 3.1, we introduce the basic notation; after some preliminaries, we prove the main stability result in Section 3.3. In Section 3.4, we give the above mentioned examples of non-stability which require some basic techniques of Γ-convergence.

3.1 Preliminaries

In this section, we introduce the basic notation and the tools employed in the rest of the chapter.

The Mosco convergence. Let X be a reflexive Banach space, (Y_n) a sequence of closed subspaces of X. Let us set

$$Y' := \{x \in X : x = w\text{-}\lim y_{n_k}, \; y_{n_k} \in Y_{n_k}, \; n_k \to +\infty\} \tag{3.3}$$

and

$$Y'' := \{x \in X \,:\, x = s\text{-}\lim y_n\,,\ y_n \in Y_n \text{ for } n \text{ large}\}; \tag{3.4}$$

Y' and Y'' are called, respectively, the *weak-limsup* and the *strong-liminf* of the sequence (Y_n) in the sense of Mosco. We say that the sequence (Y_n) converges in the sense of Mosco if $Y' = Y'' = Y$ and we call Y the Mosco limit of (Y_n). Clearly $Y'' \subseteq Y'$: as a consequence, in order to prove that $Y_n \to Y$ in the sense of Mosco, it is sufficient to prove that $Y' \subseteq Y$ (*weak-limsup condition*) and $Y \subseteq Y''$ (*strong-liminf condition*). Since Y'' is closed, the strong-liminf condition can be established proving the inclusion $D \subseteq Y''$, D being a dense subset of Y.

Let Ω' be open and bounded in $\mathbb{R}^N$, Ω_n, Ω open subsets of Ω', $p \in [1, +\infty]$. We can identify the Sobolev space $W^{1,p}(\Omega_n)$ with a closed subspace of $L^p(\Omega'; \mathbb{R}^{N+1})$ through the map

$$\begin{array}{ccc} W^{1,p}(\Omega_n) & \longrightarrow & L^p(\Omega'; \mathbb{R}^{N+1}) \\ u & \longmapsto & (u, D_1 u, \dots, D_N u) \end{array} \tag{3.5}$$

with the convention of extending u and ∇u to zero on $\Omega' \setminus \Omega_n$.

Let Y and Y_n be the closed subspaces of $L^p(\Omega'; \mathbb{R}^{N+1})$ corresponding to $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega_n)$ respectively. We say that $W^{1,p}(\Omega_n)$ converges to $W^{1,p}(\Omega)$ in the sense of Mosco if Y is the Mosco limit of the sequence (Y_n) in the space $L^p(\Omega'; \mathbb{R}^{N+1})$.

Stability of Neumann problems. Let Ω' be open and bounded in $\mathbb{R}^N$; consider the Neumann problems

$$\begin{cases} -\Delta u_n + u_n = f \\ u \in H^1(\Omega_n) \end{cases} \tag{3.6}$$

and

$$\begin{cases} -\Delta u + u = f \\ u \in H^1(\Omega) \end{cases} \tag{3.7}$$

with $f \in L^2(\Omega')$, Ω, Ω_n open subsets of Ω'; we intend (3.6) and (3.7) in the usual weak sense, that is

$$u \in H^1(\Omega_n), \quad \int_{\Omega_n} \nabla u_n \nabla \varphi + \int_{\Omega_n} u\varphi = \int_{\Omega_n} f\varphi \quad \forall \varphi \in H^1(\Omega_n)$$

and

$$u \in H^1(\Omega), \quad \int_{\Omega} \nabla u \nabla \varphi + \int_{\Omega} u\varphi = \int_{\Omega} f\varphi \quad \forall \varphi \in H^1(\Omega).$$

We say that the problems (3.6) converge to the problem (3.7) if $(u_n, \nabla u_n) \to (u, \nabla u)$ strongly in $L^2(\Omega'; \mathbb{R}^{N+1})$ under the identification (3.5).

3.2 Some auxiliary results

In this section, we prove some results that are used in the proof of the main theorem of the chapter. We begin recalling some properties of sets which satisfy the cone condition.

Consider a closed ball $B \subseteq \mathbb{R}^N$ not containing 0 and $x \in \mathbb{R}^N$. The set

$$C := x + \{\lambda y \,:\, y \in B, 0 \le \lambda \le 1\}$$

is called a *finite closed cone* in $\mathbb{R}^N$ with vertex at x. We say that two cones C and C' are congruent if there exists a rototranslation Ψ in $\mathbb{R}^N$ such that $\Psi(C) = C'$.

A *parallelepiped* with a vertex at the origin is a set of the form

$$P := \left\{ \sum_{j=1}^{N} \lambda_j y_j \,:\, 0 \le \lambda_j \le 1,\, 1 \le j \le N \right\}$$

where $y_1, \ldots, y_N$ are N linearly independent vectors in $\mathbb{R}^N$. As for the case of cones, we say that two parallelepipeds P, P' are congruent if there exists a rototranslation Ψ in $\mathbb{R}^N$ such that $\Psi(P) = P'$.

Definition 3.2.1. *Let C be a finite closed cone in $\mathbb{R}^N$ with vertex at the origin. We say that a compact set $K \subseteq \mathbb{R}^N$ satisfies the cone condition with respect to C if for all $x \in K$ there exists a finite closed cone C_x congruent to C such that $x \in C_x \subseteq K$.*

If K satisfies the cone condition with respect to a cone C, it turns out that it is the union of the closure of a finite number of Lipschitz open sets. In fact, the following result holds.

Proposition 3.2.2. *Let C be a finite closed cone in $\mathbb{R}^N$ with vertex at the origin and let $K \subseteq \mathbb{R}^N$ be a compact set with $\operatorname{diam}(K) \leq M$ which satisfies the cone condition with respect to C. Then for every $\rho > 0$, there exist a finite number $A_1, A_2, \cdots, A_m$ of compact subsets of K with $\operatorname{diam}(A_j) \leq \rho$ and a finite number $P_1, P_2, \cdots, P_m$ of congruent parallelepipeds with a vertex at the origin such that:*

(a) for all $x \in K$ there exists $1 \leq i \leq m$ with $P_i \subseteq C_x$;

(b) $K = \bigcup_{i=1}^{m} K_i$ where $K_i = \bigcup_{x \in A_i} (x + P_i)$.

The number m and the parallelepipeds $P_1, \ldots, P_m$ depend only on C, M, ρ, and not on the particular set K.

Moreover there exists $\overline{\rho} > 0$, depending only on C, such that for $\rho < \overline{\rho}$, the following facts hold for all $i = 1, \ldots, m$:

(c) for every $y \in \partial K_i$, there exists $\eta > 0$, an orthogonal coordinate system $(\xi_1, \ldots, \xi_n)$ and a Lipschitz function f such that $B_\eta(y) \cap K_i = B_\eta(y) \cap \{\xi = (\xi_1, \ldots, \xi_n) : \xi_n \leq f(\xi_1, \ldots, \xi_{n-1})\}$;

(d) $\operatorname{int}(K_i) = \bigcup_{x \in A_i} (x + \operatorname{int}(P_i))$.

Proof. Properties (a), (b) and (c) can be obtained as in the Gagliardo theorem on the decomposition of open sets with the cone property (see [2], Thm. 4.8). In particular, $\overline{\rho}$ can be chosen as the distance of the center of P_i from ∂P_i; with this choice of $\overline{\rho}$, it turns out that, if a ball B of radius $r < \frac{\overline{\rho}}{2}$ is such that $B \cap (x_1 + P_i) \neq \emptyset$ and $B \cap (x_2 + P_i) \neq \emptyset$ for some $x_1, x_2 \in A_i$, then B cannot intersect relative opposite faces of $x_1 + P_i$ and $x_2 + P_i$ respectively.

Let us turn to the proof of point (d). The inclusion

$$\bigcup_{x \in A_i} (x + \operatorname{int}(P_i)) \subseteq \operatorname{int}(K_i)$$

is immediate. Let $y \in \operatorname{int}(K_i)$ and let $r < \frac{\overline{\rho}}{2}$ be such that $B_r(y) \subseteq K_i$. There exists $x \in A_i$ such that $y \in x + P_i$. If $y \in x + \operatorname{int}(P_i)$ for some z, the result is obtained. Let us suppose that $y \in x + \partial P_i$. For every $z \in B_r(y)$, there exists $x_z \in A_i$ with $z \in x_z + P_i$. If $y \in x_z + \operatorname{int}(P_i)$, the proof is concluded; let us assume by contradiction that $y \in x_z + \partial P_i$ for all $z \in B_r(y)$. Clearly $y - x_z$ cannot belong to the same face of P_i as z varies in $B_r(y)$ because this would contradict $z \in x_z + P_i$ for all $z \in B_r(y)$. Since $B_r(y)$ cannot intersect relative opposite faces of the parallelepipeds $x + P_i$ with $x \in A_i$, we conclude that there exists a vertex v_j of P_i such that $y - x_z$ belongs to a face passing through v_j for all $z \in B_r(y)$. Let $Q_j := \{\lambda(x - v_j) : x \in P_i, \lambda > 0\}$ and let $y_n \to y$ be such that $y - y_n \in \operatorname{int}(Q_j)$. For n large enough, since $y \in x_{y_n} + \partial P_i$, we obtain $y_n \notin x_{y_n} + P_i$ which is absurd. This concludes the proof of point (d). □

Let now consider a sequence (K_n) of compact subsets of $\mathbb{R}^N$ satisfying the cone condition with respect to a given finite closed cone C with vertex at the origin. If K_n converges to a compact set K in the Hausdorff metric, clearly K satisfies the cone condition with respect to C. Let $\mathcal{P}(K_n)$ be the family of all parallelepipeds contained in K_n and congruent to the parallelepipeds $P_1, \dots, P_m$ which appear in the decomposition (b) of Proposition 3.2.2 and let $\mathcal{P}(K)$ be the analogous family for K. Define $\mathcal{P}_r(K)$ as the subset of $\mathcal{P}(K)$ consisting of parallelepipeds P such that there exists $n_k \to \infty$ and $P^k \in \mathcal{P}(K_{n_k})$ with $P^k \to P$ in the Hausdorff metric. Let us set

$$K_r := \{x \in K \,:\, x \in \mathrm{int}(P'), P' \in \mathcal{P}_r(K)\}, \tag{3.8}$$

and

$$K_s := K \setminus K_r. \tag{3.9}$$

We call the elements of K_r *regular points* of K (relative to the approximation given by (K_n)) and the elements of K_s *singular points* of K: K_r is clearly an open set.

Proposition 3.2.3. *Let C be a finite closed cone in $\mathbb{R}^N$ and let (K_n) be a sequence of compact subsets of $\mathbb{R}^N$ satisfying the cone condition with respect to C and converging to a compact set K in the Hausdorff metric. Then $\mathcal{H}^{N-1}(K_s) < +\infty$.*

Proof. Let us fix ρ smaller than the constant $\overline{\rho}$ given by Proposition 3.2.2 (which does not depend on n). By point (b) of the same proposition, we can write

$$K_n = \bigcup_{i=1}^{m} K_n^i \qquad \text{with} \qquad K_n^i := \bigcup_{x \in A_n^i} (x + P_i)$$

where $A_n^1, \dots, A_n^m$ are compact subsets of K_n with $\mathrm{diam}(A_n^i) \le \rho$ and $P_1, \dots, P_m$ are parallelepipeds with a vertex at the origin. There exists $n_k \to \infty$ such that $A_{n_k}^i \to A^i$ in the Hausdorff metric for $i = 1, \dots, m$: clearly $K_{n_k}^i$ converges to $K^i := \bigcup_{x \in A^i}(x + P^i)$ in the Hausdorff metric. Let us prove that $\mathrm{int}(K^i) \subseteq K_r$ for $i = 1, \dots, m$. Since $\mathrm{diam}(A^i) \le \rho$, by point (d) of Proposition 3.2.2, we have $\mathrm{int}(K^i) = \bigcup_{x \in A^i}(x + \mathrm{int}(P^i))$; given $x_0 \in A^i$ and $x_{n_k} \in A_{n_k}^i$ with $x_{n_k} \to x_0$, we have that $x_0 + P^i$ is the Hausdorff limit of $x_{n_k} + P^i$. Since $\mathrm{int}(x_0 + P^i) = x_0 + \mathrm{int}(P^i)$, we conclude that $\mathrm{int}(K^i) \subseteq K_r$ and so $\bigcup_{i=1}^m \mathrm{int}(K^i) \subseteq K_r$.

By point (c) of Proposition 3.2.2, we have that K^i has Lipschitz boundary; we conclude that

$$\mathcal{H}^{N-1}(K_s) = \mathcal{H}^{N-1}(K \setminus K_r) \le \sum_{i=1}^{m} \mathcal{H}^{N-1}(\partial K^i) < +\infty.$$

The proof is now complete. □

3.3 The main result

We now recall the main regularity assumption on the sequence (K_n) of compact subsets of $\mathbb{R}^N$ in order to obtain the stability result mentioned at the beginning of the chapter. We assume $N \ge 3$.

Let π be the hyperplane $x_N = 0$ in $\mathbb{R}^N$.

Definition 3.3.1. *Let C be a finite closed cone in $\mathbb{R}^{N-1}$ and let (K_n) be a sequence of compact subsets of $\mathbb{R}^N$. We say that (K_n) satisfies the C-condition if there exist constants $\delta, L_1, L_2 > 0$ such that, for all n and for all $x \in K_n$, there exists $\Phi_x : B_\delta(x) \to \mathbb{R}^N$ with:*

(a) for all $z_1, z_2 \in B_\delta(x)$:

$$L_1|z_1 - z_2| \le |\Phi_x(z_1) - \Phi_x(z_2)| \le L_2|z_1 - z_2|;$$

(b) $\Phi_x(x) = 0$ and $\Phi_x(B_\delta(x) \cap K_n) \subseteq \pi$;

(c) for all $y \in B_{\frac{\delta}{2}}(x) \cap K_n$,

$$\Phi_x(y) \in C_y \subseteq \Phi_x(B_\delta(x) \cap K_n)$$

for some finite closed cone C_y in π congruent to C.

For technical reasons, we assume that $L_1 \operatorname{diam}(C) < \frac{1}{8}\delta$: this is clearly not restrictive up to reducing C.

We can now state the main result of the chapter.

Theorem 3.3.2. *Let C be a finite closed cone in $\mathbb{R}^{N-1}$, Ω a bounded open subset of $\mathbb{R}^N$, $1 < p \leq 2$, (K_n) a sequence of compact subsets of $\mathbb{R}^N$ satisfying the C-condition and converging to a compact set K in the Hausdorff metric. Then the spaces $W^{1,p}(\Omega \setminus K_n)$ converge to $W^{1,p}(\Omega \setminus K)$ in the sense of Mosco.*

In order to prove the main theorem, we need to analyze the structure of the sets K_n and K. This is done in the following lemmas.

Lemma 3.3.3. *Let C be a finite closed cone in $\mathbb{R}^{N-1}$ and let (K_n) be a sequence of compact subsets of $\mathbb{R}^N$ converging to K in the Hausdorff metric. Suppose that (K_n) satisfies the C-condition. Then there exist $m \geq 1$ such that, for n large enough,*

$$K_n = \bigcup_{i=1}^{m} K_n^i$$

with K_n^i compact, $B_{\frac{\delta}{3}}(x_n^i) \cap K_n \subseteq K_n^i \subseteq B_{\frac{\delta}{2}}(x_n^i)$ for some $x_n^i \in K_n$ such that $x_n^i \to x^i \in K$ for all $i = 1, \ldots, m$ and $K \subseteq \bigcup_{i=1}^m B_{\frac{\delta}{3}}(x^i)$; moreover $\Phi_{x_n^i}(K_n^i)$ satisfies the cone condition with respect to C for all $i = 1, \ldots, m$.

Proof. Since K is compact, there exists a finite number of points $x^1, \ldots, x^m \in K$ such that

$$K \subseteq \bigcup_{i=1}^{m} B_{\frac{\delta}{4}}(x^i). \tag{3.10}$$

As $K_n \to K$ in the Hausdorff metric, there exist $x_n^i \in K_n$ such that $x_n^i \to x^i$ for $i = 1, \ldots, m$. For n large enough, we clearly have

$$K_n \subseteq \bigcup_{i=1}^{m} B_{\frac{\delta}{3}}(x_n^i). \tag{3.11}$$

In order to conclude the proof, it is sufficient to take K_n^i as the preimage under $\Phi_{x_n^i}$ of the union of all cones $C' \subseteq \pi$ congruent to C such that $C' \subseteq \Phi_{x_n^i}(B_\delta(x_n^i) \cap K_n)$ and $C' \cap \Phi_{x_n^i}(\overline{B}_{\frac{\delta}{3}}(x_n^i) \cap K_n) \neq \emptyset$. In fact, K_n^i is compact and the inclusion $B_{\frac{\delta}{3}}(x_n^i) \cap K_n \subseteq K_n^i$ comes directly from the definition of K_n^i and the fact that (K_n) satisfies the C-condition; moreover, the inclusion $K_n^i \subseteq B_{\frac{\delta}{2}}(x_n^i)$ comes from the assumption $L_1 \operatorname{diam}(C) < \frac{1}{8}\delta$, and by (3.11) we have $K_n = \bigcup_{i=1}^m K_n^i$. Finally, by construction, $\Phi_{x_n^i}(K_n^i)$ satisfies the cone condition with respect to C for all n and $i = 1, \ldots, m$, and by (3.10) we have $K \subseteq \bigcup_{i=1}^m B_{\frac{\delta}{3}}(x^i)$ which concludes the proof. □

Lemma 3.3.4. *Let C be a finite closed cone in $\mathbb{R}^{N-1}$ and let (K_n) be a sequence of compact subsets of $\mathbb{R}^N$ converging to K in the Hausdorff metric. Let (K_n) satisfy the C-condition and let $K_n = \bigcup_{i=1}^m K_n^i$ according to the decomposition given by Lemma 3.3.3. Then, up to a subsequence, for $i = 1, \ldots, m$, $x_n^i \to x^i \in K$, $K_n^i \to K^i \subseteq K$ in the Hausdorff metric, $\Phi_{x_n^i} \to \Phi_i$ uniformly on $B_{\frac{3}{4}\delta}(x^i)$ with*

(a)
$$K \subseteq \bigcup_{i=1}^{m} B_{\frac{\delta}{3}}(x^i);$$

(b) $$B_{\frac{\delta}{3}}(x^i) \cap K \subseteq K^i \subseteq B_{\frac{3}{4}\delta}(x^i);$$

(c) $$K = \bigcup_{i=1}^{m} K^i;$$

(d) for all $z_1, z_2 \in B_{\frac{3}{4}\delta}(x^i)$:

$$L_1|z_1 - z_2| \leq |\Phi_i(z_1) - \Phi_i(z_2)| \leq L_2|z_1 - z_2|;$$

(e) $$\Phi_i(K \cap B_{\frac{3}{4}\delta}(x^i)) \subseteq \pi.$$

Moreover, $\Phi_i(K^i)$ satisfies the cone condition with respect to C for all $i = 1, \dots, m$.

Proof. By Lemma 3.3.3, $x_n^i \to x^i \in K$ for all $i = 1, \dots, m$ and $K \subseteq \bigcup_{i=1}^m B_{\frac{\delta}{3}}(x^i)$; this proves point (a). Since $K_n \to K$ in the Hausdorff metric, up to a subsequence, $K_n^i \to K^i \subseteq K$ in the Hausdorff metric for $i = 1, \dots, m$. Fix $i \in \{1, \dots, m\}$. Note that, for n large enough, $\overline{B}_{\frac{3}{4}\delta}(x^i) \subseteq B_\delta(x_n^i)$. We deduce that $\Phi_{x_n^i}$ are well defined on $B_{\frac{3}{4}\delta}(x^i)$; since they are equicontinuous and equibounded, we may assume that $\Phi_{x_n^i} \to \Phi_i$ uniformly on $B_{\frac{3}{4}\delta}(x^i)$ with

$$L_1|z_1 - z_2| \leq |\Phi_i(z_1) - \Phi_i(z_2)| \leq L_2|z_1 - z_2|$$

for all $z_1, z_2 \in B_{\frac{3}{4}\delta}(x^i)$. This proves point (d).

Passing to the limit in the relations

$$B_{\frac{\delta}{3}}(x_n^i) \cap K_n \subseteq K_n^i \subseteq B_{\frac{\delta}{2}}(x_n^i)$$

$$K_n = \bigcup_{i=1}^{m} K_n^i$$

$$\Phi_{x_n^i}(K_n \cap B_{\frac{3}{4}\delta}(x_n^i)) \subseteq \pi,$$

we obtain points (b), (c) and (e).

Finally, it is easy to see that $\Phi_i(K^i)$ satisfies the cone condition with respect to C. In fact, fix $y \in K^i$; since $K_n^i \to K^i$ in the Hausdorff metric, there exists $y_n \in K_n^i$ with $y_n \to y$. As $\Phi_{x_n^i}(K_n^i)$ satisfies the cone condition with respect to C, there exists C_n finite closed cone in π congruent to C such that $\Phi_{x_n^i}(y_n) \in C_n \subseteq \Phi_{x_n^i}(K_n^i)$. Up to a subsequence, $C_n \to C'$ in the Hausdorff metric with C' congruent to C. Then $\Phi_i(y) \in C' \subseteq \Phi_i(K^i)$ since $\Phi_{x_n^i}(K_n^i) \to \Phi_i(K^i)$ in the Hausdorff metric. □

We can now pass to the proof of the main theorem.

Proof of Theorem 3.3.2. Let Y' and Y'' be the weak-limsup and the strong-liminf of the sequence $W^{1,p}(\Omega \setminus K_n)$ respectively. We have to prove that $Y' = Y'' = W^{1,p}(\Omega \setminus K)$.

Let us start with the inclusion

$$Y' \subseteq W^{1,p}(\Omega \setminus K). \tag{3.12}$$

Let (u_k) be a sequence in $W^{1,p}(\Omega \setminus K_{n_k})$ $(n_k \to +\infty)$, and let $v, w_1, \cdots, w_N \in L^p(\Omega)$ be such that $u_k \to v$ and $D_i u_k \to w_i$ weakly in $L^p(\Omega)$ for $i = 1, \dots, N$ with the identification (3.5). Since $K_{n_k} \to K$ in the Hausdorff metric, it is readily seen that for $i = 1, \dots, N$, $w_i = D_i v$ in the sense of distributions in $\Omega \setminus K$. Since (K_n) satisfies the C-condition, we have $\mathcal{L}^N(K) = 0$; as a consequence, we get $v = 0$ and $w_1, \dots, w_N = 0$ a.e. on K, and so we conclude that $(v, w_1, \dots, w_N)$ is the element of $L^p(\Omega; \mathbb{R}^{N+1})$ associated to a function of $W^{1,p}(\Omega \setminus K)$ according to (3.5).

We can thus pass to the inclusion

$$W^{1,p}(\Omega \setminus K) \subseteq Y''; \tag{3.13}$$

we have to prove that, given $u \in W^{1,p}(\Omega \setminus K)$, there exists $u_n \in W^{1,p}(\Omega \setminus K_n)$ such that $(u_n, \nabla u_n) \to (u, \nabla u)$ strongly in $L^p(\Omega; \mathbb{R}^{N+1})$. By standard arguments on Mosco Convergence, it is sufficient to prove that, given any subsequence n_j, there exists a further subsequence n_{j_k} and a sequence $u_k \in W^{1,p}(\Omega \setminus K_{n_{j_k}})$ such that $(u_k, \nabla u_k) \to (u, \nabla u)$ strongly in $L^p(\Omega; \mathbb{R}^{N+1})$. Thus we deduce that, in order to prove (3.13), we can reason up to subsequences.

Using the decomposition given by Lemma 3.3.3, there exists $m \geq 1$ such that

$$K_n = \bigcup_{i=1}^{m} K_n^i$$

with K_n^i compact, $B_{\frac{\delta}{3}}(x_n^i) \cap K_n \subseteq K_n^i \subseteq B_{\frac{\delta}{2}}(x_n^i)$ for some $x_n^i \in K_n$, and $\Phi_{x_n^i}(K_n^i)$ satisfying the cone condition with respect to C for all $i = 1, \ldots, m$. By Lemma 3.3.4, up to a subsequence, $x_n^i \to x^i \in K$ for all $i = 1, \ldots, m$, with $K \subseteq \bigcup_{i=1}^{m} B_{\frac{\delta}{3}}(x^i)$, and $\Phi_{x_n^i} \to \Phi_i$ uniformly on $B_{\frac{3}{4}\delta}(x^i)$ such that, for all $z_1, z_2 \in B_{\frac{3}{4}\delta}(x^i)$

$$L_1|z_1 - z_2| \leq |\Phi_i(z_1) - \Phi_i(z_2)| \leq L_2|z_1 - z_2|.$$

Moreover, $K_n^i \to K^i$ in the Hausdorff metric with

$$K = \bigcup_{i=1}^{m} K^i,$$

$B_{\frac{\delta}{3}}(x^i) \cap K \subseteq K^i \subseteq B_{\frac{3}{4}\delta}(x^i)$ and $\Phi_i(K^i)$ satisfies the cone condition with respect to C for all $i = 1, \ldots, m$. Finally, we have that

$$\Phi_{x_n^i}(K_n^i) \to \Phi_i(K^i) \tag{3.14}$$

in the Hausdorff metric for $i = 1, \ldots, m$.

We begin proving the strong-liminf condition in the particular case in which $u \in W^{1,p}(\Omega \setminus K)$, $\operatorname{supp}(u) \subset\subset B_{\frac{\delta}{3}}(x^i)$ and

$$\operatorname{supp}(u \circ \Phi_i^{-1}) \cap \pi \subseteq [\Phi_i(K^i)]_r, \tag{3.15}$$

where, according to (3.8), $[\Phi_i(K^i)]_r$ denotes the set of regular points of $\Phi_i(K^i)$ relative to the approximation (3.14). set $w := u \circ \Phi_i^{-1}$; we have $w \in W^{1,p}(\Phi_i(B_{\frac{\delta}{3}}(x) \cap \Omega) \setminus \Phi_i(K^i))$. As in Section 3.2, let $\mathcal{P}_r(\Phi_i(K^i))$ denote the family of parallelepipeds contained in $\Phi_i(K^i)$ and congruent to the parallelepipeds $P_1, \ldots, P_m$ given by Proposition 3.2.2, that are limit in the Hausdorff metric of parallelepipeds P^n congruent to $P_1, \ldots, P_m$ and contained in $\Phi_{x_n^i}(K_n^i)$. By (3.8) and (3.15) there exist $D_1, \ldots, D_t \in \mathcal{P}_r(\Phi_i(K^i))$ such that

$$\operatorname{supp}(w) \cap \pi \subseteq \bigcup_{j=1}^{t} \operatorname{int}_\pi(D_j)$$

where $\operatorname{int}_\pi(\cdot)$ denotes the interior relative to π. Let $Q_j \subseteq \operatorname{int}_\pi(D_j)$ be a parallelepiped in π such that $\operatorname{supp}(w) \cap \pi \subseteq \cup_{j=1}^{t} \operatorname{int}_\pi(Q_j)$ and let $\varepsilon > 0$ be such that, posed $U_j := \operatorname{int}_\pi(Q_j) \times]-\varepsilon, \varepsilon[$, $(j = 1, \ldots, t)$,

$$\bigcup_{j=1}^{t} U_j \subseteq \Phi_i(B_{\frac{\delta}{3}}(x^i) \cap \Omega).$$

Through a partition of unity associated to $\{U_1, \ldots, U_t, U_0\}$ with $U_0 := \mathbb{R}^N \setminus \Phi_i(K^i)$, we may write

$$w = \sum_{j=0}^{t} \psi_j w,$$

with $\psi_j \in C^\infty(U_j)$, $\mathrm{supp}(\psi_j) \subset\subset U_j$, so that

$$u = \sum_{j=0}^{t} (\psi_j \circ \Phi_i) u.$$

Note that $\mathrm{supp}((\psi_0 \circ \Phi_i)u) \cap K = \emptyset$ so that

$$(\psi_0 \circ \Phi_i) u \in W^{1,p}(\Omega \setminus K_n)$$

for n large enough, that is $(\psi_0 \circ \Phi_i)u \in Y''$. In order to conclude, it is thus sufficient to deal with the case $\mathrm{supp}(w) \subset\subset U_j$ for $j = 1, \dots, t$.

Let us fix $j \in \{1, \dots, t\}$. Set $U_j^+ := U_j \cap (\mathbb{R}^{N-1} \times]0, \varepsilon[)$, $U_j^- := U_j \cap (\mathbb{R}^{N-1} \times]-\varepsilon, 0[)$, and let $w^\pm := w_{|U_j^\pm}$. We have $w^\pm \in W^{1,p}(U_j^\pm)$: let $\widetilde{w}^\pm$ be the extension by reflection of $w^\pm$ on U_j. Note that $\mathrm{supp}(\widetilde{w}^\pm) \subset\subset U_j$. Up to a subsequence, $Q_j \subseteq \Phi_{x_n^i}(K_n^i)$ because $D_j \in \mathcal{P}_r(\Phi_i(K^i))$ and $Q_j \subseteq \mathrm{int}_\pi(D_j)$; we deduce that $U_j \setminus \Phi_{x_n^i}(K_n^i)$ has exactly two connected components that we indicate by B^+ and B^- (note that they do not depend on n for n large). As a consequence $\Phi_{x_n^i}^{-1}(U_j) \setminus K_n$ has exactly two connected components given by $\Phi_{x_n^i}^{-1}(B^+)$ and $\Phi_{x_n^i}^{-1}(B^-)$ respectively. Consider

$$v_n := \begin{cases} \widetilde{w}^+ \circ \Phi_i & \text{on } \Phi_{x_n^i}^{-1}(B^+) \\ \widetilde{w}^- \circ \Phi_i & \text{on } \Phi_{x_n^i}^{-1}(B^-). \end{cases}$$

Since $\widetilde{w}^\pm$ has compact support in U_j, we deduce that for n large enough

$$v_n \in W^{1,p}(\Omega \setminus K_n).$$

Since $K_n^i \to K^i$ in the Hausdorff metric and $\widetilde{w}^\pm \circ \Phi_i$ does not depend on n, $v_n \to u$ and $\nabla v_n \to \nabla u$ a.e. in Ω. By the Dominated Convergence Theorem, we deduce that $(v_n, \nabla v_n) \to (u, \nabla u)$ in $L^p(\Omega; \mathbb{R}^{N+1})$ under the identification (3.5). This proves $u \in Y''$ in the case u satisfies (3.15).

In order to complete the proof of the theorem, we have to see that the assumption (3.15) is not restrictive. Consider $u \in W^{1,p}(\Omega \setminus K)$. Let $\{\varphi_1, \dots, \varphi_m, \varphi_0\}$ be a C^∞ partition of unity associated to $B_{\frac{\delta}{3}}(x^1), \dots, B_{\frac{\delta}{3}}(x^m), \mathbb{R}^N \setminus K$. We can write

$$u = \sum_{i=0}^{m} \varphi_i u.$$

Since $\mathrm{supp}(\varphi_0 u) \cap K = \emptyset$, we have that $\mathrm{supp}(\varphi_0 u) \cap K_n = \emptyset$ for n large enough and so $\varphi_0 u \in W^{1,p}(\Omega \setminus K_n)$. This implies $\varphi_0 u \in Y''$. We deduce that it is not restrictive to assume $\mathrm{supp}(u) \subset\subset B_{\frac{\delta}{3}}(x^i)$ for some $i = 1, \dots, m$.

Let us consider

$$K_s := \bigcup_{i=1}^{m} \Phi_i^{-1} \left([\Phi_i(K^i)]_s \right)$$

where, according to (3.9), $[\Phi_i(K^i)]_s$ denotes the set of singular points of $\Phi_i(K^i)$ under the approximation (3.14). By Lemma 3.2.2, we obtain

$$\mathcal{H}^{N-2}(K_s) < +\infty; \tag{3.16}$$

by Theorem 3 in section 4.7.2 of [49], since $1 < p \leq 2$, we deduce that $c_p(K_s, \Omega) = 0$, where

$$c_p(K_s, \Omega) := \inf \left\{ \int_\Omega |\nabla u|^p \,:\, u \in W_0^{1,p}(\Omega),\, u \geq 1 \text{ in a neighborhood of } \mathrm{K_s} \right\}.$$

By standard properties of capacity, there exists a sequence (ψ_k) in $C_c^\infty(\mathbb{R}^N)$ with $\psi_k \to 0$ in $W^{1,p}(\mathbb{R}^N)$ and $\psi_k \geq 1$ on a neighborhood of K_s. Since

$$u = \psi_k u + (1 - \psi_k) u,$$

we deduce that the set

$$\mathcal{D} := \{v \in W^{1,p}(\Omega \setminus K) : \operatorname{supp}(v) \cap K_s = \emptyset\}$$

is dense in $W^{1,p}(\Omega \setminus K) \cap L^\infty(\Omega \setminus K)$ and hence in $W^{1,p}(\Omega \setminus K)$. As observed in Section 3.1, in order to prove (3.13), it is sufficient to check the inclusion $\mathcal{D} \subseteq Y''$. If $u \in \mathcal{D}$, we have that

$$\operatorname{supp}(u \circ \Phi_i^{-1}) \cap \Phi_i(K^i) \subseteq [\Phi_i(K^i)]_r.$$

Consider $V_1, V_2 \subseteq \pi$ open in the relative topology of π and such that

$$\operatorname{supp}(u \circ \Phi_i^{-1}) \cap \Phi_i(K^i) \subset\subset V_1 \subset\subset V_2 \subset\subset [\Phi_i(K^i)]_r;$$

let $\varepsilon > 0$ with $U_2 := V_2 \times]-\varepsilon, \varepsilon[\subseteq \Phi_i(B_{\frac{\delta}{3}}(x^i))$ and set $U_1 := V_1 \times]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$. Consider $\varphi \in C_c^\infty(\Phi_i^{-1}(U_2))$ with $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $\Phi_i^{-1}(U_1)$. Since $u \in \mathcal{D}$, we deduce $\operatorname{supp}((1-\varphi)u) \cap K = \emptyset$ that is $(1-\varphi)u \in W^{1,p}(\Omega \setminus K_n)$ for n large enough and so $(1-\varphi)u \in Y''$. Moreover, since

$$\operatorname{supp}((\varphi u) \circ \Phi_i^{-1}) \cap \pi \subseteq [\Phi_i(K^i)]_r,$$

we deduce by the previous step that $\varphi u \in Y''$. We conclude $u = \varphi u + (1-\varphi)u \in Y''$ and the theorem is proved. □

From Theorem 3.3.2 in the case $p = 2$, we may deduce the stability of the Neumann problems mentioned at the beginning of the chapter.

Corollary 3.3.5. *Let C be a finite closed cone in $\mathbb{R}^{N-1}$, (K_n) a sequence of compact subsets of $\mathbb{R}^N$ satisfying the C-condition and converging to a compact set K in the Hausdorff metric. Let Ω be an open and bounded subset of $\mathbb{R}^N$, $f \in L^2(\Omega)$, and let u_n and u be the solutions of the following Neumann problems*

$$\text{(3.17)} \qquad \begin{cases} -\Delta u_n + u_n = f \\ u \in H^1(\Omega \setminus K_n), \end{cases}$$

$$\text{(3.18)} \qquad \begin{cases} -\Delta u + u = f \\ u \in H^1(\Omega \setminus K). \end{cases}$$

set $u_n = 0$, $\nabla u_n = 0$ on $K_n \cap \Omega$, and $u = 0$, $\nabla u = 0$ on $K \cap \Omega$.

Then we have $u_n \to u$ strongly in $L^2(\Omega)$ and $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega; \mathbb{R}^N)$, so that the problems (3.17) are stable.

Proof. Let u_n be the solution of (3.17) and u the solution of (3.18). We assume the identification (3.5). From the equation (3.17), we have that $(u_n, \nabla u_n)$ is bounded in $L^2(\Omega; \mathbb{R}^{N+1})$. There exists $v \in L^2(\Omega; \mathbb{R}^{N+1})$ such that up to a subsequence, $(u_n, \nabla u_n) \to v$ weakly in $L^2(\Omega; \mathbb{R}^{N+1})$. By Theorem 3.3.2, we have that $H^1(\Omega \setminus K_n)$ converges to $H^1(\Omega \setminus K)$ in the sense of Mosco. Thus we deduce $v \in H^1(\Omega \setminus K)$; moreover, taking $\varphi \in H^1(\Omega \setminus K)$, there exists $\varphi_n \in H^1(\Omega \setminus K_n)$ with $(\varphi_n, \nabla \varphi_n) \to (\varphi, \nabla \varphi)$ strongly in $L^2(\Omega; \mathbb{R}^{N+1})$. We conclude that

$$\text{(3.19)} \qquad \begin{aligned} \int_{\Omega \setminus K} \nabla v \nabla \varphi + \int_{\Omega \setminus K} v \varphi &= \lim_n \int_{\Omega \setminus K_n} \nabla u_n \nabla \varphi_n + \int_{\Omega \setminus K_n} u_n \varphi_n = \\ &= \lim_n \int_{\Omega \setminus K_n} f \varphi_n = \\ &= \int_{\Omega \setminus K} f \varphi, \end{aligned}$$

that is $v = u$. Finally, taking $\varphi_n = u_n$ and using again (3.19), we have that

$$||u_n||_{L^2(\Omega; \mathbb{R}^{N+1})} \to ||u||_{L^2(\Omega; \mathbb{R}^{N+1})}.$$

We conclude that $(u_n, \nabla u_n) \to (u, \nabla u)$ strongly in $L^2(\Omega; \mathbb{R}^{N+1})$ and so the proof is complete. □

Remark 3.3.6. *Similarly, under the same hypotheses of Theorem 3.3.2, we can prove that the Neumann problems*

$$
\left\{\begin{array}{l} -\Delta_p u_n + |u_n|^{p-2} u_n = f \\ u_n \in W^{1,p}(\Omega \setminus K_n) \end{array}\right. \tag{3.20}
$$

where $1 < p \leq 2$, Ω *is open and bounded in* $\mathbb{R}^N$, $f \in L^p(\Omega)$ *and* $\Delta_p u_n := \mathrm{div}(|\nabla u_n|^{p-2}\nabla u_n)$, *converge to the Neumann problem*

$$
\left\{\begin{array}{l} -\Delta_p u + |u|^{p-2} u = f \\ u \in W^{1,p}(\Omega \setminus K), \end{array}\right. \tag{3.21}
$$

that is $(u_n, \nabla u_n) \to (u, \nabla u)$ *strongly in* $L^p(\Omega; \mathbb{R}^{N+1})$ *under the identification (3.5).*

Remark 3.3.7. The Mosco convergence proved in Theorem 3.3.2 is the key point in order to prove the stability of more general problems. We now briefly sketch an application to fracture mechanics in linearly elastic bodies.

For every open and bounded set $A \subseteq \mathbb{R}^N$, let us set

$$
LD^{1,2}(A) := \left\{ u \in H^1_{\mathrm{loc}}(A; \mathbb{R}^N) \,:\, E(u) \in L^2(A, \mathrm{M}^{n\times n}_{\mathrm{sym}}) \right\},
$$

where $\mathrm{M}^{n\times n}_{\mathrm{sym}}$ denotes the set of symmetric matrices of order N and $E(u)$ denotes the symmetric part of the gradient of u. Let $|M| := [\mathrm{tr}(M^2)]^{\frac{1}{2}}$ denote the standard norm in $\mathrm{M}^{n\times n}_{\mathrm{sym}}$.

Let (K_n) be a sequence of compact subsets of $\mathbb{R}^N$ satisfying the C-condition with respect to a given $(N-1)$-dimensional finite closed cone C, and converging to K in the Hausdorff metric. Let Ω be open and bounded in $\mathbb{R}^N$ and let $\partial_D\Omega$ be a Lipschitz part of $\partial\Omega$. Consider $g_n, g \in H^1(\Omega; \mathbb{R}^N)$ with $g_n \to g$ strongly and let

$$
\Gamma_n := \left\{ u \in LD^{1,2}(\Omega \setminus K_n) \,:\, u = g_n \text{ on } \partial_D\Omega \setminus K_n \right\}
$$

and

$$
\Gamma := \left\{ u \in LD^{1,2}(\Omega \setminus K) \,:\, u = g \text{ on } \partial_D\Omega \setminus K \right\}.
$$

Given the *Lamé coefficients* μ, λ, let $u_n \in LD^{1,2}(\Omega \setminus K_n)$ be the minimum of

$$
\min_{v \in \Gamma_n} \int_{\Omega\setminus K_n} \mu |E(v)|^2 + \frac{\lambda}{2} |\mathrm{tr}\, E(v)|^2 \, d\mathcal{L}^N
$$

and let $u \in LD^{1,2}(\Omega \setminus K)$ be the minimum of

$$
\min_{v \in \Gamma} \int_{\Omega\setminus K} \mu |E(v)|^2 + \frac{\lambda}{2} |\mathrm{tr}\, E(v)|^2 \, d\mathcal{L}^N.
$$

Let $K \subseteq \Omega$, and let us suppose that K is locally contained in a Lipshitz graph, that is, $K \subseteq \bigcup_{i=1}^m U_i$ with U_i open and such that there exists an orthogonal coordinate system $(\xi_1, \ldots, \xi_N)$ and a Lipschitz function $f_i(\xi_1, \ldots, \xi_{N-1})$ with $U_i \cap K \subseteq \mathrm{graph}(f_i)$. Let $\varphi_0, \varphi_1, \ldots, \varphi_m$ be a partition of unity associated to $\mathbb{R}^N \setminus K, U_1, \ldots, U_m$. By means of Korn's inequality (see for example [85]) we get $\varphi_i u \in H^1(\Omega \setminus K; \mathbb{R}^N)$ for $i = 1, \ldots, m$. Using the Mosco convergence given by Theorem 3.3.2, we have that for all $i = 1, \ldots, m$ there exists $v_n^i \in H^1(\Omega \setminus K_n; \mathbb{R}^N)$ such that $E(v_n^i) \to E(\varphi_i u)$ strongly in $L^2(\Omega; \mathrm{M}^{n\times n}_{\mathrm{sym}})$, with the convention of considering $E(v_n^i) = 0$ and $E(\varphi_i u) = 0$ on $\Omega \cap K_n$ and $\Omega \cap K$ respectively. Setting $v_n := \varphi_0 u + \sum_{i=1}^m v_n^i$, we get $v_n \in \Gamma_n$ for n large and $E(v_n) \to E(u)$ strongly in $L^2(\Omega; \mathrm{M}^{n\times n}_{\mathrm{sym}})$. By minimality of u_n, we thus deduce that $E(u_n) \to E(u)$ strongly in $L^2(\Omega; \mathrm{M}^{n\times n}_{\mathrm{sym}})$, with the convention of considering $E(u_n) = 0$ and $E(u) = 0$ on $\Omega \cap K_n$ and $\Omega \cap K$ respectively. This can be interpreted as the convergence of the equilibrium deformations for the elastic body Ω with cracks K_n and boundary displacements g_n to the equilibrium deformation relative to the crack K and the boundary displacement g.

3.4 Non-stability examples

In this section, we give two explicit examples of non-stability when the conditions of Theorem 3.3.2 are violated. In Example 1, we see that the C-condition is not sufficient in the case $p > 2$: in fact some problems related to capacity can occur which in the case $1 < p \leq 2$ were avoided thank to (3.16). In Example 2, we see that a sort of uniform "curvilinear" cone condition for the sequence (K_n) given only by points (a) and (c) in the C-condition does not guarantee the Mosco convergence of the spaces $W^{1,p}(\Omega \setminus K_n)$ even in the case $1 < p \leq 2$.

EXAMPLE 1. Let Q, Q', Q'' be the open unit cube in $\mathbb{R}^N$, $\mathbb{R}^{N-1}$, and $\mathbb{R}^{N-2}$ respectively. For every $n \geq 1$, let us set

$$K_n := \left\{ \left[0, \frac{1}{2} - \frac{1}{n}\right] \cup \left[\frac{1}{2} + \frac{1}{n}, 1\right] \right\} \times \overline{Q''} \times \left\{\frac{1}{2}\right\}.$$

(K_n) is a sequence of compact sets in $\mathbb{R}^N$ whose limit in the Hausdorff metric is

$$K = \overline{Q'} \times \left\{\frac{1}{2}\right\}.$$

Let us set $L := \{\frac{1}{2}\} \times \overline{Q''} \times \{\frac{1}{2}\}$, $S_1 := Q' \times]0, \frac{1}{2}[$ and $S_2 := Q' \times]\frac{1}{2}, 1[$.

Let C be the finite closed cone in $\mathbb{R}^{N-1}$ determined by $B_{\frac{1}{8}}(P)$ with $P := (\frac{1}{8}, \frac{1}{8}, \ldots, \frac{1}{8})$. Clearly (K_n) satisfies the C-condition.

We claim that, if $p > 2$, then the spaces $W^{1,p}(\Omega \setminus K_n)$ do not converge to $W^{1,p}(\Omega \setminus K)$ in the sense of Mosco. In fact, assuming the Mosco convergence, by Remark 3.3.6, we deduce that the Neumann problems

$$\begin{cases} -\Delta_p v + |v|^{p-2}v = f \\ v \in W^{1,p}(Q \setminus K_n) \end{cases} \tag{3.22}$$

with $f \in L^p(Q)$ converge to the problem

$$\begin{cases} -\Delta_p v + |v|^{p-2}v = f \\ v \in W^{1,p}(Q \setminus K). \end{cases} \tag{3.23}$$

Let $f = \chi_{S_2}$ and let u_n, u be the solutions of (3.22) and (3.23) respectively. We readily deduce that $u = \chi_{S_2}$; since $(u_n, \nabla u_n) \to (u, \nabla u)$ in $L^p(Q; \mathbb{R}^{N+1})$ under the identification (3.5), we obtain that $u_n \to u$ strongly in $W^{1,p}(S_i)$ for $i = 1, 2$. By strong convergence in $W^{1,p}(S_1)$, we get $u_n \to 0$ c_p-q.e. on L, while from strong convergence in $W^{1,p}(S_2)$, we deduce $u_n \to 1$ c_p-q.e. on L. Since $c_p(L, Q) \neq 0$ as $p > 2$, we get a contradiction: we conclude that the Mosco convergence does not hold.

EXAMPLE 2. Let Q, Q', Q'' be the open unit cube in $\mathbb{R}^N$, $\mathbb{R}^{N-1}$, and $\mathbb{R}^{N-2}$ respectively. Let us write $Q = Q' \times]0, 1[$. For every $n \geq 1$ let us set

$$K_n := \bigcup_{i=1}^{n-1} \left[\frac{1}{3}, \frac{2}{3}\right] \times \overline{Q''} \times \left\{\frac{i}{n}\right\}.$$

(K_n) is a sequence of compact sets in $\mathbb{R}^N$ whose limit in the Hausdorff metric is

$$K = \left[\frac{1}{3}, \frac{2}{3}\right] \times \overline{Q''} \times [0, 1].$$

Let us set $S_1 := \left]0, \frac{1}{3}\right[\times Q'' \times]0, 1[$ and $S_2 := \left]\frac{2}{3}, 1\right[\times Q'' \times]0, 1[$.

Let C be the finite close cone in $\mathbb{R}^{N-1}$ determined by $B_{\frac{1}{6}}(P)$ with $P := (\frac{1}{6}, \frac{1}{6}, \ldots, \frac{1}{6})$. Clearly there exists $\delta > 0$ such that, for all n and for all $x \in K_n$, posed

$$\Phi_x(y) := y - x,$$

$\Phi_x : B_\delta(x) \to \mathbb{R}^N$ satisfies conditions (a) and (c) of Definition 3.3.1 with respect to C. Observe that condition (b) is not satisfied: in particular, $\Phi_x(B_\delta(x) \cap K_n) \not\subseteq \pi$.

Let $1 < p \leq 2$ and let us consider the Neumann problems

$$\begin{cases} -\Delta_p v + |v|^{p-2}v = f \\ v \in W^{1,p}(Q \setminus K_n) \end{cases} \tag{3.24}$$

with $f \in L^p(Q)$. We claim that the problems (3.24) do not converge to the Neumann problem

$$\begin{cases} -\Delta_p v + |v|^{p-2}v = f \\ v \in W^{1,p}(Q \setminus K) \end{cases} \tag{3.25}$$

in the sense given in Remark 3.3.6, that is $(u_n, \nabla u_n) \not\to (u, \nabla u)$ strongly in $L^p(Q; \mathbb{R}^{N+1})$ where u_n and u are the solutions of problems (3.24) and (3.25) respectively and the identification (3.5) is assumed. This implies that $W^{1,p}(Q \setminus K_n)$ does not converge to $W^{1,p}(Q \setminus K)$ in the sense of Mosco and so it proves that point (b) in the C-condition cannot be omitted.

We employ a Γ-convergence technique. Let us consider the following functionals $F_n : L^p(Q) \to [0, \infty]$ defined by

$$F_n(z) := \begin{cases} \dfrac{1}{p}\displaystyle\int_Q |\nabla z|^p & \text{if } z \in W^{1,p}(Q \setminus K_n) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.26}$$

We will prove that, up to a subsequence, (F_n) Γ-converges with respect to the strong topology of $L^p(Q)$ to a functional F such that if $z \in L^p(Q)$ and $F(z) < +\infty$, then

$$z_{|S_i} \in W^{1,p}(S_i) \quad \text{for } i = 1, 2, \tag{3.27}$$

$$z(\cdot, x_N) \in W^{1,p}(Q') \quad \text{for a.e } x_N \in]0,1[. \tag{3.28}$$

Let us assume for the moment (3.27) and (3.28). Given $f \in L^p(Q)$, the functional

$$G(u) := \frac{1}{p}\int_Q |u|^p - \int_Q fu$$

is a continuous perturbation of F_n: as a consequence,

$$\Gamma-\lim_n (F_n + G) = F + G.$$

Note that the solution u_n of problem (3.24) is precisely the minimum of $F_n + G$: from this, we derive that for all n

$$F_n(u_n) + G(u_n) \leq 0. \tag{3.29}$$

Suppose that the problems (3.24) converge to the problem (3.25): then in particular, $u_n \to u$ strongly in $L^p(Q)$ where, as usual, u is extended to 0 on K. Note that $F(u) < +\infty$ because of (3.29) and the Γ-liminf inequality. If we choose

$$f(x) := \chi_{S_1}$$

we conclude that u is equal to 1 on S_1 and equal to 0 on S_2. With the identification (3.5), we get $u = f$. Clearly $f(\cdot, x_N) \notin W^{1,p}(Q')$ for $x_N \in]0,1[$ and so we get a contradiction. This proves that the problems (3.24) do not converge to problem (3.25).

In order to perform the previous argument by contradiction, we have to prove (3.27) and (3.28). This can be done in the following way. Let $z_n \to z$ strongly in $L^p(Q)$ with

$$F_n(z_n) \leq C < +\infty. \tag{3.30}$$

Since

$$\frac{1}{p}\int_{S_1} |\nabla z_n|^p + \frac{1}{p}\int_{S_2} |\nabla z_n|^p \le C,$$

we deduce that $z_{|S_i} \in W^{1,p}(S_i)$ for $i = 1,2$ and so we get (3.27). For a.e. $x_N \in]0,1[$, we have that $z_n(\cdot, x_N) \to z(\cdot, x_N)$ strongly in $L^p(Q')$; by (3.30) and Fatou's lemma, we have

$$\frac{1}{p}\int_0^1 \left(\liminf_n \int_{Q'} |\nabla z_n(y, x_N)|^p \, dy\right) dx_N \le C,$$

so that for a.e. $x_N \in]0,1[$, there exists $C_{x_N} > 0$ and a subsequence n_k such that

$$\frac{1}{p}\int_{Q'} \sum_{i=1}^{N-1} |D_i z_{n_k}(y, x_N)|^p \, dy \le C_{x_N}.$$

We conclude that for a.e. $x_N \in [0,1]$, $z(\cdot, x_N) \in W^{1,p}(Q')$ so that (3.28) is proved and the proof is complete.

Chapter 4

A Γ-convergence approach to stability of unilateral minimality properties in fracture mechanics and applications

In this chapter[1] we deal with the problem of stability of *unilateral minimality properties* arising in fracture mechanics, and we give an application to the study of crack propagation in composite materials.

Let K be a $(N-1)$-dimensional set contained in $\Omega \subseteq \mathbb{R}^N$, and let u be a possibly vector valued function on Ω whose discontinuities are contained in K and which is sufficiently regular outside K. We say that the pair (u, K) is a *unilateral minimizer* with respect to the energy densities f and g if

$$\int_{\Omega \setminus K} f(x, \nabla u(x))\, dx + \int_K g(x, \nu)\, d\mathcal{H}^{N-1}(x) \leq \int_{\Omega \setminus H} f(x, \nabla v(x))\, dx + \int_H g(x, \nu)\, d\mathcal{H}^{N-1}(x). \tag{4.1}$$

for every $(N-1)$-dimensional set H containing K, and for every function v whose discontinuities are contained in H and which is sufficiently regular outside H. Here ν stands for the normal vector to K and H at the point x, while $\mathcal{H}^{N-1}$ stands for the $(N-1)$-dimensional Hausdorff measure. (u, K) is said to be *unilateral* minimizer because it is a minimum only among pairs (v, H) with H larger than K.

The problem of *stability* for the unilateral minimality property (4.1) can be formulated as follows: if (u_n, K_n) is a sequence of unilateral minimizers for f and g, with $u_n \to u$ and $K_n \to K$, is (u, K) still a unilateral minimizer?

As explained in the Introduction, the unilateral minimality property (4.1) and the problem of its stability are key points in the theory of quasistatic crack evolution in elastic bodies proposed by Francfort and Marigo in [54].

The first mathematical result of stability for unilateral minimality properties was obtained by Dal Maso and Toader [45] in a two dimensional setting under a topological restriction on the admissible cracks. They consider compact cracks with a bound on the number of connected components, and converging with respect to the Hausdorff metric. An extension of this result for unilateral minimality properties involving the symmetrized gradients of planar elasticity has been done by Chambolle in [33], while an extension to higher order minimality properties in connection to quasistatic crack growth in a plate has been proved by Acanfora and Ponsiglione in [1].

[1] The results of this chapter are contained in the paper:
Giacomini A., Ponsiglione M.: A Γ-convergence approach to stability of unilateral minimality properties in fracture mechanics and applications. Preprint SISSA 2004.

A second result of stability for unilateral minimality properties was obtained by Francfort and Larsen in [53], where they give an existence result for quasistatic crack evolutions in the context of SBV functions. In the framework of *generalized antiplanar shear* (i.e. $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$), the authors consider cracks K which are rectifiable sets in $\overline{\Omega}$, and associated displacements u in $SBV(\Omega)$ with jump set $S(u)$ contained in K (see Section 4.1 for a definition of rectifiable sets and of $SBV(\Omega)$). A key point for their result is the stability for unilateral minimizers of the form $(u_n, S(u_n))$ with bulk energy given by $f(x,\xi) = |\xi|^2$ and surface energy given by $g(x,\nu) \equiv 1$. More precisely, writing the minimality property in the equivalent form

$$\int_\Omega |\nabla u_n|^2\,dx \leq \int_\Omega |\nabla v|^2\,dx + \mathcal{H}^{N-1}(S(v) \setminus S(u_n)) \qquad \text{for all } v \in SBV(\Omega) \tag{4.2}$$

(which corresponds to (4.1) with $H = S(u_n) \cup S(v)$), they prove that if $u_n \rightharpoonup u$ weakly in $SBV(\Omega)$ (see Section 4.1 for a definition), then u satisfies the same minimality property. The main tool for proving stability is a geometrical construction which they called Transfer of Jump Sets [53, Theorem 2.1].

The case in which $S(u_n)$ is replaced by a rectifiable set K_n has been treated by Dal Maso, Francfort and Toader in [44], where they consider also a Carathéodory bulk energy $f(x,\xi)$ quasiconvex and with p growth assumptions in ξ, and a Borel surface energy $g(x,\nu)$ bounded and bounded away from zero. They employ a variational notion of convergence for rectifiable sets which they called σ^p-convergence to recover a crack K in the limit (see Section 4.1), and they prove a Transfer of Jump Sets theorem for $(K_n)_{n\in\mathbb{N}}$ satisfying $\mathcal{H}^{N-1}(K_n) \leq C$ [44, Theorem 5.1] in order to prove that minimality is preserved.

In this chapter we provide a different approach to the problem of stability of unilateral minimizer based on Γ-convergence which will permit also to treat the case of varying bulk and surface energies f_n and g_n. We restrict our analysis to the scalar case. Our approach is based on the observation that the problem has a variational character. In fact, considering for a while the case of fixed energies f and g with f convex in ξ, we have that if (u_n, K_n) is a unilateral minimizer for f and g, then u_n is a minimum for the functional

$$\mathcal{E}_n(v) := \int_\Omega f(x, \nabla v(x))\,dx + \int_{S(v)\setminus K_n} g(x,\nu)\,d\mathcal{H}^{N-1}(x). \tag{4.3}$$

Then the problem of stability of unilateral minimizers can be treated in the framework of Γ-convergence which ensures the convergence of minimizers. In Section 4.4, using an abstract representation result by Bouchitté, Fonseca, Leoni and Mascarenhas [17], we prove that the Γ-limit (up to a subsequence) of the functional $\mathcal{E}_n$ can be represented as

$$\mathcal{E}(v) := \int_\Omega f(x, \nabla v(x))\,dx + \int_{S(v)} g^-(x,\nu)\,d\mathcal{H}^{N-1}(x), \tag{4.4}$$

where g^- is a suitable function defined on $\Omega \times S^{N-1}$ determined only by g and $(K_n)_{n\in\mathbb{N}}$, and such that $g^- \leq g$. If we assume that $u_n \rightharpoonup u$ weakly in $SBV(\Omega)$, then by Γ-convergence we get that u is a minimizer for $\mathcal{E}$. Suppose now that K is a rectifiable set in Ω such that $S(u) \subseteq K$ and

$$g^-(x, \nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K. \tag{4.5}$$

Then we have immediately that the pair (u, K) is a unilateral minimizer for f and g because for all pairs (v, H) with $S(v) \subseteq H$ and $K \subseteq H$ we have

$$\begin{aligned}\int_\Omega f(x, \nabla u(x))\,dx = \mathcal{E}(u) \leq \mathcal{E}(v) = \int_\Omega f(x,\nabla v(x))\,dx + \int_{S(v)} g^-(x,\nu)\,d\mathcal{H}^{N-1} \\ = \int_\Omega f(x,\nabla v(x))\,dx + \int_{S(v)\setminus K} g^-(x,\nu) \leq \int_\Omega f(x,\nabla v(x))\,dx + \int_{H\setminus K} g(x,\nu).\end{aligned}$$

The rectifiable set K satisfying (4.5) is provided in Section 4.5, where we define a new variational notion of convergence for rectifiable sets which we call *σ-convergence*, and which departs from the notion of σ^p-convergence given in [44]. The σ-limit K of a sequence of rectifiable sets $(K_n)_{n\in\mathbb{N}}$ is constructed looking for the Γ-limit $\mathcal{H}^-$ in the strong topology of $L^1(\Omega)$ of the functionals

$$\mathcal{H}_n^-(u) := \begin{cases} \mathcal{H}^{N-1}(S(u)\setminus K_n) & u \in P(\Omega), \\ +\infty & \text{otherwise}, \end{cases} \tag{4.6}$$

where $P(\Omega)$ is the space of piecewise constant function in Ω (see (4.8)). Roughly, the σ-limit K is the maximal rectifiable set on which the density h^- representing $\mathcal{H}^-$ vanishes. By the growth estimate on g it turns out that K is also the maximal rectifiable set on which the density g^- vanishes, so that K is the natural limit candidate for K_n in order to preserve the unilateral minimality property. The definition of σ-convergence involves only the surface energies $\mathcal{H}_n^-$, and as a consequence it does not depend on the exponent p and it is stable with respect to infinitesimal perturbations in length (see Remark 4.5.10). Moreover it turns out that the σ-limit K contains the σ^p-limit points of $(K_n)_{n\in\mathbb{N}}$, so that our Γ-convergence approach improves also the minimality property given by the previous approaches.

Our method naturally extends to the case of varying bulk and surface energies f_n and g_n, and this is indeed the main motivation for which we developed our Γ-convergence approach. The key point to recover effective energies f and g for the minimality property in the limit is a Γ-convergence result for functionals of the form

$$\int_\Omega f_n(x, \nabla u_n(x))\,dx + \int_{S(u_n)} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x). \tag{4.7}$$

In Section 4.4, we prove that the Γ-limit has the form

$$\int_\Omega f(x,\nabla u(x))\,dx + \int_{S(u)} g(x,\nu)\,d\mathcal{H}^{N-1}(x),$$

where f is determined only by $(f_n)_{n\in\mathbb{N}}$, and g is determined only by $(g_n)_{n\in\mathbb{N}}$, that is no interaction occurs between the bulk and the surface part of the functionals in the Γ-convergence process. A result of this type has been proved in the case of periodic homogenization (in the vectorial case, and with dependence on the trace of u in the surface part of the energy) by Braides, Defranceschi and Vitali [21].

We notice that an approach to stability in the line of Dal Maso, Francfort and Toader in the case of varying energies would have required a Transfer of Jump Sets for f_n, g_n and f, g, which seems difficult to be derived directly. Our Γ-convergence approach also provides this result (Proposition 4.6.4).

In section 4.8 we deal with the study of quasistatic crack evolution in composite materials. More precisely we study the asymptotic behavior of a quasistatic evolution $t \to (u_n(t), K_n(t))$ relative to the bulk energy f_n and the surface energy g_n. Using our stability result we prove (Theorem 4.8.1) that $t \to (u_n(t), K_n(t))$ converges to a quasistatic evolution $t \to (u(t), K(t))$ relative to the effective bulk and surface energies f and g. Moreover convergence for bulk and surface energies for all times holds. This analysis applies to the case of composite materials, i.e. materials obtained through a fine mixture of different phases. The model case is that of periodic homogenization, i.e. materials with total energy given by

$$\mathcal{E}_\varepsilon(u,K) := \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u(x)\right)\,dx + \int_K g\left(\frac{x}{\varepsilon},\nu\right)\,d\mathcal{H}^{N-1}(x),$$

where ε is a small parameter giving the size of the mixture, and f, g are periodic in x. Our result implies that a quasisistatic crack evolution $t \to (u_\varepsilon(t), K_\varepsilon(t))$ for ε small is very near to a quasistatic evolution for the homogeneous material having bulk and surface energies f_{hom} and g_{hom}, which are obtained from f and g through periodic homogenization formulas available in the literature (see for example [21]).

The chapter is organized as follows. In Section 4.1 we recall the basic definitions on SBV functions and Γ-convergence. In Section 4.2 we prove a blow up result for Γ-limits which will be employed in the proof of the main results. In Section 4.3 we prove some representation results which we use in Section 4.4 where we deal with the Γ-convergence of free discontinuity problems like (4.7). The notion of σ-convergence for rectifiable sets is contained in Section 4.5, while the main result on stability for unilateral minimizer is contained in Section 4.6. In Section 4.7 we prove a stability result for unilateral minimality properties with boundary conditions which will be employed in Section 4.8 for the study of quasistatic crack evolution in composite materials.

4.1 Preliminaries

Sets with finite perimeter. We indicate by $P(\Omega)$ the family of sets with finite perimeter in Ω, that is the class of sets $E \subseteq \Omega$ such that $1_E \in BV(\Omega)$. In view of the applications of Sections 4.3, 4.4 and 4.5, it will be useful to look at $P(\Omega)$ in term of functions, that is to use the following equivalent description:

$$P(\Omega) = \{u \in BV(\Omega) \, : \, u(x) \in \{0,1\} \text{ for a.e. } x \in \Omega\}. \tag{4.8}$$

σ^p-convergence of sets. In [44] Dal Maso, Francfort and Toader defined a variational notion of convergence for sets in $\mathbb{R}^N$ which they called *σ^p-convergence*, and that they employed for the study of quasistatic crack growth in nonlinear elasticity.

Definition 4.1.1. *Let $(K_n)_{n\in\mathbb{N}}$ and K be subsets of Ω. We say that K_n σ^p-converges in Ω to K if the following hold*

(1) if $u_h \rightharpoonup u$ weakly in $SBV^p(\Omega)$ with $S(u_h) \subseteq K_{n_h}$, then $S(u) \subseteq K$;

(2) $K = S(u)$ and there exists $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$ with $S(u_n) \subseteq K_n$.

In the same paper the authors proved the following compactness property.

Theorem 4.1.2. *If $\mathcal{H}^{N-1}(K_n) \leq C$, then up to a subsequence $K_n \to K$ in the sense of σ^p-convergence.*

4.2 Blow-up for Γ-limits

Let $1 < p < +\infty$ and let $f : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function such that

$$a_1(x) + \alpha|\xi|^p \leq f(x,\xi) \leq a_2(x) + \beta|\xi|^p, \tag{4.9}$$

where $a_1, a_2 \in L^1(\Omega)$ and $\alpha, \beta > 0$. Let us assume that

$$\xi \to f(x,\xi) \quad \text{is convex for a.e. } x \in \Omega.$$

Let B_1 be the unit ball in $\mathbb{R}^N$ with center 0 and radius 1. The following blow up result in the sense of Γ-convergence holds.

Lemma 4.2.1. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence converging to zero. Then for a.e. $x \in \Omega$ the functionals*

$$F_k(u) := \begin{cases} \int_{B_1} f(x + \rho_k y, \nabla u(y)) \, dy & u \in W^{1,p}(B_1), \\ +\infty & \textit{otherwise in } L^1(B_1) \end{cases} \tag{4.10}$$

Γ-converge in the strong topology of $L^1(B_1)$ to the functional

$$F(u) := \begin{cases} \int_{B_1} f(x, \nabla u(y)) \, dy & u \in W^{1,p}(B_1), \\ +\infty & \textit{otherwise in } L^1(B_1). \end{cases} \tag{4.11}$$

Proof. By Scorza-Dragoni Theorem there exists a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ such that $|\Omega \setminus K_n| \to 0$ and such that f restricted to $K_n \times \mathbb{R}^N$ is continuous. Let us define

$$\mathcal{N} := \Omega \setminus \{x \in \Omega \,:\, \text{there exists } n \in \mathbb{N} \text{ such that } x \text{ is of density 1 for } K_n\}. \tag{4.12}$$

We clearly have $|\mathcal{N}| = 0$, and every $x \in \Omega \setminus \mathcal{N}$ is a Lebesgue point for $f(\cdot, \xi)$ for every $\xi \in \mathbb{R}^N$.

Let us fix $x \in \Omega \setminus \mathcal{N}$ with $x \in K_n$ for some $n \in \mathbb{N}$, and let us begin with the proof of the Γ-limsup inequality. We can prove it for a dense set in $W^{1,p}(B_1)$, for example for the piecewise affine functions. So let u be piecewise affine, and let $\nabla u(y) \in \{\xi_1, \dots, \xi_m\}$ for all $y \in B_1$. Since x is of density 1 for K_n and f is continuous on $K_n \times \mathbb{R}^N$, we have that for all $\varepsilon > 0$

$$|\{y \in B_1 \,:\, |f(x + \rho_k y, \xi_i) - f(x, \xi_i)| > \varepsilon\}| \to 0.$$

Then considering as recovering sequence $u_k = u$, we get

$$\limsup_{k\to+\infty} \int_{B_1} f(x + \rho_k y, Du)\, dy \le \int_{B_1} f(x, Du)\, dy,$$

so that the inequality is proved.

Let us come to the Γ-liminf inequality. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $L^1(B_1)$ such that $u_k \to u$ strongly in $L^1(B_1)$. We can assume that $\sup_{k\in\mathbb{N}} F_k(u_k) < +\infty$, so that $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(B_1; \mathbb{R}^N)$. Let $M > 0$ be fixed, and let δ be such that $|\{|\nabla u| \ge M\}| \le \delta$. Let us consider

$$\Phi_k^M(y) := \begin{cases} \nabla u_k(y) & \text{if } |\nabla u_k(y)| \le M, \\ 0 & \text{otherwise,} \end{cases}$$

and let us denote by Φ^M its weak limit (up to a further subsequence) in $L^p(B_1; \mathbb{R}^N)$. Since by assumption on x we have that for all $\varepsilon > 0$

$$\lim_{k\to+\infty} |\{y \in B_1 \,:\, |f(x + \rho_k y, \Phi_k^M(y)) - f(x, \Phi_k^M(y))| > \varepsilon\}| = 0,$$

we obtain

$$\begin{aligned}
\liminf_{k\to+\infty} \int_{B_1} f(x + \rho_k y, \nabla u_k(y))\, dy &\ge \liminf_{k\to+\infty} \int_{B_1} f(x + \rho_k y, \Phi_k^M(y))\, dy - e(\delta) \\
&\ge \liminf_{k\to+\infty} \int_{B_1} f(x, \Phi_k^M(y))\, dy - \tilde{e}(\delta) \ge \int_{B_1} f(x, \Phi^M(y))\, dy - \tilde{e}(\delta),
\end{aligned}$$

where $\tilde{e}(\delta) \to 0$ as $\delta \to 0$. Letting $M \to +\infty$, we get $\delta \to 0$ and $\Phi^M \rightharpoonup \nabla u$ weakly in $L^p(B_1, \mathbb{R}^N)$. The result follows by lower semicontinuity since $f(x, \cdot)$ is convex. $\square$

Let us consider now $f_n : \Omega \times \mathbb{R}^N \to [0, +\infty[$ Carathéodory function satisfying the growth estimate (4.9) uniformly in n. Let us assume that for all $A \in \mathcal{A}(\Omega)$ the localized functionals

$$\mathcal{F}_n(u, A) := \begin{cases} \int_A f_n(x, \nabla u(x))\, dx & u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.13}$$

Γ-converge with respect to the strong topology of $L^1(\Omega)$ to

$$\mathcal{F}(u, A) := \begin{cases} \int_A f(x, \nabla u(x))\, dx & u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise} \end{cases} \tag{4.14}$$

for some Carathéodory function f which satisfies estimate (4.9). Using a diagonal argument we may conclude that the following theorem holds.

Theorem 4.2.2. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence converging to zero. Then for a.e. $x \in \Omega$ there exists $(n_k)_{k\in\mathbb{N}}$ such that the functionals*

$$(4.15) \qquad F_k(u) := \begin{cases} \int_{B_1} f_{n_k}(x+\rho_k y, \nabla u(y))\,dy & u \in W^{1,p}(B_1), \\ +\infty & \text{otherwise in } L^1(B_1) \end{cases}$$

Γ-converge in the strong topology of $L^1(B_1)$ to the functional

$$(4.16) \qquad F(u) := \begin{cases} \int_{B_1} f(x, \nabla u(y))\,dy & u \in W^{1,p}(B_1), \\ +\infty & \text{otherwise in } L^1(B_1). \end{cases}$$

Remark 4.2.3. In the case of periodic homogenization, i.e. in the case in which $f_n(x,\xi) := f(nx,\xi)$ with f periodic in x, it is sufficient to choose n_k in such a way that $n_k\rho_k \to +\infty$. In fact for $x = 0$ we have

$$F_k(u) := \begin{cases} \int_{B_1} f((n_k\rho_k)y, \nabla u(y))\,dy & u \in W^{1,p}(B_1), \\ +\infty & \text{otherwise in } L^1(B_1) \end{cases}$$

which still Γ-converges to (see for instance [43])

$$F(u) := \begin{cases} \int_{B_1} f_{\text{hom}}(\nabla u(y))\,dy & u \in W^{1,p}(B_1), \\ +\infty & \text{otherwise in } L^1(B_1). \end{cases}$$

In the rest of this section we prove a regularity result for the density f defined in (4.14) under additional hypothesis on f_n which will be employed in Section 4.8. Let us assume that for a.e. $x \in \Omega$

(1) $f_n(x,\cdot)$ is convex;

(2) $f_n(x,\cdot)$ is of class C^1;

(3) for all $M \geq 0$ and for all ξ_n^1, ξ_n^2 such that $|\xi_n^1| \leq M$, $|\xi_n^1| \leq M$, $|\xi_n^1 - \xi_n^2| \to 0$ we have

$$(4.17) \qquad |\nabla_\xi f_n(x,\xi_n^1) - \nabla_\xi f_n(x,\xi_n^2)| \to 0.$$

Notice that for instance $f_n(x,\xi) := a_n(x)|\xi|^p$ with $\alpha \leq a_n(x) \leq \beta$ satisfies the assumptions above. Notice moreover that by semicontinuity of Γ-limits $\xi \to f(x,\xi)$ is convex for a.e. $x \in \Omega$.

We need the following lemma which is a straightforward variant of [44, Lemma 4.9].

Lemma 4.2.4. *Let (X, A, μ) be a finite measure space, $p > 1$, $N \geq 1$, and let $H_n : X \times \mathbb{R}^N \to \mathbb{R}$ be a sequence of Carathéodory functions which satysfies the following properties: there exist a positive constant $a \geq 0$ and a nonnegative function $b \in L^{p'}(X)$, with $p' = p/(p-1)$ such that*

(1) $|H_n(x,\xi)| \leq a|\xi|^{p-1} + b(x)$ for every $x \in X$, $\xi \in \mathbb{R}^N$;

(2) for all $M \geq 0$ and for a.e. $x \in \Omega$, for all ξ_n^1, ξ_n^2 such that $|\xi_n^1| \leq M$, $|\xi_n^1| \leq M$, $|\xi_n^1 - \xi_n^2| \to 0$ we have

$$|H_n(x,\xi_n^1) - H_n(x,\xi_n^2)| \to 0.$$

Assume that $(\Phi_n)_{n\in\mathbb{N}}$ is bounded in $L^p(X,\mathbb{R}^N)$ and that $(\Psi_n)_{n\in\mathbb{N}}$ converges to 0 strongly in $L^p(X,\mathbb{R}^N)$. Then

$$(4.18) \qquad \int_X [H_n(x,\Phi_n(x) + \Psi_n(x)) - H_n(x,\Phi_n(x))]\Phi(x)\,d\mu(x) \to 0,$$

for every $\Phi \in L^p(X,\mathbb{R}^N)$.

The following regularity result on f holds.

Proposition 4.2.5. *For a.e. $x \in \Omega$ the function $\xi \to f(x, \xi)$ is of class C^1.*

Proof. Let $x \in \Omega \setminus \mathcal{N}$, where $\mathcal{N}$ is defined in (4.12). Let $\rho_k \to 0$ and let $(n_k)_{k\in\mathbb{N}}$ be a sequence such that, according to Theorem 4.2.2, $(F_k)_{k\in\mathbb{N}}$ Γ-converges with respect to the strong topology of $L^1(B_1)$ to F.

Let $(\phi_k)_{k\in\mathbb{N}}$ be a recovering sequence for the affine function $y \to \xi \cdot y$ with $\xi \in \mathbb{R}^N$. Up to a further subsequence, we can always assume that there exists $\psi \in \mathbb{R}^N$ such that

$$\frac{1}{|B_1|}\int_{B_1} \nabla_\xi f_{n_k}(x+\rho_k y, \nabla\phi_k(y))\, dy \to \psi. \tag{4.19}$$

Let $t_j \searrow 0$ and let $\eta \in \mathbb{R}^N$. By the convexity of f_{n_k} in the second variable, we have

$$\begin{aligned}\int_{B_1} f_{n_k}(x+\rho_k y, \nabla\phi_k(y)+t_j\eta) - f_{n_k}(x+\rho_k y, \nabla\phi_k(y))\, dy& \\ \le t_j \int_{B_1} \nabla_\xi f_{n_k}(x+\rho_k y, \nabla\phi_k(y)+t_j\eta)\eta\, dy&.\end{aligned} \tag{4.20}$$

By Γ-convergence we can find k_j such that

$$\frac{f(x,\xi+t_j\eta)-f(x,\xi)}{t_j} - \frac{1}{j} \le \frac{1}{|B_1|}\int_{B_1} \nabla_\xi f_{n_{k_j}}(x+\rho_{k_j} y, \nabla\phi_{k_j}(y)+t_j\eta)\eta\, dy,$$

so that we have

$$\limsup_{j\to+\infty} \frac{f(x,\xi+t_j\eta)-f(x,\xi)}{t_j} \le \frac{1}{|B_1|}\limsup_{j\to+\infty}\int_{B_1} \nabla_\xi f_{n_{k_j}}(x+\rho_{k_j} y, \nabla\phi_{k_j}(y)+t_j\eta)\eta\, dy. \tag{4.21}$$

Notice that by Lemma 4.2.4 and by (4.19) we have that

$$\begin{aligned}\lim_{j\to+\infty}\int_{B_1} \nabla_\xi f_{n_{k_j}}(x+\rho_{k_j} y, \nabla\phi_{k_j}(y)+t_j\eta)\eta\, dy& \\ = \lim_{j\to+\infty}\int_{B_1} \nabla_\xi f_{n_{k_j}}(x+\rho_{k_j} y, \nabla\phi_{k_j}(y))\eta\, dy = |B_1|\psi\eta&,\end{aligned}$$

and so for every subgradient ζ of $f(x,\cdot)$ at ξ by (4.21) we have

$$\zeta\eta \le \limsup_{j\to+\infty} \frac{f(x,\xi+t_j\eta)-f(x,\xi)}{t_j} \le \psi\eta.$$

We deduce that $\zeta = \psi$, so that $f(x,\cdot)$ is Gateaux differentiable at ξ with $\nabla_\xi f(x,\xi) = \psi$: since $f(x,\cdot)$ is convex, we get that $f(x,\cdot)$ is of class C^1. □

Remark 4.2.6. Notice that an hypothesis of *equiuniform continuity* for $(\nabla_\xi f_n(x,\xi))_{n\in\mathbb{N}}$ like (4.17) is needed in order to preserve C^1-regularity in the passage from f_n to f: in fact if $\xi \to f_n(\xi)$ are smooth convex functions uniformly converging to a nondifferentiable convex function $\xi \to f(\xi)$, the associated integral functionals Γ-converge, and this provides a counterexample.

4.3 Some integral representation lemmas

Let $a_1, a_2 \in L^1(\Omega)$, $1 < p < +\infty$, and let $\alpha, \beta > 0$. For all $n \in \mathbb{N}$ let $f_n : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$

$$a_1(x) + \alpha|\xi|^p \le f_n(x,\xi) \le a_2(x) + \beta|\xi|^p, \tag{4.22}$$

and let $g_n : \Omega \times S^{N-1} \to [0,+\infty[$ be a Borel function such that for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ and for all $\nu \in S^{N-1} := \{\eta \in \mathbb{R}^N : |\eta| = 1\}$

$$\alpha \leq g_n(x,\nu) \leq \beta. \tag{4.23}$$

In Section 4.4 we will be interested in the functionals on $L^1(\Omega) \times \mathcal{A}(\Omega)$

$$\mathcal{E}_n(u,A) := \begin{cases} \int_A f_n(x,\nabla u(x))\,dx + \int_{A\cap(S(u)\setminus K_n)} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x) & u \in SBV^p(A), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.24}$$

where $\mathcal{A}(\Omega)$ denotes the family of open subsets of Ω, and $(K_n)_{n\in\mathbb{N}}$ is a sequence of rectifiable sets in Ω such that

$$\mathcal{H}^{N-1}(K_n) \leq C. \tag{4.25}$$

In particular we will be interested in the Γ-limit in the strong topology of $L^1(\Omega)$ of $(\mathcal{E}_n(\cdot,A))_{n\in\mathbb{N}}$ for every $A \in \mathcal{A}(\Omega)$. To this extend we consider the functionals $\mathcal{F}_n : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty]$

$$\mathcal{F}_n(u,A) := \begin{cases} \int_A f_n(x,\nabla u(x))\,dx & u \in W^{1,p}(A), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.26}$$

and the functionals $\mathcal{G}_n^- : P(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty[$

$$\mathcal{G}_n^-(u,A) := \int_{A\cap(S(u)\setminus K_n)} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x) \tag{4.27}$$

defined on Sobolev and piecewise constant functions with values in $\{0,1\}$ (see (4.8)) respectively, and we will reconstruct the Γ-limit of $(\mathcal{E}_n(\cdot,A))_{n\in\mathbb{N}}$ through the Γ-limits of $(\mathcal{F}_n(\cdot,A))_{n\in\mathbb{N}}$ and $(\mathcal{G}_n^-(\cdot,A))_{n\in\mathbb{N}}$.

For the results of Section 4.6, we will need also the functionals $\mathcal{G}_n : P(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty[$

$$\mathcal{G}_n(u,A) := \int_{A\cap S(u)} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x) \tag{4.28}$$

In this section we provide some integral representation results for the Γ-limits of the functionals $\mathcal{F}_n, \mathcal{G}_n^-, \mathcal{G}_n$ and $\mathcal{E}_n$. In the following, for every functional $\mathcal{H}$ defined on $X \times \mathcal{A}(\Omega)$ with $X = L^1(\Omega)$ or $X = P(\Omega)$ with values to $[0,+\infty]$, for every $A \in \mathcal{A}(\Omega)$ and $\psi \in L^1(A)$ we will use the notation

$$\mathbf{m}_{\mathcal{H}}(A,\psi) = \inf_{u\in X}\{\mathcal{H}(u,A) : u = \psi \text{ in a neighborhood of } \partial A\}. \tag{4.29}$$

Moreover for all $x \in \mathbb{R}^N$, $a,b \in \mathbb{R}$ and $\nu \in S^{N-1}$ let $u_{x,a,b,\nu} : B_1(x) \to \mathbb{R}$ be defined by

$$u_{x,a,b,\nu}(y) := \begin{cases} b & \text{if } (y-x)\nu \geq 0, \\ a & \text{if } (y-x)\nu < 0, \end{cases} \tag{4.30}$$

where $B_1(x)$ is the ball of center x and radius 1.

The following Γ-convergence and representation result for the functionals $\mathcal{F}_n$ holds.

Proposition 4.3.1. *There exists $\mathcal{F} : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty]$ such that up to a subsequence the functionals $\mathcal{F}_n(\cdot,A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot,A)$ for every $A \in \mathcal{A}(\Omega)$. Moreover for all $u \in W^{1,p}(\Omega)$ we have that*

$$\mathcal{F}(u,A) = \int_A f(x,\nabla u(x))\,dx \tag{4.31}$$

where

$$f(x,\xi) := \limsup_{\rho\to 0^+} \frac{\mathbf{m}_{\mathcal{F}}(B_\rho(x),\xi(z-x))}{\omega_N \rho^N}, \tag{4.32}$$

$\mathbf{m}_{\mathcal{F}}$ is defined in (4.29), and ω_N is the volume of the unit ball in $\mathbb{R}^N$. Finally f is a Carathéodory function satisfying the growth conditions (4.22).

Proof. Let us consider the restriction $\tilde{\mathcal{F}}_n$ of $\mathcal{F}_n$ to $L^p(\Omega) \times \mathcal{A}(\Omega)$. Then in view of the growth estimate (4.22), by [43, Theorem 19.6] we deduce that there exists $\tilde{\mathcal{F}} : L^p(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that up to a subsequence $\tilde{\mathcal{F}}_n(\cdot, A)$ Γ-converges in the strong topology of $L^p(\Omega)$ to $\tilde{\mathcal{F}}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$.

For every $u \in L^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$ let us set

$$\mathcal{F}(u, A) := \limsup_{M \to +\infty} \tilde{\mathcal{F}}(T_M(u), A)$$

where $T_M(u) := \min\{\max\{u, -M\}, M\}$. Let us prove that along the same subsequence $\mathcal{F}_n(\cdot, A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. As for the Γ-liminf inequality, let us consider a sequence $(u_n)_{n \in \mathbb{N}}$ in $L^1(\Omega)$ with $u_n \to u$ strongly in $L^1(\Omega)$. Then for every $M > 0$ we have that $T_M(u_n) \to T_M(u)$ strongly in $L^p(\Omega)$, so that for every $A \in \mathcal{A}(\Omega)$ we have

$$\tilde{\mathcal{F}}(T_M(u), A) \le \liminf_{n \to +\infty} \tilde{\mathcal{F}}_n(T_M(u_n), A) \le \liminf_{n \to +\infty} \mathcal{F}_n(u_n, A) + e(M),$$

where $e(M) \to 0$ as $M \to +\infty$. Taking the limsup for $M \to +\infty$, we get that the Γ-liminf inequality holds.

Let us come to Γ-limsup inequality. It is sufficient to consider $u \in L^\infty(\Omega)$, since $L^\infty(\Omega)$ is a subset of $L^1(\Omega)$ dense in energy with respect to $\mathcal{F}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. A recovering sequence for u with respect to $\tilde{\mathcal{F}}_n(\cdot, A)$ and the strong topology of $L^p(\Omega)$ is a good recovering sequence for $\mathcal{F}_n(\cdot, A)$ and the strong topology of $L^1(\Omega)$, so that the Γ-limsup inequality is proved.

We have that the following facts hold:

($\mathcal{F}1$) for all $u \in L^1(\Omega)$, $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;

($\mathcal{F}2$) $\mathcal{F}(u, A) = \mathcal{F}(v, A)$ if $u = v$ on A;

($\mathcal{F}3$) $\mathcal{F}(u + C, A) = \mathcal{F}(u, A)$ for every constant C;

($\mathcal{F}4$) $\mathcal{F}(\cdot, A)$ is lower semicontinuous with respect to the strong topology of $L^1(\Omega)$;

($\mathcal{F}5$) we have the growth estimate

$$\int_A a_1(x)\, dx + \alpha \|\nabla u\|^p_{L^p(A;\mathbb{R}^N)} \le \mathcal{F}(u, A) \le \int_A a_2(x)\, dx + \beta \|\nabla u\|^p_{L^p(A;\mathbb{R}^N)}.$$

Then the theorem follows by the representation result by Buttazzo and Dal Maso [31] (see also [17, Theorem 2]). □

Let us come to the functionals $\mathcal{G}_n$ defined in (4.28). The following proposition holds.

Proposition 4.3.2. *There exists $\mathcal{G} : P(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty[$ such that up to a subsequence $\mathcal{G}_n(\cdot, A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{G}(\cdot, A)$ for all $A \in \mathcal{A}(\Omega)$. Moreover for all $u \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we have that*

$$\mathcal{G}(u, A) = \int_{A \cap S(u)} g(x, \nu)\, dx \tag{4.33}$$

with

$$g(x, \nu) := \limsup_{\rho \to 0^+} \frac{\mathbf{m}_{\mathcal{G}}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}}, \tag{4.34}$$

where $\mathbf{m}_{\mathcal{G}}$ is defined in (4.29) and $u_{x,0,1,\nu}$ is as in (4.30).

Proof. The existence of $\mathcal{G} : P(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty[$ such that up to a subsequence $\mathcal{G}_n(\cdot, A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{G}(\cdot, A)$ for all $A \in \mathcal{A}(\Omega)$ has been proved by Ambrosio and Braides [6, Theorem 3.2]. By the growth estimate on g_n we get that $\mathcal{G}$ satisfies the following properties:

($\mathcal{G}1$) for all $u \in P(\Omega)$, $\mathcal{G}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;

($\mathcal{G}2$) $\mathcal{G}(u, A) = \mathcal{G}(v, A)$ if $u = v$ on A;

($\mathcal{G}3$) $\mathcal{G}(\cdot, A)$ is lower semicontinuous with respect to the strong topology of $L^1(\Omega)$;

($\mathcal{G}4$) we have the growth estimate

$$\alpha\mathcal{H}^{N-1}(S(u) \cap A) \le \mathcal{G}(u, A) \le \beta\mathcal{H}^{N-1}(S(u) \cap A).$$

Then the representation formulas (4.33) and (4.34) come from [17, Theorem 3]. □

Let us come to the functionals $\mathcal{G}_n^-$ defined in (4.27). The following proposition holds.

Proposition 4.3.3. *There exists $\mathcal{G}^- : P(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty[$ such that up to a subsequence $\mathcal{G}_n^-(\cdot, A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{G}^-(\cdot, A)$ for all $A \in \mathcal{A}(\Omega)$. Moreover for all $u \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we have that*

$$\mathcal{G}^-(u, A) = \int_{A \cap S(u)} g^-(x, \nu)\, d\mathcal{H}^{N-1}(x) \tag{4.35}$$

with

$$g^-(x, \nu) := \limsup_{\rho \to 0^+} \frac{\mathbf{m}_{\mathcal{G}^-}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}}, \tag{4.36}$$

where $\mathbf{m}_{\mathcal{G}^-}$ is defined in (4.29) *and $u_{x,0,1,\nu}$ is as in* (4.30).

Proof. By the growth estimate (4.23) on g_n and the result of Ambrosio and Braides [6] there exists $\mathcal{G}^- : P(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that up to a subsequence $(\mathcal{G}_n^-(\cdot, A))_{n \in \mathbb{N}}$ Γ-converges in the strong topology of $L^1(\Omega)$ to $\mathcal{G}^-(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$, and such that the following properties hold:

($\mathcal{G}^-1$) for all $u \in P(\Omega)$, $\mathcal{G}^-(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;

($\mathcal{G}^-2$) $\mathcal{G}^-(u, A) = \mathcal{G}^-(v, A)$ if $u = v$ on A;

($\mathcal{G}^-3$) $\mathcal{G}^-(\cdot, A)$ is lower semicontinuous with respect to the strong convergence in $L^1(\Omega)$;

($\mathcal{G}^-4$) we have the growth estimate

$$0 \le \mathcal{G}^-(u, A) \le \beta\mathcal{H}^{N-1}(S(u) \cap A).$$

The integral representation formula (4.35) for $\mathcal{G}^-(\cdot, A)$ is given by the result of Ambrosio and Braides [6] in view of properties ($\mathcal{G}^-1$)-($\mathcal{G}^-4$) (see also Bouchitté, Fonseca, Leoni and Mascarenhas [17]). For the sequel we need also the explicit formula (4.36) for the density g^- which is not given directly by the results of [6] and [17] because of a lack of coercivity from below. So in what follows, we modify the concrete approximation $\mathcal{G}_n^-(\cdot, A)$ for $\mathcal{G}^-(\cdot, A)$ in order to get the coerciveness we need, and to obtain in the end formula (4.36).

Let us consider the functionals

$$\mathcal{G}_n^\varepsilon(u, A) := \int_{A \cap S(u)} g_n^\varepsilon(x, \nu)\, d\mathcal{H}^{N-1}(x) \tag{4.37}$$

where

$$g_n^\varepsilon(x, \nu) := \begin{cases} \varepsilon & \text{if } x \in K_n, \nu = \nu_{K_n}(x), \\ g_n(x, \nu) & \text{otherwise.} \end{cases} \tag{4.38}$$

Let us denote by $\mathcal{G}^\varepsilon(\cdot, A)$ the Γ-limit (up to a subsequence) of $\mathcal{G}_n^\varepsilon(\cdot, A)$ for all $A \in \mathcal{A}(\Omega)$. Since $\mathcal{G}^\varepsilon$ is such that for ε small

$$\varepsilon \mathcal{H}^{N-1}(S(u) \cap A) \leq \mathcal{G}^\varepsilon(u, A) \leq \beta \mathcal{H}^{N-1}(S(u) \cap A),$$

by the representation result of [19] we have that

$$\mathcal{G}^\varepsilon(u, A) = \int_{S(u)\cap A} g^\varepsilon(x, \nu)\, d\mathcal{H}^{N-1}(x),$$

where $g^\varepsilon : \Omega \times S^{N-1} \to [0, +\infty]$ is given by

$$g^\varepsilon(x, \nu) := \limsup_{\rho \to 0^+} \frac{\mathbf{m}_{\mathcal{G}^\varepsilon}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}}. \tag{4.39}$$

We have for all $u \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$

$$\mathcal{G}_n^\varepsilon(u, A) \leq \mathcal{G}_n^-(u, A) + \varepsilon \mu_n(A),$$

where $\mu_n := \mathcal{H}^{N-1} \llcorner K_n$, so that for $n \to +\infty$ by Γ-convergence we have

$$\mathcal{G}^\varepsilon(u, A) \leq \mathcal{G}^-(u, A) + \varepsilon \mu(\bar{A}), \tag{4.40}$$

where μ is the weak* limit of $(\mu_n)_{n\in\mathbb{N}}$ (up to a subsequence) in the sense of measures. Notice that (see for instance [8, Theorem 2.56]) up to a set of $\mathcal{H}^{N-1}$-measure zero we have

$$H(x) := \limsup_{\rho \to 0^+} \frac{\mu(\bar{B}_\rho(x))}{\omega_{N-1}\rho^{N-1}} < +\infty. \tag{4.41}$$

Let us prove that for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ we have

$$g^-(x, \nu) = \lim_{\varepsilon \to 0} g^\varepsilon(x, \nu), \tag{4.42}$$

where $g^-(x, \nu)$ is defined in (4.36). In fact, notice that $\{g^\varepsilon\}_\varepsilon$ is monotone decreasing in ε and that $g^- \leq g^\varepsilon$ for all $\varepsilon > 0$, so that for all x and ν

$$g^-(x, \nu) \leq \lim_{\varepsilon \to 0} g^\varepsilon(x, \nu).$$

Let us set for every $\rho > 0$, $x \in \Omega$ and $\nu \in S^{N-1}$

$$m_\rho^\varepsilon(x, \nu) := \frac{\mathbf{m}_{\mathcal{G}^\varepsilon}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}} \quad \text{and} \quad m_\rho^-(x, \nu) := \frac{\mathbf{m}_{\mathcal{G}^-}(B_\rho(x), u_{x,0,1,\nu})}{\omega_{N-1}\rho^{N-1}}.$$

Then by (4.40) we have that

$$m_\rho^\varepsilon(x, \nu) \leq m_\rho^-(x, \nu) + \varepsilon \frac{\mu(\bar{B}_\rho(x))}{\omega_{N-1}\rho^{N-1}}.$$

Taking the lim sup for $\rho \to 0^+$ we have

$$g^\varepsilon(x, \nu) \leq g^-(x, \nu) + \varepsilon H(x),$$

and so letting $\varepsilon \to 0$ we obtain for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$

$$\lim_{\varepsilon \to 0} g^\varepsilon(x, \nu) \leq g^-(x, \nu)$$

which gives (4.42). Since for all $u \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we have $\mathcal{G}^\varepsilon(u, A) \to \mathcal{G}^-(u, A)$ as $\varepsilon \to 0$, we conclude that

$$\mathcal{G}^-(u, A) = \lim_{\varepsilon \to 0} \mathcal{G}^\varepsilon(u, A) = \lim_{\varepsilon \to 0} \int_{S(u)\cap A} g^\varepsilon(x, \nu)\, d\mathcal{H}^{N-1}(x) = \int_{S(u)\cap A} g^-(x, \nu)\, d\mathcal{H}^{N-1}(x), \tag{4.43}$$

so that the representation formulas (4.35) and (4.36) hold. $\square$

Remark 4.3.4. It is immediate to check that if we replace $P(\Omega)$ in Proposition 4.3.3 by the space $P_{a,b}(\Omega) := \{u \in BV(\Omega) : u(x) \in \{a,b\} \text{ for a.e. } x \in \Omega\}$, with $a, b \in \mathbb{R}$, then the Γ-limit in the strong topology of $L^1(\Omega)$ of $\mathcal{G}_n^-(\cdot, A)$ can still be represented by the density g^- defined in (4.36).

Let us finally come to the functionals $\mathcal{E}_n$ defined in (4.24). Using the growth estimates (4.22) and (4.23) on f_n and g_n (see [21]), there exists $\mathcal{E} : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty[$ such that up to a subsequence $\mathcal{E}_n(\cdot, A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{E}(\cdot, A)$ for all $A \in \mathcal{A}(\Omega)$. For every $\varepsilon > 0$ let us set

$$\mathcal{E}_\varepsilon(u, A) := \mathcal{E}(u, A) + \varepsilon \int_{S(u)\cap A} 1 + |[u]| \, d\mathcal{H}^{N-1}.$$

For ε small, we have that $\mathcal{E}_\varepsilon$ satisfies the following properties:

($\mathcal{E}_\varepsilon$1) for all $u \in SBV^p(\Omega)$, $\mathcal{E}_\varepsilon(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;

($\mathcal{E}_\varepsilon$2) $\mathcal{E}_\varepsilon(u, A) = \mathcal{E}_\varepsilon(v, A)$ if $u = v$ on A;

($\mathcal{E}_\varepsilon$3) $\mathcal{E}_\varepsilon(\cdot, A)$ is lower semicontinuous with respect the strong topology of $L^1(\Omega)$;

($\mathcal{E}_\varepsilon$4) we have the growth estimate

$$\begin{aligned}\int_A a_1 \, dx + \varepsilon \left(\int_A |\nabla u|^p \, dx + \int_{S(u)\cap A} 1 + |[u]| \, d\mathcal{H}^{N-1} \right) &\le \mathcal{E}_\varepsilon(u, A) \\ &\le \int_A a_2 \, dx + \beta \left(\int_A |\nabla u|^p \, dx + \int_{S(u)\cap A} 1 + |[u]| \, d\mathcal{H}^{N-1} \right).\end{aligned}$$

Then by the representation result of Bouchitté, Fonseca, Leoni and Mascarenhas [17, Theorem 1] we get that

$$\mathcal{E}_\varepsilon(u, A) = \int_A f_\infty^\varepsilon(x, \nabla u(x)) \, dx + \int_{A\cap S(u)} g_\infty^\varepsilon(x, u^-(x), u^+(x), \nu) \, d\mathcal{H}^{N-1}(x)$$

with f_∞^ε and g_∞^ε satisfying the following formulas

$$f_\infty^\varepsilon(x, \xi) := \limsup_{\rho \to 0^+} \frac{\mathbf{m}_{\mathcal{E}_\varepsilon}(B_\rho(x), \xi(z - x))}{\omega_N \rho^N}, \tag{4.44}$$

and

$$g_\infty^\varepsilon(x, a, b, \nu) := \limsup_{\rho \to 0^+} \frac{\mathbf{m}_{\mathcal{E}_\varepsilon}(B_\rho(x), u_{x,a,b,\nu})}{\omega_{N-1} \rho^{N-1}}, \tag{4.45}$$

where $\mathbf{m}_{\mathcal{E}_\varepsilon}$ is defined in (4.29) and $u_{x,a,b,\nu}$ is as in (4.30).

Notice that f_∞^ε and g_∞^ε are monotone decreasing in ε, and that $\mathcal{E}_\varepsilon(\cdot, A)$ converges pointwise to $\mathcal{E}(\cdot, A)$ as $\varepsilon \to 0$ for all $A \in \mathcal{A}(\Omega)$. We conclude that the representation result for $\mathcal{E}_\varepsilon$ implies a representation result for the functional $\mathcal{E}$.

Summarizing we have that the following proposition holds.

Proposition 4.3.5. *There exists $\mathcal{E} : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ such that up to a subsequence $\mathcal{E}_n(\cdot, A)$ Γ-converges in the strong topology of $L^1(\Omega)$ to $\mathcal{E}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$. Moreover, for every $u \in SBV^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$ we have that*

$$\mathcal{E}(u, A) = \int_A f_\infty(x, \nabla u(x)) \, dx + \int_{A\cap S(u)} g_\infty(x, u^-(x), u^+(x), \nu) \, d\mathcal{H}^{N-1}(x)$$

with

$$f_\infty(x, \xi) := \lim_{\varepsilon \to 0} f_\infty^\varepsilon(x, \xi) \qquad g_\infty(x, a, b, \nu) := \lim_{\varepsilon \to 0} g_\infty^\varepsilon(x, a, b, \nu), \tag{4.46}$$

where f_∞^ε and g_∞^ε are defined in (4.44) and (4.45) respectively.

Remark 4.3.6. In the rest of the chapter we will often make use the following property which is implied by the fact that $\mathcal{E}(u,\cdot)$ is a Radon measure for every $u \in SBV^p(\Omega)$. If $(u_n)_{n\in\mathbb{N}}$ is a recovering sequence for u with respect to $\mathcal{E}_n(\cdot,\Omega)$, then $(u_n)_{n\in\mathbb{N}}$ is optimal for u with respect to $\mathcal{E}_n(\cdot,A)$ for every $A \in \mathcal{A}(\Omega)$ such that the measure $\mathcal{E}(u,\cdot)$ vanishes on ∂A.

4.4 A Γ-convergence result for free discontinuity problems

The main result of this section is the following Γ-convergence theorem concerning the functionals $\mathcal{E}_n$ defined in (4.24).

Theorem 4.4.1. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that $\mathcal{H}^{N-1}(K_n) \le C$ for all $n \in \mathbb{N}$. Let us assume that for all $A \in \mathcal{A}(\Omega)$ the functionals $\mathcal{F}_n(\cdot,A)$ and $\mathcal{G}_n^-(\cdot,A)$ defined in* (4.26) *and* (4.27) *Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot,A)$ and $\mathcal{G}^-(\cdot,A)$ respectively. Then for all $A \in \mathcal{A}(\Omega)$ the functionals $\mathcal{E}_n(\cdot,A)$ defined in* (4.24) *Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{E}(\cdot,A)$ such that for all $u \in SBV^p(\Omega)$ and $A \in \mathcal{A}(\Omega)$*

$$\mathcal{E}(u,A) = \int_A f(x,\nabla u(x))\,dx + \int_{A\cap S(u)} g^-(x,\nu)\,d\mathcal{H}^{N-1}(x) \tag{4.47}$$

where f and g^- are the densities of $\mathcal{F}$ and $\mathcal{G}^-$ according to Propositions 4.3.1 and 4.3.3.

Proof. We know that up to a subsequence the functionals $\mathcal{E}_n(\cdot,A)$ Γ-converge in the strong topology of $L^1(\Omega)$ to a functional $\mathcal{E}(\cdot,A)$ for every $A \in \mathcal{A}(\Omega)$, and that by Proposition 4.3.5 for all $u \in SBV^p(\Omega)$ and for all $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{E}(u,A) = \int_A f_\infty(x,\nabla u)\,dx + \int_{S(u)\cap A} g_\infty(x,u^-(x),u^+(x),\nu)\,d\mathcal{H}^{N-1}(x),$$

where f_∞ and g_∞ satisfy formula (4.46). The theorem will be proved if we show that for all $u \in SBV^p(\Omega)$ we have

(a) for a.e. $x \in \Omega$

$$f_\infty(x,\nabla u(x)) = f(x,\nabla u(x)); \tag{4.48}$$

(b) for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$

$$g_\infty(x,u^-(x),u^+(x),\nu_{S(u)}(x)) = g^-(x,\nu_{S(u)}(x)), \tag{4.49}$$

where $\nu_{S(u)}(x)$ is the normal to $S(u)$ at x.

The proof will be divided into four steps.

Step 1: $\mathbf{f_\infty(x,\nabla u(x)) \le f(x,\nabla u(x))}$ for a.e. $\mathbf{x \in \Omega}$.

This inequality can be derived using the explicit formulas for f_∞ and f. Let $x \in \Omega$, $\xi \in \mathbb{R}^N$, and let us fix $\varepsilon > 0$. For every $\rho > 0$ let $u_{\varepsilon,\rho} \in W^{1,p}(B_\rho(x))$ be such that $u_{\varepsilon,\rho}(z) = \xi(z-x)$ in a neighborhood of $\partial B_\rho(x)$ and

$$\mathcal{F}(u_{\varepsilon,\rho},B_\rho(x)) \le \mathbf{m}_{\mathcal{F}}(B_\rho(x),\xi(z-x)) + \varepsilon\omega_N\rho^N.$$

Then we get

$$\begin{aligned} f_\infty^\varepsilon(x,\xi) &= \limsup_{\rho\to0^+} \frac{\mathbf{m}_{\mathcal{E}_\varepsilon}(B_\rho(x),\xi(z-x))}{\omega_N\rho^N} \le \limsup_{\rho\to0^+} \frac{\mathcal{E}(u_{\varepsilon,\rho},B_\rho(x))}{\omega_N\rho^N} \\ &\le \limsup_{\rho\to0^+} \frac{\mathcal{F}(u_{\varepsilon,\rho},B_\rho(x))}{\omega_N\rho^N} \le \limsup_{\rho\to0^+} \frac{\mathbf{m}_{\mathcal{F}}(B_\rho(x),\xi(z-x))}{\omega_N\rho^N} + \varepsilon = f(x,\xi) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, we obtain that $f_\infty(x,\xi) \leq f(x,\xi)$, so that the step is concluded.

Step 2: $\mathbf{f_\infty(x, \nabla u(x)) \geq f(x, \nabla u(x))}$ for a.e. $\mathbf{x \in \Omega}$.
We can consider those $x \in \Omega$ such that u is approximatively differentiable at x, x is a Lebesgue point for $f(\cdot,\xi)$ for all $\xi \in \mathbb{R}^N$ and such that

$$f_\infty(x, \nabla u(x)) = \lim_{\rho \to 0^+} \frac{\mathcal{E}(u, B_\rho(x))}{\omega_N \rho^N} < +\infty. \tag{4.50}$$

Let moreover $(u_n)_{n\in\mathbb{N}}$ be a recovering sequence for $\mathcal{E}(u,\Omega)$: by (4.23) and since $\mathcal{H}^{N-1}(K_n) \leq C$, we have that $\mathcal{H}^{N-1}(S(u_n))$ is bounded and so up to a subsequence

$$\mu_n := \mathcal{H}^{N-1} \llcorner S(u_n) \stackrel{*}{\rightharpoonup} \mu \qquad \text{weakly* in the sense of measures}$$

for some Borel measure μ. We can assume that (see for instance [8, Theorem 2.56])

$$\limsup_{\rho \to 0^+} \frac{\mu(\bar{B}_\rho(x))}{\rho^{N-1}} = 0. \tag{4.51}$$

Let $\rho_i \searrow 0$ be such that $\mathcal{E}(u, \partial B_{\rho_i}(x)) = 0$. In view of Remark 4.3.6, for every i there exists n_i such that for $n \geq n_i$

$$\begin{aligned} \frac{\mathcal{E}(u, B_{\rho_i}(x))}{\omega_N \rho_i^N} &\geq \frac{\mathcal{E}_n(u_n, B_{\rho_i}(x))}{\omega_N \rho_i^N} - \frac{1}{i} \\ &\geq \frac{\int_{B_{\rho_i}(x)} f_n(x, \nabla u_n(x))\, dx}{\omega_N \rho_i^N} - \frac{1}{i} = \frac{1}{\omega_N} \int_{B_1} f_n(x + \rho_i y, \nabla v_n^i(y))\, dy - \frac{1}{i} \end{aligned} \tag{4.52}$$

where

$$v_n^i(y) := \frac{u_n(x+\rho_i y) - u(x)}{\rho_i}.$$

Taking into account the assumptions on x and (4.51), we can choose $(n_i)_{i\in\mathbb{N}}$ is such a way that

$$v_{n_i}^i \to \nabla u(x) \cdot y \qquad \text{strongly in } L^1(B_1) \text{ for } i \to +\infty, \tag{4.53}$$

$$(\nabla v_{n_i}^i)_{i\in\mathbb{N}} \text{ is bounded in } L^p(B_1, \mathbb{R}^N), \tag{4.54}$$

$$\lim_{i \to +\infty} \mathcal{H}^{N-1}(S(v_{n_i}^i)) = 0, \tag{4.55}$$

and

$$f_\infty(x, \nabla u(x)) = \lim_{i\to+\infty} \frac{\mathcal{E}(u, B_{\rho_i}(x))}{\omega_N \rho_i^N} \geq \liminf_{i \to +\infty} \frac{1}{\omega_N} \int_{B_1} f_{n_i}(x + \rho_i y, \nabla v_{n_i}^i(y))\, dy. \tag{4.56}$$

Moreover by a truncation argument we can assume that $(v_{n_i}^i)_{i\in\mathbb{N}}$ is uniformly bounded in $L^\infty(B_1)$, so that we get

$$\|\nabla v_{n_i}^i\|^p_{L^p(B_1,\mathbb{R}^N)} + \int_{S(v_{n_i}^i)} |[v_{n_i}^i]|\, d\mathcal{H}^{N-1} \leq C \quad \text{and} \quad \lim_{i\to+\infty} \mathcal{H}^{N-1}(S(v_{n_i}^i)) = 0.$$

Following Kristensen [63] we get that there exists $w_i \in W^{1,\infty}(B_1)$ such that $w_i \to \nabla u(x) \cdot y$ strongly in $L^1(B_1)$ as $i \to +\infty$ and such that

$$\liminf_{i\to+\infty} \int_{B_1} f_{n_i}(x+\rho_i y, \nabla v_{n_i}^i(y))\, dy = \liminf_{i\to+\infty} \int_{B_1} f_{n_i}(x + \rho_i y, \nabla w_i(y))\, dy. \tag{4.57}$$

If n_i is chosen such that the blow-up for Γ-limits given by Theorem 4.2.2 holds, we get that

$$\liminf_{i\to+\infty}\int_{B_1} f_{n_i}(x+\rho_i y,\nabla w_i(y))\,dy \geq \omega_N f(x,\nabla u(x)),$$

so that in view of (4.56) we obtain

$$f_\infty(x,\nabla u(x)) \geq f(x,\nabla u(x)).$$

Step 3: $\mathbf{g_\infty(x,u^-(x),u^+(x),\nu_{S(u)}(x)) \leq g^-(x,\nu_{S(u)}(x))}$ for $\mathcal{H}^{N-1}$-a.e. $\mathbf{x\in S(u)}$.
Up to a subsequence, we have that

$$\mu_n := \mathcal{H}^{N-1}\llcorner K_n \overset{*}{\rightharpoonup} \mu$$

weakly* in the sense of measures. Since $\mathcal{H}^{N-1}(K_n)\leq C$ we have that for $\mathcal{H}^{N-1}$-a.e. $x\in\Omega$ (see for instance [8, Theorem 2.56])

$$H(x) := \limsup_{\rho\to0^+}\frac{\mu(\bar{B}_\rho(x))}{\omega_{N-1}\rho^{N-1}} < +\infty. \tag{4.58}$$

We claim that for all $v\in P(\Omega)$ and $A\in\mathcal{A}(\Omega)$ such that $\bar{A}\subseteq\Omega$

$$\alpha\mathcal{H}^{N-1}(S(v)\cap A) \leq \mathcal{G}^-(v,A)+\mu(\bar{A}). \tag{4.59}$$

In fact we have that for all $n\in\mathbb{N}$

$$\alpha\mathcal{H}^{N-1}\left((S(v)\setminus K_n)\cap A\right) \leq \mathcal{G}_n^-(v,A)$$

so that

$$\alpha\mathcal{H}^{N-1}\left(S(v)\cap A\right) \leq \mathcal{G}_n^-(v,A)+\mu_n(A)$$

and so passing to the Γ-limit for $n\to+\infty$ we obtain that (4.59) holds.

Let us choose $x\in S(u)$ in such a way that (4.58) holds and such that

$$\limsup_{\rho\to0^+}\frac{\int_{B_\rho(x)} a_2\,dx}{\rho^{N-1}} = 0,$$

where a_2 is defined in (4.22). Let us indicate $u^-(x), u^+(x)$ and $\nu_{S(u)}(x)$ simply by u^-, u^+ and ν. Let us moreover set $[u] := u^+ - u^-$.

Following Remark 4.3.4, let us consider the functionals $\mathcal{G}_n^-$ defined in (4.27) acting on the space $P_{u^-,u^+}(\Omega) := \{u\in BV(\Omega) \,:\, u(y)\in\{u^-,u^+\}$ for a.e. $y\in\Omega\}$.

Let us fix $\varepsilon>0$. For every $\rho>0$, let $u_{\varepsilon,\rho}\in P_{u^-,u^+}(B_\rho(x))$ be such that $u_{\varepsilon,\rho}=u_{x,u^-,u^+,\nu}$ in a neighborhood of $B_\rho(x)$ and

$$\mathcal{G}^-(u_{\varepsilon,\rho},B_\rho(x)) \leq \mathbf{m}_{\mathcal{G}^-}(B_\rho(x),u_{x,u^-,u^+,\nu})+\varepsilon\omega_{N-1}\rho^{N-1}.$$

Then we get in view of (4.59)

$$\begin{aligned}
g_\infty^\varepsilon(x,u^-,u^+,\nu) &= \limsup_{\rho\to0^+}\frac{\mathbf{m}_{\mathcal{E}_\varepsilon}(B_\rho(x),u_{x,u^-,u^+,\nu})}{\omega_{N-1}\rho^{N-1}}\\
&\leq \limsup_{\rho\to0^+}\frac{\mathcal{E}(u_{\varepsilon,\rho},B_\rho(x))+\varepsilon(1+|[u]|)\mathcal{H}^{N-1}(S(u_{\varepsilon,\rho})\cap B_\rho(x))}{\omega_{N-1}\rho^{N-1}}\\
&\leq \limsup_{\rho\to0^+}\frac{\int_{B_\rho(x)}a_2\,dx+\mathcal{G}^-(u_{\varepsilon,\rho},B_\rho(x))+\frac{\varepsilon}{\alpha}(1+|[u]|)(\mathcal{G}^-(u_{\varepsilon,\rho},B_\rho(x))+\mu(\bar{B}_\rho(x)))}{\omega_{N-1}\rho^{N-1}}\\
&\leq \limsup_{\rho\to0^+}\frac{(1+\frac{\varepsilon}{\alpha}+\frac{\varepsilon}{\alpha}|[u]|)(\mathbf{m}_{\mathcal{G}^-}(B_\rho(x),u_{x,u^-,u^+,\nu})+\varepsilon\omega_{N-1}\rho^{N-1})+\frac{\varepsilon}{\alpha}(1+|[u]|)\mu(\bar{B}_\rho(x)))}{\omega_{N-1}\rho^{N-1}}\\
&\leq \left(1+\frac{\varepsilon}{\alpha}+\frac{\varepsilon}{\alpha}|[u]|\right)(g^-(x,\nu)+\varepsilon)+\frac{\varepsilon}{\alpha}(1+|[u]|)H(x).
\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain $g_\infty(x, u^-, u^+, \nu) \le g^-(x, \nu)$, so that the step is concluded.

Step 4: $\mathbf{g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) \ge g^-(x, \nu_{S(u)}(x))}$ for $\mathcal{H}^{N-1}$-a.e. $\mathbf{x \in S(u)}$.
Let us choose $x \in S(u)$ which is an approximate jump point for u,

$$g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) = \lim_{\rho \to 0^+} \frac{\mathcal{E}(u, B_\rho(x))}{\omega_{N-1}\rho^{N-1}} < +\infty, \tag{4.60}$$

and such that

$$\lim_{\rho \to 0^+} \frac{\int_{B_\rho(x)} |a_1(y)|\, dy}{\rho^{N-1}} = 0, \tag{4.61}$$

where a_1 is defined in (4.22).
Since $\mathcal{H}^{N-1}(K_n) \le C$, up to a subsequence we have

$$\mu_n := \mathcal{H}^{N-1} \llcorner K_n \overset{*}{\rightharpoonup} \mu \qquad \text{weakly* in the sense of measures}$$

for some Borel measure μ. We can assume that (see for instance [8, Theorem 2.56])

$$\limsup_{\rho \to 0^+} \frac{\mu(B_\rho(x))}{\rho^{N-1}} < +\infty. \tag{4.62}$$

Let $(u_n)_{n\in\mathbb{N}}$ be a recovering sequence for $\mathcal{E}(u, \Omega)$, and let $\rho_i \searrow 0$ be such that $\mathcal{E}(u, \partial B_{\rho_i}(x)) = 0$. For every $i \in \mathbb{N}$ there exists $n_i \in \mathbb{N}$ such that for $n \ge n_i$ we have

$$\begin{aligned} \frac{\mathcal{E}(u, B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}} &\ge \frac{\mathcal{E}_n(u_n, B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}} - \frac{1}{i} \\ &\ge \frac{\int_{B_{\rho_i}(x)\cap[S(u_n)\setminus K_n]} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x)}{\omega_{N-1}\rho_i^{N-1}} + \frac{\int_{B_{\rho_i}(x)} a_1(y)\, dy}{\omega_{N-1}\rho_i^{N-1}} - \frac{1}{i} \\ &= \frac{1}{\omega_{N-1}} \int_{B_1\cap[S(v_n^i)\setminus K_n^i]} g_n(x+\rho_i y, \nu)\, d\mathcal{H}^{N-1}(y) + \frac{\int_{B_{\rho_i}(x)} a_1(y)\, dy}{\omega_{N-1}\rho_i^{N-1}} - \frac{1}{i} \end{aligned} \tag{4.63}$$

where

$$v_n^i(y) := u_n(x+\rho_i y) \quad \text{and} \quad K_n^i := \frac{\{K_n \cap B_{\rho_i}(x)\} - x}{\rho_i}. \tag{4.64}$$

We claim that we can find w_n^i piecewise constant in B_1 such that for $n \to +\infty$

$$w_n^i \to w^i \qquad \text{strongly in } L^1(B_1),$$

where w^i is piecewise constant and $w^i = u_{0,0,1,\nu_{S(u)}(x)}$ in a neighborhood of the boundary, and such that for n large

$$\int_{B_1\cap[S(v_n^i)\setminus K_n^i]} g_n(x+\rho_i y, \nu)\, d\mathcal{H}^{N-1}(y) \ge \int_{B_1\cap[S(w_n^i)\setminus K_n^i]} g_n(x+\rho_i y, \nu)\, d\mathcal{H}^{N-1}(y) - e_i \tag{4.65}$$

with $e_i \to 0$ for $i \to +\infty$.
Using the claim, by (4.63), (4.65) and (4.61) we have that for n large

$$g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) \ge \frac{\int_{B_{\rho_i}\cap[S(z_n^i)\setminus K_n]} g_n(\zeta,\nu)\, d\mathcal{H}^{N-1}(\zeta)}{\omega_{N-1}\rho_i^{N-1}} - \hat{e}_i = \frac{\mathcal{G}_n^-(z_n^i, B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}} - \hat{e}_i$$

where $\hat{e}_i \to 0$ and

$$z_n^i(\zeta) := w_n^i\left(\frac{\zeta - x}{\rho_i}\right) \to z^i(\zeta) := w^i\left(\frac{\zeta - x}{\rho_i}\right) \qquad \text{strongly in } L^1(B_{\rho_i}(x)).$$

By the Γ-convergence assumption on $\mathcal{G}_n^-$, using Γ-liminf inequality we have that

$$g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) \geq \frac{\mathcal{G}^-(z^i, B_{\rho_i}(x))}{\omega_{N-1}\rho_i^{N-1}} - \hat{e}_i \geq \frac{\mathbf{m}_{\mathcal{G}^-}(B_{\rho_i}, u_{x,0,1,\nu_{S(u)}(x)})}{\omega_{N-1}\rho_i^{N-1}} - \hat{e}_i.$$

Letting $i \to +\infty$, and recalling the representation formula (4.36) for $g^-(x,\nu)$, we have that the result is proved.

In order to complete the proof of the step, we have to prove the claim. Since

$$\nabla v_n^i(y) = \rho_i \nabla u_n(x + \rho_i y),$$

we get by the coercivity assumption (4.22)

$$\begin{aligned}\int_{B_1} |\nabla v_n^i(y)|^p\, dy = \rho_i^p \int_{B_1} |\nabla u_n(x+\rho_i y)|^p\, dy &= \rho_i^p \frac{\int_{B_{\rho_i}(x)} |\nabla u_n(z)|^p\, dz}{\rho_i^N} \\ &\leq \frac{\rho_i^{p-1}}{\alpha}\left(\frac{\mathcal{E}_n(u_n, B_{\rho_i}(x))}{\rho_i^{N-1}} - \frac{\int_{B_{\rho_i}(x)} a_1(y)\, dy}{\rho_i^{N-1}} \right).\end{aligned}$$

Since u_n is optimal for u we have that

$$\frac{\mathcal{E}_n(u_n, B_{\rho_i}(x))}{\rho_i^{N-1}} \overset{n\to+\infty}{\longrightarrow} \frac{\mathcal{E}(u, B_{\rho_i}(x))}{\rho_i^{N-1}} \overset{i\to+\infty}{\longrightarrow} \omega_{N-1} g_\infty(x, u^-(x), u^+(x), \nu_{S(u)}(x)) < +\infty.$$

In view also of (4.61), we conclude that we can choose n_i so that for $n \geq n_i$

$$\int_{B_1} |\nabla v_n^i(y)|^p\, dy \leq C\rho_i^{p-1} \tag{4.66}$$

for some constant $C \geq 0$. By Coarea formula for BV functions (see [8, Theorem 3.40]) we get

$$\int_{u^-(x)}^{u^+(x)} \mathcal{H}^{N-1}\left(\partial^* E_n^i(t) \setminus S(v_n^i)\right) dt \leq \int_{B_1} |\nabla v_n^i|\, dy \leq \tilde{C}\rho_i^{1-\frac{1}{p}},$$

for a suitable constant $\tilde{C}$, where

$$E_n^i(t) := \{x \in B_1 \,:\, x \text{ is a Lebesgue point for } v_n^i \text{ and } v_n^i(x) > t\} \tag{4.67}$$

and ∂^* denotes the reduced boundary. By the Mean Value Theorem there exists

$$t_n^i \in [u^-(x), u^+(x)]$$

such that

$$\mathcal{H}^{N-1}\left(\partial^* E_n^i(t_n^i) \setminus S(v_n^i)\right) \leq \frac{\tilde{C}}{u^+(x) - u^-(x)}\rho_i^{1-\frac{1}{p}}. \tag{4.68}$$

We now employ a construction similar to that employed by Francfort and Larsen in their Transfer of Jump Sets Theorem [53, Theorem 2.3]. Since x is a jump point for u we have that for $i \to +\infty$

$$u(x + \rho_i y) \to u_{0,u^-(x),u^+(x),\nu_{S(u)}(x)} \qquad \text{strongly in } L^1(B_1).$$

Then we have that for n large

$$|B_1^+ \,\triangle\, E_n^i(t_n^i)| \leq e_i,$$

where $B_1^+ := \{y \in B_1 \,:\, y \cdot \nu_{S(u)}(x) \geq 0\}$, $A \triangle B := (A \setminus B) \cup (B \setminus A)$, and $e_i \to 0$ for $i \to +\infty$. By Fubini's Theorem we have

$$\int_0^{\sqrt{e_i}} \mathcal{H}^{N-1}\left((B_1^+ \setminus E_n^i(t_n^i)) \cap H^+(s)\right) ds \leq \int_{-\infty}^{+\infty} \mathcal{H}^{N-1}\left((B_1^+ \setminus E_n^i(t_n^i)) \cap H^+(s)\right) ds \leq e_i,$$

where $H^+(s) := \{y \in B_1 \,:\, y \cdot \nu_{S(u)}(x) = s\}$, and by the Mean Value Theorem we get that there exists $0 < s_n^{i,+} < \sqrt{e_i}$ such that setting $H_n^{i,+} := H^+(s_n^{i,+})$ we have

$$\mathcal{H}^{N-1}\left((B_1^+ \setminus E_n^i(t_n^i)) \cap H_n^{i,+}\right) \leq \sqrt{e_i}.$$

Similarly we obtain $-\sqrt{e_i} < s_n^{i,-} < 0$ such that setting $H_n^{i,-} := H^+(s_n^{i,-})$ we have

$$\mathcal{H}^{N-1}\left((E_n^i(t_n^i) \setminus B_1^+) \cap H_n^{i,-}\right) \leq \sqrt{e_i}.$$

Let us write $y = (y', y_N)$, where y_N is the coordinate along $\nu_{S(u)}(x)$ and y' the coordinates in the hyperplane orthogonal to $\nu_{S(u)}(x)$. Let l_i be such that for every $y \in B_1$

$$|y_N| \geq 2\sqrt{e_i} \Longrightarrow |y'| \leq 1 - l_i.$$

Let us set

$$D_n^i := (E_n^i(t_n^i) \cup \{y \in B_1 \,:\, y_N \geq s_n^{i,+}\}) \setminus \{y \in B_1 \,:\, y_N \leq s_n^{i,-}\}.$$

We set

$$w_n^i := \begin{cases} 1 & |y'| \geq 1 - l_i, y_N \geq 0, \\ 0 & |y'| \geq 1 - l_i, y_N < 0, \\ 1 & |y'| \leq 1 - l_i, y \in D_n^i, \\ 0 & \text{otherwise}. \end{cases} \tag{4.69}$$

Notice that w_i^n is piecewise constant, with $w_n^i = u_{0,0,1,\nu_{S(u)}(x)}$ in a neighborhood of the boundary, and such that

$$\int_{B_1 \cap [S(v_n^i) \setminus K_n^i]} g_n(x + \rho_i y, \nu)\, d\mathcal{H}^{N-1}(y) \geq \int_{B_1 \cap [S(w_n^i) \setminus K_n^i]} g_n(x + \rho_i y, \nu)\, d\mathcal{H}^{N-1}(y) - \tilde{e}_i \tag{4.70}$$

with $\tilde{e}_i \to 0$ for $i \to +\infty$.

In view of (4.68) and of the assumption (4.62) we have that $\mathcal{H}^{N-1}(S(w_n^i)) \leq C_i$ uniformly in n for some finite constant C_i. By Ambrosio's Compactness Theorem we get for $n \to +\infty$

$$w_n^i \to w^i \qquad \text{strongly in } L^1(B_1),$$

where w^i is piecewise constant and $w^i = u_{0,0,1,\nu_{S(u)}(x)}$ in a neighborhood of the boundary, so that the claim is proved. □

Remark 4.4.2. Theorem 4.4.1 states that in the Γ-limit process there is no interaction between bulk and surface energies, since they are constructed looking at Γ-convergence problems in Sobolev space and in the space of piecewise constant functions respectively. As a consequence, considering bulk and surface energies of the form $c_1 f_n$ and $c_2 g_n$ with $c_1, c_2 > 0$, we get in the limit $c_1 f$ and $c_2 g$ as bulk and surface energy densities. We remark that a key assumption for non interaction is given by equi-boundness of $\mathcal{H}^{N-1}(K_n)$: dropping this assumption, interaction can occur even in the case of constant densities, for example $f(\xi) := |\xi|^p$ and $g(x, \nu) \equiv 1$ (if we consider in $]0,1[$ the set $K_n := \{\frac{i}{n} \,:\, i = 1, \ldots, n-1\}$, we get as Γ-limit the zero functional). As mentioned in the Introduction, non interaction between bulk and surface energies was noticed in the case of periodic homogenization (with $K_n = \emptyset$) by Braides, Defranceschi and Vitali in [21].

In the rest of this section we employ Theorem 4.4.1 to obtain a lower semicontinuity result for SBV functions in the case of varying bulk and surface energies in the same spirit of Ambrosio's lower semicontinuity theorems [4].

From Theorem 4.4.1 we get that the following semicontinuity result holds.

Proposition 4.4.3. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that $\mathcal{H}^{N-1}(K_n) \le C$ for all $n \in \mathbb{N}$. Let us assume that for all $A \in \mathcal{A}(\Omega)$ the functionals $\mathcal{F}_n(\cdot, A)$ and $\mathcal{G}_n^-(\cdot, A)$ defined in (4.26) and (4.27) Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot, A)$ and $\mathcal{G}^-(\cdot, A)$ respectively. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^p(\Omega)$ such that*

$$u_n \rightharpoonup u \qquad \text{weakly in } SBV^p(\Omega).$$

Then for all $A \in \mathcal{A}(\Omega)$ we have

$$\int_A f(x, \nabla u(x))\, dx \le \liminf_{n\to+\infty} \int_A f_n(x, \nabla u_n(x))\, dx, \tag{4.71}$$

and

$$\int_{S(u)\cap A} g^-(x,\nu)\, d\mathcal{H}^{N-1} \le \liminf_{n\to+\infty} \int_{(S(u_n)\setminus K_n)\cap A} g_n(x,\nu)\, d\mathcal{H}^{N-1}, \tag{4.72}$$

where f and g^- are the densities of $\mathcal{F}$ and $\mathcal{G}^-$ respectively.

Moreover let us assume $\mathcal{G}_n(\cdot, A)$ defined in (4.28) Γ-converges in the strong topology of $L^1(\Omega)$ to $\mathcal{G}(\cdot, A)$. Then

$$\int_{S(u)\cap A} g(x,\nu)\, d\mathcal{H}^{N-1} \le \liminf_{n\to+\infty} \int_{S(u_n)\cap A} g_n(x,\nu)\, d\mathcal{H}^{N-1}, \tag{4.73}$$

where g is the density of $\mathcal{G}$.

Proof. By Theorem 4.4.1, we have that for all $h, k \in \mathbb{N}$ and for all $A \in \mathcal{A}(\Omega)$ the functionals

$$\mathcal{E}_n^{h,k}(u, A) := h \int_A f_n(x, \nabla u(x))\, dx + k \int_{(S(u)\setminus K_n)\cap A} g_n(x,\nu)\, d\mathcal{H}^{N-1}$$

Γ-converge in the strong topology of $L^1(\Omega)$ to

$$\mathcal{E}^{h,k}(u, A) := h \int_A f(x, \nabla u(x))\, dx + k \int_{S(u)\cap A} g^-(x,\nu)\, d\mathcal{H}^{N-1}.$$

In particular by Γ-liminf inequality we have

$$\mathcal{E}^{h,k}(u, A) \le \liminf_{n\to+\infty} \mathcal{E}_n^{h,k}(u_n, A).$$

Then we get

$$\begin{aligned}\int_A f(x, \nabla u(x))\, dx &\le \liminf_{n\to+\infty} \int_A f_n(x, \nabla u_n(x))\, dx + \frac{k}{h} \int_{(S(u_n)\setminus K_n)\cap A} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) \\ &\le \liminf_{n\to+\infty} \int_A f_n(x, \nabla u_n(x))\, dx + \frac{k}{h} C\end{aligned}$$

for some constant C independent of h and k. Since h, k are arbitrary we get that (4.71) holds. The proof of (4.72) is analogous. Finally (4.73) derives from (4.72) in the case $K_n \equiv \emptyset$. □

4.5 A new variational convergence for rectifiable sets

In this section we use the Γ-convergence results of Section 4.4 to introduce a variational notion of convergence for rectifiable sets which will be employed in the study of stability of unilateral minimality properties.

Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω, and let us assume following Ambrosio and Braides [6, Theorem 3.2] that the functionals $\mathcal{H}_n^- : P(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty)$ defined by

$$\mathcal{H}_n^-(u, A) := \mathcal{H}^{N-1}((S(u) \setminus K_n) \cap A) \tag{4.74}$$

Γ-converge with respect to the strong topology of $L^1(\Omega)$ for every $A \in \mathcal{A}(\Omega)$ to a functional $\mathcal{H}^-(\cdot, A)$, which by the representation result of Bouchitté, Fonseca, Leoni and Mascarenhas [17, Theorem 3] is of the form

$$\mathcal{H}^-(u, A) := \int_{S(u)\cap A} h^-(x, \nu)\, d\mathcal{H}^{N-1}(x) \tag{4.75}$$

for some function $h^- : \Omega \times S^{N-1} \to [0, +\infty)$.

Definition 4.5.1 (σ-convergence of rectifiable sets). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω. We say that K_n σ-converges in Ω to K if the functionals $(\mathcal{H}_n^-)_{n\in\mathbb{N}}$ defined in (4.74) Γ-converge in the strong topology of $L^1(\Omega)$ to the functional $\mathcal{H}^-$ defined in (4.75), and K is the unique rectifiable set in Ω such that*

$$h^-(x, \nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K, \tag{4.76}$$

and such that for every rectifiable set $H \subseteq \Omega$ we have

$$h^-(x, \nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in H \Longrightarrow H \tilde{\subseteq} K. \tag{4.77}$$

The following lemma, which comes directly from Definition 4.5.1, shows that σ-convergence of rectifiable sets is stable under infinitesimal perturbation in surface.

Lemma 4.5.2. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that K_n σ-converges in Ω to K. Let $(\tilde{K}_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that $\mathcal{H}^{N-1}(\tilde{K}_n \triangle K_n) \to 0$. Then $\tilde{K}_n$ σ-converges in Ω to K.*

In order to prove the main properties of σ-convergence of rectifiable sets, we need the following covering argument.

Lemma 4.5.3. *Let $H \subseteq \Omega$ be a rectifiable set with $\mathcal{H}^{N-1}(H) < +\infty$. Then for all $\varepsilon > 0$ there exist an open set $U \in \mathcal{A}(\Omega)$ and $u \in P(\Omega)$ such that $\mathcal{H}^{N-1}(H \setminus U) < \varepsilon$ and*

$$\mathcal{H}^{N-1}((S(u) \triangle H) \cap U) < \varepsilon,$$

where $\triangle$ denotes the symmetric difference of sets.

Proof. Since H is rectifiable, we have that $H = H_0 \cup \bigcup_{i\in\mathbb{N}} K_i$, where $\mathcal{H}^{N-1}(H_0) = 0$, K_i is compact, and $K_i \subseteq M_i$ for a suitable C^1 hypersurface M_i of $\mathbb{R}^N$. For all $i \in \mathbb{N}$, let us denote by $\tilde{K}_i$ the set of point x such that x has $(N-1)$-dimensional density 1 with respect to K_i. We have that $\mathcal{H}^{N-1}(H \setminus \bigcup_{i\in\mathbb{N}} \tilde{K}_i) = 0$.

Let us fix $\varepsilon > 0$. Then for all $x \in \tilde{K}_i$, there exists $\rho(x) > 0$ such that for all $\rho < \rho(x)$ we have

$$\omega_{N-1}\rho^{N-1} < (1+\varepsilon)\mathcal{H}^{N-1}\left(\tilde{K}_i \cap B_\rho(x)\right) \tag{4.78}$$

and

$$\mathcal{H}^{N-1}\left((M_i \setminus \tilde{K}_i) \cap B_\rho(x)\right) < \varepsilon\omega_{N-1}\rho^{N-1}. \tag{4.79}$$

Since M_i is of class C^1, we can assume that $\rho(x)$ is so small that $B_\rho(x) \setminus M_i$ has exactly two connected components $B_\rho^+(x)$ and $B_\rho^-(x)$ for every $\rho < \rho(x)$.

We can apply now the Vitali-Besicovitch Covering Theorem (see [8, Theorem 2.19]), and deduce that there exists a disjoint family of balls $(B_{\rho_j}(x_j))_{j\in\mathbb{N}}$ such that

$$\mathcal{H}^{N-1}\Big(H \setminus \bigcup_{j\in\mathbb{N}} B_{\rho_j}(x_j)\Big) = 0.$$

Let us choose $n \in \mathbb{N}$ such that

$$\mathcal{H}^{N-1}\Big(H \setminus \bigcup_{j=0}^{n} B_{\rho_i}(x_i)\Big) \le \varepsilon,$$

and let us set $U := \bigcup_{j=0}^{n} B_{\rho_i}(x_i)$. Let us consider $u \in P(\Omega)$ defined setting $u = 1$ in $B^+_{\rho_j}(x_j)$, $j = 1, \dots, n$, and $u = 0$ otherwise. We have that

$$(H \triangle S(u)) \cap U \subseteq \bigcup_{j\in\mathbb{N}} (M_{i_j} \setminus \tilde{K}_{i_j}) \cap B_{\rho_j}(x_j),$$

where K_{i_j} is the compact set relative to x_j, and M_{i_j} is the associated $(N-1)$-dimensional hypersurface. In view of (4.78) and (4.79) we conclude that

$$\mathcal{H}^{N-1}((H \triangle S(u)) \cap U) < \varepsilon(1+\varepsilon)\mathcal{H}^{N-1}(H),$$

so that the theorem is proved. $\square$

Let us now come to the main properties of σ-convergence for rectifiable sets. By compactness of Γ-convergence, we deduce the following compactness result for σ-convergence.

Proposition 4.5.4 (compactness). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω with $\mathcal{H}^{N-1}(K_n) \le C$. Then there exists a subsequence $(n_h)_{h\in\mathbb{N}}$ and a rectifiable set K in Ω such that K_{n_h} σ-converges in Ω to K. Moreover*

$$\mathcal{H}^{N-1}(K) \le \liminf_{n\to+\infty} \mathcal{H}^{N-1}(K_n). \tag{4.80}$$

Proof. By Proposition 4.3.3, up to a subsequence we have that for all $A \in \mathcal{A}(\Omega)$ the functionals $\mathcal{H}^-_n(\cdot, A)$ defined in (4.74) Γ-converge in the strong topology of $L^1(\Omega)$ to a functional $\mathcal{H}^-(\cdot, A)$ which can be represented through a density h^- according to (4.75).

Let us consider the class

$$\mathcal{K} := \{H \subseteq \Omega : H \text{ is rectifiable and } h^-(x, \nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in H\}.$$

Notice that $\mathcal{K}$ contains at least the empty set. Moreover for all $H \in \mathcal{K}$ we have

$$\mathcal{H}^{N-1}(H) \le L := \liminf_{n\to+\infty} \mathcal{H}^{N-1}(K_n). \tag{4.81}$$

In fact let $H \in \mathcal{K}$. Since $H = \cup_i H_i$ with H_i compact and rectifiable with $\mathcal{H}^{N-1}(H_i) < +\infty$, it is not restrictive to consider $\mathcal{H}^{N-1}(H) < +\infty$. Given $\varepsilon > 0$, by Lemma 4.5.3 we can find an open set U and a piecewise constant function $v \in P(\Omega)$ such that

$$\mathcal{H}^{N-1}(H \setminus U) < \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}\left((S(v) \triangle H) \cap U\right) < \varepsilon,$$

where $\triangle$ denotes the symmetric difference of sets. Since $h^- \le 1$ we have

$$\mathcal{H}^-(v, U) = \int_{S(v)\cap U} h^-(x, \nu)\, d\mathcal{H}^{N-1}(x) = \int_{(S(v)\setminus H)\cap U} h^-(x, \nu)\, d\mathcal{H}^{N-1}(x) < \varepsilon.$$

Let $(v_n)_{n\in\mathbb{N}}$ be a recovering sequence for v with respect to $\mathcal{H}^-(\cdot, U)$. Then we have that

$$\limsup_{n\to+\infty} \mathcal{H}^{N-1}\left((S(v_n) \setminus K_n) \cap U\right) < \varepsilon.$$

By Ambrosio's Theorem we deduce that

$$\mathcal{H}^{N-1}(H) \le \mathcal{H}^{N-1}(H \cap U) + \mathcal{H}^{N-1}(H \setminus U) \le \mathcal{H}^{N-1}(S(v) \cap U) + 2\varepsilon$$
$$\le \liminf_{n \to +\infty} \mathcal{H}^{N-1}(S(v_n) \cap U) + 2\varepsilon \le \liminf_{n \to +\infty} \mathcal{H}^{N-1}(K_n) + 3\varepsilon = L + 3\varepsilon.$$

Since ε is arbitrary we get that (4.81) holds.

Let us now consider

$$\tilde{L} := \sup\{\mathcal{H}^{N-1}(H) \,:\, H \in \mathcal{K}\} < +\infty,$$

and let $(H_k)_{k\in\mathbb{N}}$ be a maximizing sequence for $\tilde{L}$. We set

$$K := \bigcup_{k=1}^{\infty} H_k.$$

Clearly (4.80) and (4.76) hold. Moreover, since $\mathcal{H}^{N-1}(K) = \tilde{L}$ we have that (4.77) holds, and the proof is concluded. □

Remark 4.5.5. Let $\Omega := (-1,1) \times (-1,1)$ in $\mathbb{R}^2$, and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of closed sets with $K_n \to K := \{(-1,1)\} \times \{0\}$ in the Hausdorff metric and such that

$$\mathcal{H}^1 \llcorner K_n \overset{*}{\rightharpoonup} a\mathcal{H}^1 \llcorner K$$

weakly* in the sense of measures. If $a < 1$ by (4.80) we deduce that K_n σ-converges in Ω to the empty set. We stress that the condition $a \ge 1$ is not enough to guarantee that K is the σ-limit of $(K_n)_{n\in\mathbb{N}}$ because the σ-limit is also affected by the behavior of the normal vectors to K_n. In fact considering

$$K_n := \bigcup_{i=-n}^{n} \left\{\frac{i}{n}\right\} \times \left[-\frac{1}{n}, \frac{1}{n}\right]$$

we have

$$\mathcal{H}^1 \llcorner K_n \overset{*}{\rightharpoonup} 2\mathcal{H}^1 \llcorner K$$

weakly* in the sense of measures. However also in this case we have that K_n σ-converges in Ω to the empty set. In fact let us consider $u \in P(\Omega)$ such that $u = 1$ in $\Omega^+ := (-1,1) \times (0,1)$ and $u = 0$ in $\Omega^- := (-1,1) \times (-1,0)$, and let u_n be a sequence in $P(\Omega)$ such that $u_n \to u$ strongly in $L^1(\Omega)$ and with $\mathcal{H}^{N-1}(S(u_n)) \le C$. Let (e_1, e_2) be the canonical base of $\mathbb{R}^2$. By Ambrosio's theorem we get that

$$\nu[u_n]\mathcal{H}^1 \llcorner S(u_n) \overset{*}{\rightharpoonup} e_2\mathcal{H}^1 \llcorner S(u)$$

weakly* in the sense of measures. Considering the vector field φe_2 with $\varphi \in C_c^\infty(\Omega)$ we get

$$\int_{S(u_n)\setminus K_n} \varphi e_2 \cdot \nu[u_n]\, d\mathcal{H}^1 = \int_{S(u_n)} \varphi e_2 \cdot \nu[u_n]\, d\mathcal{H}^1 \to \int_K \varphi\, d\mathcal{H}^1.$$

Since φ is arbitrary, we deduce that $\liminf_{n\to+\infty} \mathcal{H}_n^-(u_n) = \liminf_{n\to+\infty} \mathcal{H}^1(S(u_n) \setminus K_n) \ge 1$. By Γ-liminf we conclude that $\mathcal{H}^-(u) = 1$ that is $h^-(x, e_2) = 1$ for $\mathcal{H}^1$-a.e. $x \in K$. Since the σ-limit of $(K_n)_{n\in\mathbb{N}}$ can be only contained in K, we deduce that the σ-limit is the empty set.

The following proposition shows that the σ-limit is a natural limit candidate for a sequence of rectifiable sets in connection with unilateral minimality properties (see the Introduction).

Proposition 4.5.6. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω with K_n σ-converging in Ω to K. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of Borel functions satisfying the growth estimates* (4.23)*, and let g^- be the energy density of the Γ-limit in the strong topology of $L^1(\Omega)$ of the functionals $(\mathcal{G}_n^-)_{n\in\mathbb{N}}$ defined in* (4.27)*. Then we have*

(4.82) $$g^-(x, \nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K,$$

and for every rectifiable set $H \subseteq \Omega$

(4.83) $$g^-(x, \nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in H \Longrightarrow H \tilde{\subseteq} K.$$

Proof. By growth estimates on g_n we have for all $u \in P(\Omega)$ and $A \in \mathcal{A}(\Omega)$

$$\alpha \mathcal{H}^-(u, A) \leq \mathcal{G}^-(u, A) \leq \beta \mathcal{H}^-(u, A).$$

The proof will follow if we prove that for every rectifiable set $H \subseteq \Omega$

$$h^-(x, \nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in H \tag{4.84}$$

is equivalent to

$$g^-(x, \nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in H. \tag{4.85}$$

Let us show that (4.84) implies (4.85), the reverse implication being similar. It is not restrictive to assume $\mathcal{H}^{N-1}(H) < +\infty$. Given $\varepsilon > 0$, by Lemma 4.5.3 we can find an open set U and a piecewise constant function $v \in P(\Omega)$ such that

$$\mathcal{H}^{N-1}(H \setminus U) < \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}\left((S(v) \triangle H) \cap U\right) < \varepsilon,$$

where $\triangle$ denotes the symmetric difference of sets. Then we get

$$\begin{aligned}\int_H g^-(x,\nu)\, d\mathcal{H}^{N-1}(x) &= \int_{H \cap U} g^-(x,\nu)\, d\mathcal{H}^{N-1}(x) + \int_{H \setminus U} g^-(x,\nu)\, d\mathcal{H}^{N-1}(x) \\ &\leq \int_{S(v)\cap U} g^-(x,\nu)\, d\mathcal{H}^{N-1}(x) + 2\beta\varepsilon \leq \beta \int_{S(v)\cap U} h^-(x,\nu)\, d\mathcal{H}^{N-1}(x) + 2\beta\varepsilon \leq 3\beta\varepsilon.\end{aligned}$$

Since ε is arbitrary we get that

$$g^-(x, \nu_H(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in H$$

so that the proof is concluded. $\square$

The following lower semicontinuity result for surface energies along sequences of rectifiable sets converging in the sense of σ-convergence will be employed in Section 4.8.

Proposition 4.5.7 (lower semicontinuity). *Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of rectifiable sets in Ω such that K_n σ-converges in Ω to K. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of Borel functions satisfying the growth estimates* (4.23)*, and let g be the associated function according to Proposition 4.3.2. Then we have*

$$\int_K g(x,\nu)\, d\mathcal{H}^{N-1}(x) \leq \liminf_{n \to +\infty} \int_{K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x).$$

Proof. Let $H \tilde{\subseteq} K$ with $\mathcal{H}^{N-1}(H) < +\infty$. Given $\varepsilon > 0$, by Lemma 4.5.3 we can find an open set U and a piecewise constant function $v \in P(\Omega)$ such that

$$\mathcal{H}^{N-1}(H \setminus U) < \varepsilon \quad \text{and} \quad \mathcal{H}^{N-1}\left((S(v) \triangle H) \cap U\right) < \varepsilon,$$

where $\triangle$ denotes the symmetric difference of sets. If $(v_n)_{n \in \mathbb{N}}$ is a recovering sequence for v with respect to $\mathcal{H}^-(\cdot, U)$ defined in (4.75) we have

$$\limsup_{n \to +\infty} \mathcal{H}^{N-1}\left((S(v_n) \setminus K_n) \cap U\right) < \varepsilon.$$

We deduce by Γ-convergence that

$$\begin{aligned}\int_H g(x,\nu)\, d\mathcal{H}^{N-1}(x) &= \int_{H \cap U} g(x,\nu)\, d\mathcal{H}^{N-1}(x) + \int_{H \setminus U} g(x,\nu)\, d\mathcal{H}^{N-1}(x) \\ &\leq \int_{S(v)\cap U} g(x,\nu)\, d\mathcal{H}^{N-1}(x) + 2\beta\varepsilon \leq \liminf_{n \to +\infty} \int_{S(v_n)\cap U} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) + 2\beta\varepsilon \\ &\leq \liminf_{n \to +\infty} \int_{K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) + 3\beta\varepsilon.\end{aligned}$$

Since ε is arbitrary we deduce

$$\int_H g(x,\nu)\,d\mathcal{H}^{N-1}(x) \le \liminf_{n\to+\infty}\int_{K_n} g_n(x,\nu)\,d\mathcal{H}^{N-1}(x),$$

and since H is arbitrary in K the proof is concluded. □

The following proposition is essential in the study of stability of unilateral minimality properties.

Proposition 4.5.8. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that K_n σ-converges in Ω to K. Let $1<p<+\infty$, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^p(\Omega)$ with $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$ and $\mathcal{H}^{N-1}(S(u_n)\setminus K_n)\to 0$. Then $S(u)\tilde{\subseteq} K$.*

Proof. Let us consider $\tilde{K}_n := S(u_n)\cap K_n$. By compactness, up to a further subsequence we have that $\tilde{K}_n$ σ-converges in Ω to a rectifiable set $\tilde{K}\tilde{\subseteq}K$. Let $\tilde{h}^-$ be the density associated to $\tilde{K}$ according to Definition 4.5.1. By lower semicontinuity given by Proposition 4.4.3 we have

$$\int_{S(u)} \tilde{h}^-(x,\nu)\,d\mathcal{H}^{N-1}(x) \le \liminf_{n\to+\infty}\mathcal{H}^{N-1}\left(S(u_n)\setminus\tilde{K}_n\right)=0.$$

We deduce that

$$\tilde{h}^-(x,\nu_{S(u)}(x))=0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x\in S(u),$$

so that by definition of σ-limit we deduce $S(u)\tilde{\subseteq}\tilde{K}\tilde{\subseteq}K$. □

The next corollary shows that our σ-limit always contains the σ^p-limit introduced by Dal Maso, Francfort and Toader in [44] to study quasistatic crack growth in nonlinear elasticity (see Section 4.1 for a definition).

Corollary 4.5.9. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω such that K_n σ-converges in Ω to K. Let $1<p<+\infty$, and let us assume that K_n σ^p-converges in Ω to some rectifiable set $\tilde{K}$. Then $\tilde{K}\tilde{\subseteq}K$.*

Proof. Recall that by definition of σ^p-convergence we have $K=S(z)$ for some $z\in SBV^p(\Omega)$, and there exists $(z_n)_{n\in\mathbb{N}}$ sequence in $SBV^p(\Omega)$ with $z_n\rightharpoonup z$ weakly in $SBV^p(\Omega)$ and $S(z_n)\tilde{\subseteq}K_n$. The result follows applying Proposition 4.5.8 to $(z_n)_{n\in\mathbb{N}}$. □

Remark 4.5.10. Notice that is general we can have that the σ^p-limit $\tilde{K}$ of $(K_n)_{n\in\mathbb{N}}$ is strictly contained in K. In fact we can consider $\Omega := (-1,1)\times(-1,1)$ in $\mathbb{R}^2$, and

$$K_n := \{(-1,1)\setminus L_n\}\times\{0\}$$

with $L_n\subseteq(-1,1)$ and $|L_n|\to 0$. In this case we get $K=(-1,1)\times\{0\}$, while if L_n is chosen in such a way that its *c_p-capacity* is big enough (see the celebrated example of the Neumann sieve, we refer to [74]) we get $\tilde{K}=\emptyset$.

This example is based on the fact that the σ^p-limit is influenced by infinitesimal perturbations of the K_n, while the set K is not. To be precise we have that if $\mathcal{H}^{N-1}(K_n\triangle K_n')\to 0$, and $K_n\to K$ in the sense of σ-convergence, then $\tilde{K}_n$ still σ-converges to K.

In Section 4.7 and Section 4.8, we will need a definition of σ-convergence in the closed set $\overline{\Omega}$.

Definition 4.5.11 (σ-convergence in $\overline{\Omega}$). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in $\overline{\Omega}$. We say that K_n σ-converges in $\overline{\Omega}$ to $K\tilde{\subseteq}\overline{\Omega}$ if K_n σ-converges in Ω' to K for every open bounded set Ω' such that $\overline{\Omega}\subseteq\Omega'$.*

Notice that to check the σ-convergence in $\overline{\Omega}$ of rectifiable sets, it is enough check σ-convergence in Ω' for just one Ω' with $\overline{\Omega}\subseteq\Omega'$.

4.6 Stability of unilateral minimality properties

In this section we apply the results of Section 4.4 and Section 4.5 to obtain the stability result of unilateral minimality properties under Γ-convergence for bulk and surface energies.

Definition 4.6.1 (unilateral minimizers). *Let $f : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function and let $g : \Omega \times S^{N-1} \to [0, +\infty[$ be a Borel function satisfying the growth estimates (4.22) and (4.23). We say that the pair (u, K) with $u \in SBV^p(\Omega)$ and K rectifiable set in Ω is a unilateral minimizer with respect to f and g if $S(u) \tilde{\subseteq} K$, and*

$$\int_\Omega f(x, \nabla u(x))\, dx \le \int_\Omega f(x, \nabla v(x))\, dx + \int_{H \setminus K} g(x, \nu),$$

for all pairs (v, H) with $v \in SBV^p(\Omega)$, H rectifiable set in Ω such that $S(v) \tilde{\subseteq} H$ and $K \tilde{\subseteq} H$.

As in the previous sections, let $f_n : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function and let $g_n : \Omega \times S^{N-1} \to [0, +\infty[$ be a Borel function satisfying the growth estimates (4.22) and (4.23).

Let us assume that the functionals $(\mathcal{F}_n(\cdot, A))_{n \in \mathbb{N}}$ and $(\mathcal{G}_n(\cdot, A))_{n \in \mathbb{N}}$ defined in (4.26) and (4.28) Γ-converge in the strong topology of $L^1(\Omega)$ to $\mathcal{F}(\cdot, A)$ and $\mathcal{G}(\cdot, A)$ for every $A \in \mathcal{A}(\Omega)$ respectively. Let f be the density of $\mathcal{F}$ according to Proposition 4.3.1 and let g be the density of $\mathcal{G}$ according to Proposition 4.3.2.

The main result of the chapter is the following stability result for unilateral minimality properties under σ-convergence of rectifiable sets (see Definition 4.5.1), and Γ-convergence of bulk and surface energies.

Theorem 4.6.2. *Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $SBV^p(\Omega)$ with $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$, and let $(K_n)_{n \in \mathbb{N}}$ be a sequence of rectifiable sets in Ω with $\mathcal{H}^{N-1}(K_n) \le C$ and such that K_n σ-converges in Ω to K.*

Let us assume that the pair $(u_n, K_n)_{n \in \mathbb{N}}$ is a unilateral minimizer for f_n and g_n. Then (u, K) is a unilateral minimizer for f and g. Moreover we have

$$\lim_{n \to +\infty} \int_\Omega f_n(x, \nabla u_n(x))\, dx = \int_\Omega f(x, \nabla u(x))\, dx. \tag{4.86}$$

Proof. By Theorem 4.4.1 we have that the functionals

$$\mathcal{E}_n(u) := \begin{cases} \int_\Omega f_n(x, \nabla u(x))\, dx + \int_{S(u) \setminus K_n} g_n(x, \nu)\, d\mathcal{H}^{N-1}(x) & u \in SBV^p(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converge with respect to the strong topology of $L^1(\Omega)$ to the functional

$$\mathcal{E}(u) := \begin{cases} \int_\Omega f(x, \nabla u(x))\, dx + \int_{S(u)} g^-(x, \nu)\, d\mathcal{H}^{N-1}(x) & u \in SBV^p(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where f and g^- are defined in (4.32) and (4.36) respectively, with $g^- \le g$.

By Proposition 4.5.8 we have $S(u) \tilde{\subseteq} K$, so that u is admissible for K, while by Proposition 4.5.6 we have that

$$g^-(x, \nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K.$$

Then the unilateral minimality of the pair (u, K) easily follows. In fact, by Γ-convergence we have that u is a minimizer for $\mathcal{E}$ and $\mathcal{E}_n(u_n) \to \mathcal{E}(u)$. Then for all pairs (v, H) with $S(v) \tilde{\subseteq} H$ and $K \tilde{\subseteq} H$ we have

$$\begin{aligned} \int_\Omega f(x, \nabla u(x))\, dx = \mathcal{E}(u) \le \mathcal{E}(v) &= \int_\Omega f(x, \nabla v(x))\, dx + \int_{S(v)} g^-(x, \nu)\, d\mathcal{H}^{N-1} \\ &= \int_\Omega f(x, \nabla v(x))\, dx + \int_{S(v) \setminus K} g^-(x, \nu) \le \int_\Omega f(x, \nabla v(x))\, dx + \int_{H \setminus K} g(x, \nu), \end{aligned}$$

so that the unilateral minimality property holds. The convergence of bulk energies (4.86) is given by the convergence $\mathcal{E}_n(u_n) \to \mathcal{E}(u)$. $\square$

Remark 4.6.3 (stability under σ^p-convergence). In the case of fixed bulk and surface energies f and g, Dal Maso, Francfort and Toader [44] proved the stability of the unilateral minimality property under σ^p-convergence for the rectifiable sets K_n (see Section 4.1 for the definition). This result readily follows by Theorem 4.6.2. In fact by Corollary 4.5.9 we have that if K_n σ^p-converges in Ω to $\tilde{K}$, then $\tilde{K}$ is contained in the σ-limit of $(K_n)_{n\in\mathbb{N}}$. Since $S(u) \tilde{\subseteq} \tilde{K}$, we get that the unilateral minimality of the pair $(u, \tilde{K})$ is implied by the unilateral minimality of (u, K).

As mentioned in the Introduction, a method for proving stability of unilateral minimality properties nearer to the approach of [44] would be to prove a generalization of the Transfer of Jump Sets by Francfort and Larsen [53, Theorem 2.1] to the case of varying energies. The following theorem based on the arguments of Section 4.4 provides such a generalization.

Theorem 4.6.4 (Transfer of Jump Sets). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in Ω with $\mathcal{H}^{N-1}(K_n) \le C$ and K_n σ-converging in Ω to K. For every $v \in SBV^p(\Omega)$ there exists $(v_n)_{n\in\mathbb{N}}$ sequence in $SBV^p(\Omega)$ with $v_n \rightharpoonup v$ weakly in $SBV^p(\Omega)$ and such that*

$$\lim_{n\to+\infty} \int_\Omega f_n(x, \nabla v_n(x))\, dx = \int_\Omega f(x, \nabla v(x))\, dx$$

and

$$\limsup_{n\to+\infty} \int_{S(v_n)\setminus K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) \le \int_{S(v)\setminus K} g(x,\nu)\, d\mathcal{H}^{N-1}(x).$$

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a recovering sequence for v with respect to $(\mathcal{E}_n)_{n\in\mathbb{N}}$ defined in (4.24). By growth estimates on f_n and g_n, and since $\mathcal{H}^{N-1}(K_n) \le C$, we get $v_n \rightharpoonup v$ weakly in $SBV^p(\Omega)$. Since no interaction between bulk and surface energies occurs in view of Theorem 4.4.1, we get that

$$\lim_{n\to+\infty} \int_\Omega f_n(x, \nabla v_n(x))\, dx = \int_\Omega f(x, \nabla v(x))\, dx$$

and

$$\lim_{n\to+\infty} \int_{S(v_n)\setminus K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1} = \int_{S(v)} g^-(x,\nu)\, d\mathcal{H}^{N-1} \le \int_{S(v)\setminus K} g(x,\nu)\, d\mathcal{H}^{N-1}$$

because $g^- = 0$ on K, and $g^- \le g$. $\square$

4.7 Stability of unilateral minimality properties with boundary conditions

In view of the application of Section 4.8, we need a stability result for unilateral minimality properties with boundary conditions.

In order to set the problem, let us consider $\partial_D\Omega \subseteq \partial\Omega$, let $f_n : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function satisfying the growth estimate (4.22), and let $g_n : \overline{\Omega} \times S^{N-1} \to [0, +\infty[$ be a Borel function satisfying the growth estimate (4.23). We consider unilateral minimality properties of the form

$$\int_\Omega f_n(x, \nabla u_n)\, dx \le \int_\Omega f_n(x, \nabla v)\, dx + \int_{H\setminus K_n} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) \tag{4.87}$$

for every $v \in SBV^p(\Omega)$ and for every rectifiable set H in $\overline{\Omega}$ such that $S^{\psi_n}(v) \tilde{\subseteq} H$. Here $(K_n)_{n\in\mathbb{N}}$ is a sequence of rectifiable sets in $\overline{\Omega}$ with $\mathcal{H}^{N-1}(K_n) \le C$, $(u_n)_{n\in\mathbb{N}}$ is a sequence in $SBV^p(\Omega)$ with $S^{\psi_n}(u_n) \tilde{\subseteq} K_n$, $\psi_n \in W^{1,p}(\Omega)$ with $\psi_n \to \psi$ strongly in $W^{1,p}(\Omega)$, and $S^{\psi_n}(\cdot)$ is defined in (1.2).

In order to treat $S^{\psi_n}(\cdot)$ as an internal jump and in order to recover the surface energy on $\partial_D\Omega$ for the minimality property in the limit, let us consider an open bounded set Ω' such that $\overline{\Omega} \subset \Omega'$ and let us consider $g'_n : \Omega' \times S^{N-1} \to [0, +\infty[$ such that

$$g'_n(x,\nu) := \begin{cases} g_n(x,\nu) & \text{if } x \in \overline{\Omega}, \\ \beta + 1 & \text{otherwise.} \end{cases} \tag{4.88}$$

Let us consider the functionals $\mathcal{G}'_n : P(\Omega') \times \mathcal{A}(\Omega') \to [0, +\infty]$ defined by

$$\mathcal{G}'_n(v, A) := \int_{S(v)\cap A} g'_n(x, \nu)\, d\mathcal{H}^{N-1}(x)$$

and let $\mathcal{G}' : P(\Omega') \times \mathcal{A}(\Omega') \to [0, +\infty]$ be their Γ-limit in the strong topology of $L^1(\Omega')$, which according to Proposition 4.3.3 is of the form

$$\mathcal{G}'(v, A) := \int_{S(v)\cap A} g'(x, \nu)\, d\mathcal{H}^{N-1}(x) \tag{4.89}$$

We clearly have $g'(x, \nu) = g(x, \nu)$ for $x \in \Omega$, where g is the surface energy density defined in (4.34), while it turns out that (see Remark 4.7.2) the surface energy given by the restriction of g' to $\partial\Omega \times S^{N-1}$ is completely determined by the functions g_n.

Let us set

$$f'_n(x, \xi) := \begin{cases} f_n(x, \xi) & \text{if } x \in \Omega, \\ \alpha|\xi|^p & \text{otherwise,} \end{cases} \tag{4.90}$$

and let f' be the energy density of the Γ-limit of the functionals on $W^{1,p}(\Omega')$ associated to f'_n according to Proposition 4.3.1. We easily have that

$$f'(x, \xi) := \begin{cases} f(x, \xi) & \text{if } x \in \Omega, \\ \alpha|\xi|^p & \text{otherwise.} \end{cases} \tag{4.91}$$

Since Ω is Lipschitz, we can assume using an extension operator that $\psi_n, \psi \in W^{1,p}(\mathbb{R}^N)$ and $\psi_n \to \psi$ strongly in $W^{1,p}(\mathbb{R}^N)$.

Before stating our stability result, we need the following Γ-convergence result.

Lemma 4.7.1. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in $\overline{\Omega}$ such that $\mathcal{H}^{N-1}(K_n) \le C$. Let us assume that the functionals*

$$\mathcal{E}'_n(v) := \begin{cases} \int_{\Omega'} f'_n(x, \nabla v(x))\, dx + \int_{S(v)\setminus K_n} g'_n(x, \nu)\, d\mathcal{H}^{N-1}(x) & \text{if } v \in SBV^p(\Omega'), \\ +\infty & \text{otherwise} \end{cases} \tag{4.92}$$

Γ-converge in the strong topology of $L^1(\Omega')$ according to Theorem 4.4.1 to

$$\mathcal{E}'(v) := \begin{cases} \int_{\Omega'} f'(x, \nabla v(x))\, dx + \int_{S(v)} g'^{-}(x, \nu)\, d\mathcal{H}^{N-1}(x) & \text{if } v \in SBV^p(\Omega'), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.93}$$

Then we have that the functionals

$$\tilde{\mathcal{E}}'_n(v) := \begin{cases} \mathcal{E}'_n(v) & \text{if } v = \psi_n \text{ on } \Omega' \setminus \overline{\Omega}, \\ +\infty & \text{otherwise} \end{cases} \tag{4.94}$$

Γ-converge in the strong topology of $L^1(\Omega')$ to

$$\tilde{\mathcal{E}}'(v) := \begin{cases} \mathcal{E}'(v) & \text{if } v = \psi \text{ on } \Omega' \setminus \overline{\Omega}, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.95}$$

Proof. Let $v \in SBV^p(\Omega')$ with $v = \psi$ on $\Omega' \setminus \overline{\Omega}$, and let $(v_n)_{n\in\mathbb{N}}$ be a recovering sequence for v with respect to the functionals $\mathcal{E}'_n$. We have that

$$\nabla v_n \to \nabla \psi \qquad \text{strongly in } L^p(\Omega' \setminus \overline{\Omega}; \mathbb{R}^N), \tag{4.96}$$

and

$$\mathcal{H}^{N-1}(S(v_n) \cap (\Omega' \setminus \overline{\Omega})) \to 0. \tag{4.97}$$

In fact we have that for all $U \in \mathcal{A}(\Omega')$ such that $\overline{U} \subseteq \Omega' \setminus \overline{\Omega}$ and $\mathcal{E}'(v, \partial U) = 0$

$$\nabla v_n \to \nabla \psi \qquad \text{strongly in } L^p(U; \mathbb{R}^N), \tag{4.98}$$

and

$$\mathcal{H}^{N-1}(S(v_n) \cap U) \to 0. \tag{4.99}$$

Let $\varepsilon > 0$ and let us consider an open set $V \in \mathcal{A}(\Omega')$ such that $\partial\Omega \subseteq V$, $\mathcal{E}'(v, \partial V) = 0$, $\int_{V \cap \Omega} |a_1|\, dx < \varepsilon$ (a_1 is defined in (4.22)),

$$\int_V f'(x, \nabla v(x))\, dx < \varepsilon \quad \text{and} \quad \int_V f'(x, \nabla \psi(x))\, dx < \varepsilon. \tag{4.100}$$

Then for n large (no interaction between bulk and surface part occurs) we have

$$\int_V f_n'(x, \nabla v_n(x))\, dx < \varepsilon. \tag{4.101}$$

Notice that

$$\int_{\Omega' \setminus \overline{\Omega}} |\nabla v_n - \nabla \psi|^p\, dx = \int_{\Omega' \setminus (\Omega \cup V)} |\nabla v_n - \nabla \psi|^p\, dx + \int_{V \setminus \overline{\Omega}} |\nabla v_n - \nabla \psi|^p\, dx$$
$$\leq \int_{\Omega' \setminus (\Omega \cup V)} |\nabla v_n - \nabla \psi|^p\, dx + \frac{2^{p-1}}{\alpha} \int_V f_n'(x, \nabla v_n(x)) + f'(x, \nabla \psi(x))\, dx + \frac{2^{p-1}}{\alpha} \int_{V \cap \Omega} 2|a_1|\, dx.$$

Since $\nabla v_n \to \nabla \psi$ strongly in $L^p(\Omega' \setminus (\Omega \cup V); \mathbb{R}^N)$, because of (4.100) and (4.101), and since ε is arbitrary, we get that (4.96) holds.

Let us come to (4.97). Up to a subsequence we have

$$\mu_n := \mathcal{H}^{N-1} \llcorner (S(v_n) \cap (\Omega' \setminus \overline{\Omega}))) \overset{*}{\rightharpoonup} \mu \qquad \text{weakly* in } \mathcal{M}_b(\Omega').$$

In view of (4.99), in order to prove (4.97) it is sufficient to show that $\mu(\partial\Omega) = 0$. Let us assume by contradiction that $\mu(\partial\Omega) \neq 0$: then there exists a cube Q_ρ of center $x \in \partial\Omega$ and edge 2ρ such that $\mathcal{E}'(v, \partial Q_\rho) = 0$ and

$$\mu(Q_\rho) > \sigma > 0. \tag{4.102}$$

Up to a translation we may assume that $x = 0$, and moreover we can assume that

$$\Omega \cap Q_\rho = \{(x', y) : x' \in (-\rho, \rho), y \in (-\rho, h(x'))\},$$

where (x', y) is a suitable orthogonal coordinate system and h is a Lipschitz function. Let $\eta > 0$ be such that setting

$$V_\eta := \{(x', y) : x' \in (-\rho, \rho),\ y \in (h(x') - \eta, h(x') + \eta)\}$$

we have $V_\eta \subseteq Q_\rho$, and $\mathcal{E}'(v, \partial V_\eta) = 0$. Let us set

$$V_\eta^- := \{(x', y) \in V_\eta : y < h(x')\} \quad \text{and} \quad V_\eta^+ := \{(x', y) \in V_\eta : y > h(x')\}.$$

By (4.102) we have that for n large

$$\mathcal{H}^{N-1}(S(v_n) \cap V_\eta^+) > \sigma. \tag{4.103}$$

Let $\hat{v}$ be the function defined on V_η obtained reflecting $v_{|V_\eta^+}$ to V_η^-: more precisely let us set

$$\hat{v} = \begin{cases} v(x', y) & \text{if } (x', y) \in V_\eta^+, \\ v(x', 2h(x') - y) & \text{if } (x', y) \in V_\eta^-. \end{cases}$$

We clearly have $v \in W^{1,p}(V_\eta)$. Let $\hat{v}_n$ be obtained in the same way from $(v_n)_{|V_\eta^+}$. Let us consider

$$w_n := v_n + \hat{v} - \hat{v}_n.$$

We have $w_n \rightharpoonup v$ weakly in $SBV^p(V_\eta)$ so that by lower semicontinuity given by Proposition 4.4.3 we get

$$\int_{S(v)\cap V_\eta} g'^{-}(x,\nu)\, d\mathcal{H}^{N-1}(x) \leq \liminf_{n\to+\infty} \int_{(S(w_n)\setminus K_n)\cap V_\eta} g_n'(x,\nu)\, d\mathcal{H}^{N-1}(x). \tag{4.104}$$

On the other hand, since $\mathcal{E}'(v, \partial V_\eta) = 0$, we have that v_n is a recovering sequence for v in V_η. In particular we get that

$$\int_{S(v)\cap V_\eta} g'^{-}(x,\nu)\, d\mathcal{H}^{N-1}(x) = \lim_{n\to+\infty} \int_{(S(v_n)\setminus K_n)\cap V_\eta} g_n'(x,\nu)\, d\mathcal{H}^{N-1}(x). \tag{4.105}$$

Formulas (4.104) and (4.105) give a contradiction because for n large by (4.103) and since $K_n \tilde{\subseteq} \overline{\Omega}$ and $S(w_n) \tilde{\subseteq} \overline{\Omega} \cap Q_\rho$ (recall that $g_n'(x,\nu) = \beta + 1$ for $x \in \Omega' \setminus \overline{\Omega}$)

$$\int_{(S(v_n)\setminus K_n)\cap V_\eta} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) - \int_{(S(w_n)\setminus K_n)\cap V_\eta} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x) > \sigma.$$

We conclude that (4.97) holds.

We are now in a position to prove the Γ-limsup inequality for $\tilde{\mathcal{E}}_n'$ and $\tilde{\mathcal{E}}'$ (the Γ-liminf is immediate from the Γ-convergence of $\mathcal{E}_n'$ to $\mathcal{E}'$ and the fact that the constraint is closed under the strong topology of $L^1(\Omega)$). Let $\varepsilon > 0$, and let $U \in \mathcal{A}(\Omega')$ be such that $\partial\Omega \subseteq U$, $\mathcal{E}'(v, \partial U) = 0$, and

$$\int_U f(x, \nabla v)\, dx < \varepsilon. \tag{4.106}$$

In view of (4.96) and (4.97) we can find $\varphi_n \in SBV^p(\Omega')$ such that $\varphi_n = \psi_n - v_n$ on $\Omega' \setminus \overline{\Omega}$, $\varphi_n = 0$ on $\Omega \setminus U$ and

$$\begin{aligned} &\varphi_n \to 0 \qquad \text{strongly in } L^1(\Omega'), \\ &\nabla\varphi_n \to 0 \qquad \text{strongly in } L^p(\Omega'; \mathbb{R}^N), \\ &\mathcal{H}^{N-1}(S(\varphi_n)) \to 0. \end{aligned}$$

Let us consider

$$\tilde{v}_n := v_n + \varphi_n.$$

We have $\tilde{v}_n = \psi_n$ on $\Omega' \setminus \overline{\Omega}$. Moreover

$$\limsup_{n\to+\infty} \int_{S(\tilde{v}_n)\setminus K_n} g_n'(x,\nu)\, d\mathcal{H}^{N-1} = \limsup_{n\to+\infty} \int_{S(v_n)\setminus K_n} g_n'(x,\nu)\, d\mathcal{H}^{N-1},$$

and using the growth estimate on f_n'

$$\begin{aligned} \limsup_{n\to+\infty} &\left| \int_{\Omega'} f_n'(x, \nabla\tilde{v}_n(x))\, dx - \int_{\Omega'} f_n'(x, \nabla v_n(x))\, dx \right| \\ &\leq \limsup_{n\to+\infty} \int_{U\cap\Omega} f_n(x, \nabla\tilde{v}_n(x)) + f_n(x, \nabla v_n(x))\, dx \\ &\leq \limsup_{n\to+\infty} \int_U a_2(x)\, dx + \left(\frac{2^{p-1}}{\alpha} + 1\right) \int_U f_n(x, \nabla v_n(x))\, dx \\ &\qquad + \frac{2^{p-1}}{\alpha} \int_U |a_1|\, dx + 2^{p-1} \int_U |\nabla\varphi_n|^p\, dx. \end{aligned}$$

By (4.106) we get

$$\limsup_{n\to+\infty} \int_U f_n(x, \nabla v_n(x))\, dx < \varepsilon.$$

Then we conclude

$$\limsup_{n\to+\infty} \left| \int_{\Omega'} f_n'(x, \nabla \tilde{v}_n(x))\, dx - \int_{\Omega'} f_n'(x, \nabla v_n(x))\, dx \right| \le e(\varepsilon)$$

with $e(\varepsilon) \to 0$ as $\varepsilon \to 0$. We deduce that

$$\limsup_{n\to+\infty} \tilde{\mathcal{E}}'(\tilde{v}_n) \le \tilde{\mathcal{E}}'(v) + e(\varepsilon),$$

with $e(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since ε is arbitrary, using a diagonal argument we have that the Γ-limsup inequality is proved. □

Remark 4.7.2. In view of Lemma 4.7.1 we can prove that the surface energy determined by the restriction of g' to $\partial\Omega$ is actually independent of the choice of Ω' and of the constant value c' of g_n' on $\Omega' \setminus \overline{\Omega}$ provided that $c' > \beta$. In fact g' is the density of the surface energy of the Γ-limit in the strong topology of $L^1(\Omega)$ of the functionals on $SBV^p(\Omega')$ defined as

$$\hat{\mathcal{E}}_n'(v) := \int_{\Omega'} f_n'(x, \nabla v(x))\, dx + \int_{S(v)} g_n'(x, \nu)\, d\mathcal{H}^{N-1}(x).$$

Following the proof of Lemma 4.7.1 (for the functionals $\mathcal{E}_n'$ with $K_n = \emptyset$), if $v = \psi$ outside $\overline{\Omega}$, we can find $(v_n)_{n\in\mathbb{N}}$ recovering sequence for v with respect to $(\hat{\mathcal{E}}_n', \Omega', c')$ such that $v_n = \psi_n$ outside $\overline{\Omega}$. Then if Ω'' is an open set such that $\overline{\Omega} \subseteq \Omega''$ we have that $(v_n)_{|\Omega'\cap\Omega''}$ is a recovering sequence also for $(\hat{\mathcal{E}}_n', \Omega'' \cap \Omega', c'')$, and we have

$$\int_{S(v)} g'(x, \nu)\, d\mathcal{H}^{N-1} = \lim_{n\to+\infty} \int_{S(v_n)} g_n(x, \nu)\, d\mathcal{H}^{N-1}.$$

We deduce that the surface energy given by the restriction of g' to $\overline{\Omega} \times S^{N-1}$ is determined only by the $g_n : \overline{\Omega} \times S^{N-1} \to [0, +\infty]$.

The stability result for unilateral minimality properties with boundary conditions under σ-convergence in $\overline{\Omega}$ for rectifiable sets (see Definition 4.5.11) and Γ-convergence of bulk and surface energies is the following.

Theorem 4.7.3. *Let $\psi_n \in W^{1,p}(\Omega)$ with $\psi_n \to \psi$ strongly in $W^{1,p}(\Omega)$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV^p(\Omega)$ with $u_n \rightharpoonup u$ weakly in $SBV^p(\Omega)$, and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of rectifiable sets in $\overline{\Omega}$ with $\mathcal{H}^{N-1}(K_n) \le C$, such that K_n σ-converges in $\overline{\Omega}$ to K, and $S^{\psi_n}(u_n) \tilde{\subseteq} K_n$.*

Let us assume that the pair (u_n, K_n) satisfies the unilateral minimality property (4.87) *with respect to f_n, g_n and ψ_n. Then (u, K) satisfies the unilateral minimality property with respect to f, g and ψ, where f is defined in* (4.32) *and g is the restriction of g' defined in* (4.89) *to $\overline{\Omega} \times S^{N-1}$. Moreover we have*

$$\lim_{n\to+\infty} \int_\Omega f_n(x, \nabla u_n(x))\, dx = \int_\Omega f(x, \nabla u(x))\, dx. \tag{4.107}$$

Proof. Since the boundary datum ψ_n is imposed just on $\partial_D\Omega$, we can consider $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$ as part of the cracks, that is we can replace in the unilateral minimality properties K_n with $K_n' := K_n \cup \partial_N\Omega$.

It is easy to prove that K_n' σ-converges in $\overline{\Omega}$ to $K \cup \partial_N\Omega$. Then the proof follows that of Theorem 4.6.2 employing the functionals $(\tilde{\mathcal{E}}_n')_{n\in\mathbb{N}}$ defined in Lemma 4.7.1 with K_n' in place of K_n. □

4.8 Quasistatic evolution of cracks in composite materials

The aim of this section is to apply the stability results of Section 4.7 to the study the asymptotic behavior of crack evolutions relative to varying bulk and surface energies f_n and g_n. As mentioned in the Introduction, this problem is inspired by the problem of crack propagation in composite materials. We restrict our analysis to the case of antiplanar shear, where the elastic body is an infinite cylinder.

Let us recall the result of Dal Maso, Francfort and Toader [44] about quasistatic crack evolution in nonlinear elasticity: it is a very general existence and approximation result concerning a variational theory crack propagation inspired by the variational model introduced by Francfort and Marigo in [54]. As already said, we consider the antiplanar case and for simplicity we neglect body and traction forces, and so we adapt the mathematical tools employed in [44] to this scalar setting.

As in the previous sections, let $\Omega \subset \mathbb{R}^N$ (which, for $N = 2$ represents a section of the cylindrical hyperelastic body) be an open bounded set with Lipschitz boundary. The family of admissible cracks is the class of rectifiable subsets of $\overline{\Omega}$, while the class of admissible displacements is given by the functional space $SBV^p(\Omega)$, where $1 < p < +\infty$. Let $\partial_D\Omega$ be a subset of $\partial\Omega$. Given $\psi \in W^{1,p}(\Omega)$, we say that the displacement u is admissible for the fracture K and the boundary datum ψ and we write $u \in AD(\psi, K)$ if

$$S(v) \tilde{\subseteq} K \qquad \text{and} \qquad v = \psi \qquad \text{on } \partial_D\Omega \setminus K.$$

This can be summarized by the notation $S^{\psi}(u) \tilde{\subseteq} K$, where $S^{\psi}(\cdot)$ is defined in (1.2).

Let $f(x,\xi) : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Carathéodory function which is convex and C^1 in ξ for a.e. $x \in \Omega$, and satisfies the growth estimate

$$a_1(x) + \alpha|\xi|^p \leq f(x,\xi) \leq a_2(x) + \beta|\xi|^p, \tag{4.108}$$

where $a_1, a_2 \in L^1(\Omega)$ and $\alpha, \beta > 0$. Let moreover $g : \overline{\Omega} \times S^{N-1} \to [0, +\infty[$ be a Borel function such that

$$\alpha \leq g(x,\nu) \leq \beta. \tag{4.109}$$

The total energy of a configuration (u, K) is given by

$$\mathcal{E}(u,K) := \int_\Omega f(x, \nabla u(x))\, dx + \int_K g(x,\nu) d\mathcal{H}^{N-1}(x).$$

We will usually refer to the first term as *bulk energy* of u and we write

$$\mathcal{E}^b(u) := \int_\Omega f(x, \nabla u(x))\, dx, \tag{4.110}$$

while we will refer to the second term as *surface energy* of K and we write

$$\mathcal{E}^s(K) := \int_K g(x,\nu) d\mathcal{H}^{N-1}(x). \tag{4.111}$$

Let us consider now a time dependent boundary datum $\psi \in W^{1,1}([0,T]; W^{1,p}(\Omega))$ (i.e. the function $t \to \psi(t)$ is absolutely continuous from $[0,T]$ to the Banach space $W^{1,p}(\Omega)$, with summable time derivative, see for instance [22]), such that for all $t \in [0,T]$

$$\|\psi(t)\|_{L^\infty(\Omega)} \leq C. \tag{4.112}$$

In [44] Dal Maso, Francfort and Toader proved the existence of an *irreversible quasistatic crack evolution* in Ω relative to the boundary displacement ψ, i.e. the existence of a map $t \to (u(t), K(t))$ where $u(t) \in AD(\psi(t), K(t))$, $\|u(t)\|_{L^\infty(\Omega)} \leq \|\psi(t)\|_\infty$ and such that the following three properties hold:

(1) *irreversibility:* $K(t_1) \tilde{\subseteq} K(t_2)$ for all $0 \le t_1 \le t_2 \le T$;

(2) *static equilibrium:* $\mathcal{E}(u(0), K(0)) \le \mathcal{E}(v, K)$ for all (v, K) such that $v \in AD(\psi(0), K)$, and

$$\mathcal{E}(u(t), K(t)) \le \mathcal{E}(v, K) \qquad \text{for all } K(t) \tilde{\subseteq} K,\ v \in AD(\psi(t), K);$$

(3) *nondissipativity:* the function $t \to \mathcal{E}(u(t), K(t))$ is absolutely continuous and

$$\frac{d}{dt}\mathcal{E}(u(t), K(t)) = \int_\Omega \nabla_\xi f(x, \nabla u(t)) \nabla \dot{\psi}(t)\, dx,$$

where $\dot{\psi}$ denotes the time derivative of $t \to \psi(t)$.

For every $n \in \mathbb{N}$ let us consider admissible bulk and surface energies $f_n : \Omega \times \mathbb{R}^N \to \mathbb{R}$ and $g_n : \Omega \times S^{N-1} \to [0, +\infty[$ for the model of [44] satisfying the growth estimates (4.108) and (4.109) uniformly in n. Let us moreover assume that f_n is such that for a.e. $x \in \Omega$ and for all $M \ge 0$

$$|\nabla_\xi f_n(x, \xi_n^1) - \nabla_\xi f_n(x, \xi_n^2)| \to 0 \tag{4.113}$$

for all ξ_n^1, ξ_n^2 such that $|\xi|_n^1 \le M$ $|\xi|_n^2 \le M$ and $|\xi_n^1 - \xi_n^2| \to 0$. We denote by $\mathcal{E}_n$, $\mathcal{E}_n^b$ and $\mathcal{E}_n^s$ the total, bulk and surface energies associated to f_n and g_n.

Let f and g be the effective energies associated to f_n and g_n in the sense of Theorem 4.7.3, i.e. let f be given by Proposition 4.3.1 and let g be the restriction to $\overline{\Omega} \times S^{N-1}$ of the function g' defined in (4.89). Notice that by Theorem 4.2.5 we have that the function $f(x, \cdot)$ is C^1: as it is also convex in ξ and satisfies the growth estimate (4.108), we have that f and g are admissible bulk and surface energies for the model of [44].

Let $t \to \psi_n(t)$ be a sequence of admissible time dependent boundary displacements such that

$$\psi_n \to \psi \qquad \text{strongly in } W^{1,1}\Big([0,T], W^{1,p}(\Omega)\Big).$$

Let $t \to (u_n(t), K_n(t))$ be a quasistatic evolution for the boundary datum ψ_n relative to the energies f_n and g_n according to [44]. The main result of this section is the following Theorem which asserts that the σ-limit in $\overline{\Omega}$ of $K_n(t)$ (see Definition 4.5.11) still determines a quasistatic crack growth with respect to the energies f and g.

Theorem 4.8.1. *There exists a quasistatic crack growth $t \to (u(t), K(t))$ relative to the energies f and g and the boundary datum ψ such that up to a subsequence (not rebelled) the following hold:*

(1) for all $t \in [0, T]$

$$K_n(t) \ \sigma\text{-converges in } \overline{\Omega} \text{ to } K(t),$$

and there exists a further subsequence n_k (depending possibly on t) such that

$$u_{n_k}(t) \rightharpoonup u(t) \qquad \text{weakly in } SBV^p(\Omega);$$

(2) for every $t \in [0, 1]$ we have convergence of total energies

$$\mathcal{E}_n(u_n(t), K_n(t)) \to \mathcal{E}(u(t), K(t)),$$

and in particular separate convergence for bulk and surface energies, i.e.

$$\mathcal{E}_n^b(u_n(t)) \to \mathcal{E}^b(u(t)) \quad \text{and} \quad \mathcal{E}^s(K_n(t)) \to \mathcal{E}^s(K(t)).$$

Proof. Notice that by nondissipativity for $t \to (u_n(t), K_n(t))$ and by growth estimates on f_n and g_n we have that there exists a constant C such that for all $t \in [0, T]$ and for all $n \in \mathbb{N}$

$$\|\nabla u_n(t)\|^p + \mathcal{H}^{N-1}(K_n(t)) + \|u_n(t)\|_{L^\infty(\Omega)} \le C. \tag{4.114}$$

We divide the proof in several steps.

Step 1: Compactness for the cracks. In view of (4.114), using a variant of Helly's theorem (see for instance [45, Theorem 6.3] for the case of Hausdorff converging compact sets), we can find a subsequence (not rebelled) of $(K_n(\cdot))_{n\in\mathbb{N}}$ and an increasing map $t \to K(t)$ such that for all $t \in [0, T]$

$$K_n(t) \ \sigma\text{-converges in } \overline{\Omega} \text{ to } K(t). \tag{4.115}$$

Step 2: Compactness for the displacements. Notice that the sequence $(u_n(t))_{n\in\mathbb{N}}$ is relatively compact in $SBV^p(\Omega)$ by (4.114). We now want to select a particular limit point of this sequence. With this aim, let us consider

$$\vartheta_n(t) := \int_\Omega \nabla_\xi f_n(x, \nabla u_n(t)) \nabla \dot{\psi}_n(t)\, dx,$$

and let us set

$$\vartheta(t) := \limsup_{n\to+\infty} \vartheta_n(t). \tag{4.116}$$

Let us see that there exists $u(t) \in SBV^p(\Omega)$ such that

$$\vartheta(t) = \int_\Omega \nabla_\xi f(x, \nabla u(t)) \nabla \dot{\psi}(t)\, dx \tag{4.117}$$

and

$$u_{n_k}(t) \rightharpoonup u(t) \qquad \text{weakly in } SBV^p(\Omega) \tag{4.118}$$

for a suitable subsequence n_k depending on t. In fact let us consider a subsequence n_k such that

$$\vartheta(t) = \lim_{k\to+\infty} \int_\Omega \nabla_\xi f(x, \nabla u_{n_k}(t)) \nabla \dot{\psi}_{n_k}(t)\, dx,$$

and

$$u_{n_k}(t) \rightharpoonup u \qquad \text{weakly in } SBV^p(\Omega).$$

By static equilibrium for $(u_n(t), K_n(t))$ we have that

$$\int_\Omega f_{n_k}(x, \nabla u_{n_k}(t))\, dx \le \int_\Omega f_{n_k}(x, \nabla v(x))\, dx + \int_{H\setminus K_{n_k}(t)} g_n(x,\nu)\, d\mathcal{H}^{N-1}(x)$$

for all $v \in AD(\psi_{n_k}(t), H)$ with $K_{n_k}(t) \tilde{\subseteq} H$. Then by Theorem 4.7.3 we get that

$$\int_\Omega f(x, \nabla u)\, dx \le \int_\Omega f(x, \nabla v(x))\, dx + \int_{H\setminus K(t)} g(x,\nu)\, d\mathcal{H}^{N-1}(x) \tag{4.119}$$

for all $v \in AD(\psi(t), H)$ with $K(t) \tilde{\subseteq} H$ and

$$\int_\Omega f_{n_k}(x, \nabla u_{n_k}(t))\, dx \to \int_\Omega f(x, \nabla u)\, dx. \tag{4.120}$$

We claim that

$$\lim_{k\to+\infty} \int_\Omega \nabla_\xi f_{n_k}(x, \nabla u_{n_k}(t)) \nabla\Phi\, dx = \int_\Omega \nabla_\xi f(x, \nabla u) \nabla\Phi\, dx \tag{4.121}$$

for all $\Phi \in W^{1,p}(\Omega)$. This has been done in [44, Lemma 4.11] in the case of fixed bulk energy, and our proof is just a variant based on the Γ-convergence results of Section 4.4 and on assumption

(4.113) which permit to deal with varying energies. Let us consider $s_j \searrow 0$ and $k_j \to +\infty$: up to a further subsequence for k_j we can assume that

$$\int_\Omega \frac{f(x, \nabla u(x) + s_j \nabla \Phi(x)) - f(x, \nabla u(x))}{s_j} \, dx - \frac{1}{j} \leq \int_\Omega \nabla_\xi f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t) + \tilde{s}_j \nabla \Phi) \nabla \Phi \, dx$$

where $\tilde{s}_j \in [0, s_j]$. This comes from lower semicontinuity for bulk energies under Γ-convergence given by Proposition 4.4.3, and by Lagrange's Theorem. By Lemma 4.2.4 we have

$$\liminf_{j \to +\infty} \int_\Omega \nabla_\xi f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t) + \tilde{s}_j \nabla \Phi) \nabla \Phi \, dx = \liminf_{j \to +\infty} \int_\Omega \nabla_\xi f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t)) \nabla \Phi \, dx,$$

so that we get

$$\int_\Omega \nabla_\xi f(x, \nabla u) \nabla \Phi \, dx \leq \liminf_{j \to +\infty} \int_\Omega \nabla_\xi f_{n_{k_j}}(x, \nabla u_{n_{k_j}}(t)) \nabla \Phi \, dx.$$

Changing Φ with $-\Phi$, we get that (4.121) is proved: setting $u(t) := u$ we deduce that (4.117) and (4.118) hold.

Step 3: Conclusion. Let us consider $t \to (u(t), K(t))$ with $u(t)$ and $K(t)$ defined in Step 2 and Step 1 respectively. In order to see that $t \to (u(t), K(t))$ is a quasistatic crack evolution we have to check the admissibility condition $u(t) \in AD(\psi(t), K(t))$ for all t, and the properties of irreversibility, static equilibrium and nondissipativity conditions with respect to f and g.

As for admissibility, this is guaranteed by (4.115) and (4.118) which ensure that $S^{\psi(t)}(u(t)) \tilde{\subseteq} K(t)$. *Irreversibility* is given by construction in Step 1, and *static equilibrium* comes from (4.119) for $t \in (0, T]$, and by Lemma 4.7.1 (where we take $K_n = \emptyset$) for $t = 0$. As for *nondissipativity*, we have that static equilibrium implies that (see [44]) for all $t \in [0, T]$

$$\mathcal{E}(u(t), K(t)) \geq \mathcal{E}(u(0), K(0)) + \int_0^t \int_\Omega \nabla_\xi f(x, \nabla u(\tau)) \nabla \dot{\psi}(\tau) \, dx \, d\tau.$$

On the other hand by lower semicontinuity given by Proposition 4.4.3 and by Proposition 4.5.7 (applied to g' from which g is obtained by restriction) we have for all $t \in [0, T]$

$$\mathcal{E}(u(t), K(t)) \leq \liminf_{n \to +\infty} \mathcal{E}_n(u_n(t), K_n(t)),$$

and by Γ-convergence given by Lemma 4.7.1 (where we take $K_n = \emptyset$)

$$\mathcal{E}(u(0), K(0)) = \lim_{n \to +\infty} \mathcal{E}_n(u_n(0), K_n(0)).$$

Hence we get for all $t \in [0, T]$ (applying also Fatou's Lemma in the limsup version)

$$\begin{aligned} \mathcal{E}(u(t), K(t)) &\leq \liminf_{n \to +\infty} \mathcal{E}_n(u_n(t), K_n(t)) \leq \limsup_{n \to +\infty} \mathcal{E}_n(u_n(t), K_n(t)) \\ &= \limsup_{n \to +\infty} \mathcal{E}_n(u_n(0), K_n(0)) + \int_0^t \vartheta_n(s) \, ds \leq \mathcal{E}(u(0), K(0)) + \int_0^t \vartheta(s) \, ds \\ &= \mathcal{E}(u(0), K(0)) + \int_0^t \int_\Omega \nabla_\xi f(x, \nabla u(\tau)) \nabla \dot{\psi}(\tau) \, dx \, d\tau \leq \mathcal{E}(u(t), K(t)), \end{aligned}$$

so that we get

$$\mathcal{E}(u(t), K(t)) = \mathcal{E}(u(0), K(0)) + \int_0^t \int_\Omega \nabla_\xi f(x, \nabla u(\tau)) \nabla \dot{\psi}(\tau) \, dx \, d\tau$$

and

$$\lim_{n \to +\infty} \mathcal{E}_n(u_n(t), K_n(t)) = \mathcal{E}(u(t), K(t)). \tag{4.122}$$

Finally by lower semicontinuity for the bulk and surface energies under weak convergence for the displacements and σ-convergence in $\overline{\Omega}$ for the cracks, we conclude that

$$\lim_{n \to +\infty} \mathcal{E}^b_n(u_n(t)) = \mathcal{E}^b(u(t)),$$

and

$$\lim_{n \to +\infty} \mathcal{E}^s_n(K_n(t)) = \mathcal{E}^s(K(t)),$$

so that the theorem is proved. $\square$

Chapter 5

Size effects on quasistatic growth of cracks

A well known fact in fracture mechanics is that *ductility* is also influenced by the size of the structure, and in particular the structure tends to become brittle if its size increases (see for example [32], and references therein). The aim of this chapter [1]. is to capture this fact for the problem of quasistatic growth of cracks in linearly elastic bodies in the framework of the variational theory of crack propagation by Francfort and Marigo [54].

In the context of *generalized antiplanar shear*, the basic total energy of the configuration displacement-crack (u, Γ) considered in [54] is given by

$$\int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(\Gamma). \tag{5.1}$$

The first term in (5.1) implies that we assume to apply linearized elasticity in the unbroken part of Ω. The second term can be considered as the work done to create Γ.

As suggested in [54], more general surface energies can be considered in (5.1), especially those of Barenblatt's type [13], and here we consider energies of the form

$$\int_{\Gamma} \varphi(|[u]|(x)) \, d\mathcal{H}^{N-1}(x), \tag{5.2}$$

where $[u](x) := u^+(x) - u^-(x)$ is the difference of the traces of u on both sides of Γ, and $\varphi : [0, +\infty[\to [0, +\infty[$ (which depends on the material) is such that $\varphi(0) = 0$. In order to get a physical interpretation of (5.2), let us set $\sigma := \varphi'$: we interpret $\sigma(|[u]|(x))$ as density of forces in x that act between the two lips of the crack Γ whose displacements are $u^+(x)$ and $u^-(x)$ respectively. Typically σ is decreasing, and $\sigma(s) = 0$ for $s \geq \bar{s}$: this means that the interaction between the two lips of the crack decreases as the opening increases, and disappears when the opening is greater than a critical length $\bar{s}$. As a consequence, φ is increasing and concave, and $\varphi(s)$ is constant for $s \geq \bar{s}$. We will then consider φ increasing, concave, with $\varphi(0) = 0$, $a = \varphi'(0) < +\infty$, and $\lim_{s \to +\infty} \varphi(s) = 1$. We can interpret

$$\int_{\Gamma} \varphi(|[u]|(x)) \, d\mathcal{H}^{N-1}(x)$$

as the work made to create Γ with an opening given by $[u]$. Assuming linearized elasticity to hold in $\Omega \setminus \Gamma$, we consider a total energy of the form

$$\|\nabla u\|^2 + \int_{\Gamma} \varphi(|[u]|) \, d\mathcal{H}^{N-1}, \tag{5.3}$$

[1]The results of this chapter are contained in the paper
A. Giacomini: Size effects on quasistatic growth of cracks. *SIAM J. Math. Anal.* to appear

where $\|\cdot\|$ denotes the L^2 norm. The problem of irreversible quasistatic growth of cracks in the cohesive case can be addressed through a *time discretization process* in analogy to what proposed in [54] for the energy (5.1).

Let $g(t)$ be a time dependent boundary displacement defined on $\partial_D\Omega \subseteq \partial\Omega$ with $t \in [0,T]$. Let $\delta > 0$ and let $I_\delta := \{0 = t_0^\delta < t_1^\delta < \dots < t_{N_\delta}^\delta = T\}$ be a subdivision of $[0,T]$ with $\max(t_{i+1}^\delta - t_i^\delta) < \delta$, and let $g_i^\delta := g(t_i^\delta)$. At time $t = 0$ we consider u_0^δ as a minimum of

$$(5.4) \qquad \|\nabla u\|^2 + \int_{S^{g(0)}(u)} \varphi(|[u]|)\, d\mathcal{H}^{N-1}.$$

Here $S^{g(0)}(u) := S(u) \cup \{x \in \partial_D\Omega : u(x) \neq g(0)(x)\}$, and for all $x \in \partial_D\Omega$ we consider $[u](x) := g(x) - \tilde{u}(x)$, where $\tilde{u}$ is the trace of u on $\partial\Omega$. We define the crack Γ_0^δ at time $t = 0$ as $S^{g(0)}(u_0^\delta)$. We also set $\psi_0^\delta := |[u_0^\delta]|$ on Γ_0^δ. The presence of $S^{g(0)}(u)$ in (5.4) indicates that the points at which the boundary displacement is not attained are considered as a part of the crack.

Supposing to have constructed Γ_i^δ and ψ_i^δ at time t_i^δ, we consider a minimum u_{i+1}^δ of the problem

$$(5.5) \qquad \|\nabla u\|^2 + \int_{S^{g_{i+1}^\delta}(u) \cup \Gamma_i^\delta} \varphi(|[u]| \vee \psi_i^\delta)\, d\mathcal{H}^{N-1},$$

where $|[u]| \vee \psi_i^\delta := \max\{|[u]|, \psi_i^\delta\}$, and define $\Gamma_{i+1}^\delta := \Gamma_i^\delta \cup S^{g_{i+1}^\delta}(u_{i+1}^\delta)$ and $\psi_{i+1}^\delta := \psi_i^\delta \vee |[u_{i+1}^\delta]|$ on Γ_{i+1}^δ.

Notice that problem (5.5) takes into account an *irreversibility condition* in the growth of the crack. Indeed, while on $S^{g_{i+1}^\delta}(u) \setminus \Gamma_i^\delta$ the surface energy which comes in minimization of (5.5) is exactly as in (5.2), on $S^{g_{i+1}^\delta}(u) \cap \Gamma_i^\delta$ the surface energy involved takes into account the previous work made on Γ_i^δ. The surface energy is of the form of (5.2) only if $|[u]| > \psi_i^\delta$, that is only if the opening is increased. If $|[u]| \le \psi_i^\delta$ no energy is gained, that is displacements of this form along the crack are in a sense surface energy free. Notice finally that the irreversibility condition involves only the modulus of $[u]$: this is an assumption which is reasonable since we are considering only antiplanar displacements. Clearly more complex irreversibility conditions can be formulated, involving for example a partial release of energy: the one we study is the first straightforward extension of the irreversibility condition given in [54] for the energy (5.1).

The *discrete in time evolution* of the crack relative to the subdivision I_δ, and the boundary datum $g(t)$ is given by $\{(u_i^\delta, \Gamma_i^\delta, \psi_i^\delta) : i = 0, \dots, N_\delta\}$.

The *irreversible quasistatic evolution* of crack relative to the boundary datum $g(t)$ is obtained as a limit for $\delta \to 0$ of $(u^\delta(t), \Gamma^\delta(t), \psi^\delta(t))$, where $u^\delta(t) := u_i^\delta$, $\Gamma^\delta(t) := \Gamma_i^\delta$ and $\psi^\delta(t) := \psi_i^\delta$ for $t_i^\delta \le t < t_{i+1}^\delta$.

As mentioned in Section 1.4, this program has been studied in detail in several papers in the case $\varphi \equiv 1$, that is for energy of the form (5.1). In all these papers ([45],[33],[53],[44]), the analysis of the limit reveals three basic properties (irreversibility, static equilibrium and nondissipativity) which are taken as definition of irreversible quasistatic growth in brittle cracks: the time discretization procedure is considered as a privileged way to get an existence result.

In the case of energy (5.3), several difficulties arise in the analysis of the discrete in time evolution, and in the analysis as $\delta \to 0$. In Section 5.1, we prove that the functional space we need in order to apply the direct method of the Calculus of Variations in the step by step minimizations (5.4), (5.5) is the space of functions with bounded variation BV: we thus consider a relaxed version of the problems, namely

$$\int_\Omega f(\nabla u)\, dx + \int_\Gamma \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} + a|D^c u|(\Omega),$$

where $a = \varphi'(0)$, f is defined in (5.9), and $D^c u$ indicates the Cantorian part of the derivative of u. An existence result for *discrete in time evolution* in this context of BV space is given in Proposition 5.1.1.

The analysis for $\delta \to 0$ presents several difficulties, the main one being the stability of the minimality property of the discrete in time evolutions. The main purpose of this chapter is to prove that these difficulties disappear as the size of the reference configuration increases, thank to the fact that the body response tends to become more and more brittle in spite of the presence of cohesive forces on the cracks. More precisely we prove this fact for the discrete evolutions in $\Omega_h := h\Omega$ for h large, and under suitable boundary displacements. The idea is to rescale displacements and cracks to the fixed configuration Ω, and take advantage from the form of the problem in this new setting. The boundary displacements on $\partial_D\Omega_h := h\partial_D\Omega$ will be taken of the form

$$g_h(t,x) := h^\alpha g\left(t, \frac{x}{h}\right), \qquad g \in AC([0,T]; H^1(\Omega)), \qquad \|g(t)\|_\infty \le C, \qquad t \in [0,T],\ x \in \Omega_h,$$

where $\alpha > 0$ and $C > 0$. We indicate by $(u^{\delta,h}(t), \Gamma^{\delta,h}(t), \psi^{\delta,h}(t))$ the piecewise constant interpolation of the discrete in time evolution of fracture in Ω_h relative to the boundary displacement g_h. Let us moreover set for every $t \in [0,T]$

$$\mathcal{E}^{\delta,h}(t) := \int_{h\Omega} f(\nabla u^{\delta,h}(t))\,dx + \int_{\Gamma^{\delta,h}(t)} \varphi(\psi^{\delta,h}(t))\,d\mathcal{H}^{N-1} + a|D^c u^{\delta,h}(t)|(\Omega_h).$$

In the case $\alpha = \frac{1}{2}$, we make the following rescaling

$$v^{\delta,h}(t,x) := \frac{1}{\sqrt{h}} u^{\delta,h}(t,hx), \qquad K^{\delta,h}(t) := \frac{1}{h}\Gamma^{\delta,h}(t), \qquad \gamma^{\delta,h}(t) := \frac{1}{\sqrt{h}}\psi^{\delta,h}(t,hx),$$

where $t \in [0,T]$ and $x \in \Omega$. The main result of the chapter is the following (see Theorem 5.2.1 for a more precise statement).

Theorem 5.0.2. *If $\delta \to 0$ and $h \to +\infty$, there exists a quasistatic evolution $\{t \to (v(t), K(t))\}$ of brittle cracks in Ω relative to the boundary displacement g in the sense of [53] (see Theorem 1.4.2) such that for all $t \in [0,T]$ we have*

$$\nabla v^{\delta,h}(t) \rightharpoonup \nabla v(t) \qquad \text{weakly in } L^1(\Omega;\mathbb{R}^N).$$

Moreover for all $t \in [0,T]$ we have

$$\frac{1}{h^{N-1}}\mathcal{E}^{\delta,h}(t) \to \|\nabla v(t)\|^2 + \mathcal{H}^{N-1}(K(t));$$

in particular $h^{-N+1}|D^c u^{\delta,h}(t)|(\Omega_h) \to 0$,

$$\frac{1}{h^{N-1}}\int_{\Omega_h} f(\nabla u^{\delta,h}(t))\,dx \to \|\nabla v(t)\|^2,$$

and

$$\frac{1}{h^{N-1}}\int_{\Gamma^{\delta,h}(t)} \varphi(\psi^{\delta,h}(t))\,d\mathcal{H}^{N-1} \to \mathcal{H}^{N-1}(K(t)).$$

Theorem 5.0.2 proves that as the size of the reference configuration increases, the response of the body in the problem of quasistatic growth of cracks tends to become brittle, so that energy (5.1) can be considered. Moreover we have convergence results for the volume and surface energies involved.

The particular value $\alpha = \frac{1}{2}$ comes out because a problem of quasistatic evolution has been considered. In fact if we consider an infinite plane with a crack-segment of length l and subject to a uniform stress σ at infinity, following Griffith's theory the crack propagates quasistatically if $\sigma = \frac{K_{IC}}{\sqrt{\pi l}}$, where K_{IC} is the critical *stress intensity factor* (depending on the material). So if the crack has length hl, the stress rescale as $\frac{1}{\sqrt{h}}$. This is precisely what we are prescribing in the case

$\alpha = \frac{1}{2}$: in fact the stress that intuitively we prescribe at the boundary can be reconstructed from ∇u_h and rescales precisely as $\frac{1}{\sqrt{h}}$.

For the proof of Theorem 5.0.2, the first step is to recognize that $(v^{\delta,h}(t), K^{\delta,h}(t), \gamma^{\delta,h}(t))$ is a discrete in time evolution relative to the boundary displacement g for a total energy of the form

$$\int_\Omega f_h(\nabla u)\, dx + \int_\Gamma \varphi_h(|[u]| \vee \gamma)\, d\mathcal{H}^{N-1} + a\sqrt{h}|D^c u|(\Omega),$$

where $\varphi_h(s) \nearrow 1$ for all $s \in [0, +\infty[$, and $f_h(\xi) \nearrow |\xi|^2$ for all $\xi \in \mathbb{R}^N$. From the fact that $\varphi_h \nearrow 1$ we recognize that the structure tends to become brittle. Bound on total energy for the discrete in time evolution is available, so that compactness in the space BV can be applied: it turns out that the limits of the displacements are of class SBV with gradient in $L^2(\Omega; \mathbb{R}^N)$. Limits for the cracks are constructed through a Γ-convergence procedure (see Lemma 5.3.2). The main point in order to see that $(v(t), K(t))$ is a quasistatic crack growth is to recover the static equilibrium condition (see point (c) of Theorem 1.4.2)

$$\|\nabla v(t)\|^2 \le \|\nabla v\|^2 + \mathcal{H}^{N-1}(S^{g(t)}(v) \setminus K(t)), \qquad v \in SBV(\Omega)$$

from the minimality properties satisfied by $(v^{\delta,h}(t), K^{\delta,h}(t), \gamma^{\delta,h}(t))$. This is done by means of a refined version of the Transfer of Jump of [53] (Proposition 5.3.5): the main difference here is that we have to deal with BV functions and we have to transfer the jump on the part of $K^{\delta,h}(t)$ where $\psi^{\delta,h}(t)$ is greater than a given small constant.

We also consider the cases $\alpha \in]0, \frac{1}{2}[$ and $\alpha > \frac{1}{2}$. It turns out that in the case $\alpha \in]0, \frac{1}{2}[$, the body is not solicited enough to create a crack, that is Ω_h tends to behave elastically: more precisely we prove (Theorem 5.2.2) that setting

$$v^{\delta,h}(t,x) := \frac{1}{h^\alpha} u^{\delta,h}(t, hx), \tag{5.6}$$

for all $t \in [0,T]$ we have that $v^{\delta,h}(t)$ converges to the displacement of the elastic problem in Ω under boundary displacement given by $g(t)$.

In the case $\alpha > \frac{1}{2}$ we have that the the body tends brutally toward *rupture*: in fact in Theorem 5.2.3 we prove that $v^{\delta,h}(0)$ given by (5.6) converges to a piecewise constant function v in Ω, so that $S^{g(0)}(v)$ disconnects Ω. This phenomenon is a consequence of the variational approach based on the search for global minimizers: as the size of Ω_h increases, cracks carry an energy of order h^{N-1}, while non rigid displacements carry an energy of greater order: in this way crack is preferred to deformation.

The chapter is organized as follows. In Section 5.1 we deal with the problem of discrete in time evolutions for cracks in the cohesive case. The main theorems are listed in Section 5.2, while in Section 5.3 we prove some results which will be employed in their proofs to which Sections 5.4, 5.5 and 5.6 are devoted. In Section 5.7 we prove a relaxation result which is used in the problem of the discrete in time evolution of cracks.

5.1 Discrete in time evolution of fractures in the cohesive case

In this section we are interested in generalized antiplanar shear of an elastic body Ω in the context of linearized elasticity and in presence of cohesive forces on the cracks.

The notion of *discrete in time evolution* for cracks relative to time dependent boundary displacement $g(t)$ has been described at the beginning of the chapter. It relies on the minimization of functionals of the form

$$\|\nabla u\|^2 + \int_{\Gamma \cup S^{g(t)}(u)} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1}, \tag{5.7}$$

with ψ a positive function defined on Γ. We now define rigorously the functional space to which the displacements belong, and the properties of Ω, Γ, ψ and $g(t)$ in order to prove an existence result for the discrete in time evolution of cracks.

Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. Let $\partial_D\Omega \subseteq \partial\Omega$ be open in the relative topology, and let $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. Let $\varphi : [0, +\infty[\to [0, +\infty[$ be increasing and concave, $\varphi(0) = 0$ and such that $\lim_{s\to+\infty} \varphi(s) = 1$. If $a := \varphi'(0) < +\infty$, we have

$$\varphi(s) \le as \qquad \text{for all } s \in [0, +\infty[.$$

Let $T > 0$, and let us consider a boundary displacement $g \in AC([0,T]; H^1(\Omega))$ such that $\|g(t)\|_\infty \le C$ for all $t \in [0,T]$. We discretize g in the following way. Given $\delta > 0$, let I_δ be a subdivision of $[0,T]$ of the form $0 = t_0^\delta < t_1^\delta < \cdots < t_{N_\delta}^\delta = T$ such that $\max_i(t_i^\delta - t_{i-1}^\delta) < \delta$. For $0 \le i \le N_\delta$ we set $g_i^\delta := g(t_i^\delta)$.

As for the space of the displacements, it would be natural following [53] to consider $u \in SBV(\Omega)$. Since $a = \varphi'(0) < +\infty$, we have unfortunately that the minimization of (5.7) is not well posed in $SBV(\Omega)$. Let us in fact consider $(u_n)_{n\in\mathbb{N}}$ minimizing sequence for (5.7): it turns out that we may assume $(u_n)_{n\in\mathbb{N}}$ bounded in $BV(\Omega)$. As a consequence $(u_n)_{n\in\mathbb{N}}$ admits a subsequence weakly* convergent in $BV(\Omega)$ to a function $u \in BV(\Omega)$. Then we have that minimizing sequences of (5.7) converge (up to a subsequence) to a minimizer of the relaxation of (5.7) with respect to the weak* topology of $BV(\Omega)$. By Proposition 5.7.1, the natural domain of this relaxed functional is $BV(\Omega)$, and that its form is

$$\int_\Omega f(\nabla u)\, dx + \int_{\Gamma \cup S^{g(t)}(u)} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} + a|D^c u|(\Omega), \tag{5.8}$$

where

$$f(\xi) := \begin{cases} |\xi|^2 & \text{if } |\xi| \le \frac{a}{2} \\ \frac{a^2}{4} + a(|\xi| - \frac{a}{2}) & \text{if } |\xi| \ge \frac{a}{2}. \end{cases} \tag{5.9}$$

In view of these remarks, we consider $BV(\Omega)$ as the space of displacements u of the body Ω, and a total energy of the form (5.8). The volume part in the energy (5.8) can be interpreted as the contribution of the elastic behavior of the body. The second term represents the work done to create the crack $\Gamma \cup S^{g(t)}(u)$ with opening given by $|[u]| \vee \psi$. The new term $a|D^c u|$ can be interpreted as the contribute of microcracks in the configuration which are considered as reversible.

Let us define the discrete evolution of the crack in this new setting. For $i = 0$, let $u_0^\delta \in BV(\Omega)$ be a minimum of

$$\min_{u \in BV(\Omega)} \left\{ \int_\Omega f(\nabla u)\, dx + \int_{S^{g_0^\delta}(u)} \varphi(|[u]|)\, d\mathcal{H}^{N-1} + a|D^c u| \right\}. \tag{5.10}$$

We set $\Gamma_0^\delta := S^{g_0^\delta}(u_0^\delta)$.

Supposing to have constructed u_j^δ and Γ_j^δ for all $j = 0, \dots, i-1$, let u_i^δ be a minimum of

$$\min_{u \in BV(\Omega)} \left\{ \int_\Omega f(\nabla u)\, dx + \int_{S^{g_i^\delta}(u) \cup \Gamma_{i-1}^\delta} \varphi(|[u]| \vee \psi_{i-1}^\delta)\, d\mathcal{H}^{N-1} + a|D^c u|, \right\} \tag{5.11}$$

where $\psi_{i-1}^\delta := |[u_0^\delta]| \vee \cdots \vee |[u_{i-1}^\delta]|$. We set $\Gamma_i^\delta := \Gamma_{i-1}^\delta \cup S^{g_i^\delta}(u_i^\delta)$.

In the following proposition we establish the existence of this discrete evolution.

Proposition 5.1.1. *Let $I_\delta = \{0 = t_0^\delta < \cdots < t_{N_\delta}^\delta = T\}$ be a subdivision of $[0,T]$ such that $\max(t_i^\delta - t_{i-1}^\delta) < \delta$. Then for all $i = 0, \dots, N_\delta$ there exists $u_i^\delta \in BV(\Omega)$ such that setting $\Gamma_{-1}^\delta := \emptyset$, $\psi_{-1}^\delta := 0$,*

$$\Gamma_i^\delta := \bigcup_{j=0}^{i} S^{g_j^\delta}(u_j^\delta), \qquad \psi_i^\delta(x) := |[u_0^\delta]|(x) \vee \cdots \vee |[u_i^\delta]|(x)$$

the following holds:

(a) $\|u_i^\delta\|_\infty \le \|g_i^\delta\|_\infty \le C$;

(b) for all $v \in BV(\Omega)$ *we have*

$$(5.12)\qquad \int_\Omega f(\nabla u_i^\delta)\,dx + \int_{\Gamma_i^\delta} \varphi(\psi_i^\delta)\,d\mathcal{H}^{N-1} + a|D^c u_i^\delta|(\Omega) \le \int_\Omega f(\nabla v)\,dx + \int_{S^{g_i^\delta}(v)\cup\Gamma_{i-1}^\delta} \varphi(|[v]| \vee \psi_{i-1}^\delta)\,d\mathcal{H}^{N-1} + a|D^c v|(\Omega),$$

where $a = \varphi'(0)$ *and* f *is defined in* (5.9);

(c) we have that

$$\int_\Omega f(\nabla u_i^\delta)\,dx + \int_{\Gamma_i^\delta} \varphi(\psi_i^\delta)\,d\mathcal{H}^{N-1} + a|D^c u_i^\delta|(\Omega) = \inf_{v\in SBV(\Omega)} \left\{ \|\nabla v\|^2 + \int_{S^{g_i^\delta}(v)\cup\Gamma_{i-1}^\delta} \varphi(|[v]| \vee \psi_{i-1}^\delta)\,d\mathcal{H}^{N-1} \right\}.$$

Proof. We have to prove that problems (5.10) and (5.11) admit solutions. Let us consider for example problem (5.11), the other being similar. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for problem (5.11). By a truncation argument we may assume that $\|u_n\|_\infty \le \|g_i^\delta\|$. Comparing u_n with g_i^δ, we get for n large

$$(5.13)\qquad \int_\Omega f(\nabla u_n)\,dx + \int_{S^{g_i^\delta}(u_n)\cup\Gamma_{i-1}^\delta} \varphi(|[u_n]| \vee \psi_{i-1}^\delta)\,d\mathcal{H}^{N-1} + a|D^c u_n|(\Omega) \le \int_\Omega f(\nabla g_0^\delta)\,dx + \int_{\Gamma_{i-1}^\delta} \varphi(\psi_{i-1}^\delta)\,d\mathcal{H}^{N-1} + 1 \le C',$$

with C' independent of n. Since there exists $d > 0$ such that $a|\xi| - d \le f(\xi)$ for all $\xi \in \mathbb{R}^N$, we deduce that $(\nabla u_n)_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega;\mathbb{R}^N)$. Moreover if $\bar{s}$ is such that $\varphi(\bar{s}) = \frac{1}{2}$ and $\bar{a}$ is such that $s \le \bar{a}\varphi(s)$ for all $s \in [0,\bar{s}]$, we have

$$(5.14)\qquad \begin{aligned}\int_{S(u_n)} |[u_n]|\,d\mathcal{H}^{N-1} &= \int_{\{|[u_n]|\le\bar{s}\}} |[u_n]|\,d\mathcal{H}^{N-1} + \|g_i^\delta\|_\infty \mathcal{H}^{N-1}(\{|[u_n]| > \bar{s}\}) \\ &\le \bar{a} \int_{|[u_n]|\le\bar{s}} \varphi(|[u_n]|)\,d\mathcal{H}^{N-1} + 2\|g_i^\delta\|_\infty \int_{|[u_n]|>\bar{s}} \varphi(|[u_n]|)\,d\mathcal{H}^{N-1} \\ &\le (\bar{a} + 2\|g_i^\delta\|_\infty)C'.\end{aligned}$$

Finally for all n

$$|D^c u_n| \le \frac{C'}{a}.$$

We conclude that $(u_n)_{n\in\mathbb{N}}$ is bounded in $BV(\Omega)$. Then there exists $u \in BV(\Omega)$ such that up to a subsequence $u_n \xrightarrow{*} u$ weakly* in $BV(\Omega)$ and pointwise almost everywhere. Let us set $u_i^\delta := u$. By Lemma 5.7.2 we deduce that

$$(5.15)\qquad \int_\Omega f(\nabla u)\,dx + \int_{S^{g_i^\delta}(u)\cup\Gamma_{i-1}^\delta} \varphi(|[u]| \vee \psi_{i-1}^\delta)\,d\mathcal{H}^{N-1} + a|D^c u|(\Omega) \le \liminf_{n\to+\infty} \int_\Omega f(\nabla u_n)\,dx + \int_{S^{g_i^\delta}(u_n)\cup\Gamma_{i-1}^\delta} \varphi(|[u_n]| \vee \psi_{i-1}^\delta)\,d\mathcal{H}^{N-1} + a|D^c u_n|(\Omega).$$

Setting $\psi_i^\delta := \psi_{i-1}^\delta \vee |[u_i^\delta]|$, we have that point *(b)* holds. Moreover $\|u_i^\delta\|_\infty \le \|g_0^\delta\|_\infty \le C$, so that point *(a)* holds. Finally point *(c)* is a consequence of Proposition 5.7.1. $\square$

Let us consider now the following piecewise constant interpolation in time:

$$u^\delta(t) := u_i^\delta, \qquad \Gamma^\delta(t) := \Gamma_i^\delta, \qquad \psi^\delta(t) := \psi_i^\delta, \qquad g^\delta(t) := g_i^\delta \qquad t_i^\delta \le t < t_{i+1}^\delta \tag{5.16}$$

with $u^\delta(T) := u^\delta_{N_\delta}$, $\Gamma^\delta(T) := \Gamma^\delta_{N_\delta}$, $\psi^\delta(T) := \psi^\delta_{N_\delta}$, and $g^\delta(T) := g(T)$.

For every $v \in BV(\Omega)$ and for every $t \in [0,T]$ let us set

$$\mathcal{E}^\delta(t,v) := \int_\Omega f(\nabla v)\,dx + \int_{S^{g^\delta(t)}(v)\cup\Gamma^\delta(t)} \varphi(|[v]| \vee \psi^\delta(t))\,d\mathcal{H}^{N-1} + a|D^c v|(\Omega). \tag{5.17}$$

Then the following estimate holds.

Lemma 5.1.2. *There exists $e_a^\delta \to 0$ for $\delta \to 0$ and $a \to +\infty$ such that for all $t \in [0,T]$ we have*

$$\mathcal{E}^\delta(t,u^\delta(t)) \le \mathcal{E}^\delta(0,u^\delta(0)) + \int_0^{t_i^\delta}\int_\Omega f'(\nabla u^\delta(\tau))\nabla \dot g(\tau)\,dx\,d\tau + e_a^\delta,$$

where t_i^δ is the step discretization point such that $t_i^\delta \le t < t_{i+1}^\delta$.

Proof. Comparing u_i^δ with $u_{i-1}^\delta + g_i^\delta - g_{i-1}^\delta$ by means of (5.12) we obtain

$$\mathcal{E}^\delta(t_i^\delta,u_i^\delta) \le \int_\Omega f(\nabla u_{i-1}^\delta + \nabla g_i^\delta - \nabla g_{i-1}^\delta)\,dx + \int_{\Gamma_{i-1}^\delta} \varphi(\psi_{i-1}^\delta)\,d\mathcal{H}^{N-1} + a|D^c u_{i-1}^\delta|(\Omega).$$

Notice that by the very definition of f the following hold:

1) if $|\nabla u_{i-1}^\delta + \nabla g_i^\delta - \nabla g_{i-1}^\delta| \ge \frac{a}{2}$ and $|\nabla u_{i-1}^\delta| \ge \frac{a}{2}$

$$f'(\nabla u_{i-1}^\delta + \nabla g_i^\delta - \nabla g_{i-1}^\delta) = f'(\nabla u_{i-1}^\delta);$$

2) if $|\nabla u_{i-1}^\delta + \nabla g_i^\delta - \nabla g_{i-1}^\delta| < \frac{a}{2}$ and $|\nabla u_{i-1}^\delta| \ge \frac{a}{2}$

$$f(\nabla u_{i-1}^\delta + \nabla g_i^\delta - \nabla g_{i-1}^\delta) \le f(\nabla u_{i-1}^\delta);$$

3) if $|\nabla u_{i-1}^\delta + \nabla g_i^\delta - \nabla g_{i-1}^\delta| \ge \frac{a}{2}$ and $|\nabla u_{i-1}^\delta| < \frac{a}{2}$

$$f(\nabla u_{i-1}^\delta + \nabla g_i^\delta - \nabla g_{i-1}^\delta) \le f(\nabla u_{i-1}^\delta) + 2(\nabla u_{i-1}^\delta, \nabla g_i^\delta - \nabla g_{i-1}^\delta) + |\nabla g_i^\delta - \nabla g_{i-1}^\delta|^2;$$

4) if $|\nabla u_{i-1}^\delta + \nabla g_i^\delta - \nabla g_{i-1}^\delta| < \frac{a}{2}$ and $|\nabla u_{i-1}^\delta| < \frac{a}{2}$

$$f(\nabla u_{i-1}^\delta + \nabla g_i^\delta - \nabla g_{i-1}^\delta) = f(\nabla u_{i-1}^\delta) + 2(\nabla u_{i-1}^\delta, \nabla g_i^\delta - \nabla g_{i-1}^\delta) + |\nabla g_i^\delta - \nabla g_{i-1}^\delta|^2.$$

Then by convexity of f we deduce

$$\mathcal{E}^\delta(t_i^\delta,u_i^\delta) \le \mathcal{E}^\delta(t_{i-1}^\delta,u_{i-1}^\delta) + \int_\Omega f'(\nabla u_{i-1}^\delta)(\nabla g_i^\delta - \nabla g_{i-1}^\delta)\,dx + R_{i-1}^{\delta,a},$$

where

$$R_{i-1}^{\delta,a} := \int_\Omega |\nabla g_i^\delta - \nabla g_{i-1}^\delta|^2\,dx + \int_{\{|\nabla u_{i-1}^\delta| \ge \frac{a}{2}\}} |f'(\nabla u_{i-1}^\delta)|\,|\nabla g_i^\delta - \nabla g_{i-1}^\delta|\,dx.$$

Then summing up from t_i^δ to t_0^δ, and taking into account (5.16) we get

$$\mathcal{E}^\delta(t,u^\delta(t)) \le \mathcal{E}^\delta(0,u^\delta(0)) + \int_0^{t_i^\delta}\int_\Omega f'(\nabla u^\delta(\tau))\nabla\dot g(\tau)\,dx\,d\tau + \int_0^{t_i^\delta} R^{\delta,a}(\tau)\,d\tau,$$

where

$$R^{\delta,a}(\tau) := \sigma(\delta)\|\nabla\dot g(\tau)\| + \int_{\{|\nabla u^\delta(\tau)|\ge\frac{a}{2}\}} |f'(\nabla u^\delta(\tau))|\,|\nabla\dot g(\tau)|\,dx$$

and

$$\sigma(\delta) := \max_{i=1,\dots,N_\delta}\int_{t_{i-1}^\delta}^{t_i^\delta}\|\nabla\dot g\|\,d\tau.$$

In order to conclude the proof it is sufficient to see that

$$\int_0^T R^{\delta,a}(\tau)\,d\tau \to 0$$

as $\delta\to 0$ and $a\to+\infty$. Notice that $\sigma(\delta)\to 0$ as $\delta\to 0$ by the absolutely continuity of $\|\nabla\dot g\|$. Let us come to the second term. Notice that $|f'(\nabla u^\delta(\tau))| = a$ on $\{|\nabla u^\delta(\tau)|\ge\frac{a}{2}\}$. Then we have to see

$$\int_0^T\int_\Omega a|\nabla\dot g(\tau)|\mathbb{1}_{\{|\nabla u^\delta(\tau)|\ge\frac{a}{2}\}}\,dx\,d\tau \to 0 \tag{5.18}$$

as $\delta\to 0$ and $a\to+\infty$. Setting $A_a^\delta(\tau) := \{x\in\Omega : |\nabla u^\delta(\tau)|(x)\ge\frac{a}{2}\}$ we have by Hölder inequality

$$\int_\Omega a|\nabla\dot g(\tau)|\mathbb{1}_{A_a^\delta(\tau)}\,dx \le a\sqrt{|A_a^\delta(\tau)|}\left(\int_{A_a^\delta(\tau)}|\nabla\dot g(\tau)|^2\,dx\right)^{\frac12}.$$

Notice that

$$\frac{a^2}{2}|A_a^\delta(\tau)| \le a\int_{A_a^\delta(\tau)}|\nabla u^\delta(\tau)|\,dx \le 2\int_{A_a^\delta(\tau)} f(\nabla u^\delta(\tau))\,dx \le C', \tag{5.19}$$

where C' depends only on g and is obtained comparing $u^\delta(\tau)$ with $g^\delta(\tau)$ by means of (5.12). We deduce that

$$\int_\Omega a|\nabla\dot g(\tau)|\mathbb{1}_{A_a^\delta(\tau)}\,dx \le \sqrt{2C'}\left(\int_{A_a^\delta(\tau)}|\nabla\dot g(\tau)|^2\,dx\right)^{\frac12} \le \sqrt{2C'}\|\nabla\dot g(\tau)\|.$$

As $\delta\to 0$ and $a\to+\infty$, by (5.19) we have that $|A_a^\delta(\tau)|\to 0$. Then by the equicontinuity of $|\nabla\dot g(\tau)|^2$ and by the Dominated Convergence Theorem, we deduce that (5.18) holds, and the proof is finished. □

5.2 The main results

Let Ω be an open, connected and bounded subset of $\mathbb{R}^N$ with Lipschitz boundary. Let $\partial_D\Omega\subseteq\partial\Omega$ be open in the relative topology, and let $\partial_N\Omega := \partial\Omega\setminus\partial_D\Omega$.

In this section we consider discrete in time evolution of cracks in a linearly elastic body whose reference configuration is given by $\Omega_h := h\Omega$, where h is positive and large. Let us assume that the cohesive forces on the cracks of Ω_h are given in the sense of Section 5.1 by a function $\varphi : [0,+\infty[\to[0,1]$ which is increasing, concave, $\varphi(0)=0$, $\varphi'(0)=a<+\infty$ and such that $\lim_{s\to+\infty}\varphi(s)=1$. Let us moreover set

$$f(\xi) := \begin{cases} |\xi|^2 & \text{if } |\xi|\le\frac{a}{2} \\ \frac{a^2}{4}+a(|\xi|-\frac{a}{2}) & \text{if } |\xi|\ge\frac{a}{2}. \end{cases} \tag{5.20}$$

Let us consider on $\partial_D \Omega_h := h\partial_D \Omega$ boundary displacements of the following particular form

$$g_h(t,x) := h^\alpha g\left(t, \frac{x}{h}\right), \tag{5.21}$$

with $g \in AC([0,T]; H^1(\Omega))$ such that $\|g(t)\|_\infty \le C$ for all $t \in [0,T]$. Given $\delta > 0$, let

$$I_\delta = \{0 = t_0^\delta < \cdots < t_{N_\delta}^\delta = T\}$$

be a subdivision of $[0,T]$ such that $\max(t_i^\delta - t_{i-1}^\delta) < \delta$, and let $\{t \to (u^{\delta,h}(t), \Gamma^{\delta,h}(t), \psi^{\delta,h}(t)) : t \in [0,T]\}$ be the piecewise constant interpolation in the sense of (5.16) of a discrete in time evolution of cracks in Ω_h relative to the boundary datum g_h and the subdivision I_δ given by Proposition 5.1.1.

Our aim is to study the asymptotic behavior of $\{t \to (u^{\delta,h}(t), \Gamma^{\delta,h}(t), \psi^{\delta,h}(t)) : t \in [0,T]\}$ as $\delta \to 0$ and $h \to +\infty$. Let us consider $h \in \mathbb{N}$ (we can consider any sequence which diverges to $+\infty$), let us fix $\delta_h \to 0$, and let us set for all $t \in [0,T]$

$$u_h(t) := u^{\delta_h,h}(t), \qquad \Gamma_h(t) := \Gamma^{\delta_h,h}(t), \qquad \psi_h(t) := \psi^{\delta_h,h}(t), \tag{5.22}$$

and let $g_h^{\delta_h}(t) := g_h(t_i^{\delta_h})$ where $t_i^{\delta_h} \in I_{\delta_h}$ is such that $t_i^{\delta_h} \le t < t_{i+1}^{\delta_h}$. Let us moreover set for every $v \in BV(\Omega)$ and for every $t \in [0,T]$

$$\mathcal{E}_h(t,v) := \int_{\Omega_h} f(\nabla v)\,dx + \int_{S^{g_h^{\delta_h}(t)}(v) \cup \Gamma_h(t)} \varphi(|[v]| \vee \psi_h(t))\,d\mathcal{H}^{N-1} + a|D^c v|(\Omega_h). \tag{5.23}$$

The asymptotic of (u_h, Γ_h, ψ_h) depends on α, and we have to distinguish three cases. The first case $\alpha = \frac{1}{2}$ was stated at the beginning of the chapter and reveals the prevalence of brittle effects as the size of the body increases. We give here the precise statement we will prove.

Theorem 5.2.1. *Let $g \in AC(0,T; H^1(\Omega))$ be such that $\|g(t)\|_\infty \le C$ for all $t \in [0,T]$. Let $\{t \to (u_h(t), \Gamma_h(t), \psi_h(t)) : t \in [0,T]\}$ be the piecewise constant interpolation given in* (5.22) *of a discrete in time evolution of cracks in Ω_h relative to the boundary data*

$$g_h(x,t) := \sqrt{h} g\left(\frac{x}{h}, t\right).$$

Then the following facts hold:

(a) there exists a constant C' dependent only on g such that for all $t \in [0,T]$

$$\frac{1}{h^{N-1}} \mathcal{E}_h(t, u_h(t)) \le C';$$

(b) for all $t \in [0,T]$

$$v_h(t,x) := \frac{1}{\sqrt{h}} u_h(t,hx) \qquad \textit{is bounded in } BV(\Omega);$$

(c) there exists a subsequence independent of t and there exists a quasistatic crack evolution $\{t \to (v(t), K(t)) : t \in [0,T]\}$ in Ω relative to boundary displacement g in the sense of Theorem 1.4.2 such that for all $t \in [0,T]$ we have

$$\nabla v_h(t) \rightharpoonup \nabla v(t) \qquad \textit{weakly in } L^1(\Omega; \mathbb{R}^N),$$

and every accumulation point v of $(v_h(t))_{h \in \mathbb{N}}$ in the weak topology of $BV(\Omega)$ is such that $v \in SBV(\Omega)$, $S^{g(t)}(v) \tilde{\subseteq} K(t)$ and $\nabla v = \nabla v(t)$. Moreover for all $t \in [0,T]$ we have*

$$\frac{1}{h^{N-1}} \mathcal{E}_h(t, u_h(t)) \to \|\nabla v(t)\|^2 + \mathcal{H}^{N-1}(K(t)); \tag{5.24}$$

in particular $h^{-N+1}|D^c u_h(t)|(\Omega_h)| \to 0$,

$$\frac{1}{h^{N-1}} \int_{\Omega_h} f(\nabla u_h(t))\, dx \to \|\nabla v(t)\|^2, \tag{5.25}$$

and

$$\frac{1}{h^{N-1}} \int_{\Gamma_h(t)} \varphi(\psi_h(t))\, d\mathcal{H}^{N-1} \to \mathcal{H}^{N-1}(K(t)). \tag{5.26}$$

Notice that in point (c) we can not assert that the sequence $(v_h(t))_{h\in\mathbb{N}}$ converges to $v(t)$ in the weak* topology of $BV(\Omega)$: this is due to the fact that $K(t)$ could disconnect Ω (in a weak sense), so that $v(t)$ is determined up to a constant on the components of $\Omega \setminus K$ which do not touch $\partial_D\Omega$.

The case $\alpha < \frac{1}{2}$ leads to a problem in elasticity in Ω_h in the sense of the following theorem.

Theorem 5.2.2. *Let $g \in AC(0,T;H^1(\Omega))$ be such that $\|g(t)\|_\infty \leq C$ for all $t \in [0,T]$. Let $\{t \to (u_h(t), \Gamma_h(t), \psi_h(t)) : t \in [0,T]\}$ be the piecewise constant interpolation given in (5.22) of a discrete in time evolution of cracks in Ω_h relative to the boundary data*

$$g_h(x,t) := h^\alpha g\left(t, \frac{x}{h}\right)$$

with $\alpha < \frac{1}{2}$. Then the following facts hold:

(a) for all $t \in [0,T]$

$$v_h(t,x) := \frac{1}{h^\alpha} u_h(t,hx) \qquad \text{is bounded in } BV(\Omega);$$

(b) there exists a subsequence independent of t such that for all $t \in [0,T]$ we have $v_h(t) \xrightarrow{*} v(t)$ weakly in $BV(\Omega)$ and*

$$\nabla v_h(t) \rightharpoonup \nabla v(t) \qquad \text{weakly in } L^1(\Omega;\mathbb{R}^N),$$

where $v(t)$ is the minimizer of

$$\min\{\|\nabla v\|^2 \,:\, v \in H^1(\Omega), v = g(t) \text{ on } \partial_D\Omega\};$$

moreover for all $t \in [0,T]$ we have

$$\frac{1}{h^{N+2\alpha-2}} \int_{\Omega_h} f(\nabla u_h(t))\, dx \to \|\nabla v(t)\|^2.$$

Finally for the case $\alpha > \frac{1}{2}$ the body goes to *rupture* at time $t = 0$, in the sense of the following theorem.

Theorem 5.2.3. *Let $g \in AC(0,T;H^1(\Omega))$ be such that $\|g(t)\|_\infty \leq C$ for all $t \in [0,T]$. Let $\{t \to (u_h(t), \Gamma_h(t), \psi_h(t)) : t \in [0,T]\}$ be the piecewise constant interpolation given in (5.22) of a discrete in time evolution of cracks in Ω_h relative to the boundary data*

$$g_h(x,t) := h^\alpha g\left(\frac{x}{h}, t\right)$$

with $\alpha > \frac{1}{2}$. Let us set $v_h(t,x) := \frac{1}{h^\alpha} u_h(t,hx)$ for all $x \in \Omega$ and for all $t \in [0,T]$.

Then $(v_h(0))_{h\in\mathbb{N}}$ is bounded in $BV(\Omega)$, and every accumulation point v of $(v_h(0))_{h\in\mathbb{N}}$ in the weak topology of $BV(\Omega)$ is piecewise constant in Ω, i.e. $v \in SBV(\Omega)$ and $\nabla v = 0$. Moreover*

$$\mathcal{H}^{N-1}(S^{g(0)}(v(0))) \leq \mathcal{H}^{N-1}(S^{g(0)}(w)) \tag{5.27}$$

for all piecewise constant function $w \in SBV(\Omega)$.

Notice that the minimality property (5.27) can be restated saying that $v(0)$ determines a minimal partition of Ω (in the sense of the perimeter of $S^{g(0)}(w)$).

5.3 Some tools for the asymptotic analysis

In this section we prove some technical propositions which will be very useful in the proofs of the main results of the chapter. More precisely, we will prove compactness results for the displacements and the cracks, and we will prove a generalization of the Transfer of Jump of [53] which will be employed in order to study what the minimality property of the discrete in time evolutions imply in the limit.

For all $h \in \mathbb{N}$ let $f_h : \mathbb{R}^N \to [0, +\infty[$ be such that for all $\xi \in \mathbb{R}^N$

$$f_h(\xi) \nearrow |\xi|^2, \qquad f_h(\xi) \geq \min\{|\xi|^2 - 1, b_h|\xi|\} \tag{5.28}$$

with $b_h \to +\infty$ as $h \to +\infty$, and let $\varphi_h : [0, +\infty[\to [0, 1]$ be increasing and such that for all $s \in [0, +\infty[$

$$\varphi_h(s) \geq \min\{c_h s, d_h\} \tag{5.29}$$

with $c_h \to +\infty$ and $d_h \nearrow 1$ for $h \to +\infty$.

5.3.1 Compactness for the displacements

In this subsection we give a compactness and lower semicontinuity result for the displacements whose proof is inspired by the proof of Ambrosio's compactness theorem (see [5]).

Proposition 5.3.1. *Let us consider the functionals*

$$F_h(u) := \sum_{i=1}^{m} \int_\Omega f_h(\nabla u_i)\, dx + \int_{S(u)} \varphi_h(|[u_1]| \vee \ldots \vee |[u_m]|)\, d\mathcal{H}^{N-1} + a_h|D^c u|(\Omega),$$

where $u = (u_1, \ldots, u_m) \in BV(\Omega; \mathbb{R}^m)$ (with f_h and φ_h defined in (5.28) and (5.29)). Let $a_h \to +\infty$ for $h \to +\infty$, and let $(u_h)_{h\in\mathbb{N}}$ be a sequence in $BV(\Omega)$. Then the following facts hold.

(a) If

$$F_h(u^h) + \|u^h\|_{L^\infty(\Omega;\mathbb{R}^m)} \leq C$$

for some $C \in [0, +\infty[$, then up to a subsequence $u^h \overset{*}{\rightharpoonup} u$ weakly in $BV(\Omega; \mathbb{R}^m)$.*

(b) If $F_h(u^h) \leq C$ for some $C \in [0, +\infty[$ and $u^h \overset{*}{\rightharpoonup} u$ weakly in $BV(\Omega; \mathbb{R}^m)$, then $u \in SBV(\Omega; \mathbb{R}^m)$,*

$$\nabla u^h \rightharpoonup \nabla u \qquad \text{weakly in } L^1(\Omega; \mathbb{R}^{m\times N}), \tag{5.30}$$

$$\|\nabla u_i\|^2 \leq \liminf_{h\to+\infty} \int_\Omega f_h(\nabla u_i^h)\, dx, \qquad i = 1, \ldots, m, \tag{5.31}$$

and

$$\mathcal{H}^{N-1}(S(u)) \leq \liminf_{h\to+\infty} \int_{S(u^h)} \varphi_h(|[u_1^h]| \vee \ldots \vee |[u_m^h]|)\, d\mathcal{H}^{N-1}. \tag{5.32}$$

Proof. As for point (a), let us prove that there exists C' independent of h such that we have

$$|Du^h(t)|(\Omega) \leq C'. \tag{5.33}$$

In fact, since for h large we have for all $\xi \in \mathbb{R}^N$

$$|\xi| - 1 \leq f_h(\xi),$$

we deduce that for all $i = 1, \dots, m$

$$\int_\Omega |\nabla u_i^h|\, dx \le \int_\Omega [f_h(\nabla u_i^h) + 1]\, dx \le C + |\Omega|,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Moreover if h is so large that $s \le 2\varphi_h(s)$ for all $s \in [0,1]$, we have for all $i = 1, \dots, m$

$$\begin{aligned}(5.34)\quad \int_{S(u_i^h)} |[u_i^h]|\, d\mathcal{H}^{N-1} &\le \int_{|[u_i^h]|<1} |[u_i^h]|\, d\mathcal{H}^{N-1} + \|u_i^h(t)\|_\infty \mathcal{H}^{N-1}\left(\left\{|[u_i^h]| \ge 1\right\}\right) \\ &\le 2\int_{|[u_i^h]|<1} \varphi_h(|[u_i^h]|)\, d\mathcal{H}^{N-1} + 2C\int_{|[u_i^h]|\ge 1} \varphi_h(|[u_i^h]|)\, d\mathcal{H}^{N-1} \le 2(1+C)C.\end{aligned}$$

Finally for all h

$$|D^c u^h|(\Omega) \le \frac{C}{a_h}.$$

We deduce that (5.33) holds, and so up to a subsequence we may suppose that $u_h \overset{*}{\rightharpoonup} u$ weakly* in $BV(\Omega; \mathbb{R}^m)$.

Let us come to point (*b*). Let us consider $u^h \in BV(\Omega)$ such that $u^h \overset{*}{\rightharpoonup} u$ weakly* in $BV(\Omega; \mathbb{R}^m)$ and $F_h(u^h) \le C$. Notice that $(\nabla u^h)_{h\in\mathbb{N}}$ is equintegrable. In fact if r_h is such that for all $|\xi| \le r_h$

$$|\xi|^2 - 1 \le b_h|\xi|$$

we get for all $i = 1, \dots, m$ and for all $E \subseteq \Omega$

$$\begin{aligned}\int_E |\nabla u_i^h|\, dx &\le \int_{\{|\nabla u_i^h|\le r_h\}\cap E} |\nabla u_i^h|\, dx + \int_{\{|\nabla u_i^h|>r_h\}\cap E} |\nabla u_i^h|\, dx \\ &\le \left(\int_{\{|\nabla u_i^h|\le r_h\}\cap E} |\nabla u_i^h|^2\, dx\right)^{\frac{1}{2}} |E|^{\frac{1}{2}} + \int_{\{|\nabla u_i^h|>r_h\}\cap E} |\nabla u_i^h|\, dx \\ &\le \left(\int_\Omega (f_h(\nabla u_i^h) + 1)\, dx\right)^{\frac{1}{2}} |E|^{\frac{1}{2}} + \frac{1}{b_h}\int_\Omega f_h(\nabla u_i^h)\, dx \le \sqrt{(C + |\Omega|)|E|} + \frac{C}{b_h}.\end{aligned}$$

This proves that ∇u_h is equintegrable. Up to a subsequence we may suppose that for all $i = 1, \dots, m$ we have

$$\nabla u_i^h \rightharpoonup g_i \qquad \text{weakly in } L^1(\Omega; \mathbb{R}^N).$$

Since $a_h \to +\infty$, we get $D^c u^h \to 0$ strongly in the sense of measures.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be bounded, Lipschitz and C^1, and for all $i = 1, \dots, m$ let us consider the measures

$$\mu_i^h(B) := D\psi(u_i^h)(B) - \int_B \psi'(u_i^h)\nabla u_i^h\, dx, \qquad \lambda_i^h(B) := \int_{S(u_i^h)\cap B} \varphi_h(|[u_i^h]|)\, d\mathcal{H}^{N-1},$$

where B is a Borel set in Ω. Notice that $\psi(u_i^h) \in BV(\Omega)$, and that by chain rule in BV (see [8, Theorem 3.96]) we have

$$D\psi(u_i^h) = \psi'(u_i^h)\nabla u_i^h\, d\mathcal{L}^N + (\psi((u_i^h)^+) - \psi((u_i^h)^-))\nu\mathcal{H}^{N-1} \llcorner S(u_i^h) + \psi'(\tilde{u}_i^h)D^c u_i^h,$$

where $\tilde{u}_i^h(x)$ is the Lebesgue value of u_i^h at x. We deduce that

$$(5.35)\qquad |D\psi(u_i^h) - \psi'(u_i^h)\nabla u_i^h\, d\mathcal{L}^N| \le ||\psi||_{\varphi_h}\lambda_i^h + ||\psi'||_\infty |D^c u_i^h|$$

where

$$||\psi||_{\varphi_h} := \sup\left\{\frac{\psi(t) - \psi(s)}{\varphi_h(|t-s|)} \,:\, t \ne s\right\}.$$

Up to a subsequence we have

$$\mu_i^h \xrightarrow{*} D\psi(u_i) - \psi'(u)g_i \, d\mathcal{L}^N, \qquad \lambda_i^h \xrightarrow{*} \lambda_i$$

weakly* in the sense of measures, and so from (5.35), by lower semicontinuity for the variations of measures (see [8, Proposition 1.62]) and since $D^c u^h \to 0$ strongly in the sense of measures we get

$$|D\psi(u_i) - \psi'(u)g_i \, d\mathcal{L}^N| \le (\sup \psi - \inf \psi)\lambda_i.$$

As a consequence of SBV characterization (see [8, Proposition 4.12]), we get that $u_i \in SBV(\Omega)$, $\nabla u_i = g_i$ and $\mathcal{H}^{N-1} \llcorner S(u_i) \le \lambda_i$ for all $i = 1, \dots, m$. We deduce that (5.30) holds.

In order to prove (5.31), for every $M > 0$ let g_i^M be the weak limit in $L^1(\Omega)$ (up to a subsequence) of $|\nabla u_i^h| \wedge M$. Since $f_h(\xi) \to |\xi|^2$ uniformly on $[0, M]$, we have

$$\|g_i^M\|^2 \le \liminf_{h \to +\infty} \int_\Omega f_h(\nabla u_i^h) \, dx.$$

Then letting $M \to +\infty$ we obtain (5.31).

Let us come to (5.32). If λ is the weak limit in the sense of measures of

$$\lambda^h(A) := \int_{S(u^h) \cap A} \varphi_h(|[u_1^h]| \vee \dots \vee |[u_m^h]|) \, d\mathcal{H}^{N-1},$$

we have that $\lambda_i \le \lambda$ for all $i = 1, \dots, m$. Since we have $\mathcal{H}^{N-1} \llcorner S(u_i) \le \lambda_i$ for all $i = 1, \dots, m$, we deduce that $\mathcal{H}^{N-1} \llcorner S(u) \le \lambda$, so that (5.32) is proved. □

5.3.2 Compactness for the cracks

This subsection is devoted to the proof of a compactness property for the cracks of the discrete in time evolutions, which is closely related to the notion of σ^p-convergence of sets defined in [44].

The convergence we propose is related to the energies which appear in the asymptotic study of the size effects, and so it depends on the cracks, but also on the bulk and surface energies, and on the rate at which the Cantor parts of the derivative of the displacements are disappearing.

Let $(K_h)_{h \in \mathbb{N}}$ be a sequence of rectifiable sets in $\Omega \cup \partial_D \Omega$, and let $f_h : \mathbb{R}^N \to [0, +\infty[$ and $\varphi_h : [0, +\infty[\to [0, 1]$ be such that (5.28) and (5.29) hold. Let γ_h be a positive function on K_h such that for all h

$$\int_{K_h} \varphi_h(\gamma_h) \, d\mathcal{H}^{N-1} < C, \tag{5.36}$$

for some $C \in [0, +\infty[$, and let $a_h \to +\infty$. Let $g_h, g \in H^1(\Omega)$ be such that $g_h \to g$ strongly in $H^1(\Omega)$. Let us set for all $u \in BV(\Omega)$

$$\mathcal{E}_h(u) := \int_\Omega f_h(\nabla u) \, dx + \int_{S^{g_h}(u)} \varphi_h(|[u]|) \, d\mathcal{H}^{N-1} + a_h |D^c u|(\Omega).$$

Then the following compactness result holds.

Proposition 5.3.2. *Up to a subsequence there exists a rectifiable set $K \tilde{\subseteq} \Omega \cup \partial_D \Omega$ such that the following facts hold:*

(a) for all subsequences $(h_k)_{k \in \mathbb{N}}$ and for all $(u_k)_{k \in \mathbb{N}}$ such that $S^{g_{h_k}}(u_k) \tilde{\subseteq} K_{h_k}$, $|[u_k]| \le \gamma_{h_k}$, $\mathcal{E}_{h_k}(u_{h_k}) \le C'$ for some $C' \in [0, +\infty[$, and $u_k \xrightarrow{*} u$ weakly in $BV(\Omega)$, we have $u \in SBV(\Omega)$, $\nabla u \in L^2(\Omega; \mathbb{R}^N)$, and $S^g(u) \tilde{\subseteq} K$;*

(b) *there exists a countable set D in $SBV(\Omega)$ such that*

$$K = \bigcup_{u \in D} S^g(u), \tag{5.37}$$

where for every $u \in D$ there exists a sequence $(u_h)_{h\in\mathbb{N}}$ in $BV(\Omega)$ with $S^{g_h}(u_h) \,\tilde{\subseteq}\, K_h$, $|[u_h]| \leq \gamma_h$, $\mathcal{E}_h(u_h) \leq C'$ for some $C' \in [0,+\infty[$, and $u_h \overset{*}{\rightharpoonup} u$ weakly in $BV(\Omega)$;*

(c) *we have*

$$\mathcal{H}^{N-1}(K) \leq \liminf_{h\to+\infty} \int_{K_h} \varphi_h(\gamma_h)\, d\mathcal{H}^{N-1}. \tag{5.38}$$

Proof. Our approach is based on Γ-convergence. In order to deal with $S^g(u)$ as an internal jump, let us consider $\tilde{\Omega} \subseteq \mathbb{R}^N$ open and bounded, such that $\overline{\Omega} \subseteq \tilde{\Omega}$, and let us set $\Omega' := \tilde{\Omega} \setminus \partial_N\Omega$.

Let us consider the functionals $\mathcal{E}'_h : BV(\Omega') \to [0,+\infty]$

$$\mathcal{E}'_h(u) := \int_{\Omega'} f_h(\nabla u)\, dx + \int_{S(u)} \varphi_h(|[u]|)\, d\mathcal{H}^{N-1} + a_h|D^c u|(\Omega'),$$

if $u \in BV(\Omega')$, $u = g_h$ on $\Omega' \setminus \Omega$, $S(u) \,\tilde{\subseteq}\, K_h$, $|[u]| \leq \gamma_h$ on K_h, and $\mathcal{E}'_h(u) = +\infty$ otherwise for $u \in BV(\Omega')$. Let us consider on $BV(\Omega')$ the strong topology of $L^1(\Omega')$. By Proposition 1.2.2, up to a subsequence, $(\mathcal{E}'_h)_{h\in\mathbb{N}}$ Γ-converges to a functional $\mathcal{E}'$. We denote this subsequence still by $(\mathcal{E}'_h)_{h\in\mathbb{N}}$, and we may suppose that the liminf in (5.38) and the liminf along this subsequence are equal.

Let us consider

$$\mathrm{epi}(\mathcal{E}') := \{(u,s) \in BV(\Omega') \times \mathbb{R} : \mathcal{E}'(u) \leq s\},$$

and let $\mathcal{D} \subseteq \mathrm{epi}(\mathcal{E}'(t))$ be countable and dense. If $\pi : BV(\Omega') \times \mathbb{R} \to BV(\Omega')$ denotes the projection on the first factor, let $D := \pi(\mathcal{D})$ and let us set

$$K := \bigcup_{u\in D} S(u).$$

Notice that by a truncation argument we may suppose that each $u \in D$ is bounded in $L^\infty(\Omega)$, and moreover that there exists $u_h \in BV(\Omega')$ such that $u_h \overset{*}{\rightharpoonup} u$ weakly* in $BV(\Omega')$ and $\mathcal{E}'_h(u_h) \leq C'$ with $C' \in [0,+\infty[$. By Proposition 5.3.1 and Γ-liminf inequality we deduce that $u \in SBV(\Omega')$ and

$$\|\nabla u\|^2 + \mathcal{H}^{N-1}(S(u)) \leq \mathcal{E}'(u). \tag{5.39}$$

Then K is precisely of the form (5.37) once we consider the restriction of u to Ω and recall that $u = g$ on $\Omega' \setminus \Omega$. Thus point (b) is proved.

Let us prove that (5.38) holds. Let $u_1, \ldots, u_k \in D$, and let $u_1^h, \ldots, u_k^h \in BV(\Omega')$ be such that $u_i^h \overset{*}{\rightharpoonup} u_i$ weakly* in $BV(\Omega')$ and

$$\lim_{h\to+\infty} \mathcal{E}'_h(u_i^h) = \mathcal{E}'(u_i), \qquad i = 1, \ldots, k. \tag{5.40}$$

Setting $u^h := (u_1^h, \ldots, u_k^h)$, by (5.40) we have

$$\sum_{i=1}^k \int_{\Omega'} f_h(\nabla u_i^h)\, dx + \int_{S(u^h)} \varphi_h(|[u_1^h]| \vee \cdots \vee |[u_k^h]|)\, d\mathcal{H}^{N-1} + a_h|D^c u^h|(\Omega') \leq \tilde{C}$$

with $\tilde{C}$ independent of h. By Proposition 5.3.1 we deduce

$$\mathcal{H}^{N-1}\left(\bigcup_{i=1}^{k} S(u_i)\right) \le \liminf_{h\to+\infty} \int_{S(u^h)} \varphi_h(|[u_1^h]| \vee \cdots \vee |[u_k^h]|)\, d\mathcal{H}^{N-1} \le \liminf_{h\to+\infty} \int_{K_h} \varphi_h(\gamma_h)\, d\mathcal{H}^{N-1}. \tag{5.41}$$

Taking the sup over all possible $u_1, \dots, u_k$ we get

$$\mathcal{H}^{N-1}(K) \le \liminf_{h\to+\infty} \int_{K_h(t)} \varphi_h(\gamma_h(t))\, d\mathcal{H}^{N-1},$$

so that (5.38) is proved. In particular by (5.36) we have that $\mathcal{H}^{N-1}(K) < +\infty$.

Let us come to point (a). Let us extend u_k and u to Ω' setting $u_k = g_{h_k}$ and $u = g$ on $\Omega' \setminus \overline{\Omega}$. By Proposition 5.3.1 we get $u \in SBV(\Omega')$ and $\nabla u \in L^2(\Omega'; \mathbb{R}^N)$. Let us see that $S(u) \,\tilde{\subseteq}\, K$. Notice that by Γ-liminf inequality we have

$$\mathcal{E}'(u) \le \liminf_{k\to+\infty} \mathcal{E}'_{h_k}(u_k) < +\infty$$

so that $(u, \mathcal{E}'(u)) \in \mathrm{epi}(\mathcal{E}')$. Let $(v_j, s_j) \in \mathcal{D}$ be such that $v_j \to u$ strongly in $L^1(\Omega')$ and $s_j \to \mathcal{E}'(u)$. By truncation, we may assume that u and v_j are uniformly bounded in L^∞. We know that $v_j \in SBV(\Omega')$ for all j, and that by (5.39)

$$\|\nabla v_j\|^2 + \mathcal{H}^{N-1}(S(v_j)) \le \mathcal{E}'(v_j). \tag{5.42}$$

By lower semicontinuity of $\mathcal{E}'$ we have

$$\mathcal{E}'(u) \le \liminf_{j\to+\infty} \mathcal{E}'(v_j).$$

Moreover since $\mathcal{E}'(v_j) \le s_j$, we deduce

$$\limsup_{j\to+\infty} \mathcal{E}'(v_j) \le \lim_{j\to+\infty} s_j = \mathcal{E}'(u),$$

so that we have $\mathcal{E}'(v_j) \to \mathcal{E}'(u) < +\infty$. By (5.42) we conclude that $v_j \rightharpoonup u$ weakly in $SBV(\Omega')$: since $S(v_j) \,\tilde{\subseteq}\, K$ for all j, and $\mathcal{H}^{N-1}(K) < +\infty$, by Ambrosio's Theorem we get $S(u) \,\tilde{\subseteq}\, K$. The proof is now complete. $\square$

Remark 5.3.3. Notice that in the case $f_h(\xi) = |\xi|^p$ ($p \in]1, +\infty[$) and $\varphi_h = 1$, and no Cantor part is considered (i.e. $a_h = +\infty$), Proposition 5.3.2 gives an alternative proof of the compactness and lower semicontinuity properties of σ^p-convergence of sets formulated in [44].

Notice moreover that the limit set of Proposition 5.3.2 is contained (up to negligible $\mathcal{H}^{N-1}$ set) in $\Omega \cup \partial_D \Omega$, so that $\partial_N \Omega$ is not involved: this is done in view of the concrete application to quasistatic crack growth, where convergence for the surface energy holds, and so a crack would never approach $\partial_N \Omega$ otherwise but transversally. This can be seen also energetically, since the displacement can choose the more convenient boundary datum on $\partial_N \Omega$ without creating a crack on this part of the boundary.

5.3.3 A generalization of the Transfer of Jump

In this subsection we prove a generalization of the Transfer of Jump Theorem of Francfort-Larsen [53] which will be useful in the proof of Theorem 5.2.1.

Let $f_h : \mathbb{R}^N \to [0, +\infty[$ and $\varphi_h : [0, +\infty[\to [0, 1]$ be such that (5.28) and (5.29) hold. Then the following proposition holds.

Proposition 5.3.4. *Let $(u_h)_{h\in\mathbb{N}}$ be a sequence in $BV(\Omega)$ such that $(\nabla u_h)_{h\in\mathbb{N}}$ is equintegrable,*

$$\sup_h \int_{S(u_h)} \varphi_h(|[u_h]|)\, d\mathcal{H}^{N-1} \le C, \quad \text{and} \quad |D^c u_h|(\Omega) \to 0. \tag{5.43}$$

Let $u \in SBV(\Omega)$ be such that $u_h \overset{*}{\rightharpoonup} u$ weakly in $BV(\Omega)$, and let $g_h, g \in H^1(\Omega)$ be such that $g_h \to g$ strongly in $H^1(\Omega)$. Then for all $v \in SBV(\Omega)$ with $\nabla v \in L^2(\Omega;\mathbb{R}^N)$ there exists $v_h \in SBV(\Omega)$ such that $\nabla v_h \to \nabla v$ strongly in $L^2(\Omega;\mathbb{R}^N)$ and*

$$\limsup_{h\to+\infty} \left[\int_{S^{g_h}(v_h)\cup S^{g_h}(u_h)} \varphi_h(|[v_h]| \vee |[u_h]|)\, d\mathcal{H}^{N-1} - \int_{S^{g_h}(u_h)} \varphi_h(|[u_h]|)\, d\mathcal{H}^{N-1} \right] \le \mathcal{H}^{N-1}(S^g(v)\setminus S^g(u)).$$

Proof. In order to deal with $S^g(u)$ as an internal jump, let us consider $\tilde\Omega \subseteq \mathbb{R}^N$ open and bounded, and such that $\overline{\Omega} \subseteq \tilde\Omega$. Let us set $\Omega' := \tilde\Omega \setminus \partial_N\Omega$. Let $v \in SBV(\Omega)$ with $\nabla v \in L^2(\Omega;\mathbb{R}^N)$ and $\mathcal{H}^{N-1}(S^g(v)) < +\infty$. Let us consider

$$w := v - g, \qquad z := u - g, \qquad z_h := u_h - g_h,$$

and let us extend w, z, z_h to Ω' setting $w = z = z_h = 0$ on $\Omega'\setminus\Omega$. In this setting, we have to find $w_h \in SBV(\Omega')$ such that $w_h \equiv 0$ on $\Omega'\setminus\Omega$, $\nabla w_h \to \nabla w$ strongly in $L^2(\Omega';\mathbb{R}^N)$, and such that

$$\limsup_{h\to+\infty} \left[\int_{S(w_h)\cup S(z_h)} \varphi_h(|[w_h]| \vee |[z_h]|)\, d\mathcal{H}^{N-1} - \int_{S(z_h)} \varphi_h(|[z_h]|)\, d\mathcal{H}^{N-1} \right] \le \mathcal{H}^{N-1}(S(w)\setminus S(z)).$$

Then the result follows considering the restriction of w_h to Ω, and setting $v_h := w_h + g_h$.

The key point in the proof is the following: for all $\varepsilon > 0$ find $\delta > 0$ and $w_h \in SBV(\Omega')$ such that $w_h \equiv 0$ on $\Omega'\setminus\Omega$,

$$\limsup_{h\to+\infty} \|\nabla w_h - \nabla w\|_{L^2(\Omega';\mathbb{R}^N)} \le \varepsilon, \tag{5.44}$$

and

$$\limsup_{h\to+\infty} \mathcal{H}^{N-1}(S(w_h)\setminus K_h^\delta) \le \mathcal{H}^{N-1}(S(w)\setminus S(z)) + \varepsilon, \tag{5.45}$$

where $K_h^\delta := \{x \in S(z_h) : |[z_h]| \ge \delta\}$. In fact if (5.45) holds, noting that by (5.43) we get $\mathcal{H}^{N-1}(K_h^\delta) \le C+1$ for h large enough, following the decomposition

$$S(w_h)\cup S(z_h) = (S(w_h)\setminus K_h^\delta) \cup (S(w_h)\cap K_h^\delta) \cup (K_h^\delta \setminus S(w_h)) \cup [S(z_h)\setminus (S(w_h)\cup K_h^\delta)],$$
$$S(z_h) = (K_h^\delta \cap S(w_h)) \cup (K_h^\delta\setminus S(w_h)) \cup (S(z_h)\setminus K_h^\delta)$$

we have (d_h is defined in (5.29))

$$\begin{aligned}
&\limsup_{h\to+\infty} \left[\int_{S(w_h)\cup S(z_h)} \varphi_h(|[w_h]| \vee |[z_h]|)\, d\mathcal{H}^{N-1} - \int_{S(z_h)} \varphi_h(|[z_h]|)\, d\mathcal{H}^{N-1} \right] \\
&\qquad \le \mathcal{H}^{N-1}(S(w)\setminus S(z)) + \varepsilon \\
&\qquad + \limsup_{h\to+\infty} \left[\int_{S(w_h)\cap K_h^\delta} \varphi_h(|[w_h]| \vee |[z_h]|)\, d\mathcal{H}^{N-1} - \int_{S(w_h)\cap K_h^\delta} \varphi_h(|[z_h]|)\, d\mathcal{H}^{N-1} \right] \\
&\qquad \le \mathcal{H}^{N-1}(S(w)\setminus S(z)) + \varepsilon + \limsup_{h\to+\infty} (1-d_h)\mathcal{H}^{N-1}(S(w_h)\cap K_h^\delta) \\
&\qquad \le \mathcal{H}^{N-1}(S(w)\setminus S(z)) + \varepsilon.
\end{aligned}$$

Letting now $\varepsilon \to 0$, and using a diagonal argument we obtain the result.

Let $\varepsilon > 0$, and let us prove that we can find $\delta > 0$ and $(w_h)_{h\in\mathbb{N}}$ such that (5.44) and (5.45) hold. Following the Transfer of Jump [53, Theorem 2.1], let us fix $G \subseteq \mathbb{R}$ countable and dense: we recall that we have up to a set of $\mathcal{H}^{N-1}$-measure zero

$$S(z) = \bigcup_{c_1,c_2\in G} \partial^* E_{c_1}(z) \cap \partial^* E_{c_2}(z),$$

where $E_c(z) := \{x \in \Omega' \,:\, x \text{ is a Lebesgue point for } z,\ z(x) > c\}$ and ∂^* denotes the essential boundary (see [8, Definition 3.60]). Let us orient ν_z in such a way that $z^-(x) < z^+(x)$ for all $x \in S(z)$, and let us consider

$$J_j := \left\{ x \in S(z) \,:\, z^+(x) - z^-(x) > \frac{1}{j} \right\},$$

with j so large that

$$\mathcal{H}^{N-1}(S(z) \setminus J_j) < \sigma,$$

where $\sigma > 0$. Let U be a neighborhood of $S(z)$ such that

$$|U| < \frac{\sigma}{j^2}, \qquad \int_U |\nabla w|^2\, dx < \sigma.$$

Following [53, Theorem 2.1], we can find a finite disjoint collection of closed cubes $\{Q_k\}_{k=1,\dots,n}$ with edge of length $2r_k$, with center $x_k \in S(z)$ and oriented as the normal $\nu(x_k)$ to $S(z)$ at x_k, such that $\bigcup_{k=1}^n Q_k \subseteq U$ and $\mathcal{H}^{N-1}(J_j \setminus \bigcup_{k=1}^n Q_k) \le \sigma$. Let H_k denote the intersection of Q_k with the hyperplane through x_k orthogonal to $\nu(x_k)$. Following [53] we can suppose that the following facts hold:

(a) if $x_k \in \Omega$ then $Q_k \subseteq \Omega$, and if $x_k \in \partial_D\Omega$, then $\partial\Omega \cap Q_k \subseteq \{y + s\nu(x_k) \,:\, y \in H_k,\, s \in [-\frac{\sigma r_k}{2}, \frac{\sigma r_k}{2}]\}$;

(b) $\mathcal{H}^{N-1}(S(z) \cap \partial Q_k) = 0$;

(c) $r_k^{N-1} < 2\mathcal{H}^{N-1}(S(z) \cap Q_k)$;

(d) $z^-(x_k) < c_k^1 < c_k^2 < z^+(x_k)$ and $c_k^2 - c_k^1 > \frac{1}{2j}$;

(e) $\mathcal{H}^{N-1}([S(z) \setminus \partial^* E_{c_k^s}(z)] \cap Q_k) < \sigma r_k^{N-1}$ for $s = 1, 2$;

(f) $\mathcal{H}^{N-1}(\{y \in \partial^* E_{c_k^s}(z) \cap Q_k \,:\, \mathrm{dist}(y, H_k) \ge \frac{\sigma}{2} r_k\}) < \sigma r_k^{N-1}$ for $s = 1, 2$;

(g) if $Q_k^+ := \{x \in Q_k \,|\, (x - x_k)\cdot \nu(x_k) > 0\}$ and $s = 1, 2$

$$\|1_{E_{c_k^s}(z))\cap Q_k} - 1_{Q_k^+}\|_{L^1(\Omega')} < \sigma^2 r_k^N;$$

(h) $\mathcal{H}^{N-1}\left((S(w) \setminus S(z)) \cap Q_k\right) < \sigma r_k^{N-1}$ and $\mathcal{H}^{N-1}(S(w) \cap \partial Q_k) = 0$.

Since $(\nabla z_h)_{h\in\mathbb{N}}$ is equintegrable, we may assume that U is chosen so that for h large

$$\sum_{k=1}^{n}\int_{Q_k}|\nabla z_h|\,dx<\frac{\sigma}{j^2}. \tag{5.46}$$

Let $\eta\in]0,1[$: we claim that there exists $\delta>0$ such that for all $k=1,\dots,n$

$$\limsup_{h\to+\infty}|Dz_h|(\{0<|[z_h]|<\delta\}\cap Q_k)\le\eta|Q_k|. \tag{5.47}$$

Let $M>0$: by (5.29) there exists $s_h\to 0$ with $\varphi_h(s_h)\to 1$ such that for h large enough

$$Ms\le\varphi_h(s)\qquad\text{for all } s\in[0,s_h].$$

Then we have

$$\begin{aligned}|Dz_h|\,(\{0<|[z_h]|<s_h\}\cap Q_k)&=\int_{\{0<|[z_h]|<s_h\}\cap Q_k}|[z_h]|\,d\mathcal{H}^{N-1}\\&\le\frac{1}{M}\int_{\{0<|[z_h]|<s_h\}\cap Q_k)}\varphi_h(|[z_h]|)\,d\mathcal{H}^{N-1},\end{aligned}$$

so that we conclude for h large

$$\begin{aligned}|Dz_h|\left(\left\{0<|[z_h]|<\delta\right\}\cap Q_k\right)&\le\frac{1}{M}\int_{\{0<|[z_h]|<s_h\}\cap Q_k)}\varphi_h(|[z_h]|)\,d\mathcal{H}^{N-1}\\&+\frac{\delta}{\varphi_h(s_h)}\int_{\{s_h\le|[z_h]|<\delta\}\cap Q_k)}\varphi_h(|[z_h]|)\,d\mathcal{H}^{N-1}\le\left(\frac{1}{M}+\frac{\delta}{\varphi_h(s_h)}\right)C,\end{aligned} \tag{5.48}$$

where C is defined in (5.43). Taking the limsup in h and choosing δ small enough and M large enough, we have that (5.47) holds.

Let δ be as in (5.47), and let us set

$$K_h^\delta:=\{x\in S(z_h)\,:\,|[z_h]|(x)\ge\delta\}.$$

Then in view of (5.46) and of (5.47), since $|D^c z_h|(\Omega')\to 0$, by the Coarea formula for BV functions (see [8, Theorem 3.40]) we have for h large enough

$$\begin{aligned}\sum_{k=1}^{n}\int_{c_k^1}^{c_k^2}\mathcal{H}^{N-1}\left(\partial^*E_c(z_h)\cap(Q_k\setminus K_h^\delta)\right)\,dc&\le\sum_{k=1}^{n}|Dz_h|(Q_k\setminus K_h^\delta)\\=\sum_{k=1}^{n}\int_{Q_k}|\nabla z_h|\,dx+\sum_{k=1}^{n}|Dz_h|(Q_k\cap\{0<|[z_h]|<\delta\})&+|D^cz_h|\left(\bigcup_{k=1}^{n}Q_k\right)\le(1+\eta)\frac{\sigma}{j^2}.\end{aligned} \tag{5.49}$$

By the Mean Value Theorem and by property (d) we get that there exist $c_k^1<c_k^h<c_k^2$, $k=1,\dots,n$ such that

$$\sum_{k=1}^{n}\mathcal{H}^{N-1}\left(\partial^*E_{c_k^h}(z_h)\cap(Q_k\setminus K_h^\delta)\right)\le 2(1+\eta)\frac{\sigma}{j}. \tag{5.50}$$

Following [53], by property (g) we have that for h large

$$\|1_{E_{c_k^h}(z_h)\cap Q_k}-1_{Q_k^+}\|_{L^1(\Omega')}\le\sigma^2r_k^N.$$

Then by Fubini's Theorem and by the Mean Value Theorem, we can find $s_k^+ \in [\frac{\sigma r_k}{2}, \sigma r_k]$ and $s_k^- \in [-\sigma r_k, -\frac{\sigma r_k}{2}]$ such that setting $H_k^+ := \{x = y + s_k^+ \nu(x_k), y \in H_k\}$ and $H_k^- := \{x = y + s_k^- \nu(x_k), y \in H_k\}$ we have

$$\mathcal{H}^{N-1}\left(H_k^+ \setminus (E_{c_k^h}(z_h) \cap Q_k)\right) + \mathcal{H}^{N-1}\left(H_k^- \cap (E_{c_k^h}(z_h) \cap Q_k)\right) \leq 2\sigma r_k^{N-1}.$$

Let R_k be the region between H_k^- and H_k^+, i.e.

$$R_k := \{x \in Q_k : x = y + s\nu(x_k),\, y \in H_k,\, s_k^2 \leq s \leq s_k^1\},$$

and let us indicate by $R_k^+ w$ the reflection in Q_k of $w_{|Q_k^+ \setminus R_k}$ with respect to H_k^+, and by $R_k^- w$ the reflection in Q_k of $w_{|Q_k^- \setminus R_k}$ with respect to H_k^-. We can now consider w_h defined in the following way

$$w_h := \begin{cases} w & \text{on } \Omega' \setminus \bigcup_{k=1}^n R_k \\ R_k^+ w & \text{on } R_k \cap E_{c_k^h}(z_h) \\ R_k^- w & \text{on } R_k \setminus E_{c_k^h}(z_h). \end{cases}$$

w_h is well defined for σ small, and $w_h = 0$ on $\Omega' \setminus \Omega$. Notice that by construction we have that for h large

$$\|\nabla w_h - \nabla w\|_{L^2(\Omega';\mathbb{R}^N)} + \sum_{k=1}^n \mathcal{H}^{N-1}\left(\left(S(w_h) \setminus K_h^\delta\right) \cap Q_k\right) \leq e(\sigma),$$

where $e(\sigma) \to 0$ as $\sigma \to 0$: the proof follows analyzing the set $S(w_h)$ inside Q_k, and it is very similar to that contained in [53, Theorem 2.1]. Since

$$S(w_h) \setminus K_h^\delta \tilde{\subseteq} [S(w) \setminus S(z)] \cup \left[S(z) \setminus \bigcup_{k=1}^n Q_k\right] \cup \bigcup_{k=1}^n \left(\left(S(w_h) \setminus K_h^\delta\right) \cap Q_k\right),$$

we deduce

$$\limsup_{h \to +\infty} \mathcal{H}^{N-1}\left(S(w_h) \setminus K_h^\delta\right) \leq \mathcal{H}^{N-1}\left(S(w) \setminus S(z)\right) + e(\sigma),$$

with $e(\sigma) \to 0$ as $\sigma \to 0$. Choosing σ small enough and using a diagonal argument, we obtain that (5.44) and (5.45) hold, and the proof is finished. □

The following proposition extends the Transfer of Jump to the case of cracks converging in the sense of Proposition 5.3.2.

Proposition 5.3.5. *Let $(K_h, \gamma_h)_{h \in \mathbb{N}}$ and K be as in Proposition 5.3.2, and let $g_h, g \in H^1(\Omega)$ be such that $g_h \to g$ strongly in $H^1(\Omega)$. Then for all $v \in SBV(\Omega)$ with $\nabla v \in L^2(\Omega;\mathbb{R}^N)$ there exists $v_h \in SBV(\Omega)$ such that $\nabla v_h \to \nabla v$ strongly in $L^2(\Omega;\mathbb{R}^N)$ and*

$$\limsup_{h \to +\infty} \left[\int_{S^{g_h}(v_h) \cup K_h} \varphi_h(|[v_h]| \vee \gamma_h)\, d\mathcal{H}^{N-1} - \int_{K_h} \varphi_h(\gamma_h)\, d\mathcal{H}^{N-1}\right] \leq \mathcal{H}^{N-1}(S^g(v) \setminus K).$$

Proof. We indicate how to modify the proof of Proposition 5.3.4 in order to get the result for $(K_h, \gamma_h)_{h \in \mathbb{N}}$ and K.

Notice that properties (5.44) and (5.45) can be extended to the case of a finite number of converging sequences: more precisely if $k = 1, \dots, m$, $(u_h^k)_{h \in \mathbb{N}}$ is a sequence in $BV(\Omega)$ such that $u_h^k \overset{*}{\rightharpoonup} u^k$ weakly* in $BV(\Omega)$, $(\nabla u_h^k)_{h \in \mathbb{N}}$ is equintegrable,

$$\int_{S^{g_h}(u_h^k)} \varphi_h(|[u_h^k]|)\, d\mathcal{H}^{N-1} \leq C, \qquad |D^c u_h^k|(\Omega) \to 0,$$

then for every $\varepsilon > 0$ and $v \in SBV(\Omega)$ with $\nabla v \in L^2(\Omega;\mathbb{R}^N)$ there exists $v_h \in SBV(\Omega)$ such that

$$\limsup_{h \to +\infty} \|\nabla v_h - \nabla v\|_{L^2(\Omega;\mathbb{R}^N)} \leq \varepsilon \tag{5.51}$$

and

$$\limsup_{h\to+\infty}\mathcal{H}^{N-1}\left(S^{g_h}(v_h)\setminus\tilde{K}_h^\delta\right)\le\mathcal{H}^{N-1}\left(S^g(v)\setminus\bigcup_{k=1}^m S^g(u^k)\right)+\varepsilon, \tag{5.52}$$

where $\tilde{K}_h^\delta:=\{x\in\bigcup_{k=1}^m S^{g_h}(u_h^k)\,:\,|[u_h^k]|(x)\ge\delta$ for some $k=1,\dots,m\}$. This can be done using the localization on the squares already employed in [53, Theorem 2.3]: on each squares Q_j we have that $\bigcup_{k=1}^m S^g(u^k)\cap Q_j$ is essentially given by $S^g(u^{\tau(j)})$ for some $\tau(j)\in\{1,\dots,m\}$.

Let us come to the Transfer of Jump for K. We recall that

$$K=\bigcup_{u\in D}S^g(u)$$

for some countable set D, and that each $u\in D$ is limit in the weak* topology of $BV(\Omega)$ of a function u_h such that $S^{g_h}(u_h)\tilde{\subseteq}K_h$, $|[u_h]|\le\gamma_h$, $\nabla u_h\rightharpoonup\nabla u$ weakly in $L^1(\Omega;\mathbb{R}^N)$,

$$\sup_h\int_{S(u_h)}\varphi_h(|[u_h]|)\,d\mathcal{H}^{N-1}\le C,\quad\text{and}\quad|D^cu_h|(\Omega)\to0.$$

Let $\varepsilon>0$ be fixed: since $\mathcal{H}^{N-1}(K)<+\infty$, we can find m such that

$$\mathcal{H}^{N-1}\left(K\setminus\bigcup_{k=1}^m S^g(u^k)\right)\le\varepsilon \tag{5.53}$$

for some $u^k\in D$, $k=1,\dots,m$. Let u_h^k be the approximation of u^k for all $k=1,\dots,m$, and let $v\in SBV(\Omega)$ with $\nabla v\in L^2(\Omega;\mathbb{R}^N)$. Then by (5.51) and (5.52) we can find $(v_h)_{h\in\mathbb{N}}$ such that

$$\limsup_{h\to+\infty}\|\nabla v_h-\nabla v\|_{L^2(\Omega;\mathbb{R}^N)}\le\varepsilon,$$

and

$$\limsup_{h\to+\infty}\mathcal{H}^{N-1}\left(S^{g_h}(v_h)\setminus\tilde{K}_h^\delta\right)\le\mathcal{H}^{N-1}\left(S^g(v)\setminus\bigcup_k S^g(u^k)\right)+\varepsilon.$$

Setting $K_h^\delta:=\{x\in K_h\,:\,\gamma_h(x)\ge\delta\}$, recalling that $\tilde{K}_h^\delta\tilde{\subseteq}K_h^\delta$ since $|[u_h^k]|\le\gamma_h$, by (5.53) we deduce that

$$\limsup_{h\to+\infty}\mathcal{H}^{N-1}\left(S^{g_h}(v_h)\setminus K_h^\delta\right)\le\mathcal{H}^{N-1}\left(S^g(v)\setminus K\right)+2\varepsilon.$$

The proof now follows exactly as in Proposition 5.3.4. □

5.4 Proof of Theorem 5.2.1

In this section we will give the proof of Theorem 5.2.1. Let $\{t\to(u_h(t),\Gamma_h(t),\psi_h(t))\,:\,t\in[0,T]\}$ be the piecewise constant interpolation of a discrete in time evolution of cracks in Ω_h relative the subdivision $I_{\delta_h}:=\{0=t_0^{\delta_h}<\dots<t_{N_{\delta_h}}^{\delta_h}=T\}$, and the boundary displacement $\sqrt{t}g(t,\frac{x}{h})$ given in (5.22). We divide the proof in several steps.

Step 1: Rescaling. For all $t\in[0,T]$ let $v_h(t)\in BV(\Omega)$ and $K_h(t)\tilde{\subseteq}\Omega\cup\partial_D\Omega$ be defined as

$$v_h(t,x):=\frac{1}{\sqrt{h}}u_h(t,hx),\qquad K_h(t):=\frac{1}{h}\Gamma_h(t). \tag{5.54}$$

Let us moreover set

$$\gamma_h(t,x):=\frac{1}{\sqrt{h}}\psi_h(t,hx)=\max_{0\le s\le t}|[v_h(s)](t,x)|,\quad t\in[0,T],x\in\Omega. \tag{5.55}$$

We notice that $\{t \to (v_h(t), K_h(t), \gamma_h(t)) : t \in [0,T]\}$ is the piecewise constant interpolation of a discrete in time evolution of cracks in Ω relative to the subdivision I_{δ_h} and boundary displacement $g(t)$ with respect to the basic total energy

$$\int_\Omega f_h(\nabla v)\,dx + \int_{S^{g^{\delta_h}(t)}(v)\cup K_h(t)} \varphi_h(|[v]| \vee \gamma_h(t))\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c v|(\Omega),$$

where $g^{\delta_h}(t) := g(t_i^{\delta_h})$ for $t_i^{\delta_h} \le t < t_{i+1}^{\delta_h}$, $a := \varphi'(0)$,

$$\varphi_h(s) := \varphi(\sqrt{h}s), \qquad s \in [0,+\infty[, \tag{5.56}$$

and

$$f_h(\xi) := \begin{cases} |\xi|^2 & \text{if } |\xi| \le \frac{a\sqrt{h}}{2} \\ \frac{a^2 h}{4} + a\sqrt{h}(|\xi| - \frac{a\sqrt{h}}{2}) & \text{if } |\xi| \ge \frac{a\sqrt{h}}{2}. \end{cases} \tag{5.57}$$

Let us recall some properties of the evolution $\{t \to (v_h(t), K_h(t), \gamma_h(t)) : t \in [0,T]\}$ which derive from Proposition 5.1.1 and that will be employed in the sequel:

(a) for all $t \in [0,T]$

$$\|v_h(t)\|_\infty \le \|g^{\delta_h}(t)\|_\infty; \tag{5.58}$$

(b) $K_h(0) = S^{g^{\delta_h}(0)}(v_h(0))$, and $S^{g^{\delta_h}(t)}(v_h(t)) \tilde{\subseteq} K_h(t)$ for all $t \in]0,T]$;

(c) for all $w \in BV(\Omega)$ we have

$$\begin{aligned} \int_\Omega f_h(\nabla v_h(0))\,dx + \int_{S^{g^{\delta_h}(0)}(v_h(0))} \varphi_h(|[v_h(0)]|)\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c v_h(0)|(\Omega) \\ \le \int_\Omega f_h(\nabla w)\,dx + \int_{S^{g^{\delta_h}(0)}(w)} \varphi_h(|[w]|)\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c w|(\Omega); \end{aligned} \tag{5.59}$$

(d) for all $w \in BV(\Omega)$ and for all $t \in]0,T]$ we have

$$\begin{aligned} \int_\Omega f_h(\nabla v_h(t))\,dx + \int_{K_h(t)} \varphi_h(\gamma_h(t))\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c v_h(t)|(\Omega) \\ \le \int_\Omega f_h(\nabla w)\,dx + \int_{S^{g^{\delta_h}(t)}(w)\cup K_h(t)} \varphi_h(|[w]| \vee \gamma_h(t))\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c w|(\Omega). \end{aligned} \tag{5.60}$$

Let us set for all $w \in BV(\Omega)$ and for all $t \in [0,T]$

$$\mathcal{F}_h(t,w) := \int_\Omega f_h(\nabla w)\,dx + \int_{S^{g^{\delta_h}(t)}(w)\cup K_h(t)} \varphi_h(|[w]| \vee \gamma_h(t))\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c w|(\Omega). \tag{5.61}$$

Notice that for all $t \in [0,T]$

$$\mathcal{F}_h(t, v_h(t)) = \frac{1}{h^{N-1}}\mathcal{E}_h(t, u_h(t)), \tag{5.62}$$

where $\mathcal{E}_h(t,u)$ is defined in (5.23).

Recalling Lemma 5.1.2, for all $t \in [0,T]$ we have

$$\mathcal{F}_h(t, v_h(t)) \leq \mathcal{F}_h(0, v_h(0)) + \int_0^{t_h} \int_\Omega f_h'(\nabla v_h(\tau)) \nabla \dot{g}(\tau)\, dx\, d\tau + e(h), \tag{5.63}$$

where $e(h) \to 0$ as $h \to +\infty$, and $t_h := t_{i_h}^{\delta_h}$ is the step discretization point of I_{δ_h} such that $t_{i_h}^{\delta_h} \leq t < t_{i_h+1}^{\delta_h}$.

Step 2: Uniform bound on the energy. There exists a constant C' independent of h such that for all $t \in [0,T]$ we have

$$\mathcal{F}_h(t, v_h(t)) + \|v_h(t)\|_\infty \leq C'. \tag{5.64}$$

In fact by (5.59) we have

$$\mathcal{F}_h(0, u_h(0)) \leq \|\nabla g(0)\|^2,$$

and by (5.60) for all $\tau \in [0,T]$

$$\int_\Omega f_h(\nabla v_h(\tau))\, dx \leq \|\nabla g^{\delta_h}(\tau)\|^2.$$

Moreover for all $\tau \in [0,T]$

$$\int_\Omega |f_h'(\nabla v_h(\tau))|^2\, dx \leq 4 \int_\Omega f_h(\nabla v_h(\tau))\, dx.$$

Taking into account (5.63) and (5.58) we deduce that (5.64) holds.

Step 3: Compactness. In view of Step 2, by Proposition 5.3.1 and Proposition 5.3.2 we have that for all $t \in [0,T]$ the displacements $(v_h(t))_{h \in \mathbb{N}}$ are relatively compact with respect to the weak* topology of $BV(\Omega)$, while the cracks $(K_h(t), \gamma_h(t))_{h\in\mathbb{N}}$ are compact in a suitable energetic sense.

Let $B \subseteq [0,T]$ be countable and dense, and such that $0 \in B$. By Proposition 5.3.1, and by Proposition 5.3.2 (with f_h and φ_h defined in (5.57) and (5.56), $a_h := a\sqrt{h}$, $\gamma_h := \gamma_h(t)$ and $g_h := g^{\delta_h}$) up to a subsequence (which we denote by the same symbol) for all $t \in B$ there exists $v(t) \in SBV(\Omega)$ and a rectifiable set $K(t) \tilde{\subseteq} \Omega \cup \partial_D \Omega$ such that the following facts hold:

(a) $v_h(t) \xrightarrow{*} v(t)$ in the weak* topology of $BV(\Omega)$, $\nabla v_h(t) \rightharpoonup \nabla v(t)$ weakly in $L^1(\Omega; \mathbb{R}^N)$, $\nabla v(t) \in L^2(\Omega;\mathbb{R}^N)$, and

$$S^{g(t)}(v(t)) \tilde{\subseteq} K(t);$$

(b) $K(s) \tilde{\subseteq} K(t)$ for all $s, t \in B$, $s \leq t$;

(c) we have

$$\mathcal{H}^{N-1}(K(t)) \leq \liminf_{h \to +\infty} \int_{K_h(t)} \varphi_h(\gamma_h(t))\, d\mathcal{H}^{N-1}; \tag{5.65}$$

(d) $K(0) = S^{g(0)}(v(0))$.

Points (a) and (c) comes directly from Proposition 5.3.1 and Proposition 5.3.2. Let us prove point (b). Let $s,t \in B$ with $s < t$. By Proposition 5.3.2 we know that there exists a countable set $D(s)$ in $SBV(\Omega)$ such that

$$K(s) = \bigcup_{u \in D} S^{g(s)}(u), \tag{5.66}$$

and such that for every $u \in D(s)$ there exists a sequence $(u_h)_{h\in\mathbb{N}}$ in $BV(\Omega)$ such that $u_h \xrightarrow{*} u$ weakly* in $BV(\Omega)$ with $S^{g^{\delta_h}(s)}(u_h) \tilde{\subseteq} K_h(s)$, $|[u_h]| \le \gamma_h(s)$, and $\mathcal{F}_h(s,u_h) \le C'$ for some $C' \in [0,+\infty[$. Let us set $v_h := u_h - g^{\delta_h}(s) + g^{\delta_h}(t)$. Since $K_h(s) \tilde{\subseteq} K_h(t)$ and $\gamma_h(s) \le \gamma_h(t)$, we have that $S^{g^{\delta_h}(t)}(v_h) \tilde{\subseteq} K_h(t)$, $|[v_h]| \le \gamma_h(t)$,

$$\int_\Omega f_h(\nabla v_h)\,dx + \int_{K_h(t)} \varphi_h(|[v_h]|)\,d\mathcal{H}^{N-1} + a\sqrt{h}|D^c v_h|(\Omega) \le \tilde{C}'$$

with $\tilde{C}'$ independent of h, and $v_h \xrightarrow{*} u - g(s) + g(t)$ weakly* in $BV(\Omega)$. We deduce that $S^{g(t)}(u - g(s) + g(t)) \tilde{\subseteq} K(t)$ that is $S^{g(s)}(u) \tilde{\subseteq} K(t)$. Then by (5.66) we obtain $K(s) \tilde{\subseteq} K(t)$.

Let us come to point (d). Notice that

$$\|\nabla v(0)\|^2 + \mathcal{H}^{N-1}(K(0)) \le \liminf_{h\to+\infty} \mathcal{F}_h(0, v_h(0)) \le \|\nabla v(0)\|^2 + \mathcal{H}^{N-1}(S^{g(0)}(v(0))), \tag{5.67}$$

the first inequality coming from point (c) and Proposition 5.3.1, the last inequality coming from the minimality property (5.59). Since $S^{g(0)}(v(0)) \tilde{\subseteq} K(0)$, by (5.67) we get that $S^{g(0)}(v(0)) = K(0)$, so that point (d) is proved.

Step 4: Recovering the static equilibrium for $K(t)$, $t \in B$. Let B be the countable and dense set defined in Step 3, and let $K(t)$ be the limit crack associated to $(K_h(t), \gamma_h(t))_{h\in\mathbb{N}}$ for all $t \in B$. In order to prove that $K(t)$ is part of an evolution in the sense of [53] with respect to the boundary data $g(t)$, we have to prove that $K(t)$ satisfies the one-sided minimality property with respect to the Griffith's energy given by point (c) of Theorem 1.4.2. This is done in this step, where also some useful convergence results for the gradient of the displacements are obtained.

Let $t \in B$, and let us consider the subsequence of $(v_h(t), K_h(t), \gamma_h(t))_{h\in\mathbb{N}}$ (which we indicate with the same symbol), the displacement $v(t)$ and the rectifiable set $K(t)$ given by Step 3. Then for all $v \in SBV(\Omega)$ we have

$$\|\nabla v(0)\|^2 + \mathcal{H}^{N-1}(S^{g(0)}(v(0)) \le \|\nabla v\|^2 + \mathcal{H}^{N-1}(S^{g(0)}(v)), \tag{5.68}$$

and for all $t \in]0,T]$

$$\|\nabla v(t)\|^2 \le \|\nabla v\|^2 + \mathcal{H}^{N-1}(S^{g(t)}(v) \setminus K(t)). \tag{5.69}$$

Moreover

$$\|\nabla v(0)\|^2 + \mathcal{H}^{N-1}(K(0)) = \lim_{h\to+\infty} \mathcal{F}_h(0, v_h(0)), \tag{5.70}$$

where $\mathcal{F}_h$ is defined in (5.61), and for all $t \in B$

$$\nabla v_h(t) 1\!\!1_{E_h(t)} \to \nabla v(t) \qquad \text{strongly in } L^2(\Omega;\mathbb{R}^N), \tag{5.71}$$

where

$$E_h(t) := \left\{ x \in \Omega \,:\, |\nabla v_h(t)| \le \frac{a\sqrt{h}}{2} \right\},$$

and

$$\|\nabla v(t)\|^2 = \lim_{h\to+\infty} \int_\Omega f_h(\nabla v_h(t))\,dx. \tag{5.72}$$

In fact (5.68) and (5.70) come from point (d) of Step 3, from the minimality property (5.59), and from Proposition 5.3.1.

Let us come to (5.69). Let $t \in [0,T]$. By Proposition 5.3.5 we have that there exists $(v_h)_{h\in\mathbb{N}}$ sequence in $SBV(\Omega)$ such that $\nabla v_h \to \nabla v$ strongly in $L^2(\Omega;\mathbb{R}^N)$ and

$$\limsup_{h\to+\infty}\left[\int_{S^{g^{\delta_h}(t)}(v_h)\cup K_h(t)} \varphi_h(|[v_h]|\vee\gamma_h(t))\,d\mathcal{H}^{N-1} - \int_{K_h(t)} \varphi_h(\gamma_h(t))\,d\mathcal{H}^{N-1}\right] \le \mathcal{H}^{N-1}(S^{g(t)}(v)\setminus K(t)).$$

Then using the minimality property (5.60) we get

$$\limsup_{h\to+\infty}\int_\Omega f_h(\nabla v_h(t))\,dx \le \|\nabla v\|^2 + \mathcal{H}^{N-1}(S^{g(t)}(v)\setminus K(t)). \tag{5.73}$$

By Proposition 5.3.1 we have that

$$\|\nabla v(t)\|^2 \le \liminf_{h\to+\infty}\int_\Omega f_h(\nabla v_h(t))\,dx, \tag{5.74}$$

and so we obtain that (5.69) holds.

Let us now come to (5.71) and (5.72). (5.72) is a direct consequence of (5.74) and (5.73) with $v = v(t)$. Finally, notice that $(\nabla v_h(t)\mathbb{1}_{E_h(t)})_{h\in\mathbb{N}}$ is bounded in $L^2(\Omega;\mathbb{R}^N)$. Since $\nabla v_h(t) \rightharpoonup \nabla v(t)$ weakly in $L^1(\Omega;\mathbb{R}^N)$ and $\nabla v(t) \in L^2(\Omega;\mathbb{R}^N)$, we get $\nabla v_h(t)\mathbb{1}_{E_h(t)} \rightharpoonup \nabla v(t)$ weakly in $L^2(\Omega;\mathbb{R}^N)$. By (5.73) with $v = v(t)$ we have

$$\limsup_{h\to+\infty}\|\nabla v_h(t)\mathbb{1}_{E_h(t)}\|^2 \le \limsup_{h\to+\infty}\int_\Omega f_h(\nabla v_h(t))\,dx \le \|\nabla v(t)\|^2,$$

so that (5.71) holds.

Step 5: Defining $K(t)$ for all $t\in[0,T]$. Since $\{t\to K(t) : t\in B\}$ is increasing by Step 3, setting

$$K^-(t) := \bigcup_{s\in B, s\le t} K(s), \qquad K^+(t) := \bigcap_{s\in B, s\ge t} K(s),$$

there exists a countable set $B' \subseteq [0,T]\setminus B$ such that we have $K^-(t)\tilde{=}K^+(t)$ for all $t\in[0,T]\setminus B'$. For all such t's let us set $K(t) := K^-(t)\tilde{=}K^+(t)$. Up to a further subsequence relative to the elements of B' (which we indicate still with the same symbol), we find $K(t)$ such that Step 3 and Step 4 hold for every $t\in B'$. Notice that

$$\{t\to K(t) : t\in[0,T]\}$$

is increasing, and for all $t\in[0,T]$ we have $\mathcal{H}^{N-1}(K(t)) \le C'$, where C' is given by (5.64).

Let $v(t)$ be a minimum of the following problem

$$\min\left\{\|\nabla v\|^2 : v\in SBV(\Omega),\ S^{g(t)}(v)\tilde{\subseteq} K(t)\right\}. \tag{5.75}$$

Notice that problem (5.75) is well posed since $K(t)$ has finite $\mathcal{H}^{N-1}$-measure, and $g(t)$ is bounded in $L^\infty(\Omega)$: moreover by strict convexity we have that $\nabla v(t)$ is uniquely determined.

Let us prove that $(v(t), K(t))$ satisfies Step 3 and Step 4 for every $t\in[0,T]$. Moreover let us see that

$$\nabla v_h(t) \rightharpoonup \nabla v(t) \qquad \text{weakly in } L^1(\Omega;\mathbb{R}^N), \tag{5.76}$$

and that every accumulation point v of $(v_h(t))_{h\in\mathbb{N}}$ in the weak* topology of $BV(\Omega)$ is such that $v \in SBV(\Omega)$, $S^{g(t)}(v) \tilde{\subseteq} K(t)$ and $\nabla v = \nabla v(t)$.

In fact let $t \notin B \cup B'$ (otherwise the result holds by construction), and let $v_{h_m}(t) \overset{*}{\rightharpoonup} v$ weakly* in $BV(\Omega)$ for some subsequence $(h_m)_{m\in\mathbb{N}}$. By Proposition 5.3.1 we get that $v \in SBV(\Omega)$, $\nabla v \in L^2(\Omega;\mathbb{R}^N)$ and $\nabla v_{h_m}(t) \rightharpoonup \nabla v$ weakly in $L^1(\Omega;\mathbb{R}^N)$.

Applying Step 3 and Step 4 to $B \cup \{t\}$, we can find (up to a further subsequence) $\tilde{K}(t)$ such that $S^{g(t)}(v) \tilde{\subseteq} \tilde{K}(t)$, $\tilde{K}(t)$ satisfies static equilibrium, and $K(s_1) \tilde{\subseteq} \tilde{K}(t) \tilde{\subseteq} K(s_2)$ for all $s_1, s_2 \in B$ with $s_1 < t < s_2$. Then we get $\tilde{K}(t) = K(t)$ up to a set of $\mathcal{H}^{N-1}$-measure zero.

Finally, in order to prove that (5.76) holds, notice that v is minimum of problem (5.75): by uniqueness we obtain $\nabla v = \nabla v(t)$ so that (5.76) holds along the entire sequence.

Step 6: Recovering the nondissipativity condition. In order to prove that

$$\{t \to (v(t), K(t)),\ t \in [0,T]\}$$

is a quasistatic crack growth in the sense of [53], that is in the sense of Theorem 1.4.2, we have just to prove the *nondissipativity* condition, that is

$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t (\nabla v(\tau), \nabla \dot{g}(\tau))_{L^2(\Omega;\mathbb{R}^N)}\, d\tau, \tag{5.77}$$

where $\mathcal{E}(t) := \|\nabla v(t)\|^2 + \mathcal{H}^{N-1}(K(t))$ for all $t \in [0,T]$. In fact *irreversibility* and *static equilibrium* are consequences of Steps 3,4,5. First of all for all $t \in [0,T]$ we have

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t (\nabla u(\tau), \nabla \dot{g}(\tau))_{L^2(\Omega;\mathbb{R}^N)}\, d\tau. \tag{5.78}$$

In fact as noticed in [59], using the minimality property (5.69), the map $\{t \to \nabla v(t)\}$ is continuous at all the continuity points of $\{t \to \mathcal{H}^{N-1}(K(t))\}$, in particular it is continuous up to a countable set in $[0,T]$. Given $t \in [0,T]$ and $k > 0$, let us set

$$s_i^k := \frac{i}{k}t, \qquad v^k(s) := v(s_{i+1}^k) \quad \text{for } s_i^k < s \leq s_{i+1}^k, \qquad i = 0, 1, \ldots, k.$$

By (5.69), comparing $v(s_i^k)$ with $v(s_{i+1}^k) - g(s_{i+1}^k) + g(s_i^k)$, it is easy to see that

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t (\nabla v^k(\tau), \nabla \dot{g}(\tau))_{L^2(\Omega;\mathbb{R}^N)}\, d\tau + e(k),$$

where $e(k) \to 0$ as $k \to +\infty$. By the continuity property of ∇v, passing to the limit for $k \to +\infty$ we deduce that (5.78) holds. On the other hand for all $t \in [0,T]$ we have that

$$\mathcal{E}(t) \leq \mathcal{E}(0) + 2\int_0^t (\nabla u(\tau), \nabla \dot{g}(\tau))_{L^2(\Omega;\mathbb{R}^N)}\, d\tau. \tag{5.79}$$

In fact by Step 4 we have that for all $t \in [0,T]$

$$\nabla v_h(t)\mathbb{1}_{E_h(t)} \to \nabla v(t) \qquad \text{strongly in } L^2(\Omega;\mathbb{R}^N), \tag{5.80}$$

where

$$E_h(t) := \left\{ x \in \Omega : |\nabla v_h(t)| \leq \frac{a\sqrt{h}}{2} \right\}.$$

By (5.63) and by the very definition of f_h we deduce

$$\begin{aligned} \mathcal{F}_h(t, v_h(t)) \leq & \mathcal{F}_h(0, v_h(0)) + 2\int_0^t (\nabla v_h(\tau)\mathbb{1}_{E_h(\tau)}, \nabla \dot{g}(\tau))_{L^2(\Omega;\mathbb{R}^N)}\, d\tau \\ & + a\sqrt{h}\int_0^t \int_{\Omega \setminus E_h(\tau)} |\nabla \dot{g}(\tau)|\, dx\, d\tau + e(h) \end{aligned} \tag{5.81}$$

where $e(h) \to 0$ as $h \to +\infty$. Notice that by (5.64) we have

$$\frac{a}{2}h|\Omega \setminus E_h(\tau)| \leq \sqrt{h}\int_{\Omega\setminus E_h(\tau)} |\nabla v_h(\tau)|\,dx \leq \frac{2}{a}\int_{\Omega\setminus E_h(\tau)} f_h(\nabla v_h(\tau))\,dx \leq \frac{2}{a}C'.$$

We deduce that

$$\sqrt{h}\int_{\Omega\setminus E_h(\tau)} |\nabla \dot g(\tau)|\,dx \leq \left(\int_{\Omega\setminus E_h(\tau)} |\nabla \dot g(\tau)|^2\,dx\right)^{\frac{1}{2}} \sqrt{h|\Omega\setminus E_h(\tau)|} \leq \frac{2\sqrt{C'}}{a}\left(\int_{\Omega\setminus E_h(\tau)} |\nabla \dot g(\tau)|^2\,dx\right)^{\frac{1}{2}} \to 0 \tag{5.82}$$

as $h \to +\infty$ by equicontinuity of $\nabla \dot g(\tau)$. Then passing to the limit for $h \to +\infty$ in (5.81), in view of (5.72), (5.65), (5.70), (5.80) and (5.82) we deduce that (5.79) holds. This proves that (5.77) holds, and so $\{t \to (v(t), K(t)) \,:\, t \in [0,T]\}$ is a quasistatic crack growth in the sense of [53].

Step 7: Convergence of bulk and surface energies. In order to conclude the proof, let us see that (5.24), (5.25) and (5.26) hold. By (5.81) we deduce that for all $t \in [0,T]$

$$\mathcal{F}_h(t, v_h(t)) \to \mathcal{E}(t),$$

so that by (5.72) and (5.65) we deduce that

$$\mathcal{H}^{N-1}(K(t)) = \lim_{h\to+\infty}\int_{K_h(t)} \varphi_h(\gamma_h(t))\,d\mathcal{H}^{N-1}, \qquad a\sqrt{h}|D^c v_h(t)|(\Omega) \to 0.$$

Theorem 5.2.1 is now completely proved in view of the rescaling (5.54), of (5.55), (5.56) and (5.62).

5.5 Proof of Theorem 5.2.2

In this section we will give the proof of Theorem 5.2.2. Let $\{t \to (u_h(t), \Gamma_h(t), \psi_h(t)) \,:\, t \in [0,T]\}$ be the piecewise constant interpolation given in (5.22) of a discrete in time evolution of fracture in Ω_h relative the subdivision $I_{\delta_h} := \{0 = t_0^{\delta_h} < \cdots < t_{N_{\delta_h}}^{\delta_h} = T\}$, and the boundary displacement $h^\alpha g(t, \frac{x}{h})$ with $\alpha \in]0, \frac{1}{2}[$. We divide the proof in several steps.

Step 1: Rescaling. We rescale u_h and Γ_h in the following way: for all $t \in [0,T]$ let $v_h(t) \in BV(\Omega)$ and $K_h(t) \tilde{\subseteq} \Omega \cup \partial_D\Omega$ be given by

$$v_h(t,x) := \frac{1}{h^\alpha}u_h(t, hx), \qquad K_h(t) := \frac{1}{h}\Gamma_h(t), \qquad t \in [0,T], x \in \Omega. \tag{5.83}$$

Let us moreover set

$$\gamma_h(t,x) := \frac{1}{h^\alpha}\psi_h(t,hx) = \max_{0\leq s\leq t} |[v_h(s)](t,x)| \qquad t \in [0,T], x \in \Omega.$$

It turns that $\{t \to (v_h(t), K_h(t), \gamma_h(t)) \,:\, t \in [0,T]\}$ is the piecewise constant interpolation of a discrete in time evolution of cracks in Ω relative to the subdivision I_{δ_h} and boundary displacement $g(t)$ with respect to the basic total energy

$$\int_\Omega f_h(\nabla v)\,dx + h^{1-2\alpha}\int_{S^{g^{\delta_h}(t)}(v)\cup K_h(t)} \varphi_h(|[v]| \vee \gamma_h(t))\,d\mathcal{H}^{N-1} + ah^{1-\alpha}|D^c v|(\Omega),$$

where $g^{\delta_h}(t) := g(t_i^{\delta_h})$ for $t_i^{\delta_h} \leq t < t_{i+1}^{\delta_h}$, $a := \varphi'(0)$,

$$\varphi_h(s) := \varphi(h^\alpha s), \qquad s \in [0,+\infty[,$$

and

$$f_h(\xi) := \begin{cases} |\xi|^2 & \text{if } |\xi| \le \frac{ah^{1-\alpha}}{2} \\ \frac{a^2h^{2(1-\alpha)}}{4} + ah^{1-\alpha}(|\xi| - \frac{ah^{1-\alpha}}{2}) & \text{if } |\xi| \ge \frac{ah^{1-\alpha}}{2}. \end{cases}$$

We have that the following facts hold:

(a) for all $t \in [0,T]$

$$\|v_h(t)\|_\infty \le \|g^{\delta_h}(t)\|_\infty \le C; \tag{5.84}$$

(b) $K_h(0) = S^{g^{\delta_h}(0)}(v_h(0))$, and $S^{g^{\delta_h}(t)}(v_h(t)) \tilde{\subseteq} K_h(t)$ for all $t \in]0,T]$;

(c) for all $w \in BV(\Omega)$ we have

$$\begin{aligned} \int_\Omega f_h(\nabla v_h(0))\,dx + h^{1-2\alpha} \int_{S^{g^{\delta_h}(0)}(v_h(0))} \varphi_h(|[v_h(0)]|)\,d\mathcal{H}^{N-1} + ah^{1-\alpha}|D^c v_h(0)|(\Omega) \\ \le \int_\Omega f_h(\nabla w)\,dx + h^{1-2\alpha} \int_{S^{g^{\delta_h}(0)}(w)} \varphi_h(|[w]|)\,d\mathcal{H}^{N-1} + ah^{1-\alpha}|D^c w|(\Omega); \end{aligned} \tag{5.85}$$

(d) for all $w \in BV(\Omega)$ and $t \in]0,T]$ we have

$$\begin{aligned} \int_\Omega f_h(\nabla v_h(t))\,dx + h^{1-2\alpha} \int_{K_h(t)} \varphi_h(\gamma_h(t))\,d\mathcal{H}^{N-1} + ah^{1-\alpha}|D^c v_h(t)|(\Omega) \\ \le \int_\Omega f_h(\nabla w)\,dx + h^{1-2\alpha} \int_{S^{g^{\delta_h}(t)}(w)\cup K_h(t)} \varphi_h(|[w]| \vee \gamma_h(t))\,d\mathcal{H}^{N-1} + ah^{1-\alpha}|D^c w|(\Omega). \end{aligned} \tag{5.86}$$

Let us set for all $v \in BV(\Omega)$ and for all $t \in [0,T]$

$$\mathcal{F}_h(t,w) := \int_\Omega f_h(\nabla w)\,dx + h^{1-2\alpha} \int_{S^{g^{\delta_h}(t)}(w)\cup K_h(t)} \varphi_h(|[w]| \vee \gamma_h(t))\,d\mathcal{H}^{N-1} + ah^{1-\alpha}|D^c w|(\Omega).$$

Notice that

$$\mathcal{F}_h(t, v_h(t)) = \frac{1}{h^{N+2\alpha-2}}\mathcal{E}(t, u_h(t)),$$

where $\mathcal{E}(t,u_h(t))$ is defined in (5.23).

By Lemma 5.1.2 we obtain for all $t \in [0,T]$

$$\mathcal{F}_h(t, v_h(t)) \le \mathcal{F}_h(0, v_h(0)) + \int_0^{t_h} \int_\Omega f_h'(\nabla v_h(\tau))\nabla \dot{g}(\tau)\,dx\,d\tau + e(h), \tag{5.87}$$

where $e(h) \to 0$ as $h \to +\infty$, and $t_h := t_{i_h}^{\delta_h}$ is the step discretization point of I_{δ_h} such that $t_{i_h}^{\delta_h} \le t < t_{i_h+1}^{\delta_h}$.

Step 2: Uniform bound on the energy. By (5.85) comparing $v_h(0)$ and $g(0)$ we have

$$\int_\Omega f_h(\nabla v_h(0))\,dx + h^{1-2\alpha} \int_{S^{g(0)}(v_h(0))} \varphi_h(|[v_h(0)]|) + ah^{1-\alpha}|D^c v_h(0)|(\Omega) \le \|\nabla g(0)\|^2.$$

By (5.86) comparing $v_h(t)$ and $g^{\delta_h}(t)$ we obtain

$$\int_\Omega f_h(\nabla v_h(t))\,dx \le \|\nabla g^{\delta_h}(t)\|^2,$$

and since we have

$$\int_\Omega |f_h'(\nabla v_h(\tau))|^2\,dx \le 4\int_\Omega f_h(\nabla v_h(\tau))\,dx,$$

by (5.87) we deduce that

$$\int_\Omega f_h(\nabla v_h(t))\,dx + h^{1-2\alpha}\int_{K_h(t)} \varphi_h(\gamma_h(t))\,d\mathcal{H}^{N-1} + ah^{1-\alpha}|D^c v|(\Omega) \le C'$$

with C' independent of h and of t. By Proposition 5.3.1 and by (5.84) we deduce that $(v_h(t))_{h\in\mathbb{N}}$ is bounded in $BV(\Omega)$, and this proves point (a).

Step 3: Convergence to the elastic solution. Let $v(t)$ be an accumulation point for $(v_h(t))_{h\in\mathbb{N}}$ in the weak* topology of $BV(\Omega)$, and let us consider $\tilde{\Omega} \subseteq \mathbb{R}^N$ open and bounded, and such that $\overline{\Omega} \subseteq \tilde{\Omega}$. Let us set $\Omega' := \tilde{\Omega}\setminus \partial_N\Omega$. Then we can extend $v_h(t)$ and $v(t)$ to Ω' setting $v_h(t) = g^{\delta_h}(t)$ and $v(t) = g(t)$ on $\Omega'\setminus\Omega$ respectively. We have $v_{h_j}(t) \stackrel{*}{\rightharpoonup} v(t)$ weakly* in $BV(\Omega')$ for a suitable $h_j \nearrow +\infty$, and

$$\int_{\Omega'} f_{h_j}(\nabla v_{h_j}(t))\,dx + h_j^{1-2\alpha}\int_{S(v_{h_j}(t))} \varphi_{h_j}(|[v_{h_j}(t)]|)\,d\mathcal{H}^{N-1} + ah_j^{1-\alpha}|D^c v_{h_j}(t)|(\Omega') \le \tilde{C} \tag{5.88}$$

with $\tilde{C}$ independent of j. In particular we have

$$\int_{\Omega'} f_{h_j}(\nabla v_{h_j}(t))\,dx + \int_{S(v_{h_j}(t))} \varphi_{h_j}(|[v_{h_j}(t)]|)\,d\mathcal{H}^{N-1} + ah_j^{1-\alpha}|D^c v_{h_j}(t)|(\Omega') \le \tilde{C}'$$

with $\tilde{C}'$ independent of j. Then by Proposition 5.3.1 we have that $v(t) \in SBV(\Omega)$,

$$\nabla v_{h_j}(t) \rightharpoonup \nabla v(t) \qquad \text{weakly in } L^1(\Omega;\mathbb{R}^N),$$

and

$$\|\nabla v(t)\|^2 \le \liminf_{j\to+\infty}\int_\Omega f_{h_j}(\nabla v_{h_j}(t))\,dx. \tag{5.89}$$

Finally, if we consider for all Borel sets $B \subseteq \Omega'$

$$\lambda_j(B) := \int_{B\cap S(v_{h_j}(t))} \varphi_{h_j}(|[v_{h_j}(t)]|)\,d\mathcal{H}^{N-1}$$

and if (up to a subsequence) $\lambda_j \stackrel{*}{\rightharpoonup} \lambda$ weakly* in the sense of measures, we deduce following Proposition 5.3.1 that

$$\mathcal{H}^{N-1}\llcorner S(v(t)) \le \lambda \qquad \text{as measures.}$$

Since by (5.88) we have $\lambda = 0$, then we have $S(v(t)) = \emptyset$, that is $v(t) \in H^1(\Omega)$ and $v(t) = g(t)$ on $\partial_D\Omega$.

Let us consider $v \in H^1(\Omega)$ with $v = g(t)$ on $\partial_D\Omega$. Comparing $v_h(t)$ with $v - g(t) + g^{\delta_h}(t)$ by minimality property (5.86) we obtain

$$\begin{aligned}\int_\Omega f_h(\nabla v_h(t))\,dx + ah^{1-\alpha}|D^c v_h(t)|(\Omega) &\le \int_\Omega f_h(\nabla v - \nabla g(t) + \nabla g^{\delta_h}(t))\,dx \\ &\le \|\nabla v - \nabla g(t) + \nabla g^{\delta_h}(t)\|^2.\end{aligned} \tag{5.90}$$

In view of (5.89) we deduce that

$$\|\nabla v(t)\|^2 \leq \|\nabla v\|^2,$$

so that $v(t)$ is a minimizer of

$$\min\{\|\nabla v\|^2 \,:\, v \in H^1(\Omega),\, v = g(t) \text{ on } \partial_D\Omega\}.$$

By strict convexity and since Ω is connected, we have that $v(t)$ is uniquely determined, and so we deduce that $v_h(t) \overset{*}{\rightharpoonup} v(t)$ weakly* in $BV(\Omega)$ and $\nabla v_h(t) \rightharpoonup \nabla v(t)$ weakly in $L^1(\Omega;\mathbb{R}^N)$.

Choosing $v = v(t)$ in (5.90) and taking the limsup in h we have

$$\limsup_{h\to+\infty} \int_\Omega f_h(\nabla v_h(t))\, dx \leq \|\nabla u(t)\|^2,$$

so that

$$\lim_{h\to+\infty} \int_\Omega f_h(\nabla v_h(t))\, dx = \|\nabla u(t)\|^2.$$

The proof of point (b) is now concluded thank to the rescaling (5.83).

5.6 Proof of Theorem 5.2.3

In this section we will give the proof of Theorem 5.2.3. Let $\{t \to (u_h(t), \Gamma_h(t), \psi_h(t)) \,:\, t \in [0,T]\}$ be the piecewise constant interpolation given in (5.22) of a discrete in time evolution of fracture in Ω_h relative the subdivision $I_{\delta_h} := \{0 = t_0^{\delta_h} < \cdots < t_{N_{\delta_h}}^{\delta_h} = T\}$, and the boundary displacement $h^\alpha g(t, \frac{x}{h})$ with $\alpha > \frac{1}{2}$.

We rescale u_h and Γ_h in the following way: for all $t \in [0,T]$ let $v_h(t) \in BV(\Omega)$ and $K_h(t) \tilde{\subseteq} \Omega \cup \partial_D\Omega$ be given by

$$v_h(t,x) := \frac{1}{h^\alpha} u_h(t, hx), \qquad K_h(t) := \frac{1}{h}\Gamma_h(t), \qquad t \in [0,T], x \in \Omega.$$

Let us moreover set

$$\gamma_h(t,x) := \frac{1}{h^\alpha}\psi_h(t,hx) = \max_{0\leq s\leq t} |[v_h(s)](t,x)|, \qquad t \in [0,T], x \in \Omega.$$

It turns out that $\{t \to (v_h(t), K_h(t), \gamma_h(t)) \,:\, t \in [0,T]\}$ is the piecewise constant interpolation of a discrete in time evolution of cracks in Ω relative to the subdivision I_{δ_h} and boundary displacement $g(t)$ with respect to the basic total energy

$$h^{2\alpha-1} \int_\Omega f_h(\nabla v)\, dx + \int_{S^{g^{\delta_h}(t)}(v)} \varphi_h(|[v]| \vee \gamma_h(t))\, d\mathcal{H}^{N-1} + a h^\alpha |D^c v|(\Omega),$$

where $g^{\delta_h}(t) := g(t_i^{\delta_h})$ for $t_i^{\delta_h} \leq t < t_{i+1}^{\delta_h}$, $a := \varphi'(0)$,

$$\varphi_h(s) := \varphi(h^\alpha s), \qquad s \in [0, +\infty[,$$

and

$$f_h(\xi) := \begin{cases} |\xi|^2 & \text{if } |\xi| \leq \frac{ah^{1-\alpha}}{2} \\ \frac{a^2 h^{2(1-\alpha)}}{4} + a h^{1-\alpha}(|\xi| - \frac{ah^{1-\alpha}}{2}) & \text{if } |\xi| \geq \frac{ah^{1-\alpha}}{2}. \end{cases}$$

Notice that by Proposition 5.1.1 we have

$$\|v_h(0)\|_\infty \leq \|g(0)\|_\infty \leq C, \tag{5.91}$$

and for all $w \in BV(\Omega)$ we have

$$(5.92)\quad h^{2\alpha-1}\int_\Omega f_h(\nabla v_h(0))\,dx + \int_{S^{g^{\delta_h}(0)}(v_h(0))} \varphi_h(|[v_h(0)]|)\,d\mathcal{H}^{N-1} + ah^\alpha|D^c v_h(0)|(\Omega)$$
$$\leq h^{2\alpha-1}\int_\Omega f_h(\nabla w)\,dx + \int_{S^{g^{\delta_h}(0)}(w)} \varphi_h(|[w]|)\,d\mathcal{H}^{N-1} + ah^\alpha|D^c w|(\Omega).$$

Comparing $v_h(0)$ and $w = -C$ by means of (5.92) we have

$$(5.93)\quad h^{2\alpha-1}\int_\Omega f_h(\nabla v_h(0))\,dx + \int_{S^{g^{\delta_h}(0)}(v_h(0))} \varphi_h(|[v_h(0)]|)\,d\mathcal{H}^{N-1} + ah^\alpha|D^c v_h(0)|(\Omega)$$
$$\leq \mathcal{H}^{N-1}(\partial_D\Omega).$$

As a consequence, since $\|v_h(0)\|_\infty \leq C$ by (5.91), following Proposition 5.3.1, we deduce that $(v_h(0))_{h\in\mathbb{N}}$ is bounded in $BV(\Omega)$. Let v be an accumulation point for $(v_h(0))_{h\in\mathbb{N}}$ in the weak* topology of $BV(\Omega)$. Let us prove that $v \in SBV(\Omega)$ and that $\nabla v = 0$: in fact we have that for all $\xi \in \mathbb{R}^N$

$$\tilde{f}_h(\xi) \leq h^{2\alpha-1} f_h(\xi)$$

where

$$\tilde{f}_h(\xi) := \begin{cases} |\xi|^2 & \text{if } |\xi| \leq \frac{ah^\alpha}{2} \\ \frac{a^2h^{2\alpha}}{4} + ah^\alpha(|\xi| - \frac{ah^\alpha}{2}) & \text{if } |\xi| \geq \frac{ah^\alpha}{2}. \end{cases}$$

We deduce that there exists C'' independent of h such that for all h

$$\int_\Omega \tilde{f}_h(\nabla v_h(0))\,dx + \int_{S(v_h(0))} \varphi_h(|[v_h(0)]|)\,d\mathcal{H}^{N-1} + ch^\alpha|D^c v_h(0)|(\Omega) \leq C''.$$

By Proposition 5.3.1, we obtain that $v \in SBV(\Omega)$ and that $\nabla v_h(0) \rightharpoonup \nabla v$ weakly in $L^1(\Omega;\mathbb{R}^N)$. By (5.93) we obtain that

$$\|\nabla v_h(0)\|_{L^1(\Omega;\mathbb{R}^N)} \leq \frac{\mathcal{H}^{N-1}(\partial_D\Omega)+1}{ah^\alpha},$$

so that we deduce $\nabla v = 0$, that is v is piecewise constant in Ω. Finally taking the limit in (5.92) with w piecewise constant, by Proposition 5.3.1 we get exactly (5.27), so that the proof of Theorem 5.2.3 is concluded.

5.7 A relaxation result

In this section, we prove a relaxation result we used in order to study the discrete in time evolution of cracks in the cohesive case. It consists of a variant of a result by Braides, Bouchitté and Buttazzo [15]: the difference here is that we have to take into account the presence of a preexisting crack with a given opening which enters in the surface part of the energy.

Let $f : \mathbb{R} \to [0,+\infty[$ be convex, $f(0) = 0$ and with superlinear growth, i.e.

$$\limsup_{|\xi|\to+\infty} \frac{f(\xi)}{|\xi|} = +\infty.$$

Let $\varphi : [0,+\infty[\to [0,+\infty[$ be increasing, concave, and such that $\varphi(0) = 0$. Notice that if $a := \varphi'(0) < +\infty$, we have

$$(5.94)\qquad \varphi(s) \leq as \quad \text{for all } s \in [0,+\infty[.$$

Let Ω be a Lipschitz bounded open set in $\mathbb{R}^N$, and let $\partial_D\Omega \subseteq \partial\Omega$ be open in the relative topology. Let Γ be a rectifiable set in $\Omega \cup \partial_D\Omega$, and let ψ be a positive function defined on Γ. Let us extend

ψ to $\Omega \cup \partial_D\Omega$ setting $\psi = 0$ outside Γ. Let $g \in W^{1,1}(\Omega)$: we may assume that g is extended to the whole $\mathbb{R}^N$, and we indicate this extension still by g.

We will study the following functional

$$F(u) := \begin{cases} \int_\Omega f(|\nabla u|)\,dx + \int_{S^g(u)\cup\Gamma} \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} & \text{if } u \in SBV(\Omega) \\ +\infty & \text{otherwise in } BV(\Omega), \end{cases} \tag{5.95}$$

where $S^g(u)$ is defined in (1.2), $a \vee b := \max\{a, b\}$ for all $a, b \in \mathbb{R}$. The functional (5.95) naturally appears (see Section 5.1) when dealing with quasistatic growth of cracks in the cohesive case, where one is required to look for its minima. We are led to compute the relaxation of F with respect to the strong topology of $L^1(\Omega)$. The relaxation in the case $\Gamma = \emptyset$ (without boundary conditions but without superlinear growth on f) has been proved in [15]. Let

$$f_1(\xi) := \inf\{f(\xi_1) + a|\xi_2| \,:\, \xi_1 + \xi_2 = \xi\}, \tag{5.96}$$

where $a := \varphi'(0)$. We have that the following result holds.

Proposition 5.7.1. *The relaxation of the functional* (5.95) *with respect to the weak* topology of $BV(\Omega)$ is given by $\overline{F} : BV(\Omega) \to [0, +\infty]$ defined as*

$$\overline{F}(u) := \int_\Omega f_1(|\nabla u|)\,dx + \int_{S^g(u)\cup\Gamma} \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} + a|D^c u|, \tag{5.97}$$

where $a = \varphi'(0)$ and f_1 is defined in (5.96).

In order to prove Proposition 5.7.1, the first step is the following lemma.

Lemma 5.7.2. *Let $\bar{F} : BV(\Omega) \to [0, +\infty]$ be defined by*

$$\overline{F}(u) := \int_\Omega f_1(|\nabla u|)\,dx + \int_{S^g(u)\cup\Gamma} \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} + a|D^c u|,$$

with $a = \varphi'(0)$ and f_1 as in (5.96). *Then $\bar{F}$ is lower semicontinuous with respect to the weak* topology of $BV(\Omega)$.*

The proof of Lemma 5.7.2 is obtained by a standard slicing argument (see for example [8, Theorem 5.4]) based on the lower semicontinuity result in dimension one. We establish this last one.

Let $I \subseteq \mathbb{R}$ be a finite union of disjoint intervals, and let $J \subseteq I$ be a countable set. Let us consider the functional

$$\mathcal{F}(\mu) := \int_I f_1(|\phi_\mu|)\,dx + \sum_{t \in S_\mu \setminus J} \varphi(|\mu(\{t\})|) + \sum_{t \in J} \varphi(|\mu(\{t\})| \vee \psi(t)) + a|\mu^c|(I) \tag{5.98}$$

defined for all $\mu \in \mathcal{M}_b(I; \mathbb{R}^k)$, i.e. μ is a bounded $\mathbb{R}^k$-valued Radon measure on I. Here ϕ_μ is the density of the absolutely continuous part μ^a of μ, S_μ is the set of atoms of μ, $\mu^c := \mu - \mu^a - \mu \llcorner S_\mu$, ψ is a strictly positive function defined on J, $a = \varphi'(0)$ and f_1 is defined in (5.96).

Lemma 5.7.3. *The functional $\mathcal{F}$ defined in* (5.98) *is lower semicontinuous with respect to the weak* convergence in the sense of measures.*

Proof. Since $\mathcal{F}$ can be obtained as the sup of functionals of the form (5.98) with J finite, we may assume that $J = \{x_1, \ldots, x_m\}$. Let $\mu_n \stackrel{*}{\rightharpoonup} \mu$ weakly* in the sense of measures, and let λ be the weak* limit (up to a subsequence) of $|\mu_n \llcorner J|$. Let $J := J_1 \cup J_2$, with

$$J_1 := \{t \in J \,:\, |\mu(\{t\})| \geq \psi(t)\}, \qquad J_2 := J \setminus J_1.$$

Let $\varepsilon > 0$ be such that
$$\bigcup_{x_i \in J_2} \bar{B}_\varepsilon(x_i) \subseteq I$$
and such that for all n
$$|\mu_n|\left(\bigcup_{x_i \in J_2} \partial \bar{B}_\varepsilon(x_i)\right) = |\mu|\left(\bigcup_{x_i \in J_2} \partial \bar{B}_\varepsilon(x_i)\right) = 0.$$
Let us set
$$I_1 := I \setminus \bigcup_{x_i \in J_2} \bar{B}_\varepsilon(x_i), \qquad I_2 := \bigcup_{x_i \in J_2} B_\varepsilon(x_i).$$
Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the restriction of $\mathcal{F}$ to $\mathcal{M}_b(I_1;\mathbb{R}^k)$ and $\mathcal{M}_b(I_2;\mathbb{R}^k)$ respectively. We have
$$\liminf_{n\to+\infty} \mathcal{F}(\mu_n) \geq \liminf_{n\to+\infty} \mathcal{F}_1(\mu_n \llcorner I_1) + \liminf_{n\to+\infty} \mathcal{F}_2(\mu_n \llcorner I_2).$$
We notice that
$$\mathcal{F}_1(\mu_n \llcorner I_1) \geq \mathcal{G}_1(\mu_n \llcorner I_1)$$
where
$$\mathcal{G}_1(\eta) := \int_{I_1} f_1(|\phi_\eta|)\,dx + \sum_{t\in S_\eta} \varphi(|\eta(\{t\})|) + a|\eta^c|(I_1)$$
for all $\eta \in \mathcal{M}_b(I_1;\mathbb{R}^k)$. By [8, Thorem 5.2] we have that
$$\mathcal{G}_1(\mu \llcorner I_1) \leq \liminf_{n\to+\infty} \mathcal{G}_1(\mu_n \llcorner I_1),$$
so that
$$\mathcal{F}_1(\mu \llcorner I_1) = \mathcal{G}_1(\mu \llcorner I_1) \leq \liminf_{n\to+\infty} \mathcal{F}_1(\mu_n \llcorner I_1).$$
On the other hand, we have
$$\mathcal{F}_2(\mu_n \llcorner I_2) = \mathcal{G}_2(\mu_n \llcorner I_2 \setminus J_2) + \sum_{t\in J_2} \varphi(|\mu_n(\{t\})| \vee \psi(t)),$$
where
$$\mathcal{G}_2(\eta) := \int_{I_2} f_1(|\phi_\eta|)\,dx + \sum_{t\in S_\eta} \varphi(|\eta(\{t\})|) + a|\eta^c|(I_2)$$
for all $\eta \in \mathcal{M}_b(I_2;\mathbb{R}^k)$. We have
$$\begin{aligned}\liminf_{n\to+\infty} \mathcal{F}_2(\mu_n \llcorner I_2) &\geq \mathcal{G}_2(\mu \llcorner I_2 \setminus J_2) + \sum_{t\in J_2} \varphi(\lambda(\{t\}) \vee \psi(t)) \\ &\geq \mathcal{G}_2(\mu \llcorner I_2 \setminus J_2) + \sum_{t\in J_2} \varphi(\psi(t)).\end{aligned} \tag{5.99}$$
We deduce
$$\mathcal{F}_2(\mu \llcorner I_2) = \mathcal{G}_2(\mu \llcorner I_2 \setminus J_2) + \sum_{t\in J_2} \varphi(\psi(t)) \leq \liminf_{n\to+\infty} \mathcal{F}_2(\mu_n \llcorner I_2),$$
and so we get
$$\mathcal{F}(\mu) = \mathcal{F}_1(\mu \llcorner I_1) + \mathcal{F}_2(\mu \llcorner I_2) \leq \liminf_{n\to+\infty} \mathcal{F}(\mu_n).$$
The proof is now concluded. □

Let us now come to the proof of Proposition 5.7.1.

Proof of Proposition 5.7.1. We can assume without loss of generality that

$$\int_\Gamma \varphi(\psi)\, d\mathcal{H}^{N-1} < +\infty. \tag{5.100}$$

Following Lemma 5.7.2, let us consider $\tilde{\Omega}$ open and bounded in $\mathbb{R}^N$ such that $\overline{\Omega} \subset \tilde{\Omega}$, and let us set $\Omega' := \tilde{\Omega} \setminus \partial_N \Omega$. Let us consider the functional

$$F'(u) := \begin{cases} \int_\Omega f(|\nabla u|)\, dx + \int_{S(u)\cup\Gamma} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} & \text{if } u \in SBV(\Omega'), u = g \text{ on } \Omega' \setminus \Omega \\ +\infty & \text{otherwise in } BV(\Omega'). \end{cases} \tag{5.101}$$

The relaxation result of Proposition 5.7.1 is equivalent to prove that the relaxation of (5.101) under the weak* topology of $BV(\Omega')$ is

$$\overline{F'}(u) := \int_\Omega f_1(|\nabla u|)\, dx + \int_{S(u)\cup\Gamma} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} + a|D^c u|(\Omega') \tag{5.102}$$

if $u \in BV(\Omega')$, $u = g$ on $\Omega' \setminus \Omega$, and $\overline{F'}(u) = +\infty$ otherwise in $BV(\Omega')$.

Following [15], it is useful to introduce the localized version of (5.101); namely for all open set $A \subseteq \Omega'$ let us set

$$F'(u, A) := \int_{A\cap\Omega} f(|\nabla u|)\, dx + \int_{A\cap(S(u)\cup\Gamma)} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} \tag{5.103}$$

if $u \in SBV(\Omega')$, $u = g$ on $\Omega' \setminus \Omega$, and $F'(u, A) = +\infty$ otherwise in $BV(\Omega')$. Let us indicate by $\overline{F'}(u, A)$ the relaxation of (5.103) under the weak* topology of $BV(\Omega')$.

Arguing as in [15, Proposition 3.3], we have that for every $u \in BV(\Omega')$, $\overline{F'}(u, \cdot)$ is the restriction to the family $\mathcal{A}(\Omega')$ of all open subsets of Ω' of a regular Borel measure. Since for all $u \in SBV(\Omega')$ with $u = g$ on $\Omega' \setminus \Omega$ and for all $A \in \mathcal{A}(\Omega')$ we have

$$\begin{aligned} &\int_{A\cap\Omega} f(|\nabla u|)\, dx + \int_{A\cap S(u)} \varphi(|[u]|)\, d\mathcal{H}^{N-1} \le F'(u, A) \\ &\quad \le \int_{A\cap\Omega} f(|\nabla u|)\, dx + \int_{A\cap S(u)} \varphi(|[u]|)\, d\mathcal{H}^{N-1} + \int_{A\cap\Gamma} \varphi(\psi)\, d\mathcal{H}^{N-1}, \end{aligned} \tag{5.104}$$

by [15, Theorem 3.1] we obtain that for all $u \in BV(\Omega')$ with $u = g$ on $\Omega' \setminus \Omega$ and for all $A \in \mathcal{A}(\Omega')$ with $A \cap \partial_D \Omega = \emptyset$

$$\begin{aligned} &\int_{A\cap\Omega} f_1(|\nabla u|)\, dx + \int_{A\cap S(u)} \varphi(|[u]|)\, d\mathcal{H}^{N-1} + a|D^c u|(A) \le \overline{F'}(u, A) \\ &\quad \le \int_{A\cap\Omega} f_1(|\nabla u|)\, dx + \int_{A\cap S(u)} \varphi(|[u]|)\, d\mathcal{H}^{N-1} + a|D^c u|(A) + \int_{A\cap\Gamma} \varphi(\psi)\, d\mathcal{H}^{N-1}. \end{aligned} \tag{5.105}$$

As a consequence of (5.105), we deduce that

$$\overline{F'}(u, \cdot) \llcorner (\Omega' \setminus (S(u) \cup \Gamma \cup \partial_D \Omega)) = f_1(|\nabla u|)\, d\mathcal{L}^N \llcorner \Omega + a|D^c u|.$$

In order to evaluate $\overline{F'}(u, \cdot) \llcorner (S(u) \cup \Gamma \cup \partial_D \Omega)$, we notice that for all $A \in \mathcal{A}(\Omega')$ and for all $u \in SBV(\Omega')$ with $u = g$ on $\Omega' \setminus \Omega$

$$\int_{A\cap\Omega} f_1(|\nabla u|)\, dx + \int_{A\cap(S(u)\cup\Gamma)} \varphi(|[u]| \vee \psi)\, d\mathcal{H}^{N-1} + a|D^c u|(A) \le F'(u, A),$$

and since the left hand side is lower semicontinuous by Lemma 5.7.2, we get that for all $u \in BV(\Omega')$ with $u = g$ on $\Omega' \setminus \Omega$

$$\int_{A\cap\Omega} f_1(|\nabla u|)\,dx + \int_{A\cap(S(u)\cup\Gamma)} \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} + a|D^c u|(A) \le \overline{F'}(u, A). \tag{5.106}$$

By outer regularity of $\overline{F'}(u,\cdot)$ we conclude that

$$\overline{F'}(u, E) \ge \int_E \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1}$$

for all Borel sets $E \subseteq S(u) \cup \Gamma \cup \partial_D\Omega$. We have to prove the opposite inequality, namely

$$\overline{F'}(u, E) \le \int_E \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1}$$

for all Borel sets $E \subseteq S(u) \cup \Gamma \cup \partial_D\Omega$. Without loss of generality we may assume that

$$\int_{S(u)} \varphi(|[u]|)\,d\mathcal{H}^{N-1} < +\infty,$$

and by a truncation argument, we can suppose that $u_{|\Omega} \in L^\infty(\Omega)$. Let K be a compact subset of $S(u) \cup \Gamma \cup \partial_D\Omega$, $\varepsilon > 0$, and let A_ε be open with $K \subseteq A_\varepsilon$ and

$$|Du|(A_\varepsilon \setminus K) < \varepsilon, \qquad \int_{(A_\varepsilon\setminus K)\cap\Gamma} \varphi(\psi)\,d\mathcal{H}^{N-1} < \varepsilon.$$

We can find $u_h \in BV(\Omega')$ with $u_h = g$ on $\Omega' \setminus \Omega$ and such that u_h is piecewise constant in Ω (that is $(u_h)_{|\Omega} \in SBV(\Omega)$ with $\nabla u_h = 0$ in Ω), $u_h \to u$ strongly in $L^\infty(\Omega)$, and $|Du_h|(A_\varepsilon \setminus K) < \varepsilon$. Since u_h is piecewise constant in Ω we have for all h

$$\overline{F'}(u_h, A_\varepsilon) \le \int_{A_\varepsilon\cap(S(u_h)\cup\Gamma)} \varphi(|[u_h]| \vee \psi)\,d\mathcal{H}^{N-1}. \tag{5.107}$$

We conclude

$$\begin{aligned}\overline{F'}(u, A_\varepsilon) &\le \liminf_{h\to+\infty} \overline{F'}(u_h, A_\varepsilon) \le \liminf_{h\to+\infty} \int_{A_\varepsilon\cap(S(u_h)\cup\Gamma)} \varphi(|[u_h]| \vee \psi)\,d\mathcal{H}^{N-1} \\ &\le \int_{K\cap(S(u)\cup\Gamma)} \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} + a|Du_h|(A_\varepsilon \setminus K) + \int_{(A_\varepsilon\setminus K)\cap\Gamma} \varphi(\psi)\,d\mathcal{H}^{N-1} \\ &\le \int_{K\cap(S(u)\cup\Gamma)} \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1} + (a+1)\varepsilon\end{aligned} \tag{5.108}$$

so that, letting $\varepsilon \to 0$ we obtain

$$\overline{F'}(u, K) \le \int_{K\cap(S(u)\cup\Gamma)} \varphi(|[u]| \vee \psi)\,d\mathcal{H}^{N-1}.$$

Since K is arbitrary in $S(u) \cup \Gamma \cup \partial_D\Omega$, the proof is concluded. □

Chapter 6

Ambrosio-Tortorelli approximation of quasistatic evolution of brittle cracks

Numerical computations concerning the model of quasistatic crack evolution proposed by Francfort and Marigo in [54] have been performed by Bourdin, Francfort and Marigo in [18]. They employ a discretization in time procedure and an approximation of the total energy proposed in 1990 by Ambrosio and Tortorelli (see [10],[11]).

In this chapter we propose a definition of irreversible quasistatic evolution for the Ambrosio-Tortorelli functional, and prove its convergence to a quasistatic crack growth in sense of Francfort and Larsen [53]. In this way also a theoretical justification of the employment of Ambrosio-Tortorelli approximation in problem of crack evolution is given [1].

6.1 The Ambrosio-Tortorelli functional.

The Ambrosio-Tortorelli functional is given by

$$F_\varepsilon(u,v) = \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2\,dx + \frac{\varepsilon}{2}\int_\Omega |\nabla v|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1-v)^2\,dx$$

where $(u,v) \in H^1(\Omega) \times H^1(\Omega)$, $0 \le v \le 1$, $0 < \eta_\varepsilon << \varepsilon$. F_ε contains an *elliptic part*

$$\int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2\,dx \tag{6.1}$$

and a *surface part*

$$MM_\varepsilon(v) := \frac{\varepsilon}{2}\int_\Omega |\nabla v|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1-v)^2\,dx \tag{6.2}$$

which is a term of Modica-Mortola type (see [72]).

The Γ-convergence result of Ambrosio and Tortorelli [10], [11] can be expressed in the following way. Let us indicate the space of Borel functions on Ω by $\mathcal{B}(\Omega)$ and let us consider on $\mathcal{B}(\Omega) \times \mathcal{B}(\Omega)$ the functionals

$$\mathcal{F}(u,v,\Omega) := \begin{cases} F(u) & u \in GSBV(\Omega), v \equiv 1 \text{ a.e. on } \Omega \\ +\infty & \text{otherwise} \end{cases}$$

[1] The results of this chapter are contained in the paper
A. Giacomini: Ambrosio-Tortorelli approximation of quasistatic evolution of brittle fractures. *Calc. Var. Partial Differential Equation* in press.

and

$$\mathcal{F}_\varepsilon(u,v,\Omega) := \begin{cases} F_\varepsilon(u,v) & (u,v) \in H^1(\Omega), 0 \le v \le 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Here $GSBV$ is defined in Subsection 1.1.2.

Theorem 6.1.1. *The functionals $(\mathcal{F}_\varepsilon)$ on $\mathcal{B}(\Omega) \times \mathcal{B}(\Omega)$ Γ-converge to $\mathcal{F}$ with respect to the convergence in measure.*

In particular, we will use several times the following fact: if $u^i_\varepsilon \in H^1(\Omega)$, $i = 1, \dots, n$, and $v_\varepsilon \in H^1(\Omega)$ are such that $\sum_{i=1}^n F_\varepsilon(u^i_\varepsilon, v_\varepsilon) + \|u^i_\varepsilon\|_\infty \le C$, there exist $u^i \in SBV(\Omega)$, $i = 1, \dots, n$ and a sequence $\varepsilon_k \to 0$ such that $u^i_{\varepsilon_k} \to u^i$ a.e., and

$$\int_\Omega |\nabla u^i|^2\,dx \le \liminf_{\varepsilon \to 0} \int_\Omega (\eta_\varepsilon + v_\varepsilon{}^2)|\nabla u^i_\varepsilon|^2\,dx, \tag{6.3}$$
$$\mathcal{H}^{N-1}\left(\bigcup_{i=1}^n S(u^i)\right) \le \liminf_{\varepsilon \to 0} \left(\frac{\varepsilon}{2}\int_\Omega |\nabla v_\varepsilon|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1-v_\varepsilon)^2\,dx\right).$$

6.2 Description of the approximation result

Let us describe how the approximation of Ambrosio and Tortorelli will be used in order to deal with the problem of quasistatic crack growth. We will define through a variational argument the following notion of quasistatic evolution for the functional F_ε (Theorem 6.3.1): for every $\varepsilon > 0$ we find a map $t \to (u_\varepsilon(t), v_\varepsilon(t))$ from $[0,1]$ to $H^1(\Omega) \times H^1(\Omega)$, $0 \le v_\varepsilon(t) \le 1$, $u_\varepsilon(t) = g(t)$, $v_\varepsilon(t) = 1$ on $\partial_D\Omega$ such that:

(a) for all $0 \le s < t \le 1$: $v_\varepsilon(t) \le v_\varepsilon(s)$;

(b) for all $(u,v) \in H^1(\Omega) \times H^1(\Omega)$ with $u = g(t)$, $v = 1$ on $\partial_D\Omega$, $0 \le v \le v_\varepsilon(t)$:

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \le F_\varepsilon(u,v); \tag{6.4}$$

(c) the energy $\mathcal{E}_\varepsilon(t) := F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))$ is absolutely continuous and for all $t \in [0,1]$

$$\mathcal{E}_\varepsilon(t) = \mathcal{E}_\varepsilon(0) + 2\int_0^t \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau))\nabla u_\varepsilon(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau;$$

(d) there exists a constant C depending only on g such that $\mathcal{E}_\varepsilon(t) \le C$ for all $t \in [0,1]$.

Condition (a) permits to recover in this regular context the fact that the crack is increasing in time: in fact, as $v_\varepsilon(t)$ determines the crack in the regions where it is near zero, the condition $v_\varepsilon(t) \le v_\varepsilon(s)$ ensures that existing cracks are preserved at subsequent times. Condition (b) reproduces the static equilibrium condition at each time, while condition (c) stands for the nondissipativity of the evolution. Condition (d) gives the necessary compactness in order to let $\varepsilon \to 0$. In the particular case in which $||g(t)||_\infty \le C_1$ for all $t \in [0,1]$, it turns out that, using truncation arguments, $||u_\varepsilon(t)||_\infty \le C_1$ for all t so that a uniform L^∞ bound is available at any time. The requirement $v_\varepsilon(t) = 1$ on $\partial_D\Omega$ for all $t \in [0,1]$ is made in such a way that, letting $\varepsilon \to 0$, the surface energy of the crack in the limit is the usual one also for the part touching the boundary $\partial_D\Omega$. The main result of the chapter (Theorem 6.3.2) is that, as $\varepsilon \to 0$, the quasistatic evolution $t \to (u_\varepsilon(t), v_\varepsilon(t))$ for the Ambrosio-Tortorelli functional converges to a quasistatic evolution for brittle fracture in the sense of [53]. More precisely, there exists a quasistatic evolution $t \to (u(t), \Gamma(t))$, $u(t) \in SBV(\Omega)$, relative to the boundary data g and a sequence $\varepsilon_n \to 0$ such that for all $t \in [0,1]$ which are not discontinuity points of $\mathcal{H}^{N-1}(\Gamma(\cdot))$ we have

$$v_{\varepsilon_n}(t)\nabla u_{\varepsilon_n}(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega, \mathbb{R}^N),$$

$$\int_{\Omega}(\eta_{\varepsilon_n}+v_{\varepsilon_n}(t)^2)|\nabla u_{\varepsilon_n}(t)|^2\,dx \to \int_{\Omega}|\nabla u(t)|^2\,dx,$$

and

$$MM_{\varepsilon_n}(v_{\varepsilon_n}(t)) \to \mathcal{H}^{N-1}(\Gamma(t)).$$

Moreover $\mathcal{E}_{\varepsilon_n}(t) \to \mathcal{E}(t)$ for all $t \in [0,1]$. We thus obtain an approximation of the total energy at any time, and an approximation of the strain, of the bulk and the surface energy at all time up to a countable set. The main step in the proof is to derive the static equilibrium property from its regularized version (6.4). Given $z \in SBV(\Omega)$, a natural way consists in constructing $z_n \in H^1(\Omega)$ and $v_n \in H^1(\Omega)$ with $z_n = g(t)$, $v_n = 1$ on $\partial_D\Omega$, $0 \le v_n \le v_n(t)$ and such that

$$\lim_n \int_{\Omega}(\eta_{\varepsilon_n}+v_n^2)|\nabla z_n|^2\,dx = \int_{\Omega}|\nabla z|^2\,dx, \tag{6.5}$$

and

$$\limsup_n\,[MM_{\varepsilon_n}(v_n) - MM_{\varepsilon_n}(v_{\varepsilon_n}(t))] \le \mathcal{H}^{N-1}\left(S(z)\setminus\Gamma(t)\right). \tag{6.6}$$

We thus need a recovery sequence both for the displacement and the crack: moreover we have to take into account the boundary conditions and the constraint $v_n \le v_n(t)$. Density results on z, such that of considering $S(z)$ polyhedral, cannot be directly applied since the set $S(z)\setminus\Gamma(t)$ could increase too much; on the other hand it is not possible to work in $\Omega\setminus\Gamma(t)$ since no regularity results are available for $\Gamma(t)$ apart from its rectifiability. It turns out that $S(z)\cap\Gamma(t)$ is the part of the crack more difficult to be regularized, and in fact all the problems in the construction of (z_n, v_n) are already present in the particular case $S(z)\subseteq\Gamma(t)$. In order to fix ideas, let us suppose to be in this situation; we solve the problem in two steps. We firstly construct $\tilde{z}_n \in SBV(\Omega)$ with $\nabla\tilde{z}_n \to \nabla z$ strongly in $L^2(\Omega;\mathbb{R}^N)$ and such that $S(\tilde{z}_n)$ is related to $u_n(t)$ and $v_n(t)$ with precise energy estimates: this is done following the ideas of [53, Theorem 2.1], that is using local reflections and gluing along the boundaries of suitable upper levels of $u_n(t)$, but we have to choose the upper levels in a more precise way. In a second time, we regularize $S(\tilde{z}_n)$ using not only $v_n(t)$, which is quite natural, but also $u_n(t)$, so that (6.5) and (6.6) hold.

6.3 The main theorems

Let $\Omega \subseteq \mathbb{R}^N$ be open, bounded and with Lipschitz boundary, and let $\partial_D\Omega \subseteq \partial\Omega$. If $g \in W^{1,1}([0,1];H^1(\Omega))$, we indicate the gradient of g at time t by $\nabla g(t)$, and the time derivative of g at time t by $\dot{g}(t)$.

Concerning the Ambrosio-Tortorelli functional, the following theorem holds.

Theorem 6.3.1. *Let $g \in W^{1,1}([0,1];H^1(\Omega))$. Then for all $\varepsilon > 0$ there exists a strongly measurable map*

$$\begin{array}{ccc}[0,1] & \longrightarrow & H^1(\Omega)\times H^1(\Omega)\\ t & \longmapsto & (u_\varepsilon(t), v_\varepsilon(t))\end{array}$$

such that $0 \le v_\varepsilon(t) \le 1$ in Ω, $u_\varepsilon(t) = g(t)$, $v_\varepsilon(t) = 1$ on $\partial_D\Omega$ for all $t \in [0,1]$, and:

(a) for all $0 \le s \le t \le 1 : v_\varepsilon(t) \le v_\varepsilon(s)$;

(b) for all $(u,v) \in H^1(\Omega)\times H^1(\Omega)$ with $u = g(0)$, $v = 1$ on $\partial_D\Omega$

$$F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) \le F_\varepsilon(u,v);$$

(c) for all $t \in]0,1]$ and for all $(u,v) \in H^1(\Omega)\times H^1(\Omega)$ with $0 \le v \le v_\varepsilon(t)$ on Ω, and $u = g(t)$, $v = 1$ on $\partial_D\Omega$

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \le F_\varepsilon(u,v);$$

(d) *the function* $t \to F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))$ *is absolutely continuous and*

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) = F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + 2\int_0^t \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau))\nabla u_\varepsilon(\tau)\nabla \dot{g}(\tau)\, dx\, d\tau.$$

The convergence result is given by the following theorem.

Theorem 6.3.2. *Let* $g \in W^{1,1}([0,1]; H^1(\Omega))$ *be such that* $\|g(t)\|_\infty \leq C$ *for all* $t \in [0,1]$*, and let* $g_h \in W^{1,1}([0,1]; H^1(\Omega))$ *be a sequence of absolutely continuous functions with* $\|g_h(t)\|_\infty \leq C$*,* $g_h(t) \in C(\overline{\Omega})$ *for all* $t \in [0,1]$ *and such that* $g_h \to g$ *strongly in* $W^{1,1}([0,1]; H^1(\Omega))$*. For all* $\varepsilon > 0$*, let* $t \to (u_{\varepsilon,h}(t), v_{\varepsilon,h}(t))$ *be a quasistatic evolution for the Ambrosio-Tortorelli functional* F_ε *with boundary data* g_h *given by Theorem 6.3.1.*

Then there exists a quasistatic evolution $t \to (u(t), \Gamma(t))$*,* $u(t) \in SBV(\Omega)$*, relative to the boundary data* g *in the sense of Theorem 1.4.2, and two sequences* $\varepsilon_n \to 0$ *and* $h_n \to +\infty$ *such that, setting* $u_n := u_{\varepsilon_n,h_n}$ *and* $v_n := v_{\varepsilon_n,h_n}$*, the following hold:*

(a) *for all* $t \in [0,1]$ *we have*

$$F_{\varepsilon_n}(u_n(t), v_n(t)) \to \mathcal{E}(t);$$

(b) *if* $\mathcal{N}$ *denotes the point of discontinuity of* $\mathcal{H}^{N-1}(\Gamma(\cdot))$*, for all* $t \in [0,1] \setminus \mathcal{N}$ *we have*

$$v_n(t)\nabla u_n(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega; \mathbb{R}^N),$$

$$\lim_n \int_\Omega (\eta_n + v_n^2(t))|\nabla u_n(t)|^2\, dx = \int_\Omega |\nabla u(t)|^2\, dx,$$

and

$$\lim_n \frac{\varepsilon_n}{2}\int_\Omega |\nabla v_n(t)|^2\, dx + \frac{1}{2\varepsilon_n}\int_\Omega (1 - v_n(t))^2\, dx = \mathcal{H}^{N-1}(\Gamma(t)).$$

Theorem 6.3.1 concerning the quasistatic evolution for the Ambrosio-Tortorelli functional is proved in Section 6.4. In Section 6.5 we prove the compactness and approximation result given by Theorem 6.3.2. An important step in the proof is given by Theorem 6.5.6 to which is dedicated the entire Section 6.6.

6.4 Quasi-static evolution for the Ambrosio-Tortorelli functional

This section is devoted to the proof of Theorem 6.3.1 where a suitable notion of quasistatic evolution in a regular context is proposed. The evolution will be obtained through a discretization in time procedure: each step will be performed using a variational argument which will give the minimality property stated in points (b) and (c).

Let $\Omega \subseteq \mathbb{R}^N$ be open, bounded and with Lipschitz boundary, and let $\partial_D\Omega \subseteq \partial\Omega$. Let $g \in W^{1,1}([0,1]; H^1(\Omega))$. Given $\delta > 0$, let N_δ be the largest integer such that $\delta N_\delta \leq 1$; for $i \geq 0$ we set $t_i^\delta = i\delta$ and for $0 \leq i \leq N_\delta$ we set $g_i^\delta = g(t_i^\delta)$. Define u_0^δ and v_0^δ as a minimum for the problem

$$\min\{F_\varepsilon(u,v) \,:\, (u,v) \in H^1(\Omega) \times H^1(\Omega), 0 \leq v \leq 1 \text{ in } \Omega, u = g_0^\delta, v = 1 \text{ on } \partial_D\Omega\}, \tag{6.7}$$

and let $(u_{i+1}^\delta, v_{i+1}^\delta)$ be a minimum for the problem

$$\min\{F_\varepsilon(u,v) \,:\, (u,v) \in H^1(\Omega) \times H^1(\Omega), 0 \leq v \leq v_i^\delta \text{ in } \Omega, u = g_{i+1}^\delta, v = 1 \text{ on } \partial_D\Omega\}. \tag{6.8}$$

Problems (6.7) and (6.8) are well posed: in fact, referring for example to problem (6.8), let (u_n, v_n) be a minimizing sequence. Since $(g_{i+1}^\delta, v_i^\delta)$ is an admissible pair, we obtain that there exists a constant $C > 0$ such that for all n

$$F_\varepsilon(u_n, v_n) \leq C.$$

Since $\varepsilon, \eta_\varepsilon > 0$, we deduce that (u_n, v_n) is bounded in $H^1(\Omega) \times H^1(\Omega)$ so that up to a subsequence $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$. We get immediately that $u = g^\delta_{i+1}$ and $v = 1$ on $\partial_D\Omega$ since $u_n = g^\delta_{i+1}$ and $v_n = 1$ on $\partial_D\Omega$ for all n; on the other hand, since $v_n \to v$ strongly in $L^2(\Omega)$, we obtain that $0 \le v \le v^\delta_i$. By semicontinuity, we have

$$F_\varepsilon(u, v) \le \liminf_n F_\varepsilon(u_n, v_n)$$

so that (u, v) is a minimum point for problem (6.8).

We note that by minimality of the pair $(u^\delta_{i+1}, v^\delta_{i+1})$, we may write

$$\begin{aligned}(6.9)\quad & F_\varepsilon(u^\delta_{i+1}, v^\delta_{i+1}) \le F_\varepsilon(u^\delta_i + g^\delta_{i+1} - g^\delta_i, v^\delta_i) = \\ &= F_\varepsilon(u^\delta_i, v^\delta_i) + 2\int_\Omega (\eta_\varepsilon + (v^\delta_i)^2)\nabla u^\delta_i \nabla(g^\delta_{i+1} - g^\delta_i)\,dx + \int_\Omega (\eta_\varepsilon + (v^\delta_i)^2)|\nabla(g^\delta_{i+1} - g^\delta_i)|^2\,dx \le \\ &\le F_\varepsilon(u^\delta_i, v^\delta_i) + 2\int_{t^\delta_i}^{t^\delta_{i+1}}\int_\Omega (\eta_\varepsilon + (v^\delta_i)^2)\nabla u^\delta_i \nabla \dot g(\tau)\,dx\,d\tau + e(\delta)\int_{t^\delta_i}^{t^\delta_{i+1}} \|\nabla \dot g(\tau)\|_{L^2(\Omega;\mathbb{R}^N)}\,d\tau,\end{aligned}$$

where

$$e(\delta) := (1 + \eta_\varepsilon)\max_{0 \le r \le N_\delta - 1}\int_{t^\delta_r}^{t^\delta_{r+1}} \|\nabla \dot g(\tau)\|_{L^2(\Omega;\mathbb{R}^N)}\,d\tau$$

is infinitesimal as $\delta \to 0$.

We now make a piecewise constant interpolation defining

$$(6.10)\qquad u^\delta_\varepsilon(t) = u^\delta_i, \quad v^\delta_\varepsilon(t) = v^\delta_i, \quad g^\delta(t) = g^\delta_i \quad \text{for } t^\delta_i \le t < t^\delta_{i+1}.$$

Note that by construction the map $t \to v^\delta_\varepsilon(t)$ is decreasing from $[0, 1]$ to $L^2(\Omega)$. Moreover, iterating the estimate (6.9), we obtain

$$\begin{aligned} F_\varepsilon(u^\delta_\varepsilon(t), v^\delta_\varepsilon(t)) \;\le\; & F_\varepsilon(u^\delta_\varepsilon(s), v^\delta_\varepsilon(s)) + 2\int_{s^\delta}^{t^\delta}\int_\Omega (\eta_\varepsilon + v^\delta_\varepsilon(\tau)^2)\nabla u^\delta_\varepsilon(\tau)\nabla \dot g(\tau)\,dx\,d\tau + \\ (6.11)\qquad & + e(\delta)\int_{s^\delta}^{t^\delta} \|\nabla \dot g(\tau)\|_{L^2(\Omega;\mathbb{R}^N)}\,d\tau \end{aligned}$$

where $s^\delta := t^\delta_i$ and $t^\delta := t^\delta_j$ are the step discretization points such that $t^\delta_i \le s < t^\delta_{i+1}$ and $t^\delta_j \le t < t^\delta_{j+1}$.

Note that by minimality of the pair $(u^\delta_\varepsilon(t), v^\delta_\varepsilon(t))$, we have

$$F_\varepsilon(u^\delta_\varepsilon(t), v^\delta_\varepsilon(t)) \le F_\varepsilon(g^\delta(t), v^\delta_\varepsilon(t))$$

so that

$$(6.12)\qquad \int_\Omega (\eta_\varepsilon + v^\delta_\varepsilon(t)^2)|\nabla u^\delta_\varepsilon(t)|^2\,dx \le \int_\Omega (\eta_\varepsilon + v^\delta_\varepsilon(t)^2)|\nabla g^\delta(t)|^2\,dx \le C_1$$

with $C_1 > 0$ independent of δ and t. In particular by (6.12) we have that

$$\|\nabla u^\delta_\varepsilon(t)\|^2_{L^2(\Omega;\mathbb{R}^N)} \le \frac{C_1}{\eta_\varepsilon}.$$

Since $u^\delta_\varepsilon(t) = g^\delta(t)$ on $\partial_D\Omega$, and $g^\delta(t)$ is uniformly bounded in $H^1(\Omega)$ for all t and δ, we get by a variant of Poincaré inequality that $u^\delta_\varepsilon(t)$ is uniformly bounded in $H^1(\Omega)$ for all t and δ.

Now we come to v^δ_ε in order to obtain some coerciveness in the space $H^1(\Omega)$. Notice that

$$\begin{aligned} 2\quad & \left|\int_0^{t^\delta}\int_\Omega (\eta_\varepsilon + v^\delta_\varepsilon(\tau)^2)\nabla u^\delta_\varepsilon(\tau)\nabla \dot g(\tau)\,dx\,d\tau\right| \le \\ \le\quad & 2\int_0^{t^\delta} \sqrt{\eta_\varepsilon + 1}\left(\int_\Omega (\eta_\varepsilon + v^\delta_\varepsilon(t)^2)|\nabla u^\delta_\varepsilon(t)|^2\,dx\right)^{\frac12} \|\nabla \dot g(\tau)\|_{L^2(\Omega;\mathbb{R}^N)}\,d\tau, \end{aligned}$$

and by (6.12), we obtain

$$\left| 2\int_0^{t^\delta} \int_\Omega (\eta_\varepsilon + v_\varepsilon^\delta(\tau)^2)\nabla u_\varepsilon^\delta(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau \right| \le C_2 \tag{6.13}$$

with $C_2 > 0$ independent of t and δ.

By (6.11) with $s = 0$, and (6.13), we deduce

$$\begin{aligned} \frac{\varepsilon}{2} & \int_\Omega |\nabla v_\varepsilon^\delta(t)|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1 - v_\varepsilon^\delta(t))^2\,dx \le \\ \le & \; F_\varepsilon(u_\varepsilon^\delta(0), v_\varepsilon^\delta(0)) + 2\int_0^{t^\delta}\int_\Omega (\eta_\varepsilon + v_\varepsilon^\delta(\tau)^2)\nabla u_\varepsilon^\delta(\tau)\nabla\dot{g}(\tau)\,dx\,d\tau + \\ & + e(\delta)\int_{s^\delta}^{t^\delta} ||\nabla \dot{g}(\tau)||_{L^2(\Omega;\mathbb{R}^N)}\,d\tau \le \\ \le & \; F_\varepsilon(u_\varepsilon^\delta(0), v_\varepsilon^\delta(0)) + C_2 + e(\delta)\int_0^1 ||\nabla\dot{g}(\tau)||_{L^2(\Omega}\,d\tau. \end{aligned}$$

We conclude that there exists $C > 0$ independent of t and δ such that for all $t \in [0,1]$

$$||v_\varepsilon^\delta(t)||_{H^1(\Omega)} \le C. \tag{6.14}$$

We now want to pass to the limit in δ as $\delta \to 0$.

Lemma 6.4.1. *There exists a sequence $\delta_n \to 0$ and a strongly measurable map $v_\varepsilon : [0,1] \to H^1(\Omega)$ such that $v_\varepsilon^{\delta_n}(t) \rightharpoonup v_\varepsilon(t)$ weakly in $H^1(\Omega)$ for all $t \in [0,1]$. Moreover, v_ε is decreasing from $[0,1]$ to $L^2(\Omega)$, and $0 \le v_\varepsilon(t) \le 1$ in Ω, $v_\varepsilon(t) = 1$ on $\partial_D\Omega$ for all $t \in [0,1]$.*

Proof. Since the map $t \to v_\varepsilon^\delta(t)$ is monotone decreasing from $[0,1]$ to $L^2(\Omega)$, and $0 \le v_\varepsilon^\delta(t) \le 1$ for all t, we deduce by a variant of Helly's compactness theorem for sequences of monotone real functions, that there exists a subsequence $\delta_n \to 0$ and a decreasing map $v_\varepsilon : [0,1] \to L^2(\Omega)$ such that for all $t \in [0,1]$ we have $v_\varepsilon^{\delta_n}(t) \to v_\varepsilon(t)$ strongly in $L^2(\Omega)$. In particular we deduce $0 \le v_\varepsilon(t) \le 1$ in Ω. By (6.14), we have that for all $t \in [0,1]$, up to a subsequence, $v_\varepsilon^{\delta_n}(t) \rightharpoonup w$ weakly in $H^1(\Omega)$; since $v_\varepsilon^{\delta_n}(t) \to v_\varepsilon(t)$ strongly in $L^2(\Omega)$, we deduce that $w = v_\varepsilon(t)$ so that $v_\varepsilon(t) \in H^1(\Omega)$, and $v_\varepsilon^{\delta_n}(t) \rightharpoonup v_\varepsilon(t)$ weakly in $H^1(\Omega)$. As a consequence, $v_\varepsilon(t) = 1$ on $\partial_D\Omega$ for all $t \in [0,1]$. Finally, v_ε is strongly measurable from $[0,1]$ to $H^1(\Omega)$ because it is weakly measurable and separably valued (see [86, Chapter V, Section 4]). □

Let us consider the sequence δ_n, and the map v_ε given by Lemma 6.4.1. We indicate $u_\varepsilon^{\delta_n}$, $v_\varepsilon^{\delta_n}$ and g^{δ_n} simply by u_ε^n, v_ε^n and g_n.

Lemma 6.4.2. *There exists a strongly measurable map $u_\varepsilon : [0,1] \to H^1(\Omega)$ such that $u_\varepsilon^n(t) \to u_\varepsilon(t)$ strongly in $H^1(\Omega)$ for all $t \in [0,1]$. In particular, $u_\varepsilon(t) = g(t)$ on $\partial_D\Omega$ for all $t \in [0,1]$.*

Proof. Let $t \in [0,1]$. We note that $u_\varepsilon^n(t)$ is the minimum of the following problem

$$\min\left\{ \int_\Omega (\eta_\varepsilon + v_\varepsilon^n(t)^2)|\nabla z|^2\,dx \;:\; z \in H^1(\Omega), z = g_n(t) \text{ on } \partial_D\Omega \right\}.$$

Since by Lemma 6.4.1 $v_\varepsilon^n(t) \to v_\varepsilon(t)$ strongly in $L^2(\Omega)$, and $g_n(t) \to g(t)$ strongly in $H^1(\Omega)$, we deduce by standard results on Γ-convergence (see [43]), that $u_\varepsilon^n(t) \rightharpoonup u_\varepsilon(t)$ weakly in $H^1(\Omega)$ where $u_\varepsilon(t)$ is the solution of the problem

$$\min\left\{ \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(t))|\nabla z|^2\,dx \;:\; z \in H^1(\Omega), z = g(t) \text{ on } \partial_D\Omega \right\}.$$

Moreover, we have also convergence of energies, that is

$$\lim_n \int_\Omega (\eta_\varepsilon + v_\varepsilon^n(t)^2)|\nabla u_\varepsilon^n(t)|^2\,dx = \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(t))|\nabla u_\varepsilon(t)|^2\,dx. \tag{6.15}$$

Since $v_\varepsilon^n(t)\nabla u_\varepsilon^n(t) \rightharpoonup v_\varepsilon(t)\nabla u_\varepsilon(t)$ weakly in $L^2(\Omega;\mathbb{R}^N)$, we obtain

$$\int_\Omega v_\varepsilon^2(t)|\nabla u_\varepsilon(t)|^2\,dx \le \liminf_n \int_\Omega v_\varepsilon^n(t)^2|\nabla u_n(t)|^2\,dx,$$

so that by (6.15) we deduce $\nabla u_\varepsilon^n(t) \to \nabla u_\varepsilon(t)$ strongly in $L^2(\Omega;\mathbb{R}^N)$. We conclude that $u_\varepsilon^n(t) \to u_\varepsilon(t)$ strongly in $H^1(\Omega)$ for all $t \in [0,1]$, and so the map $t \to u_\varepsilon(t)$ is strongly measurable from $[0,1]$ to $H^1(\Omega)$. Finally $u_\varepsilon(t) = g(t)$ on $\partial_D\Omega$ and the proof is complete. □

The following minimality property for the pair $(u_\varepsilon(t), v_\varepsilon(t))$ holds.

Proposition 6.4.3. *If $t \in]0,1]$, for every $(u,v) \in H^1(\Omega)\times H^1(\Omega)$ such that $0 \le v \le v_\varepsilon(t)$ in Ω, and $u = g(t)$, $v = 1$ on $\partial_D\Omega$, we have*

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \le F_\varepsilon(u,v).$$

Moreover, for all $(u,v) \in H^1(\Omega)\times H^1(\Omega)$ such that $u = g(0)$, $v = 1$ on $\partial_D\Omega$, we have

$$F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) \le F_\varepsilon(u,v).$$

Proof. Let us set

$$u_n := u + g_n(t) - g(t),$$

and

$$v_n := \min\{v_\varepsilon^n(t), v\};$$

we have $u_n \to u$ strongly in $H^1(\Omega)$, and $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$. Since $0 \le v_n \le v_\varepsilon^n(t)$ in Ω, and $u_n = g_n(t)$, $v_n = 1$ on $\partial_D\Omega$, by the minimality property of the pair $(u_\varepsilon^n(t), v_\varepsilon^n(t))$ we get

$$F_\varepsilon(u_\varepsilon^n(t), v_\varepsilon^n(t)) \le F_\varepsilon(u_n, v_n),$$

that is

$$\begin{aligned}\int_\Omega (\eta_\varepsilon + v_\varepsilon^n(t)^2)|\nabla u_\varepsilon^n(t)|^2\,dx + \frac{\varepsilon}{2}\int_\Omega |\nabla v_\varepsilon^n(t)|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1 - v_\varepsilon^n(t))^2\,dx \le \\ \le \int_\Omega (\eta_\varepsilon + {v_n}^2)|\nabla u_n|^2\,dx + \frac{\varepsilon}{2}\int_\Omega |\nabla v_n|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1-v_n)^2\,dx.\end{aligned} \tag{6.16}$$

Notice that

$$\frac{\varepsilon}{2}\int_\Omega |\nabla v_n|^2\,dx = \frac{\varepsilon}{2}\int_{\{v_\varepsilon^n(t)<v\}} |\nabla v_\varepsilon^n(t)|^2\,dx + \frac{\varepsilon}{2}\int_{\{v_\varepsilon^n(t)\ge v\}} |\nabla v|^2\,dx$$

so that (6.16) becomes

$$\begin{aligned}\int_\Omega (\eta_\varepsilon + v_\varepsilon^n(t)^2)|\nabla u_\varepsilon^n(t)|^2\,dx + \frac{\varepsilon}{2}\int_{\{v_\varepsilon^n(t)\ge v\}} |\nabla v_\varepsilon^n(t)|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1 - v_\varepsilon^n(t))^2\,dx \le \\ \le \int_\Omega (\eta_\varepsilon + {v_n}^2)|\nabla u_n|^2\,dx + \frac{\varepsilon}{2}\int_{\{v_\varepsilon^n(t)\ge v\}} |\nabla v|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1-v_n)^2\,dx.\end{aligned}$$

For $n \to \infty$, the right hand side is less than $F_\varepsilon(u,v)$. Let us consider the left hand side. By semicontinuity we have

$$\liminf_n \frac{\varepsilon}{2}\int_{\{v_\varepsilon^n(t)\ge v\}} |\nabla v_\varepsilon^n(t)|^2\,dx \ge \frac{\varepsilon}{2}\int_\Omega |\nabla v_\varepsilon(t)|^2\,dx,$$

and so we conclude that $F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \le F_\varepsilon(u,v)$.

For the case $t = 0$, by lower semicontinuity we get immediately the result. □

In order to obtain the proof of Theorem 6.3.1, we need the following proposition.

Proposition 6.4.4. *For all $0 \leq s \leq t \leq 1$, we have that*

$$\begin{aligned} F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) - F_\varepsilon(u_\varepsilon(s), v_\varepsilon(s)) \quad \geq \quad & 2\int_\Omega (\eta_\varepsilon + v_\varepsilon^2(t))\nabla u_\varepsilon(t)(\nabla g(t) - \nabla g(s))\,dx + \\ & -\sigma(t-s)\int_s^t ||\nabla \dot{g}(\tau)||_{L^2(\Omega;\mathbb{R}^N)}\,d\tau \end{aligned}$$

where σ is an increasing positive function with $\sigma(r) \to 0$ as $r \to 0^+$.

Proof. By Proposition 6.4.3, we have

$$F_\varepsilon(u_\varepsilon(s), v_\varepsilon(s)) \leq F_\varepsilon(u_\varepsilon(t) - g(t) + g(s), v_\varepsilon(t))$$

so that

$$\begin{aligned} F_\varepsilon(u_\varepsilon(s), v_\varepsilon(s)) \quad \leq \quad & F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) - 2\int_\Omega (\eta_\varepsilon + v_\varepsilon^2(t))\nabla u_\varepsilon(t)(\nabla g(t) - \nabla g(s))\,dx + \\ & + \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(t))|\nabla g(t) - \nabla g(s)|^2\,dx. \end{aligned}$$

Then we conclude that

$$\begin{aligned} F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) - F_\varepsilon(u_\varepsilon(s), v_\varepsilon(s)) \quad \geq \quad & 2\int_\Omega (\eta_\varepsilon + v_\varepsilon^2(t))\nabla u_\varepsilon(t)(\nabla g(t) - \nabla g(s))\,dx + \\ & -\sigma(t-s)\int_s^t ||\nabla \dot{g}(\tau)||_{L^2}\,d\tau \end{aligned}$$

where

$$\sigma(r) := (1 + \eta_\varepsilon)\max_{t-s=r}\int_s^t ||\nabla \dot{g}(\tau)||_{L^2(\Omega;\mathbb{R}^N)}\,d\tau,$$

and so the proof is complete. □

We can now prove Theorem 6.3.1.

Proof of Theorem 6.3.1. Let us consider the sequence $\delta_n \to 0$ given by Lemma 6.4.1, and let us indicate the discrete evolutions $u_\varepsilon^{\delta_n}$ and $v_\varepsilon^{\delta_n}$ defined in (6.10) simply by u_ε^n and v_ε^n. Let us denote also by $u_\varepsilon(t)$ and $v_\varepsilon(t)$ their limits at time t according to Lemma 6.4.1 and Lemma 6.4.2. We have that the maps $t \to u_\varepsilon(t)$ and $t \to v_\varepsilon(t)$ are strongly measurable from $[0,1]$ to $H^1(\Omega)$; moreover for all $t \in [0,1]$ we have $0 \leq v_\varepsilon(t) \leq 1$ in Ω, $u_\varepsilon(t) = g(t)$, $v_\varepsilon(t) = 1$ on $\partial_D\Omega$ and $t \to v_\varepsilon(t)$ is decreasing from $[0,1]$ to $L^2(\Omega)$ so that point (a) is proved. By construction we get point (b) and by Proposition 6.4.3 we get point (c).

Let us come to condition (d). Let us fix $t \in [0,1]$, and let us divide the interval $[0,t]$ in k subintervals with endpoints $s_j^k := \frac{jt}{k}$ where $j = 0, 1, \cdots, k$. Let us define $\tilde{u}_k(s) := u_\varepsilon(s_{j+1}^k)$, and $\tilde{v}_k(s) := v_\varepsilon(s_{j+1}^k)$ for $s_j^k < s \leq s_{j+1}^k$. Then, applying Proposition 6.4.4, we have

$$\begin{aligned} F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \quad \geq \quad & F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + 2\int_0^t \int_\Omega (\eta_\varepsilon + \tilde{v}_k^2(\tau))\nabla \tilde{u}_k(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau + \\ & -\sigma\left(\frac{t}{k}\right)\int_0^t ||\nabla \dot{g}(\tau)||_{L^2}\,d\tau. \end{aligned} \tag{6.17}$$

Since $t \to v_\varepsilon(t)$ is monotone decreasing from $[0,1]$ to $L^2(\Omega)$, we have that $\tilde{v}_k(s) \to v_\varepsilon(s)$ strongly in $L^2(\Omega)$ for a.e. $s \in [0,t]$; consequently, we have that $\tilde{u}_k(s) \to u_\varepsilon(s)$ strongly in $H^1(\Omega)$ as noted in Lemma 6.4.2. We conclude by the Dominated Convergence Theorem that

$$\lim_k \int_0^t \int_\Omega (\eta_\varepsilon + \tilde{v}_k^2(\tau))\nabla \tilde{u}_k(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau = \int_0^t \int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau))\nabla u_\varepsilon(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau.$$

By (6.17) we deduce that

$$(6.18)\qquad F_\varepsilon(u_\varepsilon(t),v_\varepsilon(t)) \geq F_\varepsilon(u_\varepsilon(0),v_\varepsilon(0)) + 2\int_0^t\int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau))\nabla u_\varepsilon(\tau)\nabla\dot{g}(\tau)\,dx\,d\tau.$$

On the other hand, from (6.11), and since $F_\varepsilon(u_\varepsilon^n(0),v_\varepsilon^n(0)) = F_\varepsilon(u_\varepsilon(0),v_\varepsilon(0))$ for all n, we deduce

$$(6.19)\qquad \limsup_n F_\varepsilon(u_\varepsilon^n(t),v_\varepsilon^n(t)) \leq F_\varepsilon(u_\varepsilon(0),v_\varepsilon(0)) + 2\int_0^t\int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau))\nabla u_\varepsilon(\tau)\nabla\dot{g}(\tau)\,dx\,d\tau.$$

Since by semicontinuity we have for all $t \in [0,1]$

$$F_\varepsilon(u_\varepsilon(t),v_\varepsilon(t)) \leq \liminf_n F_\varepsilon(u_\varepsilon^n(t),v_\varepsilon^n(t)),$$

by (6.18) and (6.19), we conclude that

$$(6.20)\qquad \lim_n F_\varepsilon(u_\varepsilon^n(t),v_\varepsilon^n(t)) = F_\varepsilon(u_\varepsilon(t),v_\varepsilon(t)).$$

In particular

$$F_\varepsilon(u_\varepsilon(t),v_\varepsilon(t)) = F_\varepsilon(u_\varepsilon(0),v_\varepsilon(0)) + 2\int_0^t\int_\Omega (\eta_\varepsilon + v_\varepsilon^2(\tau))\nabla u_\varepsilon(\tau)\nabla\dot{g}(\tau)\,dx\,d\tau,$$

and this proves point (d). □

Remark 6.4.5. The map $\{t \to v_\varepsilon(t),\ t \in [0,1]\}$ is decreasing from $[0,1]$ to $L^2(\Omega)$, so that v_ε is continuous with respect to the strong topology of $L^2(\Omega)$ at all points except a countable set. Since

$$\lambda(t) := \frac{\varepsilon}{2}\int_\Omega |\nabla v_\varepsilon(t)|^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (1-v_\varepsilon(t))^2\,dx$$

is monotone increasing (see Proposition 6.5.8), we conclude that $v_\varepsilon : [0,1] \to H^1(\Omega)$ is continuous with respect to the strong topology at all points except a countable set. Then we have $v_\varepsilon \in L^\infty([0,1], H^1(\Omega))$. Moreover, we have that $u_\varepsilon : [0,1] \to H^1(\Omega)$ is continuous at the continuity points of v_ε as observed in Lemma 6.4.2. We conclude that $u_\varepsilon \in L^\infty([0,1], H^1(\Omega))$.

Remark 6.4.6. The minimality property of point (c) of Theorem 6.3.1 holds indeed in this stronger form: if $t \in]0,1]$, for all $(u,v) \in H^1(\Omega) \times H^1(\Omega)$ with $0 \leq v \leq v_\varepsilon(s)$ on Ω for all $s < t$, and $u = g(t)$, $v = 1$ on $\partial_D\Omega$, we have

$$F_\varepsilon(u_\varepsilon(t),v_\varepsilon(t)) \leq F_\varepsilon(u,v).$$

In fact, if $0 \leq v \leq v_\varepsilon(s)$, by the minimality property of $(u_\varepsilon(s),v_\varepsilon(s))$ we have

$$F_\varepsilon(u_\varepsilon(s),v_\varepsilon(s)) \leq F_\varepsilon(u + g(s) - g(t), v),$$

so that, letting $s \to t$ and using the continuity of $F_\varepsilon(u_\varepsilon(\cdot),v_\varepsilon(\cdot))$ we get the result.

This stronger minimality property is the reformulation in the context of the Ambrosio-Tortorelli functional of the minimality of the cracks required in [54].

6.5 Quasi-static growth of brittle fracture

In this section, we prove that the evolution for the Ambrosio-Tortorelli functional F_ε converges as $\varepsilon \to 0$ to a quasistatic evolution of brittle cracks in linearly elastic bodies in the sense of [53].

Let $\Omega \subseteq \mathbb{R}^N$ be open, bounded and with Lipschitz boundary. Let $\partial_D\Omega \subseteq \partial\Omega$, and let us set $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. Let $g \in W^{1,1}([0,1]; H^1(\Omega))$. In order to treat in a convenient way the boundary condition as $\varepsilon \to 0$, let B be an open ball such that $\overline{\Omega} \subset B$, and let us set $\Omega' := B \setminus \partial_N\Omega$ and $\Omega_D := \Omega' \setminus \overline{\Omega}$. Let E be an extension operator from $H^1(\Omega)$ to $H^1(B)$: we indicate $Eg(t)$ still by $g(t)$ for all $t \in [0,1]$. In this enlarged context, the following proposition holds.

Proposition 6.5.1. *Let us consider the evolution $t \to (u_\varepsilon(t), v_\varepsilon(t))$ from $[0,1]$ to $H^1(\Omega) \times H^1(\Omega)$ given by Theorem 6.3.1, and let us extend $u_\varepsilon(t)$ and $v_\varepsilon(t)$ to Ω' setting $u_\varepsilon(t) = g(t)$ and $v_\varepsilon(t) = 1$ on Ω_D respectively. Then the map*

$$\begin{array}{ccc} [0,1] & \longrightarrow & H^1(\Omega') \times H^1(\Omega') \\ t & \longmapsto & (u_\varepsilon(t), v_\varepsilon(t)) \end{array}$$

is strongly measurable and the following facts hold:

(a) for all $0 \le s \le t \le 1 : v_\varepsilon(t) \le v_\varepsilon(s)$;

(b) for all $(u,v) \in H^1(\Omega') \times H^1(\Omega')$ with $u = g(0)$, $v = 1$ on Ω_D:

$$F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) \le F_\varepsilon(u,v); \tag{6.21}$$

(c) for $t \in]0,1]$ and for all $(u,v) \in H^1(\Omega') \times H^1(\Omega')$ with $0 \le v \le v_\varepsilon(t)$ on Ω', and $u = g(t)$, $v = 1$ on Ω_D:

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \le F_\varepsilon(u,v); \tag{6.22}$$

(d) the function $t \to F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))$ is absolutely continuous and

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) = F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + 2\int_0^t \int_{\Omega'} (\eta_\varepsilon + v_\varepsilon^2(\tau)) \nabla u_\varepsilon(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau. \tag{6.23}$$

Proof. Recall that for all $t \in [0,1]$ we have $u_\varepsilon(t) = g(t)$, $v_\varepsilon(t) = 1$ on $\partial_D\Omega$, and $0 \le v_\varepsilon(t) \le 1$ in Ω. The extensions to $H^1(\Omega')$ are thus well defined. We obtain a strongly measurable map $t \to (u_\varepsilon(t), v_\varepsilon(t))$ from $[0,1]$ to $H^1(\Omega') \times H^1(\Omega')$ such that $0 \le v_\varepsilon(t) \le 1$ in Ω', $u_\varepsilon(t) = g(t)$, $v_\varepsilon(t) = 1$ on Ω_D, and such that

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \le F_\varepsilon(u,v)$$

for all $(u,v) \in H^1(\Omega') \times H^1(\Omega')$ with $0 \le v \le v_\varepsilon(t)$ on Ω', $u = g(t)$, $v = 1$ on Ω_D; note in fact that the integrations on Ω_D which appear in both sides are the same. By the same reason, we get the minimality property at time $t = 0$ and deduce that the function $t \to F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))$ is absolutely continuous with

$$F_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) = F_\varepsilon(u_\varepsilon(0), v_\varepsilon(0)) + 2\int_0^t \int_{\Omega'} (\eta_\varepsilon + v_\varepsilon^2(\tau)) \nabla u_\varepsilon(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau.$$

□

From now on, we assume that there exists a constant $C > 0$ such that for all $t \in [0,1]$, $\|g(t)\|_\infty \le C$, and that there exists $g_h \in W^{1,1}([0,1], H^1(\Omega'))$ such that $\|g_h\|_\infty \le C$, $g_h \in C(\overline{\Omega'})$, and $g_h \to g$ strongly in $W^{1,1}([0,1], H^1(\Omega'))$. For every $\varepsilon > 0$ we indicate by $(u_{\varepsilon,h}, v_{\varepsilon,h})$ the evolution for the Ambrosio-Tortorelli functional relative to the boundary data g_h given by Proposition 6.5.1. The bound on the sup-norm is made in order to apply Ambrosio's compactness theorem in SBV when $\varepsilon \to 0$. Notice that we may assume by a truncation argument that $||u_{\varepsilon,h}(t)||_\infty \le ||g_h(t)||_\infty$, that is

$$||u_{\varepsilon,h}(t)||_\infty \le C. \tag{6.24}$$

We conclude that $u_{\varepsilon,h}(t)$ is uniformly bounded in $L^\infty(\Omega')$ as ε, h and t vary. Moreover we have that the following holds.

Lemma 6.5.2. *There exists a constant $C_1 \ge 0$ depending only on g such that for all $t \in [0,1]$, ε, h*

$$F_\varepsilon(u_{\varepsilon,h}(t), v_{\varepsilon,h}(t)) + \|u_{\varepsilon,h}(t)\|_\infty \le C_1. \tag{6.25}$$

Proof. Notice that $F_\varepsilon(u_{\varepsilon,h}(0), v_{\varepsilon,h}(0)) \leq F_\varepsilon(g_h(0), 1)$ so that the term $F_\varepsilon(u_{\varepsilon,h}(0), v_{\varepsilon,h}(0))$ is bounded as ε and h vary. We now derive an estimate for the derivative of the total energy. Since $0 \leq v_{\varepsilon,h}(\tau) \leq 1$ and $\eta_\varepsilon \to 0$, by Hölder inequality we get

$$\left| \int_{\Omega'} (\eta_\varepsilon + v_{\varepsilon,h}(\tau)^2) \nabla u_{\varepsilon,h}(\tau) \nabla \dot{g}_h(\tau) \, dx \right| \leq$$

$$\leq 2 \left(\int_{\Omega'} (\eta_\varepsilon + v_{\varepsilon,h}(\tau)^2) |\nabla u_{\varepsilon,h}(\tau)|^2 \, dx \right)^{\frac{1}{2}} ||\nabla \dot{g}_h(\tau)||_{L^2(\Omega';\mathbb{R}^N)};$$

since by the minimality property (6.22)

$$\int_{\Omega'} (\eta_\varepsilon + v_{\varepsilon,h}(\tau)^2) |\nabla u_{\varepsilon,h}(\tau)|^2 \, dx \leq \int_{\Omega'} (\eta_\varepsilon + v_{\varepsilon,h}(\tau)^2) |\nabla g_h(\tau)|^2 \, dx,$$

we get the conclusion by (6.23) and (6.24). □

As a consequence of (6.25), we have

$$\int_{\Omega'} (1 - v_{\varepsilon,h}(t)) |\nabla v_{\varepsilon,h}(t)| \, dx \leq \frac{\varepsilon}{2} \int_\Omega |\nabla v_{\varepsilon,h}(t)|^2 \, dx + \frac{1}{2\varepsilon} \int_\Omega (1 - v_{\varepsilon,h}(t))^2 \, dx \leq C_1,$$

so that the functions $w_{\varepsilon,h}(t) := (1 - v_{\varepsilon,h}(t))^2$ have uniformly bounded variation.

By coarea formula for BV-functions (see [8, Theorem 3.40]), we have that

$$\int_0^1 \mathcal{H}^{N-1} (\partial^* \{v_{\varepsilon,h}(t) > s\}) \, ds = \int_{\Omega'} (1 - v_{\varepsilon,h}(t)) |\nabla v_{\varepsilon,h}(t)| \, dx$$

(∂^* denotes the essential boundary) so that by the Mean Value theorem, for all $j \geq 1$ there exists $b^j_{\varepsilon,h}(t) \in [\frac{1}{2^{j+1}}, \frac{1}{2^j}]$ with

$$\frac{1}{2^{j+1}} \mathcal{H}^{N-1} \left(\partial^* \{v_{\varepsilon,h}(t) > b^j_{\varepsilon,h}(t)\} \right) \leq C_1. \tag{6.26}$$

Let us set

$$B_{\varepsilon,h}(t) := \left\{ b^j_{\varepsilon,h}(t) \, : \, j \geq 1 \right\}. \tag{6.27}$$

We now let $\varepsilon \to 0$. Let D be countable and dense in $[0,1]$ with $0 \in D$.

Lemma 6.5.3. *There exists a sequence ε_n such that for all $t \in D$ there exists $u_h(t) \in SBV(\Omega')$, $u_h(t) = g_h(t)$ on Ω_D, with*

$$u_{\varepsilon_n,h}(t) \mathbb{1}_{\{v_{\varepsilon_n,h}(t) > b^1_{\varepsilon_n,h}(t)\}} \to u_h(t) \quad \text{in } SBV(\Omega').$$

In particular for all $t \in D$ we have

$$\int_{\Omega'} |\nabla u_h(t)|^2 \, dx + \mathcal{H}^{N-1}(S(u_h(t))) + \|u_h(t)\|_\infty \leq C_1. \tag{6.28}$$

Proof. For all $t \in [0,1]$ we may apply Ambrosio's compactness Theorem 1.1.1 to the function $z_n(t) := u_{\varepsilon_n,h}(t) \mathbb{1}_{\{v_{\varepsilon_n,h}(t) > b^1_{\varepsilon_n,h}(t)\}}$: in fact $z_n(t)$ is bounded in $L^\infty(\Omega')$ and $\nabla z_n(t)$ is bounded in $L^2(\Omega')$ by (6.25), and $S(z_n(t)) \subseteq \partial_* \{v_{\varepsilon_n,h}(t) > b^1_{\varepsilon_n,h}(t)\}$ so that $\mathcal{H}^{N-1}(S(z_n(t)))$ is uniformly bounded in n by (6.26). Using a diagonal argument, there exists a subsequence such that for all $t \in D$, $z_n(t) \to u_h(t)$ in $SBV(\Omega')$; in particular, we have that $u_h(t) = g_h(t)$ on Ω_D, and by (6.25) and the Γ-liminf inequality for the Ambrosio-Tortorelli functional (6.3), we get (6.28). □

The following lemma deals with the possibility of truncating at other levels given by the elements of $B_{\varepsilon_n,h}(t)$.

Lemma 6.5.4. *Let $t \in D$ and $j \geq 1$. For every $b^j_{\varepsilon_n,h}(t) \in B_{\varepsilon_n,h}(t)$ we have that*

$$u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} \to u_h(t) \quad \text{in } SBV(\Omega').$$

Proof. Note that, up to a subsequence, $u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} \to z$ in $SBV(\Omega')$ because of Ambrosio's Theorem 1.1.1. By (6.24), we have that

$$\begin{aligned}\|u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} - u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^1_{\varepsilon_n,h}(t)\}}\|_{L^2(\Omega')} \leq \\ \leq C\left|\left\{b^j_{\varepsilon_n,h}(t) \leq v_{\varepsilon_n,h}(t) \leq b^1_{\varepsilon_n,h}(t)\right\}\right|.\end{aligned}$$

Since $v_{\varepsilon_n,h}(t) \to 1$ strongly in $L^2(\Omega')$, we conclude that

$$\left|\left\{b^j_{\varepsilon_n,h}(t) \leq v_{\varepsilon_n,h}(t) \leq b^1_{\varepsilon_n,h}(t)\right\}\right| \to 0,$$

so that

$$\begin{aligned}||z - u_h(t)||_{L^2(\Omega')} = \\ = \lim_n ||u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} - u_{\varepsilon_n,h}(t)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^1_{\varepsilon_n,h}(t)\}}||_{L^2(\Omega')} = 0,\end{aligned}$$

that is $z = u_h(t)$ and the proof is complete. □

The following lemma deals with the possibility of truncating at time s using the function $v_{\varepsilon_n,h}(t)$ for $t \geq s$.

Lemma 6.5.5. *Let $s, t \in D$ with $s \leq t$, and $j \geq 1$. Then for every $b^j_{\varepsilon_n,h}(t) \in B_{\varepsilon_n,h}(t)$ we have that*

$$u_{\varepsilon_n,h}(s)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} \to u_h(s) \quad \text{in } SBV(\Omega').$$

Proof. Up to a subsequence, by Ambrosio's Theorem, we have that

$$u_{\varepsilon_n,h}(s)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} \to z \quad \text{in } SBV(\Omega').$$

Since $v_{\varepsilon_n,h}(t) \leq v_{\varepsilon_n,h}(s)$, we have that $\{v_{\varepsilon_n,h}(t) > b^j_{\varepsilon_n,h}(t)\} \subseteq \{v_{\varepsilon_n,h}(s) > b^{j+1}_{\varepsilon_n,h}(s)\}$. Then we have

$$\|u_{\varepsilon_n,h}(s)\mathbb{1}_{\{v_{\varepsilon_n,h}(t)>b^j_{\varepsilon_n,h}(t)\}} - u_{\varepsilon_n,h}(s)\mathbb{1}_{\{v_{\varepsilon_n,h}(s)>b^{j+1}_{\varepsilon_n,h}(s)\}}\|_{L^2(\Omega')} \leq C\left|\left\{v_{\varepsilon_n,h}(t) \leq b^j_{\varepsilon_n,h}(t)\right\}\right|.$$

Since $v_{\varepsilon_n,h}(t) \to 1$ strongly in $L^2(\Omega')$, we conclude that $\left|\left\{v_{\varepsilon_n,h}(t) \leq b^j_{\varepsilon_n,h}(t)\right\}\right| \to 0$. By Lemma 6.5.4 we have

$$u_{\varepsilon_n,h}(s)\mathbb{1}_{\{v_{\varepsilon_n,h}(s)>b^{j+1}_{\varepsilon_n,h}(s)\}} \to u_h(s) \quad \text{in } SBV(\Omega'),$$

so that $z = u_h(s)$ and the proof is complete. □

We now pass to the analysis of $u_h(t)$ with $t \in D$. The following minimality property for the functions $u_h(t)$ with $t \in D$ is crucial for the subsequent results.

Theorem 6.5.6. *Let $t \in D$. Then for every $z \in SBV(\Omega')$ with $z = g_h(t)$ on Ω_D, we have that*

$$\int_{\Omega'} |\nabla u_h(t)|^2\,dx \leq \int_{\Omega'} |\nabla z|^2\,dx + \mathcal{H}^{N-1}\left(S(z) \setminus \bigcup_{s \leq t, s \in D} S(u_h(s))\right).$$

The proof is quite technical, and it is postponed to Section 6.6. We now let $h \to \infty$.

Proposition 6.5.7. *There exists $h_n \to \infty$ such that for all $t \in D$ there exists $u(t) \in SBV(\Omega')$ with $u(t) = g(t)$ on Ω_D such that $u_{h_n}(t) \to u(t)$ in $SBV(\Omega')$. Moreover, $\nabla u_{h_n}(t) \to \nabla u(t)$ strongly in $L^2(\Omega'; \mathbb{R}^N)$ and for all $z \in SBV(\Omega')$ with $z = g(t)$ on Ω_D we have*

$$\int_{\Omega'} |\nabla u(t)|^2\, dx \le \int_{\Omega'} |\nabla z|^2\, dx + \mathcal{H}^{N-1}\left(S(z) \setminus \bigcup_{s \le t, s \in D} S(u(s)) \right).$$

Proof. The compactness is given by Ambrosio's Theorem in view of (6.28). The strong convergence of the gradients and the minimality property is a consequence of the minimality property of Theorem 6.5.6 and of [53, Theorem 2.1]. □

We can now deal with ε and h at the same time.

Proposition 6.5.8. *There exists $\varepsilon_n \to 0$ and $h_n \to +\infty$ such that for all $t \in D$ there exists $u(t) \in SBV(\Omega')$ with $u(t) = g(t)$ on Ω_D such that for all $j \ge 1$*

$$u_{\varepsilon_n, h_n}(t) 1_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} \to u(t) \quad \text{in } SBV(\Omega').$$

Furthermore for all $z \in SBV(\Omega')$ with $z = g(t)$ on Ω_D we have

$$\int_{\Omega'} |\nabla u(t)|^2\, dx \le \int_{\Omega'} |\nabla z|^2\, dx + \mathcal{H}^{N-1}\left(S(z) \setminus \bigcup_{s \le t, s \in D} S(u(s)) \right),$$

and we may suppose that the functions $\lambda_{\varepsilon_n,h_n}$ converge pointwise on $[0,1]$ to an increasing function λ such that for all $t \in D$

$$\lambda(t) \ge \mathcal{H}^{N-1}\left(\bigcup_{s \le t, s \in D} S(u(s)) \right). \tag{6.29}$$

Finally, we have that for all $t \in D$

$$\int_{\Omega'} |\nabla u(t)|^2\, dx + \mathcal{H}^{N-1}(S(u(t))) + \|u(t)\|_\infty \le C_1. \tag{6.30}$$

Proof. We find ε_n and h_n combining Lemma 6.5.3 and Proposition 6.5.7, and using a diagonal argument. Passing to the second part of the proposition, notice that the functions $\lambda_{\varepsilon_n,h_n}$ are monotone increasing. In fact if $s \le t$, since $v_{\varepsilon_n,h_n}(t) \le v_{\varepsilon_n,h_n}(s)$, and $v_{\varepsilon_n,h_n}(t) = 1$ on Ω_D, by the minimality property (6.22), we have that

$$F_{\varepsilon_n}(u_{\varepsilon_n,h_n}(s), v_{\varepsilon_n,h_n}(s)) \le F_{\varepsilon_n}(u_{\varepsilon_n,h_n}(s), v_{\varepsilon_n,h_n}(t)),$$

so that

$$\lambda_{\varepsilon_n,h_n}(t) - \lambda_{\varepsilon_n,h_n}(s) \ge$$
$$\ge \int_{\Omega'} (\eta_{\varepsilon_n} + v_{\varepsilon_n,h_n}(s)^2) |\nabla u_{\varepsilon_n,h_n}(s)|^2\, dx - \int_{\Omega'} (\eta_{\varepsilon_n} + v_{\varepsilon_n,h_n}(t)^2) |\nabla u_{\varepsilon_n,h_n}(s)|^2\, dx \ge 0.$$

Moreover by (6.25) we have $0 \le \lambda_{\varepsilon_n,h_n} \le C_1$. Applying Helly's theorem, we get that there exists an increasing function λ up to a subsequence $\lambda_{\varepsilon_n,h_n} \to \lambda$ pointwise in $[0,1]$. In order to prove (6.29), let us fix $s_1, \ldots, s_m \in D \cap [0,t]$; we want to prove that

$$\lambda(t) = \lim_n \lambda_{\varepsilon_n,h_n}(t) \ge \mathcal{H}^{N-1}\left(\bigcup_{i=1}^m S(u(s_i)) \right). \tag{6.31}$$

Then taking the sup over all possible $s_1, \ldots, s_m$, we can deduce (6.29).

Consider $z_n \in SBV(\Omega', \mathbb{R}^m)$ defined as

$$z_n(x) := (u_{\varepsilon_n,h_n}(s_1), \ldots, u_{\varepsilon_n,h_n}(s_m)).$$

Notice that by (6.25), and the fact that $t \to v_{\varepsilon_n,h_n}(t)$ is decreasing in $L^2(\Omega')$, we obtain that there exists $C' > 0$ such that for all n

$$\int_{\Omega'} (\eta_{\varepsilon_n} + v_{\varepsilon_n,h_n}(t)^2)|\nabla z_n(t)|^2\, dx + \frac{\varepsilon_n}{2} \int_{\Omega'} |\nabla v_{\varepsilon_n,h_n}(t)|^2\, dx + \frac{1}{2\varepsilon_n} \int_{\Omega'} (1 - v_{\varepsilon_n,h_n}(t))^2\, dx \le C'.$$

Then we may apply (6.3) obtaining (6.31). Finally (6.30) is a consequence of (6.25) and the lower semicontinuity (6.3). The proof is now concluded. □

Let us extend the evolution $\{t \to (u(t), \Gamma(t)) : t \in D\}$ of Proposition 6.5.8 to the entire interval $[0,1]$. Let us set for every $t \in [0,1]$

$$\Gamma(t) := \bigcup_{s \in D, s \le t} S(u(s)). \tag{6.32}$$

Proposition 6.5.9. *For every $t \in [0,1]$ there exists $u(t) \in SBV(\Omega')$ with $u(t) = g(t)$ on Ω_D such that $\nabla u \in L^\infty([0,1], L^2(\Omega'; \mathbb{R}^N))$, ∇u is left continuous in $[0,1] \setminus D$ with respect to the strong topology, and such that, if Γ is as in (6.32), the following hold:*

(a) for all $t \in [0,1]$

$$S(u(t)) \subseteq \Gamma(t) \ \text{ up to a set of } \mathcal{H}^{N-1} - \text{measure } 0, \tag{6.33}$$

and if λ is as in Proposition 6.5.8

$$\lambda(t) \ge \mathcal{H}^{N-1}(\Gamma(t)); \tag{6.34}$$

(b) for all $z \in SBV(\Omega')$ with $z = g(0)$ on Ω_D

$$\int_{\Omega'} |\nabla u(0)|^2\, dx + \mathcal{H}^{N-1}(S(u(0))) \le \int_{\Omega'} |\nabla z|^2\, dx + \mathcal{H}^{N-1}(S(z)). \tag{6.35}$$

(c) for all $t \in]0,1]$ and for all $z \in SBV(\Omega')$ with $z = g(t)$ on Ω_D

$$\int_{\Omega'} |\nabla u(t)|^2\, dx \le \int_{\Omega'} |\nabla z|^2\, dx + \mathcal{H}^{N-1}(S(z) \setminus \Gamma(t)). \tag{6.36}$$

Finally,

$$\mathcal{E}(t) \ge \mathcal{E}(0) + 2\int_0^t \int_{\Omega'} \nabla u(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau \tag{6.37}$$

where

$$\mathcal{E}(t) := \int_{\Omega'} |\nabla u(t)|^2\, dx + \mathcal{H}^{N-1}(\Gamma(t)). \tag{6.38}$$

Proof. Let $t \in [0,1] \setminus D$ and let $t_n \in D$ with $t_n \nearrow t$; by (6.30) we can apply Ambrosio's Theorem obtaining $u \in SBV(\Omega')$ with $u = g(t)$ on Ω_D such that $u(t_n) \to u$ in $SBV(\Omega')$ up to subsequences. Let us set $u(t) := u$. By [53, Lemma 3.7], we have that (6.33) and (6.36) hold, and that the convergence $\nabla u(t_n) \to \nabla u$ is strong in $L^2(\Omega'; \mathbb{R}^N)$. Notice that $\nabla u(t)$ is uniquely determined by (6.33) and (6.36) since the gradient of the solutions of the minimum problem

$$\min\left\{ \int_{\Omega'} |\nabla u|^2\, dx \,:\, u = g(t) \text{ on } \Omega_D, S(u) \subseteq \Gamma(t) \ \text{ up to a set of } \mathcal{H}^{N-1} - \text{measure } 0 \right\}$$

is unique by the strict convexity of the functional. We conclude that $\nabla u(t)$ is well defined. The argument above proves that ∇u is left continuous at all the points of $[0,1] \setminus D$. It turns out that ∇u is continuous in $[0,1]$ up to a countable set. In fact let us consider $t \in [0,1] \setminus (D \cup \mathcal{N})$ where $\mathcal{N}$ is the set of discontinuities of the function $\mathcal{H}^{N-1}(\Gamma(\cdot))$. Let $t_n \searrow t$. By Ambrosio's Theorem, we have that there exists $u \in SBV(\Omega')$ with $u = g(t)$ on Ω_D such that, up to a subsequence, $u(t_n) \to u$ in $SBV(\Omega')$. Since t is a continuity point of $\mathcal{H}^1(\Gamma(\cdot))$, we deduce that $S(u) \subseteq \Gamma(t)$ up to a set of $\mathcal{H}^{N-1}$-measure 0. Moreover by [53, Lemma 3.7] we have that u satisfies the minimality property (6.36), and $\nabla u(t_n) \to \nabla u$ strongly in $L^2(\Omega'; \mathbb{R}^N)$. We deduce that $\nabla u = \nabla u(t)$, and so $\nabla u(\cdot)$ is continuous in $[0,1] \setminus (D \cup \mathcal{N})$. We conclude that $\nabla u(\cdot)$ is continuous in $[0,1]$ up to a countable set, so that $\nabla u \in L^\infty([0,1]; L^2(\Omega'; \mathbb{R}^N))$.

We have that (6.34) is a direct consequence of (6.29), while (6.35) is a consequence of (6.21) and the Γ-convergence result of Ambrosio and Tortorelli [10] and [11].

Finally, in order to prove (6.37), we can reason in the following way. Given $t \in [0,1]$ and $m > 0$, let $s_i^m := \frac{i}{m}t$ for all $i = 0, \dots, m$. Let us set $u^m(s) := u(s_{i+1}^m)$ for $s_i^m < s \leq s_{i+1}^m$. By (6.36) we have

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t \int_{\Omega'} \nabla u^m(\tau) \nabla \dot{g}(\tau)\, d\tau\, dx + o_m \tag{6.39}$$

where $o_m \to 0$ for $m \to +\infty$ because g is absolutely continuous. Since ∇u is continuous with respect to the strong topology of $L^2(\Omega'; \mathbb{R}^N)$ in $[0,1]$ up to a countable set, passing to the limit for $m \to +\infty$ we deduce that (6.37) holds, and the proof is concluded. □

We are now in a position to prove our convergence result. We need the following lemma.

Lemma 6.5.10. *Let $\tilde{\mathcal{N}}$ be the set of discontinuity points of the function λ given by Proposition 6.5.8. Then for every $t \in [0,1] \setminus \tilde{\mathcal{N}}$, and $j \geq 1$ we have that*

$$\nabla u_{\varepsilon_n,h_n}(t) \mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} \rightharpoonup \nabla u(t) \quad \text{weakly in } L^2(\Omega'; \mathbb{R}^N).$$

Proof. Let $t \in [0,1] \setminus \tilde{\mathcal{N}}$: we may suppose that $t \notin D$, since otherwise the result has already been established. Let $s \in D$ with $s < t$. We set

$$J := \inf \left\{ \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t)) |\nabla z|^2\, dx \,:\, z = g_{h_n}(s) \text{ on } \Omega_D \right\},$$

and we indicate by $w_n(s,t)$ the minimum point of this problem. Notice that $u_{\varepsilon_n,h_n}(t) - w_n(s,t)$ is the minimum for

$$K := \inf \left\{ \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t)) |\nabla z|^2\, dx \,:\, z = g_{h_n}(t) - g_{h_n}(s) \text{ on } \Omega_D \right\}.$$

Comparing $u_{\varepsilon_n,h_n}(t) - w_n(s,t)$ with $g_{h_n}(t) - g_{h_n}(s)$, we have

$$\begin{aligned} \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t)) |\nabla u_{\varepsilon_n,h_n}(t) - \nabla w_n(s,t)|^2\, dx \leq \\ \leq \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t)) |\nabla g_{h_n}(t) - \nabla g_{h_n}(s)|^2\, dx. \end{aligned} \tag{6.40}$$

Since $u_{\varepsilon_n,h_n}(s) - w_n(s,t)$ is a good test for J, we have

$$\int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t)) \nabla w_n(s,t) (\nabla u_{\varepsilon_n,h_n}(s) - \nabla w_n(s,t))\, dx = 0,$$

and so the following equality holds

$$\begin{aligned} \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t)) (|\nabla u_{\varepsilon_n,h_n}(s)|^2 - |\nabla w_n(s,t)|^2)\, dx = \\ = \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t)) (|\nabla u_{\varepsilon_n,h_n}(s) - \nabla w_n(s,t)|^2)\, dx. \end{aligned}$$

Since $v_{\varepsilon_n,h_n}(t) \le v_{\varepsilon_n,h_n}(s)$ and by minimality of $u_{\varepsilon_n,h_n}(s)$ we have

$$\begin{aligned}
\int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t))|\nabla u_{\varepsilon_n,h_n}(s)|^2\,dx + \lambda_{\varepsilon_n,h_n}(s) \le \\
\le \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(s))|\nabla u_{\varepsilon_n,h_n}(s)|^2\,dx + \lambda_{\varepsilon_n,h_n}(s) \le \\
\le \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t))|\nabla w_n(s,t)|^2\,dx + \lambda_{\varepsilon_n,h_n}(t).
\end{aligned}$$

so that

$$\begin{aligned}
(6.41)\quad \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t))(|\nabla u_{\varepsilon_n,h_n}(s) - \nabla w_n(s,t)|^2)\,dx = \\
= \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t))(|\nabla u_{\varepsilon_n,h_n}(s)|^2 - |\nabla w_n(s,t)|^2)\,dx \le \\
\le \lambda_{\varepsilon_n,h_n}(t) - \lambda_{\varepsilon_n,h_n}(s).
\end{aligned}$$

By (6.40) and (6.41), we conclude that there exists $C' > 0$ with

$$\begin{aligned}
(6.42)\quad \int_{\Omega'} (\eta_{\varepsilon_n} + v^2_{\varepsilon_n,h_n}(t))(|\nabla u_{\varepsilon_n,h_n}(t) - \nabla u_{\varepsilon_n,h_n}(s)|^2)\,dx \le \\
\le C'\|\nabla g_{h_n}(t) - \nabla g_{h_n}(s)\| + (\lambda_{\varepsilon_n,h_n}(t) - \lambda_{\varepsilon_n,h_n}(s)).
\end{aligned}$$

Then we conclude that for $b^j_{\varepsilon_n,h_n}(t) \in B_{\varepsilon_n,h_n}(t)$

$$\begin{aligned}
(6.43)\quad ||\nabla u_{\varepsilon_n,h_n}(t)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} - \nabla u_{\varepsilon_n,h_n}(s)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}}||_{L^2(\Omega';\mathbb{R}^N)} \le \\
\le o(t-s)
\end{aligned}$$

since $\lambda_{\varepsilon_n,h_n} \to \lambda$ pointwise, and t is a continuity point for λ. Recall that by Lemma 6.5.5

$$\nabla u_{\varepsilon_n,h_n}(s)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} \rightharpoonup \nabla u(s) \quad \text{weakly in } L^2(\Omega';\mathbb{R}^N).$$

Since

$$\begin{aligned}
\nabla u_{\varepsilon_n,h_n}(t)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} - \nabla u(t) = \\
= (\nabla u_{\varepsilon_n,h_n}(t)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} - \nabla u_{\varepsilon_n,h_n}(s)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}}) + \\
+ (\nabla u_{\varepsilon_n,h_n}(s)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} - \nabla u(s)) + (\nabla u(s) - \nabla u(t)),
\end{aligned}$$

by (6.43) and the left continuity of $\{\tau \to \nabla u(\tau)\}$ at the points of $[0,1] \setminus D$, we have that

$$\nabla u_{\varepsilon_n,h_n}(t)\mathbb{1}_{\{v_{\varepsilon_n,h_n}(t) > b^j_{\varepsilon_n,h_n}(t)\}} \rightharpoonup \nabla u(t) \quad \text{weakly in } L^2(\Omega';\mathbb{R}^N),$$

so that the lemma is proved. □

We are now in a position to prove the main theorem of the chapter.

Proof of Theorem 6.3.2. By Proposition 6.5.1, we may extend $(u_{\varepsilon,h}(t), v_{\varepsilon,h}(t))$ to Ω' setting $u_{\varepsilon,h}(t) = g_h(t)$ and $v_{\varepsilon,h}(t) = 1$ on Ω_D, obtaining a quasistatic evolution in Ω'. In this context, the points of $\partial_D\Omega$ where the boundary condition is violated in the limit simply become discontinuity points of the extended function. Thus we prove the result in this equivalent setting involving Ω'.

Let $\varepsilon_n \to 0$ and $h_n \to +\infty$ be the sequences determined by Proposition 6.5.8. Let us indicate $u_{\varepsilon_n,h_n}(t), v_{\varepsilon_n,h_n}(t)$ and F_{ε_n} by $u_n(t), v_n(t)$ and F_n. Moreover, let us write $B_n(t)$ and $b^j_n(t)$

for $B_{\varepsilon_n,h_n}(t)$ and $b^j_{\varepsilon_n,h_n}(t)$. Let $\{t \to (u(t),\Gamma(t)) \in SBV(\Omega'), t \in [0,1]\}$ be the evolution relative to the boundary data g given by Proposition 6.5.9; up to a subsequence, we have that $u_n(t)\mathbb{1}_{\{v_n(t)>b^j_n(t)\}} \to u(t)$ in $SBV(\Omega')$ for all $j \geq 1$ and for all t in a countable and dense subset $D \subseteq [0,1]$ with $0 \in D$. Moreover for all $t \in [0,1]$ we have that

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau, \tag{6.44}$$

where $\mathcal{E}(t) := \int_{\Omega'} |\nabla u(t)|^2\,dx + \mathcal{H}^{N-1}(\Gamma(t))$ and $\Gamma(t)$ is as in (6.32).

By point (b) of Proposition 6.5.1 and the Ambrosio-Tortorelli Theorem 6.1.1 we have

$$\lim_n F_n(u_n(0), v_n(0)) = \mathcal{E}(0). \tag{6.45}$$

For $m \geq 1$, notice that

$$\int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)\nabla \dot{g}_{h_n}(\tau)\,dx = \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)\mathbb{1}_{\{v_n(\tau)>b^m_n(\tau)\}}\nabla \dot{g}_{h_n}(\tau)\,dx + \\ + \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)\mathbb{1}_{\{v_n(\tau)\leq b^m_n(\tau)\}}\nabla \dot{g}_{h_n}(\tau)\,dx.$$

If $\tau \in [0,1]$, we have the estimate

$$\left|\int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)\mathbb{1}_{\{v_n(\tau)\leq b^m_k(\tau)\}}\nabla \dot{g}_{h_n}(\tau)\,dx\right| \leq \\ \leq \sqrt{\eta_{\varepsilon_n} + \frac{1}{2^{2m}}}\left(\int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))|\nabla u_n(\tau)|^2\,dx\right)^{\frac{1}{2}} ||\nabla \dot{g}_{h_n}(\tau)||_{L^2(\Omega';\mathbb{R}^N)} \leq \\ \leq \sqrt{\eta_{\varepsilon_n} + \frac{1}{2^{2m}}}C \to \frac{C}{2^m}.$$

Moreover, by Lemma 6.5.10 we have that for a.e. $\tau \in [0,1]$

$$\lim_n \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)\mathbb{1}_{\{v_n(\tau)>b^m_n(\tau)\}}\nabla \dot{g}_{h_n}(\tau)\,dx = \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx,$$

and we deduce that for such τ

$$\limsup_n \left|\int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)\nabla \dot{g}_{h_n}(\tau)\,dx - \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx\right| \leq \frac{C}{2^m}.$$

Since m is arbitrary, we have that for a.e. $\tau \in [0,1]$

$$\lim_n \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)\nabla \dot{g}_{h_n}(\tau)\,dx = \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx. \tag{6.46}$$

By (6.23), (6.45), (6.46) and the Dominated Convergence Theorem, we conclude that for all $t \in [0,1]$

$$\lim_n F_n(u_n(t), v_n(t)) = \mathcal{E}(0) + 2\int_0^t \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau. \tag{6.47}$$

Since by Proposition 6.5.8 we have $\liminf_n F_n(u_n(t), v_n(t)) \geq \mathcal{E}(t)$ for all $t \in D$, by (6.44) we have for all $t \in D$

$$\lim_n F_n(u_n(t), v_n(t)) = \mathcal{E}(t).$$

In particular we get for all $t \in D$

$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau, \tag{6.48}$$

and since by Proposition 6.5.9 $\nabla u(\cdot)$ and $\mathcal{H}^{N-1}(\Gamma(\cdot))$ are left continuous at $t \notin D$ and so $\mathcal{E}(\cdot)$ is, we conclude that the equality holds for all $t \in [0,T]$. Recalling all the properties stated in Proposition 6.5.9, we deduce that $\{t \to (u(t),\Gamma(t)) : t \in [0,1]\}$ is a quasistatic evolution relative to the boundary data g. In order to prove point (a), it is sufficient to see that $\liminf_n F_n(u_n(t),v_n(t)) \geq \mathcal{E}(t)$ holds for all $t \in [0,1]$. Considering $s \geq t$ with $s \in D$, we have

$$F_n(u_n(s),v_n(s)) = F_n(u_n(t),v_n(t)) + 2\int_s^t \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(\tau))\nabla u_n(\tau)\nabla \dot{g}_{h_n}(\tau)\,dx$$

so that

$$\liminf_n F_n(u_n(t),v_n(t)) \geq \mathcal{E}(s) - 2\int_s^t \int_{\Omega'} \nabla u(\tau)\nabla \dot{g}(\tau)\,dx\,d\tau.$$

Letting $s \searrow t$, since $\mathcal{E}(\cdot)$ is continuous and by (6.48), we obtain $\liminf_n F_n(u_n(t),v_n(t)) \geq \mathcal{E}(t)$, and so point (a) is now completely proved.

Let us come to point (b). By Lemma 6.5.10, we know that if $\tilde{\mathcal{N}}$ is the set of discontinuity points of λ, for all $t \in [0,1] \setminus \tilde{\mathcal{N}}$ and for all $j \geq 1$ we have $\nabla u_n(t)\mathbb{1}_{\{v_n(t)>b_k^j(t)\}} \rightharpoonup \nabla u(t)$ weakly in $L^2(\Omega',\mathbb{R}^N)$. Since

$$v_n(t)\nabla u_n(t) = v_n(t)\nabla u_n(t)\mathbb{1}_{\{v_n(t)>b_n^j(t)\}} + v_n(t)\nabla u_n(t)\mathbb{1}_{\{v_n(t)<b_n^j(t)\}},$$

we get immediately that $v_n(t)\nabla u_n(t) \rightharpoonup \nabla u(t)$ weakly in $L^2(\Omega',\mathbb{R}^N)$. For all such t, we have that

$$\liminf_n \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(t))|\nabla u_n(t)|^2\,dx \geq \int_{\Omega'} |\nabla u(t)|^2\,dx,$$

and by (6.34)

$$\liminf_n \frac{\varepsilon_n}{2}\int_{\Omega'} |\nabla v_n(t)|^2\,dx + \frac{1}{2\varepsilon_n}\int_{\Omega'} (1-v_n(t))^2\,dx \geq \mathcal{H}^{N-1}(\Gamma(t)).$$

By point (a), we have that the two preceding inequalities are equalities. In particular, λ and $\mathcal{H}^{N-1}(\Gamma(\cdot))$ coincide up to a countable set in $[0,1]$. We deduce that λ and $\mathcal{H}^{N-1}(\Gamma(\cdot))$ have the same continuity points, that is $\tilde{\mathcal{N}} = \mathcal{N}$. We conclude that for all $t \in [0,1] \setminus \mathcal{N}$ we have $v_n(t)\nabla u_n(t) \to \nabla u(t)$ strongly in $L^2(\Omega',\mathbb{R}^N)$,

$$\lim_n \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2(t))|\nabla u_n(t)|^2\,dx = \int_{\Omega'} |\nabla u(t)|^2\,dx,$$

and

$$\lim_n \frac{\varepsilon_n}{2}\int_{\Omega'} |\nabla v_n(t)|^2\,dx + \frac{1}{2\varepsilon_n}\int_{\Omega'} (1-v_n(t))^2\,dx = \mathcal{H}^{N-1}(\Gamma(t)),$$

so that point (b) is proved, and the proof of the theorem is complete. □

6.6 Proof of Theorem 6.5.6

In this section we give the proof of Theorem 6.5.6 which is an essential step in the analysis of Section 6.5. For simplicity of notation, for all $t \in D$ we write $u(t)$, $u_n(t)$ and $v_n(t)$ for $u_h(t)$, $u_{\varepsilon_n,h}(t)$ and $v_{\varepsilon_n,h}(t)$ respectively. Moreover, let us write $B_n(t)$, $b_n^j(t)$ for $B_{\varepsilon_n,h}(t)$ and $b_{\varepsilon_n,h}^j(t)$, where $B_{\varepsilon_n,h}(t)$ is defined as in (6.27).

Given $z \in SBV(\Omega')$ with $z = g_h(t)$ on Ω_D, we want to see that

$$\int_{\Omega'} |\nabla u(t)|^2\,dx \leq \int_{\Omega'} |\nabla z|^2\,dx + \mathcal{H}^{N-1}\left(S(z)\setminus\Gamma(t)\right), \tag{6.49}$$

where $g_h(t) \in H^1(\Omega') \cap C(\overline{\Omega'})$ and $\Gamma(t) = \bigcup_{s\leq t, s\in D} S(u(s))$.

The plan is to use the minimality property (6.22) of the approximating evolution, so that the main point is to construct a sequence $(z_n, v_n) \in H^1(\Omega') \times H^1(\Omega')$ such that $z_n = g_h(t)$, $v_n = 1$ on Ω_D, $0 \leq v_n \leq v_n(t)$, and such that

$$\lim_n \int_{\Omega'} (\eta_n + v_n^2)|\nabla z_n|^2 \, dx = \int_{\Omega'} |\nabla z|^2 \, dx$$

and

$$\limsup_n \left[MM_n(v_n) - MM_n(v_n(t)) \right] \leq \mathcal{H}^{N-1}\left(S(z) \setminus \Gamma(t)\right),$$

where we use the notation

$$MM_n(w) := \frac{\varepsilon_n}{2} \int_{\Omega'} |\nabla w|^2 \, dx + \frac{1}{2\varepsilon_n} \int_{\Omega'} (1-w)^2 \, dx.$$

If a sequence with these properties exists, then by property (6.22) we get the result.

We will need a density result in SBV. Let $A \subseteq \mathbb{R}^N$ be open. We say that $K \subseteq A$ is polyhedral (with respect to A), if it is the intersection of A with the union of a finite number of $(N-1)$-dimensional simplexes of S.

The following density result is proved in [38].

Theorem 6.6.1. *Assume that ∂A is locally Lipschitz, and let $u \in GSBV^p(A)$. For every $\varepsilon > 0$, there exists a function $v \in SBV^p(A)$ such that*

(a) $S(v)$ is essentially closed, i.e., $\mathcal{H}^{N-1}(\overline{S(v)} \setminus S(v)) = 0$;

(b) $\overline{S(v)}$ is a polyhedral set;

(c) $v \in W^{k,\infty}(A \setminus \overline{S(v)})$ for every $k \in \mathbb{N}$;

(d) $||v - u||_{L^p(A)} < \varepsilon$;

(e) $||\nabla v - \nabla u||_{L^p(A;\mathbb{R}^N)} < \varepsilon$;

(f) $|\mathcal{H}^{N-1}(S(v)) - \mathcal{H}^{N-1}(S(u))| < \varepsilon$.

Theorem 6.6.1 has been generalized to non-isotropic surface energies in [40]. We will use the following result.

Proposition 6.6.2. *Given $g \in H^1(B)$ and $u \in SBV(\Omega')$ with $u = g$ on $\Omega' \setminus \overline{\Omega}$, there exists $u_h \in SBV(\Omega')$ such that*

(a) $u_h = g$ in $\Omega' \setminus \overline{\Omega}$ and in a neighborhood of $\partial_D \Omega$;

(b) $\overline{S(u_h)}$ is polyhedral and $\overline{S(u_h)} \subseteq \Omega$ for all h;

(c) $\nabla u_h \to \nabla u$ strongly in $L^2(\Omega';\mathbb{R}^N)$;

(d) for all A open subset of Ω' with $\mathcal{H}^{N-1}(\partial A \cap S(u)) = 0$, we have

$$\lim_h \mathcal{H}^{N-1}(A \cap S(u_h)) = \mathcal{H}^{N-1}(A \cap S(u)).$$

Proof. Using a partition of unity, we may prove the result in the case $\Omega' := Q \times]-1,1[$, $\Omega := \{(x,y) \in Q \times]-1,1[: y > f(x)\}$, $\partial_D \Omega := \{(x,y) \in Q \times]-1,1[: y = f(x)\}$, where Q is unit cube in $\mathbb{R}^{N-1}$ and $f : Q \to \mathbb{R}$ is a Lipschitz function with values in $]-\frac{1}{2}, \frac{1}{2}[$. Let $g \in H^1(\Omega')$, and let $u \in SBV(\Omega')$ with $u = g$ on $\Omega' \setminus \Omega$.

Let $w_h := u(x - h e_N)$ where e_N is the versor of the N-axis, and let φ_h be a cut off function with $\varphi_h = 1$ on $\{y \leq f(x) + \frac{h}{3}\}$, $\varphi_h = 0$ on $\{y \geq f(x) + \frac{h}{2}\}$, and $||\nabla \varphi_h||_\infty \leq \frac{1}{h}$. Let us set

$v_h := \varphi_h g + (1-\varphi_h)w_h$. We have that $v_h = g$ in $\Omega' \setminus \overline{\Omega}$ and in a neighborhood of $\partial_D\Omega$; moreover we have

$$\nabla v_h = \varphi_h \nabla g + (1-\varphi_h)\nabla w_h + \nabla\varphi_h(g - w_h).$$

Since $\nabla\varphi_h(g-w_h) \to 0$ strongly in $L^2(\Omega';\mathbb{R}^N)$, we have $\nabla v_h \to \nabla u$ strongly in $L^2(\Omega';\mathbb{R}^N)$. Finally, for all A open subset of Ω' with $\mathcal{H}^{N-1}(\partial A \cap S(u)) = 0$, we have

$$\lim_h \mathcal{H}^{N-1}(A \cap S(v_h)) = \mathcal{H}^{N-1}(A \cap S(u)).$$

In order to conclude the proof, let us apply Theorem 6.6.1 obtaining $\tilde{v}_h$ with polyhedral jumps such that $||v_h - \tilde{v}_h||_{L^2(\Omega')} + ||\nabla v_h - \nabla\tilde{v}_h||_{L^2(\Omega';\mathbb{R}^N)} \le h^2$, $|\mathcal{H}^{N-1}(S(v_h)) - \mathcal{H}^{N-1}(S(\tilde{v}_h))| \le h$. If we set $u_h := \varphi_h g + (1-\varphi_h)\tilde{v}_h$, we obtain the thesis. □

The following lemma contains the main ideas in order to prove Theorem 6.5.6.

Lemma 6.6.3. *Let $t \in D$; given $z \in SBV(\Omega')$ with $z = g_h(t)$ on Ω_D we have that*

$$\int_{\Omega'} |\nabla u(t)|^2\,dx \le \int_{\Omega'} |\nabla z|^2\,dx + \mathcal{H}^{N-1}\left(S(z) \setminus S(u(t))\right). \tag{6.50}$$

In order to prove Lemma 6.6.3, we need several preliminary results. Let $z \in SBV(\Omega')$ be such that $z = g_h(t)$ on Ω_D. Given $\sigma > 0$, let U be a neighborhood of $S(u(t))$ such that $|U| \le \sigma$, and $||\nabla z||_{L^2(U;\mathbb{R}^N)} \le \sigma$. Let $C := \{x \in \partial_D\Omega : \partial_D\Omega \text{ is not differentiable at } x\}$. We recall that there exists a countable and dense set $A \subseteq \mathbb{R}$ such that up to a set of $\mathcal{H}^{N-1}$-measure zero

$$S(u(t)) = \bigcup_{a,b\in A} \partial^* E_a \cap \partial^* E_b$$

where $E_a := \{x \in \Omega' : u(t)(x) \ge a\}$ and ∂^* denotes the essential boundary. Consider

$$J_j := \left\{x \in S(u(t)) \setminus C : [u(t)(x)] \ge \frac{1}{j}\right\},$$

with j chosen in such a way that $\mathcal{H}^{N-1}(S(u(t)) \setminus J_j) \le \sigma$. For $x \in J_j$, let $a_1(x), a_2(x) \in A$ be such that $u^-(t)(x) < a_1(x) < a_2(x) < u^+(t)(x)$ and $a_2(x) - a_1(x) \ge \frac{1}{2j}$. Following [53, Theorem 2.1], we consider a finite disjoint collection of closed cubes $\{Q_i\}_{i=1,\dots,k}$ with center $x_i \in J_j$, radius r_i and with normal $\nu(x_i)$ such that $\bigcup_{i=1}^k Q_i \subseteq U$, $\mathcal{H}^{N-1}(J_j \setminus \bigcup_{i=1}^k Q_i) \le \sigma$, and for all $i = 1,\dots,k$, $j = 1,2$

1. $\mathcal{H}^{N-1}\left(S(u(t)) \cap \partial Q_i\right) = 0$;
2. $r_i^{N-1} \le 2\mathcal{H}^{N-1}\left(S(u(t)) \cap Q_i\right)$;
3. $\mathcal{H}^{N-1}\left(\left[S(u(t)) \setminus \partial^* E_{a_j(x_i)}\right] \cap Q_i\right) \le \sigma r_i^{N-1}$;
4. $\mathcal{H}^{N-1}\left(\left\{y \in \partial^* E_{a_j(x_i)} \cap Q_i \,:\, \mathrm{dist}(y, H_i) \ge \frac{\sigma}{2} r_i\right\}\right) < \sigma r_i^{N-1}$ where H_i denotes the intersection of Q_i with the hyperplane through x_i orthogonal to $\nu(x_i)$;
5. $\mathcal{H}^{N-1}\left(\left(S(z) \setminus S(u(t))\right) \cap Q_i\right) < \sigma r_i^{N-1}$ and $\mathcal{H}^{N-1}(S(z) \cap \partial Q_i) = 0$.

Note that we may suppose that $Q_i \subseteq \Omega$ if $x_i \in \Omega$. Moreover we may require that (see [53, Theorem 2.1] and references therein) for all $i = 1,\dots,k$ and $j = 1,2$

$$||1_{E_{a_j(x_i)} \cap Q_i} - 1_{Q_i^+}||_{L^1(\Omega')} \le \sigma^2 r_i^N. \tag{6.51}$$

Let us indicate by R_i the rectangle given by the intersection of Q_i with the strip centered at H_i with width $2\sigma r_i$, and let us set $V_i := \{y + s\nu(x_i) : y \in \partial Q_i, s \in \mathbb{R}\} \cap R_i$. Note that up to changing the strip, we can suppose $\mathcal{H}^{N-1}(\partial R_i \cap (S(u) \cup S(z))) = \emptyset$.

If $x_i \in \partial_D\Omega$, since $x_i \notin C$, we may require that

$$\partial\Omega \cap Q_i \subseteq \{x : |(x - x_i) \cdot \nu(x_i)| < \sigma r_i\}; \tag{6.52}$$

moreover, if $(Q_i^+ \setminus R_i) \subseteq \Omega$, we can assume that $g_h(t) < a_1(x_i)$ on $\partial\Omega \cap Q_i$ because $g_h(t)$ is continuous and $g_h(t)(x_i) = u^-(x_i) < a_1(x_i)$. Similarly we may require that $g_h(t) > a_2(x_i)$ on $\partial\Omega \cap Q_i$ in the case $(Q_i^- \setminus R_i) \subseteq \Omega$.

Since we can reason up to subsequences of ε_n, we may suppose that $\sum_n \varepsilon_n \leq \frac{1}{8}$. Since by (6.25) we have that $||u_n(t)||_\infty < C_1$ and $v_n(t) \to 1$ strongly in $L^2(\Omega')$, by Lemma 6.5.4 we deduce that $u_n(t) \to u(t)$ in measure. By (6.51), we deduce that for n large enough

$$|Q_i^+ \setminus E^n_{a_2(x_i)}| \leq 2\sigma^2 r_i^N, \tag{6.53}$$

where we use the notation $E^n_a := \{x \in \Omega' : u_n(t)(x) \geq a\}$. Let $G_n \subseteq]\frac{\sigma}{4} r_i, \frac{\sigma}{2} r_i[$ be the set of all s such that

$$\int_{H_i(s)} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, d\mathcal{H}^{N-1} \geq \frac{C_1}{\sigma r_i \varepsilon_n};$$

we get immediately by (6.25) that

$$|G_n| \leq \sigma r_i \varepsilon_n,$$

so that, setting $G := \bigcup_n G_n$, we have $|G| \leq \frac{\sigma}{8} r_i$ and $|]\frac{\sigma}{4} r_i, \frac{\sigma}{2} r_i[\setminus G| \geq \frac{\sigma}{8} r_i$. From (6.53), applying Fubini's Theorem we obtain

$$\int_{]\frac{\sigma}{4} r_i, \frac{\sigma}{2} r_i[\setminus G} \mathcal{H}^{N-1}\left(H_i(s) \setminus E^n_{a_2(x_i)}\right) ds \leq 2\sigma^2 r_i^N,$$

so that there exists $\overline{s} \in]\frac{\sigma}{4} r_i, \frac{\sigma}{2} r_i[\setminus G$ such that, setting $H_i^+ := H_i(\overline{s})$, we have

$$\mathcal{H}^{N-1}\left(H_i^+ \setminus E^n_{a_2(x_i) - \frac{\delta}{2}}\right) \leq 16\sigma r_i^{N-1}. \tag{6.54}$$

Moreover we have by construction

$$\int_{H_i^+} (\eta_n + v_n^2(t))|\nabla u_n|^2 \, d\mathcal{H}^{N-1} \leq K_n, \tag{6.55}$$

where K_n is of the order of $\frac{1}{\varepsilon_n}$. In a similar way, there exists $H_i^- := H_i(\tilde{s})$ with $\tilde{s} \in] - \frac{\sigma}{2} r_i, -\frac{\sigma}{4} r_i[$ and

$$\mathcal{H}^{N-1}\left(H_i^- \cap E^n_{a_1(x_i) + \frac{\delta}{2}}\right) \leq 16\sigma r_i^{N-1}, \tag{6.56}$$

and

$$\int_{H_i^-} (\eta_n + v_n^2(t))|\nabla u_n|^2 \, d\mathcal{H}^{N-1} \leq K_n \tag{6.57}$$

where K_n is of the order of $\frac{1}{\varepsilon_n}$. We indicate by $\tilde{R}_i$ the intersection of Q_i with the strip determined by H_i^+ and H_i^-.

A similar argument prove that, up to reducing Q_i (preserving the estimates previously stated), we may suppose that

$$\int_{V_i} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, d\mathcal{H}^{N-1} \leq K_n, \tag{6.58}$$

where K_n is of the order of $\frac{1}{\varepsilon_n}$.

In order to prove Lemma 6.6.3, we claim that we can suppose $z = g_h(t)$ on Ω_D and in a neighborhood $\mathcal{V}$ of $\partial_D\Omega \setminus \bigcup_{i=1}^k Q_i$, $S(z) \setminus \bigcup_{i=1}^k R_i$ polyhedral with closure contained in Ω, and

$\mathcal{H}^{N-1}((S(z) \setminus S(u(t))) \cap Q_i) \leq \sigma r_i^{N-1}$ for all $i = 1, \dots, k$. In fact, by Proposition 6.6.2, there exists $w_m \in SBV(\Omega')$ with $w_m = g_h(t)$ in $\Omega' \setminus \overline{\Omega}$ and in a neighborhood $\mathcal{V}_m$ of $\partial_D \Omega$ such that $w_m \to z$ strongly in $L^2(\Omega')$, $\nabla w_m \to \nabla z$ strongly in $L^2(\Omega'; \mathbb{R}^N)$, $\overline{S(w_m)} \subseteq \Omega$ polyhedral, and such that for all A open subset of Ω' with $\mathcal{H}^{N-1}(\partial A \cap S(z)) = 0$, we have

$$\lim_m \mathcal{H}^{N-1}(A \cap S(w_m)) = \mathcal{H}^{N-1}(A \cap S(z)).$$

Let us fix $\sigma' > 0$ and let us consider for all $i = 1, \dots, k$ a rectangle R_i' centered in x_i, oriented as R_i and such that $R_i' \subseteq \text{int}(R_i)$, $\mathcal{H}^{N-1}(\partial R_i' \cap S(z)) = 0$, $\mathcal{H}^{N-1}(S(z) \cap (\text{int}(R_i) \setminus R_i')) \leq \sigma' r_i^{N-1}$, where $\text{int}(R_i)$ denotes the interior part of R_i. Let ψ_i be a smooth function such that $0 \leq \psi_i \leq 1$, $\psi_i = 1$ on R_i' and $\psi_i = 0$ outside R_i. Setting $\psi := \sum_{i=1}^k \psi_i$, let us consider $z_m := \psi z + (1 - \psi) w_m$. Note that $z_m \to z$ strongly in $L^2(\Omega')$, $\nabla z_m \to \nabla z$ strongly in $L^2(\Omega'; \mathbb{R}^N)$, $z_m = g_h(t)$ in Ω_D and in a neighborhood $\mathcal{V}_m'$ of $\partial_D \Omega \setminus \bigcup_{i=1}^k R_i$, $S(w_m) \setminus \bigcup_{i=1}^k R_i$ is polyhedral with closure contained in Ω. Finally, for $m \to +\infty$, we have $\mathcal{H}^{N-1}(S(z_m) \setminus \bigcup_{i=1}^k Q_i) \to \mathcal{H}^{N-1}(S(z) \setminus \bigcup_{i=1}^k Q_i))$ and $\limsup_m \mathcal{H}^{N-1}(S(z_m) \cap (\text{int}(R_i) \setminus R_i')) \leq 2\mathcal{H}^{N-1}(S(z) \cap (\text{int}(R_i) \setminus R_i')) \leq 2\sigma' r_i^{N-1}$. So, if (6.50) holds for z_m, we obtain for $m \to +\infty$ that (6.50) holds also for z since σ' is arbitrary, and so the claim is proved.

We begin with the following lemma.

Lemma 6.6.4. *Let $B_n(t)$ be as in (6.27), and let us consider $b_n^2 := b_n^{j_2}(t), b_n^3 := b_n^{j_3}(t) \in B_n(t)$ with $j_2 > j_3 > 1$. Suppose that $k_n := \frac{b_n^3}{b_n^2} > 1$ and let k, b be such that $1 < k \leq k_n$, $b_n^3 \leq b$ for all n. Then setting*

$$w_n := \begin{cases} \frac{k_n}{k_n - 1}(v_n(t) - b_n^3) + b_n^3 & \text{in } \{b_n^2 \leq v_n(t) \leq b_n^3\} \\ 0 & \text{in } \{v_n(t) \leq b_n^2\} \\ v_n(t) & \text{in } \{v_n(t) \geq b_n^3\} \end{cases}$$

we have that $w_n \in H^1(\Omega')$ with $w_n = 1$ on Ω_D, $0 \leq w_n \leq v_n(t)$ in Ω' and

$$\limsup_n \left(MM_n(w_n) - MM_n(v_n(t))\right) \leq \frac{2C_1 k}{(k-1)^2} + \frac{C_1}{(k-1)(1-b)^2} + \frac{C_1 b}{(1-b)^2}, \tag{6.59}$$

where C_1 is given by (6.25). Moreover there exist $b_n^1 := b_n^{j_1}(t) \in B_n(t)$ with $j_1 > j_2 + 1$ and a cut-off function $\varphi_n \in H^1(\Omega')$ with $\varphi_n = 0$ in $\{v_n(t) \leq b_n^1\}$, $\varphi_n = 1$ on $\{v_n(t) \geq b_n^2\}$ (in particular on Ω_D) and such that

$$\lim_n \eta_n \int_{\Omega'} |\nabla \varphi_n|^2 \, dx = 0 \tag{6.60}$$

Proof. w_n is well defined in $H^1(\Omega')$, and by construction $w_n = 1$ on Ω_D and $0 \leq w_n \leq v_n(t)$ in Ω'. Let us estimate $MM_n(w_n) - MM_n(v_n)$. Since

$$\frac{\varepsilon_n}{2} \int_{\Omega'} |\nabla w_n|^2 \, dx = \frac{\varepsilon_n}{2} \int_{\{v_n(t) \geq b_n^3\}} |\nabla v_n(t)|^2 \, dx + \frac{\varepsilon_n}{2} \int_{\{b_n^2 \leq v_n(t) \leq b_n^3\}} |\nabla w_n|^2 \, dx,$$

and $MM_n(v_n(t)) \leq C_1$ by (6.25), we have that

$$\begin{aligned} & \frac{\varepsilon_n}{2} \int_{\Omega'} |\nabla w_n|^2 \, dx - \frac{\varepsilon_n}{2} \int_{\Omega'} |\nabla v_n(t)|^2 \, dx \leq \\ \leq \; & \frac{\varepsilon_n}{2} \int_{\{b_n^2 \leq v_n(t) \leq b_n^3\}} \left(\frac{k_n^2}{(k_n - 1)^2} - 1 \right) |\nabla v_n(t)|^2 \, dx - \frac{\varepsilon_n}{2} \int_{\{v_n(t) \leq b_n^2\}} |\nabla v_n(t)|^2 \, dx \leq \\ \leq \; & C_1 \left(\frac{k_n^2}{(k_n - 1)^2} - 1 \right) = \frac{C_1(2k_n - 1)}{(k_n - 1)^2} \leq \frac{2C_1 k}{(k-1)^2}. \end{aligned}$$

Moreover we have that

$$\begin{aligned}
&\frac{1}{2\varepsilon_n}\int_{\Omega'}(1-w_n)^2\,dx-\frac{1}{2\varepsilon_n}\int_{\Omega'}(1-v_n(t))^2\,dx=\\
&=\frac{1}{2\varepsilon_n}\int_{\Omega'}\left[(1-w_n)^2-(1-v_n(t))^2\right]\,dx=\\
&=\frac{1}{2\varepsilon_n}\int_{\Omega'}(v_n(t)-w_n)(2-v_n(t)-w_n)\,dx=\\
&=\frac{1}{2\varepsilon_n}\int_{\{b_n^2\le v_n(t)\le b_n^3\}}\left(v_n(t)-\frac{k_n}{k_n-1}(v_n(t)-b_n^3)-b_n^3\right)(2-v_n(t)-w_n)\,dx+\\
&+\frac{1}{2\varepsilon_n}\int_{\{v_n(t)\le b_n^2\}}v_n(t)(2-v_n(t))\,dx=\\
&=\frac{1}{2\varepsilon_n}\int_{\{b_n^2\le v_n(t)\le b_n^3\}}\frac{1}{k_n-1}(b_n^3-v_n(t))(2-v_n(t)-w_n)\,dx+\\
&+\frac{1}{2\varepsilon_n}\int_{\{v_n(t)\le b_n^2\}}v_n(t)(2-v_n(t))\,dx\le\\
&\le\frac{C_1}{(k_n-1)(1-b_n^3)^2}+\frac{C_1b_n^2}{(1-b_n^2)^2}\le\frac{C_1}{(k-1)(1-b)^2}+\frac{C_1b}{(1-b)^2}
\end{aligned}$$

because $\frac{|\{v_n(t)\le s\}|}{\varepsilon_n}\le\frac{C_1}{(1-s)^2}$. We conclude that

$$\limsup_n\,(MM_n(w_n)-MM_n(v_n(t)))\le\frac{2C_1k}{(k-1)^2}+\frac{C_1}{(k-1)(1-b)^2}+\frac{C_1b}{(1-b)^2}.$$

Let $j_1>j_2+1$: we have that $b_n^1:=b_n^{j_1}$ and b_n^2 are not in adjacent intervals, and so there exists $l>0$ with $0<l\le b_n^2-b_n^1$. Let us divide the interval $[b_n^1,b_n^2]$ in h_n intervals of the same size $I_j, j=1,\dots,h_n$, with h_n such that $\frac{\eta_n}{\varepsilon_n}h_n\to 0$. Since

$$\sum_{j=1}^{h_n}\frac{\varepsilon_n}{2}\int_{\{v_n(t)\in I_j\}}|\nabla v_n(t)|^2\,dx\le\frac{\varepsilon_n}{2}\int_{\Omega'}|\nabla v_n(t)|^2\,dx\le C_1,$$

we deduce that there exists I_n such that

$$\frac{\varepsilon_n}{2}\int_{\{v_n(t)\in I_n\}}|\nabla v_n(t)|^2\,dx\le\frac{C_1}{h_n}. \tag{6.61}$$

Let α_n,β_n be the extremes of I_n. Let us set

$$\varphi_n:=\frac{1}{\beta_n-\alpha_n}(v_n-\alpha_n)^+\wedge 1. \tag{6.62}$$

Then $\varphi_n\in H^1(\Omega)$, $\varphi_n=0$ in $\{v_n(t)\le b_n^1\}$, $\varphi_n=1$ on $\{v_n(t)\ge b_n^2\}$ (in particular on Ω_D) and by (6.61) and the choice of h_n we have that

$$\eta_n\int_{\Omega'}|\nabla\varphi_n|^2\,dx=\eta_n\int_{\{\alpha_n\le v_n(t)\le\beta_n\}}\frac{1}{(\beta_n-\alpha_n)^2}|\nabla v_n(t)|^2\,dx\le\frac{\eta_n}{\varepsilon_n}\frac{2C_1}{h_n}\frac{h_n^2}{l^2}\to 0,$$

so that the proof is complete. □

In the following lemmas, we will use the following notation: for all measurable set $B\subseteq\Omega'$ we set

$$MM_n(w,B):=\frac{\varepsilon_n}{2}\int_B|\nabla w|^2\,dx+\frac{1}{2\varepsilon_n}\int_B(1-w)^2\,dx. \tag{6.63}$$

Let b_n^1 be as in Lemma 6.6.4 and let $\delta:=\frac{1}{8j}$ so that for all $i=1,\dots,k$

$$a_1(x_i)<a_1(x_i)+\delta<a_2(x_i)-\delta<a_2(x_i).$$

Lemma 6.6.5. *For each $i = 1, \dots, k$, there exists $w_n^{2,i} \in H^1(Q_i)$ and $[\gamma_n^i - \tau_n^i, \gamma_n^i + \tau_n^i] \subseteq [a_1(x_i) + \delta, a_2(x_i) - \delta]$ such that $0 \le w_n^{2,i} \le 1$, $w_n^{2,i} = 0$ in $\{\gamma_n^i - \tau_n^i \le u_n(t) \le \gamma_n^i + \tau_n^i\} \cap Q_i$, $w_n^{2,i} = 1$ on $\left[\{u_n(t) \le a_1(x_i) + \frac{3}{4}\delta\} \cup \{u_n(t) \ge a_2(x_i) - \frac{3}{4}\delta\}\right] \cap Q_i$, and*

$$\limsup_n \sum_{i=1}^k MM_n(w_n^{2,i}, \{v_n(t) > b_n^1\}) \le o(\sigma). \tag{6.64}$$

Moreover there exists $\varphi_n^{2,i} \in H^1(Q_i)$ such that $0 \le \varphi_n^{2,i} \le 1$, $\varphi_n^{2,i} = 0$ on $\{\gamma_n^i - \frac{\tau_n^i}{2} \le u_n(t) \le \gamma_n^i + \frac{\tau_n^i}{2}\} \cap Q_i$, $\varphi_n^{2,i} = 1$ on $\left[\{u_n(t) \le \gamma_n^i - \tau_n^i\} \cup \{u_n(t) \ge \gamma_n^i + \tau_n^i\}\right] \cap Q_i$, and

$$\lim_n \eta_n \int_{Q_i \cap \{v_n(t) > b_n^1\}} |\nabla \varphi_n^{2,i}|^2 \, dx = 0. \tag{6.65}$$

Proof. For each i let us consider the strip

$$S_n^i := E^n_{a_1(x_i)+\delta} \setminus E^n_{a_2(x_i)-\delta}.$$

Let $h_n \in \mathbb{N}$ and let us divide $[a_1(x_i) + \delta, a_2(x_i) - \delta]$ in h_n intervals of the same size: there exists a subinterval with extremes α_n^i and β_n^i such that, setting $\tilde{S}_n^i := \{x \in \Omega' : \alpha_n^i \le u_n(t) \le \beta_n^i\}$,

$$\begin{aligned} &\int_{\tilde{S}_n^i \cap Q_i} \left[\sigma(\eta_n + v_n^2(t))|\nabla u_n(t)|^2 + (1-\sigma)\right] dx \le \\ &\le \frac{1}{h_n} \int_{S_n^i \cap Q_i} \left[\sigma(\eta_n + v_n^2(t))|\nabla u_n(t)|^2 + (1-\sigma)\right] dx. \end{aligned} \tag{6.66}$$

Let $\gamma_n^i := \frac{\alpha_n^i + \beta_n^i}{2}$ and $\tau_n^i := \frac{a_2(x_i) - a_1(x_i) - 2\delta}{4h_n}$. We set

$$w_n^{2,i} := \begin{cases} \frac{1}{\beta_n^i - \gamma_n^i - \tau_n^i}(u_n(t) - \gamma_n^i - \tau_n^i)^+ \wedge 1 & \text{in } \{u_n(t) \ge \gamma_n^i + \tau_n^i\} \cap Q_i \\ 0 & \text{in } \{\gamma_n^i - \tau_n^i \le u_n(t) \le \gamma_n^i + \tau_n^i\} \cap Q_i \\ \frac{1}{\gamma_n^i - \tau_n^i - \alpha_n^i}(u_n(t) - \gamma_n^i + \tau_n^i)^- \wedge 1 & \text{in } \{u_n(t) \le \gamma_n^i - \tau_n^i\} \cap Q_i. \end{cases}$$

We have that

$$\begin{aligned} &\frac{\varepsilon_n}{2} \int_{Q_i \cap \{v_n(t) > b_n^1\}} |\nabla w_n^{2,i}|^2 \, dx + \frac{1}{2\varepsilon_n} \int_{Q_i \cap \{v_n(t) > b_n^1\}} (1 - w_n^{2,i})^2 \, dx \le \\ &\le \frac{\varepsilon_n}{2} \left(\frac{4h_n^2}{\delta^2} \int_{\tilde{S}_n^i \cap (Q_i \cap \{v_n(t) > b_n^1\})} |\nabla u_n(t)|^2 \, dx \right) + \frac{1}{2\varepsilon_n} |\tilde{S}_n^i \cap (Q_i \cap \{v_n(t) > b_n^1\})|. \end{aligned}$$

Since by (6.66)

$$\int_{\tilde{S}_n^i \cap Q_i} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, dx \le \frac{1}{h_n} \left[\int_{S_n^i \cap Q_i} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, dx + \frac{1-\sigma}{\sigma} |S_n^i \cap Q_i| \right]$$

and

$$|\tilde{S}_n^i \cap Q_i| \le \frac{1}{h_n} \left[\frac{\sigma}{1-\sigma} \int_{S_n^i \cap Q_i} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, dx + |S_n^i \cap Q_i| \right]$$

we have

$$\begin{aligned} &MM_n(w_n^{2,i}, \{v_n(t) > b_n^1\}) \le \\ &\le \frac{2h_n\varepsilon_n}{\delta^2(\eta_n + (b_n^1)^2)} \left[\int_{S_n^i \cap Q_i} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, dx + \frac{1-\sigma}{\sigma} |S_n^i \cap Q_i| \right] + \\ &\quad + \frac{1}{2\varepsilon_n h_n} \left[\frac{\sigma}{1-\sigma} \int_{S_n^i \cap Q_i} (\eta_n + v_n^2(t))|\nabla u_n(t)|^2 \, dx + |S_n^i \cap Q_i| \right]. \end{aligned}$$

Summing on $i = 1, \dots, k$, recalling (6.25) and letting $d \in]0,1]$ with $\eta_n + (b_n^1)^2 \geq d^2$ for all n, we obtain

$$\sum_{i=1}^k MM_n(w_n^{2,i}, \{v_n(t) > b_n^1\}) \leq$$
$$\leq h_n \varepsilon_n \frac{2}{\delta^2 d^2} \left[C_1 + \frac{1-\sigma}{\sigma} |\cup Q_i| \right] + \frac{1}{\varepsilon_n h_n} \frac{1}{2} \left[\frac{\sigma}{1-\sigma} C_1 + |\cup Q_i| \right].$$

We choose h_n in such a way that the preceding quantity is less than (recall that $|\cup Q_i| \leq |U| < \sigma$)

$$\sqrt{\frac{1}{\delta^2 d^2} (C_1 + 1 - \sigma) \left(\frac{\sigma}{1-\sigma} C_1 + \sigma \right)}.$$

Then we obtain

$$\sum_{i=1}^k MM_n(w_n^{2,i}, \{v_n(t) > b_n^1\}) \leq \sqrt{\frac{1}{\delta^2 d^2} (C_1 + 1 - \sigma)) \left(\frac{\sigma}{1-\sigma} C_1 + \sigma \right)} = o(\sigma).$$

This prove the first part of the lemma.

Let us define $\varphi_n^{2,i}$ as $w_n^{2,i}$ but operating with the levels $\gamma_n^i - \tau_n^i \leq \gamma_n^i - \frac{\tau_n^i}{2}$ and $\gamma_n^i + \frac{\tau_n^i}{2} \leq \gamma_n^i + \tau_n^i$. Reasoning as above we obtain

$$\eta_n \sum_{i=1}^k \int_{Q_i \cap \{v_n(t) > b_n^1\}} |\nabla \varphi_n^{2,i}|^2 \, dx \leq \frac{16 \eta_n h_n}{\delta^2 d^2} (C_1 + 1 - \sigma) \to 0$$

since h_n has been chosen of the order of $\frac{1}{\varepsilon_n}$. □

Lemma 6.6.6. *Let $Q_i \subseteq \Omega$. Then there exists $w_n^{3,i} \in H^1(Q_i)$ such that $0 \leq w_n^{3,i} \leq 1$, $w_n^{3,i} = 0$ in a neighborhood of $H_i^+ \setminus E^n_{a_2(x_i) - \frac{3}{4}\delta}$ and of $H_i^- \cap E^n_{a_1(x_i) + \frac{3}{4}\delta}$, $w_n^{3,i} = 1$ on $Q_i \setminus R_i$ for n large, and*

$$\limsup_n \sum_{Q_i \subseteq \Omega} MM_n(w_n^{3,i}, \{v_n(t) > b_n^1\}) \leq o(\sigma). \tag{6.67}$$

Moreover there exists a cut-off function $\varphi_n^{3,i} \in H^1(Q_i)$ such that $\varphi_n^{3,i} = 0$ in a neighborhood of $H_i^+ \setminus E^n_{a_2(x_i) - \delta}$ and of $H_i^- \cap E^n_{a_1(x_i) + \delta}$, $\varphi_n^{3,i} = 1$ on $Q_i \setminus R_i$ for n large, $\mathrm{supp}(\nabla \varphi_n^{3,i}) \subseteq \{w_n^{3,i} = 0\}$, *and*

$$\lim_n \eta_n \int_{Q_i \cap \{v_n(t) > b_n^1\}} |\nabla \varphi_n^{3,i}|^2 \, dx = 0. \tag{6.68}$$

Proof. Let $\pi_i^\pm$ be the planes which contain $H_i^\pm$, and for $x \in \Omega'$, let $\pi_i^\pm x$ be its projection on $\pi_i^\pm$. Let us now consider $(u_n(t))_{|H_i^+}$: we set

$$\psi_n^{i,+}(y) := \frac{4}{\delta} \left(u_n(y) - a_2(x_i) + \frac{3}{4}\delta \right)^+ \wedge 1$$

Note that $\psi_n^{i,+}$ is equal to zero on $H_i^+ \setminus E^n_{a_2(x_i) - \frac{3}{4}\delta}$ and so on $\{x \in H_i^+ : u_n(t)(x) = \gamma_n^i\}$ where γ_n^i is defined as in Lemma 6.6.5. Moreover, $\psi_n^{i,+} = 1$ on $H_i^+ \cap E^n_{a_2(x_i) - \frac{\delta}{2}}$. If $d \in]0,1]$ is such that $\eta_n + (b_n^1)^2 \geq d^2$, by (6.55) we have

$$\int_{H_i^+ \cap \{v_n(t) > b_n^1\}} |\nabla \psi_n^{i,+}|^2 \, d\mathcal{H}^{N-1} \leq \frac{16}{\delta^2} \int_{H_i^+ \cap \{v_n(t) > b_n^1\}} |\nabla u_n|^2 \, d\mathcal{H}^{N-1} \leq \frac{16 K_n}{\delta^2 d^2}. \tag{6.69}$$

Let us define

$$\tilde{\psi}_n^{i,+}(y) := \frac{4}{\delta}(u_n(y) - a_2(x_i) + \delta)^+ \wedge 1$$

which is null on $H_i^+ \setminus E^n_{a_2(x_i)-\delta}$.

In a similar way we construct $\psi_n^{i,-}$ and $\tilde{\psi}_n^{i,-}$ on H_i^- which are null on $H_i^- \cap E^n_{a_1(x_i)+\frac{3}{4}\delta}$ and on $H_i^- \cap E^n_{a_1(x_i)+\delta}$ respectively. Let us set

$$w_n^{3,i,\pm}(x) := \left[\psi_n^{i,\pm}(\pi_i^\pm x) + \frac{1}{\varepsilon_n}(d_{H_i^\pm}(x) - l_n^i)^+\right] \wedge 1$$

with $\frac{l_n^i}{\varepsilon_n} \to 0$ and $\frac{\eta_n}{(l_n^i)^2} \to 0$. This is possible since $\eta_n << \varepsilon_n$. Let $A_n^i := (H_i^+ \setminus E^n_{a_2(x_i)-\frac{\delta}{2}})\times] - \varepsilon_n - l_n^i, \varepsilon_n + l_n^i[\cap\{v_n(t) > b_n^1\}$. Then we have by definition of $\psi_n^{i,\pm}$, by (6.54), (6.69) and the fact that $K_n\varepsilon_n$ is bounded in n

$$\begin{aligned}
&\limsup_n MM_n(w_n^{3,i,+}, \{v_n(t) > b_n^1\}) = \\
&= \limsup_n \left\{\frac{\varepsilon_n}{2}\int_{Q_i\cap\{v_n(t)>b_n^1\}} |\nabla w_n^{3,i,+}|^2\,dx + \frac{1}{2\varepsilon_n}\int_{Q_i\cap\{v_n(t)>b_n^1\}} (1-w_n^{3,i,+})^2\,dx\right\} \le \\
&\qquad \le \limsup_n \left\{\frac{\varepsilon_n}{2}\int_{A_n^i}\left(|\nabla\psi_n(\pi_i^+ x)|^2 + \frac{1}{\varepsilon_n^2}\right)dx + \right. \\
&\qquad\qquad \left. + \frac{1}{2\varepsilon_n}\left(2\mathcal{H}^{N-1}(H_i^+ \setminus E^n_{a_2(x_i)-\frac{\delta}{2}})(\varepsilon_n + l_n^i)\right)\right\}
\end{aligned}$$

so that we get

$$\begin{aligned}
&\limsup_n MM_n(w_n^{3,i,+}, \{v_n(t) > b_n^1\}) = \\
&\le \limsup_n \left\{\frac{\varepsilon_n}{2}\frac{16K_n}{\delta^2 d^2}2(\varepsilon_n + l_n^i) + \frac{\varepsilon_n}{2}\frac{2}{\varepsilon_n^2}(\varepsilon_n + l_n^i)\mathcal{H}^{N-1}(H_i^+ \setminus E^n_{a_2(x_i)-\frac{\delta}{2}}) + \right. \\
&\qquad \left. + \frac{\varepsilon_n + l_n^i}{\varepsilon_n}\mathcal{H}^{N-1}(H_i^+ \setminus E^n_{a_2(x_i)-\frac{\delta}{2}})\right\} \le \\
&\qquad\qquad \le 2\limsup_n \mathcal{H}^{N-1}(H_i^+ \setminus E^n_{a_2(x_i)-\frac{\delta}{2}}) \le 4\sigma r_i^{N-1}.
\end{aligned}$$

Similar calculations hold for $w_n^{3,i,-}$. Let us set $w_n^{3,i} := w_n^{3,i,+} \wedge w_n^{3,i,-}$. Then $0 \le w_n^{3,i} \le 1$, $w_n^{3,i} = 0$ in a neighborhood of $H_i^+ \setminus E^n_{a_2(x_i)-\frac{3}{4}\delta}$ and of $H_i^- \cap E^n_{a_1(x_i)+\frac{3}{4}\delta}$, $w_n^{3,i} = 1$ on $Q_i \setminus R_i$ for n large, and we have that

$$\limsup_n \sum_{Q_i \subseteq \Omega} MM_n(w_n^{3,i}, \{v_n(t) > b_n^1\}) \le o(\sigma),$$

which prove the first part of the lemma.

We define

$$\varphi_n^{3,i}(x) := \left[\tilde{\psi}_n^{i,+}(\pi_i^+ x) + \frac{1}{l_n^i}\left(d_{H_i^+}(x) - \frac{l_n^i}{2}\right)^+\right] \wedge \left[\tilde{\psi}_n^{i,-}(\pi_i^- x) + \frac{1}{l_n^i}\left(d_{H_i^-}(x) - \frac{l_n^i}{2}\right)^+\right] \wedge 1.$$

The previous calculations prove that

$$\lim_n \eta_n \int_{Q_i\cap\{v_n(t)>b_n^1\}} |\nabla\varphi_n^{3,i}|^2\,dx = 0$$

since $\dfrac{\eta_n}{(l_n^i)^2} \to 0$. Moreover $\varphi_n^{3,i} = 1$ on $Q_i \setminus R_i$ for n large. □

Lemma 6.6.7. *Suppose that $Q_i \subseteq \Omega$; then there exists $w_n^{4,i} \in H^1(\Omega')$ such that $0 \le w_n^{4,i} \le 1$, $w_n^{4,i} = 0$ in a neighborhood of V_i, $w_n^{4,i} = 1$ on Ω_D for n large and*

$$\limsup_n \sum_{Q_i \subseteq \Omega} MM_n(w_n^{4,i}) \le o(\sigma). \tag{6.70}$$

Moreover there exists a cut-off function $\varphi_n^{4,i}$ such that $\varphi_n^{4,i} = 0$ in a neighborhood of V_i, $\varphi_n^{4,i} = 1$ on Ω_D for n large, $\operatorname{supp}(\nabla \varphi_n^{4,i}) \subseteq \{w_n^{4,i} = 0\}$, *and*

$$\lim_n \eta_n \int_{\Omega'} |\nabla \varphi_n^{4,i}|^2 \, dx = 0. \tag{6.71}$$

Proof. Let us set

$$w_n^{4,i}(x) := \frac{1}{\varepsilon_n}(d_{V_i}(x) - l_n^i)^+ \wedge 1,$$

and

$$\varphi_n^{4,i}(x) := \frac{1}{l_n^i}\left(d_{V_i}(x) - \frac{l_n^i}{2}\right)^+ \wedge 1$$

where $\frac{l_n^i}{\varepsilon_n} \to 0$ and $\frac{\eta_n}{(l_n^i)^2} \to 0$. We have immediately (since $\sum_{Q_i \subseteq \Omega} \mathcal{H}^{N-1}(V_i) \le o(\sigma)$)

$$\limsup_n \sum_{Q_i \subseteq \Omega} MM_n(w_n^{4,i}) \le o(\sigma)$$

while

$$\lim_n \eta_n \int_{\Omega'} |\nabla \varphi_n^{4,i}|^2 \, dx = 0$$

since $\frac{\eta_n}{(l_n^i)^2} \to 0$. For n large enough, $w_n^{4,i} = 1$, $\varphi_n^{4,i} = 1$ on Ω_D and the proof is complete. □

We recall that $z = g_h(t)$ in a neighborhood $\mathcal{V}$ of $\partial\Omega \setminus \cup Q_i$.

Lemma 6.6.8. *Let $Q_i \cap \partial_D \Omega \neq \emptyset$ with $Q_i^+ \setminus R_i \subseteq \Omega$. Then $E^n_{a_1(x_i)+\frac{\delta}{2}} \cap Q_i \subseteq \Omega$ for all n, and there exists $w_n^{b,i,+} \in H^1(\Omega')$ with $0 \le w_n^{b,i,+} \le 1$, $w_n^{b,i,+} = 1$ on Ω_D, $w_n^{b,i,+} = 0$ in a neighborhood of*

$$V_i^{n,+} := \left[V_i \cap E^n_{a_1(x_i)+\delta}\right] \cup \left[(V_i \cap Q_i^+) \setminus \mathcal{V}\right],$$

and such that

$$\limsup_n \sum_{Q_i \cap \partial_D \Omega \neq \emptyset} MM_n(w_n^{b,i,+}, \{v_n(t) > b_n^1\}) \le o(\sigma). \tag{6.72}$$

Moreover there exists a cut-off function $\varphi_n^{b,i,+}$ such that $\varphi_n^{b,i,+} = 1$ on Ω_D, $\varphi_n^{b,i,+} = 0$ in a neighborhood of $V_i^{n,+}$, $\operatorname{supp}(\nabla \varphi_n^{b,i,+}) \subseteq \{w_n^{b,i,+} = 0\}$, *and*

$$\lim_n \eta_n \int_{\Omega' \cap \{v_n(t) > b_n^1\}} |\nabla \varphi_n^{b,i,+}|^2 \, dx = 0 \tag{6.73}$$

Proof. Note that by construction, $E^n_{a_1(x_i)+\frac{\delta}{2}} \cap Q_i \subseteq \Omega$ since $u_n(t)$ is continuous and $u_n(t) = g_h(t)$ on Ω_D. It is now sufficient to operate as in Lemma 6.6.6 and in Lemma 6.6.7. In fact, in view of (6.58), we may construct $\tilde{w}_n^{b,i,+} \in H^1(\Omega')$ such that $0 \le \tilde{w}_n^{b,i,+} \le 1$, $\tilde{w}_n^{b,i,+} = 0$ in a neighborhood of $V_i \cap E^n_{a_1(x_i)+\delta}$, $\tilde{w}_n^{b,i,+} = 1$ on Ω_D and on $V_i \setminus E^n_{a_1(x_i)+\frac{\delta}{2}}$, and such that $\limsup_n MM_n(\tilde{w}_n^{b,i,+}, \{v_n(t) > b_n^1\}) \le o(\sigma) r_i^{N-1}$. Referring to $(V_i \cap Q_i^+) \setminus \mathcal{V}$, we can reason as in Lemma 6.6.7 getting $\overline{w}_n^{b,i,+}$, such that $0 \le \overline{w}_n^{b,i,+} \le 1$, $\overline{w}_n^{b,i,+} = 0$ in a neighborhood of $(V_i \cap Q_i^+) \setminus \mathcal{V}$, $\overline{w}_n^{b,i,+} = 1$ on Ω_D, and such that $\limsup_n MM_n(\overline{w}_n^{b,i,+}) \le o(\sigma) r_i^{N-1}$.

Setting $w_n^{b,i,+} := \tilde{w}_n^{b,i,+} \wedge \overline{w}_n^{b,i,+}$, we get the first part of the thesis. Similarly, we may construct $\varphi_n^{b,i,+}$ which satisfies (6.73). □

In a similar way we can prove the following lemma.

Lemma 6.6.9. *Let $Q_i \cap \partial_D\Omega \neq \emptyset$ with $Q_i^- \setminus R_i \subseteq \Omega$. Then $Q_i \setminus E^n_{a_2(x_i)-\frac{\delta}{2}} \subseteq \Omega$ for all n, and there exists $w_n^{b,i,-} \in H^1(\Omega')$ with $0 \leq w_n^{b,i,-} \leq 1$, $w_n^{b,i,-} = 1$ on Ω_D, $w_n^{b,i,-} = 0$ in a neighborhood of*

$$V_i^{n,-} := \left[V_i \setminus E^n_{a_2(x_i)-\delta}\right] \cup \left[(V_i \cap Q_i^-) \setminus \mathcal{V}\right],$$

and such that

$$\limsup_n \sum_{Q_i \cap \partial_D\Omega \neq \emptyset} MM_n(w_n^{b,i,-}, \{v_n(t) > b_n^1\}) \leq o(\sigma). \tag{6.74}$$

Moreover there exists a cut-off function $\varphi_n^{b,i,-}$ such that $\varphi_n^{b,i,-} = 1$ on Ω_D, $\varphi_n^{b,i,-} = 0$ in a neighborhood of $V_i^{n,-}$, $\operatorname{supp}(\nabla\varphi_n^{b,i,-}) \subseteq \{w_n^{b,i,-} = 0\}$, and

$$\lim_n \eta_n \int_{\Omega' \cap \{v_n(t) > b_n^1\}} |\nabla\varphi_n^{b,i,-}|^2 \, dx = 0 \tag{6.75}$$

We can now prove Lemma 6.6.3.

Proof of Lemma 6.6.3. We employ the notation of the preceding lemmas. Following [53, Theorem 2.1], for each i let us define z_i^+ on $Q_i^+ \cup R_i$ to be equal to z on $Q_i^+ \setminus R_i$ and to the symmetrization of z with respect to $H_i(\sigma)$ on R_i. Similarly we define z_i^-.

For each $Q_i \subseteq \Omega$, let us set z_n^i to be equal to z_i^+ on $(Q_i^+ \setminus \tilde{R}_i) \cup (E^n_{\gamma_n^i} \cap \tilde{R}_i)$, and to z_i^- in the rest of Q_i.

If $Q_i \cap \partial_D\Omega \neq \emptyset$ with $Q_i^+ \setminus R_i \subseteq \Omega$, by Lemma 6.6.5 and Lemma 6.6.8 we have $E^n_{\gamma_n^i - \tau_n^i} \cap Q_i \subseteq Q_i^+$ for all n, and its closure does not intersect $\partial\Omega$. We define z_n^i to be equal to z_i^+ on $(Q_i^+ \setminus \tilde{R}_i) \cup (E^n_{\gamma_n^i} \cap \tilde{R}_i)$, and to $g_h(t)$ in the rest of Q_i. If $Q_i^- \setminus R_i \subseteq \Omega$, by Lemma 6.6.5 and Lemma 6.6.9 we have $Q_i \setminus E^n_{\gamma_n^i + \tau_n^i} \subseteq \Omega$, and its closure does not intersect $\partial\Omega$. We define z_n^i to be equal to z_i^- on $(Q_i^- \setminus \tilde{R}_i) \cup (\tilde{R}_i \setminus E^n_{\gamma_n^i})$, and to $g_h(t)$ in the rest of Q_i.

Let us now define $\tilde{z}_n$ to be equal to z outside $\bigcup_{i=1}^k R_i$, and to z_n^i inside each R_i. We have $\tilde{z}_n = g_h(t)$ on Ω_D. Note that if $Q_i \subseteq \Omega$, $H_i^+ \setminus E^n_{\gamma_n^i}$, $H_i^- \cap E^n_{\gamma_n^i}$, $V_i^{\pm}$, and $\partial^* E^n_{\gamma_n^i} \cap Q_i$ could be contained in $S(\tilde{z}_n)$. Similarly, if $Q_i \cap \partial\Omega \neq \emptyset$ and $Q_i^+ \setminus R^i \subseteq \Omega$ (the other case being similar), then $H_i^+ \setminus E^n_{\gamma_n^i}$, $V_i^{n,\pm}$ and $\partial^* E^n_{\gamma_n^i} \cap Q_i$ could be contained in $S(\tilde{z}_n)$.

By assumption on U, we have that

$$||\tilde{z}_n - z||_{L^2(\Omega')} + ||\nabla\tilde{z}_n - \nabla z||_{L^2(\Omega';\mathbb{R}^N)} \leq o(\sigma); \tag{6.76}$$

moreover, besides the possible jumps previously individuated, $\tilde{z}_n$ has in R_i polyhedral jumps which are a reflected version of the polyhedral jumps of z in Q_i. By assumption on z, we conclude that the union of these polyhedral sets $P_i(S(z))$ has $\mathcal{H}^{N-1}$ measure which is of the order of σ that is $\mathcal{H}^{N-1}(P(S(z))) \leq o(\sigma)$ where $P(S(z)) := \bigcup_{i=1}^k P_i(S(z))$.

Let $\tilde{w}_n$ be optimal for the Ambrosio-Tortorelli approximation of $[S(z) \setminus (\bigcup Q_i)] \cup P(S(z))$ (as we can find for example in [51, Lemma 3.3]), that is $\tilde{w}_n$ is null in a neighborhood of $[S(z) \setminus (\bigcup Q_i)] \cup P(S(z))$ and

$$\begin{aligned} \limsup_n MM_n(\tilde{w}_n) &\leq \mathcal{H}^{N-1}(S(z) \setminus (\cup Q_i) \cup P(S(z))) \leq \\ &\leq \mathcal{H}^{N-1}(S(z) \setminus S(u(t))) + o(\sigma). \end{aligned} \tag{6.77}$$

As in [51], let $\tilde{\varphi}_n$ be a cut-off function associated to $\tilde{w}_n$, such that

$$\lim_n \eta_n \int_{\Omega'} |\nabla\tilde{\varphi}_n|^2 \, dx = 0. \tag{6.78}$$

Let us set for all $Q_i \subseteq \Omega$

$$w_n^i := \begin{cases} \min\{\tilde{w}_n, w_n^{2,i}, w_n^{3,i}, w_n^{4,i}\} & \text{in } \tilde{R}_i \\ \min\{\tilde{w}_n, w_n^{3,i}, w_n^{4,i}\} & \text{in } R_i \setminus \tilde{R}_i \\ \min\{\tilde{w}_n, w_n^{4,i}\} & \text{outside } R_i, \end{cases}$$

and

$$\varphi_n^i := \begin{cases} \min\{\tilde{\varphi}_n, \varphi_n^{2,i}, \varphi_n^{3,i}, \varphi_n^{4,i}\} & \text{in } \tilde{R}_i \\ \min\{\tilde{\varphi}_n, \varphi_n^{3,i}, \varphi_n^{4,i}\} & \text{in } R_i \setminus \tilde{R}_i \\ \min\{\tilde{\varphi}_n, \varphi_n^{4,i}\} & \text{outside } R_i. \end{cases}$$

For all Q_i such that $Q_i \cap \partial_D \Omega \neq \emptyset$ with $Q_i^+ \setminus R_i \subseteq \Omega$, let us set

$$w_n^i := \begin{cases} \min\{\tilde{w}_n, w_n^{2,i}, w_n^{3,i,+}, w_n^{b,i,+}\} & \text{in } \tilde{R}_i \cap E^n_{\gamma_n^i} \\ \min\{w_n^{2,i}, w_n^{b,i,+}\} & \text{in } (\tilde{R}_i \setminus E^n_{\gamma_n^i}) \cup Q_i^- \\ \min\{\tilde{w}_n, w_n^{3,i,+}, w_n^{b,i,+}\} & \text{in } R_i \setminus (\tilde{R}_i \cup Q_i^-) \\ 1 & \text{in } \Omega_D \\ \min\{\tilde{w}_n, w_n^{b,i,+}\} & \text{otherwise} \end{cases}$$

and

$$\varphi_n^i := \begin{cases} \min\{\tilde{\varphi}_n, \varphi_n^{2,i}, \varphi_n^{3,i,+}, \varphi_n^{b,i,+}\} & \text{in } \tilde{R}_i \cap E^n_{\gamma_n^i} \\ \min\{\varphi_n^{2,i}, \varphi_n^{b,i,+}\} & \text{in } (\tilde{R}_i \setminus E^n_{\gamma_n^i}) \cup Q_i^- \\ \min\{\tilde{\varphi}_n, \varphi_n^{3,i,+}, \varphi_n^{b,i,+}\} & \text{in } R_i \setminus (\tilde{R}_i \cup Q_i^-) \\ 1 & \text{in } \Omega_D \\ \min\{\tilde{\varphi}_n, \varphi_n^{b,i,+}\} & \text{otherwise} \end{cases}$$

Similarly we reason for the case $Q_i^- \setminus R_i \subseteq \Omega$. By construction, for all $i = 1, \dots, k$ we have that $w_n^i, \varphi_n^i \in H^1(\Omega')$, $0 \le w_n^i, \varphi_n^i \le 1$ and $w_n^i, \varphi_n^i = 1$ on Ω_D for n large.

Note that by Lemmas 6.6.5, 6.6.6, 6.6.7, 6.6.8 and 6.6.9, and by (6.77) and (6.78), we have that

$$\limsup_n \sum_{i=1}^k MM_n(w_n^i, \{v_n(t) > b_n^1\}) \le \mathcal{H}^{N-1}(S(z) \setminus S(u(t))) + o(\sigma), \tag{6.79}$$

and

$$\lim_n \eta_n \sum_{i=1}^k \int_{\Omega' \cap \{v_n(t) > b_n^1\}} |\nabla \varphi_n^i(x)|^2 \, dx = 0. \tag{6.80}$$

We are now in a position to conclude the proof. We set

$$v_n := \min\{w_n, w_n^i, i = 1, \dots, k\}, \quad \varphi_n := \min\{\varphi_n, \varphi_n^i, i = 1, \dots, k\}.$$

Note that $\varphi_n = 0$ in a neighborhood of $S(\tilde{z}_n)$, and $\varphi_n = 1$ on Ω_D for n large. Moreover $0 \le v_n \le w_n \le v_n(t)$ in Ω' and $v_n = 1$ on Ω_D. Let $z_n := \varphi_n \tilde{z}_n$; we have $z_n \in H^1(\Omega')$ with $z_n = g_h(t)$ on Ω_D. By (6.22), we have that

$$F_{\varepsilon_n}(u_n(t), v_n(t)) \le F_{\varepsilon_n}(z_n, v_n),$$

and so

$$\int_\Omega (\eta_n + v_n(t)^2)|\nabla u_n(t)|^2\,dx \le \int_{\Omega'} (\eta_n + v_n^2)|\nabla(\varphi_n \tilde{z}_n|^2\,dx + MM_n(v_n) - MM_n(v_n(t)).$$

We may write

$$\begin{aligned}
&\int_{\Omega'} (\eta_n + v_n(t)^2)|\nabla u_n(t)|^2\,dx \le \\
&\le \int_{\Omega'} (\eta_n + 1)|\nabla \tilde{z}_n|^2\,dx + \int_{\Omega'} (\eta_n + v_n^2)(2\nabla\varphi_n \nabla\tilde{z}_n + \tilde{z}_n|\nabla\varphi_n|^2)\,dx + \\
&+ MM_n(w_n) - MM_n(v_n(t)) + \sum_{i=1}^k MM_n(w_n^i, \{v_n(t) > b_n^1\}).
\end{aligned}$$

Taking into account (6.76), (6.60), (6.80), (6.59), and (6.79), we have that passing to the limit

$$\begin{aligned}
\int_{\Omega'} |\nabla u|^2\,dx \le \int_{\Omega'} |\nabla z|^2\,dx + \mathcal{H}^{N-1}(S(z) \setminus S(u(t))) + \\
+ \frac{2Ck}{(k-1)^2} + \frac{C}{(k-1)(1-b)^2} + \frac{Cb}{(1-b)^2} + o(\sigma),
\end{aligned}$$

so that, letting $\sigma \to 0$ and then $b \to 0$, $k \to \infty$ (which is permitted choosing appropriately j_2 and j_3), we obtain the thesis. □

We can now pass to the proof of Theorem 6.5.6. Given $0 = t_1 \le t_2 \le \ldots \le t_k = t$, it is sufficient to prove that

$$\int_{\Omega'} |\nabla u(t)|^2\,dx \le \int_{\Omega'} |\nabla z|^2\,dx + \mathcal{H}^{N-1}\left(S(z) \setminus \left(\bigcup_{i=1}^k S(u(t_i))\right)\right). \tag{6.81}$$

Passing to the sup on $t_1, \ldots, t_k$, we deduce in fact the thesis. We obtain (6.81) using the same arguments of Lemma 6.6.3; defining

$$J_j := \left\{ x \in \bigcup_{m=1,\ldots,k} \left(\bigcup_{a_1,a_2 \in A^k} [\partial^* E^k_{a_1} \cap \partial^* E^k_{a_2}] \right) : \min_{l=1,\ldots,k} [u_l(x)] > \frac{1}{j} \right\},$$

where E^k_a and A_k are defined as the corresponding sets for $u(t)$, following [53], we cover J_j in such a way that for all $x_i \in J_j$ there exists l with $x_i \in S(u(t_l))$ and

$$\mathcal{H}^{N-1}\left(\left[\bigcup_{r=1,\ldots,k} S(u(t_r)) \setminus S(u(t_l)) \right] \cap Q_i \right) \le \sigma r_i^{N-1}.$$

So in each Q_i there exists $u(t_l)$ such that $\bigcup_{r=1}^k S(u(t_r)) \cap Q_i$ is essentially (with respect to the measure $\mathcal{H}^{N-1}$) $S(u(t_l)) \cap Q_i$. Recalling that $v_n(t) \le v_n(t_l)$ for all $l = 1, \ldots, k$, we have

$$\int_{\Omega'} (\eta_n + v_n(t)^2)|\nabla u_n(t_l)|^2\,dx \le \int_{\Omega'} (\eta_n + v_n(t_l)^2)|\nabla u_n(t_l)|^2\,dx \le C,$$

and so it is readily seen that the arguments of Lemma 6.6.3 can be adapted to prove (6.81).

6.7 A final remark

The approximation result can be extended to recover the case of non isotropic surface energies, i.e., energies of the form

$$\int_{\Omega} |\nabla u|^2 \, dx + \int_{\Gamma} \varphi(\nu_x) \, d\mathcal{H}^{N-1}(x) \tag{6.82}$$

where ν_x is the normal to Γ at x, and φ is a norm on $\mathbb{R}^N$. In fact all the previous arguments are based on Theorem 6.1.1 concerning the elliptic approximation and on Theorem 6.6.1 about the density of piecewise smooth functions with respect to the total energy. An elliptic approximation of Ambrosio-Tortorelli type of (6.82) has been proved in [51], while a density result of piecewise smooth functions with respect to non-isotropic surface energies has been proved in [40]. We conclude that all the previous theorems can be modified in order to treat the more general energy (6.82).

Chapter 7

A discontinuous finite element approximation of quasi-static growth of brittle cracks

In this chapter [1] we propose a discretization of the model of quasistatic crack evolution using a suitable finite element method and we give a rigorous proof of its convergence to a quasistatic evolution in the sense of Francfort and Larsen [53]. We restrict our analysis to a two dimensional setting considering only a polygonal reference configuration $\Omega \subseteq \mathbb{R}^2$.

The discretization of the domain Ω is carried out, following [75] (see also [76]), considering two parameters $\varepsilon > 0$ and $a \in]0, \frac{1}{2}[$. We consider a regular triangulation $\mathbf{R}_\varepsilon$ of size ε of Ω, i.e. we assume that there exist two constants c_1 and c_2 so that every triangle $T \in \mathbf{R}_\varepsilon$ contains a ball of radius $c_1\varepsilon$ and is contained in a ball of radius $c_2\varepsilon$. In order to treat the boundary data, we assume also that $\partial_D\Omega$ is composed of edges of $\mathbf{R}_\varepsilon$. On each edge $[x, y]$ of $\mathbf{R}_\varepsilon$ we consider a point z such that $z = tx + (1-t)y$ with $t \in [a, 1-a]$. These points are called *adaptive vertices.* Connecting together the adaptive vertices, we divide every $T \in \mathbf{R}_\varepsilon$ into four triangles. We take the new triangulation $\mathbf{T}$ obtained after this division as the discretization of Ω. The family of all such triangulations is denoted by $\mathcal{T}_{\varepsilon,a}(\Omega)$.

The discretization of the energy functional is obtained restricting the total energy to the family of functions u which are affine on the triangles of some triangulation $\mathbf{T}(u) \in \mathcal{T}_{\varepsilon,a}(\Omega)$ and are allowed to jump across the edges of $\mathbf{T}(u)$. We indicate this space by $\mathcal{A}_{\varepsilon,a}(\Omega)$. The boundary data is assumed to belong to the space $\mathcal{AF}_\varepsilon(\Omega)$ of continuous functions which are affine on every triangle $T \in \mathbf{R}_\varepsilon$.

Given the boundary data $g \in W^{1,1}([0,1], H^1(\Omega))$ with $g(t) \in \mathcal{AF}_\varepsilon(\Omega)$ for all $t \in [0,1]$, we divide $[0,1]$ into subintervals $[t_i^\delta, t_{i+1}^\delta]$ of size $\delta > 0$ for $i = 0, \dots, N_\delta$, we set $g_i^\delta = g(t_i^\delta)$, and for all $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$ we indicate by $S_D^{g_i^\delta}(u)$ the edges of the triangulation $\mathbf{T}(u)$ contained in $\partial_D\Omega$ on which $u \neq g_i^\delta$. Using a variational argument we construct a *discrete evolution* $\{u_{\varepsilon,a}^{\delta,i} : i = 0, \dots, N_\delta\}$ such that $u_{\varepsilon,a}^{\delta,i} \in \mathcal{A}_{\varepsilon,a}(\Omega)$ for all $i = 0, \dots, N_\delta$, and such that considering the *discrete crack*

$$\Gamma_{\varepsilon,a}^{\delta,i} := \bigcup_{r=0}^{i} \left[S(u_{\varepsilon,a}^{\delta,r}) \cup S_D^{g_r^\delta}(u_{\varepsilon,a}^{\delta,r}) \right],$$

the following *unilateral minimality property* holds:

$$\int_\Omega |\nabla u_{\varepsilon,a}^{\delta,i}|^2 \, dx \leq \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^1 \left(\left(S(v) \cup S_D^{g_i^\delta}(v) \right) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1} \right). \tag{7.1}$$

[1] The results of this chapter are contained in the paper
A. Giacomini, M. Ponsiglione: A discontinuous finite element approximation of quasistatic growth of brittle fractures. *Numer. Funct. Anal. Optim.* **24** (2003), 813-850.

Moreover we get suitable estimates for the discrete total energy

$$\mathcal{E}^{\delta,i}_{\varepsilon,a} := \|\nabla u^{\delta,i}_{\varepsilon,a}\|^2_{L^2(\Omega;\mathbb{R}^2)} + \mathcal{H}^1\left(\Gamma^{\delta,i}_{\varepsilon,a}\right).$$

The definition of the discrete crack ensures that $\Gamma^{\delta,i}_{\varepsilon,a} \subseteq \Gamma^{\delta,j}_{\varepsilon,a}$ for all $i \leq j$, recovering in this discrete setting the irreversibilty condition for quasistatic crack evolution. The minimality property (7.1) is the reformulation in the finite element space of the static equilibrium condition.

In order to perform the asymptotic analysis of the *discrete evolution* $\{u^{\delta,i}_{\varepsilon,a} : i = 0, \dots, N_\delta\}$, we make the piecewise constant interpolation in time $u^{\delta}_{\varepsilon,a}(t) = u^{\delta,i}_{\varepsilon,a}$ and $\Gamma^{\delta}_{\varepsilon,a}(t) = \Gamma^{\delta,i}_{\varepsilon,a}$ for all $t^\delta_i \leq t < t^\delta_{i+1}$. The main result of the chapter is the following theorem.

Theorem 7.0.1. *Let $g \in W^{1,1}([0,1], H^1(\Omega))$ be such that $\|g(t)\|_\infty \leq C$ for all $t \in [0,1]$ and let $g_\varepsilon \in W^{1,1}([0,1], H^1(\Omega))$ be such that $\|g_\varepsilon(t)\|_\infty \leq C$, $g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega)$ for all $t \in [0,1]$ and*

$$g_\varepsilon \to g \quad \text{strongly in } W^{1,1}([0,1], H^1(\Omega)). \tag{7.2}$$

Given the discrete evolution $\{t \to u^\delta_{\varepsilon,a}(t)\}$ relative to the boundary data g_ε, let $\Gamma^\delta_{\varepsilon,a}$ and $\mathcal{E}^\delta_{\varepsilon,a}$ be the associated crack and total energy.

Then there exist $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, and a quasi-static evolution $\{t \to (u(t), \Gamma(t)), t \in [0,1]\}$ relative to the boundary data g, and such that setting $u_n := u^{\delta_n}_{\varepsilon_n,a_n}$, $\Gamma_n := \Gamma^{\delta_n}_{\varepsilon_n,a_n}$, $\mathcal{E}_n := \mathcal{E}^{\delta_n}_{\varepsilon_n,a_n}$, the following hold:

(a) if $\mathcal{N}$ is the set of discontinuities of $\mathcal{H}^1(\Gamma(\cdot))$, for all $t \in [0,1] \setminus \mathcal{N}$ we have

$$\nabla u_n(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega;\mathbb{R}^2) \tag{7.3}$$

and

$$\lim_n \mathcal{H}^1(\Gamma_n(t)) = \mathcal{H}^1(\Gamma(t)); \tag{7.4}$$

(b) for all $t \in [0,1]$ we have

$$\lim_n \mathcal{E}_n(t) = \mathcal{E}(t). \tag{7.5}$$

We conclude that we have the convergence of the total energy at each time $t \in [0,1]$, and the separate convergence of bulk and surface energy for all $t \in [0,1]$ except a countable set.

In order to prove Theorem 7.0.1, we proceed in two steps. Firstly, we fix a and let $\delta \to 0$ and $\varepsilon \to 0$. We obtain an evolution $\{t \to u_a(t) : t \in [0,1]\}$ such that $\nabla u^\delta_{\varepsilon,a}(t) \to \nabla u_a(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ for all t up to a countable set and such that the following minimality property holds: for all $v \in SBV(\Omega)$

$$\int_\Omega |\nabla u_a(t)|^2\,dx \leq \int_\Omega |\nabla v|^2\,dx + \mu(a)\mathcal{H}^1\left(\left(S(v) \cup (\partial_D\Omega \cap \{v \neq g(t)\})\right) \setminus \Gamma_a(t)\right), \tag{7.6}$$

where $\mu :]0, \frac{1}{2}[\to]0, +\infty[$ is a function independent of ε and δ, such that $\mu \geq 1$, $\lim_{a \to 0} \mu(a) = 1$ and $\Gamma_a(t) := \bigcup_{s \leq t, s \in D} S(u_a(s)) \cup (\partial_D\Omega \cap \{u_a(s) \neq g(s)\})$. The minimality property (7.6) takes into account possible anisotropies that could be generated as δ and $\varepsilon \to 0$: in fact, since a is fixed, we have that the angles of the triangles in $\mathcal{T}_{\varepsilon,a}(\Omega)$ are between fixed values (determined by a), and so cracks with certain directions cannot be approximated in length. In the second step, we let $a \to 0$ and determine from $\{t \to u_a(t) : t \in [0,1]\}$ a quasi-static evolution $\{t \to u(t) : t \in [0,1]\}$ in the sense of Francfort and Larsen. Then, using a diagonal argument, we find sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, and $a_n \to 0$ satisfying Theorem 7.0.1.

The main difficulties arise in the first part of our analysis, namely when $\delta, \varepsilon \to 0$. The convergence $u^\delta_{\varepsilon,a}(t) \to u_a(t)$ in $SBV(\Omega)$ for $t \in D \subseteq [0,1]$ countable and dense is easily obtained by means of Ambrosio's Compactness Theorem. The minimality property (7.6) derives from its discrete version (7.1) using a variant of Lemma 1.2 of [53]: given $v \in SBV(\Omega)$, we construct $v^\delta_{\varepsilon,a} \in \mathcal{A}_{\varepsilon,a}(\Omega)$ such that

$$\nabla v^\delta_{\varepsilon,a} \to \nabla v \quad \text{strongly in } L^2(\Omega;\mathbb{R}^2) \tag{7.7}$$

and

$$\limsup_{\delta,\varepsilon\to 0} \mathcal{H}^1\left[\left(S(v^\delta_{\varepsilon,a}) \cup S_D^{g^\delta_\varepsilon(t)}(v^\delta_{\varepsilon,a})\right) \setminus \Gamma^\delta_{\varepsilon,a}(t)\right] \leq \tag{7.8}$$
$$\leq \mu(a)\mathcal{H}^1\left[\left(S(v) \cup \left(\partial_D\Omega \cap \{v \neq g(t)\}\right)\right) \setminus \Gamma_a(t)\right],$$

where $g^\delta_\varepsilon(t) := g_\varepsilon(t^\delta_i)$ for $t^\delta_i \leq t < t^\delta_{i+1}$. The main difference with respect to Lemma 1.2 of [53] is that we have to find the approximating functions $v^\delta_{\varepsilon,a}$ in the finite element space $\mathcal{A}_{\varepsilon,a}(\Omega)$. This can be regarded as an interpolation problem, so we try to construct triangulations $\mathbf{T}_\varepsilon \in \mathcal{T}_{\varepsilon,a}(\Omega)$ adapted to v in order to obtain (7.7) and (7.8). In all the geometric operations involved, we need to avoid degeneration of the triangles of $\mathbf{T}(u^\delta_{\varepsilon,a}(t))$ which is guaranteed from the fact that a is constant: this is the principal reason to keep a fixed in the first step. A second difficulty arises when $u_a(\cdot)$ is extended from D to the entire interval $[0,1]$: indeed it is no longer clear whether $\nabla u^\delta_{\varepsilon,a}(t) \to \nabla u_a(t)$ for $t \notin D$. Since the space $\mathcal{A}_{\varepsilon,a}(\Omega)$ is not a vector space, we cannot provide an estimate on $\|\nabla u^\delta_{\varepsilon,a}(t) - \nabla u^\delta_{\varepsilon,a}(s)\|$ with $s \in D$ and $s < t$: we thus cannot expect to recover the convergence at time t from the convergence at time s. We overcome this difficulty observing that $\nabla u^\delta_{\varepsilon,a}(t) \to \nabla \tilde{u}_a$ with $\tilde{u}_a$ satisfying a minimality property similar to (7.6) and then proving $\nabla \tilde{u}_a = \nabla u_a(t)$ by a uniqueness argument for the gradients of the solutions.

The plan of the chapter is the following. In Section 7.1 we give the basic definitions and prove some auxiliary results. In Section 7.2, we prove the existence of a discrete evolution. In Section 7.3 we prove the convergence of the discrete evolution to a quasi-static evolution of brittle cracks in the sense of Francfort and Larsen. The proof of minimality property (7.6) requires a careful analysis to which is dedicated Section 7.4. In Section 7.5 we show that the arguments of Section 7.3 can be used to improve the convergence results for the discrete in time approximation considered in [53].

7.1 The discontinuous finite element space

Let $\Omega \subseteq \mathbb{R}^2$ be a polygonal set and let us fix two positive constants $0 < c_1 < c_2$. By a *regular triangulation* of Ω of size ε we intend a finite family of (closed) triangles T_i such that $\overline{\Omega} = \bigcup_i T_i$, $T_i \cap T_j$ is either empty or equal to a common edge or to a common vertex, and each T_i contains a ball of diameter $c_1\varepsilon$ and is contained in a ball of diameter $c_2\varepsilon$.

We indicate by $\mathcal{R}_\varepsilon(\Omega)$ the family of all regular triangulations of Ω of size ε. It turns out that there exist $0 < \vartheta_1 < \vartheta_2 < \pi$ such that for all T belonging to a triangulation $\mathbf{T} \in \mathcal{R}_\varepsilon(\Omega)$, the inner angles of T are between ϑ_1 and ϑ_2. Moreover, every edge of T has length greater than $c_1\varepsilon$ and lower than $c_2\varepsilon$.

Let us fix a triangulation $\mathbf{R}_\varepsilon \in \mathcal{R}_\varepsilon(\Omega)$ for all $\varepsilon > 0$ and let $a \in]0, \frac{1}{2}[$. Let us consider a new triangulation $\mathbf{T}$ nested in $\mathbf{R}_\varepsilon$ obtained dividing each $T \in \mathbf{R}_\varepsilon$ into four triangles taking over every edge $[x,y]$ of T a knot z which satisfies

$$z = tx + (1-t)y, \qquad t \in [a, 1-a].$$

We will call these new vertices *adaptive*, the triangles obtained joining these points *adaptive triangles*, and their edges *adaptive edges* (see Fig.1).

We denote by $\mathcal{T}_{\varepsilon,a}(\Omega)$ the set of all triangulations $\mathbf{T}$ constructed in this way. Note that for all $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$ there exists $0 < c^a_1 < c^a_2 < +\infty$ such that every $T_i \in \mathbf{T}$ contains a ball of diameter $c^a_1\varepsilon$ and is contained in a ball of diameter $c^a_2\varepsilon$. Then there exist $0 < \vartheta^a_1 < \vartheta^a_2 < \pi$ such that for all triangles T belonging to a triangulation $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$, the inner angles of T are between ϑ^a_1 and ϑ^a_2. Moreover, every edge of T has length greater than $c^a_1\varepsilon$ and lower than $c^a_2\varepsilon$.

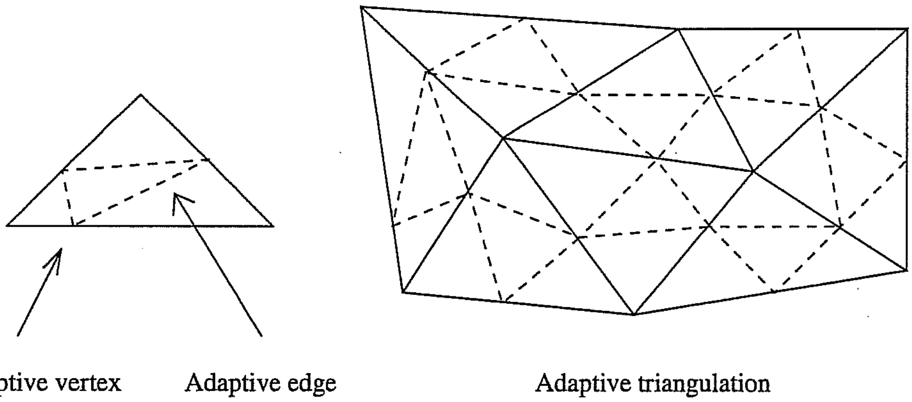

Fig. 1

We will often use the following *interpolation estimate* (see [36, Theorem 3.1.5]). If $u \in W^{2,2}(\Omega)$ and $T \in \mathbf{R}_\varepsilon$, let u_T denote the affine interpolation of u on T. We have that there exists K depending only on c_1, c_2 such that

$$\|u_T - u\|_{W^{1,2}(T)} \leq K\varepsilon \|u\|_{W^{2,2}(T)}. \tag{7.9}$$

Estimate (7.9) holds also for $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$: in this case K depends on a.

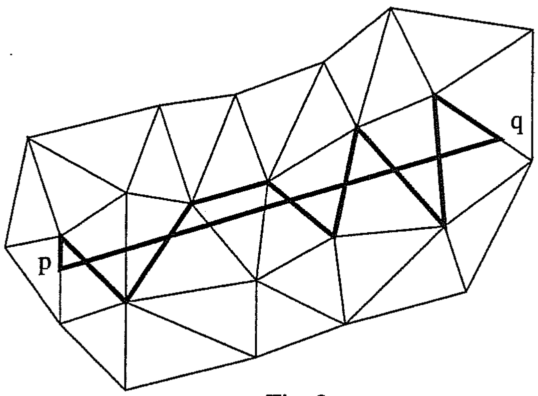

Fig. 2

In Section 7.4 we will need some elementary constructions that we collect here in some lemmas.

Lemma 7.1.1. *Let $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$, and let $l \subseteq \Omega$ be a segment with extremes p, q belonging to edges of $\mathbf{T}$. There exists a polyhedral curve Γ with extremal points p and q (see Fig.2) such that Γ is contained in the union of the edges of those $T \in \mathbf{T}$ with $T \cap l \neq \emptyset$, and such that the following properties hold:*

(1) $\Gamma = \gamma_p \cup \gamma \cup \gamma_q$, *where* γ *is union of edges of* $\mathbf{T}$ *and* γ_p, γ_q *are segments containing* p *and* q *respectively, and each one is contained in an edge of* $\mathbf{T}$*;*

(2) there exists a constant c *independent of* ϵ *(but depending on* a*) such that*

$$\mathcal{H}^1(\Gamma) \leq c\mathcal{H}^1(l).$$

Proof. Let $\{T_1, \dots T_k\}$ be the family of triangles in $\mathbf{T}$ such that the intersection with l is a segment with positive length. For every integer $1 \leq i \leq k$, let $l_i := T_i \cap l$. If l_i is an edge of T_i, we set $D_i = T_i$. Otherwise let D_i be a connected component of $T_i \setminus l_i$ such that $|D_i| \leq \frac{1}{2}|T_i|$. We claim that there exists a constant $c > 0$ independent of ϵ such that

$$\mathcal{H}^1(\partial D_i) \leq c\mathcal{H}^1(l_i). \tag{7.10}$$

We have to analyze two possibilities, namely D_i is a triangle, or D_i is a trapezoid. Suppose that D_i is a triangle and that m_i is an edge of D_i. Let α be the angle of D_i opposite to l_i. It is easy to prove that $\mathcal{H}^1(l_i) \geq \mathcal{H}^1(m_i)\sin\alpha$, and so

$$\mathcal{H}^1(l_i) \geq \frac{1}{3}\sin\alpha\mathcal{H}^1(\partial D_i).$$

Since $\vartheta_1^a \leq \alpha \leq \vartheta_2^a$, $\sin\alpha$ is uniformly bounded from below, and hence inequality (7.10) follows. If D_i is a trapezoid, since $|D_i| \leq \frac{1}{2}|T_i|$, it follows that $T_i \setminus D_i$ is a triangle such that its edges different from l_i have length greater than $\frac{1}{2}c_1^a\varepsilon$. Let α be the inner angle of $T_i \setminus D_i$ opposite to l_i. We have that

$$\mathcal{H}^1(l_i) \geq \frac{1}{2}\sin\alpha c_1^a\varepsilon \geq \frac{1}{2}\sin\alpha\frac{c_1^a}{c_2^a}\frac{1}{4}\mathcal{H}^1(\partial D_i).$$

Since $\vartheta_1^a \leq \alpha \leq \vartheta_2^a$, inequality (7.10) follows.

By (7.10), we deduce that

$$\mathcal{H}^1\left(\bigcup_{i=1}^k \partial D_i\right) \leq c\mathcal{H}^1(l);$$

moreover, since $\bigcup_{i=1}^k (\partial D_i \setminus (l_i \cap \operatorname{int}(T_i)))$ is arcwise connected and contains p, q, we conclude that there exists a curve $\Gamma \subseteq \bigcup_{i=1}^k \partial D_i$ which satisfies the thesis. $\square$

Lemma 7.1.2. *There exists a constant* $c > 0$ *such that for every segment* $l \subseteq \Omega$ *there exists* ε_0 *with the following property: for every* $\varepsilon \leq \varepsilon_0$*, setting* $\mathcal{R}(l) := \{T \in \mathbf{R}_\varepsilon : T \cap l \neq \emptyset\}$*, we have*

$$\mathcal{H}^1(\partial\mathcal{R}(l)) \leq c\mathcal{H}^1(l).$$

Proof. Let $\mathcal{N}_\varepsilon(l) := \{x \in \Omega : \operatorname{dist}(x, l) \leq c_2\varepsilon\}$. We have that $|\mathcal{N}_\varepsilon(l)| = \mathcal{H}^1(l)c_2\varepsilon + \pi c_2^2\varepsilon^2$, and hence there exists a positive constant ε_0 such that, for every $\varepsilon \leq \varepsilon_0$, we have that

$$|\mathcal{N}_\varepsilon(l)| \leq 2\mathcal{H}^1(l)c_2\varepsilon.$$

We have that $\mathcal{R}(l) \subseteq \mathcal{N}_\varepsilon(l)$, and

$$\sharp\mathcal{R}(l) \leq \frac{4}{c_1^2\pi^2}\frac{|\mathcal{N}_\varepsilon(l)|}{\varepsilon^2},$$

where $\sharp\mathcal{R}(l)$ denotes the number of triangles of $\mathcal{R}(l)$. Then, we have

$$\mathcal{H}^1(\partial\mathcal{R}(l)) \leq 3c_2\varepsilon\,\sharp\mathcal{R}(l) \leq 3c_2\varepsilon\frac{4}{c_1^2\pi^2}\frac{|\mathcal{N}_\varepsilon(l)|}{\varepsilon^2} \leq 3c_2^2\frac{4}{c_1^2\pi^2}2\mathcal{H}^1(l),$$

and so the proof is concluded. $\square$

7.2 The discontinuous finite element approximation

In this section we construct a discrete approximation of quasi-static evolution of brittle cracks in linearly elastic bodies: the discretization is done both in space and time.

From now on we suppose that Ω is a polygonal open bounded subset of $\mathbb{R}^2$, and that $\partial_D\Omega \subseteq \partial\Omega$ is open in the relative topology. For all $\varepsilon > 0$, we fix a triangulation $\mathbf{R}_\varepsilon \in \mathcal{R}_\varepsilon(\Omega)$, and suppose that $\partial_D\Omega$ is composed of edges of $\mathbf{R}_\varepsilon$ for all ε; we indicate the family of these edges by $\mathbf{S}_\varepsilon$.

We consider the following discontinuous finite element space. We indicate by $\mathcal{A}_{\varepsilon,a}(\Omega)$ the set of all u such that there exists a triangulation $\mathbf{T}(u) \in \mathcal{T}_{\varepsilon,a}(\Omega)$ nested in $\mathbf{R}_\varepsilon$ with u affine on every $T \in \mathbf{T}(u)$. For every $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$, we write $\|\nabla u\|$ for the L^2-norm of ∇u and we indicate by $S(u)$ the family of edges of $\mathbf{T}(u)$ inside Ω across which u is discontinuous. Notice that $u \in SBV(\Omega)$ and that the notation is consistent with the usual one employed in the theory of functions with bounded variation. Let us also denote by $\mathcal{AF}_\varepsilon(\Omega)$ the set of affine functions in Ω with respect to the triangulation $\mathbf{R}_\varepsilon$. Finally, given any $g \in \mathcal{AF}_\varepsilon(\Omega)$, for all $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$ we set

$$S^g_D(u) := \{\zeta \in \mathbf{S}_\varepsilon \,:\, u \neq g \text{ on } \zeta\}, \tag{7.11}$$

that is $S^g_D(u)$ denotes the edges at which the boundary condition is not satisfied. Moreover we set

$$S^g(u) := S(u) \cup S^g_D(u) \tag{7.12}$$

Let now consider $g \in W^{1,1}([0,1]; H^1(\Omega))$ with $g(t) \in \mathcal{AF}_\varepsilon(\Omega)$ for all $t \in [0,1]$. Let $\delta > 0$ and let N_δ be the largest integer such that $\delta(N_\delta - 1) < 1$; for $0 \le i \le N_\delta - 1$ we set $t^\delta_i := i\delta$, $t^\delta_{N_\delta} := 1$ and $g^\delta_i := g(t^\delta_i)$. The following proposition holds.

Proposition 7.2.1. *Let $\varepsilon > 0$, $a \in]0, \frac{1}{2}]$ and $\delta > 0$ be fixed. Then for all $i = 0, \dots, N_\delta$ there exists $u^{\delta,i}_{\varepsilon,a} \in \mathcal{A}_{\varepsilon,a}(\Omega)$ such that, setting*

$$\Gamma^{\delta,i}_{\varepsilon,a} := \bigcup_{r=0}^{i} S^{g^\delta_r}(u^{\delta,r}_{\varepsilon,a}),$$

the following hold:

(a) $\|u^{\delta,i}_{\varepsilon,a}\|_\infty \le \|g^\delta_i\|_\infty$;

(b) for all $v \in \mathcal{A}_{\varepsilon,a}(\Omega)$ we have

$$\|\nabla u^{\delta,0}_{\varepsilon,a}\|^2 + \mathcal{H}^1\left(S^{g^\delta_0}(u^{\delta,0}_{\varepsilon,a})\right) \le \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g^\delta_0}(v)\right), \tag{7.13}$$

and

$$\|\nabla u^{\delta,i}_{\varepsilon,a}\|^2 \le \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g^\delta_i}(v) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right). \tag{7.14}$$

Proof. The proof is carried out through a variational argument. Let $u^{\delta,0}_{\varepsilon,a}$ be a minimum of the following problem

$$\min\left\{\|\nabla u\|^2 + \mathcal{H}^1(S^{g^\delta_0}(u))\right\}. \tag{7.15}$$

We set $\Gamma^{\delta,0}_{\varepsilon,a} := S^{g^\delta_0}(u^{\delta,0}_{\varepsilon,a})$. Recursively, supposing to have constructed $u^{\delta,i-1}_{\varepsilon,a}$ and $\Gamma^{\delta,i-1}_{\varepsilon,a}$, let $u^{\delta,i}_{\varepsilon,a}$ be a minimum for

$$\min\left\{\|\nabla u\|^2 + \mathcal{H}^1\left(S^{g^\delta_i}(u) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right)\right\}. \tag{7.16}$$

We set $\Gamma^{\delta,i}_{\varepsilon,a} := S^{g^\delta_i}(u^{\delta,i}_{\varepsilon,a}) \cup \Gamma^{\delta,i-1}_{\varepsilon,a}$. We claim that problems (7.15) and (7.16) admit a solution $u^{\delta,i}_{\varepsilon,a}$ such that $\|u^{\delta,i}_{\varepsilon,a}\|_\infty \le \|g^\delta_i\|_\infty$ for all $i = 0, \dots, N_\delta$. We prove the claim for problem (7.16), the other case being similar. Let (u_n) be a minimizing sequence for problem (7.16): since g_i^δ is an admissible test function, we deduce that for n large

$$\|\nabla u_n\|^2 + \mathcal{H}^1\left(S^{g_i^\delta}(u_n) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right) \le \|\nabla g_i^\delta\|^2 + 1.$$

Moreover, we may modify u_n in the following way. If π denotes the projection in $\mathbb{R}$ over the interval $I := [-\|g_i^\delta\|_\infty, \|g_i^\delta\|_\infty]$, let $\tilde{u}_n \in \mathcal{A}_{\varepsilon,a}(\Omega)$ be defined on each $T \in \mathbf{T}(u_n)$ as the affine interpolation of the values $(\pi(u_n(x_1)),\ \pi(u_n(x_2)),\ \pi(u_n(x_3))$, where x_1, x_2 and x_3 are the vertices of T. Note that by construction we have for all n

$$\|\tilde{u}_n\|_\infty \le \|g_i^\delta\|_\infty, \qquad \|\nabla \tilde{u}_n\| \le \|\nabla u_n\|, \qquad S^{g_i^\delta}(\tilde{u}_n) \subseteq S^{g_i^\delta}(u_n),$$

so that $(\tilde{u}_n)$ is a minimizing sequence for problem (7.16). We conclude that it is not restrictive to assume $\|u_n\|_\infty \le \|g_i^\delta\|_\infty$.

Since $\mathbf{T}(u_n) \in \mathcal{T}_{\varepsilon,a}(\Omega)$, we have that the number of elements of $\mathbf{T}(u_n)$ is uniformly bounded. Up to a subsequence, we may suppose that there exists an integer k such that $\mathbf{T}(u_n)$ has exactly k elements $T_n^1, \dots, T_n^k$. Using a diagonal argument we may suppose that, up to a further subsequence, there exists $\mathbf{T} = \{T^1, \dots, T^k\} \in \mathcal{T}_{\varepsilon,a}(\Omega)$ such that $T_n^i \to T^i$ in the Hausdorff metric for all $i = 1, \dots, k$. Let us consider $T^i \in \mathbf{T}$, and let $\tilde{T}^i$ be contained in the interior of T^i. For n large enough, $\tilde{T}^i$ is contained in the interior of T_n^i and $(u_n)_{|\tilde{T}^i}$ is affine with $\int_{\tilde{T}^i} |\nabla u_n|^2\,dx \le C$ with C independent of n. We deduce that there exists a function u^i affine on $\tilde{T}^i$ such that up to a subsequence $u_n \to u$ uniformly on $\tilde{T}^i$. Since $\tilde{T}^i$ is arbitrary, it turns out that u^i is actually defined on T^i and

$$\int_{T^i} |\nabla u^i|^2\,dx \le \liminf_n \int_{T_n^i} |\nabla u_n|^2\,dx.$$

Let $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$ such that $u = u^i$ on T^i for every $i = 1, \dots, k$: we have

$$\|\nabla u\|^2 \le \liminf_n \|\nabla u_n\|^2.$$

On the other hand, it is easy to see that $S^{g_i^\delta}(u)$ is contained in the Hausdorff limit of $S^{g_i^\delta}(u_n)$, and that

$$\mathcal{H}^1\left(S^{g_i^\delta}(u) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right) \le \liminf_n \mathcal{H}^1\left(S^{g_i^\delta}(u_n) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right).$$

We conclude that u is a minimum point for the problem (7.16) with $\|u\|_\infty \le \|g_i^\delta\|_\infty$. We have that point (a) is proved.

Concerning point (b), by construction we get (7.13); for $i \ge 1$ we have

$$\|\nabla u_{\varepsilon,a}^{\delta,i}\|^2 + \mathcal{H}^1\left(S^{g_i^\delta}(u_{\varepsilon,a}^{\delta,i}) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right) \le \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g_i^\delta}(v) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right)$$

for all $v \in \mathcal{A}_{\varepsilon,a}(\Omega)$, so that

$$\|\nabla u_{\varepsilon,a}^{\delta,i}\|^2 \le \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g_i^\delta}(v) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right),$$

and this proves point (b). □

Remark 7.2.2. For technical reasons due to the asymptotic analysis of the discrete evolution $u_{\varepsilon,a}^{\delta,i}$ when $\delta \to 0$, $\varepsilon \to 0$ and $a \to 0$, we define $u_{\varepsilon,a}^{\delta,i}$ from $u_{\varepsilon,a}^{\delta,i-1}$ through problem (7.16) without requiring that the adaptive vertices determining $\Gamma_{\varepsilon,a}^{\delta,i-1}$ remain fixed. We just penalize their possible changes if they are used to create new cracks: in fact in this case, the surface energy increases at each change of a quantity at least of order $a\varepsilon$. As a consequence, during the step by step minimization, it could happen that some triangles $T \in \mathcal{T}_{\varepsilon,a}(\Omega)$ contain the crack $\Gamma_{\varepsilon,a}^{\delta,i}$ in their interior. This is in contrast with the interpretation of the triangles as elementary blocks for the elasticity problem, but being this situation penalized in the minimization process, we expect that it occurs rarely.

The following estimate is essential for the study of asymptotic behavior of the discrete evolution.

Proposition 7.2.3. *If $(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a})$ for $i = 0, \dots, N_\delta$ satisfies condition (b) of Proposition 7.2.1, setting $\mathcal{E}^{\delta,i}_{\varepsilon,a} := \|\nabla u^{\delta,i}_{\varepsilon,a}\|^2 + \mathcal{H}^1(\Gamma^{\delta,i}_{\varepsilon,a})$, we have for $0 \le j \le i \le N_\delta$*

$$\mathcal{E}^{\delta,i}_{\varepsilon,a} \le \mathcal{E}^{\delta,j}_{\varepsilon,a} + 2\sum_{r=j}^{i-1} \int_{t^\delta_r}^{t^\delta_{r+1}} \int_\Omega \nabla u^{\delta,r}_{\varepsilon,a} \nabla \dot{g}(\tau)\, dx\, d\tau + o^\delta, \tag{7.17}$$

where

$$o^\delta := \left[\max_{r=0,\dots,N_\delta-1} \int_{t^\delta_r}^{t^\delta_{r+1}} \|\dot{g}(\tau)\|_{H^1(\Omega)}\, d\tau \right] \int_0^1 \|\dot{g}(\tau)\|_{H^1(\Omega)}\, d\tau. \tag{7.18}$$

Proof. For all $0 \le j \le N_\delta - 1$, by construction of $u^{\delta,j+1}_{\varepsilon,a}$ we have that

$$\|\nabla u^{\delta,j+1}_{\varepsilon,a}\|^2 + \mathcal{H}^1\left(S^{g^\delta_{j+1}}(u^{\delta,j+1}_{\varepsilon,a}) \setminus \Gamma^{\delta,j}_{\varepsilon,a}\right) \le \|\nabla u^{\delta,j}_{\varepsilon,a} + \nabla(g^\delta_{j+1} - g^\delta_j)\|^2 =$$
$$= \|\nabla u^{\delta,j}_{\varepsilon,a}\|^2 + 2\int_\Omega \nabla u^{\delta,j}_{\varepsilon,a} \nabla(g^\delta_{j+1} - g^\delta_j)\, dx + \|\nabla(g^\delta_{j+1} - g^\delta_j)\|^2.$$

Notice that

$$\nabla(g^\delta_{j+1} - g^\delta_j) = \int_{t^\delta_j}^{t^\delta_{j+1}} \nabla \dot{g}(\tau)\, d\tau,$$

so that

$$\|\nabla u^{\delta,j+1}_{\varepsilon,a}\|^2 + \mathcal{H}^1\left(S^{g^\delta_{j+1}}(u^{\delta,j+1}_{\varepsilon,a}) \setminus \Gamma^{\delta,j}_{\varepsilon,a}\right) \le \tag{7.19}$$
$$\le \|\nabla u^{\delta,j}_{\varepsilon,a}\|^2 + 2\int_{t^\delta_j}^{t^\delta_{j+1}} \int_\Omega \nabla u^{\delta,j}_{\varepsilon,a} \nabla \dot{g}(\tau)\, dx\, d\tau + e(\delta) \int_{t^\delta_j}^{t^\delta_{j+1}} \|\dot{g}(\tau)\|_{H^1(\Omega)}\, d\tau,$$

where

$$e(\delta) := \max_{r=0,\dots,N_\delta-1} \int_{t^\delta_r}^{t^\delta_{r+1}} \|\dot{g}(\tau)\|_{H^1(\Omega)}\, d\tau.$$

From (7.19), we obtain that for all $0 \le j \le i \le N_\delta$

$$\|\nabla u^{\delta,i}_{\varepsilon,a}\|^2 + \mathcal{H}^1(\Gamma^{\delta,i}_{\varepsilon,a}) \le \|\nabla u^{\delta,j}_{\varepsilon,a}\|^2 + \mathcal{H}^1(\Gamma^{\delta,j}_{\varepsilon,a}) +$$
$$+ 2\sum_{r=j}^{i-1} \int_{t^\delta_r}^{t^\delta_{r+1}} \int_\Omega \nabla u^{\delta,r}_{\varepsilon,a} \nabla \dot{g}(\tau)\, dx\, d\tau + e(\delta) \int_{t^\delta_j}^{t^\delta_i} \|\dot{g}(\tau)\|_{H^1(\Omega)}\, d\tau,$$

and so the proof of point (c) is complete choosing

$$o^\delta := e(\delta) \int_0^1 \|\dot{g}(\tau)\|_{H^1(\Omega)}\, d\tau.$$

□

7.3 The convergence result

This section is devoted to the proof of Theorem 7.0.1. As in Section 7.2, let Ω be a polygonal open bounded subset of $\mathbb{R}^2$, and let $\partial_D\Omega \subseteq \partial\Omega$ be open in the relative topology. For all $\varepsilon > 0$, let $\mathbf{R}_\varepsilon \in \mathcal{R}_\varepsilon(\Omega)$ be a regular triangulation of Ω such that $\partial_D\Omega$ is composed of edges of $\mathbf{R}_\varepsilon$. As in the previous section, let $\mathcal{AF}_\varepsilon(\Omega)$ be the family of continuous piecewise affine functions with respect to $\mathbf{R}_\varepsilon$, and let $\mathcal{A}_{\varepsilon,a}(\Omega)$ be the family of functions which are affine on the triangles of some triangulation $\mathbf{T} \in \mathcal{T}_{\varepsilon,a}(\Omega)$ nested in $\mathbf{R}_\varepsilon$ and can jump across the edges of $\mathbf{T}$.

In the following, it will be useful to treat points at which the boundary condition is violated (see (7.11)) as internal jumps. Thus we consider Ω_D polygonal open bounded subset of $\mathbb{R}^2$ such that $\Omega_D \cap \Omega = \emptyset$ and $\partial\Omega \cap \partial\Omega_D = \partial_D\Omega$ up to a finite number of points; we set $\Omega' := \Omega \cup \Omega_D \cup \partial_D\Omega$. Given $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$ and $g \in \mathcal{AF}_\varepsilon(\Omega)$, we may extend g to a function of $H^1(\Omega')$ and u to a function $\tilde{u} \in SBV(\Omega')$ setting $\tilde{u} = g$ on Ω_D. In this way, recalling (7.12), we have

$$S^g(u) = S(\tilde{u}),$$

so that the violation of the boundary condition of u can be read in the set of jumps of $\tilde{u}$. Analogously, given $u \in SBV(\Omega)$ and $g \in H^1(\Omega)$, we set

$$S^g(u) := S(u) \cup \{x \in \partial_D\Omega \,:\, \gamma(u)(x) \neq \gamma(g)(x)\} \tag{7.20}$$

where γ denotes the trace operator on $\partial\Omega$. We may assume $g \in H^1(\Omega')$ using an extension operator. We can then consider $\tilde{u} \in SBV(\Omega')$ such that $\tilde{u} = u$ on Ω, and $\tilde{u} = g$ on Ω_D. In this way we have

$$S^g(u) = S(\tilde{u}) \quad \text{up to a set of } \mathcal{H}^1\text{-measure } 0.$$

Let us consider $g \in W^{1,1}([0,1], H^1(\Omega))$ such that $\|g(t)\|_\infty \le C$ for all $t \in [0,1]$ and let $g_\varepsilon \in W^{1,1}([0,1], H^1(\Omega))$ be such that $g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega)$ for all $t \in [0,1]$,

$$\|g_\varepsilon(t)\|_\infty \le C \tag{7.21}$$

for all $t \in [0,1]$, and for $\varepsilon \to 0$

$$g_\varepsilon \to g \quad \text{strongly in } W^{1,1}([0,1], H^1(\Omega)). \tag{7.22}$$

We indicate by $\{u^{\delta,i}_{\varepsilon,a},\, i = 0, \dots, N_\delta\}$ the discrete evolution relative to the boundary data g_ε given by Proposition 7.2.1, and we denote by $\mathcal{E}^{\delta,i}_{\varepsilon,a}$ its total energy as in Proposition 7.2.3.

We assume that $g(\cdot)$ and $g_\varepsilon(\cdot)$ are defined in $H^1(\Omega')$ (we still denote these extensions by $g(\cdot)$ and $g_h(\cdot)$), in such a way that (7.21) and (7.22) hold in Ω'. Let us moreover set $g^\delta_\varepsilon(t) := g_\varepsilon(t^\delta_i)$ for all $t^\delta_i \le t < t^\delta_{i+1}$ with $i = 0, \dots, N_\delta - 1$ and $g^\delta_\varepsilon(1) := g_\varepsilon(1)$.

Let us make the following piecewise constant interpolation in time:

$$u^\delta_{\varepsilon,a}(t) := u^{\delta,i}_{\varepsilon,a} \quad \text{for } t^\delta_i \le t < t^\delta_{i+1} \quad i = 0, \dots, N_\delta - 1,$$

and $u^\delta_{\varepsilon,a}(1) := u^{\delta,N_\delta}_{\varepsilon,a}$. For all $t \in [0,1]$ we define the *discrete crack* at time t as

$$\Gamma^\delta_{\varepsilon,a}(t) := \bigcup_{s \le t} S^{g^\delta_\varepsilon(s)}(u^\delta_{\varepsilon,a}(s)),$$

and the *discrete total energy* at time t as

$$\mathcal{E}^\delta_{\varepsilon,a}(t) := \|\nabla u^\delta_{\varepsilon,a}(t)\|^2 + \mathcal{H}^1\left(\Gamma^\delta_{\varepsilon,a}(t)\right).$$

We have for all $t \in [0,1]$

$$\|u^\delta_{\varepsilon,a}(t)\|_\infty \le \|g^\delta_\varepsilon(t)\|_\infty. \tag{7.23}$$

Moreover for all $v \in \mathcal{A}_{\varepsilon,a}(\Omega)$ we have

$$\|\nabla u^\delta_{\varepsilon,a}(0)\|^2 + \mathcal{H}^1\left(S^{g^\delta_\varepsilon(0)}(u^\delta_{\varepsilon,a}(0))\right) \le \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g^\delta_\varepsilon(0)}(v)\right), \tag{7.24}$$

and for all $t \in]0,1]$ and for all $v \in \mathcal{A}_{\varepsilon,a}(\Omega)$

$$\|\nabla u^\delta_{\varepsilon,a}(t)\|^2 \le \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g^\delta_\varepsilon(t)}(v) \setminus \Gamma^\delta_{\varepsilon,a}(t)\right). \tag{7.25}$$

Finally for all $0 \leq s \leq t \leq 1$ we have

$$\mathcal{E}^{\delta}_{\varepsilon,a}(t) \leq \mathcal{E}^{\delta}_{\varepsilon,a}(s) + 2\int_{s_i^{\delta}}^{t_i^{\delta}} \int_{\Omega} \nabla u^{\delta}_{\varepsilon,a}(\tau) \nabla \dot{g}_{\varepsilon}(\tau)\, dx\, d\tau + o^{\delta}_{\varepsilon}, \tag{7.26}$$

where $t_i^{\delta} \leq t < t_{i+1}^{\delta}$, $s_i^{\delta} \leq s < s_{i+1}^{\delta}$ and

$$o^{\delta}_{\varepsilon} := \left[\max_{r=0,\dots,N_{\delta}-1} \int_{t_r^{\delta}}^{t_{r+1}^{\delta}} \|\dot{g}_{\varepsilon}(\tau)\|_{H^1(\Omega)}\, d\tau\right] \int_0^1 \|\dot{g}_{\varepsilon}(\tau)\|_{H^1(\Omega)}. \tag{7.27}$$

For $s = 0$ we obtain the following estimate from above for the discrete total energy

$$\mathcal{E}^{\delta}_{\varepsilon,a}(t) \leq \mathcal{E}^{\delta}_{\varepsilon,a}(0) + 2\int_{0}^{t_i^{\delta}} \int_{\Omega} \nabla u^{\delta}_{\varepsilon,a}(\tau) \nabla \dot{g}_{\varepsilon}(\tau)\, dx\, d\tau + o^{\delta}_{\varepsilon}, \tag{7.28}$$

where $t_i^{\delta} \leq t < t_{i+1}^{\delta}$.

We study the behavior of the evolution $\{t \to u^{\delta}_{\varepsilon,a}(t),\, t \in [0,1]\}$ varying the parameters in the following way. We let firstly $\varepsilon \to 0$ and $\delta \to 0$ obtaining an evolution $\{t \to u_a(t),\, t \in [0,1]\}$ relative to the boundary data g with the minimality property (7.36); then we let $a \to 0$ obtaining a quasi-static evolution of brittle cracks $\{t \to u(t)\,, t \in [0,1]\}$ relative to the boundary data g. Finally, by a diagonal argument we deal with (δ, ε, a) at the same time.

In order to develop this program, we need some compactness, and so we derive a bound for the total energy $\mathcal{E}^{\delta}_{\varepsilon,a}$. By (7.14), we have that for all $t \in [0,1]$

$$\|\nabla u^{\delta}_{\varepsilon,a}(t)\| \leq \|\nabla g^{\delta}_{\varepsilon}(t)\| \leq \tilde{C}$$

with $\tilde{C}$ independent of δ, ε and t. We deduce for all $t \in [0,1]$

$$\mathcal{E}^{\delta}_{\varepsilon,a}(t) \leq \mathcal{E}^{\delta}_{\varepsilon,a}(0) + 2\tilde{C}^2 + o^{\delta}_{\varepsilon}$$

Notice that $\mathcal{E}^{\delta}_{\varepsilon,a}(0)$ is uniformly bounded as δ, ε vary. Moreover, by (7.23) and since $\|g_{\varepsilon}(t)\|_{\infty} \leq C$ for all $t \in [0,1]$, we have that $u^{\delta}_{\varepsilon,a}(t)$ is uniformly bounded in $L^{\infty}(\Omega)$ independently of δ, ε and a. Taking into account (7.22), we conclude that there exists C' independent of δ, ε, a such that for all $t \in [0,1]$

$$\mathcal{E}^{\delta}_{\varepsilon,a}(t) + \|u^{\delta}_{\varepsilon,a}(t)\|_{\infty} \leq C'. \tag{7.29}$$

Formula (7.29) gives the desired compactness in order to perform the asymptotic analysis of the discrete evolution.

Let now consider $\delta_n \to 0$ and $\varepsilon_n \to 0$: by (7.22) we have

$$o^{\delta_n}_{\varepsilon_n} \to 0, \tag{7.30}$$

where $o^{\delta_n}_{\varepsilon_n}$ is defined in (7.27). By Helly's theorem on monotone functions, we may suppose that there exists an increasing function λ_a such that (up to a subsequence) for all $t \in [0,1]$

$$\lambda_{n,a}(t) := \mathcal{H}^1\left(\bigcup_{s \leq t} S^{g^{\delta_n}_{\varepsilon_n}(s)}(u^{\delta_n}_{\varepsilon_n,a}(s))\right) \to \lambda_a(t). \tag{7.31}$$

Let us fix $D \subseteq [0,1]$ countable and dense with $0 \in D$.

Lemma 7.3.1. *For all $t \in D$ there exists $u_a(t) \in SBV(\Omega)$ such that up to a subsequence independent of t*

$$u^{\delta_n}_{\varepsilon_n,a}(t) \to u_a(t) \quad \textit{in } SBV(\Omega).$$

Moreover for all $t \in D$ we have

$$\|\nabla u_a(t)\|^2 + \mathcal{H}^1(S^{g(t)}(u_a(t))) + \|u_a(t)\|_{\infty} \leq C'. \tag{7.32}$$

Proof. Let us consider $t \in D$. By (7.29), we can apply Ambrosio's Compactness Theorem 1.1.1 obtaining $u \in SBV(\Omega)$ such that, up to a subsequence, $u^{\delta_n}_{\varepsilon_n,a}(t) \to u$ in $SBV(\Omega)$. Let us set $u_a(t) := u$. Using a diagonal argument, we deduce that there exists a subsequence of $(\delta_n, \varepsilon_n)$ (which we still denote by $(\delta_n, \varepsilon_n)$) such that $u^{\delta_n}_{\varepsilon_n,a}(t) \to u_a(t)$ in $SBV(\Omega)$ for all $t \in D$. In order to obtain inequality (7.32), we extend $u^{\delta_n}_{\varepsilon_n,a}(t)$ and $u_a(t)$ to Ω' setting $u^{\delta_n}_{\varepsilon_n,a}(t) := g^{\delta_n}_{\varepsilon_n}(t)$ and $u_a(t) := g(t)$ on Ω_D; since $g^{\delta_n}_{\varepsilon_n}(t) \to g(t)$ on Ω_D strongly in $H^1(\Omega_D)$, we have that $u^{\delta_n}_{\varepsilon_n,a}(t) \to u_a(t)$ in $SBV(\Omega')$, so that we can apply Ambrosio's Theorem, and derive (7.32) from (7.29). $\square$

The following result is essential for the sequel: its proof is postponed to Section 7.4.

Proposition 7.3.2. *Let $t \in D$. For all $v \in SBV(\Omega)$ we have*

$$\|\nabla u_a(t)\|^2 \le \|\nabla v\|^2 + \mu(a)\mathcal{H}^1(S^{g(t)}(v) \setminus \bigcup_{s \le t, s \in D} S^{g(s)}(u_a(s))), \tag{7.33}$$

where $\mu :]0, \frac{1}{2}[\to]0, +\infty[$ is such that $\lim_{a \to 0} \mu(a) = 1$. Moreover, $\nabla u^{\delta_n}_{\varepsilon_n,a}(t) \to \nabla u_a(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$.

We now extend the evolution $\{t \to u_a(t) : t \in D\}$ to the entire interval $[0,1]$. Let us set for all $t \in [0,1]$

$$\Gamma_a(t) := \bigcup_{s \le t, s \in D} S^{g(s)}(u_a(s)).$$

Lemma 7.3.3. *For every $t \in [0,1]$ there exists $u_a(t) \in SBV(\Omega)$ such that the following hold:*

(a) for all $t \in [0,1]$

$$S^{g(t)}(u_a(t)) \subseteq \Gamma_a(t) \quad \text{up to a set of } \mathcal{H}^1\text{-measure } 0, \tag{7.34}$$

and

$$\|\nabla u_a(t)\|^2 + \mathcal{H}^1(S^{g(t)}(u_a(t))) + \|u_a(t)\|_\infty \le C'; \tag{7.35}$$

(b) for all $v \in SBV(\Omega)$

$$\|\nabla u_a(t)\|^2 \le \|\nabla v\|^2 + \mu(a)\mathcal{H}^1\left(S^{g(t)}(v) \setminus \Gamma_a(t)\right); \tag{7.36}$$

(c) ∇u_a is left continuous in $[0,1] \setminus D$ with respect to the strong topology of $L^2(\Omega; \mathbb{R}^2)$;

(d) for all $t \in [0,1] \setminus \mathcal{N}_a$ we have that

$$\nabla u^{\delta_n}_{\varepsilon_n,a}(t) \to \nabla u_a(t) \quad \text{strongly in } L^2(\Omega, \mathbb{R}^2),$$

where $\mathcal{N}_a$ is the set of discontinuities of the function λ_a defined in (7.31).

Proof. Let $t \in [0,1] \setminus D$ and let $t_n \in D$ with $t_n \nearrow t$. By (7.32), we can apply Ambrosio's Theorem to the sequence $(u_a(t_n))$ obtaining $u \in SBV(\Omega)$ such that, up to a subsequence, $u_a(t_n) \to u$ in $SBV(\Omega)$. Let us set $u_a(t) := u$. Let us extend $u_a(t_n)$ and $u_a(t)$ to Ω' setting $u_a(t_n) := g(t_n)$ and $u_a(t) := g(t)$ on Ω_D: we have $u_a(t_n) \to u_a(t)$ in $SBV(\Omega')$. Since $\mathcal{H}^1 \llcorner S(u_a(t_n)) \le \mathcal{H}^1 \llcorner \Gamma_a(t)$ for all n, as a consequence of Ambrosio's Theorem, we deduce that $\mathcal{H}^1 \llcorner S(u_a(t)) \le \mathcal{H}^1 \llcorner \Gamma_a(t)$. This means $\mathcal{H}^1 \llcorner S^{g(t)}(u_a(t)) \le \mathcal{H}^1 \llcorner \Gamma_a(t)$, so that (7.34) holds. Moreover, for all $v \in SBV(\Omega)$, by (7.33) we may write

$$\begin{aligned} \|\nabla u_a(t_n)\|^2 &\le \|\nabla v - \nabla g(t) + \nabla g(t_n)\|^2 + \mu(a)\mathcal{H}^1\left(S^{g(t)}(v) \setminus \Gamma_a(t_n)\right) \le \\ &\le \|\nabla v - \nabla g(t) + \nabla g(t_n)\|^2 + \mu(a)\mathcal{H}^1\left(S^{g(t)}(v) \setminus \Gamma_a(t)\right) + \mu(a)\mathcal{H}^1\left(\Gamma_a(t) \setminus \Gamma_a(t_n)\right), \end{aligned} \tag{7.37}$$

so that, since by definition of $\Gamma_a(t)$ we have $\mathcal{H}^1(\Gamma_a(t) \setminus \Gamma_a(t_n)) \to 0$, we obtain that (7.36) holds; choosing $v = u_a(t)$ and taking the limsup in (7.37), we obtain that

$$\limsup_n \|\nabla u_a(t_n)\|^2 \leq \|\nabla u_a(t)\|^2,$$

and so the convergence $\nabla u_a(t_n) \to \nabla u_a(t)$ is strong in $L^2(\Omega, \mathbb{R}^2)$. Notice that $\nabla u_a(t)$ is uniquely determined by (7.34) and (7.36) since the gradient of the solutions of the minimum problem

$$\min \left\{ \|\nabla u\|^2 \,:\, S^{g(t)}(u) \subseteq \Gamma_a(t) \text{ up to a set of } \mathcal{H}^1\text{-measure } 0 \right\}$$

is unique by the strict convexity of the functional: we conclude that $\nabla u_a(t)$ is well defined. The same arguments prove that ∇u_a is left continuous at all $t \in [0,1] \setminus D$. Finally (7.35) is a direct consequence of (7.32) and of Ambrosio's Theorem, and so points (a), (b), (c) are proved.

Let us come to point (d). Let us consider $u^{\delta_n}_{\varepsilon_n,a}(t)$ with $t \notin \mathcal{N}_a$; we may suppose that $t \notin D$, since otherwise the result has already been established. By Proposition 7.3.2 with $D' := D \cup \{t\}$ in place of D, we have that, up to a subsequence, $u^{\delta_n}_{\varepsilon_n,a}(t) \to u$ in $SBV(\Omega)$ such that

$$\|\nabla u\|^2 \leq \|\nabla v\|^2 + \mu(a)\mathcal{H}^1 \left(S^{g(t)}(v) \setminus (\Gamma_a(t) \cup S^{g(t)}(u)) \right)$$

for all $v \in SBV(\Omega)$ and $\nabla u^{\delta_n}_{\varepsilon_n,a}(t) \to \nabla u$ strongly in $L^2(\Omega; \mathbb{R}^2)$. Let $s < t$ with $s \in D$; by the minimality of $u^{\delta_n}_{\varepsilon_n,a}(s)$ and by (7.29) we have

$$\begin{aligned}
\|\nabla u^{\delta_n}_{\varepsilon_n,a}(s)\|^2 \leq \|\nabla u^{\delta_n}_{\varepsilon_n,a}(t) - \nabla g^{\delta_n}_{\varepsilon_n}(t) + \nabla g^{\delta_n}_{\varepsilon_n}(s)\|^2 + \lambda_{n,a}(t) - \lambda_{n,a}(s) \leq \\
\leq \|\nabla u^{\delta_n}_{\varepsilon_n,a}(t)\|^2 + 2\sqrt{C'}\|\nabla g^{\delta_n}_{\varepsilon_n}(t) - \nabla g^{\delta_n}_{\varepsilon_n}(s)\| + \\
+ \|\nabla g^{\delta_n}_{\varepsilon_n}(t) - \nabla g^{\delta_n}_{\varepsilon_n}(s)\|^2 + \lambda_{n,a}(t) - \lambda_{n,a}(s).
\end{aligned}$$

Passing to the limit for $n \to +\infty$, recalling that $g^{\delta_n}_{\varepsilon_n}(\tau) \to g(\tau)$ strongly in $H^1(\Omega)$ for all $\tau \in [0,1]$, we deduce

$$\|\nabla u_a(s)\|^2 \leq \|\nabla u\|^2 + 2\sqrt{C'}\|\nabla g(t) - \nabla g(s)\| + \|\nabla g(t) - \nabla g(s)\|^2 + \lambda_a(t) - \lambda_a(s),$$

so that, since t is a point of continuity for λ_a, ∇u_a is left continuous at t, and g is absolutely continuous, we get for $s \to t$

$$\|\nabla u_a(t)\|^2 \leq \|\nabla u\|^2.$$

We conclude that $u_a(t)$ is a solution of

$$\min\{\|\nabla v\|^2 \,:\, S^{g(t)}(v) \subseteq \Gamma_a(t) \cup S^{g(t)}(u) \text{ up to a set of } \mathcal{H}^1\text{-measure } 0\},$$

so that $\nabla u = \nabla u_a(t)$ by uniqueness of the gradient of the solution. We deduce that $\nabla u^{\delta_n}_{\varepsilon_n,a}(t) \to \nabla u_a(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$, and so the proof is complete. $\square$

We can now let $a \to 0$.

Lemma 7.3.4. *There exists $a_n \to 0$ such that, for all $t \in D$, $u_{a_n}(t) \to u(t)$ in $SBV(\Omega)$ for some $u(t) \in SBV(\Omega)$ such that for all $v \in SBV(\Omega)$ we have*

$$\|\nabla u(t)\|^2 \leq \|\nabla v\|^2 + \mathcal{H}^1(S^{g(t)}(v) \setminus \bigcup_{s \leq t, s \in D} S^{g(s)}(u(s))). \tag{7.38}$$

Moreover, $\nabla u_{a_n}(t) \to \nabla u(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$ and

$$\|\nabla u(t)\|^2 + \mathcal{H}^1(S^{g(t)}(u(t))) + \|u(t)\|_\infty \leq C'. \tag{7.39}$$

Proof. By (7.35), applying Ambrosio's Theorem to the extensions of $u_a(t)$ to Ω' by setting $u_a(t) := g(t)$ on Ω_D, and using a diagonal argument, we find a sequence $a_n \to 0$ such that, for all $t \in D$, $u_{a_n}(t) \to u(t)$ in $SBV(\Omega)$ for some $u(t) \in SBV(\Omega)$ such that (7.39) holds.

We now prove that $u(t)$ satisfies property (7.38). Let $v \in SBV(\Omega)$. Let us fix $t_1 \le t_2 \le \ldots \le t_k = t$ with $t_i \in D$. We extend v and $u_{a_n}(t_i)$ to Ω' setting $v := g(t)$ and $u_{a_n}(t_i) := g(t_i)$ on Ω_D respectively. Since $u_{a_n}(t_i) \to u(t_i)$ in $SBV(\Omega')$ for all $i = 1, \ldots, k$, by Theorem 1.4.3 there exists $v_n \in SBV(\Omega')$ with $v_n = g(t)$ on Ω_D such that $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega'; \mathbb{R}^2)$ and

$$\limsup_n \mathcal{H}^1 \left(S(v_n) \setminus \bigcup_{i=1}^k S(u_{a_n}(t_i)) \right) \le \mathcal{H}^1 \left(S(v) \setminus \bigcup_{i=1}^k S(u(t_i)) \right). \tag{7.40}$$

By (7.33) we obtain

$$\|\nabla u_{a_n}(t)\|^2 \le \|\nabla v_n\|^2 + \mu(a_n)\mathcal{H}^1 \left(S(v_n) \setminus \bigcup_{i=1}^k S(u_{a_n}(t_i)) \right), \tag{7.41}$$

so that passing to the limit for $n \to +\infty$ and recalling that $\mu(a) \to 1$ as $a \to 0$, we obtain

$$\|\nabla u(t)\|^2 \le \|\nabla v\|^2 + \mathcal{H}^1 \left(S(v) \setminus \bigcup_{i=1}^k S(u(t_i)) \right).$$

Thus we get

$$\|\nabla u(t)\|^2 \le \|\nabla v\|^2 + \mathcal{H}^1 \left(S^{g(t)}(v) \setminus \bigcup_{i=1}^k S^{g(t_i)}(u(t_i)) \right).$$

Since $t_1, \ldots, t_k$ are arbitrary, we obtain (7.38). Choosing $v = u(t)$, taking the limsup in (7.41) and using (7.40), we obtain $\nabla u_{a_n}(t) \to \nabla u(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$. □

In order to deal with δ, ε and a at the same time, we need the following lemma.

Lemma 7.3.5. *Let $\{u(t) : t \in D\}$ be as in Lemma 7.3.4. There exist $\delta_n \to 0$, $\varepsilon_n \to 0$, and $a_n \to 0$ such for all $t \in D$ we have*

$$u^{\delta_n}_{\varepsilon_n, a_n}(t) \to u(t) \quad \text{in } SBV(\Omega).$$

Moreover, for all n there exists $\mathcal{B}_n \subseteq [0,1]$ with $|\mathcal{B}_n| < 2^{-n}$ such that for all $t \in [0,1] \setminus \mathcal{B}_n$

$$\|\nabla u^{\delta_n}_{\varepsilon_n, a_n}(t) - \nabla u_{a_n}(t)\| \le \frac{1}{n}. \tag{7.42}$$

Finally, we have that for all $v \in SBV(\Omega)$

$$\|\nabla u(0)\|^2 + \mathcal{H}^1 \left(S^{g(0)}(u(0)) \right) \le \|\nabla v\|^2 + \mathcal{H}^1 \left(S^{g(0)}(v) \right) \tag{7.43}$$

and

$$\mathcal{E}^{\delta_n}_{\varepsilon_n, a_n}(0) \to \|\nabla u(0)\|^2 + \mathcal{H}^1 \left(S^{g(0)}(u(0)) \right). \tag{7.44}$$

Proof. Let (a_n) be the sequence determined by Lemma 7.3.4. By Lemma 7.3.1, for all n there exists $(\delta^n_m, \varepsilon^n_m)$ such that for all $t \in D$ and $m \to +\infty$ we have

$$u^{\delta^n_m}_{\varepsilon^n_m, a_n}(t) \to u_{a_n}(t) \quad \text{in } SBV(\Omega),$$

and

$$\nabla u^{\delta^n_m}_{\varepsilon^n_m, a_n}(t) \to \nabla u_{a_n}(t) \quad \text{strongly in } L^2(\Omega; \mathbb{R}^2).$$

Moreover by Lemma 7.3.3 we have that $\nabla u^{\delta^n_m}_{\varepsilon^n_m,a_n} \to \nabla u_{a_n}$ quasi-uniformly on $[0,1]$ as $m \to +\infty$. Let $\mathcal{B}_n \subseteq [0,1]$ with $|\mathcal{B}_n| < 2^{-n}$ such that $\nabla u^{\delta^n_m}_{\varepsilon^n_m,a_n} \to \nabla u_{a_n}$ uniformly on $[0,1] \setminus \mathcal{B}_n$ as $m \to +\infty$. We now perform the following diagonal argument. Let $D = \{t_n,\, n \geq 1\}$. Choose m_1 such that

$$\|\nabla u^{\delta^1_{m_1}}_{\varepsilon^1_{m_1},a_1}(t_1) - \nabla u_{a_1}(t_1)\| + \|u^{\delta^1_{m_1}}_{\varepsilon^1_{m_1},a_1}(t_1) - u_{a_1}(t_1)\| \leq 1,$$

and

$$\|\nabla u^{\delta^1_{m_1}}_{\varepsilon^1_{m_1},a_1}(t) - \nabla u_{a_1}(t)\| \leq 1 \quad \text{for all } t \in [0,1] \setminus \mathcal{B}_1 \,.$$

Let m_n be such that

$$\|\nabla u^{\delta^n_{m_n}}_{\varepsilon^n_{m_n},a_n}(t_j) - \nabla u_{a_n}(t_j)\| + \|u^{\delta^n_{m_n}}_{\varepsilon^n_{m_n},a_n}(t_j) - u_{a_n}(t_j)\| \leq \frac{1}{n} \quad \text{for all } j = 1, \dots, n$$

and

$$\|\nabla u^{\delta^n_{m_n}}_{\varepsilon^n_{m_n},a_n}(t) - \nabla u_{a_n}(t)\| \leq \frac{1}{n} \quad \text{for all } t \in [0,1] \setminus \mathcal{B}_n \,.$$

We may suppose that $\delta^n_{m_n} \to 0$, $\varepsilon^n_{m_n} \to 0$. Then $(\delta^n_{m_n}, \varepsilon^n_{m_n}, a_n)$ is the sequence which satisfies the thesis. In fact by construction and taking into account (7.29), for all $t \in D$ we have $u^{\delta^n_{m_n}}_{\varepsilon^n_{m_n},a_n}(t) \to u(t)$ in $SBV(\Omega)$; moreover the set $\mathcal{B}_n$ satisfies (7.42). Notice that $u^{\delta^n_{m_n}}_{\varepsilon^n_{m_n},a^n_{m_n}}(0)$ satisfies (7.24) and so (7.43) and (7.44) follow by the Γ-convergence result of [75]. □

Let $(\delta_n, \varepsilon_n, a_n)$ be the sequence determined by Lemma 7.3.5. For all $t \in [0,1]$ let us set

$$\lambda_n(t) := \mathcal{H}^1\left(\Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)\right).$$

By Helly's theorem, we may suppose that there exist two increasing functions λ and η such that up to a subsequence

$$\lambda_n \to \lambda \quad \text{pointwise in } [0,1],$$

and

$$\lambda_{a_n} \to \eta \quad \text{pointwise in } [0,1], \tag{7.45}$$

where λ_{a_n} is defined as in (7.31). We now extend the evolution $\{t \to u(t) \,:\, t \in D\}$ to the entire interval $[0,1]$. Let us set for all $t \in [0,1]$

$$\Gamma(t) := \bigcup_{s \leq t, s \in D} S^{g(s)}(u(s)),$$

and let $\mathcal{N}$ be the set of discontinuities of $\mathcal{H}^1(\Gamma(\cdot))$. Notice that for all $t \in [0,1]$

$$\mathcal{H}^1(\Gamma(t)) \leq \lambda(t). \tag{7.46}$$

In fact if $t \in D$, let $t_1 \leq t_2 \leq \ldots \leq t_k = t$ with $t_i \in D$, consider $w_n \in SBV(\Omega'; \mathbb{R}^k)$ defined as

$$w_n(x) := (u^{\delta_n}_{\varepsilon_n,a_n}(t_1)(x), \dots, u^{\delta_n}_{\varepsilon_n,a_n}(t_k)(x)),$$

where we assume that $u^{\delta_n}_{\varepsilon_n,a_n}(t_i) = g^{\delta_n}_{\varepsilon_n}(t_i)$ on Ω_D. We have $w_n \to w := (u(t_1), \dots, u(t_k))$ in $SBV(\Omega'; \mathbb{R}^k)$, where $u(t_i) = g(t_i)$ on Ω_D. Note that for all n we have $S(w_n) = \bigcup_{i=1}^k S(u^{\delta_n}_{\varepsilon_n,a_n}(t_i))$ so that

$$\mathcal{H}^1(S(w_n)) \leq \lambda_n(t).$$

Passing to the limit for $n \to +\infty$ and applying Ambrosio's Theorem we get

$$\mathcal{H}^1\left(\bigcup_{i=1}^k S(u(t_i))\right) = \mathcal{H}^1(S(w)) \leq \liminf_n \mathcal{H}^1(S(w_n)) \leq \lambda(t);$$

we thus have

$$\mathcal{H}^1\left(\bigcup_{i=1}^{k} S^{g(t_i)}(u(t_i))\right) = \mathcal{H}^1(S(w)) \leq \lambda(t)$$

and taking the sup over all $t_1, \ldots, t_k$, we obtain (7.46) in D. The case $t \notin D$ follows since $\mathcal{H}^1(\Gamma(\cdot))$ is left continuous by definition.

Lemma 7.3.6. *For every $t \in [0,1]$ there exists $u(t) \in SBV(\Omega)$ such that the following hold:*

(a) for all $t \in [0,1]$

$$S^{g(t)}(u(t)) \subseteq \Gamma(t) \quad \text{up to a set of } \mathcal{H}^1\text{-measure } 0, \tag{7.47}$$

and for all $t \in [0,1]$ and for all $v \in SBV(\Omega)$

$$\|\nabla u(t)\|^2 \leq \|\nabla v\|^2 + \mathcal{H}^1\left(S^{g(t)}(v) \setminus \Gamma(t)\right); \tag{7.48}$$

(b) ∇u is continuous in $[0,1] \setminus (D \cup \mathcal{N})$ with respect to the strong topology of $L^2(\Omega;\mathbb{R}^2)$;

(c) if $\tilde{\mathcal{N}}$ is the set of discontinuities of the function η defined in (7.45), for all $t \in [0,1] \setminus \tilde{\mathcal{N}}$ we have that

$$\nabla u_{a_n}(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega, \mathbb{R}^2).$$

Finally

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau)\nabla \dot{g}(\tau)\, dx\, d\tau, \tag{7.49}$$

where

$$\mathcal{E}(t) := \|\nabla u(t)\|^2 + \mathcal{H}^1(\Gamma(t)).$$

Proof. The definition of $u(t)$ is carried out as in Lemma 7.3.3 considering $t \in [0,1] \setminus D$, $t_n \in D$ with $t_n \nearrow t$, and the limit (up to a subsequence) of $u(t_n)$ in $SBV(\Omega)$: (7.47) and (7.48) hold, so that point (a) is proved. It turns out that $\nabla u(t)$ is uniquely determined and that it is left continuous in $[0,1] \setminus D$. Let us consider $t \in [0,1] \setminus (D \cup \mathcal{N})$, and let $t_n \searrow t$. By Ambrosio's Theorem, we have that there exists $u \in SBV(\Omega)$ with such that, up to a subsequence, $u(t_n) \to u$ in $SBV(\Omega)$. Since t is a continuity point of $\mathcal{H}^1(\Gamma(\cdot))$, we deduce that $S^{g(t)}(u) \subseteq \Gamma(t)$ up to a set of $\mathcal{H}^1$-measure 0. Moreover by the minimality property for $u(t_n)$ and the fact $\Gamma(t) \subseteq \Gamma(t_n)$, we have that for all $v \in SBV(\Omega)$ with

$$\begin{aligned}\|\nabla u(t_n)\|^2 \leq \|\nabla v - \nabla g(t) + \nabla g(t_n)\|^2 + \mathcal{H}^1\left(S^{g(t)}(v) \setminus \Gamma(t_n)\right) \leq \\ \leq \|\nabla v - \nabla g(t) + \nabla g(t_n)\|^2 + \mathcal{H}^1\left(S^{g(t)}(v) \setminus \Gamma(t)\right),\end{aligned}$$

and so we deduce that (7.48) holds with u in place of $u(t)$, and that $\nabla u(t_n) \to \nabla u$ strongly in $L^2(\Omega;\mathbb{R}^2)$. We obtain by uniqueness that $\nabla u = \nabla u(t)$, and so $\nabla u(\cdot)$ is continuous in $[0,1] \setminus (D \cup \mathcal{N})$ and this proves point (b). Point (c) follows in the same way of point (d) of Lemma 7.3.3.

Let us come to the proof of (7.49). Given $t \in [0,1]$ and $k > 0$, let $s_i^k := \frac{i}{k}t$ for all $i = 0, \ldots, k$. Let us set $u^k(s) := u(s_{i+1}^k)$ for $s_i^k < s \leq s_{i+1}^k$. By (7.48), comparing $u(s_i^k)$ with $u(s_{i+1}^k) - g(s_{i+1}^k) + g(s_i^k)$, it is easy to see that

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u^k(\tau)\nabla \dot{g}(\tau)\, d\tau\, dx + o_k,$$

where $o_k \to 0$ as $k \to +\infty$. Since ∇u is continuous with respect to the strong topology of $L^2(\Omega;\mathbb{R}^2)$ in $[0,1]$ up to a countable set, passing to the limit for $k \to +\infty$ we deduce (7.49). □

We are now ready to prove the main result of the chapter.

PROOF OF THEOREM 7.0.1. Let D be a countable and dense set in $[0,1]$ such that $0 \in D$, and let $(\delta_n, \varepsilon_n, a_n)$ and $\{t \to u(t) \in SBV(\Omega) : t \in [0,1]\}$ be the sequence and the evolution determined in Lemma 7.3.5 and Lemma 7.3.6. Let us set

$$u_n := u^{\delta_n}_{\varepsilon_n, a_n}, \qquad \Gamma_n := \Gamma^{\delta_n}_{\varepsilon_n, a_n}, \qquad \mathcal{E}_n := \mathcal{E}^{\delta_n}_{\varepsilon_n, a_n}.$$

Let $\overline{\mathcal{N}}$ be the union of the sets of discontinuities of η and $\mathcal{H}^1(\Gamma(\cdot))$, where η is defined in (7.45). Let $\mathcal{B} := \bigcap_{k=1}^{+\infty} \bigcup_{h=k}^{\infty} \mathcal{B}_h$, where $\mathcal{B}_h$ are as in Lemma 7.3.5; since $|\bigcup_{h=k}^{\infty} \mathcal{B}_h| < 2^{-k+1}$, we have $|\mathcal{B}| = 0$. For all $t \in [0,1] \setminus (\mathcal{B} \cup \overline{\mathcal{N}})$ we claim that

$$\nabla u_n(t) \to \nabla u(t) \quad \text{strongly in } L^2(\Omega; \mathbb{R}^2). \tag{7.50}$$

In fact, since $t \notin \bigcup_{h=k}^{\infty} \mathcal{B}_h$ for some k, by Lemma 7.3.5 we have

$$\lim_n \|\nabla u^{\delta_n}_{\varepsilon_n, a_n}(t) - \nabla u_{a_n}(t)\| = 0;$$

for $t \notin \overline{\mathcal{N}}$, by Lemma 7.3.6 we have that $\nabla u_{a_n}(t) \to \nabla u(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$ and so (7.50) holds.

Since $g_{\varepsilon_n} \to g$ strongly in $W^{1,1}([0,1]; H^1(\Omega))$, we deduce that for a.e. $\tau \in [0,1]$

$$\nabla \dot{g}_{\varepsilon_n}(\tau) \to \nabla \dot{g}(\tau) \quad \text{strongly in } L^2(\Omega; \mathbb{R}^2).$$

Since $\mathcal{E}_n(0) \to \mathcal{E}(0)$ by (7.44) and $o^{\delta_n}_{\varepsilon_n} \to 0$, by semicontinuity of the energy and by (7.28) we have that for all $t \in D$

$$\mathcal{E}(t) \le \liminf_n \mathcal{E}_n(t) \le \limsup_n \mathcal{E}_n(t) \le \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau. \tag{7.51}$$

In view of (7.49), we conclude that for all $t \in D$

$$\mathcal{E}(t) = \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau,$$

and since $\nabla u(\cdot)$ and $\mathcal{H}^1(\Gamma(\cdot))$ are left continuous at $t \notin D$ and so $\mathcal{E}(\cdot)$ is, we conclude that the equality holds for all $t \in [0,1]$. As a consequence $\{t \to u(t), t \in [0,1]\}$ is a quasi-static evolution of brittle cracks. Let us prove that (7.51) is indeed true for all $t \in [0,1]$. In fact, if $t \notin D$, it is sufficient to prove

$$\liminf_n \mathcal{E}_n(t) \ge \mathcal{E}(t). \tag{7.52}$$

Considering $s \ge t$ with $s \in D$, by (7.26) we have

$$\mathcal{E}_n(s) \le \mathcal{E}_n(t) + \int_{t^{\delta_n}_{j_n}}^{s^{\delta_n}_{j_n}} \int_\Omega \nabla u_n(\tau) \nabla \dot{g}_{\varepsilon_n}(\tau)\, dx\, d\tau + o^{\delta_n}_{\varepsilon_n} \quad t^{\delta_n}_{j_n} \le t < t^{\delta_n}_{j_n+1},\ s^{\delta_n}_{j_n} \le s < s^{\delta_n}_{j_n+1},$$

so that

$$\liminf_n \mathcal{E}_n(t) \ge \mathcal{E}(s) - \int_t^s \int_\Omega \nabla u(\tau) \nabla \dot{g}(\tau)\, dx\, d\tau.$$

Letting $s \searrow t$, since $\mathcal{E}(\cdot)$ is continuous, we have (7.52) holds. By (7.51) we deduce that $\mathcal{E}_n(t) \to \mathcal{E}(t)$ for all $t \in [0,1]$, so that point (b) is proved.

We now come to point (a). Since $\lambda(t) \ge \mathcal{H}^1(\Gamma(t))$ for all $t \in [0,1]$, by (7.50) and point (b), we deduce that $\lambda = \mathcal{H}^1(\Gamma(\cdot))$ in $[0,1]$ up to a set of measure 0. Since they are increasing functions, we conclude that λ and $\mathcal{H}^1(\Gamma(\cdot))$ share the same set of continuity points $[0,1] \setminus \mathcal{N}$, and that $\lambda = \mathcal{H}^1(\Gamma(\cdot))$ on $[0,1] \setminus \mathcal{N}$. In view of (7.50), point (a) is thus established for all t except

$t \in (\mathcal{B} \cup \overline{\mathcal{N}}) \setminus \mathcal{N}$. In order to treat this case, we use the following argument. Considering the measures $\mu_n := \mathcal{H}^1 \llcorner \Gamma_n(t)$, we have that, up to a subsequence, $\mu_n \stackrel{*}{\rightharpoonup} \mu$ weakly-star in the sense of measures, and as a consequence of Ambrosio's Theorem we have $\mathcal{H}^1 \llcorner \Gamma(t) \leq \mu$ as measures. Since $t \notin \mathcal{N}$ we have $\mu_n(\mathbb{R}^2) \to \mathcal{H}^1(\Gamma(t))$, and so we deduce $\mathcal{H}^1 \llcorner \Gamma(t) = \mu$. Let us consider now $u_n(t)$; we have up to a subsequence $u_n(t) \to u$ in $SBV(\Omega)$ for some $u \in SBV(\Omega)$. Setting $u_n(t) := g^{\delta_n}_{\varepsilon_n}(t)$ and $u := g(t)$ on Ω_D, we have $u_n(t) \to u$ in $SBV(\Omega')$, and as a consequence of Ambrosio's Theorem, we get that $\mathcal{H}^1 \llcorner S^{g(t)}(u) \leq \mu = \mathcal{H}^1 \llcorner \Gamma(t)$, that is $S^{g(t)}(u) \subseteq \Gamma(t)$. By Theorem 1.4.3, we deduce that u is a minimum for

$$\min\{\|\nabla v\|^2 \,:\, S^{g(t)}(v) \subseteq \Gamma(t) \text{ up to a set of } \mathcal{H}^1\text{-measure } 0\,\},$$

and by uniqueness of the gradient we get that $\nabla u = \nabla u(t)$, so that the proof is concluded. □

7.4 Piecewise affine transfer of jump and proof of Proposition 7.3.2

The proof of Proposition 7.3.2 is based on the following proposition, which is a variant of Theorem 1.4.3 in the context of piecewise affine approximation.

Proposition 7.4.1. *Given $\varepsilon_n \to 0$, let $g^r_n \in H^1(\Omega)$ be such that $g^r_n \in \mathcal{AF}_{\varepsilon_n}(\Omega)$ and $g^r_n \to g^r$ strongly in $H^1(\Omega)$ for all $r = 0, \dots, i$. If $u^r_n \in \mathcal{A}_{\varepsilon_n,a}(\Omega)$ is such that $u^r_n \to u^r$ in $SBV(\Omega)$ for $r = 0, \dots, i$, then for all $v \in SBV(\Omega)$ with $\mathcal{H}^1\left(S^{g^i}(v)\right) < +\infty$ and $\nabla v \in L^2(\Omega;\mathbb{R}^2)$, there exists $v_n \in \mathcal{A}_{\varepsilon_n,a}(\Omega)$ such that $v_n \to v$ strongly in $L^1(\Omega)$, $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega;\mathbb{R}^2)$ and*

$$\limsup_n \mathcal{H}^1\left(S^{g^i_n}(v_n) \setminus \bigcup_{r=0}^{i} S^{g^r_n}(u^r_n)\right) \leq \mu(a)\mathcal{H}^1\left(S^{g(t)}(v) \setminus \bigcup_{r=0}^{i} S^{g^r}(u^r)\right), \tag{7.53}$$

where $\mu :]0; \frac{1}{2}[\to \mathbb{R}$ with $\lim_{a\to 0^+} \mu(a) = 1$.

In view of Proposition 7.4.1, we can now prove Proposition 7.3.2.

Proof of Proposition 7.3.2. Notice that, in order to prove (7.33), it is sufficient to prove the existence of $\mu :]0; \frac{1}{2}[\to \mathbb{R}$ with $\lim_{a\to 0^+} \mu(a) = 1$ such that, given $t \in D$, for every $0 = t_0 \leq \ldots \leq t_r \leq \ldots \leq t_i = t$, $t_r \in D$, for all $v \in SBV(\Omega)$ we have

$$\|\nabla u_a(t)\|^2 \leq \|\nabla v\|^2 + \mu(a)\mathcal{H}^1\left(S^{g(t)}(v) \setminus \bigcup_{r=0}^{i} S^{g(t_r)}(u_a(t_r))\right). \tag{7.54}$$

In fact, taking the sup over all possible $t_0, \dots, t_i$, we get (7.33).

We apply Proposition 7.4.1 considering $g^r_n := g^{\delta_n}_{\varepsilon_n}(t_r)$, $g^r := g(t_r)$, $u^r_n := u^{\delta_n}_{\varepsilon_n,a}(t_r)$, and $u^r := u_a(t_r)$ for $r = 0, \dots, i$. There exists $\mu :]0; \frac{1}{2}[\to \mathbb{R}$ with $\lim_{a\to 0^+} \mu(a) = 1$ such that for $v \in SBV(\Omega)$, there exists $v_n \in \mathcal{A}_{\varepsilon_n,a}(\Omega)$ with $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega;\mathbb{R}^2)$ and

$$\limsup_n \mathcal{H}^1\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \bigcup_{r=0}^{i} S^{g^{\delta_n}_{\varepsilon_n}(t_r)}(u^{\delta_n}_{\varepsilon_n,a}(t_r))\right) \leq$$
$$\leq \mu(a)\mathcal{H}^1\left(S^{g(t)}(v) \setminus \bigcup_{r=0}^{i} S^{g(t_r)}(u_a(t_r))\right),$$

Comparing $u^{\delta_n}_{\varepsilon_n,a}(t)$ and v_n by means of (7.25), we obtain

$$\|\nabla u^{\delta_n}_{\varepsilon_n,a}(t)\|^2 \leq \|\nabla v_n\|^2 + \mathcal{H}^1\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) \leq \tag{7.55}$$
$$\leq \|\nabla v_n\|^2 + \mathcal{H}^1\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \bigcup_{r=0}^{i} S^{g^{\delta_n}_{\varepsilon_n}(t_r)}(u^{\delta_n}_{\varepsilon_n,a}(t_r))\right),$$

so that, passing to the limit for $n \to +\infty$, we obtain that (7.54) holds. Moreover, we have that choosing $v = u_a(t)$, and taking the limsup in (7.55), we get that $\nabla u^{\delta_n}_{\varepsilon_n,a}(t) \to \nabla u_a(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$. $\square$

The rest of the section is devoted to the proof of Proposition 7.4.1. It will be convenient, as in Section 7.3, to consider Ω_D polygonal open bounded subset of $\mathbb{R}^2$ such that $\Omega_D \cap \Omega = \emptyset$ and $\partial\Omega \cap \partial\Omega_D = \partial_D\Omega$ up to a finite number of vertices; we set $\Omega' := \Omega \cup \Omega_D \cup \partial_D\Omega$. We suppose that $\mathbf{R}_\varepsilon$ can be extended to a regular triangulation of Ω' which we still indicate by $\mathbf{R}_\varepsilon$.

We need several preliminary results. First of all we need the following density result.

Proposition 7.4.2. *Given $u \in SBV(\Omega')$ with $u = 0$ on $\Omega' \setminus \overline{\Omega}$ and $\mathcal{H}^{N-1}(S(u)) < +\infty$, there exists $u_h \in SBV(\Omega')$ such that*

(a) $u_h = 0$ in $\Omega' \setminus \overline{\Omega}$;

(b) $S(u_h)$ is polyhedral, $\overline{S(u_h)} \subseteq \Omega$ and $u_h \in W^{k,\infty}(\Omega' \setminus \overline{S(u_h)})$ for all k;

(c) $u_h \to u$ strongly in $L^2(\Omega')$ and $\nabla u_h \to \nabla u$ strongly in $L^2(\Omega';\mathbb{R}^2)$;

(d) for all A open subset of Ω' with $\mathcal{H}^1(\partial A \cap S(u)) = 0$, we have

$$\lim_h \mathcal{H}^1(A \cap S(u_h)) = \mathcal{H}^1(A \cap S(u)).$$

Proof. Using a partition of unity, we may prove the result in the case $\Omega :=]-1,1[\times]0,1[$, $\Omega' :=]-1,1[\times]-1,1[$, and $\partial_D\Omega :=]-1,1[\times\{0\}$. We set $w_h(x,y) := u(x,y-h)$, and let φ_h be a cut off function with $\varphi_h = 1$ on $]-1,1[\times]-1,\frac{h}{3}[$, $\varphi_h = 0$ on $]-1,1[\times]\frac{h}{2},1[$, and $||\nabla\varphi_h||_\infty \le \frac{7}{h}$. Let us set $v_h := (1-\varphi_h)w_h$. We have that $v_h = 0$ in $\Omega' \setminus \overline{\Omega}$; moreover we have

$$\nabla v_h = (1-\varphi_h)\nabla w_h - \nabla\varphi_h w_h.$$

Since $\nabla\varphi_h w_h \to 0$ strongly in $L^2(\Omega';\mathbb{R}^2)$, we have $\nabla v_h \to \nabla u$ strongly in $L^2(\Omega';\mathbb{R}^N)$. Finally, for all A open subset of Ω' with $\mathcal{H}^1(\partial A \cap S(u)) = 0$, we have

$$\lim_h \mathcal{H}^1(A \cap S(v_h)) = \mathcal{H}^1(A \cap S(u)).$$

In order to conclude the proof, let us apply the density result by Cortesani [38] (see Theorem 6.6.1) obtaining $\tilde{v}_h$ with polyhedral jumps in Ω such that $\tilde{v}_h \in W^{k,\infty}(\Omega' \setminus \overline{S(\tilde{v}_h)})$, $||w_h - \tilde{v}_h||_{L^2(\Omega)} + ||\nabla w_h - \nabla\tilde{v}_h||_{L^2(\Omega;\mathbb{R}^2)} \le h^2$ and $|\mathcal{H}^{N-1}(S(w_h)) - \mathcal{H}^{N-1}(S(\tilde{v}_h))| \le h$. If we set $u_h := \varphi_h g + (1-\varphi_h)\tilde{v}_h$, we obtain the thesis. $\square$

Let us set $z^r_n := u^r_n - g^r_n$, and let us extend z^r_n to zero on Ω_D. Similarly, we set $z^r := u^r - g^r$, and we extend z^r to zero on Ω_D.

Let $\sigma > 0$, and let C be the set of corners of $\partial_D\Omega$. Let us fix $G \subseteq \mathbb{R}$ countable and dense: we recall that for all $r = 0, \dots, i$ we have up to a set of $\mathcal{H}^1$-measure zero

$$S(z^r) = \bigcup_{c_1,c_2 \in G} \partial^* E_{c_1}(r) \cap \partial^* E_{c_2}(r),$$

where $E_c(r) := \{x \in \Omega' \,:\, z^r(x) > c\}$ and ∂^* denotes the essential boundary (see [8]). Let us consider

$$J_j := \{x \in \bigcup_{r=0}^{i} S(z^r) \setminus C \,:\, (z^l)^+(x) - (z^l)^-(x) > \frac{1}{j} \text{ for some } l = 0, \dots, i\},$$

with j so large that $\mathcal{H}^1(\bigcup_{r=0}^{i} S(z^r) \setminus J_j) \le \sigma$. Let U be a neighborhood of $\bigcup_{r=0}^{i} S(z^r)$ such that $|U| \le \frac{\sigma}{j^2}$. Following [53, Theorem 2.1] (see Fig.3), we can find a finite disjoint collection of closed cubes $\{Q_k\}_{k=1,\dots,K}$ with center $x_k \in J_j$, edge of length $2r_k$ and oriented as the normal $\nu(x_k)$ to

$S(z^{r(k)})$ at x_k, such that $\bigcup_{k=1}^K Q_k \subseteq U$ and $\mathcal{H}^1(J_j \setminus \bigcup_{k=1}^K Q_k) \leq \sigma$. Moreover for all $k = 1, \ldots, K$ there exists $r(k) \in \{0, \ldots, i\}$ and $c_1(r(k)), c_2(r(k)) > 0$ such that

$$\mathcal{H}^1\left(\left[\bigcup_{r=0}^{i} S(z^r) \setminus S(z^{r(k)})\right] \cap Q_k\right) \leq \sigma r_k,$$

and the following hold

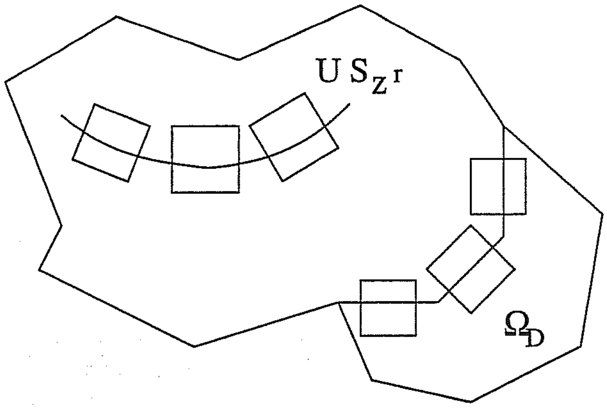

Fig. 3

(a) if $x_k \in \Omega$ then $Q_k \subseteq \Omega$, and if $x_k \in \partial_D\Omega$ then $Q_k \cap \partial_D\Omega = H_k$, where H_k denotes the intersection of Q_k with the straight line through x_k orthogonal to $\nu(x_k)$;

(b) $\mathcal{H}^1(S(z^{r(k)}) \cap \partial Q_k) = 0$;

(c) $r_k \leq c\mathcal{H}^1(S(z^{r(k)}) \cap Q_k)$ for some $c > 0$;

(d) $(z^{r(k)})^-(x) < c_1(r(k)) < c_2(r(k)) < (z^{r(k)})^+(x)$ and $c_2(r(k)) - c_1(r(k)) \geq \frac{1}{2j}$;

(e) $\mathcal{H}^1([S(z^{r(k)}) \setminus \partial^* E_{c_s(r(k))}(r(k))] \cap Q_k) \leq \sigma r_k$ for $s = 1, 2$;

(f) if $s = 1, 2$, $\mathcal{H}^1(\{y \in \partial^* E_{c_s(r(k))}(r(k)) \cap Q_k : \operatorname{dist}(y, H_k) \geq \frac{\sigma}{2} r_k\}) < \sigma r_k$;

(g) if $Q_k^+ := \{x \in Q_k \mid x \cdot \nu(x_k) > 0\}$ and $s = 1, 2$

(7.56) $$\|1_{E_{c_s(r(k))}(r(k)) \cap Q_k} - 1_{Q_k^+}\|_{L^1(\Omega')} \leq \sigma^2 r_k^2;$$

(h) $\mathcal{H}^1((S(v) \setminus S(z^{r(k)})) \cap Q_k) < \sigma r_k$ and $\mathcal{H}^1(S(v) \cap \partial Q_k) = 0$.

Let us indicate by R_k the intersection of Q_k with the strip centered in H_k with width $2\sigma r_k$, and let us set $V_k^\pm := \{x_k \pm r_k e(x_k) + s\nu(x_k) : s \in \mathbb{R}\} \cap R_k$, where $e(x_k)$ is such that $\{e(x_k), \nu(x_k)\}$ is an orthonormal base of $\mathbb{R}^2$ with the same orientation of the canonical one.

For all $B \subseteq \Omega'$, let us set

$$\mathcal{R}_n(B) := \{T \in \mathbf{R}_{\varepsilon_n} : T \cap B \neq \emptyset\}, \qquad \mathcal{T}_n^k(B) := \{T \in \mathbf{T}(z_n^{r(k)}) : T \cap B \neq \emptyset\}.$$

In the following, we will often indicate with the same symbol a family of triangles and their support in $\mathbb{R}^2$, being clear from the context in which sense has to be intended. We will consider $z_n^{r(k)}$ defined pointwise in $\Omega' \setminus \overline{S}_{z_n^{r(k)}}$ and so the upper levels of $z_n^{r(k)}$ are intended as subsets of $\Omega' \setminus \overline{S}_{z_n^{r(k)}}$.

Lemma 7.4.3. *For all $k = 1, \dots, K$ there exists $c_n^k \in [c_1(r(k)), c_2(r(k))]$ such that, setting $E_n^k := \{x \in \mathcal{R}_n(Q_k) \, : \, z_n^{r(k)}(x) > c_n^k\}$, we have*

$$\limsup_n \sum_{k=1}^{K} \mathcal{H}^1 \left((\partial_{\mathcal{R}_n(Q_k)} E_n^k) \setminus S(z_n^{r(k)}) \right) = o_\sigma, \tag{7.57}$$

and

$$\limsup_n \|1_{E_n^k} - 1_{Q_k^+}\|_{L^1(\Omega')} \le \sigma^2 r_k^2, \tag{7.58}$$

where $\partial_{\mathcal{R}_n(Q_k)}$ denotes the boundary operator in $\mathcal{R}_n(Q_k)$, and $o_\sigma \to 0$ as $\sigma \to 0$.

Proof. Note that for n large we have $\bigcup_{k=1}^K \mathcal{R}_n(Q_k) \subseteq U$, so that $|\bigcup_{k=1}^K \mathcal{R}_n(Q_k)| \le \frac{\sigma}{j^2}$. By Hölder inequality and since $\|\nabla z_n^r\| \le C'$ for all $r = 0, \dots, i$, it follows that

$$\sum_{r=0}^{i} \int_{\{\cup_k \mathcal{R}_n(Q_k) : r(k) = r\}} |\nabla z_n^r| \, dx \le \sum_{r=0}^{i} \|\nabla z_n^r\| \frac{\sqrt{\sigma}}{j} \le (i+1) C' \frac{\sqrt{\sigma}}{j}.$$

Following [53, Theorem 2.1], we can apply coarea-formula for BV-functions (see [8]) taking into account that $z_n^{r(k)}$ belongs to $SBV(\Omega')$ so that the singular part of the derivative is carried only by $S(z_n^{r(k)})$: since for n large the $\mathcal{R}_n(Q_k)$'s are disjoint, we obtain

$$\sum_{k=1}^{K} \int_{\mathbb{R}} \mathcal{H}^1 \Big((\partial E_{c,n}(r(k)) \cap \mathcal{R}_n(Q_k)) \setminus S(z_n^{r(k)}) \Big) \, dc \le (i+1) C' \frac{\sqrt{\sigma}}{j}, \tag{7.59}$$

where $E_{c,n}(r(k)) := \{x \in \Omega' \setminus \overline{S}_{z_n^{r(k)}} \, : \, z_n^{r(k)}(x) > c\}$, and so

$$\sum_{k=1}^{K} \int_{c_1(r(k))}^{c_2(r(k))} \mathcal{H}^1 \Big((\partial E_{c,n}(r(k)) \cap \mathcal{R}_n(Q_k)) \setminus S(z_n^{r(k)}) \Big) \, dc \le (i+1) C' \frac{\sqrt{\sigma}}{j}.$$

Notice that we can use the topological boundary instead of the reduced boundary of $E_{c,n}(r(k))$ in (7.59) since $z_n^{r(k)}$ is piecewise affine, and so $\partial E_{c,n}(r(k)) \setminus \partial^* E_{c,n}(r(k)) \neq \emptyset$ just for a finite number of c's. By the Mean Value Theorem we have that there exist $c_n^k \in [c_1(r(k)), c_2(r(k))]$ such that

$$\sum_{k=1}^{K} \mathcal{H}^1 \Big((\partial E_{c_n^k,n}(r(k)) \cap \mathcal{R}_n(Q_k)) \setminus S(z_n^{r(k)}) \Big) \le 2iC' \sqrt{\sigma},$$

and taking the limsup for $n \to +\infty$, we get (7.57). Let us come to (7.58). Since

$$E_{c_2(r(k)),n}(r(k)) \subseteq E_{c_n^k,n}(r(k)) \subseteq E_{c_1(r(k)),n}(r(k)),$$

by (7.56) we have that for n large

$$\|1_{E_{c_n^k,n}(r(k)) \cap Q_k} - 1_{Q_k^+}\|_{L^1(\Omega')} \le \sigma^2 r_k^2,$$

and so, since $|\mathcal{R}_n(Q_k) \setminus Q_k| \to 0$, we conclude that (7.58) holds. □

Fix $k \in \{1, \dots, K\}$, and let us consider the family $\mathcal{T}_n^k(E_n^k)$. Let us modify this family in the following way. Let $T \in \mathcal{T}_n^k(E_n^k)$; we keep it if $|T \cap E_n^k| > \frac{1}{2}|T|$, and we erase it otherwise. Let $E_n^{k,+}$ be this new family of triangles, and let $E_n^{k,-}$ be its complement in $\mathcal{T}_n^k(\mathcal{R}_n(Q_k))$.

Lemma 7.4.4. *For all $k = 1, \ldots, K$ we have*

$$\limsup_n \sum_{k=1}^{K} \mathcal{H}^1\left(\partial_{\mathcal{R}_n(Q_k)} E_n^{k,+} \setminus S(z_n^{r(k)})\right) = o_\sigma, \tag{7.60}$$

and

$$\limsup_n \|1_{E_n^{k,+}} - 1_{Q_k^+}\|_1 \leq 4\sigma^2 r_k^2, \tag{7.61}$$

where $o_\sigma \to 0$ as $\sigma \to 0$.

Proof. Let $T \in \mathcal{T}_n^k(E_n^k)$. Since $z_n^{r(k)}$ is affine on T, it follows that $T \cap E_n^k$ is either a triangle with at least two edges contained in the edges of T or a trapezoid with three edges contained in the edges of T. Let $l(T)$ be the edge inside T where $z_n^{r(k)} = c_n^k$, where c_n^k is the value determining E_n^k (we consider $l(T) = \emptyset$ if $\text{int}(T) \subseteq E_n^k$). In the case $T \in E_n^{k,+}$ as in the case $T \in E_n^{k,-}$, since the angles of the triangles of $\mathbf{T}(z_n^{r(k)})$ are uniformly bounded away from 0 and from π, arguing as in Lemma 7.1.1, we deduce that keeping or erasing T, we increase $\partial_{\mathcal{R}_n(Q_k)} E_n^k$ of a quantity which is less than $c\mathcal{H}^1(l(T))$ with c independent of ε_n. Then we have

$$\sum_{k=1}^{K} \mathcal{H}^1(\partial_{\mathcal{R}_n(Q_k)} E_n^{k,+} \setminus \partial_{\mathcal{R}_n(Q_k)} E_n^k) \leq \sum_{k=1}^{K} \sum_{T \in \mathcal{T}_n^k(E_n^k)} c\mathcal{H}^1(l(T)) \leq c \sum_{k=1}^{K} \mathcal{H}^1(\partial_{\mathcal{R}_n(Q_k)} E_n^k \setminus S(z_n^{r(k)})),$$

so that taking the limsup for $n \to +\infty$ and in view of (7.57) we deduce that (7.60) holds.

Let us come to (7.61). Note that $|\mathcal{T}_n^k(\partial Q_k^+)| \to 0$ as $n \to +\infty$. Then if $A_n^{k,+} := \{T \in \mathbf{T}(z_n^{r(k)}) : T \subseteq \text{int}(Q_k^+)\}$, for n large we have

$$|Q_k^+ \setminus E_n^{k,+}| \leq |A_n^{k,+} \setminus E_n^{k,+}| + |\mathcal{T}_n^k(\partial Q_k^+)| \leq 2|Q_k^+ \setminus E_n^k| + |\mathcal{T}_n^k(\partial Q_k^+)|,$$

where the last inequality follows by construction of $E_n^{k,+}$. Taking the limsup for $n \to +\infty$, in view of (7.58) we get

$$\limsup_n |Q_k^+ \setminus E_n^{k,+}| \leq 2\sigma^2 r_k^2.$$

The inequality $\limsup_n |E_n^{k,+} \setminus Q_k^+| \leq 2\sigma^2 r_k^2$ follows analogously. □

For all $k = 1, \ldots, K$ and $s \in \mathbb{R}$, let us set

$$H_k(s) := \{x + s\nu(x_k),\, x \in H_k\}.$$

Lemma 7.4.5. *There exist $s_n^+ \in]\frac{\sigma}{4} r_k, \frac{\sigma}{2} r_k[$ and $s_n^- \in] - \frac{\sigma}{2} r_k, -\frac{\sigma}{4} r_k[$ such that, setting $H_n^{k,+} := H_k(s_n^+)$ and $H_n^{k,-} := H_k(s_n^-)$ we have for n large enough*

$$\mathcal{H}^1(H_n^{k,+} \setminus E_n^{k,+}) \leq 20\sigma r_k, \quad \mathcal{H}^1(H_n^{k,-} \cap E_n^{k,+}) \leq 20\sigma r_k.$$

Proof. By (7.61) we can write for n large

$$\int_{\frac{\sigma}{4} r_k}^{\frac{\sigma}{2} r_k} \mathcal{H}^1(H_k(s) \setminus E_n^{k,+})\, ds \leq 5\sigma^2 r_k^2,$$

so that we get $s_n^+ \in]\frac{\sigma}{4} r_k, \frac{\sigma}{2} r_k[$ with

$$\mathcal{H}^1(H_k(s_n^+) \setminus E_n^{k,+}) \leq 20\sigma r_k.$$

Similarly we can reason for s_n^-. □

Let $S_n^{k,+}$ be the straight line containing $H_n^{k,+}$: up to replacing $H_n^{k,+}$ by the connected component of $S_n^{k,+} \cap \mathcal{R}_n(Q_k)$ to which it belongs, we may suppose that $H_n^{k,+} \setminus E_n^{k,+}$ is a finite union of segments l_j^+ with extremes A_j and B_j belonging to the edges of the triangles of $\mathcal{T}_n^k(\mathcal{R}_n(Q_k))$ such that for n large

$$\mathcal{H}^1(H_n^{k,+} \setminus E_n^{k,+}) = \mathcal{H}^1\left(\bigcup_{j=1}^{m} l_j^+\right) \leq 20\sigma r_k.$$

By Lemma 7.1.1, for all j there exists a curve L_j^+ inside the edges of the triangles of $\mathcal{T}_n^k(\mathcal{R}_n(Q_k))$ joining A_j and B_j and such that

$$\mathcal{H}^1(L_j^+) \leq c\mathcal{H}^1(l_j^+), \tag{7.62}$$

with c independent of ε_n. Let us set

$$\gamma_n^{k,+} := L_1^+ \cup B_1A_2 \cup L_2^+ \cup \cdots \cup B_{m-1}A_m \cup L_m^+.$$

Similarly, let us construct $\gamma_n^{k,-}$ relative to $H_n^{k,-} \cap E_n^+$. Note that for n large enough $\gamma_n^{k,+} \cap H_k(\sigma) = \emptyset$, $\gamma_n^{k,-} \cap H_k(-\sigma) = \emptyset$, and $\gamma_n^{k,+} \cap \gamma_n^{k,-} = \emptyset$. Let us consider the connected component $\mathcal{C}_k^+$ of $\mathcal{R}_n(Q_k) \setminus \gamma_n^{k,+}$ containing $H_k(\sigma)$. Similarly, let us consider the connected component $\mathcal{C}_k^-$ of $\mathcal{R}_n(Q_k) \setminus \gamma_n^{k,-}$ containing $H_k(-\sigma)$. For n large enough, by (7.62)

$$\mathcal{H}^1\left(\partial_{\mathcal{R}_n(Q_k)}\mathcal{C}_k^+ \setminus \bigcup_{i=1}^{m-1} B_iA_{i+1}\right) \leq c\sum_{j=1}^{m}\mathcal{H}^1(l_j^+) \leq 20c\sigma r_k. \tag{7.63}$$

A similar estimate holds for $\partial_{\mathcal{R}_n(Q_k)}\mathcal{C}_k^-$.

Let $\tilde{E}_n^{k,+}$ be the family of triangles obtained adding to $E_n^{k,+}$ those $T \in E_n^{k,-}$ such that $T \subseteq \mathcal{C}_k^+$, and subtracting those $T \in E_n^{k,+}$ such that $T \subseteq \mathcal{C}_k^-$. Let $\tilde{E}_n^{k,-}$ be the complement of $\tilde{E}_n^{k,+}$ in $\mathcal{T}_n^k(\mathcal{R}_n(Q_k))$.

We claim that there exists $C > 0$ independent of n such that for all $k = 1, \ldots, K$ and for n large

$$\mathcal{H}^1\left(\partial_{\mathcal{R}_n(Q_k)}\tilde{E}_n^{k,+} \setminus \partial_{\mathcal{R}_n(Q_k)}E_n^{k,+}\right) \leq C\sigma r_k. \tag{7.64}$$

In fact, let ζ be an edge of $\partial_{\mathcal{R}_n(Q_k)}\tilde{E}_n^{k,+} \setminus \partial_{\mathcal{R}_n(Q_k)}E_n^{k,+}$, that is ζ belongs to a triangle T that has been changed in the operation above described. Let us assume for instance that $T \in E_n^{k,-}$ and $T \subseteq \mathcal{C}_k^+$. If T' is such that $T \cap T' = \zeta$, then $T' \in E_n^{k,-}$: in fact if by contradiction $T' \in E_n^{k,+}$, then $T' \in \tilde{E}_n^{k,+}$ and so we would have $\zeta \notin \partial_{\mathcal{R}_n(Q_k)}\tilde{E}_n^{k,+}$ which is absurd. Similarly we get $T' \not\subseteq \mathcal{C}_k^+$. This means that $\zeta \subseteq \partial_{\mathcal{R}_n(Q_k)}\mathcal{C}_k^+$, and since the horizontal edges of $\gamma_n^{k,+}$ intersect by construction only elements of $E_n^{k,+}$, we deduce that $\zeta \subseteq \partial_{\mathcal{R}_n(Q_k)}\mathcal{C}_k^+ \setminus (\cup_{i=1}^m A_iB_i)$, and by (7.63) we conclude that (7.64) holds.

We can summarize the previous results as follows.

Lemma 7.4.6. *For all $k = 1, \ldots, K$ there exist two families $\tilde{E}_n^{k,+}$ and $\tilde{E}_n^{k,-}$ of triangles with $\mathcal{T}_n^k(\mathcal{R}_n(Q_k)) = \tilde{E}_n^{k,+} \cup \tilde{E}_n^{k,-}$, $Q_k^+ \setminus R_k \subseteq \tilde{E}_n^{k,+}$ and $Q_k^- \setminus R_k \subseteq \tilde{E}_n^{k,-}$, and such that*

$$\limsup_n \sum_{k=1}^{K} \mathcal{H}^1\left(\partial_{\mathcal{R}_n(Q_k)}\tilde{E}_n^{k,+} \setminus S(z_n^{r(k)})\right) = o_\sigma, \tag{7.65}$$

where $o_\sigma \to 0$ as $\sigma \to 0$. Moreover, in the case $x_k \in \partial_D\Omega$, we can modify $\tilde{E}_n^{k,+}$ or $\tilde{E}_n^{k,-}$ in such a way that $\tilde{E}_n^{k,+} \subseteq \Omega$ or $\tilde{E}_n^{k,-} \subseteq \Omega$.

Proof. We have that (7.65) follows from (7.60) and (7.64), and the fact that $\sum_{k=1}^K r_k \leq c$, with c independent of σ. Let us consider the case $x_k \in \partial_D\Omega$ with $Q_k^+ \setminus R_k \subseteq \Omega$ (the other case being similar). From (7.65) we have that for n large $\sum_{k=1}^K \mathcal{H}^1\left(\partial_{\mathcal{R}_n(Q_k)}\tilde{E}_n^{k,+} \cap Q_k^-\right) \leq o_\sigma$ because $z_n^{r(k)} = R_{\varepsilon_n} g_{h_n}(r(k))$ on Q_k^- and so there are no jumps in Q_k^-. We can thus redefine $\tilde{E}_n^{k,+}$ subtracting those triangles that are in Q_k^- obtaining again (7.65). □

We are now in position to prove Proposition 7.4.1.

Proof of Proposition 7.4.1. We work in the context of Ω'. For all $v \in SBV(\Omega')$ with $v = g^i$ on Ω_D, $\mathcal{H}^1(S(v)) < +\infty$ and $\nabla v \in L^2(\Omega';\mathbb{R}^2)$, we have to construct $v_n \in SBV(\Omega')$ such that $v_n = g_n^i$ on Ω_D, $(v_n)_{|\Omega} \in \mathcal{A}_{\varepsilon_n,a}(\Omega)$, $v_n \to v$ strongly in $L^1(\Omega')$, $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega';\mathbb{R}^2)$ and

$$\limsup_n \mathcal{H}^1\left(S(v_n)\setminus\bigcup_{r=0}^{i} S(u_n^r)\right) \le \mu(a)\mathcal{H}^1\left(S(v)\setminus\bigcup_{r=0}^{i} S(u^r)\right), \tag{7.66}$$

where we suppose that u_n^r and u^r are extended to Ω' setting $u_n^r := g_n^r$, and $u^r := g^r$ on Ω_D respectively.

We set $v = g^i + w$, where $w \in SBV(\Omega')$ with $w = 0$ on Ω_D. By density, it is sufficient to consider the case $w \in L^\infty(\Omega')$. Up to reducing U, we may assume that $\|\nabla g^i\|_{L^2(U;\mathbb{R}^2)} < \sigma$ and $\|\nabla w\|_{L^2(U;\mathbb{R}^2)} < \sigma$. Let R'_k be a rectangle centered in x_k, oriented as R_k, and such that $\overline{R'_k} \subset \mathrm{int} R_k$ and $\mathcal{H}^1(S(w)\cap(R_k\setminus R'_k)) < \sigma r_k$. We claim that there exists $w_\sigma \in SBV(\Omega')$ with $w_\sigma = w$ on $\bigcup_{k=1}^K R'_k$ and $w_\sigma = 0$ in Ω_D such that

(1) $\|w - w_\sigma\| + \|\nabla w - \nabla w_\sigma\| \le \sigma$;

(2) $\mathcal{H}^1(S(w_\sigma)\cap(Q_k\setminus R'_k)) \le o_\sigma r_k$, with $o_\sigma \to 0$ as $\sigma \to 0$;

(3) $\mathcal{H}^1(S(w_\sigma)\setminus\bigcup_{k=1}^K R_k) \le \mathcal{H}^1(S(w)\setminus\bigcup_{k=1}^K R_k) + \sigma$;

(4) $S(w_\sigma)\setminus\bigcup_{k=1}^K R_k$ is union of disjoint segments with closure contained in $\Omega\setminus\bigcup_{k=1}^K R_k$;

(5) w_σ is of class $W^{2,\infty}$ on $\Omega\setminus\left(\bigcup_{k=1}^K R_k \cup \overline{S(w_\sigma)}\right)$.

In fact, by Proposition 7.4.2, there exists $w_m \in SBV(\Omega')$ with $w_m = 0$ in $\Omega'\setminus\overline{\Omega}$ such that $w_m \to w$ strongly in $L^2(\Omega')$, $\nabla w_m \to \nabla w$ strongly in $L^2(\Omega';\mathbb{R}^2)$, $S(w_m)$ is polyhedral with $\overline{S(w_m)} \subseteq \Omega$, w_m is of class $W^{2,\infty}$ on $\Omega\setminus\left(\bigcup_{k=1}^K R_k\cup\overline{S(w_m)}\right)$, and $\lim_m \mathcal{H}^1(A\cap S(w_m)) = \mathcal{H}^1(A\cap S(w))$ for all A open subset of Ω' with $\mathcal{H}^1(\partial A\cap S(w)) = 0$. It is not restrictive to assume that $\mathcal{H}^1(S(w)\cap\partial R_k) = 0$ and $\mathcal{H}^1(S(w_m)\cap\partial R_k) = 0$ for all m. Let ψ_k be a smooth function such that $0 \le \psi_k \le 1$, $\psi_k = 1$ on R'_k and $\psi_k = 0$ outside R_k. Setting $\psi := \sum_{k=1}^K \psi_k$, let us consider $\tilde{w}_m := \psi w + (1-\psi)w_m$. Note that $\tilde{w}_m \to w$ strongly in $L^2(\Omega')$, $\nabla\tilde{w}_m \to \nabla w$ strongly in $L^2(\Omega';\mathbb{R}^2)$, $\tilde{w}_m = 0$ in Ω_D. Moreover, by capacity arguments, we may assume that $S(\tilde{w}_m)\setminus\bigcup_{k=1}^K R_k$ is a finite union of disjoint segments with closure contained in $\Omega\setminus\bigcup_{k=1}^K R_k$. Finally, for $m \to +\infty$, we have

$$\mathcal{H}^1(S(\tilde{w}_m)\setminus\bigcup_{k=1}^K R_k) \to \mathcal{H}^1(S(w)\setminus\bigcup_{k=1}^K R_k)),$$

$$\mathcal{H}^1(S(\tilde{w}_m)\cap\bigcup_{k=1}^K(Q_k\setminus R_k)) \to \mathcal{H}^1(S(w)\cap\bigcup_{k=1}^K(Q_k\setminus R_k))$$

and $\limsup_m \mathcal{H}^1(S(\tilde{w}_m)\cap(R_k\setminus R'_k)) \le 2\mathcal{H}^1(S(w)\cap(R_k\setminus R'_k)) \le 2\sigma r_k$. Then we can take $w_\sigma := \tilde{w}_m$ for m large enough.

Let $S(w_\sigma)\setminus\bigcup_{k=1}^K Q_k := \bigcup_{j=1}^m l_j$, where, by capacity arguments, we can always assume that l_j are disjoint segments with closure contained in $\Omega\setminus\bigcup_{k=1}^K Q_k$. We define a triangulation $\mathbf{T}_n \in \mathcal{T}_{\varepsilon_n,a}(\Omega')$ specifying its adaptive vertices as follows. Let us consider the families $\mathcal{R}_n(Q_k)$ and $\mathcal{R}_n(l_j)$ for $k = 1,\dots,K$ and $j = 1,\dots,m$. Note that for n large enough, $\mathcal{R}_n(Q_{k_1})\cap\mathcal{R}_n(Q_{k_2}) = \emptyset$ for $k_1 \neq k_2$, $\mathcal{R}_n(l_{j_1})\cap\mathcal{R}_n(l_{j_2}) = \emptyset$ for $j_1 \neq j_2$, and $\mathcal{R}_n(Q_k)\cap\mathcal{R}_n(l_j) = \emptyset$ for every k,j. We consider inside the triangles of $\mathcal{R}_n(Q_k)$ the adaptive vertices of $\mathbf{T}(z_n^{r(k)})$. Passing to $\mathcal{R}_n(l_j)$, by density arguments it is not restrictive to assume that l_j does not pass through the vertices of $\mathbf{R}_{\varepsilon_n}$ and that its extremes belong to the edges of $\mathbf{R}_{\varepsilon_n}$. Let $\zeta := [x,y]$ be an edge of $\mathcal{R}_n(l_j)$ such that $l_j \cap \zeta = \{P\}$. Proceeding as in [75], we take as adaptive vertex of ζ the projection of P on $\{tx + (1-t)y : t \in [a,(1-a)]\}$. Connecting these adaptive vertices, we obtain an *interpolating* polyhedral curve $\tilde{l}_j$ with

$$\mathcal{H}^1(\tilde{l}_j) \leq \mu(a)\mathcal{H}^1(l_j), \tag{7.67}$$

where μ is an increasing function such that $\lim_{a\to 0}\mu(a) = 1$. Finally, in the remaining edges, we can consider any admissible adaptive vertex, for example the middle point.

Let us define $w_n \in SBV(\Omega')$ in the following way. For all Q_k, let w_n be equal to w_σ on $\mathcal{R}_n(Q_k) \setminus R_k$, equal to the reflection of $w_{\sigma|Q_k^+\setminus R_k}$ with respect to $H_k(\sigma)$ on $\tilde{E}_n^{k,+} \cap R_k$ and equal to the reflection of $w_{\sigma|Q_k^-\setminus R_k}$ with respect to $H_k(-\sigma)$ on $\tilde{E}_n^{k,-} \cap R_k$, where $\tilde{E}_n^{k,\pm}$ are defined as in Lemma 7.4.6. On the other elements of $\mathbf{T}_n$, let us set $w_n = w_\sigma$. Notice that $w_n = 0$ on Ω_D and that inside each $\mathcal{R}_n(Q_k)$, all the discontinuities of w_n are contained in $\partial_{\mathcal{R}_n(Q_k)}\tilde{E}_n^{k,+} \cup V_k \cup P_{w_\sigma}^k$, where $P_{w_\sigma}^k$ is the union of the polyhedral jumps of w_σ in $\mathcal{R}_n(Q_k)$ and of their reflected version with respect to $H_k(\pm\sigma)$. By Lemma 7.4.6 and since $\sum_{k=1}^K \mathcal{H}^1(V_k \cup P_{w_\sigma}^k) \leq o_\sigma$ with $o_\sigma \to 0$ as $\sigma \to 0$, and $\mathcal{H}^1\left(\bigcup_{r=0}^i S(z^r) \setminus \bigcup_{k=1}^K Q_k\right) \leq 2\sigma$, we have that

$$\begin{aligned}
&\limsup_n \mathcal{H}^1\left(S(w_n) \setminus \bigcup_{r=0}^i S(z_n^r)\right) \leq \\
&\quad\leq \mathcal{H}^1\left(S(w_\sigma) \setminus \bigcup_{k=1}^K Q_k\right) + \limsup_n \mathcal{H}^1\left((S(w_n) \setminus \bigcup_{r=0}^i S(z_n^r)) \cap \mathcal{R}_n(Q_k)\right) \leq \\
&\quad\quad\leq \mathcal{H}^1\left(S(w_\sigma) \setminus \bigcup_{r=0}^i S(z^r)\right) + \mathcal{H}^1\left(\bigcup_{r=0}^i S(z^r) \setminus \bigcup_{k=1}^K Q_k\right) + o_\sigma \leq \\
&\quad\quad\quad\leq \mathcal{H}^1\left(S(w_\sigma) \setminus \bigcup_{r=0}^i S(z^r)\right) + o_\sigma,
\end{aligned}$$

and since $\|\nabla w_\sigma\|_{L^2(U;\mathbb{R}^2)} \leq o_\sigma$ we get for n large

$$\|\nabla w_n\|^2_{L^2(\bigcup_{k=1}^K \mathcal{R}_n(Q_k))} \leq o_\sigma. \tag{7.68}$$

We now want to define an interpolation $\tilde{w}_n$ of w_n on $\mathbf{T}_n$. Firstly, we set $\tilde{w}_n = 0$ on all regular triangles of Ω_D. Passing to the triangles in $\mathcal{R}_n(Q_k)$ (see fig.4), by Lemma 7.1.2, we know that for n large enough, we have

$$\mathcal{H}^1(\partial\mathcal{R}_n(V_k)) \leq c\mathcal{H}^1(V_k), \quad \mathcal{H}^1(\partial\mathcal{R}_n(P_{w_\sigma}^k)) \leq c\mathcal{H}^1(P_{w_\sigma}^k),$$

with c independent of n. If $T \in \mathcal{R}_n(V_k) \cup \mathcal{R}_n(P_{w_\sigma}^k)$, we set $\tilde{w}_n = 0$ on T; otherwise, we define $\tilde{w}_n$ on T as the affine interpolation of w_n.

Since $\nabla\tilde{w}_n$ is uniformly bounded on $\mathcal{R}_n(H_k(\pm\sigma))$, $|\mathcal{R}_n(H_k(\pm\sigma))| \to 0$ and since w_n is uniformly bounded in $W^{2,\infty}$ on the triangles contained in $\mathcal{R}_n(Q_k) \setminus \mathcal{R}_n(V_k \cup P_{w_\sigma}^k \cup H_k(\pm\sigma))$ we have by the interpolation estimate (7.9) and by (7.68)

$$\limsup_n \|\nabla\tilde{w}_n\|^2_{L^2(\bigcup_{k=1}^K \mathcal{R}_n(Q_k))} \leq o_\sigma. \tag{7.69}$$

Moreover we have

$$\limsup_n \sum_{k=1}^K \mathcal{H}^1\left(\left(S(\tilde{w}_n) \setminus \bigcup_{r=0}^k S(z_n^r)\right) \cap \mathcal{R}_n(Q_k)\right) \leq o_\sigma. \tag{7.70}$$

Let us come to the triangles not belonging to $\mathcal{R}_n(Q_k)$ for $k = 1,\dots,K$. For all $j = 1,\dots,m$, we denote by $\hat{\mathcal{R}}_n(l_j)$ the family of regular triangles that have edges in common with triangles of

$\mathcal{R}_n(l_j)$. For n large we have that $\hat{\mathcal{R}}_n(l_{j_1}) \cap \hat{\mathcal{R}}_n(l_{j_2}) = \emptyset$ for $j_1 \neq j_2$. On every regular triangle $T \notin \bigcup_{k=1}^K \mathcal{R}_n(Q_k) \cup \bigcup_{j=1}^m \hat{\mathcal{R}}_n(l_j)$, we define $\tilde{w}_n$ as the affine interpolation of w_σ. Since w_σ is of class $W^{2,\infty}$ on T and T is regular, we obtain by the interpolation estimate (7.9)

$$\|\tilde{w}_n - w_\sigma\|^2_{W^{1,2}(T)} \leq K\varepsilon_n \|w_\sigma\|_{W^{2,\infty}}. \tag{7.71}$$

Let us consider now those triangles that are contained in the elements of $\bigcup_{j=1}^m \hat{\mathcal{R}}_n(l_j)$. Following [75], we can define $\tilde{w}_n$ on every T in such a way that $\tilde{w}_n$ admits discontinuities only on $\tilde{l}_j$, and $\|\nabla \tilde{w}_n\|_{L^\infty(T)} \leq \|\nabla w_\sigma\|_\infty$. Since $|\hat{\mathcal{R}}_n(l_j)| \to 0$ as $n \to \infty$, we deduce that

$$\lim_n \|\nabla \tilde{w}_n\|^2_{L^2(\hat{\mathcal{R}}_n(l_j))} = 0. \tag{7.72}$$

Moreover by (7.67) and since $\mathcal{H}^1\left(\bigcup_{r=0}^i S(z^r) \setminus \bigcup_{k=1}^K Q_k\right) \leq 2\sigma$, we have

$$\mathcal{H}^1\left(S(\tilde{w}_n) \cap \bigcup_{j=1}^m \hat{\mathcal{R}}_n(l_j)\right) \leq \mu(a)\mathcal{H}^1\left(S(w_\sigma) \setminus \bigcup_{i=1}^k S(z^r)\right) + o_\sigma, \tag{7.73}$$

where $o_\sigma \to 0$ as $\sigma \to 0$.

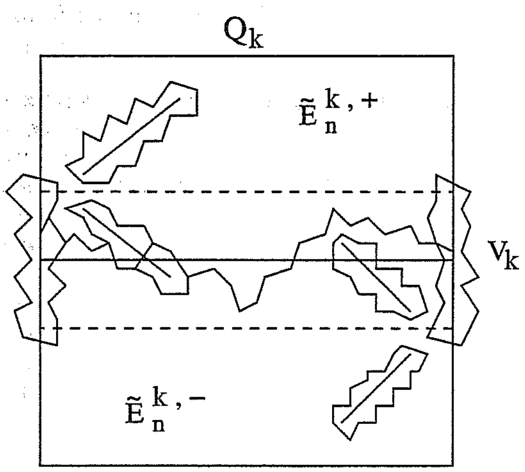

Fig. 4

We are now ready to conclude. Let us consider $\hat{w}_n \in \mathcal{A}_{\varepsilon_n,a}(\Omega)$ defined as $\hat{w}_n := g_n^i + \tilde{w}_n$. We have $\hat{w}_n \to g^i + w_\sigma$ strongly in $L^2(\Omega')$. By (7.69), (7.71), (7.72) we get

$$\limsup_n \|\nabla \hat{w}_n\|^2 \leq \|\nabla g^i + \nabla w_\sigma\|^2 + o_\sigma,$$

while by (7.70) and (7.73) we have

$$\limsup_n \mathcal{H}^1\left(S(\hat{w}_n) \setminus \bigcup_{r=0}^i S(z_n^r)\right) \leq \mu(a)\mathcal{H}^1\left(S(w_\sigma) \setminus \bigcup_{r=0}^i S(z^r)\right) + o_\sigma.$$

Letting now $\sigma \to 0$, using a diagonal argument, we conclude that Proposition 7.4.1 holds. $\square$

7.5 Revisiting the approximation by Francfort and Larsen

In this section we show how the arguments of Section 7.3 may be used to deal with the discrete in time approximation of quasi-static growth of brittle cracks proposed by Francfort and Larsen in [53]. More precisely, we prove that there is strong convergence of the gradient of the displacement (in particular convergence of the bulk energy) and convergence of the surface energy at all times of continuity of the length of the crack; moreover there is convergence of the total energy at any time.

We briefly recall the notation employed in [53]. Let I_∞ be countable and dense in $[0,1]$, and let $I_n := \{0 = t_0^n \leq \ldots \leq t_n^n = 1\}$ such that (I_n) is an increasing sequence of sets whose union is I_∞. Let $\Omega \subseteq \mathbb{R}^N$ be a Lipschitz bounded domain, and let $\partial\Omega = \partial\Omega_f^c \cup \partial\Omega_f$, where $\partial\Omega_f^c$ is open in the relative topology. Let $\Omega' \subseteq \mathbb{R}^N$ be open and such that $\overline{\Omega} \subseteq \Omega'$, and let $g \in W^{1,1}([0,1]; H^1(\Omega'))$. At any time t_k^n, Francfort and Larsen consider u_k^n minimizer of

$$\int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1}\left(S(v) \setminus \left[\bigcup_{0 \leq j \leq k-1} S(u_j^n) \cup \partial\Omega_f \right] \right)$$

in $\{v \in SBV(\Omega') : v = g(t_k^n) \text{ in } \Omega' \setminus \overline{\Omega}\}$. Setting $u^n(t) := u_k^n$ for $t \in [t_k^n, t_{k+1}^n[$, and $\Gamma^n(t) := \bigcup_{s \leq t, s \in I_n} S(u^n(s)) \cup \partial\Omega_f$, they prove that

$$\mathcal{E}^n(t) \leq \mathcal{E}^n(0) + 2\int_0^{t_k^n} \int_\Omega \nabla u^n(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau + o_n, \qquad t \in [t_k^n, t_{k+1}^n[, \tag{7.74}$$

where $\mathcal{E}^n(t) := \int_\Omega |\nabla u^n(t)|^2 \, dx + \mathcal{H}^{N-1}(\Gamma^n(t))$ and $o_n \to 0$ as $n \to +\infty$. Using Theorem 1.4.3, they obtain a subsequence of $(u^n(\cdot))$, still denoted by the same symbol, such that $u^n(t) \to u(t)$ in $SBV(\Omega')$ and $\nabla u^n(t) \to \nabla u(t)$ strongly in $L^2(\Omega'; \mathbb{R}^N)$ for all $t \in I_\infty$, with $u(t)$ a minimizer of

$$\int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{N-1}(S(v) \setminus \Gamma(t)),$$

where $\Gamma(t) := \bigcup_{s \in I_\infty, s \leq t} S(u(s)) \cup \partial\Omega_f$. The evolution $\{t \to u(t), t \in I_\infty\}$ is extended to the whole $[0,1]$ using the approximation from the left in time.

We can now use the arguments of Section 7.3. Following Lemma 7.3.6, it turns out that for all $t \in [0,1]$

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau. \tag{7.75}$$

Moreover, by the Transfer of Jump and the uniqueness argument of Lemma 7.3.3, we have that $\nabla u^n(t) \to \nabla u(t)$ strongly in $L^2(\Omega'; \mathbb{R}^N)$ for all $t \notin \mathcal{N}$, where $\mathcal{N}$ is the (at most countable) set of discontinuities of the pointwise limit λ of $\mathcal{H}^{N-1}(\Gamma(\cdot))$ (which exists up to a further subsequence by Helly's Theorem). Then we pass to the limit in (7.74) obtaining

$$\mathcal{E}(t) \leq \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau;$$

moreover, following the proof of Theorem 7.0.1, we have that for all $t \in [0,1]$

$$\mathcal{E}(t) \leq \liminf_n \mathcal{E}_n(t) \leq \limsup_n \mathcal{E}_n(t) = \mathcal{E}(0) + 2\int_0^t \int_\Omega \nabla u(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau,$$

and taking into account (7.75) we get the convergence of the total energy at any time. Since $\nabla u^n(t) \to \nabla u(t)$ strongly in $L^2(\Omega'; \mathbb{R}^N)$ for every $t \in I_\infty$, we deduce that $\lambda = \mathcal{H}^{N-1}(\Gamma(\cdot))$ on I_∞, so that the convergence of the surface energy holds in I_∞. The extension to the continuity times for $\mathcal{H}^{N-1}(\Gamma(\cdot))$ follows like in the final part of the proof of Theorem 7.0.1.

Chapter 8

Discontinuous finite elements approximation of quasistatic crack growth in nonlinear elasticity

In this chapter [1] we provide a discontinuous finite element approximation of a model of quasistatic growth of brittle cracks in nonlinear elasticity recently proposed in [44] by Dal Maso, Francfort, and Toader in the framework of the variational theory of crack propagation proposed by Francfort and Marigo in [54].

The quasistatic crack growth proposed by Dal Maso, Francfort, and Toader in [44] considers the case of nonlinear elasticity, and takes into account possible volume and traction forces applied to the elastic body. In order to describe the result of [44] (a complete description is given in Section 8.1), let us assume that the elastic body has a reference configuration given by $\Omega \subseteq \mathbb{R}^N$ open, bounded and with Lipschitz boundary. Let $\partial_D\Omega \subseteq \partial\Omega$ be open in the relative topology, and let $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. Let $\Omega_B \subseteq \Omega$, and let $\partial_S\Omega \subseteq \partial_N\Omega$ be such that $\overline{\Omega}_B \cap \partial_S\Omega = \emptyset$. Ω_B is the *brittle part* of Ω, and $\partial_S\Omega$ is the part of the boundary where traction forces are supposed to act. A crack is given by any rectifiable set in $\overline{\Omega}_B$ with finite $(N-1)$ Hausdorff measure. Given a boundary deformation g on $\partial_D\Omega$ and a crack Γ, the family of all admissible deformation of Ω is given by the set $AD(g,\Gamma)$ of all function $u \in GSBV(\Omega;\mathbb{R}^N)$ such that $S(u) \subseteq \Gamma$ and $u = g$ on $\partial_D\Omega \setminus \Gamma$. Here $S(u)$ denotes the set of jumps of u, and the equality $u = g$ is intended in the sense of traces. Requiring $u = g$ only on $\partial_D\Omega \setminus \Gamma$ means that the deformation is assumed not to be transmitted through the crack. The bulk energy considered in [44] is of the form

$$\int_\Omega W(x, \nabla u(x))\, dx,$$

where $W(x,\xi)$ is quasiconvex in ξ, and satisfies suitable regularity and growth assumptions (see (8.6) and (8.7)). Moreover the time dependent body and traction forces are supposed to be conservative with work given by

$$-\int_{\Omega\setminus\Gamma} F(t,x,u(x))\, dx - \int_{\partial_S\Omega} G(t,x,u(x))\, d\mathcal{H}^{N-1}(x),$$

where F and G satisfy suitable regularity and growth conditions (see Section 8.1). Finally the work made to produce the crack Γ is given by

$$\mathcal{E}^s(\Gamma) := \int_\Gamma k(x,\nu(x))\, d\mathcal{H}^{N-1}(x),$$

[1] The results of this chapter are contained in the paper
A. Giacomini, M. Ponsiglione: Discontinuous finite element approximation of quasistatic crack growth in nonlinear elasticity. Preprint Sissa 2004.

where $\nu(x)$ is the normal to Γ at x, and $k(x,\nu)$ satisfies standard hypotheses which guarantee lower semicontinuity (see Section 8.1). Clearly, W, F, G and k depend on the material. Let us set

$$\mathcal{E}^b(t)(u) := \int_\Omega W(x, \nabla u(x))\, dx - \int_{\Omega\setminus\Gamma} F(t,x,u(x))\, dx - \int_{\partial_S\Omega} G(t,x,u(x))\, d\mathcal{H}^{N-1}(x),$$

and

$$\mathcal{E}(t)(u,\Gamma) := \mathcal{E}^b(t)(u) + \mathcal{E}^s(\Gamma). \tag{8.1}$$

Given a boundary deformation $g(t)$ with $t \in [0,T]$ and a preexisting crack Γ_0, a quasistatic crack growth relative to g and Γ_0 is a map $\{t \to (u(t),\Gamma(t)) \,:\, t \in [0,T]\}$ such that the following conditions hold:

(1) for all $t \in [0,T]$: $u(t) \in AD(g(t),\Gamma(t))$;

(2) *irreversibility*: $\Gamma_0 \subseteq \Gamma(s) \subseteq \Gamma(t)$ for all $0 \le s \le t \le T$;

(3) *static equilibrium*: for all $t \in [0,T]$ and for all admissible configurations (u,Γ) with $\Gamma(t) \subseteq \Gamma$

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \mathcal{E}(t)(u,\Gamma);$$

(4) *nondissipativity*: the time derivative of the total energy $\mathcal{E}(t)(u(t),\Gamma(t))$ is equal to the power of external forces (see (8.23)).

In this chapter we discretize the model in the line of Chapter 7, that is using adaptive triangulations determined by the parameters $\varepsilon > 0$ and $a \in (0, \frac{1}{2})$. We restrict our analysis to a two dimensional setting considering only a polygonal reference configuration $\Omega \subseteq \mathbb{R}^2$.

The discretization of the energy functional is obtained restricting the total energy (8.1) to the family of functions u which are affine on the triangles of some triangulation $\mathbf{T}(u) \in \mathcal{T}_{\varepsilon,a}(\Omega)$ and are allowed to jump across the edges of $\mathbf{T}(u)$ contained in $\overline{\Omega}_B$. We indicate this space by $\mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$. The boundary data is assumed to belong to the space $\mathcal{AF}_\varepsilon(\Omega)$ of continuous functions which are affine on every triangle $T \in \mathbf{R}_\varepsilon$.

Let us consider a boundary datum $g_\varepsilon \in W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2))$ with $g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega)$ for all $t \in [0,T]$ (p,q are related to the growth assumptions on W,F,G) and an initial crack $\Gamma^0_{\varepsilon,a}$ (see Section 8.4). For all $u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ we indicate by $S(u)$ the edges of the triangulation $\mathbf{T}(u)$ across which u jumps, while we denote by $S^{g_\varepsilon(t)}_D(u)$ the edges of the triangulation $\mathbf{T}(u)$ contained in $\partial_D\Omega$ on which $u \ne g_\varepsilon(t)$. Let us divide $[0,1]$ into subintervals $[t^\delta_i, t^\delta_{i+1}]$ of size $\delta > 0$ for $i = 0,\dots,N_\delta$. Using a variational argument (Proposition 8.4.1), we construct a *discrete* (in time and space) *evolution* $\{(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a}) \,:\, i = 0,\dots,N_\delta\}$ such that for all $i = 0,\dots,N_\delta$ we have $u^{\delta,i}_{\varepsilon,a} \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$,

$$\Gamma^{\delta,i}_{\varepsilon,a} := \bigcup_{r=0}^{i} \left[S(u^{\delta,r}_{\varepsilon,a}) \cup S^{g_\varepsilon(t^\delta_r)}_D(u^{\delta,r}_{\varepsilon,a})\right],$$

and the following *unilateral minimality property* holds: for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$

$$\mathcal{E}^b(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}) \le \mathcal{E}^b(t^\delta_i)(v) + \mathcal{E}^s\left(\left(S(v) \cup S^{g_\varepsilon(t^\delta_i)}_D(v)\right) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right). \tag{8.2}$$

Notice that by construction $u^{\delta,i}_{\varepsilon,a} \in AD(g_\varepsilon(t^\delta_i), \Gamma^{\delta,i}_{\varepsilon,a})$. Moreover the definition of the discrete crack ensures that $\Gamma^{\delta,i}_{\varepsilon,a} \subseteq \Gamma^{\delta,j}_{\varepsilon,a}$ for all $i \le j$, recovering in this discrete setting the irreversibilty of the crack growth given in (2). The minimality property (8.2) is the reformulation in the finite element space of the equilibrium condition (3). Finally we obtain an estimate from above for $\mathcal{E}(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a})$ (see Proposition 8.4.2) which is a discrete version of (4).

In order to perform the asymptotic analysis of the *discrete evolution* $\{(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a}) : i = 0, \dots, N_\delta\}$ we make the piecewise constant interpolation in time $u^\delta_{\varepsilon,a}(t) = u^{\delta,i}_{\varepsilon,a}$ and $\Gamma^\delta_{\varepsilon,a}(t) = \Gamma^{\delta,i}_{\varepsilon,a}$ for all $t^\delta_i \le t < t^\delta_{i+1}$. Let us suppose that

$$g_\varepsilon \to g \qquad \text{strongly in } W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2))$$

(where on $W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$ we take the norm $\|u\| := \|u\|_{W^{1,p}(\Omega;\mathbb{R}^2)} + \|u\|_{L^q(\Omega;\mathbb{R}^2)}$), and that $\Gamma^0_{\varepsilon,a}$ approximate an initial crack Γ^0 in the sense of Proposition 8.3.1.

The main result of the chapter (Theorem 8.5.1) states that there exist a quasistatic evolution $\{t \to (u(t), \Gamma(t)) : t \in [0,T]\}$ in the sense of [44] relative to the boundary deformation g and the preexisting crack Γ^0 and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, such that setting

$$u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t), \qquad \Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t),$$

for all $t \in [0,T]$ the following facts hold:

(a) $(u_n(t))_{n\in\mathbb{N}}$ is weakly precompact in $GSBV^p_q(\Omega;\mathbb{R}^2)$, and every accumulation point $\tilde{u}(t)$ is such that $\tilde{u}(t) \in AD(g(t), \Gamma(t))$, and $(\tilde{u}(t), \Gamma(t))$ satisfy the static equilibrium (2); moreover there exists a subsequence $(\delta_{n_k}, \varepsilon_{n_k}, a_{n_k})_{k\in\mathbb{N}}$ of $(\delta_n, \varepsilon_n, a_n)_{n\in\mathbb{N}}$ (depending on t) such that

$$u_{n_k}(t) \rightharpoonup u(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2);$$

(b) convergence of the total energy holds, and more precisely elastic and surface energies converge separately, that is

$$\mathcal{E}^b(t)(u_n(t)) \to \mathcal{E}^b(t)(u(t)), \qquad \mathcal{E}^s(\Gamma_n(t)) \to \mathcal{E}^s(\Gamma(t)).$$

By point (a), the approximation of the deformation $u(t)$ is available only up to a subsequence depending on t: this is due to the possible non uniqueness of the minimum energy deformation associated to $\Gamma(t)$. In the case $\mathcal{E}^b(t)(u)$ is strictly convex, it turns out that the deformation $u(t)$ is uniquely determined, and we prove that (Theorem 8.6.1)

$$\nabla u_n(t) \to \nabla u(t) \qquad \text{strongly in } L^p(\Omega; M^{2\times 2}),$$

and

$$u_n(t) \to u(t) \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2).$$

The main difficulty to prove Theorem 8.5.1 consists in passing to the limit in the static equilibrium (8.2). In order to find the crack $\Gamma(t)$ in the limit, in Lemma 8.5.2 and Lemma 8.5.4 we adapt to the context of finite elements the notion of σ^p- convergence of sets formulated in [44]. This is the key tool to obtain the convergence of elastic and surface energies at all times $t \in [0,T]$ (while in [59] this was available only at the continuity points of $\mathcal{H}^1(\Gamma(t))$). In order to infer the static equilibrium of $\Gamma(t)$ from that of $\Gamma_n(t)$, we employ a generalization of the piecewise affine transfer of jumps [59, Proposition 5.1] (see Proposition 8.2.2).

The chapter is organized as follows. In Section 8.1 we describe the quasistatic crack growth of [44] precising the functional setting and the hypotheses on the elastic and surface energies involved. In Section 8.2 we introduce the finite element space, and in Section 8.3 we prove an approximation result for a preexisting crack configuration. In Section 8.4 we prove the existence of a discrete evolution, and in Section 8.5 we prove the main approximation result (Theorem 8.5.1). In Section 8.6 we treat the case of strictly convex total energy.

8.1 The quasistatic crack growth of Dal Maso-Francfort-Toader

In this section we describe the quasistatic evolution of brittle cracks proposed in [44]. They consider the case of n-dimensional nonlinear elasticity, for an arbitrary $n \ge 1$, with a quasiconvex bulk energy and with prescribed boundary deformations and applied loads, depending on time. Since we are going to approximate the case $n = 2$, we prefer to introduce the model in this particular case. For more details, we refer the reader to [44].

Let Ω be a bounded open set of $\mathbb{R}^2$ with Lipschitz boundary and let Ω_B be an open subset of Ω. Let $\partial_N\Omega \subseteq \partial\Omega$ be closed in the relative topology, and let $\partial_D\Omega := \partial\Omega \setminus \partial_N\Omega$. Let $\partial_S\Omega \subseteq \partial_N\Omega$ be closed in the relative topology and such that $\overline{\Omega}_B \cap \partial_S\Omega = \emptyset$. In the model proposed in [44], Ω_B represents the brittle part of Ω, and $\partial_D\Omega$ the part of the boundary on which the deformation is prescribed. Moreover the elastic body Ω is supposed to be subject to surface forces acting on $\partial_S\Omega$.

Admissible cracks and deformations. The set of admissible cracks is given by

$$\mathcal{R}(\overline{\Omega}_B; \partial_N\Omega) := \{\Gamma : \Gamma \text{ is rectifiable }, \Gamma \,\tilde{\subseteq}\, (\overline{\Omega}_B \setminus \partial_N\Omega),\ \mathcal{H}^1(\Gamma) < +\infty\}.$$

Here $A \,\tilde{\subseteq}\, B$ means that $A \subseteq B$ up to a set of $\mathcal{H}^1$-measure zero, and Γ rectifiable means that there exists a sequence (M_i) of C^1-manifolds such that $\Gamma \,\tilde{\subseteq}\, \bigcup_i M_i$. If Γ is rectifiable, we can define normal vector fields ν to Γ in the following way: if $\Gamma = \bigcup_i \Gamma_i$ with $\Gamma_i \,\tilde{\subseteq}\, M_i$ and $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, given $x \in \Gamma_i$, we take $\nu(x) = \nu_{M_i}(x)$, where $\nu_{M_i}(x)$ is a normal vector to the C^1-manifold M_i at x. It turns out that two normal vector fields associated to different decompositions $\bigcup_i \Gamma_i$ of Γ coincide up to their sign $\mathcal{H}^1$ almost everywhere.

Given a crack Γ, an admissible deformation is given by any function $u \in GSBV(\Omega; \mathbb{R}^2)$ such that $S(u) \,\tilde{\subseteq}\, \Gamma$.

The surface energy. The surface energy of a crack Γ is given by

$$\mathcal{E}^s(\Gamma) := \int_\Gamma k(x, \nu(x))\, d\mathcal{H}^1(x), \tag{8.3}$$

where ν is a unit normal vector field on Γ. Here $k : \overline{\Omega}_B \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $k(x,\cdot)$ is a norm in $\mathbb{R}^2$ for all $x \in \overline{\Omega}_B$ and for all $x \in \overline{\Omega}_B$ and $\nu \in \mathbb{R}^2$

$$K_1|\nu| \leq k(x,\nu) \leq K_2|\nu|, \tag{8.4}$$

where $K_1, K_2 > 0$. Notice that since k is even in the second variable, we have that the integral (8.3) is independent of the orientation given to Γ, that is independent of the particular choice of the unit normal vector field ν.

The bulk energy. Let $p > 1$ be fixed. Given a deformation $u \in GSBV^p(\Omega; \mathbb{R}^2)$ the associated *bulk energy* is given by

$$\mathcal{W}(\nabla u) := \int_\Omega W(x, \nabla u(x))\, dx, \tag{8.5}$$

where $W : \Omega \times M^{2\times 2} \to [0, +\infty)$ is a Carathéodory function satisfying

$$\text{for every } x \in \Omega : W(x,\cdot) \text{ is quasiconvex and } C^1 \text{ on } M^{2\times 2}, \tag{8.6}$$

$$\text{for every } (x,\xi) \in \Omega \times M^{2\times 2} : a_0^W|\xi|^p - b_0^W(x) \leq W(x,\xi) \leq a_1^W|\xi|^p + b_1^W(x). \tag{8.7}$$

Here $a_0^W, a_1^W > 0$, and $b_0^W, b_1^W \in L^1(\Omega)$ are nonnegative functions. Quasiconvexity of W means that for all $\xi \in M^{2\times 2}$ and for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^2)$

$$W(\xi) \leq \int_\Omega W(\xi + \nabla\varphi)\, dx.$$

If we denote by $\partial_\xi W : \Omega \times M^{2\times 2} \to M^{2\times 2}$ the partial derivative of W with respect to ξ, since $\xi \to W(x,\xi)$ is rank one convex on $M^{2\times 2}$, under the growth assumption (8.7) it turns out that (see for example [41]) there exists a positive constant $a_2^W > 0$ and a nonnegative function $b_2^W \in L^{p'}(\Omega)$, with $p' := p/(p-1)$, such that for all $(x,\xi) \in \Omega \times M^{2\times 2}$

$$|\partial_\xi W(x,\xi)| \leq a_2^W |\xi|^{p-1} + b_2^W(x). \tag{8.8}$$

By (8.7) and (8.8) the functional $\mathcal{W}$, defined for all $\Phi \in L^p(\Omega; M^{2\times 2})$ by

$$\mathcal{W}(\Phi) := \int_\Omega W(x, \Phi(x))\, dx,$$

is of class C^1 on $L^p(\Omega; M^{2\times 2})$, and its differential $\partial\mathcal{W} : L^p(\Omega; M^{2\times 2}) \to L^{p'}(\Omega; M^{2\times 2})$ is given by

$$\langle \partial\mathcal{W}(\Phi), \Psi\rangle = \int_\Omega \partial_\xi W(x, \Phi(x))\Psi(x)\, dx, \qquad \Phi, \Psi \in L^p(\Omega; M^{2\times 2}),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the spaces $L^{p'}(\Omega; M^{2\times 2})$ and $L^p(\Omega; M^{2\times 2})$. By (8.7) and (8.8), there exist six positive constants $\alpha_0^{\mathcal{W}} > 0$, $\alpha_1^{\mathcal{W}} > 0$, $\alpha_2^{\mathcal{W}} > 0$, $\beta_0^{\mathcal{W}} \geq 0$, $\beta_1^{\mathcal{W}} \geq 0$, $\beta_2^{\mathcal{W}} \geq 0$ such that for every Φ, $\Psi \in L^p(\Omega; M^{2\times 2})$

$$\alpha_0^{\mathcal{W}} \|\Phi\|_p^p - \beta_0^{\mathcal{W}} \leq \mathcal{W}(\Phi) \leq \alpha_1^{\mathcal{W}} \|\Phi\|_p^p + \beta_1^{\mathcal{W}},$$

$$|\langle \partial\mathcal{W}(\Phi), \Psi\rangle| \leq (\alpha_2^{\mathcal{W}} \|\Phi\|_p^{p-1} + \beta_2^{\mathcal{W}})\|\Psi\|_p. \tag{8.9}$$

The body forces. Let $q > 1$ be fixed. The density of applied body forces per unit volume in the reference configuration relative to the deformation u at time $t \in [0,T]$ is given by $\partial_z F(t, x, u(x))$. Here $F : [0,T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is such that:

$$\text{for every } z \in \mathbb{R}^2 : (t,x) \to F(t,x,z) \text{ is } \mathcal{L}^1 \times \mathcal{L}^2 \text{ measurable on } [0,T] \times \Omega,$$
$$\text{for every } (t,x) \in [0,T] \times \Omega : z \to F(t,x,z) \text{ belongs to } C^1(\mathbb{R}^2),$$

and satisfies the following growth conditions

$$\begin{aligned} &a_0^F |z|^q - b_0^F(t,x) \leq -F(t,x,z) \leq a_1^F |z|^q + b_1^F(t,x), \\ &|\partial_z F(t,x,z)| \leq a_2^F |z|^{q-1} + b_2^F(t,x) \end{aligned} \tag{8.10}$$

for every $(t,x,z) \in [0,T] \times \Omega \times \mathbb{R}^2$, with $a_0^F > 0$, $a_1^F > 0$ and $a_2^F > 0$, and where $b_0^F, b_1^F \in C^0([0,T]; L^1(\Omega))$, $b_2^F \in C^0([0,T]; L^{q'}(\Omega))$ are nonnegative functions, with $q' := q/(q-1)$.

In order to deal with time variations, we assume also that for every $(t,z) \in [0,T] \times \mathbb{R}^2$

$$F(t,x,z) = F(0,x,z) + \int_0^t \dot{F}(s,x,z)\, ds \qquad \text{for a.e. } x \in \Omega,$$

$$\partial_z F(t,x,z) = \partial_z F(0,x,z) + \int_0^t \partial_z \dot{F}(s,x,z)\, ds \qquad \text{for a.e. } x \in \Omega,$$

where $\dot{F} : [0,T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is such that

$$\text{for all } z \in \mathbb{R}^2 : (t,x) \to \dot{F}(t,x,z) \text{ is } \mathcal{L}^1 \times \mathcal{L}^2 \text{ measurable on } [0,T] \times \Omega,$$
$$\text{for all } (t,x) \in [0,T] \times \Omega : z \to \dot{F}(t,x,z) \text{ is of class } C^1 \text{ on } \mathbb{R}^2,$$

and satisfies the growth conditions

$$\begin{aligned} &|\dot{F}(t,x,z)| \leq a_3^F(t)|z|^{\dot{q}} + b_3^F(t,x), \\ &|\partial_z \dot{F}(t,x,z)| \leq a_4^F(t)|z|^{\dot{q}-1} + b_4^F(t,x) \end{aligned}$$

for all $(t,x,z) \in [0,T] \times \Omega \times \mathbb{R}^2$. Here $1 \le \dot{q} < q$, and $a_3^F, a_4^F \in L^1([0,T])$, $b_3^F \in L^1([0,T];L^1(\Omega))$, $b_4^F \in L^1([0,T];L^{\dot{q}'}(\Omega))$ are nonnegative functions with $\dot{q}' := \frac{\dot{q}}{\dot{q}-1}$.

Under the previous assumptions, for every $t \in [0,T]$ the functionals

$$\mathcal{F}(t)(u) := \int_\Omega F(t,x,u(x))\,dx, \qquad \dot{\mathcal{F}}(t)(u) := \int_\Omega \dot{F}(t,x,u(x))\,dx \tag{8.11}$$

are well defined on $L^q(\Omega;\mathbb{R}^2)$ and $L^{\dot{q}}(\Omega;\mathbb{R}^2)$ respectively. Moreover we have that $\mathcal{F}(t)$ is of class C^1 on $L^q(\Omega;\mathbb{R}^2)$, with differential $\partial\mathcal{F}(t) : L^q(\Omega;\mathbb{R}^2) \to L^{q'}(\Omega;\mathbb{R}^2)$ defined by

$$\langle \partial\mathcal{F}(t)(u), v\rangle = \int_\Omega \partial_z F(t,x,u(x))v(x)\,dx, \qquad u,v \in L^q(\Omega;\mathbb{R}^2),$$

where $\langle\cdot,\cdot\rangle$ denotes now the duality pairing between $L^{q'}(\Omega;\mathbb{R}^2)$ and $L^q(\Omega;\mathbb{R}^2)$. $\dot{\mathcal{F}}(t)$ is C^1 on $L^{\dot{q}}(\Omega;\mathbb{R}^2)$ with differential defined by

$$\langle \partial\dot{\mathcal{F}}(t)(u), v\rangle = \int_\Omega \partial_z \dot{F}(t,x,u(x))v(x)\,dx, \qquad u,v \in L^{\dot{q}}(\Omega;\mathbb{R}^2),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^{\dot{q}'}(\Omega;\mathbb{R}^2)$ and $L^{\dot{q}}(\Omega;\mathbb{R}^2)$. For all $u,v \in L^q(\Omega;\mathbb{R}^2)$ and for all $t \in [0,T]$ we have

$$\mathcal{F}(t)(u) = \mathcal{F}(0)(u) + \int_0^t \dot{\mathcal{F}}(s)(u)\,ds,$$

$$\langle \partial\mathcal{F}(t)(u), v\rangle = \langle \partial\mathcal{F}(0)(u), v\rangle + \int_0^t \langle \partial\dot{\mathcal{F}}(s)(u), v\rangle\,ds. \tag{8.12}$$

Moreover we have that for every $t \in [0,T]$ and for every u, $v \in L^q(\Omega;\mathbb{R}^n)$

$$\alpha_0^{\mathcal{F}}\|u\|_q^q - \beta_0^{\mathcal{F}} \le -\mathcal{F}(t)(u) \le \alpha_1^{\mathcal{F}}\|u\|_q^q + \beta_1^{\mathcal{F}},$$

$$|\langle \partial\mathcal{F}(t)(u), v\rangle| \le (\alpha_2^{\mathcal{F}}\|u\|_q^{q-1} + \beta_2^{\mathcal{F}})\|v\|_q, \tag{8.13}$$

$$|\dot{\mathcal{F}}(t)(u)| \le \alpha_3^{\mathcal{F}}(t)\|u\|_{\dot{q}}^{\dot{q}} + \beta_3^{\mathcal{F}}(t), \tag{8.14}$$

$$|\langle \partial\dot{\mathcal{F}}(t)(u), v\rangle| \le (\alpha_4^{\mathcal{F}}(t)\|u\|_{\dot{q}}^{\dot{q}-1} + \beta_4^{\mathcal{F}}(t))\|v\|_{\dot{q}}, \tag{8.15}$$

where $\alpha_0^{\mathcal{F}} > 0$, $\alpha_1^{\mathcal{F}} > 0$, $\alpha_2^{\mathcal{F}} > 0$, $\beta_0^{\mathcal{F}} \ge 0$, $\beta_1^{\mathcal{F}} \ge 0$, $\beta_2^{\mathcal{F}} \ge 0$ are positive constants, and $\alpha_3^{\mathcal{F}}, \alpha_4^{\mathcal{F}}, \beta_3^{\mathcal{F}}, \beta_4^{\mathcal{F}} \in L^1([0,T])$ are nonnegative functions.

The surface forces. The density of the surface forces on $\partial_S\Omega$ at time t under the deformation u is given by $\partial_z G(t,x,u(x))$, where $G : [0,T] \times \partial_S\Omega \times \mathbb{R}^2 \to \mathbb{R}$ is such that

$$\text{for every } z \in \mathbb{R}^2 : (t,x) \to G(t,x,z) \text{ is } \mathcal{L}^1 \times \mathcal{H}^1\text{-measurable},$$
$$\text{for every } (t,x) \in [0,T] \times \partial_S\Omega : z \to G(t,x,z) \text{ belongs to } C^1(\mathbb{R}^2),$$

and satisfies the growth conditions

$$-a_0^G(t,x)|z| - b_0^G(t,x) \le -G(t,x,z) \le a_1^G|z|^r + b_1^G(t,x),$$
$$|\partial_z G(t,x,z)| \le a_2^G|z|^{r-1} + b_2^G(t,x),$$

for every $(t,x,z) \in [0,T] \times \partial_S\Omega \times \mathbb{R}^2$. Here r is an exponent related to the trace operators on Sobolev spaces: if $p < 2$, then we suppose that $p \le r \le \frac{p}{2-p}$, while if $p \ge 2$, we suppose $p \le r$. Moreover $a_1^G \ge 0$, $a_2^G \ge 0$ are two nonnegative constants, and $a_0^G \in L^\infty([0,T];L^{r'}(\partial_S\Omega))$, $b_0^G, b_1^G \in C^0([0,T];L^1(\partial_S\Omega))$, and $b_2^G \in C^0([0,T];L^{r'}(\partial_S\Omega))$ are nonnegative functions with $r' := r/(r-1)$

We assume that for every $(t, z) \in [0,T] \times \mathbb{R}^2$

$$G(t,x,z) = G(0,x,z) + \int_0^t \dot{G}(s,x,z)ds \qquad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial_S\Omega,$$

$$\partial_z G(t,x,z) = \partial_z G(0,x,z) + \int_0^t \partial_z \dot{G}(s,x,z)ds \qquad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial_S\Omega,$$

where $\dot{G} : [0,T] \times \partial_S\Omega \times \mathbb{R}^2 \to \mathbb{R}$ is such that

$$\text{for all } z \in \mathbb{R}^2 : (t,x) \to \dot{G}(t,x,z) \text{ is } \mathcal{L}^1 \times \mathcal{H}^1\text{-measurable,}$$
$$\text{for all } (t,x) \in [0,T] \times \partial_S\Omega : z \to \dot{G}(t,x,z) \text{ belongs to } C^1(\mathbb{R}^2),$$

and satisfies the the growth conditions

$$|\dot{G}(t,x,z)| \leq a_3^G(t)|z|^r + b_3^G(t,x),$$
$$|\partial_z \dot{G}(t,x,z)| \leq a_4^G(t)|z|^{r-1} + b_4^G(t,x)$$

for all $(t,x,z) \in [0,T] \times \partial_S\Omega \times \mathbb{R}^2$. Here a_3^G, $a_4^G \in L^1([0,T])$, $b_3^G \in L^1([0,T]; L^1(\partial_S\Omega))$ and $b_4^G \in L^1([0,T]; L^{r'}(\partial_S\Omega))$ are nonnegative functions.

By the previous assumptions, the following functionals on $L^r(\partial_S\Omega; \mathbb{R}^2)$

$$\mathcal{G}(t)(u) := \int_{\partial_S\Omega} G(t,x,u(x))\, d\mathcal{H}^1(x), \quad \dot{\mathcal{G}}(t)(u) := \int_{\partial_S\Omega} \dot{G}(t,x,u(x))\, d\mathcal{H}^1(x) \tag{8.16}$$

are well defined. For every $t \in [0,T]$ we have that $\mathcal{G}(t)$ is of class C^1 on $L^r(\partial_S\Omega; \mathbb{R}^2)$ and its differential is given by

$$\langle \partial\mathcal{G}(t)(u), v\rangle = \int_{\partial_S\Omega} \partial_z G(t,x,u(x))v(x)\, d\mathcal{H}^1(x), \qquad u, v \in L^r(\partial_S\Omega; \mathbb{R}^2),$$

where $\langle\cdot,\cdot\rangle$ denotes now the duality pairing between $L^{r'}(\partial_S\Omega; \mathbb{R}^2)$ and $L^r(\partial_S\Omega; \mathbb{R}^2)$. Moreover, $\dot{\mathcal{G}}(t)$ is of class C^1 on $L^r(\partial_S\Omega; \mathbb{R}^2)$, and its differential is given by

$$\langle \partial\dot{\mathcal{G}}(t)(u), v\rangle = \int_{\partial_S\Omega} \partial_z \dot{G}(t,x,u(x))v(x)\, d\mathcal{H}^1(x)$$

for all $u, v \in L^r(\partial_S\Omega; \mathbb{R}^2)$. Finally we have

$$\mathcal{G}(t)(u) = \mathcal{G}(0)(u) + \int_0^t \dot{\mathcal{G}}(s)(u)\, ds,$$

$$\langle \partial\mathcal{G}(t)(u), v\rangle = \langle \partial\mathcal{G}(0)(u), v\rangle + \int_0^t \langle \partial\dot{\mathcal{G}}(s)(u), v\rangle\, ds,$$

for every $u, v \in L^r(\partial_S\Omega; \mathbb{R}^2)$.

Let $\Omega_S \subseteq \Omega \setminus \overline{\Omega}_B$ be open with Lipschitz boundary, and such that $\partial_S\Omega \subseteq \partial\Omega_S$; the trace operator from $W^{1,p}(\Omega_S; \mathbb{R}^2)$ into $L^r(\partial\Omega_S; \mathbb{R}^2)$ is then compact, and so there exists a constant $\gamma_S > 0$ such that

$$\|u\|_{r,\partial_S\Omega} \leq \gamma_S(\|\nabla u\|_{p,\Omega_S} + \|u\|_{p,\Omega_S}) \tag{8.17}$$

for every $u \in W^{1,p}(\Omega_S; \mathbb{R}^2)$. By the previous assumptions, we have that there exist six nonnegative constants α_0^G, $\alpha_1^{\mathcal{G}}$, $\alpha_2^{\mathcal{G}}$, β_0^G, β_1^G, $\beta_2^{\mathcal{G}}$ and four nonnegative functions $\alpha_3^{\mathcal{G}}$, $\alpha_4^{\mathcal{G}}$, $\beta_3^{\mathcal{G}}$, $\beta_4^{\mathcal{G}} \in L^1([0,T])$, such that

$$-\alpha_0^G\|u\|_{r,\partial_S\Omega} - \beta_0^G \leq -\mathcal{G}(t)(u) \leq \alpha_1^{\mathcal{G}}\|u\|_{r,\partial_S\Omega}^r + \beta_1^G,$$

$$|\langle \partial\mathcal{G}(t)(u), v\rangle| \leq (\alpha_2^{\mathcal{G}}\|u\|_{r,\partial_S\Omega}^{r-1} + \beta_2^{\mathcal{G}})\|v\|_{r,\partial_S\Omega}, \tag{8.18}$$

$$|\dot{\mathcal{G}}(t)(u)| \leq \alpha_3^{\mathcal{G}}(t)\|u\|_{r,\partial_S\Omega}^r + \beta_3^{\mathcal{G}}(t), \tag{8.19}$$

$$|\langle \partial\dot{\mathcal{G}}(t)(u), v\rangle| \leq (\alpha_4^{\mathcal{G}}(t)\|u\|_{r,\partial_S\Omega}^{r-1} + \beta_4^{\mathcal{G}}(t))\|v\|_{r,\partial_S\Omega}$$

for every $t \in [0,T]$ and $u, v \in L^r(\partial_S\Omega; \mathbb{R}^2)$.

Configurations with finite energy. The deformations on the boundary $\partial_D\Omega$ are given by (the traces of) functions $g \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$, where p, q are the exponents in (8.7) and (8.10) respectively. Given a crack $\Gamma \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$ and a boundary deformation g, the set of *admissible deformations with finite energy* relative to (g, Γ) is defined by

$$AD(g,\Gamma) := \{u \in GSBV^p_q(\Omega;\mathbb{R}^2) : S(u) \tilde{\subseteq} \Gamma, u = g \text{ on } \partial_D\Omega \setminus \Gamma\},$$

where we recall that

$$GSBV^p_q(\Omega;\mathbb{R}^2) := GSBV^p(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2),$$

and the equality $u = g$ on $\partial_D\Omega \setminus \Gamma$ is intended in the sense of traces (see [44, Section 2]).

Note that if $u \in GSBV^p_q(\Omega;\mathbb{R}^2)$, then $\mathcal{W}(u) < +\infty$ and $|\mathcal{F}(t)(u)| < +\infty$ for all $t \in [0,T]$. Moreover since $\Gamma \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$ and $S(u) \tilde{\subseteq} \Gamma$, we have that $u \in W^{1,p}(\Omega_S;\mathbb{R}^2) \cap L^q(\Omega_S;\mathbb{R}^2)$ so that $\mathcal{G}(t)(u)$ is well defined and $|\mathcal{G}(t)(u)| < +\infty$ for all $t \in [0,T]$. Notice that there exists always a deformation without crack which satisfies the boundary condition, namely the function g itself.

The total energy. For every $t \in [0,T]$, the total energy relative to the configuration (u, Γ) with $u \in AD(g,\Gamma)$ is given by

$$\mathcal{E}(t)(u,\Gamma) := \mathcal{E}^b(t)(u) + \mathcal{E}^s(\Gamma), \tag{8.20}$$

where

$$\mathcal{E}^b(t)(u) := \mathcal{W}(u) - \mathcal{F}(t)(u) - \mathcal{G}(t)(u), \tag{8.21}$$

and $\mathcal{W}$, $\mathcal{F}(t)$, $\mathcal{G}(t)$ and $\mathcal{E}^s$ are defined in (8.5), (8.11), (8.16) and (8.3) respectively. It turns out that there exist four constants $\alpha_0^{\mathcal{E}} > 0$, $\alpha_1^{\mathcal{E}} > 0$, $\beta_0^{\mathcal{E}} \geq 0$, $\beta_1^{\mathcal{E}} \geq 0$ such that

$$\mathcal{E}^b(t)(u) \geq \alpha_0^{\mathcal{E}}(\|\nabla u\|_p^p + \|u\|_q^q) - \beta_0^{\mathcal{E}}, \tag{8.22}$$

$$\mathcal{E}^b(t)(u) \leq \alpha_1^{\mathcal{E}}(\|\nabla u\|_p^p + \|u\|_q^q + \|u\|^r_{r,\partial_S\Omega}) + \beta_1^{\mathcal{E}},$$

for every $t \in [0,T]$ and $u \in GSBV^p_q(\Omega;\mathbb{R}^2)$.

The time dependent boundary deformations. We will consider boundary deformations $g(t)$ such that

$$t \to g(t) \in AC([0,T]; W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)),$$

so that

$$t \to \dot{g}(t) \in L^1([0,T]; W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)),$$

and

$$t \to \nabla\dot{g}(t) \in L^1([0,T]; L^p(\Omega; M^{2\times 2})).$$

The existence result. Let $\Gamma_0 \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$ be a preexisting crack. The next Theorem proved in [44] establishes the existence of a quasistatic evolution with preexisting crack Γ_0.

Theorem 8.1.1. *Let $\Gamma_0 \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$ be a preexisting crack. Then there exists a quasistatic evolution with preexisting crack Γ_0 and boundary deformation $g(t)$, i.e., there exists a function $t \to (u(t), \Gamma(t))$ from $[0,T]$ to $GSBV^p_q(\Omega;\mathbb{R}^2) \times \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$ with the following properties:*

(a) $(u(0), \Gamma(0))$ *is such that*

$$\mathcal{E}(0)(u(0),\Gamma(0)) = \min\{\mathcal{E}(0)(v,\Gamma) : v \in AD(g(0),\Gamma), \Gamma_0 \tilde{\subseteq} \Gamma\};$$

(b) $u(t) \in AD(g(t), \Gamma(t))$ *for all* $t \in [0,T]$;

(c) *irreversibility:* $\Gamma_0 \tilde{\subseteq} \Gamma(s) \tilde{\subset} \Gamma(t)$ *whenever* $0 \leq s < t \leq T$;

(d) *static equilibrium: for all* $t \in [0,T]$

$$\mathcal{E}(t)(u(t), \Gamma(t)) = \min\{\mathcal{E}(t)(v, \Gamma) : v \in AD(g(t), \Gamma), \Gamma(t) \tilde{\subseteq} \Gamma\};$$

(e) *nondissipativity: the function* $t \to E(t) := \mathcal{E}(t)(u(t), \Gamma(t))$ *is absolutely continuous on* $[0,T]$, *and for a.e.* $t \in [0,T]$

$$\dot{E}(t) = \langle \partial \mathcal{W}(\nabla u(t)), \nabla \dot{g}(t) \rangle - \langle \partial \mathcal{F}(t)(u(t)), \dot{g}(t) \rangle - \dot{\mathcal{F}}(t)(u(t)) \\ - \langle \partial \mathcal{G}(t)(u(t)), \dot{g}(t) \rangle - \dot{\mathcal{G}}(t)(u(t)). \tag{8.23}$$

The next theorem gives a compactness and lower semicontinuity result with respect to weak convergence in $GSBV_q^p(\Omega, \mathbb{R}^2)$ which will be often used in the next sections.

Theorem 8.1.2. *Let* $t_k \in [0,T]$ *with* $t_k \to t$, *and let* $(u_k) \subset GSBV_q^p(\Omega; \mathbb{R}^2)$, $C \in]0; +\infty[$ *such that* $S(u_k) \tilde{\subseteq} \overline{\Omega}_B$ *and*

$$\mathcal{E}^b(t_k)(u_k) + \mathcal{E}^s(S(u_k)) \leq C,$$

where $\mathcal{E}^b$ *and* $\mathcal{E}^s$ *are defined as in* (8.21) *and* (8.3). *Then there exists a subsequence* $(u_{k_h})_{h \in \mathbb{N}}$ *converging to some* u *weakly in* $GSBV_q^p(\Omega; \mathbb{R}^2)$ *such that* $S(u) \tilde{\subseteq} \overline{\Omega}_B$,

$$\mathcal{E}^b(t)(u) \leq \liminf_{h \to \infty} \mathcal{E}^b(t_{k_h})(u_{k_h}) \qquad \text{and} \qquad \mathcal{E}^s(S(u)) \leq \liminf_{h \to \infty} \mathcal{E}^s(S(u_{k_h})).$$

Proof. By (8.22) and (8.4), we have that there exists $C' \in]0, +\infty[$ such that

$$\|\nabla u_k\|_p^p + \|u_k\|_q^q + \mathcal{H}^1(S(u_k)) \leq C'.$$

Then we can apply Theorem 1.1.3 with $g(x, u_k) = |u_k|^q$, obtaining a subsequence $(u_{k_h})_{h \in \mathbb{N}}$ and $u \in GSBV^p(\Omega; \mathbb{R}^2)$ such that (1.4) holds: in particular we may assume that $u_{k_h} \to u$ pointwise a.e.. We have $u_{k_h} \to u$ strongly in $L^1(\Omega; \mathbb{R}^2)$, and by Fatou's Lemma we have that $u \in L^q(\Omega; \mathbb{R}^2)$ so that $u \in GSBV_q^p(\Omega; \mathbb{R}^2)$. We conclude $u_{k_h} \rightharpoonup u$ weakly in $GSBV_q^p(\Omega; \mathbb{R}^2)$. By [4, Theorem 3.7] we have that

$$\mathcal{E}^s(S(u)) \leq \liminf_h \mathcal{E}^s(S(u_{k_h})),$$

and by [63] we have that

$$\int_\Omega W(x, \nabla u)\, dx \leq \liminf_h \int_\Omega W(x, \nabla u_{k_h})\, dx.$$

Since by assumption the functions $z \to F(0, x, z)$ and $z \to \dot{F}(s, x, z)$ are continuous for all $s \in [0,T]$ and for a.e. $x \in \Omega$, and

$$F(t_{k_h}, x, u_{k_h}(x)) = F(0, x, u_{k_h}(x)) + \int_0^{t_{k_h}} \dot{F}(s, x, u_{k_h}(x))\, ds,$$

we have that $F(t_{k_h}, x, u_{k_h}(x)) \to F(t, x, u(x))$ for a.e. $x \in \Omega$. By Fatou's Lemma (in the limsup version) we deduce

$$\limsup_h \int_\Omega F(t_{k_h}, x, u_{k_h}(x))\, dx \leq \int_\Omega F(t, x, u(x))\, dx.$$

Since $(u_{k_h})_{|\Omega_S}$ is bounded in $W^{1,p}(\Omega_S;\mathbb{R}^2)\cap L^q(\Omega_S;\mathbb{R}^2)$, and the trace operator from $W^{1,p}(\Omega_S;\mathbb{R}^2)$ into $L^r(\Omega_S;\mathbb{R}^2)$ is compact, we get

$$\lim_h \mathcal{G}(t_{k_h})(u_{k_h}) = \mathcal{G}(t)(u),$$

and so the proof is thus concluded. $\square$

8.2 The finite element space and the Transfer of Jump Sets

Let $\Omega \subseteq \mathbb{R}^2$ be a polygonal set. We follow the notation employed in Section 7.1, where the family of regular triangulations $\mathcal{R}_\varepsilon(\Omega)$ of Ω depending on the parameter $\varepsilon > 0$, and the family of adaptive triangulations $\mathcal{T}_{\varepsilon,a}(\Omega)$ depending on the parameters $\varepsilon > 0$ and $a \in (0, \frac{1}{2})$ are introduced.

From now on for all $\varepsilon > 0$ we fix $\mathbf{R}_\varepsilon \in \mathcal{R}_\varepsilon(\Omega)$. We suppose that the brittle part Ω_B and the region Ω_S introduced before for the model of quasistatic growth of cracks are composed of triangles of $\mathbf{R}_\varepsilon$ for all ε. Moreover we suppose that $\partial_D\Omega$ and $\partial_S\Omega$ are composed of edges of $\mathbf{R}_\varepsilon$ for all ε up to a finite number of points.

We consider the following discontinuous finite element space. We indicate by $\mathcal{A}_{\varepsilon,a}(\Omega)$ the set of all $u : \Omega \to \mathbb{R}^2$ such that there exists a triangulation $\mathbf{T}(u) \in \mathcal{T}_{\varepsilon,a}(\Omega)$ nested in $\mathbf{R}_\varepsilon$ with u affine on every triangle $T \in \mathbf{T}(u)$. For every $u \in \mathcal{A}_{\varepsilon,a}(\Omega)$, we indicate by $S(u)$ the family of edges of $\mathbf{T}(u)$ inside Ω across which u is discontinuous. Notice that $u \in SBV(\Omega;\mathbb{R}^2)$ and that the notation is consistent with the usual one employed in the theory of functions of bounded variation. Let us set

$$\mathcal{AF}_\varepsilon(\Omega) := \{u : \Omega \to \mathbb{R}^2 \,:\, u \text{ is continuous and affine on each triangle } T \in \mathbf{R}_\varepsilon\}. \tag{8.24}$$

The discretization of the problem will be carried out using the space

$$\mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2) := \{u \in \mathcal{A}_{\varepsilon,a}(\Omega) \,:\, S(u) \subseteq \overline{\Omega}_B\}. \tag{8.25}$$

Given any $g \in \mathcal{AF}_\varepsilon(\Omega)$, for every $u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ let

$$S^g_D(u) := \{x \in \partial_D\Omega \,:\, u(x) \neq g(x)\}, \tag{8.26}$$

that is $S^g_D(u)$ denotes the set of edges of $\partial_D\Omega$ at which the boundary condition is not satisfied. For every $u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$, let us also set

$$S^g(u) := S(u) \cup S^g_D(u). \tag{8.27}$$

An essential tool in the approximation result of this chapter is Proposition 8.2.2 which generalizes the piecewise affine transfer of jump [59, Proposition 5.1] to the case of vector valued functions with bulk energy $\mathcal{E}^b$ and surface energy $\mathcal{E}^s$ of the form (8.21) and (8.3) respectively.

In order to deal with the surface energy $\mathcal{E}^s$ we will need the following geometric construction. Let $S \subseteq \Omega$ be a segment and let us suppose that S intersects the edges of $\mathbf{R}_\varepsilon$ at most in one point for all $\varepsilon > 0$. Let $a \in]0, \frac{1}{2}[$, and let $P = S \cap \zeta$, where $\zeta = [x, y]$ is an edge of $\mathbf{R}_\varepsilon$: we indicate with $\pi_a(P)$ the projection of P on the segment $\{tx + (1-t)y \,:\, t \in [a, 1-a]\}$. The *interpolating curve* $S_{\varepsilon,a}$ of S in $\mathbf{R}_\varepsilon$ with parameter a is given connecting all the $\pi_a(P)$'s belonging to the same triangle of $\mathbf{R}_\varepsilon$.

Lemma 8.2.1. *Under the previous assumptions, there exists a function $\eta(a)$ independent of S with $\eta(a) \to 0$ as $a \to 0$ such that*

$$\limsup_{\varepsilon \to 0} |\mathcal{E}^s(S_{\varepsilon,a}) - \mathcal{E}^s(S)| \leq \eta(a)\mathcal{E}^s(S),$$

where $\mathcal{E}^s$ is defined in (8.3).

Proof. By (8.4), we have that there exist ω and $K_3 > 0$ such that for all $x_1, x_2 \in \overline{\Omega}$ and $|\nu_1| = |\nu_2| = 1$

$$|k(x_1,\nu_1) - k(x_2,\nu_2)| \leq \omega(|x_1 - x_2|) + K_3|\nu_1 - \nu_2|,$$

where $\omega :]0,+\infty[\to]0,+\infty[$ is a decreasing function such that $\omega(s) \to 0$ as $s \to 0$. Let $T \in \mathbf{R}_\varepsilon$ be such that $T \cap S \neq \emptyset$, and let us choose $x_T \in T \cap S$ and $x_T^{\varepsilon,a} \in T \cap S_{\varepsilon,a}$. Let $c_2 > 0$ denote the characteristic constant of $\mathbf{R}_\varepsilon$ such that every $T \in \mathbf{R}_\varepsilon$ is contained in a ball of diameter $c_2\varepsilon$. Then we have

$$\begin{aligned}
&\left|\int_{S_{\varepsilon,a}\cap T} k(x,\nu_T^{\varepsilon,a})\,d\mathcal{H}^1 - \int_{S\cap T} k(x,\nu_T)\,d\mathcal{H}^1\right| \\
&\leq \left|\int_{S_{\varepsilon,a}\cap T} k(x_T^{\varepsilon,a},\nu_T^{\varepsilon,a})\,d\mathcal{H}^1 - \int_{S\cap T} k(x_T,\nu_T)\,d\mathcal{H}^1\right| + \omega(c_2\varepsilon)\mathcal{H}^1(S_{\varepsilon,a}\cap T) + \omega(c_2\varepsilon)\mathcal{H}^1(S\cap T) \\
&\leq \left|k(x_T^{\varepsilon,a},\nu_T^{\varepsilon,a})\mathcal{H}^1(S_{\varepsilon,a}\cap T) - k(x_T,\nu_T)\mathcal{H}^1(S\cap T)\right| + \omega(c_2\varepsilon)\left[\mathcal{H}^1(S_{\varepsilon,a}\cap T) + \mathcal{H}^1(S\cap T)\right],
\end{aligned}$$

where $\nu_T^{\varepsilon,a}, \nu_T$ are the (constant) normal to $S_{\varepsilon,a} \cap T$ and $S \cap T$ respectively. We have

$$\begin{aligned}
&\left|k(x_T^{\varepsilon,a},\nu_T^{\varepsilon,a})\mathcal{H}^1(S_{\varepsilon,a}\cap T) - k(x_T,\nu_T)\mathcal{H}^1(S\cap T)\right| \\
&\quad\leq k(x_T^{\varepsilon,a},\nu_T^{\varepsilon,a})\left|\mathcal{H}^1(S_{\varepsilon,a}\cap T) - \mathcal{H}^1(S\cap T)\right| + \left|k(x_T^{\varepsilon,a},\nu_T^{\varepsilon,a}) - k(x_T,\nu_T)\right|\mathcal{H}^1(S\cap T) \\
&\quad\leq K_2\left|\mathcal{H}^1(S_{\varepsilon,a}\cap T) - \mathcal{H}^1(S\cap T)\right| + \omega(|x_T^{\varepsilon,a} - x_T|)\mathcal{H}^1(S\cap T) + K_3|\nu_T^{\varepsilon,a} - \nu_T|\mathcal{H}^1(S\cap T),
\end{aligned}$$

where K_2 is defined in (8.4). We are now ready to conclude: in fact, following [75, Lemma 5.2.2], we can choose the orientation of $\nu_T^{\varepsilon,a}$ in such a way that

$$|\nu_T^{\varepsilon,a} - \nu_T|\mathcal{H}^1(S\cap T) \leq D_2 a\varepsilon, \qquad \left|\mathcal{H}^1(S_{\varepsilon,a}\cap T) - \mathcal{H}^1(S\cap T)\right| \leq D_1 a\varepsilon,$$

with $D_1, D_2 > 0$ independent of T, ε, a. Then, summing up the preceding inequalities, recalling that the number of triangles of $\mathbf{R}_\varepsilon$ intersecting S is less than $\tilde{c}\varepsilon^{-1}\mathcal{H}^1(S)$ for ε small enough, with $\tilde{c}$ independent of S and ε (see for example [59, Lemma 2.5]), we obtain

$$\limsup_{\varepsilon\to 0} |\mathcal{E}^s(S_{\varepsilon,a}) - \mathcal{E}^s(S)| \leq \rho(a)\mathcal{H}^1(S),$$

where $\rho(a) := \tilde{c}(K_2D_1 + K_3D_2)a$. In view of (8.4), we conclude that

$$\limsup_{\varepsilon\to 0} |\mathcal{E}^s(S_{\varepsilon,a}) - \mathcal{E}^s(S)| \leq K_1^{-1}\rho(a)\mathcal{E}^s(S),$$

and so the proof is concluded choosing $\eta(a) := K_1^{-1}\rho(a)$. $\square$

For all $u \in GSBV_q^p(\Omega;\mathbb{R}^2)$ and for all $g \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$, let us set

$$S^g(u) := S(u) \cup \{x \in \partial_D\Omega \,:\, u(x) \neq g(x)\}, \tag{8.28}$$

where the inequality is intended in the sense of traces. We are now in a position to state the piecewise affine transfer of jump proposition in our setting.

Proposition 8.2.2. *Let $a \in]0, \frac{1}{2}[$, and for all $i = 1, \ldots, m$ let*

$$u_\varepsilon^i \in \mathcal{A}_{\varepsilon,a}^B(\Omega;\mathbb{R}^2), \qquad u^i \in GSBV_q^p(\Omega;\mathbb{R}^2)$$

be such that

$$u_\varepsilon^i \rightharpoonup u^i \qquad \textit{weakly in } GSBV_q^p(\Omega;\mathbb{R}^2).$$

Let moreover $g_\varepsilon^i, h_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega)$, $g^i, h \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$ be such that

$$g_\varepsilon^i \to g^i, \quad h_\varepsilon \to h \qquad \textit{strongly in } W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2).$$

Then for every $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ *with* $S(v) \tilde{\subseteq} \overline{\Omega}_B$, *there exists* $v_\varepsilon \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ *such that*

$$\nabla v_\varepsilon \to \nabla v \qquad \text{strongly in } L^p(\Omega; M^{2\times 2}),$$

$$v_\varepsilon \to v \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2),$$

and such that

$$\limsup_{\varepsilon\to 0} \mathcal{E}^s\left(S^{h_\varepsilon}(v_\varepsilon) \setminus \bigcup_{i=1}^m S^{g^i_\varepsilon}(u^i_\varepsilon)\right) \le \mu(a)\mathcal{E}^s\left(S^h(v) \setminus \bigcup_{i=1}^m S^{g^i}(u^i)\right),$$

where $\mu(a)$ *depends only on* a, $\mu(a) \to 1$ *as* $a \to 0$, *and* $\mathcal{E}^s$ *is defined in* (8.3). *In particular for all* $t \in [0,T]$ *and for all* $t_\varepsilon \to t$ *we have*

$$\mathcal{E}^b(t_\varepsilon)(v_\varepsilon) \to \mathcal{E}^b(t)(v),$$

where $\mathcal{E}^b$ *is defined in* (8.21).

The proof of Proposition 8.2.2 can be obtained from that of Proposition 7.4.1, where the same result is proved in the scalar valued setting of SBV functions with

$$\mathcal{E}^b(t)(v) = \|\nabla v\|^2 \quad \text{and} \quad \mathcal{E}^s(\Gamma) = \mathcal{H}^1(\Gamma),$$

taking into account the following modifications. We can consider v scalar valued since vector valued maps can be easily dealt componentwise. Even if the surface energy is of the form (8.3), by using the density result of [40] we can still restrict ourselves to the case in which v has piecewise linear jumps outside a suitable open set U such that

$$|U| < \sigma \qquad \text{and} \qquad \mathcal{H}^1\left(\bigcup_{i=1}^m S^{g^i}(u^i) \setminus U\right) < \sigma,$$

where σ is an arbitrarily small constant. In order to approximate the piecewise linear jumps, we use Lemma 8.2.1. Finally, we are not assuming $p = 2$, and this prevents us from considering the piecewise jumps as union of disjoint segments: we overcome this difficulty choosing $v_\varepsilon = 0$ in the regular triangles which contain the intersection points, and then interpolating v outside as in [59, Proposition 5.1].

8.3 Preexisting cracks and their approximation

In Section 8.5, we will need to approximate the surface energy of a given preexisting crack Γ^0. We take the initial crack in the class

$$\text{(8.29)} \quad \boldsymbol{\Gamma}(\Omega) := \{\Gamma \tilde{\subseteq} \overline{\Omega}_B \,:\, \mathcal{H}^1(\Gamma) < +\infty, \Gamma = S^h(z) \text{ for some } h \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2) \text{ and } z \in GSBV^p_q(\Omega;\mathbb{R}^2)\}.$$

Notice that it is not restrictive to assume $h \equiv 0$. We take as discretization of $\boldsymbol{\Gamma}(\Omega)$ the following class

$$\text{(8.30)} \qquad \boldsymbol{\Gamma}_{\varepsilon,a}(\Omega) := \{\Gamma \tilde{\subseteq} \overline{\Omega}_B \,:\, \mathcal{H}^1(\Gamma) < +\infty, \Gamma = S^0(z) \text{ for some } z \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)\}.$$

We have the following approximation result.

Proposition 8.3.1. *Let* $\Gamma^0 \in \boldsymbol{\Gamma}(\Omega)$. *Then for every* $\varepsilon > 0$ *and* $a \in]0, \frac{1}{2}[$ *there exists* $\Gamma^0_{\varepsilon,a} \in \boldsymbol{\Gamma}_{\varepsilon,a}(\Omega)$ *such that*

$$\lim_{\varepsilon,a\to 0} \mathcal{E}^s(\Gamma^0_{\varepsilon,a}) = \mathcal{E}^s(\Gamma^0),$$

where $\mathcal{E}^s$ is defined in (8.3).

Moreover let $g_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega)$, $g \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)$ be such that as $\varepsilon \to 0$

$$g_\varepsilon \to g \text{ strongly in } W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2),$$

and let us consider

$$F_{\varepsilon,a}(v) := \begin{cases} \mathcal{E}^b(0)(v) + \mathcal{E}^s\left(S^{g_\varepsilon}(v) \setminus \Gamma^0_{\varepsilon,a}\right) & \text{if } v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2), \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^2), \end{cases}$$

and

$$F(v) := \begin{cases} \mathcal{E}^b(0)(v) + \mathcal{E}^s\left(S^{g}(v) \setminus \Gamma^0\right) & \text{if } v \in GSBV^p_q(\Omega;\mathbb{R}^2), S(v) \tilde{\subseteq} \overline{\Omega}_B, \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^2), \end{cases}$$

where $\mathcal{E}^b$ is defined in (8.21). *Then the family $(F_{\varepsilon,a})$ Γ-converges to F in the strong topology of $L^1(\Omega;\mathbb{R}^2)$ as $\varepsilon \to 0$ and $a \to 0$.*

Proof. Let us consider $\Gamma^0 \in \mathbf{\Gamma}(\Omega)$ with $\Gamma^0 = S^0(z)$ for some $z \in GSBV^p_q(\Omega;\mathbb{R}^2)$. Then by Proposition 8.2.2 for every $\varepsilon > 0$ and $a \in (0, \frac{1}{2})$, there exists $\tilde{z}_{\varepsilon,a} \in \mathcal{A}_{\varepsilon,a}(\Omega)$ such that for $\varepsilon \to 0$ and for all a

$$\nabla \tilde{z}_{\varepsilon,a} \to \nabla z \qquad \text{strongly in } L^p(\Omega; M^{2\times 2}),$$
$$\tilde{z}_{\varepsilon,a} \to z \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2),$$

and

$$\limsup_{\varepsilon\to 0} \mathcal{E}^s(S^0(\tilde{z}_{\varepsilon,a})) \le \mu(a)\mathcal{E}^s(S^0(z))$$

with $\mu(a) \to 1$ as $a \to 0$, where $\mathcal{E}^s$ is defined in (8.3). Let $a_i \searrow 0$, and let $\varepsilon_i \searrow 0$ be such that for all $\varepsilon \le \varepsilon_i$

$$\mathcal{E}^s(S^0(\tilde{z}_{\varepsilon,a_i})) \le \mu(a_i)\mathcal{E}^s(S^0(z)) + a_i,$$

and

$$\|\nabla \tilde{z}_{\varepsilon,a_i} - \nabla z\|_{L^p(\Omega;M^{2\times 2})} \le a_i, \qquad \|\tilde{z}_{\varepsilon,a_i} - z\|_{L^q(\Omega;\mathbb{R}^2)} \le a_i.$$

Setting

$$z_{\varepsilon,a} := \begin{cases} \tilde{z}_{\varepsilon,a_i} & \varepsilon_{i+1} < \varepsilon \le \varepsilon_i,\ a \le a_i, \\ \tilde{z}_{\varepsilon,a_{j-1}} & \varepsilon_{i+1} < \varepsilon \le \varepsilon_i,\ a_j < a \le a_{j-1},\ j \le i, \end{cases}$$

we have that

$$\lim_{\varepsilon,a\to 0} \nabla z_{\varepsilon,a} = \nabla z \qquad \text{strongly in } L^p(\Omega; M^{2\times 2}),$$
$$\lim_{\varepsilon,a\to 0} z_{\varepsilon,a} = z \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2),$$

and

$$\limsup_{\varepsilon,a\to 0} \mathcal{E}^s(S^0(z_{\varepsilon,a})) \le \mathcal{E}^s(S^0(z)).$$

Since by Theorem 8.1.2 we have $\mathcal{E}^s(S^0(z_{\varepsilon,a})) \le \liminf_{\varepsilon,a\to 0} \mathcal{E}^s(S^0(z_{\varepsilon,a}))$, we conclude that

$$\lim_{\varepsilon,a\to 0} \mathcal{E}^s(S^0(z_{\varepsilon,a})) = \mathcal{E}^s(S^0(z)).$$

Let us set for every ε, a

$$\Gamma^0_{\varepsilon,a} := S^0(z_{\varepsilon,a}).$$

We have that

$$\lim_{\varepsilon,a\to 0} \mathcal{E}^s(\Gamma^0_{\varepsilon,a}) = \mathcal{E}^s(\Gamma^0).$$

Let us come to the second part of the proof. Let us consider $(\varepsilon_n, a_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$ and $a_n \to 0$. If we prove that $(F_{\varepsilon_n,a_n})_{n\in\mathbb{N}}$ Γ-converges to F in the strong topology of $L^1(\Omega;\mathbb{R}^2)$, the proposition is proved since the sequence is arbitrary. Since we can reason up to subsequences, it is not restrictive to assume $a_n \searrow 0$.

Let us start with the Γ-limsup inequality considering $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$, with $S(v) \subseteq \overline{\Omega}_B$. For any n fixed, by Proposition 8.2.2 there exists $\tilde{v}_{\varepsilon,a_n} \in \mathcal{A}^B_{\varepsilon,a_n}(\Omega;\mathbb{R}^2)$ such that for $\varepsilon \to 0$

$$\nabla\tilde{v}_{\varepsilon,a_n} \to \nabla v \qquad \text{strongly in } L^p(\Omega;M^{2\times 2}),$$

$$\tilde{v}_{\varepsilon,a_n} \to v \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2),$$

and such that

$$\limsup_{\varepsilon\to 0} \mathcal{E}^s(S^{g_\varepsilon}(\tilde{v}_{\varepsilon,a_n}) \setminus \Gamma^0_{\varepsilon,a_n}) \leq \mu(a_n)\mathcal{E}^s(S^{g_\varepsilon}(v) \setminus \Gamma^0)$$

with $\mu(a) \to 1$ as $a \to 0$. For every $m \in \mathbb{N}$ let ε^m be such that for all $\varepsilon \leq \varepsilon^m$

$$\mathcal{E}^s(S^{g_\varepsilon}(\tilde{v}_{\varepsilon,a_m}) \setminus \Gamma^0_{\varepsilon,a_m}) \leq \mu(a_m)\mathcal{E}^s(S^g(v) \setminus \Gamma^0) + a_m,$$

and

$$\|\nabla\tilde{v}_{\varepsilon,a_m} - \nabla v\|_{L^p(\Omega;M^{2\times 2})} \leq a_m, \qquad \|\tilde{v}_{\varepsilon,a_m} - v\|_{L^q(\Omega;\mathbb{R}^2)} \leq a_m.$$

We can assume $\varepsilon^m \searrow 0$. Setting

$$v_{\varepsilon_n,a_n} := \begin{cases} \tilde{v}_{\varepsilon_n,a_m} & \varepsilon^{m+1} < \varepsilon_n \leq \varepsilon^m, \ n \geq m, \\ \tilde{v}_{\varepsilon_n,a_n} & \varepsilon^{m+1} < \varepsilon_n \leq \varepsilon^m, \ n < m, \end{cases}$$

we have that

$$\lim_n \nabla v_{\varepsilon_n,a_n} = \nabla v \quad \text{strongly in } L^p(\Omega;M^{2\times 2}),$$

$$\lim_n v_{\varepsilon_n,a_n} = v \quad \text{strongly in } L^q(\Omega;\mathbb{R}^2),$$

and

$$\limsup_n \mathcal{E}^s(S^{g_\varepsilon}(v_{\varepsilon_n,a_n}) \setminus \Gamma^0_{\varepsilon,a}) \leq \mathcal{E}^s(S^g(v) \setminus \Gamma^0).$$

Then we get

$$\begin{aligned}\limsup_n F_{\varepsilon_n,a_n}(v_{\varepsilon_n,a_n}) &\leq \limsup_n \mathcal{E}^b(0)(v_{\varepsilon_n,a_n}) + \limsup_n \mathcal{E}^s(S^{g_{\varepsilon_n}}(v_{\varepsilon_n,a_n}) \setminus \Gamma^0_{\varepsilon_n,a_n}) \\ &\leq \mathcal{E}^b(0)(v) + \mathcal{E}^s(S^g(v) \setminus \Gamma^0) = F(v),\end{aligned}$$

so that the Γ-limsup inequality holds.

Let us come to the Γ-liminf inequality. Let $v_n, v \in L^1(\Omega;\mathbb{R}^2)$ be such that $v_n \to v$ strongly in $L^1(\Omega;\mathbb{R}^2)$ and $\liminf_n F_{\varepsilon_n,a_n}(v_n) < +\infty$. By Theorem 8.1.2, we have $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \subseteq \overline{\Omega}_B$ and

$$\mathcal{E}^b(0)(v) \leq \liminf_n \mathcal{E}^b(0)(v_n).$$

Let us consider Ω_D polygonal such that $\Omega_D \cap \Omega = \emptyset$, and $\partial\Omega_D \cap \partial\Omega = \partial_D\Omega$ up to a finite number of points, and let us set

$$\Omega' := \Omega \cup \Omega_D \cup \partial_D\Omega.$$

Let us extend g_{ε_n} and g to $W^{1,p}(\Omega';\mathbb{R}^2) \cap L^q(\Omega';\mathbb{R}^2)$ in such a way that $g_{\varepsilon_n} \to g$ strongly in $W^{1,p}(\Omega';\mathbb{R}^2) \cap L^q(\Omega';\mathbb{R}^2)$, and let us also extend v_n, v to Ω' setting $v_n = g_{\varepsilon_n}$ and $v = g$ on Ω_D. We indicate these extensions with w_n and w respectively. Notice that $w_n, w \in GSBV^p_q(\Omega;\mathbb{R}^2)$, and that $S^{g_{\varepsilon_n}}(v_n) = S(w_n)$ and $S^{g_\varepsilon}(v) = S(w)$. Let us also set $z_{\varepsilon_n,a_n} = z = 0$ on Ω_D, where z_{ε_n,a_n} and z are such that $\Gamma^0_{\varepsilon_n,a_n} = S^0(z_{\varepsilon_n,a_n})$ and $\Gamma^0 = S^0(z)$. We indicate these extension by $\zeta_{\varepsilon_n,a_n}$ and ζ respectively: we have $\zeta_{\varepsilon_n,a_n}, \zeta \in GSBV^p_q(\Omega;\mathbb{R}^2)$ and $\Gamma^0_{\varepsilon_n,a_n} = S(\zeta_{\varepsilon_n,a_n})$ and $\Gamma^0 = S(\zeta)$. Then for every $\eta > 0$ we have by Theorem 8.1.2

$$\mathcal{E}^s(S(w + \eta\zeta)) \leq \liminf_n \mathcal{E}^s(S(w_n + \eta\zeta_{\varepsilon_n,a_n})).$$

Since for a.e. $\eta > 0$ we have $S(w + \eta\zeta) = S(w) \cup S(\zeta)$ and $S(w_n + \eta\zeta_{\varepsilon_n,a_n}) = S(w_n) \cup S(\zeta_{\varepsilon_n,a_n})$, we deduce that

$$\mathcal{E}^s(S^g(v) \cup \Gamma^0) \leq \liminf_n \mathcal{E}^s(S^{g_{\varepsilon_n}}(v_n) \cup \Gamma^0_{\varepsilon_n,a_n}).$$

Since by assumption $\mathcal{E}^s(\Gamma^0_{\varepsilon_n,a_n}) \to \mathcal{E}^s(\Gamma^0)$, we conclude that

$$\mathcal{E}^s(S^g(v) \setminus \Gamma^0) \leq \liminf_n \mathcal{E}^s(S^{g_{\varepsilon_n}}(v_n) \setminus \Gamma^0_{\varepsilon_n,a_n}).$$

We deduce that

$$\mathcal{E}^b(0)(v) + \mathcal{E}^s(S^g(v) \setminus \Gamma^0) \leq \liminf_n \left[\mathcal{E}^b(0)(v_n) + \mathcal{E}^s(S^{g_{\varepsilon_n}}(v_n) \setminus \Gamma^0_{\varepsilon_n,a_n})\right]$$

that is

$$F(v) \leq \liminf_n F_{\varepsilon_n,a_n}(v_n).$$

The Γ-liminf inequality holds, and so the proof is concluded. □

8.4 The discontinuous finite element approximation

In this section we construct a discrete approximation of the quasistatic evolution of brittle cracks proposed in [44] and described in the Preliminaries: the discretization is done both in space and time. Let us consider

$$g_\varepsilon \in W^{1,1}([0,T]; W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)), \quad g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega) \text{ for all } t \in [0,T],$$

where $\mathcal{AF}_\varepsilon(\Omega)$ is defined in (8.24). Let $\delta > 0$, and let N_δ be the largest integer such that $\delta(N_\delta - 1) < T$; we set $t_i^\delta := i\delta$ for $0 \leq i \leq N_\delta - 1$, $t^\delta_{N_\delta} := T$ and $g_\varepsilon^{\delta,i} := g_\varepsilon(t_i^\delta)$. Let $\Gamma^0 \in \mathbf{\Gamma}_{\varepsilon,a}(\Omega)$ be a preexisting crack in Ω, where $\mathbf{\Gamma}_{\varepsilon,a}(\Omega)$ is defined in (8.30).

Proposition 8.4.1. *Let $\varepsilon > 0$, $a \in]0, \frac{1}{2}[$ and $\delta > 0$ be fixed. Then for all $i = 0, \ldots, N_\delta$ there exists $u^{\delta,i}_{\varepsilon,a} \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ such that, setting*

$$\Gamma^{\delta,i}_{\varepsilon,a} := \Gamma^0 \cup \bigcup_{r=0}^{i} S^{g_\varepsilon^{\delta,r}}(u^{\delta,r}_{\varepsilon,a}),$$

we have for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$

$$\mathcal{E}^b(0)(u^{\delta,0}_{\varepsilon,a}) + \mathcal{E}^s\left(S^{g_\varepsilon^{\delta,0}}(u^{\delta,0}_{\varepsilon,a}) \setminus \Gamma^0\right) \leq \mathcal{E}^b(0)(v) + \mathcal{E}^s\left(S^{g_\varepsilon^{\delta,0}}(v) \setminus \Gamma^0\right), \tag{8.31}$$

and for $1 \leq i \leq N_\delta$

$$\mathcal{E}^b(t_i^\delta)(u^{\delta,i}_{\varepsilon,a}) + \mathcal{E}^s\left(S^{g_\varepsilon^{\delta,i}}(u^{\delta,i}_{\varepsilon,a}) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right) \leq \mathcal{E}^b(t_i^\delta)(v) + \mathcal{E}^s\left(S^{g_\varepsilon^{\delta,i}}(v) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a}\right). \tag{8.32}$$

Proof. Let $u^{\delta,0}_{\varepsilon,a}$ be a minimum of the following problem

$$\min_{u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)} \left\{\mathcal{E}^b(0)(u) + \mathcal{E}^s(S^{g_\varepsilon^{\delta,0}}(u) \setminus \Gamma^0)\right\}. \tag{8.33}$$

We set $\Gamma^{\delta,0}_{\varepsilon,a} := \Gamma^0 \cup S^{g_\varepsilon^{\delta,0}}(u^{\delta,0}_{\varepsilon,a})$. Recursively, supposing to have constructed $u^{\delta,i-1}_{\varepsilon,a}$ and $\Gamma^{\delta,i-1}_{\varepsilon,a}$, let $u^{\delta,i}_{\varepsilon,a}$ be a minimum for

$$\min_{u \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)} \left\{\mathcal{E}^b(t_i^\delta)(u) + \mathcal{E}^s(S^{g_\varepsilon^{\delta,i}}(u) \setminus \Gamma^{\delta,i-1}_{\varepsilon,a})\right\}. \tag{8.34}$$

We set $\Gamma^{\delta,i}_{\varepsilon,a} := \Gamma^{\delta,i-1}_{\varepsilon,a} \cup S^{g_\varepsilon^{\delta,i}}(u^{\delta,i}_{\varepsilon,a})$. It is clear by construction that (8.31) and (8.32) hold.

Let us prove that problem (8.34) admits a solution, problem (8.33) being similar. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for problem (8.34): since $g_\varepsilon^{\delta,i}$ is an admissible test function, we deduce that for n large

$$\mathcal{E}^b(t_i^\delta)(u_n) + \mathcal{E}^s\left(S^{g_\varepsilon^{\delta,i}}(u_n) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right) \le \mathcal{E}^b(t_i^\delta)(g_\varepsilon^{\delta,i}) + 1.$$

By the lower estimate on the elastic energy (8.22), we deduce that for n large

$$\alpha_0^{\mathcal{E}}\left(\|\nabla u_n\|_p^p + \|u_n\|_q^q\right) + \mathcal{E}^s\left(S^{g_\varepsilon^{\delta,i}}(u_n) \setminus \Gamma_{\varepsilon,a}^{\delta,i-1}\right) \le \mathcal{E}^b(t_i^\delta)(g_\varepsilon^{\delta,i}) + 1 + \beta_0^{\mathcal{E}}. \tag{8.35}$$

Up to a subsequence, we have that the adaptive vertices z_n^i converge to some adaptive vertices z^i, and also the values of the deformation u_n at every vertex are converging. In the end we find interpolating these values an admissible deformation $u \in \mathcal{A}_{\varepsilon,a}^B(\Omega;\mathbb{R}^2)$. Since the functional in the minimization problem (8.34) is lower semicontinuous with respect to the position of the vertices and the values of the deformation, we conclude that u is a minimum point for the problem. □

The following estimate on the total energy is essential in order to study the asymptotic behavior of the discrete evolution as $\delta \to 0$, $\varepsilon \to 0$ and $a \to 0$. Let us set $u_{\varepsilon,a}^\delta(t) := u_{\varepsilon,a}^{\delta,i}$ for all $t_i^\delta \le t < t_{i+1}^\delta$ and $i = 0,\dots,N_\delta - 1$, $u_{\varepsilon,a}^\delta(T) = u_{\varepsilon,a}^{\delta,N_\delta}$.

Proposition 8.4.2. *For all $0 \le j \le i \le N_\delta$ we have*

$$\begin{aligned}\mathcal{E}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i},\Gamma_{\varepsilon,a}^{\delta,i}) \le \mathcal{E}(t_j^\delta)(u_{\varepsilon,a}^{\delta,j},\Gamma_{\varepsilon,a}^{\delta,j}) &+ \int_{t_j^\delta}^{t_i^\delta} \langle \partial\mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau)), \nabla\dot g_\varepsilon(\tau)\rangle\, d\tau \\ - \int_{t_j^\delta}^{t_i^\delta} \dot{\mathcal{F}}(\tau)(u_{\varepsilon,a}^\delta(\tau))\, d\tau &- \int_{t_j^\delta}^{t_i^\delta} \langle \partial\mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle\, d\tau \\ - \int_{t_j^\delta}^{t_i^\delta} \dot{\mathcal{G}}(\tau)(u_{\varepsilon,a}^\delta(\tau))\, d\tau &- \int_{t_j^\delta}^{t_i^\delta} \langle \partial\mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle\, d\tau + e_{\varepsilon,a}^\delta,\end{aligned} \tag{8.36}$$

where $e_{\varepsilon,a}^\delta \to 0$ as $\delta \to 0$ uniformly in ε and a.

Proof. By the minimality property (8.32), comparing $u_{\varepsilon,a}^{\delta,i}$ with $u_{\varepsilon,a}^{\delta,i-1} - g_\varepsilon^{\delta,i-1} + g_\varepsilon^{\delta,i}$ we get

$$\begin{aligned}&\mathcal{W}(\nabla u_{\varepsilon,a}^{\delta,i}) - \mathcal{F}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i}) - \mathcal{G}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i}) + \mathcal{E}^s(S^{g_i^\delta}(u_{\varepsilon,a}^{\delta,i}) \setminus \Gamma_{i-1}^\delta) \\ &\le \mathcal{W}(\nabla u_{\varepsilon,a}^{\delta,i-1} - \nabla g_\varepsilon^{\delta,i-1} + \nabla g_\varepsilon^{\delta,i}) - \mathcal{F}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i-1} - g_\varepsilon^{\delta,i-1} + g_\varepsilon^{\delta,i}) \\ &\qquad - \mathcal{G}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i-1} - g_\varepsilon^{\delta,i-1} + g_\varepsilon^{\delta,i}).\end{aligned} \tag{8.37}$$

We have

$$\begin{aligned}\mathcal{W}(\nabla u_{\varepsilon,a}^{\delta,i-1} - \nabla g_\varepsilon^{\delta,i-1} + \nabla g_\varepsilon^{\delta,i}) &= \mathcal{W}(\nabla u_{\varepsilon,a}^{\delta,i-1}) \\ &+ \langle \partial\mathcal{W}(\nabla u_{\varepsilon,a}^{\delta,i-1} + \vartheta_{\varepsilon,a}^{\delta,i-1}(\nabla g_\varepsilon^{\delta,i} - \nabla g_\varepsilon^{\delta,i-1})), \nabla g_\varepsilon^{\delta,i} - \nabla g_\varepsilon^{\delta,i-1}\rangle \\ &= \mathcal{W}(\nabla u_{\varepsilon,a}^{\delta,i-1}) + \int_{t_{i-1}^\delta}^{t_i^\delta} \langle \partial\mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau) + v_{\varepsilon,a}^\delta(\tau)), \nabla\dot g_\varepsilon(\tau)\rangle\, d\tau,\end{aligned} \tag{8.38}$$

where $\vartheta_{\varepsilon,a}^{\delta,i-1} \in]0,1[$ and $v_{\varepsilon,a}^\delta(\tau) := \vartheta_{\varepsilon,a}^{\delta,i-1}(\nabla g_\varepsilon^{\delta,i} - \nabla g_\varepsilon^{\delta,i-1})$ for all $\tau \in [t_{i-1}^\delta, t_i^\delta[$.

Similarly we obtain

$$\mathcal{F}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i-1} - g_\varepsilon^{\delta,i-1} + g_\varepsilon^{\delta,i}) = \mathcal{F}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i-1}) + \int_{t_{i-1}^\delta}^{t_i^\delta} \langle \partial\mathcal{F}(t_i^\delta)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle\, d\tau, \tag{8.39}$$

and

$$\mathcal{G}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i-1} - g_\varepsilon^{\delta,i-1} + g_\varepsilon^{\delta,i}) = \mathcal{G}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i-1}) + \int_{t_{i-1}^\delta}^{t_i^\delta} \langle \partial\mathcal{G}(t_i^\delta)(u_{\varepsilon,a}^\delta(\tau) + z_{\varepsilon,a}^\delta(\tau)), \dot g_\varepsilon(\tau)\rangle\, d\tau, \tag{8.40}$$

where $w^\delta_{\varepsilon,a}(\tau) := \lambda^{\delta,i-1}_{\varepsilon,a}(g^{\delta,i}_\varepsilon - g^{\delta,i-1}_\varepsilon)$, $z^\delta_{\varepsilon,a}(\tau) := \nu^{\delta,i-1}_{\varepsilon,a}(g^{\delta,i}_\varepsilon - g^{\delta,i-1}_\varepsilon)$ for all $\tau \in [t^\delta_{i-1}, t^\delta_i[$, and $\lambda^{\delta,i-1}_{\varepsilon,a}, \nu^{\delta,i-1}_{\varepsilon,a} \in]0,1[$.

Since by (8.12) we have for $\tau \in [t^\delta_{i-1}, t^\delta_i[$

$$\begin{aligned}\langle \partial\mathcal{F}(t^\delta_i)(u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle - \langle \partial\mathcal{F}(\tau)(u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle \\ = \int_\tau^{t^\delta_i} \langle \partial\dot{\mathcal{F}}(s)(u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle\, ds\end{aligned}$$

we get by (8.15)

$$\begin{aligned}(8.41)\quad &\left|\langle \partial\mathcal{F}(t^\delta_i)(u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle - \langle \partial\mathcal{F}(\tau)(u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle\right| \\ &\le \int_\tau^{t^\delta_i} |\langle \partial\dot{\mathcal{F}}(s)(u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle|\, ds \\ &\le \int_\tau^{t^\delta_i} \left[\alpha^{\mathcal{F}}_4(s)\|u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)\|^{\dot q-1}_{\dot q} + \beta^{\mathcal{F}}_4(s)\right] \|\dot g_\varepsilon(\tau)\|_{\dot q}\, ds \le \gamma^{\delta,\varepsilon,a}_{\mathcal{F}}\|\dot g_\varepsilon(\tau)\|_{\dot q},\end{aligned}$$

where

$$\gamma^{\delta,\varepsilon,a}_{\mathcal{F}} := \max_{1\le i\le N_\delta}\left(\|u^{\delta,i-1}_{\varepsilon,a} + \lambda^{\delta,i-1}_{\varepsilon,a}(g^{\delta,i}_\varepsilon - g^{\delta,i-1}_\varepsilon)\|^{\dot q}_{\dot q-1}\int_{t^\delta_{i-1}}^{t^\delta_i}\alpha^{\mathcal{F}}_4(s)\,ds + \int_{t^\delta_{i-1}}^{t^\delta_i}\beta^{\mathcal{F}}_4(s)\,ds\right).$$

Similarly we obtain
(8.42)

$$\left|\langle \partial\mathcal{G}(t^\delta_i)(u^\delta_{\varepsilon,a}(\tau) + z^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle - \langle \partial\mathcal{G}(\tau)(u^\delta_{\varepsilon,a}(\tau) + z^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle\right| \le \gamma^{\delta,\varepsilon,a}_{\mathcal{G}}\|\dot g_\varepsilon(\tau)\|_{r,\partial_S\Omega},$$

where

$$\gamma^{\delta,\varepsilon,a}_{\mathcal{G}} := \max_{1\le i\le N_\delta}\left(\|u^{\delta,i-1}_{\varepsilon,a} + \nu^{\delta,i-1}_{\varepsilon,a}(g^{\delta,i}_\varepsilon - g^{\delta,i-1}_\varepsilon)\|^{r-1}_{r,\partial_S\Omega}\int_{t^\delta_{i-1}}^{t^\delta_i}a^{\mathcal{G}}_4(s)\,ds + \int_{t^\delta_{i-1}}^{t^\delta_i}b^{\mathcal{G}}_4(s)\,ds\right).$$

From (8.37), taking into account (8.38), (8.39), (8.40), (8.41), (8.42), we have

$$\begin{aligned}(8.43)\quad \mathcal{E}(t^\delta_i)(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a}) \le{}& \mathcal{E}(t^\delta_{i-1})(u^{\delta,i-1}_{\varepsilon,a}, \Gamma^{\delta,i-1}_{\varepsilon,a}) + \int_{t^\delta_{i-1}}^{t^\delta_i}\langle \partial\mathcal{W}(\nabla u^\delta_{\varepsilon,a}(\tau) + v^\delta_{\varepsilon,a}(\tau)), \nabla\dot g_\varepsilon(\tau)\rangle\,d\tau \\ &- \int_{t^\delta_{i-1}}^{t^\delta_i}\dot{\mathcal{F}}(\tau)(u^\delta_{\varepsilon,a}(\tau))\,d\tau - \int_{t^\delta_{i-1}}^{t^\delta_i}\langle \partial\mathcal{F}(\tau)(u^\delta_{\varepsilon,a}(\tau) + w^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle\,d\tau \\ &- \int_{t^\delta_{i-1}}^{t^\delta_i}\dot{\mathcal{G}}(\tau)(u^\delta_{\varepsilon,a}(\tau))\,d\tau - \int_{t^\delta_{i-1}}^{t^\delta_i}\langle \partial\mathcal{G}(\tau)(u^\delta_{\varepsilon,a}(\tau) + z^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle\,d\tau \\ &+ \gamma^{\delta,\varepsilon,a}_{\mathcal{F}}\int_{t^\delta_{i-1}}^{t^\delta_i}\|\dot g_\varepsilon(\tau)\|_{\dot q}\,d\tau + \gamma^{\delta,\varepsilon,a}_{\mathcal{G}}\int_{t^\delta_{i-1}}^{t^\delta_i}\|\dot g_\varepsilon(\tau)\|_{r,\partial_S\Omega}\,d\tau.\end{aligned}$$

Taking now $0 \leq j \leq i \leq N_\delta$, summing in (8.43) from t_j^δ to t_i^δ, we obtain

$$
\begin{aligned}
(8.44)\quad \mathcal{E}(t_i^\delta)(u_{\varepsilon,a}^{\delta,i}, \Gamma_{\varepsilon,a}^{\delta,i}) \leq \mathcal{E}(t_j^\delta)(u_{\varepsilon,a}^{\delta,j}, \Gamma_{\varepsilon,a}^{\delta,j}) &+ \int_{t_j^\delta}^{t_i^\delta} \langle \partial \mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau) + v_{\varepsilon,a}^\delta(\tau)), \nabla \dot{g}_\varepsilon(\tau)\rangle \, d\tau \\
&- \int_{t_j^\delta}^{t_i^\delta} \dot{\mathcal{F}}(\tau)(u_{\varepsilon,a}^\delta(\tau))\, d\tau - \int_{t_j^\delta}^{t_i^\delta} \langle \partial \mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle \, d\tau \\
&- \int_{t_j^\delta}^{t_i^\delta} \dot{\mathcal{G}}(\tau)(u_{\varepsilon,a}^\delta(\tau))\, d\tau - \int_{t_j^\delta}^{t_i^\delta} \langle \partial \mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau) + z_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle \, d\tau \\
&+ \gamma_{\mathcal{F}}^{\delta,\varepsilon,a} \int_{t_j^\delta}^{t_i^\delta} \|\dot{g}_\varepsilon(\tau)\|_q \, d\tau + \gamma_{\mathcal{G}}^{\delta,\varepsilon,a} \int_{t_j^\delta}^{t_i^\delta} \|\dot{g}_\varepsilon(\tau)\|_{r,\partial_S\Omega} \, d\tau.
\end{aligned}
$$

Setting

$$
\begin{aligned}
(8.45)\quad e_{\varepsilon,a}^\delta := &\int_0^1 |\langle \partial \mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau) + v_{\varepsilon,a}^\delta(\tau)), \nabla \dot{g}_\varepsilon(\tau)\rangle - \langle \partial \mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau)), \nabla \dot{g}_\varepsilon(\tau)\rangle| \, d\tau \\
&+ \int_0^1 |\langle \partial \mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle - \langle \partial \mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle| \, d\tau \\
&+ \int_0^1 |\langle \partial \mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau) + z_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle - \langle \partial \mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle| \, d\tau \\
&+ \gamma_{\mathcal{F}}^{\delta,\varepsilon,a} \int_0^1 \|\dot{g}_\varepsilon(\tau)\|_q \, d\tau + \gamma_{\mathcal{G}}^{\delta,\varepsilon,a} \int_0^1 \|\dot{g}_\varepsilon(\tau)\|_{r,\partial_S\Omega} \, d\tau,
\end{aligned}
$$

from (8.44) we formally obtain (8.36). Let us prove that $e_{\varepsilon,a}^\delta \to 0$ as $\delta \to 0$ uniformly in ε and a. By (8.32), comparing $u_{\varepsilon,a}^{\delta,i}$ with $g_\varepsilon^{\delta,i}$, and taking into account (8.22), we get for all $i = 1, \dots, N_\delta$,

$$\|\nabla u_{\varepsilon,a}^{\delta,i}\|_p + \|u_{\varepsilon,a}^{\delta,i}\|_q \leq C',$$

where

$$C' := \frac{1}{\alpha_0^\varepsilon} \max_{i=0,\dots,N_\delta} \left(\mathcal{E}^b(t_i^\delta)(g_\varepsilon^{\delta,i}) + \beta_0^\varepsilon\right).$$

Since Ω_S is Lipschitz, there exists $K_S > 0$ depending only on p, q such that

$$\|u\|_{p,\Omega_S} \leq K_S(\|\nabla u\|_{p,\Omega_S} + \|u\|_{q,\Omega_S})$$

for all $u \in W^{1,p}(\Omega_S;\mathbb{R}^2) \cap L^q(\Omega_S;\mathbb{R}^2)$. Taking into account (8.17), we obtain

$$\|u_{\varepsilon,a}^{\delta,i}\|_{r,\partial_S\Omega} \leq C''$$

for some C'' independent of δ. Since $g_\varepsilon \in W^{1,1}([0,T]; W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2))$, we obtain that for all $\tau \in [0,T]$ as $\delta \to 0$

$$
\begin{aligned}
&v_{\varepsilon,a}^\delta(\tau) \to 0 \text{ strongly in } L^p(\Omega; M^{2\times 2}),\\
&w_{\varepsilon,a}^\delta(\tau) \to 0 \text{ strongly in } L^q(\Omega;\mathbb{R}^2),\\
&z_{\varepsilon,a}^\delta(\tau) \to 0 \text{ strongly in } L^r(\partial_S\Omega;\mathbb{R}^2).
\end{aligned}
$$

Moreover $\gamma_{\mathcal{F}}^{\delta,\varepsilon,a} \to 0$ and $\gamma_{\mathcal{G}}^{\delta,\varepsilon,a} \to 0$ as $\delta \to 0$. Finally, by [44, Lemma 4.9], as $\delta \to 0$ we have that for all $\tau \in [0,T]$

$$
\begin{aligned}
&|\langle \partial \mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau) + v_{\varepsilon,a}^\delta(\tau)), \nabla \dot{g}_\varepsilon(\tau)\rangle - \langle \partial \mathcal{W}(\nabla u_{\varepsilon,a}^\delta(\tau)), \nabla \dot{g}_\varepsilon(\tau)\rangle| \to 0,\\
&|\langle \partial \mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau) + w_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle - \langle \partial \mathcal{F}(\tau)(u_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle| \to 0,\\
&|\langle \partial \mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau) + z_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle - \langle \partial \mathcal{G}(\tau)(u_{\varepsilon,a}^\delta(\tau)), \dot{g}_\varepsilon(\tau)\rangle| \to 0,
\end{aligned}
$$

uniformly in ε, a. By the Dominated Convergence Theorem, we conclude that $e_{\varepsilon,a}^\delta \to 0$ as $\delta \to 0$ uniformly in ε and a, and the proof is finished. $\square$

8.5 The approximation result

In this section we study the asymptotic behavior of the discrete evolution obtained in Section 8.4. Let us consider a given initial crack $\Gamma^0 \in \mathbf{\Gamma}(\Omega)$ where $\mathbf{\Gamma}(\Omega)$ is defined as in (8.29), and a boundary deformation $g \in W^{1,1}\big([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)\big)$. Let $\Gamma^0_{\varepsilon,a} \in \mathbf{\Gamma}_{\varepsilon,a}(\Omega)$ be an approximation of Γ^0 in the sense of Proposition 8.3.1, and let us consider

$$g_\varepsilon \in W^{1,1}\big([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)\big),$$

such that

$$g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega) \text{ for all } t \in [0,T],$$

and such that for $\varepsilon \to 0$

$$g_\varepsilon \to g \quad \text{strongly in } W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)).$$

Let

$$\{(u^{\delta,i}_{\varepsilon,a}, \Gamma^{\delta,i}_{\varepsilon,a}),\ i = 0,\dots,N_\delta\}$$

be the discrete evolution relative to the initial crack $\Gamma^0_{\varepsilon,a}$ and boundary data g_ε given by Proposition 8.4.1. We make the following piecewise constant interpolation in time:

$$u^\delta_{\varepsilon,a}(t) := u^{\delta,i}_{\varepsilon,a}, \qquad \Gamma^\delta_{\varepsilon,a}(t) := \Gamma^{\delta,i}_{\varepsilon,a}, \qquad g^\delta_\varepsilon(t) := g_\varepsilon(t^\delta_i) \qquad \text{for } t^\delta_i \le t < t^\delta_{i+1}, \tag{8.46}$$

$i = 0,\dots,N_\delta - 1$, and $u^\delta_{\varepsilon,a}(T) := u^{\delta,N_\delta}_{\varepsilon,a}$, $\Gamma^\delta_{\varepsilon,a}(T) := \Gamma^{\delta,N_\delta}_{\varepsilon,a}$, $g^\delta_\varepsilon(T) := g_\varepsilon(T)$.

By Proposition 8.4.2, for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$ we have

$$\mathcal{E}^b(0)(u^\delta_{\varepsilon,a}(0)) + \mathcal{E}^s\left(S^{g^\delta_\varepsilon(0)}(u^\delta_{\varepsilon,a}(0)) \setminus \Gamma^0_{\varepsilon,a}\right) \le \mathcal{E}^b(0)(v) + \mathcal{E}^s\left(S^{g^\delta_\varepsilon(0)}(v) \setminus \Gamma^0_{\varepsilon,a}\right),$$

and for all $t \in [t^\delta_i, t^\delta_{i+1}[$ and for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$

$$\mathcal{E}^b(t^\delta_i)(u^\delta_{\varepsilon,a}(t)) \le \mathcal{E}^b(t^\delta_i)(v) + \mathcal{E}^s\left(S^{g^\delta_\varepsilon(t)}(v) \setminus \Gamma^\delta_{\varepsilon,a}(t)\right). \tag{8.47}$$

Here $\mathcal{E}^b$ and $\mathcal{E}^s$ are defined in (8.21) and (8.3) respectively. Finally for all $0 \le s \le t \le 1$ we have

$$\begin{aligned}\mathcal{E}(t^\delta_i)(u^\delta_{\varepsilon,a}(t), \Gamma^\delta_{\varepsilon,a}(t)) \le{}& \mathcal{E}(s^\delta_j)(u^\delta_{\varepsilon,a}(s), \Gamma^\delta_{\varepsilon,a}(s)) + \int_{s^\delta_j}^{t^\delta_i} \langle \partial\mathcal{W}(\nabla u^\delta_{\varepsilon,a}(\tau)), \nabla \dot g_\varepsilon(\tau)\rangle\, d\tau \\ &- \int_{s^\delta_j}^{t^\delta_i} \dot{\mathcal{F}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) - \int_{s^\delta_j}^{t^\delta_i} \langle \partial\mathcal{F}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle\, d\tau \\ &- \int_{s^\delta_j}^{t^\delta_i} \dot{\mathcal{G}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) - \int_{s^\delta_j}^{t^\delta_i} \langle \partial\mathcal{G}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot g_\varepsilon(\tau)\rangle\, d\tau + e^\delta_{\varepsilon,a},\end{aligned} \tag{8.48}$$

where $s^\delta_j \le s < s^\delta_{j+1}$ and $t^\delta_i \le t < t^\delta_{i+1}$, $e^\delta_{\varepsilon,a}$ is defined as in (8.45), and $\mathcal{E}(t)(u,\Gamma)$ is as in (8.20). Recall that $e^\delta_{\varepsilon,a} \to 0$ as $\delta \to 0$ uniformly in ε, a.

Comparing $u^\delta_{\varepsilon,a}(t)$ with $g^\delta_\varepsilon(t)$ by (8.47), and in view of (8.9), (8.13), (8.14), (8.18), (8.19), (8.31) and (8.4), by (8.48) with $s = 0$ we deduce that there exists $C' \in]0,+\infty[$ such that for all t, δ, ε and a

$$\|\nabla u^\delta_{\varepsilon,a}(t)\|_p + \|u^\delta_{\varepsilon,a}(t)\|_q + \mathcal{H}^1(\Gamma^\delta_{\varepsilon,a}(t)) \le C'. \tag{8.49}$$

By the time dependence of $\mathcal{E}^b(\cdot,\cdot)$, in view of (8.49), by (8.47) and (8.48) we have that there exists $o^\delta_{\varepsilon,a} \to 0$ as $\delta, \varepsilon \to 0$ uniformly in a such that for all $t \in [0,T]$ and for all $v \in \mathcal{A}^B_{\varepsilon,a}(\Omega;\mathbb{R}^2)$

$$\mathcal{E}^b(t)(u^\delta_{\varepsilon,a}(t)) \le \mathcal{E}^b(t)(v) + \mathcal{E}^s\left(S^{g^\delta_\varepsilon(t)}(v) \setminus \Gamma^\delta_{\varepsilon,a}(t)\right) + o^\delta_{\varepsilon,a}, \tag{8.50}$$

and for all $0 \leq s \leq t \leq T$

$$\begin{aligned}\mathcal{E}(t)(u^\delta_{\varepsilon,a}(t),\Gamma^\delta_{\varepsilon,a}(t)) \leq& \mathcal{E}(s)(u^\delta_{\varepsilon,a}(s),\Gamma^\delta_{\varepsilon,a}(s)) + \int_s^t \langle \partial\mathcal{W}(\nabla u^\delta_{\varepsilon,a}(\tau)), \nabla \dot{g}_\varepsilon(\tau)\rangle\, d\tau \\ &- \int_s^t \dot{\mathcal{F}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) - \int_s^t \langle \partial\mathcal{F}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot{g}_\varepsilon(\tau)\rangle\, d\tau \\ &- \int_s^t \dot{\mathcal{G}}(\tau)(u^\delta_{\varepsilon,a}(\tau)) - \int_s^t \langle \partial\mathcal{G}(\tau)(u^\delta_{\varepsilon,a}(\tau)), \dot{g}_\varepsilon(\tau)\rangle\, d\tau + o^\delta_{\varepsilon,a}.\end{aligned} \tag{8.51}$$

Inequality (8.49) gives a natural precompactness of $(u^\delta_{\varepsilon,a}(t))$ in $GSBV^p_q(\Omega;\mathbb{R}^2)$. The main result of the chapter is the following.

Theorem 8.5.1. *Let $\delta > 0$, $\varepsilon > 0$, $a \in]0,\frac{1}{2}[$, and let $\{t \to (u^\delta_{\varepsilon,a}(t),\Gamma^\delta_{\varepsilon,a}(t)) : t \in [0,T]\}$ be the discrete evolution given by (8.46) relative to the initial crack $\Gamma^0_{\varepsilon,a}$ and the boundary data g_ε. Then there exist a quasistatic evolution $\{t \to (u(t),\Gamma(t))\}$ in the sense of Theorem 8.1.1 and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, such that setting $u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t)$ and $\Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)$, for all $t \in [0,T]$ the following facts hold.*

(a) For every $t \in [0,T]$, $(u_n(t))_{n\in\mathbb{N}}$ is weakly precompact in $GSBV^p_q(\Omega;\mathbb{R}^2)$, and every accumulation point $\tilde{u}(t)$ is such that $S^{g(t)}(\tilde{u}(t)) \tilde{\subseteq} \Gamma(t)$,

$$\mathcal{E}^b(t)(\tilde{u}(t)) \leq \mathcal{E}^b(t)(v) + \mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma(t)\right) \tag{8.52}$$

for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$, and

$$\mathcal{E}^b(t)(u_n(t)) \to \mathcal{E}^b(t)(\tilde{u}(t)).$$

Moreover there exists a subsequence of $(\delta_n, \varepsilon_n, a_n)_{n\in\mathbb{N}}$ (depending on t) such that

$$u_n(t) \rightharpoonup u(t) \quad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

(b) For every $t \in [0,T]$ we have

$$\mathcal{E}(t)(u_n(t),\Gamma_n(t)) \to \mathcal{E}(t)(u(t),\Gamma(t)); \tag{8.53}$$

more precisely elastic and surface energies converge separately, that is

$$\mathcal{E}^b(t)(u_n(t)) \to \mathcal{E}^b(t)(u(t)), \qquad \mathcal{E}^s(\Gamma_n(t)) \to \mathcal{E}^s(\Gamma(t)). \tag{8.54}$$

For the proof of Theorem 8.5.1 we need two preliminary steps. First of all, we fix a and study the asymptotic for $\delta, \varepsilon \to 0$ (Lemma 8.5.2), and then we let $a \to 0$ using a diagonal argument (Lemma 8.5.4).

Lemma 8.5.2. *Let a be fixed, $t \in [0,T]$, and let $\delta_n \to 0$ and $\varepsilon_n \to 0$. There exists $\Gamma_a(t) \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ and a subsequence of $(\delta_n,\varepsilon_n)_{n\in\mathbb{N}}$ (which we denote with the same symbol), such that the following facts hold:*

(a) if $w_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ is such that $S^{g^{\delta_n}_{\varepsilon_n}(t)}(w_n) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a}(t)$ and

$$w_n \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2),$$

then we have

$$S^{g(t)}(w) \tilde{\subseteq} \Gamma_a(t);$$

(b) *there exists* $\mu(a)$ *with* $\mu(a) \to 1$ *as* $a \to 0$ *such that for every accumulation point* $u_a(t)$ *of* $(u^{\delta_n}_{\varepsilon_n,a}(t))_{n\in\mathbb{N}}$ *for the weak convergence in* $GSBV^p_q(\Omega;\mathbb{R}^2)$ *and for all* $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ *with* $S(v) \tilde{\subseteq} \overline{\Omega}_B$, *we have*

$$\mathcal{E}^b(t)(u_a(t)) \le \mathcal{E}^b(t)(v) + \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(t)\right); \tag{8.55}$$

moreover

$$\lim_n \mathcal{E}^b(t)(u^{\delta_n}_{\varepsilon_n,a}(t)) = \mathcal{E}^b(t)(u_a(t)); \tag{8.56}$$

(c) *we have*

$$\mathcal{E}^s(\Gamma_a(t)) \le \liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a}(t)).$$

Proof. We now perform a variant of [44, Theorem 4.7]. Let $(\varphi_k)_{k\in\mathbb{N}} \subseteq L^1(\Omega;\mathbb{R}^2)$ be dense in $L^1(\Omega;\mathbb{R}^2)$. For every φ_k and for every $m \in \mathbb{N}$, let $v^{n,a}_{k,m}(t)$ be a minimum of the problem

$$\min\{\|\nabla v\|_p + \|v\|_q + m\|v - \varphi_k\|_1 \,:\, v \in V^n_a\},$$

where

$$V^n_a := \{v \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2),\ S^{g^{\delta_n}_{\varepsilon_n}(t)}(v) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\}.$$

Since by (8.49) we have $\mathcal{H}^1(\Gamma^{\delta_n}_{\varepsilon_n,a}(t)) \le C'$, by Theorem 1.1.3 there exists a subsequence of $(\delta_n,\varepsilon_n)_{n\in\mathbb{N}}$ (which we denote with the same symbol) such that $v^{n,a}_{k,m}(t)$ weakly converges to some $v^a_{k,m}(t) \in GSBV^p_q(\Omega;\mathbb{R}^2)$ as $n \to +\infty$ for all $k,m \in \mathbb{N}$. We set

$$\Gamma_a(t) := \bigcup_{k,m} S^{g(t)}(v^a_{k,m}(t)). \tag{8.57}$$

Let us see that $\Gamma_a(t)$ satisfies all the properties of the lemma. Clearly $\Gamma_a(t) \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ and point (c) is a consequence of Theorem 1.1.3. In particular by (8.49) we have that

$$\mathcal{H}^1(\Gamma_a(t)) \le C'. \tag{8.58}$$

Let us come to point (a). Let $w_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ be such that $S^{g^{\delta_n}_{\varepsilon_n}(t)}(w_n) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a}(t)$ and

$$w_n \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

We claim that there exists $k_m \to +\infty$ such that

$$v^a_{k_m,m}(t) \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2). \tag{8.59}$$

Then since $S^{g(t)}(v^a_{k_m,m}(t)) \subseteq \Gamma_a(t)$ for all m and in view of (8.58), we deduce that $S^{g(t)}(w) \tilde{\subseteq} \Gamma_a(t)$. Let us prove (8.59). Fixed $m \in \mathbb{N}$, let us choose k_m in such a way that

$$m\|w - \varphi_{k_m}\|_1 \to 0.$$

By minimality of $v^{n,a}_{k_m,m}(t)$ we have for all n

$$\|\nabla v^{n,a}_{k_m,m}(t)\|_p + \|v^{n,a}_{k_m,m}(t)\|_q + m\|v^{n,a}_{k_m,m}(t) - \varphi_{k_m}\|_1 \le \|\nabla w_n\|_p + \|w_n\|_q + m\|w_n - \varphi_{k_m}\|_1.$$

Passing to the limit in n, by lower semicontinuity we get for some $C \ge 0$

$$\|\nabla v^a_{k_m,m}(t)\|_p + \|v^a_{k_m,m}(t)\|_q + m\|v^a_{k_m,m}(t) - \varphi_{k_m}\|_1 \le C + m\|w - \varphi_{k_m}\|_1.$$

We deduce for $m \to +\infty$

$$\|v^a_{k_m,m}(t) - \varphi_{k_m}\|_1 \to 0,$$

which together with $\|\varphi_{k_m} - w\|_1 \to 0$ implies that

$$v^a_{k_m,m}(t) \to w \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^2).$$

Since

$$\|\nabla v^a_{k_m,m}(t)\|_p + \|v^a_{k_m,m}(t)\|_q \le C + m\|w - \varphi_{k_m}\|_1 \le C+1$$

for m large, we have that $v^a_{k_m,m}(t) \rightharpoonup w$ weakly in $GSBV^p_q(\Omega;\mathbb{R}^2)$, and this proves (8.59).

Finally, let us come to point (*b*). Let $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \subseteq \overline{\Omega}_B$, and let us fix $k_1,\dots,k_s$ and $m_1,\dots,m_r$ in $\mathbb{N}$. By Proposition 8.2.2, there exists $v_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ such that

$$\lim_n \mathcal{E}^b(t)(v_n) = \mathcal{E}^b(t)(v)$$

and

$$\limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) \le \limsup_n \mathcal{E}^s\big(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \bigcup_{i\le s,\, j\le r} S(v^{n,a}_{k_i,m_j})\big)$$
$$\le \mu(a)\mathcal{E}^s\big(S^{g(t)}(v) \setminus \bigcup_{i\le s,\, j\le r} S(v^a_{k_i,m_j})\big),$$

where $\mu(a) \to 1$ as $a \to 0$. Since the k_i's and the m_j's are arbitrary, we obtain that

$$\limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) \le \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(t)\right). \tag{8.60}$$

Let us suppose that $u^{\delta_n}_{\varepsilon_n,a}(t) \rightharpoonup u_a(t)$ weakly in $GSBV^p_q(\Omega;\mathbb{R}^2)$ along a suitable subsequence which we indicate by the same symbol. By the minimality property (8.50), comparing $u^{\delta_n}_{\varepsilon_n,a}(t)$ with v_n we get

$$\mathcal{E}^b(t)(u^{\delta_n}_{\varepsilon_n,a}(t)) \le \mathcal{E}^b(t)(v_n) + \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) + o_n, \tag{8.61}$$

with $o_n \to 0$ as $n \to +\infty$. Then we have

$$\mathcal{E}^b(t)(u_a(t)) \le \liminf_n \mathcal{E}^b(t)(u^{\delta_n}_{\varepsilon_n,a}(t))$$
$$\le \limsup_n \left(\mathcal{E}^b(t)(v_n) + \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right)\right)$$
$$\le \mathcal{E}^b(t)(v) + \limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right)$$
$$\le \mathcal{E}^b(t)(v) + \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(t)\right),$$

that is (8.55) holds. Choosing $v = u_a(t)$, passing to the limsup in (8.61), and taking into account (8.60) we obtain that

$$\limsup_n \mathcal{E}^b(t)(u^{\delta_n}_{\varepsilon_n,a}(t)) \le \mathcal{E}^b(t)(u_a(t)).$$

Since by (8.55) $\mathcal{E}^b(t)(u_a(t))$ is independent of the accumulation point $u_a(t)$, we conclude that (8.56) holds. □

Remark 8.5.3. Using Lemma 8.5.2, it is possible to construct an increasing family $\{t \to \Gamma_a(t) : t \in [0,T]\}$ and a subsequence of $(\delta_n, \varepsilon_n)_{n\in\mathbb{N}}$ such that points (*a*), (*b*) and (*c*) of Lemma 8.5.2 hold for every $t \in [0,T]$. This evolution $\{t \to \Gamma_a(t) : t \in [0,T]\}$ can be considered as an approximate quasistatic evolution, in the sense that it satisfies irreversibility, but it satisfies static equilibrium and nondissipativity up to a small error due to the fact that a is kept fixed. The presence of $\mu(a)$ in the minimality property (8.55) takes into account the anisotropy in the approximation of the surface energy: in fact, since a is kept fixed, the adaptive edges of the triangulations $\mathcal{T}_{\varepsilon,a}(\Omega)$

cannot recover all the possible directions. Notice that $\mu(a) = 1 + Ca$, where C depends only on the coercivity constants of the surface energy and on the range of the angles $\vartheta_1 \le \vartheta \le \vartheta_2$ defining the regular triangulations $\mathcal{R}_\varepsilon(\Omega)$. Using the minimality property (8.55) and following [44, Lemma 7.1] (estimate from below of the total energy) we can obtain the nondissipativity condition up to a small error, that is

$$\left|\mathcal{E}(t)(u_a(t), \Gamma_a(t)) - \mathcal{E}(0)(u_a(0), \Gamma_a(0)) - \int_0^t \vartheta_a(s)\, ds\right| \le \tilde{C} \sup_{t \in [0,T]} \|g(t)\|_{W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)} a,$$

where $\tilde{C}$ is an explicit constant depending on the coercivity constants of the bulk and surface energies and on the range of the angles of the regular triangulations $\mathcal{R}_\varepsilon(\Omega)$, and ϑ_a is defined as in (8.23).

Using the arguments of Lemma 8.5.4, it can be proved that $(u_a(t), \Gamma_a(t))$ approaches along a suitable $a_n \to 0$ a quasistatic crack growth $(u(t), \Gamma(t))$ providing in particular an approximation of the bulk and surface energies at any time, i.e. for all $t \in [0,T]$

$$\mathcal{E}^b(t)(u_{a_n}(t)) \to \mathcal{E}^b(t)(u(t)) \quad \text{and} \quad \mathcal{E}^s(\Gamma_{a_n}(t)) \to \mathcal{E}^s(\Gamma(t)).$$

However it seems difficult to obtain by this approach an explicit estimate for $|\mathcal{E}(t)(u_{a_n}(t), \Gamma_{a_n}(t)) - \mathcal{E}(t)(u(t), \Gamma(t))|$ in term of a_n.

The construction of $\{t \to \Gamma_a(t) : t \in [0,T]\}$ is the following. If $D \subseteq [0,T]$ is countable and dense, by Lemma 8.5.2 and using a diagonalization argument, we can find a subsequence of $(\delta_n, \varepsilon_n)_{n\in\mathbb{N}}$ and an increasing family $\Gamma_a(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$, $t \in D$, such that points (a), (b) and (c) hold for every $t \in D$. Let us set for every $t \in [0,T]$

$$\Gamma_a^+(t) := \bigcap_{s \ge t, s \in D} \Gamma_a(s).$$

Clearly $\{t \to \Gamma_a^+(t) : t \in [0,T]\}$ is increasing, in the sense that $\Gamma_a(s) \tilde{\subseteq} \Gamma_a(t)$ for all $s \le t$. As a consequence, the set J of discontinuity points of $\mathcal{H}^1(\Gamma_a^+(t))$ is at most countable. We can extract a further subsequence of $(\delta_n, \varepsilon_n)_{n\in\mathbb{N}}$ such that $\Gamma_a(t)$ is determined also for all $t \in J$ (notice that $\Gamma_a(t) \tilde{\subseteq} \Gamma_a^+(t)$). For all $t \in [0,T] \setminus (D \cup J)$ we set $\Gamma_a(t) := \Gamma_a^+(t)$. We have that $\Gamma_a(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$ and $\{t \to \Gamma_a(t) : t \in [0,T]\}$ is increasing.

For $t \in D \cup J$, $\Gamma_a(t)$ satisfies by construction points (a), (b) and (c) of Lemma 8.5.2. Let us consider the case $t \in [0,T] \setminus (D \cup J)$.

Concerning point (a), we have that $S^{g(t)}(w) \tilde{\subseteq} \Gamma_a(s)$ for every $s \in D \cap [t,T]$, so that passing to the intersection we get $S^{g(t)}(u_a(t)) \tilde{\subseteq} \Gamma_a(t)$.

As for point (b), considering $s \in D \cap [0,t[$, for every $v \in GSBV_q^p(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$, we have that there exists $v_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ such that

$$\lim_n \mathcal{E}^b(t)(v_n) = \mathcal{E}^b(t)(v),$$

and

$$\begin{aligned}\limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(t)\right) &\le \limsup_n \mathcal{E}^s\left(S^{g^{\delta_n}_{\varepsilon_n}(t)}(v_n) \setminus \Gamma^{\delta_n}_{\varepsilon_n,a}(s)\right) \\ &\le \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(s)\right).\end{aligned}$$

Then by minimality property (8.50) and passing to the limit in n we have

$$\mathcal{E}^b(t)(u(t)) \le \mathcal{E}^b(t)(v) + \mu(a)\mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma_a(s)\right).$$

Letting $s \to t$ we get that (8.55) holds. Reasoning as in Lemma 8.5.2, we get that also (8.56) holds.

Finally, coming to point (c), we have that for all $s \in D \cap [0,t[$

$$\liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a}(t)) \geq \liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a}(s)) \geq \mathcal{E}^s(\Gamma_a(s)),$$

so that letting $s \nearrow t$, and recalling that t is a continuity point for $\mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a}(\cdot))$, we obtain that the lower semicontinuity holds.

We can now let $a \to 0$.

Lemma 8.5.4. *There exist a map $\{t \to \Gamma(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega), t \in [0,T]\}$ and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$ such that the following facts hold:*

(a) $\Gamma^0 \tilde{\subseteq} \Gamma(s) \tilde{\subseteq} \Gamma(t)$ *for all* $0 \leq s \leq t \leq T$;

(b) for all $t \in [0,T]$, if $w_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ with $S^{g^{\delta_n}_{\varepsilon_n}(t)}(w_n) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)$ is such that

$$w_n \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2),$$

then we have

$$S^{g(t)}(w) \tilde{\subseteq} \Gamma(t);$$

(c) for all $t \in [0,T]$ and for every accumulation point $u(t)$ of $(u^{\delta_n}_{\varepsilon_n,a_n}(t))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV^p_q(\Omega;\mathbb{R}^2)$ and for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$, we have

$$\mathcal{E}^b(t)(u(t)) \leq \mathcal{E}^b(t)(v) + \mathcal{E}^s\left(S^{g(t)}(v) \setminus \Gamma(t)\right), \tag{8.62}$$

and

$$\mathcal{E}^b(t)(u(t)) = \lim_n \mathcal{E}^b(t)(u^{\delta_n}_{\varepsilon_n,a_n}(t)); \tag{8.63}$$

(d) for all $t \in [0,T]$ we have

$$\mathcal{E}^s(\Gamma(t)) \leq \liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)). \tag{8.64}$$

Proof. Let us consider $\delta_h \to 0$ and $\varepsilon_h \to 0$. Given $a \in]0, \frac{1}{2}[$ and $t \in [0,T]$, let $\Gamma_a(t)$ be the rectifiable set given by Lemma 8.5.2. Recall that by (8.57) we have

$$\Gamma_a(t) = \bigcup_{k,m} S^{g(t)}(v^a_{k,m}(t)),$$

where $v^a_{k,m}(t)$ is the weak limit in $GSBV^p_q(\Omega;\mathbb{R}^2)$ along a suitable subsequence depending on a of a minimum $v^{h,a}_{k,m}(t)$ of the problem

$$\min\{\|\nabla v\|_p + \|v\|_q + m\|v - \varphi_k\|_1 \,:\, v \in V^h_a(t)\}, \tag{8.65}$$

where $(\varphi_k)_{k\in\mathbb{N}} \subseteq L^1(\Omega;\mathbb{R}^2)$ is dense in $L^1(\Omega;\mathbb{R}^2)$ and

$$V^h_a(t) := \{v \in \mathcal{A}^B_{\varepsilon,a}(\Omega';\mathbb{R}^2), S^{g^{\delta_h}_{\varepsilon_h}(t)}(v) \subseteq \Gamma^{\delta_h}_{\varepsilon_h,a}(t)\}.$$

Let $a_n \to 0$, and let $D := \{t_j \,:\, j \in \mathbb{N}\} \subseteq [0,T]$ be countable and dense with $0 \in D$. Using a diagonal argument, up to a subsequence of $(\delta_h, \varepsilon_h)_{h\in\mathbb{N}}$, we may suppose that for all $t \in D$ and for all n

$$v^{h,a_n}_{k,m}(t) \rightharpoonup v^{a_n}_{k,m}(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

Moreover, we may assume that for all $t \in D$ and for all n

$$u^{\delta_h}_{\varepsilon_h, a_n}(t) \rightharpoonup u_{a_n}(t) \qquad \text{weakly in } GSBV^p_q(\Omega; \mathbb{R}^2)$$

with

$$\mathcal{E}^b(t)(u^{\delta_h}_{\varepsilon_h, a_n}(t)) \to \mathcal{E}^b(t)(u_{a_n}(t)).$$

By Lemma 8.5.2, we have that $u_{a_n}(t)$ satisfies the minimality property (8.55).

Up to a subsequence of $(a_n)_{n\in\mathbb{N}}$, we may suppose that for all k, m and $t \in D$ we have

$$v^{a_n}_{k,m}(t) \rightharpoonup v_{k,m}(t) \qquad \text{weakly in } GSBV^p_q(\Omega; \mathbb{R}^2), \tag{8.66}$$

and

$$u_{a_n}(t) \rightharpoonup u(t) \qquad \text{weakly in } GSBV^p_q(\Omega; \mathbb{R}^2). \tag{8.67}$$

For all $t \in D$, let us set

$$\Gamma(t) := \bigcup_{k,m} S^{g(t)}(v_{k,m}(t)). \tag{8.68}$$

By Proposition 8.2.2, in view of the minimality property (8.55) and taking into account that $\mu(a_n) \to 1$, we have that for all $v \in GSBV^p_q(\Omega; \mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$

$$\mathcal{E}^b(t)(u(t)) \leq \mathcal{E}^b(t)(v) + \mathcal{E}^s(S^{g(t)}(v) \setminus \Gamma(t)), \tag{8.69}$$

and as a consequence, we obtain

$$\mathcal{E}^b(t)(u_{a_n}(t)) \to \mathcal{E}^b(t)(u(t)).$$

We now perform the following diagonal argument. Choose $\delta_{h_0}, \varepsilon_{h_0}$ in such a way that

$$\|v^{h_0,a_0}_{0,0}(t_0) - v^{a_0}_{0,0}(t_0)\|_1 + \|u^{\delta_{h_0}}_{\varepsilon_{h_0},a_0}(t_0) - u_{a_0}(t_0)\|_1 \\ + |\mathcal{E}^b(t_0)(u^{\delta_{h_0}}_{\varepsilon_{h_0},a_0}(t_0)) - \mathcal{E}^b(t_0)(u_{a_0}(t_0))| \leq 1.$$

Supposing to have constructed $\delta_{h_n}, \varepsilon_{h_n}$, we choose $\delta_{h_{n+1}}, \varepsilon_{h_{n+1}}$ in such a way that for all $k \leq n+1$, $m \leq n+1$ and for all t_i with $1 \leq i \leq n+1$ we have

$$\|v^{h_{n+1},a_{n+1}}_{k,m}(t_i) - v^{a_{n+1}}_{k,m}(t_i)\|_1 + \|u^{\delta_{h_{n+1}}}_{\varepsilon_{h_{n+1}},a_{n+1}}(t_i) - u_{a_{n+1}}(t_i)\|_1 \\ + |\mathcal{E}^b(t_i)(u^{\delta_{h_{n+1}}}_{\varepsilon_{h_{n+1}},a_{n+1}}(t_i)) - \mathcal{E}^b(t_i)(u_{a_{n+1}}(t_i))| \leq \frac{1}{n+1}.$$

Let us set $\delta_n := \delta_{h_n}$ and $\varepsilon_n := \varepsilon_{h_n}$, and let us prove that $\Gamma(t)$ defined in (8.68) satisfies the properties of the Lemma. We have immediately that $\Gamma(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega)$.

Concerning point (d), notice that

$$\Gamma^{\delta_n}_{\varepsilon_n,a_n}(t) = \bigcup_{m,k} S^{g^{\delta_n}_{\varepsilon_n}(t)}(v^{h_n,a_n}_{k,m}(t)), \qquad \Gamma(t) = \bigcup_{m,k} S^{g(t)}(v_{k,m}(t)),$$

and that for all k, m

$$v^{h_n,a_n}_{k,m}(t) \rightharpoonup v_{k,m}(t) \qquad \text{weakly in } GSBV^p_q(\Omega; \mathbb{R}^2);$$

then (8.64) is a consequence of Theorem 1.1.3. In particular, by (8.49), we get that

$$\mathcal{H}^1(\Gamma(t)) \leq C'. \tag{8.70}$$

Let us come to point (b). Let $w_n \in \mathcal{A}^B_{\varepsilon_n,a}(\Omega;\mathbb{R}^2)$ with $S^{g^{\delta_n}_{\varepsilon_n}(t)}(w_n) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)$ be such that

$$w_n \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

For every $m \in \mathbb{N}$, let us choose k_m in such a way that

$$m\|w - \varphi_{k_m}\|_1 \to 0.$$

By minimality of $v^{h_n,a_n}_{k_m,m}(t)$ we have for all n

$$\|\nabla v^{h_n,a_n}_{k_m,m}(t)\|_p + \|v^{h_n,a_n}_{k_m,m}(t)\|_q + m\|v^{h_n,a_n}_{k_m,m}(t) - \varphi_{k_m}\|_1 \le \|\nabla w_n\|_p + \|w_n\|_q + m\|w_n - \varphi_{k_m}\|_1.$$

By construction of h_n, and in view of (8.66), we have

$$v^{h_n,a_n}_{k_m,m}(t) \rightharpoonup v_{k_m,m}(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

Then passing to the limit in n, by lower semicontinuity we get for some $C \ge 0$

$$\|\nabla v_{k_m,m}(t)\|_p + \|v_{k_m,m}(t)\|_q + m\|v_{k_m,m}(t) - \varphi_{k_m}\|_1 \le C + m\|w - \varphi_{k_m}\|_1.$$

We deduce for $m \to +\infty$

$$\|v_{k_m,m}(t) - \varphi_{k_m}\|_1 \to 0,$$

which together with $\|\varphi_{k_m} - w\|_1 \to 0$ implies that

$$v_{k_m,m}(t) \to w \quad \text{strongly in } L^1(\Omega;\mathbb{R}^2).$$

Since

$$\|\nabla v_{k_m,m}(t)\|_p + \|v_{k_m,m}(t)\|_q \le C + m\|w - \varphi_{k_m}\|_1 \le C + 1$$

for m large, we have that

$$v_{k_m,m}(t) \rightharpoonup w \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

Since $S^{g(t)}(v_{k_m,m}(t)) \subseteq \Gamma(t)$ for all m, and since $\mathcal{H}^1(\Gamma(t)) < C'$, we deduce that $S^{g(t)}(w) \tilde{\subseteq} \Gamma(t)$.

Coming to point (c), we have that (8.63) holds by construction. Moreover (8.62) holds in view of (8.69) and by the fact that $u^{\delta_n}_{\varepsilon_n,a_n}(t)$ weakly converges in $GSBV^p_q(\Omega;\mathbb{R}^2)$ to $u(t)$ defined in (8.67).

In order to prove point (a), notice that if $s \le t$ with $s,t \in D$, we have for all $k,m \in \mathbb{N}$ that

$$S^{g^{\delta_n}_{\varepsilon_n}(t)}(v^{h_n,a_n}_{k,m}(s) + g^{\delta_n}_{\varepsilon_n}(t) - g^{\delta_n}_{\varepsilon_n}(s)) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a_n}(s) \subseteq \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t),$$

and

$$v^{h_n,a_n}_{k,m}(s) + g^{\delta_n}_{\varepsilon_n}(t) - g^{\delta_n}_{\varepsilon_n}(s) \rightharpoonup v_{k,m}(s) + g(t) - g(s) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2),$$

where $v^{h,a}_{k,m}(s)$ and $v_{k,m}(s)$ are defined in (8.65) and (8.66). By point (b) we deduce that

$$S^{g(t)}(v_{k,m}(s) + g(t) - g(s)) = S^{g(s)}(v_{k,m}(s)) \tilde{\subseteq} \Gamma(t).$$

Then by the definition of $\Gamma(s)$ we get $\Gamma(s) \tilde{\subseteq} \Gamma(t)$. Finally, by the same argument, we deduce $\Gamma^0 \tilde{\subseteq} \Gamma(s)$.

In order to deal with all $t \in [0,T]$, we proceed as in Remark 8.5.3. For all $t \in [0,T] \setminus D$ let us set

$$\Gamma^+(t) := \bigcap_{s \ge t, s \in D} \Gamma(s).$$

Clearly $\Gamma^+(t) \in \mathcal{R}(\overline{\Omega}_B;\partial_N\Omega)$ and satisfies point (a), so that the set J of discontinuity points of $\mathcal{H}^1(\Gamma^+(\cdot))$ is at most countable. We can then extract a further subsequence of $(\delta_n,\varepsilon_n,a_n)_{n\in\mathbb{N}}$ such that $\Gamma(t)$ is determined also for all $t \in J \setminus D$ (notice that $\Gamma(t) \tilde{\subseteq} \Gamma^+(t)$). For all $t \in [0,T] \setminus (D \cup J)$ we set $\Gamma(t) := \Gamma^+(t)$. We have that $\Gamma(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N \Omega)$ and that $\Gamma(t)$ satisfies point (a). Let us see that $\Gamma(t)$ satisfies also points (b), (c) and (d) also for $t \in [0,T] \setminus (D \cup J)$.

Concerning point (b), for every accumulation point $u(t)$ of $(u^{\delta_n}_{\varepsilon_n,a_n}(t))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV^p_q(\Omega;\mathbb{R}^2)$, by the first part of the proof, we have that $S^{g(t)}(u(t)) \tilde{\subseteq} \Gamma(s)$ for all $s \in D$ with $s \geq t$, so that passing to the intersection, we get that $S^{g(t)}(u(t)) \tilde{\subseteq} \Gamma(t)$.

Let us come to point (c). Let

$$u_j(t) := u^{\delta_{n_j}}_{\varepsilon_{n_j},a_{n_j}}(t) \rightharpoonup u(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2)$$

along a subsequence $n_j \nearrow +\infty$. Let us set $\Gamma_j := \Gamma^{\delta_{n_j}}_{\varepsilon_{n_j},a_{n_j}}$ and $g_j := g^{\delta_{n_j}}_{\varepsilon_{n_j}}$. Up to a further subsequence there exists $s_j \in D$ with $s_j \nearrow t$, and such that setting $u_j(s_j) := u^{\delta_{n_j}}_{\varepsilon_{n_j},a_{n_j}}(s_j)$, we have

$$\|u_j(s_j) - u(s_j)\|_1 + |\mathcal{E}^b(s_j)(u_j(s_j)) - \mathcal{E}^b(s_j)(u(s_j))| \to 0. \tag{8.71}$$

We have that there exists $u^*(t) \in GSBV^p_q(\Omega;\mathbb{R}^2)$ such that up to a subsequence

$$u(s_j) \rightharpoonup u^*(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2).$$

By the minimality property (8.62) of $u(s_j)$, for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$, we have that

$$\mathcal{E}^b(s_j)(u(s_j)) \leq \mathcal{E}^b(s_j)(v - g(t) + g(s_j)) + \mathcal{E}^s(S^{g(t)}(v) \setminus \Gamma(s_j)).$$

Passing to the limit in j we have that for all $v \in GSBV^p_q(\Omega;\mathbb{R}^2)$ with $S(v) \tilde{\subseteq} \overline{\Omega}_B$

$$\mathcal{E}^b(t)(u^*(t)) \leq \mathcal{E}^b(t)(v) + \mathcal{E}^s(S^{g(t)}(v) \setminus \Gamma(t)). \tag{8.72}$$

As a consequence of the stability of this unilateral minimality property, it follows that

$$\mathcal{E}^b(s_j)(u(s_j)) \to \mathcal{E}^b(t)(u^*(t)).$$

By (8.71) we get

$$u_j(s_j) \rightharpoonup u^*(t) \qquad \text{weakly in } GSBV^p_q(\Omega;\mathbb{R}^2),$$

and

$$\mathcal{E}^b(s_j)(u_j(s_j)) \to \mathcal{E}^b(t)(u^*(t)). \tag{8.73}$$

By (8.50), comparing $u_j(t)$ with $u_j(s_j) - g_j(s_j) + g_j(t)$, taking into account that

$$S^{g_j(s_j)}(u_j(s_j)) \subseteq \Gamma_j(s_j) \subseteq \Gamma_j(t),$$

we obtain

$$\mathcal{E}^b(t)(u_j(t)) \leq \mathcal{E}^b(s_j)(u_j(s_j)) + o_j$$

where $o_j \to 0$ as $j \to +\infty$. Passing to the limit in j we have by (8.73)

$$\mathcal{E}^b(t)(u(t)) \leq \liminf_j \mathcal{E}^b(t)(u_j(t)) \leq \limsup_j \mathcal{E}^b(t)(u_j(t)) \leq \mathcal{E}^b(t)(u^*(t)).$$

By (8.72) we deduce that (8.62) holds. Moreover we have that $\mathcal{E}^b(t)(u(t)) = \mathcal{E}^b(t)(u^*(t))$ and that $\mathcal{E}^b(t)(u(t))$ is independent of the accumulation point $u(t)$. Then we deduce that (8.63) holds.

Finally, concerning point (d), we have that for all $s \in D \cap [0,t[$

$$\liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a_n}(t)) \geq \liminf_n \mathcal{E}^s(\Gamma^{\delta_n}_{\varepsilon_n,a_n}(s)) \geq \mathcal{E}^s(\Gamma(s)),$$

so that letting $s \nearrow t$ we obtain (8.64). The proof is now complete. $\square$

We can now prove Theorem 8.5.1.

PROOF OF THEOREM 8.5.1. Let $(\delta_n, \varepsilon_n, a_n)_{n\in\mathbb{N}}$ and $\{t \to \Gamma(t) \in \mathcal{R}(\overline{\Omega}_B; \partial_N\Omega), t \in [0,T]\}$ be given by Lemma 8.5.4. For all $t \in [0,T]$, let us set

$$u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t), \qquad \Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t).$$

Let us see that it is possible to choose an accumulation point $u(t) \in GSBV^p_q(\Omega;\mathbb{R}^2)$ of $(u_n(t))_{n\in\mathbb{N}}$ such that $\{t \to (u(t),\Gamma(t)) : t \in [0,T]\}$ is a quasistatic growth of brittle cracks in the sense of Dal Maso-Francfort-Toader. Let us set

$$\begin{aligned}\vartheta_n(s) := \langle \partial\mathcal{W}(\nabla u_n(s)), \nabla \dot g_{\varepsilon_n}(s)\rangle \\ - \dot{\mathcal{F}}(s)(u_n(s)) - \langle \partial\mathcal{F}(s)(u_n(s)), \dot g_{\varepsilon_n}(s)\rangle \\ - \dot{\mathcal{G}}(s)(u_n(s)) - \langle \partial\mathcal{G}(s)(u_n(s)), \dot g_{\varepsilon_n}(s)\rangle.\end{aligned}$$

By growth conditions of $\mathcal{W}, \mathcal{F}, \mathcal{G}$ and by (8.49) we have that there exists $\psi \in L^1(0,T)$ such that $\vartheta_n(s) \le \psi(s)$ for all n. Let us consider

$$\vartheta(s) := \limsup_n \vartheta_n(s).$$

By [44, Theorem 5.5 and Lemma 4.11], for every $s \in [0,T]$ there exists $u(s)$ accumulation point of $(u_n(s))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV^p_q(\Omega;\mathbb{R}^2)$ such that

$$\begin{aligned}\vartheta(s) := \langle \partial\mathcal{W}(\nabla u(s)), \nabla \dot g(s)\rangle \\ - \dot{\mathcal{F}}(s)(u(s)) - \langle \partial\mathcal{F}(s)(u(s)), \dot g(s)\rangle \\ - \dot{\mathcal{G}}(s)(u(s)) - \langle \partial\mathcal{G}(s)(u(s)), \dot g(s)\rangle.\end{aligned}$$

Applying Fatou's Lemma (in the limsup version) to (8.51) with $s = 0$, we have that

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \limsup_n \mathcal{E}(0)(u_n(0),\Gamma_n(0)) + \int_0^t \vartheta(s)\,ds.$$

By Proposition 8.3.1, we have that $\limsup_n \mathcal{E}(0)(u_n(0),\Gamma_n(0)) = \mathcal{E}(0)(u(0),\Gamma(0))$, so that we get

$$\mathcal{E}(t)(u(t),\Gamma(t)) \le \mathcal{E}(0)(u(0),\Gamma(0)) + \int_0^t \vartheta(s)\,ds.$$

Moreover, again by [44, Theorem 3.13],

$$\mathcal{E}(t)(u(t),\Gamma(t)) \ge \mathcal{E}(0)(u(0),\Gamma(0)) + \int_0^t \vartheta(s)\,ds,$$

so that

$$\mathcal{E}(t)(u(t),\Gamma(t)) = \mathcal{E}(0)(u(0),\Gamma(0)) + \int_0^t \vartheta(s)\,ds. \tag{8.74}$$

We deduce that $\{t \to (u(t),\Gamma(t)) : t \in [0,T]\}$ is a quasistatic growth of brittle cracks: in fact by Lemma 8.5.4 we get that $\Gamma(\cdot)$ is increasing, and for $t \in [0,T]$ $(u(t),\Gamma(t)) \in AD(g(t))$ and the static equilibrium holds; moreover the nondissipativity condition is given by (8.74).

Let us see that points (a) and (b) of Theorem 8.5.1 holds. By (8.49), $(u_n(t))_{n\in\mathbb{N}}$ is weakly precompact in $GSBV^p_q(\Omega;\mathbb{R}^2)$ for all $t \in [0,T]$. Moreover by Lemma 8.5.4 every accumulation point $\tilde u(t)$ of $(u_n(t))_{n\in\mathbb{N}}$ for the weak convergence in $GSBV^p_q(\Omega;\mathbb{R}^2)$ is such that $S^{g(t)}(\tilde u(t)) \subseteq \Gamma(t)$ and the minimality property (8.52) holds. Moreover we have

$$\mathcal{E}^b(t)(\tilde u(t)) = \lim_n \mathcal{E}^b(t)(u_n(t)).$$

Since $\mathcal{E}^b(t)(\tilde{u}(t))$ is independent of the particular accumulation point $\tilde{u}(t)$, we have that point (a) is proved.

Let us come to point (b). Taking into account (8.63) and (8.64), for all $t \in [0,T]$ we have

$$E(t) \leq \liminf_n E_n(t) \leq \limsup_n E_n(t) \leq E(0) + \int_0^t \vartheta(s)\, ds = E(t),$$

so that (8.53) holds. Moreover we deduce that separate convergence of elastic and surface energies holds at any time, so that (8.54) is proved. The proof is now concluded. ☐

8.6 The strictly convex case

In this section we assume that the function $W(x,\xi)$ is strictly convex in ξ for a.e. $x \in \Omega$ and that the function $F(t,x,z)$ is strictly convex in z for all $t \in [0,T]$ and for a.e. $x \in \Omega$: as a consequence, the elastic energy $\mathcal{E}^b(t,v)$ is strictly convex in v for all $t \in [0,T]$, and a stronger approximation result is available.

Theorem 8.6.1. *Let $g \in W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2))$ and let*

$$g_\varepsilon \in W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)), \qquad g_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega) \quad \text{for all } t \in [0,T]$$

be such that for $\varepsilon \to 0$

$$g_\varepsilon \to g \qquad \text{strongly in } W^{1,1}([0,T], W^{1,p}(\Omega;\mathbb{R}^2) \cap L^q(\Omega;\mathbb{R}^2)).$$

Let $\Gamma^0 \in \Gamma(\Omega)$ be an initial crack and let $\Gamma^0_{\varepsilon,a}$ be its approximation in the sense of Proposition 8.3.1. Let us suppose that

$$\begin{aligned} &W(x,\cdot) \text{ is strictly convex for a.e. } x \in \Omega, \\ &F(t,x,\cdot) \text{ is strictly convex for a.e. } (t,x) \in [0,T] \times \Omega. \end{aligned} \tag{8.75}$$

Given $\delta > 0$, $\varepsilon > 0$, $a \in]0,\frac{1}{2}[$, let $\{t \to (u^\delta_{\varepsilon,a}(t), \Gamma^\delta_{\varepsilon,a}(t)) \,:\, t \in [0,T]\}$ be the piecewise constant interpolation of the discrete evolution given by Proposition 8.4.1 relative to the initial crack $\Gamma^0_{\varepsilon,a}$ and the boundary data g_ε. Then there exists a quasistatic evolution $\{t \to (u(t),\Gamma(t)) \,:\, t \in [0,T]\}$ relative to the initial crack Γ^0 and the boundary data g in the sense of Theorem 8.1.1, and sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $a_n \to 0$, such that setting

$$u_n(t) := u^{\delta_n}_{\varepsilon_n,a_n}(t), \qquad \Gamma_n(t) := \Gamma^{\delta_n}_{\varepsilon_n,a_n}(t),$$

for all $t \in [0,T]$ the following facts hold:

(a) $\nabla u_n(t) \to \nabla u(t)$ strongly in $L^p(\Omega;M^{2\times 2})$ and $u_n(t) \to u(t)$ strongly in $L^q(\Omega;\mathbb{R}^2)$;

(b) $\mathcal{E}(t)(u_n(t),\Gamma_n(t)) \to \mathcal{E}(t)(u(t),\Gamma(t))$, and in particular elastic and surface energies converge separately, that is

$$\mathcal{E}^b(t)(u_n(t)) \to \mathcal{E}^b(t)(u(t)), \qquad \mathcal{E}^s(\Gamma_n(t)) \to \mathcal{E}^s(\Gamma(t)).$$

Proof. Let us consider the sequence $(\delta_n,\varepsilon_n,a_n)_{n\in\mathbb{N}}$ and the quasistatic growth of brittle cracks $\{t \to (u(t),\Gamma(t)) \,:\, t \in [0,T]\}$ given in Theorem 8.5.1. Under assumptions (8.75), we have that $u(t)$ is uniquely determined, because by (8.52) $u(t)$ minimizes

$$\min\{\mathcal{E}^b(t)(v) \,:\, v \in GSBV^p_q(\Omega;\mathbb{R}^2),\ S^{g(t)}(v) \subseteq \Gamma(t)\},$$

and $\mathcal{E}^b(t)(\cdot)$ is strictly convex. We conclude by point (a) of Theorem 8.5.1 that $u_n(t) \rightharpoonup u(t)$ weakly in $GSBV^p_q(\Omega;\mathbb{R}^2)$. Point (b) is a direct consequence of Theorem 8.5.1. By the convergence of the elastic energy, we deduce that

$$\lim_n \int_\Omega W(x, \nabla u_n(t))\, dx = \int_\Omega W(x, \nabla u(t))\, dx,$$
$$\lim_n \int_\Omega F(t, x, u_n(t))\, dx = \int_\Omega F(t, x, u(t))\, dx.$$

By the assumption on the strict convexity of W and F we deduce by [23]

$$\nabla u_n(t) \to \nabla u(t) \qquad \text{strongly in } L^p(\Omega; M^{2\times 2}),$$

and

$$u_n(t) \to u(t) \qquad \text{strongly in } L^q(\Omega;\mathbb{R}^2).$$

Point (a) is now proved, and the proof is concluded. □

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