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CANDIDATE:
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SUPERVISOR:
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ACADEMIC YEAR 2000-2001

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# Contents

# Introduction

Astroparticle physics and cosmology are experiencing an impressive development. The acquisition of more and more accurate observational data has recalled a great interest on these subjects, that provide a unique source of hints about the physics that lies far beyond the Standard Model. Parallely, the theoretical aspects of cosmology are also in rapid evolution along two main directions. The first one is related to the rise of radically new paradigms in model building, that has opened a wide range of new perspectives. The second is aimed at the achievement of a deeper and more detailed understanding of the "orthodox" picture that we have of the Early Universe.

This thesis is divided into two parts. In the first part, cosmological aspects of *brane models* are discussed. This class of models, in which gravity can propagate in a supplementary space while matter is bound to the usual three dimensional space, has been recently proposed as a solution to the hierarchy problem. However, it has been soon realized that new scenarios arise for cosmology in this framework. In particular, possible modifications of the standard baryogenesis schemes and of the cosmological evolution of the Universe will be discussed.

In the second part, a detailed analysis of *fermion preheating* will be carried out. The stage between the inflationary and the radiation dominated epochs of the early Universe is currently one of the most active areas of research in cosmology. In particular, nonthermal production of matter has been proven to be a substantial ingredient in the analysis of this stage. The phenomenologically most relevant effects related to the production of gravitinos at preheating will be especially highlighted.

# Part I

# Brane models

A large part of the theoretical activity of the last two decades has been motivated by the attempt to explain the large hierarchy between the gravity and the electroweak scale.

For a long time, together with technicolor, supersymmetry has been considered by far the preferred candidate for the solution of the *hierarchy problem* [1]. However, in the last few years, the scenario of large extra dimensions has recalled a wider and wider attention in this respect.

The idea that our world could have more than three spatial dimensions dates back to the twenties, when Kaluza [2] and Klein [3], trying to unify gravity and electromagnetism, postulated the existence of one extra compact dimension. Extra dimensions were revived more recently, when string theory (that has to be formulated in 10 or 11 dimensions) started to be appreciated as the most serious candidate for a consistent quantum theory of gravity. Until few years ago, however, the string scale was supposed to lie close to the observed Planck mass. The solution to the hierarchy problem was still left to low energy supersymmetry.

Developments in string theory also led, later on, to a qualitative change in the way extra dimensions were conceived. Indeed, the idea of D(irichlet)-branes played a key role in the success of models with large extra dimensions, that are also known as *brane models*. A D-brane is a dynamical object on which the ends of an open string have to be attached (for a review, see [4]). The existence of such objects allows to construct theories in which different fields can be localized on different manifolds of different dimensionality.[1]

This possibility has been exploited by Arkani–Hamed, Dimopoulos and Dvali in ref. [6] to suggest that the Standard Model degrees of freedom are constrained to a three-dimensional hyperplane (generically called *brane*), while gravity can propagate in a higher dimensional compact space (for earlier proposals in this direction, see [7, 5, 8]). This $n$-dimensional extra space, of volume $V_n$, is usually called *bulk*. The motivation for this setup is that, while the Standard Model interactions have been probed down to scales of the order of $\mathrm{TeV}^{-1}$, the Newton law has been tested only for distances larger than 1 mm. Therefore, while in this scenario the thickness of the brane cannot exceed $\mathrm{TeV}^{-1}$, the compactification radius[2] $\sim (V_n)^{1/n}$ of the extra dimensions can be as large as 1 mm. This situation allows to look at the hierarchy problem from a completely different perspective. Indeed, the observed Planck mass in this scenario will be given by

$$M_P^2 = M^{2+n}\, V_n\,.$$

Here $M$ is the so called fundamental Planck mass, that is the one that appears in the lagrangian of the full $(4+n)$-dimensional gravity. From the above relation it follows

[1]Field-theoretical mechanisms for localizing fields on topological defects were discussed in the earlier literature [5], but they received comparatively much less attention.

[2]We assume isotropic compactification for simplicity.

that it is possible to observe a very large Planck mass even with a relatively low value of $M$, provided $V_n$ is large enough. In particular, the hierarchy problem could seem to be solved by simply pushing $M$ down to the electroweak scale $M_{ew} \simeq$ TeV. By setting $M_P \simeq 10^{19}$ GeV, $M \simeq 1$ TeV, one gets

$$(V_n)^{1/n} \simeq 10^{-17+30/n}\,\text{cm}\,.$$

We see that $n = 1$ is phenomenologically unacceptable, since the compactification radius in this case is of the order of the size of the Solar System. For $n \geq 2$ , instead, the deviation from four dimensional gravity occurs at distances below the current experimental bound. The case $n = 2$ in particular raised considerable interest, since the compactification radius $(V_2)^{1/2} \simeq 100\,\mu$m is going to be probed experimentally in the near future by means of short-distance tests of the Newton law.

However, we cannot say that the scenario described above truly solves the hierarchy problem. Actually, it only allows a rephrasing of the problem in different terms. The magnitude of the compactification volume $V_n$ should indeed arise from the dynamics of the system. In units of the only dimensionful scale in the lagrangian of the model, i.e. in units of $M$, the compactification radius reads

$$(V_n)^{1/n}\, M \simeq \left(\frac{M_P}{M}\right)^{2/n} \simeq 10^{32/n} \gg 1\,.$$

As a consequence, in the context of large extra dimensions the hierarchy problem is simply restated in geometrical terms: how can the dynamics of the system generate a compactification radius so large with respect to the fundamental lenght of the theory?

One year after the proposal of [6], Randall and Sundrum [9] analyzed a model that could lead to the generation of a large hierarchy starting from only one small extra dimension. In the Randall-Sundrum model the metric is exponentially *warped* as we move along the extra dimension. This configuration is achieved by assuming a negative cosmological constant in the bulk. The latter turns out to be a slice of Anti-de Sitter space, bounded by two branes, one with positive and one with negative tension. The tensions are suitably tuned in order to make the whole system stable. The zero mode of the graviton turns out to be trapped at the position of the positive tension brane, while the Standard Model fields are assumed to be localized on the negative tension brane. Therefore, the zero mode of the graviton has a very small overlap with our brane, thus leading to the observed weakness of gravity. This model provides a solution to the hierarchy problem in the sense that, due to the exponential warp factor, the compactification radius has to be only one order of magnitude larger than the fundamental lenght in order to generate the hierarchy $M_P/M_{ew} \simeq 10^{16}$. Unfortunately, at variance with the scenario of ref. [6], it is difficult to embed the Randall-Sundrum model in a string-theoretical context.

The Randall-Sundrum setup lent itself to a further development [10]. Moving the negative tension brane to the infinity, we obtain a system consisting of a positive tension brane (on which the Standard Models is supposed to be constrained) and of an extra space of infinite extension. The zero mode of the graviton turns out to be localized on the brane, while its excited modes decouple at low energies. Although this system has almost nothing to say about the hierarchy problem, it raised a considerable interest. First of all, it shows that we could even live with an infinite extra dimension without knowing it. Moreover, it provides the background for the construction of models with several interesting cosmological consequences.

Turning to the phenomenological implications, the distinctive features of these scenarios are obviously related to gravity. For what concerns acceletor tests, experimental bounds come from the possible emission of Kaluza-Klein gravitons in high energy collisions or in virtual graviton exchange (see for instance [11]). A lower bound on $M$ ranging from several hundreds of GeVs to few TeVs is in general imposed by experiments.

The best testing grounds for the phenomenology of large extra dimension scenarios are however cosmology and astrophysics. Astrophysics provides the most restrictive bound on the fundamental Planck mass. In the case of $n = 2$ extra dimensions, this quantity has to be larger than about 31 TeV [12], if we do not want the emission of bulk gravitons to conflict with the bound on energy loss by SN1987A.[3]

Cosmology finds its range of possibilities much restricted as we move to the context of models with low scale gravity. For instance, phenomena like baryogenesis or inflation are usually assumed to take place at very high energy scales. In this new setting, only the range of energies below TeV is available to account for such phenomena. Moreover, in the scenario à la Arkani–Hamed-Dimopoulos-Dvali, even stronger bounds on the reheating temperature have to be imposed not to overproduce Kaluza-Klein modes of the graviton. As a consequence, in the case of two large extra dimensions it is difficult even to accommodate nucleosynthesis in the early cosmology of the model. On the other hand, brane models stimulated an extremely large activity in the cosmological aspects connected to General Relativity. Indeed, as first pointed out in [13], the cosmological evolution of brane models requires a detailed analysis, and in general it is not even expected that the late cosmology of these models is characterized by the usual Friedmann law $H^2 \propto \rho$. Later on, it was shown [14, 15] that, at least in models where the compact extra space is stabilized by some suitable mechanism, standard late expansion law is expected. Nevertheless, models with noncompact extra dimensions generally show some stage

[3]This limit implies that the radius of the extra dimensions, in case of isotropic compactification, has to be smaller than about $0.7\,\mu$m. As a by-product, the above bound will make extremely difficult the observation of any deviation from four-dimensional gravity in future short-distance tests of the Newton law.

of nonstandard cosmology [16, 17, 18, 19, 20, 21] and might provide some hint about the cosmological constant problem (see for instance [22, 23, 24, 25, 26, 27]). Finally, the evolution of cosmological perturbations in such scenarios is a very interesting matter not still completely settled [28, 29, 30, 31].

In the first chapter we will shortly review the reasons that led to the birth and the great success of brane models, focusing in particular on the motivations related to the hierarchy problem. We will outline the features of the two main scenarios that have been envisaged in this context, namely the Arkani–Hamed-Dimopoulos-Dvali proposal with large flat extra dimensions and the warped compactification of the Randall-Sundrum model. In the subsequent chapter we will be mainly interested in the cosmological aspects of brane models related to particle physics. In this chapter we will describe in some detail the problems that arise when dealing with cosmological issues in models with large extra dimensions, before moving to the description of an intrinsically higher dimensional scenario of baryogenesis. Finally, in chapter 3, we will consider how the cosmological aspects of General Relativity have to be reconsidered in the brane scenario. In particular, we will discuss the cosmological evolution of the compact Randall-Sundrum model, showing the relevance of the stabilization of the radius in this context and also showing how the model could potentially lead to nonstandard early cosmology.

# Chapter 1

# Brane models

There is a strong hierarchy between the three fundamental mass scales that appear in the description we currently have of our Universe. The Planck mass $M_P$ that governs gravitational interactions is about 16 orders of magnitude larger than the electroweak scale $M_{ew}$, that sets the scale of the masses of the known particles. The latter is about 15 orders of magnitude larger than the scale of the cosmological constant (provided it is actually nonvanishing, as observations seem to indicate), whose nature is still unknown.

In the last twenty years a relevant part of the theoretical activity has been devoted to the attempt to explain the first of these hierarchies, that is the smallness of the ratio $M_{ew}/M_P \simeq 10^{-16}$. The problem can be divided into two parts. The first question is why in Nature there should be such small numbers. In different words, the problem is how to get numbers as small as $10^{-16}$ starting from numbers of the order of one, that should be preferred in a unified picture of the world. The second problem is more technical, the question being how this large hierarchy can be protected against radiative corrections. Indeed, the Standard Model presents a physical cut-off $M_P$. This fact, together with the presence of quadratic divergences in the one-loop contribution to the Higgs mass, should push the latter to the Planck scale, thus destabilizing the fine-tuning of $M_{ew}/M_P$. Hence, this fine-tuning should be re-established after each order of radiative corrections.

Two scenarios have been contemplated for a long time as possible solutions to the hierarchy problem. In technicolor models (for a recent review, see [32]) the Higgs scalar is actually an effective degree of freedom describing a fermion condensate. Fermion condensation is a nonperturbative phenomenon and the scale of condensation is exponentially suppressed with respect to the fundamental scale. This is supposed to generate the hierarchy, that is kept stable under radiative corrections by chirality. Supersymmetry (for a review, see for instance [33]), for several reasons, has been considered the most favoured candidate to the solution of the hierarchy problem. In this context every particle with integer spin has a *superpartner* of half-integer spin. This leads to cancellations in the radiative corrections, that in this case do not spoil the fine-tuning of $M_{ew}/M_P$. Its origin in the bare lagrangian is however left to some nonperturbative phenomenon, such as gaugino condensation.

In this chapter we will review the properties of brane models. We will focus in particular on the Arkani–Hamed-Dimopoulos-Dvali (in section 1.1) and (in section 1.2) on the compact Randall-Sundrum scenarios. These are indeed the two most relevant frameworks for the solution of the hierarchy problem by means of the extra dimensions. The phenomenological properties of these two scenarios will also be quickly outlined in this chapter, while their cosmology, that is the main topic of this part of the present work, will be discussed in more detail in the next two chapters.

## 1.1 Large extra dimensions

In ref. [6] Arkani-Hamed, Dimopoulos and Dvali proposed a drastic change in the attitude towards the hierarchy problem. The smallness of the ratio $M_{EW}/M_P$ is an apparent effect of our ignorance about the gravitational interactions at short distances. In this new framework the value $M$ of the Planck mass that appears in the fundamental lagrangian is not much larger than the electroweak scale. Hence, the hierarchy problem is simply solved by nullification. The observed weakness of gravity arises from the topology of the spacetime. The latter is assumed to be of the kind $\mathbb{R}^4 \times M_n$, $M_n$ being a $n$-dimensional compact manifold (the *bulk*) of volume $V_n$, spanned by the coordinates $y_1, \ldots, y_n$. The internal manifold is usually assumed to be a $n$-torus with equal radii, but different alternatives have been discussed in the literature, see for instance [34, 35]. The Einstein-Hilbert action of the full theory reads

$$\mathcal{S}_{\rm EH}^{(4+n)} = -\frac{M^{2+n}}{2} \int d^4x\, d^ny\, \sqrt{-g^{(4+n)}}\, R^{(4+n)} \,. \tag{1.1}$$

In the low energy regime (that is, for distances larger than the compactification radius $\sim (V_n)^{1/n}$) we obtain the four dimensional effective action after integration over the $y$ coordinates

$$\mathcal{S}_{\rm eff}^{(4)} = -\frac{M^{2+n}\, V_n}{2} \int d^4x\, \sqrt{-g^{(4)}}\, R^{(4)} \,, \tag{1.2}$$

that leads to the identification $M_P^2 = M^{2+n}\, V_n$, where $M_P$ is the observed (reduced) Planck mass. As a consequence, the weakness of gravity is therefore due to the largeness of the extra space, where the flux lines of the gravitational field are diluted.

While gravity has not been probed at distances smaller than a millimeter (and therefore $(V_n)^{1/n}$ as large as 1 mm is phenomenologically admissible), the Standard

Model gauge forces have been accurately measured at weak scale distances. Consequently, the Standard Models particles cannot propagate freely in the bulk but have to be localized on a three-dimensional hypersurface (called *brane*) whose thickness cannot exceed $M_{ew}^{-1}$. Both field-theoretical [5, 36, 6] and string-theoretical [4] mechanisms for the localization of spin 0, 1/2 and 1 fields on submanifold exist.

This setup implies new dramatic phenomenological consequences. The most striking is probably that next accelerator experiments would directly probe quantum gravity (string?) effects. Moreover, in some cases we could even hope to find new physics at high energies by means of measurements of gravity. Indeed, at distances of the order of the compactification scale the effective lagrangian (1.2) is no longer a valid approximation, and one should use the full lagrangian (1.1) for the description of gravity. In particular, the Newton force should assume its $(3+n)$-dimensional form $\mathbf{F} \propto r^{-(2+n)}$. For $n = 2$ extra dimensions and $M = 1\,\mathrm{TeV}$, the transition to 5-dimensional gravity occurs at distances of the order of $100\,\mu\mathrm{m}$, that are going to be probed in the future experiments on the gravitational force at short distances.

Of course, also this scenario presents some drawbacks. Since the cutoff of the theory is so close to the electroweak scale, new difficulties arise when one wants to suppress rare processes that are however expected to be mediated by degrees of freedom close to the fundamental scale of gravity. Therefore, the suppression of flavor changing neutral currents [11], or of processes mediating fast proton decay [37, 38, 39], become challenging tasks in this context. Gauge coupling unification is generically lost as well, although some intrinsically higher dimensional way out from this problem has been envisaged [40].

Several bounds to the parameters of this class of models have been set by accelerator experiments. These bounds can arise either from corrections to the electroweak precision observables or from emission of gravitons in collisions. In fact, the spacing of the Kaluza-Klein masses of the graviton is given by the inverse of the compactification radius. Hence, Kaluza-Klein gravitons turn out to be very light, and have large multiplicity: even if the individual gravitons are weakly coupled to brane matter, they will be produced in a sizable amount as the energy of the scattering approaches $M$. Therefore a lower bound on $M$ of the order of few TeVs is imposed by accelerator experiments, see for instance [11].

The most stringent bounds, in the case of $n = 2$ extra dimensions, come from astrophysics. The requirement that the emission a light Kaluza-Klein modes of the graviton does not provide a too efficient cooling channel for SN1987A leads to a conservative lower bound of 31 TeV on the fundamental Planck mass in the case of $n = 2$ and of 2.75 TeV for $n = 3$ extra dimensions [12]. Other severe bounds come from cosmology, as we will discuss in detail in the next chapter.

However, what is probably the least attractive point of models with large extra dimensions lies precisely in the weakness of their motivation. As we have mentioned at the beginning of the present section, the main reason for the introduction of models with large extra dimensions was the solution to the hierarchy problem. It is easy to see, however, that this framework does not lead to a true solution to the hierarchy problem. Rather, the latter is reproposed in a new guise.

In the class models we are describing, the largeness of the observed Planck mass is related to the largeness of the compactification volume $V_n$. When measured in units of the (only) fundamental lenght scale $M^{-1}$ of the theory, the compactification radius turns out to be approximately given by

$$\frac{(V_n)^{1/n}}{M^{-1}} \simeq 10^{32/n} \left(\frac{\mathrm{TeV}}{M}\right)^{2/n} \gg 1\,. \tag{1.3}$$

Naturalness considerations would require the above quantity to be of order of unity. Indeed, in a full description of the system the volume of the internal space should arise from the fundamental lagrangian of the theory. In particular, a sector of the lagrangian should be responsible for the stabilization of the radius. The compactification radius will "naturally" be of the order of magnitude of the parameters of this sector, and this will reintroduce an hierarchy. Certainly, models with large extra dimensions allow to reformulate the problem in new (geometrical) terms, and one can take profit of this change in perspective to find new solutions [40]. Nevertheless, this point represents a good motivation for seeking for a more satisfactory solution to the hierarchy problem in the brane context.

## 1.2 The Randall-Sundrum model

An alternative scheme was proposed by Randall and Sundrum [9], and is based on so-called "warped compactification". In this model, two branes are embedded into an anti-De-Sitter five-dimensional space-time, and all the mass parameters of the five-dimensional action are approximately of the same order of magnitude. However, for moderately large values of the compactification radius (of the order of the 10 times the fundamental scale), a strong hierarchy appears between the effective gravitational scale and the mass scale of the degrees of freedom localized on one of the branes.

The setup of the Randall-Sundrum model is provided by a system with one compact extra dimension parametrized by a coordinate $y \in [-1/2, 1/2]$. The orbifold symmetry identifies points at coordinate $y$ with points at coordinate $-y$, thus effectively leading to a system defined on the segment $0 \leq y \leq 1/2$. Two branes are placed at the extrema of the segment $y = 0$ and $y = 1/2$. Vacuum energies $\Lambda$, $V_0$ and $V_{1/2}$ are included respectively in the bulk and on the two branes.

The classical action describing the above system is given by

$$\begin{aligned} S &= S_{bulk} + S_0 + S_{1/2} \\ S_{bulk} &= -\int d^4x \int_{-1/2}^{1/2} dy \sqrt{-g^{(5)}} \left\{ \frac{M^3}{2} R^{(5)} + \Lambda \right\} \\ S_i &= \int d^4x \sqrt{-g_i^{(4)}} \{\mathcal{L}_i - V_i\} \qquad (i = 0,\, 1/2)\,, \end{aligned} \tag{1.4}$$

where $\mathcal{L}_i$ represent the matter lagrangians on the two branes, $g_i^{(4)}$ is the induced metric at $y = i$ and $V_i$ is the tension of the brane placed at $y = i$.

It is possible to find a static solution to the Einstein equations for the above system in the absence of matter on the branes ($\mathcal{L}_i = 0$). Such static solution can be obtained only provided the following fine-tuning of the vacuum energies in the bulk and on the branes is imposed

$$V_0 = -V_{1/2} = -\frac{\Lambda}{m_0} = 6\, m_0\, M^3\,., \tag{1.5}$$

where $m_0$ is a (for the moment) free parameter with the dimensions of a mass. The solution preserves the four-dimensional Poincaré invariance, but leads to a *nonfactorizable* geometry along the extra dimension. Namely, the metric reads

$$ds^2 = e^{-m_0\, b_0\, |y|}\, \eta_{\mu\nu}\, dx^\mu dx^\nu - b_0^2\, dy^2\,. \tag{1.6}$$

The parameter $b_0$ arises as an integration constant and is arbitrary for the moment. We only have to assume that $b_0$ is somewhat larger than $M^{-1}$, in order to have a field-theoretical description of the bulk. Note that the spacetime in between the two three-branes is simply a slice of an $AdS_5$ geometry.

First of all, we have to identify the observed Planck mass of the four-dimensional effective theory. To this aim, we perturb the static solution (1.6) looking for massless modes of the graviton

$$ds^2 = e^{-m_0\, T(x)\, |y|} \left[\eta_{\mu\nu} + \bar{h}_{\mu\nu}(x)\right] dx^\mu dx^\nu - T^2(x)\, dy^2\,. \tag{1.7}$$

Here, $\bar{h}_{\mu\nu}$ represents tensor fluctuations about Minkowski space and is the physical graviton of the four-dimensional effective theory. The compactification radius $b_0$ will be instead the expectation value of the modulus $T\,(x)$. At the level of the present description, the modulus $T\,(x)$ is a massless mode of the system. However, as will be discussed in chapter 3, it is crucial that the $T$ modulus is stabilized at its vacuum expectation value $b_0$ with a sufficiently large mass.

The four-dimensional effective theory now follows by substituting Eq. (1.7) into the original action. The part containing the curvature scalar now reads

$$\int d^4x \int_{-1/2}^{1/2} dy\, M^3\, b_0\, e^{-2\, m_0\, b_0 |y|}\, \sqrt{-\bar{g}}\, \bar{R} \tag{1.8}$$

where $\overline{R}$ denotes the four-dimensional Ricci scalar made out of $\overline{g}_{\mu\nu}(x) = \eta_{\mu\nu}+\overline{h}_{\mu\nu}(x)$. We can explicitly perform the $y$ integral to obtain a purely four-dimensional action. From this we derive

$$M_P^2 = M^3 b_0 \int_{-1/2}^{1/2} dy\, e^{-2m_0\, b_0\, |y|} = \frac{M^3}{m_0}\left[1-\Omega_0^2\right] . \tag{1.9}$$

where we have defined

$$\Omega_0 \equiv e^{-m_0\, b_0/2} . \tag{1.10}$$

This result tells us that $M_P$ depends only weakly on $b_0$ in the large $m_0\, b_0$ limit. Without introducing an hierarchy between the parameters of the theory we will assume $M \simeq m_0 \simeq M_P$. Although the exponential $\Omega_0$ in eq. (1.9) has very little effect in determining the Planck scale, we will now see that it plays a crucial role in the determination of the masses of the particles bounded to one of the two branes.

Let us consider the action of a massive scalar field $\phi$ of mass $\mu$, localized on the brane placed at $y = 1/2$. Its action will be of the form

$$S_{1/2} \supset \int d^4x \sqrt{-g_{1/2}} \left\{ \frac{1}{2}\, g_{1/2}^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi - \frac{\mu^2}{2}\, \phi^2 \right\} . \tag{1.11}$$

Substituting Eq. (1.6) into this action yields

$$S_{1/2} \supset \int d^4x \sqrt{-\overline{g}}\, e^{-2\, m_0\, b_0} \left\{ \frac{1}{2}\, \overline{g}^{\mu\nu} e^{m_0\, b_0}\, \partial_\mu\phi\, \partial_\nu\phi - \frac{\mu^2}{2}\, \phi^2 \right\} . \tag{1.12}$$

In order to get canonically normalized kinetic terms for the scalar field $\phi$, the latter has to be redefined as

$$\phi \rightarrow e^{-m_0\, b_0/2}\, \phi . \tag{1.13}$$

After the above wave function renormalization, we obtain the effective four dimensional action of the scalar field as follows

$$S_{eff} \supset \int d^4x \sqrt{-\overline{g}} \left\{ \frac{1}{2}\, \overline{g}^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi - e^{-m_0\, b_0}\, \frac{\mu^2}{2}\, \phi^2 \right\} . \tag{1.14}$$

We thus see that the observed mass of the field $\phi$ is given by $e^{-m_0\, b_0/2}\, \mu$. This result is completely general: on the brane placed at $y = 1/2$, any mass parameter $m$ in the fundamental higher-dimensional theory will correspond to a physical mass

$$m_{\rm phys} \equiv e^{-m_0\, b_0/2}\, m \tag{1.15}$$

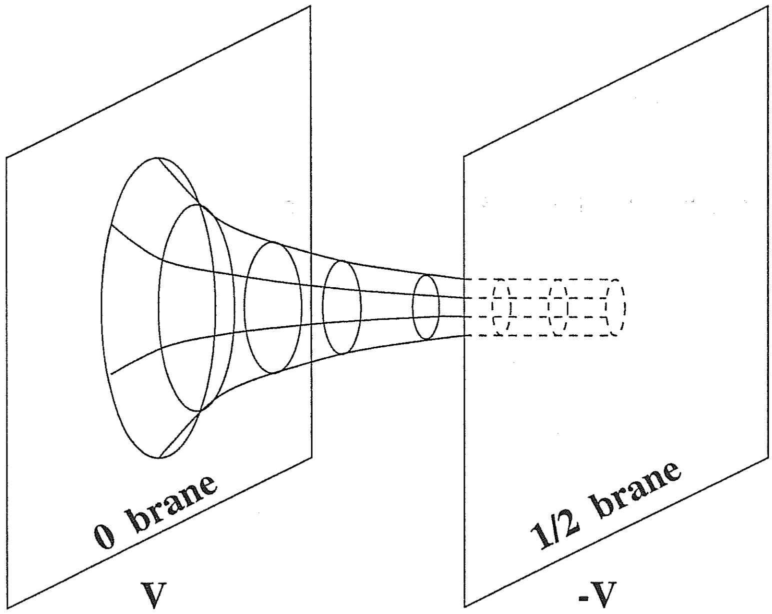


Figure 1.1: Schematic representation of the Randall-Sundrum static configuration.

when measured with the metric $\bar{g}_{\mu\nu}$, which is the metric that appears in the effective Einstein action. The origin of this rescaling resides in the fact that the redefinition (1.13) is actually related to a conformal transformation of the metric at the coordinate $y = 1/2$. Hence, all the mass scales, being related to operators that are not conformally invariant, turn out to be rescaled according to eq. (1.15).

Now, if $e^{-m_0\, b_0/2}$ is of the order of $10^{-16}$, this mechanism produces TeV physical mass scales even if the fundamental mass parameters in the lagrangian are not far from the Planck scale. Being this geometric factor an exponential, we can generate the hierarchy $M_{ew}/M_P \simeq 10^{-16}$ without a large hierarchy in the fundamental mass parameters $m_0 \simeq M \simeq \mu \simeq M_P$, $m_0\, b_0 \simeq 70$.

Notice that fields located the brane placed at $y = 0$ are not rescaled, as the *warp factor* $\exp\left(-m_0\, b_0\, |y|\right)$ evaluates to unity on the $y = 0$ brane. Therefore, if the Randall-Sundrum model has to solve the hierarchy problem, the Standard Model degrees of freedom have to be localized on the negative tension brane. The mass of the fields that are localized on the positive tension brane is naturally of the order of the observed Planck mass also from the point of view of the observer on the $y = 1/2$ brane, that interacts with them only gravitationally.

Regarding the Kaluza-Klein modes of the graviton, this setting gives rise to a new phenomenology. Indeed, it can be shown [10] that the scale of the mass of the first Kaluza-Klein graviton is of the order of $\Omega_0\, m_0 \simeq \text{TeV}$. On the other hand, its coupling to matter on the $y = 1/2$ brane is suppressed only by the TeV scale. The behavior described above is completely different from the scenario of large extra dimensions described in section 1.1, where the Kaluza-Klein mass splittings are much smaller than the weak scale and the coupling of each Kaluza-Klein mode is suppressed by the observed Planck mass.

Therefore, regarding accelerator signatures, the Randall-Sundrum model should be characterized by the emission of single graviton resonances with weak-scale coupling, whereas the Arkani–Hamed-Dimopoulos-Dvali scenario features the emission of a large multiplicity of gravitationally coupled Kaluza-Klein gravitons.

Turning to astrophysical and cosmological constraints, the Randall-Sundrum framework does not suffer from any of the stringent bounds of models with large extra dimensions (some of which will be discussed in the next chapter), that originate from the smallness of the mass of the Kaluza-Klein gravitons.

Finally, we shortly notice how the system described in the previous section has a remarkable behavior as one performs the limit $b_0 \rightarrow \infty$. In this regime the negative tension brane is removed and we are left with a brane embedded in one noncompact extra dimension. However, the Planck mass computed in eq. (1.9) is *finite* in the limit $b_0 \rightarrow \infty$. Actually, one can show [10] that the usual four dimensional Newton law governs the gravitational interaction between two particles on the brane, once their distance is larger than the $AdS$ curvature radius $m_0^{-1}$ in the bulk. Indeed, the potential generated by a particle of mass $\mu_0$ located on the brane is given by the formula

$$V\left(r\right) = \frac{m_0}{8\,\pi\,M^3}\,\frac{\mu_0}{r}\,\left(1 + \frac{1}{m_0^2\,r^2}\right)\,. \tag{1.16}$$

This noncompact Randall-Sundrum model (known as Randall-Sundrum $II$ model), having an infinite extra dimension, has a continuum of Kaluza-Klein modes of the graviton from the four dimensional point of view. However, while the zero mode is bound to the positive tension brane, the latter on the other hand repels the excited modes of the graviton. This is due to the $AdS$ geometry of the extra space. As a consequence, all the modifications to the standard four dimensional gravity are suppressed by powers of $m_0$, that is expected to be of the order of the Planck scale. This model thus provides an existence proof of how it is possible to recover an effective four dimensional standard gravity in a brane model with a *noncompact* extra dimension. Finally, notice however that this scenario does not shed any light on the issue of the hierarchy problem, being the Standard Model necessarily localized on the positive tension brane.

# Chapter 2

# Fermion localization, proton stability and baryogenesis

The early cosmology of models with low scale gravity is significantly different from the standard, four dimensional one. Due to the small scale of the cutoff of the theory, there is a narrow range of energies where phenomena such as inflation or baryogenesis should have taken place. Moreover, in models where the extra dimensions are very large, further strong bounds are imposed from the requirement that the light Kaluza-Klein modes of the graviton are not overproduced in the early stages of the evolution of the Universe. Therefore, the achievement of a phenomenologically acceptable early cosmology is a nontrivial task in this context.

On the other hand, the presence itself of the extra dimensions gives the possibility of radically new approaches to some aspects of cosmology. For instance, once the volume of the extra dimension and the positions of the branes are regarded as dynamical entities, alternative inflationary scenarios [41, 42, 43] can arise. A more challenging obstacle is constituted by the generation of the observed cosmological baryon asymmetry [39, 44, 45, 46].

This last problem is intimately related to the issue of proton stability, that is another problematic aspect of models with low scale cutoff. In fact, in the presence of a cutoff of the order of only few TeVs, baryon number violating operators up to very high dimensions have to be forbidden. Some suggestions in this respect were given in refs. [37, 38, 39]. In this chapter, we will focus on an interesting intrinsically higher dimensional mechanism proposed by Arkani–Hamed and Schmaltz [47]. In this work, a dynamical mechanism for the localization of fermions on the wall [5] is adopted: leptons and quarks are however localized at two slightly displaced positions in the extra space, and this naturally suppresses the interactions which "convert" the latter in the former.

However, the observed baryon asymmetry requires baryon number ($B$) violating interactions to have been effective in the first stages of the evolution of the Universe.

Therefore, we will discuss how this last requirement can be satisfied in a theory which adopts the idea of [47], to ensure proton stability *now* and baryon production *in the past.* The idea is that thermal corrections, which are naturally relevant at early times, may modify the localization of quarks and leptons so to weaken the mechanism that suppresses the $B$ violating interactions.

In the first section we will revise the cosmological bounds that hold in models with large extra dimensions. Then in section 2.2 we will discuss the localization of fermions on the brane and the way it can be relevant for achieving proton stability without strong fine tuning of the parameters. In section 2.3 we will finally discuss how thermal corrections could modify this picture affecting baryon number violating interactions in the early history of the brane, and eventually leading to the generation of a baryon asymmetry.

## 2.1 Cosmological bounds

One of the cornerstones of the Hot Big Bang model is Big Bang Nucleosynthesis, the process of primordial synthesis of the light elements. Theoretical predictions of the abundances of primordial elements are in good agreement with observation once the standard picture for cosmology and for the physics of electroweak and strong interactions is assumed. Big Bang Nucleosynthesis occurred when the temperature of the Universe was of few MeVs. Therefore, every new scenario of gravity and of the electroweak and strong interaction has to reproduce a "normal" Universe for temperatures below this scale. In models with large extra dimensions, the Universe is "normal" when the extra dimensions are essentially frozen and empty of energy density (more subtle issues regarding the existence of a normal expansion law in the late expansion of the Universe will however be discussed in chapter 3).

Following ref. [48], one can define a "normalcy temperature" $T_*$ such that for temperatures below $T_*$ the Universe is "normal". The normalcy temperature can be roughly identified with the reheating temperature after a period of inflation on the brane, and after the stabilization of the radii.

Since the fundamental short distance scale of the theory is $M^{-1}$, we cannot have a field-theoretical description of the system at temperatures larger than $M$. As we will see in this section, however, phenomenological bounds can push $T_*$ to values much lower than $M$.

The peculiarity of cosmology with large extra dimensions is mainly related to the emission of gravitons in the bulk. Since the mass of the Kaluza-Klein modes of the graviton is set by the inverse of the compactification radius, they will be extremely light. In the case of isotropic compactification the mass of the lightest Kaluza-Klein modes ranges from about $10^{-3}\,$eV (for $n = 2$ extra dimensions) to about $10\,$MeV (in the case $n = 6$). Therefore, they will be emitted with a very large multiplicity.

The rate of gravitons emitted into the bulk can be estimated as follows. The coupling of the graviton to matter on the brane is suppressed by an appropriate power of the fundamental Planck mass $1/M^{(n+2)/2}$. Therefore, dimensional analysis leads to the following estimate of the rate of production of $(4+n)$-dimensional gravitons produced per relativistic species ("photons") on the wall:

$$\frac{d}{dt}\frac{n_{grav}}{n_\gamma} = \langle n_\gamma \sigma_{\gamma\gamma\to grav} v\rangle \sim \frac{T^{n+3}}{M^{n+2}}\,. \tag{2.1}$$

Hence, the total number density of gravitons produced during a Hubble time starting at temperature $T_*$ is

$$\frac{n_{grav}}{n_\gamma} \sim \frac{T_*^{n+1}\,M_P}{M^{n+2}} \tag{2.2}$$

Once gravitons escaped into the bulk, they have a very low probability of returning to interact with the SM fields on the wall. The interaction between a bulk graviton and bulk matter can only take place if the graviton is within its Compton wavelength $\sim E^{-1}$ from the wall. The probability that this is the case in extra dimensions of volume $V_n$ is $\sim (E^n\,V_n)^{-1}$. If it is close to the wall, the graviton will decay into photons with a coupling suppressed by $\sim M^{-(n+2)/2}$, and therefore the width is $\sim E^{n+3}/M^{n+2}$ The total width $\Gamma$ is the product of the two factor estimated above

$$\Gamma \sim \frac{E^3}{V_n\,M^{n+2}} \sim \frac{E^3}{M_P^2}\,. \tag{2.3}$$

The above result implies that the gravitons can be very long-lived, since they cannot decay in the empty bulk . The lifetime of a graviton of energy $E$ is

$$\tau(E) \sim \frac{M_P^2}{E^3} \sim 10^{10}\text{yr} \times \left(\frac{100\text{MeV}}{E}\right)^3\,. \tag{2.4}$$

The gravitons produced at temperatures below $\sim$ 100 MeV have lifetimes larger than the present age of the Universe. As a consequence, Kaluza-Klein gravitons could overclose the Universe. Indeed, since the supplementary dimensions do not undergo cosmological expansion, the components of momentum of the bulk gravitons along these dimensions do not redshift. As a consequence, the bulk gravitons emitted at a temperature $T$ will redshift as massive matter with mass $\simeq T$. The energy density stored in the gravitons produced at temperature $T_*$ is

$$\rho_{grav} \sim T_* \times n_{grav} \sim \frac{T_*^{n+5} M_P}{M^{n+2}} \tag{2.5}$$

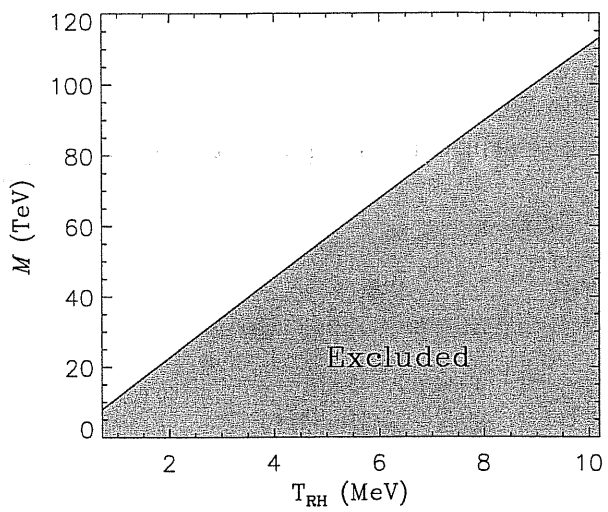


Figure 2.1: Allowed region for the normalcy temperature in the case of $n = 2$ extra dimensions, demanding that the gravitons produced at temperatures lower than $T_{RH}$ do not overclose the Universe. From [50].

which then redshifts as $R^{-3}$, $R$ being the scale factor of the Universe. For the gravitons not to overclose the Universe, we require for critical density at the present age of the Universe. This requires

$$T_* \lesssim 10^{\frac{6n-15}{n+2}} \text{MeV} \left(\frac{M}{\text{TeV}}\right) \tag{2.6}$$

A more detailed analysis leads to the constraint summarized in fig. 2.1. Notice that, for $n = 2$, $M$ has to be pushed to about 8 TeV to obtain $T_* > 0.7$ MeV, that is the minimum reheating temperature required to obtain successful Big Bang Nucleosynthesis [49].

Even stronger bounds come from the late decay of gravitons into photons which would be detected today as distortions of the diffuse photon spectrum. For $T_* \lesssim$ 100 MeV, the graviton lifetime is longer than the age of the universe by $\sim$ (100 MeV$/T_*)^3$, but a fraction $\sim (T_*/100\ \text{MeV})^3$ of them have already decayed, producing photons of energy $\sim T_*$. One can compare the predicted flux of these photons with the observational bound on the diffuse background radiation. A simple estimate gives [48]

$$T_* \lesssim 10^{\frac{6n-15}{n+5}} \text{MeV} \left(\frac{M}{\text{TeV}}\right)^{\frac{n+2}{n+5}} . \tag{2.7}$$

The results of more detailed computation [51, 50] of this bound give the following results, where the requirement $T_* > 0.7$ MeV has been imposed. For $n = 2$ extra dimensions $M$ has to be larger than 73 TeV. The bound reduces to 3.9 TeV for $n = 3$, while for $n \geq 4$ the limits are not significant.

The strongest constraint on the fundamental scale $M$ in models with $n = 2$ or $n = 3$ extra dimensions comes however from supernovae. Indeed, some of the Kaluza-Klein gravitons emitted by *all* the supernovae during the whole history of the Universe should have decayed into photons on the brane. The condition that these photons do not distort the diffuse cosmic gamma-ray background imposes the bounds $M > 84\,\text{TeV}$ for $n = 2$ extra dimensions, $M > 7\,\text{TeV}$ for $n = 3$ [52].

In any case, we can say that cosmology enforces quite strong limits on models with $n = 2$ extra dimensions, once the minimal requirement $T_* > 0.7\,\text{MeV}$ is imposed to allow the achievement of a successful nucleosynthesis. This is a really minimal requirement, since it is extremely difficult to generate the observed baryon asymmetry at such low temperatures.[1] In the next sections we will discuss a scenario that could lead to the generation of the baryon asymmetry. In the discussion of this scenario we will meet all the typical problems that arise when trying to analyze baryogenesis in the context of models with large extra dimensions.

## 2.2 Localization of fermions on a soliton and proton stability

In this section we will review a mechanism for localizing fermions proposed by Rubakov and Shaposhnikov [5]. The original analysis of ref. [5] was carried considering only one extra dimension. For simplicity we will limit ourselves to this simpler case. Generalizations to higher dimensions can be easily performed.

In subsection 2.2.1, a field theoretical mechanism for the localization of fermions on a brane will be reviewed. The brane is realized by the solitonic configuration of a five-dimensional scalar field. In the second subsection we will discuss how it is possible to achieve proton stability in this scenario, by localizing bosons and leptons at two slightly different positions.

### 2.2.1 Fermion localization

Localizing fields in the extra dimension necessitates breaking of higher dimensional translation invariance. This is accomplished in brane models by the presence of one or more branes. While at low energies the brane is usually regarded as an idealized

[1] Notice however that the above bounds apply only to models with very light Kaluza-Klein modes of the graviton. In other brane scenarios [9, 34], where the Kaluza-Klein gravitons are relatively heavy, the reheating temperature can be safely be of the order of the hundreds of GeV, and the best bounds on the scale at which gravity becomes strong are provided by accelerator experiments.

object of vanishing thickness, in this and in the following section we will consider it as an object that is extended also in its transverse direction. In particular, we will consider the thick wall generated by a spatially varying expectation value of a five-dimensional scalar field $\phi(y)$ (in order to preserve four-dimensional translational invariance, the field $\phi$ will be allowed to depend only on the extra coordinate $y$). We assume the expectation value to have the shape of a domain wall transverse to the extra dimension. The scalar potential responsible for the generation of the domain wall will be considered only in the next subsection.

Let us consider the five-dimensional action of a fermion $\psi$ with mass $m_0$ coupled to a scalar field $\phi(y)$. The five dimensional lagrangian reads[2]

$$\begin{aligned} \mathcal{L}_{\phi\psi} &= \bar{\psi}\,(i\,\partial\!\!\!/_5 + \Phi(y))\,\psi\,, \\ \Phi(y) &\equiv \frac{1}{\widetilde{M}_0^{1/2}}\,\phi(y) + m_0\,. \end{aligned} \tag{2.8}$$

We will now show that the Dirac equation for a five dimensional fermion in the background of this scalar field has a zero mode solution which corresponds to a four dimensional chiral fermion stuck at the zero of $\Phi$. A convenient representation for the gamma matrices in five dimensions is

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1,\ldots,3\,, \quad \gamma^5 = -i\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{2.9}$$

The Dirac operator is separable in a $y$-dependent part and a $x$-dependent part (we denote collectively by $x$ the coordinates in the ordinary four dimensions). As a consequence, it is convenient to expand the $\psi$ fields as

$$\psi(x,y) = \sum_n L_n(y)\; P_L\psi_n(x) + \sum_n R_n(y)\; P_R\psi_n(x)\,, \tag{2.10}$$

where the $\psi_n$ are arbitrary four-dimensional Dirac spinors and $P_{L,R} = (1 \pm i\gamma^5)/2$ are chiral projection operators.

We can define the "creation" and "annihilation" operators

$$\begin{aligned} a &= \partial_y + \Phi(y) \\ a^\dagger &= -\partial_y + \Phi(y)\,. \end{aligned} \tag{2.11}$$

In terms of these operators, the eigenvalue equation for $R_n(y)$ and $L_n(y)$ reads

$$\begin{aligned} aa^\dagger R_n(y) = \left(-\partial_y^2 + \Phi^2(y) + \partial_y\Phi(y)\right) R_n(y) &= \mu_n^2 R_n(y) \\ a^\dagger a\, L_n(y) = \left(-\partial_y^2 + \Phi^2(y) - \partial_y\Phi(y)\right) L_n(y) &= \mu_n^2 L_n(y)\,. \end{aligned} \tag{2.12}$$

[2]Since we are considering a five-dimensional system, $\widetilde{M}_0$ and $m_0$ have mass dimension 1, while $\phi$ has dimension 3/2.

The $L_n$ and $R_n$ each form an orthonormal set and for non-zero $\mu_n^2$ are related through $R_n = (1/\mu_n)\, a\, L_n$ as can be verified easily from Eq.(2.12).

Notice however that the eigenfunctions with vanishing eigenvalues $\mu_n = 0$ need not be paired.

The use of the simple harmonic oscillator notation is not accidental. For the special choice $\Phi(y) \propto y$ the operators $a$ and $a^\dagger$ become the usual creation and annihilation operators up to a normalization factor.

Expanding in $R_n\,(y)$ and $L_n\,(y)$ the action for a five dimensional Dirac fermion can be rewritten in terms of a four dimensional action for an infinite number of fermions

$$S = \int \mathrm{d}^4\mathrm{x} \left[ \bar{\psi}_L\, i\partial\!\!\!/_4\, \psi_L + \bar{\psi}_R\, i\partial\!\!\!/_4\, \psi_R + \sum_{n=1}^{\infty} \bar{\psi}_n (i\partial\!\!\!/_4 + \mu_n)\psi_n \right] . \qquad (2.13)$$

The first two terms correspond to four-dimensional component chiral fermions, they arise from the zero modes of Eq.(2.12). The third term describes an infinite tower of Dirac fermions corresponding to the modes with non-zero $\mu_n$ in the expansion.

The zero mode wave functions are easily found by integrating $a^\dagger L_0 = 0$ and $aR_0 = 0$. The solutions

$$L_0\,(y) \sim \exp\left[ -\int_0^y \Phi(s)\mathrm{d}s \right] \quad \text{and} \quad R_0\,(y) \sim \exp\left[ \int_0^y \Phi(s)\mathrm{d}s \right] , \qquad (2.14)$$

are exponentials with support near the zeros of the total mass $\Phi$.

If the extra dimension is infinite in extension, these modes cannot both be normalizable. The left-handed mode will be localized on the brane if, as in figure 2.2.2, one has $\phi\,(y > 0) > 0$ and $\phi\,(y < 0) < 0$. The right handed part remains instead delocalized in the whole space and decouples from the four-dimensional effective theory, in agreement with the index theorem on a solitonic background [53, 54]. The disappearance of the right handed fermions from the low energy regime of the theory is particularly appealing, since the Standard Model fermion content has to be limited only to left handed fields.[3] The right handed fields can also be localized if a kink–antikink solution is assumed for the scalar $\phi$ . As a result, the left fields continue to be localized on the kink, while the right ones are confined to the antikink. If the kink and the antikink are sufficiently far apart, the left handed and right handed fermions however do not interact and again the model reproducing our four dimensional world is built by fermions of a defined chirality. The fermionic content of the full dimensional theory is in this case doubled with respect to the usual one, and observers on one of the two walls will refer to the other as to a "mirror world".

[3] Concerning the cancellation of anomalies on the wall and recovering the Standard Model running of the coupling constants, see [55].

### 2.2.2 Proton stability on the brane

As we said, the localization position of the fermions depends on the vacuum configuration of the field $\phi$. We consider the simplest potential for $\phi$[4]

$$V(\phi) = -\mu_0^2 \phi^2 + \lambda_0 \phi^4 , \tag{2.15}$$

that has a solution the kink configuration

$$\phi(y) = \frac{\mu_0}{\sqrt{2\lambda_0}} \tanh(\mu_0 y) \quad , \tag{2.16}$$

that we approximate it with a straight line interpolating between the two vacua (see figure 2.2.2)

$$\phi(y) \simeq \frac{\mu_0^2}{\sqrt{2\lambda_0}} y \ , \quad |y| < \frac{1}{\mu_0} \tag{2.17}$$

$$\phi(y) \simeq \pm\frac{\mu_0^2}{\sqrt{2\lambda_0}} \ , \quad |y| > \frac{1}{\mu_0} \ , \tag{2.18}$$

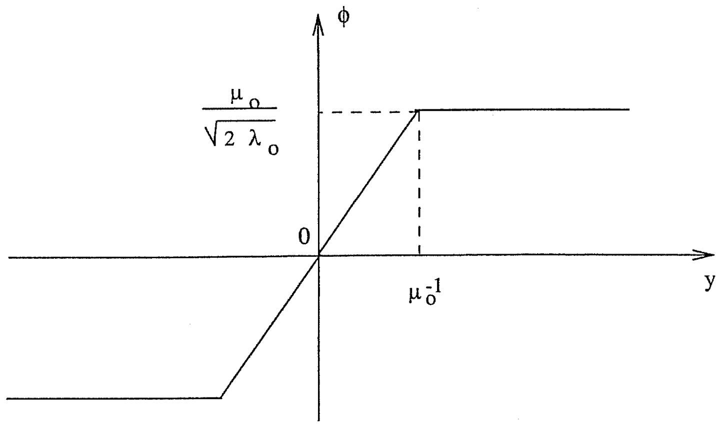


Figure 2.2: Profile of the approximation (2.17) of the kink solution (2.16).

We then see that the localization can occur only if

$$m_0 < \frac{\mu_0}{\sqrt{2\lambda_0 \widetilde{M}_0}} \ , \tag{2.19}$$

since otherwise the total fermion mass

$$\Phi(y) = \frac{1}{\widetilde{M}_0^{1/2}} \phi(y) + m_0 \tag{2.20}$$

[4]Notice that in this five-dimensional potential, $\lambda_0$ has mass dimension $-1$.

never vanishes.

If one chooses different values of the five dimensional bare mass $m_0$ for the different fermionic fields, the latter will be localized at different positions in the fifth direction. As a consequence, the wave functions of different fermions do only partially overlap, and increasing the difference between the five dimensional bare masses of two fermions results in suppressing their mutual interactions.

This idea can be adopted to guarantee proton stability. Let us give, respectively, leptons and baryons the "masses"

$$(m_0)_l = 0 \ , \ \ (m_0)_b = m_0 \ ,$$

which correspond to the localizations [5]

$$y_l = 0 \ , \ \ y_b = \frac{m_0 \sqrt{2\,\lambda_0\,\widetilde{M_0}}}{\mu_0^2} < \frac{1}{\mu_0} \, . \tag{2.21}$$

The shape of the fermion wave functions along the fifth dimension can be cast in an explicit and simple form if we consider the limit $y_b \ll 1/\mu_0$, in which the effect of the plateau for $y > 1/\mu_0$ can be neglected:

$$\begin{aligned} f_l\,(y) &= \left(\frac{\mu_0^2}{\sqrt{2\,\lambda_0\,\widetilde{M_0}\pi}}\right)^{1/4} \exp\left\{-\frac{\mu_0^2\,y^2}{2\sqrt{2\,\lambda_0\,\widetilde{M_0}}}\right\} \\ f_b\,(y) &= \left(\frac{\mu_0^2}{\sqrt{2\,\lambda_0\,\widetilde{M_0}\pi}}\right)^{1/4} \exp\left\{-\frac{\mu_0^2\,(y-y_b)^2}{2\sqrt{2\,\lambda_0\,\widetilde{M_0}}}\right\} \, . \end{aligned} \tag{2.22}$$

We assume the completely general scenario in which the Standard Model is embedded in some theory which contains some (relatively heavy) additional bosons $X$ whose interactions violate baryon number conservation. If it is the case, the $X$ bosons can be integrated out and the four fermion interaction $qq \longleftrightarrow ql$ will effectively described by

$$\int d^4x\, dy\, \frac{q\,q\,q\,l}{\Lambda\, m_X^2} \ , \tag{2.23}$$

where $m_X$ is the mass of the intermediate boson $X$ and $\Lambda$ is a parameter of mass dimension one related to the five-dimensional coupling of the $X$-particle to quarks and leptons.

[5] The last inequality in the next expression comes from (2.19). We assume quarks of different generations to be located in the same $y$ position in order to avoid dangerous FCNC mediated by the Kaluza-Klein modes of the gluons [56].

This scattering is thus suppressed by

$$I = \frac{1}{\Lambda m_X^2} \int dy \frac{\mu_0^2}{\pi \sqrt{2\lambda_0 \widetilde{M}_0}} \exp\left\{ - \frac{\mu_0^2/2}{\sqrt{2\lambda_0 \widetilde{M}_0}} \left[ y^2 + 3\,(y - y_b)^2 \right] \right\} =$$

$$= \frac{\mu_0}{\Lambda m_X^2 \sqrt{2\pi} \left(2\lambda_0 \widetilde{M}_0\right)^{1/4}} \exp\left\{ - \frac{3\left(2\lambda_0 \widetilde{M}_0\right)^{1/2}}{8} \frac{m_0^2}{\mu_0^2} \right\} . \tag{2.24}$$

Current proton stability requires $I \lesssim (10^{16}\,\mathrm{GeV})^{-2}$ , that is

$$\frac{m_0}{\mu_0} \gtrsim \frac{\sqrt{200 - 6\,\mathrm{Log}_{10}\left(\frac{\Lambda m_X^2}{\mu_0} \Big/ \mathrm{GeV}^2\right)}}{\left(2\lambda_0 \widetilde{M}_0\right)^{1/4}} . \tag{2.25}$$

The numerator in the last equation is quite insensitive to the mass scales of the model, and – due to the logarithmic mild dependence – can be safely assumed to be of order 10. For definiteness, we will thus fix it at the value of 10 in the rest of our work.

Conditions (2.19) and (2.25) give altogether

$$\frac{10\,\mu_0}{\left(2\lambda_0 \widetilde{M}_0\right)^{1/4}} \lesssim m_0 \lesssim \frac{\mu_0}{\left(2\lambda_0 \widetilde{M}_0\right)^{1/2}} , \tag{2.26}$$

that we can rewrite

$$2\lambda_0 \widetilde{M}_0 \lesssim 10^{-4} \tag{2.27}$$

$$\frac{m_0}{\mu_0} \gtrsim 10^2 . \tag{2.28}$$

We conclude that in this scenario baryon number violating interactions can be safely suppressed by assuming a mild fine-tuning (of the order of $10^{-4}$) among the fundamental parameters of the model.

## 2.3 Baryon number violation in the early Universe

In this section we will analyze the effects of temperature on the configuration outlined above. In the first subsection the thermal effects on the system are analyzed in generality. In the second subsection we discuss the consequences of such effects for the purposes of baryogenesis.

### 2.3.1 Thermal correction to the coefficients

Once the localization mechanism is incorporated in a low energy effective theory – as the system of eqs. (2.8) and (2.15) may be considered –, one can legitimately ask if thermal effects could play any significant role. In the present section we are mainly interested in any possible change in the argument of the exponential in eq. (2.24), that will be the most relevant for the purpose of baryogenesis. For this reason, we introduce the dimensionless quantity

$$a(T) = \frac{m(T)^2}{\mu(T)^2}\sqrt{2\,\lambda(T)\widetilde{M}(T)} \quad . \tag{2.29}$$

From eqs.(2.25) and (2.27), we can set $a(0) \gtrsim 100$ at zero temperature. Thermal effects will modify this value. There are however some obstacles that one meets in evaluating the finite temperature result. Apart from the technical difficulties arising from the fact that the scalar background is not constant, one problem is that nonperturbative effects may play a very relevant role at high temperature. As it is customary in theories with extra dimensions, the model we are examining is nonrenormalizable and one expects that there is a cut-off (generally related to the fundamental scale of gravity) above which it stops holding. Our considerations will thus be valid only for low temperature effects, and may only be assumed as a rough indication for what can happen at higher temperature.

Being aware of these problems, by looking at the dominant finite-temperature one-loop effects, we estimate the first corrections to the relevant parameters to be

$$\begin{aligned}
\lambda\,(T) &= \lambda_0 + c_\lambda\,\frac{T}{\widetilde{M}_0^2} && (2.30)\\
\widetilde{M}\,(T) &= \widetilde{M}_0 + c_{\widetilde{M}}\,T && (2.31)\\
m\,(T) &= m_0 + c_m\,\frac{T^2}{\widetilde{M}_0} && (2.32)\\
\mu^2\,(T) &= \mu_0^2 + c_\mu\,\frac{T^3}{\widetilde{M}_0}\ , && (2.33)
\end{aligned}$$

where the $c$'s are dimensionless coefficients whose values are related to the exact particle content of the theory.

In writing the above equations, the first of conditions (2.27) has also been taken into account. For example, both a scalar and a fermionic loop contribute to the thermal correction to the parameter $\lambda_0$. While the contribution from the former is of order $\lambda_0^2\,T$, the one of the latter is of order $T/\widetilde{M}_0^2$ and thus dominates.

Substituting eqs. (2.30) into eq. (2.29), we get, in the limit of low temperature,

$$a\,(T) \simeq a\,(0)\cdot\left[1 + \frac{T}{\widetilde{M}_0}\left(\frac{c_\lambda}{2\,\lambda_0\,\widetilde{M}_0} + \frac{c_{\widetilde{M}}}{2} + \frac{2\,c_m\,T}{m_0} - \frac{c_\mu\,T^2}{\mu_0^2}\right)\right] \quad . \tag{2.34}$$

From the smallness of the quantity $\lambda_0\,\widetilde{M_0}$ [see cond. (2.27)] we can safely assume (apart from high hierarchy between the $c$'s coefficients that we do not expect to hold) that the dominant contribution in the above expression comes from the term proportional to $c_\lambda$.

We thus simply have

$$a\,(T) \simeq a\,(0)\left(1 + c_\lambda\,\frac{T}{2\,\lambda_0\,\widetilde{M_0^2}}\right)\ . \tag{2.35}$$

We notice that the parameter $c_\lambda$, being related to the thermal corrections to the $\phi^4$ coefficient due to a fermion loop, is expected to be *negative* [57]: the first thermal effect is to decrease the value of the parameter $a(T)$, making hence the baryon number violating reactions more efficient at finite rather than at zero temperature.

### 2.3.2 Baryogenesis

We saw in the previous subsection that thermal effects may increase the rate of baryon number violating interactions of our system. This is very welcome, since a theory which never violates baryon number cannot lead to baryogenesis and thus cannot reproduce the observed Universe. Anyhow baryon number violation is only one of the ingredients for baryogenesis, and the aim of this subsection is to investigate how the above mechanism can be embedded in a more general context.

A particular scheme which may be adopted is baryogenesis through the decay of massive bosons $X$. [6] This scheme closely resembles GUT baryogenesis, but there are some important peculiarities due to the different scales of energy involved. In GUT baryogenesis the massive boson $X$, coupled to matter by the interaction $g\,X\,\psi\,\bar{\psi}$, has the decay rate

$$\Gamma \simeq \alpha\, m_x \ \ ,\ \ \alpha = \frac{g^2}{4\,\pi}\ . \tag{2.36}$$

An important condition is that the $X$ boson decays when the temperature of the Universe is below its mass (out of equilibrium decay), in order to avoid thermal regeneration. From the standard equation for the expansion of the Universe,

$$H \simeq g_*^{1/2}\,\frac{T^2}{M_p} \tag{2.37}$$

(where $g_*$ is the number of relativistic degrees of freedom at the temperature $T$), this condition rewrites

$$m_X \gtrsim g_*^{-1/2}\,\alpha\, M_p\ . \tag{2.38}$$

[6] We may think of these bosons as the intermediate particles which mediate the four fermion interaction described by the term (2.23).

If $X$ belongs to a Higgs sector, $\alpha$ can be as low as $10^{-6}$. Even in this case however the $X$ boson must be very massive. In principle this may be problematic in the theories with extra dimensions we are interested in, which have the main goal of having a very low fundamental scale.

There are some possibilities to overcome this problem. One is related to a possible deviation of the expansion of the Universe from the standard behavior. As we will discuss in detail in the next chapter, the issue of the expansion law of the brane Universe is far from being trivial. For instance, in the noncompact Randall-Sundrum model, a faster expansion rate $H \propto \rho$ rather than the standard $H \propto \sqrt{\rho}$ is expected at high energies.

However, for definiteness we will consider from now on the case of a standard Friedmann law, and discuss alternative solutions for the out of equilibrium problem. One very natural possibility is to create the $X$ particles non thermally and to require the temperature of the Universe to be always smaller than their mass $m_X$. In this way, one kinematically forbids thermal regeneration of the $X$ particles after their decay. In addition, although interactions among these bosons can bring them to thermal equilibrium, chemical equilibrium cannot be achieved.

As we will discuss in the second part of the present work, nonthermal creation of matter can be very efficient during the stage of coherent oscillations of the inflaton field after inflation [58]. Since this process is extremely model dependent, we will simply assume that, after inflation, their number density is $n_X$. To simplify our computations, we will also suppose that their energy density dominates over the thermal bath produced by the perturbative decay of the inflaton field.

Just for definiteness, let us consider a very simple model where two species of $X$ boson can decay into quarks and leptons, according to the four dimensional effective interactions

$$g\,X\,\bar{q}\,\bar{q} \quad , \quad g\,e^{-a/4}\,X\,l\,q \ , \tag{2.39}$$

where (remember the suppression given by the different localization of quarks and leptons) the quantity $a$ is defined in eq. (2.29). Again for definiteness we will consider the minimal model where no extra fermionic degrees of freedom are added to the ones present in the Standard Model. Moreover we will assume $B-L$ to be conserved, even though the extension to a more general scheme can be easily performed.

The decay of the $X$ bosons will reheat the Universe to a temperature that can be evaluated to be

$$T_{\rm rh} \simeq \left(\frac{30}{\pi^2}\,\frac{m_X\,n_X}{g_*}\right)^{1/4} \ . \tag{2.40}$$

Since we do not want the $X$ particles to be thermally regenerated after their decay, we require $T_{\rm rh} \lesssim m_X$, that can be rewritten as an upper bound on $n_X$

$$n_X \lesssim 30 \left(\frac{g_*}{100}\right) m_X^3 \ . \tag{2.41}$$

Another limit comes from the necessity to forbid the $B$ violating four fermion interaction (2.23) to erase the $B$ asymmetry that has been just created by the decay of the $X$ bosons. We thus require the interaction (2.23) to be out equilibrium at temperatures lower than $T_{\rm rh}$. From eq. (2.24) we see that we can parametrize the four fermion interaction with a coupling $g^2\, e^{-3\,a/8}/m_X^2$. Hence, the out of equilibrium condition reads

$$g^4\, e^{-3\,a/4} \lesssim g_* \frac{m_X}{M_p} \left(\frac{m_X}{T_{\rm rh}}\right)^3 \ . \tag{2.42}$$

One more upper bound on the reheating temperature comes from the out of equilibrium condition for the sphalerons. This requirement is necessary only if one chooses the theory to be $B-L$ invariant, while it does not hold for $B-L$ violating schemes. We can approximately consider the sphalerons to be in thermal equilibrium at temperatures above the electroweak scale. Thus, if $B-L$ is a conserved quantity, we will require the reheat temperature to be smaller than about 100 GeV.

If one neglects the presence of the thermal bath prior to the decay of the $X$ bosons, the very first decays will be only into couples of quarks, since the channel into one quark and one lepton is strongly suppressed by the $e^{-a(T=0)}$ factor due to the fact that the kink is not modified by any thermal correction. However, the decay process is not an instantaneous event. As it is discussed in ref. [59], if the particles produced in the very first decays thermalize very rapidly, they will create a thermal bath even when most of the energy density is still stored in the decaying particles. The temperature of this bath can even be considerably higher than the final reheating temperature. The presence of the heat bath modifies in turn the shape of the kink, as shown in the previous section, and we can naturally expect that this modification enhances the $B$ violating interaction.

If the energy density of the Universe is dominated by the $X$ bosons before they decay, one has

$$\eta_B \simeq 0.1\ (N_X\, T_{\rm rh}/m_X)\ \langle r-\bar{r}\rangle \ , \tag{2.43}$$

where $N_X$ is the number of degrees of freedom associated to the $X$ particles and $\langle r-\bar{r}\rangle$ is the difference between the rates of the decays $X \to q\, l$ and $\bar{X} \to \bar{q}\, \bar{l}$.

We denote with $X_1$ and $X_2$ the two species of bosons whose interactions (2.39) lead to baryon number violation, and parametrize by $\epsilon$ the strength of CP-violation in these interactions. Considering that $e^{-2a}$ is always much smaller than one, we get [60]

$$\langle r - \bar{r} \rangle \sim 3\, g^2\, e^{-a/2}\, \epsilon\, \mathrm{Im}\, \mathrm{I}_{SS}\, (M_{X_1}/M_{X_2}) \ , \tag{2.44}$$

where the function $\mathrm{Im}\, \mathrm{I}_{SS}(\rho) = [\, \rho^2 \, \mathrm{Log}(1 + 1/\rho^2) - 1\,] \, / \, (16\, \pi)$ can be estimated to be of order $10^{-3} - 10^{-2}$. It is also reasonable to assume $\epsilon \sim 10^{-2} - 1$.

Collecting all the above estimates, and assuming $N_X$ to be of order 10, we get

$$\eta_B \simeq \left(10^{-5} \ - \ 10^{-2}\right)\, g^2\, \frac{T_{\mathrm{rh}}}{m_X}\, e^{-a(T_{\mathrm{rh}})/2} \ . \tag{2.45}$$

From the requirement $T_{\mathrm{rh}} \lesssim m_X$ we get an upper limit on the baryon asymmetry

$$\eta_B \lesssim \left(10^{-5} \ - \ 10^{-2}\right)\, g^2\, e^{-a/2} \ , \tag{2.46}$$

where the factor $a\,(T)$ has to be calculated for a value of $T$ of the order of the reheating temperature.

We get a different limit on $\eta_B$ from the bound (2.42): assuming $m_X \sim$ TeV and $g_* \sim 100$ indeed one obtains

$$\eta_B \lesssim \left(10^{-6} \ - \ 10^{-10}\right)\, g^{2/3}\, e^{-a/4} \ . \tag{2.47}$$

Since the observed amount of baryon asymmetry is of order $10^{-10}$, even in the case of maximum efficiency of the process (that is, assuming maximal $CP$ violation and $g \sim 1$), we have that both bounds (2.46) and (2.47) imply that $a\,(T_{\mathrm{rh}})$ has to be smaller than about 40.

Unfortunately, the temperature at which the condition $a\,(T) \lesssim 40$ occurs cannot be evaluated by means of the expansion of eq. (2.35), that have been obtained under the assumption $|a\,(T) - a\,(0)| \ll a\,(0)$. On the other hand, it is remarkable that our mechanism might work with a ratio $a\,(T_d)\,/a\,(0)$ of order one. We thus expect that a successful baryogenesis may be realized for a range of the parameters of our theory which – although not evaluable through a perturbative analysis – should be quite wide and reasonable.

As we have seen in section 2.1, in scenarios with large extra dimensions and low scale gravity, the maximal temperature reached by the Universe after inflation is strongly bounded from above in order to avoid overproducing Kaluza-Klein graviton modes. Values of $T_{\mathrm{rh}}$ of the order of few MeVs (that are the maximal reheating temperatures allowed in the case of a 2 or 3 extra dimensions) would be too low for our scenario since $\eta_B$ is proportional to the ratio $T_{\mathrm{rh}}/m_X$. Hence, the observed amount of baryons would be reproduced at the price of an unnaturally small value of $a\,(T_{\mathrm{rh}})$. However, other schemes with extra dimensions exist where the bounds on $T_{\mathrm{rh}}$ are less severe. For example, in the scenarios [9, 34] the mass of the first graviton KK mode is expected to be of order TeV. The reheating temperature can thus safely be taken to be of order $10 - 100\,\mathrm{GeV}$.

An alternative way to overcome the bound (2.38) relies on the fact that, as observed in the work [59], the maximal temperature reached by the thermal bath during reheating can indeed be much higher than the final reheating temperature. This fact can be exploited for the purposes of baryogenesis in standard four dimensional cosmology [61]. In this case, even if $T_{\rm rh}$ is considerably lower than $m_X$, $X$ particles can be produced in a significant amount, and the out of equilibrium condition is easily achieved. However, the treatment of this mechanism is in our case somewhat different from the one given in ref. [61]: due to the slowness of the expansion of the Universe, the $X$ bosons will decay before the freeze out of their production. Thus, the final baryon asymmetry cannot be estimated with the use of the formulae of [61], which are valid only if the decay of the $X$ particles occurs well after their freeze out.

# Chapter 3

# Cosmological evolution of stabilized brane models

The last three years have witnessed an impressive amount of work devoted to the analysis of the cosmological expansion of brane models. This activity flourished after the work [13] has shown that the presence of extra dimensions where only gravity can propagate can have a deep impact onto the expansion law of the Universe. Later on, models with compact extra dimensions were shown [14, 15, 62] to lead to standard late cosmology if a proper mechanism for stabilizing the size of the internal space is introduced. On the other hand, in models where the extra dimension is not compact the presence of a stabilizing mechanism is clearly meaningless. Indeed, these models (of which the noncompact one proposed by Randall and Sundrum is the prototypical example) can show nonstandard cosmology at some (typically early) stage [16, 17, 18]. These models thus allow to look from a different perspective to phenomena occurred in the early Universe such as inflation [63]. On the other hand, also the details of the early cosmology of models with stabilized extra dimensions are not completely clear. Some issues, such as the behavior of the system at higher energies, are still open [64].

In the first section we will review the formalism introduced in ref. [13], outlining the results that arise if a potential that stabilizes the extra space is not introduced. These results are generally in contradiction with phenomenology, and can be summarized in the expression $H \propto \rho$ of the expansion rate of the brane, while phenomenology requires $H \propto \sqrt{\rho}$. In the subsequent section we show how this picture is modified by the presence of a stabilizing potential. In fact, the latter introduces in the bulk of extra dimensions of the amount of energy that is exactly necessary to recover the standard expansion rate. Then, in section 3.3, we will move to the specific case of the compact Randall-Sundrum model. After solving the Einstein equations, we will discuss both the low energy and the high energy regime behavior of the model. Particular attention will be paid to the interpretation of the quantities that characterize the system.

## 3.1 Exact solutions in presence of matter

In this section we revise the (by now) standard techniques for the derivation of the expansion law of the Universe in brane scenarios, as they were derived in ref. [13].

To begin with, we consider a system of one tensionless brane in an empty (4+1)-dimensional bulk. We will denote by $y \in [-1/2,\, 1/2]$ the coordinate along the extra dimension. The orbifold symmetry that identifies points at coordinate $y$ with points at coordinate $-y$ is imposed. For the moment we will consider only one brane placed at the orbifold fixed point $y = 0$. At variance with the case considered in the previous chapter, the brane is assumed to be infinitely thin (this idealized situation is a good approximation as long as the energies considered are much smaller that the inverse of the thickness of the brane).

We will denote by $M$ the fundamental Planck mass. This means that the five-dimensional Einstein-Hilbert action reads

$$\mathcal{S} = -\frac{M^3}{2} \int dt\, d^3x\, dy\, \sqrt{-g^{(5)}}\, R^{(5)} \,. \tag{3.1}$$

Since we are looking for cosmological solutions, the ansatz for the metric will be a natural extension to the present system of the Friedmann-Robertson-Walker (flat) metric

$$\begin{aligned} ds^2 &= g^{(5)}_{AB}\, dx^A\, dx^B , \qquad \text{(A,B)} \in \{0,1,2,3,5\} \\ &= n^2\,(t,\,y)\; dt^2 - a^2\,(t,\,y)\; d\vec{x}^2 - b^2\,(t,\,y)\; dy^2 \,. \end{aligned} \tag{3.2}$$

The energy-momentum tensor of the system will be assumed to be the one of an ideal fluid on the brane, while it will be vanishing in the bulk

$$T_A^B = \text{diag}\,(\rho_0\,,\, -p_0\,,\, -p_0\,,\, -p_0\,,\, 0)\; \frac{\delta\,(y)}{b\,(t,\,y)} \,. \tag{3.3}$$

The five-dimensional Einstein equations read

$$G_{AB} = \frac{1}{M^3}\, T_{AB}\,, \tag{3.4}$$

where, in terms of the metric (3.2), the nonvanishing components of the Einstein tensor read

$$G_{00} = 3\left\{\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right)-\frac{n^2}{b^2}\left(\frac{a''}{a}+\frac{a'}{a}\left(\frac{a'}{a}-\frac{b'}{b}\right)\right)\right\}, \tag{3.5}$$

$$\begin{aligned} G_{ij} &= \frac{a^2}{b^2}\delta_{ij}\left\{\frac{a'}{a}\left(\frac{a'}{a}+2\frac{n'}{n}\right)-\frac{b'}{b}\left(\frac{n'}{n}+2\frac{a'}{a}\right)+2\frac{a''}{a}+\frac{n''}{n}\right\} \\ &+\frac{a^2}{n^2}\delta_{ij}\left\{\frac{\dot{a}}{a}\left(-\frac{\dot{a}}{a}+2\frac{\dot{n}}{n}\right)-2\frac{\ddot{a}}{a}+\frac{\dot{b}}{b}\left(-2\frac{\dot{a}}{a}+\frac{\dot{n}}{n}\right)-\frac{\ddot{b}}{b}\right\}, \end{aligned} \tag{3.6}$$

$$G_{05} = 3\left(\frac{n'}{n}\frac{\dot{a}}{a}+\frac{a'}{a}\frac{\dot{b}}{b}-\frac{\dot{a}'}{a}\right), \tag{3.7}$$

$$G_{55} = 3\left\{\frac{a'}{a}\left(\frac{a'}{a}+\frac{n'}{n}\right)-\frac{b^2}{n^2}\left(\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}-\frac{\dot{n}}{n}\right)+\frac{\ddot{a}}{a}\right)\right\}, \tag{3.8}$$

where dot denotes derivative with respect to time, while prime denotes derivative with respect to $y$.

In the solving the Einstein equations we will first of all assume that the radius is static, $\dot{b}(t, y) = 0$. This is required by phenomenology: a time dependence of $b$ induces a time dependence of the compactification volume and therefore of the observed Newton constant. If $\dot{b} = 0$, one can always redefine the coordinate $y$ such that also $b' = 0$. As a consequence, we will assume $b(t, y) = b =$ constant.

From the Bianchi identity $\nabla_A G^A_B = 0$ the usual energy conservation law on the brane is derived

$$\dot{\rho}_0 + 3\frac{\dot{a}_0}{a_0}(\rho_0 + p_0) = 0, \tag{3.9}$$

where the suffix 0 is used to denote quantities evaluated at the brane at $y = 0$.

The solution of the Einstein equations is simplified by the fact that the source term $T^A_B$ is $\delta$-valued in $y$. As a consequence, the first $y$-derivatives $a'$ and $n'$ have to experience a finite jump as $y$ crosses the brane at $y = 0$. This generates quantities proportional to $\delta(y)$ in $a''$ and $n''$, and hence in $G^A_B$, that have to be matched with the $\delta$-valued part of the stress energy tensor. For the purposes of this analysis it is useful to define the jump of the $y$-derivative of $a$ (an analogous definition holds for the function $n$) across the point $y = 0$ as

$$[a']_0 = \lim_{\epsilon\to 0^+}\left[a'(+\epsilon) - a'(-\epsilon)\right], \tag{3.10}$$

such that one has for example

$$a''(y) = \hat{a''}(y) + [a']_0\,\delta(y), \tag{3.11}$$

where $\hat{a''}$ represents the nondistributional part of $a''$.

Matching the distributional parts for the Einstein tensor with the one of the energy-momentum tensor one gets the following Israel [65] junction conditions

$$\frac{[a']_0}{a_0\, b} = -\frac{1}{3\, M^3}\, \rho_0\,,$$
$$\frac{[n']_0}{n_0\, b} = \frac{1}{3\, M^3}\, (3\, p_0 + 2\, \rho_0)\;. \tag{3.12}$$

The (0, 5) Einstein equation does not have a distributional component, and is solved (remember $\dot{b} = 0$) by

$$n\,(t,\, y) = \lambda\,(t)\;\dot{a}\,(t,\, y)\;. \tag{3.13}$$

This relation introduces an unknown function of time only, and considerably simplifies the remaining equations. Note that we have a complete freedom in the choice of $\lambda$, since different $\lambda$'s correspond to different definitions of the time variable. In particular, we will take advantage of this freedom to impose that the lapse function $n$ evaluates to 1 on the brane. Therefore we will have $\lambda\,(t) = 1/\dot{a}_0\,(t)$

One can then derive the expansion law of the brane. By considering the (5, 5) Einstein equation averaged across the brane

$$\lim_{\epsilon \to 0^+}\, [G_{55}\,(+\epsilon) + G_{55}\,(-\epsilon)] = 0\,, \tag{3.14}$$

and remembering that $a'\,(+\epsilon) = -a'\,(-\epsilon)$ follows from the orbifold $y \to -y$, one obtains

$$\frac{1}{4}\,\frac{[a']_0^2}{a_0^2} + \frac{1}{4}\,\frac{[a']_0}{a_0}\,\frac{[\dot{a}']_0}{\dot{a}_0} - b^2\left(\frac{\dot{a}_0^2}{a_0^2} + \frac{\ddot{a}_0}{a_0}\right) = 0\,. \tag{3.15}$$

From the above equations, using the junction conditions (3.12) and the energy conservation law (3.9), we derive the equation

$$\frac{\dot{a}_0^2}{a_0^2} + \frac{\ddot{a}_0}{a_0} = -\frac{1}{36\, M^6}\,\rho_0\,(\rho_0 + 3\, p_0)\;. \tag{3.16}$$

Since the right hand side of eq. (3.16) in quadratic in $\rho_0$, the expansion rate of the brane turns out to be linear in the energy density, whereas in standard cosmology it should be proportional to square root of the energy. In fact, eq. (3.16) can be derived from the nonstandard Friedmann law

$$\frac{\dot{a}_0^2}{a_0^2} = \frac{\rho_0^2}{36\, M^6}\,. \tag{3.17}$$

It is worth noticing that the observed Planck mass $M_P$ does not appear in the above equation (3.17), and the expansion rate depends on the value of the fundamental Planck mass $M$. This is related to the fact that the whole derivation above was performed by simply considering the Einstein equations in a very small neighborhood of the brane, and depends therefore only on the local properties of the system. The value of the observed Planck mass, on the other hand, depends on the compactification volume, that is a global property of the system.

The behavior (3.17) is phenomenologically excluded [13]. As we will now see, other unappealing results will emerge as we consider the global properties of the solutions to the Einstein equations for this system.

The expression of the scale factor in the bulk can be obtained by solving the (0, 0) Einstein equation, using the relations (3.13), (3.12) and (3.17)

$$a(t,\, y) = a_0\,(t)\,\left(1 - \frac{\rho_0}{6\,M^3}\,b\,|y|\right)\,. \tag{3.18}$$

This solution should be globally extended to the whole bulk, $-1/2 < y < 1/2$. However, one can immediately see that the scale factor is not regular at the point $y = 1/2$, where the $y$-derivative of $a\,(t,\, y)$ (as well as the $y$-derivative of $n\,(t,\, y)$) is discontinuous. The only way to explain such a discontinuity is to place at $y = 1/2$ a second brane, whose energy $\rho_{1/2}$ gives conditions analogous to the Israel conditions (3.12). However, this implies that the energy on the new brane has to be a function of the energy on the first one. Indeed one finds

$$\begin{aligned} \rho_0\, a_0 &= -\rho_{1/2}\, a_{1/2}\,, \\ (2\,\rho_0 + 3\,p_0)\; n_0 &= -\left(2\,\rho_{1/2} + 3\,p_{1/2}\right)\, n_{1/2}\,. \end{aligned} \tag{3.19}$$

These requirements are related to a version of the Gauss' law for this system: the total charge on a compact manifold has to vanish, since there must be as many sinks as sources for the flux lines.

Such a situation in which the energy of the two branes is acausally correlated is certainly unsatisfactory, as it is unsatisfactory that a negative energy density is required on one of the two branes.

The above discussion would lead to conclude that brane models cannot generically reproduce standard cosmology. The only possibility to avoid such conclusion is to investigate the origin of eqs. (3.17) and (3.19) and see how they can be modified. Before doing so (in the next section) we show how to apply the above formalism to the Randall-Sundrum compact and noncompact models.

In the case of the Randall-Sundrum model the derivation that led to eq. (3.17) is substantially unchanged. One has only to add a (negative) cosmological constant $\Lambda$ in the bulk and a positive tension $V_0$ to the brane located at $y = 0$. That is, we have to add a term $-2\,b^2\,\Lambda/M^3$ to the right hand side of eq. (3.14) and to replace

$\rho_0$ with $\rho_0 + V_0$, $p_0$ with $p_0 - V_0$. The result is the following expansion law for the brane located at $y = 0$

$$\frac{\dot{a}_0^2}{a_0^2} = \frac{(V_0 + \rho_0)^2}{36\, M^6} + \frac{\Lambda}{6\, M^3} \,. \tag{3.20}$$

The above equation shows that the Randall-Sundrum fine tuning $\Lambda = -V_0^2/\,(6\, M^3)$ amounts to a cancellation of the effective four dimensional cosmological constant. After this cancellation, the Friedmann law reads

$$\frac{\dot{a}_0^2}{a_0^2} = \frac{V_0}{18\, M^6}\, \rho_0 + \frac{\rho_0^2}{36\, M^6} \,. \tag{3.21}$$

In the low energy regime $\rho_0 \ll 2\, V_0$ the standard expansion law $H^2 \propto \rho$ is therefore recovered. The identification

$$M_P^2 = 6\, \frac{M^6}{V_0} \tag{3.22}$$

is the same one that is derived by evaluating the strength of the Newton force in the static Randall-Sundrum noncompact model. The system thus shows no contradiction with phenomenology. Moreover, since in this model the extra dimension is infinite in extension, there are no problems in defining a smooth solution of the Einstein equation in the whole extra space.

The Randall-Sundrum scenario with a compact extra dimension, on the other hand, suffers from some of the problems that we have outlined above in the simpler case of tensionless branes in a empty bulk. First of all, the expansion rate of the system is governed by a Planck mass (3.22) that is slightly different (by a factor $(1 - \Omega_0^2)$) from the one associated to the Newton force. Due to the extreme smallness of the difference between the two Planck masses, this fact is phenomenologically irrelevant. However, it is theoretically unappealing. What is more relevant is that the energy density on the visible brane should be correlated with the energy density on the Planck brane

$$\begin{aligned} \rho_{1/2} &= -\,\Omega_0^2\, \rho_0 \,, \\ p_{1/2} &= -\,\Omega_0^2\, p_{1/2} \,, \end{aligned} \tag{3.23}$$

where $\rho_i$ and $p_i$ are, respectively, the matter[1] density and the pressure on the $i$-th brane ($i = 0\,,\, 1/2$), while $\Omega_0 = e^{-m_0\, b/2}$ was already defined in eq. (1.10).

Since $V_0 > 0$, from eq. (3.21) we see that $\rho_0 > 0$ is required. This implies that $\rho_{1/2}$, i.e. the energy density of the cosmological fluid *on our brane* should be *negative.*

[1] With matter we generically indicate any possible component which adds to the vacuum energies $V_0\,,\, V_{1/2}$ of the original static configuration.

Of course, if these results were unavoidable, the Randall-Sundrum compact model (as well as all the brane models with a compact extra space) would be in extreme trouble. We will see in the next section how this conclusion can be actually evaded.

## 3.2 The role of radion stabilization

In the previous section we have seen two problems that emerge in brane models with compact extra dimensions without matter in the bulk. The first problem is that for tensionless branes and without a cosmological constant in the bulk the expansion rate of the brane is proportional to the energy rather than to its square root. The second one is that a second brane is needed to serve as a sink for the flux lines of the gravitational field generated from the first brane. The energy density on the second brane is a function of the energy density on the first brane.

To see how to solve these problems, let us first consider the case of a tensionless brane in a bulk with vanishing cosmological constant, in a scenario à la Arkani–Hamed-Dimopoulos-Dvali. One can get a solution to both the problems outlined above by assuming some smooth distribution of energy and pressure in the bulk. In particular, in the presence of a nonvanishing value of the (5, 5) component of the stress-energy tensor $T^5_5\,(y)$, equation (3.17) acquires the form

$$\frac{\dot{a}_0^2}{a_0^2} = \frac{\rho_0^2}{36\, M^6} + \frac{T^5_5\,(y=0)}{6\, M^3}\,. \tag{3.24}$$

If

$$\frac{T^5_5\,(y=0)}{6\, M^3} = \frac{\rho_0}{3\, M_P^2} - \frac{\rho_0^2}{36\, M^6}\,, \tag{3.25}$$

then the standard expansion law will be obviously recovered on the brane. In the same way, if $T^A_B\,(y)$ assumes an appropriate form that depends on $\rho_0$ and $p_0$, it will be possible to get a global solution to the Einstein equations that can be smoothly joint at $y = 1/2$ without requiring the presence of a second brane.

This form of the stress-energy tensor in the bulk seems to be really *ad hoc.* However, as realized in [14, 62], this is exactly the kind of modification of the energy-momentum tensor that is provided by a mechanism responsible for the stabilization of the extra dimension. This mechanism can be be efficiently described by the introduction of a potential $V\,(b)$ for the radius $b$ of the extra dimension. The potential will be minimized when $b = b_0$, where $b_0$ is the radius of the extra dimension in vacuum. $V\,(b)$ can emerge after some fields with specific coupling to the branes have been integrated out. The most popular example in this respect is the Goldberger-Wise [66] mechanism for the stabilization of the extra dimension in the Randall-Sundrum model.

The presence of a stabilizing potential is also necessary to fulfil other phenomenological requirements. In fact, the excitations of the radius of the extra dimension describe from the four-dimensional point of view a scalar modulus. In absence of a stabilizing potential, this modulus will be a massless scalar mediating long range forces, that are phenomenologically excluded. As a consequence, the potential $V(b)$ has to provide a mass of at least $10^{-3}$ eV to the modulus to be compatible with the experiments.

The introduction of the stabilizing potential for $b$ is described by the addition to the action of a term

$$\mathcal{S}_b = -\int dt\, d^3x\, dy \sqrt{-g^{(5)}}\, V(b) = -\int dt\, d^3x\, dy\, b\, a^3\, n\, V(b)\ . \tag{3.26}$$

Also a kinetic term for $b$ should be needed, but we can disregard it as phenomenology requires it to be negligible. The presence of $\mathcal{S}_b$ entails the presence of new terms in the stress-energy tensor

$$\begin{aligned} T^{0}_{b\,0} &= -\frac{1}{2}\, n\, a^3\, b\, V(b)\ , \\ T^{i}_{b\,i} &= \frac{3}{2}\, n\, a^3\, b\, V(b)\ , \\ T^{5}_{b\,5} &= = \frac{1}{2}\, n\, a^3\, b\, \left[V(b) + V'(b)\right]\ . \end{aligned} \tag{3.27}$$

The radion potential close to the minimum point $b_0$ can be approximated without loss of generality by a quadratic potential

$$V(b) = M_b^5 \left(\frac{b-b_0}{b_0}\right)^2 + \mathcal{O}\left(\left(\frac{b-b_0}{b_0}\right)^3\right)\ . \tag{3.28}$$

Throughout this chapter we will assume that the radion is very heavy, that is, that the mass scale $M_b$ is extremely large.

In absence of any matter on the branes, the radion is at its minimum $b_0$. If an arbitrary amount of energy is placed on the branes, the value of $b$ will generically change, and it would start rolling away if the stabilizing potential did not keep it close to $b_0$. We denote by

$$\Delta b \equiv b(\rho) - b_0 \tag{3.29}$$

the displacement of the radion (associated to the energy $\rho$ on the branes) from its equilibrium value.

Being $V(b_0) = V'(b_0) = 0$, eqs. (3.27) imply that, while $T_b{}^5{}_5$ is of the first order in $\Delta b$, $T_b{}^0{}_0$ and $T_b{}^i{}_i$ are of the second order in $\Delta b$. The assumption of a very large $M_b$ implies that $\Delta b$ is very small. As a consequence, the presence of the radion potential will be felt by the (5, 5) Einstein equation before than by the other ones.

This observation shows how one can proceed technically in solving the equations in presence of a nonvanishing $V(b)$. Being $\Delta b$ extremely small, all the system of Einstein equations will be solved as described in the previous section. Only, we will not be allowed to use the (5, 5) equation, that is necessary to determine $\Delta b$. As a consequence, since the number of equations that have to be solved is reduced by one, also the number of constraints to the system will decrease by one. In particular, this will remove the relation between the energies on the two branes[2] (actually, a second brane is not at all necessary any more). The value of $\Delta b$ computed this way from the (5, 5) Einstein equation gives to $T^5{}_5$ exactly the correction (3.25) that allows to recover the standard Friedmann law.

## 3.3 Cosmological evolution of the Randall-Sundrum model

In this section we will account in detail for the procedure outlined above for the case of the Randall-Sundrum model with a compact extra dimension. This computation was first performed in ref. [15] in the specific case of small and constant energy on the two branes, showing that the standard Friedmann law is recovered in this regime. In [67] the full analysis was performed for arbitrary large energies of arbitrary equation of state on the two branes. In subsection 3.3.1 we solve exactly the Einstein equations for an arbitrary amount of energy with arbitrary equation of state on the two branes. In subsection 3.3.2 the effective four dimensional action of the system is analyzed, while in the following subsection the low energy behavior is shown to be analogous to the standard four dimensional one. Finally in subsection 3.3.4 the high energy behavior of the system is analyzed, concluding that it could show potential deviations from the standard cosmology. These deviations are however unlikely to have been actually occurred in the early Universe.

### 3.3.1 Solution of the Einstein equations

In the following derivation we will assume $M_b \to \infty$, that implies that $\Delta b$ is negligibly small. Therefore $b$ will be kept fixed at its equilibrium value $b_0$. The only components of the Einstein tensor we are interested in are (3.5), (3.6) and (3.7).

[2]This observation explains the origin of the second fine tuning in the static Randall-Sundrum model. Actually the two fine-tunings (1.5) are required to find a solution that is static both in the direction parallel and in the one transverse to the branes.

For the notation we refer to the one used in chapter 1 for the description of the Randall-Sundrum model.

The energy-momentum tensor of the two branes is of the form

$$\left(T_A^B\right)_{\text{brane}} = \frac{\delta\,(y)}{b_0}\,\text{diag}\,(V_0+\rho_0\,,\,V_0-p_0\,,\,V_0-p_0\,,\,V_0-p_0\,,\,0)\,+ \tag{3.30}$$
$$+\frac{\delta\,(y-1/2)}{b_0}\,\text{diag}\,\left(V_{1/2}+\rho_{1/2}\,,\,V_{1/2}-p_{1/2}\,,\,V_{1/2}-p_{1/2}\,,\,V_{1/2}-p_{1/2}\,,\,0\right)\;\;,$$

where $V_{0\,,1/2}$ are the brane tensions, while $\rho_i$ and $p_i$ are, respectively, the density and pressure of matter on the two branes with equation of state $p_i = w_i\,\rho_i\;\;(i=0\,,\,1/2)$.

First of all, we integrate the (0, 5) Einstein equation (remember $\dot{b}=0$ is assumed). This is solved either by $\dot{a}=0$ (in this case one recovers the Randall-Sundrum solution (1.6)), or by the relation (3.13) found above.

By inserting eq. (3.13) into the (0, 0) Einstein equation, we can eliminate the time-derivatives, and we obtain a simple second-order differential equation for $a^2$:

$$\left(a^2(t,y)\right)'' - 4m_0^2\,b_0^2\;a^2(t,y) = \frac{2\,b_0^2}{\lambda(t)^2}. \tag{3.31}$$

This equation has solution

$$a^2\,(t\,,\,y) = a_0^2\,(t)\,\omega_0^2\,(y) + a_{1/2}^2\,(t)\,\omega_{1/2}^2\,(y) + \frac{\omega_0^2\,(y)+\omega_{1/2}^2\,(y)-1}{2m_0^2\lambda\,(t)^2}\;\;, \tag{3.32}$$

where

$$\begin{aligned} \omega_0^2\,(y) &= \cosh\,(2\,m_0\,b_0\,|y|) - \frac{C_0}{S_0}\,\sinh\,(2\,m_0\,b_0\,|y|)\;\;, \\ \omega_{1/2}^2\,(y) &= \frac{\sinh\,(2\,m_0\,b_0\,|y|)}{S_0}\;\;, \end{aligned} \tag{3.33}$$

with $C_0 \equiv \cosh\,(m_0\,b_0)$ and $S_0 \equiv \sinh\,(m_0\,b_0)$. Eq. (3.32) relates the value of $a\,(t\,,\,y)$ in the whole space to the values on the two branes $a_0\,(t) \equiv a\,(t\,,\,0)$ and $a_{1/2}\,(t) \equiv a\,(t\,,\,1/2)$, that can be determined by solving the (0, 0) Einstein equation across the two branes. This is made by using the techniques described above in the derivation of eq. (3.12). From the symmetry $y \leftrightarrow -y$, we can write the jump conditions in the form:

$$\begin{aligned}
\frac{a'(t,0)}{a(t,0)} &= -\frac{1}{6\,M^3} b_0 \left(V_0 + \rho_0\right), \\
\frac{n'(t,0)}{n(t,0)} &= \frac{1}{6\,M^3} b_0 \left[2\left(V_0 + \rho_0\right) + 3\left(-V_0 + p_0\right)\right], \\
\frac{a'(t,\frac{1}{2})}{a(t,\frac{1}{2})} &= \frac{1}{6\,M^3} b_0 \left(V_{1/2} + \rho_{1/2}\right), \\
\frac{n'(t,\frac{1}{2})}{n(t,\frac{1}{2})} &= -\frac{1}{6\,M^3} b_0 \left[2\left(V_{1/2} + \rho_{1/2}\right) + 3\left(-V_{1/2} + p_{1/2}\right)\right] \quad . \qquad (3.34)
\end{aligned}$$

These equations lead to the following system for $a_0$, $a_{1/2}$:

$$\begin{aligned}
\left[1 + \frac{\rho_0}{6\,m_0\,M^3} - \frac{C_0}{S_0}\right] a_0^2 + \frac{a_{1/2}^2}{S_0} &= \frac{C_0 - 1}{2\,m_0^2\,\lambda^2\,S_0}\ , \\
\frac{a_0^2}{S_0} + \left[-1 + \frac{\rho_{1/2}}{6\,m_0\,M^3} - \frac{C_0}{S_0}\right] a_{1/2}^2 &= \frac{C_0 - 1}{2\,m_0^2\,\lambda^2\,S_0}\ . \qquad (3.35)
\end{aligned}$$

As expected, the system admits no solution in absence of matter on the two branes, $\rho_0 = \rho_{1/2} = 0$ . Indeed, for this choice one recovers the static Randall-Sundrum solution, which is not accounted for by the relation (3.13). When matter is instead included, the system (3.35) gives the solutions:

$$a_0^{-2}\lambda^{-2} = \frac{m_0}{3\,M^3\,(1-\Omega_0^2)} \frac{\rho_0 + \Omega_0^4\rho_{1/2} - \frac{1}{12\,m_0\,M^3}\left(1-\Omega_0^4\right)\rho_0\rho_{1/2}}{1 - \left(1-\Omega_0^2\right)\frac{\rho_{1/2}}{12\,M^3\,m_0}}\ , \qquad (3.36)$$

$$a_{1/2}^{-2}\lambda^{-2} = \frac{m_0}{3\,M^3\,(1-\Omega_0^2)} \frac{1}{\Omega_0^2} \frac{\rho_0 + \Omega_0^4\rho_{1/2} - \frac{1}{12\,M^3\,m_0}\left(1-\Omega_0^4\right)\rho_0\rho_{1/2}}{1 - \left(\Omega_0^{-2} - 1\right)\frac{\rho_0}{12\,M^3\,m_0}} \,. \qquad (3.37)$$

Since $\lambda = n_0/\dot{a}_0 = n_{1/2}/\dot{a}_{1/2}$ , we can interpret these equations as the expansion laws of the two branes. As we will see below, eqs. (3.9), (3.36), and (3.37) give standard FRW evolution on both branes at low energy.

### 3.3.2 The effective four dimensional action

We can gain some insight on the cosmology of the Randall-Sundrum model by integrating the whole action over the extra dimension $y$ . In doing so, we make use of the result (3.32). Our goal is to get an effective four dimensional action which describes the evolution of the scale factors $a_0\,(t)$ , $a_{1/2}\,(t)$ on the two branes.

We first focus on the "purely gravitational" five dimensional action, that is we integrate the Randall-Sundrum action in the absence of matter on the two branes.

The latter will be considered eventually when we deal with the equations of motion. Our starting point is thus:

$$\begin{aligned} S &= -\int dt\, d^3x\, dy\, \sqrt{-g^{(5)}} \left[ \frac{M^3 R^{(5)}}{2} + \Lambda + \frac{\delta(y)}{b_0} V_0 + \frac{\delta(y-1/2)}{b_0} V_{1/2} \right] \\ &= -\frac{M^3}{2} \int dt\, d^3x\, dy \sqrt{-g^{(5)}} \left[ R^{(5)} - 12\, m_0^2 + 12\, m_0 \left( \frac{\delta(y)}{b_0} - \frac{\delta(y-1/2)}{b_0} \right) \right], \end{aligned} \tag{3.38}$$

with the full (five dimensional) curvature scalar given by:

$$R^{(5)} = 6n^{-2} \left[ \frac{\dot{n}\,\dot{a}}{n\,a} - \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 \right] + 2b_0^{-2} \left[ \frac{n''}{n} + 3\frac{n'\,a'}{n\,a} + 3\frac{a''}{a} + 3\left( \frac{a'}{a} \right)^2 \right]. \tag{3.39}$$

Since we are interested in the evolution of the two four dimensional branes, we rewrite $n(t,\, y)$ and $a(t,\, y)$ by making use of the results of the previous sections, eqs. (3.13) and (3.32). It is then convenient to write $\sqrt{-g^{(5)}}\, R^{(5)}$ and $\sqrt{-g^{(5)}}$ in terms of $a^2$ and $\lambda$:

$$\begin{aligned} \sqrt{-g^{(5)}}\, R^{(5)} &= \frac{6b_0}{\lambda} \left( \frac{\dot{\lambda}}{\lambda} a^2 - \frac{1}{2} \frac{d\,a^2}{dt} \right) + \frac{\lambda}{2b_0} \left[ \frac{d\,(a^4)''}{dt} - 3\left(a^2\right)' \frac{d\,(a^2)'}{dt} \right], \\ \sqrt{-g^{(5)}} &= \frac{\lambda\, b_0}{4} \frac{d\,a^4}{dt}. \end{aligned} \tag{3.40}$$

With all these considerations[3], the integral over $y$ of the action (3.38) gives:

$$\begin{aligned} S = &-\frac{M^3}{2} \int d^4x \left\{ \frac{1-\Omega_0^2}{m_0} \frac{1}{1+\Omega_0^2} \left[ \frac{6}{\lambda} \left( \frac{\dot{\lambda}}{\lambda} \left(a_0^2 + a_{1/2}^2\right) - \left(a_0\, \dot{a}_0 + a_{1/2}\, \dot{a}_{1/2}\right) \right) \right] + \right. \\ &\left. + \frac{24\, m_0}{1-\Omega_0^4} \lambda \left[ \Omega_0^2 \left(a_0\, \dot{a}_0\, a_{1/2}^2 + a_{1/2}\, \dot{a}_{1/2}\, a_0^2\right) - \left(\Omega_0^4\, a_0^3\, \dot{a}_0 + a_{1/2}^3\, \dot{a}_{1/2}\right) \right] \right\}. \end{aligned} \tag{3.41}$$

By substituting $\lambda = n_0/\dot{a}_0 = n_{1/2}/\dot{a}_{1/2}$ in the last expression[4] we get:

$$\begin{aligned} S = -\frac{M_P^2}{2(1+\Omega_0^2)} \int d^4x &\left\{ n_0\, a_0^3 \frac{6}{n_0^2} \left[ \frac{\dot{n}_0\, \dot{a}_0}{n_0\, a_0} - \frac{\ddot{a}_0}{a_0} - \left( \frac{\dot{a}_0}{a_0} \right)^2 \right] + \right. \\ &+ n_{1/2}\, a_{1/2}^3 \frac{6}{n_{1/2}^2} \left[ \frac{\dot{n}_{1/2}\, \dot{a}_{1/2}}{n_{1/2}\, a_{1/2}} - \frac{\ddot{a}_{1/2}}{a_{1/2}} - \left( \frac{\dot{a}_{1/2}}{a_{1/2}} \right)^2 \right] + \\ &\left. + \frac{24\, m_0^2}{(1-\Omega_0^2)^2} \left[ \Omega_0^2 \left(a_0\, n_0\, a_{1/2}^2 + a_{1/2}\, n_{1/2}\, a_0^2\right) - \left(\Omega_0^4\, a_0^3\, n_0 + a_{1/2}^3\, n_{1/2}\right) \right] \right\}, \end{aligned} \tag{3.42}$$

[3] The calculation can be further simplified by noticing that, from the periodicity imposed in the extra space, the integral of a derivative of any function of $y$ vanishes.

[4] In this way, we substitute $\lambda(t)$ with the two degrees of freedom $n_0(t)$ and $n_{1/2}(t)$. The equations of motion of the effective four dimensional theory have thus to be supported by the constraint $n_0/\dot{a}_0 = n_{1/2}/\dot{a}_{1/2}$. This relation cannot be obtained from the action (3.42), since it is linked to the equation $G_{05} = 0$ that has no counterpart in the four dimensional effective theory.

where we used the relation (1.9) $M_P^2 = M^3\,(1-\Omega_0^2)\,/m_0$.

As we will discuss in more detail in the next section, in the low energy limit the equality $a_{1/2}(t) = \Omega_0\,a_0(t)$ and the related one $n_{1/2}(t) = \Omega_0\,n_0(t)$ hold. As a consequence, the expansion rates of the two branes are identical and the above action rewrites in the standard FRW form:

$$S = -\frac{M_P^2}{2}\int d^4x\,\bar{n}\,\bar{a}^3\,\frac{6}{\bar{n}^2}\left[\frac{\dot{\bar{n}}}{\bar{n}}\frac{\dot{\bar{a}}}{\bar{a}} - \frac{\ddot{\bar{a}}}{\bar{a}} - \left(\frac{\dot{\bar{a}}}{\bar{a}}\right)^2\right] , \tag{3.43}$$

where $\bar{a} \equiv a_0 = \Omega_0^{-1}\,a_{1/2}$ , $\bar{n} \equiv n_0 = \Omega_0^{-1}\,n_{1/2}$ .

From the effective action (3.42) we notice that the entire five dimensional system can be expressed in terms of the physics that takes place on the boundaries at $y = 0$ and $y = 1/2$ of the extra space. Notice also that the last term in the action (3.42) couples the metrics of the two walls.

Going back to the four-dimensional action (3.42), and including also matter on the two walls, we obtain the equations of motion:

$$\begin{aligned} \frac{\dot{a}_0^2}{n_0^2\,a_0^2} &= \frac{1+\Omega_0^2}{3\,M_p^2}\,\rho_0 + \frac{4\,m_0^2}{(1-\Omega_0^2)^2}\,\Omega_0^2\left(\frac{a_{1/2}^2}{a_0^2} - \Omega_0^2\right) \\ \frac{\dot{a}_{1/2}^2}{n_{1/2}^2\,a_{1/2}^2} &= \frac{1+\Omega_0^2}{3\,M_p^2}\,\rho_{1/2} + \frac{4\,m_0^2}{(1-\Omega_0^2)^2}\,\Omega_0^2\left(\frac{a_0^2}{a_{1/2}^2} - \frac{1}{\Omega_0^2}\right) , \end{aligned} \tag{3.44}$$

in addition to the relations which give energy conservation on the two branes, eqs. (3.9). We notice that, in the limit $\rho_0 \to 0$, $\rho_{1/2} \to 0$, the only solution of the above equations is the static Randall-Sundrum solution $a_{1/2} = \Omega_0 a_0$. Moreover, one can verify that eqs. (3.44) are equivalent to the equations (3.36) and (3.37) obtained from the five dimensional theory.

### 3.3.3 FRW evolution at low energy

Before interpreting the four-dimensional effective theory shown above, we come back to the static Randall-Sundrum case. We recall that in [9] the four-dimensional metric $\bar{g}_{\mu\nu}$ on both branes is defined as:

$$\bar{g}_{\mu\nu} = n(y)^{-2} g_{\mu\nu} \,. \tag{3.45}$$

The goal of this redefinition is to achieve Minkowski metric on both branes, in order to gain a simple physical interpretation of the system. An analogous procedure has to be applied also in the general case with matter on the two branes.

Generally speaking, multiplying the metric by an overall function $f$ is not equivalent to a change of the coordinate system. Thus, to have canonical normalization of the fields, the function $f$ has to be absorbed by a redefinition of the fields themselves. In order to preserve the equations of motion of the fields, we see that we cannot choose $f$ to depend on the coordinates $t$ and $x$, but it can be at most a function of $y$.

In analogy with what was done in the static case, we now wonder whether it is possible to rewrite the first four components of the five-dimensional metric in the form:

$$g_{\mu\nu}\,(t\,,\,y) = f\,(y)\;\bar{g}_{\mu\nu}\,(t)\ , \tag{3.46}$$

with $\bar{g}_{\mu\nu}$ of the standard FRW form $\text{diag}(1, -\bar{a}^2, -\bar{a}^2, -\bar{a}^2)$. This requires the ratio $n/a$ to be independent on $y$, that is $a'/a = n'/n$ for every value of $y$. From the "jump conditions" (3.34) we see that this implies $\rho + p = 0$ and, consequently, $\dot{\rho} = 0$ on the two branes. In other words, the above factorization is possible only if the two branes contain exclusively cosmological constants (in particular this is the case for the static Randall-Sundrum solution).

Anyhow, it is natural to expect that condition (3.46) is approximately recovered when the matter on the two branes has a sufficiently low energy density. This can be understood from the results of the previous sections. From eqs. (3.36) and (3.37) we have:

$$\frac{a_{1/2}^2}{a_0^2} = \Omega_0^2\,\frac{1 - \dfrac{1-\Omega_0^2}{12\,m_0\,M^3\,\Omega_0^2}\,\rho_0}{1 - \dfrac{1-\Omega_0^2}{12\,m_0\,M^3}\,\rho_{1/2}}\ . \tag{3.47}$$

If $\rho_0$ and $\rho_{1/2}$ are sufficiently small, the scale factors of the two branes are (approximately) proportional[5] by the constant factor $\Omega_0$. Since $n\,(y\,,\,t) = \lambda\,(t)\,\dot{a}\,(y\,,\,t)$, we have also $n_{1/2}\,(t) = \Omega_0 n_0\,(t)$. In particular, the ratio $n/a$ is (approximately) independent on $y$.[6]

In this low-energy limit, we can thus define the four-dimensional effective theory just as in the static Randall-Sundrum model. First, we can choose the time coordinate so that $n_0$ and $n_{1/2}$ are simultaneously time-independent. This is equivalent to setting $\lambda(t) = \lambda_0/\dot{a}_{1/2}(t)$, where $\lambda_0$ is an arbitrary factor. Then, we recover eq.(3.46) with:

$$f(0) = \lambda_0^2\;\Omega_0^{-2}, \qquad f(1/2) = \lambda_0^2, \qquad \bar{a} = \lambda_0^{-1}\Omega_0 a_0 = \lambda_0^{-1} a_{1/2}\ . \tag{3.48}$$

We can now use the freedom to fix the time coordinate, and choose a particular value of $\lambda_0$. Choosing $\lambda_0 = \Omega_0$ we recover, in the limit $\rho_0\,,\,\rho_{1/2} \to 0$, the static

[5] Notice that this relation holds exactly in the static Randall-Sundrum regime.

[6] From eqs. (3.32), (3.36), and (3.37), it is indeed possible to show that, in the low energy limit, the quantity $n'/n - a'/a$ is of the same order as $a_{1/2}/\,(\Omega_0 a_0) - 1$.

Randall-Sundrum solutions as presented in [9]. With this choice, the scale factor of the effective metrics reads $\bar{a} = a_0 = \Omega_0^{-1}\, a_{1/2}$.

We can identify the five-dimensional quantities with those measured at low energy in our brane:

- the fields must be redefined by a factor $\Omega_0$. So, for instance, the observed density is $\bar{\rho}_{1/2} = \Omega_0^4\, \rho_{1/2}$. On the other brane the canonically normalized density reads: $\bar{\rho}_0 = \rho_0$ .
- the total four dimensional effective action (3.42) acquires the form of the standard FRW action in terms of the scale factor $\bar{a}$, see eq. (3.43).
- The Hubble parameter of the low energy theory is given by $\dot{\bar{a}}/\bar{a}$. From both eq. (3.36) and eq. (3.37) we get the standard evolution law:

$$H^2 = \left(\frac{\dot{\bar{a}}}{\bar{a}}\right)^2 \simeq \frac{1}{3\,M_p^2}\,\left(\bar{\rho}_0 + \bar{\rho}_{1/2}\right) \quad , \tag{3.49}$$

while from eqs. (3.9) we recover

$$\begin{aligned} \dot{\bar{\rho}} &+ 3\frac{\dot{\bar{a}}}{\bar{a}}\,(\bar{\rho} + \bar{p}) = 0 \ , \\ \bar{\rho} &\equiv \bar{\rho}_0 + \bar{\rho}_{1/2} \ , \ \bar{p} \equiv \bar{p}_0 + \bar{p}_{1/2} \ . \end{aligned} \tag{3.50}$$

Some considerations are in order. First, we notice that at low energy, from the point of view of observers on both branes, the effective theory leads exactly to a standard four-dimensional FRW Universe. This follows from the fact that the standard Friedmann law is recovered, and that the energy densities on both branes scale with the same Hubble parameter. In particular, for what concerns observers on our brane, the matter on the positive tension brane is regarded as dark matter [15] that would completely escape any direct experimental detection (apart of course from its gravitational interactions). The gravitational effect of the matter on the brane placed at $y = 0$ is not suppressed by powers of $\Omega_0$, as it is the case for $\bar{\rho}_{1/2}$. Since the only natural mass scale of the model is the Planck scale, $\bar{\rho}_0$ must hence be fine–tuned to small values not to conflict with observations.

Then, in order to put quantitative limits on the validity of the low energy theory, we rewrite eq. (3.47) in terms of the observed matter densities:[7]

$$\frac{a_{1/2}^2}{a_0^2} = \Omega_0^2\,\frac{1 - \dfrac{\bar{\rho}_0}{10\,M_P^2\,\mathrm{TeV}^2}}{1 - \dfrac{\bar{\rho}_{1/2}}{10\,\mathrm{TeV}^4}} \quad . \tag{3.51}$$

[7]We use $m_0 \simeq M \simeq M_P$ and $M_P\,\Omega_0 \simeq$ TeV, in the spirit of the original Randall-Sundrum paper.

We see that the low energy approximation is valid as long as the observed matter densities satisfy the bounds:

$$\bar{\rho}_0 \ll 10\, M_p^2\, \mathrm{TeV}^2 \quad , \quad \bar{\rho}_{1/2} \ll 10\, \mathrm{TeV}^4 \, . \tag{3.52}$$

Finally, we notice that the Planck mass that governs the expansion law of the Universe in the low energy regime is exactly the same (1.9) one that appears in low energy effective theory, and therefore in the expression of the Newton law at distances that are large with respect to $AdS$ radius $m_0^{-1}$.

### 3.3.4 Corrections to Standard Cosmology at high energy

We now focus on the equations of motion when the low-energy conditions (3.52) are not fulfilled anymore. From what we said in the previous sections, it is clear that in this regime it is not possible any longer to have a simple interpretation of the effective four dimensional action in terms of observable quantities. However, this is not important, because we make measurements only today, in the low-energy limit. So, it is legitimate to study the evolution of the system at high energy (eqs. (3.9), (3.36), and (3.37)), and then make contact with the quantities that we observe today.[8]

We keep the previous definitions of $\bar{\rho}_i$, $\bar{p}_i$, $M_P$, and the choice $\lambda = \Omega_0/\dot{a}_{1/2}$, so that eqs.(3.37) and (3.9) read:

$$\left(\frac{\dot{a}_{1/2}}{a_{1/2}}\right)^2 = \frac{1}{3\, M_P^2}\, \frac{\bar{\rho}_0 + \bar{\rho}_{1/2} - \frac{1}{12\, m_0\, M^3}\left(\Omega_0^{-2} - \Omega_0^2\right)\bar{\rho}_0\bar{\rho}_{1/2}}{1 - \left(\Omega_0^{-2} - 1\right)\frac{\bar{\rho}_0}{12\, m_0\, M^3}} \, , \tag{3.53}$$

$$\dot{\bar{\rho}}_{1/2} + 3\,\frac{\dot{a}_{1/2}}{a_{1/2}}\left(\bar{\rho}_{1/2} + \bar{p}_{1/2}\right) = 0 \, . \tag{3.54}$$

With our ansatz for $\lambda(t)$, the warp factor on our brane is constant. So, all the Euler-Lagrange equations on our brane are the same at high and low energy (i.e., they remain exactly identical to the standard equations of physics in four dimensions). In order to close the differential system, we need an equation of evolution for $\bar{\rho}_0$. It is obtained from eqs. (3.9), (3.36), and (3.37):

$$\dot{\bar{\rho}}_0 = -3\frac{\dot{a}_{1/2}}{a_{1/2}}(\bar{\rho}_0 + \bar{p}_0)\left(1 - \frac{3(\bar{\rho}_0 + \bar{p}_0)}{2(12\, m_0\, M^3\frac{\Omega_0^2}{1-\Omega_0^2} - \bar{\rho}_0)}\right)\left(1 - \frac{3(\bar{\rho}_{1/2} + \bar{p}_{1/2})}{2(12\, m_0\, M^3\frac{\Omega_0^4}{1-\Omega_0^2} - \bar{\rho}_{1/2})}\right)^{-1} \tag{3.55}$$

[8]This remark should be important, for instance, when looking at cosmological perturbations in the early Universe. In this section, we derive only the evolution equations of the homogeneous background. When studying the perturbations, one should keep in mind that a full five-dimensional description is required at high energy.

The differences between the evolution equations for $\bar{\rho}_0$ and $\bar{\rho}_{1/2}$ (i.e., the terms in the parentheses) show explicitly that, at high energy, $\bar{\rho}_0$ is not equivalent to dark matter in our brane. This difference is due to the fact that in the high energy regime the cosmic time on our brane is not any more proportional to the time on the other brane.

Since it is assumed that $m_0 \simeq M \simeq M_P$ and that $\Omega_0 M_P \simeq$ TeV , the above equations can be cast in the more transparent form:

$$\left(\frac{\dot{a}_{1/2}}{a_{1/2}}\right)^2 = \frac{\bar{\rho}_{1/2}}{3\,M_p^2}\left(1+\frac{\bar{\rho}_0}{\bar{\rho}_{1/2}}-\frac{\bar{\rho}_0}{10\,M_p^2\,\mathrm{TeV}^2}\right)\left(1-\frac{\bar{\rho}_0}{10\,M_p^2\,\mathrm{TeV}^2}\right)^{-1}, \qquad (3.56)$$

$$\dot{\bar{\rho}}_0 = -3\frac{\dot{a}_{1/2}}{a_{1/2}}(\bar{\rho}_0+\bar{p}_0)\left(1-\frac{3(\bar{\rho}_0+\bar{p}_0)}{2(10M_P^2\,\mathrm{TeV}^2-\bar{\rho}_0)}\right)\left(1-\frac{3(\bar{\rho}_{1/2}+\bar{p}_{1/2})}{2(10\,\mathrm{TeV}^4-\bar{\rho}_{1/2})}\right)^{-1}.$$

Before analyzing the possible cosmological implications of the nonstandard evolution equations (3.56), we notice that that the above equations are singular for $\bar{\rho}_0 = 12\,\Omega_0^2\,m_0\,M^3/\,(1-\Omega_0^2)$. The origin of this singularity can be traced back in equation (3.51), where we can see that, if $\bar{\rho}_0$ gets the above value, the scale factor $a_{1/2}$ on our brane vanishes, as a consequence of the presence of a horizon in this coordinate system.

We finally discuss the implications of the equations derived in this subsection for the cosmological evolution in the early Universe.

In the regime of validity of the low-energy effective theory, $\bar{\rho}_0$ behaves as ordinary dark matter in our brane. So, the constraints that we usually have for dark matter apply to it. Although in principle we cannot say much about the physics on the 0-brane (in particular "non–standard" equations of state may be expected), we assume for simplicity that $\bar{\rho}_0$ can be decomposed into a constant term $\bar{\rho}_0^\Lambda$ ($w_0 = -1$), plus matter $\bar{\rho}_0^m$ and radiation $\bar{\rho}_0^r$ components (with $w_0 = 0, 1/3$).

For what concerns the constant component, the sum of the cosmological terms $\bar{\rho}_0^\Lambda$ and $\bar{\rho}_{1/2}^\Lambda$ is bounded by the current value of the critical density, which is of order $10^{-123}\ M_P^4$. So, the amount of fine-tuning required here is the same as in usual 4-dimensional theories:

$$\bar{\rho}_0^\Lambda + \bar{\rho}_{1/2}^\Lambda = \rho_0^\Lambda + \Omega_0^4\,\rho_{1/2}^\Lambda \leq 10^{-123}\ M_P^4. \qquad (3.57)$$

The matter and radiation components also have to be fine-tuned to small values. The best current constraint on the radiation density $\bar{\rho}_0^r$ comes from nucleosynthesis: since the observed abundances of light elements are only compatible with an effective number of neutrinos $N_{eff} = 3 \pm 1$, we see that $\bar{\rho}_0^r$ is bounded by the density of one family of relativistic neutrinos. The matter density $\bar{\rho}_0^m$ is obviously bounded by the value of the critical density today. So, in the five-dimensional theory, both $\rho_0^r$ and $\rho_0^m$ have to be fine-tuned to $\sim \Omega_0^4\ \rho_{1/2}^r$ and $\sim \Omega_0^4\ \rho_{1/2}^m$, while one may naively expect $\rho_0 \sim \rho_{1/2}$ in the early Universe.

Without the knowledge of the behavior of the Randall-Sundrum model at high energy, one may have hoped that corrections to the standard Friedmann law could have solved this problem. For example, starting from $\rho_0 \sim \rho_{1/2}$ at high energy, the equations of motion of the system could have naturally led to $\rho_0 \ll \rho_{1/2}$ at temperatures of the order of the one at which primordial nucleosynthesis occurred. This analysis shows that this is not the case. Indeed, let us assume $\bar{\rho}_{1/2} \sim \bar{\rho}_0$ at the nucleosynthesis scale ($\bar{\rho}_i \sim \mathrm{MeV}^4$) and let us consider the behavior of the system when it was close to the natural cut–off $\bar{\rho}_{1/2} \sim \mathrm{TeV}^4$. Significant deviations from the standard evolution are expected if at that epoch the energy $\bar{\rho}_0$ was almost of order $M_P^2\,\mathrm{TeV}^2$ [see eq. (3.56)]. Going backwards in time, $\bar{\rho}_0$ can increase relatively to $\bar{\rho}_{1/2}$ if $w_0 > w_{1/2}$. However, assuming radiation domination on our brane above the nucleosynthesis scale, the above requirement can be met only for $w_0 \geq 2$, which does not seem to be a realistic possibility.

# Part II

# Fermion preheating

The inflationary paradigm is nowadays widely accepted as one of the fundamental ingredients for our understanding of the early Universe. Inflation is a period of accelerated (usually quasi exponential) expansion of the physical lenghts in the Universe, that provides an elegant solution to the horizon, the flatness and the monopole problems. In the original proposal by Guth [68], such accelerated expansion was achieved by assuming that a scalar field was sitting in a false vacuum state in the primordial Universe. The potential energy of the scalar field was responsible for the quasi exponential (de Sitter) expansion of our Universe. Inflation ended via a first order phase transition, with the inflaton tunneling to its true vacuum. It was soon realized that such scenario was not viable, because the bubbles of true vacuum did not coalesce [69, 70]. The new inflationary scenario proposed by Linde [71] and by Albrecht and Steinhardt [72] does overcome these difficulties. In this scenario, the scalar field condensate is not trapped in a local minimum of its potential, but *slowly rolls* along its potential, before reaching its minimum and starting oscillating about it. During the slow roll period, the potential energy of the scalar is responsible for the de Sitter expansion of the Universe. Provided the potential is flat enough, inflation leads to the solution of the horizon, the flatness and the monopole problems. In the subsequent 20 years, several models have been proposed in the context of the slow roll scenario (for a review, see [73]), either with phenomenological or with theoretical motivations. In many cases supersymmetry is a relevant feature of these models. One reason is that, in order to solve the electroweak hierarchy problem, low energy supersymmetry is often invoked, and this implies high energy supersymmetry. Moreover, supersymmetric potentials have usually several flat directions, whose flatness is not spoiled by radiative corrections, these properties being very welcome in inflationary models. Since in the analysis of such models gravity plays a crucial role, global supersymmetry is not a good approximation, and supergravity (or superstring) embeddings are often considered in inflationary model building.

The inflationary stage leaves the Universe in an extremely cold and homogeneous state, very different from the standard Hot Big Bang conditions. Therefore, a large amount of work has been devoted to the analysis of the origin of the inhomogeneities and of the matter and radiation we observe today. For what concerns the former, it was realized soon after the proposal of ref. [68] that the de Sitter expansion can provide the seeds of the structures we observe today by amplification of the quantum fluctuation of the inflaton field [74, 75, 76, 77]. In this respect, these predictions show a very good agreement with the recent observation [78] of the acoustic peaks in the fluctuations of the cosmic microwave background radiation, that represents the most relevant observational success of the inflationary paradigm. The process by which the matter and the radiation we observe today is created after the end of inflation is known as *reheating*. During reheating, the inflaton condensate (that is the only form of energy present in the Universe after all quanta of matter have been inflated away) decays into light particles, that form the primordial radiation bath.

One important quantity in the analysis of reheating is the *reheating temperature*, that is the temperature of the radiation bath when it starts dominating the energy density of the Universe. In the context of supergravity models with gravity mediated supersymmetry breaking, the parameters of the inflationary sector have to be constrained in order to avoid an excessive reheating temperature. In fact, this would lead to an overproduction of gravitinos in the primordial thermal bath. Gravitinos can easily overclose the Universe (if they are stable) or (if they decay) spoil the successful predictions of primordial nucleosynthesis through photodissociation of the light elements, thus giving rise to the so called *gravitino problem* [79, 80, 81, 82] (notice however that gravitinos could also have beneficial effects on cosmology, see for instance [83, 84, 85]). To avoid these effects, the reheating temperature after inflation cannot be larger than $\sim 10^9$ GeV.

Such a low reheating temperature, however, leads to further cosmological problems, the most striking being probably the impossibility of obtaining a successful GUT baryogenesis [60]. This simple mechanism of baryogenesis relies on the presence in the radiation bath of some particles with baryon number violating interactions. The mass of such particles is of the order of the GUT scale $\sim 10^{16}$ GeV. Therefore, due to Boltzmann suppression, a thermal bath whose highest temperature is of the order of $\sim 10^9$ GeV will not generate them, thus rendering GUT baryogenesis unviable.

For several years reheating was thought to proceed exclusively via perturbative decay of the single quanta of inflaton during the stage of oscillations of the scalar condensate [71, 86, 87]. Later on it was realized that the coherent oscillation of the inflaton about the minimum of its potential can lead to a resonant amplification of the quantum fluctuations of other fields, thus strongly modifying the dynamics of reheating. This nonperturbative phenomenon has been called *preheating* [88], since it is usually followed by a stage of perturbative decay of the inflaton that completes the reheating (there is however some exception, see [89]).

The first analyses of this phenomenon were performed in [90, 91], but its full relevance was appreciated only a few years later in the case of production of scalars [88, 92, 58]. In these works it was realized that preheating of bosons is characterized by a very efficient and explosive creation, even when single particle decay is kinematically forbidden. This is due to the coherent oscillations of the inflaton condensate, which allow stimulated particle production into energy bands with very large occupation numbers.

Less attention was initially paid to non-perturbative production of fermions, because the efficiency of this process seemed to be strongly limited by Pauli blocking, which does not allow for occupation numbers larger than one. However, also this phenomenon turned out to be very relevant. Indeed, if one only considers the most natural couplings $\phi\bar{\psi}\psi$ and $\phi^2\chi^2$ of the inflaton $\phi$ to fermions $\psi$ or to bosons $\chi$, fermionic production occurs in a mass range much broader than the one for heavy bosons. This can "compensate" the limit imposed by Pauli blocking, as the first complete numerical calculation [93] of the inflaton decay into heavy (spin 1/2) fermions during preheating showed. The analysis of the same system was also performed analytically in refs. [94, 95].[9]

The features of particle production at preheating were soon applied to several phenomenological issues. In particular, the possibility to produce extremely massive particles even with a relatively low reheating temperature was exploited to revive GUT baryogenesis [105, 106], to produce superheavy dark matter [107] (possibly responsible for the observed flux of ultra high energy cosmic rays [108, 109]), to look at leptogenesis from a different perspective [93, 110], or to consider the possible impact of fermions produced *during* inflation on the microwave background anisotropies and on the large scale structure [111].

On the other hand, nonperturbative production of matter after inflation has also originated new difficulties for models embedded in supergravity or superstring scenarios. In particular, the results of fermion production at preheating were soon applied to non-thermal gravitino production, since the equations for the different components of the gravitino field can be reduced to the one of a spin 1/2 particle. As had also been realized in [112, 113], the transverse gravitino component is always very weakly coupled to the background, so that the production of its quanta is negligible. However, the works [114, 115] studied also nonthermal creation of the longitudinal component, arguing that it easily exceeds the limits imposed by primordial nucleosynthesis. The analyses of [114, 115] were extended in [116, 117] and followed by several related works [118, 119, 120, 121, 122, 123].

Later on, it was noticed that explicit calculations of the amount of gravitinos produced at preheating were performed only in models without supersymmetry breaking in the vacuum. Therefore, the conclusions about the production of longitudinal gravitinos in these models could be somehow misleading, since there is no longitudinal gravitino in the vacuum of the theory, but only the superpartner of the inflaton, the inflatino. Thus, one might wonder whether preheating could have actually led to a production of harmless inflatinos rather than of dangerous gravitinos. In order to discriminate between inflatino and gravitino production it was necessary to consider more realistic schemes. The simplest possibility was to consider two separate sectors, one of which drives inflation, while the second is responsible for supersymmetry breaking today. This analysis was performed in refs. [124, 125], and has actually shown that gravitino production is significantly reduced if the sector responsible for supersymmetry breaking today is coupled only gravitationally to the one responsible for inflation.

From the above discussion it is clear that the effects of preheating have deeply modified our understanding of the history of the very early Universe. More generi-

[9] Among other interesting studies on production of fermions (not all of them related to preheating) we mention [96, 97, 98, 99, 100, 101, 94, 102, 89, 103, 104].

cally, the dynamics of quantum systems in a classical background plays a key role in the study of the origin of the Universe as we observe it now, since also the inhomogeneities observed today were originated by the amplification of quantum fluctuations. In the following chapters we will describe some of the properties of such dynamics, referring in particular to refs. [95, 124, 125]. In chapter 4 we will analyze the general formalism for particle production in quantum systems of coupled bosonic and fermionic fields in a time–dependent background. Then we will move to the more specific case of preheating of fermion fields. In chapter 5 we will present an analytical study of fermion preheating after chaotic inflation. In chapter 6, finally, we will apply the formulae for a system of coupled fermions in a classical background to the case of gravitino production in a two–fields model where supersymmetry is broken gravitationally in the vacuum.

# Chapter 4

# Coupled fields in external background

The analysis of quantized systems in a classical background can be very useful for the study of various phenomena that arise in quantum theories, as for example particle production. The study of matter in external electromagnetic fields [126] dates back to the first years of quantum field theory [127, 128]. For what concerns gravity [129], the semiclassical approximation is often compulsory, due to the lack of a consistent quantum theory. Despite of this, it has been very successful in describing phenomena as particle creation from black holes [130] or the generation of the perturbations in the inflationary Universe [131].

In this chapter, we provide a formalism for the quantization of coupled fields in a classical background. This will allow us to analyze the production of quanta of matter induced by the time variation of the background, with a clear definition of the occupation numbers for the physical eigenstates. In the one field case, the procedure is well established [132, 96]. One first quantizes the system and expands the canonical hamiltonian in the creation and annihilation operators of the field. The evolution of the background creates a mixing between the positive and negative energy solutions of the field equation, which has the consequence of driving the hamiltonian non diagonal, even if one takes it to be diagonal at initial time. A diagonal form is achieved through a (time dependent) redefinition of the creation/annihilation operators of the fields. The two coefficients of this diagonalization are called Bogolyubov coefficients and can be easily related to the occupation number for the quantized field. In this chapter we will generalize this procedure to systems of more than one field, both in the bosonic and in the fermionic case. By choosing a suitable expansion of the fields we can repeat each step of the above analysis replacing the Bogolyubov coefficients with two matrices $\alpha$ and $\beta$. We can obtain a system of first order differential equations for these matrices. The expression for the occupation numbers is an easy generalization of the one valid in the one field case.

The chapter is divided into two sections, the first of which is devoted to bosons, while the second one to fermions. This second section is further divided into two parts. In the first one we consider the case in which the fermionic fields are coupled only through the "mass matrix", while in the second one we consider a more general system which will be necessary for the applications described in chapter 6.

## 4.1 System of coupled bosonic fields

In this section we consider the coupled system of $N$ real bosonic fields $\{\phi_i\}$ in a FRW background described by the action

$$S = \frac{1}{2}\int d^4x\,\sqrt{-g}\Big[g^{\mu\nu}\,\partial_\mu\phi_i\,\partial_\nu\phi_i - m^2_{ij}\phi_i\,\phi_j + \xi\,R\,\phi_i\phi_i\Big]\,. \tag{4.1}$$

We use conformal time $\eta$, such that the metric and the Ricci scalar are $g_{\mu\nu} = a^2(\eta)\,\mathrm{diag}(1,\,-1,\,-1,\,-1)$ and $R = -\,6\,a''/a^3$, where $a$ is the scale factor of the Universe and prime denotes derivative with respect to the conformal time $\eta$ (summation over repeated indices is understood). The last term describes a possible non-minimal coupling ($\xi \neq 0$) of the scalar fields to gravity.

The (symmetric) mass matrix $m^2_{ij}$ is assumed to be a function of some external (background) fields. The only assumption that we do on these external fields is that they are constant (or better, adiabatically evolving) at the very beginning[1] and at the very end of the evolution of the system. In these regimes, the matrix $m^2_{ij}$ becomes also constant and the fields which diagonalize it become free fields, whose masses are precisely given by the eigenvalues of $m^2_{ij}$. However, during the evolution the different entries of $m^2_{ij}$ are allowed to vary, and the (time dependent) eigenstates of $m^2_{ij}$ are fields whose masses change in time. These masses can change nonadiabatically and this will in general lead to particle production. The aim of this section is to give a precise definition of the occupation number and to provide the formalism to calculate it.

It is most convenient to consider the "comoving" fields $\varphi_i \equiv a\,\phi_i$. For these fields, the above action (4.1) can be rewritten as[2]

$$\begin{aligned} S &= \frac{1}{2}\int d^4x\,\left[\varphi_i'\,\varphi_i' - \varphi_i\,\Omega^2_{ij}\,\varphi_j\right]\,,\\ \Omega^2_{ij} &\equiv a^2\,m^2_{ij} + \left(-\Delta + \frac{a''}{a}\,(6\,\xi - 1)\right)\delta_{ij}\,, \end{aligned} \tag{4.2}$$

[1]We require an initial stage of adiabatic evolution to consistently define vanishing occupation numbers for the bosons at initial time.

[2]We do not necessarily need a cosmological motivation for the analysis that we perform in the rest of this section. The action (4.2) could indeed also arise in flat space, with a non-diagonal $\Omega^2_{ij}$ coming from some general interactions between the bosons $\varphi_i$ and some other background fields.

where $\Delta$ is the laplacian operator. We can also write the hamiltonian of the system, which, in terms of the fields $\varphi_i$ and their conjugate momenta

$$\Pi_i \equiv \frac{\partial \mathcal{L}}{\partial \varphi_i'} = \varphi_i' \,, \tag{4.3}$$

reads

$$H \equiv \int d^3\mathbf{x}\, \mathcal{H} = \frac{1}{2} \int d^3\mathbf{x} \left( \Pi_i \, \Pi_i + \varphi_i \, \Omega_{ij}^2 \varphi_j \right) . \tag{4.4}$$

The frequency matrix $\Omega_{ij}^2$ which enters in the above expressions is in general time dependent and non-diagonal. At any given time, it can be diagonalized by an orthogonal matrix $C$

$$C^T(\eta)\, \Omega^2(\eta)\, C(\eta) = \omega^2(\eta) \qquad \text{diagonal} . \tag{4.5}$$

We denote by $\hat{\varphi} \equiv C^T \varphi$ the bosonic fields in the basis in which the frequency matrix is diagonal. We also denote by $\omega_i^2$ the $(i,\, i)$-th entry of the diagonal matrix $\omega^2$. The set of $\omega_i$ represents the energies of the (time dependent) physical eigenstates of the system $\hat{\varphi}_i$.

We now show that the occupation numbers of these fields can be defined and computed by generalizing the usual techniques based on Bogolyubov coefficients valid in the one field case. The first step to do in this direction is to consider a basis for annihilation/creation operators $\{a_i\}$ and $\{a_i^\dagger\}$ and to perform the decompositions

$$\begin{aligned} \varphi_i(x) &= C_{ij} \int \frac{d^3\mathbf{k}}{(2\,\pi)^{3/2}} \left[ \mathrm{e}^{i\,\mathbf{k}\cdot\mathbf{x}}\, h_{jk}(\eta)\, a_k(\mathbf{k}) + \mathrm{e}^{-i\,\mathbf{k}\cdot\mathbf{x}}\, h_{jk}^*(\eta)\, a_k^\dagger(\mathbf{k}) \right] , \\ \Pi_i(x) &= C_{ij} \int \frac{d^3\mathbf{k}}{(2\,\pi)^{3/2}} \left[ \mathrm{e}^{i\,\mathbf{k}\cdot\mathbf{x}}\, \tilde{h}_{jk}(\eta)\, a_k(\mathbf{k}) + \mathrm{e}^{-i\,\mathbf{k}\cdot\mathbf{x}}\, \tilde{h}_{jk}^*(\eta)\, a_k^\dagger(\mathbf{k}) \right] . \end{aligned} \tag{4.6}$$

The reason why we explicitly factorized the matrix $C$ in these decompositions will be soon clear. Due to the fact that the fields are coupled together, the matrices $h$ and $\tilde{h}$ are generically expected to be non diagonal.

To quantize the system, we impose the equal time commutation relations

$$\left[ \varphi_i(\eta,\, \mathbf{x}) \,,\, \Pi_j(\eta,\, \mathbf{y}) \right] = i\, \delta^3(\mathbf{x} - \mathbf{y})\, \delta_{ij} \tag{4.7}$$

for the conjugate fields, and

$$\left[ a_i(\mathbf{k}) \,,\, a_j^\dagger(\mathbf{p}) \right] = \delta^3(\mathbf{k} - \mathbf{p})\, \delta_{ij} \tag{4.8}$$

for the annihilation/creation operators. We can satisfy both these relations requiring

$$\left[ h\, \tilde{h}^\dagger - h^*\, \tilde{h}^T \right]_{ij} = i\, \delta_{ij} \,, \tag{4.9}$$

as it can be easily checked from the decomposition (4.6).

From the action (4.2), one deduces the second order equations of motion

$$\varphi_i'' + \Omega_{ij}^2 \varphi_j = 0 \,. \tag{4.10}$$

However, one can achieve a system of only first order equations by setting some additional relations between the conjugate fields $\varphi_i$ and $\Pi_i$. We want these relations to generalize the one which is usually taken in the one field case, see i.e. [132]. Also we want them to allow a rewriting of the hamiltonian (4.4) in a simple form. The sets of fields where this generalization is most evident is given by $\{\hat{\varphi}_i, \hat{\Pi}_i \equiv (C^T \Pi)_i\}$. These fields are decomposed as in eqs. (4.6), only without the $C$ matrix before the integrals. In terms of these fields, the hamiltonian (4.4) reads

$$H = \int d^3\mathbf{x} \, \frac{1}{2} \left( \hat{\Pi}_i \, \hat{\Pi}_i + \omega_i^2 \, \hat{\varphi}_i \, \hat{\varphi}_i \right) , \tag{4.11}$$

since the frequency $\omega$ is diagonal. One is thus led to impose the conditions[3]

$$\begin{aligned} h &= \frac{\mathrm{e}^{-i \int^\eta \omega \, d\eta'}}{\sqrt{2\,\omega}} \alpha + \frac{\mathrm{e}^{i \int^\eta \omega \, d\eta'}}{\sqrt{2\,\omega}} \beta \,, \\ \tilde{h} &= \frac{-\, i \, \omega \, \mathrm{e}^{-i \int^\eta \omega \, d\eta'}}{\sqrt{2\,\omega}} \alpha + \frac{i\omega \, \mathrm{e}^{i \int^\eta \omega \, d\eta'}}{\sqrt{2\,\omega}} \beta \end{aligned} \tag{4.13}$$

which are indeed a natural generalization of the one which is usually taken in the one field case [132]. For one field, $\alpha$ and $\beta$ are numbers, called Bogolyubov coefficients. In our case they are $N \times N$ matrices. The analysis of the system is in our case simplified if we consider, rather then the matrices $\alpha$ and $\beta$, the combinations

$$\begin{aligned} A &\equiv \mathrm{e}^{-i \int^\eta \omega \, d\eta'} \, \alpha \,, \\ B &\equiv \mathrm{e}^{i \int^\eta \omega \, d\eta'} \, \beta \,. \end{aligned} \tag{4.14}$$

The above condition (4.9) is satisfied if the matrices $A$ and $B$ obey the relations

$$\begin{aligned} A\,A^\dagger - B^*\,B^T &= \mathbb{1} \,, \\ A\,B^\dagger - B^*\,A^T &= 0 \,. \end{aligned} \tag{4.15}$$

These relations can be imposed at the initial time, and are preserved by the evolution, as we shortly discuss. In the one field case, they reduce to the usual condition $|\alpha|^2 - |\beta|^2 = |A|^2 - |B|^2 = 1$.

[3]Equations (4.13) are written in matrix notation. In general, for any function $f(\omega_i)$ and any matrix $M$, we use the notation

$$\left(f\left(\omega\right) M\right)_{ij} \equiv f\left(\omega_i\right) M_{ij} \,, \qquad \left(M \, f\left(\omega\right)\right)_{ij} \equiv M_{ij} \, f\left(\omega_j\right) . \tag{4.12}$$

As we have said, the evolution of the system can be described by two sets of first order differential equations. The first set is obtained by inserting eqs. (4.6) into the definition of the conjugate momenta, eq. (4.3)

$$h' = \tilde{h} - \Gamma\, h\,, \tag{4.16}$$

where we have defined the matrix

$$\Gamma = C^T\, C'\,, \qquad \Gamma^T = -\Gamma\,. \tag{4.17}$$

The second set of equations is obtained by rewriting eqs. (4.10) in terms of $\varphi_i$ and $\Pi_i$

$$\tilde{h}' = -\Gamma\, \tilde{h} - \omega^2\, h\,. \tag{4.18}$$

We can now use relations (4.13) and decouple the terms proportional to $A'$ and $B'$, so to arrive to the final result

$$\begin{aligned} A' &= -i\,\omega\, A + \frac{\omega'}{2\,\omega}\, B - I\, A - J\, B\,, \\ B' &= \frac{\omega'}{2\,\omega}\, A + i\,\omega\, B - J\, A - I\, B\,, \end{aligned} \tag{4.19}$$

where we have defined the matrices

$$\begin{aligned} I &= \frac{1}{2}\left(\sqrt{\omega}\,\Gamma\,\frac{1}{\sqrt{\omega}} + \frac{1}{\sqrt{\omega}}\,\Gamma\,\sqrt{\omega}\right), \qquad I^T = -I\,, \\ J &= \frac{1}{2}\left(\sqrt{\omega}\,\Gamma\,\frac{1}{\sqrt{\omega}} - \frac{1}{\sqrt{\omega}}\,\Gamma\,\sqrt{\omega}\right), \qquad J^T = J\,. \end{aligned} \tag{4.20}$$

In the one field case, $I = J = \Gamma = 0$, and the above system reduces to the equations for the two Bogolyubov coefficients

$$\alpha' = \frac{\omega'}{2\,\omega}\, \mathrm{e}^{2\,i\int^\eta \omega\, d\eta'}\, \beta\,, \qquad \beta' = \frac{\omega'}{2\,\omega}\, \mathrm{e}^{-2\,i\int^\eta \omega\, d\eta'}\, \alpha\,, \tag{4.21}$$

already discussed in the previous literature (see i.e. [132]). In the one field case the only source of nonadiabaticity is related to a rapid change of the only frequency $\omega(\eta)$, so that the system is said to evolve adiabatically as long as the condition $\omega' \ll \omega^2$ is fulfilled. In the present case, there are more sources of nonadiabaticity, related to the fact that now the frequency $\Omega_{ij}$ is a $N \times N$ matrix. This is associated with the presence of nonvanishing matrices $I$ and $J$ in the equations of motion for the matrices $A$ and $B$.

It is straightforward to show that the above equations (4.19) preserve the normalization conditions (4.15), due to the properties $I^T = -I$ and $J^T = J$.

In the one field case, the number of particles is given by the square of the modulus of the second Bogolyubov coefficient, $|\beta|^2$. We now show that also in the multi-field case it is generally related to the matrix $B$. To see this, we decompose also the energy density operator $\mathcal{H}$ (see eq. (4.4)) in the basis of annihilation and creation operators

$$\mathcal{H} = \left(a_i^\dagger ,\, a_j\right) \left( \begin{array}{cc} \mathcal{E}_{il} & \mathcal{F}_{jl}^\dagger \\ \mathcal{F}_{im} & \mathcal{E}_{jm}^T \end{array} \right) \left( \begin{array}{c} a_l \\ a_m^\dagger \end{array} \right) . \tag{4.22}$$

From eqs. (4.6), one sees that the $N \times N$ matrices $\mathcal{E}$ and $\mathcal{F}$ which enter in this decomposition are given by

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\left(\tilde{h}^\dagger\, \tilde{h} + h^\dagger\, \omega^2\, h\right) , \\ \mathcal{F} &= \frac{1}{2}\left(\tilde{h}^T\, \tilde{h} + h^T\, \omega^2\, h\right) . \end{aligned} \tag{4.23}$$

We can now generalize the procedure adopted in the one field case. The matrix that appears in eq. (4.22) can be put in diagonal form in a basis of new (time dependent) annihilation/creation operators. Only when the hamiltonian is diagonal, each pair of (redefined) operators can be associated to a physical particle, and used to compute the corresponding occupation number. The explicit computation gives

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\left(A^\dagger\, \omega\, A + B^\dagger\, \omega\, B\right) , \\ \mathcal{F} &= \frac{1}{2}\left(A^T\, \omega\, B + B^T\, \omega\, A\right) , \end{aligned} \tag{4.24}$$

so that expression (4.22) evaluates to

$$\mathcal{H} = \frac{1}{2}\left(a^\dagger ,\, a\right) \left( \begin{array}{cc} A^\dagger & B^\dagger \\ B^T & A^T \end{array} \right) \left( \begin{array}{cc} \omega & 0 \\ 0 & \omega \end{array} \right) \left( \begin{array}{cc} A & B^* \\ B & A^* \end{array} \right) \left( \begin{array}{c} a \\ a^\dagger \end{array} \right) . \tag{4.25}$$

In terms of the redefined annihilation/creation operators[4]

$$\left( \begin{array}{c} \hat{a} \\ \hat{a}^\dagger \end{array} \right) \equiv \left( \begin{array}{cc} A & B^* \\ B & A^* \end{array} \right) \left( \begin{array}{c} a \\ a^\dagger \end{array} \right) \tag{4.28}$$

[4]The relation (4.28) is inverted through the matrix

$$\left( \begin{array}{cc} A^\dagger & -B^\dagger \\ -B^T & A^T \end{array} \right) , \tag{4.26}$$

as can be easily checked from conditions (4.15). We thus see that also the relations

$$A^\dagger\, A - B^\dagger\, B = 1 , \qquad A^\dagger\, B^* - B^\dagger\, A^* = 0 \tag{4.27}$$

hold for the whole evolution.

the hamiltonian is thus diagonal (remember that in eq. (4.5) $\omega$ was defined to be diagonal), and, after normal ordering[5], it simply reads

$$H = \int d^3\mathbf{k}\, \omega_i\, \hat{a}_i^\dagger\, \hat{a}_i\,. \tag{4.29}$$

We choose at initial time $A(\eta_0) = \mathbb{1}$, $B(\eta_0) = 0$, so that conditions (4.15) are fulfilled. We also choose the initial state of the theory to be annihilated by the operators $a_i$. At any generic time, the occupation number of the $i$-th bosonic eigenstate is given by (notice that in this expression we do not sum over $i$)

$$N_i\,(\eta) = \langle \hat{a}_i^\dagger\, \hat{a}_i \rangle = \left(B^* B^T\right)_{ii}\,. \tag{4.30}$$

In the one field case the above relation reduces to the usual $N = |\beta|^2$. We see that our choices correspond to an initial vanishing occupation number for all the bosonic fields.

## 4.2 System of coupled fermionic fields

We now consider a system of coupled fermions. We divide this analysis into two subsections. The first of them extends to the fermionic case the results obtained for bosons in the previous section. Because of the repeated analogies, the discussion is here shorter than the above one, where more details can be found. In the second subsection we study a more general system of equations, which can be also relevant when the background is not constant. In particular, these can be important for cosmology, where the expansion of the Universe provides a preferred direction in time.

### 4.2.1 The simpler case

Let us consider the coupled system of $N$ Dirac fields $\{\psi_i\}$ in a FRW background described by the action

$$S = \int d^4x\, \sqrt{-g}\, \bar{\psi}_i \left[ i\, \delta_{ij} \left( \tilde{\gamma}^\mu\, \partial_\mu + \frac{3}{2}\,\frac{\dot{a}}{a}\,\tilde{\gamma}^0 \right) - M_{ij} \right] \psi_j\,. \tag{4.31}$$

The gamma matrices $\tilde{\gamma}^\mu$ in FRW geometry are related to those $(\gamma^\mu)$ in flat space by $\tilde{\gamma}^\mu = a^{-1}\,\gamma^\mu$, where $a\,(\eta)$ is the scale factor of the Universe. As before, conformal time $\eta$ is used, and the matrix $M_{ij}$ is considered to be a function of some external background fields. The requirement that the action is hermitean forces $M$ to be

[5] Notice that the normal ordering prescription depends on time through the time-dependence of the operators $\hat{a}$, $\hat{a}^\dagger$ [96].

hermitean as well. For simplicity we will also take it to be real. We also require $M_{ij}$ to be constant (better, adiabatically evolving) at very early and late times, but we do not make any other assumption on its evolution.

After the redefinitions $X_i \equiv \psi_i\, a^{3/2}$, $m \equiv a\, M$, the action (4.31) reads

$$S = \int d^4x\, \bar{X}_i \left[i\,\delta_{ij}\,\gamma^\mu\,\partial_\mu - m_{ij}\right] X_j\,, \tag{4.32}$$

leading to the equations of motion (in matrix notation)

$$(i\,\gamma^\mu\,\partial_\mu - m)\,X = 0\,. \tag{4.33}$$

The on shell canonical hamiltonian is instead

$$H \equiv \int d^3\mathbf{x}\,\mathcal{H} = \int d^3\mathbf{x}\,\bar{X}\left[-i\,\gamma^i\,\partial_i + m\right] X\,. \tag{4.34}$$

In analogy with the bosonic case, we expand the fermionic eigenstates into a basis of creation/annihilation operators

$$X_i\,(x) = C_{ij} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \Big[U_r^{jk}\,(k,\,\eta)\,a_r^k\,(k) + V_r^{jk}\,(k,\,\eta)\,b_r^{+k}\,(-k)\Big]\,, \tag{4.35}$$

where the matrix $C$ is employed into the diagonalization of the mass matrix $m$

$$\mu \equiv C^T\,m\,C\,, \qquad \mu \text{ diagonal}\,, \quad C \text{ orthogonal}\,. \tag{4.36}$$

We also define the matrix

$$\Gamma \equiv C^T\,C'\,, \qquad \Gamma^T = -\Gamma\,, \tag{4.37}$$

and the "generalized spinors"

$$U_r^{ij} \equiv \left[\frac{U_+^{ij}}{\sqrt{2}}\,\psi_r\,, r\,\frac{U_-^{ij}}{\sqrt{2}}\,\psi_r\right]^T\,, \qquad V_r^{ij} \equiv \left[\frac{V_+^{ij}}{\sqrt{2}}\,\psi_r\,, r\,\frac{V_-^{ij}}{\sqrt{2}}\,\psi_r\right]^T \tag{4.38}$$

with $\psi_+ = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\psi_- = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ eigenvectors of the helicity operator $\sigma\cdot\mathbf{v}/|\mathbf{v}|$ (we assume the momentum to be directed along the third axis).

Let us consider a set of fields $X_i$ which satisfy the above equations (4.33). Due to the fact that the matrix $m_{ij}$ is real and symmetric, then also the fields $X_i^C \equiv \tilde{C}\,\bar{X}_i^T$ (where $\tilde{C}$ is the charge conjugation matrix[6]) are solutions of (4.33).[7] As a

[6]In our computations, we take

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \tilde{C} = i\,\gamma^0\,\gamma^2 = \begin{pmatrix} 0 & i\,\sigma_2 \\ i\,\sigma_2 & 0 \end{pmatrix}, \tag{4.39}$$

where $\sigma$ are the Pauli matrices.

[7]This may allow one to consistently define the Majorana condition $X_i^C \equiv X_i$.

consequence, one can impose the relation $U_r(k) = \tilde{C}\,\bar{V}_r^T(-k)$, or, using eqs. (4.38),

$$V_+ = U_-^*\,, \qquad V_- = -U_+^*\,. \tag{4.40}$$

Therefore, we have only to deal with the $U_\pm$ matrices. Remembering that the momentum $k$ is along the third axis, their equations of motion read

$$U_\pm' = -i\,k\,U_\mp \mp i\,\mu\,U_\pm - \Gamma\,U_\pm\,. \tag{4.41}$$

The quantization of the system requires

$$\begin{aligned} \left\{X_i(\eta,\mathbf{x})\,,X_j^\dagger(\eta,\mathbf{y})\right\} &= \delta^{(3)}(\mathbf{x}-\mathbf{y})\,\delta_{ij}\,,\\ \left\{a_{r\,i}(\mathbf{k})\,,\,a_{s\,j}^\dagger(\mathbf{p})\right\} &= \delta^{(3)}(\mathbf{k}-\mathbf{p})\,\delta_{rs}\,\delta_{ij}\,,\\ \left\{b_{r\,i}(\mathbf{k})\,,\,b_{s\,j}^\dagger(\mathbf{p})\right\} &= \delta^{(3)}(\mathbf{k}-\mathbf{p})\,\delta_{rs}\,\delta_{ij}\,. \end{aligned} \tag{4.42}$$

We can simultaneously satisfy these conditions by setting[8]

$$\begin{aligned} U_+U_+^\dagger + U_-^*\,U_-^T &= 2\,\mathbb{1}\,,\\ U_+\,U_-^\dagger - U_-^*\,U_+^T &= \,. \end{aligned} \tag{4.43}$$

These conditions can be imposed at initial time, and are preserved by the evolution of the system (as it is easily checked from eqs. (4.41)).

We define the diagonal matrix[9]

$$\omega = \sqrt{k^2+\mu^2}\,, \tag{4.44}$$

and we further expand

$$\begin{aligned} U_+ &\equiv \left(1+\frac{\mu}{\omega}\right)^{1/2} e^{-i\int^\eta \omega\,d\eta'}\,\alpha - \left(1-\frac{\mu}{\omega}\right)^{1/2} e^{i\int^\eta \omega\,d\eta'}\,\beta\\ &\equiv \left(1+\frac{\mu}{\omega}\right)^{1/2} A - \left(1-\frac{\mu}{\omega}\right)^{1/2} B\,,\\ U_- &\equiv \left(1-\frac{\mu}{\omega}\right)^{1/2} e^{-i\int^\eta \omega\,d\eta'}\,\alpha + \left(1+\frac{\mu}{\omega}\right)^{1/2} e^{i\int^\eta \omega\,d\eta'}\,\beta\\ &\equiv \left(1-\frac{\mu}{\omega}\right)^{1/2} A + \left(1+\frac{\mu}{\omega}\right)^{1/2} B\,, \end{aligned} \tag{4.45}$$

[8]Notice that all this analysis generalizes the one made in the one field case. For the latter, we follow [96].

[9]This definition is meaningful, since both $\omega$ and $\mu$ are diagonal matrices. More simply, it can be understood as a relation between their eigenvalues. See also the footnote containing eq. (4.12) for some clarification about our notation.

so that the above conditions (4.43) are satisfied if the matrices $A$ and $B$ obey the relations

$$\begin{aligned} A\,A^\dagger + B^*\,B^T &= \mathbb{1}\,, \\ A\,B^\dagger - B^*\,A^T &= 0\,. \end{aligned} \tag{4.46}$$

In the one field case, $\alpha$ and $\beta$ are numbers, called Bogolyubov coefficients. In our case they are $N \times N$ matrices. The matrices $A$ and $B$ are introduced since their equations of motion assume a simpler form than the corresponding ones for $\alpha$ and $\beta$. In the one field case, the above relations (4.46) reduce to the usual condition $|\alpha|^2 + |\beta|^2 = 1$.

For fermions, the evolution equations for the matrices $A$ and $B$ can be obtained in a more straightforward way with respect to the bosonic case. This is because eqs. (4.41) are already two sets of first order differential equations. Using the above decomposition (4.45), after some algebra we arrive to the final expressions

$$\begin{aligned} A' &= [-i\,\omega - I]\,A + \left[-\frac{\dot{\mu}\,k}{2\,\omega^2} + J\right] B\,, \\ B' &= \left[\frac{\dot{\mu}\,k}{2\,\omega^2} - J\right] A + [i\,\omega - I]\,B\,, \end{aligned} \tag{4.47}$$

where we have defined the matrices

$$\begin{aligned} 2I &\equiv \left(1+\frac{\mu}{\omega}\right)^{1/2}\Gamma\left(1+\frac{\mu}{\omega}\right)^{1/2} + \left(1-\frac{\mu}{\omega}\right)^{1/2}\Gamma\left(1-\frac{\mu}{\omega}\right)^{1/2}\,, \quad I^T = -I\,, \\ 2J &\equiv \left(1+\frac{\mu}{\omega}\right)^{1/2}\Gamma\left(1-\frac{\mu}{\omega}\right)^{1/2} - \left(1-\frac{\mu}{\omega}\right)^{1/2}\Gamma\left(1+\frac{\mu}{\omega}\right)^{1/2}\,, \quad J^T = J\,. \end{aligned} \tag{4.48}$$

One can easily verify that these equations preserve the above conditions (4.46). In the case of only one field, $I = J = 0$ holds, and eqs. (4.47) simplify to

$$\alpha' = -\frac{\dot{\mu}\,k}{2\,\omega^2}\,e^{2\,i\int^\eta \omega d\eta'}\,\beta\,, \qquad \beta' = \frac{\dot{\mu}\,k}{2\,\omega^2}\,e^{-2\,i\int^\eta \omega d\eta'}\,\alpha\,. \tag{4.49}$$

To properly define and compute the occupation number for the fermionic eigenstates, as before we expand the energy density operator $\mathcal{H}$ (eq. (4.34)) in the basis of annihilation and creation operators

$$\mathcal{H} = \left(a_i^\dagger\,,\, b_j\right)\begin{pmatrix} \mathcal{E}_{il} & \mathcal{F}_{jl}^\dagger \\ \mathcal{F}_{im} & -\mathcal{E}_{jm}^T \end{pmatrix}\begin{pmatrix} a_l \\ b_m^\dagger \end{pmatrix}. \tag{4.50}$$

Using eqs. (4.35) and (4.38), we find

$$\begin{aligned} \mathcal{E}\,(\eta) &\equiv \frac{1}{2}\Big[U_+^+\,\mu\,U_+ - U_-^+\,\mu\,U_- + U_+^+\,k\,U_- + U_+^+\,k\,U_+\Big]\,, \\ \mathcal{F}\,(\eta) &\equiv \frac{1}{2}\Big[-U_+^T\,\mu\,U_- - U_-^T\,\mu\,U_+ + U_+^T\,k\,U_+ - U_-^T\,k\,U_-\Big]\,, \end{aligned} \tag{4.51}$$

while eqs. (4.45) lead to

$$\begin{aligned} \mathcal{E} &= A^\dagger \omega A - B^\dagger \omega B\,, \\ \mathcal{F} &= -A^T \omega B - B^T \omega A\,. \end{aligned} \tag{4.52}$$

We have thus

$$\mathcal{H} = \left(a^\dagger,\, b\right) \begin{pmatrix} A^\dagger & B^\dagger \\ -B^T & A^T \end{pmatrix} \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix} \begin{pmatrix} a \\ b^\dagger \end{pmatrix}. \tag{4.53}$$

In terms of the redefined annihilation/creation operators[10]

$$\begin{pmatrix} \hat{a} \\ \hat{b}^\dagger \end{pmatrix} \equiv \begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix} \begin{pmatrix} a \\ b^\dagger \end{pmatrix} \tag{4.56}$$

the hamiltonian is thus diagonal, and, after normal ordering, it simply reads

$$H = \int d^3\mathbf{k}\, \omega_i \left(\hat{a}_i^\dagger\, \hat{a}_i + \hat{b}_i^\dagger\, \hat{b}_i\right). \tag{4.57}$$

We choose at initial time $A(\eta_0) = \mathbb{1}, B(\eta_0) = 0$, so that conditions (4.46) are fulfilled. We also choose the vacuum state of the theory to be annihilated by the initial annihilation operators $a_i$ and $b_i$. At any given time, the occupation number of the $i$-th fermionic eigenstate is given by (notice that in this expression we do not sum over $i$)

$$N_i\,(\eta) = \langle \hat{a}_i^\dagger\, \hat{a}_i \rangle = \langle \hat{b}_i^\dagger\, \hat{b}_i \rangle = \left(B^* B^T\right)_{ii}. \tag{4.58}$$

In the one field case the above relation reduces to the usual $N = |\beta|^2$. We see that our choices correspond to an initial vanishing occupation number for all the fermionic fields. Notice that particles and antiparticles have the same energy and are produced in the same amount, due to the reality conditions that we have imposed on the system. Finally, we observe that the first of conditions (4.46) guarantees that Pauli blocking is always satisfied.

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[10]The matrix in eq. (4.56) is unitary, so its inverse one is precisely given by

$$\begin{pmatrix} A^\dagger & B^\dagger \\ -B^T & A^T \end{pmatrix}, \tag{4.54}$$

as can be easily checked from conditions (4.46). We thus see that also the relations

$$A^\dagger A + B^\dagger B = \mathbb{1}\,, \qquad A^\dagger B^* - B^\dagger A^* = 0 \tag{4.55}$$

hold for the whole evolution.

### 4.2.2 A more general case

We now consider a more general action for the coupled system of $N$ fermionic fields. For future convenience, here we switch to the signature $-,+,+,+$ for the Minkowski metric, and we then work with the gamma matrices

$$\bar{\gamma}^0 = \begin{pmatrix} -i\,\mathbb{1} & 0 \\ 0 & i\,\mathbb{1} \end{pmatrix}, \qquad \bar{\gamma}^i = \begin{pmatrix} 0 & -i\,\sigma_i \\ i\,\sigma_i & 0 \end{pmatrix} \tag{4.59}$$

in flat space.

By a suitable conformal rescaling of the fermionic fields and of their masses as we did before eq. (4.32), we can again remove the scale factor of the Universe from the kinetic term for the fermions. However, we are now interested in a more generic system, so that we consider, instead of (4.32), the action

$$S = \int d^4x\, \bar{X}_m \left[\bar{\gamma}^0 \partial_0 + \bar{\gamma}^i\, N\, \partial_i + M\right]_{mn} X_n\,, \tag{4.60}$$

where $N$ and $M$ are two $N \times N$ matrices of the form

$$N \equiv N_1 + \bar{\gamma}^0\, N_2\,, \qquad M \equiv M_1 + \bar{\gamma}^0\, M_2\,. \tag{4.61}$$

The matrices $N$ and $M$ are assumed to be functions of some external fields. We consider a situation in which these fields evolve in time. This time dependence justifies the general form for the action that we want to discuss. As we will see in chapter 6, this analysis can have relevance for cosmology, where the expansion of the Universe provides a natural direction for time. However, the system (4.60) could also arise in flat space from some general interactions between the fermions $X_i$ and other background fields. As in the previous analyses, our main goal is to discuss the definition of the occupation number of the physical eigenstates of the system, and to provide the formalism to calculate it.

We list here our assumptions on the matrices $M$ and $N$. First, we require them to change adiabatically at initial times, so to consistently define the initial occupation numbers. Then, we assume $M_1 \to \text{constant}, M_2 \to 0, N \to \mathbb{1}$ at late times, so to recover a system of "standard" decoupled particles at the end (indeed, one can choose the basis of fields $X_i$ such that the matrix $M$ is diagonal at the end). The requirement of an hermitean action translates into the conditions

$$N_i^\dagger = N_i\,, \qquad M_1 = M_1^\dagger\,, \qquad M_2 = -M_2^\dagger\,. \tag{4.62}$$

For simplicity, we will only consider real matrices, so that $N_1$, $N_2$ and $M_1$ are required to be symmetric, while $M_2$ antisymmetric. Finally, we impose an additional condition, which is that the kinetic term for the fermions "squares" to the D'Alambertian operator $\Box$. If we take the equations of motion following from (4.60),

$$\left[\bar{\gamma}^0 \partial_0 + \bar{\gamma}^i\, N\, \partial_i + M\right] X = 0\,, \tag{4.63}$$

and we multiply them on the left by $[\bar{\gamma}^0\partial_0 + \bar{\gamma}^i N \partial_i - M]$, we get

$$\begin{aligned}\{\partial_0^2 - N^\dagger N \partial_i^2 + M^\dagger M + \bar{\gamma}^0 \bar{\gamma}^i (\partial_0 N) \partial_i + \\ - \bar{\gamma}^0 (\partial_0 M) + \left(\bar{\gamma}^i N M - M \bar{\gamma}^i N\right) \partial_i\} X = 0 .\end{aligned} \tag{4.64}$$

We thus require $N^\dagger N \equiv \mathbb{1}$, that is

$$N_1^2 + N_2^2 = \mathbb{1}, \qquad [N_1, N_2] = 0 . \tag{4.65}$$

Our strategy is to reduce this problem to the one we have already discussed. That is, we perform some redefinitions of the fields to put the action (4.60) into the form (4.32), where we perform the canonical quantization of the system in the way described in the previous subsection. The first of these redefinitions strongly relies on the above conditions (4.65). If $N$ is a unitary matrix, we can find a hermitean matrix $\Phi$ such that

$$N = \exp\left(2\,\Phi\,\bar{\gamma}^0\right), \qquad \Phi^\dagger = \Phi . \tag{4.66}$$

Due to the properties of the $\bar{\gamma}^0$ matrix, this amounts to

$$\cos\left(2\,\Phi\right) = N_1 , \qquad \sin\left(2\,\Phi\right) = N_2 . \tag{4.67}$$

After the redefinition $X \equiv \exp(-\bar{\gamma}^0\,\Phi)\hat{X}$, the equations of motions (4.63) acquire the form

$$\left(\bar{\gamma}^0\,\partial_0 + i\,\bar{\gamma}^i\,k_i + \hat{M}\right)\hat{X} = 0 , \tag{4.68}$$

where we have expanded the fermions into plane waves $X_i(\eta,\,\mathbf{k}) = e^{i\,\mathbf{k}\cdot\mathbf{x}}\,X_i(\eta)$ and introduced the new "mass matrix"

$$\begin{aligned}\hat{M} &= \exp\left(\bar{\gamma}^0\,\Phi\right)\left[M + \bar{\gamma}^0\partial_0\right]\exp\left(-\bar{\gamma}^0\,\Phi\right) \\ &\equiv \hat{M}_1 + \bar{\gamma}^0\,\hat{M}_2 .\end{aligned} \tag{4.69}$$

Notice that the two matrices $\hat{M}_1$ and $\hat{M}_2$ are symmetric and antisymmetric, respectively. This means that, in the one field case, one recovers the standard equation

$$\left(\bar{\gamma}^0\,\partial_0 + i\,\bar{\gamma}^i\,k_i + m\right)\hat{X} = 0 \tag{4.70}$$

for spin 1/2 fermions.

We have to perform a further redefinition of the fields.[11] Setting $\hat{X} = L\,\Xi$, we have

$$L^T \left[L\left(\bar{\gamma}^0\,\partial_0 + i\,\bar{\gamma}^i\,k_i\right) + \hat{M}_1\,L + \bar{\gamma}^0\left(\partial_0 + \hat{M}_2\right)L\right]\Xi = 0\,. \qquad (4.72)$$

The matrix $L$ can be chosen such that $(\partial_0 + \hat{M}_2)L = 0$. In particular, since $\hat{M}_2$ is antisymmetric and real, $L$ can be taken orthogonal. The equations of motion for the fields $\Xi$ can be thus cast in the form

$$\left(\bar{\gamma}^0\,\partial_0 + i\,\bar{\gamma}^i\,k_i + L^T\,\hat{M}_1\,L\right)\Xi = 0\,, \qquad (4.73)$$

that is with the identity matrix multiplying the term which depends on the momentum and without any $\bar{\gamma}^0$ dependence in the "mass" matrix.

These equations (and the respective action for the fields $\Xi_i$) are exactly of the form considered in the previous subsection, so that we can apply the quantization procedure discussed there. As before, the procedure is to canonically define the hamiltonian starting from the set of fields $\Xi$ and to expand it in a basis of creation/annihilation operators. The occupation numbers are then calculated after the diagonalization of the hamiltonian. It is possible to show that this procedure can be carried out starting from any of the basis for the fermionic fields, once the hamiltonian has been canonically defined in the basis $\Xi$. One can indeed explicitly verify that these calculations lead to the same results for the occupation numbers of the physical eigenstates.

We thus have

$$\begin{aligned} H &\equiv \bar{\Xi}\left[i\,\bar{\gamma}^i\,k_i + L^T\,\hat{M}_1\,L\right]\Xi = \bar{\hat{X}}\left[i\,\bar{\gamma}^i\,k_i + \hat{M}_1\right]\hat{X} \\ &= \bar{X}\left[i\,\bar{\gamma}^i\,k_i\,\mathrm{e}^{2\,\bar{\gamma}^0\,\Phi} + \mathrm{e}^{-\bar{\gamma}^0\,\Phi}\,\hat{M}_1\,\mathrm{e}^{\bar{\gamma}^0\,\Phi}\right]X\,, \end{aligned} \qquad (4.74)$$

depending on which basis we work. In particular, when working with the $\hat{X}_i$ or the $X_i$ fields, the explicit knowledge of the matrix $L$ is not needed. We present here the computation in the initial basis $X_i$, which we found more convenient in the numerical calculations for the application that will presented in chapter 6. In this basis, the hamiltonian (4.74) has the form

$$H = \bar{X}\left[i\,\bar{\gamma}^i\,k_i\,N + \tilde{M}_1 + \bar{\gamma}^0\,\tilde{M}_2\right]X\,, \qquad (4.75)$$

[11]Contrarily to naive expectations, the combination

$$\cos\,\Phi\,\partial_0\,\cos\,\Phi + \sin\,\Phi\,\partial_0\,\sin\,\Phi \subset \hat{M}_2 \qquad (4.71)$$

can be non vanishing at late times, even if the matrix $N$ is approaching $\mathbb{1}$. This occurs for example in the application that we discuss in chapter 6. If in that case we canonically defined the hamiltonian $H$ starting with the fields $\hat{X}$, the term (4.71) would give $H$ a contribution proportional to $\bar{\gamma}^0$ which does not vanish at late times. The procedure described in the text removes this problem.

where the matrices $\tilde{M}_1$ and $\tilde{M}_2$ can be obtained from eqs. (4.74) and (4.69). At the end of the evolution, we simply have $\tilde{M}_1+\bar{\gamma}^0\,\tilde{M}_2 = M_1$ diagonal, so that one recovers the "standard" hamiltonian for a system of $N$ decoupled fermions whose masses coincide with the ones of the equations of motion (which also become "standard").

To analyze the system during the evolution, we instead decompose the spinors $X_i$ as in eqs. (4.35) and (4.38).[12] Taking the third coordinate along the momentum $k$, the equations of motion (4.63) read:

$$U_\pm' = \mp i\,(M_1 \mp i\,M_2)\,U_\pm - i\,k\,\left(N_1 \pm i\,N_2\right)U_\mp\,. \tag{4.77}$$

It is straightforward to check that they preserve the conditions

$$U_+\,U_+^\dagger + U_-^*\,U_-^T = 2\cdot\mathbb{1}\,, \qquad U_+\,U_-^+ = U_-^*\,U_+^T\,, \tag{4.78}$$

which ensure the consistency of the canonical quantization.

We then expand the hamiltonian formally as in eq. (4.50), where now the $\mathcal{E}$ and $\mathcal{F}$ matrices read

$$\begin{aligned}
\mathcal{E} &= U_+^\dagger\,k\,[N_1 + i\,N_2]\,U_- + U_-^\dagger\,k\,[N_1 - i\,N_2]\,U_+ + \\
&\quad + U_+^\dagger\left[\tilde{M}_1 - i\,\tilde{M}_2\right]U_+ + U_-^\dagger\left[-\tilde{M}_1 - i\,\tilde{M}_2\right]U_-,\\
\mathcal{F} &= U_+^T\,k\,[-N_2 - i\,N_1]\,U_+ + U_-^T\,k\,[-N_2 + i\,N_1]\,U_- + \\
&\quad + U_+^T\left[-\tilde{M}_2 + i\,\tilde{M}_1\right]U_- + U_-^T\left[-\tilde{M}_2 + i\,\tilde{M}_1\right]U_+\,.
\end{aligned} \tag{4.79}$$

We notice the properties

$$\mathcal{E}^\dagger = \mathcal{E}\,, \qquad \mathcal{F}^T = \mathcal{F}\,. \tag{4.80}$$

The matrix entering in eq. (4.50) is hermitian, and can be diagonalized by a unitary matrix $C$

$$C\,\overline{H}\,(k,\eta)\,C^\dagger = H_d\,(k,\eta) \qquad \text{diagonal}\,, \tag{4.81}$$

such that the energy density is

$$\mathcal{H} = \left(a^+, b\right)\overline{H}\begin{pmatrix} a \\ b^\dagger \end{pmatrix} \equiv \left(\hat{a}^+, \hat{b}\right)H_d\begin{pmatrix} \hat{a} \\ \hat{b}^\dagger \end{pmatrix}. \tag{4.82}$$

[12] The charge conjugation matrix now reads $\tilde{C} = -\bar{\gamma}^0\,\bar{\gamma}^2$, so that conditions (4.40) are replaced by

$$V_+ = -i\,U_-^*\,, \qquad V_- = i\,U_+^*\,. \tag{4.76}$$

A first step in this diagonalization can be made by noticing that the matrix $\overline{H}$ can be rewritten as

$$\overline{H} = \mathcal{U}^\dagger \, \overline{H}_0 \, \mathcal{U}\,, \tag{4.83}$$

with

$$\overline{H}_0 \equiv \begin{pmatrix} -\tilde{M}_1 + i\,\tilde{M}_2 & k\,N_1 + i\,k\,N_2 \\ k\,N_1 - i\,k\,N_2 & \tilde{M}_1 + i\,\tilde{M}_2 \end{pmatrix} \tag{4.84}$$

hermitean and

$$\mathcal{U} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} U_+ & -i\,U_-^* \\ -U_- & -i\,U_+^* \end{pmatrix} \tag{4.85}$$

unitary, as it follows from eqs. (4.78).

We are not able to provide general analytical formulae for the diagonalization of the remaining matrix $\overline{H}_0$. This diagonalization can however be performed numerically. In addition, some important conclusions can be drawn from the properties of the matrix $\mathcal{H}$. Due to the relations (4.80), one can show (i.e. by counting the number of independent equations that must be satisfied) that the matrix $C$ entering in eq. (4.81) can be of the form

$$C \equiv \begin{pmatrix} I & J \\ i\,J^* & -i\,I^* \end{pmatrix} \tag{4.86}$$

(where $I$ and $J$ are $N \times N$ matrices). Unitarity of $C$ requires

$$I\,I^\dagger + J\,J^\dagger = \mathbb{1}\,, \qquad I^\dagger\,J = J^T\,I^*\,. \tag{4.87}$$

By explicitly performing the product (4.81) one realizes that the eigenvalues of the hamiltonian occur in pairs, and that $H_d$ is of the form $\mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_N, -\omega_1, -\omega_2, \ldots, -\omega_N)$. The eigenstates corresponding to each couple $\pm\,\omega_i$ are interpreted as particle and antiparticle states with the same energy. Finally, it is possible to show that particles and antiparticles are produced in the same amount. Defining the vacuum state to be annihilated by the initial annihilation operators $a_i$ and $b_i$, we have indeed (we remind that in this expression we do not sum over $i$)

$$N_i\,(\eta) = \langle \hat{a}_i^\dagger\,\hat{a}_i \rangle = \langle \hat{b}_i^\dagger\,\hat{b}_i \rangle = \left(J\,J^\dagger\right)_{ii}\,. \tag{4.88}$$

We assume that no fermionic particles are present at initial time $\overline{\eta}$. In our formalism, this is equivalent to requiring $J(\overline{\eta}) = 0$. Notice also that the unitarity condition (4.87) ensures that the Pauli principle is always fulfilled.

# Chapter 5

# Fermion preheating after chaotic inflation

As a first application of the formulae we have shown in the previous chapter, we present here an analytical study of the preheating of massive fermions after chaotic inflation.

We will consider the case of a single fermion field, therefore the formalism we will use is the well known one for the analysis of systems with one single quantum field in a classical background. In particular, we will refer to the system considered in ref. [93], where the first full numerical analysis of preheating of massive fermions after chaotic inflation in an expanding Universe was performed. This will allow us to give an analytical confirmation to the results of production of massive fermions in the expanding Universe.

In the next section we will describe the system we are going to analyze. We consider creation of very massive particles right after chaotic inflation. The coupling of the fermions to the oscillating inflaton gives them a time-dependent mass, that will lead to a nonadiabatical change of the frequency of the fermions and their consequent creation. In case of very massive fermions, the nonadiabaticity condition can be satisfied only at the moments when their total mass vanishes, and the production occurs at discrete intervals, until the amplitude of the inflaton oscillations become too small for the total fermionic mass to vanish.

In section 5.2 we derive analytical formulae for the spectra of the fermions after a generic production. This derivation (made in close analogy with the one of ref. [58] for the bosonic case) exploits the fact that the production occurs in very short intervals around the zeros of the total fermionic mass: the calculation is made possible by the fact that the occupation number can be considered as constant outside these short periods, and that the expansion of the Universe can be neglected inside them. As a result, the only physical quantities relevant for the creation are the time derivative $\phi'$ of the inflaton field and the value of the scale factor $a$ of the

Universe at each production.

In section 5.3 we consider the production in a non-expanding Universe. In this case the analytical formulae simplify considerably. In particular, they show the presence of resonance bands which are anyhow limited by Pauli blocking.

In section 5.4, the more interesting case of production in an expanding Universe is described. The creation is now very different with respect to the previous case. The expansion removes the resonance bands and the production (almost) saturates a Fermi sphere up to a maximal momentum. In section 5.4 the total energy density $\rho_X$ of produced fermions is also calculated. To this aim, a proper average of the analytical formulae has to be done, exploiting the fact that the expansion of the Universe gives the production a stochastic character. In this way one can get a "mean" function that interpolates very well between the maxima and the minima of the spectra of produced particles. The results agree extremely well with the ones of ref. [93] and extend their range of validity, showing that $\rho_X$ actually decreases for relatively small values of the mass $m_X$ of the fermions.

All this analysis neglects the backreaction of the produced fermions on the evolution of the inflaton field and of the scale factor. Despite the difficulty of a more complete treatment, backreaction effects can be understood at least in the Hartree approximation. This was also done numerically in ref. [93]. In section 5.5 we see that the analytical formulae here provided allow to understand the effects of backreaction observed in the numerical simulations.

## 5.1 Production of fermions at preheating

The system we analyze is the one considered in section 4.2.1 in the simpler case of only one fermion field. The equation of motion for the canonically normalized field $X$ thus reads

$$(i\,\gamma^\mu\,\partial_\mu - m\,(\eta))\,X = 0\,. \tag{5.1}$$

Fermion production is possible if the effective mass $m\,(\eta)$ varies nonadiabatically in function of the conformal time $\eta$. This may happen during the coherent oscillations of the inflaton $\phi$ at reheating in presence of a Yukawa interaction $\phi\bar{X}X$, such that the total effective mass is given by

$$m(\eta) = a\,(\eta)\,\left(m_X + g\phi(\eta)\right)\,. \tag{5.2}$$

From eqs. (4.49) and (4.58) it is immediate to see that in the single field case nonadiabaticity (and therefore variation in time of the occupation number) occurs whenever $m' \gtrsim m^2$. If the "bare" mass $m_X$ is very high, non-adiabaticity can be achieved if the Yukawa interaction is sufficiently strong to make the total mass (5.2) vanish. Fermions are then created whenever the inflaton field crosses the value $\phi_* \equiv -m_X/g$.

We will study the production after chaotic inflation, that is while the inflaton field coherently oscillates about the minimum of the potential

$$V = \frac{1}{2} m_\phi^2 \phi^2 , \quad m_\phi \simeq 10^{13}\,\mathrm{GeV} . \tag{5.3}$$

If one neglects the backreaction of the created particles (this effect will be considered in the last part of this chapter), then, after few oscillations, the inflaton evolves according to the formula

$$\phi(t) \simeq \frac{M_{Pl}}{\sqrt{3\pi}} \frac{\cos(m_\phi t)}{m_\phi t} , \tag{5.4}$$

where $t$ is the physical time. The presence of the $t$ at the denominator of eq. (5.4) shows the damping of the oscillations due to the expansion of the Universe. Thus, it follows that there exists a final time after which $|\phi| < m_X/g$ and the total mass no longer vanishes, so that the production ends.

To proceed with our analysis, we decompose the fermion field $X$ as in eq. (4.35). In the single field case, the equations of motion (4.41) for the $U_\pm$ functions read

$$U_\pm'(\eta) = -i\,k\,U_\mp(\eta) \mp i\,m\,U_\pm(\eta) , \tag{5.5}$$

which can be decoupled into

$$U_\pm'' + \left[\omega_k^2 \pm i\,m'\right] U_\pm = 0 , \qquad \omega^2(\eta) = k^2 + m^2 . \tag{5.6}$$

For the present analysis it will be more useful to solve the equations of motion (5.5) and (5.6) for $U_\pm$ rather than the ones for $\alpha$ and $\beta$. Then, once $U_\pm(\eta)$ are given, it will be straightforward to compute the Bogolyubov coefficients using eq. (4.45).

## 5.2 Analytical evaluation of the occupation number

In this section we calculate analytically the evolution of the Bogolyubov coefficients during the oscillations of the inflaton field after chaotic inflation.

In this derivation, we exploit the fact that, in the regime of very massive fermions we are interested in, the creation occurs only for very short intervals about the points $\phi_* \equiv -m_X/g$ where the total fermionic mass (cfr. eq. (5.2)) vanishes. As the perfect agreement with the numerical results will confirm, this consideration allows one to treat the fermionic production with the same formalism adopted in the bosonic case [58]. While far from the zeros of the total mass $m$ the Bogolyubov coefficients are essentially constant, whenever $\phi$ crosses $\phi_*$ a sudden variation occurs. Since the interval of production is very narrow, one can safely neglect the expansion of the Universe during the production and also linearize the function $\phi(\eta) \simeq \phi_* + \phi'(\eta_*)(\eta - \eta_*)$. As a consequence, the frequency $\omega$ defined in eq. (5.6) acquires the form

$$\omega^2 \simeq k^2 + a^2(\eta_*)\,\phi'^2(\eta_*)\,(\eta - \eta_*)^2 , \tag{5.7}$$

and the whole calculation strongly resembles the one for scattering through a quadratic potential.

The use of this formalism is very well established in case of production of bosons [58]. For what concerns fermions, it has been recently adopted in ref. [111] for the study of the production during inflation. Fermionic production during inflation is possible only if the coefficient $g$ of the Yukawa interaction has opposite sign with respect to the inflaton field during inflation, or the total mass (5.2) would never vanish.[1] If this is the case, it is possible to choose the value of $g$ such that the production occurs only once during inflation, while during reheating $|\phi|$ is always too small for having creation. The present derivation is strongly inspired by this work. However, we are interested in couplings for which the production occurs several times during reheating, and it is not at all guaranteed a priori that an analytical approximation may work also in this case. The following analysis not only positively answers to this question, but also provides very simple formulae valid for arbitrary number of productions.

We decompose the functions $U_\pm(\eta)$ as in eq. (4.45)

$$\begin{aligned} U_+ &= \alpha\left(1+\frac{m}{\omega}\right)^{1/2} e^{-i\int^\eta \omega_k\, d\eta} - \beta\left(1-\frac{m}{\omega}\right)^{1/2} e^{i\int^\eta \omega_k\, d\eta} , \\ U_- &= \beta\left(1+\frac{m}{\omega}\right)^{1/2} e^{i\int^\eta \omega_k\, d\eta} + \alpha\left(1-\frac{m}{\omega}\right)^{1/2} e^{-i\int^\eta \omega_k\, d\eta} . \end{aligned} \tag{5.8}$$

When $\phi$ is not very close to $\phi_*$, the adiabaticity condition $\omega' \ll \omega^2$ holds and eqs. (5.8) are a solution to the equations of motion (5.6) and (5.5), with $\alpha$ and $\beta$ constant. In general $\alpha$ and $\beta$ will be functions of $\eta$, but in most of the evolution (whenever the adiabaticity condition holds) it is a very good approximation to treat the coefficients $\alpha$ and $\beta$ as constant.

As we have said, these coefficients undergo a sudden change whenever $\phi$ crosses $\phi_*$ and then they stabilize to new (almost) constant values. Our aim is to find the values at the end of the variation in terms of the ones prior to it.

[1] Of course this constraint does not apply to our case, since the inflaton field changes sign after each half oscillation.

We have not specified the lower limit of the integrals appearing in eqs. (5.8). For present convenience we choose it to be the time $\eta_{*1}$ of the first production (that is when $\phi = \phi_*$ for the first time).

Let us consider the evolution equation (5.6) near the point $\eta_{*1}$. Since for high mass $m_X$ the fermionic production is limited to a very short interval, one can neglect the expansion of the Universe during it and write the equation for $\phi(\eta)$ in a linearized form. We can thus write

$$m(\eta) \simeq a_{*1}\, g\, \phi'_{*1}\, (\eta - \eta_{*1})\,, \qquad a_{*1} \equiv a\,(\eta_{*1})\,, \qquad \phi_{*1} \equiv \left.\frac{d\,\phi}{d\,\eta}\right|_{\eta_{*1}} . \tag{5.9}$$

Following the notation of [111], we define

$$p \equiv \frac{k}{\sqrt{g|\phi'_{*1}|a_{*1}}}\,, \qquad \tau = \sqrt{g|\phi'_{*1}|a_{*1}}\,(\eta - \eta_{*1})\,. \tag{5.10}$$

In terms of these new quantities, eqs. (5.6) can be rewritten as (in this chapter dot will denote derivative with respect to $\tau$)[2]

$$\ddot{U}_\pm + \left(p^2 \mp i + \tau^2\right) U_\pm = 0\,. \tag{5.12}$$

The point $\eta_{*1}$ is thus mapped into the origin of $\tau$ and the region of asymptotic adiabaticity is at large $|\tau|$.

In the asymptotic solutions (5.8) we can see the behaviors

$$\begin{aligned} \left(1 + \frac{m}{\omega}\right)^{1/2} &\longrightarrow \frac{p}{\sqrt{2}\tau}\,, \\ \left(1 - \frac{m}{\omega}\right)^{1/2} &\longrightarrow \sqrt{2}\,, \\ e^{\pm i \int^\eta \omega_k d\eta} &\longrightarrow \left(\frac{2\tau}{p}\right)^{\pm i p^2/2} e^{\pm i \tau^2/2} e^{\pm i p^2/4}\,, \end{aligned} \tag{5.13}$$

for $\tau \to +\infty$, and

$$\begin{aligned} \left(1 + \frac{m}{\omega}\right)^{1/2} &\longrightarrow \sqrt{2}\,, \\ \left(1 - \frac{m}{\omega}\right)^{1/2} &\longrightarrow \frac{p}{-\sqrt{2}\tau}\,, \\ e^{\pm i \int^\eta \omega_k d\eta} &\longrightarrow \left(\frac{p}{-2\tau}\right)^{\pm i p^2/2} e^{\mp i \tau^2/2} e^{\mp i p^2/4}\,, \end{aligned} \tag{5.14}$$

[2] For the production at the moment $\eta_{*n}$, when the total mass vanishes for the n-th time, eq. (5.12) must be replaced by

$$\ddot{U}_\pm + \left(p^2 \pm i\,\mathrm{sign}\,(\phi'_{*n}) + \tau^2\right) U_\pm = 0\,. \tag{5.11}$$

The effect of this replacement on the final results is reported below.

for $\tau \to -\infty$.

Equations (5.12) are solved [111] by parabolic cylinder functions $D_\lambda(z)$ [133]. More precisely, the combination that matches the asymptotic solution (5.8) at $\tau \to -\infty$ is[3]

$$U_+(\tau) = \alpha^- \sqrt{2} \left(\frac{p}{\sqrt{2}}\right)^{-ip^2/2} e^{ip^2/4} e^{-\pi p^2/8} D_{ip^2/2}(-(1-i)\tau) +$$
$$- \beta^- \sqrt{2} \left(\frac{p}{\sqrt{2}}\right)^{1+ip^2/2} e^{i(\frac{\pi}{4}-\frac{p^2}{4})} e^{-\pi p^2/8} D_{-1-ip^2/2}\left(-\left(1+i\right)\tau\right) \,. \quad (5.15)$$

In the above expression, $\alpha^-$ and $\beta^-$ denote the values of the Bogolyubov coefficients before $\phi$ crosses $\phi_*$, while the functions which multiply them are *exact* solutions of the *linearized* equation (5.12). The analytical approximation consists in considering them as solutions of the true evolution equation (5.6). For $\tau \to +\infty$ it is convenient to rewrite the solution (5.15) in terms of two different parabolic cylinder functions

$$U_+(\tau) = \left[\beta^- e^{-\pi p^2/2} + \alpha^- \frac{\sqrt{\pi}\, p\, e^{-\pi p^2/4}}{\Gamma\left(1 - i\, p^2/2\right)}\, e^{-i(\frac{\pi}{4}-\frac{p^2}{2}+\frac{p^2}{2}\ln\frac{p^2}{2})}\right] \times$$
$$\times \left\{\sqrt{2}\left(\frac{p}{\sqrt{2}}\right)^{1+i\,p^2/2} e^{i(\frac{\pi}{4}-\frac{p^2}{4})} e^{-\pi p^2/8} D_{-1-ip^2/2}((1+i)\tau)\right\} +$$
$$- \left[\beta^- \frac{\sqrt{\pi}\, p\, e^{-\pi p^2/4}}{\Gamma\left(1 + ip^2/2\right)} e^{i(\frac{\pi}{4}-\frac{p^2}{2}+\frac{p^2}{2}\ln\frac{p^2}{2})} - \alpha^- e^{-\pi p^2/2}\right] \times$$
$$\times \left\{\sqrt{2}\left(\frac{p}{\sqrt{2}}\right)^{-ip^2/2} e^{ip^2/4} e^{-\pi p^2/8} D_{ip^2/2}((1-i)\tau)\right\} . \quad (5.16)$$

In this new expression, the functions within curly brackets correspond to the asymptotic forms at $\tau = +\infty$ of the two terms of the solution (5.8). The coefficients in front of them give thus the new Bogolyubov coefficients in terms of the old ones.

All this derivation can be easily generalized when productions at successive zeros of the total mass $m$ are considered. The only important points are

(i) different values of the scale factor $a$ and of the derivative $\phi'$ at different $\eta_{*i}$'s,

(ii) a change of sign in the transfer matrix (the one which gives the new coefficients in terms of the old ones) whenever $\phi$ crosses $\phi_*$ from below to above (cfr. the footnote just before eq. (5.12)), and

[3] We deal only with the function $u_+$, since the study of $u_-$ leads to the same results.

(iii) the phase $e^{\pm i\int \omega_k d\eta}$ which accumulates between $\eta_{*1}$ and the $\eta_{*i}$ considered.

Putting all this together, one has

$$\begin{aligned} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} &= \begin{pmatrix} F_n & H_n \\ -H_n^* & F_n^* \end{pmatrix} \begin{pmatrix} \alpha_{n-1} \\ \beta_{n-1} \end{pmatrix} \qquad \text{for } n \text{ odd}, \\ H_n &\longleftrightarrow -H_n \qquad \text{for } n \text{ even}, \end{aligned} \tag{5.17}$$

where $\alpha_n, \beta_n$ are the values of the Bogolyubov coefficients after the $n$-th production, and where

$$\begin{aligned} F_n &= \sqrt{1-e^{-\pi p_n^2}}\, e^{i\left(\frac{\pi}{4}+\arg\Gamma(ip_n^2/2)-\frac{p_n^2}{2}\ln(p_n^2/2)+p_n^2/2\right)}, \\ H_n &= e^{-\pi p_n^2/2}\, e^{2i\int_{\eta_{*1}}^{\eta_{*n}} \omega_k\, d\eta}, \qquad |F_n|^2+|H_n|^2=1. \end{aligned} \tag{5.18}$$

We remind that $p_n = k/\sqrt{g|\phi'_{*n}|a_{*n}}$.

If one starts with no fermions at the beginning, one may choose $\alpha_0 = 1, \beta_0 = 0$. Then, applying successive times the "transfer" matrix (5.17), one can get the spectrum of fermions produced after every $\eta_{*n}$.

Of course our calculation reproduces the result

$$N_1 = |\beta_1|^2 = e^{-\pi p_1^2} \tag{5.19}$$

reported in ref. [111].

We numerically integrated the evolution equations for $U_\pm$, $\phi$, and the scale factor $a$.[4] The results obtained with the analytical expression (5.17) are always in very good agreement with the numerical ones. Just to give a couple of examples, we present here two cases at different regimes (we show them only with illustrative purpose, and the values of the parameters chosen have no particular significance). In figure 5.1 we present the spectrum of the fermions after two productions, that is after one complete oscillation of the inflaton field. In analogy with the bosonic case, we measure the strength of the coupling inflaton-fermions with the quantity $q \equiv g^2\,\phi_0^2/(4\,m_\phi^2)$, where $\phi_0 \simeq 0.28 M_{Pl}$ is the value of the inflaton at the beginning of reheating. In figure 5.1 we choose $q = 10^8$, while we fix the bare fermion mass to be $m_X = 100\,m_\phi$. In figure 5.2 we show instead the resulting spectrum after seven productions in the case $q = 10^4$, $m_X = 4\,m_\phi$.

## 5.3 Production in a non-expanding Universe

In the bosonic case, the study of the non-perturbative production in a non-expanding Universe has proven very useful in understanding the effects of the production. It

[4] Our starting point is at $\phi(0) = 0.28 M_{Pl}$, short after inflation, $\phi'(0) = -0.15 M_{Pl} m_\phi$ (as follows from a numerical evaluation of the inflaton alone during inflation), and $a(0) = 1$.

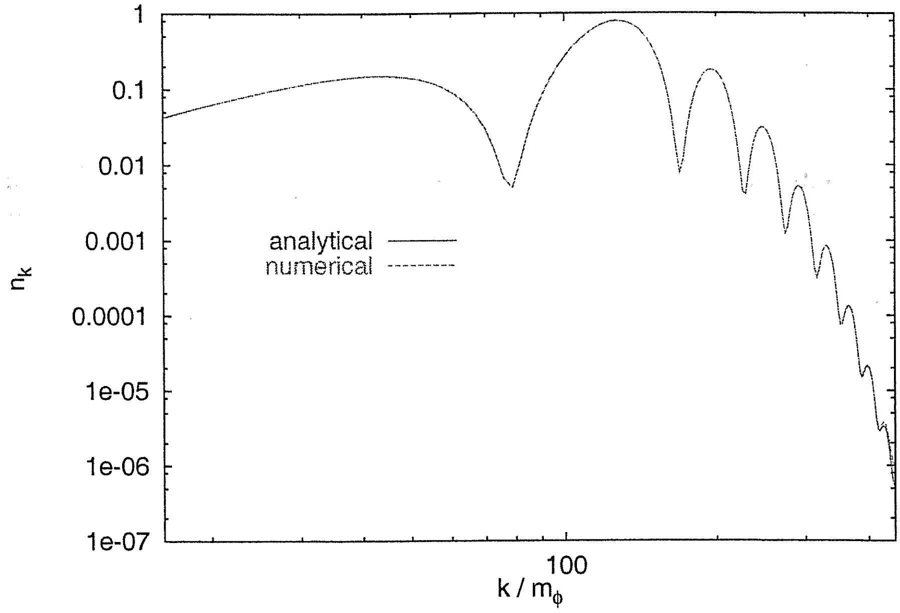


Figure 5.1: Spectrum of the fermions after two productions for $q = 10^8$ and $m_X = 100\, m_\phi$. The expansion of the Universe is taken into account.

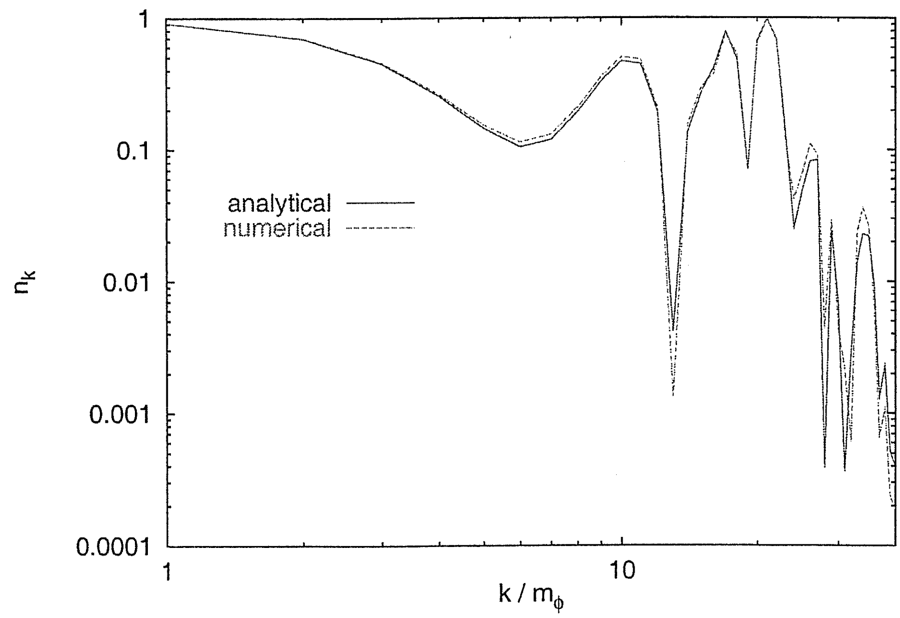


Figure 5.2: Spectrum of the fermions after seven productions for $q = 10^4$ and $m_X = 4\, m_\phi$. The expansion of the Universe is taken into account.

is shown in ref. [58] that the bosonic wave function satisfies the Mathieu equations, whose solutions are characterized by resonance bands (in momentum space) of very "explosive" and efficient production. It is then shown that, due to the expansion of the Universe, modes of a given comoving momentum $k$ cross several resonance bands during the evolution. This gives the creation the stochastic character described in the work [58]. In ref. [100] it is stated that an analogous behavior occurs also for fermions. The expansion is expected also in this case to spoil the clear picture of distinct resonance bands. This fact may help the transfer of energy to fermions, since the resonance bands in the fermionic case are anyhow limited by the Pauli principle. The expansion allows thus new modes to be occupied, and the production is no longer limited to the regions of resonance. In ref. [100] it is stated that the production should then almost completely fill the whole Fermi sphere up to a maximal momentum $k_{\rm max}$. This behavior is confirmed by the numerical results of the work [93]. In this section we will see that the analytical formulae derived in the previous section can reproduce the resonance bands, while in the next one we will discuss the effects of the expansion of the Universe.

Let us consider the matrices (we drop the index $n$ in the matrix elements since all the $p_n$'s have now the same value $p$)

$$M = \begin{pmatrix} F & G \\ -G & F^* \end{pmatrix}, \qquad T_1 = \begin{pmatrix} e^{-i\vartheta_1^2} & 0 \\ 0 & e^{i\vartheta_1^2} \end{pmatrix}, \qquad T_2 = \begin{pmatrix} e^{-i\vartheta_2^3} & 0 \\ 0 & e^{i\vartheta_2^3} \end{pmatrix}, \tag{5.20}$$

with $G \equiv e^{-\pi p^2/2}$ and $\vartheta_i^j \equiv \int_{\eta_{*i}}^{\eta_{*j}} \omega_k d\eta$ ($F$ was defined in the previous section).

Without the expansion of the Universe, the inflaton field has the periodic evolution

$$\phi(\eta \equiv t) = \phi_0 \cos{(m_\phi \eta)}, \tag{5.21}$$

and all the $\vartheta_i^j$ (5.17) are hence sums of $\vartheta_1^2$ and $\vartheta_2^3$ (remember the $\eta_{*i}$ are the moments at which the total fermionic mass vanishes).

After the generic $n$-th complete oscillation one thus has

$$\begin{pmatrix} \alpha_{2n} \\ \beta_{2n}^* \end{pmatrix} = \begin{pmatrix} e^{-i\vartheta_1^{2n+1}} & 0 \\ 0 & e^{i\vartheta_1^{2n+1}} \end{pmatrix} T_2 M^T T_1 M \cdots T_2 M^T T_1 M \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.22}$$

with the combination $\hat{O} \equiv T_2 M^T T_1 M$ repeated $n$ times.

One has thus simply to study the eigenvalue problem for $\hat{O}$ (notice $\det \hat{O} = 1$). This operator has the form

$$\begin{aligned} \hat{O} &= \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}, \\ A &= F^2 e^{i\left(\vartheta_1^2+\vartheta_2^3\right)} + G^2 e^{-i\left(\vartheta_1^2-\vartheta_2^3\right)}, \\ B &= FG e^{i\left(\vartheta_1^2+\vartheta_2^3\right)} - F^* G e^{-i\left(\vartheta_1^2-\vartheta_2^3\right)} \end{aligned} \tag{5.23}$$

and its eigenvalues are $\lambda_{1,2} = e^{\pm i\Lambda}$ with $\cos\Lambda = \operatorname{Re} A$.

Rewriting the initial condition $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ in terms of the eigenvectors of $\hat{O}$ and substituting in formula (5.22), one gets the number of produced fermions

$$N_n = |\beta_n|^2 = \frac{|B|^2}{1-(\text{Re } A)^2}\sin^2(n\Lambda) = \frac{|B|^2}{\sin^2\Lambda}\sin^2(n\Lambda) \tag{5.24}$$

after the complete $n$-th oscillation.

We notice the presence of the envelope function

$$\tilde{E} \equiv \frac{|B|^2}{1-(\text{Re}\,A)^2} = \frac{1-|A|^2}{1-(\text{Re}\,A)^2} \tag{5.25}$$

which modulates the oscillating function $\sin^2(n\Lambda)$.

With increasing $n$ this last function oscillates very rapidly and can be at all effects averaged to 1/2. One is thus left with the envelope function which shows the presence of resonance bands. The resonance bands occur where $A$ is real and $\tilde{E} \to 1$. It is easy to understand that their width exponentially decreases with increasing momenta $k$. To see this, let us consider the behavior of $\tilde{E}$ at high momenta. In this regime, the function $A$ is given by

$$A \simeq (1-e^{-\pi p^2})e^{i\phi_A}\,, \qquad p = \frac{k}{\sqrt{g|\phi'_*|}}\,, \tag{5.26}$$

where the phase $\phi_A$ can be read from eq. (5.23). Near to the points where $\cos\phi_A = 1$ the envelope function behaves like[5]

$$\tilde{E} \simeq \frac{1-(1-e^{-\pi p^2})^2}{1-(1-e^{-\pi p^2})^2\cos^2\phi_A} \simeq \frac{1}{e^{\pi p^2}(1-\cos\phi_A)+1}\,. \tag{5.27}$$

The width of the band can be defined as the distance between the two successive points at which $\tilde{E} = 1/2$. From the last expression it follows that the difference between the phases of $A$ in these two points is given by $\Delta\alpha = 2\sqrt{2}\,e^{-\pi p^2}$. Since the most rapidly varying term which contributes to the phase of $A$ is $(p^2/2)\log(p^2/2)$, the width of the band can be thus estimated to be

$$\Delta p \simeq \frac{2\sqrt{2}}{p\log(p^2/2)}\,e^{-\pi p^2}\,. \tag{5.28}$$

We show in figure 5.3 the envelope of the produced fermions in a static Universe for the parameters $q = 10^6$ and $m_X = 100\,m_\phi$. The peaks occur where $A$ is real and it is confirmed that their width decreases very rapidly at increasing momenta. Due to the fact that the last peaks plotted are indeed very sharp, the resolution of figure 5.3 does not allow to see their top at $n_k = 1$.

[5] A completely analogous behavior occurs where $\cos\phi_A = -1$.

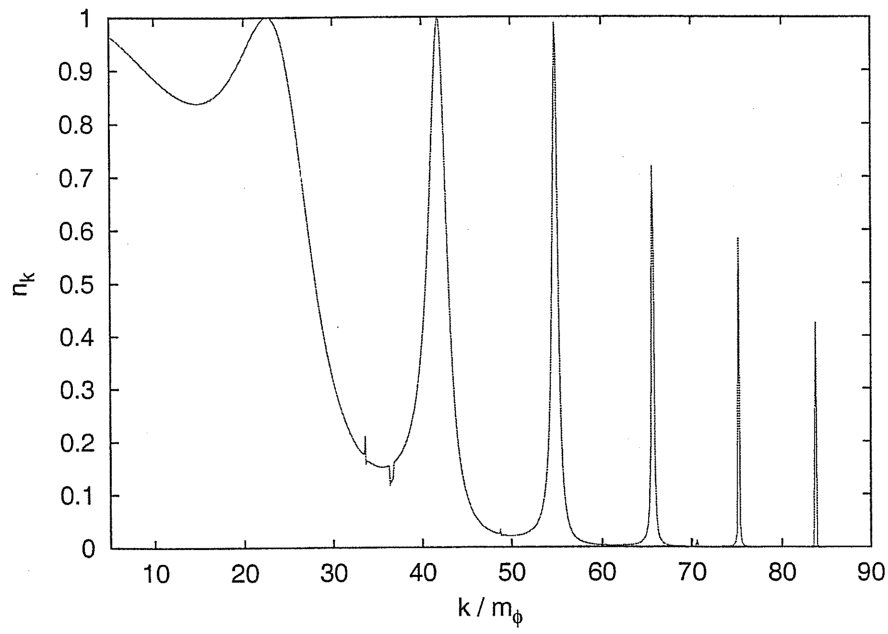


Figure 5.3: Envelope of the spectrum of the produced fermions in a static Universe. The physical parameters are $q = 10^6$ and $m_X = 100\, m_\phi$.

## 5.4 Expansion taken into account

As stated in the previous section, the resonance bands disappear when the expansion of the Universe is taken into account. In this section we will show how the occupation number varies when a nonvanishing Hubble parameter is considered. As we have seen in section 5.1, eqs. (5.17) and (5.18) give a very good agreement with the numerical results. On the other hand, the presence of phases in eq. (5.18) makes the exact analytical treatment of the occupation number impossible. Now, the same observation made in the bosonic case [58] turns very useful also to us: the phases in eq. (5.18), when the expansion of the Universe is taken into account, are not correlated among themselves. As a result, the final spectra present several high frequency oscillations about some average function. The positions of the peaks of these oscillations depend on the details of the phases. However, the "mean" function can be easily understood. Our problem can be treated as one customary does when dealing with the "random walk". Imagine one has to calculate the quantity

$$S \equiv |A_1 + A_2 + A_3 + \cdots + A_n|^2 \,, \tag{5.29}$$

where the $A_i$ are complex numbers with random phases. The "random walk recipe" indicates that the best estimation of the above quantity is achieved by summing the squares of the terms $A_i$, since the mixed products average to zero. The chaoticity of the final spectra for $n_k$ suggests that this may also be true in our case, and this is confirmed by comparison with the numerical results.

With this method, eqs. (5.17) and (5.18) turn into the much simpler relations

$$\left( \begin{array}{c} |\alpha_n|^2 \\ |\beta_n|^2 \end{array} \right) = \left( \begin{array}{cc} |F_n|^2 & |H_n|^2 \\ |H_n|^2 & |F_n|^2 \end{array} \right) \left( \begin{array}{c} |\alpha_{n-1}|^2 \\ |\beta_{n-1}|^2 \end{array} \right) , \tag{5.30}$$

where we remember

$$|F_n|^2 = 1 - e^{-\pi p_n^2} , \qquad |H_n|^2 = e^{-\pi p_n^2} , \qquad |F_n|^2 + |H_n|^2 = 1 . \tag{5.31}$$

Assuming no fermions in the initial state, and applying $n$ times this formula, it is easy to see that the occupation number after the $n$-th production is given by

$$N_n(k) = \frac{1}{2} - \frac{1}{2} \prod_{i=1}^{n} \left( 1 - 2e^{-\pi p_i^2} \right) . \tag{5.32}$$

A similar result holds for preheating of bosons, cfr. [58] where the idea of averaging on almost random phases was first introduced. In the bosonic case, one can exploit the fact that, due to the high efficiency of the production, the occupation number after the $(n+1)$-th creation is (almost) proportional to the occupation number after the $n$-th one:

$$N_{n+1}(k) \simeq \left( 1 + 2e^{\pi \kappa_n^2} \right) N_n(k) , \tag{5.33}$$

where the quantity $\kappa_n$ is analogous to our parameter $p_n$. This simplification is not possible in our case, since the Pauli principle forbids $N_n$ to be sufficiently high. However our final result, eq. (5.32), is also cast in a very simple and immediate form.

The validity of eq. (5.32) is confirmed by our numerical investigations, as we show here in one particular case. In figure 5.4 we compare the behavior of the "mean" function for the spectra with respect to the numerical one. We choose the physical parameters to be $q = 10^4, m_X = 4\, m_\phi$, and we look at the results after the seventh production (this corresponds to the choices made in figure 5.2). We see that the "mean" function interpolates very well between the maxima and the minima of the numerical spectrum, and that it can indeed be considered as a very good approximation of the actual result.

This is confirmed by figure 5.5, where we plot (for the same values adopted in figure 5.4) the quantity $n_k\, k^2$ rather than the occupation number alone. This quantity is of more physical relevance when one is interested in the total energy transferred to the fermions, since the total number density of produced fermions is

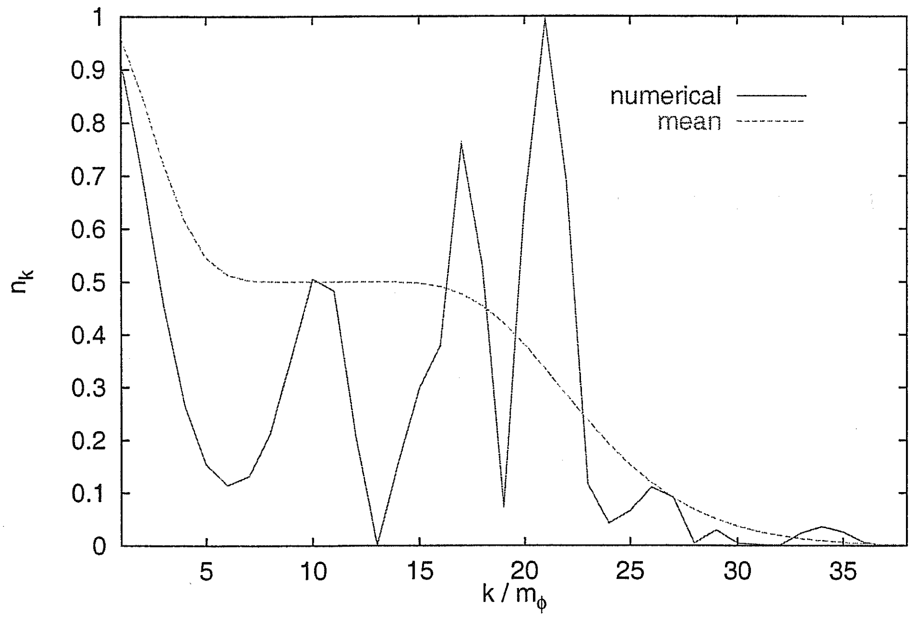


Figure 5.4: Comparison between the numerical spectrum and the "mean" function after seven productions for $q = 10^4$ and $m_X = 4\, m_\phi$.

(apart from the dilution due to the expansion of the Universe, that will be considered only in the final result)

$$N_X = \frac{2}{\pi^2} \int dk\, k^2\, n_k\,. \tag{5.34}$$

As shown in the plot, the result for $N_X$ in the numerical and in the approximated case are in very good agreement.

After checking the validity of the approximation given by the "mean" function, we adopt it to understand how the production scales when different values of the parameters $q$ and $m_X$ are considered.

Equation (5.32) allows us to give an analytical estimate of the total amount of energy stored in the fermions after the $n$-th production, and in particular after that the whole process of non-perturbative production has been completed. Notice that all the dependence on the physical parameters is in the coefficients

$$z_i \equiv \frac{k}{\left(\pi^{1/2} p_i\right)}\,, \tag{5.35}$$

where $k$ is the comoving momentum, and the only part we have to determine are the numbers

$$z_i^2 = \frac{2\sqrt{q}}{\pi} a\left(\eta_{*i}\right) \frac{\left|\phi'\left(\eta_{*i}\right)\right|}{\phi_0}\,. \tag{5.36}$$

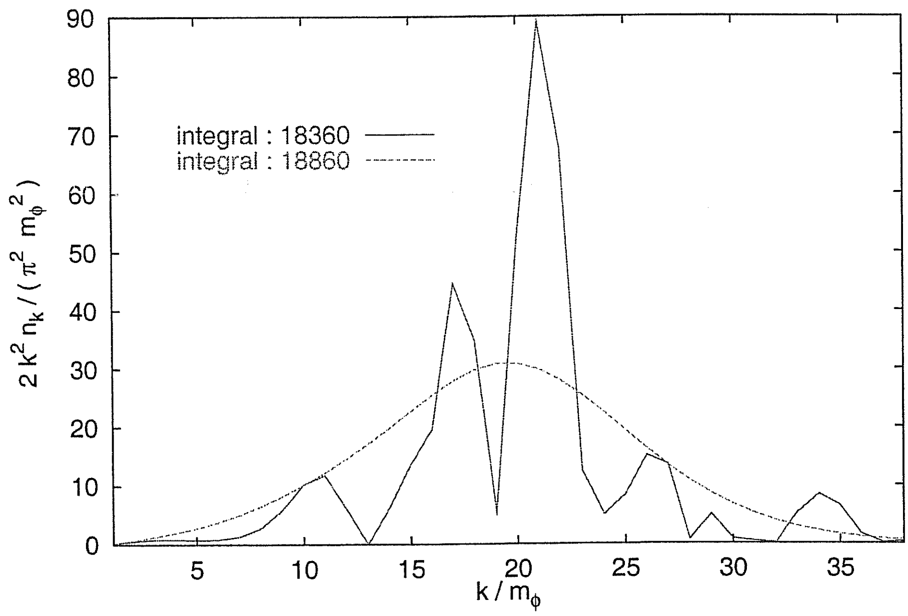


Figure 5.5: As in figure 5.4, but with the quantity $k^2 n_k$ plotted.

Here, and in the following of this section, we express the dimensionful quantities in units of $m_\phi$, the inflaton mass, apart from the inflaton field $\phi$ that is given in units of $M_{Pl}$.

We also introduce the ratio $R \equiv 2q^{1/2}/m_X$. This quantity is the most relevant, since it determines the zeros of the total fermionic mass and thus the values of the $z_i$'s. Indeed, from eq. (5.2) we see that the total mass vanishes for $\phi_* = -\phi_0/R$. It is convenient to study the production in terms of the two independent parameters $q$ and $R$ (rather than $q$ and $m_X$) since, at fixed $R$, all the spectra are the same provided we rescale $k \propto q^{1/4}$ (cfr. eq. (5.36)).

One can now proceed in two different ways, and we devote the next two subsections to each of them. First, one can evolve the equations for the inflaton field alone and find numerically the values $z_i$ for given $q$ and $R$. Inserting these values in eq. (5.32) one can get final values for the production which, as we have reported, are very close to the numerical ones. This method allows to get results which average the actual ones, and it has the advantage of being much more rapid than a full numerical evolution.

Alternatively, one can proceed with a full analytical study in order to understand the results given by the first semi-analytical method.

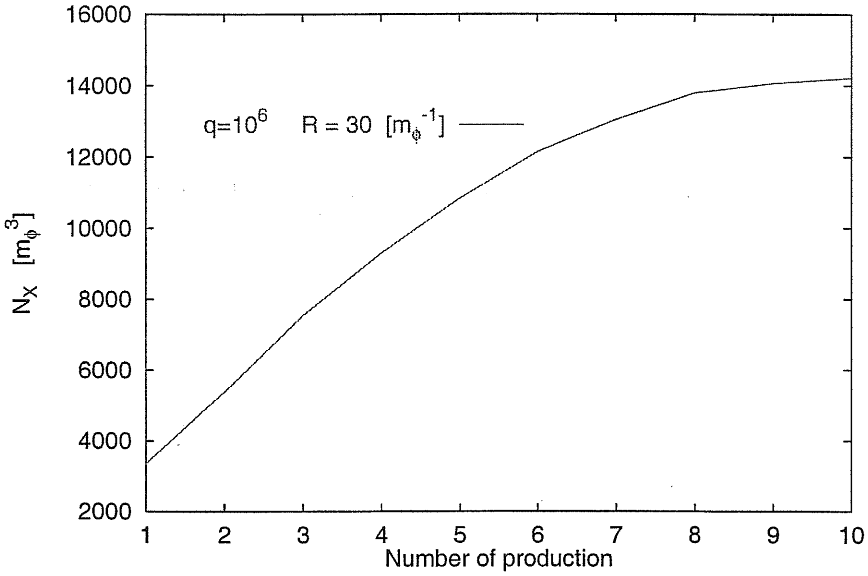


Figure 5.6: Growth of $N_X$ with the number of productions for $q = 10^6$ and $R = 30$.

### 5.4.1 Semi-analytical results

In this subsection we evaluate the analytical formula (5.32), taking the coefficients $z_i$ from a numerical evolution of the inflaton field. As we have said, this method gives results which are very close to the numerical ones, due to the fact that the "mean" function represents a very accurate averaging of the actual spectra of the produced fermions.

The first thing worth noticing is that, for each choice of $q$ and $R$, the maximum $z_i$ occurs at about half of the whole process of non-perturbative creation. It thus follows that fermions of maximal comoving momenta will be mainly produced at the half of the process. Our semi-analytical evaluations support this idea: the total energy stored in the created fermions increases slowly during the second part of the preheating. To see this, we show in figure 5.6 the numerical results for the quantity $N_X = 2\int dk\, k^2 n_k/\pi^2$ as function of the number of production. We fix the physical parameters to be $q = 10^6$ and $R = 30$, that is $m_X = 67\, m_\phi$. With this choice, the total mass vanishes 10 times, and figure 5.5 shows the results after each step of this production. We see indeed that the final productions are less efficient than the previous ones.

From eq. (5.32) it is also possible to show the evolution of the averaged spectra with the number of productions. We do this in figure 5.7. We choose the same parameters as in figure 5.6, and we show the results after each complete oscillation of the inflaton field (that is after each two productions). We observe that the

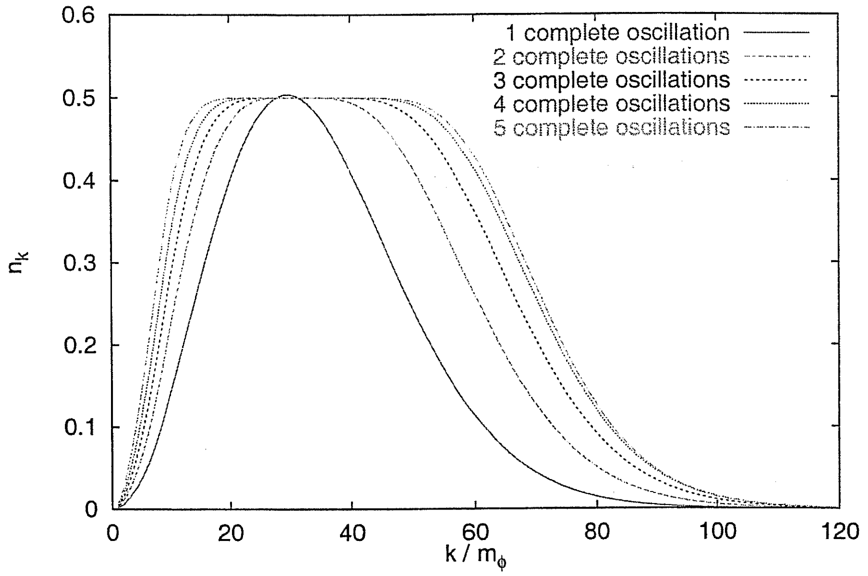


Figure 5.7: Evolution of the spectrum of produced fermions with the number of productions for $q = 10^6$ and $R = 30$.

production rapidly approaches a step function in the momentum space, i.e. there exists a maximum momentum below which Pauli blocking is saturated (notice that the value 1/2 follows from the average understood in the "mean" function), and above which $n_k \simeq 0$. The fact that the last productions do not contribute much to the total energy is also confirmed.[6]

We now turn our attention to the total energy transferred to fermions after the whole preheating process is completed. We fix the parameter $q$ to the value $10^6$ and we investigate how the total integral $N_X$ changes with different values for the parameter $R$.[7] The results are shown in figure 5.8, for $R$ ranging from 5 to 10000. For the last value the total fermionic mass changes sign more than 3700 times and a full numerical evaluation would appear very problematic. This can be done in our case, thanks to the analytical expression (5.32) found, and our results extend the validity region of the previous full numerical study [93].

In figure 5.8, the results of our semi-analytical method are also compared to the full analytical ones of the next subsection. This comparison will be discussed below.

From the scaling of $N_X$ with $R$ just reported, it is now easy to estimate the total energy transferred to fermions for generic values of $q$ and $R$. We are interested in comparing our results to the numerical ones of the work [93]. To do so, we

[6] The behavior at small $k$ is inessential, since, due to phase space suppression, this region does not significantly contribute to the total energy.

[7] As reported, the scaling of the final result with $q$ at fixed $R$ is simply understood from eq. (5.36).

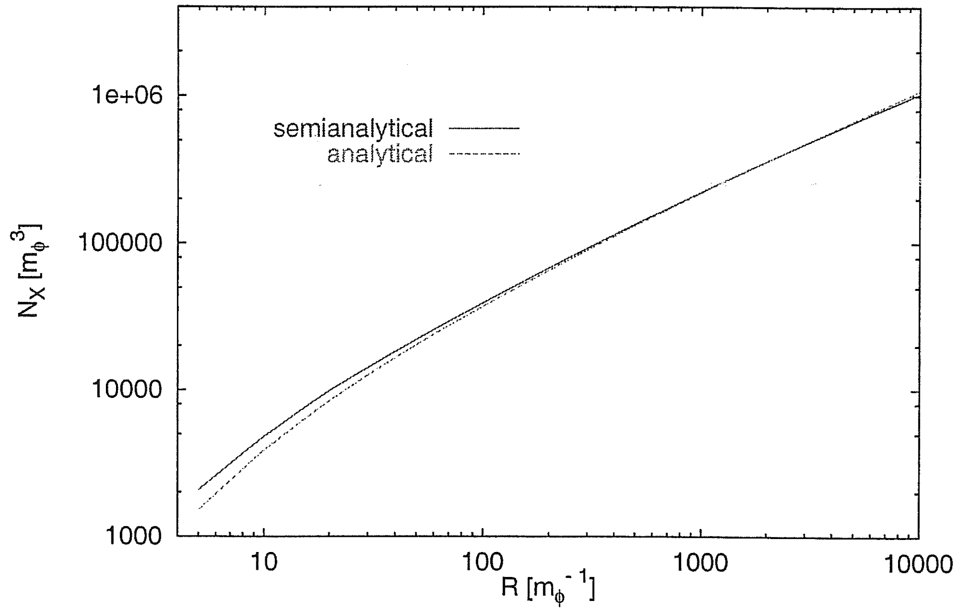


Figure 5.8: Comparison between the analytical and the semi-analytical results for $N_X$, at fixed $q = 10^6$.

consider the ratio between the energy density given to fermions and the one in the inflaton field[8]

$$\frac{\rho_X}{\rho_\phi} = \frac{2m_X}{\pi^2} \int dk\, k^2 n_k \cdot \frac{2}{m_\phi^2\, \phi_0^2}\,. \tag{5.37}$$

We present our results in figure 5.9. For comparison the numerical results of [93] are reported in figure 5.10. Notice that to compare appropriately the two results, the occupation numbers shown in fig. 5.9 should be divided by four, since in ref. [93] the number of fermions per spin degree of freedom is computed.

In figure 5.9, for any fixed $q$, the greatest plotted value for $m_X$ corresponds to the choice $R = 5$. We are not interested in extending this limit since we know that for greater $m_X$ (actually for values greater than the bound $m_X \sim \sqrt{q}/2$) the production suddenly stops. The smallest value plotted for $m_X$ (at any fixed value $q$) corresponds instead to $R = 10000$, that is to considering more than 3700 productions in the numerical evaluation of eq. (5.32).

[8]This ratio should be calculated at a time $t_{\rm end}$ at the end of preheating, when the total fermionic mass stops vanishing. Thus in the denominator of eq. (5.37) the comoving momentum $k$ should be replaced by the physical one $p = k/a$, with $a$ scale factor of the Universe at $t_{\rm end}$. Analogously, the value $\phi_0$ of the beginning of reheating should be replaced by the one at $t_{\rm end}$. However, both the replacements cancel out in the ratio, since both $\rho_X$ and $\rho_\phi$ redshift as energy densities of matter.

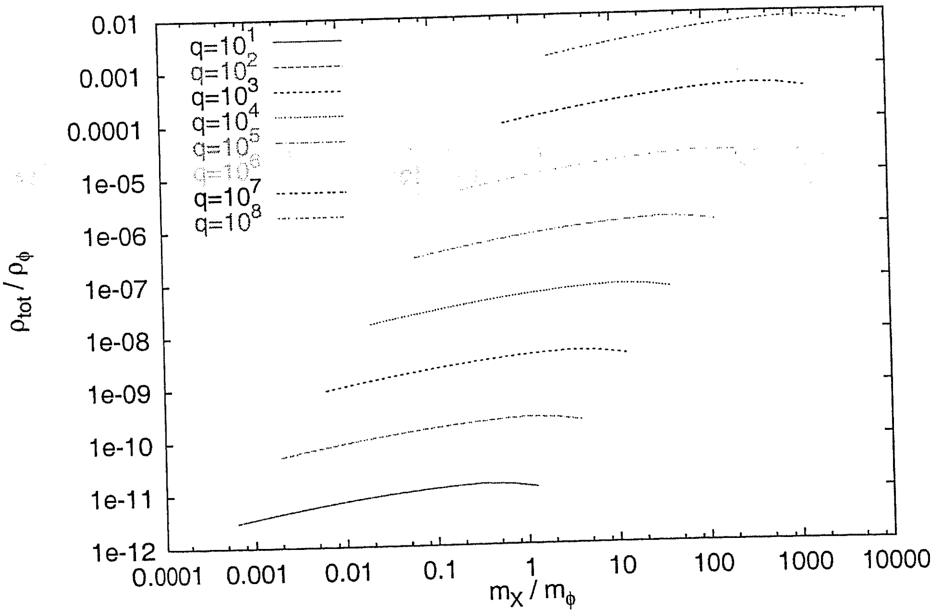


Figure 5.9: Total final energy density produced (normalized to the inflaton one) for different values of $q$ and $m_X$. Analytical result.

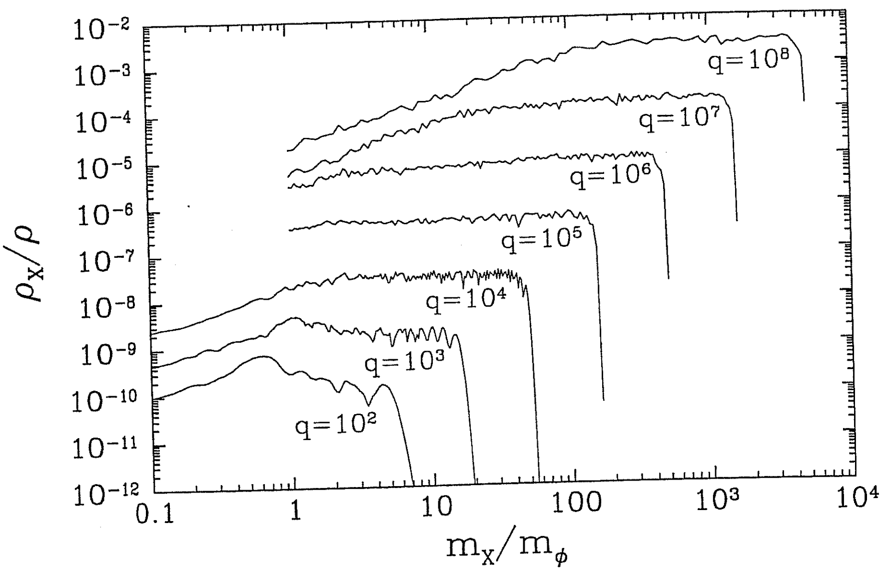


Figure 5.10: Total final energy density produced (normalized to the inflaton one) for different values of $q$ and $m_X$. Numerical result. From ref. [93].

Our final values are in good agreement with the ones of figure 5.10 in the regime of validity of the latter. The numerical results reported in that figure exhibit small fluctuations about an average function $\rho_X(m_X)$. Our results give this average function. This was expected, since the expression that we integrated, eq. (5.32), interpolates between the maxima and the minima of the numerical spectra.

The numerical results of figure 5.10 have a smaller range of validity than the ones of figure 5.9. This occurs because the numerical evolution of that work is limited to the first 20 oscillations of the inflaton field, and so fermionic production has not come to its end for small values of $m_X$. We confirm that at high values of $m_X$ (actually at small values of $R$ for any given $q$) the production depends very weakly on $m_X$. In addition, our results show a decrease of the energy transferred to fermions for smaller values of $m_X$. This behavior will be explained in details in the next subsection.

### 5.4.2 Analytical results

We want now to show that all the results presented in the previous section can be also achieved with a full analytical study of eq. (5.32).

First of all, we have to estimate the quantities $z_i$ given in eq. (5.36). To do this, it is more convenient to work in terms of the physical time $t$: after the first few oscillations, the inflaton evolution is very well approximated by the expression (remember $t$ is expressed in units of $m_\phi^{-1}$, while $\phi$ in units of $M_{Pl}$)

$$\phi(t) \simeq \frac{1}{\sqrt{3\pi}} \frac{\cos(t)}{t} \,. \tag{5.38}$$

The scale factor of the Universe follows the "matter-domination" law, and it is well approximated by $a(t) = t^{2/3}$.

The values $t_{*i}$ are determined by the condition of vanishing of the total mass of the fermions, that is, by making use of eq. (5.38),

$$\cos(t_{*i}) = -\frac{t_{*i}}{RA} \,, \tag{5.39}$$

where we remind $R \equiv (2\sqrt{q})/m_X$. The parameter $A \equiv (\sqrt{3\pi}\,\phi_0)^{-1}$ is of order one and will not play any special role in what follows. Notice that the last production occurs at $t \simeq RA$.

Hence, keeping only the dominant contribution to the derivative of $\phi$ with respect to the physical time, we get the expression

$$z_i^2 \cong \frac{2\sqrt{q}}{\pi} R^{1/3} A^{4/3} \left(\frac{t_{*i}}{RA}\right)^{1/3} \sqrt{1 - \left(\frac{t_{*i}}{RA}\right)^2} \,, \tag{5.40}$$

where we can assume $t_{*i} \simeq i\pi$.

Equation (5.40) exhibits a very good agreement with the numerical evaluation of the same quantity. It also shows that the maximal value for $z_i$ is reached at $t_{*i} = AR/2$, that is, at half of the whole process of non-perturbative creation. This was anticipated in the previous section, where we showed that the most of the fermionic production occurs in the first part of preheating.

Starting from eq. (5.40) we can also calculate the number density of produced particles

$$N_X(q,R) = \frac{2}{\pi^2}\int dk\, k^2 N(k)\,, \tag{5.41}$$

where $N(k)$ is obtained from eq. (5.32) with $n = n_{\max} = (RA)/\pi$.

For $R$ large enough, the product in eq. (5.32) can be written as the exponential of an integral. Thus, we obtain

$$\begin{aligned} N_X &= \frac{m_X}{4\pi^3}\left(\frac{2A^{4/3}}{\pi}\right)^{3/2} q^{3/4}R^{1/2} \times \\ &\times \int_0^\infty d\kappa\,\kappa^2\left\{1-\exp\left[\frac{RA}{\pi}\int_0^1 dy \log\left|1-2\exp\left(-\frac{\kappa^2}{y^{1/3}\sqrt{1-y^2}}\right)\right|\right]\right\}, \end{aligned} \tag{5.42}$$

with the substitution $\kappa = k\cdot\sqrt{\pi/\left(2A^{4/3}q^{1/2}R^{1/3}\right)}$.

The integral in $dy$ which appears in eq. (5.42) cannot be calculated analytically. Anyhow, we can approximate it by

$$\int_0^1 dy \log\left|1-2\exp\left(-\frac{\kappa^2}{y^{1/3}\sqrt{1-y^2}}\right)\right| \simeq g(\kappa)\log\left|1-2e^{-1.5\kappa^2}\right|, \tag{5.43}$$

where $g(\kappa)$ is a function of order one that, for our purposes, can be approximated by a constant $c$ in the range $0.5 \lesssim c \lesssim 1$.

Hence, the integrand within curly brackets in eq. (5.42) rewrites

$$1-\left|1-2e^{-1.5\kappa^2}\right|^{c\frac{RA}{\pi}}. \tag{5.44}$$

In the large-$R$ limit this function approximates a step function, which evaluates to one for

$$\sqrt{\frac{\pi\log 2}{2RAc}} \lesssim \kappa \lesssim \sqrt{\log\left(\frac{2RAc}{\pi\log 2}\right)} \tag{5.45}$$

and to zero for the remaining values of $\kappa$.

Since the quantity (5.44) is proportional to the occupation number $n_k$, our analytical calculation confirms the usual assumption that, after few oscillations, the fermionic production saturates the Fermi sphere up to a given maximum momentum $k_{\rm max}$. This was also shown in the previous figure 5.7. However, the present derivation gives a different scaling for $k_{\rm max}$ with respect to the previous literature [100, 93]. Indeed, from eq. (5.45) it follows (apart from proportionality factors)

$$k_{\rm max} \propto \frac{q^{1/3}}{m_X^{1/6}} \sqrt{\log\left(\frac{q^{1/2}}{m_X}\right)}, \tag{5.46}$$

which is however quite close to the result given in [93].

The origin of the above scaling can also be understood after some very simple considerations. Indeed, from the analytical formula (5.30), we notice that at high momenta $k$ the occupation number is well approximated by

$$N_n(k) \simeq \sum_{i=1}^{n} e^{-k^2/z_i^2}, \tag{5.47}$$

where we remember $z_i \propto q^{1/4} a^{1/2}(\eta_{*i}) |\phi'(\eta_{*i})|^{1/2}$.

In this last equation, we replace all the parameters $z_i$ with a mean value $\bar{z}$, so that $N_n \sim n \exp\big(-k^2/\bar{z}^2\big)$. The scaling of $\bar{z}$ with the physical parameters $q$ and $m_X$ follows from the scaling of all the $z_i$. The maximal momentum $k_{\rm max}$ is thus expected to scale as the quantity $z_i(\log n)^{1/2}$. Considering now the evolution of the inflaton field in physical time $t$, we notice that both the number $n$ of productions and the times $t_{*i}$ at which they occur are proportional to the parameter $R = q^{1/2}/(2m_X)$. Moreover, we see that the $z_i$'s scale as

$$z_i \propto q^{1/4} a_{*i} \left[\frac{d\phi}{dt}|_{*i}\right]^{1/2} \propto q^{1/4} t_{*i}^{2/3} \frac{1}{t_{*i}^{1/2}} \propto q^{1/4} R^{1/6}. \tag{5.48}$$

We thus get $k_{\rm max} \propto q^{1/4} R^{1/6} [\log R]^{1/2}$, from which the scaling (5.46) simply follows.

From eqs. (5.42) and (5.45), we obtain the final expression for the number density of the fermions created during the whole process

$$N_X(q,R) = \frac{1}{3\pi^2}\left(\frac{2A^{4/3}}{1.5\pi}\right)^{3/2} q^{3/4} R^{1/2} \left[\log\left(\frac{4Ac}{\pi \log 2}\frac{q^{1/2}}{m_X}\right)\right]^{3/2}. \tag{5.49}$$

We can now go back to figure 5.8, where this last equation (called "analytical" in the figure) is compared to the result with the semi-analytical method of the previous subsection. In plotting eq. (5.49) we chose $c = 0.78$ for the numerical factor involved. As we can see, the final results achieved with the two methods are in very good agreement with each other, thus confirming the validity of the formula (5.49).[9]

Rewriting eq. (5.49) in terms of $q$ and $m_X$ we can draw some conclusions. First, apart from a logarithmic correction, the scaling of the total energy

$$\rho_X \propto m_X N_X \propto q m_X^{1/2} \left[\log\left(\frac{4Ac}{\pi \log 2}\,\frac{q^{1/2}}{m_X}\right)\right]^{3/2} \tag{5.50}$$

is linear in $q$, as expected [93]. The dependence of $\rho_X$ on $m_X$ requires some more care: the threshold value for $m_X$ is given by the condition $R \sim 4$, that is,[10]

$$(m_X)_{\rm th} \sim \frac{\sqrt{q}}{2}\,. \tag{5.51}$$

For values of $m_X$ not much smaller than $(m_X)_{\rm th}$, the total energy depends very weakly on $m_X$, this result being in agreement with the numerical evaluation given in [93]. On the other hand, for values of $m_X$ much smaller than $(m_X)_{\rm th}$, the factor $m_X^{1/2}$ starts to dominate, and we expect it to determine the scaling of the total energy when $m_X \to 0$.

## 5.5 Backreaction

The results presented so far have been achieved neglecting the backreaction of the produced fermions on the evolution of the inflaton field and on the scale factor. This is a common approximation, since a more complete treatment (especially an analytical one) of the whole phenomenon is a very difficult task. However, the effects of the backreaction can be understood to a good degree of accuracy in the Hartree approximation.

For what concerns preheating of fermions, the Hartree approximation consists in taking into account the term

$$g\langle \bar{X} X\rangle \tag{5.52}$$

into the evolution equations for the inflaton and the scale factor. The equation for the field $\phi$ thus reads (in physical time)

$$\ddot{\phi} + 3H\dot{\phi} + m_\phi^2\phi + g\langle \bar{X} X\rangle = 0\,. \tag{5.53}$$

[9]The small discrepancy between the two curves can be attributed to the fact that $c$ is not exactly constant.

[10]The number 4 comes from the fact that the value of the inflaton at its first minimum is $\phi \simeq -0.07\,M_{Pl}$, while at beginning $\phi_0 = 0.28\,M_{Pl}$.

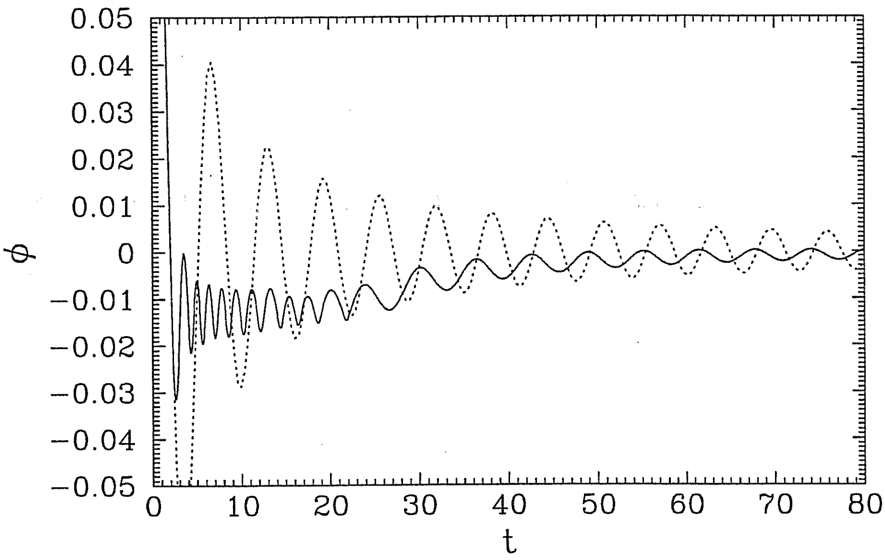


Figure 5.11: Evolution of the inflaton condensate with backreaction effect in Hartree approximation included (solid line) and neglected (dotted line). $q = 10^{10}$. Numerical result from ref. [93].

The study of this effect has been performed numerically in ref. [93], where it is shown that backreaction starts to be important for $q > q_b \sim 10^8 - 10^{10}$. Figure 5.11 shows how the evolution of the inflaton field $\phi$ is modified when backreaction is considered and $q$ is sufficiently high ($q = 10^{10}$ is used in figure 5.11). First, one observes that the amplitude of the oscillations of the inflaton is very damped already after the first production. This effect is the most obvious one, since the term (5.52) takes into account the decay of the inflaton into fermion-antifermion pairs, while in its absence the equation for $\phi$ considers only the damping due to the expansion of the Universe. The second feature that emerges from the evolution performed in ref. [93] is that at the beginning the field $\phi$ does not oscillate about the minimum of the potential $V = m_\phi^2 \phi^2/2$, but about the point $\phi_*$ where the total fermionic mass vanishes. Moreover, the frequency of these oscillations is higher than $m_\phi$.

These last two effects are due to the change in the effective potential for $\phi$ induced by the term (5.52), and disappear when the quantity $\langle \bar{X} X \rangle$ is decreased by the expansion of the Universe. Their net effect is to render the whole mechanism of preheating more efficient, since the rise in the frequency of the oscillations of $\phi$ increases the number of productions of fermions. In ref. [93] it is indeed shown that for $q = 10^8$ the total production is about 5% larger than the one without backreaction, while for $q = 10^{10}$ the increase is about 50%.

We now briefly study the evolution of the inflaton $\phi$ under eq. (5.53) by means of the analytical results presented above. We show that even a very approximate analysis confirms the numerical result that indicates in $q \simeq 10^8 - 10^{10}$ the threshold above which backreaction should be considered.

To begin with, the term (5.52) needs to be normal ordered. Doing so, one gets [93]

$$\langle \bar{X} X \rangle = \frac{2}{(2\pi a)^3} \int d^3\mathbf{k} \left(1 + \frac{ma}{\omega} - |U_-|^2\right) . \tag{5.54}$$

In terms of the Bogolyubov coefficients, this quantity evaluates to

$$\langle \bar{X} X \rangle = \frac{4}{(2\pi a)^3} \int d^3\mathbf{k} \left[|\beta|^2 \frac{am}{\omega} + \mathrm{Re}\left(\alpha\beta \frac{k}{\omega}\, e^{2i\int \omega\, d\eta}\right)\right] . \tag{5.55}$$

We see that $\langle \bar{X} X \rangle$ vanishes for $\beta = 0$. This is obvious, since backreaction starts only after fermions are produced. Some approximations can render eq. (5.55) more manageable. First, we notice that the oscillating term in the exponential averages to very small values the integral of the second term in square brackets. Second, we see that $k \ll |a\, m|$ where $\beta_k$ is significantly different from zero. From both these considerations, the integrand in eq. (5.55) can be approximated (up to the sign of $m$) by the occupation number $|\beta|^2$, so that the whole effect is (approximatively) proportional to the number of produced fermions.

The numerical results of ref. [93] show that, for the values of $q$ for which the backreaction is to be considered, its effects can be seen already in the first oscillation of the inflaton field. Since we are only interested in estimating the order of magnitude of $q_b$, we thus concentrate on the first oscillation of $\phi$, neglecting the expansion of the Universe in this short interval.

With all these approximations, eq. (5.53) rewrites

$$y'' + y + 10^{-12} q^{5/4} \frac{m}{|m|} = 0 , \tag{5.56}$$

where we have rescaled $y \equiv \phi/\phi_0$ and we remind that the time is given in units of $m_\phi^{-1}$.

This equation is very similar to the one obtained in the bosonic case [58]. The last term changes sign each time $m = 0$ and, when sufficiently high, forces the inflaton field to oscillate about the point $\phi_*$ at which the total fermionic mass vanishes. It is also responsible for the increase of the frequency of the oscillations. To see this, we assume that this term dominates over the second one of eq. (5.56) and we solve it right after the first fermionic production at the time $t_{*1}$. In the absence of the second term, eq. (5.56) is obviously solved by a segment of parabola until the time $t_{*2} \simeq t_{*1} + 2|y'(t_{*1})|/(q^{5/4} 10^{-12})$ at which $m$ vanishes again and the last term of eq. (5.56) changes sign. As long as the second term of eq. (5.56) can be neglected, the inflaton evolution proceeds along segments of parabola among successive zeros of the mass $m$. The "time" duration of these segments is expected to be of the same order of the first one, since the successive fermionic productions balance the decrease of $\langle \bar{X}X \rangle$ due to the expansion of the Universe.[11]

The period of these oscillations can thus be roughly estimated to be $T \sim 2\big(t_{*2} - t_{*1}\big)$. We see that, for $q \gtrsim 10^9$, this period is smaller than the one that the inflaton oscillations would have neglecting backreaction. Since this increase of the frequency is the main responsible for the higher fermionic production, the result $q_b \sim 10^9$ can be considered our estimate for the value of $q$ above which backreaction should be taken into account.

This result, although obtained with several approximations, is in agreement with the numerical one of ref. [93].

[11]However, after the first part of the process, the production looses its efficiency and the expansion of the Universe dominates. As we have said, the term $\langle \bar{X}X \rangle$ can then be neglected and the inflaton starts oscillating about the minimum of the tree level potential.

# Chapter 6

# Non-thermal production of gravitinos

In this chapter the full formalism described in chapter 4 is applied to non-thermal production of gravitinos in a system with two chiral superfields.

As discussed in the introduction to this part of the present work, the parameters of the inflationary sector of supergravity theories have to be restricted in order to avoid the thermal overproduction of gravitinos at reheating. These constraints, in the case of models with gravity–mediated supersymmetry breaking, can be summarized as an upper limit on the reheating temperature of the order of $10^9$ GeV (for a review, see [134]). More recently, it was shown that, during the stage of coherent oscillations of the inflaton condensate right after inflation, preheating can lead to an efficient nonthermal production of fermions [93]. In particular, it was argued [114, 115, 116, 117] that the nonthermal production of 1/2–helicity component of gravitinos can in some cases be much more efficient than the thermal one, thus worsening the gravitino problem. However, the abundance of nonthermal gravitinos was computed only in models with unbroken supersymmetry in the vacuum. As a consequence, the results about the production of longitudinal gravitinos in these models cannot be fully conclusive, since the spectrum of the theory in the vacuum does not contain the longitudinal gravitino.

In order to have some more reliable estimate of the amount of gravitinos produced at preheating, it is necessary to consider more realistic scenarios. The simplest model one can assume consists of two separate sectors, with the first mimicking the inflaton oscillations while the second breaks supersymmetry in the vacuum. In this chapter we will we discuss the situation in which the two sectors communicate only gravitationally.

This chapter is divided into six sections. In the first one we introduce all the quantities relevant for the calculation. In section 6.2 we review the results about the nonthermal production of gravitinos in models in which only one relevant superfield is considered, and we describe the motivations that led to carry out an analogous analysis in a system with two superfields. In section 6.3 we then describe the model that we are considering. We also discuss there the evolution of the scalar fields, which constitute the external background for the fermionic fields. In section 6.4 we show how to apply the formalism of chapter 4 to the calculation of the abundances of the fermions of the theory. The results are presented in the two remaining subsections. In section 6.5 we present analytical results in the case in which supersymmetry is actually unbroken in the vacuum of the theory. We show that in this case, gravitinos are only gravitationally (hence negligibly) produced. This consideration suggests that in the class of models we are considering (i.e. with the two sectors coupled only gravitationally) non-thermal gravitino production might be very inefficient in the realistic situation in which the observable supersymmetry breaking (TeV scale) is much smaller than the scale of inflation ($10^{13}$GeV). This is confirmed by the numerical results presented in section 6.6, which show that gravitino production indeed decreases as the size of supersymmetry breakdown becomes smaller.

## 6.1 Definitions

We write here the relevant equations of motion for the gravitino field and the fermionic particles to which it is coupled. We follow the conventions of subsection 4.2.2, that are the same ones that are used in ref. [117]. The starting action is the one of $D = 4, \mathcal{N} = 1$ supergravity, with four fermion interactions omitted. For simplicity, we do not consider any gauge multiplet. The lagrangian reads

$$\begin{aligned} e^{-1}\mathcal{L} \;=\; & -\frac{1}{2}\, M_{\rm P}^2\, R - g_i^{\ j}\left(\partial_\mu\,\phi^i\right)\left(\partial^\mu\phi_j\right) - V - \\ & -\frac{1}{2}\, M_{\rm P}^2\, \bar{\psi}_\mu\, R^\mu + \frac{1}{2}\, m\, \bar{\psi}_{\mu R}\, \gamma^{\mu\nu}\, \psi_{\nu R} + \frac{1}{2}\, m^*\, \bar{\psi}_{\mu L}\, \gamma^{\mu\nu}\, \psi_{\nu L} - \\ & - g_i^{\ j}\left[\bar{\chi}_j\, \not{D}\, \chi^i + \bar{\chi}^i\, \not{D}\, \chi_j\right] - m^{ij}\, \bar{\chi}_i\, \chi_j - m_{ij}\, \bar{\chi}^i\, \chi^j + \\ & + \left(2\, g_j^{\ i} \bar{\psi}_{\mu R}\, \gamma^{\nu\mu}\, \chi^j\, \partial_\nu \phi_i + \bar{\psi}_R \cdot \gamma v_L + \text{h.c.}\right). \end{aligned} \tag{6.1}$$

The first line of eq. (6.1) concerns the boson fields. The first term is the standard one of Einstein gravity, with $M_{\rm P}$ denoting the reduced[1] Planck mass ($M_{\rm P} \simeq 2.4 \cdot 10^{18}$ GeV) and $R$ the Ricci scalar. Conformal time $\eta$ is used and the Minkowski metric is taken with signature $-+++$. More explicitly, the metric and the vierbein are given by $g_{\mu\nu} = a^2(\eta)\, \eta_{\mu\nu}$, $e_\mu^b = a(\eta)\, \delta_\mu^b$, where $a$ is the scale factor of the Universe. The matter content of the theory is given by some chiral complex multiplets formed by $(\phi_i,\, \chi_i)$ and their conjugate $(\phi^i,\, \chi^i)$. $\chi_i$ is a left handed field, while $\chi^i$ a right

[1]Notice that in the present chapter we will make use of the reduced Planck mass $M_P = (8\,\pi\, G_N)^{-1}$, while in the previous one the Planck mass $M_{Pl} = G_N^{-1/2}$ was used.

handed one. The left and right projections are $P_L \equiv (1+\gamma_5)/2, P_R \equiv (1-\gamma_5)/2$. The gamma matrices in curved space $\gamma$ are related to the ones in flat space $\bar{\gamma}$ by the relation $\gamma^\mu = a^{-1}\,\bar{\gamma}^\mu$, and the realization of the latter that we are using is given in eq. (4.59). The Kähler metric is the second derivative of the Kähler potential

$$g_j{}^i = \frac{\partial}{\partial\,\phi_i}\frac{\partial}{\partial\,\phi^j}\,K\,. \tag{6.2}$$

while the scalar potential $V$ is defined below.

The second line of eq. (6.1) contains the kinetic and the mass term for the gravitino field. The first one is defined to be

$$R^\mu = e^{-1}\,\epsilon^{\mu\nu\rho\sigma}\,\gamma_5\,\gamma_\nu\,D_\rho\,\psi_\sigma\,, \tag{6.3}$$

where the covariant derivative

$$D_\mu\psi_\nu = \left(\left(\partial_\mu + \frac{1}{4}\omega_\mu^{mn}\gamma_{mn}\right)\delta_\nu^\lambda - \Gamma_{\mu\nu}^\lambda\right)\psi_\lambda\,, \tag{6.4}$$

contains the spin connection $\omega_\mu^{mn}$ and the connection $\Gamma_{\mu\nu}^\lambda$ ($\gamma_{mn} \equiv [\bar{\gamma}_m, \bar{\gamma}_n]/2$). The mass parameter $m$ is instead given by

$$m \equiv \mathrm{e}^{\frac{K}{2\,M_\mathrm{P}^2}}\,W\,, \tag{6.5}$$

and is related to the gravitino mass $m_{\tilde{G}}$ by

$$m_{\tilde{G}} = |m|\,M_\mathrm{P}^{-2}\,. \tag{6.6}$$

We then find the kinetic and mass term for the chiral fermions. The first is given by

$$D_\mu\chi_i \equiv \left(\partial_\mu + \frac{1}{4}\omega_\mu^{mn}\,\gamma_{mn}\right)\chi_i + \frac{1}{4\,M_\mathrm{P}^2}\left[\partial_j K\,\partial_\mu\phi^j - \partial^j K\,\partial_\mu\phi_j\right]\chi_i + \Gamma_i^{j\,k}\chi_j\partial_\mu\phi_k\,. \tag{6.7}$$

where $\Gamma_i^{j\,k} \equiv g^{-1}{}_i^{\,l}\,\partial^j\,g_l{}^k$ is the Kähler connection. For what concerns the part of the lagrangian (6.1) containing the fermion masses, we have defined

$$\begin{aligned} m^i &\equiv D^i m \equiv \partial^i m + \frac{1}{2\,M_\mathrm{P}^2}\partial^i K\,m\,, \\ m^{ij} &\equiv D^i\,D^j m = \left(\partial^i + \frac{1}{2\,M_\mathrm{P}^2}\partial^j K\right)m^j - \Gamma_k^{ij}\,m^k\,. \end{aligned} \tag{6.8}$$

In compact notation, the scalar potential reads

$$V \equiv -3\,M_\mathrm{P}^{-2}|m|^2 + m_i\,g^{-1}{}_j{}^i\,m^j\,. \tag{6.9}$$

The last line of eq. (6.1) describes the interactions of the gravitino with the chiral fields (i.e. with matter). The field $v_L$ is defined to be

$$v_L \equiv m^i \chi_i + (\not\partial\, \phi_i)\, \chi^j\, g_j{}^i \,. \tag{6.10}$$

As discussed in refs. [116, 117], this combination of matter fields is the goldstino (actually its left-handed component) in a cosmological context, where supersymmetry is broken both by the kinetic and the potential energies of the scalar fields. We work in the unitary gauge, where the goldstino is gauged away to zero. We also Fourier transform the fermion fields in the spatial direction, i.e. $\chi(\eta, \vec{x}) \equiv \chi(\eta) e^{i\, x^i\, k_i}$.

The gravitino field has transversal and longitudinal components. To appreciate their different behavior, one can introduce the projectors [117]

$$\begin{aligned} (P_\gamma)_i &\equiv \frac{1}{2}\left(\bar{\gamma}^i - \frac{1}{\vec{k}^2}\, k_i \left(k_j\, \bar{\gamma}^j\right)\right), \\ (P_k)_i &\equiv \frac{1}{2\, \vec{k}^2}\left(3\, k_i - \bar{\gamma}_i \left(k_j\, \bar{\gamma}^j\right)\right), \end{aligned} \tag{6.11}$$

where $k_i$ are the spatial components of the comoving momentum of the gravitino (i.e. $\partial_0 k_i = 0$) and $\vec{k}^2 \equiv k_i\, k_i$. These projectors are employed in the decomposition

$$\psi_i = \psi_i^T + (P_\gamma)_i\, \theta + (P_k)_i\, k_i\, \psi_i \,, \tag{6.12}$$

where $\theta \equiv \bar{\gamma}^i\, \psi_i$.[2] As it is discussed in detail in ref. [117], from the lagrangian (6.1) one recovers four independent equations for the gravitino components. Two of them are algebraic constraints which involve $\psi_0$, $k_i\, \psi_i$, and $\theta$. We use them to eliminate the first two combinations in favor of the last one. The other two are instead dynamical, and can be written in the form

$$\left[\bar{\gamma}^0\, \partial_0 + i\, \bar{\gamma}^i\, k_i + \frac{\dot{a}\, \bar{\gamma}^0}{2} + \frac{a\, m}{M_{\rm P}^2}\right] \psi_i^T \;=\; 0\,, \tag{6.14}$$

$$\left(\partial_0 + \hat{B} + i\, \bar{\gamma}^i\, k_i\, \bar{\gamma}^0\, \hat{A}\right) \theta - \frac{4}{\alpha\, a}\, k^2\, \Upsilon \;=\; 0\,, \tag{6.15}$$

[2]Notice that

$$\begin{aligned} k_i\, (P_\gamma)_i &= 0\,, \qquad \bar{\gamma}^i\, (P_\gamma)_i = 1\,, \\ k_i\, (P_k)_i &= 1\,, \qquad \bar{\gamma}^i\, (P_k)_i = 0\,, \\ k_i \psi_i^T &= \bar{\gamma}^i \psi_i^T = 0\,. \end{aligned} \tag{6.13}$$

where

$$\begin{aligned}
\Upsilon &= g_j{}^i \left(\chi_i\, \partial_0\, \phi^j + \chi^j\, \partial_0\, \phi_i\right)\,, \\
\underline{m} &= P_R\, m + P_L\, m^*\,, \qquad |m|^2 = \underline{m}^\dagger\, \underline{m}\,, \\
\hat{A} &= \frac{1}{\alpha}\left(\alpha_1 - \bar{\gamma}^0\, \alpha_2\right)\,, \qquad \hat{B} = -\frac{3}{2}\, \dot{a}\, \hat{A} + \frac{1}{2\, M_{\mathrm{P}}^2}\, a\, \underline{m}\, \bar{\gamma}^0 \left(1 + 3\, \hat{A}\right)\,, \\
\alpha &= 3\, M_{\mathrm{P}}^2 \left(H^2 + \frac{|m|^2}{M_{\mathrm{P}}^4}\right)\,, \\
\alpha_1 &= -M_{\mathrm{P}}^2 \left(3\, H^2 + 2\, \dot{H}\right) - \frac{3}{M_{\mathrm{P}}^2}\, |m|^2\,, \qquad \alpha_2 = 2\, a^{-1}\, \partial_0 \underline{m}^\dagger\,.
\end{aligned} \tag{6.16}$$

We use the "dot" notation $\dot{f} \equiv a^{-1}\partial_0 f$. $H \equiv \dot{a}/a$ is the Hubble expansion rate.

We notice that the transverse component of the gravitino, $\psi_i^T$, is decoupled from the longitudinal component and from matter, apart from gravitational effects due to the expanding background. In particular, transverse gravitinos are produced only gravitationally [114, 115, 116, 117, 112, 113], and for this reason we will not consider this component any longer in the rest of this chapter.

We are thus left with the gravitino longitudinal component, rewritten in terms of $\theta$, and the matter fields. In case of only one chiral supermultiplet the combination $\Upsilon$ defined above is proportional to the goldstino, and thus vanishes in the unitary gauge. This case will be discussed in the next subsection. In the more general case of $N$ chiral superfields, we have (always in the unitary gauge) $N-1$ non vanishing independent fermionic chiral fields, and one should go into a basis orthogonal to the goldstino. The equations of motion for all these fields can be of course deduced from the initial lagrangian (6.1). If only two superfields are present, one is just left with the matter field $\Upsilon$ defined above.

## 6.2 Gravitino production in the case of one superfield

The case of systems where the only one chiral superfield plays a relevant role was discussed in detail in refs. [114, 115, 116]. In this case, the equation of motion for the longitudinal component of the gravitino reads

$$\left[\bar{\gamma}^0\, \partial_0 + \bar{\gamma}^0\, \hat{B} + i\, \bar{\gamma}^i\, k_i \left(\frac{\alpha_1}{\alpha} - \bar{\gamma}^0\, \frac{\alpha_2}{\alpha}\right)\right] \theta = 0\,. \tag{6.17}$$

The presence in the equation of motion of some terms proportional to $\bar{\gamma}^0$ (besides the time derivative $\bar{\gamma}^0\, \partial_0$) shows that the field $\theta$ is not canonically normalized. Canonical normalization can be achieved by means of the redefinition (see also eq. (6.43) below)

$$\theta = \frac{2\, i\, \bar{\gamma}^i k_i}{\left(\alpha\, a^3\right)^{1/2}}\, \tilde{\theta}\,. \tag{6.18}$$

In the case of systems containing only one chiral superfield, the relation $\alpha_1^2 + \alpha_2^2 = \alpha^2$ holds. Therefore we can find a function $\varphi$ such that

$$\frac{\alpha_1}{\alpha} - \bar{\gamma}^0\, \frac{\alpha_2}{\alpha} = e^{2\,\bar{\gamma}^0\,\varphi}\,. \tag{6.19}$$

The equation of motion for the canonically normalized longitudinal gravitino thus reads

$$\begin{aligned} &\left(\bar{\gamma}^0\, \partial_0 + i\, \bar{\gamma}^i\, k_i\, e^{2\,\bar{\gamma}^0\,\varphi} + a\, M_{\tilde{\theta}}\right)\, \tilde{\theta} = 0\,, \\ &M_{\tilde{\theta}} = \left[\frac{m}{2\, M_P^2} + \frac{3}{2}\left(\frac{m}{M_P^2}\,\frac{\alpha_1}{\alpha} + H\,\frac{\alpha_2}{\alpha}\right)\right]\,. \end{aligned} \tag{6.20}$$

To get fully canonical kinetic term, we have to further redefine the field $\tilde{\theta}$ as

$$\tilde{\theta} = e^{-\bar{\gamma}^0\,\varphi}\,\hat{\theta}\,. \tag{6.21}$$

Hence, the equation of motion for the longitudinal gravitino gets the standard form

$$\begin{aligned} &\left(\bar{\gamma}^0\, \partial_0 + i\, \bar{\gamma}^i\, k_i + a\, m_\theta\right)\, \hat{\theta} = 0\,, \\ &m_\theta = M_{\tilde{\theta}} + \frac{\partial_0\,\varphi}{a}\,. \end{aligned} \tag{6.22}$$

The effective mass $m_\theta$ is time-dependent. As discussed in chapter 4, a time dependence of the effective mass leads to the production of quanta of longitudinal gravitinos. Notice that in this case the situation is slightly different from the one described in the previous chapter, since the effective mass of $\theta$ generically never happens to vanish. Nevertheless the Fermi sphere is expected to be almost saturated up to some maximum value of the comoving momentum $k_{max}$. From the discussion carried out in the previous chapter we can estimate $k_{max} \sim \sqrt{\partial_0\,(a m_\theta)}$. These estimates are confirmed by the numerical analysis performed in [116].

From eqs. (6.16), (6.19) and (6.20) it is apparent that, whereas $M_{\tilde{\theta}}$ vanishes in the limit $M_P \to \infty$, $\partial_0\,\varphi$ does not. This means that the effective, time-dependent mass of the longitudinal component of the gravitino will be nonvanishing also in the globally supersymmetric limit. In this limit, due to the equivalence theorem [135, 114, 116, 118], the longitudinal component of the gravitino is identified with the goldstino of the globally supersymmetric theory, that is spontaneously broken by the inflaton potential and kinetic energy. If the theory contains only one chiral supermultiplet, the goldstino will be necessarily the inflatino, that is the superpartner of the inflaton field. As discussed in ref. [116], gravitino production should thus be rather regarded as inflatino production in the globally supersymmetric theory.

It is interesting to consider simple models such as the ones described by the superpotentials $W = M_\phi\, \Phi^2/2$ (that leads to a quadratic potential in the limit of global supersymmetry) or $W = \sqrt{\lambda}\, \Phi^3/3$ (quartic potential)[3] . In the first case it is possible to show that $m_\theta$ varies of an amount $\sim M_\phi$ in a time $\sim M_\phi^{-1}$ during the first inflaton oscillation (besides $M_P$, $M_\phi$ is the only relevant dimensionful scale in the system), thus leading to a number density of gravitinos that can be estimated as $n_{3/2} \sim M_\phi^3$. Notice that, since the quantity $\partial_0 \varphi$ does not vanish in the limit $M_P \to \infty$ also the total number of gravitinos produced at preheating is not Planck-mass suppressed. Later on, during the stage of coherent oscillations of the inflaton, the gravitinos are diluted as $n_{3/2} \sim M_\phi^3/a^3$ (we set the scale factor of the Universe $a = 1$ at the end of inflation). The energy density of the massive inflaton condensate redshifts as matter $\rho_\phi\,(a) \sim M_\phi^2\, M_P^2/a^3$ until reheating completes (when the scale factor of the Universe is $a_{RH}$) and the inflaton energy is converted into radiation, with temperature

$$T_{RH} \sim \rho_\phi\,(a_{RH})^{1/4} \sim \frac{M_\phi^{1/2}\, M_P^{1/2}}{a_{RH}^{3/4}} \,. \tag{6.23}$$

We can therefore compute the ratio of the number of gravitinos to entropy density at reheating

$$\frac{n_{3/2}}{s} \sim \frac{M_\phi^3}{a_{RH}^3}\, \frac{1}{T_{RH}^3} \sim \frac{M_\phi\, T_{RH}}{M_P^2} \,. \tag{6.24}$$

In order not to spoil the successful Big Bang Nucleosynthesis results,

$$\frac{n_{3/2}}{s} \lesssim 10^{-12} \tag{6.25}$$

is required [82]. Using eq. (6.24), and remembering that $T_{RH}$ has to be in any case smaller than $10^9$ GeV, because of the thermal gravitino problem, one gets $n_{3/2}/s \lesssim 10^{-15}$. Therefore, in the case of a quadratic superpotential, nonthermal production is not competitive with thermal production.

[3]As it is known, the contributions from the Kähler potential to the scalar potential are very relevant for $\phi \sim M_{\rm P}$. This is a common problem for supersymmetric theories of inflation, where the $F$ terms generically spoil the flatness of the potential during the inflationary regime (for a review, see [73]; see also [136] for a recent discussion). As a consequence, the theories that we are here describing should be modified during inflation; however, we will not consider this issue here and we will still assume that the values of $M_\phi$ and of $\lambda$ are not too different from the ones imposed in "usual" chaotic inflation.

An analogous estimate in the case of cubic superpotential gives a different result

$$\frac{n_{3/2}}{s} \sim \lambda^{3/4} \sim 10^{-9} \,. \tag{6.26}$$

This value exceeds the gravitino bound by about 3 orders of magnitude. Therefore the supergravity inflationary model with cubic superpotential appears to be strongly disfavoured. Notice that the result (6.26) does not depend on the reheating temperature. The reason is that, while the energy density of the massive inflaton condensate redshifts as matter during inflaton oscillation, in the case of quartic potential the energy redshifts as radiation. In this case, gravitinos are not diluted during reheating, while in the former case they could be diluted as much as needed by suitably delaying reheating, i.e. by lowering the reheating temperature.

The fact that preheating of longitudinal gravitinos can be adequately described in the globally supersymmetric limit has been exploited in ref. [116] to simplify considerably the computations needed for the analysis of this phenomenon. This allowed to estimate the amount of gravitinos produced in more realistic supersymmetric inflationary models, in particular in hybrid inflation [137, 138]. Since in this model the energy density of the inflaton redshifts as matter during reheating, the ratio $n_{3/2}/s$ turns out to be proportional to $T_{RH}$. However, an extremely low reheating temperature (of the order of $10^5$ GeV) is needed to obey the gravitino bound. This would worsen drastically the gravitino problem.

As stated above, the results we have just described rely on the identification of the longitudinal component of the gravitino with the inflatino. However, in general we expect the inflaton field not to be responsible for the breaking of supersymmetry in the vacuum. Therefore, it is not guaranteed that what was the gravitino during inflation and reheating (that is, when supersymmetry is broken by the inflaton), is the gravitino today. It is possible to rephrase this observation by saying that all the models in which the abundance of nonthermal gravitinos was computed up to this point have unbroken supersymmetry in the vacuum. Therefore, the true gravitino is massless and the longitudinal gravitino is indeed the inflatino. Now, the gravitino problem originates by the fact that gravitinos are very long-lived, since they have only gravitational interactions, and, if they are lighter than about 20 TeV they usually decay after nucleosynthesis. Their decay products then photodissociate the nucleosynthesis products to an unacceptable level if the number density of gravitinos exceeds the bound (6.25). On the other hand, inflatinos do not necessarily interact only gravitationally, and are not dangerous relics, since they can safely decay before nucleosynthesis [139, 140]. Therefore, it is crucial to discriminate whether gravitinos or inflatinos are produced at preheating.

To get a confirmation of the results described in the present section, one needs to analyze more complicate models, where supersymmetry is broken in the vacuum. Moreover, since we require supersymmetry breaking in vacuum with vanishing cosmological constant, we will need to work in the full supergravity context, keeping a finite value for the Planck mass $M_P$. We will analyze a model consisting of two sectors: the first will mimic inflaton oscillations as early times, while the second will break supersymmetry at late times. The two sectors are coupled only gravitationally, and it is expected that any direct coupling between them will increase the final amount of nonthermal gravitinos. To compute the actual number of nonthermal gravitinos it will be necessary to use the formalism described in chapter 4, resorting to numerical techniques. Indeed, while it is possible to give an analytical description of the behavior of the system at early and at late times, a full numerical computation is needed at intermediate times, when both sectors equally contribute to the breaking of supersymmetry. In fact, there could be the possibility that, at intermediate times, the fermions efficiently produced at the initial stages oscillate into the gravitinos of our vacuum. Moreover, the behavior of the background in this regime could show further nonadiabaticity, thus inducing additional gravitino production. In the following sections we will show how the first possibility is actually not realized, while the second source of production is not very efficient.

## 6.3 Description of the model and evolution of the scalar fields

The matter content of the model we are considering is of two superfields $\Phi$ and $S$, with superpotential

$$W = \frac{m_\phi}{2}\,\Phi^2 + \mu^2\,(\beta + S) \tag{6.27}$$

and minimal Kähler potential

$$K = \Phi^\dagger\,\Phi + S^\dagger\,S\,. \tag{6.28}$$

The potential for the scalar components $\phi$ and $s$ of the superfields $\Phi$ and $S$ can be computed using eq. (6.9). We then assume that the scalar fields are real, that is (after $V$ is computed) we perform the substitutions

$$\phi = \phi^* \longrightarrow \frac{\phi}{\sqrt{2}}\,, \qquad s = s^* \longrightarrow \frac{s}{\sqrt{2}}\,. \tag{6.29}$$

In this way the real scalar fields have canonical kinetic terms.

During inflation, the field $\phi$ acts as the inflaton, while the v.e.v. of $s$ is quickly driven to $\langle s\rangle \simeq 0$. The potential is then practically the one of chaotic inflation, and $m_\phi \sim 10^{13}\,\mathrm{GeV}$ must be set to match the COBE results for the size of the CMBR fluctuations.[4]

[4] As discussed in the footnote before eq. 6.23, in the supergravity context the superpotential 6.27 does not lead to a true inflationary potential. Nevertheless we assume the value of $m_\phi$ to be the not too far from the one that is obtained in the nonsupersymmetric inflationary scenario

At the end of inflation, the field $\phi$ oscillates about the minimum $\phi = 0$. The amplitude of these oscillations is damped by the expansion of the Universe (and, later on, also by the decay of the inflaton that every realistic model must include). If only $\phi$ was present, we eventually would have unbroken supersymmetry in the minimum $\phi = 0$. The role of the $s$ field is to provide the supersymmetry breaking in the minimum. The second term in eq. (6.27) is known as the Polonyi superpotential [141], and provides a simple example on how supersymmetry can be broken in a hidden sector and transmitted to the visible one by gravity. What is remarkable of this potential is that, for particular values of the parameter $\beta$, supersymmetry is broken with a vanishing value for the cosmological constant. If indeed we take $\beta = (2-\sqrt{3})M_{\rm P}$ (for a more detailed discussion, see for example [134]), the potential $V(\phi = 0,\, s)$ vanishes in its minimum at

$$s_0 = \sqrt{2}(\sqrt{3}-1)M_{\rm P}\,. \tag{6.30}$$

For this value, the gravitino mass $m_{\tilde{G}} = {\rm e}^{K/2\,M_{\rm P}^2}\,|W|/M_{\rm P}^2$ evaluates to

$$m_{\tilde{G}} = {\rm e}^{2-\sqrt{3}}\,\frac{\mu^2}{M_{\rm P}} \simeq 1.31\,\frac{\mu^2}{M_{\rm P}}\,, \tag{6.31}$$

which is a typical result for this breaking of supersymmetry. We see that the "intermediate" scale $\mu$ must be taken of order $10^{10}$GeV to reproduce the expected gravitino mass $m_{\tilde{G}} \sim 100\,$GeV of gravity mediated breaking models.

In the following, we discuss in more details the evolution of the two scalar fields. To do this, we use physical time $t$ and work with the adimensional quantities

$$\begin{aligned} \hat{\phi} &\equiv \frac{\phi}{M_{\rm P}}\,, \qquad \hat{s} \equiv \frac{s}{M_{\rm P}}\,, \qquad \hat{\beta} \equiv \frac{\beta}{M_{\rm P}}\,, \\ \hat{t} &\equiv t\,m_\phi\,, \qquad \hat{\mu}^2 \equiv \frac{\mu^2}{M_{\rm P}\,m_\phi}\,, \qquad \hat{H} \equiv \frac{H}{m_\phi}\,, \qquad \hat{V} \equiv \frac{V}{M_{\rm P}^2\,m_\phi^2}\,, \end{aligned} \tag{6.32}$$

where we remind that $H$ and $V$ are, respectively, the Hubble constant and the scalar potential. In terms of these redefined quantities, the equations of motion for the two scalars read

$$\frac{d^2\hat{\phi}_i}{d\hat{t}^2} + 3\,\hat{H}\,\frac{d\hat{\phi}_i}{d\hat{t}} + \frac{d\hat{V}}{d\hat{\phi}_i} = 0\,, \qquad \hat{\phi}_i = \hat{\phi},\, \hat{s}. \tag{6.33}$$

We start our numerical calculations at $\hat{\phi} \simeq 1.4$, short after inflation, and with the scale factor $a$ normalized to one.

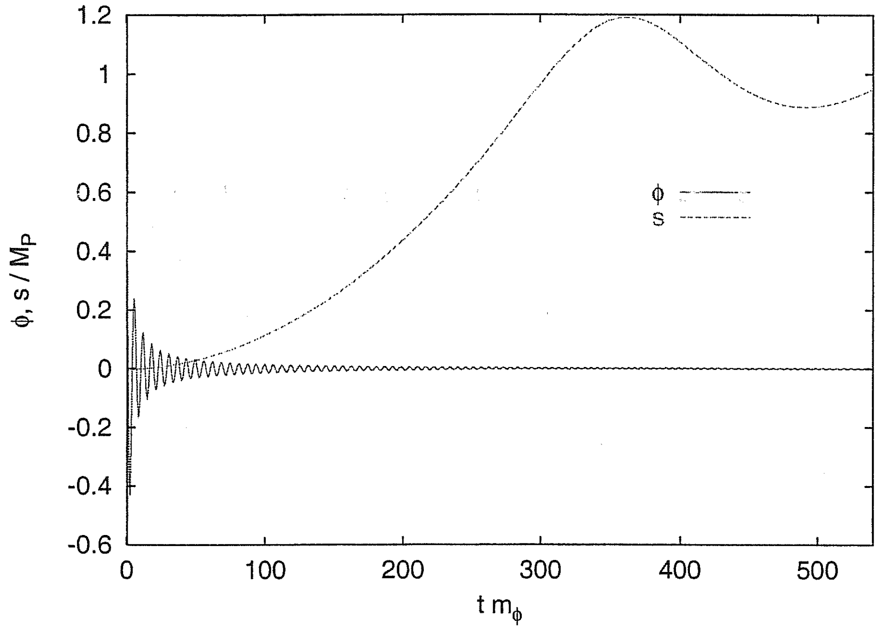


Figure 6.1: Evolution of the two scalar fields $\phi$ and $s$ for $\hat{\mu}^2 = 10^{-2}$.

We show in figure 6.1 the evolution for the two scalar fields after inflation, in the case $\hat{\mu}^2 = 10^{-2}$. As we said, initially the model reproduces the scalar potential of chaotic inflation, and thus we have

$$\hat{\phi} \simeq \sqrt{\frac{8}{3}} \frac{\cos \hat{t}}{\hat{t}}, \qquad \hat{s} \simeq 0. \tag{6.34}$$

The initial dynamics of the Polonyi field $s$ is determined by $\phi$. More precisely, we can write an effective potential $V(s)$ for it by substituting eq. (6.34) into $V(\phi,\, s)$ and then averaging over the inflaton oscillations. Expanding $V$ for both $\hat{\phi}$ and $\hat{s}$ smaller than one, we find that the potential is minimized by

$$\hat{s}_{\text{min}} \simeq \frac{\sqrt{2}\,\hat{\mu}^2[16\,\hat{\beta}\,\hat{\mu}^2 - \langle\hat{\phi}^4\rangle]}{4\,\langle\hat{\phi}^2\rangle - 16\,\hat{\beta}^2\,\hat{\mu}^4} \simeq \frac{3\,\sqrt{2}\,\hat{\beta}\,\left(\hat{\mu}^2\,\hat{t}\right)^2}{1 - 3\,\hat{\beta}^2\,\left(\hat{\mu}^2\,\hat{t}\right)}\,. \tag{6.35}$$

To be precise, the Polonyi field is always smaller than $\hat{s}_{\text{min}}$, due to the fact that the expansion of the Universe slows its motion towards the minimum of $V(s)$. However, eq. (6.35) gives a good estimate for the order of magnitude of $s$ in this initial stage.

What is most important to emphasize, is that eqs. (6.34) and (6.35) explicitly show the presence of two very different (physical) time-scales in the model we are considering. The first of them is set by the inverse inflaton mass $m_\phi^{-1}$ which is the time-scale of the oscillations of the inflaton field. The second one is given by

$\hat{\mu}^{-2} m_{\phi}^{-1}$. Equation (6.35) shows that this is the relevant time-scale for the Polonyi field in the initial stage. However, this is true also for the complete evolution of $s$. To see this, let us consider the latest times shown in figure 6.1. In this stage the amplitude of the oscillations of $\phi$ are negligible. The evolution of $s$ is not any longer influenced by the inflaton field, but it starts oscillating about the minimum of its own potential given in eq. (6.30).[5] The amplitude of these oscillations is also damped by the expansion of the Universe, while their period is related to the inverse Polonyi mass, which is now (i.e. at $\phi = 0$) given by [134]

$$m_s = \sqrt{2\sqrt{3}}\, m_{\tilde{G}} \simeq 2.4\, \hat{\mu}^2\, m_\phi\,. \tag{6.36}$$

The quantity $\hat{\mu}^2$ defines the ratio between the two scales. In figure 6.1 we have chosen, for illustrative purposes, $\hat{\mu}^2 = 10^{-2}$. However, this value is unphysical, since it would correspond to a too high supersymmetry breaking scale. Indeed, as eq. (6.31) shows, we must require $\hat{\mu}^2 \sim 10^{-11}$, if supersymmetry is supposed to solve the hierarchy problem.

While the size of $\hat{\mu}^2$ controls the supersymmetry breaking in the vacuum of the theory, both scalar fields contribute to break supersymmetry during their evolution. In particular, both their kinetic and potential energies contribute to the breaking, as emphasized in ref. [116]. This can be seen by considering the transformation law of the chiral fermions $\chi_i$ under an infinitesimal supersymmetry transformation with parameter $\varepsilon$. In our case they read

$$\delta\chi_i = -\frac{1}{2}\, P_L \left[ m_i - \frac{1}{\sqrt{2}}\, \bar{\gamma}^0\, \frac{d\phi_i}{dt} \right] \varepsilon\,, \tag{6.37}$$

where $\phi_1 = \phi,\ \phi_2 = s$.

We define the quantities

$$f_i^2 \equiv m_i^2 + \frac{1}{2} \left( \frac{d\phi_i}{dt} \right)^2\,, \tag{6.38}$$

which give a "measure" of the size of the supersymmetry breaking provided by the $F$ term associated with the $i$-th scalar field. More precisely, we will be interested in the normalized quantities

$$r_\phi \equiv \frac{f_1^2}{f_1^2 + f_2^2}\,, \qquad r_s \equiv \frac{f_2^2}{f_1^2 + f_2^2}\,, \tag{6.39}$$

which indicate the relative contribution of the two scalar fields $\phi$ and $s$.

[5] There is of course a possible moduli problem associated with these oscillations. However, we do not consider this issue here.

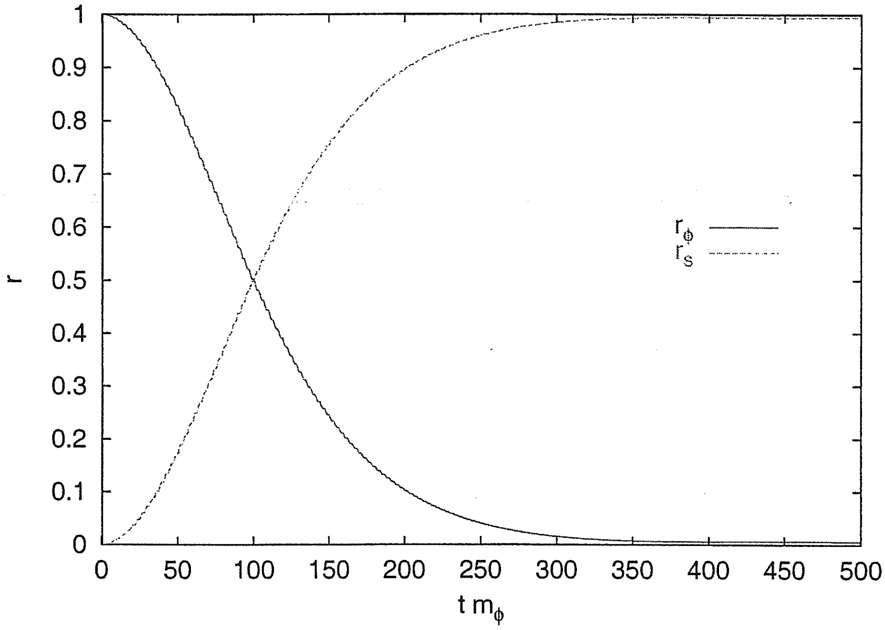


Figure 6.2: Relative contribution of the two scalar fields $\phi$ and $s$ to the supersymmetry breaking during their evolution. As in figure 6.1, $\hat{\mu}^2 = 10^{-2}$.

In figure 6.2 we have shown the evolution of $r_\phi$ and $r_s$ for the specific case $\hat{\mu}^2 = 10^{-2}$. As expected, in the initial stages only the inflaton contributes to the supersymmetry breaking, while only the Polonyi contributes at later times. The regime of equal contribution is around $\hat{t} = \hat{\mu}^{-2}$, when $\phi$ and $s$ are of the same size (cf. figure 6.1). As it should be clear from the above discussion, $r_\phi$ and $r_s$ share the identical behavior for all the choices of $\hat{\mu}^2$, once $\hat{t}$ is given in units of $\hat{\mu}^{-2}$.

## 6.4 Effective fermionic lagrangian and hamiltonian

The fermionic content of the model we are considering is of the gravitino $\psi_\mu$ and the two chiral fermions $\tilde{\phi}$ and $\tilde{s}$. In the unitary gauge, one combination of $\tilde{\phi}$ and $\tilde{s}$, the goldstino $\upsilon$, is set to zero, while the transverse component of the gravitino, $\psi_i^T$, is only gravitationally coupled to the other fields. The other two fermions $\theta$ (the longitudinal gravitino component) and $\Upsilon$ (the combination of chiral fermions orthogonal to $\upsilon$) are coupled together, as we described in section 6.1.

With some algebra, we can rewrite the initial lagrangian (6.1) in three terms

$$\mathcal{L} = \mathcal{L}_{\text{background}} + \mathcal{L}_{\psi_i^T} + \mathcal{L}_{\theta\,\Upsilon}\,. \tag{6.40}$$

The first term governs the dynamics of the scalar fields and of the scale factor of the Universe. The second describes the (decoupled) transverse gravitino component, while the third one reads

$$\begin{aligned}\mathcal{L}_{\theta\Upsilon} &= -\frac{\alpha}{4\,k^2}\,a^3\,\overline{\theta}\left[\bar{\gamma}^0\,\partial_0\theta + i\,\bar{\gamma}^i\,k_i\hat{A}\,\theta + \right. \\ &\qquad \left. -\left(\frac{3}{2}\,\dot{a}\,\bar{\gamma}^0 + \frac{3}{2\,M_{\rm P}^2}\,a\,m\right)\hat{A}\,\theta - \frac{a\,m}{2\,M_{\rm P}^2}\,\theta - \frac{4\,k^2}{\alpha\,a}\,\bar{\gamma}^0\,\Upsilon\right] + \\ &\quad -\frac{4\,a}{\alpha\,\Delta^2}\,\overline{\Upsilon}\left[\bar{\gamma}^0\,\partial_0\Upsilon - i\,\bar{\gamma}^i\,k_i\hat{A}\,\Upsilon - \frac{3}{2}\,\dot{a}\,\bar{\gamma}^0\,\hat{A}\,\Upsilon + \frac{3}{2\,M_{\rm P}^2}\,\hat{A}\,a\,m\,\Upsilon + \right. \\ &\qquad \left. +\,2\,\dot{a}\,\bar{\gamma}^0\,\Upsilon - \frac{a\,m}{2\,M_{\rm P}^2}\,\Upsilon + \frac{1}{4}\,a\,\alpha\,\Delta^2\,\bar{\gamma}^0\,\theta\right]. \end{aligned} \tag{6.41}$$

We have expanded the fermions into plane waves $X_i(\eta,\,\mathbf{k}) = \mathrm{e}^{i\,k_i x^i}\,X_i(\eta)$, where $k_i$ is the comoving momentum (i.e. $\partial_0\,k_i = 0$), and we have introduced

$$\begin{aligned}\Delta &\equiv \frac{2}{\alpha}\left[\dot{\phi}_i\,\dot{\phi}_j\,m_k m_l\left({g^{-1}}_{kl}\,g_{ij} - \delta_{ik}\,\delta_{lj}\right)\right]^{1/2} \\ &= \frac{2}{\alpha}\left(m_1\dot{\phi}_2 - m_2\dot{\phi}_1\right), \end{aligned} \tag{6.42}$$

where the second equality holds in the case of a minimal Kähler potential, $g_i^j = \delta_i^j$. The quantity $\Delta$ has no counterpart in the one chiral superfield case, and indeed it is negligible unless both the scalar fields give a sizeable contribution to the breaking of supersymmetry.

One can explicitly verify that the lagrangian (6.41) reproduces the equation of motion (6.15) for the longitudinal gravitino component, as well as the one for $\Upsilon$ that one obtains from the initial lagrangian (6.1). However, we notice that the two fields $\theta$ and $\Upsilon$ are not canonically normalized. Canonical normalization has to be imposed, if we want our fields to give invariant quantities (as for example the occupation number) in comoving units in the adiabatic regime. Among the possible redefinitions, we choose

$$\begin{aligned}\theta &= \frac{2\,i\,\bar{\gamma}^i\,k_i}{\left(\alpha\,a^3\right)^{1/2}}\,\tilde{\theta}, \\ \Upsilon &= \frac{\Delta}{2}\left(\frac{\alpha}{a}\right)^{1/2}\tilde{\Upsilon}, \end{aligned} \tag{6.43}$$

since the equations of motion look quite symmetric in terms of the new fields. In matrix form, they are exactly of the form (4.63), i.e.

$$\left(\bar{\gamma}^0\,\partial_0 + i\,\bar{\gamma}^i\,k_i\,N + M\right)X = 0\,, \tag{6.44}$$

where $X$ is the vector $(\tilde{\theta}, \tilde{\Upsilon})^T$. In our specific case, the "mass" matrix $M$ is given by

$$\begin{aligned} M &= \mathrm{diag}\left(\frac{m\,a}{2\,M_{\mathrm{P}}^2} + \frac{3}{2}\left(\frac{m\,a}{M_{\mathrm{P}}^2}\,\tilde{\alpha}_1 + \dot{a}\,\tilde{\alpha}_2\right)\,,\right. \\ &\qquad \left. -\frac{m\,a}{2\,M_{\mathrm{P}}^2} + \frac{3}{2}\left(\frac{m\,a}{M_{\mathrm{P}}^2}\,\tilde{\alpha}_1 + \dot{a}\,\tilde{\alpha}_2\right) + a\,(m_{11} + m_{22})\right)\,, \end{aligned} \tag{6.45}$$

and the $N$ matrix by

$$N \equiv N_1 + \bar{\gamma}^0\,N_2 = \begin{pmatrix} -\tilde{\alpha}_1 & 0 \\ 0 & -\tilde{\alpha}_1 \end{pmatrix} + \bar{\gamma}^0 \begin{pmatrix} -\tilde{\alpha}_2 & -\Delta \\ -\Delta & \tilde{\alpha}_2 \end{pmatrix}\,. \tag{6.46}$$

In the above equations, we have defined $\tilde{\alpha}_i \equiv \alpha_i/\alpha$. The relation $\tilde{\alpha}_1^2 + \tilde{\alpha}_2^2 \equiv 1$ which holds in the one chiral field case [114, 115] is now replaced by[6]

$$\tilde{\alpha}_1^2 + \tilde{\alpha}_2^2 + \Delta^2 \equiv 1\,. \tag{6.47}$$

We thus see that the matrices $N_1$ and $N_2$ satisfy both conditions (4.65).

The equations of motion (6.44) have a clear behavior in the low energy limit, when the two scalars of the theory settle to their minima. In this final stage one has $\tilde{\alpha}_1 = -1, \tilde{\alpha}_2 = \Delta = 0$, as it can be easily checked from the definitions listed above. As a consequence, eqs. (6.44) decouple, and each of them acquires the standard form for spin 1/2 fermions

$$\begin{aligned} \left(\bar{\gamma}^0\,\partial_0 + i\,\bar{\gamma}^i\,k_i + a\,m_{\tilde{\theta}}\right)\tilde{\theta} &= 0\,, \\ \left(\bar{\gamma}^0\,\partial_0 + i\,\bar{\gamma}^i\,k_i + a\,m_{\tilde{\Upsilon}}\right)\tilde{\Upsilon} &= 0\,, \end{aligned} \tag{6.48}$$

where the two masses are constant. In particular, notice that $m_{\tilde{\theta}} = m/M_{\mathrm{P}}^2$, which is exactly the expression that one encounters in supergravity for the gravitino mass.

We thus see that the system has all the properties assumed in section (4.2.2), so that we can apply the procedure derived there to quantize it and to define the occupation numbers of the fermionic eigenstates. Among the possible choices for the transformation matrix $\Phi$ which enters into eq. (4.66), we take

$$\Phi = \frac{1}{2}\,(\arccos\,\tilde{\alpha}_1)\begin{pmatrix} \tilde{\alpha}_2/\omega & \Delta/\omega \\ \Delta/\omega & -\tilde{\alpha}_2/\omega \end{pmatrix}\,, \tag{6.49}$$

with

$$\omega \equiv \sqrt{1 - \tilde{\alpha}_1^2} = \sqrt{\tilde{\alpha}_2^2 + \Delta^2}\,. \tag{6.50}$$

[6] When only one scalar field gives a substantial contribution to supersymmetry breaking, the quantity $\Delta$ almost vanishes, and the relation $\tilde{\alpha}_1^2 + \tilde{\alpha}_2^2 \simeq 1$ holds approximatively.

Following eq. (4.74), the hamiltonian of the system is instead given by[7]

$$H = \bar{X}\left[i\,\bar{\gamma}^i\,k_i N + \tilde{M}_1 + \bar{\gamma}^0\,\tilde{M}_2\right] X\,, \tag{6.52}$$

with

$$\begin{aligned} \tilde{M}_1 &\equiv M + \frac{\omega^2}{2}\,(Q\,M\,Q - M) + \frac{\partial_0\,\tilde{\alpha}_1}{2\,\omega}\,Q - \frac{\omega}{2}\,\tilde{\alpha}_1\,\partial_0\,Q\,, \\ \tilde{M}_2 &\equiv \frac{\omega\,\tilde{\alpha}_1}{2}\,[M,\,Q] + \frac{\omega^2}{4}\,[Q,\,\partial_0\,Q]\,. \end{aligned} \tag{6.53}$$

We have denoted

$$Q = \begin{pmatrix} \tilde{\alpha}_2/\omega & \Delta/\omega \\ \Delta/\omega & -\tilde{\alpha}_2/\omega \end{pmatrix}. \tag{6.54}$$

As we have already remarked, at late times $\tilde{\alpha}_1 = -1$, while $\tilde{\alpha}_2 = \Delta = 0$. In this regime the above hamiltonian becomes the standard one of two decoupled spin 1/2 fermions

$$H = \bar{X}\left[i\,\bar{\gamma}^i\,k_i + M\right] X\,, \tag{6.55}$$

with the standard gravitino mass for the field $\tilde{\theta}$ (cf. eq. (6.48)).

We conclude this subsection discussing the explicit diagonalization of the hamiltonian, i.e. of the matrices $\overline{H}$ and $\overline{H}_0$ entering in eqs. (4.82) and (4.83). One can now explicitly verify that the eigenvalues of the $\overline{H}_0$ matrix occur in pairs, that is they are of the form $\pm\omega_1, \pm\omega_2$. One can also verify that if $(v_1,\,v_2,\,v_3,\,v_4)$ is an eigenvector of $\overline{H}_0$ belonging to the eigenvalue $\omega$, then $(-v_3^*, -v_4^*,\,v_1^*,\,v_2^*)$ is also an eigenvector of $\overline{H}_0$ belonging to the eigenvalue $-\omega$. We then find

$$R^\dagger\,\overline{H}_0\,R = H_d\,, \qquad R = \begin{pmatrix} R_1 & -R_2^* \\ R_2 & R_1^* \end{pmatrix}, \tag{6.56}$$

where $H_d = \mathrm{diag}(\omega_1,\,\omega_2,-\omega_1,-\omega_2)$ is the matrix that we formally introduced in eq. (4.81).

[7]Notice that the matrix $\hat{M}_2$ that appears in the equations of motion (4.68) reads

$$\hat{M}_2 = [Q,\,M]\,\frac{\omega}{2} + Q\,\dot{Q}\,\frac{1-\tilde{\alpha}_1}{2}\,. \tag{6.51}$$

At late times $\hat{M}_2 \sim Q\,\dot{Q}$ does not vanish. Indeed $\tilde{\alpha}_2$ and $\Delta$ decrease at late times in a way such that the elements of $Q$ keep on oscillating with amplitude equal to unity. Therefore, as discussed in the footnote containing eq. (4.71), the fields $\hat{X}$ are not a suitable basis for the definition of the hamiltonian.

The $2 \times 2$ matrices defined by eq. (4.86) are thus given by

$$\begin{aligned} I &= \frac{1}{\sqrt{2}} \left[ R_1^\dagger U_+ + R_2^T U_- \right] , \\ J &= \frac{i}{\sqrt{2}} \left[ -R_1^\dagger U_-^* + R_2^T U_+^* \right] . \end{aligned} \tag{6.57}$$

We remind that the matrices $R_1$ and $R_2$ are obtained through the diagonalization of $\overline{H}_0$ see eq. (6.56). The matrices $U_+$ and $U_-$ are instead determined by their evolution equation (4.77). The only point left is to give more explicitly their values at the initial time $\overline{\eta}$. This can be done by setting $J = 0$ in eq. (6.57), which, as we remarked, corresponds to requiring no fermions in the initial state (see eq. (4.88)). Moreover, conditions (4.78) have to be imposed. From these requirements, we see that $U_+(\overline{\eta})$ has to fulfil

$$U_+^\dagger(\overline{\eta}) \left[ \mathbb{1} + R_2(\overline{\eta}) R_1^{*-1}(\overline{\eta}) R_1^{T-1}(\overline{\eta}) R_2^\dagger(\overline{\eta}) \right] U_+(\overline{\eta}) = 2\,\mathbb{1} . \tag{6.58}$$

In this last expression, the matrix in square brackets is hermitean and can be diagonalized with a unitary transformation. More precisely, we can set it to be equal to $V^\dagger \Lambda V$ with $\Lambda$ diagonal and real, and $V$ unitary. The initial condition for $U_+$ can thus be written

$$U_+(\overline{\eta}) = V^\dagger \sqrt{2\,\Lambda^{-1}} . \tag{6.59}$$

Finally, $U_-(\overline{\eta})$ is obtained by setting $J(\overline{\eta}) = 0$ in eq. (6.57).

## 6.5 Analytical results with unbroken supersymmetry in the vacuum

The case $\hat{\mu}^2 = 0$ is particularly interesting since some results can be worked out analytically, and since it provides some hints between the final gravitino abundance and the size of supersymmetry breaking. For $\hat{\mu}^2 = 0$ supersymmetry is unbroken in the minimum of the theory, at $\phi = s = 0$.[8] Because of this, in the vacuum of the theory the gravitino has only the transverse component.

The computation of the formulae of section 6.4 is in this case particularly simplified. The quantity $\Delta$ vanishes identically, so that the two fields $\Upsilon$ (which is always the Polonyi fermion) and $\theta$ (which is always the inflatino) are decoupled. Going back to the formalism of section (4.2.2), we find that we have to perform only the

[8] Indeed for $\hat{\mu}^2$ strictly zero the potential for the Polonyi field becomes flat for $\phi = 0$, and $s \equiv 0$ for the whole evolution.

first redefinition of the fermions, $X \equiv \exp(-\bar{\gamma}^0\,\Phi)\,\hat{X}$, where now $\Phi = \mathrm{diag}(\varphi, -\varphi)$, with

$$\cos(2\,\varphi) = \tilde{\alpha}_1\,, \qquad \sin(2\,\varphi) = \tilde{\alpha}_2\,. \tag{6.60}$$

The two redefined fields have the "standard" equations of motion and hamiltonian

$$\begin{aligned}
&\left(\bar{\gamma}^0\,\partial_0 + i\,\bar{\gamma}^i\,k_i + m_{\hat{\theta}}\right)\hat{\theta} = 0\,,\\
&\left(\bar{\gamma}^0\,\partial_0 + i\,\bar{\gamma}^i\,k_i + m_{\hat{\Upsilon}}\right)\hat{\Upsilon} = 0\,,\\
H &= \int d^3\mathbf{k}\left[\bar{\hat{\theta}}\left(i\,\bar{\gamma}^i\,k_i + m_{\hat{\theta}}\right)\hat{\theta} + \bar{\hat{\Upsilon}}\left(i\,\bar{\gamma}^i\,k_i + m_{\hat{\Upsilon}}\right)\hat{\Upsilon}\right],\\
m_{\hat{\theta}} &= m_\theta + \partial_0\varphi\,, \qquad m_{\hat{\Upsilon}} = m_\Upsilon - \partial_0\varphi\,,
\end{aligned} \tag{6.61}$$

with $m_\theta$ and $m_\Upsilon$ given in eq. (6.45).

In practice, "removing" the time dependent matrix which multiplies the momentum in the original equations for $\theta$ and $\Upsilon$ gives an additional contribution to the mass of the fields $\hat{\theta}$ and $\hat{\Upsilon}$. When $\Delta = 0$, an explicit computation of $\partial_0\varphi$ gives [117]

$$\partial_0\varphi = -a\,\frac{\dot{\tilde{\alpha}}_2}{2\,\tilde{\alpha}_1} = a\,\left(m_{11} + m\right) + 3\,a\left(H\,\frac{\dot{\phi}}{\sqrt{2}} - m_1\,m\right)\frac{m_1}{\frac{1}{2}\,\dot{\phi}^2 + m_1^2}\,, \tag{6.62}$$

where the various quantities have been introduced in section 6.1. Using the equation of motion for the inflaton field $\phi$, one can show that it is precisely $\partial_0\varphi \equiv m_\Upsilon$, so that the field $\hat{\Upsilon}$ is effectively massless. The mass for $\hat{\theta}$ is instead of the order the inflaton mass. More precisely, it has a variation of the order $m_\phi$ within the first oscillation of the inflaton (i.e. in the time $m_\phi^{-1}$) and then it stabilizes at $m_\phi$ [114, 115]. Since the fields are decoupled, the formulae for the occupation numbers (4.49) are quite simple. They show that the Polonyi fermion is not produced, while the production of the inflatino field has a cut-off at $k \sim m_\phi$, and decreases as $k^{-4}$ at big momenta.[9]

The main point of this section is that the Polonyi fermion is *not* produced at preheating for $\hat{\mu}^2$ strictly zero. When $\hat{\mu}^2 \neq 0$ the Polonyi fermion provides the longitudinal component for the gravitino, so its abundance turns out crucial to understand whether gravitinos are or are not overproduced. If one believes that the limit $\hat{\mu}^2 \to 0$ is continuous, the present analysis suggests indeed that the production of gravitinos should become smaller as $\hat{\mu}^2$ decreases. Although we do not have a rigorous proof of this continuous behavior,[10] the numerical results that we show in the next subsection strongly support this assumption.

[9] This can be explicitly seen by integrating eq. (4.49) for $\beta'$ in the limit of large $k$ and with $\alpha \simeq 1$.

[10] The problem is that the dynamics of the Polonyi field is governed by the timescale $\hat{\mu}^{-2}\,m_\phi^{-1}$, which becomes infinite in the limit $\hat{\mu}^2 \to 0$ [117].

## 6.6 Numerical results with broken supersymmetry in the vacuum

We now analyze the situation $\hat{\mu}^2 \neq 0$. As we have said, in this case the quantity $\Delta$ is generally non vanishing in the most interesting part of the evolution. As a consequence, the dynamics of the fermionic fields $\tilde{\theta}$ and $\tilde{\Upsilon}$ is coupled, i.e. we have mixed terms in their equations of motion (6.44) and in their hamiltonian (6.52). For the following discussion it is useful to explicitly write $\tilde{\Upsilon}$ in terms of the chiral fields $\chi_1$ and $\chi_2$. Combining the definitions (6.16), (6.42), and (6.49) we have, for minimal Kähler potential and real scalar fields,

$$\tilde{\Upsilon} = \frac{a^{3/2}[m_1^2 + m_2^2 + \frac{1}{2}\dot{\phi}_1^2 + \frac{1}{2}\dot{\phi}_2^2]^{1/2}}{m_1\,\dot{\phi}_2^2 - m_2\,\dot{\phi}_1^2}\left(\dot{\phi}_1\,\chi_1 + \dot{\phi}_2\,\chi_2\right), \tag{6.63}$$

where we remind that the two scalars $\phi_1$ and $\phi_2$ are the inflaton and the Polonyi field, while $\chi_1$ and $\chi_2$ the corresponding fermions. Moreover, the definition (6.10) of the goldstino now reads

$$v_L = \left(m_i - \frac{1}{\sqrt{2}}\bar{\gamma}^0\,a\,\dot{\phi}_i\right)\chi_i\,. \tag{6.64}$$

Let us first consider the initial and final stages of the evolution, where only one of the two scalar fields significantly contribute to the supersymmetry breaking and the quantity $\Delta$ is essentially vanishing. During inflation, one has $m_2,\ \dot{\phi}_1,\ \dot{\phi}_2 \simeq 0$, and the supersymmetry breaking is provided almost completely by $m_1$. Moreover, the goldstino is practically the field $\chi_1$. We remind that we are working in the unitary gauge, so that $\chi_1 \simeq v = 0$. Equation (6.63) thus rewrites

$$\tilde{\Upsilon} \simeq \frac{a^{3/2}\,|m_1|}{m_1\,\dot{\phi}_2}\,\chi_1\,\dot{\phi}_2 = a^{3/2}\,\chi_1\,, \qquad \hat{t} \ll \hat{\mu}^{-2}\,. \tag{6.65}$$

Notice the factor $a^{3/2}$ appearing in the last expression, which is a consequence of the fact that the field $\tilde{\Upsilon}$ is canonically normalized in comoving units (cf. the discussion after the lagrangian (6.41)).[11] In the late stages of the evolution, supersymmetry is instead broken by the Polonyi field, and the only non-vanishing contribution is provided by $m_2$. With the same arguments used to get eq. (6.65), one can show that[12]

$$v \propto \chi_2 = 0\,, \qquad \tilde{\Upsilon} = -a^{3/2}\,\chi_1\,, \qquad \hat{t} \gg \hat{\mu}^{-2}\,. \tag{6.66}$$

[11] One may also worry about dividing by $\dot{\phi}_2 \simeq 0$ in eq. (6.65). However, this is due to the fact that the quantity $\Upsilon$ is ill-defined in the static $\dot{\phi}_i \to 0$ limit, while $\tilde{\Upsilon}$ is not.

[12] One can also show that, in the late stages of the evolution, $|\dot{\phi}_1| \simeq |\dot{\phi}_2|$.

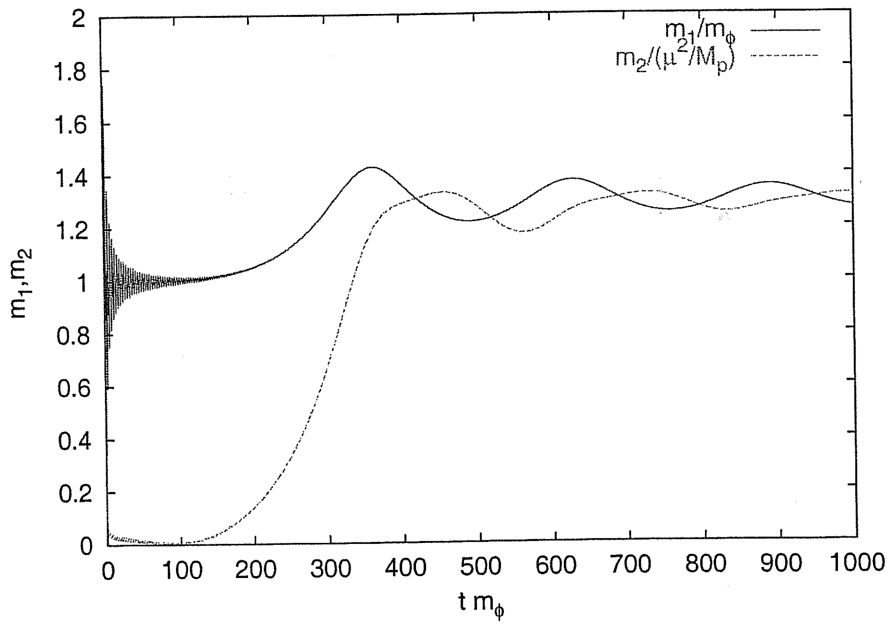


Figure 6.3: Evolution of the masses of the two fermionic eigenstates. As in figure 6.1, $\hat{\mu}^2 = 10^{-2}$. Notice the different normalizations for the two masses.

In order to compute the evolution of the occupation number of the fermions we adopt the procedure described in section 4.2.2: during the evolution of the system, the states are mixed in such a way that the hamiltonian is kept in a diagonal form. The two eigenstates obtained through this diagonalization coincide with the fields $\theta$ and $\Upsilon$ only for $\Delta = 0$. In particular, this is true at very early and late times. The safest way to make the proper identifications in these regimes is to consider the evolution of the mass eigenvalues, which always behave like in the example shown in figure 6.3. The two masses present (for $\hat{\mu}^{-2} \ll 1$) a strong hierarchy. We denote with $\psi_1$ the eigenstate with bigger mass, and with $\psi_2$ the other one. The mass of $\psi_1$ converges to the inflatino mass ($\simeq 1.31\, m_\phi$) at late times, and it is always of the order $m_\phi$. On the contrary, the mass of $\psi_2$ converges to the gravitino mass ($\simeq 1.31\, \hat{\mu}^2\, m_\phi$) in the vacuum. As it will be clear below, we can "qualitatively" identify $\psi_1$ with the inflatino and $\psi_2$ with the Polonyi fermion for the whole evolution. Although rigorous only at late times, this identification can be useful for a qualitative understanding of the system.

What is most important to us is the relation between the eigenstates ($\psi_1$, $\psi_2$) and the gravitino $\theta$ and the matter field $\Upsilon$. As we have said, the last fields coincide with the physical eigenstates only at the very beginning and at the end of the evolution.

More precisely, from the behavior of the two masses we have

$$\theta \equiv \psi_2 \qquad \text{and} \qquad \Upsilon \equiv \psi_1 \tag{6.67}$$

at late times. On the contrary, it must be

$$\theta \equiv \psi_1 \qquad \text{and} \qquad \Upsilon \equiv \psi_2 \tag{6.68}$$

at early times, since the longitudinal gravitino component $\theta$ is provided by the goldstino and supersymmetry is initially broken only by the inflaton field.

At intermediate times, the hamiltonian cannot be diagonalized with a simple rotation in "flavor" space, and $\theta$ cannot be just a simple (i.e. with only numbers as coefficients) linear combination of $\psi_1$ and $\psi_2$. However, we can gain an intuitive description of the system through the identifications

$$\begin{aligned} \theta &\sim \sqrt{r_\phi}\,\psi_1 + \sqrt{r_s}\,\psi_2\,, \\ \Upsilon &\sim -\sqrt{r_s}\,\psi_1 + \sqrt{r_\phi}\,\psi_2\,. \end{aligned} \tag{6.69}$$

The coefficients $r_\phi$ and $r_s$ give a "measure" of the relative contribution to supersymmetry breaking provided by the two scalar fields (see eq. (6.39)). These relations can thus be justified as a "generalization" of the equivalence theorem, as it was also suggested in [116, 117]. We remark that they are rigorous at early and late times (when they coincide with the identifications (6.67) and (6.68)). At intermediate times they interpolate between these two regimes and can be thus used as a qualitative description of the system.

From eqs. (6.69) we deduce the following estimates for the occupation numbers

$$\begin{aligned} N_\theta &= r_\phi\,N_1 + r_s\,N_2\,, \\ N_\Upsilon &= r_s\,N_1 + r_\phi\,N_2\,. \end{aligned} \tag{6.70}$$

The evolution of these quantities is shown in figure 6.4 for modes of comoving momentum $k = m_\phi$ and for $\hat{\mu}^2 = 10^{-2}$. Notice that (by construction) $N_\theta \equiv N_1$ at early times, while $N_\theta \equiv N_2$ at late ones. In these regimes these identifications are rigorous.

In figures 6.5 and 6.6 we plot instead the spectra of the states $\psi_1$ and $\psi_2$ in the case $\hat{\mu}^2 = 10^{-2}$ and at the times $t = 10\,m_\phi^{-1}$ (that is, after a couple of oscillations of the inflaton), $t = \hat{\mu}^{-2}\,m_\phi^{-1}$, and $t = 10\,\hat{\mu}^{-2}\,m_\phi^{-1}$.

It is apparent that most quanta of the state $\psi_1$ are produced at the very first oscillations of the inflaton field, while quanta of $\psi_2$ are mainly produced at the times when the Polonyi scalar starts oscillating. This supports the qualitative identification of $\psi_1$ with the inflatino and of $\psi_2$ with the Polonyi fermion. It is worth noticing that, for comoving momenta smaller than $m_\phi$, the increase of $N_2$ is not related to a

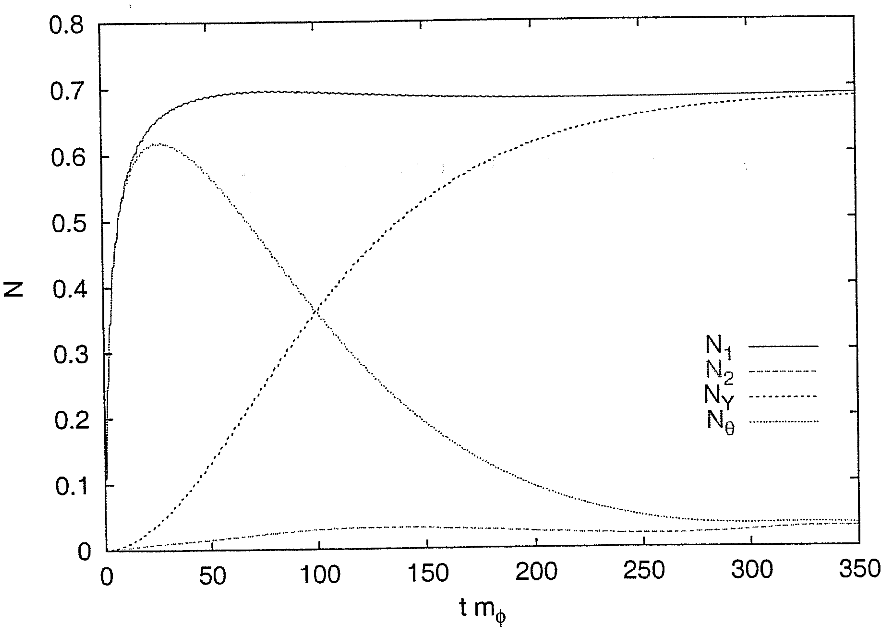


Figure 6.4: Evolution of $N_\theta$ and $N_\Upsilon$ for $\hat{\mu}^2 = 10^{-2}$ and $k = m_\phi$. See the text for details.

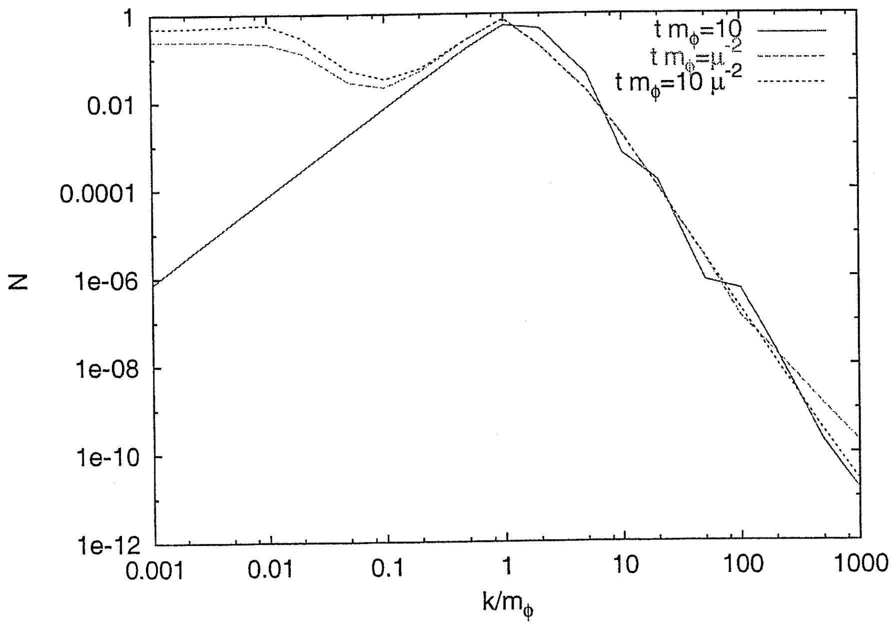


Figure 6.5: Spectrum of the state $\psi_1$ at different times for $\hat{\mu}^2 = 10^{-6}$.

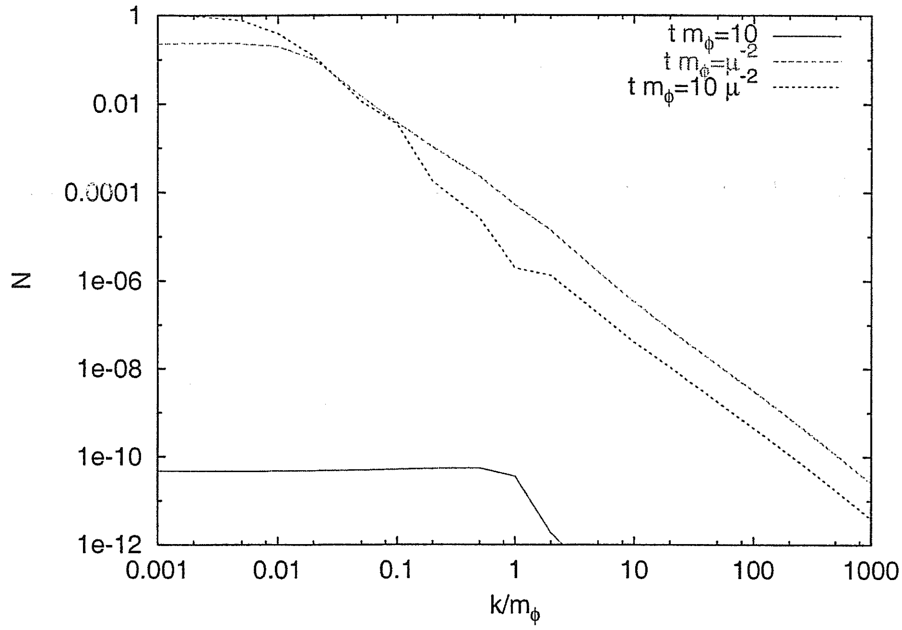


Figure 6.6: Spectrum of the state $\psi_2$ at different times for $\hat{\mu}^2 = 10^{-6}$.

"conversion" of quanta of $\psi_1$ to $\psi_2$. Indeed, the increase in $N_2(k)$ is not accompanied by a decrease in $N_1(k)$ for $k \lesssim m_\phi$.

We can now show the most important result of the present chapter, that is the spectra of $\Upsilon$ and $\theta$ at the end of the process. We present them in figures 6.7 and 6.8, respectively. They are computed at the time[13] $t = 10\,\hat{\mu}^{-2} m_\phi^{-1}$. The time required for the numerical computation increases linearly with $\hat{\mu}^2$, and the realistic case $\hat{\mu}^2 = 10^{-11}$ is far from our available resources. We thus kept $\hat{\mu}^2$ as a free parameter and we performed the explicit numerical computation only up to $\hat{\mu}^2 = 10^{-6}$. In particular, in figures 6.7 and 6.8 the spectra of the fermions produced at preheating are shown for $\hat{\mu}^2 = 10^{-2}$, $10^{-3}$, $10^{-4}$, $10^{-5}$, $10^{-6}$. The case $\hat{\mu}^2 = 10^{-11}$ can be clearly extrapolated from the ones shown in these figures.

In figure 6.7, the spectra for the state $\psi_1$ are shown. This state corresponds to the matter fermion $\Upsilon$ in the true vacuum. It is apparent that the main features of the spectrum are independent of the value of $\hat{\mu}^2$. The reason for this is that $\psi_1$ is associated to the inflatino, that is produced by the coherent oscillations of the inflaton. As we discussed above, the dynamics responsible for the production of this state is independent on the value of $\hat{\mu}^2$. Therefore the only relevant scale for

[13] In the cases $\hat{\mu} = 10^{-2} - 10^{-4}$ we have continued the evolution further, until the spectra stop evolving. We have found that the spectra shown in figure 6.7 coincide with the final ones, while $N_\theta$ very slightly decreases for $t > 10\,\hat{\mu}^{-2} m_\phi^{-1}$. Thus, we believe the results shown in figure 6.8 to provide an accurate upper bound on the final gravitino abundance.

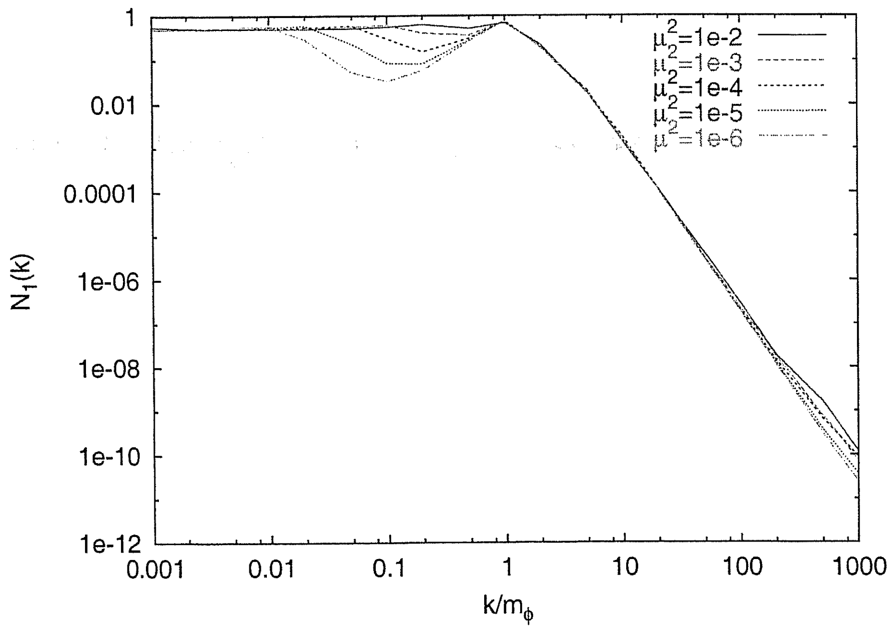


Figure 6.7: Spectrum of inflatinos at late times.

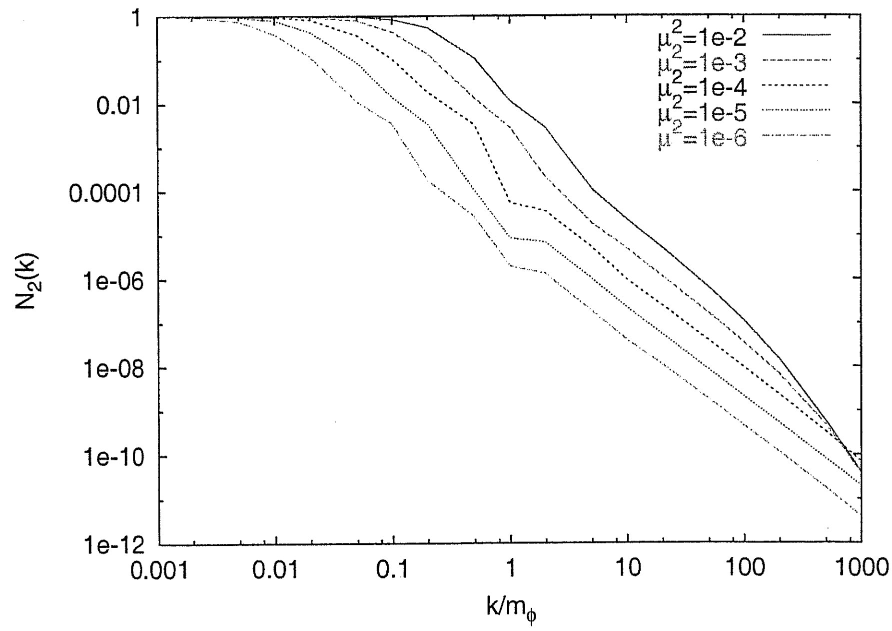


Figure 6.8: Spectrum of gravitinos at late times.

$N_1$ is $m_\phi$, and indeed the spectrum of $\psi_1$ exhibits a cut-off at comoving momentum $k \sim m_\phi$.

The spectra shown in figure 6.8 are related to the abundance of gravitinos after the fields have stabilized in their minima. We see that in this case the occupation number decreases as $\hat{\mu}^2$ becomes smaller. Indeed, the occupation number $N_2(k)$ is of order unity for comoving momenta $k$ smaller than some cut-off $k_*$. From figure 6.8 we can deduce the dependence $k_* \propto (\hat{\mu}^2)^{1/3}$ of this cut-off on the parameter $\hat{\mu}^2$. This behavior suggests that gravitinos are not produced in the limit $\hat{\mu}^2 \to 0$, confirming what we have argued in the section 6.5.

We can conclude that in the model we are considering both inflatinos and gravitinos are produced nonthermally. However, the mechanism responsible for the production of gravitinos is much less efficient that the one acting on inflatinos. Indeed, the latter is related to the dynamics of the inflaton, while the former is related to the dynamics of the Polonyi field. As a consequence, and as it is clearly confirmed by the comparison of figures 6.7 and 6.8, the number of non-thermal gravitinos is much smaller than the number of inflatinos.

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