# Liouville Field Theory, Drinfel'd-Sokolov Linear System and Riemann Surfaces

Ettore Aldrovandi

# Contents

# Introduction

This thesis deals entirely with the Liouville equation in various contexts. The Liouville equation has been playing an important rôle in mathematics since the works by Poincaré, Klein and Koebe, concerning uniformization, at the beginning of this century. For a review on uniformization from the analytic point of view see [B], which also contains an extensive bibliography.

On the physical side, it is mainly concerned with string theory and has become definitely popular among Physicists since the work by Polyakov [P] on the bosonic string. In string theory the Liouville field describes the conformal anomaly arising in the string partition function (when the sum over the metric is included). Since then, a lot of work on the quantization of the Liouville field had been undertaken, notably by Brateen, Curtright, Ghandour and Thorn, [BCGT] and Gervais and Neveu [GN]. The second reference contains also a careful treatment of the classical theory, both with open and closed string boundary conditions. A further improvement has come from the works by Knizhnik, Polyakov and Zamolodchikov [KPZ] and by Distler and Kawai [DK], where they take on the study of the quantum theory of the non-critical string and critical exponents for 2D quantum gravity are calculated.

At present, Liouville Field theory has enormous ramifications and it is a very active field of research. Of course, we are interested in the mathematical aspects. Recently, there has been an enormous interplay between Mathematics and Physics, mainly motivated by problems arising in two-dimensional physics. Liouville theory is not an exception. Zograf and Takhtadzhyan, for instance, have recently shown that the Liouville action functional is the generating functional for the accessory parameters of the Fuchsian and Schottky uniformization of a Riemann Surface [ZT], thereby proving a conjecture of Polyakov and closing a circle of ideas that dates back to Poincaré.

We have made this rather long introduction about interesting issues in literature just to point out what the general framework is. However, we must explain what this thesis is concerned with. The main point is to understand the classical phase space, namely the space of solutions of the Liouville equation. The relevance of point becomes obvious if one is willing to address the problem of a more rigorous approach to quantization. This becomes even more important when non-trivial topologies are taken into account. The relevance of this, as far as string theory and 2D-gravity are concerned, need not to be mentioned.

To better understand the classical phase space means to be able to produce a sufficiently simple and explicit parametrization of it. Since the Liouville equation defines an integrable system, this parametrization is given (or should be given) by a *free field parametrization*, namely a coordinatization of the phase space in terms of free bosonic oscillators. This is the point where we focus our attention. However, this is still a rather general scheme, so that in

this work we mainly analyze two rather different problems.

To enter in some more details, we have as main motivation the work [BBT]. As it will be explained in the introductions to the single chapters, in that paper it is shown how to construct solutions of the Toda Field equations *starting* from the datum of free bosonic oscillators. The solutions so constructed are periodic and *local*, in the sense that they commute at equal time. By construction, the phase space for these solutions is described by free fields. The natural question that arise is to see whether this procedure exhausts all the physically reasonable solutions[1]. We used the Liouville equation as a laboratory to answer to this question. The procedure we adopted is to represent a given solution as the integrability condition for a flat connection. This is the "zero-curvature" representation for the integrable systems and it is by now very standard. Then we started moving backward with respect to the logic of [BBT], in the hope to produce a free field representation. The procedure is not completely obvious, but the answer is essentially affirmative. The subtle point is that the representation is not unique, even though there is a clear symmetry that links the two interesting solutions we have found.

The motivation for the second part of the work, which is the most extended one, is still to be found in [BBT]. In this case we asked ourselves whether it is possible to generalize the techniques of [BBT] to cover the case of higher genus Riemann Surfaces. In this case we tried to reproduce the procedure of [BBT] to obtain a solution of the Liouville equation on a Riemann Surface. We found that formally the same procedure can in fact be implemented, with an important modification with respect to the case of the cylinder. On a Riemann Surfacethe modes of the would-be free oscillator are *constrained*: this is necessary in order to obtain a single valued solution and it cannot be avoided. Moreover, due to the existence of these constraints, the Poisson structure must be modified by using the machinery of Dirac brackets, a features which is definitely not present in the genus zero case. Besides this, the analogy between the two approaches is a very strict one. The two only points that are unfortunately unsolved yet are the calculation of the full exchange algebra of the chiral fields and consequently the proof of locality in the large over the surface. We believe that this requires some new insight.

Finally, some words must be spent about the subject dealt with in the last chapter, which has a rather ambiguous character. The reason of its presence is twofold. First of all, it contains some rather long sections reviewing the uniformization theory of compact Riemann Surfaces and the connection it has with the Liouville equation. It has the function of clearly stressing that *our* approach to the Liouville equation is not (at least immediately) connected with uniformization. At the same time it is there in order to give geometric support to the constructions of the first chapter and to give an outlook to their possible formulation on a compact curve. In the attempt to geometrize the transformations leading to the Drinfel'd-Sokolov differential system widely used in this thesis, we are led to discuss the Miura transformation. In this context we interpret the Miura transformation as a distribution in the holomorphic tangent bundle to a ruled surface and the Drienfel'd-Sokolov connection as a *partial connection* on a principal fibration over this surface. Both this issues are, to the extent of our knowledge, new.

The material contained in this thesis is organized as follows. The first chapter deals

[1] However pathologies are maybe always producible

with the first problem we mentioned, namely the one concerning the generality of the free field representation on a cylinder. The approach to the Liouville equation on the Riemann Surface using the method of [BBT] is developed in chapters two and three. The division is motivated only by the large dimension a unified chapter would have reached. Finally, the geometric issues inspired also by the first part and the discussion of the geometry of the Miura transformation and the Drinfel'd-Sokolov connection, are presented in the fourth chapter.

# Chapter 1

# Free Fields in Liouville Field Theory

## 1.1 Introduction

The aim of this chapter is to explain how to explicitly construct a free field representation for the classical Toda Field Theory. More precisely, starting from a given solution of the equations of motion we construct a set of Poisson commuting free fields of definite chirality, and we show that the datum of these two objects is sufficient to reconstruct the solution in question. This completes, in some sense, the ideas of [BBT]: there, it is shown that starting from the free field picture it is indeed possible to construct solutions to the Toda field equations satisfying the physical requirement of locality.

Thus we ask ourselves to what extent is this framework valid or, in other terms, if it is true that for any solution of the Toda equations of motion we can exhibit a set of Poisson commuting free Bosonic fields. This constitutes the main motivation in addressing such a problem. It should be also noticed that the representation in terms of free fields is in fact an explicit parametrization of the phase space of the theory, or at least of that part for which a free field representation can be formulated. Therefore all can be reformulated by saying that we tried to see whether this coordinatization has a global meaning.

The construction we aim at goes along the same lines as in [LS], where it is shown that the integration of the Toda equation of motion can be reduced to a factorization problem in group theory. This approach, which is by now very standard, relies heavily on the *zero curvature representation*, that is on the interpretation of the equations of motion as the integrability condition for a flat connection. The solution is constructed in terms of a properly defined factorization (with respect to the structure group) of the fundamental solution of the associated linear problem. It is in terms of this factorization that the free fields we mentioned above are constructed. In particular, they come out once the linear problem has been separated into two linear ones of definite chirality. It is customary to call them "Drinfel'd–Sokolov" (DS) linear systems [DS]. A natural question is to see how the geometric and symplectic aspects fit together. This happens in a completely non trivial way, in fact improving the Leznov–Saveliev analysis.

In summary what we shall do in the next sections is

1. to show how to produce the DS linear systems once we start from a given solution. To this aim we shall exploit the zero curvature representation.

2. to construct explicitly the free field representation and show that this representation is global, i.e. every solution can be described in this way. Thus we prove that at this point the scheme is coherent with the one in [BBT].

3. to carefully take care of the transformation of the symplectic structure in order to show that the Poisson brackets of the free fields correspond to the canonical ones of the Toda fields.

4. to outline a geometrical interpretation of the main steps of the construction, and in particular of the rôle of the DS linear system.

We shall perform most of the calculations on the Liouville model, i.e. the most simple example of Toda Field Theory, because in spite of its simplicity, it already contains many essential difficulties of the problem.

## 1.2 The Liouville Model

We consider the Liouville equation on a cylindrical space–time, where time is a straight line and space is a circle. We denote by $t, x$ the corresponding coordinates[1]. We consider also the light–cone coordinates $z_\pm = x \pm t$ and the corresponding derivatives $\partial_\pm = \frac{1}{2}(\partial_x \pm \partial_t)$. The Liouville equation is

$$\partial_+ \partial_- \varphi = e^{2\varphi} \tag{1.1}$$

Now let $\mathfrak{g}$ be the Lie algebra $sl_2(\mathbb{R})$ with the standard generators in the defining representation

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and commutation relations

$$[H, E_\pm] = \pm 2E_\pm \,, \; [E_+, E_-] = H$$

The Liouville equation is the integrability condition

$$F_{+-}(A) = \partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0$$

for the connection $A$ defined by

$$\begin{aligned} A_+ &= \partial_+ \Phi + e^{\varphi} E_+ \\ A_- &= -\partial_- \Phi + e^{\varphi} E_- \end{aligned}$$

where we put $\Phi = \frac{1}{2}\varphi H$. Thus when equation (1.1) is obeyed the connection $A$ is flat: the parallel transport along a path will depend only on its extreme points. A solution $\Psi$ of

$$(\partial_\pm \pm A_\pm)\Psi = 0 \tag{1.2}$$

[1] $x$ will be only defined up to integer multiples of $2\pi$.

will give the parallel translation of sections along any path. This operator can also be formally written with a path–ordered exponential. In particular, choosing a path moving around the cylinder at equal time we have

$$\Psi(x) = \mathbb{P}\exp(-\int^x (A_+ + A_-)\,dx)$$

where for the time being we leave unspecified boundary conditions.

We impose the equal time canonical Poisson brackets

$$\{\pi(x), \varphi(y)\} = \delta(x-y) \tag{1.3}$$

where $\pi$ is the canonical momentum conjugated with $\varphi$, i.e. $\pi(x) = \dot{\varphi}(x)$. The canonical structure we imposed on $\varphi$ leads to the following canonical stucture on the connection. The $x$–component of the connection is $A_x = A_+ + A_-$ and omitting the index $x$ we define

$$\{A(x) \overset{\otimes}{,} A(y)\} = \sum_{\mu,\nu}\{A^\mu(x), A^\nu(y)\}X_\mu \otimes X_\nu$$

where $\{X_\mu\}$ is any basis in the Lie algebra $\mathfrak{g}$. Using the canonical structure on $\varphi$ we find [B, BV]

$$\{A(x) \overset{\otimes}{,} A(y)\} = [r_{12}, A(x)\otimes 1 + 1 \otimes A(y)] \tag{1.4}$$

where the $r$–matrix is given by

$$\begin{aligned} r &= \frac{1}{2}(E_+ \otimes E_- - E_- \otimes E_+) + \lambda C \\ C &= H \otimes H + 2(E_+ \otimes E_- + E_- \otimes E_+) \end{aligned}$$

and $C$ is the Casimir element. Since $\{A(x) \overset{\otimes}{,} A(y)\}$ is a Poisson bracket, it has to satisfy the Jacobi identity and this yields relations on the $r$–matrix. Imposing the classical $r$–matrix to satisfy the classical Yang–Baxter equation (CYBE)[2]

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

allows us to fix the parameter $\lambda$. One has two solutions, namely

$$\begin{aligned} r^+ &= \frac{1}{4}(H \otimes H + 4E_+ \otimes E_-) \\ r^- &= -\frac{1}{4}(H \otimes H + 4E_- \otimes E_+). \end{aligned}$$

The importance of translating the Poisson bracket on the connection is that it permits us to calculate the Poisson bracket of the elements of the transport matrix in the form [B, BV]

$$\{\Psi(x) \overset{\otimes}{,} \Psi(x)\} = -[r^{\pm}_{12}, \Psi(x)\otimes\Psi(x)] \tag{1.5}$$

This relation is worked out using explicitly the representation of the transport matrix as a path–ordered exponential.

[2]The CYBE is a sufficient condition for the Jacobi identity to hold. The Jacobi identity only requires the element on the LHS of the CYBE to be *ad*–invariant.

## 1.3 The Leznov–Saveliev Analysis

We now explain the main points of the group theoretical analysis of (1.2), exploiting the Gauss factorization in $SL_2(\mathbb{R})$ [LS, BB] and keeping track of the gauge transformations the connection $A$ will undergo It is to be stressed that this procedure has become classical. It opens the door to the approach to integrable systems in terms of Lie Algebras.

The idea is to perform two series of gauge transformations with the aim of killing the chiral and the anti–chiral part of the connection, respectively. First of all we write $\Psi$ in two ways

$$\Psi = e^{\Phi} V = e^{-\Phi} \bar{V}$$

so that (1.2) is transformed in the following two linear problems

$$(\partial_{\pm} + A_{\pm}^{(1)})V = 0 \tag{1.6}$$

$$(\partial_{\pm} + A_{\pm}^{(2)})\bar{V} = 0 \tag{1.7}$$

where the connections $A^{(1)}$ and $A^{(2)}$ are given by

$$\begin{aligned} A_{+}^{(1)} &= 2\partial_{+}\Phi + E_{+} \\ A_{-}^{(1)} &= e^{2\varphi} E_{-} \\ A_{-}^{(2)} &= -2\partial_{-}\Phi + E_{-} \\ A_{+}^{(2)} &= e^{2\varphi} E_{+} \end{aligned}$$

Solving the above two linear problems allows us to reconstruct the solution to (1.1). Indeed we have

$$e^{-2\Phi} = V\,\bar{V}^{-1}$$

If we introduce the vectors

$$< \lambda_{+}| = \begin{pmatrix} 1 & 0 \end{pmatrix}, < \lambda_{-}| = \begin{pmatrix} 0 & 1 \end{pmatrix}, |\lambda_{+}>= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\lambda_{-}>= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we find

$$e^{-\varphi} =< \lambda_{+}|V\,\bar{V}^{-1}|\lambda_{+}> \tag{1.8}$$

The interesting point is that the vectors $\xi =< \lambda_{+}|V$ and $\bar{\xi} =< \lambda_{-}|\bar{V}$ are chiral, namely they satisfy

$$\partial_{-}\xi = \partial_{+}\bar{\xi} = 0$$

This clearly follows if one takes into account the explicit form of the two connections $A^{(1)}$ and $A^{(2)}$. Using explicitly the components of $V$ and $\bar{V}$ in the form

$$V = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix}, \bar{V} = \begin{pmatrix} \bar{\xi}_3 & \bar{\xi}_4 \\ \bar{\xi}_1 & \bar{\xi}_2 \end{pmatrix}$$

we see that relation (1.8) takes the well-known form

$$e^{-\varphi} = \xi_1\bar{\xi}_2 - \xi_2\bar{\xi}_1 \tag{1.9}$$

where the solution is represented by means of chiral (spin -1/2) conformal fields [S].

Moreover, starting from (1.3), (1.5) and the explicit representation of the transport matrix as a path–ordered exponential, we can work out the Poisson brackets of the fields $\xi$ and $\bar{\xi}$. We quote only the result, since the procedure is by now very standard[3]:

$$\begin{aligned}
\{\xi(x) \overset{\otimes}{,} \xi(y)\} &= \xi(x) \otimes \xi(y)(r^+\theta(x-y) + r^-\theta(y-x)) \\
\{\bar{\xi}(x) \overset{\otimes}{,} \bar{\xi}(y)\} &= \bar{\xi}(x) \otimes \bar{\xi}(y)(r^-\theta(x-y) + r^+\theta(y-x)) \\
\{\xi(x) \overset{\otimes}{,} \bar{\xi}(y)\} &= \xi(x) \otimes \bar{\xi}(y)\, r^- \\
\{\bar{\xi}(x) \overset{\otimes}{,} \xi(y)\} &= \bar{\xi}(x) \otimes \xi(y)\, r^+
\end{aligned} \tag{1.10}$$

The collection of the relations above is called *exchange algebra.* From the form of the exchange algebra we see that the two chiral halves are coupled between themselves in a non trivial way, thus enforcing the remarks above.

To the aim of formulating all the matter in the way we required, we notice that in (1.8) the "highest weights " $< \lambda_+|$ and $|\lambda_+ >$ are insensitive to the unipotent parts of $V$ and $\bar{V}$. The meaning is the following: we perform the two Gauss decompositions

$$\begin{aligned}
V &= N_- B_+ \\
\bar{V} &= N_+ B_-
\end{aligned}$$

where $N_-$ lies in the lower unipotent subgroup, $B_+$ in the upper Borel subgroup, $N_+$ in the upper unipotent and $B_-$ in the lower Borel. Formula (1.8) becomes

$$e^{-\varphi} =< \lambda_+|B_+\, B_-^{-1}|\lambda_+ >$$

and from the linear systems (1.6) and (1.7) we find the relations

$$\begin{aligned}
\partial_- B_+ = 0\,, \qquad & \partial_- N_-\, N_-^{-1} = -e^{2\varphi} E_- \\
\partial_+ B_- = 0\,, \qquad & \partial_+ N_+\, N_+^{-1} = -e^{2\varphi} E_+
\end{aligned}$$

The upshot is that all the chirally spurious elements have been thrown away, leaving us only with objects of definite chirality.

Looking a bit more closely to the Gauss decomposition of $V$ and $\bar{V}$ and to the relations fulfilled by $N_\pm$, a comparison with the form of $A_-^{(1)}$ and $A_+^{(1)}$ leads us to interpret $N_\pm$ as those gauge transformations that kill the anti–chiral and chiral parts of the connections $A^{(1)}$ and $A^{(2)}$ respectively. Thus the linear problems (1.6) and (1.7) get transformed into the following ones

$$(\partial_+ + C_+)B_+ = 0\,, \qquad \partial_- B_+ = 0 \tag{1.11}$$

$$(\partial_- + C_-)B_- = 0\,, \qquad \partial_+ B_- = 0 \tag{1.12}$$

where the connections $C_+$ and $C_-$ are given by

$$\begin{aligned}
C_+ &= (\partial_+\varphi + \xi_3/\xi_1)H + E_+ \\
C_- &= -(\partial_-\varphi + \bar{\xi}_4/\bar{\xi}_2)H + E_-
\end{aligned}$$

[3] details can be found, e.g. in [B]

and these explicit forms are found exploiting the Gauss decomposition of $V$ and $\bar{V}$ in the form:

$$V = \begin{pmatrix} 1 & 0 \\ \xi_3/\xi_1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 \\ 0 & \xi_1^{-1} \end{pmatrix}$$
$$\bar{V} = \begin{pmatrix} 1 & \bar{\xi}_4/\bar{\xi}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\xi}_2^{-1} & 0 \\ \bar{\xi}_1 & \bar{\xi}_2 \end{pmatrix}$$

For future consistence with the existing literature let us put

$$p = -(\partial_+\varphi + \xi_3/\xi_1)$$
$$\bar{p} = (\partial_-\varphi + \bar{\xi}_4/\bar{\xi}_2)$$

From the linear systems (1.6) and (1.7) we find the relations

$$p = \partial_x \log \xi_1 \,, \quad \bar{p} = -\partial_x \log \bar{\xi}_2$$

very reminiscent of the "bosonization" rules for the classical spin $-1/2$ fields in Liouville theory [GN, S].

Thus the original linear problem (1.2) has been reformulated into two other ones in which the chiralities are completely separated. As a last remark, let us notice that the matrices $B_\pm$ can be again factorized into simpler components in the following way:

$$B_+ = e^{K_+} M_+ = \begin{pmatrix} e^{k_+} & ue^{k_+} \\ 0 & e^{-k_+} \end{pmatrix} \,, \; B_- = e^{K_-} M_- = \begin{pmatrix} e^{-k_-} & 0 \\ ve^{k_-} & e^{k_-} \end{pmatrix}$$

and at this point it is not difficult to show, taking properly into account all the relations, that the Liouville field can be expressed as

$$e^{2\varphi} = \frac{\partial_+ u \, \partial_- v}{(1 - u\, v)^2} \tag{1.13}$$

which is the very classical formula dating back to Poincaré.

## 1.4 The Drinfel'd-Sokolov Linear Systems

The representations (1.9) and (1.13) may seem a little bit tautological, since the solution is constructed in terms of fields which at this stage have not yet an independent meaning. More precisely, the fields $\xi$ and $\bar{\xi}$ should be solution of the linear problems (1.6) and (1.7) which in turn still depend on the field $\varphi$. Thus we have to already know the solution of the Liouville equation in order for the above formulas (1.9) and (1.13) to work.

Thus in this way we do not get any explicit nor reliable representation of the space of classical solutions. To achieve this, one has to proceed differently, and this is the point where the so called Drinfel'd–Sokolov linear systems come about. The main idea is to consider them as the starting point, and then reconstruct a solution of the Liouville equation (or, more generally, of the Toda equations) having certain desired properties. This is the main content of [BBT]. Even though this is certainly not the place for a full account of that

work, we feel compelled to outline some essential steps in order to better clarify what is the relationship with what we are doing here.

As already mentioned, we consider separately the two chiral halves of the theory and introduce the so–called associated Drinfeld–Sokolov linear systems:

$$\partial_+ Q_+ - (pH - E_+)Q_+ = 0 \tag{1.14}$$
$$\partial_- Q_- + Q_-(\bar{P}H - E_-) = 0 \tag{1.15}$$

where the chiral $Q_+(x_+)$ and the anti-chiral $Q_-(x_-)$ take values in $SL_2(\mathbb{R})$. The fields $p$ and $\bar{p}$ are periodic chiral and anti-chiral respectively, and have the Poisson brackets

$$\begin{aligned} \{p(x), p(y)\} &= -\frac{1}{2}\delta'(x-y) \\ \{\bar{p}(x), \bar{p}(y)\} &= \frac{1}{2}\delta'(x-y) \\ \{p(x), \bar{p}(y)\} &= 0 \end{aligned} \tag{1.16}$$

From the solution $Q_+(x_+)$ and $Q_-(x_-)$ of equations (1.14) and (1.15) normalized by $Q_+(0) = 1$, $Q_-(0) = 1$ we define a basis $\sigma$, $\overline{\sigma}$

$$\sigma(x_+) = < \lambda_+ | Q_+(x) \tag{1.17}$$
$$\overline{\sigma}(x_+) = Q_-(x) | \lambda_+ > \tag{1.18}$$

which enjoys from (1.16) a Poisson bracket algebra analogous to the exchange algebra of the fields $\xi$ and $\bar{\xi}$ previously calculated, with important exception that this time the two chiral halves are decoupled:

$$\{\sigma(x) \overset{\otimes}{,} \overline{\sigma}(y)\} = 0$$

The aim now is to construct a periodic local solution of the Liouville equation in the form

$$e^{-\varphi(x_+,x_-)} = \sigma(x_+) M \sigma(x_-) \tag{1.19}$$

where $M$ is a constant matrix to be determined. Notice the close resemblance with the formal solution (1.8). Unlike that case, the solution (1.19) depends on the free bosonic fields $p$ and $\bar{p}$ whose algebra (1.16) is assumed a priori, so that no logical short-circuits arise. Since the fields $p(x)$ and $\bar{p}(x)$ are periodic, we can expand them in Fourier series

$$p(x) = \sum_n p_n e^{inx}, \qquad \bar{p}(x) = \sum_n \bar{p}_n e^{inx}$$

An important rôle is played by the zero modes $p_0$, $\bar{p}_0$ and by the left and right monodromy matrices

$$S = Q_+(2\pi), \quad \bar{S} = Q_-(2\pi)$$

We shall also need the constants

$$k = \sum_{n\neq 0} \frac{ip_n}{n}, \quad \bar{k} = \sum_{n\neq 0} \frac{i\bar{p}_n}{n}$$

Since

$$\sigma(x+2\pi) = \sigma(x)S\,, \quad \bar{\sigma}(x+2\pi) = \bar{S}\,\bar{\sigma}(x)$$

to get a periodic solution, we must have

$$S\ M\ \overline{S} = M$$

In order to satisfy this equation one proceeds to diagonalize the monodromy matrices

$$\begin{aligned} S &= gKg^{-1}\,, \quad K = e^{2\pi p_0 H} \\ \bar{S} &= \bar{g}^{-1}\bar{K}\bar{g}\,, \quad \bar{K} = e^{-2\pi\bar{p}_0 H} \end{aligned} \tag{1.20}$$

The condition (1.4) will be satisfied if

$$\begin{aligned} M &= g\rho\overline{g} \\ K\bar{K} &= 1 \end{aligned}$$

where $\rho \in \exp(\mathbb{R}H), \mathbb{R}\cdot H$ being the Cartan subalgebra. The second condition simply means that $p_0 = \overline{p}_0$ and will eventually be imposed. The diagonal matrix $\rho$ has to be chosen in such a way that the field of the LHS of (1.19) be local. The solution is

$$\rho = \theta\overline{\theta}, \quad \theta = e^{q-k}, \quad \overline{\theta} = e^{\bar{q}+\bar{k}}$$

and $q, \bar{q}$ are the conjugate variables of $p_0$ by $\bar{p}_0$. Therefore:

$$\{p_0,\, q\} = \{\bar{p}_0,\, \bar{q}\} = \frac{1}{4\pi}$$

Finally we can write

$$e^{-\phi(x_+,x_-)} = \psi(x_+)\,\overline{\psi}(x_-) \tag{1.21}$$

where we define the new objects (Block wave basis):

$$\psi(x_+) = \sigma(x_+)g\theta\,, \quad \bar{\psi}(x_-) = \bar{\theta}\bar{g}\bar{\sigma}(x_-)$$

They have diagonal monodromy $K$ and $\bar{K}$ respectively, and satisfy a rather complicated exchange algebra. We refer to [BBT] for the details. The important point is that

$$\{\psi(x) \overset{\otimes}{,} \overline{\psi}(y)\} = 0$$

as long as $p_0$ and $\bar{p}_0$ are considered independent. However, (1.19) represents a general local and periodic solution provided we reduce the phase space by imposing $p_0 = \bar{p}_0$.

This ends our brief review of [BBT]. What remains to do, of course, is to check whether the DS construction exhausts all the periodic and local solutions of the Liouville (or Toda) field equations. Closely related to this, there is the the question to see whether the Poisson brackets (1.3) match those for the free bosonic oscillators.

Therefore, as already explained in the introduction, we intend to "invert" the DS construction. Part of this program is the Leznov-Saveliev analysis that brought us to consider a couple of chiral and anti-chiral linear systems. What we have to do in the following is to compare the properties of the linear systems we may construct, starting from a given solution

of the Liouville equation, with those of the DS systems. In particular, we want two properties to be verified. First, the Poisson bracket algebra (1.16) must be obeyed, and secondly the oscillators must be *periodic*. Besides these, there is a third property the system should enjoy. We expect to find track of the modes $q_0\,, \bar{q}_0$ in the process of construction of the free fields. After all, the symplectic structure (1.3) is non degenerate, while the (1.16) is. This is the reason why it is necessary to introduce the conjugate modes to $p_0$ and $\bar{p}_0$ in the DS construction. Thus the next task is to see what happened to the symplectic structure (1.3) in the course of the various gauge transformations and to start the construction of the free oscillators. This is matter of the subsequent sections.

## 1.5 Symplectic Analysis

On the grounds of the Leznov–Saveliev analysis, one could expect the matrices $B_\pm$ and the fields $p$ and $\bar{p}$ to be the desired solution to the problem of finding a free oscillator description for the Liouville theory. Although very appealing, this is not the case, as we shall see in a moment. Anyway, the fields $p$ and $\bar{p}$ are the starting point for the analysis from the symplectic point of view of what we have done so far.

Thus our first task is to work out the Poisson brackets fulfilled by $p$ and $\bar{p}$. This requires some preparation, in particular we shall need the full exchange algebra obeyed by the matrices $V$ and $\bar{V}$. This exchange algebra is computed starting from the canonical Poisson brackets (1.3) and (1.4) and iteratively using (1.5) and the representation of $\Psi$ as a path–ordered exponential. The computation itself is only matter of a careful book-keeping of all the various terms and proceeds in an analogous fashion as, e.g., in [B]. The result is

$$\begin{aligned}
\{V(x) \overset{\otimes}{,} V(y)\} &= \theta(x-y)\, V(x)\otimes V(y) \\
&\quad \cdot \left(r^+ - Ad_{V(y)^{-1}\otimes V(y)^{-1}}\cdot E_+\otimes E_-\right) \\
&+ \theta(y-x)\, V(x)\otimes V(y) \\
&\quad \cdot \left(r^- + Ad_{V(y)^{-1}\otimes V(x)^{-1}}\cdot E_-\otimes E_+\right) \qquad (1.22)\\
\{V(x) \overset{\otimes}{,} \bar{V}(y)\} &= V(x)\otimes \bar{V}(y) \\
&\quad \cdot(\theta(x-y)(r^- + e^{2\varphi(y)}\, Ad_{V(y)^{-1}\otimes \bar{V}(y)^{-1}}\cdot E_-\otimes E_+) \\
&\quad +\theta(y-x)(r^- + e^{2\varphi(x)}\, Ad_{V(x)^{-1}\otimes \bar{V}(x)^{-1}}\cdot E_-\otimes E_+)) \qquad (1.23)\\
\{\bar{V}(x) \overset{\otimes}{,} \bar{V}(y)\} &= \theta(x-y)\bar{V}(x)\otimes \bar{V}(y) \\
&\quad \cdot \left(r^- + Ad_{\bar{V}(y)^{-1}\otimes \bar{V}(y)^{-1}}\cdot E_-\otimes E_+\right) \\
&+ \theta(y-x)\, \bar{V}(x)\otimes \bar{V}(y) \\
&\quad \cdot \left(r^+ Ad_{\bar{V}(y)^{-1}\otimes \bar{V}(x)^{-1}}\cdot E_+\otimes E_-\right) \qquad (1.24)
\end{aligned}$$

where we used the notation $Ad_{g\otimes h}\cdot v\otimes w = gvg^{-1}\otimes hwh^{-1}$. Some word must be spent about the use of the $\theta$–function in the above formulas. As we said at the beginning, we are considering the Liouville equation on a cylindrical space–time. Thus the use of the $\theta$–function as it stands may seem incorrect. It is properly handled with the following interpretation.

The transport matrix $\Psi$ can be thought of as the holonomy of a connection. However, in order to properly speak of a holonomy, we must fix some base point out of which all the

paths will emanate. We agree to consider a definite base point once for all and to give it the coordinates $t = 0$, $x = 0 \bmod 2\pi$. It is clear that the choice of a base point and the natural orientation are sufficient to define an order structure on $S^1$ (the slice $t = 0$ of the cylinder), and the $\theta$ is a compact way of expressing this structure. However, the transport matrix $\Psi$ is a non–periodic object and it is better defined over the universal cover. Thus we agree also that the "$x$" parametrizes also a straight line covering $S^1$ emanating from a base point [4] identified by $x = 0$.

Using the path–ordered exponential we often referred to, we produce a transport matrix satisfying the condition $\Psi(0) = 1$. The corresponding holonomy will be $T := \Psi(2\pi)$ to be called "monodromy matrix" from now on. It is clear that the initial condition $\Psi(0) = 1$ has nothing special in it, any other initial condition being perfectly good. If we set $\Psi(0) = g$ the monodromy will accordingly change to the conjugate one $T^{(g)} = g^{-1}Tg$. This remark will have great importance later on.

The complete exchange algebra we quoted encodes all the possible Poisson brackets we may need. Thus, using (1.22 –1.24) and (1.3) we calculate the Poisson brackets of the fields $p$ and $\bar{p}$. The result is the following[5]

$$\begin{aligned} \{p(x), p(y)\} &= -\frac{1}{2}\delta'(x-y) \\ \{\bar{p}(x), \bar{p}(y)\} &= \frac{1}{2}\delta'(x-y) \\ \{p(x), \bar{p}(y)\} &= -\xi_1(x)^{-2}\bar{\xi}_2(y)^{-2} \end{aligned}$$

These commutation relations make clear that $p$ and $\bar{p}$ are *not* the free oscillators we are looking for, even though the first two Poisson brackets are those of free bosonic fields; unfortunately, the third one says that the two sectors are still coupled. This is the first observation that throws away this possibility. The second concerns the periodicity. Remember that the DS construction and our setting as well require to work with periodic fields.

The fields $p$ and $\bar{p}$ we have constructed are certainly not periodic, so that they do not satisfy our requirements. The fact that they are not periodic should be clear from the behaviour of $V$ and $\bar{V}$ after a cycle around the cylinder:

$$V \longrightarrow V\,T\,, \quad \bar{V} \longrightarrow \bar{V}\,T$$

where $T$ is the monodromy we defined above.

Now, in order to construct the desired representation, We first try the following heuristic approach. If we represent the solution as in (1.8) we see that we can change the fundamental matrices $V$ and $\bar{V}$ by right multiplication with the same element $g \in SL_2(\mathbb{R})$ :

$$V \longrightarrow V\,g\,, \quad \bar{V} \longrightarrow \bar{V}\,g$$

and concerning their monodromy, this operation amounts to the conjugation $T \to g^{-1}Tg$, as already explained. The content of this apparently trivial symmetry will be fully exploited later on. For now let us content ourselves with the following simple observation. In the

[4] this time on the universal cover

[5] In this kind of calculations we report only the result, the actual calculation being completely straightforward, though very long.

expressions of $p$ and $\bar{p}$ appear the ratios of the elements in the first and second column of $V$ and $\bar{V}$ respectively. Clearly there is no point in exchanging the rôle played by those columns, namely we can equally well use the second of $V$ and the first of $\bar{V}$ . This is achieved by right multiplying with the element

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

whose effect on $V$ and $\bar{V}$ is

$$V\,w = \begin{pmatrix} \xi_2 & -\xi_1 \\ \xi_4 & -\xi_3 \end{pmatrix}\,,\ \bar{V}\,w = \begin{pmatrix} \bar{\xi}_4 & -\bar{\xi}_3 \\ \bar{\xi}_2 & -\bar{\xi}_1 \end{pmatrix}$$

Thus we obtain a new pair of candidates, to be called $p_2$ and $\bar{p}_2$ while we rename the old one to $p_1$ and $\bar{p}_1$ . The new pair is given by:

$$\begin{aligned} p_2 &= -(\partial_+\varphi + \xi_4/\xi_2) \\ \bar{p}_2 &= (\partial_-\varphi + \bar{\xi}_3/\bar{\xi}_1) \end{aligned}$$

We computed *all* the possible Poisson brackets among the elements of these two pairs and we found the following result[6]: *The pairs* $(p_1\,,\,\bar{p}_2)$, $(p_2\,,\,\bar{p}_1)$, *and* $(p_2\,,\,\bar{p}_2)$, *all satisfy the following Poisson brackets*

$$\{p(x), p(y)\} = -\frac{1}{2}\delta'(x-y) \tag{1.25}$$

$$\{\bar{p}(x), \bar{p}(y)\} = \frac{1}{2}\delta'(x-y) \tag{1.26}$$

$$\{p(x), \bar{p}(y)\} = 0 \tag{1.27}$$

(it is understood that the three pairs mentioned above should be substituted in the Poisson brackets ).

This is of course only an intermediate result, since we must address the question of periodicity with the additional problem of deciding which choice is the most suitable one for our purposes. In order to discuss the periodicity, we explicitly represent the monodromy as

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

so that after a period we have

$$\begin{aligned} \xi_1 &\to \alpha\xi_1 + \gamma\xi_2 & \qquad \bar{\xi}_3 &\to \alpha\bar{\xi}_3 + \gamma\bar{\xi}_4 \\ \xi_2 &\to \beta\xi_1 + \delta\xi_2 & \qquad \bar{\xi}_4 &\to \beta\bar{\xi}_3 + \delta\bar{\xi}_4 \\ \xi_3 &\to \alpha\xi_3 + \gamma\xi_4 & \qquad \bar{\xi}_1 &\to \alpha\bar{\xi}_1 + \gamma\bar{\xi}_2 \\ \xi_4 &\to \beta\xi_3 + \delta\xi_4 & \qquad \bar{\xi}_2 &\to \beta\bar{\xi}_1 + \delta\bar{\xi}_2 \end{aligned}$$

[6]Again the details of the calculation are not interesting: they are of the same type as the ones needed to produce the exchange algebra or the Poisson brackets for the first pair.

and the periodicity properties of the various fields are

$$\begin{aligned} p_1 &\to p_1 - \frac{\gamma}{\alpha\xi_1^2 + \gamma\xi_1\xi_2} \\ \bar{p}_1 &\to \bar{p}_1 + \frac{\beta}{\delta\bar{\xi}_2^2 + \beta\bar{\xi}_1\bar{\xi}_2} \\ p_2 &\to p_2 - \frac{\beta}{\delta\xi_1^2 + \beta\xi_1\xi_2} \\ \bar{p}_2 &\to \bar{p}_2 + \frac{\gamma}{\alpha\bar{\xi}_2^2 + \gamma\bar{\xi}_1\bar{\xi}_2} \end{aligned}$$

Notice that the periodicity transformations of $p_2$ and $\bar{p}_2$ involve the same constants as in $p_1$ and $\bar{p}_1$ but in the reverse order. Therefore we have that for the pair $(p_1\,,\,\bar{p}_2)$ we would have periodicity if the condition $\gamma = 0$ held true, while for the pair $(p_2\,,\,\bar{p}_1)$ we would have periodicity with $\beta = 0$ and finally for the pair $(p_2\,,\,\bar{p}_2)$ to have periodicity the condition $\gamma = \beta = 0$ is required. The obvious question now is: can we impose the conditions above? In order to better understand this point, let us note that setting some component of the monodromy matrix to zero is not only matter of imposing appropriate conditions on the differential equations. These conditions should be compatible with the Dynamics, that is they should not be destroyed when applying the Poisson brackets. So we should analyze the dynamical content of the monodromy matrix itself. This descends from the Poisson brackets of the transport matrix evaluated at $x = 2\pi$, that is

$$\{T \overset{\otimes}{,} T\} = -[r_{12}^{\pm}, T \otimes T] \tag{1.28}$$

This formula shows that the components of the monodromy matrix are dynamical objects. Thus we can calculate their Poisson brackets with any other dynamical variable, in particular with the various $p$'s we produced. We list the result case by case.

1. For the pair $(p_1\,,\,\bar{p}_2)$ the relevant Poisson brackets are

$$\{p(x), \gamma\} = \gamma\delta(x)\,, \quad \{\bar{p}(x), \gamma\} = -\gamma\delta(x)$$

2. For the Poisson brackets of the second pair $(p_2\,,\,\bar{p}_1)$ with $\beta$ we find

$$\{p(x), \beta\} = -\beta\delta(x)\,, \quad \{\bar{p}(x), \beta\} = \beta\delta(x)$$

3. For the last pair $(p_2\,,\,\bar{p}_2)$ the interesting Poisson brackets are those with both $\beta$ and $\gamma$:

$$\begin{aligned} \{p(x), \beta\} &= -\beta\delta(x)\,, & \{\bar{p}(x), \beta\} &= \beta\delta(x) \\ \{p(x), \gamma\} &= \gamma\delta(x)\,, & \{\bar{p}(x), \gamma\} &= -\gamma\delta(x) \end{aligned}$$

Moreover from (1.28) we have that $\{\beta\,,\,\gamma\} = 0$.

The elements $\beta$ and $\gamma$ are preserved by taking the Poisson brackets with the corresponding $p$'s, thus we could set them to zero as hamiltonian constraints on the phase space. In this

case the above Poisson brackets tell us that the interesting physical quantities, that is the fields $p$, are first class with respect to the constraints.

It is perhaps worthwhile recalling what we have done so far. Starting from a given solution of the Liouville equation, we transformed it into a integrability condition for a linear system of differential equations. In more technical terms we mapped the solution into the integrability condition for a flat connection. Hence we successively transformed the connection by means of two series of gauge transformations, in order to obtain a pair formed by a chiral and an anti-chiral connection. These two are in turn parametrized by two fields, one for each connection, which we expected to yield our desired free field representation. Therefore the entire process is considered as a coordinate transformation in the phase space. Looking more closely at the fields $p$ and $\bar{p}$ we have thus constructed, we discovered they do not completely satisfy all our requirements, in particular they do not Poisson commute with each other. Nevertheless, we produced out of the same data three other possible candidates. After checking they have indeed the correct Poisson brackets, what we are doing is to check the other properties they should enjoy. We have found that as they stand, all the pairs are in general not periodic, due to the overall non trivial monodromy. A possible way-out is to impose the periodicity as a constraint on the phase space: in this case the constraints are preserved by the Poisson brackets with the $p$'s, everything being "first class".

There is another criterion we can rely on, namely the question of the zero modes of $p$ and $\bar{p}$ and their conjugate variables. The motivation is still in the DS construction, which we would like to compare with ours. Thus we analyze the zero modes of $p$ and $\bar{p}$ and successively we shall try to find analogous quantities to $q$, $\bar{q}$.

Since in our setting the fields $p$, $\bar{p}$ are in principle not periodic, we *define* the zero modes of the various fields to be the integral over a period, namely we put

$$p_{i,0} = \int_0^{2\pi} p_i(x)\, dx\ , \quad i = 1, 2$$

and analogously, for $\bar{p}_i$ $i = 1, 2$. Using the linear systems (1.6) and (1.7) we find

$$\begin{aligned}
p_{1,0} &= \log(\alpha + \gamma \frac{\xi_2(0)}{\xi_1(0)}) \\
p_{2,0} &= \log(\delta + \beta \frac{\xi_1(0)}{\xi_2(0)}) \\
\bar{p}_{1,0} &= -\log(\delta + \beta \frac{\bar{\xi}_1(0)}{\bar{\xi}_2(0)}) \\
\bar{p}_{2,0} &= -\log(\alpha + \gamma \frac{\bar{\xi}_2(0)}{\bar{\xi}_1(0)})
\end{aligned}$$

The list so obtained must be compared with the one obtained when analyzing the periodicity. Imposing periodicity *and* equal zero modes yields the following picture.

1. With $\gamma = 0$ we have $p_1$, $\bar{p}_2$ periodic. Their zero modes are equal, respectively, to $\log \alpha$ and $-\log \alpha$. Setting them equal implies $\alpha = 1$, so that the monodromy must be

$$T = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

2. with $\beta = 0$ we have $(p_2, \bar{p}_1)$ to be the periodic pair. The zero modes are $\log\delta$ and $-\log\delta$ , so that $p_{2,0} = \bar{p}_{1,0}$ implies

$$T = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

3. Setting simultaneously $\beta = 0$, $\gamma =$ all the fields become periodic and the pair $p_{2,0}$, $\bar{p}_{2,0}$ is admissible. Equality of the zero modes translates into $\log\delta = -\log\alpha$ which implies $\delta = 1/\alpha$, so that for the monodromy we have

$$T = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}$$

Notice that in the first two cases the zero-modes turn out to be zero, a case which happens to appear as a singular one in the DS construction. This prompts us for the third possibility, but this is no more than a hint: the procedure has to be refined with a direct analysis of the monodromy.

## 1.6 Diagonalization of the Monodromy

It is clear from the preceding considerations that we should proceed to investigate whether the conditions previously found can be actually satisfied. In case the answer is affirmative, we have to determine to which *extent* these conditions hold, namely how large is the region of the phase space they select.

To this end, we use the remark we made previously about the conjugation of the monodromy matrix, that is we change the initial data of the linear systems (1.6), (1.7) by making the transformation

$$V \longrightarrow V\,g \ , \quad \bar{V} \longrightarrow \bar{V}\,g$$

with $g$ a unimodular $2 \times 2$ matrix. Accordingly, the monodromy will change as

$$T \longrightarrow T' = g^{-1}Tg$$

and we ask ourselves whether $T'$ can be put into diagonal (or triangular) form. However we are interested in doing this over the reals, possibly, and this is a problem having a very neat counterpart in group factorization theory.

We recall that a general $n \times n$ matrix $g$ is called *semisimple* if it is conjugate to a diagonal matrix (as an element of $gl_n(\mathbb{C})$). A semisimple matrix $g$ is called *elliptic* (respectively *hyperbolic*) if all its (complex) eigenvalues have modulus 1 (respectively are real and strictly greater than zero). A matrix $u$ is called *unipotent* if $u - 1$ is nilpotent.

When we consider a semisimple Lie group $G$ over $\mathbb{R}$ this classification has its counterpart in the Iwasawa $(KAN)$ decomposition: an element $g$ is elliptic if and only if it is conjugated to an element in the maximal compact subgroup $K$; it is hyperbolic if and only if it is conjugated to an element in the diagonal subgroup $A$ and it is unipotent if and only if it is conjugated to an element in the nilpotent subgroup $N$ [He]. In our case $G = SL_2(\mathbb{R})$ everything can

be checked by hand, but we prefer to stick to the general terminology in view of a possible extension to higher rank Toda theories.

Since the Iwasawa decomposition is an isomorphism of analytic manifolds and $T \in SL_2(\mathbb{R})$, we expect $T$ to fall in one and only one of these three classes. However, we would like $T$ to be hyperbolic, so that it is diagonalizable over the reals.

Thus our our aim is twofold, namely we approach the conjugation of the monodromy both by writing the algebraic equations, and by showing that in the relevant cases the resulting monodromies are in fact hyperbolic. The explicit form of the diagonalization will permit us to construct all the relevant quantities we are interested in, and to clarify the content of the already mentioned Weyl symmetry.

## Hyperbolicity of the Monodromy

We exploit the representation of $T$ as a path-ordered exponential. In all the calculations done so far we have used the most simple contour, namely the $t = 0$ circle. Correspondingly, we have

$$T = \mathbf{P} \exp(-\int_0^{2\pi} (A_+(x) + A_-(x))\, dx)$$

By writing explicitly the components the integrand reads

$$\begin{aligned} A_+ + A_- &= \partial_t \Phi + e^{\varphi}(E_+ + E_-) \\ &= \frac{1}{2}\partial_t \varphi\, H + e^{\varphi}(E_+ + E_-) \end{aligned}$$

We see that $A_+(x) + A_-(x)$ is valued in the span of $H$, $E_+ + E_-$. In the Cartan decomposition of $sl_2(\mathbb{R})$ this span is a supplementary space to the maximal compact subalgebra, but it is not itself a subalbegra. This means that the above representation is of little practical use, since even expanding the ordered exponential (by making use of Chen integrals, for instance) there is no control on which term are produced. However, instead of using the simple contour above, we can use another path homotopic to the first. This is meaningful, of course, if in doing this we do not cross any singularity of the connection. Thus for the moment we *assume* that everything is regular.

Instead of the simple path going straight from $(0,0)$ to $(2\pi,0)$[7] we choose the following one

$$\gamma_\tau = \begin{cases} (\tau,\, \tau) & \tau \in [0,\pi] \\ (\tau,\, 2\pi - \tau) & \tau \in [\pi, 2\pi] \end{cases}$$

which in the light-cone coordinates has the form

$$\gamma_\tau = \begin{cases} (2\tau,\, 0) & \tau \in [0,\pi] \\ (2\pi,\, 2\tau - 2\pi) & \tau \in [\pi, 2\pi] \end{cases}$$

Since this corresponds to a shift of $2\pi$ first in $x_+$ and then in $x_-$, we have

$$T = \Psi_-\, \Psi_+$$

[7] Remember that this path is closed!

with

$$\begin{aligned}\Psi_+ &= \mathbf{P}\exp\left(-\int_0^{2\pi} A_+(x_+, x_- = 0)\, dx_+\right)\\ \Psi_- &= \mathbf{P}\exp\left(-\int_0^{2\pi} A_+(x_+ = 2\pi, x_-)\, dx_-\right)\end{aligned}$$

Thus $\Psi_+$ is upper triangular, while $\Psi_-$ is lower triangular. Finding the explicit form of $\Psi_\pm$ amounts to solve (1.2) on the appropriate paths. The result is

$$\begin{aligned}\Psi_+ &= \begin{pmatrix} e^{-\frac{1}{2}(\varphi_1-\varphi_0)} & -e^{-\frac{1}{2}(\varphi_0+\varphi_1)}\int_0^{2\pi} e^{2\varphi(x_+,x_-=0)}dx_+ \\ 0 & e^{\frac{1}{2}(\varphi_1-\varphi_0)}\end{pmatrix}\\ \Psi_- &= \begin{pmatrix} e^{\frac{1}{2}(\varphi_2-\varphi_1)} & 0 \\ -e^{-\frac{1}{2}(\varphi_2+\varphi_1)}\int_0^{2\pi} e^{2\varphi(x_+=2\pi,x_-)}dx_- & e^{-\frac{1}{2}(\varphi_2-\varphi_1)}\end{pmatrix}\end{aligned}$$

from which it is clear that in both cases the diagonal elements are strictly positive while the off-diagonal ones are negative. We have indicated with $\varphi_0$, $\varphi_1$, $\varphi_2$ the values taken by $\varphi$ at the vertices "triangle" defined by the path. Notice that periodicity implies that $\varphi_2 = \varphi_0$. For the trace we have

$$\begin{aligned}\mathrm{tr}\, T &= 2\cosh\frac{1}{2}(\varphi_2 - 2\varphi_1 + \varphi_0) + (+.\text{ve contribution})\\ &> 2\end{aligned}$$

thus proving that in this case the monodromy $T$ is indeed hyperbolic. This statement has been proved assuming there are no singularities, but it is easy to see that extending this method to more complex "zig-zags" we can arrange for a path that avoids crossing isolated singularities and remains homotopic to the initial one. However we should exclude singularities on the $t = 0$ axis (but this case can be handled with a time shift) and accumulating singularities, continuous singular lines and all the like.

## Diagonalization

We now exploit the invariance under changing the initial data. Thus we have

$$\begin{aligned}e^{-\varphi} &= <\lambda_+|V\,\bar{V}^{-1}|\lambda_+>\\ &= <\lambda_+|(Vg)\,(\bar{V}g)^{-1}|\lambda_+>\\ &= <\lambda_+|\hat{V}\,\hat{\bar{V}}^{-1}|\lambda_+>\end{aligned}$$

where $\hat{V}$ and $\hat{\bar{V}}$ are the upper and lower triangular parts of $V$ and $\bar{V}$, respectively. The idea is to redo for these new objects what we have done so far, but now with a diagonal monodromy. If we insist for $g^{-1}Tg$ to be a diagonal matrix, there are several ways to accomplish this. For instance, a convenient choice is to put

$$g = R\,S$$

with

$$R = \begin{pmatrix} 1 & 0 \\ r & 1\end{pmatrix}, \quad S = \begin{pmatrix} 1 & s \\ 0 & 1\end{pmatrix}$$

that is, we first kill the element $\gamma$ of the monodromy matrix and afterwards we kill $\beta$. In this way $r$ satisfies a second order equation and $s$ is determined by $r$. We could have chosen the reverse order: the results are completely equivalent. What is important is that in the hyperbolic case we have two solutions for the number $r$, namely

$$r_{\pm} = \frac{-(\alpha - \delta) \pm \sqrt{\Delta}}{2\beta}$$

where

$$\Delta = (\alpha + \delta)^2 - 4 \equiv (\mathrm{tr}\, T)^2 - 4$$

is the discriminant. Thus we see that the existence of two real solutions, namely the hyperbolic character of $T$, corresponds to $T$ being hyperbolic as a Fuchsian transformation of the upper half plane $\mathbb{H}$ . The corresponding elements $s_{\pm}$ are given by

$$s_{\pm} = \mp \frac{\beta}{\sqrt{\Delta}}$$

We remark that in a possible parabolic monodromy the factor $S$ does not exist, as the discriminant is zero.

Thus we distinguish between the two solutions by putting a plus or minus sign: $g_{\pm}$. Their explicit forms are

$$g_+ = \begin{pmatrix} 1 & -\frac{\beta}{\sqrt{\Delta}} \\ r_+ & -\frac{\beta}{\sqrt{\Delta}} r_- \end{pmatrix}, \quad g_- = \begin{pmatrix} 1 & \frac{\beta}{\sqrt{\Delta}} \\ r_- & \frac{\beta}{\sqrt{\Delta}} r_+ \end{pmatrix}$$

The difference between the two is that ${g_+}^{-1} T g_+ = D$, $D = \mathrm{diag}(\lambda_+, \lambda_-)$, while ${g_-}^{-1} T g_- = \tilde{D}$, with $\tilde{D} = \mathrm{diag}(\lambda_-, \lambda_+)$, where $\lambda_{\pm}$ are the eigenvalues of $T$. We shall exploit the significance of these two possibilities in a moment.

Before further proceeding in this direction, we should mention that in general we cannot keep the previous results about the Poisson bracket algebra of the $p$'s. The reason is that when we change the initial data for the fundamental matrices $V$ and $\bar{V}$ the elements mix among themselves drastically modifying the resulting expressions for the fields $p$ and $\bar{p}$. Using a plus or minus sign to distinguish among them, we obtain

$$\begin{aligned} p_+(x) &= p_1(x) - \frac{r_+}{\xi_1(x)^2 + r_+ \xi_1(x)\xi_2(x)} \\ \bar{p}_+(x) &= \bar{p}_1(x) + \frac{1}{r_- \bar{\xi}_2(x)^2 + \bar{\xi}_1(x)\bar{\xi}_2(x)} \\ p_-(x) &= p_1(x) - \frac{r_-}{\xi_1(x)^2 + r_- \xi_1(x)\xi_2(x)} \\ \bar{p}_-(x) &= \bar{p}_1(x) + \frac{1}{r_+ \bar{\xi}_2(x)^2 + \bar{\xi}_1(x)\bar{\xi}_2(x)} \end{aligned}$$

where $p_1$ and $\bar{p}_1$ are the pair we calculated from $V$ and $\bar{V}$. Even more importantly, $g_{\pm}$ have a non trivial dynamical content, that they inherit from the one of $T$. Using the equations for the diagonalization condition, or even their explicit expression, it is possible to calculate the

Poisson brackets of the matrix elements of $g_\pm$ with any quantity like the $\xi$'s and so on. The explicit expressions do not matter here, and moreover they do not look very appealing. The important point is that this dynamical content restores the correct Poisson brackets, so that the new $p_\pm, \bar{p}_\pm$ continue to satisfy our favourite Poisson bracket algebra:

$$\begin{aligned} \{p_\pm(x), p_\pm(y)\} &= -\frac{1}{2}\delta'(x-y) \\ \{\bar{p}_\pm(x), \bar{p}_\pm(y)\} &= \frac{1}{2}\delta'(x-y) \\ \{p_\pm(x), \bar{p}_\pm(y)\} &= 0 \end{aligned}$$

Moreover these fields are periodic by construction, so that their zero modes make sense. We find

$$\begin{aligned} p_{\pm,0} &= \log \lambda_\pm \\ \bar{p}_{\pm,0} &= -\log \lambda_\mp \end{aligned}$$

so that they are equal (indeed $\lambda_- = 1/\lambda_+$).

Thus we have found two pairs of free bosonic fields satisfying all our requirements. Now we can turn to the question of the relationship between them. First we notice that we can pass from the diagonal matrix $D$ to $\tilde{D}$ via a Weyl transformation

$$\tilde{D} = wDw^{-1}, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(we confuse the group element $w$ with the element of the Weyl group it represents). It follows that $g_- w g_+^{-1}$ commutes with $T$, so that $a = g_+^{-1} g_- w$ commutes with $D$, hence is diagonal (trivial verification). This could be shown group-theoretically from the fact that it normalizes a diagonal subgroup, but in this case a direct calculation gives

$$a = \begin{pmatrix} \frac{\beta}{\sqrt{\Delta}} & 0 \\ 0 & \frac{\sqrt{\Delta}}{\beta} \end{pmatrix}$$

Now we have $V_\pm = V \cdot g_\pm$ and $\bar{V}_\pm = \bar{V} \cdot g_\pm$, and by construction

$$\begin{aligned} V_+ &\longrightarrow V_+ D \quad , \quad \bar{V}_+ \longrightarrow \bar{V}_+ D \\ V_- &\longrightarrow V_- \tilde{D} \quad , \quad \bar{V}_- \longrightarrow \bar{V}_- \tilde{D} \end{aligned}$$

but we also have

$$V_- w \longrightarrow V_- w\, D\,, \quad \bar{V}_- w \longrightarrow \bar{V}_- w\, D$$

so that the quantities $\tilde{V}_+ = V_- w$ and $\tilde{\bar{V}}_+ = \bar{V}_- w$ have the same monodromy as $V_+$, $\bar{V}_+$. Actually we have $\tilde{V}_+ = V_+ a$ and $\tilde{\bar{V}}_+ = \bar{V}_+ a$ and since $a$ is diagonal, the free bosonic fields constructed from $\tilde{V}_+$ and $\tilde{\bar{V}}_+$ are in fact equal to $p_+$ and $\bar{p}_+$. On the other hand, the fundamental matrices $\tilde{V}_+$ and $\tilde{\bar{V}}_+$ are obtained from $V_-$ and $\bar{V}_-$ multiplying with the element

$w$ which acts by exchanging the columns. Therefore the exchange of the columns of a fundamental matrix can be interpreted in terms of the action of the Weyl group and it is a symmetry, in the sense that the two representations so obtained describe the same theory. Indeed we have

$$\begin{aligned} e^{-\varphi} &= <\lambda_+|V\,\bar{V}^{-1}|\lambda_+> \\ &= <\lambda_+|V_+\,\bar{V}_+^{-1}|\lambda_+> \\ &= <\lambda_+|\tilde{V}_+ a^{-1}\,a\,\tilde{\bar{V}}_+^{-1}|\lambda_+> \\ &= <\lambda_+|V_-\,w\,w^{-1}\bar{V}_-^{-1}|\lambda_+> \\ &= <\lambda_+|V_-\,\bar{V}_-^{-1}|\lambda_+> \end{aligned}$$

so that the Liouville field $\varphi$ is insensitive to the transformation.

### Connection with the Drinfel'd-Sokolov construction

The last thing that remains to do to complete the picture is to retrieve also the conjugate variables to the zero modes. After that the contact with the DS construction will be complete. By what we have proved above, we shall consider only one set of free oscillators, say the one with the "plus" sign. Thus, in order to simplify the notation, we shall drop even the plus sign and retain the notation $V\,\bar{V}$ to mean the fundamental matrices constructed after rotating the initial data. By the notation $r\,\bar{p}$ we shall mean the new bosonic oscillators as well.

Extracting from $V$ and $\bar{V}$ the upper and lower triangular parts $\hat{V}$ and $\hat{\bar{V}}$ respectively, we have repeatedly written

$$e^{-\varphi} =< \lambda_+|\hat{V}\,\hat{\bar{V}}^{-1}|\lambda_+> \tag{1.29}$$

Now the solution will coincide with the DS one if

$$\hat{V}\,\hat{\bar{V}}^{-1} = Q_+ g\,\rho\,\bar{g}\,Q_-$$

The simpler thing to do is to examine this relation at the origin, where $Q_+(0) = Q_-(0) = 1$. On the other hand, due to the normalization condition $\Psi(0) = 1$ on the transport matrix, at the origin we have $V(0) = e^{-\Phi(0)}RS$ and $\bar{V}(0) = e^{\Phi(0)}RS$ where $RS$ have been previously introduced. Therefore, with some matrix manipulation, we have

$$\hat{V}(0) = \begin{pmatrix} e^{-\varphi_0/2} & s\,e^{-\varphi_0/2} \\ 0 & e^{\varphi_0/2} \end{pmatrix} \qquad \hat{\bar{V}}(0) = \begin{pmatrix} (1+sr)^{-1}e^{\varphi_0/2} & 0 \\ e^{-\varphi_0/2} & (1+sr)e^{-\varphi_0/2} \end{pmatrix}$$

On the other hand, the matrices $g\,\bar{g}$ that appear on the DS expression for the solution are unipotent [BBT], hence we can arrange the matrix product $\hat{V}(0)\,\hat{\bar{V}}(0)^{-1}$ in such a way that two unipotent matrices appear on the external sides. This leads to the identification of the matrix $\rho$

$$\rho = \begin{pmatrix} (1+sr)e^{-\varphi_0} & 0 \\ 0 & (1+sr)^{-1}e^{\varphi_0} \end{pmatrix}$$

However this is formal. Besides the correct matricial form, $\rho$ should have the same dynamicl content of the matrix $\rho$ that appears in the DS construction. Thus the identification will be

complete once we show that $\rho$ has the correct Poisson brackets with $p$ and $\bar{p}$, and the final check is the calculation of the Poisson bracket of $p_0$ with $\rho$. We find the following result

$$\{p_0, \rho\} = \frac{1}{4\pi}\rho \tag{1.30}$$

(remember that the zero modes of $p\,\bar{p}$ are equal). This tells us that the exponent of $\rho$ contains the conjugate variable to zero mode. Indeed, setting $\rho = \exp q_0 H$ formula (1.30) trivially implies

$$\{p_0, q_0\} = \frac{1}{4\pi}$$

which is what we wanted to prove: the exponent of $\rho$ containns the conjugate variable to the zero-mode $p_0$. Looking back at the DS construction, one may object that there the zero modes and their conjugate variables are introduced for each chirality, and only after one makes the identification. This is done to keep uncoupled the two halves of the Block wave basis, but at the very end one is obliged to set $p_0 = \bar{p}_0$ in order to have a local solution. Clearly, the two halves of the theory get coupled, but only through the zero modes [BBT]. However, we can also decide to proceed the other way around, namely first set the zero modes equal, and only after introduce one conjugate variable which definitely leads to the same conclusions.

Since here we are dealing with the reverse process that starts from a local solution, it is quite natural to directly find equal zero modes and therefore only one conjugate variable.

Now the program of finding the representation of the solution in terms of free bosonic oscillators is complete. However, we did something more than merely establish a correspondence between $\varphi$ and a pair $\{p\,,\bar{p}\}$ of fields satisfying certain conditions.

In summary, what we have established is the following. On the one hand we have the DS construction: this means that we can start from scratch with a pair of periodic fields, satisfying the desired Poisson brackets and such that a periodic local solution of the Liouville equation can be constructed. Notice that in the course of the DS construction, a problem of diagonalizability of the monodromy is explicitly solved over the reals. This implies that the DS procedure yields chiral fields with hyperbolic monodromy.

What we have been doing here is to show that the reverse track can be also followed.Thus we start from a given solution of the Liouville equation. Assuming that we are in a sufficiently regular setting, we find that it is possible to construct a pair of bosonic oscillators satisfying the desired Poisson brackets and the requirement of periodicity. What is more, the free field representation so constructed has exactly the DS form, that is besides the pair $\{p\,,\bar{p}\}$ itself, we reconstruct the entire structure of the DS linear systems, together with the additional variables needed to avoid degeneracy of the symplectic structure.

In this way we indirectly confirm that the DS construction furnishes the good solutions having hyperbolic monodromy. Indeed, even though we cannot a priori exclude the existence of periodic solutions of the Liouville equation of elliptic or parabolic character, it is conceivable that they may be of very singular nature. This is supported by some very simple explicit examples one can easily construct. On the other hand, it is apparent from our construction that in these singular cases the DS form cannot be reached, thus proving our assertion.

Finally, we have that there is a hidden symmetry of the Weyl group acting on the bosonic oscillators. This means that the correspondence between the solution $\varphi$ and the fields $\{p\,,\bar{p}\}$ is not one-to-one. Indeed in the course of the construction we exhibited a rather explicit action

of the Weyl group on the pair $\{p, \bar{p}\}$. This action yields another free field representation in the DS form. Of course the solution is insensitive to this action, so that given an admissible solution there are at least two pairs satisfying all our requirements.

# Chapter 2

# The Liouville Equation on a Riemann Surface I

## 2.1 Introduction

The aim of this chapter is to extend as far as possible the constructions of [BBT] to a general Riemann Surface. Due to the great amount of geometrical complications coming up in the whole matter, we definitely confine ourselves to the Liouville case, i.e. do not treat higher Toda models.

Here we give a brief description of what we are going to do in the sequel. First of all, we recall that a solution of the Liouville equation can be constructed in terms of one chiral and one anti-chiral spin $-1/2$ fields on the Riemann Surface. These doublets come from solutions to a pair of (chiral and anti-chiral) DS linear systems, at least locally.

Next we show how to globalize and consistently define the linear systems on the whole surface[1]. This will be done in terms of meromorphic connections on appropriate line bundles. Afterwards we take on the problem of actually solve the DS system. Being a system of differential equations on a Riemann Surface, the DS one has an associated monodromy, which is a flat cocycle on the surface or, in other terms, a representation of its fundamental group. Thus we analyze in detail this representation exploiting its geometrical significance.

Subsequently we introduce a convenient parametrization of the meromorphic connections in terms of a basis of meromorphic differentials formerly introduced by and Novikov [KN1, KN2]. This parametrization allows us to perform more explicit computations than those permitted by simply using the formalism of local charts. It also by using this basis that we introduce a symplectic form, generalizing the one already appeared in the previous chapter and in [BBT].

Finally, we address the problem of producing a single-valued solution of the Liouville equation. We prove that this is not always the case and implies constraints on the phase space. Accordingly, since the constraints turn out to be not of first class, the formalism of Dirac brackets has to be introduced.

[1] We fix mainly our attention to the chiral one.

## 2.2 The Liouville Equation

In this section we set up the formalism concerning the Liouville equation on a (compact) Riemann Surface. So let $X$ be a fixed compact Riemann Surface whose genus $g$ will be supposed to be greater or equal than two. Usually $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ will denote a certain complex atlas of $X$. Giving the complex atlas is the same thing as giving the complex structure, which will be held fixed throughout.

We write the Liouville equation in the local chart $(U_\alpha, z_\alpha)$ as

$$\partial_\alpha \bar{\partial}_\alpha \varphi_\alpha = e^{2\varphi_\alpha} \tag{2.1}$$

where $\partial_\alpha = \frac{\partial}{\partial z_\alpha}$ and $\bar{\partial}_\alpha = \frac{\partial}{\partial \bar{z}_\alpha}$ and $\varphi$ glues according to the rule[2]

$$\varphi_\beta(z_\beta) = \varphi_\alpha(f_{\alpha\beta}(z_\beta)) + \frac{1}{2}\log|f'_{\alpha\beta}(z_\beta)|^2$$

with the holomorphic change of coordinates $z_\alpha = f_{\alpha\beta}(z_\beta)$. The rule above ensures the covariance of the Liouville equation, that is it will maintain its form (2.1) in any other local patch different from the particular $(U_\alpha, z_\alpha)$ used to first define it. The way $\varphi$ glues chart by chart implies that the quantity $\exp 2\varphi_\alpha$ transforms as (1,1)-form over the surface, and it might be tempting to consider it as the Kähler form of a metric on $X$. This is indeed the case if $\exp 2\varphi$ were regular, as it happens when the solution we consider is the one coming from the uniformization of $X$. However, we will construct a class of solutions with very bad singularities at certain points of the surface. We shall keep on calling "metric" the quantity $\exp 2\varphi$, being aware that this is a slight abuse of language. The best we can expect is that our solution will furnish a true metric on the non compact curve obtained deleting some points from the original one.

Now we start the long way in constructing the solution. First of all, let us make some observation of local character. The first one, which dates back to early works on string theory (and Liouville Field Theory!) such as [GN], concerns the representation of the desired solution in terms of spin $-1/2$ doublets of definite chirality. Thinking to work in a neighbourhood of the origin in $\mathbb{C}$, we put

$$e^{-\varphi} = \sigma M \bar{\sigma} \tag{2.2}$$

where $\sigma$ and $\bar{\sigma}$ are a row and column vectors of dimension two respectively, satisfying $\bar{\partial}\sigma = \partial\bar{\sigma} = 0$. $M$ is a $2 \times 2$ constant matrix whose meaning will be clear later. It is easy to check that $\varphi$ so defined satisfies (2.1), provided $\det M = 1$ and the Wronskians of $\sigma$ and $\bar{\sigma}$ are normalized to 1 and $-1$ respectively. From a strictly local point of view the matrix $M$ is really not important. It can be set to any preferred value. On the contrary, it will play an important rôle when dealing with global constructions.

From the work [BBT] we know that these doublets $\sigma$ and $\bar{\sigma}$ can be constructed in terms of much more elementary objects, namely they can be extracted from the fundamental solution matrices to a pair of DS linear system, one for each chirality. As we already know, the advantage is that the fields figuring into the "connection matrices" of the linear systems can be much more easily handled, at least from the dynamical point of view. In the same way as

[2] It is important to keep in mind that $\varphi$ is *not* holomorphic, even though we write down only the coordinate $z$.

we did in the previous chapter, the space-time considered in [BBT] is a cylinder, which is the same thing as $\mathbb{C}^*$ , the complex plane minus the origin. Thus we can transfer the formalism of DS systems to a neighbourhood of $0 \in \mathbb{C}$ simply using the map $(t, x) \mapsto z = e^{t+ix}$. Thus we shall consider the differential equations

$$\partial \mathcal{Q} = (p\,H + E_+)\mathcal{Q} \tag{2.3}$$
$$\bar{\partial}\bar{\mathcal{Q}} = -\bar{\mathcal{Q}}(\bar{p}\,H - E_-) \tag{2.4}$$

over a disc surrounding the origin in $\mathbb{C}$. $H$ and $E_\pm$ are the standard generators already introduced in the previous chapter. We assume that (2.3) and (2.4) are completely independent *ordinary* differential equations, in the sense that $\bar{\partial}p = \partial\bar{p} = 0$. We shall follow the tradition in almost completely focusing our attention on the chiral one, namely the equation (2.3). In (2.3) and (2.4) the quantities $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ are taken to be fundamental solution matrices of the the respective differential equations, rather than vector solutions.

Still working locally, it is easy to write down the solution $\mathcal{Q}$ to (2.3). It looks like

$$\mathcal{Q} = \begin{pmatrix} \sigma_1 & \sigma_2 \\ 0 & \sigma_1^{-1} \end{pmatrix}$$

where

$$\sigma_1(z) = e^{\int^z p(w)\,dw}, \quad \sigma_2(z) = -\sigma_1(z)\int^z \sigma_1(w)^{-2}\,dw \tag{2.5}$$

The line integrals start from an unspecified initial point, at the moment. The path of integration does not matter as long as it does not encircle any possible singularity of $p$. This being the case, the rather formal solution just written down will in fact be plagued by monodromies and a more careful analysis is required. However we shall do this in connection with the definition of (2.3) on the whole surface $X$.

The solution $\bar{\mathcal{Q}}$ for the antichiral DS system (2.4) is given in the form

$$\bar{\mathcal{Q}} = \begin{pmatrix} \bar{\sigma}_1 & 0 \\ \bar{\sigma}_2 & \bar{\sigma}_1^{-1} \end{pmatrix}$$

where

$$\bar{\sigma}_1(\bar{z}) = e^{-\int^{\bar{z}} \bar{p}(\bar{w})\,d\bar{w}}, \quad \bar{\sigma}_2(\bar{z}) = \bar{\sigma}_1(\bar{z})\int^{\bar{z}} \bar{\sigma}_1(\bar{w})^{-2}\,d\bar{w} \tag{2.6}$$

and the same remarks about the solution are obviously valid.

## 2.3 The chiral DS linear system

The aim of this section is to define the DS system on the Riemann Surface $X$ and to describe its solution, whose local form has been written down in (2.5). This requires a certain series of preliminaries. For the necessary background and notations about Riemann Surfaces and related topics we mainly refer to [G1, G2].

As previously stated, we are looking for a doublet of spin $-1/2$ chiral fields to be extracted from a fundamental solution of the DS system (2.3). The first task is thus to define it in a global way. The form of $\mathcal{Q}$ implies that its columns have components of conformal weights

$(-1/2,0)$ and $(1/2,0)$ [BBT], thus our arena will be the vector bundle $V = K^{-1/2} \oplus K^{1/2}$, where $K^{1/2}$ is a square root of the canonical line bundle $K$ [F]. Since there are many possible choises for this square root, we choose one once for all. The vector bundle $V$ is clearly holomorphic.

## Meromorphic connections

The next step will be to define the DS linear system as an analytic connection on this vector bundle. This is, we believe, the correct and easiest way of setting a differential equation in a global context. Generally speaking, an analytic connection in a holomorphic vector bundle $E$ over $X$ is a map

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$$

where $\mathcal{O}_X$ is the sheaf of holomorphic functions on the Riemann Surface, $\mathcal{E}$ is the sheaf of holomorphic sections of $E$ and $\Omega^1_X$ the sheaf of holomorphic differentials on $X$, i.e. the sheaf of holomorphic sections of $K$. Analytic connections do not always exist [A]. According to a theorem of Weil, their existence is equivalent for $E$ to be a direct sum of indecomposable analytically flat bundles [A, G2]. This is certainly not the case for $V$, since $c_1(K^{1/2}) = g - 1$ and the genus $g$ is supposed to be greater or equal than two. Thus in such a case an analytic connection must be more properly defined as a map [De]

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X(*Y)$$

where now $\Omega^1_X(*Y)$ is the sheaf of holomorphic differentials, meromorphic on a subset $Y$ of $X$.[3] Allowing poles trivializes the cohomological obstructions to the existence of (analytic) connections. Therefore on these general grounds we may expect the would-be DS connection to be meromorphic.

After these general remarks let us take on a more detailed study. First of all we define our DS connection in a local chart:

$$\nabla^{DS} = \partial + \begin{pmatrix} -p(z) & 1 \\ 0 & p(z) \end{pmatrix} dz$$

Afterwards, we pretend this form to be maintained in any other local chart, that is, applying $\nabla$ to a section of $V$ should give a (meromorphic) section of $V \otimes K$. This will give conditions on the coefficients of the connection. First of all, the "1" into the connection matrix has an invariant meaning once one notices that 1 is a section of $Hom(K^{1/2}, K^{-1/2} \otimes K)$. The condition on $p$ is the following transformation rule:

$$\frac{d}{d\,z_\beta} \log k^{1/2}_{\alpha\beta} = p_\beta(z_\beta) - p_\alpha(z_\alpha(z_\beta)) \frac{d\,z_\alpha}{d\,z_\beta} \tag{2.7}$$

where $k_{\alpha\beta} = d\,z_\beta / d\,z_\alpha$ are the transition functions of $K$ and $k^{1/2}_{\alpha\beta}$ is a suitably chosen collection of square roots defining $K^{1/2}$. The condition (2.7) has the form of a relation in Čech cohomology. Indeed, introducing $c_{\alpha\beta} = \frac{d}{d\,z_\beta} \log k^{1/2}_{\alpha\beta}$, we easily have $\{c_{\alpha\beta}\} \in Z^1(\mathfrak{U}, \Omega^1_X)$ and (2.7)

[3] One can consider also replacing the sheaf $\Omega^1_X(*Y)$ with $\mathcal{M}^1_X$, the sheaf of meromorphic 1-differentials on $X$. This kind of rephrasing the classical notion of connection opens the way to the modern notion of (coherent) $\mathcal{D}_X$-modules.

takes the form $\{c_{\alpha\beta}\} = \delta\{p_\alpha\}$, where $\delta$ is the Čech coboundary map. Thus $\{p_\alpha\} \in C^0(\mathfrak{U}, \Omega^1_X)$ is the cochain whose coboundary is $\{c_{\alpha\beta}\}$. However, it is not difficult to see that the above relation cannot take place, since the cocycle $\{c_{\alpha\beta}\}$ is the one defining the Chern class of $K^{1/2}$ [G1], so that (2.7) can be realized only if we take $\{p_\alpha\} \in C^0(\mathfrak{U}, \mathcal{M}^1_X)$. This confirms our statement that an analytic connection will in general be meromorphic. This can be also seen directly in the following way [SH]. Due to (2.7), the residue of $p$ is well defined, and the difference of any two cochains satisfying (2.7) is an abelian differential whose total residue is therefore zero. Hence we can calculate the total residue in a fairly simple situation, namely considering as $p$ the analytic variable $\{-\frac{d}{d\,z_\alpha}\log h_\alpha\}$, where $\{h_\alpha\}$ is a meromorphic section of $K^{1/2}$. Thus we obtain [G1, SH]

$$\mathrm{Res}(-\partial \log h) = -\sum_{x\in X} v_x(h) = -c_1(K^{1/2}) = 1 - g$$

where $v_x(h)$ is the order of $h$ at $x \in X$. Hence the residue of any solution to (2.7) is different from zero, and we obtain a meromorphic object. Due to the fact that the transformation rule (2.7) is formally the same as the one of a connection, it is customary to call $p$ a meromorphic connection (on $K^{1/2}$). Meromorphic connections on $K$ are treated in [SH].

## Parametrizing the DS System via the KN Algebra

From the discussion above it should be evident that we have a large freedom in choosing a solution to the relation (2.7). Indeed fixing a particular solution, for instance choosing the logarithmic one, we can obtain any other by simply adding an abelian differential [SH]. This property is the analogue of the well known fact that smooth connections form an affine space modelled on the vector space of 1-forms with values in the adjoint bundle. Indeed most of the formal properties of smooth connections continue to hold in the analytic setting.

We now use the affine character of the set of meromorphic connections on $K^{1/2}$ to introduce a convenient parametrization based on a basis for abelian differentials introduced by Krichever and Novikov [KN1, KN2]. Actually, for any power $K^n$, $n \in \mathbb{Z}$, they constructed a basis for the corresponding sections $\Omega^n_X(*Y)$,[4] where $Y$ is a certain collection of points of $X$. These bases enjoy formal properties quite analogous to those of the standard Laurent basis $\{z^n\}$ on the Riemann sphere $\mathbb{P}^1$. As already stated in [KN1, KN2], and definitely proved in [KL], these bases can be used to expand analytic objects on the Riemann Surface exactly in the same fashion as we do on $\mathbb{P}^1$. Due to this fundamental property, the KN bases have been exploited in the operator approach to string theory and related subjects on higher genus Riemann surfaces [KN2, BLMR].

It is worthwhile describe the KN basis in some more detail. $Y$ will simply taken to be a two-point subset of $X$, i.e. $Y = \{P_0, P_\infty\}$. The points $P_0$ and $P_\infty$ must be in "general position", in the sense that they should not be Weierstrass points. These two points play the same rôle as the origin and the point at infinity in the Riemann sphere. In the framework of operator formalism, they are those points to which the "in" and "out" string vacua refer to. We note in passing that the entire formalism is easily extended to collections made up by more than two points [Sc], with, say, $P_1, \ldots, P_k$ "in" points and $Q_1, \ldots Q_l$ "out" points.

[4] we recall that with this symbol we mean objects holomorphic on $X \setminus Y$, meromorphic on $Y$

The actual construction goes as follows (we refer to [KN1, KN2, BLMR, KL] for a full detailed account). With minor modifications, we keep the notations of [KL]. One builds out of $Y$ a divisor $D$ of the form $D = -a_k^{(n)} P_0 - b_k^{(n)} P_\infty$ where $a_k^{(n)}, b_k^{(n)} \in \mathbb{Z}$. As usual, for a vector bundle $F$ whose sheaf of holomorphic sections is $\mathcal{F}$ and a divisor $D$, the symbol $\mathcal{F}(D)$ denotes the sheaf of meromorphic sections of $F$ of order greater or equal than $-D$. Loosely speaking, they are meromorphic sections whose poles not higher than $D$. These are the same thing as holomorphic sections of the vector bundle $F \otimes [D]$, where $[D]$ is the line bundle associated to the divisor $D$ [GH].

The idea is to carefully choose the integers in the divisor $D$ in such a way that the cohomology group $H^0(X, \Omega_X^n(D))$ be one dimensional and its generator $f_k^{(n)}$ have exactly $v_{P_0}(f_k^{(n)}) = a_k^{(n)}$ and $v_{P_\infty}(f_k^{(n)}) = b_k^{(n)}$. Apart a few exceptional cases, this can be indeed obtained, the tools being the Riemann-Roch and the Weierstrass gap theorems. The divisor is chosen as follows.

1. For $n \neq (0, 1)$ or $n = 1$, $k \neq (0, \ldots, g)$ or $n = 0$, $k \neq (-g, \ldots, 0)$ :

$$D = -(k - n)P_0 + (k + n - (2n - 1)g)P_\infty$$

2. For $n = 1$ and $1 \leqq k \leqq g$ :

$$D = -(k - 1)P_0 + (k - g)P_\infty$$

3. For $n = 1$ and $k = 0$ :

$$D = P_0 + P_\infty$$

4. For $n = 0$ and $-g \leqq k \leqq -1$ :

$$D = -kP_0 + (k + g + 1)P_\infty$$

5. For $n = 0$ and $k = 0$ :

$$D = 0$$

The third and the fourth are the exceptional cases, as the corresponding cohomology group $H^0(X, \Omega_X^n(D))$ has dimension higher than one. In the third case, the space of abelian differentials with poles of order not greater than one at both $P_0$ and $P_\infty$ is $g+1$ dimensional, while in the fourth case the corresponding space of meromorphic functions is two-dimensional. In particular, in the third case the $g + 1$ dimensional space $H^0(X, \Omega_X^1(P_0 + P_\infty))$ is generated by a basis of first kind (i.e. holomorphic) abelian differentials and a third kind one. Thus we see that, apart the overall normalization constants, further elements of choice are involved.

The nonuniqueness can be in part removed imposing a further condition. We require $\{f_k^{(n)}\}_{k \in \mathbb{Z}}$ to satisfy the following duality relation:

$$\mathrm{Res}_{P_0} f_k^{(n)} f_l^{(1-n)} = \delta_{k+l,0}$$

We set $f_0^{(0)} \equiv 1$ and $f_0^{(1)}$ to be the unique third kind differential with purely imaginary periods and residues $\pm 1$ at the poles.

Since we are mainly interested in the basis for the abelian differentials, we end this description by making explicit what is going on in this case.

$$\begin{array}{ll} \dots & \dots \\ k=-2 & D=3P_0-(g+1)P_\infty \\ k=-1 & D=2P_0-gP_\infty \\ k=0 & D=P_0+P_\infty \\ k=1 & D=(1-g)P_\infty \\ k=2 & D=-P_0+(2-g)P_\infty \\ \dots & \dots \\ k=g-1 & D=(2-g)P_0-P_\infty \\ k=g & D=(1-g)P_0 \\ k=g+1 & D=-gP_0+2P_\infty \\ \dots & \dots \end{array}$$

The range $k \notin (0,\dots,g)$ is made of second kind differentials. They have poles at $P_0$ for $k \leqq -1$ and zeroes for $k \geqq g+1$. The third kind differential corresponds, as already noted, to $k=0$, and the range $1 \leqq k \leqq g$ corresponds to the holomorphic differentials. Notice also that the order of $D$ decreases exactly by 1 at each step in $k$, so that $\Omega^n_X(*Y)$ can be nicely decomposed according to the order in $P_0$.

For historical reasons the bases $\{f^{(1)}_k\}$, $\{f^{(0)}_k\}$ ( and others in fact) are given special names. Thus we set $A_k = f^{(0)}_k$ and $\omega^k = f^{(1)}_{-k}$ for $k \in \mathbb{Z}$. The index shift [KN2, BLMR] is technical. It is done in order to have the duality in the form

$$\mathrm{Res}_{P_0}(A_k\omega^l) = \delta^l_k$$

Apart the different indexing, one can verify that the bases introduced here coincide exactly with those of [BLMR], which in turn modifies slightly those of [KN1, KN2]. Having passed through this review of the necessary properties of the KN bases, we are in position to use it to parametrize the DS connection or, which is the same thing, the connection $p$ on the spin $1/2$ line bundle. We decompose $p$ according to [BT]:

$$p = \Gamma_0 + \sum_k p_k\,\omega^k \tag{2.8}$$

where $\Gamma_0$ is a reference connection to keep fixed throughout. The constants $p_k$ are to be considered as "modes" of the connections much in the same way as we do in the usual case when expanding on the basis $\{e^{inx}\}$. It is convenient to take as $\Gamma_0$ the logarithmic connection previously introduced:

$$\Gamma_0 = -\partial \log h$$

for $h$ a meromorphic section of $K^{1/2}$. If $K^{1/2}$ represents an odd theta characteristic, the section $h$ can be chosen holomorphic [F].

We shall tacitly assume, in the sequel, that the sum in (2.8) is *finite*. This allows for the singularities in the solutions of the DS system to be no worse than those of Baker-Akhiezer type [Du]. However, they can also be much more better, depending modes are non zero in the expansion (2.8). For instance, if the singularity is *regular*, that is only simple poles in (2.8) are allowed, we can get solutions with polar behaviour at the singularities.

What we shall do next is to glue the local solution (2.5) to the DS system into a global one, and to exploit the KN parametrization to obtain explicit formulas where possible.

## 2.4 Solution of the DS system

The aim of this section is to build up a global solution to the chiral DS system. Our starting point will be the local solution (2.5): we shall try to pairwise glue different copies of it relative to different charts at points where these charts intersect. This is a very simple minded approach close to the one adopted in [SH].

Before entering into the details of our specific problem, let us see what we can expect in general. As we already explained, the DS connection is a map

$$\nabla^{DS} : \mathcal{V} \longrightarrow \mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_X(*Y)$$

where $Y = \{P_0, P_\infty\}$ and $\mathcal{V}$ is the sheaf of holomorphic section of the chosen vector bundle $V = K^{-1/2} \oplus K^{1/2}$. With respect to the covering $\mathfrak{U}$ it is clearly a collection of meromorphic differential equations. Thus for any open set $U_\alpha \in \mathfrak{U}$ we can exhibit a fundamental solution $\mathcal{Q}_\alpha$ of the differential equation $\nabla^{DS} = 0$. This requires a choice of the integration constants in (2.5). Once this has been done, $\mathcal{Q}_\alpha$ is a local frame for $V$ on $U_\alpha$, but when changing local chart, on $U_\alpha \cap U_\beta$ we have the gluing law

$$\mathcal{Q}_\alpha T^\vee_{\alpha\beta} = \begin{pmatrix} k_{\alpha\beta}^{-1/2} & 0 \\ 0 & k_{\alpha\beta}^{1/2} \end{pmatrix} \cdot \mathcal{Q}_\beta$$

where $T^\vee_{\alpha\beta}$ is a constant matrix. It takes into account the fact that $\mathcal{Q}_\alpha$ and the RHS of the above equation both solve the differential equation $\nabla^{DS}\mathcal{Q} = 0$ on $U_\alpha \cap U_\beta$ with respect to $z_\alpha$. The dual sign "$^\vee$" is for the time being only matter of notation.

To have consistency on the triple intersections the matrices $\{T^\vee_{\alpha\beta}\}$ must satisfy the cocycle condition:

$$T^\vee_{\alpha\beta} T^\vee_{\beta\gamma} = T^\vee_{\alpha\gamma}$$

Changing our choice of the local frames $\{\mathcal{Q}_\alpha\}$ yields a new collection $\{\mathcal{Q}'_\alpha\}$ such that

$$\mathcal{Q}_\alpha = \mathcal{Q}'_\alpha C_\alpha$$

for appropriate constant matrices $\{C_\alpha\}$. This in turn produces a new cocycle $T'^\vee_{\alpha\beta}$ such that

$$T'^\vee_{\alpha\beta} = C_\alpha^{-1} T^\vee_{\alpha\beta} C_\beta$$

and therefore describes the same cohomology class.

Thus we interpret the collection $\{T^\vee_{\alpha\beta}\}$ as defining a flat rank 2 vector bundle or, in other terms, a rank 2 local system. The local system, in turn, is a representation of the fundamental group into the relevant structural group. Also, using the Čech point of view to deal with vector bundles, the gluing rule given above can be reinterpreted by saying that the collection $\{\mathcal{Q}_\alpha\}$ of fundamental matrices realizes a bundle morphism between $T^\vee$ and $V$. However, these statements are rather sloppy, since we did not take into account the singularities of the

connection at all. By doing this, we shall discover that the local system is properly setted on the Riemann Surface with the points of $Y$ deleted. Therefore the fundamental group involved is the one of the punctured surface. What we are going to do here and in the next section is to build a collection of local fundamental matrices for the DS connection, and subsequently describe in detail the associated monodromy, i.e. the local system.

In order to proceed let us consider the covering $\mathfrak{U} = (U_\alpha, z_\alpha)$. Fix a collection $\{Q_\alpha\}$ of points of $X$, one for each open neighbourhood $U_\alpha$. It is clear, refining the covering if necessary, that we can always arrange things in such a way that $P_0$ belongs to $U_0$ and not to any other member of the covering, and the same happens for $P_\infty$. For every $\alpha$ and $Q \in Q_\alpha$ define

$$\mathcal{L}_\alpha(Q) := \int_{Q_\alpha}^{Q} p_\alpha(z_\alpha)\, dz_\alpha$$

where the integration goes along any path contained in $U_\alpha$ and joining $Q_\alpha$ and $Q$. Thus the local solution (2.5) is written as

$$\sigma_{1\alpha}(Q) = e^{\mathcal{L}_\alpha(Q)}, \quad \sigma_{2\alpha}(Q) = -e^{\mathcal{L}_\alpha(Q)} \int_{Q_\alpha}^{Q} e^{-2\mathcal{L}_\alpha(z_\alpha)}\, dz_\alpha \tag{2.9}$$

Now, to glue two solutions, consider the following situation. Let the point $Q$ belong to $U_\alpha \cap U_\beta$ and consider another point $Q'$ still belonging to the intersection $U_\alpha \cap U_\beta$ and lying, say, on the path from $Q_\alpha$ to $Q$. Then we have

$$\begin{aligned}
\mathcal{L}_\alpha(Q) &= \int_{Q_\alpha}^{Q} p_\alpha(z_\alpha)\, dz_\alpha \\
&= \left(\int_{Q_\alpha}^{Q'} + \int_{Q'}^{Q}\right) p_\alpha\, dz_\alpha \\
&= \int_{Q_\alpha}^{Q'} p_\alpha\, dz_\alpha + \int_{Q'}^{Q} p_\beta\, dz_\beta + \int_{Q'}^{Q} d\log\left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2} \\
&= \int_{Q_\alpha}^{Q'} p_\alpha\, dz_\alpha + \int_{Q'}^{Q_\beta} p_\beta\, dz_\beta + \int_{Q_\beta}^{Q} p_\beta\, dz_\beta + \log\left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2}\Bigg|_{Q'}^{Q} \\
&= \mathcal{L}_\beta(Q) + \log\left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2}(Q) + b_{\alpha\beta}
\end{aligned}$$

with

$$b_{\alpha\beta} = \int_{Q_\alpha}^{Q'} p_\alpha\, dz_\alpha - \int_{Q_\beta}^{Q'} p_\beta\, dz_\beta - \log\left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2}(Q')$$

so that we get the transformation rule:

$$\sigma_{1\alpha}(Q) = c_{\alpha\beta}\left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2}(Q)\,\sigma_{1\beta}(Q) \tag{2.10}$$

having posed $c_{\alpha\beta} = e^{b_{\alpha\beta}}$. The transformation rule (2.10) is meaningful due to the fact that $b_{\alpha\beta}$ (or $c_{\alpha\beta}$) is a number, i.e it does not depend on the point $Q'$ used to calculate it. This is

easily seen simply choosing another point $Q''$ to calculate it. Indeed notice that if $Q'$ and $Q''$ both lie in $U_\alpha \cap U_\beta$ by using (2.7) we obtain

$$\int_{Q''}^{Q'} p_\alpha \, dz_\alpha = \int_{Q''}^{Q'} p_\beta \, dz_\beta + \int_{Q''}^{Q'} d\log\left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2}$$

which suffices to prove the assertion.

On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ it is easy to verify the cocycle condition:

$$c_{\alpha\beta} c_{\beta\gamma} = c_{\alpha\beta}$$

simply observing that $c_{\alpha\beta}$ can be rewritten in the form

$$c_{\alpha\beta} = e^{\mathcal{L}_\alpha(Q') - \mathcal{L}_\beta(Q')} k_{\alpha\beta}(Q'))^{1/2}$$

Thus we have constructed a collection $\{c_{\alpha\beta}\}$ with values in $\mathbb{C}^*$ satisfying the cocycle condition on our Riemann Surface $X$. However this construction cannot be considered complete since we have not discussed the possible singularities of the local solutions.

There are two classes of singular points that can arise in the course of the construction. The set $Y = \{P_0, P_\infty\}$, where the poles of the KN basis are located, and the points at which the section $h$ we used to construct the reference connection $\Gamma_0$ may attain zeros (or poles). Let us discuss first the second class. Its singular points, possibly different from those contained in $Y$, arise in the following way. Assume $K^{1/2}$ is an odd theta characteristic, so that $h$ is holomorphic. Denoting by $(h)$ the associated divisor, we have $deg\,(h) = c_1(K^{1/2}) = g - 1$, hence $h$ has $g - 1$ zeros, not necessarily distinct. Let $a$ be one of them. If the order of $h$ at $a$ is $k$, with respect to a local coordinate $z$ vanishing at $a$ we have $\Gamma_0 = -\frac{k}{z} + O(1)$, hence $\sigma_1$ is of the form $\sigma_1 \sim z^{-k}$. This of course means that $\sigma_1$ is meromorphic at $a$, $k$ being positive. On the other hand, $\sigma_2$ contains ${\sigma_1}^{-2}$, which results to be holomorphic, hence does not create troubles under integration (compare with (2.9)). Thus, at zeros of $h$, our solution will have pole-like singularities, remaining single-valued, though.

The singularities attained at $Y$ are more serious. There, the connection can have poles (as well as zeros) of arbitrary but finite order (recall our assumption on (2.8) to be a finite sum). This means that the solution behaves like exponential of a polynomial in $1/z$, thus producing an essential singularity. Beside this, depending on the coefficient of $f_0^{(1)}$, the logarithmic singularity will be in effect, yielding a multi-valued $\sigma_1$ in the vicinity of $P_0$ and $P_\infty$. We conclude that our construction of gluing local expressions for $\sigma_1$ is meaningful only after we delete from $X$ the two point set $Y$.

Finally we should discuss what happens if we change the reference collection from $\{Q_\alpha\}$ to $\{Q'_\alpha\}$. It does not present any difficulty to see that $\{c_{\alpha\beta}\}$ changes by a coboundary, that is

$$c_{\alpha\beta} \longrightarrow c'_{\alpha\beta} = a_\alpha \, c_{\alpha\beta} {a_\beta}^{-1}$$

where the non-zero complex number $a_\alpha$ is given by $a_\alpha = \exp(\int_{Q_\alpha}^{Q'_\alpha} p_\alpha)$.

In summary, the differential equation

$$\partial\sigma_1 = p\,\sigma_1$$

can be solved on the non compact Riemann Surface $X' = X \setminus Y$. The solution involves the 1-cocycle $\{c_{\alpha\beta}\}$ with values in $\mathbb{C}^*$, and we denote by $C$ the corresponding cohomology class in $H^1(X', \mathbb{C}^*)$. By the usual correspondence, $C$ is a flat line-bundle over $X'$.[5] Thus the object we constructed is actually a meromorphic[6] section of $C \otimes K^{1/2}$.

Before going any further in this direction, we take on the construction of $\sigma_2$. We use the same scheme as before, namely we consider the point $Q$ lying in the intersection $U_\alpha \cap U_\beta$ together with the paths joining it to the reference points $Q_\alpha$ and $Q_\beta$. Just as we did before, we use the established gluing rule

$$\sigma_{1\alpha}(Q) = c_{\alpha\beta}\, k_{\alpha\beta}(Q)^{-1/2}\, \sigma_{1\beta}(Q)$$

to change local coordinate when reaching a point $Q' \in U_\alpha \cap U_\beta$ on the composite path joining $Q_\alpha$ and $Q_\beta$. Thus now we have

$$\begin{aligned}
\sigma_{2\alpha}(Q) &= -\sigma_1(Q) \int_{Q_\alpha}^{Q'} \sigma_1(z_\alpha)^{-2}\, dz_\alpha - \sigma_1(Q) \int_{Q'}^{Q} \sigma_1(z_\alpha)^{-2}\, dz_\alpha \\
&= -\sigma_1(Q) \int_{Q_\alpha}^{Q'} \sigma_1(z_\alpha)^{-2}\, dz_\alpha - \sigma_1(Q) c_{\alpha\beta}^{-2} \int_{Q'}^{Q} \sigma_1(z_\beta)^{-2}\, dz_\beta \\
&= -c_{\alpha\beta}\, k_{\alpha\beta}(Q)^{-1/2}\, \sigma_{1\beta}(Q) \int_{Q_\alpha}^{Q'} \sigma_1(z_\alpha)^{-2}\, dz_\alpha \\
&\quad -c_{\alpha\beta}^{-1}\, k_{\alpha\beta}(Q)^{-1/2}\, \sigma_{1\beta}(Q) \left( \int_{Q_\beta}^{Q} - \int_{Q_\beta}^{Q'} \right) \sigma_1(z_\beta)^{-2}\, dz_\beta
\end{aligned}$$

hence

$$\sigma_{2\alpha}(Q) = d_{\alpha\beta}\, k_{\alpha\beta}(Q)^{-1/2}\, \sigma_{1\beta}(Q) + c_{\alpha\beta}^{-1}\, k_{\alpha\beta}(Q)^{-1/2}\, \sigma_{2\beta}(Q) \tag{2.11}$$

with

$$d_{\alpha\beta} = -c_{\alpha\beta} \int_{Q_\alpha}^{Q'} e^{-2\mathcal{L}_\alpha(z_\alpha)} dz_\alpha + c_{\alpha\beta}^{-1} \int_{Q_\beta}^{Q'} e^{-2\mathcal{L}_\beta(z_\beta)} dz_\beta$$

Again, the number $d_{\alpha\beta}$ does not depend on the point used to calculate it. Therefore, taking into account the previous remarks about the singularities, the transformation rule (2.11) is well-defined on $X'$. It is apparent that the field $\sigma_2$ does not simply transform as a spin $-1/2$ one: $\sigma_1$ and $\sigma_2$ got mixed upon changing the local chart. In particular the conformal weight, so to speak, is still preserved, while there is a flat matrix mixing the components. Thus we obtained

$$\begin{pmatrix} \sigma_{1_\alpha} \\ \sigma_{2_\alpha} \end{pmatrix} = k_{\alpha\beta}^{-1/2} \begin{pmatrix} c_{\alpha\beta} & 0 \\ d_{\alpha\beta} & c_{\alpha\beta}^{-1} \end{pmatrix} \cdot \begin{pmatrix} \sigma_{1_\beta} \\ \sigma_{2_\beta} \end{pmatrix}$$

or, passing to the collection $\{\mathcal{Q}_\alpha\}$ of fundamental matrices

$$\mathcal{Q}_\alpha = \begin{pmatrix} k_{\alpha\beta}^{-1/2} & 0 \\ 0 & k_{\alpha\beta}^{1/2} \end{pmatrix} \cdot \mathcal{Q}_\beta \cdot \begin{pmatrix} c_{\alpha\beta} & d_{\alpha\beta} \\ 0 & c_{\alpha\beta}^{-1} \end{pmatrix}$$

[5] Since $C$ is flat, the argument according to which all the vector bundles over a non compact curves are trivial does not apply.

[6] $\sigma_1$ is meromorphic on $X'$ because of the zeros of $h$. However, in the vicinity of the punctures it is to beconsidered as holomorphic, since those points do not belong to $X'$ any longer.

From now on we shall call $\{T_{\alpha\beta}\}$ the collection of matrices

$$T_{\alpha\beta} = \begin{pmatrix} c_{\alpha\beta} & 0 \\ d_{\alpha\beta} & c_{\alpha\beta}^{-1} \end{pmatrix}$$

constructed in that way, while we have

$$T_{\alpha\beta}^{\vee} = \begin{pmatrix} c_{\alpha\beta}^{-1} & -d_{\alpha\beta} \\ 0 & c_{\alpha\beta} \end{pmatrix} = {}^{t}T_{\alpha\beta}^{-1}$$

for the matrices $\{T_{\alpha\beta}^{\vee}\}$ introduced above. The notation has been chosen in such a way that $T^{\vee}$ is indeed the dual of $T$, once one proves that the cocycle condition $T_{\alpha\beta}T_{\beta\gamma} = T_{\alpha\gamma}$ holds. Writing the cocycle condition on $\{T_{\alpha\beta}\}$ yields the following one on $d$:

$$d_{\alpha\gamma} = c_{\beta\gamma}\, d_{\alpha\beta} + c_{\alpha\beta}^{-1}\, d_{\beta\gamma}$$

It remains to verify whether $\{d_{\alpha\beta}\}$ does satisfy the condition above. This is simply checked as

$$\begin{aligned} d_{\alpha\gamma} &= -c_{\alpha\gamma}\int_{Q_\alpha}^{Q'} e^{-2\mathcal{L}_\alpha(z_\alpha)}dz_\alpha + c_{\alpha\gamma}^{-1}\int_{Q_\gamma}^{Q'} e^{-2\mathcal{L}_\gamma(z_\gamma)}dz_\gamma \\ &= -c_{\alpha\beta}\,c_{\beta\gamma}\int_{Q_\alpha}^{Q'} e^{-2\mathcal{L}_\alpha(z_\alpha)}dz_\alpha + c_{\alpha\beta}^{-1}\,c_{\beta\gamma}^{-1}\int_{Q_\gamma}^{Q'} e^{-2\mathcal{L}_\gamma(z_\gamma)}dz_\gamma \\ &\quad + c_{\beta\gamma}c_{\alpha\beta}^{-1}\int_{Q_\beta}^{Q'} e^{-2\mathcal{L}_\beta(z_\beta)}dz_\beta - c_{\beta\gamma}c_{\alpha\beta}^{-1}\int_{Q_\beta}^{Q'} e^{-2\mathcal{L}_\beta(z_\beta)}dz_\beta \\ &= c_{\beta\gamma}\,d_{\alpha\beta} + c_{\alpha\beta}^{-1}\,d_{\beta\gamma} \end{aligned}$$

on the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$.

It remains to examine what happens when changing the reference point collection. This has been already checked for the diagonal entries. Passing to the reference collection $\{Q'_\alpha\}$, we have that $\{d_{\alpha\beta}\}$ transforms as

$$d_{\alpha\beta} = a_\alpha^{-1}a_\beta^{-1}d'_{\alpha\beta} + c'_{\alpha\beta}f_\alpha a_\beta^{-1} - {c'_{\alpha\beta}}^{-1}f_\beta a_\alpha^{-1}$$

so that this relation can be recast in the matrix form

$$T_{\alpha\beta} = \begin{pmatrix} a_\alpha & 0 \\ f_\alpha & a_\alpha^{-1} \end{pmatrix} \cdot T'_{\alpha\beta} \cdot \begin{pmatrix} a_\beta & 0 \\ f_\beta & a_\beta^{-1} \end{pmatrix}^{-1}$$

thus showing that the cocycle $\{T_{\alpha\beta}\}$ is replaced by a cohomologous one.

Thus the matrices $\{T_{\alpha\beta}\}$ do form an $SL_2(\mathbb{C})$-valued cocycle. We denote by $T$ the corresponding cohomology class in $H^1(X', SL_2(\mathbb{C}))$. $T$ is a rank-2 flat $SL_2(\mathbb{C})$ vector bundle on $X'$, or, as we already said, a rank-2 local system [G1, G2]. The solution of the DS system we have just produced is to be properly interpreted as a section of $K^{-1/2} \otimes T$[7] on $X'$.

The next step is to describe in more detail the structure of the flat bundle $T$ so constructed. Due to the fact that the cocycle $T$ is associated to the lack of single-valuedness of the local determinations of the solution, it is natural to refer to $T$ as the monodromy data of the DS system.

[7] Here we refer to the section obtained gluing the first row of the local fundamental matrices.

## 2.5 Analysis of the Monodromy

The flat bundle $T$ determined as monodromy of the DS system is the same thing as a representation of the fundamental group $\pi_1(X')$ into $SL_2(\mathbb{C})$ (up to conjugation). This follows from the general fact that for any connected manifold $M$ and any (Lie) group $G$ we have

$$H^1(M, G) \cong Hom(\pi_1(M), G)/G$$

where the quotient is taken with respect to the action of $G$ on itself by conjugation [KT1, G1]. Given an element $F$ of $H^1(M, G)$, which is a flat bundle on $M$ we denote by $\hat{F}$ the corresponding element in the other space and call it "the characteristic representation associated with $F$". Since later on we shall slightly use the explicit form of the isomorphism, it is perhaps worthwhile to describe at least the way how it is constructed.

The main point is to describe the fundamental group from within a Čech setting. This is done in the following way. Fix a base-point on the manifold and a suitable good-covering of which we fix a certain element $U_0$ containing the base point. Given a closed path $\gamma$, we cover it with a chain $(U_0, U_1, \ldots, U_n, U_0)$ of open sets starting and ending at $U_0$. Homotopic paths correspond to chains such that we can pass from one to the other by a finite sequence of simple operations consisting in replacing a pair $(U_i, U_{i+1})$ of consecutive elements by a triple $(U_i, U_j, U_{i+1})$ with non void intersection. The inverse path $\gamma^{-1}$ correspond to the chain $(U_0, U_n, \ldots, U_1, U_0)$, and so on. Given the cocycle $g = \{g_{ij}\}$ the corresponding element in $Hom(\pi_1(M), G)$ is the one assigning to the chain $(U_0, U_1, \ldots, U_n, U_0)$, the group element $g_{01} \cdot g_{12} \cdots g_{n-1\,n} \cdot g_{n0}$. It is clear that this operation is well-defined up to conjugation and compatible with refinements of the covering (full details in [G1]).

After this short digression, let us come back to our monodromy $T$. Considered as a vector bundle, it has triangular transition functions, so from that point of view it is an extension

$$0 \longrightarrow C^{-1} \longrightarrow T \longrightarrow C \longrightarrow 0$$

where the flat line-bundle $C$ appears as a quotient. In view of the isomorphism just described it is a representation taking place in the (lower) Borel subgroup of $SL_2(\mathbb{C})$. This fact allows us to separately analyze the components of the representation.

Concerning the flat line bundle $C$, in view of the isomorphism mentioned above, we can consider it as an element of

$$Hom(H_1(X'), \mathbb{C}^*)$$

This follows from the fact that being $\mathbb{C}^*$ commutative, the homomorphisms of the fundamental group to $\mathbb{C}^*$ factor through $\pi_1(X')/[\pi_1(X'), \pi_1(X')] \cong H_1(X')$. This is of course a somewhat complicated way of retrieving the isomorphism

$$H^1(X', \mathbb{C}^*) \cong Hom(H_1(X'), \mathbb{C}^*)$$

into the present context.

We recall that the fundamental group of the non compact surface $X'$ is generated by elements $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, $c_0$, $c_\infty$, satisfying the relation

$$\prod_{i=1}^{g} [a_i, b_i] = c_0 c_\infty$$

where the $a_i$ and $b_i$ are the usual cycles around the handles while $c_0$ and $c_\infty$ are two cycles encircling the to punctures $P_0$ and $P_\infty$. The projection onto $H_1(X', \mathbb{Z})$ kills the commutator subgroup, thus in the first homology group the only relation we have is $c_0 = c_\infty^{-1}$, so that it is freely generated by the symbols $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, $c_0$ (we make the slight abuse of language of denoting with the same letter both an element of the fundamental group and its image in the first homology group). Incidentally this also says that the dimension of $H^1(X', \mathbb{C}^*)$ over $\mathbb{C}$ is $g + 1$.

Thus the line bundle $C$ is the same thing as a character of the first homology group and it is determined by its value on the generators $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, $c_0$ of the group. Below we shall see that the parametrization provided by the KN basis gives a formula for the character $\hat{C}$ so determined. Before doing this, we still remain at this level of generality to treat the off-diagonal term in the monodromy.

We recall that $T$ is the extension of $C$ by $C^{-1}$. Generally, on a manifold $M$, the extension class represented by the exact sequence of vector bundles

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

is an element of the cohomology group $H^1(M, \underline{Hom}(\mathcal{E}', \mathcal{E}''))$ or $H^1(M, \mathcal{E}'' \otimes \mathcal{E}'^\vee)$. Here we mean cohomology groups with values in the appropriate sheaves of sections. In our case of flat bundles this translates into the fact that the extension class represented by $T$ is an element of the cohomology group

$$H^1(X', \mathcal{C}^{-2})$$

We hope the notation will not be confusing. This is the first cohomology group of $X'$ with values in the sheaf of *locally constant* sections $\mathcal{C}^{-2}$ of the flat bundle $C^{-2}$. For flat bundles sheaves of locally constant sections are of course meaningful: they are $\mathbb{C}_{X'}$-modules, instead of being $\mathcal{O}_{X'}$-modules [G2].

This is of course rather abstract, and one would be pleased in finding some more reliable characterization. In particular, the first thing is to actually determine the desired object in $H^1(X', \mathcal{C}^{-2})$. This is done by explicitly tracking the various isomorphisms in term of the coordinate covering, but we already did it when verifying the identity:

$$d_{\alpha\gamma} = c_{\beta\gamma}\, d_{\alpha\beta} + c_{\alpha\beta}^{-1}\, d_{\beta\gamma}$$

Following [G2], we modify it introducing the 1-cochain $\{S_{\alpha\beta}\} = \{c_{\alpha\beta}\, d_{\alpha\beta}\}$, thereby obtaining the identity

$$S_{\alpha\gamma} = c_{\beta\gamma}^2\, S_{\alpha\beta} + S_{\beta\gamma}$$

This identity is in fact the cocycle condition for the 1-cochain $\{S_{\alpha\beta}\}$ with values in the locally constant sections of the flat line bundle $C^{-2}$.[8] However we should verify that our procedure indeed defines an element of a cohomology group. In other words, a new cocycle $\{S'_{\alpha\beta}\}$ differing from the previous one by a coboundary, should be associated essentially to the same data for our differential equation. The cochain relation $S' - S = \delta(f)$ has the explicit form

$$S'_{\alpha\beta} = S_{\alpha\beta} + f_\beta - c_{\alpha\beta}^2 f_\alpha$$

[8] See [G2] for the use of the Čech formalism with coefficients in sheaves of modules.

and suppose such an $\{f_\alpha\}$ to be given. Dividing by $c_{\alpha\beta}$ and taking into account the explicit form for $d_{\alpha\beta}$ previously found, we see that shifting by a coboundary precisely amounts to a change in the integration constants in the indefinite integrals defining $\{\sigma_{2\alpha}\}$. In other words, this is the same as changing the initial points in the integral defining $\{\sigma_{2\alpha}\}$ in (2.9).

Thus the off-diagonal element of the monodromy has a geometrical interpretation in its own. We stress that this is due to the triangular character of $T$.

This completes the analysis of the monodromy of the DS differential system at this level. However, we have gained only a little, since we still lack any really workable expression for the various quantities we introduced, the various flat bundles and all the like. To this end we find it convenient to introduce the KN parametrization.

We recall that the parametrization for the '$p$' in the DS connection was

$$p = \Gamma_0 + \sum_n p_n \omega^n$$

with $\Gamma_0$ the logarithmic connection $\Gamma_0 = -\partial \log h$. Now we plug this expansion in the local expression (2.9) for $\sigma_1$. We keep the usual definitions and notations for the covering $\mathfrak{U}$ and the reference points $\{Q_\alpha\}$. With an easy integration we find

$$\sigma_{1\alpha}(Q) = e^{\mathcal{L}_\alpha(Q)} = h_\alpha(Q)^{-1} h_\alpha(Q_\alpha) \exp\left(\sum_n p_n \int_{Q_\alpha}^{Q} \omega_n\right)$$

$h_\alpha$ is the determination of $h$ in the chart $U_\alpha$. Thus we have an expression of this kind for each element of the covering. Consequently we get in this way an expression for the 1-cocycle $c$ we constructed out of $\sigma_1$. Specifically we obtain

$$c_{\alpha\beta} = h_\alpha(Q_\alpha) h_\beta(Q_\beta)^{-1} \exp\left(\sum_n p_n \int_{Q_\alpha}^{Q_\beta} \omega_n\right)$$

Now we exploit the description of the fundamental group of $X'$ by means of chains of open sets we briefly outlined above. Fix a base point $Q_0 \in X'$ and an open set $U_0$ containing it. We consider a path $\gamma$ and a covering chain $(U_{\alpha_0}, U_{\alpha_1}, \cdots, U_{\alpha_n}, U_{\alpha_0})$, $U_{\alpha_0} = U_0$. According to our discussion above, the character $\hat{C}$ associated with the cocycle $\{C_{\alpha\beta}\}$ is given by the formula

$$\hat{C}(\gamma) = \prod_{i=0}^{n} c_{\alpha_i \alpha_{i+1}}$$

which, together with explicit form for the 1-cocycle quoted above yields

$$\hat{C}(\gamma) = \exp\left(\sum_n p_n \int_\gamma \omega_n\right) \tag{2.12}$$

Being a character of the first homology group of $X'$, $\hat{C}$ is defined by its values on the generators:

$$\begin{aligned} \hat{C}(a_i) &= \exp(2\pi i\, p_n \sum_n M_i^n) \\ \hat{C}(b_i) &= \exp(2\pi i\, p_n \sum_n N_i^n) \end{aligned}$$

with

$$\begin{aligned} M_i^n &= \frac{1}{2\pi i}\oint_{a_i}\omega_n \\ N_i^n &= \frac{1}{2\pi i}\oint_{b_i}\omega_n \end{aligned}$$

We stress that the numbers $\{M_i^n,\, N_i^n;\, n\in\mathbb{Z}, i=1,\ldots,g\}$ are in principle explicitly computable, using the concrete expression for the KN basis [BLMR, KL] in terms of the $\theta$-function and the Prime-form [F] of the Riemann Surface $X$, the compact completion of $X'$. The resulting expressions would be necessarily quite complicated. In spite of this, we can see that for each cycle $a_i$ , $b_i$, $i=1,\ldots,g$ the numbers $M_i^n$ , $M_i^n$ are zero for all but a finite number of values of the index $n\in\mathbb{Z}$. The number of these values is $2g+1$. This is due to the fact that the dimension of the space of second kind differentials modulo exact differentials is $2g$ [L]. Here we have used the name "second kind differential" to mean those abelian differentials which are not of third kind (thus including the first kind ones). Obviously, the third kind has also non zero periods, in general, thus yielding the number $2g+1$.

The value of $\hat{C}$ on the cycle around the puncture $P_o$ has instead a fairly simple form:

$$\hat{C}(c_0)=\exp\left(\sum_n p_n\oint_\gamma\omega_n\right)=e^{2\pi i p_0}$$

as follows from the fact that $\omega_0$ is the only element in the KN basis having non zero residues at the points $P_0$ and $P_\infty$.

This description of the character $\hat{C}$ closely resembles the one of the characters of the first homology group of a compact Riemann Surface obtained while one is describing the flat bundles by means of their factor of automorphy [G3, G4]. Since we have shown that the dimension of $H^1(X',\mathbb{C}^*)$ is $g+1$, and the index for the modes runs over $\mathbb{Z}$, it is clear that the parametrization we obtained is largely redundant.

Now we have to characterize the off-diagonal element of the monodromy in the same terms as we did for $C$. Before we introduce explicitly the KN basis into the formulas for $d=\{d_{\alpha\beta}\}$, we would like to recast it into a different form, perhaps giving $d$ a nicer geometrical significance. To this end, notice that the transformation rule (2.10) for $\sigma_1$ can be rewritten in the following form

$$e^{-2\mathcal{L}_\alpha(z_\alpha)}dz_\alpha = c_{\alpha\beta}^{-2}e^{-2\mathcal{L}_\beta(z_\beta)}dz_\beta$$

and can be interpreted as the definition for each $\alpha$, $\beta$ of a 1-form $\phi_{\alpha\beta}$ :

$$\phi_{\alpha\beta}=\begin{cases} e^{-2\mathcal{L}_\alpha}dz_\alpha & \text{on } U_\alpha \\ c_{\alpha\beta}^{-2}e^{-2\mathcal{L}_\beta}dz_\beta & \text{on } U_\beta \end{cases}$$

On the intersection $U_\beta\cap U_\gamma$ we have

$$\begin{aligned} \phi_{\alpha\beta} &= c_{\alpha\beta}^{-2}e^{-2\mathcal{L}_\beta}dz_\beta \\ &= c_{\alpha\beta}^{-2}c_{\beta\gamma}^{-2}e^{-2\mathcal{L}_\gamma}dz_\gamma \\ &= c_{\alpha\gamma}^{-2}e^{-2\mathcal{L}_\gamma}dz_\gamma \\ &= \phi_{\alpha\gamma} \end{aligned}$$

Thus the two prolongations from $U_\alpha$ agree on the intersection $U_\beta \cap U_\gamma$ and we can drop the second index: $\phi_{\alpha\beta} \to \phi_\alpha$. With this process we have produced a 1-form defined on the "star" of the open set $U_\alpha$, where the star is made by $U_\alpha$ itself together with all the open sets having non void intersection with it[9]. Keeping in mind the explicit expression for $d$ stated soon after the gluing law (2.11) and the definition of $S$, we find

$$S_{\alpha\beta} = \int_{(Q_\beta, Q_\alpha)} \phi_\beta = -c_{\alpha\beta}^2 \int_{(Q_\alpha, Q_\beta)} \phi_\alpha$$

where $(Q_\alpha, Q_\beta)$ is the 1-simplex joining $Q_\alpha$ with $Q_\beta$. Now introducing the KN basis and recalling the expression of $\sigma_1$ in terms of that we have previously exploited, we immediately obtain

$$\phi_\alpha(Q) = h_\alpha(Q)^2 h_\alpha(Q_\alpha)^{-2} \exp\left(-2\sum_n p_n \psi_n(Q_\alpha, Q)\right)$$

where we introduced $\psi_n(Q, Q') := \int_Q^{Q'} \omega_n$. We remark the apparence of the term $h^2$ into the above expression: as the spin bundle we chose at the very beginning is an odd theta characteristic, its holomorphic section $h$ squared is a linear combination of the holomorphic abelian differentials of $X$, with coefficients the derivatives of the corresponding $\theta$-function at the origin of the Jacobian torus of $X$ [F]. Thus the forms we are going to construct exploiting our parametrization have a very concrete meaning.

Using the expression found above for the form $\phi_\alpha$, we have the following formula for $S_{\alpha\beta}$ :

$$S_{\alpha\beta} = -h_\beta(Q_\beta)^{-2} e^{2\sum_n p_n \psi_n(Q_\alpha, Q_\beta)} \int_{(Q_\alpha, Q_\beta)} e^{-2\sum_n p_n \psi_n(Q_\alpha, z)} h(z)^2$$

From here, a formula for $d$ can be obtained:

$$d_{\alpha\beta} = -h_\alpha(Q_\alpha)^{-1} h_\beta(Q_\beta)^{-1} e^{\sum_n p_n \psi_n(Q_\alpha, Q_\beta)} \int_{(Q_\alpha, Q_\beta)} e^{-2\sum_n p_n \psi_n(Q_\alpha, z)} h(z)^2$$

Now we are ready to produce a global object representing $S$ directly depending on the (classes of) paths on $X'$. This being the aim, we have to come back for a while at the general level, since we need a different characterization for the cohomology group $H^1(X', \mathcal{C}^{-2})$. We quote from [G2] the following theorem: on any connected manifold $M$ carrying a flat bundle $F$, there is an isomorphism

$$H^1(M, \mathcal{F}) \cong H^1(\pi_1(M), \hat{F})$$

where $\mathcal{F}$ is the sheaf of locally constant sections of $F$, $\hat{F}$ is the associated characteristic representation and the space at the RHS is the group of *crossed homomorphisms* of $\pi_1(M)$ into the representation space of $\hat{F}$ [10] modulo the trivial ones.

We recall that a crossed homomorphism $u$ of a group $\Pi$ into a $\Pi$-module $V$ is a map $u : \Pi \to V$ satisfying

$$u(xy) = y^{-1} \cdot u(x) + u(y)$$

[9] The reason for the term "star" becomes evident if one introduces the concept of the "nerve" of the covering $\mathfrak{U}$ [SH]

[10] The space over which the representation $\hat{F}$ takes place can be taken to be $\mathbb{C}^r$ (or $\mathbb{R}^r$) where $r$ is the rank of $F$.

$x, y \in \Pi$, where the dot stands for the action of $\Pi$ on $V$. The space of all crossed homomorphism is denoted by $Z^1(\Pi, V)$. The trivial crossed homomorphisms (i.e. the coboundaries) are those given by

$$u(x) = v - x^{-1} \cdot v$$

for $v \in V$. The space of the coboundaries is denoted by $B^1(\Pi, V)$. Thus $H^1(\Pi, V) = Z^1(\Pi, V)/B^1(\Pi, V)$.

It is now worth describing the explicit form of the stated isomorphism $H^1(M, \mathcal{F}) \cong H^1(\pi_1(M), \hat{F})$. We use the representation of $\pi_1(M)$ we are now familiar with, namely the one given by $\pi_1(\mathfrak{U}, U_0)$. For a cocycle $\{A_{\alpha\beta}\} \in Z^1(\mathfrak{U}, \mathcal{F})$, the corresponding element in $Z^1(\pi_1(\mathfrak{U}, U_0), \hat{F})$ is given by

$$\begin{aligned} \hat{A}_\gamma &= (\hat{F}_{\alpha_1\alpha_2} \cdots \hat{F}_{\alpha_{p-1}\alpha_p})^{-1} \cdot A_{\alpha_0\alpha_1} + (\hat{F}_{\alpha_2\alpha_3} \cdots \hat{F}_{\alpha_{p-1}\alpha_p})^{-1} \cdot A_{\alpha_1\alpha_2} + \cdots \\ &\quad \cdots + \hat{F}^{-1}_{\alpha_{p-1}\alpha_p} \cdot A_{\alpha_{p-2}\alpha_{p-1}} + A_{\alpha_{p-1}\alpha_p} \end{aligned}$$

where $\gamma$ is the chain $(U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_p})$ based at $U_0$. It is a standard matter of tracing all the relations to verify that this correspondence is well defined on the classes and goes through when taking the direct limit on the coverings. We refer to [G2] for the details.

Now we apply this formalism to our case where $S = \{S_{\alpha\beta}\} \in Z^1(\mathfrak{U}, \mathbb{C}^{-2})$. Exploiting the formula above, we obtain

$$\hat{S}_\gamma = -h_0(Q_0)^{-2} e^{2\sum_k p_k \int_\gamma \omega_k} \int_\gamma e^{-2\sum_k p_k \psi_k(Q_0, z)} h(z)^2 \tag{2.13}$$

where $\gamma$ is a path on $X'$ starting and ending at $Q_0$ covered by an appropriate chain of open sets. Some comments are in order here. The integral on the RHS of (2.13), as it stands, should be properly defined on the universal cover of $X'$. This is due to the fact that the integrand is multivalued on the surface. Thus formula (2.13) can be read in two ways. Interpreting the integral on the RHS as an integral over the universal covering space, (2.13) becomes an instance of the fact that elements of the group $H^1(\pi_1(M), \hat{F})$ can be represented by means of differential 1-forms on the universal covering space of $M$ "twisted" by the flat factor of automorphy $\hat{F}$ [G2]. On the other hand, (2.13) says that the LHS can be used to *define* the integral on the RHS, thereby giving a full meaning to a way of naively continuing the integrals in (2.9), outside their domain of definition, along a complete path on the surface.

Formula (2.13) is inferred inserting the expressions for $S_{\alpha\beta}$ and $c_{\alpha\beta}$ we obtained previously into the general formula we quoted above. Here are the first few steps. Consider for instance three open sets $U_\alpha, U_\beta, U_\gamma$. According to the quoted prescription we must consider

$$c^2_{\beta\gamma} S_{\alpha\beta} + S_{\beta\gamma}$$

which, using the forms $\{\phi_\alpha\}$ previously introduced, reads

$$\begin{aligned} -c^2_{\beta\gamma} \int_{(Q_\alpha, Q_\beta)} \phi_\beta - \int_{(Q_\beta, Q_\gamma)} \phi_\gamma &= -c^2_{\beta\gamma} \int_{(Q_\alpha, Q_\beta)} \phi_\beta - c^2_{\beta\gamma} \int_{(Q_\beta, Q_\gamma)} \phi_\beta \\ &= -c^2_{\beta\gamma} \int_{(Q_\alpha, Q_\gamma)} \phi_\beta \end{aligned}$$

so that, inserting the explicit expression for $\phi_\beta$, we find

$$h_\gamma(Q_\gamma)^{-2}e^{2\sum_n p_n\psi_n(Q_\alpha,Q_\gamma)}\int_{(Q_\alpha,Q_\gamma)}e^{-2\sum_k p_k\psi_k(Q_\alpha,z)}h(z)^2$$

Now add $U_\delta$ at the end of the chain. The relevant quantity now is

$$c^2_{\gamma\delta}c^2_{\beta\gamma}S_{\alpha\beta}+c^2_{\gamma\delta}S_{\beta\gamma}+S_{\gamma\delta}$$

and, using the result for three sets, we reexpress it as

$$-c^2_{\gamma\delta}c^2_{\beta\gamma}\int_{(Q_\alpha,Q_\gamma)}\phi_\beta-c^2_{\gamma\delta}\int_{(Q_\gamma,Q_\delta)}\phi_\gamma$$

but the forms $\phi_\gamma$ and $c^2_{\beta\gamma}\phi_\beta$ are the same on $U_\beta\cap U_\gamma$ so that the last expression can be written in term of the integral of a unique form we continue to call $\phi_\gamma$ :

$$c^2_{\gamma\delta}c^2_{\beta\gamma}S_{\alpha\beta}+c^2_{\gamma\delta}S_{\beta\gamma}+S_{\gamma\delta}=-c^2_{\gamma\delta}\int_{(Q_\alpha,Q_\delta)}\phi_\gamma$$

Plugging in again the expression in terms of the KN basis we find the same formula as the one relative to the three sets, except for the index shift:

$$h_\delta(Q_\delta)^{-2}e^{2\sum_n p_n\psi_n(Q_\alpha,Q_\delta)}\int_{(Q_\alpha,Q_\delta)}e^{-2\sum_k p_k\psi_k(Q_\alpha,z)}h(z)^2$$

... and so on. It is clear that we obtain the formula we claimed for above.

With the integral formula for $\hat{S}$ so obtained, it is very easy to verify the cocycle condition. Indeed, denoting by $\gamma_1\cdot\gamma_2$ the element of $\pi_1(X')$ obtained by first running along $\gamma_1$ and afterwards along $\gamma_2$, we have

$$\hat{S}_{\gamma_1\cdot\gamma_2}=-h_0(Q_0)^{-2}e^{2\sum_k p_k\int_{\gamma_1}\omega_k}e^{2\sum_k p_k\int_{\gamma_2}\omega_k}\int_{\gamma_1\cdot\gamma_2}e^{-2\sum_k p_k\psi_k(Q_0,z)}h(z)^2$$

rewriting the integral in $\gamma_1\cdot\gamma_2$ as

$$\int_{\gamma_1}e^{-2\sum_k p_k\psi_k(Q_0,z)}h(z)^2+e^{-2\sum_k p_k\int_{\gamma_1}\omega_k}\int_{\gamma_2}e^{-2\sum_k p_k\psi_k(Q_0,z)}h(z)^2$$

we arrive at the desired relation

$$\hat{S}_{\gamma_1\cdot\gamma_2}=e^{2\sum_k p_k\int_{\gamma_2}\omega_k}\hat{S}_{\gamma_1}+\hat{S}_{\gamma_2}$$

It remains to mention that $\dim H^1(\pi_1(X'),\hat{C}^{-2})=2g$. This is shown in the subsequent appendix, exploiting a sort of differential calculus specially suited for dealing with such algebraic cocycles.

## Appendix: Application of the free differential calculus

Here we calculate the dimension of the group $H^1(\pi_1(X'), \hat{C}'^{-2})$; we use Fox's differential calculus on free groups, in the same spirit as in [Go]. Fox's differential calculus is a non commutative version of differential calculus for words in a free group. It turns out that this allows for a very easy description of the algebraic cocycles on a group.

Given a group $\Pi$, a cocycle a cocycle $u$ with values in a $\Pi$-module $V$ is a map $u : \Pi \to V$ satisfying

$$u(xy) = u(x) + x \cdot u(y)$$

for $x, y \in \Pi$. This is the same definition as adopted in the main text, upon substituting $u(x) \to u(x^{-1})$. This one, however, is much more convenient for the present purposes.

Among the possible $\Pi$-modules consider the group ring $M = \mathbb{Z}\Pi$, that is the set of formal sums $\sum_{x\in\Pi} m_x\, x$. The $\Pi$-module structure is given by the left action of $\Pi$ on $\mathbb{Z}\Pi$, which extends to an action of $\mathbb{Z}\Pi$ onto itself. The augmentation $\epsilon : \mathbb{Z}\Pi \to \mathbb{Z}$ sends the element $\sum_{x\in\Pi} m_x\, x$ to $\sum_{x\in\Pi} m_x$. Thus $M$ can be endowed with a $\mathbb{Z}\Pi$-bimodule structure with respect to the action $m \mapsto m_1 \cdot m\, \epsilon(m_2)$. In this framework a derivation of $M$ corresponds to a $\mathbb{Z}$-linear map $\delta : \mathbb{Z}\Pi \to \mathbb{Z}\Pi$ satisfying $\delta(m_1 \cdot m_2) = \delta(m_1)\epsilon(m_2) + m_1 \cdot \delta(m_2)$.

Restricting to $\Pi$, a derivation is just a 1-cocycle of $\Pi$ with values in $\mathbb{Z}\Pi$. If $\delta$ is a derivation, then $x \mapsto \delta(x)u$, $u \in \mathbb{Z}\Pi$, is again a derivation. Thus the set of all derivations is a right $\mathbb{Z}\Pi$-module.

Now let $\Pi$ be a free group on $n$ generators $x_1, \ldots, x_n$. Fox's result is that the set of all derivations is generated by $n$ elements $\partial_i = \partial/\partial x_i$, $i = 1, \ldots, n$, which satisfy $\partial_i(x_j) = \delta_{ij}$. Restricting the formulas to $\Pi$, we have the relation

$$\delta(ab) = \delta(a) + a\,\delta(b).$$

This renders an easy task the calculation of Fox derivations of any word in the generators. For instance in the case of two generators $a$, $b$ we derive

$$\begin{array}{ll}
\frac{\partial}{\partial a}a^{-1} = -a^{-1} & \\
\frac{\partial}{\partial a}(ab) = 1 & \frac{\partial}{\partial b}(ab) = a \\
\frac{\partial}{\partial a}(aba^{-1}) = 1 - aba^{-1} & \frac{\partial}{\partial b}(aba^{-1}) = a \\
\frac{\partial}{\partial a}(aba^{-1}b^{-1}) = 1 - aba^{-1} & \frac{\partial}{\partial b}(aba^{-1}b^{-1}) = a - aba^{-1}b^{-1}
\end{array}$$

One also derives a set of rules quite similar to those of usual differential calculus. In particular, for $u \in \mathbb{Z}\Pi$, we have a "mean value theorem" in the form

$$u - \epsilon(u) = \sum_{i=1}^{n} (\partial_i u)(x_i - 1)$$

The interest in such a concept is that there is a close relation between Fox's differential calculus and group cohomology.

Let $V$ be a $\Pi$-module by means of $\phi : \Pi \to GL(V)$. This is the same thing as the choice of $n$ automorphisms $\phi(x_1), \ldots, \phi(x_n)$ of $V$. Thus the representation $\phi$ determines by linearity a ring homomorphism $\mathbb{Z}\Pi \to End(V)$. Now, if $u : \Pi \to V$ is a cocycle, it extends to a map $u : \mathbb{Z}\Pi \to V$ satisfying the cocycle identity

$$u(xy) = u(x)\epsilon(y) + \phi(x)u(y)$$

where now $x$, $y \in \mathbb{Z}\Pi$. Using the mean value theorem we have

$$\begin{aligned} u(w) &= u(w - \epsilon(w)) \\ &= u\left(\sum_{i=1}^{n}(\partial_i w)(x_i - 1)\right) \\ &= \sum_{i=1}^{n}\phi(\partial_i w)u(x_i) \end{aligned}$$

for any $w \in \mathbb{Z}\Pi$. Conversely, for any $(u_1, \ldots, u_n) \in V^n$ the position

$$u(w) = \sum_{i=1}^{n}\phi(\partial_i w)u_i$$

defines a cocycle $u : \mathbb{Z}\Pi \to V$. Thus we get an isomorphism $Z^1(\Pi, V_\phi) \cong V^n$ (depending on $\phi$).

Suppose now that for a group $\pi$ we have $\pi \cong \Pi/\mathcal{R}$, where $\mathcal{R} \subset \Pi$ is the normal subgroup generated by some relation $R_j$ in $\Pi$. Suppose that $V$ is a $\pi$-module via $\phi : \pi \to GL(V)$. Thus $\phi$ lifts to $\Pi$ and $\mathcal{R} = ker\phi$. The cocycles of $\pi$ are identified with those of $\Pi$ who kill the relations. Therefore, in view of the preceding calculations, the cocycle $u : \Pi \to V$ descends on $\pi$ if and only if it is the space of those $(u_1, \ldots, u_n) \in V^n$ such that

$$\sum_{i=1}^{n}\phi(\partial_i R)u_i = 0$$

for any $R \in \mathcal{R}$.

We now apply this formalism to our case, so that $\pi$ is the fundamental group of $X'$, $\Pi$ is the free group on generators $a_1, b_1, \ldots, a_g, b_g, c_0, c_\infty$ and $\mathcal{R}$ is generated by the single relation

$$R = \prod_{i=1}^{g}[a_i, b_i]\, c_0^{-1}c_\infty^{-1}$$

The rôle of $\phi$ is played by $\hat{C}'^{-2}$, and $V \cong \mathbb{C}$. For the sake of brevity we put $\chi \equiv \hat{C}'^{-2}$.

By what we have just said the space of cocycles $u : \Pi \to \mathbb{C}$ is isomorphic to $\mathbb{C}^{2g+2}$, and since we have only one relation, we must expect dim $Z^1(\pi\,,\,\chi) = 2g+1$. Using the rules given above for the free differential calculus, we calculate

$$\begin{aligned} \frac{\partial}{\partial a_i}R &= \prod_{k=1}^{i-1}[a_k, b_k]\,(1 - a_i b_i a_i^{-1}) \\ \frac{\partial}{\partial b_i}R &= \prod_{k=1}^{i-1}[a_k, b_k]\,(a_i - [a_i, b_i]) \\ \frac{\partial}{\partial c_0}R &= -\prod_{k=1}^{g}[a_k, b_k]\, c_0^{-1} = -Rc_\infty \\ \frac{\partial}{\partial c_\infty}R &= -R \end{aligned}$$

Thus, given $(u_1, v_1, \ldots, u_g, v_g, w_0, w_\infty) \in \mathbb{C}^{2g+2}$, the relation is

$$\sum_{i=1}^{g} (\chi(\partial_{a_i} R)u_i + \chi(\partial_{b_i} R)v_i) + \chi(\partial_{c_0} R)w_0 + \chi(\partial_{c_\infty} R)w_\infty = 0$$

Since $\chi \equiv \hat{C}'^{-2}$ is a character we get

$$\begin{aligned} \chi(\partial_{a_i} R) &= \chi(1 - a_i b_i a_i^{-1}) = 1 - \chi(b_i) \\ \chi(\partial_{b_i} R) &= \chi(a_i - [a_i, b_i]) = \chi(a_i) - 1 \\ \chi(\partial_{c_0} R) &= -\chi(Rc_\infty) = -\chi(c_\infty) = -\chi(c_0)^{-1} \\ \chi(\partial_{c_\infty} R) &= -1 \end{aligned}$$

So that finally the relation takes the form

$$\sum_{i=1}^{g} ((1 - \chi(b_i))u_i + (\chi(a_i) - 1)v_i) - \chi(c_0)^{-1} w_0 - w_\infty = 0$$

This shows that the space of cocycles of the fundamental group of $X'$ with values in $\mathbb{C}$ (with respect to the character $\hat{C}'^{-2}$) is an hyperplane in $\mathbb{C}^{2g+2}$. Incidentally this also yields very simply the value taken on the path encircling $P_\infty$ in terms of the others.

Finally, to complete the argument, notice that in general the trivial crossed homomorphisms descend to the group $\pi$, due to the fact that $\phi$ kills the relations. In our case this space is just a one dimensional line, thus yielding $\dim H^1(\pi\,, \chi) = 2g$.

## 2.6 The DS connection and its monodromy

Here we briefly discuss the nature of the map associating the DS connection to its monodromy, in particular with respect to the parametrization given by the KN basis. As far as any possible coordinatization is concerned, the relevant parametrization is the one for the meromorphic connections on the line bundle $K^{1/2}$. Let us call $\mathcal{C}$ this set. This is an affine space modelled over the vector space of meromorphic abelian differentials on the Riemann Surface $X$. Thus by suitably adding appropriate differentials, we can reach the situation where the poles of the connection are located at the points of $Y$.

We temporarily put $\mathcal{M}^1 = \Omega^1_X(*Y)$. Fixing the reference connection $\Gamma_0 \in \mathcal{C}$ allows us to make the identification $\mathcal{C} \cong \mathcal{M}^1 \equiv T_{\Gamma_0}\mathcal{C}$, the tangent space at $\Gamma_0$. By what we have done previously we have the correspondence

$$\Phi : \mathcal{M}^1 \longrightarrow H^1(X', \mathbb{C}^*)$$

associating to the connection $p = \Gamma_0 + \sum_n p_n \omega^n$ the character $\hat{C}$ as given in (2.12). This correspondence can be thought as a control panel, by means of which we produce any character in $H^1(X', \mathbb{C}^*)$ tuning the modes $\{p_n\}$ of the connection.

It is indeed rather clear that the map $\Phi$ is onto. As we already mentioned, it is very likely to have a huge kernel. For instance, by the way the character is constructed, if $f \in \mathcal{M}^0 = \mathcal{O}_X(*Y)$, then we have $\Phi(\partial f) = 1$.

This is of course not the end of the story, since the monodromy representation actually takes place in a larger space. Let us denote by $B$ the Borel subgroup of $SL_2(\mathbb{C})$ made by lower triangular matrices. By what we have said in the main text, the monodromy is an element in the space

$$H^1(X', B) \cong Hom(\pi_1(X'), B)/B$$

of (classes of) extensions of flat line bundles by their duals. We have already given some detail on its structure when computing the resulting monodromy from the given DS connection. Here we add that while as a non-abelian cohomology space it in principle only a pointed set (and certainly not a group), the representation on the RHS gives it an interesting structure of (singular) algebraic variety [Go]. It is rather obvious that the sequence

$$H^1(X', B) \xrightarrow{\pi} H^1(X', \mathbb{C}^*) \longrightarrow 0$$

is exact: simply lift any cocycle $C = \{c_{\alpha\beta}\}$ to the trivial extension, for instance. Furthermore, since the extension of $C$ by $C^{-1}$ is an element of the cohomology group $H^1(\pi_1(X'), \hat{C}^{-2})$, this last one, in turn, is to be considered as the *fibre* of the map $\pi$. Thus all the fibres are isomorphic as vector spaces, whereas their algebraic structure changes while moving the point $C$ on the "base" $H^1(X', \mathbb{C}^*)$.

It is interesting to see, by the way, that this algebraic structure can be retrieved also in the realm of non-abelian cohomology [Ma]. Here, however, the situation is such that all the steps can be verified by hand. Consider the exact sequence

$$1 \longrightarrow N \longrightarrow B \longrightarrow \mathbb{C}^* \longrightarrow 1$$

where $N$ is the subgroup of $B$ of unipotent (lower triangular with 1 on the diagonal) matrices. It is obviously abelian and normal in $B$. Any class $c$ in $H^1(X', \mathbb{C}^*)$ determines a twisted sheaf $N^{(c)}$ as follows. Given $c \in \mathbb{C}^*$, let $\tilde{c}$ be any lifting to $B$, that is, a matrix of the form

$$\begin{pmatrix} c & x \\ 0 & c^{-1} \end{pmatrix}$$

where $x$ is any complex number. Then, if $u_\alpha \in \Gamma(U_\alpha, N)$, the $\{u_\alpha\}$ is a section of $N^{(c)}$ if

$$u_\alpha = \tilde{c}_{\alpha\beta} u_\beta \tilde{c}_{\alpha\beta}^{-1} \quad \text{on} \quad U_\alpha \cap U_\beta$$

Notice that conjugation is simply the shift $u \to c^{-2}u$, so that we directly see the independence of the chosen lifting. In a general situation this independence is implied by the normality (and the abelianess) of the subgroup. Thus we see that, passing to the components, the restriction morphisms of the twisted sheaf $N^{(c)}$ are such that

$$u_\alpha = c_{\alpha\beta}^{-1} u_\beta$$

This is precisely the way how the restriction morphisms of the sheaf $\mathcal{C}^{-2}$ work. Thus we have $N^{(c)} \cong \mathcal{C}^{-2}$. It is also worth noticing that the first cohomology group with values in $N^{(c)}$, besides being of topological nature, acquires in this way the algebraic significance of parametrizing the possible liftings $c \rightsquigarrow \tilde{c}$ which are cocycles. The possibility of lifting a

cocycle into a cocycle is equivalent in the general situation to the vanishing of the second cohomology group with values in $N^{(c)}$ [Ma]. However in our case this can be checked by hand.

After this remark concerning the algebraic meaning of the groups $H^1(\pi_1(X'), \hat{C}^{-2})$, we add some more comments about the correspondence with the DS system. Let us put for simplicity $M = H^1(X', \mathbb{C}^*)$ and $Z = H^1(X', B)$. Thus we have the map

$$\pi : Z \longrightarrow M$$

which is onto and whose fibres are the groups $H^1(\pi_1(X'), \hat{C}^{-2}) \cong H^1(X', \mathcal{C}^{-2})$. Notice that $M \cong (\mathbb{C}^*)^{2g+1}$. Also note that during our analysis we have incidentally computed the dimension of $Z$ to be $4g + 1$ (dimension of the base $M$ plus the dimension of the fibre).

Thus the correct scheme is that the monodromy gives a map

$$\tilde{\Phi} : \mathcal{M}^1 \longrightarrow Z$$

which composed with $\pi$ yields $\Phi$. This should be evident from the actual construction of the monodromy representation.

Here we would like to stress that it is possible to drift vertically in $Z$ just by using the kernel of $\Phi$. Suppose that $\omega$ and $\omega + \delta\omega$ are such that the resulting characters are equal:

$$\hat{C}_\gamma(\omega) = \hat{C}_\gamma(\omega + \delta\omega)$$

for any $\gamma \in \pi_1(X')$. Referring to the explicit expression (2.13) we obtain a new crossed homomorphism given by:

$$\hat{S}'_\gamma = -h_0(Q_0)^{-2}\hat{C}_\gamma(\omega)^2 \int_\gamma e^{-2\int_{Q_0}^z (\omega+\delta\omega)} h(z)^2 \tag{2.14}$$

This can be accomplished by adding to $\omega$ exact (meromorphic) differentials. It can be proved directly that the above formula defines a crossed homomorphism.

As a last item, we pass to the infinitesimal level to put this variation in a more geometric perspective. The tangent space to $Z$ has been extensively described [Go, JM]. Given a semisimple Lie group $G$, the tangent space to $Hom(\pi_1, G)/G$ at the smooth point $[\phi]$ is given by the cohomology group $H^1(\pi_1, Ad_\phi \mathfrak{g})$, where by $Ad_\phi \mathfrak{g}$ we mean that $\mathfrak{g}$ is to be considered as a $\pi_1$-module via the composition of $\phi$ with the Adjoint action of $G$.

Our case is sufficiently simple to permit a direct verification. Dropping the hats ^ for simplicity of notation, by elementary variation of

$$\phi(\gamma) = \begin{pmatrix} C_\gamma & 0 \\ C_\gamma^{-1} S_\gamma & C_\gamma^{-1} \end{pmatrix}$$

we get

$$u(\gamma) = \phi(\gamma)^{-1}\dot{\phi}(\gamma) = \begin{pmatrix} C_\gamma^{-1}\dot{C}_\gamma & 0 \\ \dot{S}_\gamma - 2C_\gamma^{-1}\dot{C}_\gamma S_\gamma & -C_\gamma^{-1}\dot{C}_\gamma \end{pmatrix}$$

with $\gamma \in \pi_1$. Putting $\delta C_\gamma = C_\gamma^{-1}\dot{C}_\gamma$, it is immediate to verify that from the cocycle condition

$$u(\gamma_1\gamma_2) = Ad_{\phi(\gamma_2)^{-1}} u(\gamma_1) + u(\gamma_2)$$

we have the rules:

$$\begin{aligned} \delta C_{\gamma_1\gamma_2} &= \delta C_{\gamma_1} + \delta C_{\gamma_2} \\ \dot{S}_{\gamma_1\gamma_2} &= C_{\gamma_2}{}^2 \dot{S}_{\gamma_1} + \dot{S}_{\gamma_2} + \delta C_{\gamma_2} C_{\gamma_2}{}^2 S_{\gamma_1} \end{aligned}$$

From this we see that a variation such that $\delta C = 0$ again yields a crossed homomorphism in $Z^1(\pi_1, \hat{C}^{-2})$, thus proving that these are precisely the vertical tangent vectors to $Z$, as wanted.

# Chapter 3

# The Liouville Equation on a Riemann Surface II

## 3.1 Poisson structure

Here we define a Poisson structure on the space $\Omega^1_X(*Y) =: \mathcal{M}^1_X$. With respect to the parametrization $\Gamma_0 + \mathcal{M}^1_X$ it can be considered as a Poisson structure on the space $\mathcal{C}$ of meromorphic connections on $K^{1/2}$and hence also on the space of DS connections on $V = K^{-1/2} \oplus K^{1/2}$.

We need some preliminaries. The first is essentially a remark about a global coordinatization of $X$ obtained using the KN framework. On the Riemann Surface one can in fact introduce a global (Euclidean) time coordinate $\tau$ by means of

$$\tau(Q) = \mathrm{Re} \int_{Q_0}^{Q} \omega^0$$

where $Q_0$ is a fixed point of $X$. Recall that $\omega^0$ is the unique third kind differential with purely imaginary periods, so that this definition is well posed. As $\tau$ varies in $\mathbb{R}$, there is a fibration $X \to \mathbb{R}$. We call $C_\tau$ the inverse image of $\tau \in \mathbb{R}$ by this fibration. It is impostant to notice that these "equal time" contours are not always connected. Indeed, as $\tau$ increases, the $C_\tau$ will encounter the $2g$ zeros of $\omega^0$ and split into disconnected cycles which afterwards rejoin. Since as $\tau \to \pm\infty$ the contour $C_\tau$ approaches two small circles surrounding $P_0$ and $P_\infty$, this gives the picture of a string entering at $\tau = -\infty$ at $P_0$, interacting and then exiting at $\tau = +\infty$ at $P_\infty$.

Even if not connected, the $C_\tau$'s are nevertheless homologous to each other, so that if $\omega$ is an Abelian differential holomorphic in the region whose boundary is $C_\tau - C_{\tau'}$, then

$$\int_{C_\tau} \omega = \int_{C_{\tau'}} \omega$$

From this it is clear that

$$\mathrm{Res}_{P_0} \omega = \frac{1}{2\pi i} \int_{C_\tau} \omega$$

where the integral can be taken over *any* $C_\tau$.

Now consider $\mathcal{M}_X^0 := \mathcal{O}_X(*Y)$. Clearly the holomorphic differential $\partial$ gives a map

$$\partial : \mathcal{M}_X^0 \longrightarrow \mathcal{M}_X^1$$

and at the level of the respective KN bases we have

$$\partial A_n = \sum_m \gamma_{mn}\omega^m$$

The cocycle $\gamma_{mn}$ is "local" in the sense that the sum above is finite [KN1].

Notice that by virtue of the duality relation we have

$$\frac{1}{2\pi i}\int_{C_\tau} A_m \partial A_n = \sum_l \gamma_{ln}\frac{1}{2\pi i}\int_{C_\tau} A_m\omega^l = \gamma_{mn}$$

Thus the symbol $\gamma_{mn}$ is manifestly antisymmetric.

Now can write down the Poisson structure. First we write it very simply in terms of the coordinates $\{p_n\}$ as:

$$\{p_n, p_m\} = \gamma_{nm} \tag{3.1}$$

This is to be interpreted in the following way. A Poisson bracket is a bilinear structure on the ring of functions satisfying the well-known properties. In our case, if $\omega$ is a point in $\mathcal{M}_X^1$, we have that $\{p_n\}$ is the set of its coordinates with respect to the KN basis. By using the duality relation we obtain

$$p_n = \frac{1}{2\pi i}\int_{C_\tau} A_n\omega$$

so that $p_n$ is the value on $\omega$ of the elementary function given as the evaluation of the dual basis vector $A_n$. This is, of course, a customary reinterpretation of the notion of coordinate function. Actually the structure we have just introduced can be taken as a bracket (in fact a symplectic structure) on $\mathcal{M}_X^0/\mathbb{C}$ via the relation [JKL]

$$\{f, g\} = \frac{1}{2\pi i}\int_{C_\tau} f\partial g$$

The quotient by $\mathbb{C}$ is important. Due to the presence of $A_0 \equiv 1$ and the third kind differential $\omega^0$ in the KN bases, the Poisson Bracket is degenerate. Indeed we have

$$\{p_0, p_m\} = 0$$

for each $m \in \mathbb{Z}$. This degeneracy is quite analogous to the one in the genus zero case when $\gamma_{nm} = -in\delta_{n+m,0}/4\pi$.

The structure we have so far introduced has the following remarkable feature. The symbol $\gamma_{nm}$, considered as an infinite matrix, has $2g$ eigenvectors. These are the vectors $\vec{M}_i = \{M_i^n\}$ and $\vec{N}_i = \{N_i^n\}$ whose components have been defined in a preceding section. The above assertion relies on

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_{a_i} \partial A_m = \sum_n \gamma_{nm}\frac{1}{2\pi i}\int_{a_i}\omega^n \\ &= \sum_n M_i^n\gamma_{nm} \end{aligned}$$

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_{b_i} \partial A_m \\ &= \sum_n N_i^n \gamma_{nm} \end{aligned}$$

Using these vectors as a notational tool, we can write, for short

$$\int_{a_i} \omega = \vec{M}_i \cdot \vec{p} \ , \quad \int_{b_i} \omega = \vec{N}_i \cdot \vec{p}$$

for the action of $H_1(X,\mathbb{Z})$ on $\mathcal{M}_X^1$, where as usual $\omega = \sum_n p_n \omega^n$. It follows that these functionals are "conserved" with respect to the introduced Poisson bracket. Indeed we have for any $k \in \mathbb{Z}$

$$\{p_k, \vec{M}_i \cdot \vec{p}\} = \sum_n \{p_k, p_n\} M_i^n = \sum_n \gamma_{kn} M_i^n = 0$$

and the same happens for $\vec{N}_i \cdot \vec{p}$.

However, due to our convention of avoiding to write down formulas for the anti-chiral part, it is understood that a "mirror image" of the Poisson structure just described exists for the other half of the theory. According to the spirit of the DS construction we are trying to emulate, the two halves Poisson commute:

$$\{p_n, \bar{p}_m\} = 0$$

for every $n, m \in \mathbb{Z}$.

## 3.2 Disentangling the monodromy

Here we address the problem of putting together the two halves of the theory. Thus we should bear in mind that what we have done so far for the chiral half is to be replied for the antichiral one. Supposing we have done this, the idea is to exploit (2.2) to achieve the result. However in the present global context the matrix $M$ we put there will play a crucial rôle.

To begin with, recall that the monodromy matrices $\{T_{\alpha\beta}\}$ we constructed have the form

$$T_{\alpha\beta} = \begin{pmatrix} c_{\alpha\beta} & 0 \\ d_{\alpha\beta} & c_{\alpha\beta}^{-1} \end{pmatrix}$$

and correspondingly the 1-cocycle for the anti-chiral systems reads

$$\bar{T}_{\alpha\beta} = \begin{pmatrix} \bar{c}_{\alpha\beta} & 0 \\ \bar{d}_{\alpha\beta} & \bar{c}_{\alpha\beta}^{-1} \end{pmatrix}$$

as one can easily verify starting from (2.6). Now, the quantity $\exp(-\varphi)$ should transform with conformal weight $(-1/2, -1/2)$, thus we localize (2.2) and write the local component in each chart as

$$e^{-\varphi_\alpha} = \sigma_\alpha M_\alpha \bar{\sigma}_\alpha$$

insisting for these objects to transform with the rule

$$e^{-\varphi_\alpha} = k_{\alpha\beta}^{-1/2} \bar{k}_{\alpha\beta}^{-1/2} e^{-\varphi_\beta}$$

Taking into account the transformation rule for the row vector $\sigma$ and the corresponding one for the column vector $\bar{\sigma}$, this happens if and only if

$$M_\beta = {}^tT_{\alpha\beta}\, M_\alpha\, \bar{T}_{\alpha\beta} \tag{3.2}$$

holds. Writing explicitly the components of $\{M_\alpha\}$ as

$$M_\alpha = \begin{pmatrix} x_\alpha & y_\alpha \\ u_\alpha & v_\alpha \end{pmatrix}$$

and working out the matrix products furnishes the following conditions

$$\begin{aligned} c_{\alpha\beta}^{-1} v_\alpha &= \bar{c}_{\alpha\beta} v_\beta && (3.3)\\ c_{\alpha\beta} y_\alpha + d_{\alpha\beta} v_\alpha &= \bar{c}_{\alpha\beta} y_\beta && (3.4)\\ \bar{c}_{\alpha\beta}^{-1} u_\alpha - \bar{d}_{\alpha\beta} v_\beta &= c_{\alpha\beta}^{-1} u_\alpha && (3.5)\\ c_{\alpha\beta} x_\alpha + d_{\alpha\beta} u_\alpha &= \bar{c}_{\alpha\beta}^{-1} x_\alpha - \bar{d}_{\alpha\beta} y_\beta && (3.6) \end{aligned}$$

The correct way of stating (3.2) is to say that $\{M_\alpha\}$ is a locally constant section of the (flat) bundle $T^\vee \otimes \bar{T}^\vee$, and since flat bundles admit global locally constant sections only under certain restrictive conditions[G2], it is clear that equations (3.3)–(3.6) imply conditions on the monodromies $T$ and $\bar{T}$. Let us now examine the conditions in more detail.

Condition (3.3) takes the form

$$\bar{c}_{\alpha\beta} = v_\alpha\, c_{\alpha\beta}^{-1}\, v_\beta^{-1} \tag{3.7}$$

that is, the cocycle $\{\bar{c}_{\alpha\beta}\}$ is cohomologous to $c_{\alpha\beta}^{-1}$, so that at the level of cohomology classes, the bundle $\bar{C}$ is the inverse to $C$.

Now consider condition (3.4). Using the relation between $C$ and $\bar{C}$ just proven, we can write (3.4) as

$$d_{\alpha\beta} = c_{\alpha\beta}^{-1} \frac{y_\beta}{v_\beta} - c_{\alpha\beta} \frac{y_\alpha}{v_\alpha}$$

hence

$$S_{\alpha\beta} = \frac{y_\beta}{v_\beta} - c_{\alpha\beta}^2 \frac{y_\alpha}{v_\alpha} \tag{3.8}$$

that is the 1-cocyle $\{S_{\alpha\beta}\}$ must be a coboundary. In exactly the same way we find

$$\bar{S}_{\alpha\beta} = \frac{u_\beta}{v_\beta} - \bar{c}_{\alpha\beta}^2 \frac{u_\alpha}{v_\alpha} \tag{3.9}$$

so that also the 1-cocycle $\{\bar{S}_{\alpha\beta}\}$ must be a coboundary.

Finally, condition (3.6) gives a relation on the determinants

$$\det M_\alpha = \det M_\beta \tag{3.10}$$

However, since at the beginning of this chapter we noticed that the determinant of $M$ must be normalized, condition (3.10) tells us that the "locally constant section" $\det M$ must be globally set to 1 in order to have a solution of the Liouville equation.

Apart from the last one, the first three conditions impose very severe restrictions on the monodromies of the two sectors. The first one imposes the flat line bundles $C$ and $\bar{C}$ to be one the inverse of the other, while the other two conditions are equivalent to both the left and right monodromies be diagonalizable. In other terms, the extension classes represented by the chiral and anti-chiral monodromies must be trivial. Before further exploring the consequences of these conditions, we would like to mention they could also have been deduced by means of the following alternative reasoning.

First of all, the local systems $T$ and $\bar{T}$ determine representations of the fundamental group of $X'$ into the group $B$ of lower triangular $2 \times 2$ matrices. Thus it is not difficult to verify that the single-valuedness of the solution is the same thing as the equivalence of the representations $T^\vee$ and $\bar{T}$ in $SL_2(\mathbb{C})$. In other words, they must define the *same* point (orbit) in $Hom(\pi_1(X'), SL_2(\mathbb{C}))/SL_2(\mathbb{C})$. One may also wonder why representation $T^\vee$ should intervene instead of $T$. The reason is very simple. We can think of the expression $\sigma M \bar{\sigma}$ as a quadratic form on the local systems $T$ and $\bar{T}$[1]. Thus $T$ and $\bar{T}$ are put in a duality relation by this quadratic form, so that $\bar{T}$ turns out to be equivalent (in all possible senses) to the dual $T^\vee$ of $T$.

Therefore the relation to be studied is

$$M\,\bar{T}_\gamma = T^\vee_\gamma\, M \qquad \gamma \in \pi_1(X')$$

where $M$ is a certain matrix in $SL_2(\mathbb{C})$. Writing out the components as

$$T^\vee_\gamma = \begin{pmatrix} \hat{C}_\gamma^{-1} & -\hat{C}_\gamma^{-1}\hat{S}_\gamma \\ 0 & \hat{C}_\gamma \end{pmatrix} \quad \bar{T}_\gamma = \begin{pmatrix} \hat{\bar{C}}_\gamma & 0 \\ \hat{\bar{C}}_\gamma^{-1}\hat{\bar{S}}_\gamma & \hat{\bar{C}}_\gamma^{-1} \end{pmatrix} \quad M = \begin{pmatrix} x & y \\ u & v \end{pmatrix}$$

we get the following four conditions

$$\hat{C}_\gamma = \hat{\bar{C}}_\gamma^{-1} \tag{3.11}$$

$$y\hat{C}_\gamma^{-1} - v\hat{C}_\gamma^{-1}\hat{S}_\gamma = y\hat{\bar{C}}_\gamma^{-1} \tag{3.12}$$

$$u\hat{\bar{C}}_\gamma + v\hat{\bar{C}}_\gamma^{-1}\hat{S}_\gamma = u\hat{C}_\gamma \tag{3.13}$$

$$x\hat{\bar{C}}_\gamma + y\hat{\bar{C}}_\gamma^{-1}\hat{S}_\gamma = x\hat{C}_\gamma^{-1} - u\hat{C}_\gamma^{-1}\hat{S}_\gamma \tag{3.14}$$

The first one is already in the desired form. Using it we recast (3.12) into

$$\hat{S}_\gamma = \frac{y}{v} - \frac{y}{v}\hat{C}_\gamma^2 \tag{3.15}$$

while (3.13) yields

$$\hat{\bar{S}}_\gamma = \frac{u}{v} - \frac{u}{v}\hat{\bar{C}}_\gamma^2 \tag{3.16}$$

The fourth condition turns out to be an unessential identity. Therefore we have obtained that both $\hat{S}$ and $\hat{\bar{S}}$ must be the trivial crossed homomorphisms, exactly as we did before, as well as the condition on $\hat{C}$ and $\hat{\bar{C}}$ to be mutual inverses.

[1] It is clear that if we insist on $\sigma M \bar{\sigma}$ to transform globally with conformal weight $(-1/2, -1/2)$, the quadratic form must be thought to affect only the flat factors.

Now, what we have been doing above is to show that the chiral and anti-chiral local systems are equivalent *if and only if* both the monodromies are diagonalizable and hence mutual inverses. The "if" part of this statement is almost trivial. However, concerning both the "if" and the "only if" part, one has to explicitly notice that once the extension classes are trivial there is an explicit change of basis putting the cocycles $T$ and $\bar{T}$ into diagonal form. For instance, if for the cocycle $\{S_{\alpha\beta}\}$ we have the relation $S_{\alpha\beta} = f_\beta - c_{\alpha\beta}^2 f_\alpha$, then

$$T_{\alpha\beta} = \begin{pmatrix} c_{\alpha\beta} & 0 \\ d_{\alpha\beta} & c_{\alpha\beta}^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -f_\alpha & 1 \end{pmatrix} \cdot \begin{pmatrix} c_{\alpha\beta} & 0 \\ 0 & c_{\alpha\beta}^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ f_\beta & 1 \end{pmatrix}$$

and the same holds for the anti-chiral half. Quite obviously, once we put both the local systems into diagonal form, condition (3.11) implies that $\bar{T} = T^{-1}$.

There are a few remarks worth doing immediately. The first one is that requiring single-valuedness of the solution is a constraint on the "modes" $p_k$ and $\bar{p}_k$, or, in other words, is a constraint on the phase space of the system. This should be clear from the expression (2.13) of the crossed homomorphism $\hat{S}$ in terms of the elements of the KN basis. Due to the fact that in the KN basis there are non-exact differentials, it is quite natural to expect $\hat{S}$ not to be zero in cohomology. This is to be contrasted with the "genus zero" case of [BBT], where it is shown that the monodromies can always be disentangled. In our case the phase space of the solutions of the system happens to be strictly smaller than the (direct) sum of the vector spaces of the chiral and anti-chiral 1-forms with poles on $P_0$ and $P_\infty$.

The second remark is about the collection $\{M_\alpha\}$ or, equivalently, about the intertwiner $M$. Given trivial cocycles $\{S_{\alpha\beta}\} = \delta\{f_\alpha\}$ and $\{\bar{S}_{\alpha\beta}\} = \delta\{\bar{f}_\beta\}$, we can try to use conditions (3.7), (3.8), (3.9) and (3.10) to reconstruct the collection $\{M_\alpha\}$. Thus we write

$$M_\alpha = \begin{pmatrix} \frac{1}{v_\alpha} + f_\alpha \bar{f}_\alpha v_\alpha & f_\alpha v_\alpha \\ \bar{f}_\alpha v_\alpha & v_\alpha \end{pmatrix}$$

where we have used the condition $\det M_\alpha = 1$ to determine the element we had previously called $x_\alpha$. If we forget about the dependence on the modes for a while, we see that the collection $\{M_\alpha\}$ is determined up to the determination of $\{t_\alpha\}$ and the rescaling of $\{f_\alpha\}$ and $\{\bar{f}_\alpha\}$. In particular, the collection $\{t_\alpha\}$ is a section of the (trivial) bundle $C \otimes \bar{C}$, thus can be globally rescaled by any (non-zero) factor. Now suppose we have been able to satisfy the coboundary conditions on $S$ and $\bar{S}$. This means that there are "admissible" subsets of the modes, so to speak, in terms of which we get some $f$ and $\bar{f}$. On the contrary, $\{t_\alpha\}$ still *remains undetermined*. The same result will be also obtained looking at the characteristic representations. We omit the details, which are completely analogous. With the convention that omitting the index of the local chart indicates a quantity concerned with the representation space picture, the intertwiner takes the form:

$$M = \begin{pmatrix} \frac{1}{v} + f\bar{f}v & fv \\ \bar{f}v & v \end{pmatrix}$$

Therefore, even when it is possible to have a diagonal monodromy, the solution is determined up to the choice of a (non vanishing) parameter.

Due to the way we determined it, each element of the collection $\{M_\alpha\}$ is actually factorizable as

$$M_\alpha = \begin{pmatrix} 1 & f_\alpha \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{v_\alpha} & 0 \\ 0 & v_\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \bar{f}_\alpha & 1 \end{pmatrix}$$

and a corresponding formula holds for the intertwiner. Having in mind the genus zero DS construction [BBT] reviewed in a previous chapter, it is tempting to introduce the notation

$$M_\alpha = G_\alpha \rho_\alpha \bar{G}_\alpha$$

and maybe infer that $\rho_\alpha$ plays the same rôle as the $\rho$ of the DS construction.

Anyway, without making any identification, we can write the monodromy as

$$T_{\alpha\beta} = {}^tG_\alpha^{-1}\, K_{\alpha\beta}\, {}^tG_\beta \ , \quad \bar{T}_{\alpha\beta} = \bar{G}^{-1}\, \bar{K}_{\alpha\beta}\, \bar{G}_\beta$$

so that

$$\begin{aligned} {}^tT_{\alpha\beta}\, M_\alpha\, \bar{T}_{\alpha\beta} &= G_\beta\, K_{\alpha\beta}\, \rho_\alpha\, \bar{K}_{\alpha\beta}\, \bar{G}_\beta \\ &= M_\beta \end{aligned}$$

if and only if

$$\rho_\beta = \rho_\alpha\, K_{\alpha\beta}\, \bar{K}_{\alpha\beta}$$

This is a rewriting of the main relations (3.7), (3.8), (3.9) in a form very similar to the one appearing in the DS construction. We shall comment further on this correspondence later on. Needless to say, all the relation written above for the open covering of the surface can be rewritten for the intertwiner with the obvious modifications.

## 3.3 The constraints

At the end of the previous section we suggested that the condition about the diagonalizability of the monodromy for both the chiral and the anti-chiral DS systems should be considered as constraints on the phase space.

From a geometrical point of view, diagonalizability is equivalent to the crossed homomorphisms $\hat{S}$ and $\hat{\bar{S}}$ to be coboundaries:

$$\hat{S}_\gamma = f - \hat{C}_\gamma^2 f \tag{3.17}$$

$$\hat{\bar{S}}_\gamma = \bar{f} - \hat{\bar{C}}_\gamma^2 \bar{f} \tag{3.18}$$

for any cycle $\gamma \in \pi_1(X')$. The complex numbers $f, \bar{f}$ can be taken as parameters.

When inserting the parametrization given by the KN algebra, using (2.13), for the condition on the chiral part we obtain

$$\frac{h_0(Q_0)^{-2}}{f} \int_\gamma e^{-2\sum_k p_k \psi_k(Q_0,z)} h(z)^2 = 1 - e^{-2\sum_k p_k \int_\gamma \omega_k} \tag{3.19}$$

A similar formula is obtained for the antichiral part. Formulae (3.17) and (3.19) make rather difficult to describe analytically the region of the phase space where the cocycle $\hat{S}$ happens to be trivial.

We stress that condition (3.19) can indeed be fulfilled. According to our digression about the monodromy of the DS system, we know that we can use the elements of the KN basis to drift vertically in $H^1(X', B)$ over a fixed point in $H^1(X', \mathbb{C}^*)$. An identical remark is to be made for the anti-chiral half.

In addition we should impose the condition on the diagonal parts of the monodromy to be mutual inverses:

$$\hat{C}_\gamma = \hat{\bar{C}}_\gamma^{-1} \tag{3.20}$$

This amounts to set

$$\sum_n p_n \int_\gamma \omega^n = \sum_n \bar{p}_n \int_\gamma \bar{\omega}^n \pmod{2\pi i} \tag{3.21}$$

but each member of this equality has zero Poisson bracket with each mode, as we have already noticed. Formula (3.21) can be imposed on the generators thus yielding $2g + 1$ conditions. We anticipate that in the following we shall change the symplectic structure, so that even though we expect no problems as for the possibility of setting (3.20), formulae (3.20) and (3.21) should be rediscussed in conjunction with this modification.

Now we proceed emulating the genus zero case [BBT], that is, first we get rid of the off-diagonal terms of the monodromies and afterwards we impose (3.20). However, in our present case diagonalizability is not automatic, and we impose (3.17) and (3.18)as constraints on the phase space. More precisely we introduce "functions" $\mathcal{F}_i, \mathcal{F}_{\bar{\imath}}$ for $i, \bar{\imath} = 1, \ldots, 2g + 1$

$$\mathcal{F}_i = \hat{S}_i - f(1 - \hat{C}_i^2) \tag{3.22}$$

$$\mathcal{F}_{\bar{\imath}} = \hat{\bar{S}}_i - \bar{f}(1 - \hat{\bar{C}}_i^2) \tag{3.23}$$

with the convention that the index range $\{1, \ldots, g\}$ corresponds to the "$a$" generators, the range $\{g + 1, \ldots, 2g\}$ to the "$b$" generators and $i = 2g + 1$ to the circle $c_0$ around $P_0$. The convention on the index $\bar{\imath}$ is the same.

Now observe that the already exploited simple fact

$$\sum_m \{p_n, p_m\} \int_\gamma \omega^m = \sum_m \gamma_{nm} \int_\gamma \omega^m = -\int_\gamma \partial A_n = 0$$

for any cycle $\gamma$, implies the following brackets

$$\begin{aligned} \{\hat{C}_i, \hat{C}_j\} &= 0 \\ \{\hat{C}_i, \hat{S}_j\} &= 0 \end{aligned}$$

for $i = 1, \ldots, 2g + 1$. The same holds for the corresponding brackets in the anti-chiral sector. On the contrary, the brackets:

$$\{\hat{S}_i, \hat{S}_j\}\,, \quad \{\hat{\bar{S}}_i, \hat{\bar{S}}_j\}$$

are different from zero. Their actual value does not matter here; moreover, it cannot be calculated in closed simple form. However, chiral and anti-chiral quantities do commute. We have

$$\{\mathcal{F}_i, \mathcal{F}_j\} = \{\hat{S}_i, \hat{S}_j\} \tag{3.24}$$

$$\{\mathcal{F}_{\bar{\imath}}, \mathcal{F}_{\bar{\jmath}}\} = \{\hat{\bar{S}}_i, \hat{\bar{S}}_j\} \tag{3.25}$$

$$\{\mathcal{F}_i, \mathcal{F}_{\bar{\jmath}}\} = 0 \tag{3.26}$$

These formulas tell us that the constraints are not first class, that is they are not reproduced by the Poisson brackets. This becomes evident if one tries to write down explicit expressions for the above brackets.

In such a case, the mechanism of Dirac brackets is at hand [HRT]: for any two functions $F, G$ on the phase space we define their Dirac bracket as

$$\{F,G\}_* = \{F,G\} - \{F,\mathcal{F}_a\}C^{-1}_{ab}\{\mathcal{F}_b,G\}$$

where the summation convention is in effect, and $a = (i,\bar{\imath})$, while the matrix $C_{ab}$ is given by

$$C_{ab} = \{\mathcal{F}_a,\mathcal{F}_b\}$$

so that it turns out to be an antisymmetric block-diagonal matrix:

$$\{C_{ab}\} = \begin{pmatrix} \{\mathcal{F}_i,\mathcal{F}_j\} & 0 \\ 0 & \{\mathcal{F}_{\bar{\imath}},\mathcal{F}_{\bar{\jmath}}\} \end{pmatrix}$$

The Dirac bracket is constructed in such a way that everything commutes with the constraints:

$$\begin{aligned} \{F,\mathcal{F}_c\}_* &= \{F,\mathcal{F}_c\} - \{F,\mathcal{F}_a\}C^{-1}_{ab}\{\mathcal{F}_b,\mathcal{F}_c\} \\ &= \{F,\mathcal{F}_c\} - \{F,\mathcal{F}_a\}\delta_{ac} \\ &= 0 \end{aligned}$$

Now we point out an inconsistency between the geometrical and dynamical sides of the theory which is easily solved by using the Dirac brackets. Suppose we are on the constrained surface $\mathcal{F}_a = 0$, $a = (i,\bar{\imath})$. We can rewrite $\mathcal{F}_i = 0$ as

$$f = \frac{\hat{S}_i}{(1-\hat{C}_i^2)}\,, \quad i = 1,\ldots,2g+1$$

and we must have

$$0 = \{f,f\} = \left\{\frac{\hat{S}_i}{(1-\hat{C}_i^2)},\frac{\hat{S}_j}{(1-\hat{C}_j^2)}\right\}$$

for *any* $i,j = 1,\ldots,2g+1$. Notice that this must be true even if we decided to give the parameters $f, \bar{f}$ a non trivial Poisson structure. However, the RHS is equal to

$$\frac{1}{(1-\hat{C}_i^2)}\frac{1}{(1-\hat{C}_j^2)}\{\hat{S}_i,\hat{S}_j\}$$

which is different from zero, thus leading to an inconsistency. If we use the Dirac brackets we must subtract

$$\begin{aligned} \left\{\frac{\hat{S}_i}{(1-\hat{C}_i^2)},\mathcal{F}_a\right\}C^{-1}_{ab}\left\{\mathcal{F}_b,\frac{\hat{S}_j}{(1-\hat{C}_j^2)}\right\} &= \frac{1}{(1-\hat{C}_i^2)}\frac{1}{(1-\hat{C}_j^2)}\{\hat{S}_i,\mathcal{F}_k\}C^{-1}_{kl}\{\mathcal{F}_l,\hat{S}_j\} \\ &= \frac{1}{(1-\hat{C}_i^2)}\frac{1}{(1-\hat{C}_j^2)}\{\hat{S}_i,\hat{S}_j\} \end{aligned}$$

so that

$$\left\{ \frac{\hat{S}_i}{(1-\hat{C}_i^2)}, \frac{\hat{S}_j}{(1-\hat{C}_j^2)} \right\}_* \equiv 0$$

holds identically, yielding a consistent relation. The same again happens for the anti-chiral quantities.

Since now we have modified the Poisson bracket, in principle some relation should be rediscussed, notably those involving the conserved quantities $\vec{M}_i$. According to our present convention about the index $i$, from now on $\vec{M}_i$ stands for what we have previously indicated with $\{\vec{M}_i\,, \vec{N}_i\}$. Using the usual relations we readily see that

$$\{p_n, \vec{M}_i \cdot \vec{p}\} = 0$$

for every mode $p_n$. This implies that $\vec{M}_i \cdot \vec{p}$ commutes with the constraints, so that the quantities $\vec{M}_i \cdot \vec{p}, \vec{M}_i \cdot \vec{p}$ are conserved also with respect to the new bracket $\{\ ,\ \}_*$. It follows that the condition (3.20) is compatible also with the Dirac brackets.

## 3.4 The free parameter $v$

We remarked that the intertwiner between the two monodromy representations is determined up to the choice of a value for the non-vanishing parameter $v$ (or $v_\alpha$ if we refer to the local determination $M_\alpha$). We also put in evidence the close formal analogy between our construction of the solution and the genus zero one in [BBT].

At the same time, we should point out an important difference, which consists in the following. In genus zero there is so much degeneracy that we may be led to confuse different objects. Indeed in that case we have $p_0$ to be both the conserved moment along the circle *and* the degenerate variable for the Poisson structure. Accordingly, we add the coordinate $q_0$ to remove degeneracy.

This scheme does not pass through in the case we are dealing with. Indeed we have $2g+1$ conserved quantities corresponding to the homology cycles, but the Poisson bracket is degenerate only along the one-dimensional space spanned by $p_0$. On the other hand, as we have repeatedly observed, the matrix $\rho$ depends on only one free parameter. Paradoxically it is this difference that makes us able to handle things, because the degeneracy of the canonical structure does not grow with the genus of the surface. In practice, the conserved moments have to be coupled in order to get a single-valued solution, but to cure the degeneracy we have only one variable to look at, so that things remain relatively simple.

We might be tempted to remove the degeneracy of the Poisson bracket via the same procedure as in the genus zero case. Let us show that this is at least non contradictory. First of all, we observe that the properties of the KN basis are such that when getting closer and closer to the punctures it resembles the standard Laurent basis on $\mathbb{P}^1 \setminus \{P_0, P_\infty\}$.[2] Thus, when very close to a puncture, say $P_0$, we can use the standard $\{z^n\}$ basis. Even more, on the contour $C_\tau$, which is now very close to $P_0$, we can make use of the basis $\{e^{inx}\}$. This results in a reshuffling of the modes, but after a moment's thought we conclude that the modes $p_0\,, \bar{p}_0$

[2] $\mathbb{P}^1 \setminus \{P_0, P_\infty\}$ is conformally equivalent to the cylinder

are not touched. Thus $p_0\,,\bar{p}_0$ are the modes that go in front of $1/z$ and $1/\bar{z}$ respectively, and they can be identified with the zero modes of the true DS construction in genus zero.

Now, by the standard construction, we get two variables $q_0\,,\bar{q}_0$ coupled to them:

$$\{p_0, q_0\} = \{\bar{p}_0, \bar{q}_0\} = \frac{1}{4\pi} \tag{3.27}$$

At the same time, $q_0\,,\bar{q}_0$ do not couple with the rest of the "local" modes[3] Since the localization procedure did not touch the matrix $\rho$ we can put as usual

$$1/v = \theta\bar{\theta} = e^{q_0+\bar{q}_0}$$

Now if we change again basis and move to the KN one, we still have (3.27) to hold with respect to the "global" Poisson bracket we defined on the KN basis. The point is still that the zero modes do not get mixed with the other ones.

However, having previously introduced the Dirac bracket, we have to verify that (3.27) continues to hold also with respect to it. This is a trivial task, since it is immediate to verify that

$$\{p_0, \mathcal{F}_a\} = \{\bar{p}_0, \mathcal{F}_a\} = 0$$

so that we have

$$\begin{aligned} \{p_0, q_0\}_* &= \{p_0, q_0\} \\ &= \frac{1}{4\pi} \\ \{\bar{p}_0, \bar{q}_0\}_* &= \frac{1}{4\pi} \end{aligned}$$

as wanted.

Finally, we should actually couple the conserved quantities:

$$\hat{C}_\gamma = \hat{\bar{C}}_\gamma^{-1}$$

This operation couples the two chiral sectors, which up to now have been taken completely separated. However, the solution commutes with this last constraint:

$$\{e^{-\varphi(z,\bar{z})}, \hat{C}_\gamma - \hat{\bar{C}}_\gamma^{-1}\}_* = 0$$

exactly as in [BBT]. This follows almost trivially from the fact that

$$\{\hat{C}_\gamma - \hat{\bar{C}}_\gamma^{-1}, \mathcal{F}_a\} = 0$$

so that the Dirac brackets is actually equal to the usual one, and from the observation that the new contributions due to $q_0$ and $\bar{q}_0$ cancel between the two sectors.

Thus the field $e^{-\varphi(z,\bar{z})}$ commutes with the last constraint (3.20) which can indeed be imposed. Thus we get a single-valued yet singular solution to the Liouville equation. We ask ourselves whether it is also local in the sense of [BBT], that is whether the field $e^{-\varphi}$ has

[3] Notice that here we are considering the bracket given locally around $P_0$ by the DS construction.

zero equal-time Poisson brackets. Of course, since we are on a Riemann Surface, we should specify very carefully what "equal-time" means. According to what we have said about the global coordinatization of the surface, the equal-time slices are exactly the contours $C_\tau$. Unfortunately, the calculation of the Poisson bracket

$$\{e^{-\varphi(z)}, e^{-\varphi(z')}\}_*\Big|_{C_\tau}$$

for *every* $C_\tau$ requires the full exchange algebra of the fields $\sigma$ and , which is presently not explicitly calculable, at least not in terms of the fields themselves and an appropriate $r$-matrix. However, by reducing to the standard Laurent basis, we can certainly calculate the Poisson bracket in the vicinity of a puncture. In this case, using the genus zero calculations [BBT], the solution is expected to be local in the sense specified above.

# Chapter 4

# Geometry of the Liouville Equation

## 4.1 Introduction

The purpose of this chapter is to furnish some geometric support to what we have done concerning the zero-curvature representation of the Liouville equation. At the same time, it can be considered as an outlook to a possible treatment of Toda Field theory in a more geometric context. Thus, even though single parts of what we are going to say are rather well-known in the literature – in same cases well-established, as is the case for uniformization theory – the entire context is meant to be an outline of the relations among Toda Field Theory (notably Liouville Field Theory), uniformization of curves and Virasoro or/and $W_n$ symmetries on them. We shall observe, in passing, how the approach we have followed to the solution of the Liouville equation refers to a very singular geometric context. We shall give a possible explanation of the gauge transformation leading to the Drinfel'd-Sokolov system starting from the zero-curvature representation. It is to be mentioned that a precise geometric rôle for the Toda Field equations has been already advocated, for instance in [GM], while there already exists a certain amount of literature concerning the relation between uniformization and pure $W_n$-gravity, see for example [Z, BFK, GLM, ZT].

Concerning the notation, we drop all the previously used one, except that for the field $\varphi$ and the labelling of local coordinates. We shall refer to the Riemann Surface mainly with the symbol $C$ or the word "curve"[1], whereas the word "surface" will be reserved to a compact manifold of complex dimension two.

## 4.2 An outline of uniformization theory

Here we briefly review the approach to uniformization theory of curves based on pseudogroups and differential equations [G1, G5]. Although it does not yield such strong results as the one based on discontinuous actions of Fuchsian groups [B, FK], in recent times it has become popular among physicists because it is essentially based on *symmetries*. We shall stick in the sequel to the genus $g > 1$ case.

[1] when not explicitly using the words "Riemann Surface".

"To uniformize means to represent parametrically by single-valued holomorphic or meromorphic functions" [B]. In the present case this means to find a collection $\{A_\alpha\}$, subordinate to the atlas $\{U_\alpha, z_\alpha\}$, of functions:

$$A_\alpha \ : \ U_\alpha \longrightarrow \ V_\alpha \subset \mathbb{P}^1$$

which are required to be local homeomorphisms and such that:

$$A_\alpha(z_\alpha) = \frac{a_{\alpha\beta}A_\beta(z_\beta) + b_{\alpha\beta}}{c_{\alpha\beta}A_\beta(z_\beta) + d_{\alpha\beta}} \tag{4.1}$$

In this way, the new complex atlas $\{U_\alpha, A_\alpha \circ z_\alpha\}$ is such that all local charts are connected through projective transformations. One immediately observes that the collection $\{A_\alpha\}$ can be thought of as a section of a flat $PSL(2,\mathbb{C})$ bundle on $C$. Secondly, projective transformations $\{f_{\alpha\beta}\}$ are characterized by the property:

$$\{f_{\alpha\beta}, z_\beta\} = 0$$

where

$$S(f) = \{f, z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \tag{4.2}$$

is the Schwarzian derivative. These remarks are used in [G1] to explicitly construct the uniformizing atlas in terms of sections of adequate vector bundles. The result is as follows. A projective structure (subordinate to the complex one on $C$) is constructed taking a section $\{\xi_\alpha = (\xi_{1\alpha}, \xi_{2\alpha})\}$ of the bundle

$$T \otimes K^{-1/2}$$

where $K^{1/2}$ is a square root of the canonical bundle and $T$ is a flat $SL(2,\mathbb{C})$ bundle, that is an element $T \in H^1(C, SL(2,\mathbb{C}))$. The coordinate sections $\{A_\alpha\}$ are constructed as:

$$A_\alpha(z_\alpha) = \frac{\xi_{1\alpha}(z_\alpha)}{\xi_{2\alpha}(z_\alpha)} \tag{4.3}$$

and $\xi_{1\alpha}$ , $\xi_{2\alpha}$ are two independent solutions of the differential equation

$$\frac{d^2}{dz_\alpha^2}\xi_{\alpha,i} + \frac{1}{2}u_\alpha(z_\alpha)\xi_{\alpha,i}(z_\alpha) = 0 \ \ i = 1,2 \tag{4.4}$$

with normalized Wronskian. The equation in (4.4) glues coherently on $C$ as $\{u_\alpha(z_\alpha)\}$ is a projective connection [G1, HS], that is a 1-coboundary for the 1-cocycle $S_{\alpha\beta} = \{f_{\alpha\beta}, z_\beta\}dz_\beta^2$, namely:

$$\{f_{\alpha\beta}, z_\beta\}dz_\beta^2 = u_\beta(z_\beta)dz_\beta^2 - u_\alpha(z_\alpha)dz_\alpha^2$$

By Serre duality, $H^1(C, K^2) = 0$, so that projective connections certainly exist and are in one-to-one correspondence with projective structures [G1]. We stress that *all* projective structures arise in the way so outlined [G1]. It is also worthwhile to write down the transformation rule:

$$\begin{cases} \xi_{1\alpha}(z_\alpha) = \left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2}(a_{\alpha\beta}\xi_{1\beta}(z_\beta) + b_{\alpha\beta}\xi_{2\beta}(z_\beta)) \\ \xi_{2\alpha}(z_\alpha) = \left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2}(c_{\alpha\beta}\xi_{1\beta}(z_\beta) + d_{\alpha\beta}\xi_{2\beta}(z_\beta)) \end{cases} \tag{4.5}$$

What is more interesting for our purposes is the description of the above structure in terms of an analytically flat bundle, also discovered by Gunning. Rewrite (4.5) as:

$$\xi_\alpha(z_\alpha) = k_{\alpha\beta}^{-1/2} T_{\alpha\beta}\xi_\beta(z_\beta)$$

and differentiate

$$k_{\alpha\beta}^{1/2}\xi_\alpha(z_\alpha) = T_{\alpha\beta}\xi_\beta(z_\beta)$$

with respect to $z_\beta$. The result is:

$$\frac{d}{dz_\beta}k_{\alpha\beta}^{1/2}\xi_\alpha(z_\alpha) + k_{\alpha\beta}^{-1/2}\frac{d\xi_\alpha}{dz_\alpha} = T_{\alpha\beta}\frac{d\xi_\alpha}{dz_\beta}$$

so that the collection $\{F_\alpha\}$ of matrices:

$$F_\alpha(z_\alpha) := \left(\xi_\alpha'(z_\alpha), \xi_\alpha(z_\alpha)\right)$$

realizes the transformation:

$$F_\alpha(z_\alpha)e_{\alpha\beta}(z_\beta) = T_{\alpha\beta}F_\beta(z_\beta) \tag{4.6}$$

where

$$e_{\alpha\beta}(z_\beta) = \begin{pmatrix} k_{\alpha\beta}^{-1/2} & 0 \\ \frac{d}{dz_\beta}k_{\alpha\beta}^{1/2} & k_{\alpha\beta}^{1/2} \end{pmatrix} \tag{4.7}$$

are the transition functions of a vector bundle $E \to C$ which fits into the exact sequence

$$0 \longrightarrow K^{1/2} \longrightarrow E \longrightarrow K^{-1/2} \longrightarrow 0 \tag{4.8}$$

The extension given in (4.8) is a non-zero element of the cohomology group $H^1(C, (K^{-1/2})^\vee \otimes K^{1/2}) = H^1(C, K) \cong H^0(C, \mathcal{O}_C)^\vee \cong \mathbb{C}$. That the extension class of $E$ cannot be trivial follows from the fact that it is represented by the cocycle:

$$\sigma_{\alpha\beta} = k_{\alpha\beta}^{-1/2}\frac{d}{dz_\beta}k_{\alpha\beta}^{1/2} = \frac{d}{dz_\beta}\log k_{\alpha\beta}^{1/2} \tag{4.9}$$

and the equation $\sigma = \delta\, h$ does not admit holomorphic solution. This has been already observed concerning the DS connections. Moreover, $\sigma$ essentially defines the Chern class of $K^{1/2}$ so that it cannot be zero in cohomology. Notice that a *meromorphic* solution of $\sigma = \delta\, h$ trivializes the extension class through the automorphism $H = \{H_\alpha\}$ where:

$$H_\alpha = \begin{pmatrix} 1 & 0 \\ h_\alpha & 1 \end{pmatrix}$$

The bundle $E$ can be realized up to an obvious automorphism, as the bundle of 1-jets of $-1/2$-differentials on $C$. Because of this reason we shall often refer to it as the jet-bundle. The interest in $E$ lies in the fact that it is flat in the sense of Weil theorem [G2, A]: $c_1(\det E) = 0$ and $E$ is indecomposable. Hence it supports an analytic connection, which is the image

through $F = \{F_\alpha\}$ of the trivial connection $\partial$ on $T$. With obvious manipulations we obtain from $\nabla = F^{-1} \circ \partial \circ F$ that $\nabla = \partial_\alpha + \Lambda_\alpha$ with

$$\Lambda_\alpha = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2}u_\alpha(z_\alpha) & 0 \end{pmatrix} \tag{4.10}$$

The importance of this relies on the fact that now the flat bundle $T$ is reconstructed as the monodromy data of the flatness equation:

$$\partial F_\alpha = F_\alpha \Lambda_\alpha(z_\alpha) \tag{4.11}$$

We end with two remarks. The connection $\Lambda_\alpha$ has the typical form of the differential operators appearing in the study of KdV hierarchies [DS, BV]. The second remark is that $F_\alpha$ should be considered as a flat coordinate frame for the dual bundle $E^\vee \to C$.

## 4.3 Connection with Liouville equation

Once one is able to uniformize the curve $C$, a solution of the Liouville equation

$$\partial_z \partial_{\bar{z}} \varphi = e^{2\varphi} \tag{4.12}$$

is easily written down. It is possible to use the projective atlas $\{U_\alpha, A_\alpha \circ z_\alpha\}$ to pull-back on $C$ the Poincaré metric on the hyperbolic disk:

$$ds^2 = \frac{|dz|^2}{(1 - |z|^2)^2}$$

to obtain the very classical formula:

$$e^{2\varphi_\alpha} = \frac{|\partial A_\alpha|^2}{(1 - |A_\alpha(z_\alpha)|^2)^2} \tag{4.13}$$

One can verify that (4.13) defines a $(1,1)$ form by using the projective transformation (4.1). Of course, in order to have a tensorial object in (4.13), the projective transformations are reduced from $PSL(2,\mathbb{C})$ to $PSU(1,1)$. Usually in Physics one prefers to keep the two chiral halves independent, by writing:

$$e^{2\varphi_\alpha} = \frac{\partial A_\alpha(z_\alpha) \bar{\partial} B_\alpha(\bar{z}_\alpha)}{(1 - A_\alpha(z_\alpha) B_\alpha(\bar{z}_\alpha))^2}$$

It is obvious that in order to obtain a metric from this formula, one has to put constraints on the monodromy of $\{B_\alpha\}$, which results to be conjugate to that of $\{A_\alpha\}$. If one in addition requires reality of $e^{2\varphi} dz d\bar{z}$, the flat cocycles are again reduced to $PSU(1,1)$ with the additional freedom of taking $B_\alpha(\bar{z}_\alpha)$ to be a projective transformation of $\overline{A_\alpha(z_\alpha)}$.

All the quantities arising in the uniformization context are very likely to acquire a physical meaning, as we now show. First of all, consider the (improved) energy-momentum tensor:

$$T_\alpha(z_\alpha) = e^{\varphi_\alpha} \partial^2 e^{-\varphi_\alpha} = (\partial \varphi_\alpha)^2 - \partial^2 \varphi_\alpha \tag{4.14}$$

It is holomorphic on-shell. Plugging (4.13) into its expression, we find:

$$T_\alpha(z_\alpha) = -\frac{1}{2}\{A_\alpha(z_\alpha), z_\alpha\} \tag{4.15}$$

so that we identify it with the projective connection $-\frac{1}{2}u_\alpha(z_\alpha)$. Moreover, using the spin $-1/2$ realization of the projective atlas $A_\alpha = \xi_{1\alpha}/\xi_{2\alpha}$, we can write the Liouville field as:

$$e^{-\varphi_\alpha} = \xi_{2\alpha}\bar{\xi}_{1\alpha} - \xi_{1\alpha}\bar{\xi}_{2\alpha} \tag{4.16}$$

with

$$\xi_{1\alpha} = \frac{A_\alpha}{(\partial A_\alpha)^{1/2}} \quad , \quad \xi_{2\alpha} = \frac{1}{(\partial A_\alpha)^{1/2}} \tag{4.17}$$

This follows from

$$u_\alpha(z_\alpha) = \{A_\alpha(z_\alpha), z_\alpha\}$$

and

$$\{A_\alpha, z_\alpha\} = -2(\partial A_\alpha)^{1/2}\frac{d^2}{dz_\alpha^2}\left((\partial_\alpha A_\alpha)^{-1/2}\right) \tag{4.18}$$

where, of course, these last equations are valid whenever the $\{A_\alpha\}$ is a projective coordinate system. Notice that in terms of $Z_\alpha \equiv A_\alpha \circ z_\alpha$ the spin $-1/2$ fields have the simple form:

$$\xi_{1\alpha} = Z_\alpha dZ_\alpha^{-1/2} \quad , \quad \xi_{2\alpha} = dZ_\alpha^{-1/2}$$

Equation (4.16) can be generalized to "higher spins" as:

$$e^{-2j\varphi_\alpha} = \left(\xi_{2\alpha}\bar{\xi}_{1\alpha} - \xi_{1\alpha}\bar{\xi}_{2\alpha}\right)^{2j}$$

with $j \in \mathbb{Z}/2$ [D'H, A-G]. In terms of a projective coordinate system, the RHS is simply a polynomial of the appropriate degree (Eichler cohomology classes, [G1]).

We cannot conclude this section without mentioning the "naive bosonization" of the spin fields [S, GN]. Contrariwise to all the above introduced objects, it is a construction that has not an immediately covariant character. Dropping the index $\alpha$ for a while, we set:

$$\xi_1 = e^{-\psi} \quad , \quad \bar{\partial}\psi = 0$$

It is immediate to find the relation:

$$-\partial^2\psi + (\partial\psi)^2 = -\frac{1}{2}\{A, z\} = T(z) \tag{4.19}$$

We call it *Miura transformation*. It is matter of coordinate patching to prove that in order (4.19) to be consistent, $\partial\psi$ should transform as a holomorphic connection on $K^{-1/2}$ and, as we proved in chapter 2 discussing the DS, such objects do not exist. Thus, paying attention to the residues, $\partial\psi$ can be taken to be meromorphic with simple poles. This is still not enough, since in a coordinate transformation $\xi_1$ and $\xi_2$ mix among themselves according to (4.5), and this is compatible with (4.19) if and only if the cocycle $T = \{T_{\alpha\beta}\}$ is made of triangular matrices, which is hardly acceptable, if $T$ has to be a flat bundle arising from (or yielding) uniformization. The reason is that $T$, as a flat bundle, represents the fundamental group of $C$ and since everything is regular, this representation must be stable in the sense of Geometric Invariant Theory, that is "not contained in a proper parabolic subgroup". This remark clarifies the reason why the geometric situation described by the DS system we have analyzed before is to be considered as a singular one. Later on in this chapter we shall illustrate a possible correct geometrical setting for the Miura transformation.

## 4.4 Zero-curvature representation

The approach outlined above relies entirely on holomorphic concepts. We are nevertheless interested in analyzing also a differential geometric counterpart which also gives a different approach to uniformization. What we are going to do here is to properly set the zero-curvature equations of Chapter 1 on our curve $C$. In particular we shall consider the linear system (1.6) with covariant derivatives:

$$\nabla_{\pm} = (\partial_{\pm} + A_{\pm}^{(1)})$$

We recall that:

$$A_+ = \begin{pmatrix} \partial_+\varphi & 1 \\ 0 & -\partial_+\varphi \end{pmatrix} \quad , \quad A_- = \begin{pmatrix} 0 & 0 \\ e^{2\varphi} & 0 \end{pmatrix}$$

Now consider the vector bundle $V = K^{-1/2} \oplus K^{1/2}$ over $C$ and the general $SL(2,\mathbb{C})$ connection:

$$\nabla_A = d + Adz + \bar{A}d\bar{z}$$

with

$$\begin{cases} A = pH + aE_+ + uE_- \\ \bar{A} = qH + \mu E_+ + gE_- \end{cases}$$

By using explicitly the transformation property of a connection, it is not difficult to verify that $\nabla_A$ indeed defines a connection on $V$ *provided* the various parameters transform according to the rules:

1. $$p_\beta = p_\alpha \frac{dz_\alpha}{dz_\beta} + \frac{d}{dz_\beta} \log k_{\alpha\beta}^{-1/2}$$

   so that $\{p_\alpha\}$ is a connection on $K^{-1/2}$ (it is *not* assumed to be holomorphic!);

2. $a_\alpha = a_\beta$, thus $\{a_\alpha\}$ is a scalar;

3. $u_\alpha = u_\beta k_{\alpha\beta}^2$, so that $\{u_\alpha\}$ is a quadratic differential;

4. $q_\alpha = \bar{k}_{\alpha\beta} q_\beta$, $\{q_\alpha\}$ is a $(0,1)$ form on $C$;

5. $g_\alpha = k_{\alpha\beta}\bar{k}_{\alpha\beta} g_\beta$, $\{g_\alpha\}$ is an ordinary metric on the tangent bundle of $C$;

6. $\mu_\alpha = k_{\alpha\beta}^{-1}\bar{k}_{\alpha\beta}\mu_\beta$, that is $\{\mu_\alpha\}$ is a 1-form with values in $K^{-1} = T_C$ or, in other parlance, a Beltrami differential.

However this is too general and of little utility. The linear system above corresponds to the case (recall that $e^{2\varphi}$ is a *metric* on $C$!)"

$$a \equiv 1 \quad , \quad u = q = \mu = 0$$

For reason that will be clear later, it will be convenient to consider the case where also $u$ and $\mu$ are non-zero. Let us start by imposing the flatness condition in the simplest case. We find the equations:

$$\partial_{\bar{z}} p = g \tag{4.20}$$

$$p = \frac{1}{2}\partial_z \log g \tag{4.21}$$

if $u = \mu = 0$. This is nothing but the Liouville equation:

$$\frac{1}{2}\partial_z\partial_{\bar{z}} \log g = g$$

or, if $g = e^{2\varphi}$:

$$\partial_z\partial_{\bar{z}}\varphi = e^{2\varphi}$$

in a more familiar form. Two remarks are in order. In eqn. (4.21) we have that $p$ is the connection on $K^{1/2}$ compatible with the fibre metric $\sqrt{g}$. In addition the other equation (4.20) says that the metric $g$ has constant negative curvature. The second remark is about flatness of $\nabla_A$. This is not a contradiction, since the bundle that supports the flat connection, as a *holomorphic one*, is different from $V$. We can introduce *any* metric $g$ on $V$ and then the constant curvature one will be set by the flatness condition. Hence we can suppose that $p$ is the metric connection from the very beginning.

Therefore the connection $\nabla_A$ on the bundle $V$ describes the uniformization of the curve $C$. This has been already noticed by Hitchin [Hi1]. In his terminology, $V$ with the connection $\nabla_A$ is the simplest example of what he calls a Higgs bundle, namely a vector bundle $V \to C$ together with a $(1,0)$-form $\Phi$ with values in $\mathrm{End}V$ – i.e. a section of $\mathrm{End}V \otimes K$ – which is holomorphic with respect to the $(0,1)$ part $\bar{\partial}_B$ of a given metric connection $B$ on $V$ and such that the *Higgs bundle equations* or *self-duality equations* are satisfied:

$$F(B) + [\Phi, \Phi^*] = 0 \tag{4.22}$$

$$\bar{\partial}_B\Phi = 0 \tag{4.23}$$

$F(B) = \nabla_B^2$ is the curvature of $\nabla_B$. The adjoint in (4.22) is taken with respect to a metric in $V$, but, since in (4.22) appears only a (graded) Lie bracket, everything could be defined in terms of principal bundles without even referring to vector bundles. The eqns. (4.22) and (4.23) imply that the connection $\nabla_A = \nabla_B + \Phi + \Phi^*$ is flat. If we give $V = K^{-1/2} \oplus K^{1/2}$ the diagonal metric

$$\begin{pmatrix} \sqrt{g} & 0 \\ 0 & 1/\sqrt{g} \end{pmatrix}$$

it is immediate to verify that:

$$\nabla_B = \partial_B + \bar{\partial}$$

with

$$\partial_B = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} \quad , \quad \Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dz$$

so that

$$\Phi^* = \begin{pmatrix} 0 & 0 \\ g & 0 \end{pmatrix} d\bar{z}$$

and the Higgs bundle equations are precisely (4.20) and (4.21). With this notion at hand, it is easier to discuss the more general case. Indeed, we keep $\nabla_B$ fixed and put:

$$\Phi = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} dz$$

so that

$$\Phi^* = \begin{pmatrix} 0 & g^{-1}\bar{u} \\ g & 0 \end{pmatrix} d\bar{z}$$

namely, $\mu = g^{-1}\bar{u}$. Equation (4.23) becomes:

$$\bar{\partial} u = 0$$

so that $u$ is a holomorphic quadratic differential. By elementary Riemann-Roch $\dim H^0(C, K^2) = 3g - 3$ [G1] and we also know that holomorphic quadratic differentials parametrize infinitesimal deformations of the complex structure of $C$. This can also be seen as follows. The self-duality equation (4.22) gives

$$\partial_{\bar{z}} p = g - g^{-1} u\bar{u} \tag{4.24}$$

which is at first sight very different from the Liouville equation. Nevertheless we have ([Hi1], theorem 11.2) that the metric

$$\hat{g} = u dz^2 + \left(g + \frac{u\bar{u}}{g}\right) dz d\bar{z} + \bar{u} d\bar{z}^2$$

has constant (negative) curvature and satisfies the standard Liouville equation in the new coordinates given by:

$$dZ = \sqrt{g}\left(dz + \frac{\bar{u}}{g} d\bar{z}\right)$$

Its Kähler form is given by:

$$\hat{\omega} = \left(g - \frac{u\bar{u}}{g}\right) dz \wedge d\bar{z}$$

## 4.5 Back to the jet-bundle

After this rather long digression, we can join the two pictures. Given our connection $\nabla_A$ on $V$, since $\bar{\partial}_A = \partial + \Phi^* d\bar{z}$ is automatically integrable, there is a new holomorphic bundle $C^\infty$-equivalent to $V$, such that the connection one gets on it has $(0,1)$ part simply given by $\bar{\partial}$. It is simple to verify that this holomorphic bundle is indeed $E \to C$ previously introduced. The $C^\infty$-bundle morphism locally given by:

$$\psi_\alpha = \begin{pmatrix} 1 & 0 \\ p_\alpha & 1 \end{pmatrix}$$

maps $V$ into $E$. The connection on $E$ is obtained through

$$\nabla_\Lambda = \psi \circ \nabla_A \circ \psi^{-1}$$

and it is given by:

$$\begin{aligned} \Lambda_\alpha &= \begin{pmatrix} 0 & 1 \\ -\partial p_\alpha + p_\alpha^2 & 0 \end{pmatrix} \\ \bar{\Lambda}_\alpha &= \begin{pmatrix} 0 & 0 \\ g_\alpha - \bar{\partial} p_\alpha & 0 \end{pmatrix} \end{aligned}$$

$\nabla_\Lambda$ is flat precisely when the Liouville equation is obeyed. According to the fact that $E$ is analytically flat, when (4.20) and (4.21) hold we have:

$$\bar{\Lambda}_\alpha \equiv 0 \quad , \quad \bar{\partial}\Lambda_\alpha = 0$$

that is the connection is analytic. Moreover, on-shell we have the equality:

$$\Lambda_\alpha = \begin{pmatrix} 0 & 1 \\ T_\alpha(z_\alpha) & 0 \end{pmatrix}$$

With the position $T_\alpha(z_\alpha) = -\frac{1}{2}u_\alpha(z_\alpha)$ we are back again to the uniformization framework. It is maybe interesting to briefly see how things get modified by the presence of a non-zero quadratic differential. Still using $\psi$, in this case we obtain:

$$\begin{aligned} \Lambda_\alpha &= \begin{pmatrix} 0 & 1 \\ -\partial p_\alpha + p_\alpha^2 + u_\alpha & 0 \end{pmatrix} \\ \bar{\Lambda}_\alpha &= \begin{pmatrix} -p_\alpha g_\alpha^{-1}\bar{u}_\alpha & g_\alpha^{-1}\bar{u}_\alpha \\ g_\alpha - g_\alpha^{-1}\bar{u}_\alpha p_\alpha^2 - \bar{\partial}p_\alpha & p_\alpha g_\alpha^{-1}\bar{u}_\alpha \end{pmatrix} \end{aligned}$$

The connection $\nabla_\Lambda$ is flat precisely when $p$ is metric and (4.24) is satisfied:

$$\begin{aligned} p &= \frac{1}{2}\partial \log g \\ \bar{\partial}p &= g - g^{-1}u\bar{u} \end{aligned}$$

The connection assumes a more familiar form if we introduce the Beltrami differential $\mu_\alpha = g_\alpha^{-1}\bar{u}_\alpha$. The quantity

$$-\partial p_\alpha + p_\alpha^2 + u_\alpha$$

can be still identified with the energy-momentum tensor, so we call it $T_\alpha$. With this position:

$$\Lambda_\alpha = \begin{pmatrix} 0 & 1 \\ T_\alpha(z_\alpha) & 0 \end{pmatrix} \tag{4.25}$$

$$\bar{\Lambda}_\alpha = \begin{pmatrix} \frac{1}{2}\partial\mu & \mu \\ \mu T - \frac{1}{2}\partial^2\mu & -\frac{1}{2}\partial\mu \end{pmatrix} \tag{4.26}$$

and the flatness condition takes the form:

$$(\bar{\partial} - \mu\partial - 2\partial\mu)T = -\frac{1}{2}\partial^3\mu \tag{4.27}$$

so that the stress-energy tensor is no more holomorphic. Equation (4.27) is very well known as it is (up to constant factor) the expression of the conformal Ward identity [BFK, Z, DeBGo]. This more complicated structure describes simultaneously the uniformization and the variation of the complex structure. In fact, if $s$ is a flat section for this more general connection, from

$$\nabla_\Lambda s = 0$$

we can obtain equations for only the first component:

$$\begin{aligned}\left(\partial^2 - T\right)\eta_{1,2} &= 0\\ \left(\bar{\partial} - \mu\partial + \frac{1}{2}\partial\mu\right)\eta_{1,2} &= 0\end{aligned}$$

where $\eta_{1,2}$ are two linearly independent solutions of this system. Putting $Z = \eta_1/\eta_2$ we find:

$$T = -\frac{1}{2}\{Z, z\}$$

and

$$\bar{\partial}Z = \mu\partial Z$$

so that $Z$ is a solution of the Beltrami equation. Since from the Higgs eqns. (4.22) and (4.23) we have seen that the non-zero quadratic differential situation can be always reduced to the ordinary one by changing conformal class, from now on we stick on the simpler case.

We end this section with the following remark. The connection $\nabla_A$ is a strict analogous to the system (1.6) of Chapter 1. There we have shown that it is possible to simultaneously kill the antichiral part while putting the rest in a triangular form. In the present situation we have realized the first step and discussed the relative geometry. In the next section we are going to illustrate whether and when it is possible to accomplish also the last step in a non trivial geometry. This will give us the occasion to discuss a possible geometrical framework for the Miura transformation.

## 4.6 Reaching the DS triangular form

Consider again the holomorphic bundle $E \to C$ with the analytic connection $\nabla : \mathcal{E} \to \mathcal{E}\otimes\Omega^1_C$ given by

$$\nabla = \partial + \begin{pmatrix} 0 & 1 \\ T(z) & 0 \end{pmatrix} dz$$

Suppose $s$ to be a local holomorphic section of $E$ such that $\nabla s = 0$. In coordinates we have

$$\partial s_1 + s_2 = 0 \tag{4.28}$$

$$\partial s_2 + T(z)s_1 = 0 \tag{4.29}$$

Restoring the index $\alpha$ for the local chart, if we define

$$w_\alpha(z_\alpha) = \frac{s_{2\alpha}(z_\alpha)}{s_{1\alpha}(z_\alpha)}$$

equations (4.28) and (4.29) become

$$-\partial w_\alpha + w_\alpha^2 = T_\alpha(z_\alpha) \tag{4.30}$$

$$\partial s_{1\alpha} + w_\alpha s_{1\alpha} = 0 \tag{4.31}$$

so that we have reobtained the Miura transformation. Equation (4.31) is a rewriting of the naive bosonization formula we had previously encountered.

As a further comment, let us notice that locally, on the intersection $U_\alpha \cap U_\beta$ we have

$$w_\beta = w_\alpha k_{\alpha\beta}^{-1} + \frac{d}{dz_\beta} \log k_{\alpha\beta}^{-1/2}$$

so that $\{w_\alpha\}$ fulfills the correct transformation rules under change of coordinates. However, we must give a geometrical interpretation to the operations we have performed. This can be accomplished in several ways, according to the personal taste, all leading to the same conclusions. We shall touch each one of them.

First of all, notice that taking the ratio of the two coordinates in the fibre means essentially to consider the projective space associated to it. Thus, what we are doing probably is to pass from $E$ to its projectivization $\pi : \mathbb{P}(E) \to C$. This is the bundle having $\mathbb{P}^1$ as typical fibre and its fibre $\mathbb{P}(E)_p$ over a point $p \in C$ is the projective space $\mathbb{P}(E_p)$, where $E_p$ is the fibre of $E$ over $p \in C$ [GH, Ko]. In the sequel we shall call $X = \mathbb{P}(E)$, the total space of the projective bundle. Using an established terminology, it is a *ruled surface* [Ha], namely a complex surface $X \xrightarrow{\pi} C$ having $\mathbb{P}^1$ as fibre and admitting a section $\sigma : C \to X$. This canonical section $\sigma$ is constructed as follows. In the case at hand, $K^{1/2}$ is a subbundle of $E$ and it is a line bundle. It defines the section $\sigma$ simply by taking:

$$\sigma(p) = \left\{\text{image in } \mathbb{P}(E_p) \text{ of } K_p^{1/2}\right\}$$

This argument is general, as any (rank 2) holomorphic vector bundle has a line subbundle [G2]. Moreover, any smooth ruled surface arise as a projectivization of a rank 2 vector bundle over a curve [Ha].

The above settles the global geometry as needed in the following. There are two ways by means of which we can see the global aspects of what we have done in (4.28)-(4.31). At first we can start from local considerations. It is better to rewrite (4.30) and (4.31) giving a different name to $s_{1\alpha}$ there. Thus, we have

$$-\partial w_\alpha + w_\alpha^2 = T_\alpha(z_\alpha) \tag{4.32}$$

$$\partial v_\alpha + w_\alpha(z) v_\alpha = 0 \tag{4.33}$$

which are obtained from (4.28)-(4.29) through the change of variables [Ar]:

$$\begin{aligned} s_{1\alpha} &= v_\alpha \\ s_{2\alpha} &= w_\alpha v_\alpha \end{aligned} \tag{4.34}$$

Thus $v_\alpha$ is clearly to be interpreted as the coordinate along the line in the fibre whose image in the associated projective space is given by $w_\alpha$. If we consider for a moment (4.28)-(4.29) as an ordinary differential equation in $\mathbb{C}^2$, (4.32)-(4.33) is its transformation through the blow-up at the origin [Ar]. It is maybe worthwhile to spend a few words on what a blow up is.

Consider a vector space $V$ over $\mathbb{C}$ and its associated projective space $\mathbb{P}(V)$. Over $\mathbb{P}(V)$ we have the tautological sequence

$$0 \longrightarrow L \longrightarrow V \times \mathbb{P}(V) \longrightarrow Q \longrightarrow 0$$

where the line-bundle $L$ is defined as

$$L = \{(l, v) \in \mathbb{P}(V) \times V | v \in l\}$$

that is, the fibre of $L$ over $l \in \mathbb{P}(V)$ is $l$ itself considered as a line in $V$. If we call $\phi$ the map

$$\phi : L \hookrightarrow \mathbb{P}(V) \times V \to V$$

This defines $L$ as the blow up of $V$ at the origin. If $v \in V$ and $v \neq 0$, then $\phi^{-1}(v)$ consists of one point, namely $(l, v)$ s.t. $v \in l$. On the other hand, $\phi^{-1}(0) = \mathbb{P}(V)$. It is not difficult to see that $L$ is defined by the equations

$$x_i y_j = x_j y_i \quad i, j = 1, \ldots, \dim V \tag{4.35}$$

where the $\{x_i\}$ are the coordinates in $V$ and the $[y_i]$ are homogeneous coordinates in $\mathbb{P}(V)$. Specializing to dim $V = 2$, we see that (4.34) is exactly (4.35) specialized to an affine chart of $\mathbb{P}^1$. According to this construction, the coordinate $v_\alpha$ appearing in (4.33) is the coordinate along the fibre of $L$. As $L \backslash \{\text{zero section}\}$ and $\mathbb{C}^2 \backslash \{0\}$ are isomorphic, (4.32)-(4.33) is obtained from (4.28)-(4.29) through a change of coordinates. However, $z$ is a parameter varying over the curve $C$, so that our definition should be "relative" over $C$. Clearly the definition goes through also in this case and we have the tautological sequence

$$0 \longrightarrow L(E) \longrightarrow \pi^* E \longrightarrow Q \longrightarrow 0$$

over $X = \mathbb{P}(E)$. The *tautological line bundle* [Ko] is defined as

$$L(E) = \{(l_p, v) \in \pi^* E \,|\, v \in l_p \ , \ p \in \mathbb{C}\}$$

Notice that $\pi^* E$, when restricted to the fibre $\mathbb{P}(E)_p$ is trivial:

$$\pi^* E \,|_{\mathbb{P}(E)_p} \cong \mathbb{P}(E)_p \times E_p$$

We can think of $L(E)$ as the blow up of $E$ with respect to the zero section. It follows from the above considerations that outside their respective zero sections $L(E)$ and $E$ are isomorphic. Thus we can regard the Miura transformation, together with (4.33), as the transformation of the differential equation associated to the analytic connection when passing to the coordinates given by the tautological bundle over the ruled surface. Hence the coordinate $v_\alpha$ is a coordinate on the fibre $L(E)$. Now the point is: can we consider (4.32) and (4.33) as some sort of connection? Moreover, we have given so far an abstract analysis of the differential equation determined by the connection. The point is to include locally flat *frames* for the connection, that is, bases of actual solutions of $\nabla = 0$. This brings us to elaborate another point of view about the local formulæwe have written down. In addition, it will be possible to give a meaning to the procedure of triangularization of the connection performed in Chapter 1.

When dealing with frames, it is convenient to consider principal bundles, so let $P$ be the holomorphic principal bundle over $C$ to which $E$ is associated. It has $SL(2, \mathbb{C})$ as typical fibre. The ruled surface $X$ is also associated to $P$ due to the fact that $\mathbb{P}^1 \cong SL(2, \mathbb{C})/B$, where $B$ is the "upper" Borel subgroup of $SL(2, \mathbb{C})$. At this point, it is necessary to set a convention.

In order to let the coset structure of $\mathbb{P}^1$ explicitly appear, we identify $g \in SL(2,\mathbb{C})$ with a (properly normalized) basis in $\mathbb{C}^2$ as $g = (f_1, f_2)$ and we define the projection

$$SL(2,\mathbb{C}) \longrightarrow \mathbb{P}^1$$

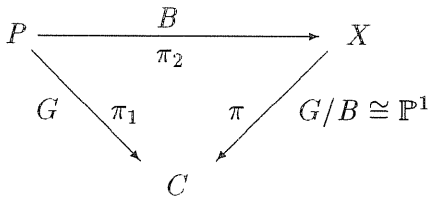
to be the map

$$g = (f_1, f_2) \rightsquigarrow [f_1]$$

where $[f_1]$ denotes the line through $f_1 \in \mathbb{C}^2$. This is the same convention as in [Gr]. It is such that the stabilizer subgroup is exactly $B$. Locally on $C$, the projection $(f_1, f_2) \to [f_1]$ defines a principal fibration

$$SL(2,\mathbb{C}) \xrightarrow{B} \mathbb{P}^1$$

The $\mathbb{C}^*$ factor corresponding to dilations of $f_1$ defines the tautological line bundle $L$ over $\mathbb{P}^1$. This can be globalized to yield the diagram




where

$$G = SL(2,\mathbb{C})$$

Both $P \to C$ and $P \to X$ are principal fibrations [KoN]. According to the local situation, the $\mathbb{C}^*$ factor globally defines the tautological bundle $L(E)$. Adapting to this holomorphic case some classical notion of differential geometry, we have that the connection in $E$ can be described in terms of a connection 1-form $\omega$ on $P$ which locally has the form

$$\omega = \theta_g + \mathrm{Ad}_{g^{-1}} \cdot \Lambda(z)dz \tag{4.36}$$

where $g \in SL(2,\mathbb{C})$, $\theta_g = g^{-1}dg$ is the Maurer-Cartan form and

$$\Lambda(z) = \begin{pmatrix} 0 & 1 \\ T(z) & 0 \end{pmatrix} \tag{4.37}$$

The equation

$$\omega = 0$$

defines the horizontal distribution $H_p$ in $T_P$ (we mean the holomorphic tangent bundle). With respect to the situation depicted in the diagram above we have the sequences:

$$0 \longrightarrow V_P \longrightarrow T_P \longrightarrow \pi_1^* T_C \longrightarrow 0$$
$$0 \longrightarrow V_{\pi_2} \longrightarrow T_P \longrightarrow \pi_2^* T_X \longrightarrow 0$$
$$0 \longrightarrow V_X \longrightarrow T_X \longrightarrow \pi^* T_C \longrightarrow 0$$

Writing the connection 1-form in terms of the standard basis of the Lie algebra as

$$\omega = \omega^0 H + \omega^+ E_+ + \omega^- E_-$$

it is apparent that $\omega^-$ kills the vertical subbundle $V_{\pi_2}$, so that it is horizontal with respect to the fibration $\pi_2 : P \to X$ and it can be considered as a form on $X$. Using the coordinate $w$, already introduced before, for the fibre of $X$, we can use the following coordinates on $P$:

$$(z, g) = (z, v(w)b)$$

where $b \in B$ and $\sigma(w)$ is the lifting $\sigma : \mathbb{P}^1 \to SL(2, \mathbb{C})$ [G]

$$\sigma(w) = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$$

One immediately finds the expression for $\omega^-$ as

$$\omega^- = dw + \left(T(z) - w^2\right) dw \tag{4.38}$$

which should be compared with (4.32).

Of course this is not the end of the story. There are two points which have not yet been clarified. The first one is the meaning of the condition

$$\omega^- = 0 \tag{4.39}$$

on $X$. The second one is about the rôle of the upper triangular part of the connection. To discuss the first point, we notice that (4.39) defines the image in $T_X$ of the horizontal distribution $H_P$. If we consider $\omega^-$ as a form on $P$, it is clear that $H_P \subset \ker \omega^-$. On the other hand, if $\eta \in X$ such that $\omega^-(\eta) = 0$ and $\eta \notin \pi_{2*} H_P$ were to exist, it should be $\eta = \pi_{2*}\tilde{\eta}$ with $\tilde{\eta} \in V_{\pi_2}$, and in that case $\omega(\tilde{\eta}) = 0$ on $P$. Therefore $\pi_{2*} H_P = \ker \omega^-$ in $T_X$.

This is in some sense a typical situation which has been given the name of "partial connection" [KT2]. It happens when a distribution in the total space of a principal bundle is horizontal and equivariant without being supplementary to the vertical tangent bundle. In this case, instead of being isomorphic to the tangent space of the base space of the fibration, it determines only a distribution thereof [KT2].

Thus we have found the (complex) differential geometric meaning of the Miura transformation. At the infinitesimal level, it defines the image, in the tangent bundle to the ruled surface $X$, of the horizontal distribution of the partial connection on $\pi_2 : P \to X$. However, we are very suspicious as for the existence of the leaves of the foliation determined by $\omega^- = 0$.

The final point is to perform the change of coordinates in $P$ and look at the behaviour of $\omega$. We retain the lift $\sigma : \mathbb{P}^1 \to SL_2(\mathbb{C})$. We can consider the map

$$\lambda : (z, w) \longrightarrow (z, \sigma(w)) \tag{4.40}$$

as a local section to $\pi_2 : P \to X$, hence

$$(z, w, b) \longrightarrow (z, \sigma(w)b) \tag{4.41}$$

is local coordinate chart for $P$ with respect to $X$. Using (4.36) and inserting (4.41) we find

$$\omega = \theta_b + \mathrm{Ad}_{b-1} \cdot (\sigma(w)^{-1} d\sigma(w) + \mathrm{Ad}_{\sigma(w)^{-1}} \cdot \Lambda(z) dz)$$

The expression inside the brackets is $\lambda^*\omega$ and it is given by

$$\lambda^*\omega = \omega^- E_- + \begin{pmatrix} w & 1 \\ 0 & -w \end{pmatrix} dz \tag{4.42}$$

and, quite obviously,

$$\lambda^*\omega|_{\omega^-=0} = \begin{pmatrix} w & 1 \\ 0 & -w \end{pmatrix} dz \tag{4.43}$$

which has the Drinfel'd–Sokolov form. These statements are clearly local and hold where a solution to (4.39) is possible. In this case, $w$ is given as a function of $z$ as a local section of $\pi : X \to C$, so that, pulling back to $C$, what remains of the connection 1-form $\omega$ appears as a connection 1-form over $C$ to yield the analytic "covariant derivative"

$$\nabla^{DS} = \partial + \begin{pmatrix} w(z) & 1 \\ 0 & -w(z) \end{pmatrix} dz \tag{4.44}$$

The fact that one obtains derivation operators at each step in (4.42), (4.43), (4.44) can be easily seen by plugging vectors in the appropriate slots of the differential forms and writing the resulting differential equations.

## 4.7 Comments

The previous results end the long run of this final chapter. It should be rather clear from the mathematics outlined above why (4.44) appears as the gauge transformed connection of (4.36) when we perform the Gauss factorization of the local frame:

$$V = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 \\ 0 & \xi_1^{-1} \end{pmatrix}, \quad w = \xi_3/\xi_1$$

Since this factorization hides behind it the locus $\omega^- = 0$, one does not see at first sight that this transformation holds true only along a solution of the equation defining the Miura transformation. Thus we expect to encounter problems if we are willing to keep (4.44) to hold globally on $C$. These problems will arise first of all from the fact that $\omega^- = 0$ may not be globally defined on $X$. This is due to the fact that $w(z)$ is possibly not single-valued. This may happen indeed as it can be seen from the expression of $w$ in terms of the coordinates of the local frame: this one is twisted by the flat cocycle $\{T_{\alpha\beta}\}$ upon changing coordinates.

Secondly, even if we might be able to solve $\omega^- = 0$, we have already noticed that the solution must have poles, so that the gauge transformed connection $\nabla^{DS}$ will be singular. We have seen in the course of chapter 2 that a singular connection has particular features, and that the sheaf of its horizontal sections lives on the curve with the singular points deleted.

Now, it is also clear that in the simple topology of chapter 1 the factorization can be done in such a way that it is a true gauge transformation. There are no ruled surfaces nor

uniformizing bundles, and there is only one class of non contractible loops. Therefore the ratio $\xi_3/\xi_1$ can be arranged to be single-valued.

Nevertheless, we believe that the geometric considerations developed here still offer support to what we have done on the cylinder. Indeed the ratio $\xi_3/\xi_1$ is better interpreted as a map into the space $SL_2(\mathbb{R})/B \cong \mathbb{RP}^1$. In this way, it is given a meaning also at those points where $\xi_1$ may vanish.

# Conclusions

In this thesis we have treated different problems related to the Liouville equation. Thus, in order to draw some conclusions and to give an outlook of the possible developments, it is necessary to stick to each different aspect.

As far as the problem of finding the free field representation is concerned, we succeded in doing it (for the case of the Liouville equation) and we found two different solutions, showing how to interpret this phenomenon in terms of the action of the Weyl group of $SL_2(\mathbb{R})$. Moreover, we did not merely find a pair of fields satisfying the desired Poisson brackets, but we also showed that the fields we constructed are exactly those coming from the Drinfel'd-Sokolov construction.

One may wonder about the case of higher rank groups or, what is the same, about Toda Field equations. It is not difficult to show that also in this case a free field representation can indeed be constructed starting from the zero-curvature representation. The point is to prove the hyperbolic character of the monodromy, which is a necessary condition in order to put things in the DS scheme. The problem is to find a general proof, possibly working for every Lie Algebra, besides having proofs made by "ad hoc" calculations, which in the $A_n$ series, at least, can indeed be found. It is natural to conjecture the existence of a variety of solutions, acted upon by the Weyl group of the algebra they refer to.

The picture we made above obviously refers to genus zero. Concerning the case of higher genus, there are problems still to be solved yet in the Liouville case. We found in this work that the DS construction works almost parallel with the one of genus zero, except some important differences, notably the fact that the generalized free oscillators we took as our starting point are in fact constrained. Besides trying to have a more precise characterization of the allowed zones on the phase space of the theory, the precise meaning of this should be discussed more thoroughly.

Another (maybe more) important point, is to find an expression for the exchange algebra on the Riemann Surface. Those people working in integrable system know that the exchange algebra can be taken as the starting point for the quantization. Unfortunately, there seems not to be any method at our disposal to calculate the exchange algebra on the Riemann Surface in full generality. The meaning of this should be: to fix a general contour $C_\tau$, and having defined the $\sigma$ and $\bar{\sigma}$ as line integrals starting from a fixed point on the surface, to calculate their Poisson bracket when they reach the contour $C_\tau$. As we already said, it is not possible to accomplish this at the moment, also because of the action of the fundamental group.

These are in our opinion the main points in the rank two case. Other possible developments are the extension to higher rank bundles, so obtaining an analogous scheme for the

Toda Field equations.

Due to the point of view we have taken here, a natural question about the higher genus theory would be about the possibility of inverting this generalized DS construction. This is almost entirely matter of speculation. The material we have placed in the last chapter go in this direction. Indeed what we have done is to formulate a zero-curvature representation and tried to find a way to reach the form of the DS connection. It is clear that the geometric side is much more complicated, and indeed the interpretation we gave for the DS connection is very subtle. Thus we imagine that a parallel analysis of the symplectic features would raise enormous difficulties.

On the other hand, there are many other interesting aspects of the scheme we have outlined. First of all, the extension to higher ranks. It has been already noticed that higher jet bundles may offer a possible geometric explanation for the $W_n$-symmetries. It is thus interesting to see the connection with the differential geometric picture given by the zero curvature representation of the Toda Fields equations. These last ones enter into a class of connections studied in Differential and/or Algebraic Geometry in their connections with problems of uniformization of higher dimensional bounded symmetric domains and variations of Hodge structures. Furthermore, it is also interesting to relax the regularity assumptions (i.e. holomorphicity) made in the last chapter. Indeed, a uniformization theory of "branched projective structures" exists, and it is much more richer from the point of view of the geometry of vector bundles. Therefore, it is maybe more interesting to make contact with the flat representation of the Toda Field equations using this more general structures.

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