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Noncommutative Geometry and String Field Theory

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Chapter 1

Introduction

String theory is so far one of the most successful attempts to unify general relativity and quantum mechanics [1, 2]. There are various arguments why string theory should be the ultimate theory describing completely our universe. Unfortunately, any direct experimental evidence in favor of string theory is still lacking.

One of the main motors which stimulates the ongoing research in string theory is its internal beauty. It goes hand in hand with modern mathematics, sometimes even anticipating it. Mathematics is the language in which God wrote the world, as one can deduce from its unexpected effectiveness in describing the real world. Perhaps even more surprising is the fact that string theory is unexpectedly effective in describing various branches of modern mathematics. From time to time string theorists learn some new part of mathematics and they often discover that it nicely fits into string theory. They are able to establish a dictionary between mathematical and physical terms which provides us with a simple and intuitive understanding of deep mathematical theorems.

Among notable examples, which have been brought down from the heaven of mathematics to the earth of string theory in recent years are: noncommutative geometry, K-theory and derived categories. The main theme of this thesis is the first one in this list — noncommutative geometry.

The idea of replacing the ordinary coordinates by noncommutative ones

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1.1}$$

where $\theta^{\mu\nu}$ is real antisymmetric matrix is of course not new. It has been proposed already in 1947 by Snyder [3] in analogy with what does quantum mechanics when replaces the coordinate and the canonical momentum by noncommuting operators.

The major mathematical developments in the noncommutative geometry came in the 1980's in the work of Alain Connes and his collaborators [4]. The basic idea is quite simple. According to a classical theorem of Gelfand and Najmark all topological properties of a manifold are encoded in a commutative C^* algebra of complex functions on this manifold. The idea of noncommutative geometry is to translate first all the geometric notions to the algebraic ones, and then try to abandon the assumption of commutativity of the basic algebra.

First time the noncommutative geometry entered string theory, was in 1986, when Edward Witten used it as a guiding principle in constructing a covariant string field theory [5]. String field theory can be viewed either as an interacting field theory for infinitely many excitations of a string or as a way of calculation of string amplitudes using string propagators and vertices. As opposed to conventional methods of calculation on-shell string diagrams, string field theory necessarily goes off-shell. It is precisely this aspect which enables the study of nonperturbative physics like tachyon condensation.

Noncommutative geometry started to become popular among string theorists only after 1997 when Connes, Douglas and Schwarz [6] shown in the matrix model that the inclusion of constant Neveu-Schwarz B-field background effectively makes the spacetime noncommutative. After the subsequent research it was clearly established that one gets a noncommutative field theory as a low energy effective world-volume theory on a D-brane with nonzero parallel B-field. Much of the early developments have been summarized and many original material added in a seminal paper by Seiberg and Witten in 1999 [7]. There is also a recent review by Douglas and Nekrasov [8].

The organization of this thesis follows to some extent the evolution of the authors interests. Apart of explaining our work we tried to provide some introductory material to make the whole thesis more readable and self contained. For obvious reasons we could not have been very detailed neither in the review part, nor in the presentation of our work. We refer the interested reader to several good review articles. Concerning our original work, we tried to clearly explain the main ideas postponing sometimes possible generalizations, technical details and various speculations to the published articles.

The thesis is organized as follows: In chapter 2 we explain basic ideas of noncommutative field theory, its stringy origin and various interesting properties. We explain two of our papers, one with Bonora and Tomasiello [28] on anomalies, and another one with Bonora, Sheikh-Jabbari and Tomasiello [36] on the possibility of having $SO(N)$ and $Sp(N)$ gauge groups. Chapter 3 is devoted to the cubic string field theory. As there are not as many reviews as on the other subjects, we shall be slightly more detailed. We will discuss various formulations and approaches, and relations between them. We also explain the Sen's conjectures which triggered much of the recent developments. In chapter 4 we explain some properties of the string field algebra, the star product, the wedge states and in particular the identity state. Results here are mostly original. Chapter 5 discusses some approaches to the problem of finding the exact solution for the tachyon condensate, which we consider to be an interesting and important problem. One section of that chapter is based on our paper [86], the other parts are new results. Chapter 6 sort of merges the two noncommutativities encountered above, it adds the B field into the string field theory. It is based on our paper [91] with some updates, especially regarding the K-theory. Finally chapter 7 deals with the most mathematical aspects of noncommutative geometry and applies much of the fancy techniques to the problem of tachyon condensation on the torus. It is a simplified version of our paper with Krajewski [123].

Chapter 2

Noncommutative field theory

Noncommutative field theory as a nonlocal modification of an ordinary field theory has been already studied for some time. It has become much more popular subject, when people realized that it arises as a certain low energy effective theory on D-brane in the presence of a constant Neveu-Schwarz two form B-field. Subsequently, there was an explosion of literature on various stringy and field theoretic properties. Let us look first on its stringy origin.

2.1 Stringy origin

Let us consider an open string ending on a Dp-brane in flat space with constant flat metric $g_{\mu\nu}$ and constant $B_{\mu\nu}$ Neveu-Schwarz background. The worldsheet action of the string is

$$S = \frac{1}{4\pi\alpha'} \int d\sigma \, d\tau \, \left(g_{\mu\nu} \partial_a X^\mu \partial^a X^\nu - 2\pi i \alpha' B_{\mu\nu} \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu \right), \tag{2.1}$$

where $\mu, \nu = 0, 1, \ldots, d-1$ are target space indices ($d = 10$ or $d = 26$ depending on whether we are studying superstring or bosonic string theory). Indices $a, b = 1, 2$ are worldsheet indices.

The second term in the action is a total derivative, it does not affect the equations of motion, only the boundary conditions. For μ along the Dp-brane, i.e. $0 \leq \mu \leq p-1$, the boundary conditions are

$$g_{\mu\nu} \partial_\sigma X^\nu + 2\pi i \alpha' B_{\mu\nu} \partial_\tau X^\nu |_{\partial\Sigma} = 0, \tag{2.2}$$

along the remaining directions we have the Dirichlet condition $\partial_\tau X|_{\partial\Sigma} = 0$. Here $\partial\Sigma$ denotes the boundary of the worldsheet $\sigma = 0, \pi$. Note that in the limit $B_{\mu\nu} \to 0$ we have p Neumann and $d-p$ Dirichlet boundary conditions.

One can calculate the propagator with these boundary conditions, for our purposes we quote only the result for the points on the boundary of the worldsheet

$$\langle \mathrm{T} X^\mu(\tau) X^\nu(\tau') \rangle = -\alpha' G^{\mu\nu} \log(\tau-\tau')^2 + \frac{i}{2}\theta^{\mu\nu}\varepsilon(\tau-\tau'), \tag{2.3}$$

where $\varepsilon(\tau) = 1$ for $\tau > 0$ and $\varepsilon(\tau) = -1$ for $\tau < 0$. The coefficients $G^{\mu\nu}$ and $\theta^{\mu\nu}$ are the effective metric and noncommutativity seen by open strings. They are given in terms of closed string metric and the B-field by

$$G_{\mu\nu} = g_{\mu\nu} - (2\pi\alpha')^2 (Bg^{-1}B)_{\mu\nu}, \tag{2.4}$$

$$\theta^{\mu\nu} = -(2\pi\alpha')^2 \left(\frac{1}{g+2\pi\alpha' B} B \frac{1}{g-2\pi\alpha' B} \right)^{\mu\nu}. \tag{2.5}$$

The Heisenberg operators associated to the string coordinate on the boundary of the worldsheet satisfy simple commutation relation

$$[X^\mu(\tau), X^\nu(\tau)] = \lim_{\varepsilon\to 0} (X^\mu(\tau)X^\nu(\tau-\varepsilon) - X^\nu(\tau)X^\mu(\tau-\varepsilon)) = i\theta^{\mu\nu}. \tag{2.6}$$

This noncommutativity is at the heart of noncommutative field theory, for many purposes however, it is more convenient to derive low energy effective theory without the recourse to Heisenberg operators of the string.

Let us now look at the string amplitudes at the tree level. The three gluon amplitude is proportional to the conformal field correlator

$$\begin{aligned} \langle \xi^1 \cdot \partial X e^{ip^1\cdot X(\tau_1)} \xi^2 \cdot \partial X e^{ip^2\cdot X(\tau_2)} \xi^3 \cdot \partial X e^{ip^3\cdot X(\tau_3)} \rangle &\sim \frac{1}{(\tau_1-\tau_2)(\tau_2-\tau_3)(\tau_3-\tau_1)} \\ \cdot (\xi^1\cdot\xi^2 p^2\cdot\xi^3 + \xi^1\cdot\xi^3 p^1\cdot\xi^2 + \xi^2\cdot\xi^3 p^3\cdot\xi^1 &+ 2\alpha' p^3\cdot\xi^1 p^1\cdot\xi^2 p^2\cdot\xi^3) \\ \cdot e^{-\frac{i}{2}(p_i^1\theta^{ij}p_j^2\varepsilon(\tau_1-\tau_2) + p_i^2\theta^{ij}p_j^3\varepsilon(\tau_2-\tau_3) + p_i^3\theta^{ij}p_j^1\varepsilon(\tau_3-\tau_1))}. \end{aligned} \tag{2.7}$$

Fixing the $SL(2,\mathbb{R})$ invariance, or alternatively including correctly the worldsheet ghosts, leads to the elimination of the denominator $(\tau_1-\tau_2)(\tau_2-\tau_3)(\tau_3-\tau_1)$. Using the momentum conservation $p^1+p^2+p^3=0$ to simplify a bit the expression, leads finally to the amplitude

$$\mathcal{M} \sim (\xi^1\cdot\xi^2 p^2\cdot\xi^3 + \xi^1\cdot\xi^3 p^1\cdot\xi^2 + \xi^2\cdot\xi^3 p^3\cdot\xi^1 + 2\alpha' p^3\cdot\xi^1 p^1\cdot\xi^2 p^2\cdot\xi^3) e^{-\frac{i}{2}p_i^1\theta^{ij}p_j^2}. \tag{2.8}$$

This amplitude is clearly reproduced by the field theoretic action

$$\frac{1}{4G_o^2}\int \mathrm{Tr}\, \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu}, \tag{2.9}$$

where G_o is the effective coupling constant

$$G_o = g_o \left(\frac{\det G}{\det(g + 2\pi\alpha' B)} \right)^{\frac{1}{4}} \tag{2.10}$$

determined by the comparison with the Dirac-Born-Infeld action for slowly varying fields. The noncommutative field strength is defined by

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \hat{A}_\mu * \hat{A}_\nu - \hat{A}_\nu * \hat{A}_\mu . \tag{2.11}$$

The star product $f * g$ of any two functions on the spacetime is defined by

$$\begin{aligned} f(x) * g(x) &= e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\zeta^\nu}} f(x+\xi) g(x+\zeta)\Big|_{\xi=\zeta=0} \\ &= f(x) e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} g(x) \\ &= fg + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f \partial_\nu g + \mathcal{O}(\theta^2). \end{aligned} \tag{2.12}$$

This star product is called Moyal-Weyl product or Moyal product for short. As an illustrative example we can readily calculate

$$\begin{aligned} x^\mu * x^\nu - x^\nu * x^\mu &= i\theta^{\mu\nu}, \\ e^{ikx} * e^{ipx} &= e^{-\frac{i}{2}\theta^{\mu\nu}k_\mu p_\nu} e^{i(k+p)x}. \end{aligned} \tag{2.13}$$

One could repeat the above reasoning for arbitrary tree amplitude with arbitrary external states and to all orders in α'. We always arrive to the conclusion, that the only effect of the noncommutativity is to replace ordinary products by the noncommutative $*$ product. For the loop amplitudes it is more complicated, it turns out that a single star product is not enough and we need to introduce additional products [9, 10].

2.2 Gauge symmetry

We have just seen that the action for pure noncommutative Yang-Mills is (omitting the hats)

$$\frac{1}{4G_o^2} \int \operatorname{Tr} F_{\mu\nu} * F^{\mu\nu}, \tag{2.14}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu * A_\nu - A_\nu * A_\mu . \tag{2.15}$$

This action is invariant under the gauge transformations

$$\begin{aligned} \delta A_\mu &= \partial_\mu \lambda + A_\mu * \lambda - \lambda * A_\mu, \\ \delta F_{\mu\nu} &= F_{\mu\nu} * \lambda - \lambda * F_{\mu\nu}. \end{aligned} \tag{2.16}$$

The fields $A_\mu, F_{\mu\nu}$ and the gauge parameter λ can be complex valued functions or matrix valued functions. The invariance of the action follows from the basic identity

$$\int \mathrm{Tr}\, f * g = \int \mathrm{Tr}\, g * f. \tag{2.17}$$

When the noncommutative gauge field is an antihermitian matrix $N \times N$ it is clearly a noncommutative generalization of $U(N)$ field. Let us look how is this condition preserved under gauge transformation. More generally suppose that $A_\mu = A_\mu^a T^a$ is an element of some Lie algebra with basis T^a. Then under a noncommutative gauge transformation

$$\begin{aligned} \delta A_\mu &= \partial_\mu \lambda + A_\mu^a * \lambda^b T^a T^b - \lambda^b * A_\mu^a T^b T^a \qquad (2.18)\\ &= \partial_\mu \lambda + \frac{1}{2}(A_\mu^a * \lambda^b + \lambda^b * A_\mu^a)\,[T^a, T^b] + \frac{1}{2}(A_\mu^a * \lambda^b - \lambda^b * A_\mu^a)\{T^a, T^b\}, \end{aligned}$$

the transformed field will belong to the same Lie algebra, only if does the anticommutator $i\{T^a, T^b\}$. This fact is true only for $u(N)$ algebra. It seems that there is no way to construct noncommutative theory with other gauge groups. There is however a somewhat more exotic possibility proposed in our work with Bonora, Sheikh-Jabbari and Tomasiello [36], and independently by Jurčo, Schraml and Wess [37]. To explain it, we need first to present some basics of the Seiberg-Witten map.

Seiberg and Witten in their celebrated paper [7] discussed an apparent paradox that open strings in the low energy regime can lead both to the ordinary and noncommutative gauge theory. Indeed, there are various ways of regularizing the worldsheet theory in the presence of background gauge field. Point splitting regularization leads to noncommutative gauge invariance as described above, Pauli-Villars regularization leads to an ordinary gauge invariance. As the resulting theory should not depend on how we treat the infinities, they deduced, there should be a map which maps noncommutative theory and its gauge invariance to the ordinary and vice versa. Such a map has to satisfy

$$\hat{A}(A) + \hat{\delta}_{\hat{\lambda}} \hat{A}(A) = \hat{A}(A + \delta_\lambda A). \tag{2.19}$$

We have restored here the hats to indicate the noncommutative quantities. This can be solved perturbatively in $\theta^{\mu\nu}$ with the result

$$\begin{aligned} \hat{A}^\mu(A) &= A^\mu - \frac{i}{4}\theta^{\nu\rho}\{A_\nu, \partial_\rho A^\mu + F^\mu_\rho\} + \mathcal{O}(\theta^2), \\ \hat{\lambda}(\lambda, A) &= \lambda + \frac{i}{4}\theta^{\mu\nu}\{\partial_\mu\lambda, A_\nu\} + \mathcal{O}(\theta^2). \qquad (2.20) \end{aligned}$$

Obviously the gauge groups of ordinary and noncommutative gauge theories are different. However, the equivalence between these two theories requires only, that physical configuration spaces — the sets of gauge orbits — match. From another point of view, we can expand all the star products in the noncommutative Yang-Mills theory and we get an ordinary Yang-Mills theory with an infinite series of higher order terms:

$$S_{NCYM} = S_{YM} + \textit{higher order terms}. \tag{2.21}$$

The noncommutative gauge theory can thus be viewed as an efficient and elegant way of dealing with such ordinary theories with higher order terms.

The solution for the Seiberg-Witten map has been found recently to all orders by Liu and Michelson [11, 12] and by Okawa and Ooguri [13].

2.3 Perturbative properties

Since the times of Feynman the perturbation theory has become one of the most popular tools in theoretical physics. It is therefore not surprising, that with the advent of noncommutative field theory people tried to repeat old text-book calculations in this new setting. Pioneering calculations have been performed by Filk in 1996 [14], who obtained the Feynman rules and understood the role of planarity of the Feynman diagrams. Interesting UV/IR mixing phenomena in non-planar diagrams have been discovered by Minwalla, Van Raamsdonk and Seiberg [15]. Thorough discussion of one loop renormalization of noncommutative Yang-Mills theories [18] can be found in the papers by Bonora and Salizzoni [19], by Martin and Sanchez-Ruiz [20] and references therein. Unitarity has been studied recently in [22]. More references on unitarity, Wilsonian renormalizability and thermal aspects can be found in the review [8].

2.3.1 Feynman rules

For simplicity let us turn our attention to the noncommutative scalar field theory only. The results can be straightforwardly extended to the case of fermions and gauge fields. Noncommutative analog of ordinary ϕ^3 theory has the action

$$S = \int d^px \left[\frac{1}{2}\partial\phi * \partial\phi + \frac{1}{2}m^2\phi * \phi + \frac{1}{3!}g\,\phi * \phi * \phi \right]. \tag{2.22}$$

From the definition of the star product follows

$$\begin{aligned} \int d^px\, \phi * \phi &= \int d^px\, \phi\phi, \\ \int d^px\, \partial\phi * \partial\phi &= \int d^px\, \partial\phi\partial\phi, \end{aligned} \tag{2.23}$$

the propagator is thus the same as in the commutative theory. The vertex is easily obtained from the identity

$$\int d^p x\, \phi * \phi * \phi = \int\!\!\int\!\!\int d^p k_1\, d^p k_2\, d^p k_3\, (2\pi)^p \delta(k_1 + k_2 + k_3) \tilde{\phi}(k_1) \tilde{\phi}(k_2) \tilde{\phi}(k_3) e^{-\frac{i}{2} k_1 \times k_2}. \tag{2.24}$$

Here we have introduced a convenient notation $k_1 \times k_2 = k_1^\mu \theta^{\mu\nu} k_2^\nu$. The resulting vertex is

$$V = g \frac{1}{2} \left(e^{-\frac{i}{2} k_1 \times k_2} + e^{\frac{i}{2} k_1 \times k_2} \right) = g \cos \left(\frac{k_1 \times k_2}{2} \right). \tag{2.25}$$

Note that the vertex exhibits Bose symmetry under the interchange of the external lines due to the momentum conservation and antisymmetry of the $\times$ product.

For more complicated diagrams there is an elegant way of organizing the calculation. It is the 't Hooft's double line notation [16, 17], which was originally developed to study the large N_c limit of gauge theories. All lines are replaced by double lines or alternatively by thin strips which are connected by the vertices so that the two lines never join or cross each other. It means, that the strips are glued together to produce smooth surface with boundaries which resembles somewhat string worldsheet diagrams. All vertices carry a factor of

$$V(k_1, k_2, \ldots, k_n) = e^{-\frac{i}{2} \sum_{i<j} k_i \times k_j}. \tag{2.26}$$

Note that this is not symmetric under an interchange of external lines, it only becomes symmetric after summing over different orders of $n-1$ lines.

To give an example of how does it work, let us look on the noncommutativity factor in the one loop bubble diagram. The equivalence of the two methods of Feynman diagram calculation is best exemplified by the identity

$$\cos^2 \left(\frac{k \times l}{2} \right) = \frac{1}{2} + \frac{1}{2} \cos(k \times l). \tag{2.27}$$

The left hand side is the noncommutativity contribution in the straightforward calculation. On the right hand side the first and second terms correspond to the planar and nonplanar diagrams in the double line notation respectively. In general one can prove, that for any planar diagram the phase factor depends only on the external momenta.

2.3.2 One loop renormalization

To demonstrate how does the noncommutativity affects the one loop renormalization, we will repeat the calculation [15] of the quadratic part of the effective action in the Euclidean noncommutative ϕ^4 theory in $d = 4$. On the tree level we have

$$\Gamma_0^{(2)} = p^2 + m^2. \tag{2.28}$$

The one-loop corrections are

$$\begin{aligned}\Gamma^{(2)}_{1planar} &= \frac{g^2}{3(2\pi)^4}\int \frac{d^4k}{k^2+m^2}, \\ \Gamma^{(2)}_{1nonplanar} &= \frac{g^2}{6(2\pi)^4}\int \frac{d^4k}{k^2+m^2}e^{ik\times p}.\end{aligned} \tag{2.29}$$

Introducing the Schwinger parameters to regularize the divergences we get

$$\begin{aligned}\Gamma^{(2)}_{1planar} &= \frac{g^2}{48\pi^2}\int \frac{d\alpha}{\alpha^2}e^{-\alpha m^2}, \\ \Gamma^{(2)}_{1nonplanar} &= \frac{g^2}{96\pi^2}\int \frac{d\alpha}{\alpha^2}e^{-\alpha m^2 - \frac{p\circ p}{\alpha}},\end{aligned} \tag{2.30}$$

where we denote $p \circ q = -p_\mu \theta^2_{\mu\nu} q_\nu$. The ultraviolet divergence $\alpha \to 0$ is regularized by introducing the cutoff factor $e^{-\frac{1}{\Lambda^2\alpha}}$. The divergent parts of the diagrams are thus

$$\begin{aligned}\Gamma^{(2)}_{1planar} &= \frac{g^2}{48\pi^2}\left(\Lambda^2 - m^2\log\frac{\Lambda^2}{m^2}\right) + \mathcal{O}(1), \\ \Gamma^{(2)}_{1nonplanar} &= \frac{g^2}{96\pi^2}\left(\Lambda^2_{eff} - m^2\log\frac{\Lambda^2_{eff}}{m^2}\right) + \mathcal{O}(1),\end{aligned} \tag{2.31}$$

where

$$\Lambda^2_{eff} = \frac{1}{\frac{1}{\Lambda^2} + p\circ p}. \tag{2.32}$$

The planar diagram is the same as its commutative counterpart (for higher point functions there would be simple phase factors given by the external momenta). The nonplanar part shows however rather peculiar behavior. It is finite in the limit $\Lambda \to \infty$, provided $p \neq 0$. Clearly the ultraviolet limit $\Lambda \to \infty$ and the infrared limit $p \to 0$ do not commute. This is the famous UV-IR mixing.

The one-loop effective action to the quadratic order takes the form

$$\begin{aligned}S^{(2)}_{1PI} = & \int d^4p\,\frac{1}{2}\phi(p)\phi(-p)\left[p^2 + M^2 + \frac{g^2}{96\pi^2\left(p\circ p + \frac{1}{\Lambda^2}\right)}\right. \\ & \left. -\frac{g^2M^2}{96\pi^2}\log\frac{1}{M^2\left(p\circ p + \frac{1}{\Lambda^2}\right)} + \cdots\right],\end{aligned} \tag{2.33}$$

where M is the renormalized mass

$$M^2 = m^2 + \frac{g^2\Lambda^2}{48\pi^2} - \frac{g^2 m^2}{48\pi^2}\log\frac{\Lambda^2}{m^2} + \cdots. \tag{2.34}$$

There are two limiting cases. First let us take the limit $p \to 0$, i.e. $p \circ p << \frac{1}{\Lambda^2}$, then

$$S^{(2)}_{1PI} = \int d^4p\, \frac{1}{2}\phi(p)\phi(-p)(p^2 + M'^2). \tag{2.35}$$

In the second case, when we take first the limit $\Lambda \to \infty$, i.e. $p \circ p >> \frac{1}{\Lambda^2}$, we get

$$S^{(2)}_{1PI} = \int d^4p\, \frac{1}{2}\phi(p)\phi(-p)\left[p^2 + M^2 + \frac{g^2}{96\pi^2 p\circ p} - \frac{g^2 M^2}{96\pi^2}\log\frac{1}{M^2 \circ p} + \cdots\right]. \tag{2.36}$$

We see that after removing the cutoff the amplitude has a pole at $p = 0$. It is not clear, at least to the author, whether this infrared divergence can cause some problems with the standard renormalization procedure. At the one-loop level there should be no problems. If we recall the way noncommutative field theory is derived from the string theory we get a natural physical interpretation of the divergence. It corresponds to the propagation of some massless degrees of freedom of the closed string. However unlike the naive expectations, they do not correspond neither to graviton, NS-NS B-field, nor the dilaton. They are sort of collective excitations. Further discussion can be found in [21, 23]. The problem deserves further investigation since it can help us to understand the outstanding question, how the closed strings emerge from the open strings. Noncommutative field theory resembles in many respects open string field theory and it knows indeed about the closed strings.

2.4 Anomalies

We have seen in the section 2.2 that the principle of gauge symmetry can be very naturally embedded into noncommutative field theory. On the other hand we know from the ordinary field theory, that on the quantum level there are certain problems with chiral fermions [24, 25]. Generically they lead to anomalies and unless their quantum numbers cause the anomalies to cancel, the theory looses its gauge invariance and hence consistency.

The noncommutative field theories arising in the string theory naturally contains fermions and thus make the study of possible anomalies relevant [26, 27, 28, 29, 30]. To anticipate, for the $U(N)$ noncommutative gauge theory the result is as one might have expected. For other gauge groups like $SO(n)$ and $Sp(n)$ it probably does not have much meaning, it is not even clear how to formulate the dynamics. An interesting paper by Intriligator and Kumar [29] appeared recently, where they discuss the product group $U(N_1) \times U(N_2) \times \cdots \times U(N_k)$, which admits chiral fermions in the bifundamental representation.

To start, we need to discuss the possible couplings of fermions to the gauge field. We shall follow to some extent [27]. The gauge transformation law (2.16), as one can easily check, satisfies the closure condition

$$\delta_{\lambda_1}\delta_{\lambda_2} - \delta_{\lambda_2}\delta_{\lambda_1} = \delta_{[\lambda_1,\lambda_2]}. \tag{2.37}$$

When we couple fermions, this condition *uniquely* determines the gauge transformation of a fermion to have one of the three forms

$$\begin{aligned} &\text{a)} \quad \delta\psi = -\lambda * \psi, && (2.38)\\ &\text{b)} \quad \delta\psi = \psi * \lambda, && (2.39)\\ &\text{c)} \quad \delta\psi = -[\lambda, \psi]_*. && (2.40) \end{aligned}$$

Associated covariant derivatives take the form

$$\begin{aligned} &\text{a)} \quad D_\mu\psi = \partial_\mu\psi + A_\mu * \psi, && (2.41)\\ &\text{b)} \quad D_\mu\psi = \partial_\mu\psi - \psi * A_\mu, && (2.42)\\ &\text{c)} \quad D_\mu\psi = \partial_\mu\psi + [A_\mu, \psi]_*. && (2.43) \end{aligned}$$

We will consider here only the possibility (a), since the results for (b) follow simply from the observation that

$$\psi^a *_\theta A_\mu^{ab} = A_\mu^{ab} *_{-\theta} \psi^a. \tag{2.44}$$

Concerning the case (c), we know that in the commutative case there is no anomaly for the fermions in the adjoint representation. One can argue that neither in the noncommutative case there is an anomaly.

The singlet anomaly takes the form

$$\langle\, \mathrm{Tr}\left[\partial_\mu J^5_\mu + \left[A_\mu, J^5_\mu\right]_*\right] \rangle = 2i\frac{1}{n!}\left(\frac{i}{4\pi}\right)^n \varepsilon^{\mu_1\ldots\mu_{2n}}\, \mathrm{Tr}\, F_{\mu_1\mu_2} * F_{\mu_3\mu_4} * \ldots * F_{\mu_{2n-1}\mu_{2n}}. \tag{2.45}$$

The trace is taken only over the $U(N)$ indices. In some sense it resembles both the singlet and multiplet covariant anomaly in the commutative case.

Now let us turn our attention to the consistent anomaly. The effective action coming from integrating over fermions in the background gauge field is given by

$$e^{-W[A]} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\int \bar{\psi}(i\not{D}-m)\psi}. \tag{2.46}$$

Under the gauge transformation of the form (2.16) the effective action changes as

$$\delta W[A] = X_\lambda W[A], \tag{2.47}$$

where

$$X_\lambda = \int (\partial_\mu \lambda + A_\mu * \lambda - \lambda * A_\mu).\frac{\delta}{\delta A_\mu} \tag{2.48}$$

Note that one is free to put ordinary or star product between the bracket and the variational derivative, due to the presence of the integral sign. By simple calculation, analogous to the commutative case, one can verify that these operators do satisfy simple commutation relations

$$X_\lambda X_{\lambda'} - X_{\lambda'} X_\lambda = X_{[\lambda,\lambda']_*}. \tag{2.49}$$

The integrated anomaly

$$\mathcal{A}[\lambda] = X_\lambda W[A] = \int \lambda(x)\mathcal{A}(x) \tag{2.50}$$

should therefore satisfy the Wess-Zumino consistency condition [31]

$$X_\lambda \mathcal{A}[\lambda'] - X_{\lambda'}\mathcal{A}[\lambda] = \mathcal{A}[[\lambda,\lambda']_*]. \tag{2.51}$$

This condition can be elegantly expressed [32] in the BRS formalism as

$$s\mathcal{A}[c] = 0, \tag{2.52}$$

where s is the BRST operator which acts on the gauge field A and the ghost c as

$$\begin{aligned} sA &= dc + A * c - c * A, \\ sc &= -c * c. \end{aligned} \tag{2.53}$$

From here on, we shall use the formalism of differential forms, we can deal with them as usual, but we never use the relation $\omega_1\omega_2 = (-)^{k_1 k_2}\omega_2\omega_1$, for any k_i–forms ω_i.

The strategy is to solve the noncommutative Wess-Zumino consistency condition (2.52) using appropriate generalization of the descent equations [32, 33, 34, 35].

If one tries to derive for noncommutative YM theories descent equations similar to those of the commutative case, at first sight this seems to be impossible. In fact the standard expression one starts with, $\mathrm{Tr}(F \ldots F)$, in the commutative case should be replaced by $\mathrm{Tr}(F * \ldots * F)$. However one notices that it would be both closed and invariant if we were allowed to permute cyclically the terms under the trace symbol. In fact, terms differing by a cyclic permutation differ by a total derivative of the form $\theta^{ij}\partial_i \ldots$. Such terms could of course be discarded upon integration. However, the spirit of the descent equations requires precisely to work with unintegrated objects.

The way out is then to define a bi–complex which does the right job. It is defined as follows. Consider the space of traces of $*$ products of such objects as A, dA, c, dc. The space of cochains is now this space, modulo the circular relation

$$\mathrm{Tr}(E_1 * E_2 * \ldots * E_n) \approx \mathrm{Tr}(E_n * E_1 * \ldots * E_{n-1})(-1)^{k_n(k_1+\ldots+k_{n-1})}, \tag{2.54}$$

where E_i is any of A, dA, c, dc, and k_i is the form order of E_i.

This is naturally a bicomplex (let us call it $\mathcal{C}$), since we have two differential operators which preserve the circular relation (2.54). These are the exterior derivative d and the BRST cohomology operator s. Note that in our conventions d and s commute.

We can now start the usual machinery of consistent anomalies, reducing the problem to a cohomological one. In a noncommutative even D–dimensional space we start with $\mathrm{Tr}(F*F*\ldots*F)$ with n entries, $n = D/2+1$. In the complex $\mathcal{C}$ this expression is closed and BRST–invariant. Then it is easy to prove the descent equations:

$$\begin{aligned}
\mathrm{Tr}(F * F * \ldots * F) &= d\Omega^0_{2n+1} \\
s\Omega^0_{2n+1} &= d\Omega^1_{2n} \\
s\Omega^1_{2n} &= d\Omega^2_{2n-1}
\end{aligned} \tag{2.55}$$

and so on. Here the Chern–Simons term can be represented in $\mathcal{C}$ by

$$\Omega^0_{2n+1} = n\int_0^1 dt\,\mathrm{Tr}(A * F_t * F_t * \ldots * F_t), \tag{2.56}$$

where we have introduced a parameter t, $0 \le t \le 1$, and the traditional notation $F_t = tdA + t^2 A * A$. The anomaly can instead be represented by

$$\Omega^1_{2n} = n\int_0^1 dt(t-1)\,\mathrm{Tr}(dc*A*F_t*\ldots*F_t+dc*F_t*A*\ldots*F_t+\cdots+dc*F_t*F_t*\ldots*A), \tag{2.57}$$

where the sum under the trace symbol includes $n-1$ terms. Finally

$$\Omega^2_{2n-1} = n\int_0^1 dt\frac{(t-1)^2}{2}\,\mathrm{Tr}(dc * dc * A * F_t * \ldots * F_t + \ldots), \tag{2.58}$$

where the dots represent $(n-1)(n-2)-1$ terms obtained from the first one by permuting in all distinct ways dc, A and F_t, keeping track of the grading and keeping dc fixed in the first position.

The only trick to be used in proving the above formulas is to assemble terms in such a way as to form the combination $dA + 2tA * A = \frac{dF_t}{dt}$, and then integrate by parts.

In four dimensions the anomaly takes the form

$$\Omega_4^1 = -\frac{1}{2}\operatorname{Tr}(dc * A * dA + dc * dA * A + dc * A * A * A). \tag{2.59}$$

This anomaly, once it is integrated over, coincides in the form with the result of explicit path-integral calculations [27]. The approach based on the descent equations determines, of course, the anomaly only up to some arbitrary coefficient, which has to be fixed by other means.

2.5 Noncommutative $SO(n)$ and $Sp(n)$ gauge theories

As we have already anticipated in section 2.2, to construct noncommutative versions of $SO(n)$ and $Sp(n)$ gauge theories we need something more elaborated than in the $U(n)$ case. To get some idea of one could expect, consider the finite noncommutative $U(n)$ gauge transformation

$$A' = U^{-1} * (d + A*)U, \tag{2.60}$$

where $U^{-1} * U = 1$. To preserve the antihermicity of the gauge field one needs also $U^\dagger = U^{-1}$. Due to the presence of the star product, the noncommutative gauge group element U is not a function on $\mathbb{R}^d$ valued in ordinary Lie group $U(n)$. To construct noncommutative $SO(n)$ and $Sp(n)$ gauge theories, we will go one step further. We will have to abandon the requirement, that the infinitesimal gauge transformation and the gauge field itself are Lie algebra valued.

To start with, we will work in a setting in which θ has to be thought of as a parameter. Accordingly, we will consider $\mathcal{A}_\theta$ as an algebra of (possibly formal) power series in θ. This algebra has an anti–automorphism r defined by

$$(.)^r : f(x, \theta) \mapsto f^r(x, \theta) \equiv f(x, -\theta). \tag{2.61}$$

This map reduces to the identity on the generators x^μ and reverses the order in the product: $(x_1^\mu * \ldots * x_n^\mu)^r = (x_n^\mu)^r * \ldots * (x_1^\mu)^r$.

First of all, we consider our groups as subgroups of $U(n)$. In other words we keep the usual antihermiticity condition on the $u(n)$–valued connections A and gauge transformations λ. To fix our conventions we will use Greek letters for space-time indices and i and j for matrix (group) indices. Here, for later use, we write down explicitly the hermiticity condition:

$$\begin{aligned} A^*_{ij}(x,\theta) &= -A_{ji}(x,\theta), \\ \lambda^*_{ij}(x,\theta) &= -\lambda_{ji}(x,\theta). \end{aligned} \tag{2.62}$$

Our defining condition for the $NCSO(n)$ connections and gauge transformations is to take the gauge connections and transformations satisfying the following constraints:

$$\begin{aligned} A^r_{ij}(x,\theta) &= -A_{ji}(x,\theta), \\ \lambda^r_{ij}(x,\theta) &= -\lambda_{ji}(x,\theta). \end{aligned} \tag{2.63}$$

Let us comment on these constraints. First of all, it is easy to see that they are preserved by gauge transformations. One can see it componentwise. Alternatively, rewrite (2.63) in the concise form $A = -(A^t)^r$ and $\lambda = -(\lambda^t)^r$, i.e. t is the matrix transposition. Define $((.)^t)^r \equiv (.)^{rt}$; one can show that the rt map, in close analogy to the hermitian conjugation $(.)^\dagger$, enjoys the property

$$(f * g)^{rt} = g^{rt} * f^{rt}. \tag{2.64}$$

The proof is now formally similar to the usual one for $U(n)$: $(\lambda * A - A * \lambda)^{rt} = A^{rt} * \lambda^{rt} - \lambda^{rt} * A^{rt} = -(\lambda * A - A * \lambda)$.

The second comment we wish to make is that the constraints we introduced are natural if one recalls that in noncommutative gauge theories the map $-(\cdot)^{rt}$ is nothing but complex conjugation; our theory is the charge-conjugation invariant version of the usual one. More explicitly, as discussed in [39], indeed the charge conjugation operator is

$$A^c = -A^{rt}. \tag{2.65}$$

One can write an explicit solution of (2.63) as:

$$\mathcal{A}_\mu = \frac{1}{2}(A_\mu - A^{rt}_\mu) = \frac{1}{2}(A_\mu + A^c_\mu). \tag{2.66}$$

This notation may be ambiguous and we hasten to specify that when (2.66) is used we understand that A_μ transforms with gauge parameter $\Lambda = \frac{1}{2}(\lambda - \lambda^{rt})$[1]. More precisely, our $\mathcal{A}_\mu$ enjoys the noncommutative gauge transformations generated by Λ:

$$\mathcal{A}_\mu \to \mathcal{A}'_\mu = U_*^{-1}(\Lambda) * \mathcal{A}_\mu * U_*(\Lambda) - U_*^{-1}(\Lambda) * \partial_\mu U_*(\Lambda), \tag{2.67}$$

[1] In the ordinary commutative case, this is the way to 'reduce' a unitary connection to an orthogonal one, [40], Prop.6.4.

where

$$
\begin{aligned}
U_*(\Lambda) &\equiv 1 + i\Lambda - \frac{1}{2}\Lambda * \Lambda + \cdots, \\
U_*^{-1}(\Lambda) &= U_*(-\Lambda), \qquad U_*^{-1} * U_* = 1.
\end{aligned} \tag{2.68}
$$

As we see it is immediate that our $NCSO(n)$ gauge fields are charge conjugation invariant.

Thirdly, we anticipated above that under the (2.63), connections and gauge parameters do not turn out to be $so(n)$–valued. Nevertheless (2.63) introduces restrictions on the matrix functions A_{ij}. To see what they are, let us write (2.63) more explicitly

$$
\begin{aligned}
A_{ij}(x,\theta) &= -A_{ji}(x,-\theta), \\
\lambda_{ij}(x,\theta) &= -\lambda_{ji}(x,-\theta).
\end{aligned} \tag{2.69}
$$

Inserting a power expansion in θ for A

$$
A^\mu(x,\theta) = A_0^\mu(x) + i\theta_{\nu\rho}A_1^{\mu\nu\rho}(x) + \ldots, \tag{2.70}
$$

we see that (2.63) implies that $A_0, A_2, \ldots$ are antisymmetric and $A_1, A_3 \ldots$ symmetric. The hermiticity condition (2.62) imposes that all the coefficients $A_0, A_1, \ldots$ be real. The same conclusions hold for the power expansion of λ.

Up to now, $A_0, A_1, \ldots$ are unrestricted, except for the just mentioned constraint. However, if we want to make connection with string theory, $A_1, A_2, \ldots$ are expected not to introduce new degrees of freedom, but to be functionally dependent on A_0. The simplest proposal is to regard them as given by the Seiberg–Witten map [7]:

$$
A^\mu(A_0) = A_0^\mu - \frac{i}{4}\theta^{\nu\rho}\{A_{0\nu}, \partial_\rho A_0^\mu + F_{0\rho}{}^\mu\} + \mathcal{O}(\theta^2); \tag{2.71}
$$

(the presence of i is due the fact that Seiberg and Witten use hermitean connections rather than anti-hermitean ones, as we do). This is indeed consistent: the term linear in θ is symmetric if the constant part is antisymmetric. In fact, one can also see that the next term is antisymmetric, and so on; so we have complete accord with (2.70).

This is related to the further subtle issue of fixing θ to a particular value. In this case, of course the approach we have taken so far – considering θ as a formal parameter – loses its validity, and the very definition of r is in jeopardy. However, thanks to the fact that $A_1, A_2, \ldots$ depend on A_0, even when one puts θ to a particular value, A is not the most general $U(n)$ field;

our constraint becomes more involved but is still there. If we invert the map to obtain $A_0(A)$, the constraint can be formulated simply as

$$A_0(A) = -A_0^t(A). \tag{2.72}$$

So we could say that our theory is the image of the Seiberg–Witten map restricted to the $SO(n)$ case.

It is now easy to introduce similar definitions for noncommutative $Sp(n)$. One imposes in this case the condition $JA^r = -A^t J$, where $J = \varepsilon \otimes \mathrm{Id}_n$, where $\varepsilon = i\sigma_2$. This constraint is preserved by gauge transformations that satisfy the same condition.

One could think that the group $SU(n)$ could be tackled in a similar way: by defining a constraint like $\mathrm{Tr}(A + A^{rt}) = \mathrm{Tr}(A + A^r) = 0$. However, this would not be gauge invariant. So, even by using the r map, it is not possible to define a $NCSU(n)$ gauge theory.

To define a Yang–Mills $NCSO(n)$ theory, let $A = A(x, \theta)$ satisfy the constraint (2.63). The action we propose is the usual one

$$S = -\frac{1}{4} \int d^d x F_{ij}^{\mu\nu} F_{ji\mu\nu}. \tag{2.73}$$

The action (2.73) is naturally gauge invariant under $NCSO(n)$ and positive. It reduces to the usual one for $SO(n)$ in the $\theta = 0$ case.

It is not clear to us, how to impose the condition (2.71) in a consistent way. If one tries not to impose it, one faces a lot of additional degrees of freedom which are not present in the commutative case and again it is not clear what happens to them. Some ideas have been presented in [38].

It is rather straightforward to introduce matter fields in this context in a coherent way. First one has to found a generalization of the Seiberg-Witten map to such fields. Then it is easy to see that our constraint $\psi^{rt} = -\psi$ is consistent with this map. Detailed form of the map can be found in [36].

Finally we would like to discuss a subtle issue whether the proposed noncommutative gauge theories can be obtained as a low energy limit of string theory. Normally $SO(n)$ and $Sp(n)$ theories are obtained from $U(n)$ theory by an orientifold projection. The noncommutativity comes from the background B-field. One is tempted to say that the noncommutative $SO(n)$ and $Sp(n)$ theories can be obtained by combining orientifold and the B-field. But alas, to proceed along these lines one needs D-branes parallel to the orientifold and the B-field along them. But these are just the components of the B-field which are projected out. Therefore in our paper [36] we tried to suggest other possibilities. One is that according to [41, 42, 43, 44], certain quantized constant B-field is allowed. What is projected out are only the fluctuations. Second possibility is to consider a step-like B-field across the orientifold. Here it is not obvious that it is a consistent string theory background. Perhaps both cases are two different descriptions of the same phenomenon. Details can be found in [36]. Slightly different approach to the construction of noncommutative $SO(n)$ and $Sp(n)$ theories is presented in [45].

2.6 Noncommutative solitons

One of the remarkable aspects of noncommutative theories, is that they allow for particularly nice solitons. Let us consider first the case of a single scalar field in two dimensions with an action

$$S = \int d^2x \left(\frac{1}{2}\partial\phi\partial\phi + V(\phi) \right). \tag{2.74}$$

Rescale the coordinates satisfying $[x_1, x_2] = i\theta$ to get $[\tilde{x}_1, \tilde{x}_2] = i$. In terms of the new variables the action becomes

$$S = \int d^2\tilde{x} \left(\frac{1}{2}\partial\phi\partial\phi + \theta V(\phi) \right). \tag{2.75}$$

In the limit of large θ we can neglect the kinetic term and we can look for stationary solutions. Suppose that the potential $V(\phi)$ takes the form

$$V(\phi) = \frac{1}{2}m^2\phi * \phi + \frac{1}{3}g\phi * \phi * \phi + \cdots. \tag{2.76}$$

The equation of motion $V'(\phi) = 0$ reads

$$m^2\phi + g\phi * \phi + \cdots = 0. \tag{2.77}$$

In the commutative theory the only solution would be a constant. The noncommutativity allows however for more interesting possibilities. If we can find a ϕ_0 such that

$$\phi_0 = \phi_0 * \phi_0, \tag{2.78}$$

then we can look for the stationary point in the form $\phi = c\phi_0$. The constant c is determined as a solution of the algebraic equation

$$m^2c + gc^2 + \cdots = 0. \tag{2.79}$$

The simplest example of a solution to the projector equation (2.78) is

$$\phi_0(x_1, x_2) = 2e^{-\frac{x_1^2+x_2^2}{\theta}}. \tag{2.80}$$

Elegant and simple construction of all solutions has been found by Gopakumar, Minwalla and Strominger [46]. It is based on a simple observation that $[x_1, x_2] = i\theta$ is formally identical to the basic commutation relation $[q, p] = i\hbar$ of quantum mechanics.

When quantizing a classical system, we want to associate to a given function on the classical phase space an operator in a Hilbert space. There is a well known ambiguity in this procedure, to the function $p^n q^m$ we can associate many operators, distinguished by the ordering of the operators $\hat{q}$ and $\hat{p}$. One particularly convenient prescription is so called Weyl or symmetric quantization. To a given function $f(q,p)$ on $\mathbb{R}^2$ it associates an operator $O_f(\hat{q},\hat{p})$ in the Hilbert space $\mathcal{H}$ by the formula

$$O_f(\hat{q},\hat{p}) = \frac{1}{(2\pi)^2}\int d^2k\, \tilde{f}(k_q,k_p) e^{-i(k_q\hat{q}+k_p\hat{p})}, \tag{2.81}$$

where the tilde denotes a Fourier transform. Its important property is, that it defines an isomorphism between the algebra of functions on the plane with the Moyal product (2.12), and an algebra of operators on $\mathcal{H}$

$$O_f O_g = O_{f*g}. \tag{2.82}$$

Second important property is

$$\frac{1}{2\pi}\int dp\, dq\, f(q,p) = \mathrm{Tr}_{\mathcal{H}}\, O_f. \tag{2.83}$$

In this correspondence the problem of finding all solutions to the equation $\phi_0 * \phi_0 = \phi_0$ is equivalent to finding all projectors in the Hilbert space. This problem is easily solved by the Fock construction, introducing the annihilation and creation operators:

$$\begin{aligned} a &= \frac{\hat{q}+i\hat{p}}{2}, \\ a^\dagger &= \frac{\hat{q}-i\hat{p}}{2}. \end{aligned} \tag{2.84}$$

The basis of all linear operators is given by

$$|m\rangle\langle n| =: \frac{a^{\dagger m}}{\sqrt{m!}} e^{-a^\dagger a} \frac{a^n}{\sqrt{n!}} : . \tag{2.85}$$

With a little effort the normal ordered operators can be rewritten in the Weyl ordered form and one can read of the corresponding function. The simplest projector and its Moyal plane function equivalent is

$$|0\rangle\langle 0| \simeq 2e^{-\frac{r^2}{\theta}}. \tag{2.86}$$

More general rank one projectors corresponding to the spherical functions are

$$|n\rangle\langle n| \simeq 2(-1)^n L_n\left(\frac{2r^2}{\theta}\right) e^{-\frac{r^2}{\theta}}, \tag{2.87}$$

where $L_n(r)$ are Laguerre polynomials. Finally the most general solution is of the form

$$U\left(\sum_n a_n |n\rangle\langle n|\right) U^\dagger, \qquad a_n \in \{0,1\}, \tag{2.88}$$

where U is an arbitrary unitary operator.

Let us anticipate that these kinds of noncommutative solitons will be later understood in the context of string theory as D-branes of lower dimension [109, 89, 92]. One could ask the question what happens to these noncommutative solitons under the Seiberg–Witten map, i.e. when one wants to express them in the commutative coordinates. This has been studied by Hashimoto and Ooguri [47], who have shown that the soliton gets squeezed to the delta-function support. This fits nicely with the stringy picture of D-branes as infinitely thin objects.

Chapter 3

Cubic string field theory

The standard perturbative definition of string theory tells us only how to calculate the on-shell S-matrix elements. In particle physics the on-shell matrix elements are the only thing that is measured. One can develop a systematic perturbative method to calculate all processes using purely the on-shell data. In practice, nowadays, almost everybody uses the quantum field theory which is a very powerful off-shell extension. Its advantages are obvious: it gives simpler and more systematic prescription for perturbative computation of amplitudes and more importantly it allows us to use semiclassical methods.

In the same spirit the field theory of strings is a certain off-shell extension and can be used in principle to calculate in a systematic fashion all string amplitudes. For quite a long time people thought that the string field theory is a rather complicated tool to obtain results already known. However in recent years this general attitude has changed. The string field theory came to play very important role, since it can tell us what happens to the tachyon in the open bosonic string theory, and to the tachyon on non-BPS branes and in the brane-antibrane systems in superstring theory.

This can be easily understood. The tachyon (just as the Higgs field in the standard model) signals a instability of the vacuum. It develops a constant vacuum expectation value (VEV). To calculate this VEV and the form of the theory around the correct vacuum, one needs to know amplitudes at zero momentum, which is for the tachyon an off-shell momentum.

First versions of string field theory have been constructed in the light cone gauge. These theories will not be reviewed here, since they are not suitable for the study of the tachyon condensation due to the lack of Lorentz invariance and have been superseded by better theories. Here we will concentrate on manifestly Lorentz invariant and gauge invariant Witten's open cubic string field theory [5]. Another, so called background independent or boundary string field theory was also developed by Witten and refined by Shatashvili [52, 53]. It has led to some remarkable exact results about the tachyon condensation but for the lack of space we refer the interested reader to the literature [68, 69, 70].

Witten has succeeded in formulating the covariant string field theory in an attractive language of noncommutative differential geometry. His theory has several very nice features: it is manifestly covariant, gauge invariant and cubic. Subsequent works by Gross and Jevicki [48, 49] put the theory on a firmer footing, when they expressed everything in terms of Fock space oscillators of the first quantized string theory. Later LeClair, Peskin and Preitschopf [50, 51] have given another rigorous definition of the Witten's theory by expressing all terms in the SFT action as correlators in the two dimensional CFT.

The string field theory has become very popular recently mainly due to the conjectures of Ashoke Sen [54, 55]. Roughly speaking, according to his conjectures, the D-branes arise as solitons in open string field theory. The perturbative vacuum of the string field theory is a space-filling D25-brane, which is unstable and decays into some new nonperturbative vacuum which is called closed string vacuum, since it is believed to have only the closed string excitations. Some aspects of the conjecture have been tested, most importantly in the papers [56, 57, 68, 69, 70].

Finally we would like to mention two reviews on the subject, an older one by Thorn [58] and a quite recent one by Ohmori [59], which also discusses a lot about the tachyon condensation.

3.1 Witten's original formulation

In this section we introduce the Witten's covariant cubic string field theory [5] for the open bosonic string. The basic dynamical variable is the string field which in the Schrödinger representation is given by

$$\Psi\left[X(\sigma), c(\sigma)\right] = \langle\, X(\sigma), c(\sigma) | \Psi\,\rangle. \tag{3.1}$$

It is a straightforward generalization of the quantum mechanical wave function of a particle to the string case. Later we will find convenient to work directly with the state $|\Psi\rangle$ of the Hilbert space of the first quantized string theory.

The string field action has an attractive form of noncommutative Chern-Simons action

$$S = -\frac{1}{g^2}\int\left(\frac{1}{2}\Psi * Q\Psi + \frac{1}{3}\Psi * \Psi * \Psi\right), \tag{3.2}$$

where g is the open string coupling constant and Q is the BRST operator. The integration and the associative star product will be defined later. The beauty of the formulation lies in the noncommutative geometric interpretation of these objects.

The ghost number of the string field corresponds formally to the form degree, the associative star multiplication $*$ to the wedge product $\wedge$, and the BRST operator Q to the exterior differential d. This means that the star product acts additively on the ghost number and the operator Q has the properties

$$Q^2 = 0, \tag{3.3}$$

$$\int Q\Psi = 0, \tag{3.4}$$

$$Q(A*B) = QA*B+(-)^A A*QB. \tag{3.5}$$

Clearly also Q increases the ghost number by one, just as the exterior differential d increases the form degree. Let us anticipate that these formal relations could be also satisfied by other choices for Q for example the combination of ghost modes $c_0 + \frac{1}{2}(c_2 + c_{-2})$. According to recent conjectures [95, 96] such actions could correspond to string field theories around the nonperturbative vacua without open strings.

Very nice feature of the action is its manifest gauge invariance

$$\delta\Psi = Q\Psi + \Psi * \Lambda - \Lambda * \Psi, \tag{3.6}$$

where Λ is a string field of ghost number zero, which plays a role of a gauge parameter. The ordinary space-time gauge symmetry is a consequence of this stringy gauge invariance, for interacting theory the relationship is however not so straightforward.

The star product of two string fields is defined as gluing of the right part of the first string with the left part of the second string. Explicitly it is given by the formula

$$(\Psi_1 * \Psi_2)(X_0(\sigma)) = \int \mathcal{D}X_1(\sigma)\mathcal{D}X_2(\sigma)\, \Psi_1(X_1(\sigma))\Psi_2(X_2(\sigma)) \tag{3.7}$$
$$\prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta(X_1(\sigma)-X_0(\sigma))\delta(X_2(\sigma)-X_1(\pi-\sigma))\delta(X_0(\pi-\sigma)-X_2(\pi-\sigma)).$$

In this formula and for the rest of this subsection we are ignoring the ghost degrees of freedom. We will come back to them later.

One may readily check that the star product is associative. The integration is defined for a ghost number 3 string field by

$$\int \Psi = \int \mathcal{D}X(\sigma) \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta(X(\sigma) - X(\pi-\sigma))\Psi(X(\sigma)). \tag{3.8}$$

The 3-string interaction term can be written easily as

$$\begin{aligned}\langle \Psi_1, \Psi_2, \Psi_3 \rangle &= \int \Psi_1 * \Psi_2 * \Psi_3 \\ &= \int \mathcal{D}X_1(\sigma)\mathcal{D}X_2(\sigma)\mathcal{D}X_3(\sigma)\, \Psi_1(X_1(\sigma))\Psi_2(X_2(\sigma))\Psi_3(X_3(\sigma)) \qquad (3.9) \\ &\prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta(X_2(\sigma) - X_1(\pi-\sigma))\delta(X_3(\sigma) - X_2(\pi-\sigma))\delta(X_1(\sigma) - X_3(\pi-\sigma)).\end{aligned}$$

We can further simplify the formula (3.7) by integrating over the delta functions

$$\begin{aligned}(\Psi_1 * \Psi_2)(X_0(\sigma)) &= \int \mathcal{D}Y_L(\sigma)\, \Psi_1\Big(\theta\Big(\frac{\pi}{2}-\sigma\Big)\, X_0(\sigma) + \theta\Big(\sigma-\frac{\pi}{2}\Big)\, Y_L(\pi-\sigma)\Big) \\ &\quad \Psi_2\Big(\theta\Big(\frac{\pi}{2}-\sigma\Big)\, Y_L(\sigma) + \theta\Big(\sigma-\frac{\pi}{2}\Big)\, X_0(\sigma)\Big), \qquad (3.10)\end{aligned}$$

where the functional integration $\mathcal{D}Y_L(\sigma)$ is over half of the string $0 \leq \sigma < \pi/2$ only.

It is empirically established fact that string field theory works well when expanded into Fourier modes. Let us therefore expand the string coordinate $X(\sigma)$

$$X(\sigma) = x_0 + \sqrt{2}\sum_{n=1}^{\infty} x_n \cos n\sigma. \tag{3.11}$$

The star product then takes the form

$$\begin{aligned}(\Psi_1 * \Psi_2)(\vec{x}) &= \int \mathcal{D}y\, \Psi_1\Big(\frac{1}{2}(1-iXC)\vec{x} + \frac{1}{2}(1+iXC)C\vec{y}\Big) \\ &\quad \Psi_2\Big(\frac{1}{2}(1-iXC)\vec{y} + \frac{1}{2}(1+iXC)\vec{x}\Big), \qquad (3.12)\end{aligned}$$

where the matrices X and C were introduced by Gross and Jevicki [48] and have the following form

$$\begin{aligned}X_{nm} &= \frac{i}{\pi}(-1)^{\frac{n-m-1}{2}}\left(1-(-1)^{n+m}\right)\left(\frac{1}{n+m} + \frac{(-1)^m}{n-m}\right), \\ X_{0m} &= -\frac{\sqrt{2}\,i}{\pi m}(-1)^{\frac{m-1}{2}}\left(1-(-1)^m\right), \\ C_{nm} &= (-1)^n\delta_{nm}. \qquad (3.13)\end{aligned}$$

They satisfy the relations

$$X = X^\dagger = -X^T, \quad X^2 = 1, \quad XC = -CX. \tag{3.14}$$

These relations imply that $\frac{1}{2}(1 \pm iXC)$ are projectors which is crucial for proving associativity in this setting. What remains, is to specify the integration measure in (3.12). The choice is to certain extent arbitrary. We will take

$$\mathcal{D}Y = \prod_{n=0}^{\infty} dy_n \, \delta\Big(Y\left(\frac{\pi}{2}\right) - X\left(\frac{\pi}{2}\right)\Big), \tag{3.15}$$

which is compatible with associativity. Let us see what happens when we take first two modes only

$$\begin{aligned}
(\Psi_1 * \Psi_2)(x_0, x_1) &= \int dy_0 \, dy_1 \, \delta(y_0 - x_0) \\
\Psi_1 &\left(\frac{1}{2}x_0 + \frac{\sqrt{2}}{\pi}x_1 + \frac{1}{2}y_0 + \frac{\sqrt{2}}{\pi}y_1, \frac{1}{2}x_1 + \frac{\sqrt{2}}{\pi}x_0 - \frac{1}{2}y_1 - \frac{\sqrt{2}}{\pi}y_0\right) \\
\Psi_2 &\left(\frac{1}{2}y_0 + \frac{\sqrt{2}}{\pi}y_1 + \frac{1}{2}x_0 - \frac{\sqrt{2}}{\pi}x_1, \frac{1}{2}y_1 + \frac{\sqrt{2}}{\pi}y_0 + \frac{1}{2}x_1 - \frac{\sqrt{2}}{\pi}x_0\right) \\
&= \int dy_1 \, \Psi_1\left(x_0 + \frac{\sqrt{2}}{\pi}(x_1 + y_1), \frac{1}{2}x_1 - \frac{1}{2}y_1\right) \\
&\quad \Psi_2\left(x_0 + \frac{\sqrt{2}}{\pi}(y_1 - x_1), \frac{1}{2}y_1 + \frac{1}{2}x_1\right).
\end{aligned} \tag{3.16}$$

As a result of the truncation this star product is not associative. If we however restrict ourselves to x_0 or x_1 modes separately, we get meaningful results. First ignoring x_1 mode we have the most trivial commutative star product

$$(\psi_1 * \psi_2)(x_0) = \psi_1(x_0)\psi_2(x_0). \tag{3.17}$$

From here on, we shall denote the string field truncated to a single mode by lower case ψ. For functionals $\psi_{1,2}$ independent of x_0 we get another product

$$(\psi_1 * \psi_2)(x_1) = \int dy_1 \, \psi_1\left(\frac{x_1 - y_1}{2}\right) \psi_2\left(\frac{y_1 + x_1}{2}\right). \tag{3.18}$$

Both of these products are commutative and associative. The second one has the same form (up to a normalization factor) as the first one after Fourier transformation. Indeed

$$(\psi_1 * \psi_2)(k_1) = 2\sqrt{2\pi}\, \psi_1(k_1)\psi_2(k_1). \tag{3.19}$$

Our conventions for the Fourier transformation are those standard used in quantum mechanics

$$\begin{aligned}\psi(x) &= \frac{1}{\sqrt{2\pi}}\int \psi(k)e^{ikx}dk,\\ \psi(k) &= \frac{1}{\sqrt{2\pi}}\int \psi(x)e^{-ikx}dx.\end{aligned}\tag{3.20}$$

3.2 Fock representation

Although the Schrödinger representation is very convenient for defining the star multiplication of string fields, it is not always good for practical calculations and for making things rigorous. Therefore already in 1986 Samuel [60], Cremmer, Schwimmer and Thorn [61] and Gross and Jevicki [48, 49] tried successfully to re-formulate the Witten's string field theory in terms of oscillators on a Fock space of the first quantized string theory.

In analogy with the quantum mechanics of a particle, the Schrödinger functional is a x-representation of some vector in Hilbert space $\mathcal{H}$

$$\Psi\left[X(\sigma)\right] = \langle\, X(\sigma)|\Psi\,\rangle,\tag{3.21}$$

where again we are ignoring ghosts. The string field integration is a linear map to real numbers and can thus be represented by some bra-vector $\langle I| \in \mathcal{H}^*$

$$\int \psi = \langle\, I|\psi\,\rangle.\tag{3.22}$$

Inserting in the RHS the resolution of identity

$$1 = \int \mathcal{D}X(\sigma)|X(\sigma)\rangle\langle X(\sigma)|,\tag{3.23}$$

we see that from (3.8) follows immediately the Schrödinger representation of the state $|I\rangle$

$$\langle\, X(\sigma)|I\,\rangle = \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta(X(\sigma) - X(\pi-\sigma)).\tag{3.24}$$

The 3-string interaction, just as the integration, can be represented by a bra-vector $\langle V| \in \mathcal{H}^*\otimes\mathcal{H}^*\otimes\mathcal{H}^*$

$$\int \Psi_1 * \Psi_2 * \Psi_3 = \langle V||\Psi_1\rangle|\Psi_2\rangle|\Psi_3\rangle.\tag{3.25}$$

It is a machine that given three string fields produces a real number. We will need a further property

$$\langle\, \mathrm{bpz}(\Psi_1)|\Psi_2 * \Psi_3\,\rangle = \langle V||\Psi_1\rangle|\Psi_2\rangle|\Psi_3\rangle, \tag{3.26}$$

where the BPZ conjugation is defined by

$$\langle\, \mathrm{bpz}(\Psi)|X(\sigma)\,\rangle = \Psi[X(\pi-\sigma)]. \tag{3.27}$$

Recalling the mode expansion (3.11)

$$x(\sigma) = x_0 + \sqrt{2}\sum_{n=1}^{\infty} x_n \cos n\sigma, \tag{3.28}$$

we can construct the Hilbert space using the creation and annihilation operators $a_n^\dagger, a_n$ defined by the relations

$$\begin{aligned} x_n &= \frac{i}{\sqrt{2n}}(a_n - a_n^\dagger), \\ p_n &= -i\frac{\partial}{\partial x_n} = \sqrt{\frac{n}{2}}(a_n + a_n^\dagger) \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} x_0 &= \frac{i}{2}(a_0 - a_0^\dagger), \\ p_0 &= -i\frac{\partial}{\partial x_0} = (a_0 + a_0^\dagger). \end{aligned} \tag{3.30}$$

The original construction [48, 49] of the 3-string vertex $\langle V|$ is quite complicated, we will follow later different route using the CFT techniques. However it is pedagogical and interesting to look how far one can get along the most obvious path. We shall derive the Fock space representation for the two truncated commutative star products (3.17) and (3.18).

3.2.1 Truncated star product I.

As a first example we consider the star product (3.17)

$$(\psi_1 * \psi_2)(x) = \psi_1(x)\psi_2(x). \tag{3.31}$$

Let us start with the 3-string interaction

$$\begin{aligned} \langle V||\psi_1\rangle|\psi_2\rangle|\psi_3\rangle &= \langle\, bpz(\psi_1)|\psi_2 * \psi_3\,\rangle \\ &= \int dx\, \psi_1(x)\psi_2(x)\psi_3(x) \\ &= \frac{1}{\sqrt{2\pi}}\int dp_1 dp_2 dp_3\, \delta(p_1+p_2+p_3)\, \psi_1(p_1)\psi_2(p_2)\psi_3(p_3), \end{aligned} \tag{3.32}$$

from which we will determine the star product. We shall evaluate (3.32) on the overcomplete basis of coherent states of the form

$$|\psi\rangle = e^{la^\dagger}|0\rangle. \tag{3.33}$$

As the above star product can be studied on its own, we adopt more symmetric definition of the creation and annihilation operators

$$\begin{aligned} a^\dagger &= \frac{1}{\sqrt{2}}(p+ix), \\ a &= \frac{1}{\sqrt{2}}(p-ix). \end{aligned} \tag{3.34}$$

The momentum and coordinate representations are easily found to be

$$\begin{aligned} \psi(p) &= \frac{1}{\sqrt[4]{\pi}} e^{-\frac{p^2}{2}+\sqrt{2}pl-\frac{l^2}{2}}, \\ \psi(x) &= \frac{1}{\sqrt[4]{\pi}} e^{-\frac{x^2}{2}+\sqrt{2}ixl+\frac{l^2}{2}}. \end{aligned} \tag{3.35}$$

The result of the gaussian integrations is simply

$$\int dx\, \psi_1(x)\psi_2(x)\psi_3(x) = \frac{1}{\sqrt[4]{\pi}}\sqrt{\frac{2}{3}}\, e^{\frac{1}{6}\sum l_i^2 - \frac{2}{3}(l_1l_2+l_2l_3+l_3l_1)}. \tag{3.36}$$

Now we would like to express $\langle V|$ as a state in the tensor product $\mathcal{H}^*\otimes\mathcal{H}^*\otimes\mathcal{H}^*$ of three Fock spaces. Let us try an ansatz

$$\langle V| = \kappa\,\langle 0|\, e^{\frac{1}{2}\sum N_{ij}a_ia_j} \tag{3.37}$$

for some real 3×3 matrix N and normalization factor κ. Imposing cyclicity of the vertex and symmetry of the matrix N we are left with only two parameters λ, μ. The matrix N takes the form

$$\begin{pmatrix} 2\lambda & \mu & \mu \\ \mu & 2\lambda & \mu \\ \mu & \mu & 2\lambda \end{pmatrix}. \tag{3.38}$$

Evaluating the LHS of (3.32) for states (3.33) using the above vertex gives

$$\langle 0|e^{\frac{1}{2}\sum N_{ij}a_ia_j} e^{\sum l_i a_i^\dagger}|0\rangle = e^{\lambda\sum l_i^2 + \mu(l_1l_2+l_2l_3+l_3l_1)}. \tag{3.39}$$

Matching (3.36) and (3.39) we get

$$\begin{aligned} \lambda &= \frac{1}{6}, \\ \mu &= -\frac{2}{3}. \end{aligned} \tag{3.40}$$

The normalization factor is clearly

$$\kappa = \frac{1}{\sqrt[4]{\pi}}\sqrt{\frac{2}{3}}. \tag{3.41}$$

Note that this matrix N has eigenvalues $1, 1, -1$ and satisfies $N^2 = 1$. This is a generic feature of the full string field theory vertex.

Now we would like to derive an oscillator formula for the star product. We start with a trivial identity

$$|\psi\rangle = {}_b\langle 0|e^{a^\dagger b}|\psi\rangle_b \otimes |0\rangle_a, \tag{3.42}$$

where $a^\dagger$ and $b^\dagger$ are creation operators generating two Fock spaces $\mathcal{H}_a, \mathcal{H}_b$ respectively. The identity can be easily checked for coherent states. Then

$$\begin{aligned} |\psi_2\rangle * |\psi_3\rangle &= {}_4\langle 0|e^{a_1^\dagger a_4}|\psi_2 * \psi_3\rangle_4 \otimes |0\rangle_1 \\ &= {}_{2,3,4}\langle V|e^{-a_1^\dagger a_4^\dagger}|\psi_2\rangle \otimes |\psi_3\rangle \otimes |0\rangle_4 \otimes |0\rangle_1 \\ &= \kappa \cdot {}_{2,3}\langle 0|e^{\lambda(a^{\dagger 2}+a_2^2+a_3^2)-\mu a^\dagger(a_2+a_3)+\mu a_2 a_3}|0\rangle_1 \otimes |\psi_2\rangle \otimes |\psi_3\rangle. \end{aligned} \tag{3.43}$$

In writing the second line we have used the crucial fact following from the definition (3.32) that

$$\langle V||\psi_1\rangle|\psi_2\rangle|\psi_3\rangle = \langle\, \psi_1^c|\psi_2 * \psi_3\,\rangle, \tag{3.44}$$

where the upper script c denotes conjugation defined as the ordinary complex conjugation in the x-representation.

One might be also interested in the identity of the star algebra. In the x-representation it is the function $\psi(x) = 1$ which when translated into the Fock space takes the following form

$$|I\rangle = \sqrt{2}\,\pi^{\frac{1}{4}}e^{-\frac{1}{2}a^\dagger a^\dagger}|0\rangle. \tag{3.45}$$

Note also that it is related to the integration on the algebra since

$$\langle\, I|\psi\,\rangle = \int dx\,\psi(x). \tag{3.46}$$

3.2.2 Truncated star product II.

As the second example we consider the star product (3.18)

$$(\psi_1 * \psi_2)(x_1) = \int dy\,\psi_1\left(\frac{x-y}{2}\right)\psi_2\left(\frac{x+y}{2}\right), \tag{3.47}$$

which has been studied extensively in [87, 88]. It takes particularly simple form in the momentum representation

$$(\psi_1 * \psi_2)(k) = 2\sqrt{2\pi}\,\psi_1(k)\psi_2(k). \tag{3.48}$$

Since this star product arises in the odd sector of the string we have

$$\langle\, bpz(\psi)|x\,\rangle = \psi(-x) \tag{3.49}$$

and therefore

$$\begin{aligned} \langle V||\psi_1\rangle|\psi_2\rangle|\psi_3\rangle &= \langle\, bpz(\psi_1)|\psi_2 * \psi_3\,\rangle \\ &= \int dx\,\psi_1(-x)\int dy\,\psi_2\left(\frac{x-y}{2}\right)\psi_3\left(\frac{x+y}{2}\right) \\ &= 2\int dx_1dx_2dx_3\,\delta(x_1+x_2+x_3)\,\psi_1(x_1)\psi_2(x_2)\psi_3(x_3), \\ &= 2\sqrt{2\pi}\int dp\,\psi_1(p)\psi_2(p)\psi_3(p). \end{aligned} \tag{3.50}$$

Again evaluating the 3-string interaction on the coherent states we get

$$\int dp\,\psi_1(p)\psi_2(p)\psi_3(p) = \frac{1}{\sqrt[4]{\pi}}\sqrt{\frac{2}{3}}\,e^{-\frac{1}{6}\sum l_i^2-\frac{2}{3}(l_1l_2+l_2l_3+l_3l_1)}. \tag{3.51}$$

From this we can read of the matrix N appearing in (3.37). It is just the opposite to what we found before

$$\begin{aligned} \lambda &= -\frac{1}{6}, \\ \mu &= +\frac{2}{3}. \end{aligned} \tag{3.52}$$

The star product looks finally as

$$|\psi_2\rangle * |\psi_3\rangle = \kappa \cdot {}_{2,3}\langle 0|e^{\lambda(a^{\dagger 2}+a_2^2+a_3^2)+\mu a^\dagger(a_2+a_3)+\mu a_2a_3}|0\rangle_1 \otimes |\psi_2\rangle \otimes |\psi_3\rangle, \tag{3.53}$$

where κ is the same constant as before. The identity, which in the p-representation is the function $\psi(p) = \frac{1}{2\sqrt{2\pi}}$, takes the following form

$$|I\rangle = \left(\frac{1}{2\sqrt{2\pi}}\right)\sqrt{2}\,\pi^{\frac{1}{4}}e^{+\frac{1}{2}a^\dagger a^\dagger}|0\rangle. \tag{3.54}$$

3.2.3 The full three vertex

Let us now present without a derivation the results for the three vertex, in the full string field theory specified by the star product (3.7). The product itself is defined through the vertex by the formula

$$\Psi_1 * \Psi_2 = \mathrm{bpz}\left(\langle V|\Psi_1 \otimes \Psi_2\right), \tag{3.55}$$

where bpz denotes the BPZ conjugation in conformal field theory, which looks as

$$\mathrm{bpz}\, \phi_n = (-1)^{n+h}\phi_{-n} \tag{3.56}$$

for a primary field ϕ of dimension h. To calculate the BPZ conjugation of some operator, one has to reverse the order, replace the modes according to the above rule, and for total number n of Grassman fields b and c put the sign $(-1)^{[\frac{n}{2}]}$. This accounts for the sign change when one wants to put the Grassman fields in their original order. The three vertex is given here

$$\begin{aligned} \langle V| &= \left(\frac{3\sqrt{3}}{4}\right)^3 \delta\left(p^{(1)}+p^{(2)}+p^{(3)}\right)\langle\tilde{0}|\otimes\langle\tilde{0}|\otimes\langle\tilde{0}| \times \\ &\quad \times \exp\left(\sum_{m,n=0}^{\infty}\frac{1}{2}\alpha_n^{(r)\mu}N_{nm}^{rs}\alpha_m^{(s)\nu}g_{\mu\nu}+\sum_{m=0,n=1}^{\infty}c_n^{(r)}X_{nm}^{rs}b_m^{(s)}\right) \end{aligned} \tag{3.57}$$

The vacuum $\langle\tilde{0}|$ is related to the standard $SL(2,\mathbb{R})$ invariant vacuum $\langle 0|$ through $\langle\tilde{0}| = \langle 0|c_{-1}^{(i)}c_0^{(i)}$ where $i = 1,2,3$ label one of the three Fock spaces in the tensor product. As usual $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$. The Neumann coefficients N_{nm}^{rs}, X_{nm}^{rs} are defined in terms of the 6-string Neumann functions $\bar{N}_{nm}^{rs}, 1 \leq r, s \leq 6$ through

$$\begin{aligned} N_{nm}^{rs} &= \frac{1}{2}\left(\bar{N}_{nm}^{rs}+\bar{N}_{nm}^{r(s+3)}+\bar{N}_{nm}^{(r+3)s}+\bar{N}_{nm}^{(r+3)(s+3)}\right), \\ X_{nm}^{rs} &= n(-1)^{r-s+1}\left(\bar{N}_{nm}^{rs}-\bar{N}_{nm}^{r(s+3)}\right). \end{aligned} \tag{3.58}$$

All coefficients have a cyclic symmetry under $r \to (r \bmod 3)+1$, $s \to (s \bmod 3)+1$. The 6-string Neumann functions are given by

$$\begin{aligned} N_{nm}^{rs} &= \frac{1}{nm}\oint_{z^r}\frac{dz}{2\pi i}\oint_{z^s}\frac{dw}{2\pi i}\frac{1}{(z-w)^2}(-1)^{n(r-1)+m(s-1)}\left[f(z)\right]^{n(-1)^r}\left[f(w)\right]^{m(-1)^s}, \\ N_{n0}^{rs} &= \frac{1}{n}\oint_{z^r}\frac{dz}{2\pi i}\oint_{z^s}\frac{dw}{2\pi i}\frac{1}{z-w}(-1)^{n(r-1)+(s-1)}\left[f(z)\right]^{n(-1)^r}\frac{f'(w)}{f(w)} \end{aligned} \tag{3.59}$$

with

$$f(z) = \frac{z(z^2-3)}{3z^2-1} \tag{3.60}$$

and

$$z^i = \left\{ \sqrt{3},\, \frac{1}{\sqrt{3}},\, 0,\, -\frac{1}{\sqrt{3}},\, -\sqrt{3},\, \infty \right\}. \tag{3.61}$$

For a table of these coefficients we refer the interested reader to [102]. This integral representation is good enough for standard computation with a symbolic manipulator like *Mathematica* or *Maple*. For calculations by hand or to very high levels (big n or m) with computer, it is much more efficient to use the recursive formulas from [48, 49]. Their coefficients are related to ours by

$$X^{rs}_{nm} = n\tilde{N}^{rs}_{nm}. \tag{3.62}$$

The large level behaviour of these coefficients has been studied by Romans in [104].

3.3 CFT approach

We are now going to describe a beautiful approach to string field theory developed by LeClair, Peskin and Preitschopf in [50, 51]. The basic idea is to express all the terms in the field theory action as correlators in the two dimensional conformal field theory (CFT) on the string worldsheet. The highlights of the approach is its conceptual simplicity, manifest symmetry of the string field theory vertex and an elegant proof of gauge invariance. The latter requires 'Generalized gluing and resmoothing theorem', which justifies the use of geometric intuition in string field theory. The approach can be also used to derive the oscillator form of the vertex, discussed in the preceding subsection, in a simpler and more straightforward manner.

The string field $|\Psi\rangle$ is an element of the first quantized string theory Hilbert space $\mathcal{H}$, which is a state space in the combined worldsheet matter and ghost CFT. In the two dimensional CFT there is a state–operator correspondence between states and operators given by the relation

$$|\Psi\rangle = \Psi(0)|0\rangle. \tag{3.63}$$

The n-point vertex is given by for any $n \geq 1$

$$\langle\, \Psi_1, \Psi_2, \ldots, \Psi_n \,\rangle = \langle\, f^{(n)}_1 \circ \Psi_1(0)\, f^{(n)}_2 \circ \Psi_2(0)\, \ldots\, f^{(n)}_n \circ \Psi_n(0)\,\rangle, \tag{3.64}$$

which we are going to write also as

$$\begin{aligned} \langle\, \Psi_1, \Psi_2, \ldots, \Psi_n \,\rangle &= \int \Psi_1 * \Psi_2 * \cdots * \Psi_n \\ &= \langle V_{12\ldots n}||\Psi_1\rangle \otimes |\Psi_2\rangle \otimes \cdots \otimes |\Psi_n\rangle. \end{aligned} \tag{3.65}$$

The $f_k^{(n)}$'s are complex maps given below. The symbol $f \circ \Psi$ denotes the conformal transform of $\Psi(0)$ by f. For a primary field of dimension d it has the explicit form $f \circ \Psi = f'(0)^d \Psi(f(0))$, for non–primary fields it has more complicated form involving additional terms with higher derivatives of f.

The construction of maps $f_k^{(n)}$ proceeds as follows. We represent the string worldsheet as the complex upper half plane. The radial coordinate $|z|$ represents the time evolution, and therefore the part of the world-sheet before the moment of the interaction at $|z| = 1$ is the unit upper half disk. The past infinity is represented by a point in the origin. By a mapping

$$h(z) = \frac{1+iz}{1-iz} \tag{3.66}$$

we map it to the unit right-half-disk $\{|w| \leq 1, \mathrm{Re}\, w \geq 0\}$. Then by power mapping $(.)^{\frac{2}{n}}$ we shrink[1] it to a wedge of $\frac{360^\circ}{n}$ and rotate it by this angle k times in the clockwise direction. Finally we map it back to the upper half plane. Putting everything together we arrive to

$$f_k^{(n)} = h^{-1}\left(e^{-\frac{2\pi i k}{n}} (h(z))^{\frac{2}{n}}\right). \tag{3.67}$$

3.3.1 Relation to Witten's formulation

Now we would like to argue that the above prescription for the string field vertices is indeed equivalent to the original Witten's one in terms of Schrödinger representation. First let us consider a two vertex

$$\begin{aligned}
\langle \Psi_1, \Psi_2 \rangle &= \langle I \circ \Psi_1(0)\, \Psi_2(0) \rangle \\
&= \langle \mathrm{bpz}(\Psi_1) | \Psi_2 \rangle \\
&= \int \mathcal{D}X(\sigma) \, \langle \mathrm{bpz}(\Psi_1) | X(\sigma) \rangle \langle X(\sigma) | \Psi_2 \rangle \\
&= \int \mathcal{D}X_1(\sigma) \mathcal{D}X_2(\sigma) \prod_{0 \leq \sigma \leq \pi} \delta(X_2(\sigma) - X_1(\pi - \sigma))\, \Psi_1(X_1(\sigma)) \Psi_2(X_2(\sigma)),
\end{aligned} \tag{3.68}$$

where $I(z) = -\frac{1}{z}$ is the inversion map, which is actually equal to $f_1^{(2)}$. Let us give an interpretation to this result. Suppose, that we are representing world-sheets of two strings as upper half disks with insertion of a local operator at the origin. Performing a path integral over each half disk with Neumann boundary conditions on the real axis produces a functional of the boundary conditions on the unit semi-circle. Now mapping the whole upper half plane

[1] It is actually shrinking for $n > 2$. For $n = 2$ it leaves it invariant and for $n = 1$ it expands the half-disk to the full disk.

to the unit disk using $h(z)$, the upper half disk is mapped to the right half disk with insertion at point 1. Gluing two such right half disks along the imaginary axis $[-i, i]$ is tantamount to performing the above integral with delta functions. One could argue that, in more generality, gluing two half disks so that $[0, i]$ of one disk gets glued with $[-i, 0]$ of the other disk corresponds again to the functional integral over the boundary conditions with the delta function overlap (3.7).

Let us be more explicit. Take for instance the three vertex, which is given by a path integral over the trisected region. One can factorize the integrand into three parts which depend only on the fields in the three regions. The whole path integral can be therefore written as a path integral over the boundary conditions on the three inner boundaries and over the three inner regions. The bulk path integrals give just the Schrödinger functionals of the boundary conditions $X_{ij}(\sigma)$ between the regions i and j, where $i, j = 1, 2, 3$. Since these boundaries are half-strings it is natural to parameterize them by $\sigma \in [0, \frac{\pi}{2}]$. The remaining integral over the boundary conditions

$$\begin{aligned} \langle \Psi_1, \Psi_2, \Psi_3 \rangle \ = \ & \int \mathcal{D}X_{12}(\sigma)\mathcal{D}X_{23}(\sigma)\mathcal{D}X_{31}(\sigma) \\ & \Psi_1[X_{31}(\sigma), X_{12}(\sigma)]\Psi_2[X_{12}(\sigma), X_{23}(\sigma)]\Psi_3[X_{23}(\sigma), X_{31}(\sigma)] \end{aligned} \tag{3.69}$$

can be easily recast in the original Witten's form

$$\begin{aligned} & \int \mathcal{D}X_1(\sigma)\mathcal{D}X_2(\sigma)\mathcal{D}X_3(\sigma)\, \Psi_1[X_1(\sigma)]\Psi_2[X_2(\sigma)]\Psi_3[X_3(\sigma)] \\ & \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta(X_2(\sigma) - X_1(\pi - \sigma))\delta(X_3(\sigma) - X_2(\pi - \sigma))\delta(X_1(\sigma) - X_3(\pi - \sigma)), \end{aligned}$$

where now the variable $\sigma \in [0, \pi]$ parameterizes the whole string.

3.3.2 Fock representation from the CFT approach

We have seen in the section 3.2 that the direct construction of the Fock space representation of the string field theory vertex is quite complicated. Actually Gross and Jevicki [48, 49] also used some sort of CFT approach to solve their overlap equations for the vertex. The arguments given in [50] leads rather quickly to the right answer.

Calculating the vertex (3.64) for N arbitrary states one needs to evaluate expectation values of products of operators like

$$f \circ a_{-n} = \oint \frac{dz}{2\pi i} z^{-n} f'(z)\partial_z X(f(z)), \tag{3.70}$$

and similarly for ghosts. The expectation value in (3.64) is calculated in a standard way using the Wick theorem. All one needs are the contractions

$$f_r \circ [\ldots a_{-n} \ldots] \, f_s \circ [\ldots a_{-m} \ldots] = \oint \frac{dz}{2\pi i} z^{-n} f_r'(z) \oint \frac{dw}{2\pi i} w^{-m} f_s'(w) \frac{-1}{(f_r(z) - f_s(w))^2}. \tag{3.71}$$

Now it is clear that using these contractions is effectively equivalent to having a N-vertex (3.65) with

$$\langle V_{12\ldots n}| = \langle 0| \otimes \langle 0| \otimes \cdots \otimes \langle 0| \, e^{-\frac{1}{2}\sum_{r,s}\sum_{m,n\geq 1} a_n^r N_{nm}^{rs} a_m^s}, \tag{3.72}$$

where

$$N_{nm}^{rs} = \frac{1}{nm} \oint \frac{dz}{2\pi i} z^{-n} f_r'(z) \oint \frac{dw}{2\pi i} w^{-m} f_s'(w) \frac{-1}{(f_r(z) - f_s(w))^2}. \tag{3.73}$$

For an extension to non-zero momentum we refer the reader to [50]. One can check that this result agrees with that obtained by Gross and Jevicki.

3.4 Sen's conjectures

The open bosonic string field theory living in 26 dimensions has been thought for a long time to be internally inconsistent due to the presence of a tachyon of mass $m^2 = -\frac{1}{\alpha'}$ in its spectrum. In some of the early works it has been proposed, that the negative mass squared of the tachyon indicates an instability of the vacuum against the decay into some nonperturbative vacuum [62, 63, 64]. In a sense, the fate of the tachyon should be the same as of the Higgs field in the standard model. Later on, the vacuum has been explicitly constructed in a level truncation approximation [65, 66] but its physical meaning remained unclear.

A major breakthrough in the whole string theory came in 1995 with the discovery of D-branes by Polchinski [67]. They are extended objects on which the open strings can end and which carry conserved Ramond-Ramond (R-R) charges. In the bosonic theory there are no R-R charges, nevertheless one can still imagine having D-branes, extended objects with open strings attached on. Since they do not carry any conserved charge, they are in general unstable, which is revealed by the presence of the tachyon on their world-volume.

Ashoke Sen's major insight was, that the perturbative vacuum of the open bosonic string theory is the unstable D25 brane. The instability causes it to decay into some other vacuum, called a closed string vacuum, which is from the perspective of the open string field theory a nonperturbative state.[2] Based on this picture, Sen made three particular conjectures:

- The difference between the energy of the unstable and the perturbatively stable vacuum state ΔE should be equal to the rest mass $T_{25}V_{25}$ of the unstable D25-brane, where V_{25} and T_{25} are its volume and the tension respectively. This difference can be calculated from the string field theory as minus the string field action per unit time (which is the potential), at the critical point.

- There are translationally noninvariant vacua which correspond to the lower-dimensional branes. They are lump configurations in the tachyon and other string fields with exactly the right energies to be interpreted as D-branes.

- The perturbatively stable vacuum is the closed string vacuum only and hence there should be no open string excitations around.

The first conjecture has been tested to a rather good accuracy by the level truncation method in [56, 57] and we shall briefly sketch the calculation. It has been proved exactly in the framework of background independent string field theory (BSFT) in [68, 69, 70].

The second conjecture has been also tested numerically in [74, 76] and again exactly in the BSFT in the references mentioned above. Further references can be found in the review [59]. A different and particularly nice approach to test this conjecture is via background B-field [89, 90, 91, 92] to be discussed in chapter 6.

The third conjecture remains least understood, though enough evidence has been already gathered [93, 94].

The biggest puzzle at the moment seems to be, whether there are closed strings in this vacuum and how should they emerge. Whether they should arise only at the quantum level [71], as some sort of classical configurations [89] or by somehow enlarging the state space of open string fields to some singular configurations [72, 73].

Let us now briefly sketch for the pedagogical reasons how the numerical proof of the first conjecture in the Witten's cubic string field theory goes. It is based on the level truncation approximation, a technique devised by Kostelecký and Samuel [65, 66] in 1988. The idea is quite simple, the string field is expanded into a basis of L_0 eigenstates and only first few levels are

[2]Actually the closed string vacuum is again unstable due to the presence of the *closed* string tachyon, but it is probably out of reach of the open string field theory.

kept nonzero. The level of L_0 eigenvector is defined as its L_0 eigenvalue plus one.

The zeroth level truncation of the string field leaves behind just one term $|\Psi\rangle = tc_1|0\rangle$. Here c_1 is a mode of the ghost field $c(z)$ and t is the tachyon wave function. In full generality, we could have

$$|\Psi\rangle = \int d^{26}p\, t(p) c_1|0,p\rangle. \tag{3.74}$$

Due to the fact, that the star product of two zero momentum states is again a zero momentum state, we can consistently restrict our attention to zero momentum, that is t constant. Now we are going to evaluate the string field action

$$S[\Psi] = -\frac{1}{g^2}\left(\frac{1}{2}\langle\,\Psi, Q\Psi\,\rangle + \frac{1}{3}\langle\,\Psi, \Psi * \Psi\,\rangle\right). \tag{3.75}$$

The kinetic term is

$$\langle\,\Psi, Q\Psi\,\rangle = t^2\langle\,0|c_{-1}Qc_1|0\,\rangle = -t^2\langle\,0|c_{-1}c_0c_1|0\,\rangle = -t^2 \tag{3.76}$$

due to the standard normalization $\langle\, c_{-1}c_0c_1\,\rangle = 1$. To calculate the cubic term, one may proceed by several ways. One way is to use the explicit form of the vertex (3.57) and get

$$\begin{aligned}\langle\,\Psi, \Psi * \Psi\,\rangle &= t^3\langle V_3|\, c_1^{(1)}|0\rangle \otimes c_1^{(2)}|0\rangle \otimes c_1^{(3)}|0\rangle \\ &= \left(\frac{3\sqrt{3}}{4}\right)^3 t^3.\end{aligned} \tag{3.77}$$

Another way is to use the CFT prescription

$$\langle\,\Psi, \Psi * \Psi\,\rangle = t^3\langle\, f_1^{(3)} \circ c(0)\, f_2^{(3)} \circ c(0)\, f_3^{(3)} \circ c(0)\,\rangle, \tag{3.78}$$

and to calculate

$$\begin{aligned} f_1^{(3)} \circ c(0) &= \frac{3}{8}\, c(-\sqrt{3}), \\ f_2^{(3)} \circ c(0) &= \frac{3}{8}\, c(\sqrt{3}), \\ f_3^{(3)} \circ c(0) &= \frac{3}{2}\, c(\sqrt{0}). \end{aligned} \tag{3.79}$$

Then we need the correlator

$$\langle\, c(x)c(y)c(z)\,\rangle = -(x-y)(y-z)(z-x) \tag{3.80}$$

to get

$$\langle\, c(-\sqrt{3})\, c(\sqrt{3})\, c(\sqrt{0})\,\rangle = 2\cdot 3\cdot \sqrt{3}. \tag{3.81}$$

Again we arrive at the same result (3.77). Apart of the terms we have just calculated, there is an overall volume factor V_{26} from the zero modes. Putting everything together we get

$$S[t] = -\frac{V_{26}}{g^2}\left(-\frac{1}{2}t^2 + \frac{1}{3}\left(\frac{3\sqrt{3}}{4}\right)^3 t^3\right). \tag{3.82}$$

Local minimum of this action is at

$$t = t_c = \left(\frac{4}{3\sqrt{3}}\right)^3 \simeq 0.456. \tag{3.83}$$

The tension of the D25-brane is known to be [2]

$$T_{25} = \frac{1}{2\pi^2 g^2}, \tag{3.84}$$

which has been also ingeniously derived from string field theory by Sen [55]. Plugging the critical value of t into the action we get

$$\frac{S[t_c]-S[0]}{T_{25}V_{26}} \simeq 0.684, \tag{3.85}$$

which is not too far from one. The conjecture is thus confirmed in zeroth level approximation to nearly 70%.

Let us also sketch how it works at level two. We will need the explicit results later. One can easily show, that all odd level components of the string field can be consistently set to zero due to the twist symmetry. Another simplification comes from the fact, that due to the gauge symmetry of string field action, we may (perhaps we should) fix the gauge. Simple choice is the Siegel gauge $b_0|\Psi\rangle = 0$, which can always be reached perturbatively. The tachyon field then has the form

$$\Psi = \left[tc_1 + uc_{-1} + v\frac{1}{\sqrt{13}}L_{-2}c_1\right]|0\rangle. \tag{3.86}$$

With a bit of effort one can calculate the string field action by one of above methods

$$S[\Psi] = -\frac{1}{2}t^2 - \frac{1}{2}u^2 + \frac{1}{2}v^2 + \tag{3.87}$$
$$+\frac{1}{3}\left(\frac{3\sqrt{3}}{4}\right)^3\left[t^3 + \frac{11}{9}t^2u - \frac{5\sqrt{13}}{9}t^2v + \frac{19}{81}tu^2 + \frac{7\cdot 83}{243}tu^2 - \frac{110\sqrt{13}}{243}tuv\right].$$

Here we keep only interaction terms whose total level is at most 4, hence the name (2,4) approximation. One could do also (2,6) approximation, which is slightly better, but that improvement is not worth the effort. One may prefer then to go directly to level (4,8).

Let us look for the stationary points. There is just one good, which somehow survives from one level to another, as the level is increased, and which also satisfies some sort of additional BRST condition discussed by Hata and Shinohara [103]. The results from numerical calculation are

$$t_c \simeq 0.542,\ u_c \simeq 0.173,\ v_c \simeq 0.187. \tag{3.88}$$

Evaluating the string field action for this solution we get

$$\frac{S[t_c]}{T_{25}V_{26}} \simeq 0.949, \tag{3.89}$$

which is indeed very close to one, confirming the Sen's conjecture to 95%. This calculation has been performed originally [56] also at level (4,8) with the result 98.6% of the expected answer and later by Moeller and Taylor [57] in the Fock basis up to level (10,20) confirming it to 99.91%.

There have been also checks done for the lump solution [74, 75, 76]. The level truncation has to be a bit modified, however, to incorporate correctly the momenta. The best results have been obtained for compactified space, where one can define the level truncation procedure quite naturally.

Similar analysis for the case of the tachyon on non-BPS branes and in the brane-antibrane pairs in the Berkovits' and Witten's superstring field theory has been done in [77, 78, 79, 80, 81].

Chapter 4

Properties of the string field algebra

In the preceding chapter we have introduced various formulations or descriptions of cubic string field theory, each of them had its own advantages and drawbacks. We have also explained the famous Sen's conjectures, which up to date has been confirmed within the standard cubic string field theory only numerically. To go beyond these numerical calculations, perhaps one needs to understand better the nature of the string field algebra and to develop some new techniques.

The goal of the present chapter is to understand better an important commutative subalgebra formed by the so called wedge states introduced by Rastelli and Zwiebach [82]. This family incorporates for instance the $SL(2,\mathbb{R})$ invariant vacuum $|0\rangle$, the identity element $|\mathcal{I}\rangle$ of the algebra and a state denoted as $|\infty\rangle$, which is a projector in the algebra. The latter state plays a key role in the description of the lumps around the tachyon vacuum [96].

We shall describe in detail the construction of wedge states and their star multiplication properties. We also introduce so called wedge states with insertions, which also can be multiplied exactly, they will turn out to be useful in the study of the identity wedge state and anomalies associated with it. Then we turn our attention to the problem of behaviour of the coefficients entering the definition of the wedge states. The results are quite remarkable, although for our main goal they are not very useful. Finally we study how is the level expansion good for star multiplication of wedge states.

4.1 Wedge states

4.1.1 Finite conformal transformation

Let us recall first some facts about finite conformal transformations. A primary field of conformal weight d transforms under finite conformal transformation f as

$$f \circ \Psi = [f'(z)]^d\, \Psi(f(z)). \tag{4.1}$$

We would like to rewrite this transformation rule using the Virasoro generators L_n of the conformal group in the form

$$[f'(z)]^d\, \Psi(f(z)) = U_f \Psi(z) U_f^{-1}, \tag{4.2}$$

where

$$U_f = e^{\sum v_n L_n}. \tag{4.3}$$

To determine the coefficients v_n we note that

$$U_f \Psi(z) U_f^{-1} = e^{\mathrm{ad}_{\sum v_n L_n}} \Psi(z), \tag{4.4}$$

where as usually $\mathrm{ad}_X Y = [X, Y]$. We may prove an important identity

$$\left(\mathrm{ad}_{\sum v_n L_n}\right)^k \Psi(z) = (v(z)\partial_z + dv'(z))^k \Psi(z), \tag{4.5}$$

for any $k \in \mathbb{N}$, where we set

$$v(z) = \sum v_n z^{n+1}. \tag{4.6}$$

The proof for $k = 1$ can be easily performed for example by expanding $\Psi = \sum \frac{\Psi_n}{z^{n+d}}$ and using the commutation relations

$$[L_m, \Psi_n] = ((d-1)m - n)\Psi_{m+n}. \tag{4.7}$$

Then for $k > 1$ it follows trivially. We thus see that in general

$$U_f \Psi(z) U_f^{-1} = e^{v(z)\partial_z + dv'(z)} \Psi(z). \tag{4.8}$$

Our task is therefore for a given $f(z)$ to find a solution $v(z)$, such that for any $\Psi(z)$ and any d

$$e^{v(z)\partial_z + dv'(z)} \Psi(z) = [f'(z)]^d\, \Psi(f(z)). \tag{4.9}$$

A priori it is not even clear that such $v(z)$ exists. Let us insert into the left hand side the identity $e^{-v\partial} e^{v\partial}$. One may easily check that

$$e^{v(z)\partial_z} \Psi(z) = \Psi(e^{v(z)\partial_z} z). \tag{4.10}$$

Since $e^{v(z)\partial_z + dv'(z)}e^{-v(z)\partial_z}$ is just an ordinary function we have to take such $v(z)$ that

$$e^{v(z)\partial_z} z = f(z). \tag{4.11}$$

From that follows another important relation

$$v(z)\partial_z f(z) = v(f(z)). \tag{4.12}$$

The proof is simple:

$$v(z)\partial_z f(z) = e^{v(z)\partial_z} v(z)\partial_z z = e^{v(z)\partial_z} v(z) = v(f(z)). \tag{4.13}$$

For consistency we should be able to show also

$$e^{v(z)\partial_z + dv'(z)}e^{-v(z)\partial_z} = [f'(z)]^d, \tag{4.14}$$

for any d. In order to prove it let us define for $t \in [0,1]$

$$\begin{aligned} f_t(z) &= e^{tv(z)\partial_z} z \\ X_t(z) &= e^{tv(z)\partial_z + dtv'(z)}e^{-tv(z)\partial_z} \end{aligned} \tag{4.15}$$

Let us derive a differential equation for X_t:

$$\begin{aligned} \partial_t X_t(z) &= dv'(f_t(z))X_t(z) \\ &= d\frac{\partial_t\partial_z f_t(z)}{\partial_z f_t(z)} X_t(z). \end{aligned} \tag{4.16}$$

Integrating this equation from 0 to t we obtain

$$X_t(z) = [f_t'(z)]^d \tag{4.17}$$

which for $t = 1$ completes our proof.

For a given $f(z)$ we can formally determine $v(z)$ from (4.11). Plugging the Laurent expansion of f we get all the coefficients v_n recursively. If f vanishes in the origin $f(0) = 0$ and is holomorphic nearby, then only v_n with $n \geq 0$ are nonzero.

For some purposes it is convenient to separate out the global scaling component v_0. This is easily achieved by writing

$$f(z) = f'(0)\frac{f(z)}{f'(0)} \tag{4.18}$$

and using the composition rule

$$U_{f\circ g} = U_f U_g. \tag{4.19}$$

It follows that

$$U_f = e^{v_0 L_0} e^{\sum_{n\geq 1} v_n L_n}, \tag{4.20}$$

where

$$\begin{aligned} e^{v_0} &= f'(0) \\ e^{\sum_{n\geq 1} v_n z^{n+1}\partial_z} &= \frac{f(z)}{f'(0)}. \end{aligned} \tag{4.21}$$

4.1.2 The definition of wedge states

The wedge states form a subset of more general surface states. The former are characterized by a specific choice of conformal mapping

$$f_r(z) = h^{-1}\left(h(z)^{\frac{2}{r}}\right) = \tan\left(\frac{2}{r}\arctan(z)\right), \tag{4.22}$$

which maps half-disk in the upper half plane into a half-disk in the right half-plane, then shrinks or expands it into a wedge of angle $\frac{360°}{r}$. Finally it maps the wedge back into the upper half-plane.

The wedge states themselves are defined by the requirement

$$\langle f_r|\phi\rangle = \langle f_r \circ \phi \rangle, \qquad \forall\phi. \tag{4.23}$$

From this follows a simple formula for the wedge states

$$\langle f_r| = \langle 0|U_{f_r}. \tag{4.24}$$

We will denote the kets for these states in the sequel mostly as $|r\rangle$ and sometimes as $|\frac{360°}{r}\rangle$. The one point function (4.23) on the disk can be alternatively calculated on the disk via

$$\langle f_r \circ \phi \rangle_{\text{half-plane}} = \langle F_r \circ \phi \rangle_{\text{disk}}, \tag{4.25}$$

where $F_r(z) = h(z)^{\frac{2}{r}}$.

From the results of [50] we know that we can apply any conformal transformation, not necessarily $SL(2,\mathbb{C})$, to any correlator. Since only $SL(2,\mathbb{C})$ transformations map the complex plane into itself in a single valued manner, a general mapping $f(z)$ will carry the plane into a Riemann surface with branch points. Evaluation of conformal field theory correlators on a general Riemann surface has to be defined, the most natural choice is to evaluate the propagators $\langle XX \rangle$ and $\langle bc \rangle$ by mapping them back to the plane. It would seem that we have not gained anything, the bonus comes later when we glue together various pieces of Riemann surfaces.

By simple mapping $z \to z^{\frac{r}{2}}$ the disk correlator can be viewed as ordinary one point function on the Riemann surface with total neighborhood angle πr.

BPZ contraction in (4.23) can in general be viewed as two point function on the disk, where at point 1 we insert the vertex operator creating the state ϕ and in -1 the vertex operator for the wedge state. The functional integral over the left and right half-disk separately with fixed boundary condition on the line segment separating them, produces two Schrödinger functionals for these two states. The functional integral over the boundary between the half-disks represents the BPZ contraction itself.

From all of this discussion it should be clear that gluing in half-disk with insertion of the vertex operator for the wedge state (which we do not know explicitly) is equivalent to gluing a piece of Riemann surface of total neighborhood angle $\pi(r-1)$.

The star multiplication of two wedge states readily follows. The three vertex contracted with two wedge states $|r\rangle, |s\rangle$ and one auxiliary state $|\phi\rangle$

$$\langle V||r\rangle \otimes |s\rangle \otimes |\phi\rangle \tag{4.26}$$

can be represented first as Riemann surface of total neighborhood angle 3π with three insertions. By the above mentioned equivalence we can replace the half-disks with vertex operators for the wedge states by parts of the Riemann surfaces of angles $\pi(r-1)$ and $\pi(s-1)$. Gluing them together produces a surface with total angle $\pi(r+s-2)$. Equating $r+s-2=t-1$ gives $t=r+s-1$ and thus the desired composition rule

$$|r\rangle * |s\rangle = |r+s-1\rangle. \tag{4.27}$$

Let us give some concrete examples of the wedge states

$$\begin{aligned}
|1\rangle &= e^{L_{-2}-\frac{1}{2}L_{-4}+\frac{1}{2}L_{-6}-\frac{7}{12}L_{-8}+\frac{2}{3}L_{-10}-\frac{13}{20}L_{-12}+\cdots}|0\rangle \\
|2\rangle &= |0\rangle \\
|3\rangle &= e^{-\frac{5}{27}L_{-2}+\frac{13}{486}L_{-4}-\frac{317}{39366}L_{-6}+\frac{715}{236196}L_{-8}-\frac{17870}{14348907}L_{-10}+\cdots}|0\rangle \\
|4\rangle &= e^{-\frac{1}{4}L_{-2}+\frac{1}{32}L_{-4}-\frac{1}{128}L_{-6}+\frac{7}{3072}L_{-8}-\frac{1}{1536}L_{-10}+\cdots}|0\rangle \\
|\infty\rangle &= e^{-\frac{1}{3}L_{-2}+\frac{1}{30}L_{-4}-\frac{11}{1890}L_{-6}+\frac{1}{1260}L_{-8}+\frac{34}{467775}L_{-10}+\cdots}|0\rangle.
\end{aligned} \tag{4.28}$$

To avoid the confusion, the vacuum $|0\rangle$ is a wedge state $|2\rangle$. Wedge state with $r=0$ simply does not exist. For general r we have

$$\begin{aligned}
|r\rangle &= \exp\left(-\frac{r^2-4}{3\,r^2}L_{-2}+\frac{r^4-16}{30\,r^4}L_{-4}-\frac{(r^2-4)\,(176+128\,r^2+11\,r^4)}{1890\,r^6}L_{-6}+\right. \\
&\quad \left.+\frac{(r^2-4)\,(4+r^2)\,(16+32\,r^2+r^4)}{1260\,r^8}L_{-8}+\cdots\right)|0\rangle.
\end{aligned} \tag{4.29}$$

4.1.3 Wedge states with insertions

Let us take a primary field $\mathcal{P}(z)$ of dimension d and a point x inside the unit circle.[1] The wedge states with insertion are defined by

$$\langle f_{r,\mathcal{P},x}| = \langle 0| I \circ \mathcal{P}(x)\, U_{f_r}, \tag{4.30}$$

where $Iz = -1/z$. More generally we can add any number of insertions. We shall need the following simple property

$$\begin{aligned} \langle\, f_{r,\mathcal{P},x}|\phi\,\rangle &= \langle\, h \circ I \circ \mathcal{P}(x)\ h^{\frac{2}{r}} \circ \phi(0)\,\rangle_{\text{disk}} \\ &= \langle\, h^{\frac{r}{2}} \circ I \circ \mathcal{P}(x)\ h \circ \phi(0)\,\rangle_{\text{Riemann-surface}} \end{aligned} \tag{4.31}$$

We thus see that the effect of the vertex operator for the wedge state with insertion is again to replace this half-disk with a piece of Riemann surface of total angle $\pi(r-1)$ and inserting an operator $\mathcal{P}$ at point

$$h^{\frac{r}{2}} \circ I(x) = e^{ir \arctan x + i\frac{\pi r}{2}}. \tag{4.32}$$

This equality is actually valid for the standard definition of the function $\arctan x$ in the complex plane. However to appreciate the geometric picture, it is better to temporarily think of x as sitting in the line segment $(-1,1)$ of the real axis. Let us now calculate the star product

$$U_r^\dagger \mathcal{P}_1(x)|0\rangle * U_s^\dagger \mathcal{P}_2(y)|0\rangle. \tag{4.33}$$

Again we consider the Witten vertex as Riemann surface obtained by gluing three half-disks, corresponding to the states $U_r^\dagger \mathcal{P}_1(x)|0\rangle, U_s^\dagger \mathcal{P}_2(y)|0\rangle$ and $|\phi\rangle$ in clockwise order. We can replace two of them according to the above rule. Finally we wish to interpret this three vertex as a BPZ contraction of ϕ and a wedge state with (two) insertions. To find the insertion points we have to match simply

$$\begin{aligned} e^{is \arctan y + i\frac{\pi s}{2}} &= e^{it \arctan y' + i\frac{\pi t}{2}}, \\ e^{ir \arctan x + i\frac{\pi r}{2} + i\pi(s-1)} &= e^{it \arctan x' + i\frac{\pi t}{2}} \end{aligned} \tag{4.34}$$

where $t = r + s - 1$. The solution is

$$\begin{aligned} x' &= \cot\left(\frac{r}{t} \operatorname{arccot} x\right), \\ y' &= -\cot\left(\frac{s}{t} \operatorname{arccot}(-y)\right), \end{aligned} \tag{4.35}$$

[1] One may try to go outside of the unit circle by an analytic continuation, but it is quite problematic. Our formulas show clearly that for $x \to \pm i$ the level truncation breaks down, the star product itself is singular. There are two branch cuts starting at $\pm i$ and going to infinity. Across these branch cuts the star product would vary discontinuously and therefore it would fail to be a good product.

where we define $\operatorname{arccot} x = \frac{\pi}{2} - \arctan x$, so that x' is continuous as we vary x across zero. For general complex x and y in the unit circle it is more convenient to use the formula

$$\begin{aligned} x' &= I \circ h^{-1} \circ (\cdot)^{\frac{r}{t}} \circ h \circ I(x), \\ y' &= I \circ h^{-1} \circ (\cdot)^{\frac{s}{t}} \circ h \circ I(y), \end{aligned} \tag{4.36}$$

where one must be careful about the proper definition of the rational powers. The mapping $h \circ I$ takes the unit disk into left half plane. The power then, in the case of x', maps $e^{-i\pi r/2} \to e^{-i\pi r/t}$ and in the case of y' it sends $e^{i\pi s/2} \to e^{i\pi s/t}$.

Having found out the insertion points it is quite simple to work out also the normalization factors coming from the transformation law of the primary fields $\mathcal{P}_1$ and $\mathcal{P}_2$. Altogether, we arrive at

$$U_r^\dagger \mathcal{P}_1(x)|0\rangle * U_s^\dagger \mathcal{P}_2(y)|0\rangle = \left(\frac{r}{t}\cdot\frac{1+x'^2}{1+x^2}\right)^{d_1} \left(\frac{s}{t}\cdot\frac{1+y'^2}{1+y^2}\right)^{d_2} U_{r+s-1}^\dagger \mathcal{P}_1(x')\mathcal{P}_2(y')|0\rangle. \tag{4.37}$$

4.1.4 Sliver states with insertions

The sliver state plays a particular role in the vacuum string field theory, originally proposed by Rastelli, Sen and Zwiebach [95]. The sliver itself corresponds to the D25-brane, and its various modifications to lower dimensional branes. These lower dimensional solutions has been constructed by these people via oscillator methods [95, 96] based on the ideas of Kostelecký and Potting [83], in the formalism of split string field theory in [97, 100, 101] and in [98] by boundary CFT methods.

Our original idea was to construct these solutions using deformation of the sliver by the operator $e^{ikX(0)}$, i.e. to consider some (infinite) linear combination of the states of the form

$$U_\infty'^\dagger|0,k\rangle, \tag{4.38}$$

where in general U_r' stands for U_r with the global scaling component $\left(\frac{2}{r}\right)^{L_0}$ thrown away. Therefore $U_\infty'^\dagger|0,k\rangle$ is a well defined finite state in the level expansion.

From the formula (4.37) follows for $r \to \infty$

$$U_r'^\dagger|0,k\rangle * U_r'^\dagger|0,l\rangle = 2^{\frac{\alpha'}{4}(k^2+l^2)} \left(\frac{2}{r}\right)^{2\alpha' kl} U_{2r-1}'^\dagger|0,k+l\rangle. \tag{4.39}$$

We see that although the two factors on the left hand side approach finite unambiguous limit, the result is zero or infinite depending on the sign of kl. Therefore our initial guess of representing the lump solutions using these states was not correct. It would be interesting if the construction in [98] could be interpreted somehow along these lines.

Ghost insertions are also interesting. One can again calculate

$$U_r'^{\dagger} c_1|0\rangle * U_r'^{\dagger} c_1|0\rangle = -\frac{8}{r} U_{2r-1}'^{\dagger} c_1 c_0|0\rangle. \tag{4.40}$$

At first sight it looks great since

$$QU_r'^{\dagger} c_1|0\rangle = -U_r'^{\dagger} c_0 c_1|0\rangle \tag{4.41}$$

and it seems that $\psi = \frac{r}{8} U_r'^{\dagger} c_1|0\rangle = \frac{1}{4} U_r^{\dagger} c_1|0\rangle$ solves the string field equation of motion $Q\psi + \psi * \psi = 0$. Unfortunately the presence of $1/r$ on the right hand side of (4.40) spoils this formal argument. Similar studies of ghost number one excitations around the sliver were performed by David [84].

4.2 Identity string field

In this section we would like to turn our attention to the identity element of the string field algebra. In general, identity element of any algebra is quite an important object. It may or may not exist. For the string field star algebra we shall give an explicit construction bellow. However, since we are lacking a mathematically satisfactory definition of the algebra itself we cannot say whether the identity actually belongs to the space or not. To give a good definition of the algebra one can require finite norm for example, but then the problem is shifted to finding a good norm. Obviously for the closure of the algebra we need the product of two vacuum states $|0\rangle$ to have again a finite norm. Even for the case of the canonical norm $||\psi|| = \sqrt{\langle\,\psi|\psi\,\rangle}$ it is not clear to us.

Let us now forget about the problems whether the identity should belong to the algebra or not and let us describe its various forms. In the Witten's formulation it was clearly the functional (3.24)

$$\langle\, X(\sigma)|I\,\rangle = \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta(X(\sigma) - X(\pi-\sigma)). \tag{4.42}$$

It is a one line computation to verify that it is the identity for the star product (3.7). To write it in the Fock space, first we need to use the mode expansion (3.11) to get

$$\langle\, X(\sigma)|I\,\rangle = \prod_{n=1,3,\ldots} \delta(x_n). \tag{4.43}$$

Then after expressing the coherent states $|x_n\rangle$ using the creation operators we get by a simple calculation

$$\begin{aligned} |I\rangle &= \int \mathcal{D}X(\sigma)\,|X(\sigma)\rangle\langle X(\sigma)|I\rangle \\ &= e^{-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n a_n^\dagger a_n^\dagger}|0\rangle. \end{aligned} \tag{4.44}$$

Clearly it is a natural generalization we found in section 3.1.2 for the truncated star product identity. Treating carefully the ghosts one gets in the oscillator approach [49]

$$|I\rangle = \frac{i}{4} b(i)b(-i)e^{\sum_{n=1}^{\infty}(-1)^n(-\frac{1}{2}a_n^\dagger a_n^\dagger + c_{-n}b_{-n})}c_1 c_0|0\rangle. \tag{4.45}$$

From the geometrical representation of the star product discussed in preceding section 4.1 it is clear that the identity is the wedge state $|360°\rangle$. This can be seen somehow indirectly also from the formalism presented in section 3.1.3. Since the identity is also integration, it should satisfy

$$\langle I|\Psi\rangle = \langle f_0^{(2)} \circ \Psi\rangle = \langle 0|U_{f_0^{(2)}}\Psi\rangle \tag{4.46}$$

and therefore indeed

$$|I\rangle = U^\dagger_{360°}|0\rangle. \tag{4.47}$$

It looks explicitly as

$$|I\rangle = e^{L_{-2}-\frac{1}{2}L_{-4}+\frac{1}{2}L_{-6}-\frac{7}{12}L_{-8}+\frac{2}{3}L_{-10}-\frac{13}{20}L_{-12}+\cdots}|0\rangle. \tag{4.48}$$

We have calculated the higher level terms in the exponent exactly up to L_{-100} term, the results are plotted in the graph Fig. A. It is quite surprising that up to the level 20 the coefficients are less or around one, but then start to diverge faster than any exponential. This divergence should be however viewed as some combinatorial divergence related to unfortunate ordering of the Virasoro generators. Indeed, a nice alternative form of the identity has been found by Ellwood et. al. [105]

$$|I\rangle = \left(\prod_{n=2}^{\infty} e^{-\frac{2}{2^n}L_{-2^n}}\right) e^{L_{-2}}|0\rangle, \tag{4.49}$$

in which higher level terms have manifestly well behaved coefficients.

Finally let us note, that one can easily perform an explicit calculation in level truncation to show, that various forms of the identity (4.45), (4.48) and (4.49) are indeed in mutual agreement.

4.2.1 Conservation laws for the identity

Virasoro conservation laws

Let us recall the derivation of the conservation laws due to Rastelli and Zwiebach [82]. We start with a global coordinate $\tilde{z}$ on the 1-punctured disk, associated to the identity $\langle I|$. For any holomorphic vector field $\tilde{v}(\tilde{z})$ we have the basic identity

$$\langle I| \oint_C d\tilde{z}\, \tilde{v}(\tilde{z})\tilde{T}(\tilde{z}) = 0, \tag{4.50}$$

where C is a contour encircling the puncture. Passing to the local coordinate z around the puncture we get

$$\langle I| \oint_C dz\, v(z) \left(T(z) - \frac{c}{12} S(f^{360°}(z), z)\right) = 0, \tag{4.51}$$

where

$$S(f^{360°}(z), z) = 6(1+z^2)^{-2} = 6\sum_{m=1}^{\infty} m(-1)^{m-1} z^{2(m-1)} \tag{4.52}$$

is the Schwarzian derivative reflecting the non-tensor character of the energy momentum tensor when the central charge c is nonzero. For a particular choice of the vector field $v(z) = z^{n+1} - (-1)^n z^{-n+1}$, which is holomorphic everywhere in the global coordinate $\tilde{z}$ except the puncture, we get

$$\begin{aligned} K_{2n}|I\rangle &= -\frac{c}{2} n(-1)^n |I\rangle, \\ K_{2n+1}|I\rangle &= 0, \end{aligned} \tag{4.53}$$

where we define

$$K_n = L_n - (-1)^n L_{-n}. \tag{4.54}$$

The same identities can be derived for the b ghost, in that case, there is no anomaly however.

Let us further comment on some applications of the formulas we have obtained. First one can rewrite the state $T(z)|I\rangle$ in a form which is manifestly well defined in the level expansion and perform the geometric sums provided $|z| > 1$.

$$T(z)|I\rangle = \frac{c}{2}\frac{1}{(1+z^2)^2}|I\rangle + \frac{1}{z^2}L_0|I\rangle + \frac{1}{z^2}\sum_{n\geq 1}\left(z^n + (-1)^n z^{-n}\right) L_{-n}|I\rangle. \tag{4.55}$$

From these identities, and those for the b ghost, one can easily check the overlap equations

$$\begin{aligned}\left(T(z)-\frac{1}{z^4}T(-1/z)\right)|I\rangle &= 0,\\ \left(b(z)-\frac{1}{z^4}b(-1/z)\right)|I\rangle &= 0.\end{aligned} \tag{4.56}$$

Conservation of the c-ghost

We start from the identity

$$\langle I|\oint_C dz\,\phi(z)c(z)=0, \tag{4.57}$$

where $\phi(z)$ is a quadratic differential holomorphic everywhere except at the puncture located at the origin, and C is a contour encircling the puncture. The $\phi(z)$ transforms as follows

$$\tilde{\phi}(\tilde{z})=\left(\frac{dz}{d\tilde{z}}\right)^2\phi(z). \tag{4.58}$$

We shall pass from the local coordinate z around the puncture to the global coordinate on the 1-punctured disk

$$\tilde{z}=\frac{2z}{1-z^2}. \tag{4.59}$$

For the particular choice of the quadratic differentials

$$\begin{aligned}\phi_{2n}(z) &= \frac{1}{z^2}\left(z^n-\left(-\frac{1}{z}\right)^n\right)^2,\\ \phi_{2n+1}(z) &= \frac{1}{z^2}\left(z^{2n+1}-\left(-\frac{1}{z}\right)^{2n+1}-(-1)^n\left(z-\frac{1}{z}\right)\right),\end{aligned} \tag{4.60}$$

where $n\geq 1$, the transformed differentials are

$$\tilde{\phi}_{2n}(\tilde{z}) = \frac{4}{\tilde{z}^{2n+2}}\left(\sum_{k=1,3,5,\ldots}\binom{n}{k}(1+\tilde{z}^2)^{\frac{k-1}{2}}\right)^2, \tag{4.61}$$

$$\tilde{\phi}_{2n+1}(\tilde{z}) = -\frac{2}{\tilde{z}^{2n+3}}\frac{1}{1+\tilde{z}^2}\left(-(-1)^n\tilde{z}^{2n}+\sum_{k=0,2,4,\ldots}\binom{2n+1}{k}(1+\tilde{z}^2)^{\frac{k}{2}}\right).$$

All the sums here are finite due to the combinatorial factors which are defined to be zero whenever the lower entry is bigger than the upper entry.

The quadratic differentials expressed in the global coordinate system $\tilde{z}$ are holomorphic in the whole complex plane except zero, in particular they do not have any singularity at $\pm i$. Therefore one may derive from (4.57) the conservation laws

$$\begin{aligned} C_{2n}|I\rangle &= (-1)^n C_0|0\rangle, \\ C_{2n+1}|I\rangle &= (-1)^n C_1|0\rangle. \end{aligned} \tag{4.62}$$

where in general we define

$$C_k = c_k + (-1)^k c_{-k}. \tag{4.63}$$

Let us remark that a naive conservation law based on the quadratic differential

$$\phi(z) = \frac{1}{z^2}\left(z^n - \left(-\frac{1}{z}\right)^n\right) \tag{4.64}$$

fails, since $\tilde{\phi}(\tilde{z})$ does have have poles in $\pm i$. This is actually a simple manifestation of the midpoint anomalies.

Again, as we have done for the energy momentum tensor, we can rewrite the state $c(z)|I\rangle$ in a form which is manifestly well defined in the level expansion provided $|z| > 1$,

$$c(z)|I\rangle = -z\frac{1-z^2}{1+z^2}\,c_0|I\rangle + \frac{z^2}{1+z^2}\,(c_1 - c_{-1})\,|I\rangle + z\sum_{n\geq 1}\left(z^n - \left(-\frac{1}{z}\right)^n\right) c_{-n}|I\rangle. \tag{4.65}$$

The single poles for $z \to \pm i$ were first found by other means by [107, 108]. From this formula, it is a simple exercise to verify the overlap equation

$$\big(c(z) - z^2 c(-1/z)\big)\,|I\rangle = 0. \tag{4.66}$$

Another observation we can make is about $c_L|I\rangle$, where

$$c_L = \frac{1}{2}c_0 + \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(c_{2k+1} + c_{-(2k+1)}\right). \tag{4.67}$$

We can easily calculate that

$$c_L|I\rangle = \frac{1}{2}c_0|I\rangle + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}c_{-(2k+1)}|I\rangle + \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}C_1|I\rangle, \tag{4.68}$$

which is divergent due to the last term. This fact rules out the possibility of relating solutions to the vacuum and Witten's cubic string field theories through some type of $(c_L - Q_L)|I\rangle$ shift.

Current conservation laws

For completeness let us also consider conservation laws for currents. Let us take in general a holomorphic current $J(z)$ having the following OPE with the stress tensor

$$T(z)J(0) = \frac{2q}{z^3} + \frac{J(0)}{z^2} + \frac{\partial J(0)}{z}. \qquad (4.69)$$

Under finite conformal transformations it transforms as

$$\frac{dw}{dz}\tilde{J}(w) = J(z) - q\frac{d^2w}{dz^2}\left(\frac{dw}{dz}\right)^{-1}. \qquad (4.70)$$

The anomalous constant q is zero for the BRST current and ∂X currents, for the ghost number current it is $-\frac{3}{2}$. Following the same procedure as above, one may derive conservation laws

$$\begin{aligned} H_0|I\rangle &= 0, \\ H_{2k+1}|I\rangle &= 0, \\ H_{2k}|I\rangle &= (-1)^k 2q|I\rangle, \end{aligned} \qquad (4.71)$$

where

$$H_k = J_k + (-1)^k J_{-k}. \qquad (4.72)$$

4.2.2 Anomalous properties of the identity

One particularly puzzling aspect [82] of the identity is the following: We know that c_0 acts as a derivation on the string field algebra. Therefore we can write formally

$$c_0|\Psi\rangle = c_0\left(|I\rangle * |\Psi\rangle\right) = c_0|I\rangle * |\Psi\rangle + |I\rangle * c_0|\Psi\rangle = c_0|I\rangle * |\Psi\rangle + c_0|\Psi\rangle,$$

from which follows that

$$c_0|I\rangle * |\Psi\rangle = 0 \qquad (4.73)$$

for any string field $|\Psi\rangle$. Naively one would conclude (taking $|\Psi\rangle = |I\rangle$) that $c_0|I\rangle = 0$, but that is manifestly not true. One could possibly imagine several ways out:

- $|I\rangle$ is not a true identity on all states,
- c_0 is not a true derivation on the whole algebra
- even though $c_0|I\rangle \neq 0$, still we have $c_0|I\rangle * |\Psi\rangle = 0$ for any 'well behaved' state $|\Psi\rangle$

- simply $c_0|I\rangle * |\Psi\rangle$ is not well defined in the level expansion

We will argue for the last possibility, but in some limited sense all the explanations are true.

The derivations of the fact that $|I\rangle$ is the identity and that c_0 is a derivation on the algebra are quite firm when one restricts on well behaved states. Again it is difficult to say what is a well defined state, but those which contain finitely many levels certainly are. We have checked numerically that the identity is an identity on many states, it seems that it is an identity even on itself and other wedge states, which might be somehow problematic.

To check the third possibility it is best to look first at some example. Let us calculate

$$c_0|I\rangle * |0\rangle.$$

In general there are many ways to do the calculation. The most naive way would be to truncate the identity to some maximal level, legally use the c_0 conservation to arrive at $c_0(|I\rangle * |0\rangle) - |I\rangle * c_0|0\rangle$ which is indeed very close to zero, since the identity $|I\rangle$ works well for the states $|0\rangle, c_0|0\rangle$. A sort of 'canonical' way of calculation suggested in [82] is to re-order $c_0|I\rangle$ to have only the ghost c_1 acting on the vacuum and Virasoro generators acting on it from the left. Then one can use the recursive relations of [49, 60, 82] for the Virasoro generators, to reduce the expression to the linear combination of terms

$$L_{-a}L_{-b}\ldots(c_1|0\rangle * |0\rangle).$$

Actually we can perform this calculation exactly even with some sort of regularization. As a first step let us commute the c-ghost to the vacuum. In more generality, we will do it for an arbitrary wedge state instead of the identity and for convenience work with the bra vectors

$$\langle r|c_0 = \langle 0|U_r c_0 = \langle 0|\left(U_r c_0 U_r^{-1}\right)U_r. \tag{4.74}$$

The factor in the bracket is readily

$$\begin{aligned} U_r c_0 U_r^{-1} &= \oint \frac{dz}{2\pi i}\frac{1}{z^2}U_r c(z)U_r^{-1} \\ &= -4r^2\sum_n c_n \oint \frac{dw}{2\pi i}\frac{1}{w^{n-1}}\frac{(1+w^2)^{r-2}}{[(1+iw)^r-(1-iw)^r]^2} \\ &= c_0 + \frac{r^2-4}{3}c_2 + \frac{r^4-10r^2+24}{15}c_4 + \frac{10r^6-168r^4+945r^2-1732}{945}c_6 + \cdots \end{aligned} \tag{4.75}$$

For general r it looks difficult to find a closed expression, for the identity state which corresponds to $r=1$ the sum can be easily performed

$$U_r c_0 U_r^{-1} = -\frac{i}{2}\left(c(i)-c(-i)\right). \tag{4.76}$$

Therefore

$$c_0|I\rangle = \frac{i}{2}\,U_1^\dagger\,(c(i) - c(-i))\,|0\rangle. \tag{4.77}$$

This is already in the form for which we know how to calculate the star products exactly. Recall however that the insertion points $\pm i$ are singular. A natural regularization from the point of view of level expansion is to replace

$$c(i) - c(-i) \quad \longrightarrow \quad c(ai) - c(-ai), \tag{4.78}$$

where $a < 1$ is approaching unity from bellow.[2] Let us calculate in more generality

$$\begin{aligned} \frac{i}{2}\,U_1^\dagger\,(c(ia) - c(-ia))\,|0\rangle * U_s^\dagger|0\rangle &= \frac{is}{2}\frac{1-a^2}{1+x'^2(ia)}U_s^\dagger c(x'(ia))|0\rangle \\ &\quad -\frac{is}{2}\frac{1-a^2}{1+x'^2(-ia)}U_s^\dagger c(x'(-ia))|0\rangle, \end{aligned} \tag{4.79}$$

where

$$x'(ia) = -i\frac{(a-1)^{\frac{2}{t}} + (a+1)^{\frac{2}{t}}}{(a-1)^{\frac{2}{t}} - (a+1)^{\frac{2}{t}}}. \tag{4.80}$$

This has well defined limit for $a \to \pm 1$, it is either i or $-i$ respectively. The prefactors require little bit more care, since they are limits of $0/0$ type.

For $s > 2$, the whole expression is well defined and we get

$$\lim_{a\to 1^-} \frac{i}{2}\,U_1^\dagger\,(c(ia) - c(-ia))\,|0\rangle * U_s^\dagger|0\rangle = 0. \tag{4.81}$$

For $s = 2$ we get

$$\lim_{a\to 1^-} \frac{i}{2}\,U_1^\dagger\,(c(ia) - c(-ia))\,|0\rangle * |0\rangle = i(c(i) - c(-i))|0\rangle. \tag{4.82}$$

For $s < 2$ at least one of the two prefactors in (4.79) is divergent. The conclusion of the above calculation is that the result of the calculation $c_0|I\rangle *$ $|0\rangle$ is highly sensitive to the method used. From the mathematical point of view, $c_0|I\rangle$ does not belong to the star algebra. If we want to have the derivation c_0 defined on all of the algebra, we should conclude, that neither the identity $|I\rangle$ belong to the algebra. Alternatively we can think of the

[2] Another interesting regularization would be to keep $r \neq 1$, but that is technically rather cumbersome, due to the presence of nontrivial contour integral in (4.75). Yet another possibility is to replace U_1 with U_r for $r \neq 1$ in (4.79), that is almost trivial modifications of our calculations.

identity belonging to the algebra, but then the derivative c_0 maps some elements of the algebra out of it.

One could imagine other ways how to study the object $c_0|I\rangle$. We would like to warn the reader of the following problem

$$\begin{aligned} \left[\oint \frac{dz}{2\pi i}\frac{c(z)}{z^2}\right]|I\rangle &= c_0|I\rangle, \\ \oint \frac{dz}{2\pi i}\left[\frac{c(z)}{z^2}|I\rangle\right] &= -c_0|I\rangle, \end{aligned} \tag{4.83}$$

where the brackets in the second case mean, that we are evaluating the expression, i.e. calculating it in the level expansion through formula (4.65). Both integrals are along small contours around the origin. To get identical results, we would need to include the points $\pm i$ inside the contour of integration. The problem can be traced back to the fact, that to make sense of $c(z)|I\rangle$ without analytic continuation, we need to remain outside of the unit circle.

Finally we would like to address the issue of the 'integrated' anomaly. Let us consider

$$\langle V_{123}|(c_0^{(1)}+c_0^{(2)}+c_0^{(3)})|I\rangle \otimes |\Psi_2\rangle \otimes |\Psi_3\rangle, \tag{4.84}$$

where for simplicity $|\Psi_{2,3}\rangle$ are two ghost number one ghost fields. This is equal to

$$\begin{aligned} &\langle \mathrm{bpz}\,\Psi_3|\Big(c_0|I\rangle * |\Psi_2\rangle + |I\rangle * c_0|\Psi_2\rangle - c_0\left(|I\rangle * |\Psi_2\rangle\right)\Big) \\ &= \langle \mathrm{bpz}\,\Psi_3|\Big(c_0|I\rangle * |\Psi_2\rangle\Big), \end{aligned} \tag{4.85}$$

which we call 'integrated' anomaly. In level truncation we can safely use the cyclicity to get further

$$-\langle I|c_0\Big(|\Psi_2\rangle * |\Psi_3\rangle\Big) = \langle\, \mathrm{bpz}\,\Psi_3|c_0|\Psi_2\,\rangle - \langle\, \mathrm{bpz}\, c_0\Psi_3|\Psi_2\,\rangle = 0. \tag{4.86}$$

This is as it should be, since our starting expression (4.84) is also manifestly zero in the level expansion. The moral is that while $c_0|I\rangle * |\Psi_2\rangle$ itself is ill defined, its bpz inner products with well behaved states can be consistently set to zero.

4.2.3 Application to the tachyon condensate

There is one remarkably simple application of the above results to the study of tachyon condensation. Under a general variation, the string field action should be invariant at the critical point

$$\delta S = -\frac{1}{g^2}\Big[\langle\, \delta\Psi | Q\Psi \,\rangle + \langle\, \delta\Psi | \Psi * \Psi \,\rangle\Big] = 0. \tag{4.87}$$

Let us take a particular variation $\delta\Psi = C_1|I\rangle$. The last term in (4.87) vanishes for the same reason as above, since c_0 and C_1 are both derivations and have the same property under the BPZ conjugation. Therefore we are led to the conclusion that the tachyon condensate should satisfy

$$\langle\, I | C_1 Q | \Psi \,\rangle = 0. \tag{4.88}$$

Since Q annihilates the identity, we may take the anticommutator $\{C_1, Q\}$. By a somewhat lengthy calculation we can calculate

$$\begin{aligned} \{Q, C_1\}|I\rangle &= -4\sum_{k=1}^{\infty}(-1)^k\left[2kc_{-2k}C_1 - (2k+1)c_{-(2k+1)}C_0\right]|I\rangle \\ &\quad + (7c_{-1}c_0 - c_0c_1)\,|I\rangle. \end{aligned} \tag{4.89}$$

Evaluating the left hand side of (4.88) for the tachyon string field truncated at level 2

$$\Psi = \left[tc_1 + uc_{-1} + v\frac{1}{\sqrt{13}}L_{-2}c_1\right]|0\rangle, \tag{4.90}$$

we find that (4.88) is equivalent to

$$t - 7u + \sqrt{13}\,v = 0. \tag{4.91}$$

Plugging in the values for the tachyon condensate at level two [56] ($t \simeq 0.542, u \simeq 0.173, v \simeq 0.187$) we get that it is satisfied to a very high accuracy

$$\frac{t - 7u + \sqrt{13}v}{t} = 0.0097. \tag{4.92}$$

Comparison with other quantities calculated at this level shows that such a high precision is likely to be coincidence, nevertheless as an explicit check of our computation it is welcome.

4.3 Behaviour of the wedge state coefficients

For the purposes of this section it is convenient to define the wedge states as

$$\langle r| = \langle 0|U'_{f_r}, \tag{4.93}$$

where now with slight abuse of notation we change the normalization of functions $f_r(z)$

$$f_r(z) = \frac{r}{2} h^{-1}\left(h(z)^{\frac{2}{r}}\right) = \frac{r}{2}\tan\left(\frac{2}{r}\arctan(z)\right) \tag{4.94}$$

to achieve $f_r'(0) = 1$. Unitary operator U_f' is given by

$$U_f = e^{\sum_{n\geq 1} v_n L_n} \tag{4.95}$$

and differs from U_f defined above by the absence of the factor $\exp(v_0 L_0)$ which is in the definition of the wedge state immaterial. The v_n's are defined by the Laurent expansion of a vector field (4.11) generating the conformal transformation

$$v(z) = \sum v_n z^{n+1}. \tag{4.96}$$

As shown above, the vector field $v(z)$ satisfies important equation

$$v(z)\partial_z f(z) = v(f(z)) \tag{4.97}$$

known in the mathematical literature under the name of Julia's equation. Given a function f holomorphic in the neighborhood of the origin one can always look for analytic solutions $v(z)$ in terms of formal power series (FPS). The solution is unique up to an overall constant which can be fixed for f of the form

$$f(z) = z + \sum_{n=m}^{\infty} b_n z^n, \qquad b_m \neq 0, m \geq 2 \tag{4.98}$$

by requiring

$$v(z) = b_m z^m + \sum_{n=m+1}^{\infty} c_n z^n. \tag{4.99}$$

Note that precisely with this normalization the function $v(z)$ satisfies also the (4.11). Such a unique solution is called the iterative logarithm and denoted either as f_* or $\operatorname{logit} f$. Interesting problem is when this FPS has finite radius of convergence.

It has been proved that if f is a meromorphic function, regular at the origin and having the expansion (4.98) then the formal power series f_* has a positive radius of convergence only if

$$f(z) = \frac{z}{1+bz}, \qquad b \in \mathbb{C} \tag{4.100}$$

This theorem is implied by the results of I.N. Baker and P. Erdős and E. Jabotinsky, see [106].

Let us see how this result applies to our wedge states. All of the functions f_r are holomorphic near the origin, but only those with $r = \frac{2}{k}$, $k \in \mathbb{Z}$ are meromorphic in the whole complex plane. Apart from the vacuum state $|180°\rangle$ all the other $|360°\rangle, |540°\rangle, \ldots$ thus correspond to divergent FPS with zero radius of convergence.

What about the other wedge states? One can establish following general properties of the iterative logarithm:

$$\operatorname{logit} f = -\operatorname{logit} f^{-1}, \tag{4.101}$$

$$\operatorname{logit}(\phi^{-1} \circ f \circ \phi) = \frac{1}{\phi'}\left((\operatorname{logit} f) \circ \phi\right), \tag{4.102}$$

where $\phi(z)$ is an analytic function with $\phi(0) = 0$ and $\phi'(0) \neq 0$. From these two relations follows (by taking $\phi(z) = \frac{r}{2}z$)

$$\operatorname{logit} f_{\frac{4}{r}} = -\frac{2}{r} \circ (\operatorname{logit} f_r) \circ \frac{r}{2}. \tag{4.103}$$

We thus obtain for the Laurent coefficients of $v^{(r)} = \operatorname{logit} f_r$ important relation

$$v_k^{(\frac{4}{r})} = -\left(\frac{2}{r}\right)^k v_k^{(r)}, \tag{4.104}$$

which can be readily checked for the explicit expression (4.29). We see that the FPS $\operatorname{logit} f_r$ and $\operatorname{logit} f_{\frac{4}{r}}$ have both either zero or finite radius convergence simultaneously. Summarizing, the FPS corresponding to the vector field $v(z)$ has zero radius of convergence for all $r = \frac{2}{k}$ and $r = 2k$ for $k \in \mathbb{Z}, k > 1$. By a limiting procedure this applies in particular to the interesting sliver state $|\infty\rangle$.

The absence of any finite radius of convergence means that starting from a certain level, some of the coefficients v_n start to grow faster than any exponential. This rather surprising result is confirmed by the actual calculation of the coefficients up to v_{100} which we have plotted for several wedge states in the appendix A. All the coefficients were calculated exactly using the recursive formula following from (4.11).

To summarize we have shown that the Laurent expansion of $v(z)$ has zero radius of convergence and therefore the function $v(z)$ has an essential singularity at zero. The series itself can be trusted as asymptotic only. The success of level truncation for the star products of wedge states appears to be analogous to the situation in QED, where at first few orders the perturbation theory works perfectly well, but at higher orders it breaks down. From the graphs in the appendix one can see that for the coefficients up to about v_{20} the coefficients decrease exponentially, this is the basic reason why the low order calculations works well. To carry out higher order calculations it is therefore necessary to use the oscillator representation.

Finally let us comment on one technical aspect of the calculation. To calculate e.g. the 100-th derivative at zero of the function f_r for generic r directly, is beyond the capacity of any computer. We can derive however the following useful formula for n odd

$$\frac{d^n}{dx^n}\tan\left(\frac{2}{r}\arctan x\right)|_{x=0} = \sum_{k=1,3,5,\ldots,n}\left(\frac{2}{r}\right)^k \frac{2^{k+1}(2^{k+1}-1)}{(k+1)!}B_{k+1}F(n,k), \tag{4.105}$$

where

$$F(n,k) = \sum_{m_i|\sum m=\frac{n-k}{2}} \frac{1}{(2m_1+1)\ldots(2m_k+1)} \tag{4.106}$$

can be easily calculated recursively. B_n's are the Bernoulli numbers. For n even the derivative is obviously zero.

4.4 Miscellaneous

From the definition of the star product one can easily obtain formulas for star product of vacuum state with any other state from the Fock space

$$\begin{aligned} |0\rangle * |\psi\rangle &= U_3^\dagger e^{-\frac{1}{\sqrt{3}}L_{-1}}\left(\frac{4}{3}\right)^{L_0} e^{-\frac{1}{\sqrt{3}}L_1}U_3|\psi\rangle, \\ |\psi\rangle * |0\rangle &= U_3^\dagger e^{\frac{1}{\sqrt{3}}L_{-1}}\left(\frac{4}{3}\right)^{L_0} e^{\frac{1}{\sqrt{3}}L_1}U_3|\psi\rangle. \end{aligned} \tag{4.107}$$

These formulas make perfectly sense in the level expansion, since if $|\psi\rangle$ contains finitely many levels, the whole expression can be calculated to any given level exactly, in finitely many steps. We have also checked it independently on several examples.

It is however difficult to understand from them the multiplication rule $|0\rangle * U_r^\dagger|0\rangle = U_{r+1}^\dagger|0\rangle$. One would need a formula for

$$U_3 U_r^\dagger|0\rangle. \tag{4.108}$$

Note, that a dagger makes much difference. It is almost trivial, that

$$U_r^\dagger U_s^\dagger|0\rangle = U_{\frac{rs}{2}}^\dagger|0\rangle, \tag{4.109}$$

which follows from simple composition of maps. To understand what (4.108) is[3], one may calculate analogous quantities for simpler finite conformal transformation operators

$$U_{\frac{z}{\sqrt{1-2tz^2}}} = e^{tL_2},$$
$$U_{\sqrt{z^2+2s}} = e^{sL_{-2}}. \quad (4.110)$$

By brute force series expansion one may derive

$$e^{tL_2}e^{sL_{-2}}|0\rangle = e^{\frac{s}{1-4ts}L_{-2}}|0\rangle. \quad (4.111)$$

Alternative proof is to write

$$\begin{aligned} e^{tL_2}e^{sL_{-2}}|0\rangle &= e^{tL_2}e^{sL_{-2}}e^{-tL_2}|0\rangle = U_{\frac{z}{\sqrt{1-2tz^2}}\circ\sqrt{z^2+2s}\circ\frac{z}{\sqrt{1+2tz^2}}}|0\rangle = \\ &= U_{\sqrt{z^2+\frac{2s}{1-4st}}}\left(\frac{1+4st}{1-4st}\right)^{\frac{L_0}{2}}|0\rangle = e^{\frac{s}{1-4ts}L_{-2}}|0\rangle \end{aligned} \quad (4.112)$$

The weak and a bit unclear point in this derivation is the seeming arbitrariness in inserting the factor e^{-tL_2}. Would we get the same answer by inserting other factors ?

Let us also mention that there is yet another formula which one can easily derive and which reads

$$e^{tL_2}e^{sL_{-2}}|0\rangle = e^{s(L_{-2}+4tL_0+4t^2L_2)}|0\rangle. \quad (4.113)$$

It is derived by viewing the first factor as a finite conformal transformation acting on the Virasoro generator in the second factor. It does not seem however relevant too much for our purposes.

4.5 Star products in level expansion

In this section we would like to collect some numerical results showing, how well the level expansion works for star products. We have performed some explicit checks at level 20, where one of the states was particularly simple, and some other checks at level 16 which confirmed the composition law (4.27) obtained by the gluing ideas described in section 4.1. We have written for that purpose a computer program in *Mathematica* which is based on the Virasoro conservation laws for the three vertex [60, 49, 82].

[3]Ashoke Sen has suggested, that it should be possible to get the answer by using the gluing theorem of [51].

4.5.1 Some level 20 calculations

First let us present the results, how good is the identity state $|I\rangle = |360°\rangle$ acting on some basic states

$$\begin{aligned}
|0\rangle * |360°\rangle &= |0\rangle + 0.00008L_{-2}|0\rangle - 0.00007L_{-3}|0\rangle - 0.00068L_{-4}|0\rangle + \\
&\quad +0.00039L_{-2}L_{-2}|0\rangle + \cdots \\
L_{-2}|0\rangle * |360°\rangle &= 0.9987L_{-2}|0\rangle - 0.0001L_{-3}|0\rangle - 0.0001L_{-4}|0\rangle + \\
&\quad +0.0007L_{-2}L_{-2}|0\rangle + \cdots \\
L_{-2}L_{-2}|0\rangle * |360°\rangle &= 0.0054L_{-2}|0\rangle - 0.0001L_{-3}|0\rangle + 0.0002L_{-4}|0\rangle + \\
&\quad +0.9967L_{-2}L_{-2}|0\rangle + \cdots \\
L_{-4}|0\rangle * |360°\rangle &= 0.0035L_{-2}|0\rangle - 0.0005L_{-3}|0\rangle + 0.9967L_{-4}|0\rangle + \\
&\quad +0.0002L_{-2}L_{-2}|0\rangle + \cdots
\end{aligned} \tag{4.114}$$

Now let us present the results for the products of the vacuum $|0\rangle$ and other wedge states to verify the composition rule $|r\rangle * |s\rangle = |r + s - 1\rangle$.

$$\begin{aligned}
|180°\rangle * |120°\rangle &= |0\rangle - 0.25006L_{-2}|0\rangle + 0.00197L_{-3}|0\rangle + 0.03132L_{-4}|0\rangle + \\
&\quad +0.03126L_{-2}L_{-2}|0\rangle + \cdots \\
|180°\rangle * |\infty\rangle &= |0\rangle - 0.32085L_{-2}|0\rangle + 0.00563L_{-3}|0\rangle + 0.03294L_{-4}|0\rangle + \\
&\quad +0.05137L_{-2}L_{-2}|0\rangle + \cdots \\
|180°\rangle * |720°\rangle &= |0\rangle - 38723.7L_{-2}|0\rangle - 22117.4L_{-3}|0\rangle - 12233.8L_{-4}|0\rangle + \\
&\quad +34414.4L_{-2}L_{-2}|0\rangle + \cdots
\end{aligned} \tag{4.115}$$

The first two products should be compared with the wedge states

$$\begin{aligned}
|90°\rangle &= |0\rangle - 0.25L_{-2}|0\rangle + 0.03125L_{-4}|0\rangle + 0.03125L_{-2}L_{-2}|0\rangle + \cdots \\
|\infty\rangle &= |0\rangle - 0.33333L_{-2}|0\rangle + 0.03333L_{-4}|0\rangle + 0.05556L_{-2}L_{-2}|0\rangle
\end{aligned} \tag{4.116}$$

We see that the agreement is quite good (within 0.23%) for the state $|90°\rangle$ but is considerably worse (within 7.5%) for the $|\infty\rangle$. The last product indicates that the state $|720°\rangle$ does not have much sense in the level expansion.

The state $|\infty\rangle$ is rather special, so we have performed further check. Particularly nice one, is testing also the identity (4.37). One could use the formula to find

$$L_{-2}|0\rangle * |\infty\rangle = \left(\frac{2}{\pi}\right)^4 U'^{\dagger}_{\infty} T\left(\frac{2}{\pi}\right)|0\rangle \tag{4.117}$$

$$\simeq 0.164L_{-2}|0\rangle + 0.105L_{-3}|0\rangle + 0.067L_{-4}|0\rangle - 0.055L_{-2}L_{-2}|0\rangle$$

and compare it with result of the level truncation calculation

$$L_{-2}|0\rangle * |\infty\rangle = 0.180L_{-2}|0\rangle + 0.110L_{-3}|0\rangle + 0.067L_{-4}|0\rangle - 0.058L_{-2}L_{-2}|0\rangle. \tag{4.118}$$

4.5.2 Level 16 calculations

$$\begin{aligned}
|360^\circ\rangle * |360^\circ\rangle &= |0\rangle + 1.00386L_{-2}|0\rangle - 0.50098L_{-4}|0\rangle + 0.49723L_{-2}L_{-2}|0\rangle + \cdots \\
|\infty\rangle * |\infty\rangle &= |0\rangle - 0.36150L_{-2}|0\rangle + 0.03338L_{-4}|0\rangle + 0.06549L_{-2}L_{-2}|0\rangle + \cdots \\
|\infty\rangle * |360^\circ\rangle &= |0\rangle - 0.32656L_{-2}|0\rangle + 0.00267L_{-3}|0\rangle + 0.03148L_{-4}|0\rangle + \\
&\quad +0.05365L_{-2}L_{-2}|0\rangle + \cdots \\
|360^\circ\rangle * |\infty\rangle &= |0\rangle - 0.32656L_{-2}|0\rangle - 0.00267L_{-3}|0\rangle + 0.03148L_{-4}|0\rangle + \\
&\quad +0.05365L_{-2}L_{-2}|0\rangle + \cdots \\
|\infty\rangle * |120^\circ\rangle &= |0\rangle - 0.33434L_{-2}|0\rangle - 0.00564L_{-3}|0\rangle + 0.03394L_{-4}|0\rangle + \\
&\quad +0.05587L_{-2}L_{-2}|0\rangle + \cdots \\
|720^\circ\rangle * |120^\circ\rangle &= |0\rangle - 2147.14L_{-2}|0\rangle + 1327.72L_{-3}|0\rangle - 553.046L_{-4}|0\rangle + \\
&\quad +2074.33L_{-2}L_{-2}|0\rangle + \cdots \\
|120^\circ\rangle * |120^\circ\rangle &= |0\rangle - 0.28708L_{-2}|0\rangle + 0.03348L_{-4}|0\rangle + 0.04122L_{-2}L_{-2}|0\rangle + \cdots \\
|180^\circ\rangle * |120^\circ\rangle &= |0\rangle - 0.25008L_{-2}|0\rangle + 0.00246L_{-3}|0\rangle + 0.03135L_{-4}|0\rangle + \\
&\quad +0.03127L_{-2}L_{-2}|0\rangle + \cdots \\
|180^\circ\rangle * |360^\circ\rangle &= |0\rangle + 0.00010L_{-2}|0\rangle - 0.00008L_{-3}|0\rangle - 0.00109L_{-4}|0\rangle + \\
&\quad +0.00066L_{-2}L_{-2}|0\rangle + \cdots \\
|180^\circ\rangle * |\infty\rangle &= |0\rangle - 0.31966L_{-2}|0\rangle + 0.00668L_{-3}|0\rangle + 0.03293L_{-4}|0\rangle + \\
&\quad +0.05096L_{-2}L_{-2}|0\rangle + \cdots \\
|180^\circ\rangle * |720^\circ\rangle &= |0\rangle - 1876.75L_{-2}|0\rangle - 1163.92L_{-3}|0\rangle - 567.608L_{-4}|0\rangle + \\
&\quad +1851.82L_{-2}L_{-2}|0\rangle + \cdots
\end{aligned} \tag{4.119}$$

One can compare this results obtained in level expansion with the exact answer. The errors for products with $|\infty\rangle$ state is smallest for $|120^\circ\rangle$ state: 0.3% at level 2 and 1.8% at level 4. The biggest error is with another $|\infty\rangle$ state. It is 8.4% at level 2 and 18% at level 4.

The errors for the product of $|360^\circ\rangle$ are again biggest for the $|\infty\rangle$ state with 2.1% at level 2 and 5.6% at level 4. The errors in the product of the identity with itself are 0.39% or 0.55% respectively.

The errors for $|0\rangle * |120^\circ\rangle$ are the lowest of all of the examples: 0.03% and 0.33% respectively.

The moral is that the wedge composition rule works better for states closer to the vacuum. It works worse for the identity and the worst for the $|\infty\rangle$ state.

Chapter 5

Towards the exact tachyon condensate

Since the original formulation of the Sen's conjectures [54] there has been significant progress in understanding the nonperturbative aspects of the string field theory. Initially the existence of translationally invariant vacuum with conjectured energy density was established numerically [56, 57] to a rather high accuracy by the level expansion method [65, 66] in the Witten's cubic string field theory [5, 48, 49, 60]. More recently the Sen's conjecture has been proved rigorously in the framework of background independent string field theory [68, 69, 70]. Nevertheless it seems worth continuing to look for the exact tachyon condensate in the original cubic string field theory since it can teach us many things [85].

Various insights into the nature of the tachyon condensate has already been obtained in [55, 56, 82, 83, 103, 85, 86, 87, 88]. Sen observed that only the components of the string field associated with the ghosts b, c and the matter Virasoro generators L_m acquire expectations value [55]. Then Sen and Zwiebach [56] found that only the even levels components play role, the odd levels can be set consistently to zero. Rastelli and Zwiebach [82] discussed the necessity of choosing a gauge. Perhaps one could avoid choosing it, but then the numerical computation works much worse, since some spurious fields get large values. They also introduced the wedge states, from which the sliver in particular, seems to be quite important. Kostelecký and Potting [83] devised a recursive procedure how to find the analytic solution. Their method has been later fruitfully applied in the vacuum string field theory [96]. Hata and Shinohara [103] noted, that the correct tachyon condensate is not only a stationary point of the action, but that it has to satisfy the full equation of motion $Q\Psi + \Psi * \Psi = 0$. Indeed, the numerical solution found by Sen and Zwiebach does obey these conditions. An interesting $\mathbb{Z}_4$ symmetry transforming the ghosts into antighosts

$$b_{-n} \to -nc_{-n}, \quad c_{-n} \to \frac{1}{n} b_{-n} \tag{5.1}$$

has been found by Zwiebach [85]. It also leads to nontrivial predictions for the coefficients of the condensate.

We have found [86] another infinite set of quadratic constraints coming from the anomalous symmetries of the string field vertex. At level n there are in total n such constraints on the coefficients of the string field which are in reasonable agreement with the explicit results from level truncation scheme. We shall describe them later in detail. It would be very interesting if one could find along this line even further symmetries which would then fix all the coefficients completely. In section 4.2.3 we have described linear constraint on the condensate following from the anomalous property of the identity string field (to be published).

Toy model for the tachyon condensation has been studied by De Smet and Raeymaekers [87, 88], which leads to rather nontrivial nonlinear second order differential equations. It is quite remarkable, since it shows, that in the true string field theory, there should be some mechanism (perhaps gauge invariance?) that makes the tension to have the right value and not some random transcendental number.

The most concrete suggestion, however, which could lead possibly to the exact form of the condensate was made in [91, 92]. It is based on some simple ideas from noncommutative geometry, and is described in detail in section 6.3. Roughly speaking, the idea is to look for the condensate in the form of pure-gauge-like ansatz. Later in this chapter we shall discuss the possibilities of verifying the conjecture.

5.1 Constraints from the anomalous symmetries

From [48, 49, 104, 82] we know that the 3-vertex $\langle V|$ satisfies certain identities, which are in fact conservation laws. They can be derived for general n-vertex most effectively using the method developed in [82]. We have illustrated the method on the 1-vertex, which is the identity, in section 4.2.1. For us will be important in particular the following identities for n even

$$\langle V| \sum_{i=1}^{3} (L_{-n}^{(i)} - L_{n}^{(i)}) \quad = \quad 3k_n^x \langle V|,$$

$$\langle V|\sum_{i=1}^{3}(J_{-n}^{(i)}+J_{n}^{(i)}) = 3(h_n^{gh}+3\delta_{n,0})\langle V|, \tag{5.2}$$

where L_n and J_n denote matter Virasoro and ghost current generators respectively. The constants k_n^x and h_n^{gh} take for n even the following values

$$\begin{aligned} k_n^x &= \frac{13\cdot 5}{27}\cdot\frac{n}{2}(-1)^{\frac{n}{2}}, \\ h_n^{gh} &= -(-1)^{\frac{n}{2}}. \end{aligned} \tag{5.3}$$

For n odd there would be extra signs between the generators in (5.2) and the right hand side would vanish. We are not interested in this case since it will not lead to any information about the tachyon condensate. Note that the additional term on the right hand side of the second equation in (5.2) accounts for the nontensor character of the ghost number current.

Let us study now the variation of the string field action

$$S[\Psi] = -\frac{1}{\alpha' g_o^2}\left(\frac{1}{2}\langle\,\Psi, Q\Psi\,\rangle+\frac{1}{3}\langle V||\Psi\rangle\otimes|\Psi\rangle\otimes|\Psi\rangle\right) \tag{5.4}$$

under the infinitesimal variations of the string field

$$\begin{aligned} \delta\Psi &= (L_{-n}-L_n-k_n^x)\Psi, \\ \delta\Psi &= (J_{-n}+J_n-h_n^{gh}-3\delta_{n,0})\Psi \end{aligned} \tag{5.5}$$

respectively. Under these variations the cubic term in the action is obviously invariant due to the invariance of the vertex (5.2). On the other hand we know that the total action should also be invariant as long as Ψ satisfies equations of motion. Combining these two facts we get from the kinetic term[1]

$$\begin{aligned} \langle\Psi|[Q,L_n]|\Psi\rangle &= -k_n^x\langle\Psi|Q|\Psi\rangle, \\ \langle\Psi|[Q,J_n]|\Psi\rangle &= h_n^{gh}\langle\Psi|Q|\Psi\rangle. \end{aligned} \tag{5.6}$$

Let us note that both commutators on the left hand side are modes of conformal primary fields, the latter being minus the BRST current J^B.

Explicit checks

To compare the above formulas with the results obtained in level expansion scheme in [56, 57] one should first of all impose the Siegel gauge condition

[1] With some abuse of notation, in this section only, we use $\langle\Psi|$ as a shorthand for $\langle \mathrm{bpz}\,\Psi|$.

$b_0|\Psi\rangle = 0$ on the string field and simplify the commutators. For the first equation of (5.6) one has simply

$$[Q, L_n] = -nc_0 L_n + \cdots \tag{5.7}$$

where the dots stand for terms which do not contribute. For the second equation one can use a little trick. Write the left hand side as

$$\langle\Psi|[Q, J_n]|\Psi\rangle = -\langle\Psi|\{J_n^B, b_0\}c_0|\Psi\rangle \tag{5.8}$$

where we used the facts that $[Q, J_n] = -J_n^B$ and $b_0|\Psi\rangle = 0$. The anticommutator can be easily evaluated using the operator product expansion (see e.g. [2]). Both formulas (5.6) thus simplify in the Siegel gauge to

$$\begin{aligned}
\langle\Psi|c_0 L_n|\Psi\rangle &= \frac{1}{n}k_n^x\langle\Psi|c_0 L_0^{tot}|\Psi\rangle, \\
\langle\Psi|c_0(nJ_n + L_n^{tot})|\Psi\rangle &= -h_n^{gh}\langle\Psi|c_0 L_0^{tot}|\Psi\rangle,
\end{aligned} \tag{5.9}$$

where L_n^{tot} denotes the total Virasoro generator. These identities can be easily checked for the numerical values obtained in [56, 57]. Let us define $r_n^{L,J}$ to be the ratio of the left and right hand sides of the first or second equation of (5.9) respectively. Then inserting for simplicity the values for the string field coefficients from [56] obtained at the level (4,8) we get the following results

$$\begin{aligned}
r_2^L &= 1.069, \qquad r_4^L = 1.044, \\
r_2^J &= 1.004, \qquad r_4^J = 0.939.
\end{aligned}$$

We see that the above identities are preserved within 7%. This can be compared with the value of the potential which is for the same values about 1.4% away from the expected value. This discrepancy in the errors by a factor of five does not necessarily mean that there are mistakes neither in the derivation nor in the numerical evaluation. In fact we know that the convergence properties of the level truncation approximation depends rather strongly on what kind of calculation we are doing. In an unpublished work we have studied the properties of the string field algebra unity $|I\rangle$ in the level truncation using the universal recursive methods of [82]. Keeping only terms up to level 8 in the unity $|I\rangle$ and during the whole calculation we got for example

$$\begin{aligned}
L_{-2}|0\rangle * |I\rangle &= 0.990L_{-2}|0\rangle + 0.108L_{-2}^{tot}|0\rangle - 0.196L_{-2}^{tot}L_{-2}^{tot}|0\rangle + \cdots, \\
L_{-2}^{tot}|0\rangle * |I\rangle &= 0.990L_{-2}^{tot}|0\rangle + 0.009L_{-2}^{tot}L_{-2}^{tot}|0\rangle + \cdots,
\end{aligned} \tag{5.10}$$

where the dots stand for terms which are relatively smaller or of higher levels where one can understand bigger errors. Looking at these values one might wonder whether after all the string algebra unity is unity also for the state $L_{-2}|0\rangle$. The experience from calculations at lower levels where the errors are much bigger suggests that it really converges, hopefully to the correct state. The fact that calculations involving matter Virasoro generators converge much more slowly can be easily traced back to the presence of the Virasoro anomaly.

5.2 Partial integration identities

Now let us turn our attention to one particular type of identities, which might be potentially helpful to find the exact condensate. There is a half-string formalism in the string field theory, which for the lack of space and time is omitted from this thesis. Essentially it is careful implementation of the original Witten's ideas on the geometric nature of the star product. The right part of the first string gets glued with the left half of the second string and the distinction between the degrees of freedom associated with the left and right parts of the string is kept manifestly. This formalism turned out to be quite useful in constructing exact solutions in the matter sector of vacuum string field theory [97, 100, 101].

In this section we would like to present some consequences of these ideas, when translated into the CFT approach and look how it works with the level truncation. Suppose we are given any local current $J(\sigma)$, constructed out of the world sheet fields X, b and c, and integrate it over the left and right parts of the string respectively

$$\begin{aligned} J_L &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} J(\sigma)d\sigma, \\ J_R &= \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} J(\sigma)d\sigma. \end{aligned} \tag{5.11}$$

One can also put inside the integral arbitrary measure $f(\sigma)$, if one wish so. From the Witten's definition of the star product it is clear that

$$\begin{aligned} J_L(\Psi_1 * \Psi_2) &= (J_L\Psi_1) * \Psi_2, \\ J_R(\Psi_1 * \Psi_2) &= (-1)^{J\cdot\Psi_1}\Psi_1 * (J_R\Psi_2), \end{aligned} \tag{5.12}$$

since the operator acts only on one part of the string. The sign factor $(-1)^{J\cdot\Psi_1}$ is equal to -1 if both J and Ψ_1 are Grassman odd, otherwise it is one. We see also that in general

$$J_0(\Psi_1 * \Psi_2) = (J_L\Psi_1) * \Psi_2 + (-1)^{J\cdot\Psi_1}\Psi_1 * (J_R\Psi_2). \tag{5.13}$$

If J_0 happens to be a derivation on the algebra, i.e. it satisfies the Leibnitz rule with respect to the star product

$$J_0(\Psi_1 * \Psi_2) = (J_0\Psi_1) * \Psi_2 + (-1)^{J\cdot\Psi_1}\Psi_1 * (J_0\Psi_2), \tag{5.14}$$

then we get by subtracting these to equations

$$(J_R\Psi_1) * \Psi_2 + (-1)^{J\cdot\Psi_1}\Psi_1 * (J_L\Psi_2) = 0. \tag{5.15}$$

This equation is usually referred to as the partial integration identity.

In the sequel we will be interested in the holomorphic currents of conformal dimension h with the expansion

$$J(z) = \sum \frac{J_m}{z^{m+h}}, \tag{5.16}$$

which in the (σ, τ) coordinates looks as

$$J(\sigma) = \sum J_m \cos(m\sigma) \tag{5.17}$$

We can easily rewrite the definition of J_L and J_R using the modes

$$\begin{aligned} J_L &= \frac{1}{2}J_0 + \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left[J_{2k+1} + J_{-(2k+1)}\right], \\ J_R &= \frac{1}{2}J_0 - \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left[J_{2k+1} + J_{-(2k+1)}\right]. \end{aligned} \tag{5.18}$$

Explicit checks

For our explicit check we have chosen the identity

$$L_R(|0\rangle * |0\rangle) = |0\rangle * L_R|0\rangle, \tag{5.19}$$

which, using the definition of $L_{L,R}$ and the derivation property of K_{2k+1}, can be seen to be equivalent to

$$\begin{aligned} \frac{1}{2}L_0\left(|0\rangle * |0\rangle\right) &= L_L|0\rangle * |0\rangle \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}L_{-(2k+1)}|0\rangle * |0\rangle. \end{aligned} \tag{5.20}$$

This can be further rewritten as

$$\frac{1}{2}L_0|120°\rangle = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}X^{2,1}_{2k+1,n}L_{-n}|120°\rangle. \tag{5.21}$$

Let us look first at the $L_{-2}|0\rangle$ component. We should test whether

$$1 = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(-2X^{2,1}_{2k+1,0} + \frac{27}{5}X^{2,1}_{2k+1,2}\right) \tag{5.22}$$

In the Fig. 5.1 we have plotted the corresponding partial sums. The upper value of the sum $k = N$ in fact plays role of the level, at which we truncate the input. Looking at graph, we see that it converges extremely slowly. At

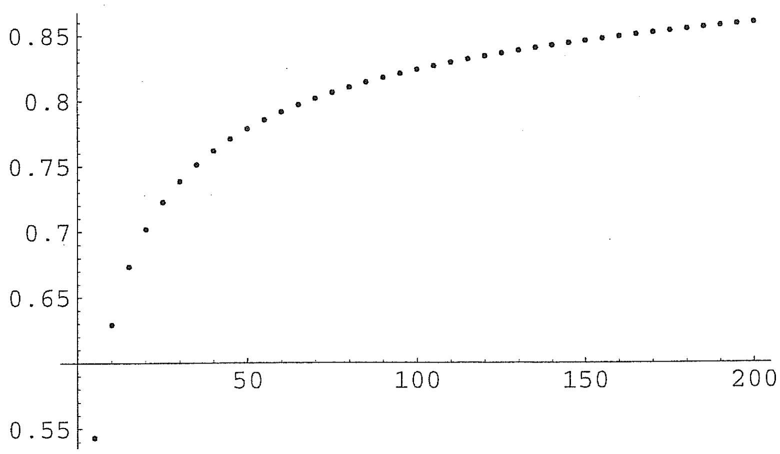

Figure 5.1: Coefficient of $L_{-2}|0\rangle$ in the product $L_L|0\rangle * |0\rangle$, normalized to unity, calculated at level N.

level 200 we get that the partial sum is equal to 0.859872, and that is actually not very high precision. We can, however, understand the rate of convergence and even to prove that it converges to 1 with much better precision. What we need, is the large k behaviour of the Neumann coefficients studied by Romans [104]

$$\begin{aligned} X^{2,1}_{2k+1,0} &= -\frac{1}{\sqrt{3}}(-1)^k b_{2k+1} \\ &\simeq (-1)^{k+1}\frac{2^{\frac{2}{3}}}{\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)}(2k+1)^{-\frac{1}{3}} \end{aligned} \tag{5.23}$$

and for $X^{2,1}_{2k+1,2}$ we get similar behaviour.

$$X^{2,1}_{2k+1,2} \simeq -\frac{2}{9}X^{2,1}_{2k+1,0} \tag{5.24}$$

The large k contribution to the sum looks as

$$\frac{16\,2^{\frac{2}{3}}}{5\,\sqrt{3}\,\pi\,\Gamma\left(\frac{2}{3}\right)\,(1+2\,n)^{\frac{4}{3}}}. \tag{5.25}$$

This sum is related to the Riemann zeta function and numerically can be calculated easily with arbitrary precision. If we sum all the terms from $k=201$ to ∞ we get an estimate of the error 0.140115, which when summed with our previous result gives 0.999987, which is indeed very close to the expected one. Qualitatively, without knowing anything about the Riemann function, one can find by replacing the sum by integral, that the error goes with the truncation level as $N^{-1/3}$.

We can also study the $L_{-3}|0\rangle$ component. At level N it is given by the expression

$$\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(-\frac{5}{27}X^{2,1}_{2k+1,0}+X^{2,1}_{2k+1,3}\right) \tag{5.26}$$

and for $N=200$ it takes value -0.00999, which is not very big, nevertheless it is only about four times less, compared to what we get at level 2. We

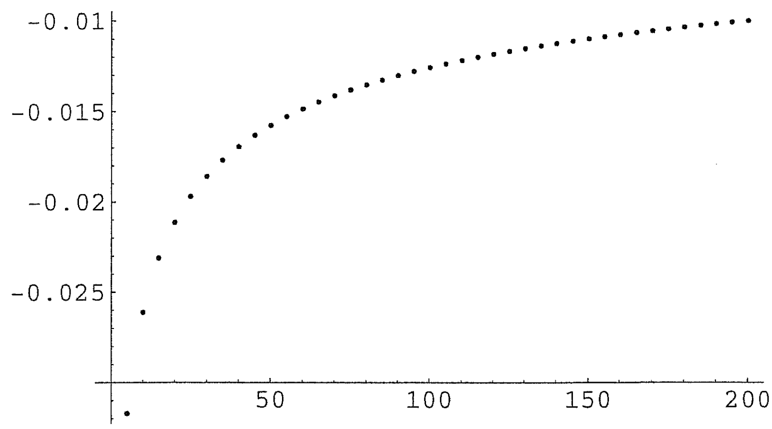

Figure 5.2: Coefficient of $L_{-3}|0\rangle$ in the product $L_L|0\rangle * |0\rangle$, calculated at truncation level N.

have again plotted the graph of partial sums in Fig. 5.2. The analysis of the error can be done as in the preceding case yielding hopefully the right answer. Again in this case the the error goes as $N^{-1/3}$. It means, that in order to increase the precision by a factor of 10, we need to increase the truncation level by a factor of 1000. To conclude, in practice these are kind of calculations, that are impossible to do numerically, unless one knows how to calculate the error.

5.3 Pure gauge like ansatz ?

Now we shall discuss one particular approach which could lead to the solution of the equation of motion for the string field

$$Q\Psi + \Psi * \Psi = 0. \tag{5.27}$$

We may try to look for solutions in the form

$$\Psi = U * QV. \tag{5.28}$$

In section 6.3 we present some arguments which lead quite naturally to this ansatz. They are based on some analogies with noncommutative field theory and on how does it describe lower dimensional branes. So we postpone the motivation there, as is more logical, and now we shall concentrate on possible solutions with this ansatz. Plugging the ansatz into the equation we get obviously

$$\begin{aligned} 0 &= QU * QV + U * QV * U * QV \\ &= QU * QV + Q(U * V) * U * QV - QU * V * U * QV. \end{aligned} \tag{5.29}$$

Experience with the star products, wedge states and the identity indicates, that it is unlikely to find some nontrivial solution to the equations $U * V = V * U = 1$ that would be analogs of pure large-gauge fields of Chern-Simons theory with nonvanishing action. In fact, there seem to be other possibilities. One particular way of solving the above equation is to solve the following system of equations

$$\begin{aligned} Q(U * V) &= 0, \\ V * U &= p, \\ U * p &= U, \end{aligned} \tag{5.30}$$

where p is some projector which satisfies

$$p * p = p, \qquad Qp = 0. \tag{5.31}$$

Finally we have to require $QU \neq 0$ and $QV \neq 0$. Actually, one can find many solutions, for which the action vanishes. The most natural candidate for p is the wedge state $|\infty\rangle$. Note, that all U's, which solve the above equation, can always be written as $U = \tilde{U} * p$. Therefore our system of equations to solve got reduced to

$$\begin{aligned} Q\left(\tilde{U} * |\infty\rangle * V\right) &= 0, \\ V * \tilde{U} * |\infty\rangle &= |\infty\rangle. \end{aligned} \tag{5.32}$$

As far as the naive counting of degrees of freedom is concerned, we can have plenty of solutions. Suppose, that our truncated Hilbert space has n basis elements. Then we have $2n$ unknowns, and certainly less then $2n$ equations, since Q automatically annihilates some states, and likewise the $|\infty\rangle$ state in the second equation reduces the number of independent equations. The real worry is, whether for some reason all these solutions are not trivial. One can try to solve these equations numerically, it may a bit problematic, however, since the star products involving Virasoro generators do not converge perfectly and neither the convergence properties of the sliver are excellent. Instead, it would be better to develop some kind of geometric intuition, which could tell us, what $\tilde{U}$ and V are.

Chapter 6

String field theory with nonzero B-field

In the second chapter we have seen how turning on a nonzero NS-NS background B-field makes the low energy effective field theory noncommutative. We have seen, that even if we start with a $U(1)$ gauge theory living on a single brane, or scalar sector thereof, we can have nontrivial soliton solutions in any number of dimensions. The noncommutativity complicates things a bit, but on the other hand, it allows for completely new phenomena, which can be given rather elegant description.

In the last three chapters we have discussed the open cubic string field theory and its relation to the physics of tachyon condensation. It is a powerful tool for the study of nonperturbative physics which is still not understood in all its details. It does not happen so often in science, that one devises a numerical algorithm, make a conjecture about its output and still is not able to understand why does it give the conjectured results.

What we would like to do in the present chapter, is to put the two things together. We will study the open string field theory in the presence of nonzero B-field. At first, it might look as a silly idea, to merge two complicated things, one of them being purely understood. Nevertheless, Harvey, Kraus, Larsen and Martinec [89] found, that the combined theory allows for a new and dramatically simpler description of D-branes, than the ordinary string field theory.

The plan for this chapter is the following. First we will work in the limit of large B-field, as was originally done first. We will give our derivation of the Witten's factorization [90], which clearly establishes the link between the traditional string field theory approach and the effective approach of [89]. Then we review the construction of D-branes in this setting and explain how do the right tensions of D-branes come out. We sketch a relation to algebraic K-theory. In the next chapter we add two complications at once, we shall discuss how to construct exact and approximate solitons when the background manifold is a torus and when the B-field stays finite.

6.1 Witten's factorization

6.1.1 String field algebra at nonzero B-field

The basic elements of the cubic string field theory developed mainly in [5, 48, 49, 50, 51] has been presented in chapter 3. The modification required by turning on a nonzero B-field is quite simple and straightforward. It has been discussed in [110, 111].

All the degrees of freedom of string field theory are contained in the string field

$$\Psi = \int d^{26}p\,(t(p)c_1 + A_\mu(p)\alpha^\mu_{-1}c_1 + \cdots)|0,p\rangle, \tag{6.1}$$

which is an element of the Fock space of the first quantized string theory. It is governed by the Chern-Simons type of an action

$$S[\Psi] = -\frac{1}{\alpha' G_o^2}\left(\frac{1}{2}\langle\,\Psi, Q\Psi\,\rangle + \frac{1}{3}\langle\,\Psi, \Psi * \Psi\,\rangle\right). \tag{6.2}$$

The noncommutative star multiplication formulated in the operator language in [48, 49, 60] has been written explicitly for the case of B-field background by Sugino and by Kawano and Takahashi in [110, 111]. The modification enters the three vertex (3.57) as follows

$$\begin{aligned}\langle V| &= \left(\frac{3\sqrt{3}}{4}\right)^3 \delta(p^{(1)}+p^{(2)}+p^{(3)})\langle\tilde{0}|\otimes\langle\tilde{0}|\otimes\langle\tilde{0}| \times \\ &\quad\times \exp\left(\sum_{m,n=0}^{\infty}\frac{1}{2}\alpha_n^{(r)\mu}N^{rs}_{nm}\alpha_m^{(s)\nu}G_{\mu\nu} + \sum_{m=0,n=1}^{\infty} c_n^{(r)}X^{rs}_{nm}b_m^{(s)} - \frac{i}{2}\theta_{\mu\nu}p^{(1)\mu}p^{(2)\nu}\right).\end{aligned} \tag{6.3}$$

As usual we denote $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$. The Neumann coefficients N^{rs}_{nm}, X^{rs}_{nm} and the vacua $\langle\tilde{0}|$ do not get modified. As in chapter 2, the B-field enters through the effective open string coupling constant, open string metric and the noncommutativity parameter [7]

$$G_o = g_o\left(\frac{\det G}{\det(g+2\pi\alpha' B)}\right)^{\frac{1}{4}}, \tag{6.4}$$

$$G_{\mu\nu} = g_{\mu\nu} - (2\pi\alpha')^2(Bg^{-1}B)_{\mu\nu}, \tag{6.5}$$

$$\theta^{\mu\nu} = -(2\pi\alpha')^2\left(\frac{1}{g+2\pi\alpha' B}B\frac{1}{g-2\pi\alpha' B}\right)^{\mu\nu}, \tag{6.6}$$

which we repeat here for the readers convenience. These effective parameters also appear in the formula for Virasoro generators (and therefore in the BRST charge Q) and in the commutation relations for the Fock space generators

$$
\begin{aligned}
[\alpha^\mu_m, \alpha^\nu_n] &= m\delta_{m+n,0} G^{\mu\nu}, \\
[x^\mu, x^\nu] &= \theta^{\mu\nu}, \\
[p^\mu, x^\nu] &= -i\, G^{\mu\nu}.
\end{aligned} \tag{6.7}
$$

6.1.2 Large B-field limit

Now take the limit $B \to \infty$ keeping fixed all closed string parameters (including the open string coupling constant g_o but not the effective one G_o). To make things more transparent set $B = tB_0$ and take $t \to \infty$ as in [90]. The effective parameters clearly depend on t as

$$
\begin{aligned}
G_o &\sim G_{o0} t^{r/2}, \\
G^{\mu\nu} &\sim G_0^{\mu\nu} t^{-2}, \\
\theta^{\mu\nu} &\sim \theta_0^{\mu\nu} t^{-1},
\end{aligned} \tag{6.8}
$$

where r denotes the rank of the B-field and for the brevity let us assume that it is maximal. Altogether the t dependence enters at two places: First in the commutation relations (6.7) for the Hilbert space operators and then also explicitly in the definition (6.3) of the star product.

To see the change in the structure of the string field algebra we have to rescale all Fock space operators in such a way that their commutation relations don't depend on t. (We are then sure that we are studying different star products on the same space). The rescaling which does that is

$$
\begin{aligned}
\alpha^\mu_m &\to \tilde{\alpha}^\mu_m = t\,\alpha^\mu_m \qquad (m \neq 0), \\
p^\mu &\to \tilde{p}^\mu = t^{3/2} p^\mu, \\
x^\mu &\to \tilde{x}^\mu = t^{1/2} x^\mu,
\end{aligned} \tag{6.9}
$$

After this rescaling the exponent in the vertex (6.3) takes the simple form

$$
\begin{aligned}
&\sum_{m,n=1}^{\infty} \frac{1}{2} \tilde{\alpha}^{(r)\mu}_n N^{rs}_{nm} \tilde{\alpha}^{(s)\nu}_m G_{0\mu\nu} + \frac{1}{\sqrt{t}} \sum_{n=1}^{\infty} \sqrt{\frac{\alpha'}{2}} \tilde{\alpha}^{(r)\mu}_n (N^{rs}_{n0} + N^{rs}_{0n}) \tilde{p}^{(s)\nu} G_{0\mu\nu} + \\
&+ \frac{1}{t} \alpha' \tilde{p}^{(r)\mu} N^{rs}_{00} \tilde{p}^{(s)\nu} G_{0\mu\nu} + \sum_{m=0,n=1}^{\infty} c^{(r)}_n X^{rs}_{nm} b^{(s)}_m - \frac{i}{2} \theta_{0\mu\nu} \tilde{p}^{(1)\mu} \tilde{p}^{(2)\nu}.
\end{aligned} \tag{6.10}
$$

We see that in the large t limit the terms which couple α oscillators with momenta p vanish but the whole star product nevertheless remains nontrivial.

Now the generic string field is the sum of terms

$$a\, e^{ik^\mu x^\nu G_{\mu\nu}}|0\rangle = a\, e^{i\tilde{k}^\mu \tilde{x}^\nu G_{0\mu\nu}}|0\rangle, \tag{6.11}$$

where $a \in \mathcal{A}_0$ is in the zero momentum subalgebra. It is then obvious that the star product respects the tensor product structure.

$$a_1 e^{i\tilde{k}_1^\mu \tilde{x}^\nu G_{0\mu\nu}}|0\rangle * a_2 e^{i\tilde{k}_2^\mu \tilde{x}^\nu G_{0\mu\nu}}|0\rangle = (a_1 * a_2) e^{-\frac{i}{2}\tilde{k}_1^\mu \tilde{k}_2^\nu \theta_{0\mu\nu}} e^{i(\tilde{k}_1+\tilde{k}_2)^\mu \tilde{x}^\nu G_{0\mu\nu}}|0\rangle. \tag{6.12}$$

Recalling the structure of the BRST operator it is also obvious that after this rescaling in the limit $t \to \infty$ all the terms with momentum operators vanish and therefore it acts only on the $\mathcal{A}_0$ component. In conclusion the full string field algebra looks as

$$\mathcal{A} = \mathcal{A}_0 \otimes \mathcal{A}_1, \tag{6.13}$$

where $\mathcal{A}_0$ is the complicated stringy subalgebra of the string states of zero momentum in the noncommutative directions and nonzero momentum in the commutative ones. The second factor $\mathcal{A}_1$ is the algebra generated by the functions e^{ikx} using the Moyal product. Its precise content, K-theory and physical applications in the important cases of (compactified) Moyal plane and noncommutative torus will be our primary concern in the next section and then in the chapter 7.

6.2 D-branes as noncommutative solitons

6.2.1 HKLM construction and D-brane tensions

In this section we shall consider mainly the case of a flat noncommutative Minkowski space with metric $g_{\mu\nu} = \eta_{\mu\nu}$. Some preliminary remarks on the torus will be given later in this section, a thorough treatment is postponed to the following chapter 7. Let us further assume, for simplicity, that the rank of the B-field is two. For the algebra $\mathcal{A}_1$ of functions on the noncommutative plane we take the Schwarz space $\mathcal{S}(\mathbb{R}^2)$. The associated algebra of Weyl ordered operators generates the algebra of the trace-class operators whose norm closure is the algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable Hilbert space $\mathcal{H}$ [119, 124]. This algebra does not contain the identity, we may wish to add it by hand. This formally corresponds to the one point compactification of the Moyal plane. Thus we have up to an isomorphism

$$\mathcal{A}_1 = \mathcal{K} \oplus \mathbb{C}\mathcal{I}. \tag{6.14}$$

The K_0 group of this algebra which will play some role later is

$$K_0(\mathcal{A}_1) = \mathbb{Z} \oplus \mathbb{Z}. \tag{6.15}$$

For a general algebra it is defined as the additive group of formal differences of certain equivalence classes of projectors. For a detailed exposition see [112].

Let us discuss now some solutions to the string field equation of motion in the background of large B-field. From the action (6.2) it takes the form

$$Q\Psi + \Psi * \Psi = 0. \tag{6.16}$$

The basic solution is $\Psi = \mathcal{A}_0 \otimes \mathcal{I}$. It is the famous solution found by Sen and Zwiebach [56, 57], which was described in detail in section 3.4. The solution was supposed to be translationally invariant and so the trivial $\mathcal{A}_1$ part of the solution has been suppressed. This solution corresponds to a D25-brane, or alternatively can be viewed as describing the decay of the D25-brane into the closed string vacuum. The value of the string field action per unit time[1] is (using the Sen's conjecture for $B = 0$)

$$S[\mathcal{A}_0 \otimes \mathcal{I}] = 2\pi\alpha' BM, \tag{6.17}$$

in accord with the Sen's conjecture (see section 3.4) applied to the case of large B-field background . Here M stands for the D25-brane mass in the absence of any B-field

$$M = \frac{1}{2\pi^2}\frac{1}{\alpha' g_o^2}\int \sqrt{g}\, d^{25}x. \tag{6.18}$$

The factor $2\pi\alpha' B$ comes from the effective open string coupling constant and from the normalization of the inner product

$$\langle 0, 0 | c_{-1} c_0 c_1 | 0, 0\rangle = \int \sqrt{G}\, d^{26}x, \tag{6.19}$$

and accounts precisely for the change in the mass of the D25-brane due to the background B-field. Note that there are some subtleties since the mass M diverges. To make it finite, we should introduce some finite volume cutoff, which however spoils the structure of the algebra. Nevertheless the simplicity of the GMS construction [46], as we have seen in section 2.6, partially justifies this slightly heuristic treatment. Somewhat more careful treatment of the noncommutative torus will be given later.

As was noticed in [109, 89] on the level of the low energy action and by Witten [90] from the string field theory point of view one can get whole family of new solutions of the form $\mathcal{A}_0 \otimes \rho$ where $\rho \in \mathcal{A}_1$ is any projector.

[1]Throughout this section we are interested in time independent configurations and hence the word action will mean the action per unit time.

Suppose now for a while that ρ is a projector onto a finite n dimensional subspace of $\mathcal{H}$. For all of these solutions one can easily calculate the value of the string field action using the Sen's conjecture for the D25 brane without any B-field. Let us list some of them in the suggestive form

Solution	Value of the action	Interpretation
$A_0 \otimes \mathcal{I}$	$2\pi\alpha' BM$	$D25 \to vac$
$A_0 \otimes (\mathcal{I} - \rho)$	$\left(2\pi\alpha' B - n\frac{\alpha'}{R^2}\right) M$	$D25 \to nD23$
$A_0 \otimes \rho$	$n\frac{\alpha'}{R^2} M$	$nD23 \to vac$

In order to get finite results we had to regularize the area of the Moyal plane (in closed string metric) to be $(2\pi R)^2$. We also used the formula $\int d^2x\, \rho(x) = 2\pi\theta n$ from [46]. The values of the action for the above solutions exactly correspond to the decay energies between various D-brane systems. This leads to the interpretation listed in the last column.

6.2.2 K-theory interpretation

Sum of any two solutions (or two projectors) is a solution (or a projector) only if the two projectors are orthogonal. Algebraic K-theory introduces an additive group K_0 of projectors by tensoring the algebra with an infinite matrix $\mathcal{M}_\infty(\mathbb{C})$. This in physical terms corresponds to adding Chan-Paton factors, which is the same as having more branes on top of each other.

Any stack of branes can be characterized by a projector. For example n D25-branes and m D23-branes corresponds to a projector in $\mathcal{M}_\infty(\mathcal{A}_1)$ of the form

$$\begin{pmatrix} \mathcal{I} & & & & & \\ & \mathcal{I} & & & & \\ & & \ddots & & & \\ & & & \rho_{m_1} & & \\ & & & & \rho_{m_2} & \\ & & & & & \ddots \end{pmatrix}, \tag{6.20}$$

where there is n times the identity $\mathcal{I}$ of the algebra $\mathcal{A}_1$ on the diagonal, and some number of projectors ρ_{m_i} of rank m_i, such that

$$\sum_i m_i = m. \tag{6.21}$$

K-theory tells us how to sum projectors of the form (6.20). If the projectors are not orthogonal, then it creates a block diagonal matrix, where in the first block sits the first projector and in the second is the second one. This block diagonal matrix is again of the form (6.20).[2] The next step in the K-theory is to construct a Grothendick group K_0 from this additive semigroup. It is just a set of equivalence classes of pairs (A, B) where the equivalence relation is $(A, B) \sim (A \oplus C, B \oplus C)$. Thus for instance $\mathcal{I} - \rho$ is well defined and is a short cut for $(\mathcal{I}, \rho)$. The element (A, B) is most naturally described as a stack of D-branes A decaying into another stack of D-branes B. All these K_0 elements look good when inserted into the string field action. It gives just the energy released in the decay.

The situation in IIB theory is different, in that case the K_0 classes correspond to conserved charges of the brane configurations. In the bosonic theory there are no conserved charges for the branes. The K-theory classes characterize merely various stationary points of the string field action. For related discussion see [127].

Various other aspects of K-theory and string theory can be found in [113, 114, 115, 116, 117, 118] and in the Komaba lectures [124].

6.2.3 Noncommutative torus

It may seem that the K-theory we were talking about, is just an artefact of the 'hand-made' algebra (6.14). By looking at the example of the 'more realistic' noncommutative torus[3] we shall try to convince the reader that there is something deeper going on.

The relevant algebra $\mathcal{A}_1$ for the noncommutative torus is the well known rotational algebra A_θ. Its K_0 group is the same as for the compactified Moyal plane above. Unfortunately in this case the beautiful construction [46] of all the projectors breaks down even though one still has a homomorphism from the algebra $\mathcal{A}_1$ to the space of bounded operators by an analog of the Weyl quantization formula. Some representatives of all the equivalence classes of projectors were nevertheless constructed by Rieffel [120]. The Powers-Rieffel projector on the torus $[0, 2\pi]^2$ takes the form (in the representation by ordinary functions)

$$p(x_1, x_2) = 2\cos(x_1) g\left(e^{i(x_2 + \frac{\theta}{2})}\right) + f\left(e^{ix_2}\right), \tag{6.22}$$

where f and g are two functions satisfying certain relations. These can be chosen to be sufficiently smooth if one wishes. The trace on the noncommutative algebra $\mathcal{A}_1$ in the representation by ordinary functions with the Moyal

[2] Actually, the theory works only with equivalence classes of the projectors, and so the order of $\mathcal{I}$'s and ρ's does not matter.

[3] This part should be regarded as a preliminary discussion only, it is based on our early work [91]. Thorough treatment will be presented in chapter 7.

product is just an ordinary integral over the torus (normalized by the total area) which gives precisely $\frac{\theta}{2\pi}$. From our point of view the only problematic feature of these solutions is that they are not well localized in one direction (in this case x_1). This prevents us from looking at those solutions as codimension two lump representing lower dimensional brane. Nevertheless the experience with the Moyal plane indicates, that there should exist also well localized solutions with straightforward physical interpretation. Indeed, we will find them in section 7.4. More discussion about the Powers-Rieffel projector can be found in [125].

General theorem due to Pimsner and Voiculescu when combined with the Rieffel's construction [112, 120] states that the range of the trace on projections in $\mathcal{A}_1$ is exactly $(\mathbb{Z} + \frac{\theta}{2\pi}\mathbb{Z}) \cap [0,1]$. The unusual normalization factor $\frac{1}{2\pi}$ comes from requiring the standard form (6.12) of the star product. Calculating the string field action for the solution $A_0 \otimes p$ one gets

$$S[A_0 \otimes p] = 2\pi\alpha' BM \,\mathrm{Tr}\, p = 2\pi\alpha' BM \left(m - \frac{\theta}{2\pi} n\right), \qquad (6.23)$$

where $m, n \in \mathbb{Z}$ are such that

$$m - \frac{\theta}{2\pi} n \in [0,1]. \qquad (6.24)$$

We see that for $m = 1$ and $n \in \mathbb{N}$ not too large (such that the projector exists) we get *precisely* the same values as those for the Moyal plane above. It is perhaps curious to note that the theorem also asserts that even without introducing the Chan-Paton factors one can describe the decay of $m > 1$ D25 branes into an appropriate number of D23 branes. This is not true for the Moyal plane case.

As we said above one may have doubts about the role of K-theory on the Moyal plane. But here on the noncommutative torus in order to find a single example of a projector we had to use the K-theoretical sources. Strikingly these projectors lead to the correct masses of D-branes, exactly as the GMS projectors. The projectors in noncommutative geometry are primarily used to define projective modules — a noncommutative generalization of vector bundles — which are naturally classified by K-theory.

To end up this discussion we would like to make the following interesting remark. The string field action is the (secondary) Chern-Simons class of the noncommutative bundle defined by the connection which is the string field. In the large B-field limit when the algebra factorizes the action becomes equal up to a factor to the Chern class of a completely different noncommutative bundle over the torus specified by the choice of the projector p. We believe that further investigations may reveal beautiful interplay between these objects in noncommutative geometry.

6.3 Proposal for the exact solution of the tachyon potential

The proposal is based on the following simple observation: All the decays of D25 brane that are described in the large B-field limit by taking a nonzero projector $\rho \in \mathcal{A}_1$ are related to each other by a nonunitary isometry[4]. Of course it doesn't mean that they are in the same K-theory class since this isometry doesn't belong to $\mathcal{A}_1$. To give an example consider the solutions describing the decays $D25 \to vac$ and $D25 \to (n)D23$ with projectors $\mathcal{I}$ and $\mathcal{I} - \rho$ respectively.

The isometry U which relates them as follows

$$\begin{aligned} \mathcal{I} - \rho &= UU^\dagger, \\ \mathcal{I} &= U^\dagger U, \end{aligned} \tag{6.25}$$

can be found in some cases explicitly. If we take for instance $\rho = |0\rangle + |1\rangle + \cdots + |n-1\rangle$, then U and $U^\dagger$ are the ordinary shift operators

$$\begin{aligned} U &= \sum_{m=0}^{\infty} |m+n\rangle\langle m|, \\ U^\dagger &= \sum_{m=0}^{\infty} |m\rangle\langle m+n|. \end{aligned} \tag{6.26}$$

The operator U is clearly noncompact (and it is neither unity) so it does not belong to $\mathcal{A}_1$. Thus the projectors $\mathcal{I}$ and $\mathcal{I} - \rho$ do not have to belong to the same K-theory class. They would, however, if we were working with the algebra of all bounded operators.

Note that string field solutions representing the above decays are related by a formula which formally looks like a string field gauge transformation

$$A_0 \otimes (\mathcal{I} - \rho) = U(Q + A_0 \otimes \mathcal{I})U^\dagger. \tag{6.27}$$

The first term on the right hand side gives of course zero contribution since Q doesn't act on $\mathcal{A}_1$. More useful relation is obtained by multiplying with $U^\dagger$ and U on the left and right respectively

$$A_0 \otimes \mathcal{I} = U^\dagger(Q + A_0 \otimes (\mathcal{I} - \rho))U. \tag{6.28}$$

[4] An operator U for which $U^\dagger U$ is projector is called a partial isometry. Then automatically $UU^\dagger$ is a projector. If $U^\dagger U = \mathcal{I}$ then U is called an isometry.

Our conjecture is as follows: Since the decays $D25 \to vac$, $D25 \to nD23$ and so on are related by gauge-like isometry transformation, it is natural to expect that in the full string theory also the trivial process $D25 \to D25$ described by the zero string field is related to the others in a similar way. Thus we expect

$$A_0 = V^\dagger * QV \tag{6.29}$$

for some V, which acts on $\mathcal{A}_0$ and satisfies

$$\begin{aligned} V^\dagger * V &= \mathcal{I}, \\ Q(V * V^\dagger) &= 0, \end{aligned} \tag{6.30}$$

where the star is now the stringy product (not the Moyal one) and the dagger means the usual star involution of the string field algebra. The last equation (which is of course also satisfied by U) was added in order to fulfill the equation of motion. Note that both conditions (6.30) could be replaced simply by $V * V^\dagger = \mathcal{I}$ but this is not favored by our analogy. It is straightforward to check the string field equations of motion (6.16) provided one can use the associativity of the algebra. This is however not a priori clear since V (in analogy with U) doesn't appear to be an element of the algebra. Indeed, it is well known that when one tries to add some elements to the so far not properly defined string algebra one runs into problems with associativity anomalies [121, 122] and identity related anomalies [82] discussed at length in chapter 4.

Chapter 7

Exact solitons on noncommutative tori

Noncommutative field theory turns out to be a very useful tool in describing various features of string theory. It shares a lot of its characteristic features. One notable example is the ultraviolet – infrared mixing discussed in section 2.3. As we have seen in previous chapters the construction of D-branes as solitons in string field theory is quite complicated. It has been discovered by Harvey, Kraus, Larsen and Martinec [89] that this construction can be drastically simplified by turning on spatial noncommutativity. Originally it has been performed in the limit of large B-field. Later on it was realized in [92] that one can actually relax the assumption of large B-field by turning on appropriate gauge field which kills all the covariant derivatives in the effective action.

Originally all these studies has been performed on the so called noncommutative or Moyal plane. They were based on a simple observation, described in section 2.6., that the algebra of functions on the plane vanishing at infinity with Moyal product is isomorphic to the algebra of compact operators on a separable Hilbert space. This allowed to use elementary methods of quantum mechanics for constructing the noncommutative solitons. For the introduction to these topics the reader is referred to lectures [124].

An obvious task is to extend the previous analysis to the case of the noncommutative torus. There are several reasons why this might be useful. There are stringent experimental bounds on the presence of the B-field in the physical uncompactified directions. So if any B-field background is physically relevant to string theory it should appear in the compactified directions. The torus is of course the simplest possibility. On the other hand from mathematical point of view it is already quite nontrivial and nicely illustrates many new mathematical concepts. Other examples of understood noncommutative spaces include the fuzzy sphere which is more complicated since it is a curved space. The solitons on the noncommutative torus has been studied in [91, 125, 126, 127, 142, 123, 143, 144, 145].

In this chapter we will present the construction of exact noncommutative solitons on noncommutative two torus in the effective description of string field theory. Although the precise form of the effective action is unknown we have enough information (from gauge invariance and some following from Sen's conjectures) that enables us to construct the exact solitons and confirm their interpretation as D-branes. The word 'exact' means that they are true solutions of equation of motion at finite noncommutativity, i.e. not only in the limit of infinite noncommutativity. In the course of the presentation we will see that problem reduces to well studied problems in the mathematics and we will give brief exposition of the notions and methods used in noncommutative geometry. This chapter is based on our work [123] with Thomas Krajewski.

The plan of this chapter is as follows. After setting the conventions we will recall the properties of the effective action obtained by integrating out (on the classical level) all the fields except the tachyon and the gauge field. We outline the general strategy for solving the equations of motion. The main problem will turn out to be related to problem of finding constant curvature connection on a projective module. We will see how one can nicely solve all the problems using the bimodule technique. Finally we will look at the problem what can be done without gauge fields, i.e. when one does not want to turn them up. One can find approximate solitons which as we will see are related in the large torus limit to those found on the noncommutative plane.

7.1 Effective description of string field theory

We start by considering the open bosonic string theory propagating in the closed string background of the form $\mathcal{M} \times \mathbb{T}^d$ where $\mathcal{M}$ is arbitrary 26-d dimensional manifold. As discussed in section 2.1, the effect of turning on a constant B-field along a flat submanifold, in this case the torus $\mathbb{T}^d$, is neatly described by replacing ordinary products by the noncommutative star products defined as

$$f * g = f\, e^{\frac{i}{2}\theta^{ij}_{Moyal}\overleftarrow{\partial}_i \overrightarrow{\partial}_j} g, \tag{7.1}$$

where θ^{ij}_{Moyal} is an antisymmetric real matrix.

One also has to replace the ordinary closed string metric g_{ij} and coupling constant g_s by effective open string parameters G_{ij} and G_s. All these effective parameters are related to the closed ones through

$$\begin{aligned}\frac{1}{G+2\pi\alpha'\Phi} &= -\frac{\theta_{Moyal}}{2\pi\alpha'}+\frac{1}{g+2\pi\alpha' B},\\ G_s &= g_s\left(\frac{\det(G+2\pi\alpha'\Phi)}{\det(g+2\pi\alpha' B)}\right)^{\frac{1}{2}},\end{aligned} \tag{7.2}$$

where we allow in more generality for a nonzero Seiberg-Witten two form Φ [7, 133], which always appears in the combination with the gauge field strength[1] $iF+\Phi$. Its appearance is related to a freedom in the regularization of the worldsheet theory.

All the fields are regarded as functions of the commutative coordinates x^a and are valued in the noncommutative algebra $\mathcal{A}_\theta$. In more concrete terms, this algebra is generated by functions $U_i = e^{i\frac{x^i}{R}}$ having the commutation relations

$$U_i * U_j = e^{-\frac{i}{R^2}\theta^{ij}_{Moyal}} U_j * U_j. \tag{7.3}$$

For simplicity we take all radii of the torus equal to R, but all what follows is easily adapted to any constant metric.

To make contact with the standard conventions for the noncommutative torus we shall omit the stars and set

$$2\pi\theta^{ij} = -\frac{\theta^{ij}_{Moyal}}{R^2}. \tag{7.4}$$

The algebra is spanned by $U_{\vec{n}} = \prod_{i=1}^{d} U_i^{n_i}$ where $\vec{n}\in\mathbb{Z}^d$.

The standard partial derivatives with respect to the coordinates x^i are derivations of the algebra $\mathcal{A}_\theta$, i.e. they also satisfy the Leibniz rule for the star product.

The ordinary integral over the torus yields a trace on the algebra, i.e. two elements of $\mathcal{A}_\theta$ commute under the integral. In accordance with the standard trace that can be found in the mathematical literature, we choose to normalize it as follows,

$$\frac{1}{(2\pi R)^d}\int_{\mathbb{T}^d}\sum_{\vec{n}\in\mathbb{Z}^d} a_{\vec{n}} U_{\vec{n}} = a_{\vec{0}}. \tag{7.5}$$

Accordingly, this trace will be referred to as the "normalized integral". We refrain from calling it the trace, since one also considers matrices with entries in $\mathcal{A}_\theta$. On the algebra $M_N(\mathcal{A}_\theta)$, one introduces the ordinary linear form Tr

[1] We are using geometric conventions where A and F are antihermitian. The relation with [7] is $A^{SW} = iA^{here}$.

from $M_N(\mathcal{A}_\theta)$ to $\mathcal{A}_\theta$ as the sum of the diagonal elements. It is not a trace since the property $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ is lost when the algebra is not commutative. However, this property is true after integration.

The effective action is obtained by integrating out all the fields from open string field theory except the tachyon T and the gauge field A_μ. It takes the general form

$$S = \frac{c}{G_s}\int_{\mathcal{M}} d^{26-d}x \int_{\mathbb{T}^d} \sqrt{\det G}\ \mathrm{Tr}\left[\frac{1}{2}f(T^2-1)G^{\mu\nu}D_\mu T D_\nu T - V(T^2-1)\right.$$
$$\left. -\frac{1}{4}(2\pi\alpha')^2 h(T^2-1)(iF_{\mu\nu}+\Phi_{\mu\nu})(iF^{\mu\nu}+\Phi^{\mu\nu}) + \cdots\right], \quad (7.6)$$

where $\mu, \nu = 1, 2, \ldots, 26$ and $c = g_s T_{25}$ is a B independent constant. The tachyon field T is a $\mathcal{A}_\theta$-valued function on $\mathcal{M}$ and transforms in the adjoint representation. The gauge field A_μ is an antihermitian matrix of functions with values in $\mathcal{A}_\theta$ and its curvature is defined as usual. This action is invariant under the standard noncommutative gauge transformations.

The explicit form of the effective action is not completely known. It may contain higher order terms that are represented in (7.6) by dots. The latter are constructed using products of higher order covariant derivatives of the tachyon field and of the curvature tensor. Fortunately their explicit form is not necessary in order to apply the method we shall describe below.

Furthermore, the functions f, h, V are only required to satisfy certain conjectured properties. The following ones are not intended to be the complete list but merely those we shall need in what follows,

$$\begin{array}{rclrclr} V(0) &=& 0, & h(0) &=& 0, & f(0)=0, \\ V(-1) &=& 1, & h(-1) &=& 1, & \\ V'(0) &=& 0, & h'(0) &=& 0. & \end{array} \quad (7.7)$$

The tachyon potential is normalized in such a way that at the closed string vacuum $T = 1$ it has a minimum equal to zero and at $T = 0$ it has a local maximum equal to 1. The conditions for $h(0)$ and $f(0)$ reflect Sen's conjecture that at the closed string vacuum all kinetic terms of physical excitations do vanish. This conjecture has been recently tested numerically [93, 94] and was also a starting point for the vacuum string field theory [95, 96]. The condition on $h(-1)$ is just a normalization. What is less clear is the physical meaning of $h'(0) = 0$. It follows nevertheless from the Dirac-Born-Infeld extension [134, 135] (see also [136, 137]) which implies $h = V$. Among the higher derivative terms, those constructed solely from the curvature F and the tachyon T (without derivatives) will be quite important later. Because of the Dirac-Born-Infeld extension, their coefficients $h_{(n)}$ also satisfy $h_{(n)}(0) = 0$ and $h'_{(n)}(0) = 0$.

The action (7.6) leads to two equations of motion obtained by variation with respect to tachyon and gauge fields. They are obtained by expanding the functions f, g and V in power series and collecting the coefficients of δT and δA_μ after repeated integrations by parts and cyclic permutations of the fields. For the sake of brevity, we do not give their explicit form here. Indeed, they are quite cumbersome due to the noncommutativity between the fields and their variations. Instead, we list a few conditions on the fields that are sufficient in order to solve the equations of motion

$$\begin{aligned} V'(T^2-1)T &= 0, \\ D_\mu T &= 0, \\ D_\alpha F_{\mu\nu} &= 0, \\ \int_{\mathbb{T}^d} \left[\delta h(T^2-1)(iF_{\mu\nu}+\Phi_{\mu\nu})(iF^{\mu\nu}+\Phi^{\mu\nu})\right] &= 0, \\ &\cdots \\ \int_{\mathbb{T}^d} \left[\delta h_{(n)}(T^2-1)(iF+\Phi)^{2n}\right] &= 0, \\ &\cdots \quad . \end{aligned} \tag{7.8}$$

These conditions imply the equations of motion whatever the higher order terms are. Indeed, the first one ensures that the contribution to δT arising from the tachyonic potential is identically zero. After integration by parts, the second and the third one imply that any contribution containing covariant derivatives of the fields vanish. Finally the last ones are the remaining contributions from the monomials in the curvature and the tachyon field without any covariant derivative.

Thanks to the condition $V'(0) = 0$ the first equation is solved by taking T to be any projector which we write as $T = 1 - P$. Therefore the second equation becomes

$$dP + AP - PA = 0. \tag{7.9}$$

We are using the coordinate free notation of differential geometry, which is defined by the same formulas as in the commutative case for Lie algebra valued forms (see the comments in section 2.4).

Most general solution is easily obtained by decomposing A into its "matrix elements"

$$A = PAP + (1-P)A(1-P) + PA(1-P) + (1-P)AP \tag{7.10}$$

and plugging it in the equation (7.9). Result is simply

$$A = A_{||} + A_{\perp} + PdP + (1-P)d(1-P), \tag{7.11}$$

where $A_{||}$ and $A_{\perp}$ are arbitrary elements of $P\mathcal{A}_\theta P$ and $(1-P)\mathcal{A}_\theta(1-P)$ algebras respectively. It means that they are subject to the constrains

$$PA_{||}P = A_{||}, \tag{7.12}$$
$$(1-P)A_{\perp}(1-P) = A_{\perp}. \tag{7.13}$$

We have solved the equation $DT = 0$ and we are left with the last equations involving the curvature $F = dA + A^2$. After an easy computation we get

$$F = P(dA_{||} + A_{||}^2 + dPdP)P + \tag{7.14}$$
$$+(1-P)(dA_{\perp} + A_{\perp}^2 + dPdP)(1-P). \tag{7.15}$$

It turns out to be diagonal. This fact is quite crucial since it allows us to satisfy identically the last set of equations in (7.8) for the tachyon. To see that, let us expand h in power series, bearing in mind that $h(0) = h'(0) = 0$,

$$h(T^2-1) = \sum_{n\geq 2} \frac{1}{n!} h^{(n)}(0) \left(T^2-1\right)^n, \tag{7.16}$$

so that its variation is easily computed by varying all monomials. The variation of a monomial of degree n gives rise to $2n$ terms

$$\delta\left(T^2-1\right)^n = \sum_{0\leq k\leq n-1} \left(T^2-1\right)^k \left(T\delta T + \delta T T\right)\left(T^2-1\right)^{n-1-k}. \tag{7.17}$$

Although δT is completely arbitrary, when $T = 1-P$ is a projector, this formula simplifies into

$$\delta\left(T^2-1\right)^n = (-1)^n \left(T\delta T\left(T^2-1\right) + \left(T^2-1\right)\delta T T\right) \tag{7.18}$$

and therefore, using (7.7)

$$\delta h(T^2-1) = -\left((1-P)\delta T P + P\delta T(1-P)\right), \tag{7.19}$$

so that it vanishes identically when multiplied with a diagonal quantity and integrated. The same analysis remains true for the higher order terms involving $h_{(n)}$ so that we can conclude that the last set of equations in (7.8) is satisfied.

We are thus left with the only remaining equation $D_\alpha F_{\mu\nu} = 0$. A further simplification occurs because of $D_\alpha P = 0$. This allows to solve the last equation as follows. Suppose we can find gauge fields $A_{||}, A_{\perp}$ (with the above properties) such that

$$P(dA_{||} + A_{||}^2 + dPdP)P = \lambda_{||}P, \tag{7.20}$$
$$(1-P)(dA_{\perp} + A_{\perp}^2 + dPdP)(1-P) = \lambda_{\perp}(1-P) \tag{7.21}$$

with $\lambda_{||}, \lambda_{\perp}$ being numbers. Then the compatibility condition (7.9) clearly implies

$$D_\alpha F = \lambda_{||} D_\alpha P + \lambda_{\perp} D_\alpha (1-P) = 0, \tag{7.22}$$

which ensures that the full set of equations of motion is satisfied, whatever the higher order terms are. The existence of such gauge fields as well as some explicit constructions will be given in the next section.

7.2 Constant curvature connections

To solve (7.20) and (7.21) we have to make contact with some available results in the mathematical literature. In fact, they are equivalent to the existence of a constant curvature connections on the projective modules determined by P and $1-P$. On the noncommutative tori this has been extensively studied, yet formulated in a rather different way, using functional analytic methods. This way of presenting projective modules over noncommutative tori and the associated constant curvature connections have been proposed by A. Connes more than twenty years ago [128].

To proceed, let us first recall that a projector P in $M_N(\mathcal{A}_\theta)$ determines a finitely generated projective module $\mathcal{E} = P\mathcal{A}_\theta^N$. Alternatively, one can consider $\mathcal{E}$ as the subspace of elements ξ of $\mathcal{A}_\theta^N$ that fulfill $P\xi = \xi$. This is naturally a right $\mathcal{A}_\theta$ module since it is stable by multiplication by elements of $\mathcal{A}_\theta$ on the right.

Projective modules are fundamental objects in noncommutative geometry since they provide a generalization of vector bundles. Moreover, one can define the analogue of a covariant derivative and its curvature for these modules. In fact, a covariant derivative ∇_μ associated to the partial derivative ∂_μ is just a linear map from $\mathcal{E}$ to itself satisfying the Leibniz rule

$$\nabla_\mu (P\xi f) = \nabla_\mu (P\xi) f + P\xi \partial_\mu f \tag{7.23}$$

for any $P\xi \in \mathcal{E}$ and $f \in \mathcal{A}_\theta$. It is not difficult to construct a covariant derivative just by taking the ordinary partial derivative and projecting the result onto $\mathcal{E}$ using P. This defines the covariant derivative ∇_μ^0 by

$$\nabla_\mu^0 (\xi) = P\partial_\mu (P\xi) \tag{7.24}$$

for any $P\xi \in \mathcal{E}$.

To the module $\mathcal{E}$ we also associate its algebra of endomorphisms $\mathrm{End}(\mathcal{E})$ which is the algebra of linear maps from $\mathcal{E}$ to itself that commute with the right action of $\mathcal{A}_\theta$. It is isomorphic to the algebra of matrices of the form

$PAP \in M_N(\mathcal{A}_\theta)$, or equivalently, that satisfy $PAP = A$, the sum and product being the ordinary operations on matrices but the unit is the projector P instead of 1.

It can be easily shown that all connections are of the form

$$\nabla_\mu = \nabla^0_\mu + A_\mu, \tag{7.25}$$

where A_μ is a matrix satisfying $PA_\mu P = A_\mu$. In a language more familiar to physicists A_μ may be identified with the gauge field.[2] The gauge transformations are the unitary elements of End$(\mathcal{A}_\theta)$, that is those $g \in$End$(\mathcal{E})$ that fulfill $gg^* = g^*g = P$. Note that one usually imposes that A_μ be antihermitian and it transforms in the ordinary way under gauge transformations.

The curvature tensor is defined as the commutator of the covariant derivatives,

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu]. \tag{7.26}$$

Unlike the covariant derivatives, it commutes with the right action of $\mathcal{A}_\theta$, so that it may be identified with an element of End$(\mathcal{A}_\theta)$.

If we compute the curvature of the connection defined in (7.25), we get

$$F = P(dA + A^2 + dPdP)P, \tag{7.27}$$

which is just the left hand side of (7.20) and which we would like to be proportional to P. Since P is the identity of End$(\mathcal{E})$, gauge fields whose curvature is proportional to P are known in the mathematical literature as "constant curvature connections".

Shifting back to the notations we have been using in the previous section, let us consider our projector P corresponding to the tachyon field. This projector determines a projective module, but it has no reason, *a priori*, to admit a constant curvature connection. However, it is known that in the two dimensional case, for θ irrational, [130, 132], any projective module is either free or is isomorphic to the Heisenberg modules which admit a constant curvature connection. For higher dimensional tori the situation is slightly complicated, interested reader is referred to our paper [123].

In the two dimensional case, this isomorphism can be described quite explicitly using the techniques developed in [128]. Let us fix an irrational number $\theta \in]0,1[$ so that the algebra $\mathcal{A}_\theta$ is generated by two unitary elements U_1 and U_2 such that

$$U_1U_2 = e^{2i\pi\theta}U_2U_1. \tag{7.28}$$

[2]The reader should note, that what we call A_μ in this section was actually $A_{||\mu}$ in the section preceding.

θ is the dimensionless parameter related to the ordinary Moyal deformation parameter through $\theta_{Moyal} = -2\pi\theta R^2$, R being the radius of the torus.

It is known [120] that gauge equivalence classes of projectors in $\mathcal{A}_\theta$, in the two dimensional case and when θ is irrational, are parameterized by the values of the normalized integral. More precisely, if P is a projector, then there are two integers p and q such that

$$\frac{1}{(2\pi R)^2}\int_{\mathbb{T}^2} P = p + q\theta. \qquad (7.29)$$

Obviously, if two projectors are gauge equivalent, they yield the same integral, thus the same numbers p and q. Conversely, given any two integers p and q such that $p + q\theta \in [0,1]$, then there is a projector in $\mathcal{A}_\theta$ such that (7.29) holds and any two such projectors are equivalent. In more physical terms, this means that p and q parameterize our vacua.

Let us now focus, for simplicity, on the case $p = 0$ and $q = 1$. Thus we are working within the gauge equivalence class of the Powers-Rieffel projector, which is the simplest possible example of a non trivial projector in $\mathcal{A}_\theta$. We denote by P a projector within this class, $P\mathcal{A}_\theta$ is then a right $\mathcal{A}_\theta$-module ($\mathcal{A}_\theta$ acting by multiplication) which can be equivalently described as follows [128].

Let $S(\mathbb{R})$ be the space of complex valued function on $\mathbb{R}$ which decrease fast at infinity. We define a right action of $\mathcal{A}_\theta$ by

$$\begin{aligned}(\psi U_1)(t) &= \psi(t+\theta),\\ (\psi U_2)(t) &= e^{-2i\pi t}\psi(t),\end{aligned} \qquad (7.30)$$

for any function ψ. Accordingly, this turns $S(\mathbb{R})$ into a right module which is in fact finitely generated and projective.

Covariant derivatives ∇_1 and ∇_2 are defined by

$$\begin{aligned}\nabla_1\psi(t) &= -\frac{i}{\theta R}t\psi(t),\\ \nabla_2\psi(t) &= \frac{1}{2\pi R}\frac{d}{dt}\psi(t),\end{aligned} \qquad (7.31)$$

which satisfy the Leibniz rule with respect to the right action of $\mathcal{A}_\theta$. The curvature of this connection is constant,

$$F_{12} = [\nabla_1, \nabla_2] = -\frac{i}{\theta_{Moyal}}, \qquad (7.32)$$

and we shall comment on the physical meaning of this value in the next section.

To describe explicitly the isomorphism between the projective module determined by the projector P and $S(\mathbb{R})$, as well as the pull back of the connection, it is convenient to use the so-called "bimodule construction".

In general terms, suppose that we are given two algebras $\mathcal{A}$ and $\mathcal{B}$ and a $(\mathcal{B},\mathcal{A})$-bimodule $\mathcal{M}$. This means that $\mathcal{M}$ is a vector space equipped with a left action of $\mathcal{B}$ and a right action of $\mathcal{A}$ and that these two actions commute, i.e.

$$(b\psi)a = b(\psi a) \tag{7.33}$$

for any $\psi \in \mathcal{M}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Besides, let us assume that $\mathcal{M}$ comes equipped with two scalar products, $(\cdot,\cdot)_\mathcal{A}$ and $(\cdot,\cdot)_\mathcal{B}$. The first one takes its values in $\mathcal{A}$ and is $\mathcal{A}$-linear whereas the second one takes its values in $\mathcal{B}$ and is $\mathcal{B}$-linear.

We finally assume that they are compatible,

$$(\psi,\xi)_\mathcal{B}\zeta = \psi(\xi,\zeta)_\mathcal{A} \tag{7.34}$$

for any elements ψ, ξ and ζ of $\mathcal{M}$. Then one easily shows that if

$$(\psi,\psi)_\mathcal{B} = 1, \tag{7.35}$$

then

$$P = (\psi,\psi)_\mathcal{A} \tag{7.36}$$

is a projector in $\mathcal{A}$. Indeed,

$$\begin{aligned} P^2 &= (\psi,\psi)_\mathcal{A}(\psi,\psi)_\mathcal{A} \\ &= (\psi,\psi(\psi,\psi)_\mathcal{A})_\mathcal{A} \\ &= (\psi,(\psi,\psi)_\mathcal{B}\psi)_\mathcal{A} \\ &= (\psi,\psi)_\mathcal{A}. \end{aligned} \tag{7.37}$$

The use of Morita equivalence to construct projectors has been first proposed in [120] and in fact it lead Rieffel to the discovery of his projector. It generalizes the partial isometry introduced in [46] which cannot be applied to the noncommutative torus. Indeed, the normalized trace of any projector obtained by a partial isometry should be one, but then the projector is the identity.

Let us now fix $\mathcal{A} = \mathcal{A}_\theta$, $\mathcal{B} = \mathcal{A}_{-1/\theta}$ and $\mathcal{M} = S(\mathbb{R})$. The left action of $\mathcal{A}_{-1/\theta}$ is

$$\begin{aligned} V_1 f(t) &= e^{2i\pi t/\theta} f(t), \\ V_2 f(t) &= f(t+1) \end{aligned} \tag{7.38}$$

and the two scalar products are defined as

$$(\psi,\chi)_{\mathcal{A}} = \theta \sum_{m_1,m_2\in\mathbb{Z}^2} \int_{\mathbb{R}} dt\overline{\psi(t+m_1\theta)}\chi(t)e^{2i\pi m_2 t}U_1^{m_1}U_2^{m_2}, \quad (7.39)$$

$$(\psi,\chi)_{\mathcal{B}} = \sum_{n_1,n_2\in\mathbb{Z}^2} \int_{\mathbb{R}} dt\psi(t-n_1)\overline{\chi(t)}e^{\frac{2i\pi n_2 t}{\theta}}V_1^{n_1}V_2^{n_2}. \quad (7.40)$$

The compatibility condition follows from Poisson resummation. Note that these formulas can be extended to a much broader context [127] and [130].

Besides, it is known that there exists an element ψ_0 in $S(\mathbb{R})$ such that $(\psi_0,\psi_0)_{\mathcal{B}} = 1$ [120]. The construction of such a ψ_0 is actually a non trivial task and it is this condition that is at the origin of the strange choice of functions involved in the Powers-Rieffel projector. Thanks to the bimodule technique, we end up with a projector $P_0 = (\psi_0,\psi_0)_{\mathcal{A}}$ whose trace is θ.

We know that all projectors of trace θ in $\mathcal{A}_\theta$ are gauge equivalent. Thus the projector P we started with is related to P_0 by

$$P = uP_0u^*, \quad (7.41)$$

where $u \in \mathcal{A}$ is unitary, i.e. $uu^* = u^*u = 1$. This also implies that

$$P = (u\psi_0, u\psi_0)_{\mathcal{A}}. \quad (7.42)$$

Writing $\psi = u\psi_0$, this allows us to define a map S from $P\mathcal{A}_\theta$ to $S(\mathbb{R})$ by

$$S(Pa) = \psi Pa \quad (7.43)$$

for any $Pa \in \mathcal{A}_\theta$. It is a right $\mathcal{A}_\theta$-module homomorphism which turns out to be invertible. Indeed, its inverse is given by

$$S^{-1}(\xi) = (\psi,\xi)_{\mathcal{A}}. \quad (7.44)$$

Therefore, it establishes a right $\mathcal{A}_\theta$-module homomorphism between $P\mathcal{A}_\theta$ and $S(\mathbb{R})$.

Using the covariant derivatives on $S(\mathbb{R})$ defined in (7.31) we define a covariant derivative $S^{-1}\nabla_\mu S$ on the algebra $\mathcal{A}$. In terms of gauge fields, we have

$$\begin{aligned} S^{-1}\nabla_\mu S(Pa) &= (\psi, \nabla_\mu(\psi Pa))_{\mathcal{A}} \\ &= (\psi,\nabla_\mu\psi)_{\mathcal{A}}Pa + (\psi,\psi)_{\mathcal{A}}\partial(Pa) \\ &= A_\mu Pa + P\partial_\mu(Pa), \end{aligned} \quad (7.45)$$

with

$$A_\mu = (\psi, \nabla_\mu \psi)_{\mathcal{A}} (\psi, \psi)_{\mathcal{A}}. \tag{7.46}$$

It can be checked by direct computation, using the Leibniz rule to derive the scalar products, that the curvature associated to A_μ is proportional to P,

$$F_{\mu\nu} = P[\partial_\mu P, \partial_\nu P]P + P\partial_\mu A_\nu P - P\partial_\nu A_\mu P + [A_\mu, A_\nu] = -\frac{i}{\theta_{Moyal}} \varepsilon_{\mu\nu} P. \tag{7.47}$$

Obviously this construction can be generalized to projectors with other traces and in higher dimensions. One can thus construct, with a high level of explicitness, gauge fields compatible with given projectors that extremize the effective action (7.6). Note that the above formula (7.46) has already appeared in the physical literature in [138] in the study of noncommutative instantons.

Let us end this section by a comment on the gauge theoretical aspect of the procedure we followed. First we have found a non trivial extremum of the potential which is just the projector $T_P = 1 - P$. The latter induces spontaneous symmetry breaking since it is not invariant under the full gauge group G. The subgroup H_P of all unitary elements u such that $uT_Pu^* = T_P$ is the unbroken subgroup and the compatibility condition (7.9) just means that we are only considering a gauge theory with the little group H_P as gauge group. From this interpretation, it follows that all projectors are equally good. Indeed, trading P for uPu^* is just a gauge transformation and it leaves the physics invariant. Our truly independent solutions are in fact equivalence classes of projectors. On the noncommutative tori, the latter are known to form a discrete set.

7.3 D-brane interpretation

To verify that the soliton solutions we have found correspond to D-branes, we can calculate their tension as in [89, 92, 124]. We shall specialize to the solitons on the noncommutative two torus for which we were able to make everything rather explicit. As they are localized in two directions they should represent codimension two branes, i.e. D23-branes.

Let us take first for the Seiberg-Witten parameter Φ the simplest choice $\Phi = -B = -\frac{1}{\theta_{Moyal}}$ which sets to zero all the terms in the action containing $iF + \Phi$. The only contribution to the tension then comes only from the potential term. Using

$$\frac{\sqrt{\det G}}{G_s} = \frac{2\pi\alpha'}{g_s|\theta|_{Moyal}} \tag{7.48}$$

we get for the action evaluated on our solution

$$S = -\frac{2\pi\alpha' c}{g_s |\theta|_{Moyal}} \int_{\mathbb{T}^d} P \int_{\mathcal{M}} d^{26-d}x \sqrt{\det g_{\mathcal{M}}}, \tag{7.49}$$

since

$$V(T^2 - 1) = V(-P) = V(0) + (V(-1) - V(0))P = P. \tag{7.50}$$

As we have discussed above the projectors on the algebra $\mathcal{A}_\theta$ are classified according to their trace which belongs to $(\mathbb{Z} + \theta\mathbb{Z}) \cap [0,1]$. For a projector P of normalized trace θm we get the tension

$$T = \frac{2\pi\alpha' c}{g_s |\theta|_{Moyal}} (2\pi R)^2 \theta m = (2\pi)^2 \alpha' m T_{25} = m T_{23}, \tag{7.51}$$

which identifies our soliton to be describing m D23-branes. Curiously the mathematical theorem tells us that the soliton number cannot exceed $\frac{1}{\theta}$. Physically it is welcome since the total energy of the D23-branes should be smaller than the energy of the original unstable D25-brane.

We may ask what would happen if we had chosen different value for the parameter Φ. Would we end up with the right tension? The value we took before was special in the sense that all terms containing the curvature vanished. If we take different value for Φ we have to know all terms in the action which do not contain derivatives. Fortunately these are precisely those terms provided by the Dirac-Born-Infeld action [134, 135]

$$S = \frac{c}{G_s} \int_{\mathcal{M}} d^{26-d}x \int_{\mathbb{T}^d} \left[-V(T^2 - 1) \sqrt{\det(G + 2\pi\alpha'(iF + \Phi))} + \cdots \right]. \tag{7.52}$$

Clearly the $1 - P$ part of F will not contribute. We can use formula

$$\sqrt{\det(a + bP)} = \sqrt{\det a}\,(1 - P) + \sqrt{\det(a+b)}\,P. \tag{7.53}$$

It is then simple exercise to check using (7.2) that

$$\frac{\sqrt{\det(G + 2\pi\alpha'(\theta^{-1}_{Moyal} + \Phi))}}{G_s} = \frac{\sqrt{|\det \theta^{-1}_{Moyal}|}}{g_s} \tag{7.54}$$

is actually Φ independent and when we evaluate the action we get the same tensions as before.

The reader may wonder what is the physical interpretation of the gauge fields $A_{||}$ and $A_\perp$. Actually with an action of the form (7.52), where $V(T^2-1)$

multiplies also all the higher order terms, all equations of motion will be satisfied for any $A_\perp$ and also the action will not depend on it. On the other hand $A_{||}$ is fixed by the formula (7.46) up to a gauge transformation. Further insight might be obtained in the small θ limit when the solitons are more and more sharply peaked. Then $A_{||}$ is a gauge field localized on the D23-brane and $A_\perp$ lives in the whole bulk. Since the action does not depend at all on $A_\perp$ we conclude that it is an unphysical degree of freedom, which is in accord with the fact that there are no open string degrees of freedom in the closed string vacuum surrounding the D23-brane.

7.4 Approximate solutions without gauge fields and selfdual projectors

In the previous sections we were solving exactly the equations of motion by turning on an appropriate gauge field. Obviously one may ask whether it is possible to satisfy the equations without turning on the gauge field at all. This should indeed be possible as suggest the results in CSFT [74, 75, 76] and BSFT [68, 69]. In the effective field theory approach we are pursuing here we are then limited only to approximate solutions neglecting higher derivatives.

Natural procedure is to first exclude all the derivative terms and search for the minima of the potential. As is known [46] nontrivial solutions of $V'(\phi) = 0$ can be constructed using projectors. The next step is to take into account the kinetic term, which reduces to

$$S_{kin}[P] = \frac{1}{\lambda^2} \int P \partial_\mu P \partial_\mu P, \tag{7.55}$$

when evaluated on a projector P. Note that we have summarized all relevant information on the function f and the effective couplings into the "coupling constant" λ. Whereas all projectors are ground states of the potential, only a few of them will minimize the correction given by the kinetic term. Let us thus try to find the extrema of (7.55) on the space of all projectors.

This problem has been studied in [139]. Let us first derive the equation of motion. If P is a projector, then $P + \delta P$ is a projector (at the first order) iff $\delta P = [\delta a, P]$ with $\delta a \in \mathcal{A}$. Accordingly, the equation of motion are

$$P \Delta P - \Delta P\, P = 0, \tag{7.56}$$

where Δ is the standard laplacian. It is worthwhile to notice that this equation takes a particular form because we search for the extrema of (7.55) on the space of projectors. If we were instead working on the space of all scalar fields, the equations of motion would be completely different and would include also a nontrivial dependence on the function f appearing in the effective action (7.6). To our knowledge it is not clear that the extrema of the full action (in the absence of gauge field and neglecting the higher order derivatives of the tachyon) can be obtained by first searching for the zeros of the potential and then minimizing the kinetic term on these zeros.

However, the problem of minimizing the kinetic term on the space of projectors is an interesting question since it involves some topologically stable solutions in two dimension. In fact this action admits a topological bound

$$\frac{1}{\lambda^2}\int P\partial_\mu P\partial_\mu P \geq \frac{1}{\lambda^2}\left|i\epsilon_{\mu\nu}\int P\partial_\mu P\partial_\nu P\right|. \tag{7.57}$$

This is easily derived from the relation

$$\int(\partial PP)^*\partial PP = \int P\partial_\mu P\partial_\mu P - i\epsilon_{\mu\nu}\int P\partial_\mu P\partial_\nu P, \tag{7.58}$$

where $\partial = \partial_1 - i\partial_2$ and from the corresponding equality involving $\overline{\partial}$.

This inequality (7.57) is similar to the inequality arising in four dimensional Yang-Mills theory and is interpreted as follows. The space of projectors is not connected and on each of its connected components the action is bound by the LHS.

It is an easy exercise to show that the LHS is invariant under a small deformation of the projector. In fact, when θ is irrational, two projectors lie in the same connected component iff they have the same trace [120]. Furthermore, if the normalized trace of P is $p+q\theta$, then

$$\frac{\epsilon_{\mu\nu}}{(2\pi R)^2}\int P\partial_\mu P\partial_\nu P = 2i\pi q, \tag{7.59}$$

so that q is an analogue of a two dimensional instanton number. Indeed, it has been obtained in [139] as a topological bound in the study of a noncommutative generalization of a non-linear σ-model. It is part of a general theory that encompasses both this model and the ordinary non-linear field theory with values in S^2. Because S^2 is homeomorphic to the space of rank one projectors in $M_2(\mathbb{C})$, it is easy to write the kinetic term of the standard non-linear σ model and (7.59) is nothing but the winding number of the corresponding map from S^2 into itself. This means that it measures the homotopy class of this map, i.e. it is an element of $\pi_2(S^2)$.

In the context of noncommutative geometry, the rôle of the homotopy groups is played by the K-theory of the algebra, (i.e. classes of projectors and unitary elements of matrix algebra over the algebra of coordinates).

They provide, when suitably differentiated and integrated, quantities that are stable under small deformations. In a more abstract language, this is formulated through the pairing of the cyclic cohomology of the algebra (i.e. all the possible ways to differentiate and integrate in a "suitable way" elements of the algebra) with its K-theory (i.e. noncommutative analogues of homotopy classes of vector bundles and gauge transformation). This theory is fully developed in the treatise [4] and the recent review [141].

Turning back to the problem of minimizing (7.55), it follows from (7.58) that the bound will be saturated iff $\partial P\, P = 0$ (antiself-duality equation) or $\overline{\partial} P\, P = 0$ (self-duality equation). Because of the nonlinear constraint arising form the fact that P must be a projector, these equations are not easy to solve.

If P is a self-dual projector, then $1 - P$ is an antiself-dual and vice-versa, so that it is sufficient to look only for the former. Fortunately, this non-linear problem can be turned into a linear one using the bimodule technique.

Using the notation of section 3, we recall that it allows to construct a projector in $\mathcal{A}$ provided we have an element $\psi \in \mathcal{M}$ such that $(\psi, \psi)_{\mathcal{B}} = 1$. For simplicity, we restrict our discussion to the homotopy class of the Powers-Rieffel projector, so that we know that all projectors in this class are obtained through unit vector in the bimodule $\mathcal{N} = \mathcal{S}(\mathbb{R})$. It follows that the resulting projector will satisfy

$$\frac{1}{(2\pi R)^2} \int P = \theta, \qquad \text{and} \qquad \frac{\epsilon_{\mu\nu}}{(2\pi R)^2} \int P \partial_\mu P \partial_\nu P = 2i\pi. \tag{7.60}$$

Note that the map from from unit vectors to projectors is not one to one; two unit vectors yield the same projector iff they differ by a gauge transformation. Indeed, if ψ and χ are unit vectors such that

$$P = (\psi, \psi)_{\mathcal{A}} = (\chi, \chi)_{\mathcal{A}}, \tag{7.61}$$

then

$$\psi = (\psi, \chi)_{\mathcal{B}}\, \psi \qquad \text{and} \qquad \chi = (\chi, \psi)_{\mathcal{B}}\, \chi \tag{7.62}$$

and the gauge transformation is $u = (\chi, \psi)_{\mathcal{B}}$ which belongs to the gauge group (i.e. the group of unitary elements of $\mathcal{B}$).

A natural way to construct a unit vector ψ is to start with an arbitrary element $\chi \in \mathcal{E}$ whose norm $(\chi, \chi)_{\mathcal{B}}$ is invertible. Then standard mathematical techniques (holomorphic functional calculus, for instance) allow us to define the square root of $(\chi, \chi)_{\mathcal{B}}$ so that

$$\psi = \left((\chi, \chi)_{\mathcal{B}}\right)^{-1/2} \chi \tag{7.63}$$

is a unit vector.

The main difficulty of this approach is to determine whether the norm (χ, χ) is invertible or not. For instance, if $\theta > 1$, we know that $(\chi, \chi)_{\mathcal{B}}$ is not invertible, otherwise we would have constructed a projector of trace $\theta > 1$ in $\mathcal{A}_\theta$, which is impossible. On the other hand, for $\theta = 1/n$, $\mathcal{B}$ is commutative and $(\chi, \chi)_{\mathcal{B}}$ is invertible iff this function does not vanish.

In trading the projectors for the vectors $\chi \in \mathcal{E}$, we have introduced spurious gauge degrees of freedom. In fact, two such vectors yield the same projector iff they differ by a complex (not necessarily unitary) gauge transformation. By definition such an element belongs to the group of invertible elements of $\mathcal{B}$. Therefore, we have to identify the vectors that yield self-dual projectors but differ only by a complex gauge transformation.

Let us introduce the complex covariant derivatives

$$\nabla = \nabla_1 - i\nabla_2 \tag{7.64}$$

associated to the covariant derivatives ∇_1 and ∇_2 introduced in section 3.

Now we have all the tools to solve the self-duality equation. In fact the projector P associated to a vector χ of invertible norm satisfies the equation

$$\overline{\partial} P P = 0 \tag{7.65}$$

if and only if there is $\rho \in \mathcal{B}$ such that

$$\nabla \chi = \rho \chi. \tag{7.66}$$

The proof follows from an explicit computation of $\overline{\partial} P \, P$ in terms of χ. This equation is a first order linear equation in χ and allows thus all the powerful methods of linear functional analysis to be applied.

The complex gauge group acts on ρ as

$$\rho \longrightarrow g\rho g^{-1} + g\partial g^{-1}, \tag{7.67}$$

so that ρ is nothing but a complex gauge field. Moreover, if $\lambda \in \mathcal{B}$ can be deduced from ρ by a complex gauge transformation

$$\lambda = g^{-1}\rho g + g^{-1}\overline{\partial} g, \tag{7.68}$$

then χ satisfies the linear problem associated to ρ iff $g\chi$ satisfies the linear problem associated to λ. This means that we have to solve the linear problem only on the orbit of the action of the complex gauge group on the complex connection. In other words, if we find a subset of $\mathcal{B}$ that intersects each orbit at least once, it is sufficient to solve linear problem for those values of ρ.

The general problem of the study of these orbit spaces is a rather difficult problem which is tantamount to the study of the moduli spaces of complex vector bundles over the noncommutative torus. In the special case we are considering here, one can show that constants intersect each orbit at least once, thus we have to solve the equation[3]

$$\nabla f = \frac{\lambda}{2\pi R} f, \tag{7.69}$$

with $\lambda \in \mathbb{C}$. This is very easy because (7.69) is just the differential equation

$$f^{'}(t) + \frac{2\pi t}{\theta} f(t) - \lambda i f(t) = 0, \tag{7.70}$$

whose solution is the gaussian

$$f_\lambda(t) = A e^{-\frac{\pi t^2}{\theta} + \lambda i t}, \tag{7.71}$$

A being an arbitrary constant, which is absolutely inessential since it cancels in the expression for the projector. Projector based on this function has been for the first time constructed by Boca [140]. The self-duality property was recognized in [139].[4] It has been also studied in [127].

Besides, one can show that two values of λ differ by a gauge transformation iff they belong to the same class in $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, where τ is the modular parameter of our initial torus [139]. Here we have set $\tau = i$, but it is an easy exercise to work with a general value of τ and observe the covariance under transformation in $SL_2(\tau)$. Therefore, the torus parameterizes all instantons in the homotopy class of the Powers-Rieffel projector.

7.5 Moyal plane limit of Boca's projector

The Powers-Rieffel projector [120] discussed in the context of tachyon condensation in [91, 125, 126, 127] is fully legitimate one in the sense that one can find exact solitons based on this projector. Nonetheless one may wish to have projector which does reduce to the nice GMS solitons [46] in the large torus limit keeping θ_{Moyal} fixed. The Powers-Rieffel projector is also not suitable for orbifolding [127].

[3]We are now using the symbol f instead of χ to indicate that the constant ρ is now complex number and to be in accord with references [127, 123].

[4]Note that the Boca's projector shares this self-duality property with its Moyal plane counterpart, the simplest GMS soliton. Indeed, in the operator correspondence $\bar{\partial} PP \sim [a, |0\rangle\langle 0|]\,|0\rangle\langle 0| = 0$.

A projector which overcomes these difficulties was constructed by Boca [140] using the bimodule technique starting from the gaussian function $f = e^{-\frac{\pi t^2}{\theta}}$. He has proved for certain range of θ that $b = \langle f, f\rangle_{\mathcal{B}}$ is invertible. Then one can easily check that

$$P_\theta = \langle b^{-\frac{1}{2}} f, b^{-\frac{1}{2}} f\rangle_{\mathcal{A}} \tag{7.72}$$

is a projector. For special values of $\theta = 1/q, q \in \mathbb{Z}$ the projector can be expressed explicitly in terms of Jacobi's theta functions. For example for q even one has

$$P_{\frac{1}{q}} = \frac{\sum_{r,s=0}^{q-1} e^{-\frac{\pi i r s}{q}} \vartheta^{(q)}_{\frac{s}{q},\frac{r}{2}}(U_2, \frac{iq}{2}) \vartheta^{(q)}_{\frac{r}{q},\frac{s}{2}}(U_1, \frac{iq}{2})}{q\vartheta(U_2^q, \frac{iq}{2})\vartheta(U_1^q, \frac{iq}{2})}, \tag{7.73}$$

where

$$\begin{aligned} \vartheta^{(N)}_{\frac{a}{N},b}(U,\tau) &= \sum_m e^{\pi i \tau (m+\frac{a}{N})^2 + 2\pi i (m+\frac{a}{N}) b} U^{Nm+a}, \\ \vartheta(U,\tau) &= \sum_m e^{\pi i \tau m^2} U^m. \end{aligned} \tag{7.74}$$

Note that the above formula (7.73) makes sense as it stands since the denominator turns out to be central element in the algebra. One can translate back this expression in the language of ordinary functions and the Moyal star product. With some effort one can check that in the large torus limit keeping θ_{Moyal} fixed (which is equivalent to $q \to \infty$), the above projector goes to the basic GMS soliton

$$P = |0\rangle\langle 0| \simeq 2e^{-\frac{x_1^2+x_2^2}{|\theta_{Moyal}|}}. \tag{7.75}$$

In this limit the theta functions in the denominator do not contribute. In the numerator the sums over r and s factorize, after some rearrangement one can replace them again by Jacobi's theta function, use its duality property and obtain the result with all the factors right.

An alternative way is to start with the formula for $b^{-1/2} f$ obtained in [127]. For small θ it simply reduces to

$$b^{-1/2} f \sim \sqrt[4]{\frac{2}{\theta}} f. \tag{7.76}$$

From the formula (7.39) we first calculate

$$\begin{aligned} \sqrt{\frac{2}{\theta}} \langle f, f\rangle_{\mathcal{A}} &= \theta \sum_{m,n} e^{-\frac{\pi\theta}{2}(m^2 - 2imn + n^2)} e^{im\frac{y}{R}} * e^{in\frac{x}{R}} \\ &= \theta\, \vartheta\left(\frac{x}{2\pi R}, \frac{i\theta}{2}\right) \vartheta\left(\frac{y}{2\pi R}, \frac{i\theta}{2}\right). \end{aligned} \tag{7.77}$$

Note that in the second line the product between the theta functions is the ordinary commutative one. The theta function is the basic one defined by

$$\vartheta(\nu,\tau)=\sum_m e^{\pi i\tau m^2+2\pi i m\nu} \tag{7.78}$$

with the familiar modular transformation property

$$\vartheta\left(\frac{\nu}{\tau},-\frac{1}{\tau}\right)=(-i\tau)^{\frac{1}{2}}e^{\pi i\nu^2\tau}\vartheta(\nu,\tau). \tag{7.79}$$

From that simply follows the asymptotic behavior

$$\langle b^{-\frac{1}{2}}f,b^{-\frac{1}{2}}f\rangle_{\mathcal{A}}\sim 2e^{-\frac{x_1^2+x_2^2}{|\theta_{Moyal}|}}, \tag{7.80}$$

which is just as for the GMS soliton.

Appendix A

Behaviour of the wedge state coefficients

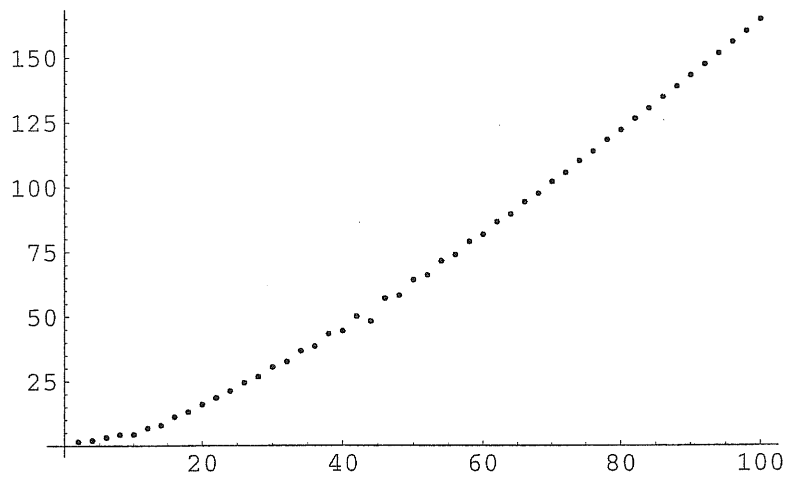

Figure A.1: Plot of $\log |v_{2n}|$ for the first 50 coefficients in the definition of the wedge state $|720°\rangle$

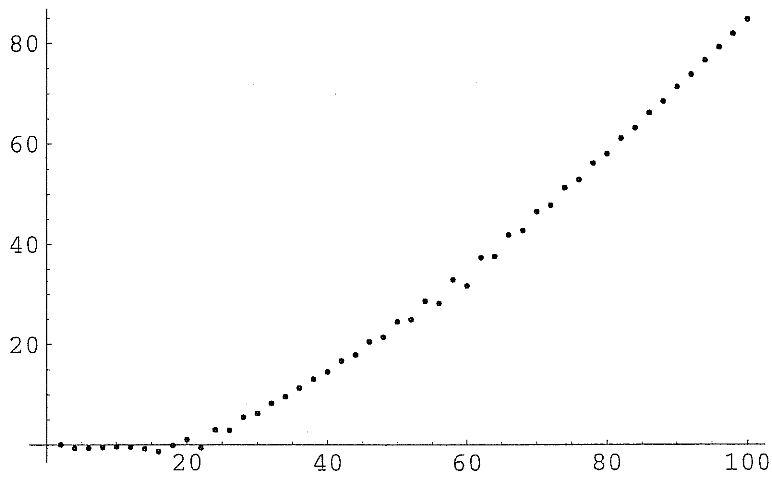

Figure A.2: Plot of $\log |v_{2n}|$ for the first 50 coefficients in the definition of the wedge state $|360°\rangle$

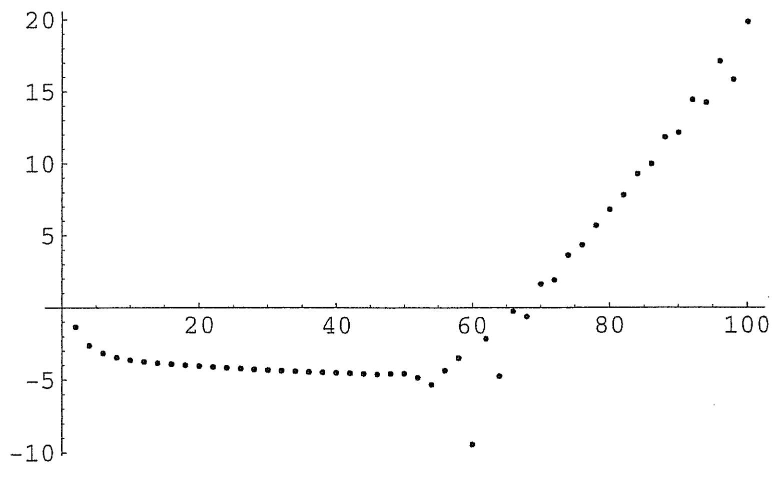

Figure A.3: Plot of $\log |v_{2n}|$ for the first 50 coefficients in the definition of the wedge state $|240°\rangle$

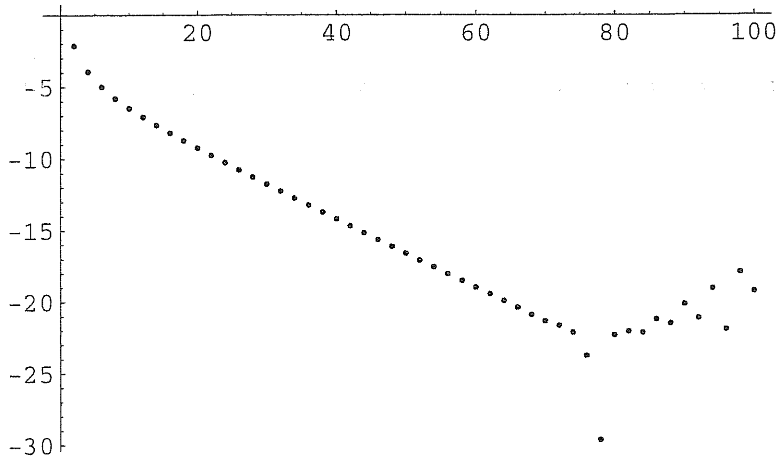

Figure A.4: Plot of $\log|v_{2n}|$ for the first 50 coefficients in the definition of the wedge state $|144°\rangle$

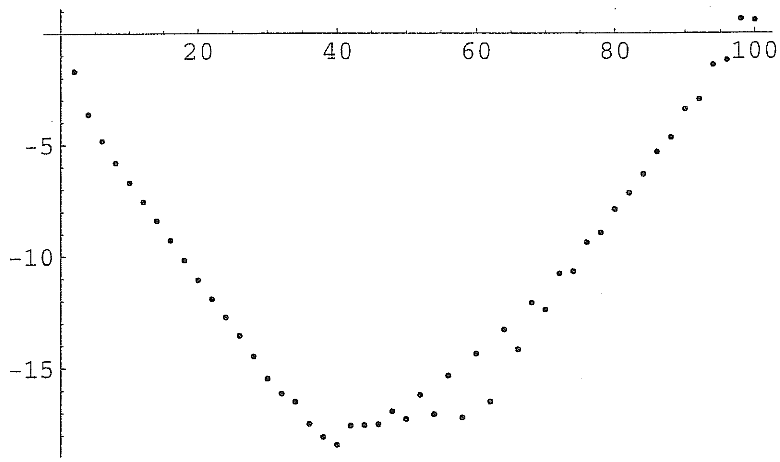

Figure A.5: Plot of $\log|v_{2n}|$ for the first 50 coefficients in the definition of the wedge state $|120°\rangle$

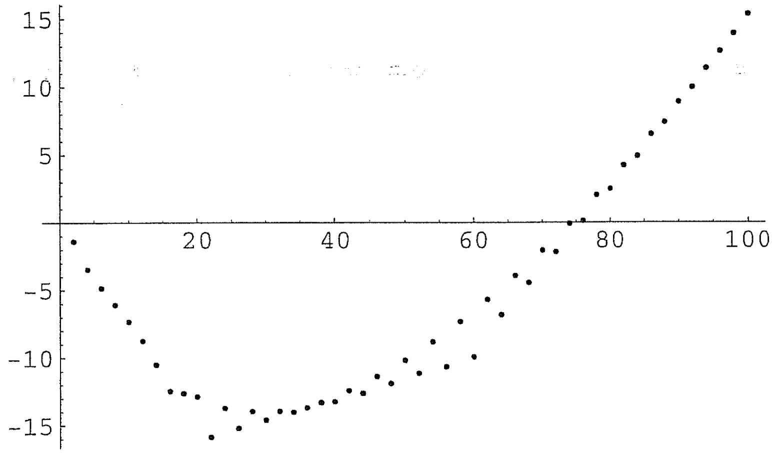

Figure A.6: Plot of $\log|v_{2n}|$ for the first 50 coefficients in the definition of the wedge state $|90°\rangle$

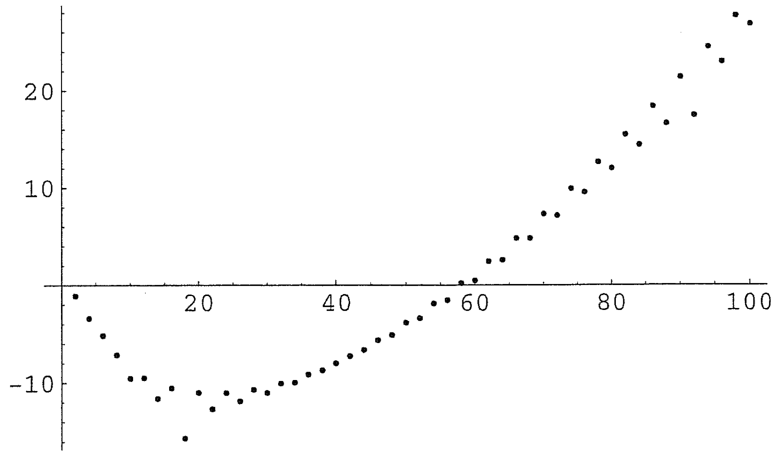

Figure A.7: Plot of $\log|v_{2n}|$ for the first 50 coefficients in the definition of the wedge state $|\infty\rangle$

Appendix B

Commutators

In this appendix we list some useful commutators and anticommutators which we have honestly calculated using the standard operator product expansion techniques. Here L's stand for Virasoro generators with a central charge c. J stands for the ghost number current and J^B is the BRST current. The equation (B.4) is correct only for total or ghost Virasoro, otherwise it would be zero.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{B.1}$$

$$[L_m, c_n] = -(2m+n)c_{m+n} \tag{B.2}$$

$$[L_m, b_n] = (m-n)b_{m+n} \tag{B.3}$$

$$[L_m, J_n] = -nJ_{m+n} - \frac{3}{2}m(m+1)\delta_{m+n,0} \tag{B.4}$$

$$[J_m, c_n] = c_{m+n} \tag{B.5}$$

$$[J_m, b_n] = -b_{m+n} \tag{B.6}$$

$$[J_m, J_n] = m\delta_{m+n,0} \tag{B.7}$$

$$\{J^B_m, b_n\} = mJ_{m+n} + L_{m+n} + \frac{3}{2}m(m+1)\delta_{m+n,0} \tag{B.8}$$

$$[J^B_m, J_n] = 2mnc_{m+n} - J^B_{m+n} \tag{B.9}$$

$$\{J^B_m, c_n\} = -\sum_{l=-\infty}^{\infty} lc_{-l}c_{m+n+l} \tag{B.10}$$

$$\{J^B_m, J^B_n\} = -2mn\sum_{l=-\infty}^{\infty} lc_{-l}c_{m+n+l} \tag{B.11}$$

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