ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Physics Of Extra Dimensions

Thesis submitted for the degree of Doctor Philosophiæ

under the supervision of Professors:

Antonio Masiero & Seifallah Randjbar-Daemi

Candidate:

Rula Tabbash

Academic Year 2000–2001

SISSA - SCUOLA
NTERNAZIONALE
SUPERIORE
STUDI AVANZATI

TRIESTE
Via Beirut 2-4

TRIESTE

SISSA – ISAS
International School for Advanced Studies
Trieste – Italy

Physics Of Extra Dimensions

Thesis submitted for the degree of Doctor Philosophiæ

under the supervision of Professors:

Antonio Masiero & Seifallah Randjbar-Daemi

Candidate:

Rula Tabbash

Academic Year 2000–2001

dedication...

Preface

The idea of believing in and looking for a unique source to existence have been one of the oldest continuously paused questions in the history of man kind. The idea goes back to at least to the Sumero-Semitic syncretism of the 3rd millennium BC, when the process of identification (unification) of deities had already taken place before the majority of records were written. The concept of having a single creator God, *Tiamat*, is documented in the Babylonian Creation story "*Enuma Elish*" more than four thousand years ago.[1]

A search for simplification in understanding an obviously sophisticated and detailed phenomena happening in mother Nature is, in many senses, a search for identification (unification), and vise versa. Many of the human species named under physicists believe in this equivalence.

The first step towards a better "understanding" was by introducing the concept of a *cause* of an action, which was eventually translated by Newton (1665) into the more concrete concept of *force*.[2] To date we are aware of the presence of four kind of distinct forces in Nature at very low energies; the electromagnetic, weak, strong, and gravitational forces. The attempt to unify these forces at higher energies (though not the only interesting problem of physics today) by deriving their equations of motion from a single simple renormalizable action was pursued by realizing the covariance of their physical laws under certain symmetries. Those symmetries reduce the number of free parameters in the theory and therefore enhances its predictivity; this is one aspect of the simplification one sought after.

The present Standard Model (SM) of elementary particles (Abdus-Salam, Glashow, and Weinberg; Nobel Prize 1979) succeeded very well in unifying the strong and electroweak interactions under the gauge group $SU(3)_C \times$

[1] A translation of *Enuma Elish* (before 2000 BC) can be found in [1].

[2] Thanks to gauge theories, now we also know the cause of a force!.

$SU(2)_L \times U(1)_Y$ at energy scales around 100 GeV.

No understanding of a similar unification of gravity with the other forces at accessible energies to accelerators is achieved to date. The only promising quantum theory of gravity, so far, is superstring/M theory and its mathematical consistency requires the space-time to have ten/eleven dimensions. The low energy field theoretical description of string theory can be used in order to explain our four-dimensional world using the standard Kaluza-Klein compactification of the extra dimensions explained in the proceeding chapters.

Coming back to the SM; its main building blocks are: Action principle, Quantum Field Theory, Poincaré and gauge invariances, and the justified assumption of living in $(3+1)$-dimensional space-time.

The community of modern theoretical particle physicists is clearly impressed by this model (for an obvious reason that its predictions fit the experimental data with an impressive precision [2]), and believes the assumptions under which it is made are to be valid at low energies and the concepts it is based on are to be generalized to theories where gravity is included.

Despite the experimental success[3] of the standard model as a theory describing the strong and electroweak interactions of elementary particles, this model is not theoretically satisfactory for various reasons. There are two main sources of theoretical dissatisfaction; the first has to do with the model itself (when it comes to explaining, flavor, and charge quantization, for example), and the second arises when the standard model is discussed within a more general context where the fourth fundamental force of nature, gravity, is present; leading subsequently to an instability of the weak scale at the quantum level (caused by the quadratic divergences in the Higgs $(\text{mass})^2$).

Therefore, there are obvious urges to go beyond the standard model: the need to explain neutrino oscillations (for instance), quantum instability of the electroweak scale, and the hope for a quantum theory of gravity to unify with the standard model (or its extension). Further simplification is yet to be worked for.

The electroweak quantum instability, known as the *hierarchy problem* between the electroweak and gravity scales, was the main motivation to start

[3] In the recent past there has been dissatisfaction from the experimental point of view, as well, since the standard model is in shortage of explaining, for instance, neutrino oscillations [3] and the recent data of $g-2$ of the Muon [4]. Further more, there is no hint for the existence of an elementary scalar field in Nature, while the Higgs sector is very essential in the SM (see [5] for possible future detection of the standard model Higgs particle).

searching for new physics beyond the standard model. These searches lead to the birth of many hoping-to-be-physical [4] theories like technicolor [6], grand unified theories [7], supersymmetry [8], and recently models with large extra dimensions [9].

By coincidence,[5] also the most promising theory to quantize gravity and unify it with the other gauge forces (through superstring/M theory) seem to require both extra dimensions, beyond the known four, and supersymmetry as crucial ingredients for its consistency.

This strongly hints to that, hoping for a unique theory, the hierarchy problem may be solved in a theory in higher dimensions than 4 with broken supersymmetry.

This thesis concerns itself to a good extend with solving the hierarchy problem within the context of extra dimensions.

In string theory, the extra six-dimensional space is squeezed and pressed (or *compactified*) into a manifold of a tiny volume. The original compactification scale of this theory is of order M_P^{-1} which is extremely small and it would be impossible for any machine to detect a modification of the gravitational law at such energies. With such a scale, the world will definitely appear four-dimensional without any hint to the presence of dimensions beyond the known four. Inspired by string theory, though not as mathematically rigorous, a recent interest in extra-dimensional model has been revived by Dvali *et al* [9],[6] who pointed out that the modification to Newton's low by introducing large extra dimensions could be a valid possibility since no tests of gravity has been carried out to distances much below 1 millimeter (see for instance [15, 16]).

The activities so far fall into two categories: models as [9] based on the original idea of Kaluza and Klein [17, 18] which consider a tensor product (factorizable metric) of the four-dimensional world with the compact space; and alternatives to compactification which consider non-compact extra dimensions, [19], with *warped* (non-factorizable) metric as in [20]. The new proposals were all aimed to a solution to the hierarchy problem by lowering

[4] I mean by a "physical" theory here the one which is able to fit with present experimental data, to reproduce all what the standard model is able to reproduce at $\sim$ 100GeV, and not to contradict with the standard cosmological scenario after Big Bang Nucleosynthesis. In addition it should have some extra predictive power to SM preferably testable at LHC.

[5] It may be realized in the future that it is not a mere coincidence. Who knows?.

[6] Credits of associating a physical meaning to extra dimensions prior to [9] go to [10]-[14].

the gravity scale from 10^{19}GeV down to few TeV. In Kaluza-Klein (KK) like models, the particles are free to propagate inside the internal compact space which should be small enough not to lead to phenomena contradicting the presence knowledge of particle physics and big bang nucleosynthesis. There is an infinite number of images of the internal space at each point of the four-dimensional world. When the extra dimensions are non-compact, the only way to avoid long-range observable effects is to localize the fields on a thin three-dimensional wall (brane). In this case, there will not be infinite images of the brane world, and in principle one can have only one single brane on which ordinary matter is localized.[7]

In both categories, it turns out that lowering the gravity, or the fundamental, scale of the theory does not by itself solve the problem. It rather addresses it in a different way as it introduces other sources of fine-tuning into the theory like for instance explaining the size of the manifold made by the extra dimensions or the mechanism which traps the matter fields and gauge forces on a brane. Let alone that there is again a huge difference between few TeV and M_P. However, as we shall explain later, the hierarchy problem is not having a low energy cutoff close to the weak scale, but rather having no quadratic or significant quantum correction to the classically computed physical quantities.

There are generic difficulties which face models with extra dimensions as upon compactifying down to four dimensions one may in general get new degrees of freedom added to the spectrum of the Standard Model. The new states can be purely from the gravitational sector, or have Standard Model Kaluza-Klein excitations in addition (depending on whether the SM interactions are written directly in four dimensions, using the induced metric, or written fully in D dimensions). In any case, the new states might lead to detectable modifications of the existing accelerator data and cosmological observations [21]. This leads to imposing judicious bounds on the parameters of these theories. Whether these bounds are implemented or not, the theories with large extra dimensions experience difficulties in realizing complementary scenarios like the standard Cosmological one. For instance, imposing an upper bound on the reheating temperature in order to avoid overproduction of Kaluza-Klein modes of the graviton, and discrete symmetries in order to pre-

[7]A special case of only one image of the 4-dimensional space-time can also be considered, within the context of Kaluza-Klein compactification, if ordinary matter fields are forced to be localized around a specific point in the internal space [9]. We will not discuss this possibility here.

vent a fast proton decay make it difficult to construct a baryogenesis model. In addition, recovering the standard Friedmann-Robertson-Walker Universe in 4 dimensions starting from higher-dimensional Einstein's equations is not straightforward whether the compactification is standard or warped.

In both factorizable geometry and non, it is important that the internal space has no effects interfering with the SM precision tests. This of course does not happen naturally, and, as mentioned above, bounds on the parameters of the model should be imposed.

So far no full and consistent model has been constructed, neither in tensor compactification nor in the warp one, however the phenomenological and cosmological implications have been studied extensively, and attempts towards creating a theoretically appealing model persist. In this report we will review the tools and basic concepts in constructing Kaluza-Klein theories and Brane world scenarios.

plan of the thesis work

This report is organized as follows: In chapter 1 an introduction to the Kaluza-Klein theory is presented including the very basic concepts on which the Kaluza-Klein picture and Brane-world scenarios are based. It is far from being a complete review, neither in topics covered nor in list of references, however it is meant to serve as a background material mainly for Chapter 4.

In Chapter 2, the idea of warped extra dimensions and localization of matter fields on a hyperspace in $4+d$ dimensions is reviewed with the help of some examples taken from the literature.

Chapter 3 discusses some of the physical implications of theories with extra dimensions, mainly the modification of the Cosmological evolution of the Universe and the fast proton decay. The last part if this chapter contains an example of generating baryon number violation in theories with extra dimensions and low gravity scales.

Chapter 4 is devoted to a solution to the hierarchy problem, where an understanding of a single source responsible to both electroweak symmetry breaking and standard model fermion chirality is provided.

The references are listed according to the order in which they appear first in the text. Unfortunately, it was not possible to refer to all the interesting and important papers written in this subject due to the limitation of space and time allowed for writing and submitting this dessertation. Therefore, only "samples" of such contributions to the literature could be provided.

publications

This thesis is partially based on the papers:

1. Rula Tabbash
   "A new approach to the hierarchy problem", [hep-ph/0107213].

2. Rula Tabbash
   "Compact hyperbolic manifolds as internal worlds", [hep-ph/0104233].

3. G. R. Dvali, S. Randjbar-Daemi, R. Tabbash
   "The origin of spontaneous symmetry breaking in theories with large extra dimensions", [hep-ph/0102307]. To appear in Phys.Rev.D.

4. Avijit Mukherjee, Rula Tabbash
   "Geometric bounds on Kaluza-Klein masses", Eur.Phys.J.C20 (2001) 193, [hep-ph/0007177].

5. Antonio Masiero, Marco Peloso, Lorenzo Sorbo, Rula Tabbash
   "Baryogenesis versus proton stability in theories with extra dimensions", Phys.Rev.D62 (2000) 063515, [hep-ph/ 0003312].

Contents

Chapter 1

Kaluza-Klein Picture

1.1 Historical remark

Although having seven heavens is not a particularly new idea, the first to be published in a scientific journal, proposing our observed world to be an effective theory of a fundamental theory existing in more than four dimensions, was Nordström's [22] back in 1914. Having no general relativity at that time, Gunnar Nordström wrote down Maxwell's equations in 5 dimensional space-time, and by wrapping the fifth dimension on a circle, he reduced the equations to Maxwell-Nordström electromagnetic-gravitational theory in 4 dimensions.

In 1918, Hermann Weyl introduced the concept of gauge invariance [23] in the first attempt to unify electromagnetism and gravitation in a geometric context.

A year later, in 1919, the mathematician Theodor Kaluza proposed [17] obtaining a four dimensional Einstein-Maxwell theory starting from Einstein's gravity equations in five dimensions. Assuming the five-dimensional manifold, W, to be a product of a 4-dimensional space-time M_4 and a circle S^1, $W = M_4 \times S^1$, the fifteen components of the metric can be decomposed from a four-dimensional point of view into 10 describing the gravity tensor, four forming the components of a $U(1)$ gauge field, and one degree of freedom representing a scalar field. By Fourier expanding those fields, retaining only the zero modes, and integrating along the circle S^1 one obtains a theory in four dimensions which is invariant under both four-dimensional general coordinate transformations, and a $U(1)$ gauge transformations.

In his original work, Kaluza assumed the zero mode of the scalar field to be constant, $\phi^0 = 1$. In any case, the value of ϕ^0 had to be positive in order to insure the proper relative sign of the Einstein and Maxwell terms so that the energy is positive. This in turn means that the fifth dimension must be space like. This can also be easily understood in terms of causality; clearly a compact time-like dimension would lead to closed time-like curves. The abelian gauge symmetry arising in four dimensions upon compactification originates from the isometry of the circle. Those last two point, the requirement that the extra dimensions must be space-like, and that the isometry of the compact space results in a gauge symmetry (generally non-abelian) of the effective action are general arguments [24].

Seven years later, Oskar Klein used Kaluza's idea in an attempt [18] to explain the underlying quantum mechanics of Schrödinger equation by deriving it from a five-dimensional space in which the Planck constant is introduced in connection with the periodicity along the closed fifth dimension. In this paper he also discusses the size of the compactified circle, getting closer to giving the extra dimensions a physical meaning than his predecessors did. In a separate work [25], still in 1926, Klein proposed[1] a relativistic generalization of Schrödinger's equation by starting from a massless wave equation in five dimensions and arriving at four-dimensional Klein-Gordon equation for individual harmonics.

Afterwards, many people adopted Klauza's idea, starting from Einstein early last century and continuing by several physicists today. During this period, the idea of having new dimensions to propagate in inspired many to write the first complete models for Lagrangians unifying Yang-Mills and gravity theories [27], supergravity in 11 dimensions [28], and superstring theories which has to be considered in 10 dimensions for the theory to be anomaly free and hence consistent at the quantum level.

Supergravity in eleven dimensions is special for at least three reasons: it is unique, the maximum number of additional dimensions to four on which a supergravity theory can be constructed is seven (otherwise the theory would contain massless particles of spin greater than two [29]), and moreover seven extra dimensions is the minimum number for having the standard model gauge group $SU(3) \times SU(2) \times U(1)$ generated from the isometry group of the internal space (which could for instance $\mathbb{C}P^2 \times S^2 \times S^1$). The first to

[1]Credit to this approach goes also to V. Fock and H. Mandel, though others worked for and achieved the same aim as well [26].

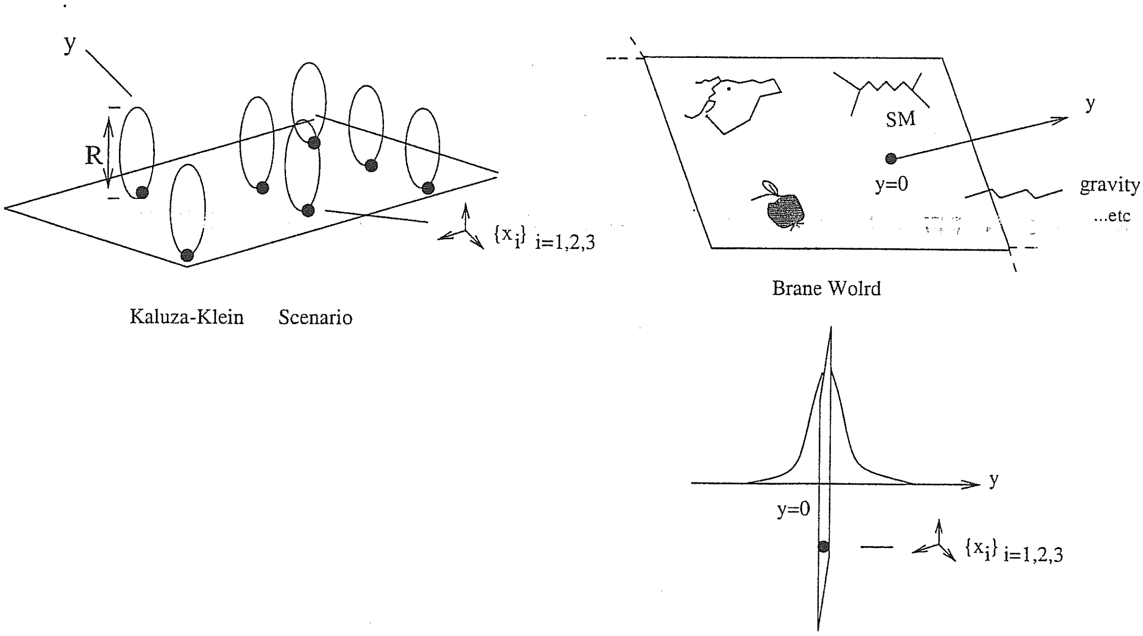

Figure 1.1: Kaluza-Klein large extra dimensions versus Brane world.

give a physical meaning to extra dimensions were Scherk and Schwarz in 1975 [10], in an attempt to put string theory (which is consistent only in ten dimensions) in contact with the four-dimensional world by assuming that the six extra dimensions are curled up into a tiny small size which renders them unobserved. An alternative to compactification was introduced in 1984 by Wetterich [19] and others [30, 31, 32] by considering non-compact internal spaces as a possible solution to the chirality problem [33, 34] in the context of Kaluza-Klein theories.

1.2 Introduction

In the original Kaluza-Klein framework, the particles are free to move in the compact space, as well as the 4-dimensional one, and hence the volume of the new dimensions should be small in order not to undesirably interfere with the present observations and existing experimental data, since the new idea of $4+d$ dimensions has a strong potential to naturally modify the expansion of the Universe and the cross sections of elementary particle interactions. Upon compactification, the eigenvalues of the Laplacian (or the Dirac operator) on the internal will have the interpretation of masses in four dimensions. An illustrative example is the case of a product of a four-dimensional Minkowski space and a circle $M_4 \times S^1$. The wave function of a scalar field can be expanded in Fourier series along S^1,

$$\phi(x,y) = \sum_{n\in\mathbb{Z}} e^{in\frac{y}{R}} \phi_n(x) ,$$

where y is the coordinate on S^1, of radius R, and n is the eigenvalue of the one-dimensional angular momentum operator. The Klein-Gordon equation of a massless ϕ can therefore be written as

$$p^\mu p_\mu = -p_0^2 + \underline{p}^2 = -\frac{n^2}{R^2} ,$$

where p_μ is the 4-dimensional momentum. Clearly, from a M_4 point of view, each Kaluza-Klein mode $\phi_n(x)$ is seen as a separate particle with mass^2 equal to n^2/R^2. Below energy scales $1/R$, only massless modes with $n = 0$ can be excited and hence the low energy physics is effectively four-dimensional. This is just a naïve conclusion, and we will see later on various bounds and conditions which apply to realistic Kaluza-Klein type scenarios in order to indeed achieve phenomenologically acceptable models in four dimensions.

As discussed in the introduction, the modern interest in having extra dimensions is focused more on finding a solution to the hierarchy problem rather than searching for detectable hints to string theory at low scales. If the volume of the compactified extra space is characterized by scale larger than M_p^{-1}, the $(4+d)$-dimensional gravity scale, will be automatically lowered, could even be few TeVs, and the difference between the fundamental gravity and electroweak scales will be smaller. As usual, the 4-dimensional Planck scale, $M_P = 10^{19}$GeV, is related to the fundamental gravity scale, M, of the $(4+d)$-dimensional space via the volume of the internal space V_d. Having the gravity action in $4+d$-dimensions and integrating over the coordinates of the internal space,

$$S = \frac{1}{\kappa^2}\int d^{4+d}z\sqrt{g(z)}\mathcal{R}_{4+d} \Rightarrow S_{\text{eff}} = \frac{V_d}{\kappa^2}\int d^4x\sqrt{g_4(x)}\mathcal{R}_4 ,$$

where $\kappa^2 = M^{-d-2}$, we get

$$M_P^2 = V_d M^{d+2} . \tag{1.1}$$

V_d is usually proportional to the radius of compactification, $V_d \propto R^d$.[2]

Having a fundamental gravity scale in $4 + d$ dimensions much different from M_P means that Newton's low for gravity should be modified at small distances $r \ll R$ to be

$$V(r) = \frac{m_1 m_2}{M^{d+2}} \frac{1}{r^{1+d}} ,$$

where for larger distances, $r \gg R$, the potential recovers its usual $1/r$ dependence

$$V(r) = \frac{m_1 m_2}{M^{d+2} R^d} \frac{1}{r} .$$

As can be deduced from (1.1), or equivalently

$$R = 10^{-17+\frac{30}{n}} (\frac{1\mathrm{TeV}}{M})^{1+\frac{2}{n}} \mathrm{~cm} ,$$

achieving $M \sim 1$TeV requires assuming bigger compactification radii than the ones used in string theory, i.e. $R \gg 10^{-19}\mathrm{GeV}^{-1}$. Having lowered the fundamental gravity scale M to few a TeV, the gauge hierarchy problem will be re-addressed under as the problem explaining the size of the internal space.

For $M \sim$TeV, this radius can be as big as $10^{13}\mathrm{GeV}^{-1}$ for two extra dimensions, $10^8\mathrm{GeV}^{-1}$ for three, $10^5\mathrm{GeV}^{-1}$ for four, and so on. However, the radius should not exceed the millimeter since we do have tests of Newton's low up to this scale [16, 160]. The case of one extra dimensions for a Kaluza-Klein type scenario is excluded for low M since it requires the compactification radius to be very big, 10^9Km!. This problem becomes milder as the number of extra dimensions increases, however it may be desirable to avoid this shortcoming even for a small number of them. The primary problem is not explaining the smallness of these radii, which is obviously a fine-tuning problem, but rather avoiding their undesirable contributions to well studied observables (the masses of the KK excitation can be very light).

As said above, the Kaluza-Klein idea is that there is a multi-dimensional world of the low energy physics which could be responsible for the internal symmetries that we observe. The scale of this structure is assumed to be small enough to render it unobservable at present energies; below 10^3GeV. The motivation is to obtain a geometric interpretation of internal quantum

[2]This relation is not generic to all manifolds, although it is very common. For example, for a compact hyperbolic manifold, there will be an additional exponential dependence of the volume on the radius of compactification.

numbers, as electric charge, and to consider them conceptually the same as the energy and momentum.

In these theories the starting point is a $(4+d)$-dimensional space-time, and assuming general coordinate invariance adopting the $(4+d)$-dimensional Ricci scalar, with perhaps some covariant coupling to "matter", as the Lagrangian.

One argues then that the ground state values of the fields (including the metric tensor) must solve the classical field equations and must have as much symmetries as possible.[3] The goal is to find a suitable ground state configuration. This ground state should hence have the geometry of a product space $M_4 \times K$, where M_4 is a 4-dimensional space of a Minkowskian signature, and K is a d-dimensional compact space of an Euclidean signature and a small size. Both M_4 and K should admit groups of motion.

In the case of M_4 being a Minkowski space, its group of motion is the Poincaré group in four dimensions. The internal space K should admit a compact group G.

The masses of the bosonic particle (or fermionic) will be determined by the eigenvalues of the Laplacian (or Dirac) operator on K, since the operator can be written as a direct sum of the operators on M_4 and K.

The zero modes of those operators, if any, would correspond to massless particles and these are important from the four-dimensional point of view since they will represent the effective Lagrangian. The modes (excitations) which involve propagation on K will generally have large masses, proportional to the scale of the size of K.

All excitations of any particle will be classified in irreducible representations of the ground state symmetry. If this symmetry is smaller than the Lagrangian symmetry, we end up with spontaneous symmetry breaking.

Finding the eigenvalues of the Laplacian ∇^2 (or those of $\not{D}$) is not a straightforward problem. In fact, there is no analytic expression for the eigenvalues of these two operators on a generic compact manifold (although lower and upper bounds on these values exist for a large number of them, see [35]). However, things can be made simpler when a specific choice for K is considered.

In this chapter the basic concepts on which the Kaluza-Klein idea is based are briefly reviewed. Topics cover harmonic expansion on coset spaces, effective Lagrangians, stable compactification, and the issue of chiral fermions.

[3]Since we know from experience that more symmetrical states tend to have lower energies.

The chapter is concluded with a remark on possible geometric estimates for masses of Kaluza-Klein fields on a generic compact space.

1.3 Coset spaces

Here we will focus on the most symmetrical category of compact manifolds, the coset spaces G/H, [4] where G is a group (here a Lie group) of motion of K (which leave the metric invariant) and H is a subgroup of $O(d)$ (the tangent space group) [36].An example of a coset space G/H is $S^2 = SU(2)/U(1)$. The Kaluza-Klein reduction procedure for a generic manifold has been worked out in [37]-[39].

Let G be generated by

$$[Q_{\hat{a}}, Q_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}} Q_{\hat{c}} \, ,$$

where $Q_{\hat{a}} = \{Q_{\bar{a}}, Q_a\}$, $Q_{\bar{a}}$ being the generators of H. Note that, $d \equiv \dim(G/H) = \dim G - \dim H$.

We assume that G/H is reductive, *i.e.*

$$[Q_{\bar{a}}, Q_b] = C_{\bar{a}b}{}^{c} Q_c \, ,$$

and symmetric;

$$[Q_a, Q_b] = C_{ab}{}^{\bar{c}} Q_{\bar{c}} \, .$$

Let y^i parameterize the coordinates on $K = G/H$, and let L_y be the representative of each equivalence class (coset) of G with respect to its subgroup H (for example one can chose$L_y = e^{y^a Q_a}$).

A generic transformation by G carries L_y into $L_{y'}$

$$g L_y = L_{y'} h \quad , \quad h \in H \, .$$

This equation can be solved unambiguously for y' and h as a function of y and g.

To find the d-bein, construct the 1-form $L_y^{-1} dL_y$ which necessarily belongs to the Lie algebra of G and hence can be expanded as

$$L_y^{-1} dL_y = e^a(y) Q_a + e^{\bar{a}}(y) Q_{\bar{a}} \, .$$

[4] Since, after all, the purpose of Kaluza-Klein picture is to explain internal symmetries in terms of the symmetry of the compactified space.

Knowing how L_y transforms under G, one can derive, in a straight forward way, the the transformation rule for $e^a(y)$ and show that they are the vielbeins on G/H and that $e^{\bar{a}}(y)$ are the connection form on H. They are [40]

$$e^a(y') = e^b(y)D_a{}^a(h^{-1}) + (g^{-1}dg)^{\hat{b}}D_{\hat{b}}{}^a(L_y h^{-1}) \tag{1.2}$$

$$\begin{aligned} e^{\bar{a}}(y') &= e^{\bar{b}}(y)D_{\bar{b}}{}^{\bar{a}}(h^{-1}) + (hdh^{-1})^{\bar{a}} \\ &+ (g^{-1}dg)^{\hat{b}}D_{\hat{b}}{}^{\bar{a}}(L_y h^{-1}) \, . \end{aligned} \tag{1.3}$$

Firstly, note that $d_y g = 0$, and hence $e^a(y)$ satisfies the requirement for an invariant d-bein. Later on we shall show that the x-dependent g-transformations of G/H will give rise to the Kaluza-Klein local gauge transformations. For the time being, we will assume that g is independent of coordinates. The equation 1.2 indicates an ordinary tangent space rotation of the e^a's, provided H is embedded in $SO(d)$, the tangent space group of G/H.

In Kaluza-Klein theory, g can depend on the coordinates of M_4, and therefore generally $d_x g \neq 0$. To understand the second term in (1.2) one has to take into account the fluctuations of the ground state. Among these there will be a Yang-Mills vector, $A^{\hat{a}}$, which undergoes the usual inhomogeneous transformations of a gauge potential. It turns out that the second term in (1.2) is canceled precisely by the inhomogeneous term in the transformations of $A^{\hat{a}}$ [36] (see the next section). [5]

Similarly, the last term in (1.3) is canceled by the transformations of $A^{\bar{a}}$, and hence (1.3) indicates that $e^{\bar{a}}$ is a connection form on G/H.

In many cases of interest, the $(4+d)$-dimensional metric is coupled to a gauge field associated with a group U. When it is possible to embed H into U, this gauge fields generally acquire non zero, but G invariant, values on G/H.

Let $q_{\bar{c}}$ denote the image of $Q_{\bar{c}}$ in the Lie algebra of U. Then the gauge potential 1-form

$$A = \frac{1}{\rho} e^{\bar{c}} q_{\bar{c}} \, ,$$

[5]The metric on K can be expressed as usual in terms of the d-beins, $g_{mn}(y) = e^a_m(y)e^a_n(y)$ $(n, m; a = 1, ...d)$. The d-beins form a set of linearly independent 1-forms $e^a(y) = dy^m e^a_m(y)$. When the space admits a group of motion G (leaving g_{mn} invariant), to each element $g \in G$ there corresponds a transformation $y \to y'$ such that $e^a(y') = e^b(y)D_b{}^a$ where $D \in O(d)$.

and it can be shown that its associated field strength,

$$F = \frac{1}{2\rho} e^a \wedge e^b C^{\bar{c}}_{ab} q_{\bar{c}} \ ,$$

satisfied the Yang-Mills equation [37]. The 2-form F is invariant under the left translations of G/H associated with the gauge transformations corresponding to the embedding of H in U.

Example: $S^4 = SO(5)/SO(4)$

Consider $S^4 = SO(5)/SO(4)$, parametrized by coordinates

$$\begin{aligned} u^m(y) &= a\frac{2y^m}{1+y^2} \ , \quad m = 1,2,3,4 \\ u^5(y) &= a\frac{1-y^2}{1+y^2} \ , \end{aligned}$$

where $y^2 = (y^1)^2 + ... + (y^4)^2$, $u^m u^m + u^5 u^5 = a^2$, and l is an arbitrary number. One can chose L_y, the 5×5 orthogonal matrix to be [41]

$$L_y = \begin{pmatrix} \delta_{mn} - \frac{2y^n y^m}{1+y^2} & \frac{2y^m}{1+y^2} \\ -\frac{2y^m}{1+y^2} & \frac{1-y^2}{1+y^2} \end{pmatrix} \ .$$

Now we can construct $L_y^{-1} dL_y$, which is a 5×5 antisymmetric matrix in the Lie algebra of $SO(5)$ by decomposing it into a convenient basis. Chose $\{Q_{b5}, Q_{bc}\}$, $b, c = 1, ..., 4$ such that $\{Q_{bc}\}$ are the generators of $SO(4)$. Then

$$L_y^{-1} dL_y = e^b Q_{b5} + \frac{1}{2} A^{bc} Q_{bc} \ ,$$

where

$$e^b = \frac{2a}{1+y^2} dy^b \ , \tag{1.4}$$

$$A^{bc} = y^{[b} e^{c]} \ . \tag{1.5}$$

The $SO(5)$ invariant metric of S^4 can be derived from (1.4) by identifying the 4-beins ${e_m}^b$ as the coefficients of dy^b. It is

$$g_{mn} = \frac{4a^2}{1+y^2} \delta_{mn} \ .$$

It is worthwhile to note that projecting the self-dual part of A^{bc} in (1.5) yields to the $SU(2)$ instanton configuration [41]

$$A^i = -\frac{1}{2}\eta^i_{bc}A^{bc} = -\frac{2\eta^i_{bm}y^b}{1+y^2}dy^m \quad , \quad i = 1,2,3 \; , \tag{1.6}$$

η^i_{bc} being the 't Hooft symbols [42]. A^i is an instanton on an S^4 of radius a. This instanton can be shown to be a solution of the Einstein-Yang-Mills equations of motion in a background metric $M_4 \times S^4$ [41]. We will be using examples of similar instanton/monopole backgrounds later on.

1.4 Harmonic expansion

A generic function $\phi_j(g)$ on G which belongs to a unitary irreducible representation of it can be represented by the expansion

$$\phi_j(g) = \sum_n \sum_{p,q} \sqrt{d_n} D^n_{jpq}(g)\phi^n_{qp} \; . \tag{1.7}$$

$D^n_{pq}(g)$ is a unitary matrix of dimension d_n, and the sum is over all irreducible representations of G.[6]

Now for functions on a coset space G/H, the expansion is subject to some restriction. The concern here is with multiplet of functions, $\phi_i(g)$, which have the property

$$\phi_i(hg) = \mathbb{D}_i{}^j(h)\phi_j(g) \; ,$$

where $h \in H$ and $\mathbb{D}$ is some definite representation of H. So, for the case of G/H, only terms in (1.7) which satisfy

$$D^n(hg) = \mathbb{D}(h)D^n(g)$$

should be included. In other words, one should keep those representations of G which upon restriction to H they produce the representation $\mathbb{D}$ of H. Moreover, in Kaluza-Klein theories, the coefficients of expansion $\phi^n_{qp}(x)$ generally depend on M_4.

The formula for expanding a function on G/H hence is [36, 40]

$$\phi_j(x,y) = \sum_n \sum_{p,q} \sqrt{\frac{d_n}{d_{\mathbb{D}}}} D^n_{jpq}(L_y^{-1})\phi^n_{qp}(x) \; ,$$

[6] This is a generalized form of Fourier transform.

where $d_{\mathbb{D}}$ is the dimension of $\mathbb{D}$. The notation means that from the matrix D^n_{ipq} we take only the rows which satisfy

$$D^n_{ipq}(hL_y^{-1}) = \mathbb{D}_{ij}(h)D^n_{jpq}(L_y^{-1}) \ .$$

To see how the four-dimensional fields transform, and hence to find out the internal symmetry of the 4-dimensional effective action, we should look at the transformations of $\phi^n_{qp}(x)$. Under the transformation $y \rightarrow y'$ induced by G-action on G/H,

$$\phi_j(x,y) \rightarrow {\phi'}_j(x,y') = \mathbb{D}_{ji}(h)\phi_i(x,y) \ .$$

Using L_y transformations we find that

$$\phi^n_{qp}(x) \rightarrow {\phi'}^n_{qp}(x) = D^n_{jpq}(g)\phi^n_{qp}(x) \ .$$

Therefore the $4d$ fields transform under G, which is the Kaluza-Klein gauge group (referred to by physicists as the isometry group)

1.5 Effective Lagrangians

First of all, let us fix the notations which we will use here onwards, unless otherwise indicated. We denote the coordinates in the $4+d$ dimensions by

$$z^M = (x^\mu, y^m) \ ,$$

where the Greek middle alphabet indices take the values 0,1,2,3 and their Latin counterparts take the values 4,5,...d. The internal space is parametrized by y^m, and the four-dimensional one by x^μ. The tangent space metric is $\eta_{AB} = \mathrm{diag}(-1,1,1,...,1)$.

The metric in $4+d$ dimensions is

$$g_{MN} = {E_M}^A {E_N}^B \eta_{AB} \ ,$$

where ${E_M}^A$ are the vielbeins.

To see how the isometry group result in an internal gauge symmetry of the effective action, let us start with the following ansatz for the vielbeins in $4+d$ dimensions [43, 44]

$${E_M}^A(x,y) = \begin{pmatrix} {e_\mu}^\alpha(x) & -{A_\mu}^{\hat{b}}(x){D_{\hat{b}}}^a(L_y) \\ 0 & {e_m}^a(y) \end{pmatrix} \ .$$

This ansatz is compatible with the 4-dimensional general coordinate transformations $x^\mu \to x'^\mu$, and the Yang-Mills transformations $y^m \to y'^m$ with the associated frame rotations $D_a{}^b$

$$
\begin{aligned}
e_\mu{}^\alpha(x') &= \frac{\partial x^\nu}{\partial x'^\mu} e_\nu{}^\alpha(x) , \\
E_\mu{}^a(x', y') &= \left(\frac{\partial x^\nu}{\partial x'^\mu} E_\nu{}^b(x,y) + \frac{\partial y^n}{\partial x'^\mu} e_n{}^b(y) \right) D_b{}^a(h^{-1}) , \qquad (1.8) \\
e_m{}^a(y') &= \frac{\partial y^n}{\partial y'^m} e_n{}^b(y) D_b{}^a(h^{-1}) ,
\end{aligned}
$$

$e_\mu{}^\alpha(x)$ and $e_n{}^b(y)$ are the vielbeins on M_4 and $K = G/H$ respectively. The equation (1.8) implies, in the view of the ansatz above,

$$
-A'^{\hat{b}}_\mu(x') D_{\hat{b}}{}^a(L_{y'}) = \left(-\frac{\partial x^\nu}{\partial x'^\mu} A^{\hat{c}}_\nu(x) D_{\hat{c}}{}^b(L_y) + \frac{\partial y^n}{\partial x'^\mu} e_n{}^b(y) \right) D_b{}^a(h^{-1}) . \quad (1.9)
$$

Extracting the coefficient of dy'^m and dx'^μ from the formulae (1.2) and (1.3) one gets

$$
\frac{\partial y^m}{\partial x^\mu} = -(g^{-1}\partial_\mu g)^{\hat{b}} K_{\hat{b}}{}^m(y) ,
$$

where $K_{\hat{b}}{}^m(y)$ is the Killing vector defined by [40]

$$
K_{\hat{b}}{}^m(y) = D_{\hat{b}}{}^c(L_y) e_c{}^m(y) .
$$

After finding the transformation rule for K, it is simple to derive the formula for $\partial y'^m / \partial x^\mu$

$$
\frac{\partial y'^m}{\partial x^\mu} = -(g^{-1}\partial_\mu g)^{\hat{b}} K_{\hat{b}}{}^m(y') .
$$

Going back to (1.9) one finds that

$$
A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} \left(g A_\nu(x) g^{-1} - g^{-1}\partial_\nu g \right) ,
$$

where $A_\mu = A_\mu{}^{\hat{a}} Q_{\hat{a}}$. This is precisely the transformation rule to be expected for a Yang-Mills potential.

Now we arrive to the point where we derive the four-dimensional effective action of the gravitational one in $4 + d$ dimensions,

$$
S = \int d^{4+d}z \det e(x,y) \mathcal{R} ,
$$

where $\mathcal{R}$ is the Ricci scalar curvature of the $(4+d)$-dimensional space-time. Upon substituting the metric ansatz into $\mathcal{R}$ (in the case of a zero torsion) we get

$$\mathcal{R} = \mathcal{R}_4 - \frac{1}{4} F_{\alpha\beta}{}^{\hat{a}} F_{\alpha\beta}{}^{\hat{b}} D_{\hat{a}}{}^{c}(L_y) D_{\hat{b}}{}^{c}(L_y) + \mathcal{R}_d \; ,$$

where $\mathcal{R}_4$ is the usual 4-dimensional Ricci scalar, and $\mathcal{R}_d$ is the constant curvature of G/H. Upon integrating over the internal space, the Yang-Mills emerges since

$$\frac{1}{V_d} \int d^d y \; \det[e(y)] \; D_{\hat{a}}{}^{c}(L_y) D_{\hat{b}}{}^{c}(L_y) = d \; \delta_{\hat{a}\hat{b}}$$

and therefore we get massless Yang-Mills fields and graviton.

1.6 Compactification & Stability

As said earlier, Scherk and Schwarz first attempted in [10] to put string theory in contact with the observed world by compactifying the extra dimensions. The important factor at this point was to actually show that the product of a four-dimensional Minkowsky space-time with a compact internal space is a solution of the equations of motion. This was done again by Scherk and Schwarz [45] by proposing the concept of *spontaneous compactification.*

Until then, compactification seemed an arbitrary condition imposed on the Kaluza-Klein picture. The idea of spontaneous compactification in [10, 45] was showing that the ground state of an Einstein gravity theory coupled to some matter fields (could be Yang-Mills fields [46, 47], non-linear sigma models [48], or anitsymmetric tensor gauge fields [49]) occurs in which the geometry factorizes into the product of a four-dimensional space-time of a constant curvature with a two-dimensional sphere with a magnetic monopole background. In this solution, the field strength of the Yang-Mills potential contributes to the stress-energy tensor in the right-hand side of the six-dimensional Einstein equations and generates the curvature of S^2.

Remarkably, it has been long realized that topological considerations are central to the question of stability.

The existence of the solution is again not enough to consider it physical. It should be moreover stable, in the sense that there should not be tachyons in small fluctuations around the background solution. The issue of classical stability of the $(4+d)$-dimensional Einstein-Yang-Mills theories with an arbitrary gauge group was explored by Randjbar-Daemi, Salam and strathdee [50] and Schellekens [51].

All stable compactifications on d-dimensional spheres $S^d = SO(d+1)/SO(d)$ with a symmetric topologically non-trivial classical gauge field which is embedded in an H-subgroup (here $SO(d)$) of the Yang-Mills gauge group have been classified by Schellekens in [52] and they occur only for d =2,4,5,6,8,9,10,12, and 16.

In the following we briefly review [53] which provides a explanation of the procedure used to probe the classical stability through an example of spontaneous compactification of a six-dimensional Einstein-Maxwell theory. More general treatment of spontaneous compactification on generic symmetric coset spaces and Yang-Mills theories is provided in [37].

The procedure for checking the classical stability of compactification against small perturbations, in words, is done by

- checking if the ansatz of the desired form $M_4 \times K$ is a solutions of the bosonic equations of motion.
- performing small perturbations around the background solution, and computing the physical states (the ones which couple to conserved currents).
- showing that the spectrum does not contain tachyons or negative metric states.

There may be other ways, however we restrict ourselves in this section to the above one.

More discussion regarding the size of the internal space is in section 3.5.

1.6.1 The vacuum solution

Consider an action of gravity in D dimensions coupled to an $U(1)$ gauge field, F_{MN}, and a cosmological constant, Λ,

$$S = \int d^D x \sqrt{-G} \left(\frac{1}{\kappa^2} \mathcal{R} - \frac{1}{2g^2} \mathrm{Tr} F^2 + \Lambda \right) , \qquad (1.10)$$

where $\kappa^2 = M^{-d-2}$ is the d-dimensional Newton's constant. The field equations of (1.10) are

$$\frac{1}{\kappa^2}\mathcal{R}_{MN} = \frac{1}{g^2}\mathrm{Tr}F_{MR}F_N{}^R - \frac{1}{D-2}G_{MN}\left(\frac{1}{2g^2}\mathrm{Tr}F^2 + \Lambda\right) , \tag{1.11}$$

$$\nabla_M F^{MN} = 0 . \tag{1.12}$$

Now consider solutions in $D = 6$ of the form $M_4 \times K$, where M_4 is the flat 4-dimensional Minkowski space and K is a compact manifold. Throughout this section K will be taken to be S^2. Furthermore we shall assume that the gauge field configuration A will be non-vanishing only on K. One can of course think of many other choices for K.

The flatness of the Minkowski space implies

$$\frac{1}{2g^2}\mathrm{Tr}F^2 + \Lambda = 0 ,$$
$$\mathcal{R}_{mn} = \frac{\kappa^2}{g^2}\mathrm{Tr}F_{mr}F_n{}^r , \tag{1.13}$$

where m, n are indices in K. Our problem is now to find solutions of Yang-Mills equations in K which also solve the Einstein equation (1.13).

The ansatz for solutions to (1.13), for the case $K = S^2$, are

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + a^2\left(d\theta^2 + \sin\theta\ d\varphi^2\right) , \tag{1.14}$$

where a is the radius of S^2 to be determined, and

$$A_\varphi = -\frac{n}{2}(\cos\theta \mp 1) . \tag{1.15}$$

Here $n \in \mathbb{Z}$, because of the patching on the upper and lower hemispheres, and $-(+)$ indicate the expression on the upper (lower) hemispheres. (θ, φ) are the coordinates on S^2.

Substituting the ansatz back into (1.13) reduces the equations into algebraic equations between κ, a, g, and Λ:

$$g^2 = \frac{1}{8}n^2\kappa^4\Lambda ,$$

$$a^2 = \frac{1}{8}n^2\frac{\kappa^2}{g^2} . \tag{1.16}$$

1.6.2 Fluctuations and gauge fixing

The spectrum of the four-dimensional theory is obtained by looking at the small fluctuations around the backgrounds (1.14) and (1.15),

$$g_{MN}(x,y) = \bar{g}_{MN} + \kappa h_{MN}(x,y) \ ,$$

$$A_M = \bar{A}_M + V_M(x,y) \ ,$$

where $\bar{g}_{MN}$ should be read off (1.14), and $\bar{A}_M$ is (1.15).

To study the stability of the above configuration, one must study the response of the system to some external physical disturbance. This is usually done by coupling the perturbations h_{AB} and V_A to appropriate sources T_{AB} and J_A respectively. The sources are constrained to respect the symmetries of (1.10). The new action (1.10) is

$$S' = S + \int d^6 z \sqrt{-\bar{g}} \{ \frac{1}{2} T_{AB} h_{AB} + J^A V_A \} \ .$$

The constraints on the sources read

$$\nabla^B T_{AB} - \kappa \bar{F}_{AB} J^B = 0 \quad , \quad D^A J_A = 0$$

We chose the gauge fixing

$$\nabla^B (h_{AB} - \frac{1}{2} \eta_{AB} h_{CC}) = 0 \quad , \quad D^A V_A = 0 \ .$$

The next step is to solve the linearized equations of motion, taking into account the constraints on the sources and the gauge fixing, for h_{AB} and V_A in terms of the sources T_{AB} and J_A.

The linearized equations of motion then take the form [53]

$$\nabla^2 h_{AB} - \frac{1}{4} \eta_{AB} \nabla^2 h_{CC} + \bar{\mathcal{R}}_{BC} h_{AC} + \bar{\mathcal{R}}_{AC} h_{BC}$$
$$+ \kappa \bar{F}_{BC} \nabla_{[A} V_{C]} + \kappa \bar{F}_{AC} \nabla_{[B} V_{C]} - \kappa \eta_{AB} \bar{F}_{CD} \nabla_{[C} V_{D]} + T_{AB} = 0 \ ,$$

$$\nabla^2 V_A + \bar{\mathcal{R}}_{AB} V_B + \kappa \nabla_C (h_{AB} \bar{F}_{BC} + \bar{F}_{AB} h_{BC} + \frac{1}{2} h_{BB} \bar{F}_{CA}) + J_A = 0 \ .$$

1.6.3 Solutions to the linearized equations

To find the solutions for the above equations, it is useful to expand the fluctuations and sources in spherical harmonics of S^2 using the procedure given in section 1.4.

First, all fields should be decomposed into irreducible representations of the H-subgroup, in this case represented by the $SO(2)$ rotations (or a $U(1)$). Those representations are labeled by the λ (named *isohelicity* [53]). Let $\phi_\lambda(x,\theta,\varphi)$ be a typical one.

$$\phi_\lambda(x,\theta,\varphi) = \sum_{l\geq|\lambda|} \sqrt{2l+1} \sum_m D^l_{\lambda m}(L^{-1}_{\theta\varphi})\phi^l_{\lambda m(x)} ,$$

where $D^l_{\lambda m}$ belong to the $(2l+1)$-dimensional unitary irreducible representation of $SU(2)$. λ can be an integer or half an integer, and $m = -\lambda, ..., +\lambda$. One can chose $L_{\theta\varphi} = \mathrm{e}^{\frac{i}{2}\varphi\sigma_3}\mathrm{e}^{\frac{i}{2}\theta\sigma_2}\mathrm{e}^{-\frac{i}{2}\varphi\sigma_3} \in SU(2)$ ($\sigma_{1,2,3}$ are Pauli matrices), in other words

$$L^{-1}_{\theta\varphi} = \begin{pmatrix} \cos\frac{\theta}{2} & -\mathrm{e}^{i\varphi}\sin\frac{\theta}{2} \\ \mathrm{e}^{-i\varphi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} . \tag{1.17}$$

To see an example of how the summation in the harmonic expansion is done, consider the representation $l = 1/2$. The coefficients of $\phi^{1/2}_{1/2,-1/2}$, for instance, can be read from (1.17) $-\mathrm{e}^{i\varphi}\sin\frac{\theta}{2}$. The general rule have been explained in section 1.4.

The only left issue is to search for tachyons or negative norm states. Among the fields h_{AB} and V_A there will be following fields (the $SO(2)$ irreducible pieces) which have a definite isohelicity:

$$\begin{aligned}
&h_{++} = \tfrac{1}{2}(h_{55} - h_{66} - 2ih_{56}) && \lambda = 2,\\
&h_{+-} = \tfrac{1}{2}(h_{55} + h_{66}) && \lambda = 0,\\
&h_{\mu+} = \sqrt{\tfrac{1}{2}}(h_{\mu5} - ih_{\mu6}) && \lambda = 1,\\
&V_+ = \sqrt{\tfrac{1}{2}}(V_5 - iV_6) && \lambda = 1,\\
&h_{\mu\nu}, V_{\mu\nu} && \lambda = 0,\\
&h_{--} = h^*_{++}, h_{-+}, ...etc. &&
\end{aligned}$$

The non-zero masses are the following labeled by their spin $0, 1, 2$ (the technical details of the calculations were neatly worked out in [53]),

$$
\begin{array}{ll}
M_0^2 = (l-1)(l+2)/a^2, & l \geq 2 \\
M_{0\pm}^2 = [2l(l+1)+1+\pm\sqrt{1+12l(l+1)}]/2a^2, & l \geq 0 \\
M_{1\pm}^2 = [l(l+1)+\pm\sqrt{2l(l+1)}]/a^2, & l \geq 1 \\
M_2^2 = l(l+1)/a^2, & l \geq 0
\end{array}
$$

and it can be checked that no tachyons are present in this model, and hence it is expected to be stable against small perturbations. There will be in addition six massless states; the graviton ($\lambda = \pm 2$), a massless "photon" ($\lambda = \pm 1$), and a Yang-Mills triplet ($\lambda = \pm 1, l = 1$).

It is worthwhile to emphasis as a final point in this section that there is a mixing of the six-dimensional metric and Maxwell fields as pointed out in [53]. This mixing is contrary to the common assumption that Kaluza-Klein gauge fields originate purely from the metric. This argument is expected to hold whenever matter fields which participate in the spontaneous compactification carry the appropriate quantum numbers [53].

1.7 Chiral fermions

Upon compactifying a Kaluza-Klein type theory with fermions, one expects in general two kinds of difficulty. In order for the resulting fermions in 4 dimensions to be chiral, they should be massless to start with. Hence the first requirement for the Dirac operator on the compact internal manifold is to have at least one zero mode, while not all manifolds admit harmonic spinors. Also, the index of the Dirac operator is often (not always) zero and hence the 4-dimensional theory will be non-chiral.

Moreover, the irreducible spin representation in $4+d$ ($d \geq 1$) dimensions is always, greater than the irreducible one in four dimensions.[7] Also the number of zero modes, though finite [54], can lead to more massless fermions than desired in four dimensions. So, one generally expects more degrees of freedom, than the ones present in the standard model, to result in the four-dimensional effective action after compactification. Many of those extra degrees of freedom can be eliminated once the issue of achieving chiral fermions is settled.

[7] Which is $2^{\frac{n}{2}}$ for $n \in 2\mathbb{Z}$ and $2^{\frac{n-1}{2}}$ for $n \in 2\mathbb{Z}+1$.

One way to eventually achieve a chiral Lagrangian in four dimensions (another way is be by using by orbifold compactification) stemming from a Kaluza-Klein field theory is by coupling the fermions in $4+d$ dimensions to a stable non-trivial background, like a magnetic monopole, as was proposed by Randjbar-Daemi, Salam and Strathdee [53, 55]. The problem of obtaining left-right asymmetry of fermion quantum numbers was discussed by Witten [34].

The necessity for a non-trivial background to get chiral fermions from extra dimensions can be explained in the following simple example. Consider a manifold, W, in 6 dimensions where $M_4 \times S^2$. The Dirac equation on a generic manifold is

$$\not{D}_W \psi(x,y) = 0 \ .$$

With the appropriate choice of Dirac matrices, the above equation can be written as

$$\not{D}_4 \psi(x,y) + \not{D}_{S^2} \psi(x,y) = 0 \ .$$

In the absence of a background gauge field

- $\not{D}_{S^2}\psi = 0$ has no *regular* solutions.

- Index$\not{D}_{S^2} = 0$.

The first point can be understood easily, with and without going into explicit computations, since, according to Lichnerowicz's [56] theorem, all positively curved smooth compact manifolds, including S^2, do not admit harmonic spinors. In order for the resulting fermion in 4 dimensions to be chiral, it should be massless to start with. Hence the first requirement for the Dirac operator on the compact internal manifold is to have at least one zero mode. The existence of fermion zero modes by couplings to gauge fields with non-trivial topology was pointed out in [46, 57]. Also, the index of the Dirac operator is often, not always, zero (as in the case of S^2).

Now let us couple $\psi(x,y)$ to a magnetic monopole background described in equation (1.15). [8] Consider the coupled Dirac operator on S^2

$$\not{D}_{S^2} = \Gamma^m E^\alpha_m (\partial_\alpha - \frac{1}{2}\omega_{\alpha[k,l]}\Sigma^{kl} - i\bar{A}_\alpha(y)) \ .$$

[8] For simplicity, we will do the computations on the upper hemisphere only, while taking into account the consistent patching.

As can be seen from (1.15), the background solution is proportional to the spin connection ω_α ($\omega_\theta = 0$, and $\omega_\varphi = -(\cos\theta - 1)$).

Now let $\psi(x, y)$ be a Weyl spinor in 6 dimensions. Using the chirality matrix, γ_5, in four dimensions, ψ can be written as:

$$\psi = \frac{1+\gamma_5}{2}\psi + \frac{1-\gamma_5}{2}\psi = \psi_R + \psi_L \ . \tag{1.18}$$

The Dirac equation in a monopole background on the upper hemisphere hence simplifies to

$$\left(\partial_\theta + \frac{i}{\sin\theta} - \frac{n+1}{2\sin\theta} + \frac{n+1}{2}\,\mathrm{ctg}\theta\right)\psi_R = 0 \ ,$$

$$\left(-\partial_\theta + \frac{i}{\sin\theta} - \frac{n-1}{2\sin\theta} + \frac{n-1}{2}\,\mathrm{ctg}\theta\right)\psi_L = 0 \ .$$

It can be easily checked that regular solutions exist only for one of the chiralities ψ_L and for $n \neq 0$.[9]

1.8 Mass estimates for KK excitations

As we mentioned in the introduction, the eigenvalue problem of the Laplacian, ∇^2, is not known on a generic compact space. However, we know that the spectrum of ∇^2 on compact Riemannian manifolds is discrete, bounded from below, and the eigenvalues (counted with multiplicity) are ordered: $0 = \lambda_0 \leq \lambda_n \leq \lambda_{n+1}$. Moreover, in the mathematics literature there are many results on the lower bounds on the first eigenvalue, λ_1, of the Laplace-Beltrami operator, $\nabla^2 \equiv \frac{1}{\sqrt{g}}\partial_m(\sqrt{g}g^{mn}\partial_n)$, (Laplacian acting on scalars) on a compact manifold.[10]

These general bounds can be helpful in the estimating the masses of Kaluza-Klein excitations, mainly of the graviton and scalar fields, in the cases where it is difficult to perform explicit computations of the specturm.

The lower bounds on the first eigenvalue of the Laplacian translate into lower bounds on the 4-dimensional tree-level masses of the bosonic particles arising from compactification.

[9] And *vise versa* for a negative non-zero n.

[10] Upper bounds on those eigenvalues have also been worked out (for a review see [35] and references thererin), however these are not much of a physical relevance since the upper bounds depend generally on the level of excitation.

Lichnerowicz theorem enables us to to impose similar lower bounds on fermion masses, and also to exclude tree level massless fermions, unless coupled to a non-trivial background (as explained in section 1.7), for positively curved internal spaces.

The lower bound on the first non-zero eigenvalue of ∇^2 acting on scalars, $\nabla_K^2 \phi_n = -\lambda_n \phi_n$, on a generic compact manifold of a scalar curvature bounded from below by $(d-1)\zeta$ ($\zeta \in \mathbb{R}$) is [58]

$$\lambda_1 + \max\{-(d-1)\zeta, 0\} \geq \frac{\pi^2}{4\sigma^2} , \tag{1.19}$$

where σ is the diameter of the manifold (the longest distance). The fundamental parameter for masses arising from compactification is hence σ (this can be understood by observing that it is possible to change the spectrum of the Laplacian by deforming the manifold, and yet keeping its volume fixed). Usually σ is greater or equal than the characteristic scale of the volume of K (see for example [59]), and hence the estimates of the Kaluza-Klein excitations depending on mere dimensional analysis (on complicated manifolds where explicit computations can not be done) has to be somehow lowered.

In most cases, specially for a space of a constant curvature, one can relate σ to the volume of the manifold, and hence rewrite the bounds in terms of the volume instead (e.g. in S^d and compact hyperbolic manifolds). The inequality (1.19) translates effectively into a statement about the bounds on the 4-dimensional masses of the lowest excitations, m_1^2. It is obvious that when the Ricci curvature, $\mathcal{R}_d$, of K is non-negative, then one recovers the standard scenario:

$$\lambda_1 \geq \frac{\pi^2}{4\sigma^2},$$

where in the standard Kaluza-Klein scenario, as in [9], σ is identified with the compactification radius, R (*e.g.* for a circle, $\sigma = 2\pi R$).

When $\mathcal{R}_d < 0$,

$$\lambda_1 \geq \frac{\pi^2}{4\sigma^2} - (d-1)|\zeta| , \tag{1.20}$$

the lower bound may not in this case always hold [60], specially if some particular tuning between σ and the volume scale of K is needed (in order to address the hierarchy problem, for instance [59]).

The observed massless fermions in four dimensions are nothing but the zero modes of $\not{D}_K$ (which lie in ker$\not{D}_K$). The argument is based on the relation between the squared Dirac operator and the scalar curvature,

$$\not{D}_K^2 = \nabla^* \nabla + \frac{1}{4} \mathcal{R}_d \ , \qquad (1.21)$$

where $\nabla^*\nabla$ is the connection Laplacian and is a positive operator. It has been pointed out by Lichnerowicz [56] using (1.21) that manifolds with a positive curvature do not admit harmonic (massless) spinors. This can be easily seen by sandwiching (1.21) for $\not{D}^2 = 0$ inside an complete orthonormal set of wave functions and considering a constant curvature case, one gets

$$|\nabla \psi(y)|^2 + \frac{1}{4} \mathcal{R}_d |\psi(y)|^2 = 0 \ ,$$

obviously the above equation has no solutions for $\mathcal{R}_d > 0$ on a compact space.

As it is the case for the Laplacian, the eigenvalues of the Dirac operator on a compact space are discrete. Therefore, the eigenvalues of the squared Dirac operator are discrete and positive, and in addition any eigenvalue, ν_q^2,is bounded from below by the curvature [61], including ν_1^2,

$$\nu_1^2 \geq \frac{d}{4(d-1)} \lambda'_1 \ , \qquad (1.22)$$

where λ'_1 is the first eigenvalue of the Yamabe operator,

$$L \equiv \frac{4(d-1)}{d-2} \nabla_K^2 + \mathcal{R}_d \ .$$

with ∇_K^2 being the positive Laplacian acting on functions. This means that the lower bounds on the massive spin 0 and spin 1/2 excitations are related. For constant curvature $\mathcal{R}_d = (d-1)\zeta$ For $\mathcal{R}_d \geq 0$

$$\nu_1^2 \geq \left(\frac{d}{d-2} \right) \frac{\pi^2}{4\sigma^2} + \frac{d}{4(d-1)} \mathcal{R}_d,$$

and for $\mathcal{R}_d < 0$

$$\nu_1^2 \geq \frac{d}{d-2} \frac{\pi^2}{4\sigma^2} - \frac{d(5d-6)}{4(d-1)(d-2)} |\mathcal{R}_d|.$$

Chapter 2

Warped Non-compact Extra Dimensions

2.1 Introduction

The notion of warped product of Riemannian manifolds was first introduced by Bishop and O'Neill in 1969 [62], and it was shown that such solutions can be of a physical relevance [19], [30, 31, 32], and, was recently revived in [64, 20, 63]. The recent literature is rich of new scenarios in non-compact spaces, like [65]-[74].[1]

The non-direct product spaces (non-factorizable geometry) of relevance are solutions to the equations of motion in which the metric of space-time is a product of an internal space and a scalar function of the extra coordinates (the warp factor), and the metric ansatz representing the space $M \otimes_f K$ is

$$ds^2 = f(y)^2 g_{\mu\nu} dx^\mu dx^\nu + g_{mn}(y) dy^m dy^n \ ,$$

where the *warp factor*, f, is a smooth positive function $f : K \to \mathbb{R}^+$. The Ricci scalar is:

$$\mathcal{R}_{n+d} = \frac{1}{f^2} \left\{ \mathcal{R}_n - \frac{n}{2} f \nabla^2 f - \frac{n}{4}(n-1)|\nabla f|^2 \right\} + \mathcal{R}_d \ , \tag{2.1}$$

where $\mathcal{R}_n$ and $\mathcal{R}_d$ are the intrinsic curvatures of M and K respectively. $n = \dim M$ and is usually taken to to be equal to 4.

[1] For recent reviews see [75, 76, 77].

The space M is hence an $(n-1)$-dimensional extended object inside the full mother manifold with a tension λ and has generally the action

$$S_M = \lambda \int_M d^n x \left[-\det(\partial_\mu z^M(x)\partial_\nu z^N(x)\eta_{MN})\right]^{\frac{1}{2}} .$$

The warping space is usually taken to be non-compact with an infinite volume. [2] However, using a damping warp factor as a solution of the equations of motion leads to a finite volume of the internal non-compact space [32] by "effectively" changing the measure as we shall see later on (as in [20, 63]). Therefore, it is perhaps more natural to use warped geometry in the presence of non-compact internal spaces.

The idea [11, 12, 13] of living on a distributional *brane* source embedded in a five-dimensional (or higher) manifold with non-compact transverse direction has in principle a welcoming environment in superstring theory, D-branes [78]. A D-branes is a BPS states carrying Ramond-Ramond charge, nicely stable, and naturally admits localized modes on its world-volume. There are also BPS flat domain wall solutions [79]-[86], as well as curved [87, 88], in five-dimensional gauged supergravity.

A classification of all (non D-brane) the possible generalization of Green-Schwarz superstring action to the of a p-dimensional extended object was given in [89] where it was also shown that topological extenions may arise in the p-brane supersymmetry algebra [90]. In performing the classification, the symmetries of the world-volume of the p-brane (reparametrization invariance and Sigel symmetry) were used to show that gauge fixed version of the brane world-volume action coupled to a $(p+1)$-form field will have some amount of supersymmetry of the mother manifold will projected, depending on the dimension of the embedding space and the number p. In this classification the supersymmetric 3-brane (4-dimensional world-volume) can occur only in 2 and 4 extra dimensions. The supermultiplet will be certainly confined to the brane. Making a non-supersymmetric gauge theoretical analogy to such a classification is obviously a difficult task, however it remains desirable until proven impossible!.

This chapter discusses some examples of warped solutions to Einstein and Einstein-Yang-Mills systems, as well as the conventional methods for

[2] A topological space is said to be non-compact if it can not be represented by a union of a countable number of compact subsets. A compact subset is a set which is closed (contains the limits of all its convergent sequences), and bounded (is contained in a sphere of $\mathbb{R}^d$). Non-compact spaces can also have a finite volume, *e.g.*.

localizing gravity and fermions on a domain wall with a warped non-compact transverse space.

2.2 Junction conditions

Consider a simple case of a one extra dimensions $d = 1$,

$$S = \int d^4x\; dy \sqrt{g(x,y)} \left(\frac{1}{\kappa^2} \mathcal{R}_5 + \Lambda \right) + \lambda \int d^4x dy\; \sqrt{\bar{g}(x,y)} \delta(y) \;, \qquad (2.2)$$

where the last term represents a δ-function like co-dimension 1 sources, the simplest form of a *brane*, which is hypersurfaces in 5-dimensions characterized by $y = 0$. The brane has the induced metric $\bar{g}(x)_{\mu\nu} = g_{\mu\nu}(x, y = 0)$, and a tension λ. The ansatz for the background solution is $M_4 \times \mathbb{R}$ as in [20]

$$ds^2 = \mathrm{e}^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \;. \qquad (2.3)$$

Einstein equations of this system are

$$\mathcal{R}_{MN} - \frac{1}{2} g_{MN} (\mathcal{R} - \frac{2}{M^3} \Lambda) = \frac{1}{4M^3} \sqrt{-\bar{g}(x)} \bar{g}_{\mu\nu}(x) \delta^\mu_M \delta^\nu_N \delta(y) \;,$$

where $\mathbb{R}$ is parametrized by $y \in (\infty, +\infty)$.

Away from the origin, the above equations are just the standard 5-dimensional Einstein equations with a cosmological constant, and they reduce to

$$\mathcal{R}_{MN} = \frac{\Lambda}{(d+2) M^{d+2}} g_{MN} \;.$$

At the origin, the solutions for the equations require proper matching at the location of the source. This will obviously impose a relation between the tension of the brane λ and the warp factor σ, and consequently the cosmological constant Λ. The procedure of matching differs according to the model, however this feature still holds.

The discontinuity at $y = 0$ is characterized by the extrinsic curvature, $K_{\mu\nu}$, which contains information of how the hypersurface is embedded in the higher-dimensional space-time. The extrinsic curvature is defined as

$$K_{\mu\nu} = -\eta^{\rho\mu} \nabla_\nu n_\rho \quad , \quad \perp_{\mu\nu} = n_\nu n_\nu \;,$$

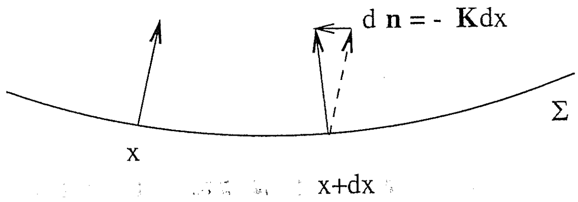

Figure 2.1: The extrinsic curvature measure the deformation of a figure (here the *brane*) in lying in a spacelike hypersurface, Σ, that takes place when each point in the figure is carried out forward unit interval of proper time "normal" to the hypersurface out into the enveloping spacetime.

where n_ρ is a normal vector in the direction of the extra dimension(s), defining the orthogonal complement components of the induced metric on the brane $\perp_{\mu\nu}= g_{\mu\nu}(x,y) - \bar{g}_{\mu\nu}(x)$.

Assuming that the extrinsic curvature varies smoothly within the neighborhood $[-\epsilon, +\epsilon]$ of the constant hypersurface, the discontinuity as measured at scales larger than the thickness 2ϵ is given by the jump in $K_{\mu\nu}$ across the discontinuity which is described by the quantity $K^+_{\mu\nu} - K^-_{\mu\nu}$ or equally by

$$[K_{\mu\nu}] \equiv \int_{-\epsilon}^{+\epsilon} d\epsilon \; n^\rho \partial_\rho K_{\mu\nu} \; . \tag{2.4}$$

One can relate the normal derivatives of $K_{\mu\nu}$ to the the usual (intrinsic) curvature with in the thin brane limit (the transverse derivatives are negligible compared to the normal ones) [91] in the formula,

$$K'_{\mu\nu} \equiv n^\rho \partial_\rho K_{\mu\nu} = \eta^\gamma_\mu \eta^\lambda_\nu \mathcal{R}_{\gamma\lambda} \; ,$$

from which one can deduce that the dominant components of the background Ricci tensor $\mathcal{R}_{\mu\nu}$ will be given by the asymptotic formula

$$\mathcal{R}_{\mu\nu} \simeq K'_{\mu\nu} + K' \perp_{\mu\nu}$$

where $K' = \perp^\rho{}_\sigma \mathcal{R}^\sigma{}_\rho$. Hence

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} \simeq K'_{\mu\nu} - \eta_{\mu\nu} K' \; .$$

Matching at $y = 0$ of the equations of motion requires

$$[K_{\mu\nu}] = \frac{1}{M^{d+2}} \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} d\epsilon [(d+2) T_{\mu\nu} - g_{\mu\nu} g^{\rho\gamma} T_{\rho\gamma}] \; . \tag{2.5}$$

The above condition is a higher-dimensional version of the so called Israel matching conditions [92]. [3]

The matching condition at the origin of the solutions to the equations of motion coming from (2.2) reads

$$3\sigma' g_{\mu\nu} = 4\frac{\lambda}{M^3} g_{\mu\nu} \quad \text{at} \quad z = 0 \, ,$$

where the prime refers to a derivative with respect to y. The above equation yields to

$$\sigma(y) = \frac{\lambda}{12M^3}|y| \, . \tag{2.6}$$

The equations of motion are satisfied, in the case of a flat four-dimensional space, when

$$\sigma'^2 = -\frac{\Lambda}{12M^3} \, , \tag{2.7}$$

where the additive integration constant is omitted as it just amounts to an overall rescaling of x^μ's. Clearly, Λ has to be negative indicating that the five-dimensional space is an AdS_5.[4]

The equations (2.6) and (2.7) imply that the tension of the brane and the cosmological constant are related by

$$\lambda^2 = -12M^3\Lambda \, .$$

In other words, the argument of the warp factor in (2.3) is [20, 63]

$$\sigma(y) = \sqrt{\frac{-\Lambda}{12M^3}}|y| \, . \tag{2.8}$$

This indicates a genuine singularity in the curvature, as can be seen by substituting the warp factor (2.8) in (2.1). Therefore, this solution should be embedded in a string theoretical context for it to make sense.

Apart from the singularity, proving an ansatz to be a solution for the equations of motion is not at all enough, it should also be proved stable at least classically.

[3]Further discussion can be found in [91, 93, 94].

[4]The notation used here, as in [53], is such that $\Lambda < 0$ indicates an anti-de Sitter space.

The stability can be insured if the brane is a BPS state with exact supersymmetry in the full space-time, and/or carries a conserved charge thorough its coupling to a four form. In fact, the stability of the brane-models have been questioned in some recent papers, as [95]-[100], and it is far from being obvious.

2.3 Randall-Sundrum scenario

The Randall-Sundrum model [63] where the hierarchy problem is addressed is an extension of the above simple example. Here is how it works. Let us add another brane to (1.10) which has a tension λ' and consider the warp ansatz (2.3) of $M_4 \times S^1/Z_2$, where S^1/Z_2 has a length r_c and parametrized by the angular coordinate $\phi \in [-\pi, \pi]$ (related to the old coordinate by $y = r_c\phi$).

The background metric (2.3) would satisfy the equations of motion of the action with the two branes

$$\frac{\sigma'^2}{r_c^2} = \frac{-\Lambda}{12M^3} \quad , \quad \frac{\sigma''}{r_c} = \frac{\lambda}{12M^3}\,\delta(\phi) - \frac{\lambda'}{12M^3}\delta(\phi - \pi) \; ,$$

if similar conditions to the above discussed are fulfilled. There is a different way in obtaining the matching conditions here, coordinate dependent, and it is done by making use of the Z_2 symmetry of the internal space. The consistency of the first equation with the orbifold symmetry $\phi \to -\phi$ implies (2.8). Which makes sense only if $\Lambda < 0$, indicating again that the 5-dimensional space-time between the two branes is an anti-de Sitter space AdS_5.

The second equation of motion makes sense when

$$\lambda' = -\lambda = \sqrt{3M^3\Lambda} = 6M^3k \quad ; \quad k \equiv \sqrt{\frac{-\Lambda}{12M^3}} \; . \tag{2.9}$$

The five-dimensional gravity scale M is again linked to M_P through the "volume" of the internal space

$$M_P^2 = M^3 r_c \int_{-\pi}^{\pi} d\phi\; \mathrm{e}^{-2kr_c|\phi|} = \frac{M^3}{k}(1 - \mathrm{e}^{-2kr_c\pi}) \; . \tag{2.10}$$

Assuming the bulk curvature Λ to be less than M^5, and $kr_c \gg 1$, one deduces that M_P depends weekly on r_c, unlike the case in Kaluza-Klein type models (1.1).

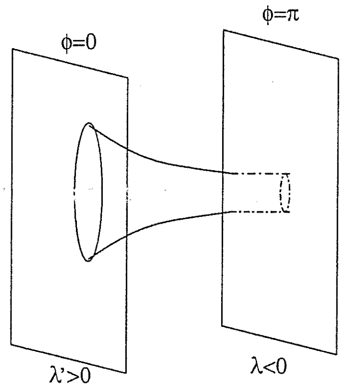

Figure 2.2: A sketch of Randall-Sundrum two-brane model with a damping warp factor towards the brane with a negative tension.

If we live on the brane located at $\phi = \pi$, then mass scales in the 4-dimensional theory will be lowered generally by the value of the metric at the location which is a factor of $e^{-kr_c\pi}$. This will remove any fine tuning in generating the hierarchy between the Planck and electroweak scales since $M_P \sim e^{k\pi r_c} M_{EW}$.

The masses of the KK excitations are quantized with gaps $\Delta m \sim k e^{-kr_c}$ [63, 76, 101]. From the point of view of an observer on the brane with $\lambda < 0$, the KK gravitons will appear to have physical masses of order $\sim k$, *i.e.* of order TeV, while their dimensionful couplings to the above matter will be characterized by mass scales of order $(M^3/k)^{1/2}$, which is roughly the weak scale.

Solving the hierarchy problem by living on the brane with negative tension does not seem to be satisfactory. The reason is that the weak energy principle characterized by the energy momentum tensor $T_{\mu\nu}\zeta^\mu\zeta^\nu \geq 0$ (for an observer with 4-velocity ζ^μ) will be violated [102] and hence one may expect unphysical modes to appear in the spectrum, like modes with arbitrary large negative energies [103], which puts the stability of the model under question.

There are in the literature however stable vacua with negative cosmological constant, mainly within a supersymmetry context, as then one of Maldacena [104] for instance which is not meant to describe the world as it is, however did shed a light on a deeper understanding to the relation between general relativity and conformal field theory beyond the gauge principle.

The above model [20] resembles in its general setup the one in [105] of the eleven-dimensional supergravity compactified on a manifold with boundaries, this configuration was shown to be relevant to the strong coupling limit of the $E_8 \times E_8$ heterotic string theory [105]. In five dimensions, the effective strongly coupled heterotic string theory is shown to be a gauged version of $N = 1$ five-dimensional supergravity with four-dimensional boundaries which are identified with a pair of 3-brane world-volumes [64]. These brains couple to a 4-form field which is the dual of the cosmological constant. There have been various attempts to provide a supersymmetric realization to the Randall-Sundrum [20] scenario [5], however no explicit link with string theory has yet been established.

Living on the brane with positive tension has the advantage of having a 4-dimensional localized gravity [63], and the hierarchy can be generated when the electroweak or super symmetry breaking is transfered to us by some mechanism on the hidden brane of negative tension [108]. Further discussion regarding anti-de Sitter/Conformal field theory correspondence in the above two brane model is present in [121]. Aother phenomenological and cosmological aspects of brane-world models can be found in [109]-[121].

2.4 General solutions with Yang-Mills fields

Here we present a generalization of the general solutions for spontaneous compactification on symmetric spaces [37] to warp compactification developed in [72]. As seen in section 1.3, the coupling to Yang-Mills fields may lead to interesting physics and could also be necessary for the consistency of the theory. The most relevant point about the presence of a gauge fields background configuration is that it is crucial in obtaining chiral fermions, as well as localizing them on a brane.

The case for pure gravity was worked out in [32],

Consider the background ansatz, discussed in [72],

$$ds^2 = e^{A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{B(r)}g_{mn}dy^m dy^n + dr^2 \ , \qquad (2.11)$$

for an Einstein-Yang-Mills system in $D = d_1 + d_2 + 1$ dimensions, where we are thought to live on a d_1-dimensional world-volume of a flat brane with a d_2-dimensional compact internal space, K, and another one-dimensional non-compact internal space, $\mathbb{R}^+$.

[5]See for example see [106, 107], as well as other references in this section.

This ansatz is fairly general since several models can be regarded as special cases of (2.11) by choosing specific forms of the functions A and B. [6] Let us also assume that $K = G/H$ and that the Yang-Mills gauge group contains H as explained in section 1.3. On such spaces, one can construct the solutions [37, 72]

$$F^a_{mp}F^{ak}_n = \frac{\mathrm{e}^B}{d_2}F^2 g_{mn} \ ,$$

where $F^2 = g^{mk}g^{nq}F^a_{mn}F^a_{kq}$ must be constant by virtue of G-invariance.

The starting action is

$$S = \int d^D x\sqrt{-G}\left(\frac{1}{\kappa^2}\mathcal{R} - \frac{1}{2g^2}\mathrm{Tr}F^2 + \Lambda\right) + \lambda\int d^{d_1}x\sqrt{\bar{g}(x)}$$

where the last term represents a δ-function like simple *brane*-source which depends only on r, $\bar{g}(x)_{\mu\nu}$ is the induced metric, λ is the tension of the d_1-dimensional world-volume brane. The *brane* is chosen to be located at $r = 0$.

Around the origin, the solutions for the equations of motion will require relating the brane tension, λ, together with the values of A' and B' in a similar manner as presented in sections 2.2 and 2.3. More to such treatment can be found in [66, 67].

Away from $r = 0$ the equations of motion are [72]

$$A'' + \frac{d_1}{2}A'^2 + \frac{d_2}{2}A'B' = \frac{4\kappa^2}{D-2}\left(-\Lambda + \frac{F^2}{4g^2}\mathrm{e}^{-2B}\right) \ ,$$

$$B'' + \frac{d_2}{2}B'^2 + \frac{d_1}{2}A'B' = \frac{4\kappa^2}{D-2}\left(-\Lambda - \frac{2D-d_1-4}{d_2}\frac{F^2}{4g^2}\mathrm{e}^{-2B}\right) + a\mathrm{e}^{-B} \ ,$$

$$d_1A'' + d_2B'' + \frac{d_1}{2}A'^2 + \frac{d_2}{2}B'^2 = \frac{4\kappa^2}{D-2}\left(-\Lambda + \frac{F^2}{4g^2}\mathrm{e}^{-2B}\right) \ ,$$

where a is the scalar curvature of K. The above equations are valid everywhere apart from the region $r = 0$ where the brane is positioned in $\mathbb{R}^+$.

When K is Ricci flat $a = 0$ (e.g. a torus or a Calabi-Yau), the solutions for $F^2 = 0$ are

$$A(r) = f_-\log[z'(r) + f_+\mathrm{z(r)}] \ , \quad A(r) = s_-\log[z'(r) + s_+\mathrm{z(r)}] \ ,$$

[6]For instance the case for which $d_2 = 0$ and $A = c|r|$ $(c < 0)$ is the Randall-Sundrum model [63]; d_1 and $A = B = c|r|$ $(c < 0)$ is [66]; $d_2 = 1$, $A = c|r|$ $(c < 0)$, as well as other higher-dimensional solutions which localize gravity [69, 67].

where

$$f_{\mp} = \frac{2}{D-1}[1 \mp \sqrt{\frac{d_2(D-2)}{d_1}}] \ , \ s_{\mp} = \frac{2}{D-2}[1 \mp \sqrt{\frac{d_1(D-2)}{d_2}}] \ ,$$

and

$$z = \mathrm{Re}(\alpha e^{\gamma r} + \beta e^{-\gamma r}) \ , \ \gamma = \sqrt{\frac{-(D-1)\kappa^2\Lambda}{2(D-2)}} \ .$$

These solutions are generalizations of the ones in [13, 32, 69]. For $\Lambda < 0$ a simple solution is

$$A = B = -\sqrt{-\frac{8\kappa^2\Lambda}{(D-1)(D-2)}}\, r \ .$$

One can also extend the fifth direction to $r \in (-\infty, +\infty)$ for which the solution is $A = B = -c|r|$, and the behavior of the solutions is similar to the warp behavior in [63]. For $a \neq 0$, solutions can be be $AdS_{d_1+1} \times G/H$ $(\Lambda < 0)$ [55, 53] with $A = -cr$ (or $A = -c|r|$ for $r \in (-\infty, +\infty)$) and $B =$ const. (as in [69]).

The solutions of both cases $a = 0$, $a \neq 0$, are compatible with the symmetry of Einstein equations $r \to -r$ and lead to the localization of gravity. The presence of the brane as a singular source is essential as a consequence of the junction condition (2.5).

In section 2.5.2 we will see how these types of solutions can be used to localize chiral fermions on the brane.

2.5 Localization of Matter

The conventional way to avoid long-range observable effects of a non-compact internal space is to localize the fields on a thin 3-dimensional wall (brane). In this case, there will not be infinite images of the brane world, and in principle one can have only one single brane on which ordinary matter is localized.

A particle will be localized if its wave function dependence on the extra dimensions has an amplitude resembling a distribution function sharply localized around a particular point in the internal space, and this requires a specific non-trivial dependence of this wave function on the coordinates of the internal space.

In the direct product compactification, the zero modes are usually constant functions over the extra space, unless they are coupled to a background configuration, as in [12].[7] However in the warped geometry they may have non-trivial dependence on the transverse space through the warp factor. [8]

An equivalent way to read whether a particle is "trapped" or not, and this is done by looking at its propagator in the full space-time. If its pole is of a scale much larger than the typical low energy scale on the brane, then the zero modes (light particles) will be bound to the brane and the bulk modes will not be excited at those low energies.

The first example of localizing gravity on a domain wall was given in [63] fermions in [12, 123], and abelian gauge fields in [124]. The following section will provide examples to these mechanisms.

2.5.1 Gravity

An interesting example of the warped solutions is to consider living on the positive tension brane explained in section 2.3 and to push the one of negative tension to infinity by taking the limit $r_c \to \infty$. In this case it is convenient to parameterize the fifth dimension with $y = \phi r_c$. It is interesting precisely because this configuration "traps" the massless graviton on the brane. To see this, let us write the linearized Einstein equations for the graviton, $h_{\mu\nu}(x,y) = h_{\mu\nu}(x)h(y)$, using for convenience a change of variable $z = \mathrm{sgn}[y](\mathrm{e}^{k|y|} - 1)$, the gauge $\partial^\mu h_{\mu\nu} = 0$, $h^\mu_\mu = 0$, and a proper normalization for $h(y)$, [63]

$$\left[-\frac{1}{2}\partial_z^2 + \frac{15k^2}{8(k|z|+1)^2} - \frac{3k}{2}\delta(z)\right] h(z) = m^2 h(z) \ .$$

Looking at the above equation one expects 1) a single bound state supported by the δ-function trapped in $z = 0$, and 2) a continuous spectrum due to the non-compactness of the space, no mass gap, and asymptotic behavior as plane-waves. Their amplitudes should be suppressed near $z = 0$ due to the

[7]There are no explicitly worked out examples for localizing fermions on a hyperspace in Kaluza-Klein field-theoretic compactification. Attempts can be found in [9].

[8]It is perhaps worth mentioning that the theory of fermions in a non-trivial background in a non-compact space with direct product can resemble a theory with warped product, with the warp factor being the wave function of the normalized zero modes in the internal space, see section 2.5.2.

potential barrier. Further details related to the linearized gravity in a brane scenario have been worked out in [125].

The normalized zero mode of this operator, corresponding to $m^2 = 0$, is [63, 101]

$$h_0(z) = \frac{1}{k(|z| + 1/k)^{3/2}} ,$$

and is trapped in the potential, representing the brane. This zero mode, unlike in the KK picture, depends non-trivially on the extra coordinate, luckily with a decreasing dependence. This observation will be replicated in section 2.5 where the wave function of the localized fermion zero modes will have similar features.

The continuum modes are

$$h_m(z) = N_m(|z| + 1/k)^{1/2} \left[Y_2(m(|z| + 1/k)) + \frac{4k^2}{\pi m^2} J_2(m(|z| + 1/k)) \right] ,$$

where m is the mass of the mode, Y_2 and J_2 are Bessel functions, and N_m is a normalization constant.[9] The potential generated by the exchange of the zero and continuum modes behaves as [63]

$$V(r) = G_N \frac{m_1 m_2}{r} \left(1 + \frac{1}{r^2 k^2} \right) ,$$

where G_N is the four-dimensional Newton's constant. So, at scales lower than k, the scenario mimics a brane with pure 4-dimensional gravity (it was shown, however, that the five-dimensional gravity will be again manifest at very large distances, and that the above potential will be a good approximation in a finite region only [103]).

Close to the brane, the wave function of the continum modes is suppressed by a factor $\sqrt{m/k}$, and hence their coupling to matter on the brane is weak for small m and their production will be insignificant at low energies [127].

At large z, the massive KK gravitaional modes have strong coupling away from the brane, as their wave function behaves in this limit as $\sim e^{imz/k}$. However, the overlap between the wave functions of the zero mode and the continum modes [63] and this will ensure that the zero mode exchange and their self-couplings are four-dimensional and that there is no ultimate coupling of the KK excitations back to the matter on the brane. It was shown in

[9]Further details related to the linearized gravity in a brane scenario have been worked out in [125], and more about gravitational trapping solutions can be found in [99, 126].

[128] in an explicit way that the gravitational interaction of [63] correspond to 4-dimensional general relativity. It was shown in [129]-[133] that the standard Friedmann expansion of the Universe can be recovered under certain conditions.

2.5.2 Fermions

Localizing fermion fields together with gravity, in the manner explained above, turns out to be impossible without additional Yukawa type couplings [134]. Equivalently, a non-trivial vacuum configuration is required. The essence of the localization here is that the kernel of the diagonalized Dirac operator along the fifth direction is modified in the presence of a non-trivial background in the extra space, very much the same as explained in section 1.7. Actually the presence of this background is equally essential in the warped geometry as in the direct one for obtaining chiral fermions on the brane. For example, the coupled Dirac operator to a background configuration in five dimensions [12] admits two zero modes, one of them turns out to be normalizable while the other is not. Hence, all modes which couple to the non-renormalizable zero-mode will decouple from the action.[10]

Let us first see why a Yukawa coupling to fermions is needed in a localized gravity context as well by reviewing [134]. Consider the relatively general metric ansatz (2.11), the Dirac equation on the $(D = d_1+d_2+1)$-dimensional space $M_4 \otimes_f K \times \mathbb{R}^+$ coupled to a gauge field is

$$\Gamma^A e_A^M(\partial_M - \Omega_M + A_M)\Psi(x, y, r) = 0, \; , \tag{2.12}$$

where E_A^M is the vielbeins, and $\Omega_M = \frac{1}{2}\Omega_{M[A,B]}\Sigma^{AB}$ is the spin connection, $\Sigma_{AB} = \frac{1}{4}[\Gamma_A, \Gamma_B]$, and A_M is the gauge fields of a gauge group G. The

[10]The fermion localization procedure is very similar to the one of lattice gauge theories. The story began with the realization that dimensional regularization explicitly violates the chiral gauge symmetry as does the other regularization procedures as Pauli-Villars and zeta function teachniques. A lattice regularization was seeked, however the naïve approach suffered from a doubler problem [135] as was shown that it is impossible to formulate a gauge theory with continuous chiral symmetry on a lattice without doubling the species of fermions. The concept of domain wall fermions was introduced [136] in order to solve this problem by simulating the behavior of chiral fermions in an even dimension $2n$ by considering a lattice theory of interacting massive fermions in $2n + 1$ dimensions. The localization mechanism in the theories of extra dimensions is not very different. For comprehensive and analytical description of chiral fermions on lattice see [137, 138] and for a recent review [139].

non-vanishing components of the spin connection is

$$\Gamma_\mu = \frac{1}{4}A' e^{A/2}\delta^a_\mu \Gamma_r \Gamma_a \, , \tag{2.13}$$

$$\Gamma_m = \frac{1}{4}A' e^{B/2}\delta^{\underline{a}}_m \Gamma_r \Gamma_{\underline{a}} + \omega_m \, , \tag{2.14}$$

where Γ matrices are the constant Dirac matrices, and $\omega_m = \frac{1}{8}\omega_{m[\underline{a},\underline{b}]}[\Gamma_{\underline{a}}, \Gamma_{\underline{b}}]$ is the spin connection on K derived from the metric $g_{mn}(y) = e^{\underline{a}}_m e^{\underline{b}}_n \delta_{\underline{ab}}$. Assuming a background gauge field configuration only in the internal space K and the absence of other Yang-Mills couplings the fermion the Dirac equation (2.12) becomes

$$\left\{ e^{A/2}\partial\!\!\!/_M + \Gamma_r\left(\partial_r + \frac{d_1}{4}A' + \frac{d_2}{4}B'\right) + e^{-B/2}D\!\!\!\!/_K \right\}\Psi = 0 \, , \tag{2.15}$$

where $D\!\!\!\!/_K$ is the Dirac operator on K in the background gauge field A_m, which is assumed to admit zero modes (see section 1.7). Let the zero modes of K be $\psi(y)$, then

$$\Psi(x,y,r) = \psi(y)\zeta(r)\chi(x) \, ,$$

where ζ and χ satisfy

$$\partial\!\!\!/_M \chi(x) = 0 \quad , \quad \zeta(r) = \exp\left[-\frac{d_1}{4}A - \frac{d_2}{4}B\right] .$$

The effective action in d_1 dimensions is hence

$$S_M = \bar{\chi}(x)\partial\!\!\!/_M \chi(x) \int drdy\sqrt{g(y)}\, e^{-A/2}\psi^\dagger(y)\psi(y) \, . \tag{2.16}$$

On the other hand, the d_1-dimensional Newton's constant is

$$G^{-1}_{d_1} = G^{-1}_D V_K \int dr \exp\left[\frac{d_1-2}{2}A + \frac{B}{2}\right] , \tag{2.17}$$

where V_K is the volume of the compact manifold K. The localization of gravity requires a finite G_{d_1}, while localizing fermions occurs only when (2.16) converges. However, the integrals (2.17) and (2.16) do not simultaneously converge in the case of an exponential warp factor $A, B \propto -|r|$ so far considered in the literature since the function $\zeta(r)$ diverges.

Now let us introduce a scalar field Φ to the model, through the Yukawa coupling $\bar{\Psi}\Phi\Psi$

$$\left\{ e^{A/2}\partial\!\!\!/_M + \Gamma_r \left(\partial_r + \frac{d_1}{4}A' + \frac{d_2}{4}B' \right) + g\Phi + e^{-B/2}D\!\!\!\!/_K \right\} \Psi = 0 \, .$$

For the purpose of localization, as will be seen from the example given below, the dynamics of Φ is irrelevant. What matters is that $\lim_{|r|\to\infty} \Phi = |\phi|\epsilon(r)$ where $\epsilon(r)$ is the sign function and $\phi \equiv \langle\Phi\rangle$ (*i.e.* behaves as a kink at $\pm\infty$). Assuming this together with imposing the chirality condition $\Gamma^r\Psi = +\Psi$ or $\Gamma^r = 1$ (in the case when $d_1 + d_2$ is even) the Dirac equation away from the origin becomes

$$\partial\!\!\!/_M \chi(x) = 0 \quad , \quad D\!\!\!\!/_K \psi(y) = 0 \, ,$$
$$\left(\partial_r + \frac{d_1}{4}A' + \frac{d_2}{4}B' + g|\phi|\epsilon(r) \right) \zeta(r) = 0 \, ,$$

which admits the solution

$$\Psi(x, y, r) = \exp\left[- \left(\frac{d_1}{4}A + \frac{d_2}{4}B \right) - g|\phi|\epsilon(r) \right] \psi(y)\chi(x) \, .$$

The condition for having localized fermions is hence

$$-\frac{A}{2} - 2g|\phi|r\epsilon(r) \; < \; 0 \, . \tag{2.18}$$

This can be achieved for large enough values of $g|\phi|$.

The chirality of the localized (normalizable) zero modes will be determined by the solutions of $D\!\!\!\!/_K\psi(y) = 0$ in the presence of the background. K will be even dimensional if $M = M_4$, and hence the solutions to the Dirac equation on K will have a definite chirality. Moreover, in this case ψ and χ will have the same chirality, and the number of chiral fermions will be equal to the index of $D\!\!\!\!/_K$ which is the difference between the number of negative and positive chirality zero modes.

Rubakov-Shaposhnikov mechanism

Now let us present an illustrative example of fermion localization in the presence of a background [12] in five dimensions. The idea of [12] is a special case of the above with $A = B = 0$ and no compact space K. The condition

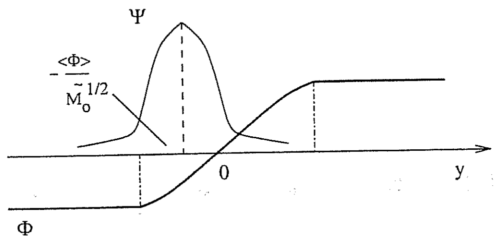

Figure 2.3: Profile of the scalar field Φ and of the fermionic field Ψ along the fifth dimension. The fermionic field, here assumed with vanishing bare five-dimensional mass, is localized where its total mass vanishes.

(2.18) is automatically satisfied when Ψ is in the positive region $\epsilon = +1$. As explained above, we necessarily need a scalar field Φ which acquires a non-zero , and varying,vacuum expectation value only along the extra space. Obviously, $\phi(y)$ breaks the full translational invariance, as it is needed to have a preferred direction orthogonal to the wall.

The vacuum expectation value of Φ can be regarded as background gauge field, as in [140], which has a domain-wall configuration, *e.g.* a kink, and this will provide an elegant dynamical origin for the spontaneous breaking of the 5-dimensional translation invariance.

The fermionic field will localize, as will be explained below, where its total mass

$$m_{tot} = m_0 + g\phi(y)$$

vanishes (m_0 is the bare fermionic mass in the five-dimensional theory), on a wall with three spatial dimensions characterized by a particular position y in the transverse direction.

For definiteness, we consider the theory described by the action

$$S = \int d^4x dy \; \Psi(x,y) \left[i\partial\!\!\!/_4 + i\partial\!\!\!/_5 + \frac{1}{\widetilde{M}_0^{1/2}} \phi(y) + m_0 \right] \Psi(x,y) \; , \qquad (2.19)$$

where the subscript "0" indicates the value of the parameter at zero temperature (we will discuss finite temperature effects later on), and the fields and the parameters have the following mass dimensions

$$[\phi] = \frac{3}{2} \; , \; [\Psi] = 2 \; , \; [m_0] = \left[\widetilde{M_0} \right] = 1 \; . \qquad (2.20)$$

For the time being we will ignore the dynamics of the field Φ, since the localization mechanism is concerned only with the background configuration. It is possible, however, that the configuration can decay due to the dynamics of the field Φ, like its interaction with fermions, thermal and other effects. We will discuss thermal effects in chapter 3.7, and we shall assume the stability of the vacuum configuration regarding what follows. At this point the shape of the configuration does not matter, what matters is that it is not constant in y, $\phi'(y) \neq 0$.

The coupled Dirac equation will hence be [11]

$$\left(i\gamma^\mu\partial_\mu + \gamma^5\partial_5 + m_{tot}\right)\psi = 0 \,. \tag{2.21}$$

The spinor Ψ can be decomposed, in a Lorentz non-invariant way in 5 dimensions, into a left and right handed fermion with respect to the chirality matrix γ_5 in four dimensions as in (1.18). The Fourier expansion can be carried out [12]

$$\begin{aligned}\Psi(x,y) &= \sum_n L_n(y)P_L\psi_n(x) + \sum_n R_n(y)P_R\psi_n(x) \;, \\ \bar{\Psi}(x,y) &= \sum_n \bar{\psi}_n(x)P_R L_n^*(y) + \sum_n \bar{\psi}_n(x)P_L R_n^*(y) \;,\end{aligned} \tag{2.22}$$

where $P_{L,R} = (1 \pm i\gamma^5)/2$. Since the kernel of the Dirac is equal to the kernel of its square, let us look at the square of the equation (2.21) in our search for the zero modes

$$\left(\not{D}_4^2 - \partial_5^2 - i\gamma^5 m'_{tot} + m_{tot}^2\right)\Psi = 0 \;, \tag{2.23}$$

where the "′" denotes a derivative with respect to y. The equation (2.23) can be rewritten using (2.22) as

$$\left(-\partial_5^2 + m_{tot}^2 - m'_{tot}\right)L_n = \mu_{n,L}^2 L_n \;, \tag{2.24}$$

$$\left(-\partial_5^2 + m_{tot}^2 + m'_{tot}\right)R_n = \mu_{n,R}^2 R_n \;, \tag{2.25}$$

[11] We follow here the notation of [70] concerning Dirac matrices,...*etc.*

[12] We perform a discrete expansion here in order to illustrate the idea in an easier way, while we keep in mind that the length of the fifth dimension is to be take to infinity at the end. One can alternatively keep the length big "enough", and not necessarily infinite, however this will be at the cost of introducing an inexplicably small mass scale to the theory.

where $\mu_{n,L}$ and $\mu_{n,R}$ are eigenvalues of the four-dimensional Dirac operator $\not{D}_4$ related to the eigenfunctions $P_L\psi_n$ and $P_R\psi_R$ respectively.

Define

$$a \equiv \partial_5 + m_{tot} \; ; \quad a^\dagger \equiv -\partial_5 + m_{tot}. \tag{2.26}$$

Therefore

$$\begin{aligned} a^\dagger a &= -\partial_5^2 + m_{tot}^2 - m'_{tot} \, , \\ aa^\dagger &= -\partial_5^2 + m_{tot}^2 + m'_{tot}. \end{aligned} \tag{2.27}$$

In this notation, the equation (2.24) and (2.25) can be written as

$$a^\dagger a L_n = \mu_{n,L}^2 L_n, \tag{2.28}$$

$$aa^\dagger R_n = \mu_{n,R}^2 R_n. \tag{2.29}$$

The eigenfunctions L_n and R_n can be normalized to form two sets of orthonormal functions. Note that the operators $aa^\dagger$ and $a^\dagger a$ commute only when $\phi(y)$ is constant in y. Also multiplying (2.28) by a from the left shows that aL_n is an eigenfunction of the operator $aa^\dagger$ with eigenvalue $\mu_{n,L}^2$ for n different from zero. So, for $n \neq 0$ one can write $\mu_{n,L} = \mu_{n,R} \equiv \mu_n$, or in other words

$$R_n = \frac{1}{\mu_n} a L_n \quad , \quad L_n = \frac{1}{\mu_n} a^\dagger R_n. \tag{2.30}$$

The above relation does not hold for the zero modes L_0 and R_0 corresponding to $\mu_0 = 0$. The zero mode wave functions are found by integrating the two equations $aL_0 = 0$ and $a^\dagger R_0 = 0$. The solutions are

$$\begin{aligned} L_0 &\sim \exp\left[-\int^y ds \, m_{tot}(s)\right] \, , \\ R_0 &\sim \exp\left[\int^y ds \, m_{tot}(s)\right] \, . \end{aligned} \tag{2.31}$$

If, for example, $\phi(y)$ has a kink configuration as in the figure 2.3 and if the extra dimension is infinite, only the left-handed mode L_0 is normalizable and is localized around the zero of its total mass m_{tot}. R_0 will not be normalizable and its coupling to other fields will always be suppressed by the length of the fifth direction. The effective four-dimensional theory will hence massless chiral fermionic field (in this case left-handed). The action (2.19) can now be expressed in terms of the 4-dimensional fields, using the orthogonality of $\{L_n\}$ and $\{R_n\}$, as

$$S = \int d^4x \left[\bar{\psi}_{0,L} i\gamma^\mu \partial_\mu \psi_{0,L} + \bar{\psi}_{0\,R} i\gamma^\mu \partial_\mu \psi_{0,R} + \sum_{n=1}^{\infty} \bar{\psi}_n \left(i\gamma^\mu \partial_\mu + \mu_n \right) \psi_n \right] . \tag{2.32}$$

The first two terms correspond to 4-dimensional two-component massless chiral fermions, and arise from the zero modes of equations (2.28) and (2.29). The third term describes an infinite tower of Dirac fermions corresponding to the (Kaluza-Klein) modes with non-zero μ_n in the expansion (2.22). If the extra space is sufficiently small, these modes decouple from the low energy theory.

Let us now consider a specific shape for the configuration, as this will serve for chapter 3.7. Let the field Φ have the following Lagrangian

$$\mathcal{L}_\Phi = \frac{1}{2} \partial_M \Phi \partial^M \Phi - (-\mu_0^2 \Phi^2 + \lambda_0 \Phi^4) \delta(y) \,, \tag{2.33}$$

where μ_0 and λ_0 have mass dimensions 1 and -1 respectively. A solution of the equations of motion is

$$\phi(y) = \frac{\mu_0}{\sqrt{2\lambda_0}} \tanh[\mu_0 \, y] \,, \tag{2.34}$$

which can be approximated with a straight line interpolating between the two vacua (see figure 2.4)

$$\begin{aligned} \phi(y) &\simeq \frac{\mu_0^2}{\sqrt{2\lambda_0}} \, y \ , \ \ |y| < \frac{1}{\mu_0} \\ \phi(y) &\simeq \pm \frac{\mu_0^2}{\sqrt{2\lambda_0}} \ , \ \ |y| > \frac{1}{\mu_0} \, . \end{aligned} \tag{2.35}$$

The localization can occur only if

$$m_0 < \frac{\mu_0}{\sqrt{2\,\lambda_0\,\widetilde{M}_0}} \,, \tag{2.36}$$

since otherwise the total fermion mass m_{tot} never vanishes.

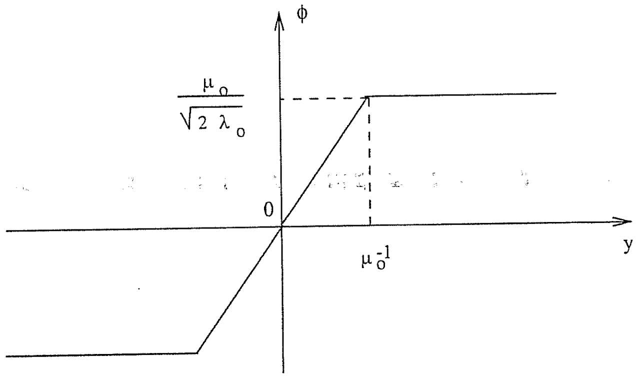

Figure 2.4: A linearized approximation to the kink solution of (2.34).

It can be shown that, from the four-dimensional point of view, a left handed chiral massless fermionic field results from the localization mechanism, if the above configuration (2.34) is assumed for the scalar ϕ. The right handed part remains instead de-localized in the whole space. This is not a problem since it is customary to limit the Standard Model fermionic content only to left handed fields. The right handed fields can also be localized if a kink–antikink solution is assumed for the scalar ϕ. As a result, the left fields continue to be localized on the kink, while the right ones are confined to the antikink. If the kink and the antikink are sufficiently far apart, the left handed and right handed fermions however do not interact and again the model reproducing our 4-dimensional world must be built by fermions of a defined chirality. The fermionic content of the full dimensional theory is in this case doubled with respect to the usual one, and observers on one of the two walls will refer to the other as to a "mirror world". The presence of this kink–antikink configuration may be required by stability consideration if thermal effects are considered.

In order to give mass to the fermions, some other scalar field acting as a Higgs in the four-dimensional theory must be considered. As it is shown in [70], the mechanism described above could give an explanation to the hierarchy among the Yukawa couplings responsible for the fermionic mass matrix. If indeed one chooses different five-dimensional bare masses for the different fermionic fields, their wave functions will only partially overlap, as a consquence, and increasing the distance between them in the transverce direction to the brane results in suppressing their mutual interaction.

Chapter 3

Living with Extra Dimensions

3.1 Introduction

Imagining our Universe being extended to more than four dimensions at energies around TeV may imply dramatic experimental consequences. First of all, the force of gravity is expected to become comparable to the other gauge forces at around TeV which would enable LHC and NLC [1] to probe the quantum structure of gravity. This could also be checked by the new experiments measuring gravity at sub-millimeter distances [15, 16]. Secondly, since the masses of the Kaluza-Klein excitations are typically of order of inverse the compactification scale, they will be excited by being exchanged or emitted at such energies and will be expected to appear in collider experiments, Cosmological, and Astrophysical environments [21]. Further, a multi-dimensional Universe will generally evolve different from the usual Friedmann-Robertson-Walker expansion law. [2] To fit the expected contribution of the infinitely new degrees of freedom within the known experimental and observational data, it is necessary to bounds on the parameters of the models with extra dimensions, both compact or non, mainly lower bounds on the fundamental scale of gravity whether this scale is set by the size of the compact dimensions, the characteristic scale which enters in the localization of matter transverse a non-compact space, or other relevant model dependent scales. In any case,

[1]LHC (CERN) has a center of mass energy around 14 TeV and is scheduled to operate in 2005, while NLC (SLAC), the anticipated e^+e^- collider, has a center of mass energy 500-1500 GeV. For reviews see [155, 156].

[2]See [157] for a breif summary of the present status of the standard Big Bang model, and [158, 159] for the recent interesting developments.

the larger the number of the extra dimensions is the looser the bounds become.

In the following we review some of the rather established bounds imposed on models with large extra dimensions, [3] with more emphasis on constraints arising from compatibility with Cosmological observations. After having done so, we go on to present examples where the standard big bang Cosmology can be recovered in Kaluza-Klein compactification schemes and in Randall-Sundrum type scenarios. Then we move on to discuss briefly the issue of stabilizing the scale of compactification, proton stability in theories with extra dimensions, and finally baryogenesis with low scale gravity. We conclude this chapter by presenting a model for baryogenesis in 5 dimensions.

3.2 General bounds

The most stringent bound [4] on the size of the compact space comes from the emission of the supernova SN1987A core of large fluxes of Kaluza-Klein gravitons which affects its energy release [165, 142] (see [21] for a review). For $n = 2$, these constrains turn out to be $R < 0.9 \times 10^{-4}$mm and for $n = 3$, $R < 1.9 \times 10^{-7}$mm [144]. Collider and other bounds on the size of compact extra dimensions can be found in [21], and a more recent analysis in [142]-[151]. Higher-dimensional operators suppressed by small scales are model dependent and they do not significantly modify the SM cross sections and precision observables for a gravity scale greater than 1TeV [21]. [5] The bounds on non-compact warped, Randall-Sundrum type, scenarios are less elaborated and more model-dependent; such constraints can be found in [145, 153, 154].

KK gravitons

The existence of light Kaluza-Klein gravitons is a general feature of compact large extra dimensions, although their coupling properties to ordinary

[3]Discussion about searches for extra dimensions in future colliders can be found in [160]-[163].

[4]It was claimed racently nucleon-nucleon gravi-bremsstrahlung in the early Universe could give a stronger bound that SN1987A provides [164].

[5]A serious problem in Kaluza-Klein large radii compactification is proton decay. For that, a TeV gravity scale will cause a serious problem [152], unless a specific symmetry is imposed.

matter is more model-dependent.[6] Each KK graviton mode interacts with the ordinary matter through the four-dimensional energy-momentum tensor and its coupling is gravitationally week, suppressed by $1/M_P$, as can be read from

$$S_n = M^{d+2} \int d^{d+4}z \; \left[\partial h(x) e^{iq_n y}\right]^* \left[\partial h(x) e^{iq_n y}\right] + \int d^4x \; h(x) T(x)$$

which shows that upon integrating over y the volume of the compact space will appear in front of the first term producing M_P^2. The proper rescaling gives a coupling M_P^{-1} with the energy momentum tensor (ordinary matter).

Despite the gravitationally weak coupling, there are a large multiplicity of those massive gravitons below an energy level, E, which is enormous $\sim (ER)^d$, and their combined effect is much stronger than the gravitational suppression. One of the typical processes for producing a KK graviton is

$$e^+e^- \to \gamma + \text{graviton} .$$

The graviton produced here will have the form of a missing energy. The total cross section is of order α/M_P^2 and is

$$\sigma(e^+e^- \to \gamma + \text{graviton}) \sim \frac{\alpha}{M_P^2}(ER)^d \sim \frac{\alpha}{E^2}\left(\frac{E}{M}\right)^{d+2}$$

which becomes comparable to the electromagnetic cross section at energies $E \sim M$ (for further details of this and others, like hadronic processes, we refer the reader to [151]).

Beside their possible detection in colliders, through their emission or exchange, they can be produced at high temperature sometime in the early Universe, and if they decay after the Big Bang Nucleosynthesis (BBN) they will distort the Cosmic Microwave Background Radiation. Not only that, their abundant number and large mass will change the expansion of the Universe, and this may be done during the BBN by slowing the expansion down since they red shift as matter, $\sim R^{-3}$ rather than radiation $\sim R^{-4}$. Furthermore, their energy density may over-close the Universe if their abundance is comparable to the photon abundance at early times. These obstacles are enough to impose a strong upper bound on the temperature of the Universe, T_*, after which it should behave as the usual Friedmann-Robertson-Walker (FRW) Universe [21].

[6] For the manifestation of these states in a string theory context see for example [165].

We explain below how these bounds arise (more or less along the lines of [76]).

At high enough temperature $T \gg R^{-1}$, the creation rate per unit time and volume of a KK graviton of mass $m_n \lesssim T$, based on dimensional analysis,[7] $\Gamma \sim \frac{T^6}{M_P}$, where the factor M_P^{-1} is the strength of the coupling of the KK graviton to matter. The estimate for the total rate of KK graviton production would be

$$\frac{dn}{dt} \sim \frac{T^6}{M_P}(TR)^d \sim T^4 \left(\frac{T}{M}\right)^{2+d} .$$

Assuming that after a certain temperature, T_*, after which the Universe evolves in the standard way (presumably during the reheating after inflation) $H = T_*^2/M_P^*$, where $M_P^{*2} = M_P/(1.66g_*^{1/2}) \sim 10^{18}$GeV, and g_* is the effective number of relativistic degrees of freedom, one finds that the number of KK gravitons created in a Hubble time per relativistic species (photons) is

$$\frac{n_{KK}(T_*)}{n_\gamma} \sim M_P^* \frac{T_*^{1+d}}{M^{2+d}},$$

which is fairly large, and requires $n_{KK} \ll n_\gamma$.

A stronger upper bound on the temperature T_* comes from slowing the expansion of the Universe at temperature below T_*, since KK gravitons produced after this temperature red-shift as non-relativistic particles. At the time of BBN their mass density per energy density of photons is of order

$$\frac{\rho_{KK}(1\text{MeV})}{\rho_\gamma} \sim \frac{T_*}{1\text{MeV}} \times \frac{T_*^{d+1} M_P^*}{M^{d+2}} .$$

Requiring $\rho_{KK} \ll \rho_\gamma$ yields to the following upper bound on the reheating temperature (or *normalcy* temperature) [21]

$$T_* \lesssim 10^{\frac{6d-9}{d+2}} \text{ MeV } \frac{M}{1\text{TeV}} .$$

The upper bound reads for $M = 1$TeV in two dimensions 10MeV which it too severe. The bound relaxes somehow for $d = 6$ and is $T_* \lesssim 1$GeV.

[7]Strictly speaking, assuming a four-dimensional KK gravity production at scales much higher than R^{-1} makes little sense, since at such high scales the Universe appears multi-dimensional presuming the general coordinate invariance is restored at high energies (the break down of the isometry of the full space, and the formation of a domain wall, is usually due to a spontaneous breaking characterized by R^{-1}).

Finally we come to the most constraining Cosmological bound which is the risk of overclosing the Universe. The energy density of the massive KK gravitons should not exceed the critical energy density today which correspond to $\sim 3 \times 10^{-9}$GeV. The life time of a graviton with an energy E is

$$\tau(E) = \frac{M_P^2}{E^3} \sim 10^{10}\text{yr.} \left(\frac{10^2\text{MeV}}{E}\right)^3 .$$

Gravitons produced at temperatures below 10^2MeV have lifetime at least as the age of the Universe. The energy density stored in the gravitons produced at a temperature T_* is

$$\frac{\rho_{KK}(T_*)}{\rho_\gamma} \sim T_* n_{KK} \simeq \frac{T_*^{d+1} M_P}{M^{d+2}}$$

which gives for $\rho_{KK} \ll \rho_\gamma$

$$T_* \lesssim 10^{\frac{6d-15}{d+2}} \text{ MeV } \frac{M}{1\text{TeV}}$$

and this constrains T_* severely from above by a value of 1.7MeV, 0.3GeV, and 2GeV for 2, 4, and 6 extra dimensions with fundamental gravity scale $M \simeq 10$TeV!. This bound suggest that $M > 10$TeV for this scenario to be compatible with BBN in the case of 2 extra dimensions.

3.3 Kaluza-Klein Cosmology

It is obvious that the evolution of the Universe as a system in more than four dimensions will be different than the standard Friedmann-Robertson-Walker scenario.

The question to pause is whether or not recovering the standard Cosmological expansion is possible and under which conditions. The answer to this question is yes, it is possible for the four-dimensional Universe to evolve according to the usual Freedmann expansion law if the internal space is small enough compared to the size of the Universe [166]-[171]. Other Cosmological consequences were also discussed as massive relics [172], inflation [168], and other finite temperature effects in [173, 174]. In the few past years the some of the above issues were re-examined in addition to some new cosmological and phenomenological aspects [175]-[192].

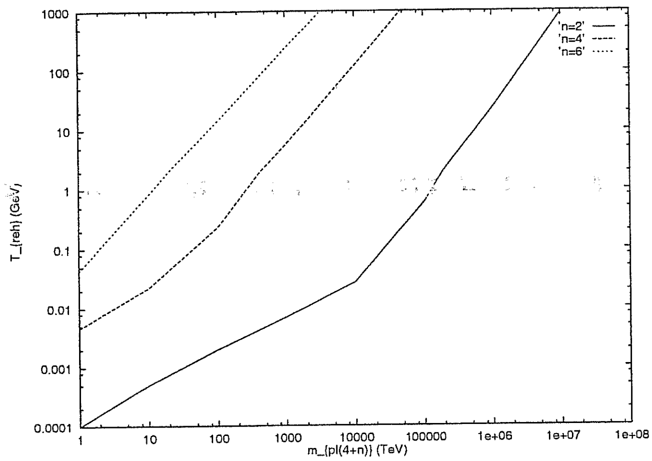

Figure 3.1: The maximum allowed normalcy temperature T_* as a function of the fundamental gravity scale M for various number of extra dimensions (taken from [152]).

Let us look at what happens to a Universe living in D dimensions with the following metric

$$g_{MN} = \begin{pmatrix} -1 & & \\ & R(t)^2 \tilde{g}_{\mu\nu}(x) & \\ & & a(t)^2 \tilde{g}_{mn}(y) \end{pmatrix} , \tag{3.1}$$

where $\tilde{g}_{\mu\nu}$ and $\tilde{g}_{mn}$ are the metrics on the d_1-dimensional space M ($d_1 = 4$ eventually) and the d_2-dimensional compact one K respectively. Further, we assume the Universe to have the perfect fluid description in D dimensions, *i.e.*

$$T_{MN} = \begin{pmatrix} -\rho & & \\ & p\tilde{g}_{\mu\nu} & \\ & & p'\tilde{g}_{mn} \end{pmatrix} , \tag{3.2}$$

where the energy density ρ and the pressures p, p' may depend on time, but not the space coordinates x^μ, y^m. They are also assumed to satisfy the equation of state $\rho = (D-1)p$,[8] $p = p'$.

[8]This equation of state is not well justified. For a generic treatment see [171].

The Einstein equations for the scale factor R and a in terms of the entries of the energy-momentum tensor can be derived from the action

$$S = \int d^{d_1}x d^{d_2}y \sqrt{-g} \left(\frac{1}{\kappa^2} \mathcal{R} + \Lambda \right) . \tag{3.3}$$

For $\Lambda = 0$ they are

$$(d_1 - 1)\frac{\ddot{R}}{R} + d_2 \frac{\ddot{a}}{a} = -\kappa^2 \rho ,$$

$$\frac{\mathcal{R}_M}{R^2} + \frac{d}{dt}\left(\frac{\dot{R}}{R}\right) + \left((d_1 - 1)\frac{\dot{R}}{R} + d_2 \frac{\dot{a}}{a} \right) \frac{\dot{R}}{R} = \frac{\kappa^2}{D-1}\rho ,$$

$$\frac{\mathcal{R}_K}{a^2} + \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) + \left((d_1 - 1)\frac{\dot{R}}{R} + d_2 \frac{\dot{R}}{R} \right) \frac{\dot{a}}{a} = \frac{\kappa^2}{D-1}\rho ,$$

where $\mathcal{R}_M$ and $\mathcal{R}_K$ are the curvature scalars of M and K respectively. Looking at the equations of motion (setting $\Lambda = 0$) it was shown in [170] that, for a flat M, the typical distance $R(t)$ between two point on M increases monotonically from 0 to ∞ at every value of R_0, while a, the radius of K, increases from zero to a maximum then drops back to zero. A similar varying behavior of the compactification scale has been noted in [169] [9] by taking into account combined thermal and quantum mechanical fluctuations effects, and the instability feature remains to be accounted for. Therefore a stabilization mechanism is needed to set a at the desired value whatever it is chosen to be in a short enough period, not only for keeping the fundamental scale of gravity low enough, but also in order not to cause any conflict with the variations of the four-dimensional Newton's constant κ^2 [193] and big bang nucleosynthesis [194].

Assuming that such a localization mechanism exists and that its net effect is to induce a constant effective pressure $p_c = \frac{1}{a_c^2}$ the independent four-dimensional Einstein equations to be solved at low energies become [170]

$$\frac{d}{dt}\left(\frac{\dot{R}}{R}\right) + (d_1 - 1)\frac{\dot{R}^2}{R^2} = \kappa^2 \rho \quad , \quad (d_1 - 1)\frac{\ddot{R}}{R} = \kappa^2 \rho$$

which clearly resemble the evolution of FRW Universe.

[9]There is a major difference in the two results as in [169] the compact space tends to increase after a certain critical length, in contrast with [170].

As pointed out in [170], the temperature profile of the early Universe is radically changed due to the presence of the extra dimensions. If all the dimensions were expanding, the Universe would have only cooled down by time. However, if the extra space undergoes a period of expansion and contraction, the D-dimensional Universe will start off very hot, cools down to a certain limit, afterwhich it reheats until it becomes effectively the observed four-dimensional Universe.

The energy momentum tensor (3.2) was shown in [171] to be derived from a free energy function of free bosonic gas in thermal equilibrium. It was shown that at energies below the compactification mass scale, the Universe expands as usually observed in four dimensions while the radius of compactification remains constant.[10] This is done by adding a source term to (3.3)

$$\Gamma = \int d^4x d^d y \sqrt{-g} \left(\frac{1}{\kappa^2} \mathcal{R} + \Lambda \right) + \Gamma_1 \ ,$$

where Γ_1 represents the contribution of matter fields, vacuum fluctuations,...*etc.* Its variations defines the energy-momentum tensor

$$\delta\Gamma_1 = \int d^4x d^d y \sqrt{-g} \frac{1}{2} \delta g^{MN} T_{MN} \ .$$

The stress tensor T_{MN} is as usual expected to have the same invariances as the metric. In [171] the full manifold is considered $R^1 \times S^3 \times S^d$, where the 3-dimensional space is taken to be closed. The case of flat Universe can be obtained by taking $R \to \infty$ after a careful transition from the discrete to continuum limit. The symmetry of T_{MN} in this case is $O(1,3) \times O(d+1)$. The action with this metric ansatz is hence

$$\Gamma = \int dt \ \frac{\Omega_3 \Omega_d}{\kappa^2} \left[\frac{6\dot{R}^2}{R^2} + 6d \frac{\dot{R}}{R} \frac{\dot{a}}{a} + d(d-1) \frac{\dot{a}^2}{a^2} \right] + U \ , \qquad (3.4)$$

where Ω_3 and Ω_d denote the spatial volumes

$$\Omega_3 = 2\pi^2 R^3 \quad , \quad \Omega_d = (2\pi)^{\frac{d+1}{2}} \frac{a^d}{\Gamma(\frac{d+1}{2})} \ ,$$

and U is the effective "potential" which incorporates a classical gravity terms arising from $\mathcal{R}+\Lambda$, from the one-loop quantum part Γ_1, and from the thermal

[10] For alternative expansion, *e.g.* inflation, see for instance [168, 184, 186].

part depending on the entropy S,

$$U = \frac{\Omega_3 \Omega_d}{\kappa^2}\left[-\frac{6}{R^2} - d\frac{(d-1)}{a^2} + \Lambda\right] + U_{1-loop} \,. \tag{3.5}$$

The Free energy to be computed is obtained by Legendre transform of U

$$U_{1-loop} = F + TS = \Omega_3 \Omega_d \rho(R, a, S) \,, \tag{3.6}$$

where the temperature is given by $T = \partial U / \partial S$. Let the pressures p and p' entries in (3.2) be defined by the thermodynamic identity

$$d\left[\Omega_3 \Omega_d \rho(R, a, S)\right] = TdS - \Omega_d p d\Omega_3 - \Omega_3 p' d\Omega_d \,,$$

then the conservation of energy $\nabla_M T^{MN} = 0$ implies the conservation of entropy.[11] Since we have no a priori idea of what could the equation of state for Kaluza-Klein Cosmology be, let us suppose that ρ, p, and p' are due to a gas of non-interacting bosons in thermal equilibrium. The free energy of this system can be written as follows

$$\begin{aligned} \beta F &= \frac{1}{2} \ln \det\left[-\Delta + \mu^2\right] \\ &= \sum_{r,m,n=0}^{\infty} D_{mn} \ln\left[\left(\frac{2\pi r}{\beta}\right)^2 + m\frac{m+2}{R^2} + n\frac{n+d-1}{a^2} + \mu^2\right] \end{aligned} \tag{3.7}$$

where μ is a mass parameter, $\beta = (kT)^{-1}$, and Δ denotes the Laplacian on the full compact manifold $S^1 \times S^3 \times S^d$ and the three first terms in (3.7) represent its eigenvalues. D_{mn} is the multiplicity factor related to each of the eigenvalues and is equal to the dimension of $O(1,3) \times O(d+1)$ representation

$$D_{mn} = (m+1)^2 \, (2n + d - 1)\frac{(n+d-2)!}{(d-1)!n!} \,.$$

[11]Recall that in the usual treatment [195] the energy density and pressure must satisfy the integrability condition

$$\frac{\partial^2 S(V,T)}{\partial T \partial V} = \frac{\partial^2 S(V,T)}{\partial V \partial T} \,,$$

where $dS(V,T) \equiv dQ/T = [\rho(T)dV + p(T)dV + Vdp]/T$ which imply $dp/dT = (\rho(T) + p(T))/T$. Since the equations of motion are $d/dt[R^3(\rho + p)/T] = 0$, the entropy will be constant in time.

The divergent sum (3.7) can be regularized by making use of the identity

$$\ln X = \frac{d}{ds} X^s |_{s=0} = \frac{d}{ds}\left[\frac{1}{\Gamma(-s)}\int_0^\infty dt t^{-s-1} e^{-tX}\right]_{s=0}$$

The free energy can thus be rearranged as

$$\beta F = \frac{d}{ds}\left[\frac{1}{\Gamma(-s)}\int_0^\infty dt\, t^{-s-1} e^{-t\mu^2} \sigma_1(4\pi^2 t/\beta^2)\sigma_3(t/R^2)\sigma_d(t/a^2)\right]_{s=0} ,$$

where the functions $\sigma_{1,3,d}$ are defined by

$$\begin{aligned}
\sigma_1(4\pi^2 t/\beta^2) &= 2\sum_0^\infty e^{-16\pi^2 t/\beta^2} \\
\sigma_3(t/R^2) &= \sum_0^\infty (m+1)^2 e^{-m(m+2)t/R^2} \\
\sigma_d(t/a^2) &= \sum_0^\infty (2n+d-1)\frac{(n+d-2)!}{(d-1)!n!} e^{-n(n+d-1)t/a^2} .
\end{aligned}$$

Note that each of the σ function can be written in terms of the Jacobi Theta function

$$\theta_3(0, iu/\pi) \equiv \sum_{n=-\infty}^\infty e^{-\tau n^2} = \sqrt{\frac{\pi}{\tau}}\, e^{-\pi^2 n^2/\tau}$$

which has away from $\tau = 0$ the modular property

$$\theta_3(z/\tau, -\tau^{-1}) = \sqrt{\frac{\tau}{i}}\, e^{i\pi z^2/\tau}\theta_3(z,\tau) .$$

The sums (3.8) hence converge and define analytic functions on the half plane Re$t > 0$ and are singular at $t = 0$

$$\sigma_1 \sim t^{-1/2} , \; \sigma_3 \sim t^{-3/2} , \; \sigma_d \sim t^{-d/2} .$$

For σ_1, the zero temperature limit corresponds to $\sigma_1 \simeq \sqrt{4\pi t/\beta^2}$, and the flat four-dimensional space limit of the energy density $F/(2\pi^2 R)$ corresponds to $\sigma_3 \simeq \frac{1}{4}\sqrt{\pi}R^3 t^{-3/2}$.

Now, in a regime in which $R \gg T^{-1} > a$ and a relativistic gas of particles, $\mu < T$, the formula for the free energy converges at $t = 0$ (see [171] for the technical details) and the "s" regularization can be removed, and the approximate expression becomes

$$F \simeq \Omega_3 \left(\sigma_d a^{-4} - \frac{1}{90}\pi^2 \beta^{-4} + \dots \right) ,$$

where

$$\sigma_d = -\frac{1}{32\pi^2} \int_0^\infty du\, u^{-3} \sigma_d(u) .$$

The 1-loop contribution to the potential U related to F by (3.6) therefore is

$$U_{1-\text{loop}} \simeq \Omega_3 \frac{\sigma_d}{a^4} + S^{4/3} \frac{\tau}{R} \quad , \quad \tau = \frac{3}{4} \left(\frac{45}{4\pi^4} \right)^{1/3}$$

The full potential (3.5) up to one loop is therefore

$$U = \frac{\Omega_3 \Omega_d}{\kappa^2} \left[-\frac{6}{R^2} - d\frac{(d-1)}{a^2} + \Lambda \right] + \sigma_d \frac{\Omega_3}{a^4} + \tau S^{4/3} R + \zeta m ,$$

where the last term ζm is identified by the energy for a constant a,

$$E = -\frac{\Omega_3 \Omega_d}{\kappa^2} \frac{\dot{R}^2}{R^2} + U . \tag{3.8}$$

The time evolution of the D-dimensional Universe is governed by the Euler-Lagrange equations derived from (3.4). Is is shown that for an appropriate choice of Λ and the parameters in U, it is possible to obtain a solution where the internal radius is constant while the large radius evolves as in the standard Cosmology [171]. For $\dot{a} = 0$ the equations of motion become

$$\partial_t (R\dot{R}) - \frac{1}{2}\dot{R}^2 = \frac{1}{12} \frac{\kappa^2}{\Omega_3 \Omega_d} R \frac{\partial U}{\partial R} \tag{3.9}$$

$$R^{-1} \partial_t (R^2 \dot{R}) - \dot{R}^2 = \frac{\kappa^2}{6d} \frac{R^2}{\Omega_3 \Omega_d} a \frac{\partial U}{\partial a} \tag{3.10}$$

The compatibility of (3.8), (3.9), and (3.10) requires

$$R\frac{\partial U}{\partial R} - \frac{2}{d} a \frac{\partial U}{\partial a} = E - U ,$$

and this constraint allows us to determines Λ and a in terms of κ^2 which upon substituting back into (3.10) gives

$$\partial_t(R\dot{R}) = -1 \quad \text{or} \quad R^2 = t\,(t_0 - t) \ , \tag{3.11}$$

where t_0 is computed from (3.8), $t_0^2 = (\tau/3\pi^2)\kappa^2 S^{4/3}/\Omega_d$.

As a summary, there are solutions for Einstein equations in the regime $R \gg T^{-1} > a$ where the size of the internal space remain constant, and the 4-dimensional Universe evolves according to the conventional Friedmann-Robertson-Walker Cosmology, as (3.11) clearly indicates, for a closed Universe in a radiation dominated era. It remains to check the classical stability, at least, of the value of a by solving the equations for small perturbations around the desired value.

3.4 Randall-Sundrum Cosmology

Here we review an example [130] of the evolution of a brane-world embedded in five dimensions, and under which condition the FRW expansion can be recovered. The time evolution of the scale factor, is worked out explicitly for the a specific choice of the equation of state $p = \omega\rho$ ($\omega =$ const.), and it is shown that the standard cosmological evolution can be obtained [130].

Since we are interested in a FRW metric for the four-dimensional Universe, we start from the following maximally symmetric three-dimensional metric ansatz

$$ds^2 = -n^2(\tau, y)d\tau^2 + a^2(\tau, y)\tilde{g}_{ij}dx^i dx^j + b^2(\tau, y)dy^2 \ ,$$

and we assume the brane as a hypersurface in five dimensions defined by $y = 0$.

Upon substituting this ansatz into the five-dimensional Einstein equations, the non-vanishing components of the Einstein tensor

$G_{MN} \equiv \mathcal{R}_{MN} - \frac{1}{2}\mathcal{R}\, g_{MN}$ are [130]

$$G_{00} = 3\left\{\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right) - \frac{n^2}{b^2}\left[\frac{a''}{a}+\frac{a'}{a}\left(\frac{a'}{a}-\frac{b'}{b}\right)\right] + \kappa\frac{n^2}{a^2}\right\}, \tag{3.12}$$

$$\begin{aligned} G_{ij} &= \frac{a^2}{b^2}\gamma_{ij}\left\{\frac{a'}{a}\left(\frac{a'}{a}+2\frac{n'}{n}\right) - \frac{b'}{b}\left(\frac{n'}{n}+2\frac{a'}{a}\right) + 2\frac{a''}{a} + \frac{n''}{n}\right\} \\ &\quad + \frac{a^2}{n^2}\tilde{g}_{ij}\left\{\frac{\dot{a}}{a}\left(-\frac{\dot{a}}{a}+2\frac{\dot{n}}{n}\right) - 2\frac{\ddot{a}}{a} + \frac{\dot{b}}{b}\left(-2\frac{\dot{a}}{a}+\frac{\dot{n}}{n}\right) - \frac{\ddot{b}}{b}\right\} - \kappa\tilde{g}_{ij}, \end{aligned} \tag{3.13}$$

$$G_{05} = 3\left(\frac{n'}{n}\frac{\dot{a}}{a} + \frac{a'}{a}\frac{\dot{b}}{b} - \frac{\dot{a}'}{a}\right), \tag{3.14}$$

$$G_{55} = 3\left\{\frac{a'}{a}\left(\frac{a'}{a}+\frac{n'}{n}\right) - \frac{b^2}{n^2}\left[\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}-\frac{\dot{n}}{n}\right)+\frac{\ddot{a}}{a}\right] - \kappa\frac{b^2}{a^2}\right\}, \tag{3.15}$$

where a prime stands for a derivative with respect to y while a dot denotes the derivative with respect to τ.

The energy-momentum tensor can be decomposed into two parts; the bulk matter source and the brane matter one

$$T^A{}_B = \check{T}^A_B|_{\text{bulk}} + \tilde{T}^A_B|_{\text{brane}},$$

where $\check{T}^A_B|_{\text{bulk}}$ is the energy-momentum tensor of the bulk matter which we assume to have the form

$$\check{T}^A_B|_{\text{bulk}} + \text{diag}(-\rho_B, p_B, p_B, p_B, p_T),$$

where the energy density ρ_B and the pressures p_B and p_T are independent of the coordinate y. Later we will be interested in the special case of a Cosmological constant $-\rho_B = p_B = p_T$.

The second term $\tilde{T}^A_B|_{\text{brane}}$ corresponds to the matter content on the brane, i.e at $y = 0$. According to our metric ansatz, the matter description on the brane is the one of a perfect fluid in four dimensions

$$\tilde{T}^A_B|_{\text{brane}} = \frac{\delta y}{b}\text{diag}(-\rho_b, p_b, p_b, p_b, 0),$$

where ρ_b and p_b are, respectively, the energy density and pressure on the brane and are functions of time only. The last entry being zero means physically that there is no flow of ordinary matter along the fifth direction, and consequently the vanishing of G_{05} and this leads in turn that (3.12) and (3.15) become

$$F' = 2\frac{a'a^3}{3}\kappa^2 \check{T}^0_0 \tag{3.16}$$

$$\dot{F} = 2\frac{\dot{a}a^3}{3}\kappa^2 \check{T}^5_5 \ , \tag{3.17}$$

where F is a function of τ and y defined by

$$F(\tau, y) = \frac{a'^2 a^2}{b^2} - \frac{\dot{a}^2 a^2}{n^2}\kappa a^2 \ . \tag{3.18}$$

Integrating (3.16) over y, keeping in mind that $\rho_B{}' = 0$, we obtain

$$F + \frac{\kappa^2}{6} a^4 \rho_B + C = 0 \ ,$$

where C is a constant of integration.

Now, by taking the time derivative of (3.16), the y derivative of (3.17), and assuming $-\rho_B = p_T$ we see that

$$(a' - \dot{a})a^3 \frac{d}{dt}\rho_B = 0$$

which indicates that ρ_B is independent of time for $a' - \dot{a} \neq 0$, the thing we will assume from now on. [12] This will indicate that $\dot{C} = 0$.

Coming back to (3.13), G_{ij} can be written using the Bianchi identity $\nabla_A G^{A0} = 0$ as

$$\partial_\tau \left(\frac{F'}{a'}\right) = \frac{2}{3}\dot{a}a^2 \tilde{g}_{ij} G^j_i$$

which is identically satisfied when $-\rho_B = p_B$ as can be seen using (3.16).

Hence, when the bulk source is a Cosmological constant, any set of functions a, n, and b satisfying (3.18) or more explicitly

$$\left(\frac{\dot{a}}{na}\right)^2 = \frac{1}{6}\kappa^2 \rho_B + \left(\frac{a'}{ba}\right)^2 - \frac{\kappa}{a^2} + \frac{C}{a^4} \tag{3.19}$$

[12] If a is, for example, an exponential function of y and τ as $e^{gy\tau}$ this assumption is no more valid.

together with $G_{05} = 0$ will locally be solution of Einstein equations in five dimensions, away from $y = 0$.

At the position of the brane, $y = 0$, the junctions condition should be taken into account (see section 2.2) which here take the form

$$\frac{[a']}{a(0)b(0)} = -\frac{\kappa^2}{3}\rho_b \tag{3.20}$$

$$\frac{[n']}{n(0)b(0)} = -\frac{\kappa^2}{3}(3p_b + \rho_b)\ , \tag{3.21}$$

where $[Q] = Q(0^+) - Q(0^-)$ defines the jump of the function Q across $y = 0$.

Assuming the symmetry $y \to -y$ (for simplicity, which is not necessarily imply the presence of another brane), the junction condition (3.20) can be used to compute a' at the two sides of the brane, and by continuity, when $y \to 0$, one sees that the equation (3.19) implies the generalization of the first Friedmann equation (after setting $n(0) = 1$ by a suitable change of time):

$$\frac{\dot{a}(0)^2}{\dot{a}(0)^2} = \frac{\kappa^2}{6}\rho_B + \frac{\kappa^4}{36}\rho_b^2 + \frac{C}{a(0)^2} - \frac{\kappa}{a(0)^2}\ . \tag{3.22}$$

Note from the above equation that the bulk energy density enters linearly while the brane energy density enters quadratically, and the Cosmological evolution depends on the constant C, an effective radiation term, which is determined by the initial conditions can be constrained by BBN [130].

Equation (3.22) is enough to study the cosmological evolution on the brane whatever the metric outside looks like and whether or not $\dot{b} = 0$.

However, let us assume $\dot{b} = 0$ and set it to $b = 1$. Using the equation $G_{05} = 0$ one can express n in terms of a according to

$$\frac{\dot{a}}{n} \equiv \alpha(t)\ ,$$

where α depends solely on time. Inserting this into (3.16) leads to

$$\alpha^2 + \kappa - (aa')' = \frac{\kappa^2}{3}a^2\rho_B$$

which applies in the bulk at both sides of the brane. Integrating it over y gives

$$a^2 = A\cosh(\sigma y) + B\sinh(\sigma y)\ , +C$$

with

$$\sigma = \sqrt{-\frac{2\kappa^2}{3}\rho_B} \ ,$$

for $\rho_B < 0$.

For $\rho_B > 0$, the solution is

$$a^2 = A\cos(\sigma y) + B\sin(\sigma y) + C \ ,$$

with

$$\sigma = \sqrt{\frac{2\kappa^2}{3}\rho_B} \ .$$

For $a = 0$ it is

$$a^2 = (\alpha^2 + \kappa)y^2 + Sy + E \ .$$

where the functions $A, B, C, S,$ and E are functions of time (or constants) and can be determined in terms of the input parameters of the theory.

In order to see more explicitly the Cosmological evolution on the brane, let us decompose the energy density on the brane into two parts:

$$\rho_b = \rho + \rho_\Lambda \ ,$$

where ρ_Λ is a constant, and ρ stands for the contribution of matter on the brane. Substituting in (3.22) we get

$$\frac{\dot{a}(0)^2}{\dot{a}(0)^2} = \frac{\kappa^2}{6}\rho_B + \frac{\kappa^4}{36}\rho_\Lambda^2 + \frac{\kappa^4}{18}\rho_\Lambda\rho + \frac{\kappa^4}{36}\rho^2\frac{C}{a(0)^2} - \frac{\kappa}{a(0)^2} \ .$$

If we chose

$$\frac{\kappa^2}{6}\rho_B + \frac{\kappa^4}{36}\rho_\Lambda^2 = 0 \ , \qquad (3.23)$$

we see that for $\rho \ll \rho_\Lambda$ the standard Cosmology is recovered [129, 130] (the solution for the case $\rho = 0$ is also in [196]).

Assuming the equations of state to be $\rho = \omega p$, the energy density and pressure become

$$\rho = \rho_c \left(\frac{a(0)}{a_c}\right)^{-q} \quad , \quad q = 3(1+\omega) \ ,$$

where ρ_c and a_c are constants. The scale factor describing the time evolution on the brane, when (3.23) is satisfied, behaves as [130]

$$a(0,t) = a_c \kappa^{2/q} \rho_c^{1/2} \left(\frac{q^2}{72} \kappa^2 \rho_\Lambda t^2 + \frac{q}{6} t \right)^{1/q} ,$$

and therefore the evolution of the Universe will be non-conventional $a(t) \sim t^{1/q}$ at early times while at late times it is described by the standard Cosmology $a(t) \sim t^{2/q}$ (further discussion can be found in [130]).

3.5 Stability of the scale of compactification

So far, the examples explained in sections 3.3 and 3.4 represent equilibrium configurations, what remains is to check their stability against the perturbations of the metric and other background fields.

The procedure of checking classical perturbative stability of a solution to Einstein equations was explained in section 1.6 under1.6.3. There, it was argued that one way to see whether the solution is stable or not is by looking at the spectrum and making sure it contains no tachyons or ghosts. The absence of such modes would indicate that the radius of compactification would only oscillate around the background value and that the configuration, say $M \times K$, will not acquire a shape different from the initial ansatz. [13]

For example, the theory provided in section 1.6 (of reference[53]) is free of both tachyons and negative norm fields, in addition to having a fixed value of the radius in of S^2 in terms of the other parameters in the theory (equation (1.16)),[14] and therefore the radius of compactification is expeted to be oscilating around the value $(8M^4 g^2/n^2)^{-1}$. It is may also be helpful to note that in this model there is no massless mode in the gravity sector to associate with the *radion*, the radius of compactification.

In general, this not the case. The radion is usually massless and it has no potential (or a flat one). Therefore, there is no reason why the scale

[13]It was argued in [197] that even a manifold which is stable against all deformations will become unstable if the temperature is raised to a critical value, which is independent of the shape of the manifold, as on rather general grounds the free energy could become increasingly negative above the critical temperature. The phase transition due to this phenomena indicates the necessity of having time-dependence for the field associated with the evolution of the compact space.

[14]There is no analogue for this relation in the original Kaluza-Klein compactification performed on a circle since S^1 is flat.

of its vacuum expectation value should be fixed at a specific point which is moreover close to the electroweak scale. When put in a cosmological context, the value of the radius should not vary much as this may have strong impact on the Universe as observed today. Therefore, the ground state state of the radion field has to have a determind value for at least two reasons. The first is that there is no experimental evidence for a variation of the fundamental constants [194]; according to observations the internal space(s) should be static or nearly static at least from the time of nucleosynthesis. The second is that we wish to have a lower gravity scale, preferably as low as few TeV, as a cutoff for the standard model of particle physics. Therefore, a good stabilization mechanism for the radion should be a part of any realistic model.

In superstring theories, the fundamental physical constants are related to the vacuum expectation values of the moduli fields which are defined by the shape and size of internal space(s) of compactification. In these theories, some stabilization mechanisms are proposed, for example in [198, 199], and recently it was shown that the stabilization can be enhanced due to the coupling of the dilaton, the string coupling constant $e^{\langle \phi \rangle}$ to the kinetic energy of ordinary matter fields [219]. In the context of Kaluza-Klein theory this issue was, and is still, subject to numerous investigations (see for instance [200]-[219] for tensor product compactification, and [220]-[229] for warped).

Another way to stabilize the radion, than the one provided in [53], is by generating a potential for the radion, through an *ad hoc* coupling to a scalar field for example (as in [220]), and checking whether this potential has a minimum which could correspond to the desired value of the radion. [15] In fact, the background solution for the scalar field in [220] is, in a sense, similar to the Yang-Mills background (1.15), and the relation determening the value of the radion in [220] in terms of the values of the minima of scalar potential and other paramters of the model is somehow analogous to equation (1.16).

In [217] the issue of stabilization was discussed within a cosmological time evolution context, which is essentially the same idea. Again a potential was introduced for this purpose, with the background metric similar to (3.1), and it was argued that the fluctuations of the radion were oscillatory with positive frequency, around a value r_0, and this subsequently lead to the understanding that r_0 is a minumum of the potential. Of course one has to check that *all* physical modes of the model have the feature.

[15] In [220] the back reaction of the scalar field into the metric was not taken into account, for a discussion about this point see [227].

We believe that the method in section 1.6 is nice since it does not involve *ad hoc* potentials to stabilize the radion as usually alternative stability checks require. It may even be that higher order corrections result in a potential [230] which could lead to a better understanding to the origin of stabilization potentials, which may also shed some light on the way how supersymmetry is broken in superstring theories, and to possible relation of the value of the radion at the minimum with the rest of parameters in the theory. This problem remains challenging.

3.6 Proton stability

Another problematic issue arising from models with large extra (and also infinite) dimensions is the potentially too fast proton decay. Generally, the proton decay rate due to gravitational effects of, *e.g.* dimension 5 operators, is given by

$$\tau_p^{-1} \sim \frac{m_p^5}{M_P^4}.$$

which we know from the bounds on proton life time it should satisfy $\tau_p \lesssim 10^{33}$ years. This implies a bound, too well known in Grand Unified Theories, that the smallest mass which could substitute M_P and does not lead to an unacceptably fast proton decay is $10^{15} - 10^{16}$GeV. Therefore one does generally expect a problem with proton decay at low fundamental scales $\sim$TeV.

If the particle mediating proton decay, say via the dimension five non-renormalizable operator $qqql/m_X$, has a mass m_X $\sim$TeV range, it should be incredibly weakly coupled in order not to lead to proton decay. A way to suppress such an operator could be by imposing certain symmetries [152] (see also [231]-[235]. Since global symmetries are are expected to be broken by quantum effects, those symmetries better be gauged. Moreover, the procedure of orbifold compactification with fixed point (which is used by some in order to achieve chiral fermions, project out unwanted states,...*etc*) is allowed only in a string theory context, and therefore a neater way to achive acceptable proton decay rate could be by choosing an appropriate Yang-Mills gauge group in D-dimensions together with an appropriate compact space. It is possible to construct examples of such compactification where operators leading to proton decay are forbidden because the quantum numbers of quarks and leptons can not form invariant couplings under the unbroken symmetry (could be the unbroken part of the gauge group, the isometry group, or both) as can be checked with the $U(6)$ model in chapter 4. This possibility will be discussed elsewhere.

We review here a simple mechanism for proton stability suggested in [70], as this will serve in explaining chapter 3.7. [16] The same idea of fermion localization [12] explained in section 2.5.2 can be adopted to guarantee proton stability [70]: quarks and leptons are localized at two slightly displaced point along the fifth direction, and this suppressed their mutual interaction. To see how, let us give leptons and baryons the following 5-dimensional masses

$$(m_0)_l = 0 \ , \ (m_0)_b = m_0 \ , \tag{3.24}$$

which correspond to the localizations [17]

$$y_l = 0 \ , \ y_b = \frac{m_0\sqrt{2\lambda_0 \widetilde{M}_0}}{\mu_0^2} < \frac{1}{\mu_0} \ . \tag{3.25}$$

The shape of the fermion wave functions along the fifth dimension is the one in (2.31) up to a normalization. It can be cast in an explicit and simple form if we consider the limit $y_b \ll 1/\mu_0$, in which the effect of the plateau for $y > 1/\mu_0$ can be neglected: [18]

$$\begin{aligned} f_l(y) &= \left(\frac{\mu_0^2}{\sqrt{2\lambda_0 \widetilde{M}_0 \pi}}\right)^{1/4} \exp\left\{-\frac{\mu_0^2 y^2}{2\sqrt{2\lambda_0 \widetilde{M}_0}}\right\} \\ f_b(y) &= \left(\frac{\mu_0^2}{\sqrt{2\lambda_0 \widetilde{M}_0 \pi}}\right)^{1/4} \exp\left\{-\frac{\mu_0^2 \left(y - y_b\right)^2}{2\sqrt{2\lambda_0 \widetilde{M}_0}}\right\} \ . \end{aligned} \tag{3.26}$$

We assume the Standard Model to be embedded in some theory which, in general, contains some additional bosons X whose interactions violate baryon number conservation. If it is the case, the four fermion interaction $qq \longleftrightarrow ql$

[16] See also [122, 152, 178] for alternative suggestions.

[17] The last inequality in the next expression comes from (2.36). We assume quarks of different generations to be located in the same y position in order to avoid dangerous FCNC mediated by the Kaluza-Klein modes of the gluons [189].

[18] This is also the limit in which the approximation (2.35) is valid.

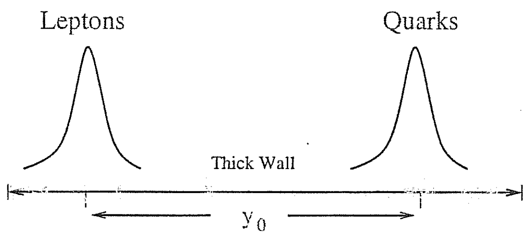

Figure 3.2: The wave functions of quarks and leptons in the transverse space to the brane world.

can be effectively described by

$$\int d^4x\, dy\, \frac{q\,q\,q\,l}{\Lambda\, m_X^2}\ , \tag{3.27}$$

where m_X is the mass of the intermediate boson X and Λ is a parameter of mass dimension one related to the five-dimensional coupling of the X-particle to quarks and leptons. This scattering is thus suppressed by [19]

$$\begin{aligned} I &= \frac{1}{\Lambda m_X^2} \int dy \frac{\mu_0^2}{\pi\sqrt{2\lambda_0 \widetilde{M}_0}} \exp\left\{ -\frac{\mu_0^2/2}{\sqrt{2\lambda_0\widetilde{M}_0}} \left[y^2 + 3\,(y-y_b)^2\right] \right\} \\ &= \frac{\mu_0}{\Lambda m_X^2 \sqrt{2}\,\pi \left(2\lambda_0 \widetilde{M}_0\right)^{1/4}} \exp\left\{ -\frac{3\left(2\lambda_0\, \widetilde{M}_0\right)^{1/2}}{8} \frac{m_0^2}{\mu_0^2} \right\} \ . \end{aligned} \tag{3.28}$$

Current proton stability [236] requires $I \sim (10^{16}\,\mathrm{GeV})^{-2}$, that is

$$\frac{m_0}{\mu_0} \gtrsim \frac{\sqrt{200 - 6\,\mathrm{Log}_{10}\left(\frac{\Lambda m_X^2}{\mu_0} / \mathrm{GeV}^2\right)}}{\left(2\lambda_0 \widetilde{M}_0\right)^{1/4}} \ . \tag{3.29}$$

The numerator in the last equation is quite insensitive to the mass scales of the model, and – due to the logarithmic mild dependence – can be safely

[19]From the approximation (2.35), only the squared difference of the five-dimensional masses affects the suppression factor. For this reason, the above choice $(m_0)_l = 0$ was only done in order to simplify notation and does not have any physical meaning.

assumed to be of order 10. For definiteness, we will thus fix it at the value of 10 in the rest of this chapter. Conditions (2.36) and (3.29) give altogether

$$\frac{10\,\mu_0}{\left(2\,\lambda_0\,\widetilde{M}_0\right)^{1/4}} \lesssim m_0 \lesssim \frac{\mu_0}{\left(2\,\lambda_0\,\widetilde{M}_0\right)^{1/2}} \,, \tag{3.30}$$

that we can rewrite

$$\begin{aligned} 2\lambda_0\tilde{M}_0 &\lesssim 10^{-4} \\ \frac{m_0}{\mu_0} &\gtrsim 10^2 \quad . \end{aligned} \tag{3.31}$$

Notice that the last limit in equations (3.31) is stronger than the one given in [70] where proton stability is achieved if the ratio of the massive scales of the model is of order 10. However, in ref. [70] the field Φ simply scales linearly as a function of y, while we expect that whenever a specific model is assumed, conditions analogous to our (2.36) and (3.31) should be imposed [237].

3.7 Baryogenesis with Low Scale Gravity

In this section we discuss a mechanism for generating baryon number violation in a model with one extra dimensions. As explained in section 3.6, the Rubakov-Shaposhnikov localization mechanism [12] was used in [70] in order to insure proton stability at zero temperature.[20] The baryon number is almost conserved at zero temperature due to the slight overlap of the wave functions of quarks and lepton in the internal space due to their different localization along the fifth direction. The suppression decreases due to finite temperature effects leading to bayon number violation at earlier times in the history of the Universe.

Here, we review [237] where the idea of [70] is adopted for generating operators leading to baryon number violation at finite temperature, while such operators remain appropriately suppressed at zero temperature.

[20]Beside providing an way around fast proton decay in theories with low fundamental scales, this idea has further applications as one may reproduce the entries of the Cabibbo-Kobayashi-Maskawa matrix just placing quarks of different generations at different and appropriately chosen positions (this requires the presence of at least two extra dimensions, as shown in [238]) without assuming any hierarchy in the original Yukawa couplings of the fermions to the Higgs field. Discussion for collider signature is in [239].

The starting point is to assume that quarks and leptons are now (*i.e.* at zero temperature) sufficiently apart in the extra space. However, we wonder if finite temperature effects could change this picture increasing the interactions between quarks and leptons at early times. This would render proton stability now compatible with baryogenesis in the early Universe. An exact computation of the corrections on a solitonic background presents some technical difficulties. Moreover, relying on a perturbative analysis at a scale close to the cut-off of the theory may be unsafe. For this reason, we do not give a precise final value for the temperature necessary to achieve the observed baryon asymmetry. We limit ourselves to a perturbative analysis made on dimensional arguments, which anyhow indicates that finite temperature effects should indeed increase the interactions between baryons and leptons.

As it is well known, baryon number violation alone is not sufficient for baryogenesis. To present a more complete analysis we discuss a particular model reminiscent of GUT baryogenesis. In doing so, we meet another problem typical of these brane scenarios. Due to the low energy densities involved, the expansion rate of the Universe is always very small. If baryogenesis originates from the decay of a boson X, the out of equilibrium condition requires m_X much higher than the physical cut-off of brane models. This problem can be overcome for example if the temperature of the Universe never exceeds m_X and if these bosons are created non-thermally (for instance at preheating). This and some other options are briefly discussed in the last section.

3.7.1 Thermal correction to the coefficients

Once the localization mechanism is incorporated in a low energy effective theory – as the system described above may be considered –, one can legitimately ask if thermal effects could play any significant role. We are mainly interested in any possible change in the argument of the exponential in (3.28), that will be the most relevant for the purpose of baryogenesis. For this reason, we introduce the dimensionless quantity

$$a(T) = \frac{m(T)^2}{\mu(T)^2}\sqrt{2\,\lambda(T)\widetilde{M}(T)} \quad . \tag{3.32}$$

From (3.29) and (3.31), we can set $a(0) \simeq 100$ at zero temperature. Thermal effects will modify this value. There are however some obstacles that one meets in evaluating the finite temperature result. Apart from some technical difficulties arising from the fact that the scalar background is not constant, the main problem is that nonperturbative effects may play a very relevant role at high temperature. As it is customary in theories with extra dimensions, the model described by (2.19) and (2.33) is nonrenormalizable and one expects that there is a cut-off (generally related to the fundamental scale of gravity) above which it stops holding. Our considerations will thus be valid only for low temperature effects, and may be assumed only as an indication for what is expected to happen at higher temperature.

Being aware of these problems, by looking at the dominant finite-temperature one-loop effects, we estimate the first corrections to the relevant parameters to be

$$\begin{aligned} \lambda\,(T) &= \lambda_0 + c_\lambda\,\frac{T}{\widetilde{M}_0^2} \\ \widetilde{M}\,(T) &= \widetilde{M}_0 + c_{\widetilde{M}}\,T \\ m\,(T) &= m_0 + c_m\,\frac{T^2}{\widetilde{M}_0} \\ \mu^2\,(T) &= \mu_0^2 + c_\mu\,\frac{T^3}{\widetilde{M}_0}\ , \end{aligned} \tag{3.33}$$

where the c's are dimensionless coefficients whose values are related to the exact particle content of the theory.

In writing the above equations, the first of conditions (3.31) has also been taken into account. For example, both a scalar and a fermionic loop contribute to the thermal correction to the parameter λ_0. While the contribution from the former is of order $\lambda_0^2 T$, the one of the latter is of order $T/\widetilde{M}_0^2$ and thus dominates. [21]

Substituting equations (3.34) into eq. (3.32), we get, in the limit of low temperature,

$$a\,(T) \simeq a\,(0)\cdot\left[1+\frac{T}{\widetilde{M}_0}\left(\frac{c_\lambda}{2\,\lambda_0\,\widetilde{M}_0}+\frac{c_{\widetilde{M}}}{2}+\frac{2\,c_m\,T}{m_0}-\frac{c_\mu\,T^2}{\mu_0^2}\right)\right]\ . \tag{3.34}$$

[21] Notice also that with our choice (3.24) loops with internal leptons dominate over loops with internal quarks, since the former have vanishing 5-dimensional bare mass and thus are not Boltzmann suppressed. However, although this choice is the simplest one, one may equally consider the most general case where all the fermions have a nonvanishing five-dimensional mass.

From the smallness of the quantity $\lambda_0 \widetilde{M_0}$ (see (3.31)) we can safely assume (apart from high hierarchy between the c's coefficients that we do not expect to hold) that the dominant contribution in the above expression comes from the term proportional to c_λ.

We thus simply have

$$a\,(T) \simeq a\,(0) \left(1 + c_\lambda \, \frac{T}{2\,\lambda_0\,\widetilde{M_0^2}} \right) \ . \qquad (3.35)$$

We notice that the parameter c_λ, being related to the thermal corrections to the ϕ^4 coefficient due to a fermion loop, is expected to be *negative* [218]: the first thermal effect is to decrease the value of the parameter $a(T)$, making hence the baryon number violating reactions more efficient at finite rather than at zero temperature.

There is another effect which may be very crucial at finite temperature, linked to the stability of the Z_2 symmetry. When a temperature is turned on, we generally expect the formation of a fermion–antifermion condensate $\langle \bar{\psi}\,\psi \rangle \neq 0$. If it is the case, the Yukawa coupling $\Phi\,\bar{\psi}\,\psi$ in the Lagrangian (2.19) renders one of the two vacua unstable. While this leads to an instantaneous decay of the kink configuration, a kink–antikink system could have a sufficiently long lifetime provided the two objects are enough far apart.

Let us conclude this section with an important remark. While in this discussion we have considered only the localization of fermions along one extra dimension, almost everything we have said can be generalized if two or more additional dimensions are present [22]. The index theorem guarantees [240, 241] indeed the possibility of localizing fermions on a topological defect of an arbitrary dimension. Just to give an example, let us consider the localization on a Nielsen–Olesen [242] vortex in the case of two extra–dimensions, as it is discussed in [9]. Also in this case, one can localize quarks and leptons at two different positions (actually along different circles about the center of the vortex). Once again, proton stability requires conditions completely analogous to conditions (3.31) here discussed. Of course the calculation of thermal corrections gives different results, since the dimension of the couplings of the model changes according to the number of spatial dimensions. However, also in the case of the two-dimensional vortex, the qualitative result turns out to be identical to what has been derived in the one-dimensional

[22] This is mandatory in the scenario [9], since the presence of only one large extra dimension is phenomenologically excluded.

case: the most significant effect comes from the variation of the coefficient λ of the ϕ^4 interaction, and it is in the direction of enhancing the quark–lepton interaction with increasing temperature.

3.7.2 A model for baryogenesis

We saw in the previous section that thermal effects may increase the rate of baryon number violating interactions of the system. This is very welcome, since a theory which never violates baryon number cannot lead to baryogenesis and thus can hardly reproduce the observed Universe. Anyhow baryon number violation is only one of the ingredients for baryogenesis, and the aim of this section is to investigate how the above mechanism can be embedded in a more general context.

A particular scheme which may be adopted is baryogenesis through the decay of massive bosons X. [23] This scheme closely resembles GUT baryogenesis, but there are some important peculiarities due to the different scales of energy involved. In GUT baryogenesis the massive boson X , coupled to matter by the interaction $g\, X\, \psi\, \bar{\psi}$, has the decay rate

$$\Gamma \simeq \alpha\, m_x \quad , \quad \alpha = \frac{g^2}{4\,\pi} \quad . \tag{3.36}$$

An important condition is that the X boson decays when the temperature of the Universe is below its mass (out of equilibrium decay), in order to avoid thermal regeneration. From the standard equation for the expansion of the Universe,

$$H \simeq g_*^{1/2}\, \frac{T^2}{M_{\rm P}} \tag{3.37}$$

(where g_* is the number of relativistic degrees of freedom at the temperature T), this condition rewrites

$$m_X \gtrsim g_*^{-1/2}\, \alpha\, M_{\rm P} \quad . \tag{3.38}$$

If X is a Higgs particle, α can be as low as 10^{-6} . Even in this case however the X boson must be very massive. In principle this may be problematic in

[23] We may think of these bosons as the intermediate particles which mediate the four fermion interaction described by the term (3.27).

the theories with extra dimensions we are interested in, which have the main goal of having a very low fundamental scale.

There are some possibilities to overcome this problem. One is related to a possible deviation of the expansion of the system from the standard Friedmann law. This is a concrete possibility, since the exact expansion law is very dependent on the particular brane model one is considering and on the fact that the size of the compact dimension is or is not stabilized. For example, we will show in the next chapter that the Randall-Sundrum model [20] with a stabilized radion can have (depending on the energy density on the zero brane) an expansion rate higher than the standard one for temperatures close to the cut-off scale of the system (that is TeV). This accelerated expansion could in principle favor the out of equilibrium condition for the X bosons.

However this issue is very dependent on the specific cosmological scenario adopted, and one may be interested in more general solutions for the out of equilibrium problem. One very natural possibility is to create the X particles non thermally and to require the temperature of the Universe to have been always smaller than their mass m_X. In this way, one kinematically forbids regeneration of the X particles after their decay. In addition, although interactions among these bosons can bring them to thermal equilibrium, chemical equilibrium cannot be achieved.

Nonthermal creation of matter has raised a considerable interest in the last years. In particular, the mechanism of preheating has proven quite successful, as we have discuss in the first part of this work. The efficiency of preheating has been exploited in the work [243] to revive GUT baryogenesis in the context of standard four-dimensional theories. Here, we will not go into the details of the processes that could have lead to the production of the X bosons. Rather, we will simply assume that, after inflation, their number density is n_X. To simplify our computations, we will also suppose that their energy density dominates over the thermal bath produced by the perturbative decay of the inflaton field.[24]

[24] An alternative way to overcome the bound (3.38) relies on the fact that, as observed in the works [244, 245], the maximal temperature reached by the thermal bath during reheating can indeed be much higher than the final reheating temperature. In this case, even if $T_{\rm rh}$ is considerably lower than m_X, X particles can be produced in a significant amount, and the out of equilibrium condition is easily achieved. However, the treatment of this mechanism is in our case somewhat different from the one given in [244]: due to the slowness of the expansion of the Universe, the X bosons will decay before the freeze

Just for definiteness, let us consider a very simple model where there are two species of X boson which can decay into quarks and leptons, according to the 4-dimensional effective interactions [25]

$$g\, X\, \bar{q}\, \bar{q} \quad , \quad g\, e^{-a/4}\, X\, l\, q \;, \tag{3.39}$$

where (remember the suppression given by the different localization of quarks and leptons) the quantity a is defined in eq. (3.32). Again for definiteness we will consider the minimal model where no extra fermionic degrees of freedom are added to the ones present in the Standard Model. Moreover we will assume $B-L$ to be conserved, even though the extension to a more general scheme can be easily performed.

The decay of the X bosons will reheat the Universe to a temperature that can be evaluated to be

$$T_{\rm rh} \simeq \left(\frac{30}{\pi^2} \frac{m_X\, n_X}{g_*} \right)^{1/4} \;. \tag{3.40}$$

Since we do not want the X particles to be thermally regenerated after their decay, we require $T_{\rm rh} m_X$, that can be rewritten as an upper bound on n_X

$$n_X \lesssim 30 \left(\frac{g_*}{100} \right) m_X^3 \;. \tag{3.41}$$

Another limit comes from the necessity to forbid the B violating four fermion interaction (3.27) to erase the B asymmetry that has been just created by the decay of the X bosons. We thus require the interaction (3.27) to be out of equilibrium at temperatures lower than $T_{\rm rh}$. From eq. (3.28) we see that we can parameterize the four fermion interaction with a coupling $g^2\, e^{-3\,a/8}/m_X^2$. Hence, the out of equilibrium condition reads

$$g^4\, e^{-3\,a/4} \lesssim g_* \frac{m_X}{M_{\rm P}} \left(\frac{m_X}{T_{\rm rh}} \right)^3 \;. \tag{3.42}$$

out of their production. The final baryon asymmetry cannot be estimated with the use of the formulae of [244], which are valid only if the decay of the X particles occurs well after their freeze out.

[25] Here one can not really adopt a GUT regime, since in unified theories the quarks and leptons are in the same multiplets which is not consistent with the idea of separation in the extra space.

One more upper bound on the reheating temperature comes from the out of equilibrium condition for the sphalerons. This requirement is necessary only if one chooses the theory to be $B-L$ invariant, while it does not hold for $B-L$ violating schemes. We can approximately consider the sphalerons to be in thermal equilibrium at temperatures above the electroweak scale. Thus, if $B-L$ is a conserved quantity, we will require the reheat temperature to be smaller than about 100 GeV.

If one neglects the presence of the thermal bath prior to the decay of the X bosons, the very first decays will be only into couples of quarks, since the channel into one quark and one lepton is strongly suppressed by the $e^{-a(T=0)}$ factor due to the fact that the kink is not modified by any thermal correction. However, the decay process is not an instantaneous event. It is shown in [244] that the particles produced in the very first decays are generally expected to thermalize very rapidly, so to create a thermal bath even when most of the energy density is still stored in the decaying particles. [26] The temperature of this bath can even be considerably higher than the final reheating temperature. The presence of the heat bath modifies in turn the shape of the kink, as shown in the previous section, and we can naturally expect that this modification enhances the B violating interactions.

If the energy density of the Universe is dominated by the X bosons before they decay, one has

$$\eta_B \simeq 0.1 \; (N_X \, T_{\rm rh}/m_X) \; \langle r - \bar{r} \rangle \; , \tag{3.43}$$

where N_X is the number of degrees of freedom associated to the X particles and $\langle r - \bar{r} \rangle$ is the difference between the rates of the decays $X \to q\, l$ and $\bar{X} \to \bar{q}\, \bar{l}$.

We denote with X_1 and X_2 the two species of bosons whose interactions (3.39) lead to baryon number violation, and parameterize by ϵ the strength of CP-violation in these interactions. Considering that e^{-2a} is always much smaller than one, we get [246]

$$\langle r - \bar{r} \rangle \sim 3\, g^2\, e^{-a/2}\, \epsilon \, {\rm Im}\, {\rm I}_{SS}\, (M_{X_1}/M_{X_2}) \; , \tag{3.44}$$

where the function ${\rm Im}\, {\rm I}_{SS}(\rho) = [\, \rho^2 \; Log(1 + 1/\rho^2) - 1\,] \,/\, (16\, \pi)$ can be estimated to be of order $10^{-3} - 10^{-2}$. It is also reasonable to assume $\epsilon \sim 10^{-2} - 1$.

[26] As shown in [244], what is called the reheating temperature is indeed the temperature of the thermal bath when it starts to dominate. After the first decays, the temperature of the light degrees of freedom can be even much higher than $T_{\rm rh}$.

Collecting all the above estimates, and assuming N_X to be of order 10, we get

$$\eta_B \simeq \left(10^{-5} - 10^{-2}\right) g^2 \frac{T_{\rm rh}}{m_X} e^{-a(T_{\rm rh})/2} \ . \tag{3.45}$$

From the requirement $T_{\rm rh} \lesssim m_X$ we get an upper limit on the baryon asymmetry

$$\eta_B \lesssim \left(10^{-5} - 10^{-2}\right) g^2 e^{-a/2} \ , \tag{3.46}$$

where the factor $a(T)$ has to be calculated for a value of T of the order of the reheating temperature.

We get a different limit on η_B from the bound (3.42): assuming $m_X \sim$ TeV and $g_* \sim 100$ indeed one obtains

$$\eta_B \lesssim \left(10^{-6} - 10^{-10}\right) g^{2/3} e^{-a/4} \ . \tag{3.47}$$

Since the observed amount of baryon asymmetry is of order 10^{-10}, even in the case of maximum efficiency of the process (that is, assuming maximal CP violation and $g \sim 1$), the bounds (3.46) and (3.47) imply that $a(T_{\rm rh}) \sim 40$. Unfortunately, the temperature at which the condition $a(T) \sim 40$ occurs cannot be evaluated by means of the expansion of eq. (3.35), that have been obtained under the assumption $|a(T) - a(0)| \ll a(0)$. On the other hand, it is remarkable that our mechanism may work with a ratio $a(T_d)/a(0)$ of order one. We thus expect that a successful baryogenesis may be realized for a range of the parameters of this model which – although not possible to evaluate through a perturbative analysis – should be quite wide and reasonable.

As we have discussed in the previous chapter, in scenarios with large extra dimensions and low scale gravity, the maximal temperature reached by the Universe after inflation is strongly bounded from above in order to avoid overproducing Kaluza-Klein graviton modes, which may eventually contradict cosmological observations [21]. For instance, in models with two large extra dimensions the reheating temperature cannot exceed much 1MeV (unless the fundamental scale M is unnaturally high.

This value is too low for the scenario we are describing since η_B is proportional to the ratio $T_{\rm rh}/m_X$, and hence the observed amount of baryons would be reproduced at the price of an unnaturally small value of $a(T_{\rm rh})$. However, other schemes with extra dimensions exist where the bounds on $T_{\rm rh}$ are less severe. For example, in the proposals [20, 59] the mass of the first graviton KK mode is expected to be of order TeV. The reheating temperature can thus safely be taken to be of order $10 - 100\,\mathrm{GeV}$.

There are of course several possible baryogenesis schemes alternative to the one just presented (see for instance [247, 248]). A possible option which also requires a minimal extension to the Standard Model could be to achieve the baryon asymmetry directly through the 4 fermions interactions $q + q \leftrightarrow q + l$ in the thermal primordial bath. The out of equilibrium condition may be provided by the change of the kink as the temperature of the bath decreases. [27] What may be problematic is the source of CP violation which may lead the creation of the baryon asymmetry. A possibility in this regard may be provided by considering a second Higgs doublet, but the whole mechanism certainly deserves a deep analysis by itself.

[27] This condition may be easily achieved due of the exponential dependence of the rate of this process on the temperature, see equations (3.28) and (3.35).

Chapter 4

Gauge Hierarchy, Electroweak Spontaneous Symmetry Breaking, & Fermion Chirality Linked

4.1 Introduction

In the SM, the hierarchy of mass scales is present at both classical and quantum levels. At the tree level, there is a huge difference between the scales associated with the electroweak and the gravitational interactions, $M_W/M_p \sim 10^{-17}$.

If quantum corrections were not to significantly alter the value of the Higgs $(\text{mass})^2$ computed classically, one could simply consider the number 10^{-17} above as one of the many extreme ratios existing in nature (like the mass ratio of a feather and of an elephant). However, there are huge quadratic corrections to the Higgs $(\text{mass})^2$ at the quantum level due to the fact that the Higgs particle is described by a fundamental scalar field. These corrections change the classical value of the Higgs $(\text{mass})^2$ by many orders of magnitude, and adjusting the value back to its classical one requires a fine-tuning of order 10^{-34} in the case of a gravitational cutoff, and another of order 10^{-26} in GUTs.

Supersymmetry is a very good example where the problem is nicely solved at the quantum level, in fact once the SUSY breaking scale is set classically, usually taken to be around few TeV, only logarithmic corrections will alter this value. The key principle for the absence of quadratic divergences in this theory is that the Higgs mass is protected by the symmetry above the cutoff. Being in a multiplet with chiral fermions, the Higgs is deemed to be massless, due to chiral symmetry, as long as SUSY is unbroken.

The idea of large extra dimensions explains *perturbative stability* of the weak scale versus the Planck scale,[1], by lowering the cut-off of the theory. Nevertheless, it does not address the issue of *sensitivity* of the Higgs mass to the ultraviolet cut-off. This is an important issue from the point of view of the low energy calculability, and is the central point to be addressed in [249].

In the chapter we present the work [249], where the above key principle is employed, though without supersymmetry, to explicit models where the Higgs mass is protected by a gauge symmetry in $4+d$ dimensions. It is argued that identifying the electroweak Higgs particle with the extra components of the gauge field in $4+d$ dimensions provides a solution to the hierarchy problem.[2]

The absence of ultraviolate quadratic divergences is due to to the manifistation of exact gauge symmetry at energies beyond the cutoff.[3] The higher-dimensional gauge symmetry plays the role of the "protector" in this case forbidding the Higgs $(\text{mass})^2$ from receiving cut off dependent (and hence local) corrections. This symmetry is spontaneously broken by compactification.

The idea is implemented within explicit models which also provide a link between fermion chirality and electroweak symmetry breaking in four dimensions.

[1] Provided that a convincing stabilization mechanism for the radius of compactification is found.

[2] An alternative approach was discussed in [250], where the Higgs mass is controlled by a higher-dimensional extended supersymmetry, spontaneously broken globally by Scherk-Schwarz mechanism. An alternative to the Higgs mechanism was suggested in [251]. The idea can also be thought of as performing a compactification on a manifold with the first betti number $b_1(Y) = 0$ (as for the case of compact hyperbolic manifolds of $d > 3$, see [59, 35, 60]).

[3] Soon after the proposal in [249], several models appeared, [252, 253, 254], using the same idea in explaining the finiteness of the Higgs mass.

4.2 The proposal

After compactifying extra dimensions on a monopole background, via mechanism of [50], some of the extra components of the gauge fields become tachyonic and spontaneously break the electroweak symmetry. Their quantum numbers are identical to those of SM Higgs doublet. Notice that the monopole background is essential for generating the families of chiral fermions in four-dimensions, and therefore is doing a double job. If the tachyonic mass is a tree level effect the natural scale of the symmetry breaking is $\sim 1/a$, the inverse radius of extra compact space, since the only source of spontaneous breaking of the higher-dimensional gauge invariance is the compactification itself. In other words, since in the infinite volume limit $a \to \infty$ the full higher-dimensional gauge invariance must be recovered, the weak scale must go as $M_W \sim 1/a$. Thus in this case the size of extra dimensions to which gauge fields can propagate should be $a \sim 1/\text{TeV}$ (as in [14]). However, it is important to stress that in order for the theory not to become infinitely strongly coupled above the compactification scale, the cut-off M must be lowered as in [9], possibly via increase of the volume of some additional dimensions to which only gravity can spread. This issue will not be discussed here.

The models constructed in [249] were in the direction of answering the following question: *"How close can one get the to the Standard Model, in four dimensions, by Kaluza-Klein compactifying large extra dimensions of an Einstein Yang-Mills theory coupled to fermions in $4+d$ dimensions?"*.

Eventually, the effective action of a "good" theory should provide us in four dimensions with:

- chiral fermions in 4 dimensions.
- standard model gauge group (or a group containing it), spontaneously broken to $U(1)_{em}$.
- fermions and Higgs particles in the correct representations of the standard model gauge group.
- a good solution to the hierarchy problem.
- correct quark and lepton masses.
- suppressed proton decay.

In the following, examples will explicitly given where the first four points mentioned above can be realized, with some hints to achieving the last two.

In a an attempt to approaching the answer of the above question, two principal ideas were proposed: Identifying the electroweak Higgs with the extra components of the Yang-Mills field as a solution for the hierarchy problem, and linking fermion chirality to the spontaneous electroweak symmetry breaking.

Approaching the hierarchy problem:

The suggestion to solving the hierarchy problem is that the electroweak Higgs particle is identified with the extra components of the gauge field in $4+d$ dimensions.

Note that this identification will not be meaningful, from the point of view of explaining the absence of quadratic divergences to the Higgs mass, unless it is proven that those extra components (which are a scalar in four dimensions) possess a ϕ^4 potential with a negative bilinear term so to derive the usual spontaneous symmetry breaking described by the SM. Not only that, this scalar filed should also have the appropriate Yukawa couplings with the fermions in the right SM group representations.

Suppose that we start from an Einstein Yang-Mills theory coupled to fermions on a $4+d$ dimensional manifold, W, $W = M_4 \times Y_d$. The field content to start with consists of a graviton, $g_{MN}(x,y)$, a gauge field $A^a_M(x,y)$ in the algebra of a Lie group G, and a fermion $\psi(x,y)$. Where $M, N = 0, 1, 2, ..., d+3$, $a = 1, ...\text{dim}G$, x and y are the coordinates on M_4 and Y_d respectively. We take Y to be a compact manifold with a typical volume of order TeV^d, and $\times$ to indicate a tensor product. Arguing that the Higgs particle is

$$H(x) \equiv A_\alpha(x) \qquad \alpha \in Y$$

implies a solution for the quantum instability of the electroweak scale as was first pointed out by [255, 256].[4]

Classically, the Planck scale is related to the fundamental $4+d$ dimensional gravity scale, as we mentioned previously, by the relation

$$M_P = a^{\frac{d}{2}}\, M^{\frac{d}{2}+1}$$

[4]An earlier attempt for obtaining spontaneous symmetry breaking in six dimensions is in [257].

When $Y_d = Y_1 \times Y_2 \times ... \times Y_n$, the a^d should be replaced by the product of the volumes of each manifold. As an example, we take

$$W = M_4 \times S^2 \times \mathbb{C}P^2$$

as in the model we discussed in [249] for quarks and leptons. In this example we have the gravity scale $M \sim 10^4$TeV, and hence new physics is expected to show off at around $1/a \sim 1$TeV. Whether one considers the cutoff to be M, $1/a$, or the scale at which the gauge coupling in 10 dimensions becomes strong (in our case this happens at around 1TeV as well), the hierarchy between the weak scale and the cutoff, Λ, is much milder than the one in the ordinary gravity or grand unified theories:

$$\frac{m_H}{\Lambda} \sim 10^{-5} - -10^{-1}$$

Quantum mechanically, the absence of quadratic, or large, divergences as the ones present in the standard model [5] can be understood via the argument of symmetry, as in the case of SUSY, however with no fundamental scalar to start with. The gauge symmetry in this case is spontaneously broken due to the presence of a topologically non-trivial background,[6] as will be explained later on, however, at energies larger than the compactification scale, it is recovered and the Higgs field is massless being a component of the massless gauge field. In other words the Higgs mass^2, m_H^2, can not be larger than $1/a$

$$m_H^2 \ = \ \frac{1}{a^2}\, f(Ea)$$

where E is the common energy scale, and a is the typical radius of compactification. We conjecture that $\lim_{E\to\infty} f(Ea) = 0$. In fact, it was shown in [256, 261, 262] that the function $f(Ea)$ is exponentially damping at energies higher than $1/a$. Finding the explicit form of f in our case is technically more complicated due to the presence of a monopole background, however we believe that the $\lim_{E\to\infty} f(Ea)$ will always be finite.

[5] $\delta m_H^2 = \frac{1}{8\pi^2}(\lambda_H^2 - \lambda_t^2)\Lambda^2 +$log.div. + finite term, where λ_H^2 and λ_t^2 are the self coupling of the Higgs and its coupling to the top quark respectively.

[6] In [258, 259], similar ideas have been used to study dynamical breaking of supersymmetry in the context of type I string theory. In [260] the idea of achieving spontaneous breaking of the C, P, and rotational symmetries by topological defects in the internal space was discussed.

Linking chirality and SSB:

As discussed in an earlier chapter, the only way to get chiral fermions in Kaluza-Klein type field theories couple them to a topologically non-trivial background [53, 34]. This in principle changes both the index and the kernel of $\not{D}$ and hence allows for achieving a chiral theory, as the standard model, in four dimensions. Examples will be shown in details in the following sections.

4.3 The background solution

As discussed earlier, the background solutions should satisfy the Einstein and Yang-Mills classical equations of motion. Although the background we are going to use will solve the field equations of any generally covariant and gauge invariant action containing the metric and the Yang-Mills fields only, for the sake of simplicity we start from Einstein-Yang-Mills system in D-dimensions. The action is given by

$$S = \int d^D x \sqrt{-G} \left(\frac{1}{\kappa^2} \mathcal{R} - \frac{1}{2g^2} \mathrm{Tr} F^2 + \lambda + \bar{\psi} i \not{\nabla} \psi \right)$$

where ψ is in some representation of the gauge group G. This action can be the low energy string field theory action with the λ-term induced by some mechanism. The presence of λ in our discussion is required if we insist on having product spaces like $M_1 \times M_2 \times ...$ as a solution of the classical bosonic field equations, where one of the factors in the product is flat, e.g. the flat 4-dimensional Minkowski space (as discussed earlier). Our argument about chirality is not sensitive to the flatness of any of the factors in the product. The presence of tachyons, however, depends on the definition of a mass operator. This is different for example in AdS^d and $(\mathrm{Minkowski})^d$.

The bosonic field equations are (1.11) and (1.12). In this paper we shall consider solutions of the form $M_4 \times K$, where M_4 is the flat 4-dimensional Minkowski space and K is a compact manifold. Therefore the equations of motion (1.13) will be used.

The internal space K will be mostly taken to be either S^2 or $S^2 \times \mathbb{C}P^2$. Furthermore we shall assume that the gauge field configuration A will be non-vanishing only on K. One can of course think of many other choices for K.

For $K = \mathbb{C}P^1 \times \mathbb{C}P^2$ the metric is given by

$$ds^2 = a_1^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \frac{4a_2^2}{1+\zeta^\dagger\zeta} d\bar{\zeta}^a \left(\delta^{ab} - \frac{\zeta^a \bar{\zeta}^b}{1+\zeta^\dagger\zeta}\right) d\zeta^b \tag{4.1}$$

where a_1 and a_2 are the radii of $\mathbb{C}P^1$ and $\mathbb{C}P^2$ respectively, and $\zeta = (\zeta^1, \zeta^2)$ is a pair of local complex coordinates in $\mathbb{C}P^2$. The $\mathbb{C}P^2$ metric is the standard *Fubini-Study* metric. There are two facts about $\mathbb{C}P^2$ which are of importance for our present discussion. The first is the isometry group $SU(3)$ of $\mathbb{C}P^2$. Together with the invariance group $SU(2)$ of the metric of S^2, $SU(3)$ will form part of the gauge group in M_4. $SU(3)$ will be identified with the strong interaction color gauge group. The low energy 4-dimensional gauge group will be $\tilde{G} \times SU(2) \times SU(3)$, where $\tilde{G}$ is the subgroup of the D-dimensional gauge group G which leaves the background solution invariant. Note that even with $G = U(1)$ we can obtain a 4-dimensional gauge theory with a gauge group $U(1) \times SU(2) \times SU(3)$. Although such a solution can produce chiral fermions in a non-trivial representation of $U(1) \times SU(2) \times SU(3)$, it is not possible, however, to obtain the correct Standard Model spectrum of leptons, quarks, and the Higgs fields. For this we need a bigger G. We shall discuss this point in a greater detail in a later section.

The second important fact about $\mathbb{C}P^2$ is that in the absence of a background $U(1)$ gauge field it is not possible to have globally well defined spinor field on it. This is principally due to the fact that the complex coordinates ζ do not cover $\mathbb{C}P^2$ globally. We need at least three patches (U, ζ), (U', ζ'), and (U'', ζ''), where in $U \bigcap U'$ we have the transition rule $\zeta'_1 = \frac{1}{\zeta_1}$ and $\zeta'_2 = \frac{\zeta_2}{\zeta_1}$. It needs some work to show that the two chiral spinors of the tangent space $O(4)$ of $\mathbb{C}P^2$ can not be patched consistently on the overlap. We shall give some more details of this later on.

To write the solution of the Yang-Mills equations on $K = \mathbb{C}P^1 \times \mathbb{C}P^2$ we first work out the spin connection on K. It is given by

$$\Omega = -(\cos\theta - 1) d\varphi \frac{\tau^3}{2} + \begin{pmatrix} \frac{1}{2}\omega^i\sigma^i & 0 \\ 0 & -\frac{3}{2}\omega\sigma_3 \end{pmatrix} \tag{4.2}$$

where the first factor refers to $\mathbb{C}P^1$and the second, which is a 4×4 matrix, refers to $\mathbb{C}P^2$. Here τ^3 as well as σ^i and σ_3 are Pauli matrices. Also the expressions are valid on the upper hemisphere on $\mathbb{C}P^1$ and the local patch (U, ζ) on $\mathbb{C}P^2$. The expressions for ω^i and ω can be read from the Fubini-Study metric (4.1) on $\mathbb{C}P^2$. We shall not need the explicit expression for ω^i.

The one for ω is given by

$$\omega(\zeta,\bar{\zeta}) = \frac{1}{2\,(1+\zeta^{\dagger}\zeta)}\left(\zeta^{\dagger}d\zeta - d\zeta^{\dagger}\zeta\right) \tag{4.3}$$

Note that $d\omega$ is the self dual Kähler form on $\mathbb{C}P^2$. It is thus an instanton type solution of the Yang-Mills equation in $\mathbb{C}P^2$.

It is important note from (4.2) that the $\mathbb{C}P^2$ spin-connection takes its values in the subgroup $SU(2)\times U(1)$ of the tangent space $SO(4)$. Furthermore, under $SO(4)\to SU(2)\times U(1)$ the two chiral spinors of $O(4)$ decompose according to

$$2_+ = 2_0 \tag{4.4}$$

$$2_- = 1_{-\frac{3}{2}} + 1_{\frac{3}{2}} \tag{4.5}$$

where the subscripts indicate the $U(1)$-charges. Using this fact one can understand why spinors are not globally well defined on $\mathbb{C}P^2$. The point is that in the overlap of two patches (U,ζ) and (U',ζ') we have

$$\omega(\zeta') = \omega(\zeta) - id\varphi \tag{4.6}$$

where φ is defined by $\zeta_1 = |\zeta_1|e^{i\varphi}$. For 2_- to be globally well defined $1_{\pm 3/2}$ should patch according to the rule $\psi'(\zeta') = e^{\pm\frac{3}{2}i\varphi}\psi(\zeta)$. We thus obtain transition functions which are anti-periodic under $\varphi\to\varphi+2\pi$. Coupling a background gauge field proportional to ω can change this. With a little more work one can show that a similar obstruction also prevents $2_+ = 2_0$ from being well defined.

Now we are in a position to write our solution of the Yang-Mills equation on $\mathbb{C}P^1\times\mathbb{C}P^2$. It is easy to show that the ansatz

$$A = \frac{n}{2}(\cos\theta - 1)d\varphi + qi\omega \tag{4.7}$$

where $n = \mathrm{diag}(n_1, n_2, \ldots)$ and $q = \mathrm{diag}(q_1, q_2, \ldots)$ are matrices in the Cartan-subalgebra of G. The consistent patching of spinors requires that $n_1, n_2, \ldots$ be integers and $q_1, q_2, \ldots$ be one half of an odd integer. Note that the substitution of the above ansatz in the Einstein equations will require that the radii a_1 and a_2 of $\mathbb{C}P^1$ and $\mathbb{C}P^2$ are quantized.

As mentioned in the beginning of this section our ansatz for the background configuration solves the field equations derived from any generally covariant and gauge invariant Lagrangian in $D = 10$, which contains the metric and the Yang-Mills potentials only. Such an effective Lagrangian will contain infinite number of parameters and therefore the relationship between the radii and other parameters will be more involved.

4.4 Chiral fermions

It is a well known fact that in order to obtain chiral fermions in $D = 4$ we need topologically non-trivial background gauge fields on $\mathbb{C}P^1 \times \mathbb{C}P^2$. Our solution for the Yang-Mills equations consist of magnetic monopole on S^2 and the potential for the Kähler form on $\mathbb{C}P^2$. The Kähler form defines a topologically non trivial line bundle on $\mathbb{C}P^2$.

Consider the $D = 10$ fermion Lagrangian

$$\mathcal{L} = \bar{\psi} i \nabla\!\!\!\!/ \psi \tag{4.8}$$

where

$$\nabla_{\hat{M}} \psi = (\partial_{\hat{M}} + \omega_{\hat{M}} - iA_{\hat{M}})\psi \ , \quad \hat{M} = 0, 1, ..., 9 \tag{4.9}$$

$\omega_{\hat{M}}$ and $A_{\hat{M}}$ are, respectively, the $SO(1,9)$ and the Lie algebra valued spin and gauge connections. We analyze the fermion problem in two steps. In the first step we write the manifold as $M_6 \times \mathbb{C}P^2$. Correspondingly we write the $D = 10$ Dirac matrices as

$$\begin{aligned} \hat{\Gamma}_a &= \Gamma \times \gamma_a \qquad a = 6, 7, 8, 9 \\ \hat{\Gamma}_A &= \Gamma_A \times 1 \qquad A = 0, 1, ..., 5 \end{aligned}$$

where γ_a and Γ_A are respectively 4×4 and 8×8 Dirac matrices satisfying

$$\begin{aligned} \{\gamma_a, \gamma_b\} &= 2\delta_{ab} \\ \{\Gamma_A, \Gamma_B\} &= 2\eta_{AB} \end{aligned}$$

and $\Gamma = \Gamma_0 \Gamma_1 ... \Gamma_5$.

Substituting these Γ's into $\mathcal{L}$ and recalling that the geometry has factorized form we obtain

$$\mathcal{L} = \bar{\psi} \Gamma i \nabla\!\!\!\!/_{\mathbb{C}P^2} \psi + \bar{\psi} i \nabla\!\!\!\!/_{M_6} \psi \tag{4.10}$$

The chiral fermions on M_6 will originate from those modes for which

$$\not{\nabla}_{\mathbb{C}P^2} \psi = 0 \tag{4.11}$$

Those ψ's which are not annihilated by $\not{\nabla}_{\mathbb{C}P^2}$ will give rise to massive fermionic modes on M_6. The standard way to analyze (4.11) is to operate one more time with $\not{\nabla}_{\mathbb{C}P^2}$ on it. Using the background connections (4.2) and (4.7) we obtain

$$(\nabla^2 - \frac{3}{2})\ \psi_+ = 0 \tag{4.12}$$

$$\{\nabla^2 + (q\ \sigma_3 - \frac{3}{2})\}\ \psi_- = 0 \tag{4.13}$$

where

$$\nabla \psi_+ = (d + i\omega^r \frac{\sigma^r}{2} + \omega\ q)\ \psi_+ \tag{4.14}$$

$$\nabla \psi_- = \{d + \omega(q - \frac{3}{2}\sigma_3)\}\ \psi_- \tag{4.15}$$

and

$$\psi_\pm = \frac{1 \pm \hat{\gamma}_5}{2}\ \psi \quad , \quad \hat{\gamma}_5 = \gamma_6\gamma_7\gamma_8\gamma_9$$

The Kähler instanton ω is given by equation (4.3). Since $\nabla^2 \leq 0$ (4.12) will have no non-zero solutions. Thus fermions of ψ_+ type will all be non-chiral and massive. Equation (4.13), on the other hand, can have solutions. Their existence depends on the eigenvalues of q. Clearly for $q = 3/2$ we have only one solution with $\sigma_3 = +1$. For $q = +5/2$ we obtain 3 solutions with $\sigma_3 = +1$. They form a triplet of the isometry group $SU(3)$ of $\mathbb{C}P^2$. For $q = -5/2$ and $\sigma_3 = -1$ we obtain a 3^* of $SU(3)$. These are the only type of solutions we need to consider.

Next we study the M_6 Dirac Lagrangian

$$\mathcal{L} = \bar{\psi} i \not{\nabla}_{M_6} \psi \tag{4.16}$$

where ψ is assumed to be a solution of (4.13). We shall assume that the $D = 10$ spinor is chiral and has positive chirality. Then the spinor of ψ_-

type will have negative $D=6$ chirality. We choose the $D=6$ Γ matrices to be

$$\begin{aligned}\Gamma_\alpha &= \Gamma_\alpha \times \tau_1 & \alpha=0,1,2,3 \\ \Gamma_4 &= \Gamma_5 \times \tau_1 & \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \\ \Gamma_5 &= 1 \times \tau_2 & \end{aligned} \tag{4.17}$$

and $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Inserting the Γ_A's in (4.16) we obtain

$$\mathcal{L} = \bar{\psi} i \nabla_{M_6} \psi + \frac{i}{\sqrt{2}} \left\{ \bar{\psi}(\gamma_5+1) D_- \psi + \bar{\psi}(\gamma_5-1) D_+ \psi \right\} \tag{4.18}$$

where

$$D_\pm \psi = e^m_\pm \left(\partial_m + \frac{i}{2}\omega_m (n-\gamma_5) \right) \psi \tag{4.19}$$

$e^m_\mp$ are the $U(1)$ components of an orthonormal frame on S^2 and ω_m is the corresponding spin connection ($\omega_\theta = 0$, $\omega_\varphi = -\cos\theta + 1$ in the upper hemisphere and $\omega_\varphi = -\cos\theta - 1$ in the lower hemisphere). Decomposing $\psi = \psi_L + \psi_R$, where $\psi_L = \dfrac{1-\gamma_5}{2}\psi$, we obtain the analogue of (4.12, 4.13) for the Dirac operator on $\mathbb{C}P^1$

$$\left\{ \nabla^2 - \frac{1}{2}(1-n) \right\} \psi_R = 0$$

$$\left\{ \nabla^2 - \frac{1}{2}(1+n) \right\} \psi_L = 0$$

$n=1$ produces one ψ_R while $n=-2$ gives rise to two ψ_L which form a doublet of the Kaluza-Klein $SU(2)$.

4.5 General rules for Higgs-type tachyons

To obtain the spectrum of the effective theory in 4-dimensions we need to expand the functions about our background solution in harmonics on $\mathbb{C}P^1 \times \mathbb{C}P^2$. These include fluctuations of the gravitational, Yang-Mills, as well as fermionic fields. The techniques of doing such analysis have been developed long ago. In this paper we shall ignore the gravitational fluctuations and consider only the Yang-Mills and fermionic fields. The full set of linearized gravity Yang-Mills equations can be found in [50]. In the same paper it was shown that there are tachyonic modes in the components of the gauge field fluctuations tangent to S^2. Here, we would like to show that the rule to identify the tachyonic modes given in [50] for $G \equiv SU(3)$ is in fact quite general and applies to any gauge group G. It should be emphasized that neglecting the gravitational fluctuations is justified as they will not mix with the gauge field fluctuations of interest for us.

In general, we should write $A = \bar{A} + V$ where $\bar{A}$ is the background solution and V depends on the coordinates of M_4, S^2 and $\mathbb{C}P^2$. Our first interest is in the fields which are tangent to S^2. It is these fields, which if develop a tachyonic vacuum expectation value, can break $SU(2)$, provided such modes are singlets of $SU(3)$ isometry of $\mathbb{C}P^2$.

We suppress the $\mathbb{C}P^2$ dependence of these fields and denote by $V_{\underline{1}}$ and $V_{\underline{2}}$ their components with respect to an orthonormal frame on S^2. It is convenient to use the "helicity" basis on S^2 defined by

$$V_\pm = \frac{1}{\sqrt{2}}(V_{\underline{1}} \mp i\, V_{\underline{2}})$$

$V_\pm$ are matrices in the Lie algebra of G. What governs their mode expansion on S^2 is their isohelicities. This is basically the effective charge of $V_\pm$ under the combination of $U(1)$ transformations which leave our background configuration invariant. These charges can be evaluated in the same way which was done in [50]. For the sake of simplicity, let us assumes $G = U(N)$ and assume that charge matrices n and q introduced in (4.7) are diagonal $N \times N$ matrices. Then $V_\pm$ are $N \times N$ matrices with elements $V_{\pm i}{}^j$, $i, j = 1, ...N$. Their isohelicities, $\lambda(V_{\pm i}{}^j)$, are given by

$$\lambda(V_{\pm i}{}^j) = \pm 1 + \frac{1}{2}(n_i - n_j)$$

Note that there is a hermiticity relation

$$V_{+i}{}^j = \left(V_{-j}{}^i\right)^*$$

The harmonic expansion of $V_{+i}{}^j$ on S^2 will produce an infinite number of Kaluza-Klein modes. These expansions are defined by

$$V_\pm(x, \theta, \varphi) = \sum_{l \geq |\lambda_\pm|} \sqrt{\frac{2l+1}{4\pi}} \sum_{m \leq |l|} V_\pm^{lm}(x)\; D^l_{\lambda_\pm, m}(\theta, \varphi) \tag{4.20}$$

$D^l_{\lambda_\pm,m}(\theta,\varphi)$ are $2l+1$-dimensional unitary matrices.

The tachyonic modes are generally contained in the leading terms with $l=|\lambda_\pm|$. The effective 4-dimensional mass^2 of $V_\pm^{lm}(x)$ obtains contributions from the appropriate Laplacian acting on S^2 and $\mathbb{C}P^2$. $V_\pm$ are charged scalar fields on $\mathbb{C}P^2$. We shall analyze their dependence on the $\mathbb{C}P^2$ coordinates in the next section. Here we shall consider the S^2 contribution to their masses. The condition for this contribution to be tachyonic is expressed in the following simple rule

$$M^2(V_{+i}{}^j) < 0 \quad \text{if} \quad \lambda(V_{+i}{}^j) \leq 0$$

Likewise

$$M^2(V_{-i}{}^j) < 0 \quad \text{if} \quad \lambda(V_{-i}{}^j) \geq 0$$

To prove these claims let us make more detailed analysis.

Since we are assuming $V_\pm$ are independent of the $\mathbb{C}P^2$ coordinates, their mass term comes from the expansion of $\mathrm{Tr}F_{mn}F^{mn}$, where m,n indicate indices tangent to S^2. The cubic and the quadratic parts in $\mathrm{Tr}F_{mn}F^{mn}$ will produce the interaction terms in the Higgs potential. We have

$$\begin{aligned}\mathrm{Tr}F_{mn}F^{mn} &= \mathrm{Tr}\bar{F}_{mn}\bar{F}^{mn} + \mathrm{Tr}(D_+V_- - D_-V_+)^2 \\ &- 4\,i\,\mathrm{Tr}\bar{F}_{+-}[V_-,V_+] - 2\,i\,\mathrm{Tr}(D_+V_- - D_-V_+)[V_-,V_+] \\ &+ \mathrm{Tr}[V_-,V_+]^2\end{aligned}$$

where the covariant derivatives are defined by

$$D_mV_n = \nabla_mV_n - i[\bar{A}_m, V_n]$$

∇_m denotes the ordinary Riemannian covariant derivative on S^2. Now, since for $\lambda_+ \leq 0$ $(\lambda_+ \geq 0)$ $D_-D^{l=|\lambda_+|}_{\lambda_+,m} = 0 = D_+D^{l=|\lambda_-|}_{\lambda_-,m}$ we see that such modes will be annihilated by $D_\pm$ and thus the S^2 contribution to their $D=4$ action is given by

$$S = -\frac{1}{2g^2}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \; \mathrm{Tr}\left\{4D_\mu V_+D^\mu V_- - 4i\bar{F}_{+-}[V_-,V_+] + [V_-,V_+]^2\right\}$$

The mass terms hence come from $-4iTr\bar{F}_{+-}[V_-,V_+]$ term only.

To proceed it is convenient to choose the Cartan-Weyl basis for the Lie algebra of G. Let Q_j denote the basis of the Cartan subalgebra, E_α and

$E_{-\alpha} = E_\alpha^\dagger$ the generators outside the Cartan subalgebra. The only part of the algebra needed for the evaluation of the mass terms is

$$[Q_j, E_\alpha] = \alpha_j E_\alpha$$

In this basis we can write

$$V_\pm = V_\pm^\alpha E_\alpha + (V_\mp^\alpha)^* E_{-\alpha} + V_\pm^j Q_j$$

It is easy to see that

$$\lambda(V_\pm^\alpha) = \pm 1 + p.\alpha \tag{4.21}$$

where $p.\alpha = p^j \alpha_j$ and p^j are defined by

$$\frac{1}{2} n = p^j Q_j$$

To simplify the discussion consider the case when only one $\lambda(V_+^\alpha) \leq 0$. Set the remaining modes to zero. Of course this is not a loss of generality. In this case $V_+ = V_+^\alpha E_\alpha$ and $V_- = (V_+^\alpha)^* E_{-\alpha}$. The mass term then becomes

$$\begin{aligned} \mathrm{Tr}\left(-4i\bar{F}_{+-}[V_-, V_+]\right) &= -4i\mathrm{Tr}V_+[\bar{F}_{+-}, V_-] \\ &= \frac{4}{a_1^2}\, p.\alpha\, |V_+^\alpha|^2 \mathrm{Tr} E_\alpha E_{-\alpha} \end{aligned}$$

where we inserted $\bar{F}_{+-} = -\frac{i}{a^2} p^j Q_j$. The kinetic part of the action for V_+^α thus becomes

$$S_2 = -\frac{2Tr(E_\alpha E_{-\alpha})}{g^2} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \, \sin\theta \left\{ \partial_\mu V_+^\alpha \partial^\mu V_+^\alpha)^* + \frac{p.\alpha}{a_1^2} |V_+^\alpha|^2 \right\}$$

Substituting $p.\alpha = \lambda(V_+^\alpha) - 1$ we obtain the mass of V_+^α in terms of its isohelicity as (recall that our signature is $(-, +, +, ...)$)

$$m^2 = \frac{\lambda - 1}{a_1^2} \tag{4.22}$$

which is negative for $\lambda \leq 0$. Similar reasoning can be applied if for some V_-^α the corresponding isohelicity $\lambda(V_-^\alpha)$ is non-negative.

This rule gives us an easy way of identifying possible tachyonic modes which can act as Higgs scalars in the $D = 4$ effective theory.

4.6 Examples

In this section we shall ignore the $\mathbb{C}P^2$ part and give some examples of a $D = 6$ gravity Yang-Mills theories which produce standard model type Higgs sectors upon compactification to $D = 4$. Leptons and quarks will be included in the next sections. We basically need to choose the gauge group G and assign magnetic charges n.

The notation is always

$$\bar{A} = \frac{n}{2}(\cos\theta \mp 1)d\varphi \tag{4.23}$$

where $n = \text{diag}(n_1, n_2, ...)$ is in the Lie algebra of G, $-(+)$ give the expression for $\bar{A}$ in the upper (lower) hemispheres.

4.6.1 Tachyons

$\boldsymbol{G = SU(3)}$

$$n = \text{diag}(n_1, n_2, -n_1 - n_2), \qquad n_1, n_2 \in \mathbb{Z} \tag{4.24}$$

The isohelicities can be assembled in a 3×3 matrix

$$\lambda(V_\pm) = \begin{pmatrix} \pm 1 & \pm 1 + \frac{1}{2}(n_1 - n_2) & \frac{1}{2}(2n_1 + n_2) \\ \pm 1 - \frac{1}{2}(n_1 - n_2) & \pm 1 & \pm 1 + \frac{1}{2}(n_1 + 2n_2) \\ \pm 1 - \frac{1}{2}(2n_1 + n_2) & \pm 1 - \frac{1}{2}(n_1 + 2n_2) & \pm 1 \end{pmatrix} \tag{4.25}$$

Using the results of section 4.4 we see that in order to obtain left handed doublets and right handed singlets we had to take $(n_1, n_2) = (1, 1)$. With these values of n_1 and n_2, ${V_{-1}}^3$ and ${V_{-2}}^3$ will contain tachyonic modes in the leading term of their expansion on S^2.

In this example the $SU(2) \times U(1)$ subgroup of $SU(3)$ is unbroken and the tachyonic Higgs ${V_{-1}}^3$ and ${V_{-2}}^3$ form a doublet of $SU(2)$ with $U(1)$ charge of $3/2$. We denote this doublet by ϕ. Its isohelicity is $+1/2$. Therefore it will also be a doublet of the Kaluza-Klein isometry of S^2. One can integrate the (θ, φ) dependence of ϕ on S^2 and work out its $D = 4$ effective action. The result is

$$\mathcal{L} = -\frac{1}{2g^2}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \, \text{Tr} F_{MN} F^{MN}$$

$$= -\frac{1}{4g_1^2} F^{8\ 2}_{\mu\nu} - \frac{1}{4g_2^2} F^{r\ 2}_{\mu\nu} - \frac{1}{4e^2} W^{r\ 2}_{\mu\nu}$$

$$-\mathrm{Tr}\left\{ \nabla_\mu \phi^\dagger \nabla^\mu \phi - \frac{3}{2a_1^2}\phi^\dagger\phi + 2g_1^2(\phi^\dagger\phi)^2 \right\} \tag{4.26}$$

where we have regarded ϕ as a 2×2 complex matrix, and

$$\nabla_\mu \phi = \partial_\mu \phi - \frac{3}{2} i V_\mu^8 \phi - i V_\mu^r \frac{\sigma^r}{2} \phi - i W_\mu^r \phi \frac{\tau^r}{2}$$

where V_μ^8, V_μ^r, and W_μ^r are respectively the $U(1)$, $SU(2)_L$, and the Kaluza-Klein $SU(2)_R$ gauge fields. g_1, g_2, and e are their respective couplings. Some calculation show that

$$g_2 = \frac{1}{2\sqrt{\pi}} \frac{g}{a_1} = \sqrt{3} g_1 \tag{4.27}$$

The Kaluza-Klein gauge coupling e can also be expressed in terms of the fundamental scales g and a_1.

In the next section we shall work out the Yukawa couplings for this model as well.

$G = U(6)$

With $n = \mathrm{diag}(n_1, ..., n_5, n_6)$ we can again work out the table of isohelicities for $V_\pm$. We shall see in section 4.6.2 that in order to obtain one family of leptons and quarks we need to take $n = \mathrm{diag}(-2, 1, 1, -2, 1, 1)$. Note that since the group is $U(6)$ rather than $SU(6)$, n is not traceless. $\lambda(V_\pm)$ is given by

$$\lambda(V_+) = \begin{pmatrix} +1 & -\frac{1}{2} & -\frac{1}{2} & +1 & -\frac{1}{2} & -\frac{1}{2} \\ +\frac{5}{2} & +1 & +1 & +\frac{5}{2} & +1 & +1 \\ +\frac{5}{2} & +1 & +1 & +\frac{5}{2} & +1 & +1 \\ +1 & -\frac{1}{2} & -\frac{1}{2} & +1 & -\frac{1}{2} & -\frac{1}{2} \\ +\frac{5}{2} & +1 & +1 & +\frac{5}{2} & +1 & +1 \\ +\frac{5}{2} & +1 & +1 & +\frac{5}{2} & +1 & +1 \end{pmatrix} \tag{4.28}$$

Since $V_- = V_+^\dagger$, therefore $\lambda({V_-}_i{}^j) = -\lambda({V_+}_j{}^i)$.

The tachyonic modes are contained in ${V_{+1}}^i$, and ${V_{+4}}^i$ where $i = 2, 3, 5, 6$. They will all be doublets of the Kaluza-Klein $SU(2)$. They also transform under some representation of the unbroken part of $U(6)$, which is $SU(2) \times SU(2) \times U(1)^3 \times U(1)'$, which is generated by diag$(0, \frac{\sigma^i}{2}, 0, 0, 0)$, $i = 1, 2, 3$; diag$(0, 0, 0, 0, \frac{\sigma^i}{2})$; diag$(-2, 1, 1, 0, 0, 0)$; diag$(0, 0, 0, -2, 1, 1)$; diag$(1, 1, 1, -1, -1, -1)$; and the 6×6 unit matrix 1_6 which generates $U(1)'$. The tachyonic Higgs will be neutral under this $U(1)'$, therefore their tree level vacuum expectation value will not break it. Under $U(6) \to SU(2) \times SU(2) \times U(1)^3$ we have

$$\underline{6} = (1,1)_{(-2,0,1)} + (2,1)_{(1,0,1)} + (1,1)_{(0,-2,-1)} + (1,2)_{(0,1,-1)} \tag{4.29}$$

The quantum numbers of the relevant Higgs tachyons will be

$$\begin{aligned} {V_{+1}}^i &\sim (2,1)_{(-3,0,0)} \quad i = 2, 3 && (4.30) \\ {V_{+4}}^t &\sim (1,2)_{(0,-3,0)} \quad t = 5, 6 && (4.31) \end{aligned}$$

As we said earlier, the tachyonic modes in all these fields will be in the doublet representation of the Kaluza-Klein $SU(2)$. The vacuum expectation value of the fields ${V_{+1}}^i \sim (2,1)_{(-3,0,0)}$ and ${V_{+4}}^t \sim (1,2)_{(0,-3,0)}$ will give masses to the quarks and leptons respectively. In section 4.7 we shall show that the leading term in their expansion on $\mathbb{C}P^2$ is a singlet of $SU(3)$ and therefore their masses receive no contribution from the dependence on the $\mathbb{C}P^2$ coordinates. Thus they remain tachyonic. The other tachyonic fields, namely ${V_{+1}}^i$, $i = 5, 6$; ${V_{+4}}^t$, $t = 2, 3$, would induce Yukawa couplings between quarks and leptons. We shall show that in fact the leading term in their harmonic expansion on $\mathbb{C}P^2$ is a triplet of $SU(3)$. Thus the vacuum expectation value of these fields can break the color $SU(3)$. We will determine the conditions to avoid this.

4.6.2 Fermions

We consider the two examples of the previous section.

$G = SU(3)$

Here we assume that $D = 6$ and there is no $\mathbb{C}P^2$ factor. Let us take ψ in $\underline{3}$ of $SU(3)$ and $n = \text{diag}(1, 1, -2)$. According to our rules this will produce two right handed singlets of the Kaluza-Klein $SU(2)$ which we denote by $SU(2)_K$ and a left handed doublet. The singlets will form a doublet of $SU(2)_G \subset SU(3)$ and the doublet of $SU(2)_K$ will be a singlet of $SU(2)_G$. Thus under $SU(2)_K \times SU(2)_G \times U(1)$ where $U(1) \subset SU(3)$ we have $(1, 2_R)_{1/2} + (2_L, 1)_1$. The $D = 4$ Yukawa and gauge couplings can be easily worked out. The result is

$$\begin{aligned}
\mathcal{L}_F &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \, \bar{\psi} i \nabla\!\!\!\!/ \, \psi \\
&= \bar{\lambda}_L \, i\gamma^\mu \left(\partial_\mu - i g_1 V_\mu^8 - i e W_\mu^i \frac{\tau^i}{2} \right) \lambda_L \\
&\qquad + \bar{\lambda}_R \, i\gamma^\mu \left(\partial_\mu - i \frac{g_1}{2} V_\mu^8 - i g_2 V_\mu^i \frac{\sigma^i}{2} \right) \lambda_R \\
&\qquad - 2 g_1 \left\{ \bar{\lambda}_L \phi (i\sigma_2) \lambda_R - \bar{\lambda}_R (i\sigma_2) \phi^\dagger \lambda_L \right\}
\end{aligned}$$

where $\lambda_L = (2_L, 1)_1$ and $\lambda_R = (1, 2_R)_{1/2}$.

This expression together with the bosonic part given in equation (4.16) give the total effective $D = 4$ action for the $SU(3)$ example. Although this example leads to interesting chiral and Higgs spectrum in $D = 4$ can not be considered satisfactory. It has both perturbative and global chiral anomalies in $D = 6$. The perturbative anomalies can be eliminated with the standard Green Schwarz mechanism[263]. To apply this mechanism [264] we need first to introduce an antisymmetric rank two potential together with three right handed $D = 6$ $SU(3)$ singlets to kill the pure gravitational anomaly which is given by $\mathcal{R}^4$ term in the anomaly 8− form. The remaining terms in the anomaly 8-form factorize appropriately in order to be canceled by a judicious transformation of the antisymmetric potential. This mechanism does not cancel the global anomalies [265] whose presence is due to the fact that $\pi_6(SU(3)) = Z_6$ is non zero. To kill these ones we need to introduce further $SU(3)$ multiplets or to change the gauge group altogether and chose to a gauge group like E_6 which has a trivial $\pi_6(E_6)$.

$G = U(6)$

Now assume $D = 10$ and choose ψ to be in $\underline{6}$ of $U(6)$ and $q = \text{diag}(5/2, 5/2, 5/2, 3/2, 3/2, 3/2)$. As before n will be taken to be $n = \text{diag}(-2, 1, 1, -2, 1, 1)$. According to the results of the previous section with respect to the isometry group $SU(2) \times SU(3)$ we have the following chiral fermions

$$(2_L, \underline{3}) + (1_R, \underline{3}) + (1_R, \underline{3}) + (2_L, 1) + (1_R, 1) + (1_R, 1)$$

Clearly the first three triplets are candidates for $\begin{pmatrix} u \\ d \end{pmatrix}_L$, u_R and d_R. The last two pieces can be identified with the leptons $\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$ and e_R.[7]

These multiplets also transform in the following representation of the unbroken $SU(2) \times SU(2) \times U(1)^3 \subset G$

$$\begin{aligned} (2_L, \underline{3}) &\sim (1, 1)_{(-2,0,1)} \\ (1_R, \underline{3}) + (1_R, \underline{3}) &\sim (2, 1)_{(1,0,1)} \end{aligned}$$

$$(2_L, 1) \sim (1, 1)_{(0,-2,-1)} \quad \text{and} \quad (1_R, 1) \sim (1, 2)_{(0,1,-1)}$$

The Yukawa coupling between the quarks will be through the Higgs field ${V_{+1}}^i$ given in (4.30), while the electron will get its mass through coupling to ${V_{+4}}^t$. Thus our construction leads to a multi Higgs theory in which the quarks and leptons obtain their masses from their Yukawa couplings to different Higgs scalars. Note also that there is no common $U(1)$ under which both Higgs multiplets are charged. The hypercharge coupling in our model is different from the standard electroweak theory.

4.7 Higgs like Tachyons on $\mathbb{C}P^2 \times \mathbb{C}P^1$

If the total space-time dimension is $D = 6$ the masses of the Higgs like tachyons are given by (4.22). In the case of a $D = 10$ theory we need to take into account the contribution of $\mathbb{C}P^2$ part as well. The fields $V_\pm$ are like scalar fields on $\mathbb{C}P^2$ which are charged with respect to the $\mathbb{C}P^2$ part of the background gauge field (4.7), viz, $iq\omega$. The $\mathbb{C}P^2$ contribution to the masses

[7] We have an extra right handed singlet in the lepton sector. This can be removed by choosing the last entry in q to be for instance $-1/2$ or the last entry in n to be 0. In this way the unbroken subgroup of $U(6)$ will be $SU(2) \times U(1) \times U(1)'$.

of $V_\pm$ come from the commutator term in the $\mathbb{C}P^2$ covariant derivative of $V_\pm$, i.e.

$$\begin{aligned} DV_\pm &= dV_\pm - i[i\omega q, V_\pm] \\ &= dV_\pm + \omega[q, V_\pm] \end{aligned}$$

To be specific let us consider the example of the $U(6)$ model for which $q = \mathrm{diag}(5/2, 5/2, 5/2; 3/2, 3/2, 3/2)$. Write

$$V = \left(\begin{array}{c|c} v & u \\ \hline \tilde{u} & \tilde{v} \end{array}\right) \tag{4.32}$$

where v, $\tilde{v}$, u, and $\tilde{u}$ each is a 3×3 matrix. Then

$$[q, V] = \left(\begin{array}{c|c} 0 & u \\ \hline -\tilde{u} & 0 \end{array}\right)$$

This indicates that out of the Higgs fields given in equations (4.30–4.31) the ones which give masses to quarks and leptons, namely, ${V_{+1}}^i$ and ${V_{+4}}^t$ (which lie respectively inside v and $\tilde{v}$ in the above notation), do not couple to the background ω field on $\mathbb{C}P^2$. The leading term in their harmonic expansion on $\mathbb{C}P^2$ will be a constant (independent of the coordinates of $\mathbb{C}P^2$). Their masses will be tachyonic and will be given by (4.22) for $\lambda = -\frac{1}{2}$, i.e. $M^2 = -\frac{3}{2}\frac{1}{a_1^2}$.

The remaining fields ${V_{+1}}^t$ and ${V_{-i}}^4$ on the other hand are located inside u and they couple to the background ω-field. Their masses will receive contribution from $\mathbb{C}P^2$ and in principle can become non-tachyonic. To verify this we need to evaluate the eigenvalues of $\nabla^2_{\mathbb{C}P^2}$ on these fields. Their covariant derivatives are

$$\begin{aligned} D{V_{+1}}^t &= d{V_{+1}}^t + \omega {V_{+1}}^t \\ D{V_{-4}}^i &= d{V_{-4}}^i + \omega {V_{-4}}^i \end{aligned}$$

Since they couple with the same strength to the ω-field they will receive the same contribution from $\nabla^2_{\mathbb{C}P^2}$. It turns out that the leading term in the expansion of any of these fields on $\mathbb{C}P^2$ is a triplet of $SU(3)$ and D^2 acting on it is $-\frac{1}{a_2^2}$. Thus the total mass^2 of such modes will be

$$-\frac{3}{2}\frac{1}{a_2^2} + \frac{1}{a_2^2} = \frac{1}{a_1^2}\left(-\frac{3}{2} + \frac{a_1^2}{a_2^2}\right)$$

If a_1 and a_2 were independent we could choose $(\frac{a_1}{a_2})^2 \geq \frac{3}{2}$ and make these fields non-tachyonic. If we insist on the validity of the background Einstein equations then the ratio of $\frac{a_1}{a_2}$ will be fixed. Equation (1.13) leads to $(\frac{a_1}{a_2})^2 = \frac{12}{17}$. [8] With this value unfortunately the above mass^2 is still negative. The vacuum expectation value of these fields will break the color $SU(3)$.

One way to change the ratio $\frac{a_1}{a_2}$ is to couple a $U(1)$ gauge field to gravity in $D = 10$. This $U(1)$ will *not* couple to anything else. In particular the fermions will be neutral under it, so the spectrum of the chiral fermions will be unaltered. Its sole effect will be to add an extra term to the right hand side of Einstein equations. In particular (1.13) will be replaced by

$$\mathcal{R}_{\hat{m}\hat{n}} = \frac{\kappa^2}{g^2}\mathrm{Tr}F_{\hat{m}\hat{r}}F_{\hat{n}}{}^{\hat{r}} + \frac{\kappa^2}{g'^2}\mathrm{Tr}F'_{\hat{m}\hat{r}}F'_{\hat{n}}{}^{\hat{r}}$$

where F' and g' refer to the extra $U(1)$ system. Now if we set

$$A' = \frac{n'}{2}(\cos\theta - 1)d\varphi + q'i\omega$$

where n' and q' are real numbers, the ratio of a_1/a_2 will turns out to be

$$\frac{a_1^2}{a_2^2} = \frac{36 + 3n'^2\frac{g^2}{g'^2}}{51 + 2q'^2\frac{g^2}{g'^2}}$$

There is a big range of parameters for which $a_1/a_2 \geq 3/2$.

4.8 Other scalars

The components of the gauge field fluctuations tangent to $\mathbb{C}P^2$ will also give rise to infinite tower of Kaluza-Klein modes which will be scalars fields in $D = 4$. These modes will belong to unitary representations of $SU(2)\times SU(3)$.

[8] To obtain $(\frac{a_1}{a_2})^2 = \frac{12}{17}$ we need to use the following results, which can be obtained by straightforward calculation,
$\mathcal{R}(S^2) = \frac{1}{a_1^2}1_{2\times2}$, $\mathcal{R}(\mathbb{C}P^2) = \frac{3}{2}\frac{1}{a_1^2}1_{4\times4}$, $\mathrm{Tr}F^2_{S^2} = 6\frac{1}{a_1^4}$, and $\mathrm{Tr}F^2_{\mathbb{C}P^2} = \frac{51}{2}\frac{1}{a_2^2}$.

If there is a tachyon Higgs among them they will break $SU(3)$. We need to verify that this does not happen. To this end we denote these fields by V_a, where a is tangent to $\mathbb{C}P^2$, and write those terms in the bilinear part of $\mathrm{Tr}F_{MN}F^{MN}$ which contains V_a. In this section we are considering only the $U(6)$ model. The V_a are 5×5 Hermitian matrices. After some manipulation and the imposition if the $D = 10$ background gauge condition $D_M V^M = 0$, the bilinear terms of interest to us can be written as

$$S_2 = -\frac{1}{2g^2} \int d^{10}x \mathrm{Tr}\{2V_a(-\partial^2 - D_m^2 - D_{\hat{m}}^2 + \frac{3}{2}\frac{1}{a_2^2})V^a + 4iV^a[\bar{F}_{ab}, V^b]\} \tag{4.33}$$

where D_m and $D_{\hat{m}}$ are respectively the covariant derivatives on S^2 and $\mathbb{C}P^2$ and

$$D_m V_a = \partial_m V_a - \frac{i}{2}\omega_m[n, V_a] \tag{4.34}$$

$$D_{\hat{m}} V_a = \nabla_{\hat{m}} V_a - \frac{i}{2}\omega_{\hat{m}}[q, V_a] \tag{4.35}$$

$\nabla_{\hat{m}}$ is the Riemann covariant derivative on $\mathbb{C}P^2$. The contribution of D_m^2 on each $SU(2)$ mode of V_a will simply be $\frac{1}{a_1^2}[l(l+1) - \lambda^2]$, $l \geq |\lambda|$ where λ represents the isohelicities of various components of V_a, $\lambda(V_{ai}{}^j) = \lambda(V_{+i}{}^j) - 1$, where $\lambda(V_{+i}{}^j)$ are given in equation (4.28).

To work out the contributions of $D_{\hat{m}}^2$ and the commutator term $[\bar{F}_{ab}, V^b]$, we represent V_a as in (4.32), i.e.

$$V_a = \left(\begin{array}{c|c} v_a & u_a \\ \hline \tilde{u}_a & \tilde{v}_a \end{array}\right) \tag{4.36}$$

where v_a, $\tilde{v}_a$, u_a, $\tilde{u}_a$ each is a 3×3 matrix. Then

$$[q, V_a] = \left(\begin{array}{c|c} 0 & u_a \\ \hline -\tilde{u}_a & 0 \end{array}\right) \tag{4.37}$$

This indicates that the commutator terms in (4.33) and (4.35) do not contribute to $D_{\hat{m}} v_a$ and $D_{\hat{m}} \tilde{v}_a$. Thus $D_{\hat{m}}$ acting on these fields is just the Riemannian Laplacian acting on vectors and its contribution to the masses of these fields will be non-tachyonic.

The only fields we need to be concerned about are those in u_a. To analyze the contribution of these terms we introduce 2 complex $SU(2)$ vectors u_α and u'_α defined by

$$\begin{cases} u_{\underline{1}} = \frac{1}{\sqrt{2}}(u_6 + iu_7) \\ u_{\underline{2}} = \frac{1}{\sqrt{2}}(u_8 + iu_9) \end{cases} \qquad \begin{cases} u'_{\underline{1}} = \frac{1}{\sqrt{2}}(u_6 - iu_7) \\ u'_{\underline{2}} = \frac{1}{\sqrt{2}}(u_8 - iu_9) \end{cases} \tag{4.38}$$

where 6, 7, 8, and 9 are directions tangent to $\mathbb{C}P^2$. In terms of these new fields the u_a part of (4.35) can be rewritten as

$$D_{\hat{m}} u_\alpha = (\partial_{\hat{m}} + i\omega^i_{\hat{m}} \frac{\sigma^i}{2} - i\frac{5}{2}\omega_{\hat{m}}) u_\alpha \tag{4.39}$$

$$D_{\hat{m}} u'_\alpha = (\partial_{\hat{m}} + i\omega^i_{\hat{m}} \frac{\sigma^i}{2} - i\frac{5}{2}\omega_{\hat{m}}) u'_\alpha \tag{4.40}$$

The contribution of $D^2_{\hat{m}}$ on u_α and u'_α will again be positive.

Finally we need to evaluate the contribution of $2i\text{Tr}V^a[\bar{F}_{ab}, V^b]$ to the masses of u_α and u'_α. After some calculation this turns out to be

$$2i\text{Tr}V^a[\bar{F}_{ab}, V^b] = \frac{2}{a_2^2}\text{Tr}(u^\dagger_\alpha u_\alpha - u'^\dagger_\alpha u'_\alpha) \tag{4.41}$$

It is seen that the contribution of this term to the u_α mass is non-tachyonic. However, it makes a negative contribution to the mass^2 of u'_α field. Upon substitution of the above in (4.35) we find out that the negative contribution in (4.41) is off-set by the $\frac{3}{2}\frac{1}{a_2^2}$ term in equation (4.33), with the result that u'_α is also non-tachyonic.

We thus conclude that all the tachyonic Higgs are singlets of $SU(3)$ and doublets of $SU(2)$.

4.9 Massless scalars and loop-induced hierarchy

So far we have been discussing tachyonic mass of the scalar particles at the tree level of the effective 4 dimensional theory. The natural scale of this mass and therefore also of the symmetry breaking is the compactification scale. This is few order of magnitude above the electroweak symmetry breaking scale of a 200 hundred GeV. It will be very desirable if we could find a mechanism to lower the scale of the tachyonic mass. An obvious idea is if the tree level mass of the scalars is zero and they obtain their tachyonic value as a consequence of loop effects. Our theory is of course a non renormalizable one, at least in conventional sense. However, the Higgs mass is controlled by $1/a$ due to higher dimensional gauge invariance. Our main point is that the sign of the one loop induced effective mass will depend on the imbalance between the contribution of fermions and bosons. By a judicious choice of the fermionic degrees of freedom this sign can be made tachyonic. Any way whatever the justification the first step in implementing this idea is to find tree level massless scalars in the spectrum of the effective four dimensional theory. Unlike the massless chiral fermions whose presence is dictated by the topology of the gauge field in compact subspace, to verify the existence of the massless scalars in the spectrum requires more detailed analysis of the mass spectrum and should be carried out separately for each case. In this section we give an example of a model in $D = 10$ in which a monopole background on the $S^2 \times S^{'2} \times S^{''2}$ internal space leads to massless scalars transforming non trivially under the $SU(2) \times SU(2) \times SU(2)$ isometry group of the internal space. This example which was is only for illustrative purpose and is not going to be used for a realistic model building.

We start from a $U(N)$ gauge theory in 10 dimensions and consider a solution of equations (1.13) in which the internal space is $S^2 \times S^{'2} \times S^{"2}$. In the notation of previous section we denote the magnetic charge matrices on the three S^2's by n n' and n". Denoting all the quantities on $S'2$ with a prime our ansatz for the gauge field becomes

$$A = \frac{n}{2}(cos\theta - 1)d\phi + \frac{n'}{2}(cos\theta' - 1)d\phi' + \frac{n''}{2}(cos\theta'' - 1)d\phi''$$

The structure of the charge matrices will determine the unbroken subgroup of $U(N)$. As before we shall take them to be $N \times N$ diagonal real matrices.

The scalars of interest for us are those components of the fluctuations of the vector potential which are tangent to $S^2 \times S^{'2} \times S^{"2}$ and are in the directions of perpendicular to the Cartan subalgebra of $U(N)$. Consider the field V_{-i}^{j} tangent to S^2.

The masses of these fields can be calculated using the appropriate modification of equation (4.33). The result is

$$S_2 = -\frac{1}{2g^2}\int d^{10}x\Big\{(V^j_{-i})^*(-\partial^2 - D^2 - D'^2 - D''^2 + \frac{1}{a^2})V^j_{-i} \\ -\frac{1}{a^2}(V^j_{-i})^*(n_i - n_j)V^j_{-i}\Big\} \qquad (4.42)$$

where D^2, D'^2 and D''^2 are the appropriate Laplacian on the three S^2's. The eigenvalues of these Laplacians are basically determined from the isohelicities of V^j_{-i} which are given by

$$\lambda(V^j_{-i}) = -1+\tfrac{1}{2}(n_i-n_j), \qquad \lambda'(V^j_{-i}) = \tfrac{1}{2}(n'_i-n'_j), \qquad \lambda''(V^j_{-i}) = \tfrac{1}{2}(n''_i-n''_j)$$

Similar expressions can be written for the bilinear parts of the fields tangent to S'^2 and S''^2.

For our illustrative example we consider an n matrix which has only the elements n_1 and n_2 different from zero and such that $n_1 - n_2 \geq 2$. Then $\lambda(V^2_{-1}) \geq 0$ and according to our general rule the leading mode in this field can be tachyonic. The question we would like to answer is if by an appropriate choice of magnetic charges we can make the mass of this field to vanish. It is not difficult to write down the formula for the masses of the infinite tower of modes of V^2_1. These are given by

$$a^2M^2 = l(l+1) - \lambda^2 + \frac{a^2}{a'^2}(l'(l'+1) - \lambda'^2) + \frac{a^2}{a''^2}(l''(l''+1) - \lambda''^2) + 1 - (n_1 - n_2)$$

To verify the existence of a massless mode first we employ the background equations (1.13) to obtain the ratios

$$\frac{a^2}{a'^2} = \frac{\mathrm{Tr}n^2}{\mathrm{Tr}n'^2}, \quad \text{and} \quad \frac{a^2}{a''^2} = \frac{\mathrm{Tr}n^2}{\mathrm{Tr}n''^2}.$$

It is seen that for the choice of $n'_1 - n'_2 = n_1 - n_2$, $\mathrm{Tr}n^2 = \mathrm{Tr}n'^2$ and $n''_1 - n''_2 = 0$ the leading mode is indeed massless. For this choice there will of course be a similar massless mode in the fluctuations V'^2_{-1} tangent to S'^2. The $SU(2) \times SU(2) \times SU(2)$ quantum numbers of these modes will be $(l = \frac{1}{2}(n_1 - n_2) - 1, l' = \frac{1}{2}(n_1 - n_2), 0)$, and $l = (\frac{1}{2}(n_1 - n_2), l' = \frac{1}{2}(n_1 - n_2) - 1, 0)$, respectively. We can make all other modes to have positive masses by appropriate choices of the remaining magnetic charges.

Chapter 5

Concluding Remark

Considering the possibility of living in a higher-dimensional space-time seriously opens new gates towards understanding the present problems of particle physics, mainly the unification of gravity with the other fundamental forces, and the gauge hierarchy problem. So far, the dimensionality of our space-time is an assumption based on our observational, technical, and probably mental limitations. The new colliders, like LHC, will be able to test this possibility and detect a possible modification of Newton's low at energies around few TeV.

The last three years has witnessed an enormous development in extra-dimensional model building. It is very desirable to significantly increase the predictability of the new scenarios as a compensation of having extra dimensions. In particular, it would be nice if these models are free of the same problems the standard model has in four dimensions at low energies.

The hope is that, eventually, most of these models can be somehow a low energy effective field theoretical description of a supertring theory. Alternatively, one can be more modest and seek a consistent theory in a field theory context trying to make the best of this approach.

Interestingly enough, by going to higher dimensions one can link some of the features of the standard model together. The fermion chirality and electroweak spontaneous symmetry breaking can have the same origin: a non-trivial Yang-Mills background in the internal space. What is even more interesting is that the mechanism in which the SM gauge group is spontaneously broken does not involve a fundamental scalar field, the thing which leads to the absence of quadratic corrections to the classical value of Higgs mass square.

There are certainly many other interesting issues to discuss. Unfortunately, it was impossible to incorporate all the work done in this subject as the recent literature consists of more than 1,000 article written from 1998 until now, touching various sides and implications of the new thrilling idea, while the total number of papers, old and, probably exceeds 2,000. Therefore, only examples could be provided.

As perhaps all new subjects in physics, the models with extra dimensions seem to open new problems and not to completely solve any. It remains to check the classical and semiclassical stability for many of the scenarios with both categories warped and factorizable geometry. The issue of semiclassical stability was not discussed here, and can be found in [266]-[269] for Kaluza-Klein compactification, and for instance [100] for brane-worlds. Furthermore, the stabilization of the compactification scale close to the electroweak scale is an open problem although some attempts exist [217, 220]. Other aspects not discussed here include unification with large extra dimensions [270], brane-world in [9] scenario, "fat" branes [271], the cosmological constant problem (see [76] for a review),...*etc.*, in addition to many of the older literature which are covered in [269] and [26].

Chapter 6

Acknowledgement

Bibliography

[1] S. Dalley, "Myths from mesopotamia, creations, the flood, Gilgamesh and others", a new translation, Oxford Paperbacks, Oxford Un. Press, ISBN 0-19-281789-2.

[2] [LEP Collaborations], "A combination of preliminary electroweak measurements and constraints on the standard model," hep-ex/0103048. For a review, see H. Przysiezniak, "Precision electroweak measurements," hep-ex/0101008.

[3] T. Kajita [Super-Kamiokande Collaboration], Nucl. Phys. Proc. Suppl. **77** (1999) 123.

[4] H. N. Brown *et al.* [Muon g-2 Collaboration], Phys. Rev. Lett. **86** (2001) 2227.

[5] M. Carena *et al.*, arXiv:hep-ph/0010338.

[6] For a recent review see: K. D. Lane, "Technicolor 2000," hep-ph/0007304.

[7] C. Kounnas, A. Masiero, D. V. Nanopoulos and K. A. Olive, Singapore, Singapore: World Scientific (1984) 425 P. (International School For Advanced Studies Lecture Series, 2).

[8] H. P. Nilles, Phys. Rept. **110** (1984) 1.

[9] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. **B429** (1998) 263.

[10] J. Scherk and J. H. Schwarz, Phys. Lett. B **57** (1975) 463.

[11] K. Akama, Lect. Notes Phys. **176** (1982) 267.

[12] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **125** (1983) 136.

[13] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **125** (1983) 139.

[14] I. Antoniadis, Phys. Lett. B **246** (1990) 377.

[15] J. C. Price, in Proc. of Int. Symp. on Experimental gravitational Physics, ed. P. F. Michelson, Guangzhou, China (World Sientific, Singapor, 1988).

[16] J. C. Long, H. W. Chan and J. C. Price, Nucl. Phys. B **539** (1999) 23.

[17] Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Klasse 966 (1921).

[18] O. Klein, Nature **188** (1926) 516.

[19] C. Wetterich, Nucl. Phys. B **242** (1984) 473.

[20] L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370.

[21] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Rev. D **59** (1999) 086004.

[22] G. Nordström, Phys. Zeitsch. **15** (1914) 504.

[23] H. Weyl, Sitzunberger. Akademie der Wissenschaften Berlin, (1918) 465. An English presentation of this paper can be found in L. O'Raifeartaigh and N. Straumann, hep-ph/9810524.

[24] B. S. De Witt, in "Relativity, Groups, and Topology", Eds. C. De Witt and B. S. De Witt, Gordon and Breach, New York (1964).

[25] O. Klein, Z. F. Physik **37** (1926) 895.

[26] "Modern Kaluza-Klein Theories", (Edited by T. Appelquist, A. Chodos, and P. Freund), Frontiers In Physics v. 65, Addison-Wesley Pub. Com. (1987).

[27] R. Kerner, Annales Poincare Phys. Theor. **9** (1968) 143.

[28] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. B **76** (1978) 409.

[29] W. Nahm, Nucl. Phys. B **135** (1978) 149.

[30] B. de Wit and H. Nicolai, Phys. Lett. B **148** (1984) 60.

[31] P. van Nieuwenhuizen and N. P. Warner, Commun. Math. Phys. **99** (1985) 141.

[32] S. Randjbar-Daemi and C. Wetterich, Phys. Lett. B **166** (1986) 65.

[33] C. Wetterich, Nucl. Phys. B **222** (1983) 20; Nucl. Phys. B **223** (1983) 109.

[34] E. Witten, "Fermion Quantum Numbers In Kaluza–Klein Theory", The Proc. of Second Shelter Island Meeting (1983) 227.

[35] A. Mukherjee and R. Tabbash, Eur. Phys. J. C **20** (2001) 193.

[36] Personal notes of S. Randjbar-Daemi; J. Strathdee, "Symmetry In Kaluza-Klein Theory," IC-82/228 Based on lectures given at Summer School 'Supergravity 82', Trieste, Italy, Sep 6-18, 1982.

[37] S. Randjbar-Daemi and R. Percacci, Phys. Lett. B **117** (1982) 41.

[38] R. Percacci and S. Randjbar-Daemi, J. Math. Phys. **24** (1983) 807.

[39] J. Scherk and J. H. Schwarz, Nucl. Phys. B **153** (1979) 61.

[40] A. Salam and J. Strathdee, Annals Phys. **141** (1982) 316.

[41] S. Randjbar-Daemi, A. Salam and J. Strathdee, Nucl. Phys. B **242** (1984) 447.

[42] G. 't Hooft, Phys. Rev. D **14** (1976) 3432

[43] J. F. Luciani, Nucl. Phys. B **135** (1978) 111.

[44] E. Witten, Nucl. Phys. B **186** (1981) 412.

[45] E. Cremmer and J. Scherk, Nucl. Phys. B **103** (1976) 399.

[46] Z. Horvath, L. Palla, E. Cremmer and J. Scherk, Nucl. Phys. B **127** (1977) 57.

[47] Z. Horvath and L. Palla, Nucl. Phys. B **142** (1978) 327.

[48] C. Omero and R. Percacci, Nucl. Phys. B **165** (1980) 351.

[49] P. G. Freund and M. A. Rubin, Phys. Lett. B **97** (1980) 233.

[50] S. Randjbar-Daemi, A. Salam and J. Strathdee, Phys. Lett. B **124** (1983) 345.

[51] A. N. Schellekens, Nucl. Phys. B **248** (1984) 706;

[52] A. N. Schellekens, Nucl. Phys. B **262** (1985) 661; "Boson Mass Spectrum Of Kaluza-Klein Theories On Hyperspheres,"

[53] S. Randjbar-Daemi, A. Salam and J. Strathdee, Nucl. Phys. B **214** (1983) 491.

[54] H. B. Lawson and M. Michelsohn, "Spin Geometry", Princeton University Press, (1989).

[55] S. Randjbar-Daemi, A. Salam and J. Strathdee, Phys. Lett. B **132** (1983) 56.

[56] A. Lichnerowicz, "Spineurs harmonique", C.R. Acad. Sci. Paris Sér . A **257** (1963) 7.

[57] L. Palla, "Spontaneous Compactification", in Proc. of 1978 Tokyo Conf. on High Energy Physics, Eds. S. Homma et al (Phys. Society of Japan, 1979) 629.

[58] P. Li and S.-T. Yau, "Estimates of Eigenvalues of a Compact Riemannian Manifold", in "Geometry of the Laplace Operator", Proc. Sym. Pure Maths. (AMS) Vol. 36, (1980) 205.

[59] N. Kaloper, J. March-Russell, G. D. Starkman and M. Trodden, Phys. Rev. Lett. **85** (2000) 928.

[60] R. Tabbash, in the Proc. of Cairo International Conference on High Energy Physics (2001) .

[61] O. Hijazi, Commun. Math. Phys. **104**, (1986) 151; C. Bär, Math. Ann. **293**, (1992) 39.

[62] R. L. Bishop and B. O'Neill, "Manifolds of Negative Curvature",Trans. Amer. Math. Soc. **145** (1969) 1.

[63] L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 4690.

[64] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, Phys. Rev. D **59** (1999) 086001.

[65] A. G. Cohen and D. B. Kaplan, Phys. Lett. B **470** (1999) 52.

[66] T. Gherghetta and M. Shaposhnikov, Phys. Rev. Lett. **85** (2000) 240;

[69] T. Gherghetta, E. Roessl and M. E. Shaposhnikov, Phys. Lett. B **491** (2000) 353.

[67] I. Olasagasti and A. Vilenkin, Phys. Rev. D **62** (2000) 044014

[68] G. R. Dvali, hep-th/0004057.

[69] T. Gherghetta, E. Roessl and M. Shaposhnikov, Phys. Lett. B **491** (2000) 353.

[70] N. Arkani-Hamed and M. Schmaltz, Phys. Rev. D **61** (2000) 033005.

[71] J. Chen, M. A. Luty and E. Ponton, JHEP **0009** (2000) 012.

[72] S. Randjbar-Daemi and M. Shaposhnikov, Phys. Lett. B **491** (2000) 329

[73] M. Giovannini, H. Meyer and M. E. Shaposhnikov, "Warped compactification on Abelian vortex in six dimensions," hep-th/0104118.

[74] M. Ito, "Various types of five dimensional warp factor and effective Planck scale," hep-th/0109140.

[75] P. Nath, Part. Nucl. Lett. **104** (2001) 7.

[76] V. A. Rubakov, "Large and infinite extra dimensions: An Introduction," hep-ph/0104152.

[77] R. Dick, Class. Quant. Grav. **18** (2001) R1.

[78] J. Polchinski, "TASI lectures on D-branes," hep-th/9611050.

[79] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, Nucl. Phys. B **552** (1999) 246

[80] K. Behrndt and M. Cvetic, Phys. Lett. B **475** (2000) 253

[81] A. Chamblin and G. W. Gibbons, Phys. Rev. Lett. **84** (2000) 1090.

[82] R. Kallosh and A. D. Linde, JHEP **0002** (2000) 005

[83] K. Behrndt and S. Gukov, Nucl. Phys. B **580** (2000) 225

[84] K. Behrndt and M. Cvetic, Phys. Rev. D **61** (2000) 101901

[85] K. Behrndt, Phys. Lett. B **487** (2000) 30

[86] A. Ceresole, G. Dall'Agata, R. Kallosh and A. Van Proeyen, "Hypermultiplets, domain walls and supersymmetric attractors," hep-th/0104056.

[87] K. Behrndt, G. Lopes Cardoso and D. Lust, Nucl. Phys. B **607** (2001) 391.

[88] G. Lopes Cardoso, G. Dall'Agata and D. Lust, "de Sitter BPS domain wall solutions in four- and five-dimensional gauged supergravity," hep-th/0104156.

[89] J. A. De Azcarraga and P. K. Townsend, Phys. Rev. Lett. **62** (1989) 2579.

[90] A. Achucarro, J. M. Evans, P. K. Townsend and D. L. Wiltshire, Phys. Lett. B **198** (1987) 441.

[91] R. A. Battye and B. Carter, Phys. Lett. B **509** (2001) 331.

[92] W. Israel, Nuovo Cimento **B44** (1966) 1.

[93] C. W. Misner, K. S. Throne, and J. A. Wheeler, "Gravitation", Freeman and Company, San Francisco (1973).

[94] V. A. Berezin, V. A. Kuzmin and I. I. Tkachev, Phys. Rev. D **36** (1987) 2919.

[95] P. Binetruy, J. M. Cline and C. Grojean, Phys. Lett. B **489** (2000) 403.

[96] S. Alexander, Y. Ling and L. Smolin, "A thermal instability for positive brane cosmological constant in the Randall-Sundrum cosmologies," hep-th/0106097.

[97] S. Kobayashi, K. Koyama and J. Soda, "Thick brane worlds and their stability," hep-th/0107025.

[98] M. Bouhmadi-Lopez and A. Zhuk, "Comments on conformal stability of brane-world models," hep-th/0107227.

[99] A. Karch and L. J. Randall, JHEP **0105**, 008 (2001)

[100] D. Ida, T. Shiromizu and H. Ochiai, "Semiclassical instability of the brane-world: Randall-Sundrum bubbles," hep-th/0108056.

[101] J. Lykken and L. J. Randall, JHEP **0006** (2000) 014.

[102] E. Witten, "The cosmological constant from the viewpoint of string theory," hep-ph/0002297.

[103] R. Gregory, V. A. Rubakov and S. M. Sibiryakov, Phys. Rev. Lett. **84** (2000) 5928.

[104] J. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231 [Int. J. Theor. Phys. **38** (1998) 1113].

[105] P. Horava and E. Witten, Nucl. Phys. B **475** (1996) 94

[106] E. Bergshoeff, R. Kallosh and A. Van Proeyen, Fortsch. Phys. **49** (2001) 625.

[107] A. Karch and L. J. Randall, JHEP **0106** (2001) 063

[108] T. Gherghetta and A. Pomarol, Nucl. Phys. B **586** (2000) 141; Nucl. Phys. B **602** (2001) 3.

[109] W. D. Goldberger and M. B. Wise, Phys. Lett. B **475** (2000) 275.

[110] M. Sasaki, T. Shiromizu and K. Maeda, Phys. Rev. D **62** (2000) 024008

[111] I. I. Kogan, S. Mouslopoulos, A. Papazoglou, G. G. Ross and J. Santiago, Nucl. Phys. B **584** (2000) 313.

[112] J. Lesgourgues, S. Pastor, M. Peloso and L. Sorbo, Phys. Lett. B **489** (2000) 411.

[113] S. W. Hawking, T. Hertog and H. S. Reall, Phys. Rev. D **62** (2000) 043501.

[114] J. E. Kim and B. Kyae, Phys. Lett. B **486** (2000) 165 [hep-th/0005139].

[115] C. Csaki, M. Graesser, L. J. Randall and J. Terning, Phys. Rev. D **62** (2000) 045015.

[116] M. G. Santos, F. Vernizzi and P. G. Ferreira, Phys. Rev. D **64** (2001) 063506.

[117] I. Brevik, K. A. Milton, S. Nojiri and S. D. Odintsov, "Quantum (in)stability of a brane-world AdS(5) universe at nonzero temperature," hep-th/0010205.

[118] M. G. Santos, F. Vernizzi and P. G. Ferreira, "Isotropization and instability of the brane," hep-ph/0103112.

[119] K. Ghoroku and A. Nakamura, "Stability of Randall-Sundrum braneworld and tachyonic scalar," hep-th/0103071.

[120] P. D. Mannheim, Phys. Rev. D **63** (2001) 024018.

[121] N. Arkani-Hamed, M. Porrati and L. J. Randall, JHEP **0108** (2001) 017.

[122] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. **B436** (1998) 257.

[123] G. W. Gibbons and D. L. Wiltshire, Nucl. Phys. B **287** (1987) 717.

[124] G. R. Dvali and M. A. Shifman, Phys. Lett. B **396** (1997) 64.

[125] S. B. Giddings, E. Katz and L. J. Randall, JHEP **0003** (2000) 023.

[126] A. Karch and L. J. Randall, Phys. Rev. Lett. **87** (2001) 061601

[127] A. Hebecker and J. March-Russell, Nucl. Phys. B **608** (2001) 375.

[128] J. Garriga and T. Tanaka, Phys. Rev. Lett. **84** (2000) 2778.

[129] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B **565** (2000) 269.

[130] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B **477** (2000) 285.

[131] E. E. Flanagan, S. H. Tye and I. Wasserman, Phys. Rev. D **62** (2000) 044039.

[132] C. Csaki, M. Graesser, C. Kolda and J. Terning, Phys. Lett. B **462** (1999) 34.

[133] J. M. Cline, G. D. Moore and G. Servant, Phys. Rev. D **60** (1999) 105035.

[134] S. Randjbar-Daemi and M. E. Shaposhnikov, Phys. Lett. B **492** (2000) 361.

[135] L. H. Karsten, Phys. Lett. B **104** (1981) 315.

[136] D. B. Kaplan, Phys. Lett. B **288** (1992) 342.

[137] R. Narayanan and H. Neuberger, Nucl. Phys. B **412** (1994) 574; H. Neuberger, Phys. Rev. Lett. **81** (1998) 4060.

[138] S. Randjbar-Daemi and J. Strathdee, Phys. Lett. B **348** (1995) 543; Phys. Rev. D **51** (1995) 6617; Nucl. Phys. B **443** (1995) 386; Nucl. Phys. B **466** (1996) 335; Nucl. Phys. B **461** (1996) 305; Phys. Lett. B **402** (1997) 134; C. D. Fosco and S. Randjbar-Daemi, Phys. Lett. B **354** (1995) 383.

[139] For a recent review see: H. Neuberger, "Exact chiral symmetry on the lattice," hep-lat/0101006.

[140] D. B. Kaplan and M. Schmaltz, Phys. Lett. B **368** (1996) 44.

[141] S. Cullen, M. Perelstein and M. E. Peskin, Phys. Rev. D **62** (2000) 055012.

[142] L. J. Hall and D. R. Smith, Phys. Rev. D **60** (1999) 085008.

[143] A. Perez-Lorenzana, "Theories in more than four dimensions," hep-ph/0008333.

[144] S. Hannestad and G. Raffelt, Phys. Rev. Lett. **87** (2001) 051301.

[145] M. Besancon, hep-ph/0106165.

[146] T. Han, J. D. Lykken and R. Zhang, Phys. Rev. D **59** (1999) 105006.

[147] I. Antoniadis and K. Benakli, hep-ph/0108174.

[148] V. Barger, T. Han, C. Kao and R. J. Zhang, Phys. Lett. B **461** (1999) 34.

[149] C. Adloff *et al.* [H1 Collaboration], Phys. Lett. B **479** (2000) 358.

[150] S. Sigurdsson, "Experimental hints of gravity in large extra dimensions?," astro-ph/0107169.

[151] G. F. Giudice, R. Rattazzi and J. D. Wells, Nucl. Phys. B **544** (1999) 3.

[152] K. Benakli and S. Davidson, Phys. Rev. D **60** (1999) 025004.

[153] T. G. Rizzo, hep-ph/0011139.

[154] H. Davoudiasl, J. L. Hewett and T. G. Rizzo, Phys. Rev. Lett. **84** (2000) 2080; Phys. Lett. B **473** (2000) 43; Phys. Rev. D **63** (2001) 075004; Phys. Lett. B **493** (2000) 135; "Gravi-burst: Super-GZK cosmic rays from localized gravity," hep-ph/0010066.

[155] H. Murayama and M. E. Peskin, Ann. Rev. Nucl. Part. Sci. **46** (1996) 533

[156] M. E. Peskin, "Physics goals of the linear collider," hep-ph/9910521.

[157] E. W. Kolb and M. S. Turner, Eur. Phys. J. C **15** (2000) 125.

[158] A. G. Riess *et al.* [Supernova Search Team Collaboration], Astron. J. **116** (1998) 1009.

[159] S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], Astrophys. J. **517** (1999) 565.

[160] J. C. Long, "Laboratory Seach for Extra-Dimensional Effects in the Sub-millimeter Regime"; A. Kapitulnik, "Experimental Tests of Gravity Below 1 mm", *Talks presented at the International Coference on Physics Beyind Four Dimensions*, ICTP, Triest, Italy (2000).

[161] E. A. Mirabelli, M. Perelstein and M. E. Peskin, Phys. Rev. Lett. **82** (1999) 2236.

[162] K. Cheung, "Mini-review on collider signatures for extra dimensions," hep-ph/0003306.

[163] T. G. Rizzo, "Probes of universal extra dimensions at colliders," hep-ph/0106336.

[164] M. Fairbairn, Phys. Lett. B **508** (2001) 335.

[165] S. Cullen and M. Perelstein, Phys. Rev. Lett. **83** (1999) 268.

[166] A. Chodos and S. Detweiler, Phys. Rev. D **21** (1980) 2167.

[167] P. G. Freund, Nucl. Phys. B **209** (1982) 146.

[168] Q. Shafi and C. Wetterich, Phys. Lett. B **129** (1983) 387.

[169] M. A. Rubin and B. D. Roth, Nucl. Phys. B **226** (1983) 444.

[170] D. Sahdev, Phys. Lett. B **137** (1984) 155.

[171] S. Randjbar-Daemi, A. Salam and J. Strathdee, Phys. Lett. B **135** (1984) 388.

[172] E. W. Kolb and R. Slansky, Phys. Lett. B **135** (1984) 378.

[173] E. Alvarez and M. Belen Gavela, Phys. Rev. Lett. **51** (1983) 931.

[174] K. Maeda, Phys. Lett. B **138** (1984) 269.

[175] K. D. Krori, P. Borgohain and K. Das, Gen. Rel. Grav. **22** (1990) 791.

[176] U. Bleyer, M. Mohazzab and M. Rainer, "Dynamics of dimensions in factor space cosmology," gr-qc/9508035.

[177] U. Gunther and A. Zhuk, Class. Quant. Grav. **15** (1998) 2025.

[178] G. Shiu and S. H. Tye, Phys. Rev. D **58** (1998) 106007.

[179] G. R. Dvali and A. Y. Smirnov, Nucl. Phys. B **563** (1999) 63.

[180] N. Kaloper and A. D. Linde, Phys. Rev. D **59** (1999) 101303.

[181] J. M. Cline, C. Grojean and G. Servant, Phys. Rev. Lett. **83** (1999) 4245.

[182] D. J. Chung and K. Freese, Phys. Rev. D **61** (2000) 023511.

[183] G. R. Dvali and S. H. Tye, Phys. Lett. B **450** (1999) 72.

[184] N. Arkani-Hamed, S. Dimopoulos, N. Kaloper and J. March-Russell, "Early inflation and cosmology in theories with sub-millimeter dimensions," hep-ph/9903239.

[185] A. Melchiorri, F. Vernizzi, R. Durrer and G. Veneziano, Phys. Rev. Lett. **83** (1999) 4464.

[186] N. Arkani-Hamed, S. Dimopoulos, N. Kaloper and J. March-Russell, Nucl. Phys. B **567** (2000) 189.

[187] A. Riotto, Phys. Rev. D **61** (2000) 123506.

[188] D. J. Chung and K. Freese, Phys. Rev. D **62** (2000) 063513.

[189] A. Delgado, A. Pomarol and M. Quiros, JHEP **0001** (2000) 030.

[190] K. Akama and T. Hattori, Mod. Phys. Lett. A **15** (2000) 2017.

[191] S. Hannestad, Phys. Rev. D **64** (2001) 023515.

[192] G. D. Starkman, D. Stojkovic and M. Trodden, Phys. Rev. D **63** (2001) 103511; "Homogeneity, flatness and 'large' extra dimensions," hep-th/0106143.

[193] G. Steigman, Ann. Rev. Nucl. Part. Sci. **29** (1979) 313.

[194] E. W. Kolb, M. J. Perry and T. P. Walker, Phys. Rev. D **33** (1986) 869.

[195] S. Weinberg, "Gravitation and Cosmology", John Wiley & Sons Inc. (1972).

[196] N. Kaloper, Phys. Rev. D **60** (1999) 123506.

[197] P. Candelas and S. Weinberg, Nucl. Phys. B **237** (1984) 397.

[198] N. V. Krasnikov, Phys. Lett. B **193** (1987) 37.

[199] T. R. Taylor, Phys. Lett. B **252** (1990) 59.

[200] K. Kikkawa, T. Kubota, S. Sawada and M. Yamasaki, Nucl. Phys. B **260** (1985) 429; Phys. Lett. B **144** (1984) 365.

[201] Y. Okada, Phys. Lett. B **150** (1985) 103.

[202] T. Koikawa and M. Yoshimura, Phys. Lett. B **150** (1985) 107.

[203] M. Gleiser, S. Rajpoot and J. G. Taylor, Annals Phys. **160** (1985) 299.

[204] Y. Okada, Nucl. Phys. B **264** (1986) 197.

[205] F. S. Accetta, M. Gleiser, R. Holman and E. W. Kolb, Nucl. Phys. B **276** (1986) 501.

[206] K. Maeda, Class. Quant. Grav. **3** (1986) 233; Class. Quant. Grav. **3** (1986) 651; Phys. Lett. B **186** (1987) 33.

[207] L. M. Sokolowski and Z. A. Golda, Phys. Lett. B **195** (1987) 349.

[208] M. Szydlowski, Phys. Lett. B **215** (1988) 711.

[209] V. A. Berezin, G. Domenech, M. L. Levinas, C. O. Lousto and N. D. Umerez, Gen. Rel. Grav. **21** (1989) 1177.

[210] Y. A. Kubyshin, V. A. Rubakov and I. Tkachev, Int. J. Mod. Phys. A **4** (1989) 1409.

[211] M. Demianski, M. Szydlowski and J. Szczesny, Gen. Rel. Grav. **22** (1990) 1217.

[212] L. Amendola, E. W. Kolb, M. Litterio and F. Occhionero, Phys. Rev. D **42** (1990) 1944.

[213] R. Holman, E. W. Kolb, S. L. Vadas and Y. Wang, Phys. Rev. D **43** (1991) 995.

[214] U. Bleyer and A. Zhuk, Class. Quant. Grav. **12** (1995) 89.

[215] E. Carugno, M. Litterio, F. Occhionero and G. Pollifrone, Phys. Rev. D **53** (1996) 6863.

[216] T. Banks and M. Dine, Phys. Rev. D **50** (1994) 7454.

[217] N. Arkani-Hamed, S. Dimopoulos and J. March-Russell, Phys. Rev. D **63** (2001) 064020.

[218] K. R. Dienes, E. Dudas, T. Gherghetta and A. Riotto, Nucl. Phys. B **543** (1999) 387.

[219] G. Huey, P. J. Steinhardt, B. A. Ovrut and D. Waldram, Phys. Lett. B **476** (2000) 379.

[220] W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. **83** (1999) 4922.

[221] L. Mersini, Mod. Phys. Lett. A **16** (2001) 1583.

[222] U. Gunther and A. Zhuk, Phys. Rev. D **61** (2000) 124001; Class. Quant. Grav. **18** (2001) 1441.

[223] J. Garriga, O. Pujolas and T. Tanaka, Nucl. Phys. B **605** (2001) 192.

[224] U. Mahanta, "Radion stabilization in the Randall-Sundrum model with quadratic and quartic potentials," hep-ph/0006350.

[225] S. Kalyana Rama, Phys. Lett. B **495** (2000) 176.

[226] U. Gen and M. Sasaki, "Radion on the de Sitter brane," gr-qc/0011078.

[227] G. W. Gibbons, R. Kallosh and A. D. Linde, JHEP **0101** (2001) 022.

[228] S. Bae and H. S. Lee, "Constraints on the radion from vacuum stability in the Randall-Sundrum theory," hep-ph/0011275.

[229] Z. Chacko and P. J. Fox, Phys. Rev. D **64** (2001) 024015.

[230] S. Randjbar-Daemi and M. H. Sarmadi, Phys. Lett. B **151** (1985) 343.

[231] F. Feruglio, "Grand unification in extra dimensions and proton decay," hep-ph/0105321.

[232] G. Altarelli and F. Feruglio, Phys. Lett. B **511** (2001) 257.

[233] A. Aranda and C. D. Carone, Phys. Rev. D **63** (2001) 075012.

[234] F. C. Adams, G. L. Kane, M. Mbonye and M. J. Perry, Int. J. Mod. Phys. A **16** (2001) 2399.

[235] S. J. Huber and Q. Shafi, Phys. Lett. B **498** (2001) 256.

[236] M. Shiozawa *et al.* [Super-Kamiokande Collaboration], Phys. Rev. Lett. **81** (1998) 3319.

[237] A. Masiero, M. Peloso, L. Sorbo and R. Tabbash, Phys. Rev. D **62** (2000) 063515.

[238] E. A. Mirabelli and M. Schmaltz, Phys. Rev. D **61** (2000) 113011.

[239] N. Arkani-Hamed, Y. Grossman and M. Schmaltz, Phys. Rev. D **61** (2000) 115004.

[240] R. Jackiw and C. Rebbi, Phys. Rev. D **13** (1976) 3398.

[241] E. J. Weinberg, Phys. Rev. D **24** (1981) 2669.

[242] H. B. Nielsen and P. Olesen, Nucl. Phys. B **61** (1973) 45.

[243] E. W. Kolb, A. D. Linde and A. Riotto, Phys. Rev. Lett. **77** (1996) 4290.

[244] D. J. Chung, E. W. Kolb and A. Riotto, Phys. Rev. D **60** (1999) 063504.

[245] G. F. Giudice, E. W. Kolb and A. Riotto, Phys. Rev. D **64** (2001) 023508.

[246] D. V. Nanopoulos and S. Weinberg, Phys. Rev. D **20** (1979) 2484.

[247] A. Mazumdar and A. Perez-Lorenzana, "Affleck-Dine baryogensis in large extra dimensions," hep-ph/0103215; R. Allahverdi, K. Enqvist, A. Mazumdar and A. Perez-Lorenzana, "Baryogenesis in theories with large extra spatial dimensions," hep-ph/0108225.

[248] L. E. Ibanez and F. Quevedo, JHEP **9910** (1999) 001.

[249] G. Dvali, S. Randjbar-Daemi and R. Tabbash, "The origin of spontaneous symmetry breaking in theories with large extra dimensions," hep-ph/0102307.

[250] R. Barbieri, L. J. Hall and Y. Nomura, Phys. Rev. D **63** (2001) 105007. For further explanation see A. Masiero, C. A. Scrucca, M. Serone and L. Silvestrini, "Non-local symmetry breaking in Kaluza-Klein theories," hep-ph/0107201.

[251] M. Shaposhnikov and P. Tinyakov, Phys. Lett. B **515** (2001) 442.

[252] N. Arkani-Hamed, A. G. Cohen and H. Georgi, Phys. Lett. B **513** (2001) 232.

[253] L. J. Hall, Y. Nomura and D. R. Smith, "Gauge-Higgs unification in higher dimensions," hep-ph/0107331;

[254] I. Antoniadis, K. Benakli and M. Quiros, "Finite Higgs mass without supersymmetry," hep-th/0108005.

[255] N. V. Krasnikov, Phys. Lett. B **273** (1991) 246.

[256] H. Hatanaka, T. Inami and C. S. Lim, Mod. Phys. Lett. A **13** (1998) 2601.

[257] N. S. Manton, Nucl. Phys. B **158** (1979) 141.

[258] C. Bachas, "A way to break supersymmetry," hep-th/9503030; "Magnetic supersymmetry breaking," hep-th/9509067.

[259] R. Blumenhagen, L. Görlich, B. Körs and D. Lüst, JHEP **0010**, (2000) 006; Fortsch. Phys. **49**, (2001) 591.

[260] M. Sakamoto and S. Tanimura, "Spontaneous breaking of the C, P, and rotational symmetries by topological defects in extra two dimensions," hep-th/0108208.

[261] H. Hatanaka, Prog. Theor. Phys. **102** (1999) 407.

[262] Y. Hosotani, Annals Phys. **190** (1989) 233.

[263] M. Green and J. Schwarz, Phys. Lett **149B** (1984) 117.

[264] S. Randjbar-Daemi, Abdus Salam, E. Sezgin, and J. A. Strathdee, Phys. Lett **151B** (1985) 351.

[265] E. Witten, Phys. Lett **117B** (1982) 324.

[266] E. Witten, Nucl. Phys. B **195** (1982) 481.

[267] A. N. Schellekens, Nucl. Phys. B **250** (1985) 252.

[268] J. A. Frieman and E. W. Kolb, Phys. Rev. Lett. **55** (1985) 1435.

[269] "Supergravities in Diverse Dimensions", v.2 , (Edited by M. Abdus Salam and E. Sezgin), Elsevier Science Publishers B.V., and World Sientific Publishing CO. PTE. LTD. (1989), and references therein.

[270] K. R. Dienes, E. Dudas and T. Gherghetta, Phys. Lett. B **436** (1998) 55; Nucl. Phys. B **537** (1999) 47.

[271] A. De Rujula, A. Donini, M. B. Gavela and S. Rigolin, Phys. Lett. B **482** (2000) 195.