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TRIESTE - ITALY

Thesis submitted for the degree of Doctor Philosophiae

Aspects of Stringy Dynamics in External Fields

Candidate

Ciprian Acatrinei

Supervisor

Prof. Roberto Iengo

ACADEMIC YEAR 1999-2000

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Contents

1 Introduction

Preamble (Motivation)

The dynamics of various systems in presence of external fields is an old subject. It literally gave rise to electromagnetic and gravitational theories, thus to modern physics. Later it was used to describe quantum mechanical systems interacting with classical radiation fields, preparing the way for field quantization. Even when a quantum theory of fields and a systematic prescription for calculating quantum effects order by order became available, the dynamics in classical backgrounds remained an useful tool. It sometimes provided nonperturbative insights [1], or manifestly gauge invariant methods of calculation.

Backgrounds were also used to manifestly break a symmetry and see how the theory behaves under the new conditions. Alternatively, they were put to work in order to test the stability of a theory, as we will later see. Another aspect which will be discussed here is the use of a given background to test a duality.

By definition, the behaviour of a given object placed in an external field can provide information both on the nature of the field and on the properties of the probe (with respect to that field). In the first instance we will use an open string as a probe, to be tested by an electromagnetic field. We will study the properties of the string, especially the interplay between its tail of (free) oscillators and its charged end, which couples to the electromagnetic field. Next we will move to gravitational backgrounds, for which the test particles will be closed strings and D0-branes. Here we will try to see whether the duality between the standard description of closed strings and the matrix model one stands the addition of a nontrivial gravitational background. In some sense, the leitmotif will be the interplay between dynamics in spaces with different dimensionalities: the bulk/boundary relationship for open strings, and the agreement between a ten-dimensional string theory and the matrix $(0+1)$-dimensional quantum mechanics of D-particles. In what follows we detail the context in which each of these investigations will be performed.

Contexts

1) It has long been known that charged particle-antiparticle pairs should emerge from the vacuum once an electric field disturbs it [1]. We will be concerned with the similar effect in open string theory, whose dynamics in various backgrounds became of interest more recently. Open strings coupled through their end-points to external electromagnetic fields have been studied by various authors [3, 4, 5, 6, 7]. In particular, the induced Born-Infeld [2] effective action for the external field was obtained in [3, 4], whereas [5, 7] discussed the possibility of string pair creation through the Schwinger mechanism. In reference [5] bosonic strings in weak electric fields were considered, whereas reference [7] studied both bosonic and fermionic strings,

for electric field strengths up to the value (in natural units) of the string tension. The above quantum effects were studied by calculating a one-loop effective action (whose imaginary part is relevant for the pair creation process) in a background electric field. The calculation of [7] relied on canonical methods. Although powerful and elegant, they needed as an input previous results concerning the eigenmodes and spectrum of the string in electromagnetic backgrounds [4]. Our contribution will be to make the calculation through a more direct, path integral, method. We will not need to rely on the results of [4], but rather derive the shift in the spectrum as a by-product of our calculations. We will also include magnetic fields in this type of analysis, observing that they affect the pair creation rate in a nontrivial way, due to the presence of the Born-Infeld (BI) term.

Also, we demonstrate the emergence of the BI term for charged strings (reference [7] showed it only in the case of globally neutral strings). The derivation of the BI term presented here shows transparently its stringy origin, complementing the original work of [3, 4].

2) Our next subject is still concerned with open strings in electromagnetic backgrounds. The difference is that we now generalize the string action, allowing the string tension to depend both on the space-time and on the world-sheet direction. This will allow us, using two different limits in the 'tension configuration space', both to recover standard relativistic string theory results and to use open strings with one charged end-point to simulate dissipative dynamics of the above end-point. In the later case, in which excitations in the space-like directions propagate slowly (nonrelativistically) along the string, the pair creation of such open strings has the same form as the one for point particles put in a Caldeira-Leggett [9] oscillator bath. This extends previous work [10, 11] in which it was argued that a string can be used to replace the various ways [8, 9] previously used to describe quantum dissipative dynamics. Namely, the Caldeira-Leggett [9] approach, in which the dissipation is introduced through the coupling to an infinite set of oscillators whose frequencies and masses satisfy a definite relation (a kind of spectral condition) is well reproduced in our framework.

3) Our third and last topic moves the stage into more string theoretical topics. We will be concerned with the interaction of D-particles (D0-branes). Their dynamics [30, 31] can be described through the exchange of closed strings, or alternatively through loops of open strings (and inserting various vertex operators on the world sheet to account for further interactions). This description is ten-dimensional, involving ten-dimensional supergravity in the long distance limit (in fact eleven dimensional). Another description, the matrix model (which might be called holographic since it relates a higher dimensional field theory to a supersymmetric quantum mechanical model), was proposed in reference [32], based on earlier suggestions [33]. Those two radically different paradigms were found in agreement in many nontrivial circumstances, involving either two-body [34] or three-body [35, 36] dynamics (for a

temporary doubt eventually fixed see [37]). Nevertheless, all the tests the conjecture passed took place in flat space-time. Gravitational perturbations were not included (however, see [38, 39]), although, in addition to making the tests more difficult, they also rise a conceptual issue. Namely, closed string theory includes gravity explicitely, whereas the matrix model does not. Now, one problem of string theory is its background-dependent formulation: few things are known outside flat space-time. One may then ask the following: If one introduces a gravitational background in closed string theory, what kind of modification ('gravitational' or not) of the matrix model would hopefully account for this gravitational deformation. It turns out, and this is our final topic, that a similar gravitational background introduced in the matrix model formulation makes it reproduce string theory results. Thus, matrix models do not need gravity to account for closed strings in flat backgrounds, but apparently (and maybe unfortunately) they need it explicitely in order to reproduce curved backgrounds dynamics.

Structure of the thesis

Section 2 presents a review of point particles and open strings in electromagnetic backgrounds (hopefully from an unitary point of view). Nothing is new in this section, except maybe some interpretations of known results.

Section 3 is concerned with the effective action for an open string placed in a constant electromagnetic field background. The imaginary part of this action is calculated, thus allowing the evaluation of the rate at which string pair production takes place. Several comments are made about the role of the BI action, both in principle (it is a natural consequence of the coupling of pointlike charges to neutral extended objects, and as such lies at the heart of the calculations, also) and in practice (its effect on the pair production rate). This section is based on reference [12].

Section 4 extends the calculations of the previous one in the more general setting of strings with anisotropic tension. Several limits for the tensions of those strings are studied. The connection between one of them and dissipative dynamics is explained; the pair production rate (PPR) is calculated in such a limit, and some speculations are made about its relevance for thin superconductor physics. The main reference for this section is [13].

In Section 5 we go to gravitational backgrounds, more precidely to the field generated by a gravitational (shock) wave [40, 41, 42, 43, 44, 45]. After a brief introduction to various aspects related to the analysis to be presented, we present the string computation, which encompasses the interaction of two D-particles in presence of a gravitational wave. It is followed by the formulation of the corresponding $SU(2)$ Yang-Mills matrix model in the gravitational wave background. The amplitude calculated in this framework is shown to agree with the closed string theory one. The material presented in this section is taken from [14].

The Appendices are organised as follows: Appendix A presents some old and new facts about the diagonalization of the electromagnetic field strength in dimensions higher than four. Appendix B reviews the boundary state formalism, which can be used to make independent checks for some of the calculations in sections 2 and 3; a formula useful in this context is proved in Appendix C. Appendix D lists some string propagators and useful identities, needed in section 5, whereas Appendix E briefly reviews the gravitational wave metric and presents the trajectory of a point particle evolving in such a background.

2 Point particles and open strings in electromagnetic backgrounds

In this section, we present a brief overview of various aspects of the dynamics of point particles and open strings under the influence of external electric and magnetic fields. Although this review is by no means complete, it will prepare the stage for the next sections. In addition, it is hoped it can offer a glance at this fascinating topic.

2.1 Particles

2.1.1 Propagators

We first discuss the nonrelativistic case. The action is $S = \int dt L(x_i, t, A_\mu)$, with the Lagrangian

$$L = \frac{1}{2}\left(\frac{dx_i}{dt}\right)^2 - eA_i v_i - eA_0. \tag{1}$$

The propagator can be computed either as the Green function for the corresponding Schrödinger equation

$$\left[\left(i\frac{\partial}{\partial t} - eA_0\right) + \frac{h^2}{2m}\left(i\frac{\partial}{\partial x_i} - eA_i\right)^2\right] G(\vec{x'}, t'|\vec{x}, t) = \delta(\vec{x'} - \vec{x})\delta(t' - t), \tag{2}$$

or from the path integral

$$G(\vec{x'}, t'|\vec{x}, t) = \int_{\vec{x}(t)=\vec{x}, \vec{x}(t')=\vec{x'}} D\vec{x} \quad e^{i\int dtL}. \tag{3}$$

For a constant magnetic field B the propagator reads, in p spatial dimensions,

$$G(\vec{x'}, t'|\vec{x}, t) = \left(\frac{m}{ih(t' - t)}\right)^{p/2} \frac{\omega(t' - t)/2}{\sin\omega(t' - t)/2} \quad e^{iS_{cl}(\vec{x'}, \vec{x}, t' - t)}. \tag{4}$$

The first prefactor is a free particle one, whereas the second is of an harmonic oscillator type, with frequency $\omega = \frac{eB}{m}$. We do not exhibit $S_{cl}(\vec{x'}, \vec{x}, t' - t)$; it can be easily found, either by plugging the solutions of the equations of motion in the expression for the action, or by solving the appropriate Hamilton-Jacobi equation.

The propagator in constant electric fields, due to the linear potential involved, is trivially the same as the one for a free particle, with the exception of the $e^{iS_{cl}}$ part. It will not interest us in what follows.

The propagator for a relativistic spinless particle can be obtained from the nonrelativistic one through the use of an additional invariant parameter λ (used by Fock [16] and Feynman [15], and in a different set-up also by Schwinger [1]), as follows.

The Klein-Gordon equation

$$(i\frac{\partial}{\partial x_\mu} - eA_\mu)^2\Psi(x) = m^2\Psi(x) \tag{5}$$

can be thought of as a Schrödinger-like equation for the function $\phi(x,\lambda)$:

$$i\frac{\partial}{\partial\lambda}\phi(x,\lambda) = \frac{1}{2}(i\frac{\partial}{\partial x_\mu} - eA_\mu(x))^2\phi(x,\lambda), \tag{6}$$

which has solutions of the form $\phi(x,\lambda) = e^{im^2\lambda/2}\Psi(x)$, with $\Psi(x)$ satisfying (5). We can now represent the propagator for (6) as a path integral over trajectories $x_\mu(\lambda)$ (parametrized by λ) in a $(d+1)$-dimensional space-time, with Lagrangian $L' = \frac{1}{2}(\frac{dx_\mu}{d\lambda})^2 + e\frac{dx_\mu}{d\lambda}A_\mu$. Thus we are allowed to use (4) to evaluate this $(d+1)$-dimensional propagator.

Then, we can obtain the d-dimensional, on-shell, relativistic propagator by Fourier transforming the $(d+1)$-dimensional one:

$$G(x'',x';m) = \int_0^\infty d\lambda e^{im^2\lambda/2}\int Dx_\mu e^{\int d\lambda L'}. \tag{7}$$

One remark is in order for closed d-dimensional particle trajectories (corresponding to loop computations in field theory). If after a 'proper time' λ the particle ends at the starting point, an additional factor $\frac{1}{\lambda}$ should be included in the integrand of equation (7), because the particle can now start its journey at any point of the loop (we consequently divide by the translational symmetry along the circle). We will see in equation (9) that this factor can be obtained directly using a more 'field theoretical' approach.

For relativistic particles with spin, the same construction as above is possible, relating the relativistic propagator to the nonrelativistic one. Here, still, the construction is hampered by the dificulties encountered in constructing a path integral for *nonrelativistic* particles with spin. We will not be concerned with that, since the Schwinger-like procedure we will present later avoids these difficulties (in loop calculations).

2.1.2 Spectrum

We will discuss separately the cases of magnetic and electric fields, for nonrelativistic particles and then for relativistic fields.

A nonrelativistic particle put into a magnetic fields exhibits the well-known Landau levels spectrum, which means a $(2n+1)\frac{B}{m}$ term in the energy, in addition to the squares of the momenta laying outside the magnetic field plane. If the particle has spin, an additional term $-\vec{s}\cdot\vec{B}$ appears; $\vec{s}$ is the spin 'vector'. This coupling lowers the energy, and is in fact responsible for the apparition of tachyonic excitations for relativistic particles with spin higher than one, as we will see.

The spectrum of a nonrelativistic particle in an electric field is continuous, since the electric field just acts with a constant force on the particle.

For relativistic particles in magnetic fields the discussion is similar, at least at the heuristic level. In fact the analysis proceeds along the same lines as for the nonrelativistic particle in magnetic fields. Namely, one first separates the directions along which the particle behaves as a free one, and in the end remains with an harmonic oscillator type problem. The resulting spectrum is

$$E_n^2 = p_\perp^2 + m^2 + (2n+1)eB - g_s e\vec{B}\cdot\vec{S}. \tag{8}$$

In (8), $p_\perp$ is the momentum transverse to the magnetic field plane, g_s, e, and m are, respectively, the gyromagnetic ratio, the charge, and the mass of the particle. One can thus see that (for zero transverse momentum and in the first Landau level $n=0$) the value of the energy becomes negative for spin $s \geq 1$ if $g_s > \frac{1}{s}$. In fact, as it was argued in [24], $g_s = 2$ is the natural value. Thus, the spectrum includes excitations with negative energy, which can destabilize the theory.

A particular case in which such a tachyonic mode appears is provided by Yang-Mills theories [23]. There too, the instability can be traced back to the nonminimal coupling ($g_s > \frac{1}{s}$) of the electromagnetic field to excitations with spin greater than or equal to one [24].

For relativistic particles in electric fields, the discussion is more involved. One can repeat the previous steps to end up with an harmonic oscillator with imaginary frequency (due to the Minkowski signature of the time direction). This inverted harmonic oscillator potential (somehow reminescent of instanton methods) suggests an instability, which in fact really appears. This is discussed in the following paragraph.

2.1.3 Pair creation

We now present a brief description of the Schwinger effect, namely the tunneling of charged particle-antiparticle pairs out of the vacuum in presence of an external electric field. The rate at which those pairs are created can be read out from the imaginary part of the vacuum free energy, imaginary part which appears due to the presence of the electric field.

The vacuum free energy in an external generic electromagnetic field $F_{\mu\nu}$ is the logarithm of the vacuum-to-vacuum transition amplitude $e^{-iW(F_{\mu\nu})(vac)} =< 0|e^{-i\hat{H}\times(time)}|0>$. For a static field, $W_{vac}(F_{\mu\nu}) = \mathcal{E}_{vac}(F_{\mu\nu}) \times (time)$, where $(time)$ is the total time interval. Reexpressing the vacuum free energy either through equation (7), or *à* *la* Schwinger (which means starting from the field theory effective action, taking $t = \lambda/2$ to be the so-called Schwinger proper time and using $log(a/b) = \int_0^\infty \frac{dt}{t}(e^{-at} - e^{-bt})$), one obtains

$$W_{vac} = \int_0^\infty \frac{dt}{t} Tre^{-t\hat{H}}. \tag{9}$$

The trace is evaluated by means of the path integral

$$z = Tre^{-t\hat{H}} = \int DX_0 D\vec{X} e^{-S(X_0, \vec{X}, F_{\mu\nu})}. \tag{10}$$

If the vacuum energy $\mathcal{E}_{vac}(F_{\mu\nu}) = Re\mathcal{E}_{vac}(F_{\mu\nu}) - i\frac{\Gamma}{2}$ gets a nonvanishing imaginary part $\frac{\Gamma}{2}$, this induces the vacuum decay with rate per unit volume $\gamma = \frac{\Gamma}{V}$:

$$\gamma = -2Im \int_0^\infty \frac{dt}{t} \int' DX_0 D\vec{X} e^{-S(X_0, \vec{X}, F_{\mu\nu})}. \tag{11}$$

The prime means that we have factored out the zero mode part of the action which, upon integration, gives precisely the space-time volume.

The partition function (10) is evaluated with the help of the (Wick rotated) equation (4):

$$z = \frac{1}{4\pi} \frac{eE}{i\, sin(eET)}. \tag{12}$$

Evaluating also the imaginary part of the t integral, which appears due to the poles exhibited by (12) we get the Schwinger pair creation (vacuum decay) rate

$$\gamma = \frac{e^2 E^2}{8\pi^3} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} \quad e^{-\pi n \frac{m^2}{eE}}. \tag{13}$$

The $n = 1$ term was obtained before Schwinger [1], at least by Sauter [17].

We stress that the factor $\frac{eE}{sin(eET)}$ corresponds to the partition function of an harmonic oscillator with imaginary frequency iE. In presence of a magnetic field, (10) would include a harmonic oscillator partition function $\frac{eB}{sh(eBT)}$, now with real frequency B. This would not produce any instability, since it does not have poles, thus it does not induce an imaginary part of the vacuum free energy. On the other end, since $\frac{x}{sh(x)} < 1, \forall x > 0$, one can easily show that a non-zero magnetic field would always reduce the pair creation rate.

We are going to explore all these issues in detail, in the context of string theory, in the next two sections.

In order to study fermionic pair creation, one takes advantage of the relation

$$Tr \log(\not{D} - m) = Tr \log(-\not{D} - m) = \frac{1}{2} Tr \log(-\not{D}^2 - m^2) \tag{14}$$

and squares the Dirac operator $\not{D} = \gamma_\mu(i\partial_\mu - eA_\mu)$:

$$\not{D}^2 = D^2 + \frac{e}{2}F_{\mu\nu}\Sigma_{\mu\nu} \tag{15}$$

($\Sigma_{\mu\nu}$ are the commutators of γ-matrices). This shows that the spin $\frac{1}{2}$ result differs with respect to the spin 0 case only by the trace (over Dirac indices) of the last term on the right hand side of equation (15). This gives an additional $cos(eEt)$ factor ($ch(eBt)$ for the magnetic field). Due to that factor the addition of a magnetic field B can now increase the pair production rate. A similar effect will be observed for superstrings in section 3.

2.1.4 Induced non-commutativity

It can be shown [18] that in the limit $\frac{B}{m} \to \infty$. the coordinates coupled by the magnetic field do no commute anymore, but now obey the algebra

$$[\hat{X}_i, \hat{X}_j] = i\ B^{-1}_{ij}. \tag{16}$$

2.2 Strings

2.2.1 Action, boundary conditions, mode expansion

The main reference for this subsection is [4]. For a bosonic open string living on a Euclidean world-sheet, coupled to a $U(1)$ gauge field, the action reads:

$$\begin{aligned} S &= \frac{M^2}{2}\int_0^t d\tau \int_0^l d\sigma[(\frac{\partial X^\mu}{\partial\tau})^2 + (\frac{\partial X^\mu}{\partial\sigma})^2] \\ &\quad -iq_1 F_{\mu\nu}\int_0^t d\tau \left[X_\mu \frac{\partial X_\nu}{\partial\tau}\right]_{\sigma=0} - iq_2 F_{\mu\nu}\int_0^t d\tau \left[X_\mu \frac{\partial X_\nu}{\partial\tau}\right]_{\sigma=l}. \end{aligned} \tag{17}$$

M^2 denotes the string tension, which will also be sometimes denoted by the standard $\frac{1}{\alpha'}$; q_1 and q_2 are the magnitudes of the charges situated at the end-points of the string. This action being invariant under a rescaling of both σ and τ by a factor of l, we set $l = 1$.

The equations of motion in the bulk are the free ones

$$\frac{\partial^2 X_n^0}{\partial\tau^2} - \frac{\partial^2 X_n^0}{\partial\sigma^2} = 0 \tag{18}$$

whereas the external field modifies the boundary conditions at the string's ends

$$\frac{\partial X_\mu}{\partial\tau} = (-)^{i+1} q_i \frac{F_{\mu\nu}}{M^2}\frac{\partial X_\nu}{\partial\tau} \qquad i = 1, 2. \tag{19}$$

We now present a set of solutions of (18) satisfying the boundary conditions (19) in the case in which the external field is block diagonal, say applied in the plane $1-2$. For a magnetic field $F_{12} = B$ the following combinations

$$X_+ = \frac{1}{\sqrt{2}}(X_1 + iX_2), \quad X_- = \frac{1}{\sqrt{2}}(X_1 - iX_2) \tag{20}$$

are useful (for electric fields no i should appear in (20)). These new coordinates satisfy decoupled boundary conditions

$$\frac{\partial X_+}{\partial \tau} = (-)^{i+1} q_i \frac{B}{M^2}\frac{\partial X_+}{\partial \tau} \quad i = 1, 2. \tag{21}$$

and the opposite sign for X_-. The modes obeying (18) and (21) have the form

$$\psi_n = \frac{1}{\sqrt{n-f}} cos[(n-f)\sigma + f_1] e^{-(n-f)\tau} \tag{22}$$

where $\pi f = f_1 + f_2$, $f_i = arctan(q_i B)$. The properties of the above modes, and their usefulness in studying strings in constant magnetic backgrounds, are detailed in [4].

2.2.2 Propagators

We now find the propagator on the disc, satisfying (18) and (19). Conformally mapping the disc to the half-plane, the boundary conditions (19) take the form (we have introduced the derivatives ∂ and $\bar{\partial}$, with respect to the complex linear combinations of τ and σ):

$$g_{ij}(\partial - \bar{\partial})X_j + 2\pi\alpha' F_{ij}(\partial + \bar{\partial})X_j\Big|_{x=\bar{x}} = 0 \tag{23}$$

The propagator satisfying (18) and (23) reads [4, 3] (see also [28], from which the form presented here is taken)

$$\langle x^i(z) x^j(z')\rangle = -\alpha'\Big[g^{ij}\log|z-z'| - g^{ij}\log|z-\bar{z}'| + G^{ij}\log|z-\bar{z}'|^2 + \frac{1}{2\pi\alpha'}\theta^{ij}\log\frac{z-\bar{z}'}{\bar{z}-z'} + D^{ij} \tag{24}$$

Here

$$\begin{aligned} G^{ij} &= \left(\tfrac{1}{g+2\pi\alpha' B}\right)^{ij}_S = \left(\tfrac{1}{g+2\pi\alpha' B} g \tfrac{1}{g-2\pi\alpha' B}\right)^{ij}, \\ G_{ij} &= g_{ij} - (2\pi\alpha')^2 (Bg^{-1}B)_{ij}, \\ \theta^{ij} &= 2\pi\alpha' \left(\tfrac{1}{g+2\pi\alpha' B}\right)^{ij}_A = -(2\pi\alpha')^2 \left(\tfrac{1}{g+2\pi\alpha' B} B \tfrac{1}{g-2\pi\alpha' B}\right)^{ij}, \end{aligned} \tag{25}$$

where $(\ )_S$ and $(\ )_A$ denote the symmetric and antisymmetric part of the matrix. B is the matrix with components F_{ij}. The constants D^{ij} in (24) can depend on

B but are independent of z and z'; they play no essential role and can be set to a convenient value.

Open string vertex operators are inserted on the boundary of the half-plane Σ. So to get the relevant propagator, one restricts equation (24) to real z and z', which are denoted by τ and τ'. Evaluated at these boundary points, the propagator is

$$\langle x^i(\tau)x^j(\tau')\rangle = -\alpha' G^{ij}\log(\tau-\tau')^2 + \frac{i}{2}\theta^{ij}\epsilon(\tau-\tau'), \tag{26}$$

where D^{ij} were set to a convenient value. $\epsilon(\tau)$ is the function that is 1 or -1 for positive or negative τ.

In equation (26) G_{ij} is interpreted as [28] the effective metric seen by the open strings.

2.2.3 Noncommutativity of the string end-points coordinates

The coefficient θ^{ij} in the propagator also has a simple intuitive interpretation [27, 28]. In conformal field theory, one can compute commutators of operators from the short distance behavior of operator products by interpreting time ordering as operator ordering. Interpreting τ as time, we see that

$$[x^i(\tau), x^j(\tau)] = T\left(x^i(\tau)x^j(\tau^-0) - x^i(\tau)x^j(\tau^+0)\right) = i\theta^{ij}. \tag{27}$$

That is, x^i are coordinates on a noncommutative space with noncommutativity parameter θ. We see that noncommutativity of the string end-point coordinates emerges for any value of the magnetic field to string tension ratio (contrary to the point particle case, were $\frac{B}{m}$ had to go to infinity).

2.2.4 Spectrum

One can now find the expression of the Virasoro generators L_n. L_0 will provide the mass spectrum. Using the notation

$$f = arctg(B/M^2) \tag{28}$$

we get the spectrum (L_0 also gets shifted by a term $\frac{1}{2}f(1-f)$ in order to reduce the Virasoro algebra to the free open string one; this term adds to the usual -1 shift):

$$E^2 = \frac{1}{2}f(1-f) + f b_0^+ b_0 - f\sum_{n=1}^{\infty}(a_n^+ a_n - b_n^+ b_n) + L_0^{free}. \tag{29}$$

Above, a and b are the harmonic oscillator annihilation operators associated with the directions coupled by F_{12}, whereas L_0^{free} is the usual free bosonic spectrum

$$L_0^{free} = -1 + \sum_{n=1}^{\infty} n(a_n^+ a_n + b_n^+ b_n) + L_0^{\perp}. \tag{30}$$

$L_0^\perp$ represents the contribution to the spectrum of the coordinates transverse to the magnetic field plane. One can easily see that the vacuum and the states resulting from the use of the a_1^+ oscillator can become tachyonic in strong enough magnetic fields. Those states forming the first Regge trajectory have the spectrum [22] ($n = a_1^+ a_1$)

$$E^2 = -1 + \frac{1}{2} f(1-f) + (1-f)n. \tag{31}$$

Thus, tachyonic states appear for

$$n < \frac{1 - \frac{1}{2} f(1-f)}{(1-f)}. \tag{32}$$

The previous result can be extended to supersymmetric open strings; this is a more interesting situation, since in this case there is no tachyon in zero external field. Now the fermionic oscillators d and $\hat{d}$ associated with the coupled coordinates also enter into play. In the Ramond sector the spectrum reads [22]

$$E_R^2 = f(b_0^+ b_0 + d_0^+ d_0) - f \sum_{n=1}^{\infty} (a_n^+ a_n - b_n^+ b_n + d_n^+ d_n - \hat{d}_n^+ \hat{d}_n) + L_{free}^R. \tag{33}$$

In the equation above

$$L_R^{free} = \sum_{n=1}^{\infty} n(a_n^+ a_n + b_n^+ b_n + d_n^+ d_n + \hat{d}_n^+ \hat{d}_n) + L_0^\perp. \tag{34}$$

One can easily see that there are no tachyons in the Ramond sector.

For the Neveu-Schwarz sector [22]

$$E_{NS}^2 = f b_0^+ b_0 - f \sum_{n=1}^{\infty} (a_n^+ a_n - b_n^+ b_n) - f \sum_{n=\frac{1}{2}}^{\infty} (d_n^+ d_n - \hat{d}_n^+ \hat{d}_n) + L_{NS}^{free} \tag{35}$$

L_{NS}^{free} being the usual free spectrum. Again, we find tachyonic states on the first Regge trajectory. It is worth stressing that now the presence of tachyons is due exclusively to the magnetic field; the superstring had no tachyon to start with. This tachyonic part of the spectrum, which indicates an instability, made people speculate about possible phase transitions in high magnetic fields, for both particles and strings.

Thus, the open string spectrum in presence of a magnetic field contains tachyonic excitations [22], which destabilize the theory. These instabilities, like the similar tachyonic mode appearing in Yang-Mills theories [23], can be traced back to the nonminimal coupling of the electromagnetic field to excitations with spin greater than or equal to one [24], in a way similar to the one we used in subsection 2.1.2.

Our expedient (and provisional) remedy to these instabilities will be to 'Higgs' the theory by stretching the string between some different, parallel, D-branes. The Dirichlet boundary conditions along the coordinates orthogonal to the branes then produce an additional mass term, which can overcome the destabilizing contribution of the magnetic field.

The spectrum in electric fields can be obtained by analytically continuing the results found in pure magnetic fields. The shift will now become complex, not just negative. We will have more to say about this at the end of Section 3.

3 Pair production rate and Born-Infeld action

Our aim in this section is to study a nonperturbative aspect of the dynamics of open strings, namely the tunneling of string-antistring pairs out of the vacuum, in presence of an external (fixed) electromagnetic field coupled to the string end-points. The role of the induced Born-Infeld term will also be explored in this context.

We will evaluate the string pair production rate by calculating a 1-loop effective action, using the Schwinger proper time representation. The trace involved in our considerations will be evaluated via a path integral (this represents the novelty, as well as the advantage, of our approach).

3.1 Generalities

To calculate the rate at which the vacuum decays in presence of an external field we will use the same method we employed in the first section for point particles. Here we briefly review it restricting our attention to the case in which the relevant degreees of freedom are open strings.

The vacuum free energy in an external field $F_{\mu\nu}$ is again the logarithm of the vacuum-to-vacuum transition amplitude $e^{-iW(F_{\mu\nu})_{(vac)}} =< 0|e^{-i\hat{H}\times(time)}|0 >$. For a static field, $W_{vac}(F_{\mu\nu}) = \mathcal{E}_{vac}(F_{\mu\nu}) \times (time)$, where $(time)$ is the total time interval. Reexpressing the vacuum free energy *à la* Schwinger, and taking the Hamiltonian to be the one for open strings, one obtains

$$W_{vac} = \int_0^\infty \frac{dt}{t} Tr e^{-t\hat{H}_{string}}. \tag{36}$$

We notice that from Schwinger's field theoretical stand-point, we have rather to postulate equation (36) for strings, since we lack a second-quantized description for them. On the other hand, the first-quantized approach of Feynman-Fock works as well here as it worked for point particles in (7). The trace in (36) will be evaluated by means of a (suitably normalised) path integral

$$Tr e^{-t\hat{H}_{string}} = \int DX_0 D\vec{X} e^{-S(X_0,\vec{X},F_{\mu\nu})}. \tag{37}$$

The action $S(X_0, \vec{X}, F_{\mu\nu})$ (discussed below) includes the external electromagnetic field, which couples to the end-points of the string. Due to that coupling, the vacuum energy $\mathcal{E}_{vac}(F_{\mu\nu})$ gets an imaginary part, $\frac{\Gamma}{2}$, which induces the vacuum decay. The decay rate per unit volume, $\gamma = \frac{\Gamma}{V}$, reads

$$\gamma = -2Im \int_0^\infty \frac{dt}{t} \int' DX_0 D\vec{X} e^{-S(X_0,\vec{X},F_{\mu\nu})}. \tag{38}$$

The prime means that we have factored out the zero mode part of the action which, upon integration, gives precisely the space-time volume.

For a bosonic open string living on a Euclidean world-sheet, coupled to a $U(1)$ gauge field, the action to be used in equations (37,38) reads:

$$
\begin{aligned}
S \; = \; & \frac{M^2}{2}\int_0^t d\tau \int_0^l d\sigma \left[(\frac{\partial X^\mu}{\partial \tau})^2 + (\frac{\partial X^\mu}{\partial \sigma})^2 \right] \\
& -iq_1 F_{\mu\nu}\int_0^t d\tau \left[X_\mu \frac{\partial X_\nu}{\partial \tau} \right]_{\sigma=0} - iq_2 F_{\mu\nu}\int_0^t d\tau \left[X_\mu \frac{\partial X_\nu}{\partial \tau} \right]_{\sigma=l},
\end{aligned}
\tag{39}
$$

where M^2 denotes the string tension, whereas q_1 and q_2 are the magnitudes of the charges situated at the end-points of the string. This action being invariant under a rescaling of both σ and τ by a factor of l, we set $l = 1$.

We remark that, in order to avoid confusion with Schwinger's proper time, we have not used T to indicate the tension, but M^2. This shows explicitely that the string tension has the dimension of a mass to the square, M being the string scale mass. Later, we will also use $\frac{1}{\alpha'}$ to denote the string tension.

Thus, our main task is to evaluate the following path integral

$$
Z = \int DXe^{-S(X_\mu, F_{\mu\nu})}, \tag{40}
$$

with $S(X_\mu, F_{\mu\nu})$ given by equation (39).

3.2 Path integral evaluation

3.2.1 Free path integral

We start with the zero electromagnetic field case [13]. This is useful in order to obtain the correct normalization for our general path integral (40), and in order to get a general understanding of the effects of the string extension.

i) Result

If $F_{\mu\nu} \equiv 0$, eq.(37) factorizes into products of free path integrals along each space-time direction. For a generic uncoupled coordinate X we have to evaluate $\int DXe^{-S}$, with

$$
S = \frac{M^2}{2}\int_0^t d\tau \int_0^1 d\sigma \left[(\frac{\partial X}{\partial \tau})^2 + (\frac{\partial X}{\partial \sigma})^2 \right]. \tag{41}
$$

Taking the boundary conditions to be periodic along τ and Neumann along σ ,

$$
X(t+\tau, \sigma) = X(\tau, \sigma), \tag{42}
$$

$$
\left. \frac{\partial X}{\partial \sigma} \right|_{\sigma=0,1} = 0, \tag{43}
$$

which means expanding $X(\tau,\sigma)$ as follows

$$X(\tau,\sigma)=\sum_{n\in Z}\sum_{k\in N}X_{nk}cos(k\pi\sigma)exp(2\pi in\frac{\tau}{t}), \tag{44}$$

one gets [13] the result

$$Z_{free}\equiv\int DXe^{-S}=\sqrt{\frac{M^2}{4\pi}}e^{\frac{\pi}{6}\frac{1}{t}}\prod_{n\geq 1}\frac{1}{1-e^{-4\pi n\frac{1}{t}}}. \tag{45}$$

Through a modular transformation one can reexpress (45) as follows:

$$Z_{free}=\sqrt{\frac{M^2}{4\pi}\frac{2}{t}}e^{\frac{\pi}{24}t}\prod_{k\geq 1}\frac{1}{1-e^{-\pi kt}}. \tag{46}$$

ii) Evaluation

The explicit proof of equation (45) follows.

Inserting the mode expansion (44), the action becomes

$$S=M^2\frac{tl}{2}\left(\sum_{n>0,k>0}|X_{nk}|^2[\frac{4\pi^2}{t^2}n^2+\frac{\pi^2}{l^2}k^2]+\sum_{n>0,k=0}|X_{n0}|^2[2\frac{4\pi^2}{t^2}n^2]+\sum_{n=0,k>0}X_{0k}^2[\frac{1}{2}\frac{\pi^2}{l^2}k^2]\right)$$

We dropped the zero-mode $(n=0,k=0)$ in the path integral, since it just gives the space-time volume. Then

$$Z_{free}=\prod_{k>0}\frac{1}{\sqrt{\frac{\pi t}{4l}k^2M^2}}\prod_{n>0}\frac{1}{\frac{4\pi l}{t}n^2M^2}\prod_{k>0}\prod_{n>0}\frac{1}{M^2\frac{lt}{2\pi}(\frac{4\pi^2}{t^2}n^2+\frac{\pi^2}{l^2}k^2)}. \tag{47}$$

Using - only for the index n, corresponding to the Fourier transform along τ - the free particle normalization

$$\prod_{n>0}\frac{1}{4\pi}\frac{t}{m}\frac{1}{n^2}=\sqrt{\frac{m}{2\pi t}}.$$

and, subsequently, the Euler factorization of $\frac{\sinh x}{x}$:

$$\prod_{n=1}^{\infty}(1+\frac{x^2}{n^2})=\frac{\sinh\pi x}{\pi x}.$$

we obtain

$$Z_{free}=\sqrt{\frac{M^2l}{2\pi t}}e^{-\pi\frac{t}{2l}\sum_{k>0}k}\prod_{k>0}\frac{1}{1-e^{-2\pi k\frac{t}{2l}}}. \tag{48}$$

Now we make use of the transformation properties of the Dedekind eta function

$$\eta(x) = e^{i\pi\frac{x}{12}}\prod_1^\infty(1-e^{2\pi inx}) = \frac{1}{\sqrt{-ix}}\eta(-\frac{1}{x}) \tag{49}$$

and get in the end:

$$\int DXe^{-S} = \sqrt{\frac{M^2}{4\pi}}e^{-\frac{\pi}{2}\frac{t}{l}\sum_{k>0}k}e^{-\frac{\pi}{12}\frac{t}{2l}}e^{\frac{\pi}{12}\frac{2l}{t}}\prod_{n>1}\frac{1}{1-e^{-4\pi n\frac{l}{t}}}.$$

The factor $e^{-\frac{\pi}{2}\frac{t}{l}\sum_{k>0}k}e^{-\frac{\pi}{12}(\frac{t}{2l})}$ becomes one if we use the Riemann ζ-function regularization $\sum_{k=1}^{\infty} = lim_{s\to 1}\zeta(-s) = -\frac{1}{12}$, as in string theory. The remaining exponential term is important. We will see that it amounts to normal order the Hamiltonian in the boundary state formalism, and it gives a contribution which we will interpret in a thermodynamical context in the next section.

iii) 'Higgsing' the theory.

Instead of the Neumann boundary conditions previously encountered for the generic uncoupled coordinate X, one could use Dirichlet boundary conditions for the ends of the string, imagining they stay on two parallel D-branes situated at a relative distance d. That means asking

$$X(\sigma = 0) = 0, \qquad X(\sigma = 1) = d. \tag{50}$$

The corresponding mode expansion is

$$X(\tau,\sigma) = d\,\sigma + \sum_{n\in Z}\sum_{k>0} X_{nk}\sin(k\pi\sigma)exp(2\pi in\frac{\tau}{t}), \tag{51}$$

and it modifies the final result of the path integration in only one way: an additional exponential factor $e^{-\frac{t}{2}M^2d^2}$ appears in (45). This is equivalent to a mass term $\frac{1}{2}M^2d^2$, whose role will be to stabilize the theory in presence of a magnetic field. This term can be inserted at any step of the calculation; we will come back to it later.

3.2.2 Path integral in presence of constant external electromagnetic fields

We wish to study an electromagnetic field $F_{\mu\nu}$ as general as possible. Nevertheless, by rotations and boosts, one can block-diagonalize the matrix representing it. This is the subject of Appendix A. Once $F_{\mu\nu}$ is diagonalized, we are left with path integrals along pairs of coupled coordinates.

Choosing the $0-1$ plane, we evaluate

$$Z_{01} = \int DX_0 DX_1 e^{-S(X_0,X_1,F_{01})}, \tag{52}$$

the action $S(X_0, X_1, F_{01})$ being eq. (39) now restricted to $\mu = 0, 1$:

$$\begin{aligned} S &= \frac{M^2}{2}\int_0^t d\tau \int_0^1 d\sigma[(\frac{\partial X^\mu}{\partial \tau})^2 + (\frac{\partial X^\mu}{\partial \sigma})^2] \\ &-iq_1 E_1 \int_0^t d\tau \left[X_0 \frac{\partial X_1}{\partial \tau}\right]_{\sigma=0} - iq_2 E_2 \int_0^t d\tau \left[X_0 \frac{\partial X_1}{\partial \tau}\right]_{\sigma=1} . \end{aligned} \tag{53}$$

We remark that one can treat in this way the more general case in which different field strengths $E_{1,2}$ are applied to the two string end-points. Subsequently the two charges q_1 and q_2 will be included into the field strength value, through the more compact notation $q_{1,2}E_{1,2} \to E_{1,2}$.

Developing in the same interaction-independent Fourier basis as in the free case (44), the action (53) becomes

$$S = S(0) + \sum_{n=1}^{\infty} S(n),$$

with

$$S(0) = -\frac{t}{2}\sum_{k>0}\left[(X_{0k}^0)^2 \frac{M^2}{2}\pi^2 k^2 - (X_{0k}^1)^2 \frac{M^2}{2}\pi^2 k^2\right]$$

and

$$S(n > 0) = \mathbf{X}^\dagger A \mathbf{X}.$$

The term $\mathbf{X}^\dagger A\mathbf{X}$ encodes the modes which couple due to the electric field:

$$\mathbf{X}^\dagger = (X_{-n,0}^0, X_{-n,1}^0, \dots X_{-n,k}^0, \dots, X_{-n,0}^1, X_{-n,1}^1, \dots X_{-n,k}^1, \dots),$$

$$A = \begin{pmatrix} a_0 & 0 & 0 & \cdot & \cdot & C_1 & C_2 & C_1 & \cdot & \cdot \\ 0 & a_1 & 0 & \cdot & \cdot & C_2 & C_1 & C_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \\ D_1 & D_2 & D_1 & \cdot & \cdot & b_0 & 0 & 0 & \cdot & \cdot \\ D_2 & D_1 & D_2 & \cdot & \cdot & 0 & b_1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \end{pmatrix}, \tag{54}$$

where $C_1 = -D_1 = -2\pi n(E_1 + E_2)$, $C_2 = -D_2 = -2\pi n(E_1 - E_2)$, whereas $a_0 = -M^2 t\frac{4\pi^2 n^2}{t^2}$, $a_{k>0} = -M^2\frac{t}{2}(\frac{4\pi^2}{t^2}n^2 + \pi^2 k^2)$, and $b_k = -a_k$, $\forall k \geq 0$. The appearance of nonzero a_k terms for $k \geq 1$ is due to the finite spatial extension of the string. One can prove that

$$\begin{aligned} Det(A) &= \prod_{i=1}^{\infty} a_i b_i \times \left[1 - C_1 D_1 \left(\sum_{(i-j)even} \frac{1}{a_i b_j} \right) - C_2 D_2 \left(\sum_{(i-j)odd} \frac{1}{a_i b_j} \right) \right. \\ & \left. + (C_1^2 - C_2^2)(D_1^2 - D_2^2) \left(\sum_{(i-j)odd,(k-l)odd} \frac{1}{a_i a_j b_k b_l} \right) \right] . \end{aligned} \tag{55}$$

The infinite product $\prod_{i=1}^{\infty} a_i b_i$ corresponds to the case in which the external field is zero; it is thus given by the square of equation (45). The term in square brackets represents the correction due to the presence of the electric field. Evaluating it with the help of the identities (valid for some complex x)

$$\frac{1}{2x^2} + \sum_{k=2,4,6\ldots} \frac{1}{x^2 + k^2} = \frac{\pi}{4x} \coth(\frac{\pi x}{2}) \tag{56}$$

$$\sum_{k=1,3,5\ldots} \frac{1}{x^2 + k^2} = \frac{\pi}{4x} \tanh(\frac{\pi x}{2}), \tag{57}$$

which are needed in order to perform the sums in equation (55), and remembering that the path integral is given by the inverse of the determinant, we obtain the following partition function:

$$Z_{01} = \frac{M^2}{4\pi} e^{\frac{\pi}{3}\frac{1}{t}} \prod_{n\geq 1} \frac{[(1 - E_1^2)(1 - E_2^2)]^{-1}}{[1 - \frac{(1+E_1)(1+E_2)}{(1-E_1)(1-E_2)} e^{-4\pi n \frac{1}{t}}][1 - \frac{(1-E_1)(1-E_2)}{(1+E_1)(1+E_2)} e^{-4\pi n \frac{1}{t}}]}. \tag{58}$$

The string tension M^2 was absorbed into E: $E/M^2 \to E$, in order to express the electric field in terms of the natural units available.
Using the convenient notation $\epsilon = \epsilon_1 + \epsilon_2$, with $\epsilon_j = arcthE_j$, $j = 1, 2$, as well as ζ-function regularizing the divergent product $\prod_{n\geq 1}[(1 - E_1^2)(1 - E_2^2)]$ via $\sum_{k\geq 1} 1 = lim_{s\to 0}\zeta(s) = -\frac{1}{2}$, we finally obtain

$$Z_{01} = \frac{M^2}{4\pi} e^{\frac{\pi}{3}\frac{1}{t}} \sqrt{(1 - E_1^2)(1 - E_2^2)} \prod_{n=1}^{\infty} \frac{1}{[1 - e^{2\epsilon - 4\pi n \frac{1}{t}}][1 - e^{-2\epsilon - 4\pi n \frac{1}{t}}]}. \tag{59}$$

Under the ζ-function regularization, the divergent infinite product, to which *all* the string oscillation modes along σ do contribute, has metamorphosed into the Born-Infeld term $\sqrt{(1 - E_1^2)(1 - E_2^2)}$. We see that this happens not only in the case of globally neutral strings [7] but rather, as usual from T-duality/D-branes arguments, each string end-point has associated with it a separate BI action, no matter what its charge is.

It is sometimes useful to recast (59) in a different form (switching from the closed string channel to the open string one). Using the transformation properties of the Dedekind eta function

$$\eta(x) = e^{i\pi\frac{x}{12}} \prod_{1}^{\infty} (1 - e^{2\pi i n x}) = \frac{1}{\sqrt{-ix}} \eta(-\frac{1}{x}),$$

and of the first Θ-function

$$\Theta_1(v|\tau) = 2q^{\frac{1}{8}} \sin \pi v \prod_{n=1}^{\infty} (1-q^n)(1-e^{2\pi i v} q^n)(1-e^{-2\pi i v} q^n) = -e^{-i\frac{v^2}{\tau}} \frac{1}{\sqrt{-i\tau}} \Theta_1(\frac{v}{\tau}|-\frac{1}{\tau}),$$

equation (59) becomes

$$Z_{01} = (E_1 + E_2) \frac{M^2}{4\pi} e^{\frac{\pi}{12} t} \frac{e^{-\frac{t}{2\pi}\epsilon^2}}{sin(\epsilon \frac{t}{2})} \prod_{n>1} \frac{1}{[1 - e^{(i\epsilon - \pi n)t}][1 - e^{-(i\epsilon + \pi n)t}]}. \tag{60}$$

A linear $(E_1 + E_2)$ factor appears in front instead of the BI term.

Both (59) and (60) display poles, which signal the string pair production. They are

$$t_p(k) = \frac{2k\pi}{\epsilon}; \; k = 0, 1, 2, \ldots \tag{61}$$

A more detailed discussion of the role of the poles will be presented in the next section, in a more general context.

The path integrals along the other coupled directions are obtained from (59) or (60). The Z_{23} path integral, for the 2 and 3 directions coupled by a magnetic eigenvalue B, for instance, is obtained by replacing $\epsilon_{1,2} \to i f_{1,2}$ $(i = \sqrt{-1}\,)$ in (59) [1, 7]. Now $f_{1,2} = arctg B_{1,2}$ and $f = f_1 + f_2$ (like in section 2).

Z_{23} does not exhibit poles, as expected:

$$\begin{aligned} Z_{23} &= \frac{M^2}{4\pi} e^{\frac{\pi}{3}\frac{1}{t}} \sqrt{(1 + B_1^2)(1 + B_2^2)} \prod_{n>1} \frac{1}{[1 - e^{2if - 4\pi n \frac{1}{t}}][1 - e^{-2if - 4\pi n \frac{1}{t}}]} \\ &= \frac{M^2}{4\pi} e^{\frac{\pi}{12} t} (B_1 + B_2) \frac{e^{\frac{t}{2\pi} f^2}}{sh(f \frac{t}{2})} \prod_{n>1} \frac{1}{[1 - e^{(f - \pi n)t}][1 - e^{-(f + \pi n)t}]}. \end{aligned} \tag{62}$$

One has to take into account also the uncoupled directions and the ghosts. The contribution of one free coordinate, denoted Z_{free}, has been displayed in (45, 46); the ghosts cancel the stringy part of two free coordinates and give

$$z_g^2 = (Z_{free})^{-2} \times \frac{M^2}{2\pi t}. \tag{63}$$

(Of course, if there are no free coordinates the ghosts do not cancel anything; one must just put $d = 0$ in equation (64) below.) Using equations (45) or (46), (63), (59) or (60), and (62), one can write down the whole partition function,

$$Z = z_g^2 \times (Z_{free})^d \times Z_{01} \times Z_{23} \times Z_{45} \times \ldots, \tag{64}$$

in D dimensions out of which d are left uncoupled by the electromagnetic field.

We quote now the result of a longer calculation, performed in D dimensions without previous diagonalization of $F_{\mu\nu}$. It says that the partition function is just the one without electromagnetic field present, times a product of BI-like factors:

$$Z_{(F_{\mu\nu})} = Z_{(F=0)} \times \prod_{n=1}^{\infty} [Det(\eta_{\mu\nu} - cth(2\pi n/t) F_{\mu\nu})]^{-1}. \tag{65}$$

This shows the importance of the BI action for the whole problem treated here: (65) encodes all the effects of the external field, in particular the way it distorts the open string spectrum. Eq. (65) arises from the determinant of a matrix analogous to (54) (now containing $D \times D$ infinite blocks) upon summing over the σ-oscillators of the string. Remarkably enough, after extensive use of (56,57), each infinite off-diagonal block gets replaced by a simple term $F_{\mu\nu} cth(2\pi n/t)$, whereas on the principal diagonal one gets the metric tensor $\eta_{\mu\nu}$, thus giving equation (65).

It is interesting to relate equation (65) to equation (59), in the case of a block-diagonal $F_{\mu\nu}$. This is done by expressing the hyperbolic cotangent function in terms of exponential functions, and noticing that a part of the factors obtained in this way cancel the similar ones from $Z_{(F=0)}$. The above intermediate steps can be easily read out from the form of equation (102). This shows the way in which the Born-Infeld-like factors in (65) generate (after the mentioned cancellations with the free partition function factors) the true BI action, displayed in (59). For non-diagonal $F_{\mu\nu}$'s the same procedure can be applied. The result is the general BI determinant $Det(\eta_{\mu\nu} - F_{\mu\nu})$ under the square root of equation (59).

One can now go back into the calculation and notice that the true origin of the BI action is the existence of the constant off-diagonal factors $C_{1,2}$ and $D_{1,2}$ in (54). The form of those factors is due to the string being charged only at the end-points. Thus, the physical origin of the BI action for the electromagnetic field is the interplay between the dynamics of charged point-particles (the string end-points) and the dynamics of a neutral object having spatial extension in one more dimension (the string tail, with its uncoupled oscillators).

3.3 Pair production rate

We can now estimate the rate at which open strings would be produced out of the vacuum in a given background. Let us consider first the case of a pure electric field. Thus now the D-dimensional partition function Z (64) reads $Z(t) = z_g^2 \times (Z_{free})^{d=(D-2)} \times Z_{01}$. Using the formula

$$\frac{1}{x - i\epsilon} = P(\frac{1}{x}) + i\pi\delta(x) \tag{66}$$

in order to evaluate

$$Im \int \frac{dt}{t} Z(t), \tag{67}$$

as well as equations (45,60,63), one obtains the pair production rate in presence of an electric field in D dimensions [7, 13]

$$\gamma(E) \equiv \sum_{k=1}^{\infty} \gamma_k = \pi \sum_k (-)^{k+1}(E_1 + E_2) e^{-k\epsilon \frac{M^2}{4\pi}} \frac{\epsilon}{k\pi} [Z_{free}]^{D-2}. \tag{68}$$

In the previous formula, Z_{free} is given by eq. (46), in which t is replaced now by $\frac{2k\pi}{\epsilon}$, i.e. its value at the $k-$th pole (cf. equation (61)). The term γ_1 dominates the sum, and we will later concentrate on it. (One can easily see that the pair creation rate is zero for neutral strings, when $E_1 + E_2 = 0$).

The result (68) gets corrected in two ways in presence of a magnetic field. First, the electric-like eigenvalue $\mathcal{E}$ may change in presence of other components of $F_{\mu\nu}$, as discussed in Appendix A. Thus we assume $F_{\mu\nu}$ to be in block-diagonal form, with $\mathcal{E} \equiv E$. Second, the presence of non-zero magnetic-like eigenvalues (which thus do not interfere anymore with the electric field) changes the form of the production rate. This happens because the function $Z(t)$ in (67) changes, being now equal to $Z = z_g^2 \times (Z_{free})^{D-4} \times Z_{01} \times Z_{23}$, cf. equation (64). We turn our attention to that point. We consider only one non-zero magnetic eigenvalue B (the analysis proceeds identically and independently for several B's). Then each term γ_k in the production rate (68) gets corrected, $\gamma_k(E, B) = \gamma_k(E) \times \delta_k$, with the following correction factor

$$\delta_k(f) = \sqrt{1 + \frac{B^2}{M^4}} \prod_{n=1}^{\infty} \frac{(1 - x^n)^2}{(1 - e^{2if}x^n)(1 - e^{-2if}x^n)} = \frac{1}{\cos f} \prod_{n=1}^{\infty} \frac{(1 - x^n)^2}{(1 - 2x^n \cos 2f + x^{2n})}. \tag{69}$$

x is evaluated at the pole of order k: $x = e^{-4\pi/t_p(k)}$. We now discuss the behaviour of $\delta_1 \equiv \delta$ as a function of the magnetic field.

In the absence of the BI term, $\delta \leq 1$; the magnetic field (even if it lays along two directions orthogonal to the electric field one, thus not affecting the value of the electric-like eigenvalue) would always decrease the PPR, as is the case for bosonic (Klein-Gordon) point-particles. Nevertheless, this $\sqrt{1 + (\frac{B}{M^2})^2}$, stringy, contribution triggers a qualitative change; the pair production can be enhanced by the magnetic field, although by a small factor, $\sqrt{2}$ at most per string end-point, if we do not allow the fields to be greater than the string tension. However, if we allow B to take any value (and we are allowed to do so, since in principle only the electric field should stay smaller than the string tension) the PPR increases indefinitely as $B \to \infty$. This enhancement of the PPR is somehow unexpected in a bosonic theory, and further understanding of it would be required. One path towards a better understanding of this phenomenon might pass through the Seiberg-Witten rescaling of the open string coupling constant [28], since their rescaling factor is just the BI term.

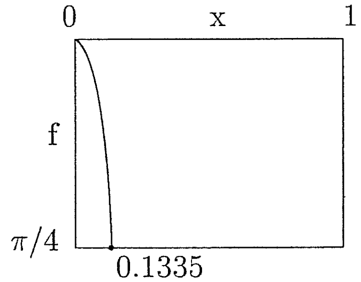

fig.1 Modification of the pair creation rate due to a magnetic field

In the figure above, the coordinate axes are $x = e^{-4\pi/t_p}$, $f = arctg(B)$, with $t_p = 2\pi/\epsilon$. The part of the figure at the left of the curve connecting $(0,0)$ and $(x \simeq 0.1335, f = \pi/4)$ corresponds to an increase of the PPR, with a maximum at $(x = 0, f = \pi/4)$, where an enhancement by a factor of $\sqrt{2}$ is obtained. On the above mentioned curve (and for $f = 0$) the PPR equals the pure electric field one, whereas on its right it is smaller. We included in the figure above only magnetic fields smaller or equal to the string tension. It is easy to see what happens if they overcome this value (which represents a barrier only for electric fields). We just note that, as $f \to \frac{\pi}{2}$, i.e. as $B \to \infty$, the PPR diverges! This is the reason for which we have not tried to extend the vertical axis of the above figure to values of f near $\pi/2$.

As detailed in the next section, a dissipative, non-relativistic, limit of string theory can be obtained [11, 13], by taking the velocity v of propagation of excitations along space-like coordinates to be much smaller than the velocity of light $c = 1$ along the time coordinate. In this case, the enhancement due to the BI term may become dramatic: it is of the form $\sqrt{1 + (\frac{B}{vM^2})^2}$, with $v \ll 1$.

Supersymmetric strings

In the case of supersymmetric strings the tachyon disappears from the free spectrum, hence also the exponential increase it would produce in equations (46, 68). For zero magnetic field the fermionic contribution was evaluated in [7] and contains the sum over the even spin structures of powers of Θ- and η-functions; in particular, it cancels the $e^{-k\epsilon}$ factor in (68). An additional magnetic field further corrects the contribution of each of the spin structures s by a factor

$$\delta_k^{(s)}(f) = \prod_{n>0} \frac{(1 + (-)^s 2x^n \cos 2f + x^{2n})}{(1 - (-)^s x^n)^2}, \quad s = 2, 3, 4, \tag{70}$$

with n integer for $s = 2$ and half-integer for $s = 3, 4$; x is evaluated at the k-th pole, as in (69). These corrections can increase the PPR, now independently of the BI term. This effect is just the fermionic string counterpart of what happens in the case of pointlike Dirac fermions. The BI factor remains unchanged and no further poles appear in the path integral. The PPR reads

$$\gamma_{superstring}(\epsilon, f) = \sum_{k=1}^{\infty} \gamma_k \delta_k(f) \times \sum_{s=2,3,4} [Z^{(s)}_{free}]^{D-2} \times \delta^{(s)}_k(i\epsilon) \times \delta^{(s)}_k(f), \tag{71}$$

where γ_k, δ_k, and $\delta^{(s)}_k$ are the ones from eqs.(68), (69), and (70), respectively. $Z^{(s)}_{free} = \sqrt{\Theta_s(t)/\eta(t)}$ is the contribution per uncoupled fermionic direction.

A temporary fix for the magnetic field induced instabilities

We have now to remember that if we stretch the string along one direction - in order to stabilize the theory - an additional factor $e^{-\frac{t_p}{2}M^2d^2} = e^{-\frac{\pi}{\epsilon}M^2d^2}$ appears. It corresponds to a mass term $\frac{1}{2}M^2d^2$, whose role is to compensate the possibly negative contribution [22, 4] of (cf. Eq. (31), written in string tension units):

$$\alpha' m^2 = -\frac{f^2}{2\pi^2} - \frac{f}{\pi}(n - 1/2) + (n-1) \geq -1, \tag{72}$$

$n = a_1^\dagger a_1$ being the number operator for the string modes lying on the first Regge trajectory. Thus, it is enough to take $\frac{1}{2}M^2d^2 = 1$; this provides a damping factor for the PPR of the form $e^{-\frac{\pi}{\epsilon}}$. Nevertheless, for big electric fields ($E \to 1$, or $\epsilon \to \infty$) this does not influence much the PPR, and the previous conclusions hold.

(Of course, our way to 'cure' the instability is provisional, as far as the stability of the system composed of two parallel branes is not discussed.)

Starting to uncover the spectrum from the path integral

One final remark is in order. Using eqs. (62) and (46) one sees that in the critical dimension $D = 26$ and in the limit $t \to \infty$ the leading term in the partition function is of the form $e^{\pi t(\frac{f^2}{2\pi^2} - \frac{f}{2\pi} + 1)}$, modulo a prefactor. The coefficient of $(-t)$ in the exponent gives precisely the lower tachyonic mass ($n = 0$) in equation (72), as it should. This check confirms that the path integral encodes - at least in principle - all the information about the spectrum.

One could continue and obtain the mass of the next state, and so on, and one could even hope to get the whole spectrum of the string in external electromagnetic fields from the path integral, in the same way in which one obtains the harmonic oscillator spectrum from its partition function. This would also provide information on the string wave functional. We have not pursued this issue any further.

The spectrum in electric fields is obtained by replacing f by $i\epsilon$. This makes m^2 not only become negative, but also aquire an imaginary part. We notice that the transition from magnetic to electric fields is again mediated by a simple imaginary factor, which however affects now the inverse tangent of the field strength value (for open strings this seems to be the relevant quantity, not the field strength itself).

We end with two speculative remarks: first, one might imagine those stretched strings to lie between two brane universes, maybe playing a cosmological role - perhaps being a kind of dark matter. Second, the calculations presented here could have some relevance for QCD mesons, or for vortices in superconductors.

4 Dissipative limit

In this section we will investigate another possible use of open strings, namely in the description of quantum dissipative dynamics. This will be implemented by slightly generalizing the standard string theory action, allowing the string tension to depend on the space-time and world-sheet direction. We will consider strings with only one end carrying electric charge. Then, once the tail of oscillators is integrated out, the charged end-point will exhibit a kind of dissipative dynamics. We explain in detail the meaning of the above in the first subsection. Next, we calculate the pair production rate for such strings, and see it is similar to the one calculated for point particles endowed with dissipative dynamics.

4.1 Nonrelativistic limit and quantum dissipation

We wish now to explain in more detail which situations we are going to adress, and to see how both relativistic and dissipative nonrelativistic dynamics arise from our formalism. We take all the space directions to be on equal footing and the electric field along X_1. The action, now restricted to the 0-1 plane, is:

$$S = -\frac{M^2}{2}\int_0^t d\tau \int_0^l d\sigma[(\frac{\partial X^0}{\partial\tau})^2 + (\frac{\partial X^0}{\partial\sigma})^2] + \frac{M'^2}{2}\int_0^t d\tau \int_0^l d\sigma[(\frac{\partial\vec{X}}{\partial\tau})^2 + v^2(\frac{\partial\vec{X}}{\partial\sigma})^2]$$
$$-iE\int_0^t d\tau \left[X_0\frac{\partial X_1}{\partial\tau}\right]_{\sigma=l} . \qquad (73)$$

For $v = 1$ and $M^2 = M'^2$, eq (73) is the standard way of writing the worldsheet action in string theory. But we have in mind possible applications to dissipative quantum dynamics in nonrelativistic situations, thus we consider also $v < 1$ and, for generality, $M^2 \neq M'^2$ (although the precise relation between M^2 and M'^2 will not be needed in the following). The dimensionless, velocity-like, parameter v is the main new element of our approach. It will unable us to recover standard, relativistic, string theory results for $v = 1$, as well as a nonrelativistic, dissipative, type of dynamics for $v \ll 1$.

We stress that not the difference between M^2 and M'^2 is essential in our approach (this could have been done also for point-particles, taking an anisotropic mass - a mass depending on the space direction; by rescaling appropriately the 'fields' X_i one could then restore isotropy). Instead, the important feature is the possibility to have a parameter $v \neq 1$, which is specific to a string theory: we use the additional space-like extension σ of the string world-sheet to introduce this new element into play.

Once we have introduced the action in the form (73), we make some changes of variables which will help us either to understand better the situation or to write more explicit formulae.

We rescale the world-sheet coordinates by $\tau \to M^2 l\tau, \sigma \to \sigma l$, so that the limits of integration become:

$$T = \frac{t}{M^2 l} \qquad \Delta\sigma = 1. \tag{74}$$

T gets the dimension of a length to the square, and allows the explicit connection with Schwinger's 'proper time' formalism. The case of quantum dissipation for nonrelativistic situations corresponds to the limit:

$$M^2 T >> 1 >> M^2 vT. \tag{75}$$

We see what that means: the oscillations of X_0 along σ are much suppresed compared to those along T. For $\vec{X}$ the reverse happens. So time (X_0) is "rigid", but space ($\vec{X}$) not at all. We will compute the decay rate in these limits, while keeping $M'^2 v$ fixed. M^2 will set the scale.

We now specialize to the (2+1)-dimensional case, that is $\vec{X} = (X_1, X_2, X_3)$, where $X_{1,2}$ describe the physical transverse oscillations of the string, whereas we take $X_3 = \sigma b$, with b the intermembrane distance. Thus, X_3 contributes to the action with the term:

$$\int_0^T d\tau \int_0^1 d\sigma \frac{1}{2} M^2 M'^2 v^2 (\frac{\partial b\sigma}{\partial\sigma})^2 = \frac{1}{2} T M^2 M'^2 v^2 b^2. \tag{76}$$

The partition function will contain as a factor the exponential of minus this term, and (76) will play a role similar to the square rest mass of a string. Thus, we identify

$$\mathcal{E}_0 = \sqrt{\frac{1}{2} M^2 M'^2 v^2 b^2} \tag{77}$$

as being the rest energy of a string.

The physics of the situation is quite transparent, and shows clearly why this limit has to be called 'nonrelativistic'. We have already seen that in the free action for X_0, the oscillations along τ (the 'kinetic term') dominate over those along σ. Along the spatial directions the opposite happens. This is due just to $M^2 >> \frac{1}{T} \sim \mathcal{E}_0^2$ and is similar to what happens in string theory when $\frac{1}{\alpha'} \sim \frac{1}{l_s^2} >> \mathcal{E}^2$, where $\mathcal{E}$ is the available energy. Then, the stringy massive modes are frozen, in the same way in which the oscillations of X_0 along σ are frozen in our problem. Now, since τ has the dimensionality of time to the square, and the energy scale of our problem is $\mathcal{E}_0$, the natural time scale will be $time = \mathcal{E}_0 \tau$. In consequence, we have

$$d(time) = \mathcal{E}_0 d\tau \qquad d(X_3) = b d\sigma. \tag{78}$$

Using the previous equation in the free action, we obtain the velocities with which X_0 and $X_{1,2}$ signals propagate, respectively:

$$v_{prop}^2(X_0) = \frac{M^2}{M'^2} \frac{1}{v^2} \qquad v_{prop}^2(X_1) = \frac{M^2}{M'^2}. \tag{79}$$

To prove equation (79) we notice that *after* the rescaling (74), and using (77) and (78), the integrand of the first term in equation (73) becomes

$$S_0 = -\frac{1}{2}\left[(\frac{\partial X^0}{\partial \tau})^2 + M^4(\frac{\partial X^0}{\partial \sigma})^2\right] \sim \frac{\mathcal{E}^2}{2}\left[(\frac{\partial X^0}{\partial(time)})^2 + \frac{M^2}{M'^2}\frac{1}{v^2}(\frac{\partial X^0}{\partial(space)})^2\right] \tag{80}$$

Similarly, the second term in (73) reads now

$$S_1 = \frac{M^2M'^2}{2}\left[\frac{1}{M^4}(\frac{\partial \vec{X}}{\partial \tau})^2 + v^2(\frac{\partial \vec{X}}{\partial \sigma})^2\right] \sim \frac{\mathcal{E}^2}{2}\left[\frac{M'^2}{M^2}(\frac{\partial \vec{X}}{\partial(time)})^2 + v^2(\frac{\partial \vec{X}}{\partial(space)})^2\right] \tag{81}$$

From the above two equations we can read the velocity of propagation in real space and time of the excitations along X_0 and X_1, and see that they are given by (79).

Furthermore, for our system the length scale is given by b, whereas a typical time scale is given by $\mathcal{E}_0^{-1}$. Thus a typical speed is of the order of

$$v_{typical} = b\mathcal{E}_0. \tag{82}$$

It is now easy to see that the relation between these three velocities, thanks to the limit (75), is the following:

$$v_{prop}(X_0) >> v_{typical} >> v_{prop}(X_{1,2}). \tag{83}$$

The first inequality means that the propagation of time excitations is practically instantaneous (the time is Galilean) - a nonrelativistic situation. The second one says that the propagation of excitations of space-like coordinates is very slow compared to the typical velocity - or that strings are seen as very long. This corresponds to pure dissipative dynamics, since in this limit one obtains the Caldeira-Leggett action ([10, 11]), as we now explain.

We wish first to see qualitatively why this nonrelativistic limit is related to the Caldeira-Leggett type quantum dissipative dynamics. In [9], dissipative dynamics was obtained by integrating out a thermal bath made of oscillators, on which a spectral condition has been imposed. The way we can reobtain a 'termal bath' is very simple: we just rewrite $X(\sigma,\tau)$ as $X_\sigma(\tau)$. Then, we see that only $X_{1,2}$ depend on σ, so the integration over a thermal bath is replaced here by path integrating over σ. The finite spatial extension of the strings amounts to an infinity of harmonic oscillators which can form a suitable bath. On the other hand, X_0 is independent (in the nonrelativistic limit!) of σ, so the time is singled out as a good coordinate for a pointlike object from the beginning. This is what we need, since we want to have the same time coordinate along the string to obtain nonrelativistic quantum mechanics, whereas integrating along the spatial extension will provide us with a dissipative dynamics for the point particle which remains after [10, 11]. In CL language, time is already a macroscopical cordinate, whereas the space-like coordinates of the string's charged end are not; they can be made so only if accompanied by a dissipative term. What is remarkable is the fact that the CL 'spectral condition' is not needed. The string seems to *automatically* provide such a constraint. Of course, *some* constraint is to be expected because all the dynamics is encoded in a continuum Lagrangian with fewer parameters than the 'many-body' CL one (for various approaches, see [10, 11, 9]).

For completeness, we now present a calculation [11] which shows explicitely the way the Caldeira-Legget description of dissipative dynamics can be simulated by open strings.

We begin by writing, quite generally, the (Euclidean) action for a free open string

$$\mathcal{S}_E = \int_0^t d\tau \int_0^1 d\sigma \left(\frac{\rho}{2} \left(\frac{\partial X}{\partial \tau} \right)^2 + \frac{\nu}{2} \left(\frac{\partial X}{\partial \sigma} \right)^2 \right). \tag{84}$$

$X = X_\mu(\tau, \sigma)$ are the space-time coordinates of the string, whereas τ plays again the role of the Schwinger (Euclidean) proper time parameter and σ parametrizes the extension of the string. The boundary conditions are again periodic in τ and Neumann in σ: $\partial_\sigma X|_{\sigma=0} = \partial_\sigma X|_{\sigma=1} = 0$. The manifold spanned by the string during its motion in τ is thus an annulus. ρ plays the role of the mass density and ν that of an elastic constant.

We now derive the dynamics of the end point of this string, say at $\sigma = 0$, denoting $q(t) \equiv X(\tau, \sigma = 0)$, by deriving the end point's effective action, $S(q)$ (which is going to be of Caldeira-Leggett type [9]). We begin by Fourier analysing in σ

$$X(\tau, \sigma) = \sum_{k=0}^{\infty} X_k(\tau) \cos(k\pi\sigma), \tag{85}$$

and further Fourier analyse in τ: $X_k(\tau) = \sum_{n=-\infty}^{\infty} X_{kn} e^{i\frac{2\pi n}{t}\tau}$. The Euclidean action is now

$$\mathcal{S}_E = \frac{\pi^2}{4} t\nu \sum_{k=0} k^2 X_{k0}^2 + \frac{\pi^2}{2} t \sum_{k=0} \sum_{n=1} \left(\rho \left(\frac{2n}{t} \right)^2 + k^2 \nu \right) |X_{kn}|^2, \tag{86}$$

and the effective end-point action is obtained by tracing out all string degrees of freedom, except those for the end point. We thus compute the constrained functional integral

$$e^{-S(q)} = \mathcal{N} \int \prod_{k=0} \left\{ dX_{k0} \prod_{k=1} d^2 X_{kn} \right\} e^{-\mathcal{S}_E} \, \delta(\sum_{k=0} X_{k0} - q_0) \prod_{n=1} \delta(\sum_{k=0} X_{kn} - q_n), \tag{87}$$

having introduced $q(t) = \sum_{n=-\infty}^{+\infty} q_n e^{i\frac{2\pi n}{t}\tau}$. Now for each n we evaluate the constrained functional integral, by replacing $X_{0n} = q_n - \sum_{k=1} X_{kn}$. We have

$$e^{-S(q_n)} = e^{-\frac{1}{2}\rho t \left(\frac{2\pi n}{t}\right)^2 |q_n|^2} \mathcal{N} \int \prod_{k=1} d^2 X_{kn} \exp\{-X_{kn}^* B_{kl} X_{ln} - X_{kn}^* V_k - V_k^* X_{kn}\}, \tag{88}$$

where we have introduced the matrix $(k,l=1,2,\ldots,\infty)$

$$B_{kl}=a(k)\delta_{kl}+a(0) \tag{89}$$

and the vector $V_k=a(0)q_n$, which is actually independent of k. Here

$$a(k)=\frac{1}{2}t\left(\rho\left(\frac{2\pi}{t}\right)^2+\pi^2k^2\nu\right). \tag{90}$$

Completing the square, we find

$$e^{-S(q_n)}=e^{-\frac{1}{2}\rho t\left(\frac{2\pi}{t}\right)^2|q_n|^2}\exp\{V_k^*B_{kl}^{-1}V_l\}. \tag{91}$$

One can check that the inverse matrix is

$$B_{kl}^{-1}=\frac{1}{a(k)}\delta_{kl}-\frac{a(0)}{a(k)Qa(l)}, \tag{92}$$

where we have set $Q=\sum_{k=0}\frac{a(0)}{a(k)}$. We get, therefore

$$S(q_n)=\frac{1}{\sum_{k=0}1/a(k)}|q_n|^2. \tag{93}$$

Going now to the continuum limit in σ, which corresponds to the case where the inner circular border of the annulus shrinks to zero and mathematically to the limit $\nu{\to}0$ (but keeping $\rho\nu$ finite), we replace the sum over k with an integral, to get (regardless of the sign of n)

$$S(q_n)=\frac{\nu t/2}{\int_0^\infty dx/\left(\frac{\rho}{\nu}\left(\frac{2\pi n}{t}\right)^2+x^2\right)}|q_n|^2=\eta 2\pi|n||q_n|^2. \tag{94}$$

(The general result, for arbitrary ρ and ν, is $S(q_n)=\eta 2\pi th(\sqrt{\frac{\rho}{\nu}}2\pi n/t)|q_n|^2$, but we will not need it here). $\eta=\sqrt{\rho\nu}$ is now to be identified with the friction coefficient of the CL formulation of quantum dissipation [9]. Interestingly, we obtain pure dissipative dynamics for the end-point. By Fourier transformation of this expression we get the standard form for the periodic case

$$S(q)=\frac{\eta\pi}{4t^2}\int_0^t d\tau\int_0^t d\tau'\left(\frac{q(\tau)-q(\tau')}{\sin[\pi(\tau-\tau')/t]}\right)^2. \tag{95}$$

4.2 Path integral evaluation

4.2.1 Free case ($E = 0$)

The evaluation of the path integral for one decoupled coordinate X, whose action reads

$$S = \int_0^t d\tau \int_0^l d\sigma [\frac{M^2}{2}(\frac{\partial X}{\partial \tau})^2 + \frac{M^2 v^2}{2}(\frac{\partial X}{\partial \sigma})^2]. \tag{96}$$

follows along the same lines as the one for the usual relativistic string. The result is

$$\int DXe^{-S} = \sqrt{\frac{M^2 l}{2\pi t}} e^{-\pi \frac{tv}{2l} \sum_{k>0} k} \prod_{k>0} \frac{1}{1 - e^{-2\pi k \frac{tv}{2l}}}. \tag{97}$$

Now we make use of the transformation properties of the Dedekind eta function

$$\eta(x) = e^{i\pi \frac{x}{12}} \prod_1^\infty (1 - e^{2\pi i n x}) = \frac{1}{\sqrt{-ix}} \eta(-\frac{1}{x}) \tag{98}$$

and get at the end:

$$\int DXe^{-S} = \sqrt{\frac{M^2 v}{4\pi}} e^{\frac{\pi}{6} \frac{l}{vt}} \prod_{n>1} \frac{1}{1 - e^{-4\pi n \frac{l}{vt}}}. \tag{99}$$

The exponential term $e^{\frac{\pi}{6} \frac{l}{vt}}$ is important. It amounts to the normal ordering of the Hamiltonian in the boundary state formalism, and it gives a contribution which we can interpret in a thermodynamical context.

This is so because, if we make the identification $1/t = \beta^{-1}$, were β^{-1} means temperature now (τ has been already taken to be Euclidean), we can interpret the path integral as a partition function of a system in thermal equilibrium:

$$\int DXe^{-S} = Z_0 = \sum_n e^{-\beta E_n} = e^{-\beta F} = e^{-W}.$$

In the limit $t \to 0$, the specific heat of this system [26] is then given by ($U = \frac{\partial W}{\partial \beta}$):

$$C = \left(\frac{\partial U}{\partial \beta^{-1}}\right)_{V=cst} = \frac{\pi}{3} \frac{l}{v} \beta^{-1},$$

which is positive (in our case $W = -\frac{\pi}{12} \frac{2l}{vt}$) and raises linearly with the temperature β^{-1}. This indicates that

$$c = \frac{\partial C}{\partial \beta^{-1}} = \frac{\pi}{3} \frac{l}{v} \tag{100}$$

is a temperature independent physical parameter, which could in principle be measured. It will appear in our result for string pair production in the nonrelativistic case, when the string could be identified with a vortex line.

We may say that a system with an infinity of degrees of freedom, like a string, can be characterized by a peculiar specific heat in a thermodynamical context. Thus, our final result is

$$\int DXe^{-S} = \sqrt{\frac{M^2 v}{4\pi}} e^{\frac{1}{2t}c} \prod_{n \geq 1} \frac{1}{1 - e^{-4\pi n \frac{l}{vt}}}. \tag{101}$$

4.2.2 Interacting case ($E \neq 0$)

We now evaluate the path integral for the interacting case, for the case of a string charged only at one end (such that integrating out the tail one could get dissipative dynamics for the end-point).

The interaction term is quadratic, so that we could hope to eliminate it by a suitable linear transformation of X_0 and X_1. This does not work directly, since our interaction is a boundary term, not a bulk term.

Our strategy will be to decompose in Fourier modes, and evaluate the determinant in momentum space. All these steps are identical to the ones taken in order to evaluate the path integral in the previous section, although they are slightly more coplicated algebraically. We thus reproduce here only the result, and will provide subsequently another derivation via the boundary state formalism. The path integral

$$Z = \int DX_0 DX_1 e^{-[\int_0^t d\tau \int_0^l d\sigma [-\frac{M^2}{2}(\frac{\partial X^0}{\partial \tau})^2 - \frac{M^2}{2}(\frac{\partial X^0}{\partial \sigma})^2 + \frac{M'^2}{2}(\frac{\partial X^1}{\partial \tau})^2 + \frac{M'^2 v^2}{2}(\frac{\partial X^1}{\partial \sigma})^2] - i \int_0^t d\tau [E X^0 \frac{\partial X^1}{\partial \tau}]_{\sigma = l}]}$$

can be thus evaluated with the result

$$\begin{aligned} Z &= (Z_{free}) \textstyle\prod_{n \geq 1} \left(1 - \frac{E^2}{M^2 M'^2 v} cth(2\pi n \frac{l}{t}) cth(2\pi n \frac{l}{vt})\right)^{-1} \\ &= [1 - \tfrac{E^2}{M^2 M'^2 v}]^{\frac{1}{2}} \sqrt{\tfrac{M^2 M'^2 v}{(4\pi)^2}} e^{\frac{\pi}{6}\frac{l}{t}(1+\frac{1}{v})} \\ &\times \textstyle\prod_{n \geq 1} \frac{1}{[1 + e^{-4\pi n \frac{l}{t}(1+\frac{1}{v})} - \frac{1+\frac{E^2}{M^2 M'^2 v}}{1-\frac{E^2}{M^2 M'^2 v}}(e^{-4\pi n \frac{l}{t}} + e^{-4\pi n \frac{l}{vt}})]}. \end{aligned} \tag{102}$$

We have ζ-function regularised the divergent sum $\sum_{k>0} 1 = lim_{s \to 0} \zeta(s) = -\frac{1}{2}$.

4.3 Boundary state formalism evaluation

We are going to repeat the former computation, this time through the boundary state formalism (BSF), that is, implementing in an operatorial way the boundary conditions at the string's end-points in presence of the electric field. A review is presented in Appendix B. The result eq.(225) can be written as:

$$\int DX^0(\sigma,t)DX^1(\sigma,t)e^{-S} = N(E) < B_E(l)|B_0 >= N(E) < B_E|e^{-lH}|B_0 >, \quad (103)$$

where $N(E)$ is a normalization factor to be obtained later by comparing with the path integral result (102). The boundary state $|B_E >$ is obtained in eq.(224) of Appendix B. Using the notation

$$\frac{E}{M^2} = A \qquad 2\pi\frac{l}{t} = l_0$$
$$\frac{E}{M'^2 v} = B \qquad 2\pi\frac{l}{vt} = l_1$$

to simplify the writing, we have:

$$< B_E(l)|B_0(0) >=< 0|exp\sum_{n\geq 1}[\frac{AB+1}{AB-1}(X^1_n\tilde{X}^1_n e^{-2nl_1} - X^0_n\tilde{X}^0_n e^{-2nl_0})$$
$$+\frac{2}{AB-1}(AX^0_n\tilde{X}^1_n - BX^1_n\tilde{X}^0_n)e^{-nl_0}e^{-nl_1}]exp\sum_{n\geq 1}[X^0_{-n}\tilde{X}^1_{-n} - X^1_{-n}\tilde{X}^0_{-n}]|0 > \quad (104)$$

where l plays now the role of 'time'. The former expression is thus given by an infinite product of terms of the form

$$< 0|e^{\kappa b\tilde{b}}e^{\lambda a\tilde{a}}e^{\mu a\tilde{b}}e^{\nu b\tilde{a}}e^{\rho a^\dagger\tilde{a}^\dagger}e^{\sigma b^\dagger\tilde{b}^\dagger}|0 >,$$

the operators $a, a^\dagger$ and $b, b^\dagger$ satisfying the usual harmonic oscillator commutation relations. Using the formula (proved in Appendix C)

$$< 0|e^{\kappa b\tilde{b}}e^{\lambda a\tilde{a}}e^{\mu a\tilde{b}}e^{\nu b\tilde{a}}e^{\rho a^\dagger\tilde{a}^\dagger}e^{\sigma b^\dagger\tilde{b}^\dagger}|0 >= \frac{1}{(1-\kappa\sigma)(1-\lambda\rho) - \mu\nu\rho\sigma} \quad (105)$$

and substituting back , we get

$$Z = N(E) < B_E(l)|B_0 >= N(E)e^{\frac{\pi}{12}\frac{2l}{t}}e^{\frac{\pi}{12}\frac{2l}{vt}}$$
$$\times\prod_{n>0}\frac{1}{1+e^{-4\pi n\frac{l}{t}(1+\frac{1}{v})} - \frac{1+\frac{E^2}{M^2M'^2v}}{1-\frac{E^2}{M^2M'^2v}}(e^{-4\pi n\frac{l}{t}} + e^{-4\pi n\frac{l}{vt}})}. \quad (106)$$

In order to reproduce the path integral result (102) we have to take $N(E) = N_0\sqrt{1-\frac{E^2}{M^2M'^2v}}$, $N_0 = \frac{\sqrt{M^2M'^2v}}{4\pi}$. The term $\sqrt{1-\frac{E^2}{M^2M'^2v}}$ is going to be just the Born-Infeld action, specialized to our case.

It is interesting to point out the origin of the various factors. If we have a closer look at the free string computation , we can see that $\frac{(M^2M'^2v)^{\frac{1}{2}}}{4\pi}$ comes from the zero modes along σ, i.e. it is related to the pointlike component of our object, not to its stringy excitations. Also, it is the part of the free string partition function *not* cancelled by the ghosts. The exponential term has already been interpreted as a specific heat contribution, characterizing the way energy is distributed among the string modes (degrees of freedom). Finally, $\sqrt{1-\frac{E^2}{M^2M'^2v}}$ is a boundary term (the electric field acts on the world sheet boundary, not on the bulk), and it has a long story. Its form gives rise - for the relativistic string - to the Born-Infeld action. That factor has already been obtained in a variety of ways, for instance doing the path integral (after integrating out the bulk) in configuration space [3], or through operatorial methods, developing in a Fourier basis adapted to the form of the interaction [4].

At this point we can, as in the previous section, take into account additional magnetic fields by using (102), with the appropriate substitutions. As remarked in the previous section, the BI factor corresponding to a magnetic eigenvalue B would then be $\sqrt{1+\frac{B^2}{v^2T^2}}$, which leads (in the limit (75)) to a dramatic increase of the pair creation rate.

4.4 Poles

4.4.1 General pole equation and particular limits

From now on, we switch to the notation of the rescaling (74). That simply amounts to set $l = 1$ and replace $t = M^2T$. First, let us consider M^2, M'^2, v arbitrary. In general, we see (102) that we have poles in T whenever

$$\frac{E^2}{M^2M'^2v} = th2\pi n\frac{1}{M^2T}\cdot th2\pi n\frac{1}{vM^2T}. \tag{107}$$

This is the general case from which, taking various limits, we obtain different physical situations.

1) We can look at the case of the relativistic string $M^2 = M'^2$, $v = 1$. Poles are located, for a given n, at:

$$\frac{E}{M^2} = th2\pi n\frac{1}{M^2T}, \tag{108}$$

as in equation (61). This is the situation studied in [7], and in the previous section. For either small E or small $\frac{1}{M^2T}$ we get

$$T = \frac{2\pi n}{E}. \tag{109}$$

2) A second possibility is $\frac{1}{M^2T} >> 1, \frac{1}{vM^2T} >> 1$, either for the relativistic invariant case or in general. In that case the pole is independent of T, and is located at the 'critical' value of E:

$$\frac{E^2}{M^2M'^2v} = 1. \tag{110}$$

That means that in this limit pair creation can not take place, unless the string breaks already at the classical level.

3) If we take the limit $\frac{1}{vM^2T} >> 1$ and either $\frac{1}{M^2T} << 1$ or E small, we get

$$T = \frac{2\pi n M'^2 v}{E^2}. \tag{111}$$

We will see that this case corresponds to a nonrelativistic string, from which dissipative point particle quantum mechanics can be obtained by integrating out the string degrees of freedom [11]. We notice that the position of the pole goes now like E^{-2}, not like E^{-1}. We will have more to say about this situation later.

4.4.2 Another method

We note that it is possible to obtain the expression for the poles also by solving the Euclidean equations of motion, subject to the boundary conditions established in Appendix B. Our expression for the poles (107) is just the consistency condition needed in order for the boundary conditions at $\sigma = 0, 1$ to be satisfied by both X_0 and X_1, as we show below:

Using the bulk equations of motion

$$\frac{\partial^2 X^0_n}{\partial\sigma^2} = \omega_n^2 X^0_n \qquad \frac{\partial^2 X^1_n}{\partial\sigma^2} = \frac{1}{v^2}\omega_n^2 X^1_n$$

we obtain ($\omega_n = \frac{2\pi n}{M^2T}$) the general form of the solution:

$$X^0_n(\sigma) = A_0 ch\omega_n\sigma + B_0 sh\omega_n\sigma \quad X^1_n(\sigma) = A_1 ch\frac{\omega_n}{v}\sigma + B_1 sh\frac{\omega_n}{v}\sigma.$$

Using $\frac{\partial X^0_n}{\partial\sigma}|_{\sigma=0} = \frac{\partial X^1_n}{\partial\sigma}|_{\sigma=0} = 0$ we get $B_0 = B_1 = 0$. Finally, the boundary conditions at $\sigma = l \equiv 1$ (see Appendix A) impose the restriction:

$$A_0\omega_n sh(\omega_n) = \frac{\omega_n E}{M^2} A_1 ch(\frac{\omega_n}{v})$$

$$A_1 \frac{\omega_1}{v} sh(\frac{\omega_n}{v}) = \frac{\omega_n E}{M'^2 v} A_0 ch(\omega_n).$$

For consistency then:

$$\frac{E^2}{M^2 M'^2 v} = th(2\pi n \frac{1}{M^2 T}) th(2\pi n \frac{1}{v M^2 T})$$

which is our former pole condition. In fact, the possibility of having a classical solution for a particular T for which the boundary conditions at both ends can be satisfied implies that the (Euclidean) action of the corresponding modes is zero. Thus the gaussian integration over those modes produces a singularity.

4.5 Comparison with previous results

We wish now to compare the result we have obtained with previous ones. First we remark that we can rewrite our partition function Z in the form:

$$\begin{aligned} Z &\sim \prod_{n=1}^{\infty} \frac{1}{\frac{E^2}{M^2 M'^2 v} - th 2\pi n \frac{1}{M^2 T} th 2\pi n \frac{1}{v M^2 T}} \\ &= \prod_{n=1}^{\infty} det \begin{vmatrix} th(2\pi n \frac{1}{M^2 T}) & -\frac{E}{M^2} \\ \frac{E}{M'^2 v} & -th(2\pi n \frac{1}{v M^2 T}) \end{vmatrix}^{-1} \\ &= \prod_{n=1}^{\infty} det \left(g^{\mu\nu} th(2\pi n \frac{1}{v_\mu M_\mu^2 T}) - \frac{F_{\mu\nu}}{M_\mu^2 v_\mu} \right)^{-1}, \end{aligned}$$

where M_μ^2, v_μ are the string tension and velocity parameter for the coordinate X_μ. This is similar to the discussion at the end of subsection 3.2.2, and to equation (102). There we saw that the BI action emerges from expressions like the above one, through the interplay with free oscillator modes. However, the equation above shows another way in which the Born-Infeld action appears directly: namely when the hyperbolic tangent above goes to one (which is the second limit in section 4.5). Of course, we have proved the equality in the last line only for one electric field F_{01} along the X_1 direction. Nevertheless, the formula can be extended to a general constant $F_{\mu\nu}$, the generalization requiring to put an antisymmetric matrix into its block diagonal form. This way of presentation is appropriate for studying various particular cases.

We made our calculations on a cylinder, of circumference t and length l. Through a conformal mapping, it can be transformed into an annulus, on which some of the previous calculations have been done. We remark that the limit $l \to \infty$ (the present $M^2T \to 0$) corresponds to the shrinking of the interior circle of the annulus to zero radius, thus obtaining a disk (with a puncture).

4.5.1 Relativistic string

If we take a relativistic string, $M^2 = M'^2$ and $v = 1$, and keep M^2T finite, our results should be the same as the ones of [7]. To check this, we rewrite our partition function Z (along the directions X_0 and X_1) in terms of η and Θ functions:

$$Z = -\frac{i}{2\pi}E\frac{\eta(\tau = 2i\frac{1}{M^2T})}{\Theta_1(u = -\frac{i}{2\pi}ln(\frac{1+\frac{E}{M^2}}{1-\frac{E}{M^2}})|\tau)}. \tag{112}$$

To prove that, put $e^{2\pi iu} = e^w = \frac{1+\frac{E}{M^2}}{1-\frac{E}{M^2}}$ and $q = e^{-4\pi\frac{1}{M^2T}} = e^{2\pi i\tau}$. Remembering that

$$\Theta_1(u|\tau) = 2q^{\frac{1}{8}}\sin\pi u\prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi iu}q^n)(1-e^{-2\pi iu}q^n)$$

we can rewrite our two-dimensional partition function as:

$$Z = \sqrt{1-\frac{E^2}{M^4}}\frac{M^2}{4\pi}2\sin\pi u\frac{\eta(\tau = 2i\frac{1}{M^2T})}{\Theta_1(u = -\frac{i}{2\pi}ln(\frac{1+\frac{E}{M^2}}{1-\frac{E}{M^2}})|\tau)}.$$

Using the relationship between u and $\frac{E^2}{M^4}$ and the fact that $\sin\pi u = -i\sinh\frac{w}{2}$ we obtain (112).

The free one-dimensional partition function (along a space-like direction, X_2 say) becomes

$$Z_2 = \sqrt{\frac{M'^2v}{4\pi}}\eta(\frac{2i}{M^2T}).$$

Now, if we take into account the other 24 free dimension and the ghosts, and also use the way η and Θ functions behave under modular transformations, we obtain the full 26-dimensional bosonic partition function

$$Z_{26} = -\frac{i}{2\pi}E\frac{M^{26}}{(4\pi)^{13}}\eta^{-21}(\frac{2i}{M^2T})\Theta_1^{-1}(u|\frac{2i}{M^2T}). \tag{113}$$

This is exactly the Bachas and Porrati amplitude for the case of an open bosonic string with a non-zero charge only at one end. Although a calculation involving superstrings would give rise to additional factors in the amplitude, it does not change the pole structure [7], which is in fact given by eq.(112). For this reason we restrict ourselves to the bosonic case. As discussed in ref [7], this amplitude has poles in T, due to the zeroes of the θ-function Θ_1. These poles induce the imaginary part in the r.h.s. of eq.(3) and thus the vacuum decay rate.

4.5.2 Born-Infeld action

Let us take the relativistic string case again, but now supplement it with the limit $M^2T \to 0$ ($l \to \infty$) - as we said, the annulus is shrinking to a disk. Then, since $th(\infty) = 1$, we see that our expression for Z reduces to the Born-Infeld action. In this way we reobtain the result of [3]. In this case there is no pair production, at least for $E < M^2$. We remark that in our case the string has a charge only at one end, whereas in [3] it had equal and opposite charges at the two ends, both coupled to the electric field.

We notice that it has been also observed in ref [11] that the Born-Infeld action can be obtained by tracing out the string degrees of freedom in the relativistic case and the limit $l \to \infty$.

In this limit, we have already seen that there are no poles (except for $E \to E_{critical}$), hence no pair production. Indeed, in the limit $M^2T \to 0$ the 26-dimensional partition function (26) reduces to

$$\frac{M^{26}}{(4\pi)^{13}}(e^{4\pi\frac{1}{M^2T}})\sqrt{1-\frac{E^2}{M^4}}$$

which is (modulo a constant) a Born-Infeld action.

4.6 Vacuum Decay Amplitude in a Dissipative Context

We proceed now with the nonrelativistic case, where instead $v << 1$ will be a crucial condition. We put everything together to evaluate the decay rate , which is given by

$$\gamma = -2Im\int_0^\infty \frac{dT}{T}\int DX_0 DX_1 DX_2 e^{-S},$$

multiplied by the rest mass factor (cf. section 4.1) coming from the fact that our strings are stretched along X_3. We will work with the rescaled variable $T = \frac{t}{M^2}$. We obtain the vacuum transition amplitude (over unit space and time, since we have already substracted the zero mode - cf. eq. (38))

$$\gamma = -2Im\int_0^\infty \frac{dT}{T}\sqrt{\frac{M^2(M'^2v)^2}{(4\pi)^3}e^{\frac{\pi}{6}\frac{1}{M^2T}(1+\frac{2}{v})}e^{-\frac{1}{2}M^2M'^2v^2b^2T}}\sqrt{1-\frac{e^2E^2}{M^2M'^2v}}$$

$$\times\prod_{n>0}\frac{1}{[1-e^{-4\pi n\frac{1}{M^2vT}}][1+e^{-4\pi n\frac{1}{M^2T}(1+\frac{1}{v})}-\frac{1+\frac{E^2}{M^2M'^2v}}{1-\frac{E^2}{M^2M'^2v}}(e^{-4\pi n\frac{1}{M^2T}}+e^{-4\pi n\frac{1}{M^2vT}})]}. \tag{114}$$

In order to get the rate of pair production we will focus on the imaginary part - due to the presence of poles - of the former expression.

Taking first the limits $M^2T >> 1$ and $M^2vT << 1$ while keeping M^2 and M'^2v fixed we remain with :

$$\gamma = -2Im \int_0^\infty \frac{dT}{T} \sqrt{\frac{M^2(M'^2v)^2}{(4\pi)^3}} e^{\frac{\pi}{6}\frac{1}{M^2T}(1+\frac{2}{v})} e^{-\frac{1}{2}M^2M'^2v^2b^2T} \sqrt{1 - \frac{e^2E^2}{M^2M'^2v}}$$

$$\times \prod_{n>0} \frac{1}{[1 - \frac{1+\frac{E^2}{M^2M'^2v}}{1-\frac{E^2}{M^2M'^2v}}(e^{-4\pi n\frac{1}{M^2T}})]}. \tag{115}$$

One should stress the crucial role of the limit (75): in the infinite product, it formally makes the two coupled coordinates collapse into a single, peculiar, infinite product (with the dependence on the electric field built in); it thus makes the two sets of harmonic oscillators behave like a single, peculiar one. One might think about this new set as a single 'coordinate', with the electric field accounting somehow for a nonzero chemical potential. In brief, in presence of an electric field, (75) inextricably mixes the directions coupled by the field, giving rise to a new type of dynamics. This is what makes the above equation have the pole structure of (111). Also, the above limit is to be contrasted with the zero electric field limit, in which the two coupled coordinates separate into two independent infinite products.

We take into account only the first, dominant pole ($n = 1$) and evaluate the residue at this pole using the expansion:

$$\frac{1}{1 - e^{\frac{2E^2}{M^2M'^2v} - 4\pi\frac{1}{M^2T}}} \simeq \frac{1}{-\frac{2E^2}{M^2M'^2v} + 4\pi\frac{1}{M^2T}} = \frac{\frac{-M^2M'^2v}{2E^2}T}{T - 2\pi\frac{M'^2v}{E^2}}.$$

Thus we have a pole for $T = T_P = \frac{2\pi M'^2v}{E^2}$. Using now the identity $\frac{1}{x-i\epsilon} = P(\frac{1}{x}) + i\pi\delta(x)$ in order to obtain the imaginary part of the T-integral in equation (115), we get the vacuum decay rate:

$$\gamma = \frac{1}{8\pi}\frac{M^2(M'^2v)^{\frac{3}{2}}}{E}\sqrt{1 - \frac{E^2}{M^2M'^2v}} e^{\frac{E^2}{6M^2M'^2v^2}} e^{\frac{\pi^2}{12}\frac{M^2M'^2v}{E^2}} e^{-\pi M^2(M'^2)^2v^3\frac{b^2}{E^2}}. \tag{116}$$

We remark that we have used the transformation properties of the Dedekind η-function, equation (49).

We reinterpret the exponents in terms of physical quantities. The quantum dissipation coefficient is $\eta = M'^2v$ (see [9, 11]). The rest energy of the nucleated object is $\mathcal{E}_0 = \sqrt{\frac{1}{2}M^2M'^2v^2b^2}$ (see eq.(77)), thus $\pi M^2(M'^2)^2v^3\frac{b^2}{E^2} = \mathcal{E}_0^2 \cdot T_P$. Further, we write

$$\frac{\pi^2}{12}\frac{M^2M'^2v}{E^2} \equiv -\Delta\mathcal{E}^2 \cdot T_P$$

(with $\Delta\mathcal{E}^2 = -\frac{\pi M^2}{24}$), reabsorbing it into a redefinition of the rest energy: $\mathcal{E}^2 = \mathcal{E}_0^2 + \Delta\mathcal{E}^2$. Concerning the first term, we rewrite it in terms of the temperature derivative of the specific heat, $c = \frac{\pi}{3M^2 v}$ (see eq.(100)):

$$\frac{E^2}{6M^2 M'^2 v^2} = c\frac{E^2}{4\pi\eta}.$$

In physical applications, both $\mathcal{E}^2$ and c will be taken as physical parameters to be determined experimentally.

We note one further point : in order to fix the normalization of the electric field E we have to remember that in the Schwinger method the space-time trajectory of a particle is described by a path integral action S which includes a term $\frac{1}{4}\int_0^T d\tau(\frac{\partial X_0}{\partial\tau})^2$. Thus in eq.(116) we have to rescale X_0, which ultimately implies rescaling $E \rightarrow E\sqrt{2}$. Finally we get the decay rate :

$$\gamma = \frac{1}{8\sqrt{2}\pi}\frac{M^2(\eta)^{\frac{3}{2}}}{E} e^{\frac{E^2}{2\pi\eta}c} e^{-\pi\eta\frac{\mathcal{E}_0^2}{E^2}}. \tag{117}$$

We note that our result has the same form as equation (13) of the second reference in [25]. It is remarkable that while there a cut-off on the frequency for which dissipation occurs has been introduced by hand, here the way we calculate the path-integral starting with an underlying string theory takes care of everything.

It is also interesting to compare the last term of the result (117) to the exponential term in equation (13). The exponent of this suppresion term, in the dissipative case (117), includes the inverse square of the electric field $\frac{\mathcal{E}_0^2}{E^2}$, and not its inverse $\frac{m^2}{E}$, as in the non-dissipative regime (13).

We now discuss the conditions in which a production rate should be observable. For that, we would need $T\mathcal{E}^2 \sim 1$. Due to our assumption $M^2 T >> 1$ (with $T = T_P \sim \frac{\eta}{E^2}$) we get the condition

$$\mathcal{E}^2 \sim \frac{E^2}{\eta} << M^2. \tag{118}$$

Now, using the fact that $\frac{E^2}{M^2\eta} << 1 << \frac{E^2}{M^2\eta v}$ - which follows from the pole equation and the nonrelativistic limit - we end up with:

$$v << \frac{\mathcal{E}^2}{M^2} << 1. \tag{119}$$

Other relationships are possible among our parameters. For instance the relation $M^2 vT << 1 << M^2 T$ could make us infer that $M^2 T\sqrt{v} \sim 1$ which is a reasonable assumption. These constraints can be further refined if we assume a definite relationship between M^2 and M'^2 ($M^2 \sim M'^2$, or $M^2\sqrt{v} \sim M'^2$, for instance). However, we prefer not to add any further assumption for the time being. We just stress again that in order for the pair creation of vortices in a thin superconductor to be observable we need (118) to be satisfied. We postpone further elaborations for the time when experimental results will be in sight.

5 D0-branes in gravitational wave background

5.1 Overview

We now switch to gravitational backgrounds, instead of the electromagnetic ones we investigated up to now. Namely, we will study the interaction of two D0-branes (D-particles) in the space-time of a gravitational (shock) wave. We first do the calculation in string theory and find that, at large distances from the shock-wave source, the $O(v^4)$ term in the amplitude (v is the relative velocity of the two D-particles) is an α'-independent function of the interbrane separation b. The amplitude is therefore that of supergravity - for large b, only closed-string massless modes contribute. We then show how the same result is obtained in the matrix model (at small b) by setting up the formulation of the dimensionally reduced super Yang-Mills theory in the curved background of the shock wave. This will also provide a nontrivial check of the matrix model conjecture [32], in a curved background. In the rest of this introduction we discuss in more detail the two approaches.

Within the closed-string theory analysis, we take the graviton to be the source for the gravitational wave metric, since it is the simplest massless particle of this theory. We then write the amplitude for the scattering of two D-particles and two gravitons describing the incoming and outgoing massless source. The process in which we are interested emerges as a *pinching* limit of the full amplitude, in which the two graviton vertex operators collide in the world-sheet, representing particles with parallel momenta. The full amplitude in this pinching limit describes the interaction of two D-branes with themselves and with the shock-wave.

In this computational framework, the D-particles are special boundary states at the ends of a cylindrical world-sheet, and the string amplitude is obtained by computing the correlator of the two graviton vertices on the cylinder (in ten dimensional uncompactified space-time). The D-particles are treated in the eikonal approximation. However the computation of the amplitude is not quite conventional because of the peculiarities of the three-body kinematics. After summing over the spin-structures we look at the expansion in the relative velocity v of the D-particles.

The first result is that, up to the fourth order in v, the leading singularities in the momentum transfer from the source-gravitons are a function of the interbrane distance b that is the same for arbitrary b and therefore independent from α', see eq. (155). This result generalizes well-known properties of brane interaction to the case of a non-trivial background. The leading singularities in the momentum transfer correspond to the leading powers of the transverse distance r_{T} from the shock-wave source. In particular, the fourth order term in v gets contribution from the leading behavior of the shock-wave gravitational field, that is $O({r_{\mathrm{T}}}^{-6})$, whereas the second order term is sub-leading and $O({r_{\mathrm{T}}}^{-8})$. We will focus on the fourth order term.

Of course, the long interbrane-distance regime is dominated by the massless modes of the closed string and is therefore the supergravity (and M theory) result.

On the other hand, we can consider the same process in the short interbrane-distance regime where we expect the theory to be described by the dimensionally reduced $d = 10$ super Yang-Mills theory corresponding to the exchange of massless open-string states. The gauge group is $SU(2)$ because we want to describe the interaction of two D-particles. The novelty is here that the super Yang-Mills theory must be written in a gravitational background. This frames the problem in the language of the matrix model, for which the D-particles are the fundamental degrees of freedom, in the rather unconventional case of a curved background. The final amplitude describes the motion of two D-particles in the space-time of the shock wave.

By performing the small v and large $r_{\rm T}$ expansion of the matrix-model result, the amplitude of the closed-string computation will be reproduced, see eq. (206).

5.2 String calculation

Consider two D-particles located at $\vec{Y}_1^{\perp}$ and $\vec{Y}_2^{\perp}$ and moving with velocities $\vec{v}_1$ and $\vec{v}_2$, respectively, where $\vec{Y}_{(i)}^{\perp}\cdot\vec{v}_{(i)} = 0$. We consider the frame where $\vec{v}_1+\vec{v}_2 = \vec{Y}_1^{\perp}+\vec{Y}_2^{\perp} = 0$. We call $\vec{v} = \vec{v}_1 - \vec{v}_2$ and $\vec{b} = \vec{Y}_1^{\perp} - \vec{Y}_2^{\perp}$, where by definition $\vec{b}\cdot\vec{v} = 0$. Their interaction with an external gravitational field - generated by the source graviton moving along the direction z at a transverse distance $\vec{r}_{\rm T}$ (by definition $\vec{r}_{\rm T}$ is orthogonal to z) from the center of mass of the two D-particles - is dominated by a term proportional to $1/q^2$, where q is the momentum transfer by the gravitational field to the system of the two D-particles. The momentum $\vec{q}$ can be separated into a part q_z along the direction of motion of the external graviton (i.e., the shock-wave direction), and the remaining orthogonal directions, $\vec{q}_{\rm T}$. In the eikonal approximation, $q_0 = q_z$, which implies $q^2 = q_{\rm T}^2$.

We illustrate the above set-up with the following figure:

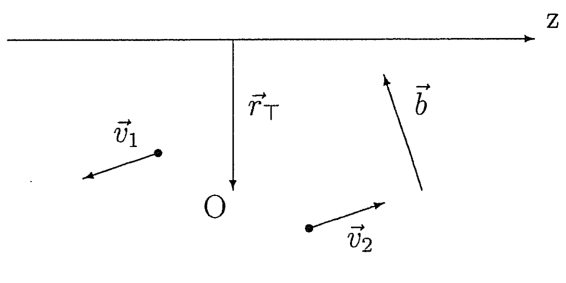

figure 2: Set-up for string calculation

In the above figure, 'O' represents the origin of the system of coordinates, $\vec{b}$ is orthogonal to the (parallel) velocities of the two D-particles and measures their separation in that hyperplane, whereas $\vec{r}_{\rm T}$ is orthogonal to the shock-wave (z) axis.

In the frame chosen, the motion of the center of mass of the particles is factorized out. The effect of the external source graviton on it is taken into account by disconnected diagrams—which we do not consider—in which the source couples independently first to one of the two D-particles and then to the other. Here we focus instead on the relative motion of the two D-particles as they interact with the gravitational field.

We write the amplitude in $d = 10$ as

$$a\,(r_T, b, v) = \int \frac{\mathrm{d}q_z\, \mathrm{d}^{d\text{-}2}\vec{q}_\mathrm{T}}{(2\pi)^{d-1}}\, e^{i\vec{r}\cdot\vec{q}_\mathrm{T}} \int \mathrm{d}\ell \int \mathrm{d}^2 w\, \mathrm{d}^2 z\ \mathcal{A}\,(z, w;\, \ell)\,, \tag{120}$$

where (up to an overall constant)

$$\mathcal{A}\,(z, w;\, \ell) = \sum_S\, (\pm)_S\, \langle v_2, Y_2| e^{-\ell H}\, V_k(z,\bar{z})\, V_p(w,\bar{w}) |v_1, Y_1\rangle_S \tag{121}$$

contains the interaction of the two D-particles with the two gravitons, the vertices of which are given by

$$V_p(z,\bar{z}) = \varepsilon^{(p)}_{\mu\nu} \Big[\partial X^\mu(z) + ip\cdot\psi(z)\psi^\mu(z)\Big] \Big[\bar{\partial} X^\nu(\bar{z}) + ip\cdot\bar{\psi}(\bar{z})\bar{\psi}^\nu(\bar{z})\Big] e^{ip\cdot X}\,, \tag{122}$$

and similarly for the other one at w and carrying momentum k. The index S stands for the three even spin structures of the fermionic propagators. The amplitude (121), in the case of $\vec{v} = 0$, has been considered in [46].

For later use, we define the two vectors q and P to be

$$q_0 = k_0 + p_0\,, \quad \vec{q} = \vec{k} + \vec{p} \quad \text{and} \quad P_0 = k_0 - p_0\,, \quad \vec{P} = \vec{k} - \vec{p}\,. \tag{123}$$

The D-particle kets or bras are given by a tensor product of center-of-mass and string-oscillator states. The D-particle center-of-mass state in space-time is described by a Fourier transform in terms of the eigenstates of the energy-momentum transfer Q.

One has further to integrate over the world-line of the D-particle, which in the eikonal approximation is the straight line $Y^\mu(t)$ parameterized by $Y(t) = (t, \vec{R}(t))$ with $\vec{R} = \vec{v}\,t + \vec{Y}_\perp$. We thus get

$$|v, Y_\perp\rangle = \int \mathrm{d}t \int \frac{\mathrm{d}^d Q}{(2\pi)^d}\, e^{i\,(Q_0 t - \vec{Q}\cdot\vec{R}(t))} |v, Q\rangle = \int \frac{\mathrm{d}^{d\text{-}1}\vec{Q}}{(2\pi)^{d\text{-}1}}\, e^{i\,\vec{Q}\cdot\vec{Y}_\perp} |v, Q_0 = \vec{v}\cdot\vec{Q}, \vec{Q}\rangle\,, \tag{124}$$

Q is here the momentum transferred to the brane, Q_0 being thus the energy transfer. Notice that in the eikonal approximation we have the constraint $Q_{0i} = \vec{v}_i\cdot\vec{Q}_i \equiv v_i Q^i_L$. This is similar to what holds for the momentum transfer from the source graviton (whose velocity is equal to 1 along the z-direction) namely $q_0 = q_z$.

In eq. (121) the Hamiltonian H transports the D-particle at $\vec{Y}_2^{\perp}$ and is given by

$$H = H_{\text{CM}} + H_{\text{OS}} = \frac{1}{2} Q_2^2 + H_{\text{OS}} , \tag{125}$$

where H_{OS} contains the string oscillators.

Let us also define the normalized correlators

$$\langle V_k(z,\bar{z})\, V_p(w,\bar{w})\rangle_S = \frac{\langle v_2, Y_2|e^{-\ell H}\, V_k(z,\bar{z})\, V_p(w,\bar{w})|v_1, Y_1\rangle_S}{\langle v_2, Y_2|e^{-\ell H_{\text{OS}}}|v_1, Y_1\rangle_S} , \tag{126}$$

where

$$\langle v_2, Y_2|e^{-\ell H_{\text{OS}}}|v_1, Y_1\rangle_S = \frac{\vartheta_S\,(iv/\pi)\,\vartheta_S^3\,(0)}{\vartheta_1'^4\,(0)} , \tag{127}$$

and therefore

$$\mathcal{A}\,(z, w;\, \ell) = \sum_S\, (\pm)_S\, \langle V_k(z,\bar{z})\, V_p(w,\bar{w})\rangle_S\, \frac{\vartheta_S\,(iv/\pi)\,\vartheta_S^3\,(0)}{\vartheta_1'^4\,(0)} . \tag{128}$$

Since we are not interested in spin effects, we can consider only the term proportional to $\varepsilon^{(p)} \cdot \varepsilon^{(k)}$. The two-graviton correlation function is then

$$\begin{aligned}
\langle V_k(z,\bar{z})\, V_p(w,\bar{w})\rangle_S \;=\;& \varepsilon^{(p)}_{\mu\lambda}\varepsilon^{(k)}_{\nu\rho}\, \Big[\langle \partial X^\mu \partial X^\lambda\rangle\langle \bar{\partial} X^\nu \bar{\partial} X^\rho\rangle \\
-\;& \langle \psi^\mu \psi^\lambda\rangle_S \langle \bar{\psi}^\nu \bar{\psi}^\rho\rangle_S \langle p\cdot\psi\, p\cdot\bar{\psi}\, k\cdot\psi\, k\cdot\bar{\psi}\rangle_S \\
+\;& \langle \partial X^\mu \partial X^\lambda\rangle\langle \bar{\psi}^\nu \bar{\psi}^\rho\rangle_S \langle p\cdot\bar{\psi}\, k\cdot\bar{\psi}\rangle_S \\
+\;& \langle \bar{\partial} X^\mu \bar{\partial} X^\lambda\rangle\langle \psi^\nu \psi^\rho\rangle_S \langle p\cdot\psi\, k\cdot\psi\rangle_S\Big] \langle e^{ik\cdot X(z)}\, e^{ip\cdot X(w)}\rangle .
\end{aligned} \tag{129}$$

We look for the singularities as we take the limit $z \to w$:

$$\begin{aligned}
\langle V_k(z,\bar{z})\, V_p(w,\bar{w})\rangle_S \;\to\;& \frac{\varepsilon^{(p)}\cdot\varepsilon^{(k)}}{(4\pi)^2} \Bigg\{ \frac{1}{|z-w|^4} \left(1 + O(q^2)\right) \\
&- \frac{1}{|z-w|^2} \Big[\langle k\cdot\psi\, k\cdot\bar{\psi}\rangle_S \langle p\cdot\psi\, p\cdot\bar{\psi}\rangle_S \\
&+ \langle k\cdot\psi\, p\cdot\bar{\psi}\rangle_S \langle k\cdot\bar{\psi}\, p\cdot\psi\rangle_S + O(q^4)\Big]\Bigg\} \langle e^{ik\cdot X(z)}\, e^{ip\cdot X(w)}\rangle .
\end{aligned} \tag{130}$$

As we shall see, we can neglect in (130) $O(q^2)$ terms in the quartic pole and $O(q^4)$ terms in the quadratic pole.

While the second term, which comes from the fermionic correlators, is already a quadratic pole, the first one, which comes from the bosonic correlators, is quartic and, in order to contribute, it must be multiplied by terms containing a factor

$|z-w|^2$. In fact, it is the quadratic pole times a factor $|z-w|^{q^2/(4\pi)}$, contained in $\langle e^{ik\cdot X} e^{ip\cdot X}\rangle$, that gives rise to the singularity $1/q^2$ for which we are looking:

$$\int d^2(z-w)|z-w|^{-2+q^2/(4\pi)} = \frac{8\pi^2}{q^2} \,. \tag{131}$$

We split the center-of-mass modes X_0 from the string-oscillator modes in the correlator:

$$\langle e^{ik\cdot X(z)}\, e^{ip\cdot X(w)}\rangle = \langle e^{ik\cdot X_0(z)}\, e^{ip\cdot X_0(w)}\rangle \langle e^{ik\cdot X(z)}\, e^{ip\cdot X(w)}\rangle_{\rm OS} \,, \tag{132}$$

and analyze their contributions separately.

5.2.1 String center-of-mass modes

Let us first consider the string center-of-mass modes. They are given by

$$X_0^\mu(z) = X_0^\mu - iQ^\mu \,\mathrm{Im}\, z \,, \tag{133}$$

where $\mathrm{Im}\, z$ plays the role of proper time of closed string propagating from one D-particle to the other one.

Because

$$\begin{aligned} \langle \vec{Q}_2, Q_2^0 | e^{ik\cdot X(0)} &= \langle \vec{Q}_2 - \vec{k}, Q_2^0 - k_0 | \\ e^{ip\cdot X(0)} | \vec{Q}_1, Q_1^0\rangle &= |\vec{Q}_1 + \vec{p}, Q_1^0 + p_0\rangle \,, \end{aligned} \tag{134}$$

we have two conservation laws given by the sandwiching of the external states:

$$\begin{aligned} \langle \vec{Q}_2 - \vec{k}, Q_2^0 - k_0 | \vec{Q}_1 + \vec{p}, Q_1^0 + p_0\rangle &= \delta^{(d-1)}\left(\vec{Q}_2 - \vec{k} - \vec{Q}_1 - \vec{p}\right) \\ &\quad \times \delta\left(Q_2^0 - k_0 - Q_1^0 - p_0\right) \,. \end{aligned} \tag{135}$$

Notice that the energy conservation gives $\vec{Q}_2 \cdot \vec{v}_2 - \vec{Q}_1 \cdot \vec{v}_1 = q_z$.

According to (126), for

$$Z_{\rm CM} \equiv \langle e^{ik\cdot X_0(z)}\, e^{ip\cdot X_0(w)}\rangle = \langle v_2, Y_2 | e^{-\ell H_{\rm CM}}\, e^{ik\cdot X_0(z)} e^{ip\cdot X_0(w)} | v_1, Y_1\rangle \,, \tag{136}$$

we have

$$\begin{aligned} Z_{\rm CM} &= \int \frac{d^{d-1}\vec{Q}_1}{(2\pi)^{D-1}} \int \frac{d^{d-1}\vec{Q}_2}{(2\pi)^{D-1}} \exp\left[i\vec{Q}_1 \cdot Y_\perp^1 - i\vec{Q}_2 \cdot Y_\perp^2\right] \\ &\quad \times \exp\left\{-\frac{\ell}{2}\left[\vec{Q}_2^2 - (\vec{v}_2 \cdot Q_2)^2\right] + \mathrm{Im}\, z\left(\vec{k}\cdot\vec{Q}_2 - k_0\, \vec{v}_2 \cdot \vec{Q}_2\right) \right. \\ &\quad \left. + \mathrm{Im}\, w\left(\vec{p}\cdot\vec{Q}_1 - p_0\, \vec{v}_1 \cdot \vec{Q}_1\right)\right\} \delta^{(d)}\left(Q_2 - Q_1 - k - p\right) \,. \end{aligned} \tag{137}$$

We replace by means of the δ-functions $\vec{Q}_2 = \vec{Q}_1 + \vec{q}$; we set, by neglecting $\vec{q}\cdot\vec{v}/v$, $Q_L = -q_z/v$ and $\vec{Q}_1 = \vec{Q}$. Finally, we integrate over the components $\vec{Q}_\perp$ orthogonal to $\vec{v}$ and find

$$\begin{aligned} Z_{\text{CM}} &= -\frac{1}{v} e^{-i\vec{q}\cdot\vec{b}/2} \int \frac{\mathrm{d}^{d-2}\vec{Q}_\perp}{(2\pi)^{d-2}} \\ &\quad \times \exp\left[-\frac{\ell}{2}\vec{Q}_\perp^2 + \left(\operatorname{Im} z\,\vec{k} + \operatorname{Im} w\,\vec{p} + i\vec{b}\right)\cdot\vec{Q}_\perp\right] \\ &\quad \times \int \frac{\mathrm{d}q_z}{2\pi} \exp\left[-\frac{\ell}{2}\left[1 + (\operatorname{Im} z + \operatorname{Im} w)\frac{v_z}{\ell}\right]\left(\frac{q_z}{v}\right)^2\right], \end{aligned} \tag{138}$$

which, after integration in $\vec{Q}_\perp$ and q_z, for small v_z and $q_{\text{T}} \to 0$, gives

$$Z_{\text{CM}} = \ell^{-(d-1)/2}\, e^{\left(\operatorname{Im} z\vec{k} + \operatorname{Im} w\vec{p} + i\vec{b}\right)^2/2\ell}\left[1 - \frac{v_z}{2\ell}(\operatorname{Im} z + \operatorname{Im} w)\right], \tag{139}$$

where we have dropped overall factors of 2π.

We retain a term proportional to $|z-w|^2$ from the expansion of the exponential in (139), and finally obtain

$$\begin{aligned} Z_{\text{CM}} &= -\ell^{-(d-1)/2}\, e^{-\vec{b}^2/2\ell}\left\{1 - \frac{v_z}{2\ell}(\operatorname{Im} z + \operatorname{Im} w)\right. \\ &\quad \left. + |z-w|^2\left[\frac{k_0^2}{4\ell} - \frac{1}{16\ell^2}(\vec{P}\cdot\vec{b})^2\right]\right\}. \end{aligned} \tag{140}$$

5.2.2 Oscillator modes

The oscillator part is given by the expectation value of the exponential factors

$$\begin{aligned} \langle e^{ik\cdot X(z)}\, e^{ip\cdot X(w)}\rangle_{\text{OS}} &= e^{-\langle [k\cdot X(z) + p\cdot X(w)]\rangle^2\rangle_{\text{OS}}/2} \\ &= \left|\frac{\vartheta_1(z-w)}{\vartheta_1(z-\bar{w})}\right|^{q^2/4\pi} \left|\frac{\vartheta_1^2(z-\bar{w})}{\vartheta_1(z-\bar{z})\,\vartheta_1(w-\bar{w})}\right|^{k_0^2/4\pi}, \end{aligned} \tag{141}$$

which in the $z \to w$ limit gives

$$\langle e^{ik\cdot X(z)}\, e^{ip\cdot X(w)}\rangle_{\text{OS}} \longrightarrow \left[1 + |z-w|^2\frac{k_0^2}{2\pi}\,\partial_w^2 \ln\vartheta_1(w-\bar{w})\right]\left|\frac{\vartheta_1(z-w)}{\vartheta_1(z-\bar{w})}\right|^{q^2/4\pi}, \tag{142}$$

where we have kept terms up to $|z-w|^2$ as required.

The fermionic correlator can be written (up to $O(q^2)$) as

$$\begin{aligned} \langle k\cdot\psi\, k\cdot\bar{\psi}\rangle_S \langle p\cdot\psi\, p\cdot\bar{\psi}\rangle_S &+ \langle k\cdot\psi\, p\cdot\bar{\psi}\rangle_S \langle k\cdot\bar{\psi}\, p\cdot\psi\rangle_S = \\ &\left(\frac{1}{4\pi}\right)^2\left\{k_0^2\frac{(\vec{q}_{\text{T}}\cdot\vec{v})^2}{v^2}\left[Q_S^2 - R_S^2 - P_S^2\right] + q^2 k_0^2\left[Q_S P_S + P_S^2\right]\right. \\ &- \left. q^2\,\frac{(\vec{P}\cdot\vec{v})^2}{4v^2}\left[P_S^2 - Q_S P_S\right] + k_0\, q^2\,\frac{\vec{P}\cdot\vec{v}}{v} R_S P_S\right\}, \end{aligned} \tag{143}$$

where the spin-structure dependent functions Q_S, R_S and P_S are given in the appendix.

We perform now the integration

$$\int \mathrm{d}^2 z \int \mathrm{d}^2 w = \int \mathrm{d}^2 (z-w) \int \mathrm{d}\,\mathrm{Re}\, w \int \mathrm{d}\,\mathrm{Im}\, w \,. \tag{144}$$

Every term in (143) is $O(q^2)$. Unless we find an additional singularity coming from the integration over $\mathrm{Im}\, w$, the q^2 in front of all terms will cancel the $1/q^2$ coming from the integration over $(z-w)$ and we would be left without the $1/q^2$ pole in which we are interested. We therefore look for terms behaving like $(w-\bar{w})^{-1-q^2/4\pi}$. After summing over the spin structures, a factor of this form could arise from terms in (143) proportional to Q_S^2, R_S^2, $Q_S P_S$ and $R_S P_S$. The function P_S^2 does not contain the required term. The contributions coming from the terms proportional to Q_S^2 and R_S^2 are equal and therefore cancel. Therefore, there are only two terms in (143) that can give rise to the required power of v and $1/\vec{q}_{\mathrm{T}}^{\,2}$:

$$\begin{aligned} \langle k\cdot\psi\, k\cdot\bar{\psi}\rangle_S \langle p\cdot\psi\, p\cdot\bar{\psi}\rangle_S \;&+\; \langle k\cdot\psi\, p\cdot\bar{\psi}\rangle_S \langle k\cdot\bar{\psi}\, p\cdot\psi\rangle_S \\ &\longrightarrow\; \left(\frac{1}{4\pi}\right)^2 \left\{ k_0\, q^2\, \frac{\vec{P}\cdot\vec{v}}{v} R_S P_S + q^2 \left[k_0^2 + \frac{(\vec{P}\cdot\vec{v})^2}{4v^2} \right] Q_S P_S \right\} . \end{aligned} \tag{145}$$

The sum over the spin structures yields, for $v \to 0$,

$$\begin{aligned} \sum_S (\pm)_S\, R_S\, P_S \frac{\vartheta_S(-iv/\pi)\,\vartheta_S^3(0)}{\vartheta_1'^4(0)} &= \frac{1}{2}\sum_S (\pm)_S \frac{\vartheta_S(w-\bar{w})\,\vartheta_S^2(0)}{\vartheta_1^2(w-\bar{w})\vartheta_1'^2(0)} \\ &\quad \times \left[e^v\, \vartheta_S(w-\bar{w}-iv/\pi) - e^{-v}\, \vartheta_S(w-\bar{w}+iv/\pi) \right] \\ &\longrightarrow \frac{i}{\pi}\frac{v^3}{(2\pi)^2}\frac{\vartheta_1'(w-\bar{w})}{\vartheta_1(w-\bar{w})} , \end{aligned} \tag{146}$$

and

$$\begin{aligned} \sum_S (\pm)_S\, Q_S\, P_S \frac{\vartheta_S(-iv/\pi)\,\vartheta_S^3(0)}{\vartheta_1'^4(0)} &= \frac{1}{2}\sum_S (\pm)_S \frac{\vartheta_S(w-\bar{w})\,\vartheta_S^2(0)}{\vartheta_1^2(w-\bar{w})\vartheta_1'^2(0)} \\ &\quad \times \left[e^v\, \vartheta_S(w-\bar{w}-iv/\pi) + e^{-v}\, \vartheta_S(w-\bar{w}+iv/\pi) \right] \\ &\longrightarrow 2\left(\frac{iv}{2\pi}\right)^4 \frac{\vartheta_1''(w-\bar{w})}{\vartheta_1(w-\bar{w})} , \end{aligned} \tag{147}$$

after dropping a total derivative, which does not contribute because the contributions at the integration limits cancel against each other.

5.2.3 The final amplitude

We can now collect all the relevant terms by putting in evidence the integral over z around the quadratic pole and writing everything else for $z = w$. The integrand does not depend on $\mathrm{Re}\, w$ and its integration gives a factor 1. We thus obtain

$$
\begin{aligned}
a\left(r_{\mathrm{T}}, b, v\right) \simeq & -\int \frac{\mathrm{d}^{d-2} \vec{q}_{\mathrm{T}}}{(2\pi)^{d-1}} e^{i \vec{r} \cdot \vec{q}_{\mathrm{T}}} e^{-\vec{b}^2/2\ell} \int \mathrm{d}\ell\, \ell^{-(d-1)/2} \\
& \times \int \mathrm{d}^2(z-w)\, |z-w|^{-2+q^2/(4\pi)} \int_0^{\ell} \mathrm{d}\, \mathrm{Im}\, w \left[F_2(w, \bar{w}) + F_4(w, \bar{w})\right] .
\end{aligned} \tag{148}
$$

In eq. (148), $F_4(w, \bar{w})$ comes from the bosonic correlator, the term proportional to $|z-w|^{-4}$ in (130), times the terms proportional to $|z-w|^2$ in (140) and (142) which compensate for the quartic pole, and is given by

$$
F_4(w, \bar{w}) = \left(\frac{iv}{2\pi}\right)^4 \left[\frac{k_0^2}{2\pi} \partial_w^2 \ln \vartheta_1(w-\bar{w}) + \frac{k_0^2}{4\ell} - \frac{1}{16} \frac{\vec{b} \cdot \vec{P}}{\ell^2}\right] , \tag{149}
$$

where the factor in front comes from the expansion in v of the $\vartheta_1(iv/2\pi)$ obtained after summing over the spin structures (see formulas in the appendix).

The term $F_2(w, \bar{w})$ in eq. (148) originates from the fermionic correlators, the term proportional to $|z-w|^{-2}$ in (129), and it has two terms, one that originates from (147) and one from the product of the fermionic correlator (146) and the term proportional to v_z in (139):

$$
\begin{aligned}
F_2(w, \bar{w}) = & -2 \frac{k_0^2 q^2}{(4\pi)^2} \left[\left(1 + \frac{v_z^2}{v^2}\right) \left(\frac{iv}{2\pi}\right)^4 \frac{\vartheta_1''(w-\bar{w})}{\vartheta_1(w-\bar{w})} \right. \\
& \left. - \frac{v_z}{v} \left(\frac{iv^3}{4\pi^3}\right) \frac{\vartheta_1'(w-\bar{w})}{\vartheta_1(w-\bar{w})} \left(\frac{v_z}{\ell}\right) \mathrm{Im}\, w \right] \left[\vartheta_1(w-\bar{w})\right]^{-q^2/(4\pi)} .
\end{aligned} \tag{150}
$$

We use

$$
\vartheta_1'(w-\bar{w}) \left[\vartheta_1(w-\bar{w})\right]^{-q^2/(4\pi)-1} = \frac{2\, i\, \pi}{q^2} \frac{\partial}{\partial\, \mathrm{Im}\, w} \left[\vartheta_1(w-\bar{w})\right]^{-q^2/(4\pi)} , \tag{151}
$$

after which, the integrals over $\mathrm{Im}\, w$ give (at the leading order in $q^2 \to 0$)

$$
\int_0^{\ell} \partial_w^2 \ln \vartheta_1(w-\bar{w})\, \mathrm{d}\, \mathrm{Im}\, w = -\pi , \tag{152}
$$

$$
\int_0^{\ell} \mathrm{Im}\, w \frac{\partial}{\partial\, \mathrm{Im}\, w} \left[\vartheta_1(w-\bar{w})\right]^{-q^2/(4\pi)} \mathrm{d}\, \mathrm{Im}\, w = -\ell \tag{153}
$$

and

$$
\int_0^{\ell} \frac{\vartheta_1''(w-\bar{w})}{\left[\vartheta_1(w-\bar{w})\right]^{1+q^2/(4\pi)}} \mathrm{d}\, \mathrm{Im}\, w = -\frac{8\, \pi^2}{q^2} \tag{154}
$$

respectively. In eq. (154), only the upper limit of integration contributes and we have used the relation $\vartheta_1(2i\ell - y) = e^{2\pi\ell+2\pi i y}\vartheta_1(-y)$.

At this point, the integration over $\mathrm{d}^2(z-w)$ in (148) gives the desired quadratic pole and we are left with the final result

$$a\,(r_{\mathrm{T}}, b, v) \;\simeq\; k_0^2 \int \frac{\mathrm{d}^{d-2}\vec{q}_{\mathrm{T}}}{(2\pi)^{d-2}} \frac{1}{\vec{q}_{\mathrm{T}}^{\,2}}\, e^{i\vec{r}\cdot\vec{q}_{\mathrm{T}}} \times \int_0^\infty \mathrm{d}\ell \left\{ \frac{11}{4} v^4 - 2\, v_{\mathrm{T}}^2\, v^2 - b_z^2 \frac{v^4}{4\ell} \right\} \ell^{-(d-1)/2} e^{-b^2/2\ell}\,, \tag{155}$$

where we have dropped overall factors, the knowledge of which would require fixing the absolute normalization of the string amplitude. As discussed in the introduction, the first non-vanishing term is of order $O(v^4)$ and its functional behavior in b is independent from α'.

Notice that, because of the three-body kinematics, the amplitude (155) is $O(v^4)$, whereas the interaction of two D-particles without shock wave is $O(v^3)$ [20].

A final comment. The first term on the right-hand side of eq. (143) would also give an α'-independent behavior in b, when summed over spin structure and expanded at the order $O(v^2)$. However, since this term is proportional to $(\vec{q}_{\mathrm{T}}\cdot\vec{v})^2/q_{\mathrm{T}}^2$, it is sub-leading for $r_{\mathrm{T}} \to \infty$, and we do not keep it into account in this paper.

5.3 Matrix model calculation

The action in the matrix model is given by the dimensionally reduced $d = 10$ and $N = 1$ super Yang-Mills action. We are going to compute the one-loop contribution to the effective action in a suitable background.

The gauge-fixed bosonic action is

$$S_B = \int \mathrm{d}t \,\left[\frac{1}{2}\, \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \mathrm{Tr}\,(D_\mu^B A^\mu)^2 \right]\,, \tag{156}$$

where

$$F_{0i} = \partial_t X_i \quad \text{and} \quad F_{ij} = i\,[X_i, X_j]\,, \tag{157}$$

since

$$A^i \equiv X^i \quad \text{and} \quad A^0 = 0\,. \tag{158}$$

The action (156) is expanded around the classical background field B_μ by separating A_μ into B_μ and the fluctuation X_μ which is integrated out in the path integral. The background covariant derivative is defined by

$$D_\mu^B A_\nu = \partial_\mu A_\nu + i\,[B_\mu, A_\nu]\,. \tag{159}$$

The ghost action is

$$S_C = \int \mathrm{d}t\ \mathrm{Tr}\ \bar{C}(D^B)^2 C\,, \tag{160}$$

where the covariant derivative D can be taken equal to D^B because we are only interested in the quadratic part of the ghost action.

Similarly, the fermionic action is given by

$$S_F = \int \mathrm{d}t \ \mathrm{Tr}\ \bar{\Psi}\Gamma^\mu D^B_\mu \Psi\,. \tag{161}$$

For the case of two D-particles, the fields

$$X^i = X^i_a \frac{\tau^a}{2}\,, \quad C = C_a \frac{\tau^a}{2} \quad \text{and} \quad \Psi = \psi_a \frac{\tau^a}{2} \tag{162}$$

take value in the space of the gauge group $SU(2)$ (and τ^a are Pauli's matrices).

In the eikonal approximation, and before introducing the shock wave, the background field B is taken to be

$$\vec{B} = \begin{pmatrix} \vec{v}_1\, t + \vec{b}_1 & 0 \\ 0 & \vec{v}_2\, t + \vec{b}_2 \end{pmatrix} \quad \text{and} \quad B_0 = 0\,, \tag{163}$$

where $\vec{v}_i$ and $\vec{b}_i$ are the velocities and positions of the two D-particles. In the frame of reference where $\vec{v}_1 + \vec{v}_2 = 0$ and $\vec{b}_1 + \vec{b}_2 = 0$, we can then take

$$\vec{B} = \left(\vec{v}\, t + \vec{b}\right) \frac{\tau_3}{2}\,, \tag{164}$$

where now $\vec{v}_1 = -\vec{v}_2 = \vec{v}/2$, $\vec{b}_1 = -\vec{b}_2 = \vec{b}/2$; v and b are the relative velocity and distance in the motion of the two D-particles. As we did for the string case, we have thus factorized out the motion of the center of mass. Notice that the action of the matrix-valued background field on the matrix valued quantum field ϕ (where ϕ can be A^μ, C or ψ) is $B \circ \phi = [B, \phi]$.

5.3.1 Kinematics

The space-time of the shock wave moving along z from right to left is described by the metric element

$$\mathrm{d}s^2 = 2\, \mathrm{d}U\, \mathrm{d}V + h\, \mathrm{d}U^2 + \mathrm{d}\vec{x}^2_{\mathrm{T}} \tag{165}$$

where the light-cone variables (U, V) are defined to be

$$U \equiv (z+t)/\sqrt{2} \quad \text{and} \quad V \equiv (z-t)/\sqrt{2}\,. \tag{166}$$

The coefficient h is given by

$$h = f(\vec{r}_{\mathrm{T}})\delta(U)\,, \tag{167}$$

where $\vec{r}_{\mathrm{T}}$ is the distance from the center of mass of the two D-particles to the shock wave. For a source graviton in $d = 10$ and with an energy k_0

$$f(\vec{r}_{\mathrm{T}}) = \frac{8\sqrt{2}\, G_N k_0}{\pi^3} \frac{}{r^6_{\mathrm{T}}}\,, \tag{168}$$

where G_N is Newton's constant.

¿From now on, we take U as the evolution variable.

The metric components must be thought as a matrix of $SU(2)$ depending on the positions $\vec{Y}_i$ of the two D-particles. However,

$$\begin{pmatrix} f(\vec{Y}_1)\delta(U) & 0 \\ 0 & f(\vec{Y}_2)\delta(U) \end{pmatrix} = \begin{pmatrix} f(\vec{r}_\top)\delta(U) & 0 \\ 0 & f(\vec{r}_\top)\delta(U) \end{pmatrix} \left[1 + O\left(\frac{b_z}{r_\top}\right)\right] , \quad (169)$$

and, for $b_z \ll r_\top$, we can neglect the higher order terms and take the metric to be proportional to the identity matrix.

Because of the non-flat external metric, care must be taken in lowering and rising indices. In particular, in going from the background gauge fields with lower indices to the background coordinates with upper indices, we must use

$$B_U = h\, B^U + B^V \quad \text{and} \quad B_V = B^U \quad (170)$$

while $\vec{B}_\top = \vec{B}^\top$.

We now fix B^U, B^V and $\vec{B}^\top$, by taking into account the trajectories of the D-particles.

The trajectory of a particle moving in the shock-wave background is (see the appendix)

$$\begin{cases} V &= U\, w + w_0 - \frac{1}{2} f\theta(U) \\ \vec{x}_\top &= \vec{v}_w\, U + \vec{b}_w\, . \end{cases} \quad (171)$$

By substituting $U = (z+t)/\sqrt{2}$ in (171), we reproduce the D-particle trajectories (for $i = 1, 2$) before the shock $z^{(i)} = v_z^{(i)}\, t + b_z^{(i)}$ and $\vec{x}_\top^{(i)} = \vec{v}_\top^{(i)} t + \vec{b}_\top^{(i)}$, by the assignment

$$w^{(i)} = \frac{v_z^{(i)} - 1}{v_z^{(i)} + 1}, \quad w_0^{(i)} = \frac{\sqrt{2}\, b_z^{(i)}}{v_z^{(i)} + 1}, \quad (172)$$

and

$$\vec{v}_w^{(i)} = \vec{v}_\top^{(i)} \frac{\sqrt{2}}{v_z^{(i)} + 1}, \quad \vec{b}_w^{(i)} = \vec{b}_\top^{(i)} - \vec{v}_\top^{(i)} \frac{b_z^{(i)}}{v_z^{(i)} + 1}\, . \quad (173)$$

The matrix-valued $\vec{B}_\top = (\vec{x}_\top^{(1)} - \vec{x}_\top^{(2)})\, \tau_3/2$ is thus found to be

$$\vec{B}_\top = \left(\frac{\sqrt{2}\, \vec{v}_\top}{1 - v_z^2/4} U + \vec{b}_\top + \frac{\vec{v}_\top\, b_z\, v_z}{4 - v_z^2} \right) \frac{\tau_3}{2}\, . \quad (174)$$

The part of the background proportional to the identity is irrelevant for our computation concerning the relative motion of the D-particles.

Next, we have

$$\frac{B_{(i)}^U + B_{(i)}^V}{\sqrt{2}} = z_{(i)} = \frac{U + V_{(i)}}{\sqrt{2}} \quad (175)$$

and, since $B_0 = 0$ implying $B_U = B_V$, we impose

$$h\, B^U_{(i)} + B^V_{(i)} = B^U_{(i)} \,. \tag{176}$$

By solving these constraints at the leading order in h, we get the matrix-valued

$$B_U = B_V = \frac{v_z\, U + b_z/\sqrt{2}}{1 - v_z^2/4} \left(1 + \frac{h}{2}\right) \frac{\tau_3}{2} \,. \tag{177}$$

Notice the absence of the shift in the trajectory, that is present in the one particle problem; it cancels out in the two-body relative motion. Also, we will see that we only need the leading first-order terms in the velocity in (174) and (177).

Derivatives in the curved background are often complicated. However, an important property of the shock-wave metric is that, since $\partial_V g^{V\lambda} = \partial_V \delta(U) = 0$, we have that

$$D^B_\nu g^{\nu\lambda} = g^{\nu\lambda} D^B_\nu \tag{178}$$

with great simplifications in the computation. Moreover, $\sqrt{g} = 1$ and the covariant derivative is the usual one. The derivative

$$\partial_U = -\partial_V \tag{179}$$

because the fields do not depend on z.

Similarly, for the fermions we can pass the covariant derivative D^B_μ through the Γ-matrices, and use

$$\mathrm{Tr}_\gamma \ln(\Gamma_\mu D^\mu) = \frac{1}{2}\ \mathrm{Tr}_\gamma \ln(\Gamma_\mu D^\mu)^2 = \frac{1}{2}\ \mathrm{Tr}_\gamma \ln\left[D^2 + \frac{1}{2}\Sigma_{\mu\nu} F^{\mu\nu}\right] \tag{180}$$

even in the shock-wave space-time (the spin connection is zero). Eq. (180) simplifies the evaluation of the fermionic path integral.

5.3.2 The quadratic action

Because we are interested in a one-loop computation, we need only the part of the action that is quadratic in the fields. The bosonic action is then

$$S_B = \int \mathrm{d}U\ \mathrm{Tr}\ X^\mu \left[-\delta^\nu_\mu D^2 - 2iF^\nu_\mu\right] X_\nu \,. \tag{181}$$

In eq. (181), the background field strength F^μ_ν is independent from U. As we shall see, because the result is already proportional to F^4, we only need the part of F that is leading order in v and which is given by $(\mu, \nu = U, V, \top)$

$$F^\nu_\mu = \begin{pmatrix} 2v_z & 0 & \sqrt{2}v_\top \\ 0 & -2v_z & -\sqrt{2}v_\top \\ \sqrt{2}v_\top & -\sqrt{2}v_\top & 0 \end{pmatrix} + h \begin{pmatrix} v_z & -2v_z & 0 \\ 0 & -v_z & 0 \\ 0 & -\sqrt{2}v_\top & 0 \end{pmatrix} . \tag{182}$$

Notice that in writing (182), we used $\partial_V h = 0$ as well as $\partial_U h \simeq 0$ (consistently with (179)) since terms proportional to $\delta'(U)$ would eventually multiply functions of U^2 in the final amplitude and would not contribute.

In order to compute the amplitude for the scattering of the two D-particles in the external field of the shock wave, we need

$$\begin{aligned} a\,(r_\top, b, v) &= \ln \int [\mathrm{d}X][\mathrm{d}\bar{C}][\mathrm{d}C][\mathrm{d}\Psi][\mathrm{d}\bar{\Psi}]e^{-S_B - S_C - S_F} = -\frac{1}{2}\,\mathrm{Tr}\,\ln\left(-\delta^\nu_\mu D^2 - 2iF^\nu_\mu\right) \\ &+ \mathrm{Tr}\,\ln\left(-D^2\right) + \frac{1}{4}\,\mathrm{Tr}\,\ln\left(-D^2 - \frac{1}{2}\Sigma_{\mu\nu}F^{\mu\nu}\right). \end{aligned} \tag{183}$$

The operator traces can be written in terms of the Schwinger representation by introducing a parameter s. The amplitude (183) thus becomes

$$a\,(r_\top, b, v) = a_B + a_F\,, \tag{184}$$

where

$$a_B = -\frac{1}{2}\int \mathrm{d}U \int_0^\infty \frac{\mathrm{d}s}{s} \lim_{U_{1,2}\to U} \mathcal{W}_g(s, U_1, U_2)\,\left(\mathrm{Tr}_{\mu\nu}\, e^{2isF^\nu_\mu} - 2\right) \tag{185}$$

for the bosonic part, and

$$a_F = +\frac{1}{4}\int \mathrm{d}U \int_0^\infty \frac{\mathrm{d}s}{s} \lim_{U_{1,2}\to U} \mathcal{W}_g(s, U_1, U_2)\,\left(\mathrm{Tr}_\gamma\, e^{s\Sigma_{\mu\nu}F^{\mu\nu}/2}\right) \tag{186}$$

for the fermionic one.

In (185) and (186) we have separated between parenthesis the part that does depend on U and defined

$$\mathcal{W}_g(s, U_1, U_2) = \langle U_1|\, e^{sD^2} |U_2\rangle\,, \tag{187}$$

the kernel of the scalar propagator in the shock wave space-time, where, to leading order in h, D^2 is given by

$$\begin{aligned} D^2 &= 2\,\partial_U\partial_V - 2B_U B_V - \vec{B}_\top\cdot\vec{B}_\top + h\,B_V B_V - h\,\partial^2_V \\ &= -2\,\partial^2_U - 2v^2U^2 - b^2 + h\left[-\partial^2_U - \frac{b_z^2}{2}\right]. \end{aligned} \tag{188}$$

In writing (188), we have used (174), (177) and (179) after replacing for V and $x_\top$ the values on the trajectory. In the the term proportional to h, we have neglected contributions proportional to v because this part is going to be multiply by v^4. The operator D^2 acts on the components ϕ_a of the matrix-valued field $\phi = \phi^a\,\tau_a/2$ and B_U, B_V and $B_\top$ in eq. (188) are the coefficients in front of $\tau_3/2$ in eq. (177) that remain after performing the trace over the gauge group. Terms linear in B_μ do not contribute to the trace since $B \propto \tau_3$.

The kernel (187) can be expanded around the flat space-time ($h = 0$) part:

$$\mathcal{W}_g(s,U) \equiv \lim_{U_1,U_2\to U} \mathcal{W}_g(s,U_1,U_2) = \mathcal{W}_\eta(s,U) + h\,\Omega(s,U) + O(h^2)\,, \tag{189}$$

where

$$\mathcal{W}_\eta(s,U) = \lim_{U_{1,2}\to U} \langle U_1|e^{-s(\partial_U^2+2v^2U^2+b^2)}|U_2\rangle \tag{190}$$

is just the kernel for the harmonic oscillator, that is

$$\mathcal{W}_\eta(s,U) = \sqrt{\frac{v}{2\pi\sin 4vs}}\,e^{-sb^2}\,e^{vU^2(\cos 4vs-1)/\sin 4vs}\,. \tag{191}$$

For the flat space-time case, (183) reproduces the known result [20, 31, 34] for the phase shift of two interacting D-particles. In fact, traces of odd powers of F^μ_ν vanish and we have that

$$\mathrm{Tr}_{\mu\nu}\, e^{2isF^\nu_\mu} - 2 = (10-2) + 2\,(\cos 4\,vs - 1) \tag{192}$$

and

$$\mathrm{Tr}_\gamma\, e^{s\Sigma_{\mu\nu}F^{\mu\nu}/2} = 16\cos 2\,vs\,, \tag{193}$$

since $\mathrm{Tr}_{\mu\nu}\, F^2 = 8v^2$ and

$$\mathrm{Tr}_\gamma\, (\Sigma_{\mu\nu}F^{\mu\nu})^2 = -(16\times 2)\,\mathrm{Tr}_{\mu\nu}\, F^2\,, \tag{194}$$

where the factor 16 comes from the Dirac trace.

By taking the traces in (185) and (186) we thus find

$$a\,(b,v) \simeq \int_0^\infty \frac{\mathrm{d}s}{s}\int \mathrm{d}U\,\mathcal{W}_\eta(s,U)\left[4\cos 2\,vs - \cos 4\,vs - 3\right], \tag{195}$$

where

$$\int \mathrm{d}U\,\mathcal{W}_\eta(s,U) = \frac{e^{-sb^2}}{2\sin 2\,sv}\,. \tag{196}$$

In the light-cone formalism, we reproduce the formulas of refs. [20, 31, 34] with $2\,v$ in the place of v there. The different factor in front of v is absorbed in the overall normalization since the use of these formulas for string theory makes sense only up to $O(v^4)$ (actually, in the flat case, $O(v^3)$, due to eq. (196)).

As we shall see, contrary to the flat space-time case, the amplitude in the shock-wave background is proportional to $h = f(r_\mathrm{T})\delta(U)$, and therefore the integration over U yields $\int \mathrm{d}U\,h\,\mathcal{W} \simeq f(r_\mathrm{T})e^{-sb^2}$ instead of (196). In agreement with the string amplitude (155), the amplitude in the curved background is thus $O(v^4)$ rather than $O(v^3)$, as in the flat case.

5.3.3 The scattering amplitude

In order to compute (183) in the non-flat metric, we must expand the exponential functions in (185) and (186) in powers of h. To the leading order in h, we have that

$$\mathrm{Tr}_{\mu\nu}F^2 = 8\left(v^2 + h\,v_z^2\right)\,,\quad \mathrm{Tr}_{\mu\nu}F^4 = 32\left(v^4 + 2h\,v_z^2 v^2\right)\,. \tag{197}$$

As in the flat-space case above, (194) holds together with

$$\mathrm{Tr}_\gamma\,(\Sigma_{\mu\nu}F^{\mu\nu})^4 = -(16\times 16)\,\mathrm{Tr}_{\mu\nu}F^4 \tag{198}$$

and it is true in general that $(\mathrm{Tr}\ F^2)^2 = 2\,\mathrm{Tr}\ F^4$.

We can thus expand to fourth order the exponential functions and write that

$$-\frac{1}{2}\left(\mathrm{Tr}_{\mu\nu}\,e^{2isF^\nu_\mu} - 2\right) + \frac{1}{4}\left(\mathrm{Tr}_\gamma\,e^{s\Sigma_{\mu\nu}F^{\mu\nu}/2}\right) = -\frac{s^4}{4}\,\mathrm{Tr}\ F^4\,, \tag{199}$$

where the constant terms as well as those quadratic in the velocity have cancelled as it happens in the flat-metric case.

By means of (199), we can now write (183) as

$$a\,(r_\mathrm{T}, b, v) \simeq \int \mathrm{d}U \int \frac{\mathrm{d}s}{s}\left[\mathcal{W}_\eta(s,U) + h\,\Omega(s,U)\right]\left(-\frac{s^4}{4}\,\mathrm{Tr}\ F^4\right)\,, \tag{200}$$

where $\Omega(s,U)$ is given by

$$\begin{aligned}\Omega(s,U) &= -\lim_{U_{1,2}\to U} s\left[\partial^2_{U_1} + \frac{b_z^2}{2}\right]\mathcal{W}_\eta(s,U_1,U_2)\\ &= -s\left[\frac{1}{4s} + \frac{b_z^2}{2}\right]\mathcal{W}_\eta(s,U)\,,\end{aligned} \tag{201}$$

and $\mathcal{W}_\eta$ by (191).

Collecting all terms linear in h yields, apart for an overall factor:

$$a\,(r_\mathrm{T}, b, v) \simeq 4\int \mathrm{d}U\, f(\vec{r}_\mathrm{T})\delta(U)\int \mathrm{d}s\ s^{5/2}e^{-sb^2}\left\{\frac{7}{4}v^4 - 2\,v^2\,v_\mathrm{T}^2 - s\,b_z^2\,\frac{v^4}{2}\right\}\,, \tag{202}$$

where we have now taken (191) at $v = 0$.

5.3.4 Comparison with the string result

Going back to the string computation, the amplitude (155) is what we want to compare with the matrix theory result in (202). After changing $\ell \to 1/2s$, setting $d = 10$, and normalizing the incoming and outgoing source-graviton states each by the usual factor $1/\sqrt{k_0}$, we obtain that the string amplitude is (up to an overall constant)

$$a\left(r_{\mathrm{T}}, b, v\right) \simeq f(\vec{r}_{\mathrm{T}}) \int \mathrm{d}s\, s^{5/2} e^{-sb^2} \left\{\frac{11}{4}\, v^4 - 2\, v_{\mathrm{T}}^2\, v^2 - s\, b_z^2\, \frac{v^4}{2}\right\} . \tag{203}$$

since

$$f(\vec{r}_{\mathrm{T}}) \,\propto\, k_0 \int \frac{\mathrm{d}^8 \vec{q}_{\mathrm{T}}}{(2\pi)^8} \frac{1}{\vec{q}_{\mathrm{T}}^{\,2}}\, e^{i\vec{r}_{\mathrm{T}}\cdot\vec{q}_{\mathrm{T}}} . \tag{204}$$

After integrating (202) over U, we see that the matrix model result (up to an overall constant) becomes equal to that of string theory except for the numerical factor $7/4$ instead of $11/4$ in front of the v^4 term. This missing term comes from the inclusion of the Jacobian arising from the δ-function constraining the D-particles to lie on their respective trajectories, which in our formalism amounts to implementing (175) and (176) for both. In order to enforce this constraint, we replace the integration over U by

$$\begin{aligned}
\int \mathrm{d}U \;\rightarrow\; & \int \mathrm{d}U \prod_{i=1,2} \int \mathrm{d}B^U_{(i)} \mathrm{d}B^V_{(i)}\, \delta\left(B^U_{(i)} + B^V_{(i)} - \sqrt{2}\, z^{(i)}\right) \delta\left(B^U_{(i)}(1-h) - B^V_{(i)}\right) \\
& \simeq \frac{1}{4} \int \mathrm{d}U\, (1+h) \\
& \times \prod_{i=1,2} \int \mathrm{d}B^U_{(i)} \mathrm{d}B^V_{(i)}\, \delta\left(B^U_{(i)} - \frac{1}{\sqrt{2}}\left(1+\frac{h}{2}\right) z^{(i)}\right) \delta\left(B^V_{(i)} - \frac{1}{\sqrt{2}}\left(1-\frac{h}{2}\right) z^{(i)}\right) .
\end{aligned} \tag{205}$$

The extra (leading in h) factor $1+h$ in front, after multiplication by $-s^4\,\mathrm{Tr}\,F^4/4$, provides the missing term $4s^4v^4h$ that adds a term v^4 to (203) and makes the matrix-model result agree with that of string theory, since $7/4+1 = 11/4$, and we finally obtain that

$$a\left(r_{\mathrm{T}}, b, v\right) \simeq f(\vec{r}_{\mathrm{T}})\, \frac{v^2}{b^7} \left[\frac{11}{4}\, v^2 - 2\, v_{\mathrm{T}}^2 - \frac{7}{4} \frac{b_z^2}{b^2}\, v^2\right] . \tag{206}$$

A Diagonalization of $F_{\mu\nu}$

A non-zero background field $F_{\mu\nu}$ can be block-diagonalized in any dimension D: $F_{01} = -F_{10} = \mathcal{E}$, $F_{23} = -F_{32} = \mathcal{B}_1$, $F_{45} = -F_{54} = \mathcal{B}_2$, etc., and $F_{ij} = 0$ for $i \neq j \pm 1$. $\mathcal{E}$ represents the electric-like eigenvalue of $F_{\mu\nu}$, whereas the $\mathcal{B}$'s are the magnetic-like ones. The PPR rate being a relativistic invariant, one can calculate it either by first diagonalizing $F_{\mu\nu}$ and path integrating subsequently, or by path integrating directly. The second approach will be mentioned later (eq. 65). We will use a block-diagonalized $F_{\mu\nu}$ and we will obtain the PPR rate as a function of its eigenvalues. It is thus of interest to study their dependence on the initial, in general non-diagonal, field strength.

The most important effect of non-diagonal F_{ij}'s is to change $\mathcal{E}$ - which enters the exponential factor of the PPR. Inserting an imaginary factor in front of the electric components F_{0j}, in order to use the Euclidean metric in the eigenvalue equation $det(F_{\mu\nu} - \eta_{\mu\nu}\lambda) = 0$, $F_{\mu\nu}$ reads, in $D = 4$,

$$F = \begin{pmatrix} 0 & iE & 0 & 0 \\ -iE & 0 & b & 0 \\ 0 & -b & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix}. \tag{207}$$

B and b are the components of the magnetic field parallel, respectively orthogonal, to the electric field E; $\mathcal{E}$, the real eigenvalue of F, is

$$2(\mathcal{E}^2)_{1,2} = \sqrt{(B^2 + b^2 - E^2)^2 + 4E^2B^2} - (B^2 + b^2 - E^2). \tag{208}$$

For $b = E$, $\mathcal{E}$ decreases with respect to the case $b = 0$; it vanishes if $B = 0$. In five dimensions, for a field strength tensor with non-zero components

$$F_{01} = iE, \quad F_{12} = b, \quad F_{34} = B, \tag{209}$$

and if $b = E$, the electric-like eigenvalue is zero for any B. This happens because, inside the matrix given by (209), the block containing E and b does not have lines or columns in common with the block containing the B's.

In general, for an electric field parallel to the x-axis and a purely magnetic block partially diagonalized (with the F_{1j}'s left in their original form)

$$F = \begin{pmatrix} 0 & iE & 0 & 0 & 0 & 0 & \cdot \\ -iE & 0 & b_1 & b_2 & b_3 & b_4 & \cdot \\ 0 & -b_1 & 0 & B_1 & 0 & 0 & \cdot \\ 0 & -b_2 & -B_1 & 0 & 0 & 0 & \cdot \\ 0 & -b_3 & 0 & 0 & 0 & B_2 & \cdot \\ 0 & -b_4 & 0 & 0 & -B_2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \tag{210}$$

the magnetic components fall into two classes: 1) The F_{1j}'s, called here b's; they decrease the electric eigenvalue $\mathcal{E}$ and might even cancel it (as in eq.209), unless they sit above a 2×2 block containing a non-zero B (as in eq.207). 2) The other components, the B's; they do not influence $\mathcal{E}$ in absence of the b's; for non-zero b's, they temper the reduction of $\mathcal{E}$ those produce. At fixed F_{1j}'s, $\mathcal{E}$ grows when the B's are increased. This is easily seen up to $D = 5$ space-time dimensions and, in principle, also up to $D = 9$, by finding analytic expressions for the eigenvalues. In ten dimensions the characteristic equation of the matrix F becomes of degree five and is not any more solvable by radicals. We have tested numerically various cases for D from 6 to 10, and the conclusions above held. The F_{1j}'s reduce $\mathcal{E}$ and the production rate, whereas the other magnetic components temper their decreasing effect if their corresponding planes intersect. This is probably true in any space-time dimension. Moreover, in higher dimensions the effect of a given variation of one single F_{ij} is less important than a similar change in lower dimensions. The other numerous components provide a kind of inertia.

One can also fill the empty off-diagonal magnetic part of F (e.g. the 4×4 matrix containing B_1 and B_2 in (210)). Increasing those components might decrease or increase $\mathcal{E}$, but their influence is small, being supressed by at least one order of magnitude with respect to their initial variation.

For completeness we also prove that , given an antisymmetric matrix

$$F = \begin{pmatrix} 0 & F_{01} & F_{02} & F_{03} & F_{04} & F_{05} & \cdots \\ F_{10} & 0 & F_{12} & F_{13} & F_{14} & F_{15} & \cdots \\ F_{20} & F_{21} & 0 & F_{23} & F_{24} & F_{25} & \cdots \\ F_{30} & F_{31} & F_{32} & 0 & F_{34} & F_{35} & \cdots \\ F_{40} & F_{41} & F_{42} & F_{43} & 0 & F_{45} & \cdots \\ F_{50} & F_{51} & F_{52} & F_{53} & F_{54} & 0 & \cdots \\ . & . & . & . & . & . & \cdots \end{pmatrix} \tag{211}$$

$$F_{ij} = -F_{ji}, \tag{212}$$

we have

$$Det(F)_{2n\times 2n} = \left[\frac{1}{2^n} \sum_{a_i} \varepsilon_{a_1 a_2 \ldots a_{2n}} F^{a_1 a_2} F^{a_3 a_4} \ldots F^{a_{2n-1} a_{2n}} \right]^2 . \tag{213}$$

This allows us to write

$$Det(F - \lambda I) = \sum_{k=1}^{n} \left(\lambda^{2(n-k)} \left[\frac{1}{2^k} \sum_{a_1 \ldots a_{2k}} \varepsilon_{a_1 a_2 \ldots a_{2k}} F^{a_1 a_2} F^{a_3 a_4} \ldots F^{a_{2k-1} a_{2k}} \right]^2 \right) , \tag{214}$$

$$Det(F)_{(2n+1)\times(2n+1)} = 0. \tag{215}$$

Proof of equation (213):

note We take here n to be even, and the matrix to be $n \times n$.

important Better to do it first on an explicit example (8×8, say).

Each term of the determinant (213) contains a product of n F_{ij}'s, with a total of $2n$ indices, which can be *split* into two subsets, each containing once and only once each of the n indices.

(The splitting is made in the follwing way,

$1 \to i_1$

$2 \to i_2$ if $i_1 \neq 2$, otherwise take next index,

$3 \to i_3$ if $i_1, i_2 \neq 2, 3$ otherwise take next index,

$4 \to i_4$ if $i_{1,2,3} \neq 2, 3, 4$, etc.,

and it is not unique.)

Now, for each splitting, we get two separate products of $n/2$ F's., which we take as our basic units, denoted by $F_1^{(n/2)}$ and $F_1^{(n/2)}$. Form with them F_1^2, F_2^2 and $2F_1F_2$. Each one of these is present in the determinant: the first two by construction are just a choice of elements symmetric with respect to the principal diagonal, whereas the last has not anymore that symmetry, and hence appears twice (since the 'mirror' partner is also a term of the determinant). Thus, our determinant has the form (213).

B Boundary state formalism

We review here the boundary state formalism [21], applying it to our case. The strategy is the following: We first establish the boundary conditions along σ, in the open string channel. We then switch to the closed string channel and obtain the operatorial boundary conditions there. Our amplitude will be the scalar product between the state satisfying them at $\sigma = 0$ and at $\sigma = l$. Since the uncoupled case (X_2) is easily obtainable from the coupled one, we restrict to the action along X_0 and X_1:

$$S = \int_0^t d\tau \int_0^l d\sigma[-\frac{M^2}{2}(\frac{\partial X^0}{\partial \tau})^2 - \frac{M^2}{2}(\frac{\partial X^0}{\partial \sigma})^2 + \frac{M'^2}{2}(\frac{\partial X^1}{\partial \tau})^2 + \frac{M'^2 v^2}{2}(\frac{\partial X^1}{\partial \sigma})^2] - i\int_0^t d\tau [EX^0 \frac{\partial X^1}{\partial \tau}]_{\sigma=l}$$

The boundary conditions at $\sigma = l$ are:

$$(M^2 \frac{\partial X_n^0}{\partial \sigma} + iE\frac{\partial X_n^1}{\partial \tau})_{\sigma=l} = 0 \tag{216}$$

$$(M'^2 v^2 \frac{\partial X_n^1}{\partial \sigma} + iE\frac{\partial X_n^0}{\partial \tau})_{\sigma=l} = 0. \tag{217}$$

At $\sigma = 0$ we just put $E = 0$. Assuming now periodic boundary conditions along τ, $X(\tau, \sigma) = X(t + \tau, \sigma)$, we Fourier expand X with respect to τ :

$$X(\tau, \sigma) = \sum_{nZ} e^{i\omega_n \tau} X_n(\sigma) \quad \text{where} \quad \omega_n = \frac{2\pi}{t} n.$$

The boundary conditions for the X_n's are given by the Fourier transform of eqs.(32-33):

$$\frac{\partial X_n^0}{\partial \sigma}|_{\sigma=0} = 0 \qquad M^2 \frac{\partial X_n^0}{\partial \sigma}|_{\sigma=l} - \omega_n E X_n^1 |_{\sigma=l} = 0 \tag{218}$$

$$\frac{\partial X_n^1}{\partial \sigma}|_{\sigma=0} = 0 \qquad M'^2 v^2 \frac{\partial X_n^1}{\partial \sigma}|_{\sigma=l} - \omega_n E X_n^0 |_{\sigma=l} = 0. \tag{219}$$

The free equations of motion (there is no electric field in the bulk) are:

$$\frac{\delta S(n)}{\delta X_{-n}^0(\sigma)} = -M^2 \omega_n^2 X_n^0 + M^2 \frac{\partial^2 X_n^0}{\partial \sigma^2} = 0$$

$$\frac{\delta S(n)}{\delta X_{-n}^1(\sigma)} = M'^2 \omega_n^2 X_n^1 - M'^2 v^2 \frac{\partial^2 X_n^1}{\partial \sigma^2} = 0.$$

Until now, we spoke about the periodic propagation of an open string and consequently developed X on open-string modes. Now we reverse the picture (i.e. the roles of Euclidean τ and σ) and look at a closed string parametrised by τ, propagating in the 'time' σ, for σ from 0 to l. Hence let us develop X in closed-string modes:

$$X(t,\sigma) = X_0 + \frac{i}{\sqrt{4\pi}}\sum_{n\geq 1}\frac{1}{\sqrt{n}}(e^{i\omega_n\tau-\omega_n\sigma\frac{1}{v}}X_n - e^{-i\omega_n\tau+\omega_n\sigma\frac{1}{v}}X_{-n}+ \tag{220}$$

$$+\tilde{X}_n e^{-i\omega_n\tau-\omega_n\sigma\frac{1}{v}} - \tilde{X}_{-n}e^{i\omega_n\tau+\omega_n\sigma\frac{1}{v}}).$$

Using that we obtain the boundary conditions for the closed string modes:

$$X_n^0(l) + \tilde{X}_{-n}^0(l) = -\frac{E}{M^2}(X_n^1(l) - \tilde{X}_{-n}^1(l)) \qquad \forall n \tag{221}$$

$$X_n^1(l) + \tilde{X}_{-n}^1(l) = -\frac{E}{M'^2v}(X_n^0(l) - \tilde{X}_{-n}^0(l)) \qquad \forall n. \tag{222}$$

At $\sigma = 0$ the right-hand-side vanishes. For $E = 0$ we get the boundary condition for an uncoupled coordinate X:

$$X_n(\sigma = 0, l) + \tilde{X}_{-n}(\sigma = 0, l) = 0 \qquad \forall n. \tag{223}$$

Next we find for the boundary state $|B_E>$ satisfying (37,38) the expression (up to a normalization constant $N^{\frac{1}{2}}(E)$):

$$|B_E>= exp\left\{\frac{1}{\frac{E^2}{M^2M'^2v}-1}\sum_{n>0}[(1+\frac{E^2}{M^2M'^2v})(X_{-n}^1\tilde{X}_{-n}^1 - X_{-n}^0\tilde{X}_{-n}^0)\right.$$

$$\left. +2E(\frac{X_{-n}^0\tilde{X}_{-n}^1}{M^2} - \frac{X_{-n}^1\tilde{X}_{-n}^0}{M'^2v})]\right\}|0> \tag{224}$$

the state and the operators X being taken at $\sigma = l$. At $\sigma = 0$ this becomes

$$|B_0>= exp\sum_{n\geq 1}(X_{-n}^0\tilde{X}_{-n}^0 - X_{-n}^1\tilde{X}_{-n}^1)|0> .$$

Now, the quantity of interest is, up to a normalization factor $N(E)$:

$$< B_E(l)|B_0>=< B_E|e^{-lH}|B_0>= N^{-1}(E)\int DX^0(\sigma,t)DX^1(\sigma,t)e^{-S} \tag{225}$$

which we are going to calculate in the main body of the paper, obtaining thus the required path integral.

C One useful formula

We prove here the equality - eq.(105) in the text:

$$< 0|e^{\kappa b\tilde{b}}e^{\lambda a\tilde{a}}e^{\mu a\tilde{b}}e^{\nu b\tilde{a}}e^{\rho a^\dagger \tilde{a}^\dagger}e^{\sigma b^\dagger \tilde{b}^\dagger}|0 >= \frac{1}{(1-\kappa\sigma)(1-\lambda\rho)-\mu\nu\rho\sigma} \tag{226}$$

where a,$\tilde{a}$,b,$\tilde{b}$ are independent annihilation operators, while $a^\dagger$,$\tilde{a}^\dagger$,$b^\dagger$,$\tilde{b}^\dagger$ are the corresponding creation operators.

We expand the exponentials in power series in the left-hand-side of (42) and keep only the non-zero terms. Then, by using the harmonic oscillator normalization $< 0|a^n(a^\dagger)^n|0 >= n!$, as well as a trick - differentiating and then resuming - we obtain what follows:

$$< 0|e^{\kappa b\tilde{b}}e^{\lambda a\tilde{a}}e^{\mu a\tilde{b}}e^{\nu b\tilde{a}}e^{\rho a^\dagger \tilde{a}^\dagger}e^{\sigma b^\dagger \tilde{b}^\dagger}|0 >=$$

$$= \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} < 0|\frac{(\kappa b\tilde{b})^s}{s!}\frac{(\lambda a\tilde{a})^r}{r!}\frac{(\mu a\tilde{b})^m}{m!}\frac{(\nu b\tilde{a})^m}{m!}\frac{(\rho a^\dagger\tilde{a}^\dagger)^{m+r}}{(m+r)!}\frac{(\sigma b^\dagger\tilde{b}^\dagger)^{m+s}}{(m+s)!}|0 >=$$

$$= \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\kappa^s}{s!}\frac{\lambda^r}{r!}\frac{\mu^m}{m!}\frac{\nu^m}{m!}\frac{\rho^{m+r}}{(m+r)!}\frac{\sigma^{m+s}}{(m+s)!}[(r+m)!(s+m)!]^2 =$$

$$= \sum_{m=0}^{\infty}\frac{1}{(m!)^2}(\mu\nu)^m\frac{d^m}{d\kappa^m}(\frac{1}{1-\kappa\sigma})\frac{d^m}{d\lambda^m}(\frac{1}{1-\lambda\rho}) =$$

$$= \sum_{m=0}^{\infty}(\mu\nu)^m(\rho\sigma)^m\frac{1}{(1-\kappa\sigma)^{m+1}}\frac{1}{(1-\lambda\rho)^{m+1}} =$$

$$= \frac{1}{(1-\kappa\sigma)(1-\lambda\rho)-\mu\nu\rho\sigma}.$$

For $\mu = \nu = 0$ we also obtain as a particular case:

$$< 0|e^{\kappa b\tilde{b}}e^{\lambda a\tilde{a}}e^{\rho a^\dagger \tilde{a}^\dagger}e^{\sigma b^\dagger \tilde{b}^\dagger}|0 >= \frac{1}{(1-\kappa\sigma)(1-\lambda\rho)}.$$

D String propagators

The bosonic propagator is given by (see, for instance, [47])

$$\langle X^\mu(z) X^\nu(w)\rangle = -\eta^{\mu\nu} G(z,w) - S^{\mu\nu} G(z,\bar{w})\,, \tag{227}$$

where the metric signature is given by $\eta^{\mu\nu} = (-1, \delta^{ij})$,

$$G(z,w) = \frac{1}{4\pi} \ln \left| \frac{\vartheta_1(z-w)}{\vartheta_1'(0)} \right| - \frac{\mathrm{Im}^2(z-w)}{2\,\mathrm{Im}\,\tau}\,, \tag{228}$$

$S^{00} = -1$ and $S^{ij} = -\delta^{ij}$. Everywhere, we write

$$\vartheta_i(x) \equiv \vartheta_i(x \mid \tau) \tag{229}$$

where $\tau = 2i\ell$.

The fermionic propagator at velocity $v = 0$ is

$$\langle \psi^\mu(z) \psi^\nu(w)\rangle_S = -\frac{\eta^{\mu\nu}}{4\pi} P_S \tag{230}$$

where

$$P_S = \frac{\vartheta_S(z-w)\vartheta_1'(0)}{\vartheta_1(z-w)\vartheta_S(0)}\,. \tag{231}$$

At $v \neq 0$ we have that (here the direction 1 is parallel and the directions i, j are orthogonal to $\vec{v}$) [48]

$$\begin{aligned}
\langle \psi^0(z)\psi^0(w)\rangle_S &= \langle \psi^1(z)\psi^1(w)\rangle_S = \frac{1}{4\pi} Q_S \\
\langle \psi^0(z)\psi^1(w)\rangle_S &= \langle \psi^1(z)\psi^0(w)\rangle_S = \frac{1}{4\pi} R_S \\
\langle \psi^i(z)\psi^j(w)\rangle_S &= \frac{\delta^{ij}}{4\pi} P_S\,,
\end{aligned} \tag{232}$$

where the spin-structure-dependent functions are defined as

$$Q_S = \frac{1}{2}\left[e^{v} \frac{\vartheta_S(w - iv/\pi)\,\vartheta_1'(0)}{\vartheta_S(-iv/\pi)\,\vartheta_1(w)} + e^{-v} \frac{\vartheta_S(w + iv/\pi)\,\vartheta_1'(0)}{\vartheta_S(-iv/\pi)\,\vartheta_1(w)} \right] \tag{233}$$

$$R_S = \frac{1}{2}\left[e^{v} \frac{\vartheta_S(w - iv/\pi)\,\vartheta_1'(0)}{\vartheta_S(-iv/\pi)\,\vartheta_1(w)} - e^{-v} \frac{\vartheta_S(w + iv/\pi)\,\vartheta_1'(0)}{\vartheta_S(-iv/\pi)\,\vartheta_1(w)} \right]. \tag{234}$$

In performing the sum over the spin structure, it is useful to use Riemann's identity:

$$\frac{1}{2}\sum_S (\pm)_S\, \vartheta_S(u)\vartheta_S(v)\vartheta_S(w)\vartheta_S(s) = \vartheta_1(u_1)\vartheta_1(v_1)\vartheta_1(w_1)\vartheta_1(s_1)\,, \tag{235}$$

where

$$\begin{aligned}
u_1 &= (u+v+w+s)/2 \qquad & v_1 &= (u+v-w-s)/2 \\
w_1 &= (u-v+w-s)/2 \qquad & s_1 &= (u-v-w+s)/2\,.
\end{aligned} \tag{236}$$

E Shock-wave metric

The space-time of the shock wave is defined by the metric element (165) which gives a metric tensor (its signature is such that $\eta_{00} = -1$) with components:

$$g_{UU} = h\,, \quad g_{UV} = 1\,, \quad g_{VV} = 0\,, \quad g_{ij} = \delta_{ij} \tag{237}$$

and

$$g^{UU} = 0\,, \quad g^{UV} = 1\,, \quad g^{VV} = -h\,, \quad g^{ij} = \delta^{ij}\,, \tag{238}$$

where $h = f(r_{\mathrm{T}})\delta(U)$.

Similarly, the einbein, necessary in writing the fermionic action, is given by

$$e^U_U = e^V_U = 1\,, \quad e^U_V = 0\,, \quad e^V_U = \frac{h}{2}\,, \quad e^j_i = \delta^i_j \tag{239}$$

where $e^a_\mu \eta_{ab} e^b_\nu = g_{\mu\nu}$. The spin connection is zero: $\omega_{ab\mu} = 0$.

The action for the motion of one particle in the shock-wave metric is found by varying the metric line element (165) and it is given by

$$S = \int \mathrm{d}s \left[2\frac{\mathrm{d}U}{\mathrm{d}s}\frac{\mathrm{d}V}{\mathrm{d}s} + f\delta(U)\left(\frac{\mathrm{d}U}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}\vec{x}_{\mathrm{T}}}{\mathrm{d}s}\right)^2 \right]. \tag{240}$$

The trajectory is therefore

$$\left\{ \begin{array}{rcl} V &=& U\,w + w_0 - \frac{1}{2} f\theta(U) \\ U &=& s \\ \vec{x}_{\mathrm{T}} &=& \vec{v}_w\,U + \vec{b}_w\,, \end{array} \right. \tag{241}$$

where the coefficients are given in section 5 by eqs. (172) and (173).

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