



ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

LIMITS OF MINIMUM PROBLEMS WITH CONVEX OBSTACLES FOR VECTOR VALUED FUNCTIONS

Candidate: Enrico Vitali

Supervisor: Prof. Gianni Dal Maso

Thesis submitted for the degree of Doctor Philosophiae

academic year 1991/92

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INTRODUCTION

In this thesis we deal with some questions related to the asymptotic behaviour of a sequence of minimum problems with convex obstacles for functionals of the calculus of variations in the setting of vector valued Sobolev functions.

A classical problem consists in minimizing the Dirichlet integral $\int_{\Omega} |Du|^2 dx$ when u takes prescribed values on the boundary of the domain Ω and is constrained to lie above a fixed function ψ (compatible with the boundary conditions).

More precisely, given an open subset Ω of \mathbf{R}^n ($n \geq 1$) and a function $\psi: \Omega \rightarrow [-\infty, +\infty]$, we consider the problem

$$(I.1) \quad \min_{u \in \mathcal{K}} \int_{\Omega} |Du|^2 dx,$$

where

$$(I.2) \quad \mathcal{K} = \{u \in H_0^1(\Omega) : u \geq \psi \text{ a.e. on } \Omega\}$$

($H_0^1(\Omega)$ denotes the usual space of (real) Sobolev functions with zero boundary conditions). A more general class of constraints corresponds to sets \mathcal{K} of the form

$$(I.3) \quad \mathcal{K} = \{u \in H_0^1(\Omega) : \psi \leq u \leq \varphi \text{ a.e. on } \Omega\},$$

where $\varphi: \Omega \rightarrow [-\infty, +\infty]$ and $\psi \leq \varphi$ on Ω . In both cases, if $\mathcal{K} \neq \emptyset$ the existence of a solution can be easily obtained by the direct method of the calculus of variations.

Let us point out the particular case when $\varphi = \psi = 0$ on a closed subset E of Ω , and $\psi = -\infty$, $\varphi = +\infty$ on $\Omega \setminus E$. Then $u \in \mathcal{K}$ means that $u = 0$ a.e. on E , and the variational inequality corresponding to (I.1) is actually an equation on $\Omega \setminus E$ with zero boundary conditions:

$$(I.4) \quad \begin{cases} -\Delta u = 0 & \text{on } \Omega \setminus E, \\ u \in H_0^1(\Omega \setminus E). \end{cases}$$

In order to be able to treat also constraints given on “thin” sets, such as lines in \mathbf{R}^2 or surfaces in \mathbf{R}^3 , the condition $\psi \leq u \leq \varphi$ will be actually taken in the sense “quasi everywhere” (q.e.) with respect to the usual $H_0^1(\Omega)$ -capacity.

A quite natural question which arises in several situations is the study of the asymptotic behaviour of a sequence of problems (I.1) corresponding to a sequence (\mathcal{K}_h) of sets, as far as the convergence of both the minimum values and the minimizers is concerned. For instance, given closed subsets E_h of Ω ($h \in \mathbf{N}$), the corresponding problems (I.4) constitute a sequence of Dirichlet problems in a perforated domain.

In such cases it happens to be convenient to deal with problems on a fixed vector space of functions (the whole $H_0^1(\Omega)$), replacing $\int_{\Omega} |Du|^2 dx$ by

$$\int_{\Omega} |Du|^2 dx + G_{\mathcal{K}_h}(u),$$

where

$$(I.5) \quad G_{\mathcal{K}_h}(u) = \begin{cases} 0, & \text{if } u \in \mathcal{K}_h, \\ +\infty, & \text{otherwise.} \end{cases}$$

This simple trick leads to the study of a sequence of functionals on $H_0^1(\Omega)$. A suitable setting to frame this analysis is the context of Γ -convergence, introduced in the '70's by De Giorgi (see [20] for a thorough introduction). Indeed, it is a natural notion of variational convergence for functionals which guarantees, under mild assumptions, the convergence of the minimum points and of the minimum values. Another important aspect is that it provides a general compactness theorem which allows, in the applications, to start with an abstract limit functional.

A natural extension to vector valued Sobolev functions, though in the setting of convex problems, can be obtained by considering sets of the form

$$(I.6) \quad \mathcal{K} = \{u \in H_0^1(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for q.e. } x \in \Omega\},$$

where K is a multifunction from Ω to \mathbf{R}^m ($m \geq 1$) with non-empty closed convex values. With a view to applications, $\int_{\Omega} |Du|^2 dx$ in (I.1) is now replaced by a general energy $F(u) = \int_{\Omega} W(x, Du) dx$, where $W(x, \cdot)$ is a quadratic form satisfying suitable boundedness and coerciveness conditions. For instance, if $n = m = 3$ and $W(x, \eta) = \frac{\lambda}{2} |\text{tr}_{\frac{1}{2}}(\eta^T + \eta)|^2 + \mu |\frac{1}{2}(\eta^T + \eta)|^2$ ($\lambda, \mu > 0$; η 3×3 matrix), then $F(u)$ is the usual energy of linearized elasticity, while, if E is a closed subset of Ω , a functional $G_{\mathcal{K}}$ (see (I.5)), for $\mathcal{K} = \{u \in H_0^1(\Omega, \mathbf{R}^3) : u = 0 \text{ q.e. on } E\}$, may represent the effect of rigid inclusions in the elastic body.

As in the scalar case (see Chapter III for detailed references) our aim is to determine the general form of the limit of a sequence $(F + G_{\mathcal{K}_h})$, with \mathcal{K}_h as in (I.6) and

$G_{\mathcal{K}_h}$ defined in (I.5). Actually, we shall be interested in sequences of the form $(F + G_h)$, where (G_h) lives in a more general class \mathcal{G} closed under Γ -convergence.

According to the foregoing outline, the thesis is divided into three chapters.

In the first one we take into account the sets \mathcal{K} of the form (I.6). The main result of this chapter (Theorem 4.1) is their characterization as the closed subsets of $H_0^1(\Omega, \mathbf{R}^m)$ which are stable under convex combinations with C^1 -coefficients. We point out that the usual methods applied for the scalar case, i.e., $m = 1$ (see [6]), do not work if $m > 1$, since they are based on the order structure of \mathbf{R} (think, e.g., of truncation and monotonicity methods). A straightforward application (Theorem 5.3) is that the limit of minimum problems with obstacles corresponding to sets \mathcal{K}_h ($h \in \mathbf{N}$), is still an obstacle problem of the same type, corresponding to a set \mathcal{K} , if the sequence (\mathcal{K}_h) converges in the sense of Mosco (see [36]) to \mathcal{K} .

In the second chapter we give an integral representation theorem for the class \mathcal{G} mentioned above and hence, in particular, to the limits of minimum problems with obstacles. More precisely, we shall consider an explicit dependence of the functional on the domain, so that an element G of \mathcal{G} is a functional from $H^1(\Omega, \mathbf{R}^m) \times \mathcal{A}(\Omega)$ to $[0, +\infty]$, where Ω is a fixed open subset of \mathbf{R}^n and $\mathcal{A}(\Omega)$ denotes the family of all its open subsets. G is required to satisfy the following properties:

- (i) (lower semicontinuity) for every $A \in \mathcal{A}(\Omega)$ the function $G(\cdot, A)$ is lower semicontinuous on $H^1(\Omega, \mathbf{R}^m)$;
- (ii) (measure property) for every $u \in H^1(\Omega, \mathbf{R}^m)$ the set function $G(u, \cdot)$ is the trace of a Borel measure on Ω ;
- (iii) (locality property) $G(u, A) = G(v, A)$ whenever $u, v \in H^1(\Omega, \mathbf{R}^m)$, $A \in \mathcal{A}(\Omega)$, and $u|_A = v|_A$;
- (iv) (C^1 -convexity) for every $A \in \mathcal{A}(\Omega)$ the function $G(\cdot, A)$ is convex on $H^1(\Omega, \mathbf{R}^m)$ and, in addition, $G(\varphi u + (1 - \varphi)v, A) \leq G(u, A) + G(v, A)$ for every $u, v \in H^1(\Omega, \mathbf{R}^m)$ and for every $\varphi \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$, with $0 \leq \varphi \leq 1$ on Ω .

If $G \in \mathcal{G}$, then (Theorem 6.5), for every $u \in H^1(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$

$$G(u, A) = \int_A g(x, u(x)) d\mu + \nu(A),$$

where μ and ν are suitable positive Borel measures on Ω and $g: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ is a Borel function, convex and lower semicontinuous in the second variable.

Finally, in Chapter 3 we study the class \mathcal{G} from the point of view of its convergence properties, proving in particular, its closure (Theorem 4.1). Moreover, we single out a special subclass which is relevant for the applications to Dirichlet problems in perforated domains in linearized elasticity and which enjoys a nice closure property.

In the first two chapters we shall work, more generally, in the space $W^{1,p}(\Omega, \mathbf{R}^m)$, with $1 < p < +\infty$.

Chapter I

A CHARACTERIZATION OF C^1 -CONVEX SETS IN SOBOLEV SPACES

Introduction

A closed valued multifunction $F: \Omega \rightarrow \mathbf{R}^m$ on an open subset Ω of \mathbf{R}^n is a map F from Ω into the set of all closed subsets of \mathbf{R}^m . The set

$$(0.1) \quad \mathcal{F} = \{u \in L^p(\Omega, \mathbf{R}^m) : u(x) \in F(x) \text{ for a.e. } x \in \Omega\}$$

of all L^p selections of a closed valued multifunction F is closed and *decomposable*, i.e., if u, v are two functions in \mathcal{F} and χ is the characteristic function of some measurable subset of Ω , then $\chi u + (1 - \chi)v$ belongs to \mathcal{F} .

F. Hiai and H. Umegaki proved in [32] that all closed decomposable subsets \mathcal{F} of $L^p(\Omega, \mathbf{R}^m)$, $1 \leq p < +\infty$, can be written as in (0.1) for a suitable closed valued multifunction F . If, in addition, \mathcal{F} is convex in $L^p(\Omega, \mathbf{R}^m)$, then $F(x)$ is convex in \mathbf{R}^m for a.e. $x \in \Omega$.

This result can not be extended directly to the case of Sobolev spaces, because the only decomposable subsets of a Sobolev space are those composed of just one function.

In this chapter we show that, in the case of the Sobolev space $W_0^{1,p}(\Omega, \mathbf{R}^m)$, the notion of decomposability may be replaced by the notion of C^1 -convexity, introduced in a more general context by G. Bouchitté and M. Valadier [10] (under the slightly different name of C^1 -stability) in order to study the commutativity properties of the operations of integration and infimum.

A subset \mathcal{K} of $W_0^{1,p}(\Omega, \mathbf{R}^m)$ is said to be C^1 -convex if for every pair of functions $u, v \in \mathcal{K}$ the convex combination $\alpha u + (1 - \alpha)v$ belongs to \mathcal{K} , whenever $\alpha \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$ and $0 \leq \alpha \leq 1$.

The main result of this chapter (Theorem 4.1) is that a closed subset \mathcal{K} of $W_0^{1,p}(\Omega, \mathbf{R}^m)$, $1 < p < +\infty$, is C^1 -convex if and only if there exists a multifunction $K: \Omega \rightarrow \mathbf{R}^m$, with closed convex values, such that

$$(0.2) \quad \mathcal{K} = \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \Omega\},$$

where *p-q.e.* means *quasi everywhere* (with respect to the intrinsic $W_0^{1,p}(\Omega)$ -capacity) and the pointwise value $u(x)$ of u is defined, for *p-q.e.* $x \in \Omega$, as the limit, as $r \rightarrow 0^+$, of the average of u (with respect to Lebesgue measure) on the ball $B(x, r)$ with center x and radius r (see [43]).

In the case $m = 1$ this result can be obtained by using the methods of [6], which are based on the order structure of \mathbf{R} . To treat the case $m > 1$ we have to use different techniques.

The multifunction K which appears in (0.2) is not uniquely determined by \mathcal{K} . Our proof, however, provides a quasi lower semicontinuous multifunction K that, in addition, can be written in the form

$$(0.3) \quad K(x) = \text{cl}\{u_k(x) : k \in \mathbf{N}\},$$

where (u_k) is a suitable sequence in $W_0^{1,p}(\Omega, \mathbf{R}^m)$ and cl denotes the closure in \mathbf{R}^m . We then prove (Corollary 3.4) that the value $K(x)$ of a multifunction K satisfying (0.2) and (0.3) are uniquely determined by \mathcal{K} for quasi every $x \in \Omega$.

Since all sets \mathcal{K} of the form (0.2) are closed and C^1 -convex, to prove our main result we have only to show that for every closed and C^1 -convex subset \mathcal{K} of $W_0^{1,p}(\Omega, \mathbf{R}^m)$ there exists a convex valued multifunction $K: \Omega \rightarrow \mathbf{R}^m$ satisfying (0.2). It is not restrictive to assume that $0 \in \mathcal{K}$.

The construction of K is quite easy. Since $\mathcal{K} \cap L^\infty(\Omega, \mathbf{R}^m)$ is dense in \mathcal{K} (Proposition 3.7), we can choose a sequence (u_k) in $\mathcal{K} \cap L^\infty(\Omega, \mathbf{R}^m)$ dense in \mathcal{K} for the strong topology of $W_0^{1,p}(\Omega, \mathbf{R}^m)$ and we just define $K(x)$ as in (0.3). The inclusion

$$\mathcal{K} \subseteq \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \Omega\}$$

is then obvious. To prove the opposite inclusion, we approximate the convex set $K(x)$ from the interior by means of the polyhedral sets $C_k(x)$ defined as the convex hull in \mathbf{R}^m of the finite sets $\{u_1(x), \dots, u_k(x)\}$. First we prove the inclusion

$$(0.4) \quad \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in C_k(x) \text{ for } p\text{-q.e. } x \in \Omega\} \subseteq \mathcal{K},$$

and then we show that every function $u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$, with $u(x) \in K(x)$ for *p-q.e.* $x \in \Omega$, can be approximated, in the strong topology of $W_0^{1,p}(\Omega, \mathbf{R}^m)$, by a sequence of functions v_k such that $v_k(x) \in C_k(x)$ for *p-q.e.* $x \in \Omega$. As \mathcal{K} is closed, this fact together with (0.4) shows that

$$\{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \Omega\} \subseteq \mathcal{K}$$

and concludes the proof of (0.2).

The most difficult step is the proof of (0.4), which is obtained by showing that the set of all convex combinations $v = \sum_i \varphi^i u_i$ of the functions u_1, \dots, u_k with C^1 coefficients $\varphi^1, \dots, \varphi^k$ is dense in the set

$$\{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in C_k(x) \text{ for } p\text{-q.e. } x \in \Omega\}$$

for the strong topology of $W_0^{1,p}(\Omega, \mathbf{R}^m)$. This density result would be trivial if the vectors $u_1(x), \dots, u_k(x)$ were uniformly linearly independent in \mathbf{R}^m (i.e., if the norm of the exterior product $u_1(x) \wedge \dots \wedge u_k(x)$ had a positive lower bound).

Unfortunately, it is not easy to reduce the problem to this simple case. Therefore, we prefer to use a different approach. First we prove an analogous result in the space $W^{1,\infty}((\mathbf{R}^m)^{k+1}, \mathbf{R}^m)$ of Lipschitz functions. More precisely, denoting by M_k the set of vectors $(\xi, \xi_1, \dots, \xi_k) \in (\mathbf{R}^m)^{k+1}$ such that $\xi \in \text{co}\{\xi_1, \dots, \xi_k\}$, we prove (Theorem 2.8) that the function $u(\xi; \xi_1, \dots, \xi_k) = \xi$ can be approximated in the weak* topology of $W^{1,\infty}(M_k, \mathbf{R}^m)$ by convex combinations $v = \sum_i \varphi^i u_i$ of the functions $u_i(\xi; \xi_1, \dots, \xi_k) = \xi_i$ with Lipschitz coefficients $\varphi^i(\xi; \xi_1, \dots, \xi_k)$. Finally, we obtain the desired result on $W_0^{1,p}(\Omega, \mathbf{R}^m)$ by a standard superposition argument (Theorem 2.9).

The proof of the density theorem in the Lipschitz case relies on some results on Lipschitz parametrizations of convex sets recently obtained by A. Ornelas [37] and S. Lojasiewicz, Jr. [33].

In the last section we apply our main result to prove a stability result for the class of convex sets of the form (0.2). More precisely, in Theorem 5.3 we prove that, if (\mathcal{K}_h) is a sequence of closed convex subsets of $W_0^{1,p}(\Omega, \mathbf{R}^m)$ of the form

$$\mathcal{K}_h = \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K_h(x) \text{ for } p\text{-q.e. } x \in \Omega\},$$

which converges to a set \mathcal{K} in the sense of U. Mosco (see [36]), then the limit set \mathcal{K} can be written in the form (0.2) for a suitable multifunction $K: \Omega \rightarrow \mathbf{R}^m$ with closed convex values. This allows us to prove that, in this case, the limit of the minimum problems with obstacles

$$\min\{\Phi(u) : u \in W_0^{1,p}(\Omega, \mathbf{R}^m), u(x) \in K_h(x) \text{ for } p\text{-q.e. } x \in \Omega\}$$

is an obstacle problem of the same type, under the usual convexity and coerciveness assumptions on the functional Φ .

Another application of our result concerns the problem of the integral representation of C^1 -convex local functionals on $W^{1,p}(\Omega, \mathbf{R}^m)$ and will be given in the next

chapter. This will be the crucial step in the proof of a general integral representation theorem for the “relaxed obstacle problems” (i.e., variational limits of obstacle problems, see [22], [18], [8], [23]) in the vector-valued case.

1. Notation and preliminaries

Throughout this chapter p is a fixed real number, $1 < p < +\infty$, and m, n are two fixed positive integers. If $B \subseteq \mathbf{R}^n$ is a Borel set we denote its Lebesgue measure by $|B|$. The notation *a.e.* stands for almost everywhere with respect to the Lebesgue measure.

If d is a positive integer, the $(d-1)$ -dimensional simplex Σ_d is defined by

$$\Sigma_d = \{\lambda \in \mathbf{R}^d : \lambda^1 + \cdots + \lambda^d = 1, \lambda^i \geq 0\},$$

where $\lambda = (\lambda^1, \dots, \lambda^d)$. For every $x \in \mathbf{R}^d$ and $r > 0$ we set

$$B(x, r) = \{y \in \mathbf{R}^d : |y - x| < r\}.$$

If $B \subseteq \mathbf{R}^d$ is a Borel set, we denote its Lebesgue measure by $|B|$.

For any open subset Ω of \mathbf{R}^n the space $L^p(\Omega, \mathbf{R}^m)$ is endowed with the norm

$$\|u\|_{L^p(\Omega, \mathbf{R}^m)} = \left(\int_{\Omega} |u|^p dx \right)^{1/p}.$$

Let $W^{1,p}(\Omega, \mathbf{R}^m)$ be the Banach space of all functions $u \in L^p(\Omega, \mathbf{R}^m)$ with first order distributional derivative Du in $L^p(\Omega, \mathbf{R}^{mn})$, endowed with the norm

$$\|u\|_{W^{1,p}(\Omega, \mathbf{R}^m)} = (\|u\|_{L^p(\Omega, \mathbf{R}^m)}^p + \|Du\|_{L^p(\Omega, \mathbf{R}^{mn})}^p)^{1/p}.$$

The closure of $C_0^1(\Omega, \mathbf{R}^m)$ in $W^{1,p}(\Omega, \mathbf{R}^m)$ will be denoted by $W_0^{1,p}(\Omega, \mathbf{R}^m)$. \mathbf{R}^m will be omitted, in the notation, if $m = 1$. If $u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$, then u can be considered naturally as an element of $W^{1,p}(\mathbf{R}^n, \mathbf{R}^m)$ by setting $u = 0$ outside Ω .

For every compact set $K \subseteq \Omega$ we define the p -capacity of K with respect to Ω by

$$\text{cap}_p(K, \Omega) = \inf \{ \|\varphi\|_{W^{1,p}(\Omega, \mathbf{R})}^p : \varphi \in C_0^\infty(\Omega), \varphi \geq 1 \text{ on } K \}.$$

The definition is extended to all subsets of Ω as external capacity in the usual way (see, for example, [15] and [43]). Let us note that the family of the sets of p -capacity zero does not depend on Ω , i.e., if $E \subseteq \Omega$ then $\text{cap}_p(E, \Omega) = 0$ if and only if $\text{cap}_p(E, \mathbf{R}^n) = 0$.

Let E be a subset of \mathbf{R}^n . If a statement depending on $x \in \mathbf{R}^n$ holds for every $x \in E \setminus N$, where N is a set with p -capacity zero, then we say that it holds *p -quasi everywhere* (p -q.e.) on E .

A function $f: \Omega \rightarrow \mathbf{R}^m$ is said to be $\text{cap}_p(\cdot, \Omega)$ -quasi continuous in Ω if for every $\varepsilon > 0$ there exists a set $E \subseteq \Omega$ with $\text{cap}_p(E, \Omega) < \varepsilon$ such that the restriction of f to $\Omega \setminus E$ is continuous. One can verify that f is $\text{cap}_p(\cdot, \Omega)$ -quasi continuous if and only if f is $\text{cap}_p(\cdot, \mathbf{R}^n)$ -quasi continuous (hence, we shall drop the prefix $\text{cap}_p(\cdot, \Omega)$).

It is well known (see, for instance, [27]) that for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ there exists a quasi continuous representative of u , which is uniquely defined quasi everywhere in Ω , and which is given by

$$\lim_{r \rightarrow 0^+} \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y) dy$$

for p -q.e. $x \in \Omega$. Throughout this work we shall use such a quasi continuous representative to individuate the pointwise values of an element of $W^{1,p}(\Omega, \mathbf{R}^m)$. Moreover, we may also assume that the quasi continuous representative we are going to choose is Borel measurable.

It turns out that for every subset E of Ω

$$\text{cap}_p(E, \Omega) = \inf \{ \|u\|_{W^{1,p}(\Omega)}^p : u \in W_0^{1,p}(\Omega), u \geq 1 \text{ q.e. on } E \}.$$

Actually this infimum is attained by a unique function which is called the *capacitary potential* of E . It turns out that this function takes its values in $[0, 1]$.

If F is a multivalued function from a subset E of \mathbf{R}^n to \mathbf{R}^m and x_0 is a point of Ω , we say that F is *lower semicontinuous in x_0* if for every open subset G of \mathbf{R}^m with $G \cap F(x_0) \neq \emptyset$ there exists a neighbourhood U of x_0 such that for every $y \in U \cap E$ we have $G \cap F(y) \neq \emptyset$. It is clear how the notion of *quasi lower semicontinuous* multifunction can be defined.

Finally, let us recall the notion of Hausdorff distance. For $\xi \in \mathbf{R}^m$ and $E \subseteq \mathbf{R}^m$ we set $d(\xi, E) = \inf \{ |\xi - \eta| : \eta \in E \}$. If X and Y are any subsets of \mathbf{R}^m , let us define

$$\rho(X, Y) = \max \left(\sup_{y \in Y} d(y, X), \sup_{x \in X} d(x, Y) \right).$$

The restriction of ρ to the family of all non-empty compact subsets of \mathbf{R}^m is the Hausdorff distance.

2. Lipschitz projections onto convex sets and a density result

In this section we derive some useful results from the theory of Lipschitz parametrization of convex sets by means of projections (see [37] and [33]). A first application is in the proof of the basic density result expressed in Theorem 2.9.

We first recall two results we shall use in the sequel. Let us indicate the convex hull of a subset A of \mathbf{R}^m by $\text{co } A$.

Lemma 2.1. *For every $A, B \subseteq \mathbf{R}^m$ we have $\rho(\text{co } A, \text{co } B) \leq \rho(A, B)$.*

Proof. Given $\xi \in \text{co } A$ we can write $\xi = \alpha^1 \xi_1 + \cdots + \alpha^k \xi_k$, where $\alpha \in \Sigma_k$ and $\xi_i \in A$ for every i . It is easy to see that $d(\xi, \text{co } B) \leq \max_{1 \leq i \leq k} d(\xi_i, \text{co } B)$. Therefore

$$\sup_{\xi \in \text{co } A} d(\xi, \text{co } B) \leq \sup_{\xi \in A} d(\xi, \text{co } B) \leq \sup_{\xi \in A} d(\xi, B).$$

The analogous inequality with the roles of A and B reversed concludes the proof. \square

Lemma 2.2. *Let Ω be an open subset of \mathbf{R}^n , let $u \in W^{1,p}(\Omega, \mathbf{R}^m)$, and let $f: \mathbf{R}^m \rightarrow \mathbf{R}$ be a Lipschitz function with Lipschitz constant L . If $f \circ u \in L^p(\Omega, \mathbf{R})$, then $f \circ u \in W^{1,p}(\Omega, \mathbf{R})$ and*

$$(2.1) \quad |D_i(f \circ u)| \leq L |D_i u| \quad \text{a.e. on } \Omega$$

for $i = 1, \dots, n$.

Proof. The result is classical if f is a C^1 function (see, for instance, [35] Theorem 3.1.9). In the general case it follows by approximating f with a sequence of C^1 functions with Lipschitz constants bounded by L . \square

Throughout this section let k be a fixed positive integer. Let \mathcal{C} be the family of all non-empty compact convex subsets of \mathbf{R}^m and consider the map from $\mathbf{R}^m \times \mathcal{C}$ to \mathbf{R}^m taking (ξ, C) into the orthogonal projection of ξ onto C . It turns out that this map is non-expansive as a function of $\xi \in \mathbf{R}^m$, but as a function of $C \in \mathcal{C}$ it is not Lipschitzian with respect to the Hausdorff distance.

Indeed, given $(\xi, \xi_1, \dots, \xi_k) \in (\mathbf{R}^m)^{k+1}$, let $\Pi(\xi; \xi_1, \dots, \xi_k)$ be the orthogonal projection of ξ onto $\text{co}\{\xi_1, \dots, \xi_k\}$. The following example shows that $\Pi(\xi, \cdot): (\mathbf{R}^m)^k \rightarrow \mathbf{R}^m$ is not locally Lipschitzian.

Let $m = 2$ and $k = 2$. Let us consider the set S of the points $(x, y) \in \mathbf{R}^2$ with $x \geq 0$ and $(x - 1)^2 + y^2 > 1$. If $\xi = (2, 0)$, $\xi_1 = (0, 0)$, $\xi_2 = (x, y)$ it turns out that $\Pi(\xi; \xi_1, \xi_2) = (\frac{2x^2}{x^2+y^2}, \frac{2xy}{x^2+y^2})$ for every $\xi_2 \in S$. As $\frac{\partial}{\partial x}(\frac{2xy}{x^2+y^2})$ is unbounded near $(0, 0)$, the map $\Pi(\xi; \xi_1, \cdot)$ is not lipschitzian on S in any neighbourhood of the origin.

Nevertheless, making use of a selection of a suitable Lipschitz multivalued projection, it is possible to find a Lipschitz function on $(\mathbf{R}^m)^{k+1}$ with properties analogous to those of the projection Π .

This result is based on the following theorem (see [37] and [33]).

Theorem 2.3. *Let \mathcal{C} be the family of all non-empty compact convex subsets of \mathbf{R}^m . Then there exists a map $P: \mathbf{R}^m \times \mathcal{C} \rightarrow \mathbf{R}^m$ satisfying the following properties:*

(i) *there exists a constant $L > 0$ such that*

$$|P(\xi, C) - P(\xi', C')| \leq L(|\xi - \xi'| + \rho(C, C'))$$

for every $\xi, \xi' \in \mathbf{R}^m$ and $C, C' \in \mathcal{C}$;

(ii) *$P(\xi, C) \in C$ for every $\xi \in \mathbf{R}^m$, $C \in \mathcal{C}$, and $P(\xi, C) = \xi$, if $\xi \in C$;*

(iii) *$d(\xi, C) \leq |\xi - P(\xi, C)| \leq \sqrt{3}d(\xi, C)$ for every $\xi \in \mathbf{R}^m$ and $C \in \mathcal{C}$.*

Proof. Let us consider the multivalued map \mathcal{P} from $\mathbf{R}^m \times \mathcal{C}$ to \mathbf{R}^m defined as

$$\mathcal{P}(\xi, C) = C \cap \overline{B(\xi, r)},$$

where $\overline{B(\xi, r)}$ is the closed ball of center ξ and radius $r = \sqrt{3}d(\xi, C)$. By Lemma 1 in [37] (see also [33], Theorem 1), for every (ξ, C) and (ξ', C') ,

$$\rho(\mathcal{P}(\xi, C), \mathcal{P}(\xi', C')) \leq (1 + \sqrt{3})|\xi - \xi'| + 3\rho(C, C').$$

Hence, \mathcal{P} is lipschitzian. Let $\mathcal{S}: \mathcal{C} \rightarrow \mathbf{R}^m$ be the Steiner point selection, which is Lipschitz continuous on all of \mathcal{C} with Lipschitz constant bounded by m (see, e.g., [40]). Then the function $P = \mathcal{S} \circ \mathcal{P}$ satisfies (i), (ii), and (iii). \square

Corollary 2.4. *There exists a Lipschitz function $P_k: (\mathbf{R}^m)^{k+1} \rightarrow \mathbf{R}^m$ satisfying the following properties:*

(i) *$P_k(\xi; \xi_1, \dots, \xi_k) \in \text{co}\{\xi_1, \dots, \xi_k\}$ for every $\xi, \xi_1, \dots, \xi_k \in \mathbf{R}^m$, and $P_k(\xi; \xi_1, \dots, \xi_k) = \xi$, if $\xi \in \text{co}\{\xi_1, \dots, \xi_k\}$;*

(ii) *$d(\xi, \text{co}\{\xi_1, \dots, \xi_k\}) \leq |\xi - P_k(\xi; \xi_1, \dots, \xi_k)| \leq \sqrt{3}d(\xi, \text{co}\{\xi_1, \dots, \xi_k\})$ for every $\xi, \xi_1, \dots, \xi_k \in \mathbf{R}^m$.*

Proof. If P is the function given in Theorem 2.3, we define

$$P_k(\xi; \xi_1, \dots, \xi_k) = P(\xi, \text{co}\{\xi_1, \dots, \xi_k\}).$$

Properties (i) and (ii) follow immediately, while Lemma 2.1 ensures that P_k is lipschitzian. \square

The next result follows immediately from Lemma 2.2.

Corollary 2.5. *Let Ω be an open subset of \mathbf{R}^n , and let u, u_1, \dots, u_k be functions in $W_0^{1,p}(\Omega, \mathbf{R}^m)$. Then, $P_k(u; u_1, \dots, u_k) \in W_0^{1,p}(\Omega, \mathbf{R}^m)$.*

Remark 2.6. We underline that in this corollary we cannot replace P_k with the orthogonal projection Π introduced above. Indeed, if $m = 2$ and $k = 2$, let us consider the functions

$$u(t) = (2, 0), \quad u_1(t) = (0, 0), \quad u_2(t) = \left(\frac{t^2}{2} \sin^2 \frac{1}{t}, t\right)$$

for $t \in \Omega =]0, 1[$. It turns out that $u_2(t) \in S$ for every $t \in \Omega$, where S is the set introduced in the example preceeding Theorem 2.3. Hence

$$\Pi(u(t); u_1(t), u_2(t)) = \left(\frac{2t^2 \sin^4 \frac{1}{t}}{t^2 \sin^4 \frac{1}{t} + 4}, \frac{4t \sin^2 \frac{1}{t}}{t^2 \sin^4 \frac{1}{t} + 4} \right),$$

which does not belong to $H^1(]0, 1[, \mathbf{R}^2)$.

Remark 2.7. Let $u, u_1, \dots, u_k \in W_0^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ and let $C_k(x) = \text{co}\{u_1(x), \dots, u_k(x)\}$. If $u(x) \in C_k(x)$ for a.e. $x \in \Omega$, then $u(x) \in C_k(x)$ for p -q.e. $x \in \Omega$. Indeed, since P_k is lipschitzian, the function $P_k(u; u_1, \dots, u_k)$ is quasi-continuous and by assumption $P_k(u; u_1, \dots, u_k) = u$ a.e. on Ω . By well known properties of Sobolev functions (see [27]) this implies $P_k(u; u_1, \dots, u_k) = u$ p -q.e. on Ω , hence $u(x) \in C_k(x)$ for p -q.e. $x \in \Omega$.

The projection P_k obtained in Corollary 2.4 will now be used for the proof of the following theorem, on which the density result stated in Theorem 2.9 is based.

Let M_k be the set in $(\mathbf{R}^m)^{k+1}$ defined by

$$M_k = \{(\xi, \xi_1, \dots, \xi_k) \in (\mathbf{R}^m)^{k+1} : \xi \in \text{co}\{\xi_1, \dots, \xi_k\}\},$$

and Λ be the multivalued map from M_k into Σ_k given by

$$\Lambda(\xi; \xi_1, \dots, \xi_k) = \{ \lambda \in \Sigma_k : \xi = \lambda^1 \xi_1 + \dots + \lambda^k \xi_k \}.$$

We point out that Λ does not admit a lipschitzian selection, nor even a continuous one. In fact, let $m = 1$, $k = 2$, and consider, for any $t \in]0, 1[$, the sets

$$T_t = \{ (\xi, \xi_1, \xi_2) \in \mathbf{R}^3 : \xi_1 = 0, \xi_2 \in \mathbf{R} \setminus \{0\}, \xi = t\xi_2 \} \subseteq M_2.$$

It is clear that $\Lambda(\xi; \xi_1, \xi_2) = \{(1-t, t)\}$ for every $(\xi; \xi_1, \xi_2) \in T_t$. As $(0, 0, 0)$ is an accumulation point of T_t , every continuous selection of Λ should assume in $(0, 0, 0)$ the value $(1-t, t)$. But, clearly, it is impossible to satisfy this requirement for every $t \in]0, 1[$. Hence there are no continuous selections of Λ .

However, we can find a suitable approximation $f_\varepsilon(\xi; \xi_1, \dots, \xi_k)$ of ξ such that $(\xi, \xi_1, \dots, \xi_k) \mapsto \Lambda(f_\varepsilon(\xi; \xi_1, \dots, \xi_k); \xi_1, \dots, \xi_k)$ has a lipschitzian selection. More precisely, the following result holds.

Theorem 2.8. *For every $\varepsilon > 0$ there exists a function $f_\varepsilon: (\mathbf{R}^m)^{k+1} \rightarrow \mathbf{R}^m$ such that*

- (i) $|f_\varepsilon(\xi; \xi_1, \dots, \xi_k) - \xi| < \varepsilon$ for every $(\xi, \xi_1, \dots, \xi_k) \in (\mathbf{R}^m)^{k+1}$;
- (ii) f_ε is lipschitzian on $(\mathbf{R}^m)^{k+1}$, with Lipschitz constant depending on k and m , but not on ε ;
- (iii) $f_\varepsilon(\xi; \xi_1, \dots, \xi_k) \in \text{co}\{\xi_1, \dots, \xi_k\}$ for every $(\xi, \xi_1, \dots, \xi_k) \in M_k$; moreover, there exists a locally Lipschitz function $\lambda_\varepsilon: M_k \rightarrow \Sigma_k$ (with Lipschitz constant depending also on ε) such that $f_\varepsilon(\xi; \xi_1, \dots, \xi_k) = \sum_{i=1}^k \lambda_\varepsilon^i(\xi; \xi_1, \dots, \xi_k) \xi_i$ for every $(\xi, \xi_1, \dots, \xi_k) \in M_k$.

Proof. The proof will be by induction on k . The case $k = 1$ is trivial. Fixed $k > 1$, let us assume that the theorem is true for $k-1$ and let us prove it for k . Let us fix $\varepsilon > 0$. For every $i = 1, \dots, k$ and $(\xi, \xi_1, \dots, \xi_k) \in (\mathbf{R}^m)^{k+1}$ we define

$$P_{i,k}(\xi; \xi_1, \dots, \xi_k) = P_{k-1}(\xi; \xi_1, \dots, \widehat{\xi_i}, \dots, \xi_k),$$

where $P_{k-1}: (\mathbf{R}^m)^k \rightarrow \mathbf{R}^m$ is the projection given by Corollary 2.4, and the notation $\widehat{\xi_i}$ means that the variable ξ_i is omitted. Let us consider the function $h_{i,\varepsilon}: (\mathbf{R}^m)^{k+1} \rightarrow \mathbf{R}^m$ defined by

$$h_{i,\varepsilon}(\xi; \xi_1, \dots, \xi_k) = g_\varepsilon(P_{i,k}(\xi; \xi_1, \dots, \xi_k); \xi_1, \dots, \widehat{\xi_i}, \dots, \xi_k),$$

where $g_\varepsilon: (\mathbf{R}^m)^k \rightarrow \mathbf{R}^m$ denotes the function f_ε corresponding to the integer $k-1$, given by the inductive assumption. As $P_{i,k}(\xi; \xi_1, \dots, \xi_k)$ belongs to $\text{co}\{\xi_1, \dots, \widehat{\xi}_i, \dots, \xi_k\}$, from the properties of g_ε we obtain that $h_{i,\varepsilon}(\xi; \xi_1, \dots, \xi_k) \in \text{co}\{\xi_1, \dots, \widehat{\xi}_i, \dots, \xi_k\}$ and that $|h_{i,\varepsilon} - P_{i,k}| < \varepsilon$. In particular $h_{i,\varepsilon}(\xi; \xi_1, \dots, \xi_k)$ is a 2ε -approximation of ξ whenever $|P_{i,k}(\xi; \xi_1, \dots, \xi_k) - \xi| < \varepsilon$ (or, by Corollary 2.4(ii), whenever $d(\xi, \text{co}\{\xi_1, \dots, \widehat{\xi}_i, \dots, \xi_k\}) < \varepsilon/\sqrt{3}$).

To obtain a global approximation satisfying the requirements of the theorem, we shall modify the projection $J_k: (\xi, \xi_1, \dots, \xi_k) \mapsto \xi$ by using the functions $h_{i,\varepsilon}$ for $i = 1, \dots, k$, through a suitable partition of unity. To this purpose let us introduce the following level sets in $(\mathbf{R}^m)^{k+1}$:

$$\begin{aligned} U_i &= \{|P_{i,k} - J_k| < 2\varepsilon\} \quad (i = 1, \dots, k), \\ U_0 &= \cap_{i=1}^k \{|P_{i,k} - J_k| > \varepsilon/4\}. \end{aligned}$$

These sets form an open covering of $(\mathbf{R}^m)^{k+1}$. Moreover, we consider the level sets

$$\begin{aligned} V_i &= \{|P_{i,k} - J_k| < \varepsilon\} \subseteq U_i \quad (i = 1, \dots, k), \\ V_0 &= \cap_{i=1}^k \{|P_{i,k} - J_k| > \varepsilon/2\} \subseteq U_0, \end{aligned}$$

which still constitute an open covering of $(\mathbf{R}^m)^{k+1}$. Let $\eta_\varepsilon: [0, +\infty) \rightarrow [0, 1]$ be defined by

$$\eta_\varepsilon(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq \varepsilon, \\ 2 - \frac{t}{\varepsilon}, & \text{if } \varepsilon \leq t \leq 2\varepsilon, \\ 0, & \text{if } t \geq 2\varepsilon. \end{cases}$$

We set now

$$\begin{aligned} \varphi_{i,\varepsilon} &= \eta_\varepsilon \circ |P_{i,k} - J_k| \quad (i = 1, \dots, k), \\ \varphi_{0,\varepsilon} &= \prod_{i=1}^k (1 - \eta_{\varepsilon/4} \circ |P_{i,k} - J_k|). \end{aligned}$$

It is clear that

$$\varphi_{i,\varepsilon} = \begin{cases} 1 & \text{on } V_i, \\ 0 & \text{on } (\mathbf{R}^m)^{k+1} \setminus U_i, \end{cases}$$

for $i = 0, \dots, k$. As η_ε and $P_{i,k}$ are lipschitzian, it turns out that $\varphi_{i,\varepsilon}$ is lipschitzian on $(\mathbf{R}^m)^{k+1}$. Moreover, the Lipschitz constant satisfies the estimate

$$\text{Lip}(\varphi_{i,\varepsilon}) \leq \frac{c}{\varepsilon} \quad i = 0, \dots, k.$$

Here and in the rest of the theorem c stands for a constant depending only on k and m , which can change from a line to another. Now we are in a position to define the desired partition of unity. Since $\varphi = \sum_{i=0}^k \varphi_{i,\varepsilon} \geq 1$ on $(\mathbf{R}^m)^{k+1}$, we can set

$$\psi_{i,\varepsilon} = \frac{\varphi_{i,\varepsilon}}{\varphi} \quad (i = 0, \dots, k).$$

It is easy to see that $(\psi_{i,\varepsilon})_i$ is a partition of unity in $(\mathbf{R}^m)^{k+1}$ and that $\psi_{i,\varepsilon} = 0$ on the complement of U_i . Moreover,

$$\text{Lip}(\psi_{i,\varepsilon}) \leq \frac{c}{\varepsilon}.$$

Let us define

$$f_\varepsilon = \psi_{0,\varepsilon} J_k + \sum_{i=1}^k \psi_{i,\varepsilon} h_{i,\varepsilon} = J_k + \sum_{i=1}^k \psi_{i,\varepsilon} (h_{i,\varepsilon} - J_k).$$

Let us now verify properties (i) through (iii), taking into account the analogous properties of g_ε guaranteed by the inductive assumption. As to (i), note that on U_i , for $i = 1, \dots, k$ we have

$$(2.2) \quad |h_{i,\varepsilon} - J_k| \leq |h_{i,\varepsilon} - P_{i,k}| + |P_{i,k} - J_k| \leq 3\varepsilon.$$

This implies that (i) is satisfied with 3ε instead of ε (hence, actually, what we have defined above is $f_{3\varepsilon}$). As $\psi_{i,\varepsilon} = 0$ on the complement of U_i , by (2.2) we get

$$\begin{aligned} \text{Lip}(\psi_{i,\varepsilon}(h_{i,\varepsilon} - J_k)) &\leq \text{Lip}(\psi_{i,\varepsilon}) \sup_{U_i} |h_{i,\varepsilon} - J_k| + \sup_{U_i} \psi_{i,\varepsilon} \text{Lip}(h_{i,\varepsilon} - J_k) \\ &\leq \frac{c}{\varepsilon} \cdot 3\varepsilon + c = 4c \end{aligned}$$

for every $i = 1, \dots, k$, so (ii) is proved, too. Let us come to (iii); it is clear that, if $(\xi, \xi_1, \dots, \xi_k) \in M_k$, then $f_\varepsilon(\xi; \xi_1, \dots, \xi_k) \in \text{co}\{\xi_1, \dots, \xi_k\}$. Consider now the term $\psi_{0,\varepsilon} J_k$. On U_0 we have $|P_{i,k} - J_k| > \frac{\varepsilon}{4}$ for every $i = 1, \dots, k$. Hence, if $(\xi, \xi_1, \dots, \xi_k) \in M_k \cap U_0$, (ii) in Corollary 2.4 implies

$$(2.3) \quad d(\xi, \text{co}\{\xi_1, \dots, \widehat{\xi}_i, \dots, \xi_k\}) > \frac{\varepsilon}{4\sqrt{3}}, \quad i = 1, \dots, k.$$

As $\xi \in \text{co}\{\xi_1, \dots, \xi_k\}$, the previous inequality shows that

$$(2.4) \quad \text{co}\{\xi_1, \dots, \xi_k\} \neq \bigcup_{i=1}^k \text{co}\{\xi_1, \dots, \widehat{\xi}_i, \dots, \xi_k\}.$$

Let H be the affine subspace generated by ξ_1, \dots, ξ_k . By the Carathéodory Theorem, (2.4) implies that H has dimension $k - 1$, hence ξ_1, \dots, ξ_k are affinely independent. Moreover, by (2.3) the open ball in H with center ξ and radius $\frac{\varepsilon}{4\sqrt{3}}$ is contained in $\text{co}\{\xi_1, \dots, \xi_k\}$. Therefore,

$$|(\xi_1 - \xi_k) \wedge (\xi_2 - \xi_k) \wedge \dots \wedge (\xi_{k-1} - \xi_k)| \geq c\varepsilon^{k-1},$$

where \wedge denotes the exterior product. Thus, $\xi = \sum_{i=1}^k \mu^i(\xi; \xi_1, \dots, \xi_k) \xi_i$, where $\mu^i(\xi; \xi_1, \dots, \xi_k)$ is given by

$$\frac{|(\xi_1 - \xi_k) \wedge \dots \wedge (\xi_{i-1} - \xi_k) \wedge (\xi - \xi_k) \wedge (\xi_{i+1} - \xi_k) \wedge \dots \wedge (\xi_{k-1} - \xi_k)|}{|(\xi_1 - \xi_k) \wedge \dots \wedge (\xi_{k-1} - \xi_k)|}.$$

It follows that μ^i is locally lipschitzian on $M_k \cap U_0$.

By the inductive assumption there exists a locally lipschitzian function $\nu_\varepsilon: M_{k-1} \rightarrow \Sigma_{k-1}$ such that $g_\varepsilon(\xi; \xi_1, \dots, \xi_{k-1}) = \sum_{j=1}^{k-1} \nu_\varepsilon^j(\xi; \xi_1, \dots, \xi_{k-1}) \xi_j$; hence

$$\begin{aligned} h_{i,\varepsilon}(\xi; \xi_1, \dots, \xi_k) &= g_\varepsilon(P_{i,k}(\xi; \xi_1, \dots, \xi_k); \xi_1, \dots, \widehat{\xi_i}, \dots, \xi_k) \\ &= \sum_{\substack{j=1 \\ j \neq i}}^k \nu_{i,\varepsilon}^j(\xi; \xi_1, \dots, \xi_k) \xi_j, \end{aligned}$$

where

$$\nu_{i,\varepsilon}^j(\xi; \xi_1, \dots, \xi_k) = \begin{cases} \nu_\varepsilon^j(P_{i,k}(\xi; \xi_1, \dots, \xi_k); \xi_1, \dots, \widehat{\xi_i}, \dots, \xi_k), & \text{if } 1 \leq j < i, \\ \nu_\varepsilon^{j-1}(P_{i,k}(\xi; \xi_1, \dots, \xi_k); \xi_1, \dots, \widehat{\xi_i}, \dots, \xi_k), & \text{if } i < j \leq k. \end{cases}$$

Finally, for every $(\xi, \xi_1, \dots, \xi_k) \in M_k$ we have

$$f_\varepsilon(\xi; \xi_1, \dots, \xi_k) = \psi_{0,\varepsilon} \sum_{j=1}^k \mu^j \xi_j + \sum_{i=1}^k \psi_{i,\varepsilon} \sum_{\substack{j=1 \\ j \neq i}}^k \nu_{i,\varepsilon}^j \xi_j = \sum_{j=1}^k \lambda_\varepsilon^j \xi_j,$$

where $\lambda_\varepsilon^j = \psi_{0,\varepsilon} \mu^j + \sum_{\substack{i=1 \\ i \neq j}}^k \psi_{i,\varepsilon} \nu_{i,\varepsilon}^j$ are the desired lipschitzian coefficients. \square

The foregoing theorem is the main tool in the proof of the following basic density result.

Theorem 2.9. *Let Ω be a bounded open subset of \mathbf{R}^n . Let u_1, \dots, u_k be functions of $W_0^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ and let $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ with $u(x) \in \text{co}\{u_1(x), \dots, u_k(x)\}$ for a.e. $x \in \Omega$. Then $u \in W_0^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$, and there exists a sequence of functions $\varphi_h: \mathbf{R}^n \rightarrow \Sigma_k$ of class $C^\infty(\mathbf{R}^n, \mathbf{R}^k)$ such that*

$$\sum_{i=1}^k \varphi_h^i u_i \xrightarrow{h} u$$

in the strong topology of $W_0^{1,p}(\Omega, \mathbf{R}^m)$.

Proof. The boundedness of u follows from the boundedness of u_1, \dots, u_k . For every $h \in \mathbf{N}$ define $v_h = g_h(u; u_1, \dots, u_k)$, where g_h is the function f_ε given by Theorem 2.8 for $\varepsilon = 1/h$. Hence, by properties (i) and (ii) of g_h and by Lemma 2.2, we have that $v_h \in W^{1,p}(\Omega, \mathbf{R}^m)$, $|v_h - u| < 1/h$ a.e. in Ω , and $\|Dv_h\|_{L^p(\Omega, \mathbf{R}^{mn})}$ is bounded uniformly

with respect to h . This implies that (v_h) converges to u weakly in $W^{1,p}(\Omega, \mathbf{R}^m)$. By (iii) of Theorem 2.8, for every $h \in \mathbf{N}$ the function v_h can be written in the form

$$v_h = \sum_{i=1}^k \lambda_h^i(u; u_1, \dots, u_k) u_i,$$

where λ_h^i are locally lipschitzian. Thus, since u, u_1, \dots, u_k are bounded and belong to $W^{1,p}(\Omega, \mathbf{R}^m)$, the composite functions $\lambda_h^i(u; u_1, \dots, u_k)$ belong to $W^{1,p}(\Omega, \mathbf{R}^m)$. Therefore, $v_h \in W_0^{1,p}(\Omega, \mathbf{R}^m)$ and u , being the weak limit of (v_h) , belongs to $W_0^{1,p}(\Omega, \mathbf{R}^m)$, too. Defining u, u_1, \dots, u_k as the zero function on $\mathbf{R}^n \setminus \Omega$, the extended functions belong to $W^{1,p}(\mathbf{R}^n, \mathbf{R}^m) \cap L^\infty(\mathbf{R}^n, \mathbf{R}^m)$, thus $\lambda_h^i(u; u_1, \dots, u_k) \in W_{\text{loc}}^{1,p}(\mathbf{R}^n)$. This shows that v_h belongs to the set \mathcal{H} of all the convex combinations of u_1, \dots, u_k with coefficients in $W_{\text{loc}}^{1,p}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. This is a subset of $W_0^{1,p}(\Omega, \mathbf{R}^m)$ by the boundedness of Ω and of u_1, \dots, u_k . Therefore, u is in the weak closure of \mathcal{H} , and, since \mathcal{H} is convex, u belongs to the strong closure of \mathcal{H} , too. Thus, we can find a sequence (w_h) in \mathcal{H} converging to u strongly in $W_0^{1,p}(\Omega, \mathbf{R}^m)$. Let

$$w_h = \sum_{i=1}^k \nu_h^i u_i$$

with $\nu_h: \mathbf{R}^n \rightarrow \Sigma_k$ and $\nu_h \in W_{\text{loc}}^{1,p}(\mathbf{R}^n, \mathbf{R}^k)$.

Denote by (ρ_j) a sequence of non-negative mollifiers, and let $\psi_{h,j} = \nu_h * \rho_j$. Then,

$$\begin{aligned} \psi_{h,j} &\in C^\infty(\mathbf{R}^n, \mathbf{R}^k), & \psi_{h,j}(x) &\in \Sigma_k \text{ for a.e. } x \in \mathbf{R}^n, \\ \psi_{h,j} &\xrightarrow{j} \nu_h & \text{strongly in } W^{1,p}(\Omega, \mathbf{R}^k). \end{aligned}$$

If $w_{h,j} = \sum_{i=1}^k \psi_{h,j}^i u_i$, then for every $h \in \mathbf{N}$

$$w_{h,j} \xrightarrow{j} w_h \quad \text{strongly in } W_0^{1,p}(\Omega, \mathbf{R}^m).$$

By a diagonalization argument we can assert the existence of a sequence (φ_h) which satisfies all the requirements of the theorem. \square

3. Some properties of closed and C^1 -convex subsets of $W_0^{1,p}(\Omega, \mathbf{R}^m)$

In this section Ω is an arbitrary open subset of \mathbf{R}^n .

Definition 3.1. A subset \mathcal{H} of $W^{1,p}(\Omega, \mathbf{R}^m)$ is said to be C^1 -convex if $\alpha u + (1 - \alpha)v \in \mathcal{H}$ whenever $u, v \in \mathcal{H}$ and $\alpha \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$, with $0 \leq \alpha \leq 1$.

Remark 3.2. A closed subset \mathcal{H} of $W^{1,p}(\Omega, \mathbf{R}^m)$ is C^1 -convex if and only if for every finite family $(u_i)_{i \in I}$ of elements of \mathcal{H} and for every family $(\alpha^i)_{i \in I}$ of non-negative functions in $C^1(\Omega) \cap W^{1,\infty}(\Omega)$, such that $\alpha^i \geq 0$ and $\sum_i \alpha^i = 1$, the convex combination $\sum_i \alpha^i u_i$ belongs to \mathcal{H} . Indeed, assume that \mathcal{H} is C^1 -convex. Let $u_1, \dots, u_s \in \mathcal{H}$ and $\alpha^1, \dots, \alpha^s \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$, with $\alpha^i \geq 0$ and $\sum_{i=1}^s \alpha^i = 1$. It would be clear by induction that $\sum_{i=1}^s \alpha^i u_i \in \mathcal{H}$ if we had $\alpha^i > \varepsilon$ for every $i \in I$ and for a suitable $\varepsilon > 0$. Since \mathcal{H} is closed, we can reduce our problem to this case by considering the coefficients $\alpha_\varepsilon^i = (\alpha^i + \varepsilon)/(1 + s\varepsilon)$, and then by taking the limit as ε goes to 0.

Proposition 3.3. Let \mathcal{K} be a convex subset of $W_0^{1,p}(\Omega, \mathbf{R}^m)$. Then there exists a closed valued multifunction K from Ω to \mathbf{R}^m , unique up to sets of p -capacity zero, such that

- (i) for every $u \in \mathcal{K}$ we have $u(x) \in K(x)$ for p -q.e. $x \in \Omega$;
- (ii) if H is a closed valued multifunction from Ω to \mathbf{R}^m such that for every $u \in \mathcal{K}$ we have $u(x) \in H(x)$ for p -q.e. $x \in \Omega$, then $K(x) \subseteq H(x)$ for p -q.e. $x \in \Omega$.

Moreover, K satisfies the following properties:

- (iii) K is quasi lower semicontinuous and $K(x)$ is convex for p -q.e. $x \in \Omega$;
- (iv) if (u_k) is a countable dense subset of \mathcal{K} , then

$$(3.1) \quad K(x) = \text{cl}\{u_k(x) : k \in \mathbf{N}\} = \text{cl}\left(\bigcup_{k=1}^{\infty} C_k(x)\right) \quad \text{for } p\text{-q.e. } x \in \Omega,$$

where $C_k(x) = \text{co}\{u_1(x), \dots, u_k(x)\}$.

Proof. The uniqueness of K follows immediately from properties (i) and (ii). Let (u_k) be a countable dense subset of \mathcal{K} . For every $k \in \mathbf{N}$ we fix a quasi continuous representative of u_k , and for every $x \in \Omega$ we define

$$K(x) = \text{cl}\{u_k(x) : k \in \mathbf{N}\}.$$

Let us prove (i). Given $u \in \mathcal{K}$, there exists a subsequence (v_h) of (u_k) which converges to u in $W_0^{1,p}(\Omega, \mathbf{R}^m)$; hence, a further subsequence converges to u pointwise p -q.e. on Ω . Therefore, $u(x) \in K(x)$ for p -q.e. $x \in \Omega$.

To prove (ii) we observe that $u_k(x) \in H(x)$ for p -q.e. $x \in \Omega$ and for every $k \in \mathbf{N}$. As H is closed valued, this clearly implies $K(x) \subseteq H(x)$ for p -q.e. $x \in \Omega$.

Let us now turn to (iii). Fix $\varepsilon > 0$ and let $A_\varepsilon \subseteq \Omega$ with $\text{cap}_p(A_\varepsilon, \Omega) < \varepsilon$ such that $u_k|_{\Omega \setminus A_\varepsilon}$ is continuous for every $k \in \mathbf{N}$. For any $x_0 \in \Omega \setminus A_\varepsilon$ and for any open set G such that $G \cap K(x_0) \neq \emptyset$ there exists $k_0 \in \mathbf{N}$ with $u_{k_0}(x_0) \in G \cap K(x_0)$. By the continuity of $u_{k_0}|_{\Omega \setminus A_\varepsilon}$ there exists a neighbourhood U of x_0 such that $u_{k_0}(x) \in G \cap K(x)$ for every $x \in U \setminus A_\varepsilon$. Therefore, $K|_{\Omega \setminus A_\varepsilon}$ is lower semicontinuous in x_0 . Let us verify the convexity of $K(x)$ for p -q.e. $x \in \Omega$. By the convexity of \mathcal{K} and property (i) there exists a set $N \subseteq \Omega$ with $\text{cap}_p(N, \Omega) = 0$ such that for every $x \in \Omega \setminus N$, for every $\lambda \in [0, 1] \cap \mathbf{Q}$, and for every $h, k \in \mathbf{N}$, $\lambda u_h(x) + (1 - \lambda)u_k(x) \in K(x)$. It is now enough to note that $\{\lambda u_h(x) + (1 - \lambda)u_k(x) : h, k \in \mathbf{N}, \lambda \in [0, 1] \cap \mathbf{Q}\}$ is dense in $\{\lambda \xi_1 + (1 - \lambda)\xi_2 : \xi_1, \xi_2 \in K(x), \lambda \in [0, 1]\}$. This concludes the proof of (iii).

Finally, (iv) now follows by noticing that for every $k \in \mathbf{N}$ and for p -q.e. $x \in \Omega$, $C_k(x) \subseteq K(x)$ by the convexity of $K(x)$ given in (iii). \square

Corollary 3.4. *Let \mathcal{K} be a closed and convex subset of $W_0^{1,p}(\Omega, \mathbf{R}^m)$. Then, up to sets of p -capacity zero, there exists at most one multifunction K from Ω to \mathbf{R}^m such that*

- (i) $K(x) = \text{cl}\{u_h(x) : h \in \mathbf{N}\}$ for a suitable sequence (u_h) in $W_0^{1,p}(\Omega, \mathbf{R}^m)$,
- (ii) $\mathcal{K} = \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \Omega\}$.

We conclude this section by proving that if \mathcal{K} is a closed and C^1 -convex subset of $W_0^{1,p}(\Omega, \mathbf{R}^m)$, in Proposition 3.3 the sequence (u_k) can be chosen in $\mathcal{K} \cap L^\infty(\Omega, \mathbf{R}^m)$. To this aim we need the following two lemmas.

Lemma 3.5. *Let $T_k : \mathbf{R}^m \rightarrow \mathbf{R}^m$ be the orthogonal projection onto the ball $\overline{B(0, k)}$, i.e.,*

$$T_k(\xi) = \frac{k}{|\xi| \vee k} \xi = \begin{cases} \xi, & \text{if } |\xi| \leq k, \\ k \frac{\xi}{|\xi|}, & \text{if } |\xi| \geq k, \end{cases}$$

where $a \vee b = \max\{a, b\}$. If $u \in W^{1,p}(\Omega, \mathbf{R}^m)$, then $T_k \circ u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ and the sequence $(T_k \circ u)$ converges to u in the strong topology of $W^{1,p}(\Omega, \mathbf{R}^m)$.

Proof. Let us fix $u \in W^{1,p}(\Omega, \mathbf{R}^m)$. As T_k is lipschitzian, the function $T_k \circ u$ belongs to $W^{1,p}(\Omega, \mathbf{R}^m)$ by Lemma 2.2. Clearly, the sequence $(T_k \circ u)$ converges to u in $L^p(\Omega, \mathbf{R}^m)$; moreover, since T_k has Lipschitz constant 1, again by Lemma 2.2 we get

$$\begin{aligned} \int_{\Omega} |D(T_k \circ u) - Du|^p dx &\leq 2^{p-1} \int_{\{|u| \geq k\}} (|D(T_k \circ u)|^p + |Du|^p) dx \\ &\leq 2^p \int_{\{|u| \geq k\}} |Du|^p dx. \end{aligned}$$

As k goes to $+\infty$ we obtain that $(D(T_k \circ u))$ converges to Du in $L^p(\Omega, \mathbf{R}^{mn})$. \square

Lemma 3.6. *Let $u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$ and let ψ be a measurable function from Ω to $[0, 1]$ such that $\psi u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$. Then there exists a sequence (ψ_h) of functions in $C^\infty(\mathbf{R}^n)$, with $0 \leq \psi_h \leq 1$, such that $(\psi_h u)$ converges to ψu weakly in $W_0^{1,p}(\Omega, \mathbf{R}^m)$.*

Proof. Let $\rho \in C_0^\infty(B(0, 1))$ with $\int_{B(0, 1)} \rho(x) dx = 1$ and $\rho \geq 0$. For every $h \in \mathbf{N}$ set $\rho_h(x) = h^n \rho(hx)$ for $x \in \mathbf{R}^n$. Define u and ψ equal to 0 outside Ω . Clearly, $\psi u \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^m)$. For every $h \in \mathbf{N}$ define $\psi_h(x) = (\psi * \rho_h)(x)$. It is well known that $\psi_h \in C^\infty(\mathbf{R}^n)$ and $0 \leq \psi_h \leq 1$. Since (ψ_h) converges to ψ a.e. on \mathbf{R}^n , we have that $(\psi_h u)$ converges to ψu strongly in $L^p(\mathbf{R}^n, \mathbf{R}^m)$. Therefore, in order to prove that the sequence $(\psi_h u)$ converges to ψu weakly in $W_0^{1,p}(\Omega, \mathbf{R}^m)$ it is enough to show that the sequence $(D(\psi_h u))$ is bounded in $L^p(\Omega, \mathbf{R}^{mn})$. This will be obtained by proving that $(u \otimes D\psi_h)$ is bounded in $L^p(\Omega, \mathbf{R}^{mn})$. For $x \in \Omega$ it turns out that

$$\begin{aligned} (u \otimes D\psi_h)(x) &= \int_{\mathbf{R}^n} \psi(x-y)u(x) \otimes D\rho_h(y) dy \\ &= \int_{\mathbf{R}^n} \psi(x-y)u(x) \otimes D\rho_h(y) dy - \int_{\mathbf{R}^n} \psi(x)u(x) \otimes D\rho_h(y) dy, \end{aligned}$$

as $\int_{\mathbf{R}^n} D\rho_h(y) dy = 0$. It follows that

$$\begin{aligned} (3.2) \quad |u(x) \otimes D\psi_h(x)| &\leq \int_{\mathbf{R}^n} |u(x) - u(x-y)| |D\rho_h(y)| dy \\ &\quad + \int_{\mathbf{R}^n} |\psi(x-y)u(x-y) - \psi(x)u(x)| |D\rho_h(y)| dy \end{aligned}$$

for every $x \in \Omega$. Set

$$I = \int_{\Omega} \left(\int_{\mathbf{R}^n} |u(x) - u(x-y)| |D\rho_h(y)| dy \right)^p dx.$$

By the Hölder inequality and the inequality $\int_{\mathbf{R}^n} |D\rho_h(y)| dy \leq ch$, with c independent of h , we get

$$\begin{aligned} I &\leq \int_{\Omega} \left(\int_{\mathbf{R}^n} |u(x) - u(x-y)|^p |D\rho_h(y)| dy \right) \left(\int_{\mathbf{R}^n} |D\rho_h(y)| dy \right)^{p-1} dx \\ &\leq (ch)^{p-1} \int_{\Omega} \left(\int_{\mathbf{R}^n} |u(x) - u(x-y)|^p |D\rho_h(y)| dy \right) dx \\ &= (ch)^{p-1} \int_{\mathbf{R}^n} \left(\int_{\Omega} |u(x) - u(x-y)|^p dx \right) |D\rho_h(y)| dy. \end{aligned}$$

The L^p -estimate for the difference quotients (see, for example, [29] Lemma 7.23) yields

$$\begin{aligned} I &\leq (ch)^{p-1} \int_{\mathbf{R}^n} \left(\int_{\Omega} |Du(x)|^p dx \right) |y|^p |D\rho_h(y)| dy \\ &\leq (ch)^{p-1} h^{-p} \int_{\Omega} |Du(x)|^p dx \int_{\mathbf{R}^n} |D\rho_h(y)| dy \leq c^p \int_{\Omega} |Du(x)|^p dx. \end{aligned}$$

In view of the fact that $\psi u \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^m)$, the same argument gives

$$\int_{\Omega} \left(\int_{\mathbf{R}^n} |\psi(x-y)u(x-y) - \psi(x)u(x)| |D\rho_h(y)| dy \right)^p dx \leq c^p \int_{\Omega} |D(\psi u)|^p dx.$$

By (3.2) the sequence $(\int_{\Omega} |u \otimes D\psi_h|^p dx)$ is bounded in $L^p(\Omega, \mathbf{R}^{mn})$, and the proof is so accomplished. \square

Proposition 3.7. *Let Ω be bounded. Let \mathcal{K} be a closed and C^1 -convex subset of $W_0^{1,p}(\Omega, \mathbf{R}^m)$. If $\mathcal{K} \cap L^\infty(\Omega, \mathbf{R}^m) \neq \emptyset$, then $\mathcal{K} \cap L^\infty(\Omega, \mathbf{R}^m)$ is dense in \mathcal{K} .*

Proof. Since $\mathcal{K} \cap L^\infty(\Omega, \mathbf{R}^m) \neq \emptyset$, it is not restrictive to assume that $0 \in \mathcal{K}$. Given $u \in \mathcal{K}$, consider the truncated functions $T_k \circ u$ of Lemma 3.5. Since the sequence $(T_k \circ u)$ converges to u in $W_0^{1,p}(\Omega, \mathbf{R}^m)$, to prove the corollary it is enough to show that $T_k \circ u \in \mathcal{K}$ for every $k \in \mathbf{N}$. By Lemma 3.6 for any fixed $k \in \mathbf{N}$ there exists a sequence (ψ_h) in $C^\infty(\mathbf{R}^n)$, with $0 \leq \psi_h \leq 1$, such that the sequence $(\psi_h u)$ converges to $T_k \circ u$ weakly in $W_0^{1,p}(\Omega, \mathbf{R}^m)$. Since Ω is bounded, $0 \in \mathcal{K}$, and \mathcal{K} is C^1 -convex, we have that $\psi_h u \in \mathcal{K}$ for every h . It follows that $T_k \circ u \in \mathcal{K}$, as \mathcal{K} is weakly closed. \square

4. The main result

The following theorem is the main result of the paper.

Theorem 4.1. *Let Ω be a bounded open subset of \mathbf{R}^n and let \mathcal{K} be a closed subset of $W_0^{1,p}(\Omega, \mathbf{R}^m)$ with $1 < p < +\infty$. Then, \mathcal{K} is C^1 -convex if and only if there exists a multifunction $K: \Omega \rightarrow \mathbf{R}^m$ with closed convex values such that*

$$(4.1) \quad \mathcal{K} = \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \Omega\}.$$

Moreover, K can be chosen as in Corollary 3.4(i).

To simplify the exposition of the proof, we consider previously the following results.

Lemma 4.2. *Let (w_k) be a sequence of functions in $W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ converging in $L^\infty(\Omega, \mathbf{R}^m)$ to a function $w \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$. Then there exists a sequence (v_k) in $W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that $v_k(x) \in \text{co}\{w_1(x), \dots, w_k(x)\}$*

for a.e. $x \in \Omega$ and (v_k) converges to w strongly in $W^{1,p}(\Omega, \mathbf{R}^m)$. If, in addition, the sequence w_k is contained in $W_0^{1,p}(\Omega, \mathbf{R}^m)$, then the functions v_k belong to $W_0^{1,p}(\Omega, \mathbf{R}^m)$, too.

Proof. Let us argue by induction on the dimension m . We will use the notation $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Let $m = 1$. It is not restrictive to assume $w = 0$. It is easy to see that the sequence

$$v_k(x) = [w_1(x) \wedge \operatorname{ess\,sup}_{\Omega} w_k] \vee \operatorname{ess\,inf}_{\Omega} w_k$$

satisfies our requirements.

Let us now prove the lemma for the dimension $m > 1$ assuming that it holds true for $m - 1$. Again it is not restrictive to assume $w = 0$.

Let $0 < t \leq 1/2$. Clearly, since $tw_1 \rightarrow 0$ as t tends to 0^+ , it is enough to prove the existence of a sequence (v_k) such that

$$(4.2) \quad v_k(x) \in \operatorname{co}\{w_1(x), \dots, w_k(x)\} \quad \text{for a.e. } x \in \Omega,$$

$$(4.3) \quad v_k \rightarrow tw_1 \quad \text{strongly in } W^{1,p}(\Omega, \mathbf{R}^m).$$

As the proof is very technical, we work first under the additional assumption that there exists $\delta > 0$ such that

$$(4.4) \quad |w_1| \geq \delta \quad \text{on } \Omega.$$

Consider the covering of $\mathbf{R}^m \setminus \{0\}$ consisting of the cones A_1, \dots, A_{2m} defined by

$$\begin{aligned} A_i &= \{\xi \in \mathbf{R}^m : \xi^i > |\xi|/m\} \quad (i = 1, \dots, m), \\ A_{m+i} &= \{\xi \in \mathbf{R}^m : \xi^i < -|\xi|/m\} \quad (i = 1, \dots, m), \end{aligned}$$

where ξ^i denotes the i -th coordinate of ξ . In view of the $L^\infty(\Omega)$ -convergence we can suppose that for every $k \geq 2$

$$(4.5) \quad |w_k| \leq \frac{t\delta}{2m}.$$

Under these assumptions, for every $i \in \{1, \dots, 2m\}$ and for every $x \in \Omega$ with $w_1(x) \in A_i$, the hyperplane

$$H_i(x) = \{\xi \in \mathbf{R}^m : \xi^i = tw_1^i(x)\}$$

intersects the line segment between $w_k(x)$ and $w_1(x)$ in a point $z_{i,k}(x)$. The idea is to reduce the problem to the dimension $m - 1$ by considering the sequence $(z_{i,k})$, which converges to tw_1 on the moving hyperplane H_i .

To this aim let us first find for every fixed i , with $1 \leq i \leq m$, a suitable extension $\zeta_{i,k}$ of $z_{i,k}$ to the whole Ω (the case $m < i \leq 2m$ will be treated analogously). It turns out that

$$z_{i,k} = w_1 + \frac{(1-t)w_1^i}{w_1^i - w_k^i}(w_k - w_1)$$

on the set $\{x \in \Omega : w_1(x) \in A_i\}$. Let

$$B_i = \{\xi \in \mathbf{R}^m : \xi^i > |\xi|/2m\} \supset \supset A_i \cap \partial B(0,1),$$

and let $\varphi_i \in C_0^\infty(B_i)$ with $0 \leq \varphi_i \leq 1$ on B_i , $\varphi_i = 1$ on $A_i \cap \partial B(0,1)$, and $\varphi_i = 0$ on $B_i \cap B(0, \frac{1}{2})$ (this latter property will be used later). Since, by (4.4) and (4.5),

$$w_1^i - w_k^i \geq (1-t)\frac{\delta}{2m} \geq \frac{\delta}{4m} \quad \text{on } \{x \in \Omega : \varphi_i\left(\frac{w_1}{|w_1|}\right) \neq 0\} \subseteq \{x \in \Omega : \frac{w_1}{|w_1|} \in B_i\},$$

we can define on the whole Ω the function

$$\gamma_{i,k} = \frac{(1-t)w_1^i}{(w_1^i - w_k^i)} \varphi_i\left(\frac{w_1}{|w_1|}\right) = \frac{(1-t)w_1^i}{(w_1^i - w_k^i) \vee (\delta/4m)} \varphi_i\left(\frac{w_1}{|w_1|}\right).$$

We note that $\gamma_{i,k}$ is a function in $W^{1,p}(\Omega)$. Moreover, it takes its values in $[0,1]$; indeed, by (4.4) and (4.5) we have

$$0 \leq \gamma_{i,k} \leq \frac{(1-t)w_1^i}{(w_1^i - w_k^i)} \leq \frac{w_1^i - \frac{t\delta}{2m}}{w_1^i - \frac{t\delta}{2m}} = 1$$

on $\{x \in \Omega : \varphi_i\left(\frac{w_1}{|w_1|}(x)\right) \neq 0\} \subseteq \{x \in \Omega : \frac{w_1}{|w_1|}(x) \in B_i\}$. Thus the function

$$\zeta_{i,k} = w_1 + \gamma_{i,k}(w_k - w_1)$$

is an extension of $z_{i,k}$, belongs to $W^{1,p}(\Omega, \mathbf{R}^m)$, and satisfies the following properties:

$$(4.6) \quad \zeta_{i,k}(x) \in \text{co}\{w_1(x), \dots, w_k(x)\} \quad \text{for a.e. } x \in \Omega,$$

$$(4.7) \quad \zeta_{i,k} \xrightarrow[k]{} \zeta_i \quad \text{in } L^\infty(\Omega, \mathbf{R}^m),$$

where

$$\zeta_i = \left[1 - (1-t)\varphi_i\left(\frac{w_1}{|w_1|}\right)\right]w_1.$$

We note that for a.e. $x \in \Omega$ all vectors $\zeta_{i,k}(x)$ of the sequence lie on the same hyperplane $\{\xi \in \mathbf{R}^m : \xi^i = \zeta_i^i(x)\}$ ($\zeta_i^i(x)$ denotes the i -th component of $\zeta_i(x)$). Let $\widehat{\zeta}_{i,k}$ and $\widehat{\zeta}_i$ be the functions with values in \mathbf{R}^{m-1} obtained from $\zeta_{i,k}$ and ζ_i by dropping the i -th component. By (4.6) and (4.7) we are able to apply the inductive hypothesis to $(\widehat{\zeta}_{i,k})_k$ and $\widehat{\zeta}_i$. Therefore, there exists a sequence $(\widehat{v}_{i,k})_k$ in $W^{1,p}(\Omega, \mathbf{R}^{m-1}) \cap L^\infty(\Omega, \mathbf{R}^{m-1})$ such that

$$\begin{aligned} \widehat{v}_{i,k}(x) &\in \text{co}\{\widehat{\zeta}_{i,1}(x), \dots, \widehat{\zeta}_{i,k}(x)\} \quad \text{for a.e. } x \in \Omega, \\ \widehat{v}_{i,k} &\xrightarrow[k]{} \widehat{\zeta}_i \quad \text{strongly in } W^{1,p}(\Omega, \mathbf{R}^{m-1}). \end{aligned}$$

Denote by $v_{i,k} \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ the functions obtained by adding ζ_i^i to $\widehat{v}_{i,k}$ as i -th component. It is clear that

$$\begin{aligned} v_{i,k}(x) &\in \text{co}\{\zeta_{i,1}(x), \dots, \zeta_{i,k}(x)\} \subseteq \text{co}\{w_1(x), \dots, w_k(x)\} \quad \text{for a.e. } x \in \Omega, \\ v_{i,k} &\xrightarrow[k]{} \zeta_i \quad \text{strongly in } W^{1,p}(\Omega, \mathbf{R}^m). \end{aligned}$$

For every $i \in \{1, \dots, 2m\}$ let $\psi_i \in C_0^\infty(A_i)$ with $0 \leq \psi_i \leq 1$ and $\sum_i \psi_i = 1$ on $\partial B(0, 1)$. Finally, for every $k \in \mathbf{N}$ define

$$v_k = \sum_{i=1}^{2m} \psi_i \left(\frac{w_1}{|w_1|} \right) v_{i,k}.$$

It turns out that $v_k(x) \in \text{co}\{w_1(x), \dots, w_k(x)\}$ for a.e. $x \in \Omega$ and that (v_k) converges to tw_1 strongly in $W^{1,p}(\Omega, \mathbf{R}^m)$; indeed the limit is

$$\sum_{i=1}^{2m} \psi_i \left(\frac{w_1}{|w_1|} \right) [1 - (1-t)\varphi_i \left(\frac{w_1}{|w_1|} \right)] w_1 = \sum_{i=1}^{2m} \psi_i \left(\frac{w_1}{|w_1|} \right) [1 - (1-t)] w_1 = tw_1.$$

Thus, the sequence (v_k) satisfies (4.2) and (4.3) under the additional assumption that $|w_1| \geq \delta$ on Ω for a suitable $\delta > 0$.

In the general case, to obtain such a sequence (v_k) it is enough to find, for every $\varepsilon > 0$, a sequence $(v_k^\varepsilon)_k$ such that

$$\begin{aligned} v_k^\varepsilon(x) &\in \text{co}\{w_1(x), \dots, w_k(x)\} \quad \text{for a.e. } x \in \Omega, \\ v_k^\varepsilon &\xrightarrow[k]{} z^\varepsilon \quad \text{strongly in } W^{1,p}(\Omega, \mathbf{R}^m), \end{aligned}$$

where $\|z^\varepsilon - tw_1\|_{W^{1,p}(\Omega, \mathbf{R}^m)}$ tends to 0 as $\varepsilon \rightarrow 0$.

Fix $\varepsilon > 0$ and let $\delta = \delta(\varepsilon) > 0$ such that

$$\int_{A_\delta} (|w_1|^p + |Dw_1|^p) dx \leq \varepsilon ,$$

where $A_\delta = \{x \in \Omega : |w_1(x)| \leq 2\delta\}$. Consider now only those indices k which satisfy (4.5), and replace in the preceding argument $\gamma_{i,k}$ with the function

$$\gamma_{i,k} = \frac{(1-t)w_1^i}{(w_1^i - w_k^i) \vee (\delta/4m)} \varphi_i\left(\frac{w_1}{|w_1| \vee 2\delta}\right) = \frac{(1-t)w_1^i}{(w_1^i - w_k^i)} \varphi_i\left(\frac{w_1}{|w_1| \vee 2\delta}\right)$$

(recall that $\varphi_i = 0$ on $B_i \cap B(0, \frac{1}{2})$). Thus, as one can easily check, we obtain a sequence $(v_{i,k})_k$ such that

$$\begin{aligned} v_{i,k}(x) &\in \text{co}\{w_1(x), \dots, w_k(x)\} \quad \text{for a.e. } x \in \Omega, \\ v_{i,k} &\xrightarrow[k]{} \zeta_i \quad \text{strongly in } W^{1,p}(\Omega, \mathbf{R}^m), \end{aligned}$$

where now

$$\zeta_i = [1 - (1-t)\varphi_i\left(\frac{w_1}{|w_1| \vee 2\delta}\right)]w_1.$$

Finally, for every $k \in \mathbf{N}$ define

$$v_k^\varepsilon = (1 - \chi_\delta(w_1)) \sum_{i=1}^{2m} \psi_i\left(\frac{w_1}{|w_1| \vee \delta}\right) v_{i,k} + \chi_\delta(w_1) w_1 ,$$

where (ψ_i) is the same partition of unity employed above and χ_δ is a function in $C_0^\infty(\mathbf{R}^m)$ satisfying the following conditions: $0 \leq \chi_\delta \leq 1$, $\chi_\delta = 1$ on $B(0, \delta)$, $\chi_\delta = 0$ on the complement of $B(0, 2\delta)$, and $|D\chi_\delta| \leq 2/\delta$ on $B(0, 2\delta)$. Thus, by the properties of χ_δ , it is clear that

$$v_k^\varepsilon(x) \in \text{co}\{w_1(x), \dots, w_k(x)\} \quad \text{for a.e. } x \in \Omega.$$

Furthermore, (v_k^ε) converges in the strong topology of $W^{1,p}(\Omega, \mathbf{R}^m)$ to the function

$$z^\varepsilon = (1 - X_\delta) \sum_{i=1}^{2m} \Psi_{i,\delta} (1 - (1-t)\Phi_{i,\delta}) w_1 + X_\delta w_1 ,$$

where $X_\delta = \chi_\delta(w_1)$, $\Psi_{i,\delta} = \psi_i\left(\frac{w_1}{|w_1| \vee \delta}\right)$, and $\Phi_{i,\delta} = \varphi_i\left(\frac{w_1}{|w_1| \vee 2\delta}\right)$. Note that we have $z^\varepsilon = tw_1$ on $\Omega \setminus A_\delta = \{x \in \Omega : |w_1(x)| > 2\delta\}$. Take now into account that on A_δ we

have $|w_1| \leq 2\delta$ and, by Lemma 2.2, $|D(\frac{w_1}{|w_1|\sqrt{\delta}})| \leq \frac{1}{\delta} |Dw_1|$. Therefore we obtain

$$\begin{aligned} \int_{A_\delta} |z^\varepsilon|^p dx &\leq c \int_{A_\delta} |w_1|^p dx \\ \int_{A_\delta} |w_1 \otimes DX_\delta|^p dx &= \int_{A_\delta} |w_1 \otimes [(D\chi_\delta)(w_1) \cdot Dw_1]|^p dx \leq c \int_{A_\delta} |Dw_1|^p dx \\ \int_{A_\delta} |w_1 \otimes D\Psi_{i,\delta}|^p dx &\leq \int_{A_\delta} (|w_1| |D\psi_i(\frac{w_1}{|w_1|\sqrt{\delta}})| \frac{|Dw_1|}{\delta})^p dx \leq c \int_{A_\delta} |Dw_1|^p dx \\ \int_{A_\delta} |w_1 \otimes D\Phi_{i,\delta}|^p dx &\leq c \int_{A_\delta} |Dw_1|^p dx, \end{aligned}$$

where c denotes a constant, independent of ε and δ , which can change from line to line. Hence

$$\int_{A_\delta} (|z^\varepsilon|^p + |Dz^\varepsilon|^p) dx \leq c \int_{A_\delta} (|w_1|^p + |Dw_1|^p) dx.$$

Therefore

$$\begin{aligned} \|z^\varepsilon - tw_1\|_{W^{1,p}(\Omega, \mathbf{R}^m)}^p &= \int_{A_\delta} |z^\varepsilon - tw_1|^p dx + \int_{A_\delta} |D(z^\varepsilon - tw_1)|^p dx \\ &\leq c \int_{A_\delta} (|w_1|^p + |Dw_1|^p) dx \leq c\varepsilon. \end{aligned}$$

This concludes the proof of the lemma in the case $W^{1,p}(\Omega, \mathbf{R}^m)$. The case $W_0^{1,p}(\Omega, \mathbf{R}^m)$ can be obtained from the case $W^{1,p}(\Omega, \mathbf{R}^m)$ by applying Theorem 2.9. \square

To apply Lemma 4.2 we shall need the following Dini-type lemma.

Lemma 4.3. *Let E be a compact subset of \mathbf{R}^n and let (H_k) be a sequence of lower semicontinuous multifunctions from E to \mathbf{R}^m with closed values. Assume that (H_k) is increasing with respect to inclusion, i.e., $H_k(x) \subseteq H_{k+1}(x)$ for every k and x . Let $u \in C^0(E, \mathbf{R}^m)$ such that*

$$u(x) \in \text{cl}\left(\bigcup_{k=1}^{\infty} H_k(x)\right) \quad \text{for every } x \in E.$$

Then, for every $r > 0$ there exists $h \in \mathbf{N}$ such that $B(u(x), r) \cap H_k(x) \neq \emptyset$ for every $k \geq h$ and for every $x \in E$.

Proof. It is not restrictive to assume $u(x) = 0$ for every $x \in E$. Fix $r > 0$. For any $x \in E$ there exists $k = k(x)$ such that $B(0, r) \cap H_k(x) \neq \emptyset$. By the lower semicontinuity

of H_k there exists a neighbourhood $U(x)$ of x such that $B(0, r) \cap H_k(y) \neq \emptyset$ for every $y \in U(x) \cap E$. Since E is compact there exists a finite number of points $x_1, \dots, x_s \in E$ such that

$$E \subseteq \bigcup_{i=1}^s U(x_i).$$

Defining $h = \max_{1 \leq i \leq s} k(x_i)$, we have that $B(0, r) \cap H_k(x) \neq \emptyset$ for every $x \in E$ and for every $k \geq h$, since $H_j(x) \subseteq H_{j+1}(x)$ for every $j \in \mathbb{N}$. \square

Proof of Theorem 4.1. Suppose $\mathcal{K} \neq \emptyset$, otherwise there is nothing to prove. It is not difficult to verify that it is not restrictive to assume $0 \in \mathcal{K}$. Clearly, if there exists a closed and convex valued multifunction K from Ω to \mathbf{R}^m such that (4.1) holds, then \mathcal{K} is C^1 -convex.

Assume now that $\mathcal{K} \subseteq W_0^{1,p}(\Omega, \mathbf{R}^m)$ is closed and C^1 -convex. Let K be the closed and convex valued multifunction given by Proposition 3.3 (i.e., the least closed valued multifunction containing the functions of \mathcal{K} among its selections). By (3.1) and by Proposition 3.7 we have

$$K(x) = \text{cl}\{u_k(x) : k \in \mathbb{N}\} = \text{cl}\left(\bigcup_{k=1}^{\infty} C_k(x)\right) \quad \text{for } p\text{-q.e. } x \in \Omega,$$

where (u_k) is a sequence in $\mathcal{K} \cap L^\infty(\Omega, \mathbf{R}^m)$ dense in \mathcal{K} and $C_k(x) = \text{co}\{u_1(x), \dots, u_k(x)\}$. By the definition of $K(x)$ we have

$$(4.8) \quad \mathcal{K} \subseteq \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \Omega\}.$$

Let us prove now the opposite inclusion. First we prove that for every $k \in \mathbb{N}$

$$(4.9) \quad \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in C_k(x) \text{ for } p\text{-q.e. } x \in \Omega\} \subseteq \mathcal{K}.$$

Let $u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$ such that $u(x) \in C_k(x)$ for p -q.e. $x \in \Omega$. Then, by the Density Theorem 2.9 there exists a sequence (φ_h) in $C^\infty(\mathbf{R}^n, \mathbf{R}^k)$, with $\varphi_h(x) \in \Sigma_k$ for every $x \in \Omega$, such that

$$\sum_{i=1}^k \varphi_h^i u_i \xrightarrow{h} u \quad \text{strongly in } W_0^{1,p}(\Omega, \mathbf{R}^m).$$

As \mathcal{K} is C^1 -convex and closed, $u \in \mathcal{K}$. Thus, (4.9) holds.

Fix now $u \in W_0^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that $u(x) \in K(x)$ for p -q.e. $x \in \Omega$. Clearly, u and all the functions u_k can be considered as defined with value 0 outside Ω . Let $\varepsilon > 0$ be fixed. Then, there exists an open set $A_\varepsilon \subseteq \mathbf{R}^n$, with $\text{cap}_p(A_\varepsilon, \mathbf{R}^n) < \varepsilon$ such that all the functions $u_k|_{\mathbf{R}^n \setminus A_\varepsilon}$, $u|_{\mathbf{R}^n \setminus A_\varepsilon}$ are continuous. In particular, the multifunction C_k is continuous on $\mathbf{R}^n \setminus A_\varepsilon$ with respect to the Hausdorff metric (recall Lemma 2.1). By applying Lemma 4.3, for every $h \in \mathbf{N}$ there exists $k_h \in \mathbf{N}$ such that

$$B(u(x), 1/\sqrt{3}h) \cap C_{k_h}(x) \neq \emptyset \quad \forall x \in \bar{\Omega} \setminus A_\varepsilon.$$

Define

$$z_h = P_{k_h}(u; u_1, \dots, u_{k_h}) \in W_0^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m),$$

where P_{k_h} is the projection defined in Corollary 2.4. From the same corollary it follows that

$$(4.10) \quad z_h(x) \in B(u(x), 1/h) \cap C_{k_h}(x) \quad \forall x \in \Omega \setminus A_\varepsilon.$$

Let ζ_ε be the capacitary potential of the set A_ε , i.e., the solution of the minimum problem

$$\min\{\|\zeta\|_{W^{1,p}(\mathbf{R}^n)}^p : \zeta \in W^{1,p}(\mathbf{R}^n), \zeta = 1 \text{ } p\text{-q.e. on } A_\varepsilon\}.$$

It is easy to prove, by a truncation argument, that $0 \leq \zeta_\varepsilon \leq 1$ p -q.e. in \mathbf{R}^n . Let us define

$$\begin{aligned} w_h^\varepsilon &= \zeta_\varepsilon u_1 + (1 - \zeta_\varepsilon) z_h && \text{on } \Omega, \\ w^\varepsilon &= \zeta_\varepsilon u_1 + (1 - \zeta_\varepsilon) u && \text{on } \Omega. \end{aligned}$$

Thus, w_h^ε and w^ε belong to $W_0^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$, and by (4.10) the sequence $(w_h^\varepsilon)_h$ converges to w^ε in $L^\infty(\Omega, \mathbf{R}^m)$. By Lemma 4.2 there exists a sequence (v_h) in $W_0^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that

$$(4.11) \quad \begin{aligned} v_h(x) &\in \text{co}\{w_1^\varepsilon(x), \dots, w_h^\varepsilon(x)\} && \text{for a.e. } x \in \Omega, \\ v_h &\rightarrow w^\varepsilon && \text{strongly in } W_0^{1,p}(\Omega, \mathbf{R}^m). \end{aligned}$$

Since $\text{co}\{w_1^\varepsilon(x), \dots, w_h^\varepsilon(x)\} \subseteq \text{co}\{u_1(x), \dots, u_{k_h}(x)\}$ for a.e. $x \in \Omega$, by (4.9) the function v_h belongs to \mathcal{K} for every $h \in \mathbf{N}$. As \mathcal{K} is closed, (4.11) implies that $w^\varepsilon \in \mathcal{K}$. Finally, $u \in \mathcal{K}$, since (w^ε) converges to u strongly in $W_0^{1,p}(\Omega, \mathbf{R}^m)$ for $\varepsilon \rightarrow 0$.

Thus, we have proved that

$$(4.12) \quad \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \Omega\} \subseteq \mathcal{K}.$$

To conclude the proof of the theorem, let us consider $u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$ with $u(x) \in K(x)$ for p -q.e. $x \in \Omega$. Since $0 \in \mathcal{K}$, by (4.8) we have $0 \in K(x)$ for p -q.e. $x \in \Omega$. Hence, $T_k(u(x)) = \frac{k}{|u(x)| \vee k} u(x) \in K(x)$ for p -q.e. $x \in \Omega$ and for every $k \in \mathbf{N}$. By Lemma 3.5

$$\begin{aligned} T_k \circ u &\in W_0^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m), \\ T_k \circ u &\rightarrow u \quad \text{strongly in } W_0^{1,p}(\Omega, \mathbf{R}^m). \end{aligned}$$

By (4.12) we have that $T_k \circ u \in \mathcal{K}$ for every $k \in \mathbf{N}$; therefore $u \in \mathcal{K}$. \square

5. Extension of the main result and an application to variational problems

First we consider the extension of Theorem 4.1 to the case of arbitrary open subsets of \mathbf{R}^n .

Theorem 5.1. *Let Ω be an open subset of \mathbf{R}^n (not necessarily bounded). Let \mathcal{K} be a closed subset of $W_0^{1,p}(\Omega, \mathbf{R}^m)$ with $1 < p < +\infty$. Then, \mathcal{K} is C^1 -convex if and only if there exists a multifunction K from Ω to \mathbf{R}^m with closed convex values such that*

$$(5.1) \quad \mathcal{K} = \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \Omega\}.$$

Moreover, K can be chosen as in Corollary 3.4(i).

Proof. Clearly the sets \mathcal{K} of the form (5.1) are C^1 -convex. Conversely, let \mathcal{K} be C^1 -convex. To prove that (5.1) holds for a suitable multifunction K , it is enough to consider the case $\Omega = \mathbf{R}^n$. Indeed, for any given open subset Ω of \mathbf{R}^n , \mathcal{K} can be naturally considered as a subset of $W^{1,p}(\mathbf{R}^n, \mathbf{R}^m)$; therefore, once the case $\Omega = \mathbf{R}^n$ is proven,

$$\begin{aligned} \mathcal{K} &= \{u \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \mathbf{R}^n\} \\ &= \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \mathbf{R}^n\}, \end{aligned}$$

where in the last equality we have taken into account that $\mathcal{K} \subseteq W_0^{1,p}(\Omega, \mathbf{R}^m)$. Now, (5.1) follows by noticing that $0 \in K(x)$ for p -q.e. $x \in \mathbf{R}^n \setminus \Omega$.

Assume now $\Omega = \mathbf{R}^n$ and $\mathcal{K} \neq \emptyset$; we can suppose that $0 \in \mathcal{K}$. For every $h \in \mathbf{N}$ define $B_h = B(0, h)$ and

$$\mathcal{K}_h = \{u \in W_0^{1,p}(B_{h+1}, \mathbf{R}^m) : \exists v \in \mathcal{K} \quad v = u \text{ a.e. on } B_h\}.$$

We can apply Theorem 4.1 to \mathcal{K}_h obtaining

$$(5.2) \quad \mathcal{K}_h = \{u \in W_0^{1,p}(B_{h+1}, \mathbf{R}^m) : u(x) \in K_h(x) \text{ for } p\text{-q.e. } x \in B_{h+1}\},$$

where $K_h(x)$ is closed and convex, and satisfies condition (i) of Corollary 3.4. We claim that for every $h \in \mathbf{N}$

$$(5.3) \quad K_h(x) = K_{h+1}(x) \quad \text{for } p\text{-q.e. } x \in B_h.$$

Let φ_h be a function of $C_0^\infty(B_{h+1})$, with $\varphi_h = 1$ on B_h , $0 \leq \varphi_h \leq 1$, and $|D\varphi_h| \leq 2$. Let us prove that $K_h(x) \subseteq K_{h+1}(x)$ for p -q.e. $x \in B_h$. Fix $u \in \mathcal{K}_h$ and let $v \in \mathcal{K}$ such that $v = u$ on B_h ; then $\varphi_{h+1}v \in \mathcal{K}_{h+1}$, so that

$$u(x) = \varphi_{h+1}(x)v(x) \in K_{h+1}(x) \quad \text{for } p\text{-q.e. } x \in B_h.$$

Since u is arbitrary in \mathcal{K}_h , by applying Proposition 3.3 (ii) with $K(x) = K_h(x)$ and

$$H(x) = \begin{cases} K_{h+1}(x), & \text{if } x \in B_h, \\ \mathbf{R}^m, & \text{if } x \in B_{h+1} \setminus B_h, \end{cases}$$

we get $K_h(x) \subseteq K_{h+1}(x)$ for p -q.e. $x \in B_h$.

To prove the opposite inclusion, we argue analogously applying again Proposition 3.3 (ii), now with $K(x) = K_{h+1}(x)$ and

$$H(x) = \begin{cases} K_h(x), & \text{if } x \in B_h, \\ \mathbf{R}^m, & \text{if } x \in B_{h+2} \setminus B_h. \end{cases}$$

Let us define $K(x)$ for every $x \in \mathbf{R}^n$ by setting $K(x) = K_h(x)$ if x belongs to $B_h \setminus B_{h-1}$. By (5.3), for every $h \in \mathbf{N}$ we have

$$K(x) = K_h(x) \quad \text{for } p\text{-q.e. } x \in B_h.$$

Let us prove now that (5.1) holds. Given $u \in \mathcal{K}$, for every $h \in \mathbf{N}$ we have that $\varphi_h u \in \mathcal{K}_h$, hence $u(x) \in K_h(x) = K(x)$ for p -q.e. $x \in B_h$. It follows that $u(x) \in K(x)$ for p -q.e. $x \in \mathbf{R}^n$.

On the other hand, consider $u \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^m)$ with $u(x) \in K(x)$ for p -q.e. $x \in \mathbf{R}^n$. Since $0 \in \mathcal{K}$, we have $0 \in K_h(x)$ for p -q.e. $x \in B_{h+1}$ and for every $h \in \mathbf{N}$. Moreover, $K(x) = K_{h+2}(x)$ for p -q.e. $x \in B_{h+2}$. Then, $\varphi_{h+1}(x)u(x) \in K_{h+2}(x)$ for p -q.e. $x \in B_{h+3}$; by (5.2) it turns out that $\varphi_{h+1}u \in \mathcal{K}_{h+2}$. Let $v \in \mathcal{K}$ be such that $v = \varphi_{h+1}u$ on B_{h+2} . Thus, $\varphi_h u = \varphi_h v \in \mathcal{K}$, being $\varphi_h v$ a convex combination of v and 0 . Finally, the convergence of the sequence $(\varphi_h u)$ to u in $W^{1,p}(\mathbf{R}^n, \mathbf{R}^m)$ yields that $u \in \mathcal{K}$. \square

In the following theorem we drop the Dirichlet boundary condition imposed to the functions of the set \mathcal{K} ; correspondingly we have to require suitable regularity assumptions to the boundary $\partial\Omega$ of Ω . We recall that, if $u \in W^{1,p}(\Omega, \mathbf{R}^m)$, then

$$(5.4) \quad \lim_{r \rightarrow 0^+} \frac{1}{|B(x, r)|} \int_{B(x, r)} |u(x) - u(y)| dy = 0$$

for p -q.e. $x \in \Omega$ (see, e.g., [27]). If $v, w \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^m)$ are two extensions of u , and x is a point of $\partial\Omega$ where v and w satisfy (5.4), then the condition

$$\liminf_{r \rightarrow 0^+} \frac{|B(x, r) \cap \Omega|}{|B(x, r)|} > 0$$

guarantees that $v(x) = w(x)$. Hence, if, for instance, $\partial\Omega$ is bounded and lipschitzian, then the pointwise values of u on $\partial\Omega$ are uniquely determined, up to sets of p -capacity zero, as the values of the quasi continuous representative of any $W^{1,p}$ extension of u .

Theorem 5.2. *Let Ω be an open subset of \mathbf{R}^n whose boundary is bounded and lipschitzian. Let \mathcal{K} be a closed subset of $W^{1,p}(\Omega, \mathbf{R}^m)$. Then \mathcal{K} is C^1 -convex if and only if there exists a closed and convex valued multifunction K from $\bar{\Omega}$ to \mathbf{R}^m such that*

$$(5.5) \quad \mathcal{K} = \{u \in W^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \bar{\Omega}\}.$$

Moreover, K can be chosen of the form $K(x) = \text{cl}\{u_h(x) : h \in \mathbf{N}\}$, where (u_h) is a sequence in $W^{1,p}(\Omega, \mathbf{R}^m)$.

Proof. Clearly, it is enough to prove that every closed and C^1 -convex subset \mathcal{K} of $W^{1,p}(\Omega, \mathbf{R}^m)$ is of the form (5.5). Define

$$\widehat{\mathcal{K}} = \{u \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^m) : u|_{\Omega} \in \mathcal{K}\}.$$

By Theorem 5.1 there exists a closed and convex valued multifunction K on \mathbf{R}^n such that

$$\widehat{\mathcal{K}} = \{u \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \mathbf{R}^n\}.$$

It is easy to see that $K(x) = \mathbf{R}^m$ for p -q.e. $x \in \mathbf{R}^n \setminus \bar{\Omega}$. Let us verify that (5.5) holds for this K . Let $u \in \mathcal{K}$, and let $v \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^m)$ be an extension of u . Then, $v \in \widehat{\mathcal{K}}$, so that $v(x) \in K(x)$ for p -q.e. $x \in \mathbf{R}^n$, hence $u(x) \in K(x)$ for p -q.e. $x \in \bar{\Omega}$.

Conversely, let $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ such that $u(x) \in K(x)$ for p -q.e. $x \in \bar{\Omega}$. Consider an extension $v \in W^{1,p}(\mathbf{R}^n, \mathbf{R}^m)$ of u . Since $K(x) = \mathbf{R}^m$ for p -q.e. $x \in \mathbf{R}^n \setminus \bar{\Omega}$, we have $v(x) \in K(x)$ for p -q.e. $x \in \mathbf{R}^n$. Therefore, $v \in \widehat{\mathcal{K}}$ and we can conclude that $u = v|_{\Omega} \in \mathcal{K}$. \square

We conclude this section by applying our representation result (Theorem 5.1) to prove the closedness, under the convergence in the sense of Mosco, of the class of the subsets of $W_0^{1,p}(\Omega, \mathbf{R}^m)$ of the form (5.1).

We recall that, if Ω is an open subset of \mathbf{R}^n and \mathcal{K}_h is a sequence of subsets of $W_0^{1,p}(\Omega, \mathbf{R}^m)$, the *strong lower limit* $s\text{-}\liminf_{h \rightarrow \infty} \mathcal{K}_h$ of the sequence (\mathcal{K}_h) is the set of all $u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$ with the following property: there exist $k \in \mathbf{N}$ and a sequence (u_h) converging to u strongly in $W_0^{1,p}(\Omega, \mathbf{R}^m)$ such that $u_h \in \mathcal{K}_h$ for every $h \geq k$. Moreover, the *weak upper limit* $w\text{-}\limsup_{h \rightarrow \infty} \mathcal{K}_h$ of the sequence (\mathcal{K}_h) is the set of all $u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$ with the following property: there exist a sequence (u_k) converging to u weakly in $W_0^{1,p}(\Omega, \mathbf{R}^m)$ and a subsequence (\mathcal{K}_{h_k}) of (\mathcal{K}_h) such that $u_k \in \mathcal{K}_{h_k}$ for every $k \in \mathbf{N}$.

If $s\text{-}\liminf_{h \rightarrow \infty} \mathcal{K}_h = w\text{-}\limsup_{h \rightarrow \infty} \mathcal{K}_h = \mathcal{K}$ we say that the sequence \mathcal{K}_h converges to \mathcal{K} in the sense of Mosco (see [36]).

Theorem 5.3. *Let \mathcal{K}_h be a sequence of subsets of $W_0^{1,p}(\Omega, \mathbf{R}^m)$ such that*

$$\mathcal{K}_h = \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K_h(x) \text{ for } p\text{-q.e. } x \in \Omega\},$$

where K_h are closed and convex valued multifunctions from Ω to \mathbf{R}^m . Then there exists a closed and convex valued multifunction K from Ω to \mathbf{R}^m such that

$$s\text{-}\liminf_{h \rightarrow \infty} \mathcal{K}_h = \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \Omega\}.$$

In particular, if \mathcal{K}_h converges to \mathcal{K} in the sense of Mosco, then \mathcal{K} can be written in the form (5.1).

Proof. By Theorem 5.1, it is enough to prove that $s\text{-}\liminf_{h \rightarrow \infty} \mathcal{K}_h$ is a closed and C^1 -convex subset of $W_0^{1,p}(\Omega, \mathbf{R}^m)$. Whereas the closedness is a general property of the strong lower limit, the C^1 -convexity comes from the fact that the same property holds for the sets \mathcal{K}_h . \square

The following corollary is an immediate consequence of the previous theorem and of Theorem B in Section 2 of [36].

Corollary 5.4. *Let \mathcal{K}_h and K_h be as in Theorem 5.3 and let $\Phi: W_0^{1,p}(\Omega, \mathbf{R}^m) \rightarrow \mathbf{R}$ be a continuous convex functional satisfying the inequality*

$$\Phi(u) \geq a \|u\|_{W_0^{1,p}(\Omega, \mathbf{R}^m)}^p - b \quad \forall u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$$

for suitable constants $a > 0$ and $b \geq 0$. Assume that K_h converges to a set K in the sense of Mosco. Then there exists a closed and convex valued multifunction K from Ω to \mathbf{R}^m (independent of Φ) such that the minimum values

$$\min\{\Phi(u) : u \in W_0^{1,p}(\Omega, \mathbf{R}^m), u(x) \in K_h(x) \text{ for } p\text{-q.e. } x \in \Omega\}$$

converge, as $h \rightarrow \infty$, to the minimum value

$$\min\{\Phi(u) : u \in W_0^{1,p}(\Omega, \mathbf{R}^m), u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \Omega\}.$$

For other applications of the convergence in the sense of Mosco we refer to [36] and [5].

Chapter II

INTEGRAL REPRESENTATION FOR A CLASS OF C^1 -CONVEX FUNCTIONALS

Introduction

This chapter contains an integral representation theorem for a class of convex local functionals which arise in the study of the asymptotic behaviour of a sequence of minimum problems with obstacles for vector valued Sobolev functions.

Given an open subset Ω of \mathbf{R}^n and $1 < p < +\infty$, let $W^{1,p}(\Omega, \mathbf{R}^m)$ be the usual space of Sobolev functions with values in \mathbf{R}^m , and let $\mathcal{A}(\Omega)$ be the family of all open subsets of Ω . The functionals $G: W^{1,p}(\Omega, \mathbf{R}^m) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ we are going to consider are assumed to satisfy the following properties:

- (i) (lower semicontinuity) for every $A \in \mathcal{A}(\Omega)$ the function $G(\cdot, A)$ is lower semicontinuous on $W^{1,p}(\Omega, \mathbf{R}^m)$;
- (ii) (measure property) for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ the set function $G(u, \cdot)$ is (the trace of) a Borel measure on Ω ;
- (iii) (locality property) $G(u, A) = G(v, A)$ whenever $u, v \in W^{1,p}(\Omega, \mathbf{R}^m)$, $A \in \mathcal{A}(\Omega)$, and $u|_A = v|_A$;
- (iv) (C^1 -convexity) for every $A \in \mathcal{A}(\Omega)$ the function $G(\cdot, A)$ is convex on $W^{1,p}(\Omega, \mathbf{R}^m)$ and, in addition, $G(\varphi u + (1 - \varphi)v, A) \leq G(u, A) + G(v, A)$ for every $u, v \in W^{1,p}(\Omega, \mathbf{R}^m)$ and for every $\varphi \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$ with $0 \leq \varphi \leq 1$ on Ω .

This set of conditions is motivated by the study of the limit behaviour, as $h \rightarrow \infty$, of a sequence of convex obstacle problems of the form

$$(0.1) \quad \min \left\{ \int_{\Omega} W(x, Du(x)) dx : u \in H_0^1(\Omega, \mathbf{R}^m), u(x) \in K_h(x) \text{ for } p\text{-q.e. } x \in A \right\},$$

where $W(x, \xi)$ is quadratic with respect to ξ , A is an open subset of Ω with $A \subset\subset \Omega$, and $K_h(x)$ is a closed convex subset of \mathbf{R}^m for every $h \in \mathbf{N}$ and for every $x \in \Omega$. By using Γ -convergence techniques it is possible to prove (see Chapter III) that the limit problem can always be written in the form

$$\min\left\{\int_{\Omega} W(x, Du(x)) dx + G(u, A) : u \in H_0^1(\Omega, \mathbf{R}^m)\right\},$$

with G satisfying the conditions considered above.

In this chapter we are concerned only with the properties of G that can be deduced from (i)–(iv). The main result (Theorem 6.5) is that every functional G satisfying (i)–(iv) can be written in the form

$$(0.2) \quad G(u, A) = \int_A f(x, u(x)) d\mu + \nu(A),$$

where μ and ν are positive Borel measures and $f: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ is a Borel function, convex and lower semicontinuous in the second variable.

This result will be used in the next chapter to provide a detailed description of the structure of the limits of sequences of obstacle problems of the form (0.1) under various assumptions on $W(x, \xi)$ and $K_h(x)$.

Conditions (i)–(iii) are not enough to obtain an integral representation of the form (0.2). Indeed, even convex functionals depending on the gradient, like $G(u, A) = \int_A |Du|^p dx$, satisfy (i)–(iii). Condition (iv) is the most important one, and is responsible for an integral representation of the form (0.2), i.e., without terms depending on the gradient. This notion of convexity, also used, e.g., in [38] and [24], is strictly related to the notion of C^1 -stability introduced by G. Bouchitté and M. Valadier in [10], whose results are frequently used here.

For a general survey on integral representation theorems in L^p , $W^{1,p}$, and BV we refer to [12]. See also [1], [2], [9], [4], [28], [3] for more recent results.

In the scalar case (i.e., $m = 1$), integral representations on $W^{1,p}(\Omega)$ of the form (0.2), connected with limits of obstacle problems, can be found in [25], [22], [8], [18], [24] under suitable convexity conditions, and in [19] under monotonicity assumptions.

Although the final statement in the vector case is exactly the same as in the scalar case, the proof is completely different, since all arguments used in the papers mentioned above rely on the order structure of \mathbf{R} , involving truncations and monotonicity methods.

The main tools for the proof in the vector case are some technical results obtained in the previous chapter, based on the theory of Lipschitz parametrization of convex sets

developed in [33] and [37]. In particular we shall use the following result (Theorem 2.9 in Chapter I): given a finite number of functions u_1, \dots, u_k in $W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$, their convex combinations with smooth coefficients form a dense subset in the set of all $W^{1,p}$ -selections of the polyhedral multivalued function $x \mapsto \text{co}\{u_1(x), \dots, u_k(x)\}$, where co denotes the convex hull.

The first step (Theorem 3.7) of our result deals with the integral representation of the functional G on the set of all $W^{1,p}$ -selections of such polyhedral multifunctions.

In Theorem 5.4 we extend the integral representation of G to all the functions of $W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ which satisfy, up to sets of p -capacity zero, a suitable “obstacle condition” of the form $u(x) \in K(x)$, which is necessary (but not sufficient) for the finiteness of the functional. We note, incidentally, that the main difficulty in the proof of our result lies in the fact that the functional G is not assumed to be finite everywhere, in view of the applications to obstacle problems.

The restriction to $L^\infty(\Omega, \mathbf{R}^m)$, originated by the need of taking products of $W^{1,p}$ -functions, is dropped in Section 6. Moreover, the “obstacle condition”, given up to sets of p -capacity zero, is shown to be equivalent to the condition $u(x) \in K(x)$ almost everywhere with respect to a suitable measure (Proposition 6.3). This allows us to obtain the integral representation (0.2) for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and for every $A \in \mathcal{A}(\Omega)$.

In the last section (Theorem 7.3) we prove that, if G is quadratic or positively p -homogeneous, then so is f .

1. Notation and preliminaries

Throughout this chapter m, n are two fixed positive integers, p is a fixed real number, $1 < p < +\infty$, and Ω is an open subset of \mathbf{R}^n , possibly unbounded. We shall denote by $\mathcal{A}(\Omega)$ the family of the open subsets of Ω and by $\mathcal{B}(\Omega)$ the family of its Borel subsets.

If d is a positive integer, for every $x \in \mathbf{R}^d$ and $r > 0$ we set $B_r(x) = \{y \in \mathbf{R}^d : |y - x| < r\}$, while $\overline{B}_r(x)$ denotes the closure of $B_r(x)$. We recall that the $(d - 1)$ -dimensional simplex Σ_d is defined by

$$\Sigma_d = \{\lambda \in \mathbf{R}^d : \lambda^1 + \dots + \lambda^d = 1, \lambda^i \geq 0\},$$

where $\lambda = (\lambda^1, \dots, \lambda^d)$. If C is a convex subset of \mathbf{R}^d , we denote by $\text{ri } C$ its relative interior and by ∂C its relative boundary. In particular, $\text{ri } \Sigma_d = \{\lambda \in \mathbf{R}^d : \lambda^1 + \dots + \lambda^d = 1, \lambda^i > 0\}$ and $\partial \Sigma_d = \Sigma_d \setminus \text{ri } \Sigma_d$.

A subset A of Ω is said to be p -quasi open (resp. p -quasi closed) if for every $\varepsilon > 0$ there exists an open set A_ε with $\text{cap}_p(A_\varepsilon, \Omega) < \varepsilon$ such that $A \cup A_\varepsilon$ (resp. $A \setminus A_\varepsilon$) is an open set (resp. closed set).

A positive Borel measure μ on Ω is said to be *absolutely continuous* with respect to the p -capacity if $\mu(B) = 0$ whenever $B \in \mathcal{B}(\Omega)$ and $\text{cap}_p(B, \Omega) = 0$.

If E is a subset of Ω and $F: E \rightarrow \mathbf{R}^m$ is a multivalued function, i.e., F maps E into the set of all subsets of \mathbf{R}^m , then we say that F is *lower semicontinuous at a point* x_0 of E if for every open subset G of \mathbf{R}^m with $G \cap F(x_0) \neq \emptyset$ there exists a neighborhood U of x_0 such that for every $y \in U$ we have $G \cap F(y) \neq \emptyset$. We say that F is *upper semicontinuous in* x_0 if for every neighborhood G of $F(x_0)$ there exists a neighborhood U of x_0 such that $F(y) \subseteq G$ whenever $y \in U$. We say that F is *quasi lower semicontinuous* (resp. *quasi upper semicontinuous*) on Ω if for every $\varepsilon > 0$ there exists a set $E \subseteq \Omega$ with $\text{cap}_p(E, \Omega) < \varepsilon$ such that the restriction of F to $\Omega \setminus E$ is lower semicontinuous (resp. upper semicontinuous).

Measurability. Let (X, \mathcal{M}) be a measurable space. If μ is a positive measure on (X, \mathcal{M}) we denote by \mathcal{M}_μ the standard μ -completion of \mathcal{M} and we still denote by μ the completed measure. If μ is σ -finite the \mathcal{M}_μ -measurability is equivalent to the μ -measurability in the Carathéodory sense. Moreover, $\widehat{\mathcal{M}}$ will denote the universal completion of \mathcal{M} , i.e., the intersection $\bigcap_\mu \mathcal{M}_\mu$ for all positive finite measures μ ; equivalently, the intersection can be extended to all positive σ -finite measures μ (see [14] Ch.III, parag. 4).

It is easy to verify that every quasi continuous function $u: \Omega \rightarrow \mathbf{R}$ is μ -measurable (i.e., \mathcal{B}_μ -measurable) for every positive Borel measure μ which is absolutely continuous with respect to the p -capacity.

For convenience, we state here two results which will play an important role to prove the measurability of certain functions. They can be deduced from [14], Theorem III.23 and Theorem III.22, respectively.

Theorem 1.1. (Projection Theorem) *Let (X, \mathcal{M}) be a measurable space. If G is an element of $\mathcal{M} \otimes \mathcal{B}(\mathbf{R}^d)$, then the projection $\text{pr}_X(G)$ belongs to $\widehat{\mathcal{M}}$.*

Theorem 1.2. (Aumann-von Neumann Selection Theorem) *Let X be a topological space and let F be a multivalued function from X to \mathbf{R}^d . If the graph of F belongs to $\mathcal{B}(X) \otimes \mathcal{B}(\mathbf{R}^d)$, then there exists a $\widehat{\mathcal{B}}(X)$ -measurable function which is a selection of F on the set $\{x \in X : F(x) \neq \emptyset\}$.*

2. A class of C^1 -convex functionals: preliminary properties

Let us first introduce our class \mathcal{G}_p of C^1 -convex local functionals.

Definition 2.1. Let \mathcal{G}_p be the class of all functionals $G: W^{1,p}(\Omega, \mathbf{R}^m) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ satisfying the following properties:

- (i) (lower semicontinuity) for every $A \in \mathcal{A}(\Omega)$ the function $G(\cdot, A)$ is lower semicontinuous on $W^{1,p}(\Omega, \mathbf{R}^m)$;
- (ii) (measure property) for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ the set function $G(u, \cdot)$ is the trace of a Borel measure on Ω ;
- (iii) (locality property) $G(u, A) = G(v, A)$ whenever $u, v \in W^{1,p}(\Omega, \mathbf{R}^m)$, $A \in \mathcal{A}(\Omega)$, and $u|_A = v|_A$;
- (iv) (C^1 -convexity) for every $A \in \mathcal{A}(\Omega)$ the function $G(\cdot, A)$ is convex on $W^{1,p}(\Omega, \mathbf{R}^m)$ and, in addition, $G(\varphi u + (1 - \varphi)v, A) \leq G(u, A) + G(v, A)$ for every $u, v \in W^{1,p}(\Omega, \mathbf{R}^m)$ and for every $\varphi \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$, with $0 \leq \varphi \leq 1$ on Ω .

Example 2.2. Let $K: \Omega \rightarrow \mathbf{R}^m$ be any multifunction with closed convex values and let $G: W^{1,p}(\Omega, \mathbf{R}^m) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ be the “obstacle functional” defined by

$$G(u, A) = \begin{cases} 0, & \text{if } u(x) \in K(x) \text{ for } p\text{-q.e. } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then G satisfies all conditions of Definition 2.1, hence $G \in \mathcal{G}_p$. As mentioned in the introduction, it will be proved in the next chapter that all functionals which arise in the study of limits of obstacle problems of the form (0.1) still belong to the class \mathcal{G}_p .

Let μ and ν be two positive Borel measures on Ω and let $f: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ be a Borel function such that $f(x, \cdot)$ is convex and lower semicontinuous on \mathbf{R}^m for μ -a.e. $x \in \Omega$. If μ is absolutely continuous with respect to the p -capacity, then the functional

$$G(u, A) = \int_A f(x, u(x)) d\mu + \nu(A)$$

belongs to the class \mathcal{G}_p . In both examples the lower semicontinuity follows easily from well known properties of the quasi continuous representatives of Sobolev functions (see, for instance, [43], Lemma 2.6.4).

Remark 2.3. Given a functional G of the class \mathcal{G}_p , let us consider the following extension to $W^{1,p}(\Omega, \mathbf{R}^m) \times \mathcal{B}(\Omega)$:

$$(2.1) \quad G(u, B) = \inf\{G(u, A) : A \in \mathcal{A}(\Omega), B \subseteq A\}.$$

It turns out (see [26], Theorem 5.6) that condition (ii) of Definition 2.1 is equivalent to the assumption that, for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$, the extension (2.1) of $G(u, \cdot)$ is a Borel measure on Ω .

In the sequel, when dealing with Borel sets, we shall always consider the extension of G given by (2.1).

From property (iii) in Definition 2.1 and from (2.1) it follows that $G(u, B) = G(v, B)$ for every $B \in \mathcal{B}(\Omega)$ and for every u, v in $W^{1,p}(\Omega, \mathbf{R}^m)$ which coincide in a neighborhood of B .

Note that, if G is the obstacle functional of Example 2.2, then, in general, its extension given by (2.1) does not satisfy

$$G(u, B) = \begin{cases} 0, & \text{if } u(x) \in K(x) \text{ for } p\text{-q.e. } x \in B, \\ +\infty, & \text{otherwise,} \end{cases}$$

for every $B \in \mathcal{B}(\Omega)$. For instance, if $n = m = 1$ and $\Omega =]-2, 2[$, let us consider the obstacle functional $G: W^{1,p}(\Omega, \mathbf{R}^m) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ defined by

$$G(u, A) = \begin{cases} 0, & \text{if } u(x) \geq x^2 \text{ for } p\text{-q.e. } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the extension (2.1) gives $G(1, [0, 1]) = +\infty$, although $1 \geq x^2$ for every $x \in [0, 1]$.

Remark 2.4. If G is a functional of \mathcal{G}_p and $A \in \mathcal{A}(\Omega)$, then for every finite family $(u_i)_{i \in I}$ of elements of $W^{1,p}(\Omega, \mathbf{R}^m)$ and for every family $(\varphi^i)_{i \in I}$ of non-negative functions in $C^1(\Omega) \cap W^{1,\infty}(\Omega)$ such that $\sum_i \varphi^i = 1$ in Ω , we have $G(\sum_i \varphi^i u_i, A) \leq \sum_i G(u_i, A)$. Indeed, let $u_1, \dots, u_r \in W^{1,p}(\Omega, \mathbf{R}^m)$, $\varphi^1, \dots, \varphi^r \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$, with $\varphi^i \geq 0$ and $\sum_{i=1}^r \varphi^i = 1$. It would be clear, by induction, that $G(\sum_{i=1}^r \varphi^i u_i, A) \leq \sum_{i=1}^r G(u_i, A)$ if we had $\varphi^i > \varepsilon$ for every $i = 1, \dots, r$ and for a suitable $\varepsilon > 0$. Since G is lower semicontinuous, we can reduce our problem to this case by considering the coefficients $\varphi_\varepsilon^i = (\varphi^i + \varepsilon)/(1 + r\varepsilon)$.

We also notice that, by using the definition (2.1) of G on Borel sets, property (iv) holds for $A \in \mathcal{B}(\Omega)$, too.

Given $G \in \mathcal{G}_p$, we now generalize to Borel sets the locality property (iii) for G (Proposition 2.6). As a consequence we can single out that part of the functional G which is absolutely continuous with respect to the p -capacity (Proposition 2.8).

For the proof of the locality property on Borel sets we need the following remark, which, for future convenience, we state in a slightly more general form than actually needed here.

Lemma 2.5. *Let $s > 0$ and $T_s: \mathbf{R}^m \rightarrow \mathbf{R}^m$ be the orthogonal projection onto the ball $\overline{B}_s(0)$, i.e.,*

$$T_s(\xi) = \frac{s}{|\xi| \vee s} \xi = \begin{cases} \xi, & \text{if } |\xi| \leq s, \\ s \frac{\xi}{|\xi|}, & \text{if } |\xi| \geq s. \end{cases}$$

If $u \in W^{1,p}(\Omega, \mathbf{R}^m)$, then $T_s \circ u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ and the sequence $(T_s \circ u)$ converges, in the strong topology of $W^{1,p}(\Omega, \mathbf{R}^m)$, to 0 as s goes to 0^+ and to u as s goes to $+\infty$.

Proof. Since T_s is lipschitzian, for every fixed $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ we have $T_s \circ u \in W^{1,p}(\Omega, \mathbf{R}^m)$ by Lemma 2.2 of Chapter I. We prove only the convergence as s tends to 0^+ , the other part being analogous. Since $(T_s \circ u)$ converges to 0 strongly in $L^p(\Omega, \mathbf{R}^m)$, there is only to verify the same kind of convergence for $(D(T_s \circ u))$. Let $\sigma \geq s$; since orthogonal projections have Lipschitz constant 1, the pointwise estimate in Lemma 2.2 of Chapter I yields

$$\begin{aligned} \int_{\Omega} |D(T_s \circ u)|^p dx &= \int_{\{|u| \leq \sigma\}} |D(T_s \circ u)|^p dx + \int_{\{|u| > \sigma\}} |D(\frac{s}{\sigma}(T_\sigma \circ u))|^p dx \\ &\leq \int_{\{|u| \leq \sigma\}} |Du|^p dx + (\frac{s}{\sigma})^p \int_{\Omega} |Du|^p dx. \end{aligned}$$

The conclusion now follows taking first the limit as s tends to 0^+ and then the limit as σ tends to 0^+ . \square

Proposition 2.6. (Locality Property on Borel Sets) *Let $u, v \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ and $B \in \mathcal{B}(\Omega)$. If $u = v$ p -q.e. on B and $G(u, B), G(v, B) < +\infty$, then $G(u, B) = G(v, B)$.*

Proof. Step 1. Assume B is quasi open. For every $h \in \mathbf{N}$, let A_h be an open set with $\text{cap}_p(A_h, \Omega) < 1/h$ and such that $B_h = B \cup A_h$ is open. Let w_h be the capacity

potential of A_h and $u_h = u + w_h(v - u)$. It turns out that $u_h = v$ p -q.e. on B_h . Moreover, (u_h) converges to u in $W^{1,p}(\Omega, \mathbf{R}^m)$ since (w_h) converges to 0 in $W^{1,p}(\Omega)$ and $0 \leq w_h \leq 1$ for every $h \in \mathbf{N}$.

Since by assumption $G(u, B) < +\infty$ and $G(v, B) < +\infty$, it follows that for every given $\varepsilon > 0$ there exist an open set A and a compact set K with $K \subseteq B \subseteq A$ such that $G(u, A \setminus K) < \varepsilon$ and $G(v, A \setminus K) < \varepsilon$.

By the lower semicontinuity of G on open sets we get

$$(2.2) \quad G(u, B) \leq G(u, A) \leq \liminf_{h \rightarrow \infty} G(u_h, A) \leq \liminf_{h \rightarrow \infty} [G(u_h, A \cap B_h) + G(u_h, A \setminus K)].$$

From the locality property of G on open sets it follows that

$$(2.3) \quad G(u_h, A \cap B_h) = G(v, A \cap B_h) \leq G(v, A) < G(v, B) + \varepsilon.$$

By approximating w_h in $W^{1,p}(\Omega)$ with a sequence of equibounded functions of $\mathcal{C}_0^1(\Omega)$, the semicontinuity and C^1 -convexity of G (properties (i) and (iv)) imply that

$$(2.4) \quad G(u_h, A \setminus K) \leq G(u, A \setminus K) + G(v, A \setminus K).$$

Hence, $G(u_h, A \setminus K) < 2\varepsilon$, and, by (2.2) and (2.3), $G(u, B) \leq G(v, B) + 3\varepsilon$. Since ε is arbitrary, we can conclude that $G(u, B) \leq G(v, B)$. Interchanging the roles of u and v , we obtain the opposite inequality. This proves the theorem when B is quasi open.

Step 2. Let now B be a Borel subset of Ω . For every $h \in \mathbf{N}$ let us define $B_h = \{x \in \Omega : |u(x) - v(x)| < 1/h\}$ and

$$u_h = u + \frac{1/h}{|v - u| \vee (1/h)} (v - u) = u + T_{1/h} \circ (v - u).$$

Clearly, $u_h = v$ p -q.e. on B_h . By Lemma 2.5 we have the convergence of u_h to u in $W^{1,p}(\Omega, \mathbf{R}^m)$. At this point we can introduce the sets A and K as in Step 1 and proceed in the same way replacing the locality property of G on the open sets with the locality property on the quasi open sets proved in Step 1. We have only to remark about the estimate (2.4). Let us notice that it is enough to consider $B \subset\subset \Omega$, hence we can choose $A \subset\subset \Omega$; for every $\Omega' \subset\subset \Omega$ the coefficient in the convex combination between u and v defining u_h can be approximated in $W^{1,p}(\Omega')$ by an equibounded sequence of functions of $C^1(\overline{\Omega'})$. As G is C^1 -convex and local on open sets, this suffices to get (2.4) as above. \square

Let us consider the function $\nu_0: \mathcal{B}(\Omega) \rightarrow [0, +\infty]$ defined by

$$(2.5) \quad \nu_0(B) = \inf\{G(v, B) : v \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)\}$$

for every $B \in \mathcal{B}(\Omega)$. Moreover, for every $A \in \mathcal{A}(\Omega)$ we define

$$\text{dom}G(\cdot, A) = \{u \in W^{1,p}(\Omega, \mathbf{R}^m) : G(u, A) < +\infty\}.$$

Then the following proposition holds.

Proposition 2.7. *For every $\Omega' \in \mathcal{A}(\Omega)$ with $\text{dom}G(\cdot, \Omega') \cap L^\infty(\Omega, \mathbf{R}^m) \neq \emptyset$, the restriction of ν_0 to $\mathcal{B}(\Omega')$ is a positive finite Borel measure.*

Proof. It is clear that ν_0 is an increasing function, $\nu_0(\emptyset) = 0$ and, in view of the definition of $G(u, \cdot)$ on $\mathcal{B}(\Omega)$, that $\nu_0(B) = \inf\{\nu_0(A) : A \in \mathcal{A}(\Omega'), A \supseteq B\}$. Hence, by Proposition 5.5 and Theorem 5.6 in [26], we have only to prove that ν_0 is superadditive, subadditive, and inner regular on $\mathcal{A}(\Omega')$. The superadditivity comes immediately from the definition of ν_0 and from the additivity of G in the second variable. Let us now prove that for every $A_1, A_2, A'_2 \in \mathcal{A}(\Omega')$ with $A'_2 \subset\subset A_2$ we have

$$(2.6) \quad \nu_0(A_1 \cup A'_2) \leq \nu_0(A_1) + \nu_0(A_2).$$

We can assume that $\nu_0(A_1) + \nu_0(A_2) < +\infty$. Then, for every $\varepsilon > 0$ there exist two functions u_1, u_2 in $W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that

$$\nu_0(A_1) + \frac{\varepsilon}{2} > G(u_1, A_1) \quad \nu_0(A_2) + \frac{\varepsilon}{2} > G(u_2, A_2).$$

Let $\varphi \in C_0^1(A_2)$, with $\varphi = 1$ on a neighborhood of $\overline{A'_2}$ and $0 \leq \varphi \leq 1$. We set $u = (1 - \varphi)u_1 + \varphi u_2$. By Remark 2.3 it follows that

$$\begin{aligned} \nu_0(A_1 \cup A'_2) &\leq G(u, A_1 \cup A'_2) \\ &\leq G(u_1, A_1 \setminus A_2) + G(u_2, \overline{A'_2}) + G((1 - \varphi)u_1 + \varphi u_2, (A_2 \setminus \overline{A'_2}) \cap A_1). \end{aligned}$$

Furthermore, the C^1 -convexity of G permits to estimate the last term in the above inequality by $G(u_1, (A_2 \setminus \overline{A'_2}) \cap A_1) + G(u_2, (A_2 \setminus \overline{A'_2}) \cap A_1)$. Hence,

$$\nu_0(A_1 \cup A'_2) \leq G(u_1, A_1) + G(u_2, A_2) < \nu_0(A_1) + \nu_0(A_2) + \varepsilon.$$

Thus we obtain (2.6). This inequality will give the subadditivity of ν_0 once inner regularity will be proved.

Since $\text{dom}G(\cdot, \Omega') \cap L^\infty(\Omega, \mathbf{R}^m) \neq \emptyset$, we can find $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that $G(u, \Omega') < +\infty$. Therefore, given $A \in \mathcal{A}(\Omega')$ and $\varepsilon > 0$ there exists $A'' \in \mathcal{A}(\Omega')$ with $A'' \subset\subset A$ and $G(u, A \setminus \overline{A''}) \leq \varepsilon$; it follows that $\nu_0(A \setminus \overline{A''}) \leq \varepsilon$. Let $A' \in \mathcal{A}(\Omega')$ such that $A'' \subset\subset A' \subset\subset A$. By (2.6) we have

$$\nu_0(A) \leq \nu_0(A') + \nu_0(A \setminus \overline{A''}) \leq \nu_0(A') + \varepsilon.$$

We conclude that $\nu_0(A) = \sup\{\nu_0(A') : A' \in \mathcal{A}(\Omega'), A' \subset\subset A\}$, i.e., the inner regularity of ν_0 . \square

Proposition 2.8. *For every $A \in \mathcal{A}(\Omega)$ and $u \in \text{dom}G(\cdot, A) \cap L^\infty(\Omega, \mathbf{R}^m)$, the function $G(u, \cdot) - \nu_0(\cdot)$ is a positive Borel measure on A which is absolutely continuous with respect to the p -capacity.*

Proof. Let $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that $G(u, A) < +\infty$. As ν_0 is a finite Borel measure on A (Proposition 2.7) and $\nu_0(\cdot) \leq G(u, \cdot)$ by (2.5), we conclude that $G(u, \cdot) - \nu_0(\cdot)$ is a positive Borel measure on A . Let us fix $B \in \mathcal{B}(A)$ with $\text{cap}_p(B, \Omega) = 0$. For every $v \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ with $G(v, B) < +\infty$, we have $v = u$ p -q.e. on B ; hence, by Proposition 2.6 we conclude that $G(v, B) = G(u, B)$. Since

$$\nu_0(B) = \inf\{G(v, B) : v \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m), G(v, B) < +\infty\}$$

it follows that $\nu_0(B) = G(u, B)$, i.e., $G(u, B) - \nu_0(B) = 0$. \square

We now conclude this section by giving a basic estimate for G on the convex hull of a finite number of functions in $W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$.

Proposition 2.9. *Let $u, u_1, \dots, u_k \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$. Assume that $u(x) \in \text{co}\{u_1(x), \dots, u_k(x)\}$ for a.e. $x \in \Omega$. Then*

$$G(u, B) \leq \sum_{i=1}^k G(u_i, B)$$

for every Borel set B in Ω .

Proof. In view of the definition of G on Borel sets, it is enough to prove the inequality for every open set B with $B \subset\subset \Omega$. Hence, let B be such a set. By means of the Density

Theorem 2.9 in Chapter I, we can easily find a sequence of functions $\varphi_h: \mathbf{R}^n \rightarrow \Sigma_k$ such that $\varphi_h \in C^\infty(\mathbf{R}^n, \mathbf{R}^k)$ and

$$\sum_{i=1}^k \varphi_h^i u_i \xrightarrow{h} u$$

strongly in $W^{1,p}(A, \mathbf{R}^m)$, where A is a neighborhood of \overline{B} . Then, from the lower semicontinuity of G and the locality property on open sets we obtain

$$G(u, B) \leq \liminf_{h \rightarrow \infty} G\left(\sum_{i=1}^k \varphi_h^i u_i, B\right).$$

The conclusion follows now from the C^1 -convexity of G and from Remark 2.4. \square

3. Integral representation on moving polytopes

The aim of this section is the integral representation of the functionals in \mathcal{G}_p when restricted to the pointwise convex hull of a finite number of functions in $W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ (see Theorem 3.7).

Let $G \in \mathcal{G}_p$, $k \in \mathbf{N}$, and $u_1, \dots, u_k \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ be fixed. Throughout this section we assume that $G(u_i, \Omega) < +\infty$ for every $i = 1, \dots, k$. We point out that our proof first produces a kind of integral representation of G on the functions of the form $u = \sum_{i=1}^k \psi^i u_i$, where $\psi: \Omega \rightarrow \Sigma_k$, with the integrand depending on the coefficient ψ rather than on the function u itself (Theorem 3.3). For a constant ψ , this result is contained in the following lemma.

Let ν_0 be the set function introduced in (2.5). Under the present assumptions, Proposition 2.7 tells us that ν_0 is a finite Borel measure on Ω . Let μ be a positive finite Borel measure on Ω with $\text{supp } \mu = \Omega$ such that μ is absolutely continuous with respect to the p -capacity and $\mu(\cdot) \geq \sum_{i=1}^k (G(u_i, \cdot) - \nu_0(\cdot))$ (in view of Proposition 2.8, such a μ can be obtained, for example, by adding to $\sum_{i=1}^k (G(u_i, \cdot) - \nu_0(\cdot))$ the positive measure $f d\mathcal{L}^n$, where $f \in L^1(\Omega)$, $f > 0$ on Ω , and \mathcal{L}^n is the n -dimensional Lebesgue measure).

Lemma 3.1. For every $x \in \Omega$ and $\lambda \in \Sigma_k$ we define $u_\lambda(x) = \sum_{i=1}^k \lambda^i u_i(x)$ and

$$(3.1) \quad g(x, \lambda) = \limsup_{r \rightarrow 0^+} \frac{G(u_\lambda, B_r(x)) - \nu_0(B_r(x))}{\mu(B_r(x))}.$$

Then

- (i) for every $x \in \Omega$ the function $g(x, \cdot)$ is convex and continuous in Σ_k ;
- (ii) for every $\lambda \in \Sigma_k$ the function $g(\cdot, \lambda)$ is Borel measurable on Ω ;
- (iii) $G(u_\lambda, B) = \int_B g(x, \lambda) d\mu + \nu_0(B)$ for every $\lambda \in \text{ri}\Sigma_k$ and $B \in \mathcal{B}(\Omega)$.

Proof. From the definition of μ and ν_0 and from the convexity of G , it follows immediately that $0 \leq g \leq 1$ on $\Omega \times \text{ri}\Sigma_k$, and that $g(x, \cdot)$ is convex on $\text{ri}\Sigma_k$ for every $x \in \Omega$. Then, by Theorem 10.3 in [39], for every $x \in \Omega$ the function $g(x, \cdot)$ can be extended in one and only one way to a continuous convex function, still denoted by g , on the whole of Σ_k . Hence, $0 \leq g \leq 1$ on $\Omega \times \Sigma_k$ and (i) holds.

Let us proof (ii). If α is a positive Borel measure on Ω , the function $r \mapsto \alpha(B_r(x))$ is left continuous for every $x \in \Omega$. This implies that the upper limit which appears in (3.1) can equivalently be taken as $r \rightarrow 0^+$ with $r \in \mathbf{Q}$. Moreover, the function $x \mapsto \alpha(B_r(x))$ is lower semicontinuous for every $r > 0$ and hence Borel measurable, too. It follows that the function $g(\cdot, \lambda)$ is Borel measurable for every $\lambda \in \text{ri}\Sigma_k$. For $\lambda \in \partial\Sigma_k$, $g(\cdot, \lambda)$ is the pointwise limit of a sequence $g(\cdot, \lambda_n)$ with $\lambda_n \in \text{ri}\Sigma_k$; therefore, $g(\cdot, \lambda)$ is Borel measurable on Ω for every $\lambda \in \Sigma_k$.

By the Besicovitch differentiation theorem (see, e.g., [43], Section 1.3), we have

$$G(u_\lambda, B) = \int_B g(x, \lambda) d\mu + \nu_0(B)$$

for every $B \in \mathcal{B}(\Omega)$ and for every $\lambda \in \text{ri}\Sigma_k$. □

Before extending the previous result to non constant λ 's, we observe that the following selection lemma holds.

Lemma 3.2. Let $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that $u(x) \in \text{co}\{u_1(x), \dots, u_k(x)\}$ for a.e. $x \in \Omega$. Then, there exists a $\widehat{\mathcal{B}}(\Omega)$ -measurable function $\psi: \Omega \rightarrow \Sigma_k$ such that

$$(3.2) \quad u(x) = \sum_{i=1}^k \psi^i(x) u_i(x) \quad \text{for } p\text{-q.e. } x \in \Omega.$$

Proof. Let us fix some quasi continuous Borel measurable representatives of u, u_1, \dots, u_k (see Section 1). Let $\Lambda(x) = \{\lambda \in \Sigma_k : u(x) = \sum_{i=1}^k \lambda^i u_i(x)\}$ for every $x \in \Omega$. Λ is a multivalued function from Ω to Σ_k with non-empty closed values for p -q.e. $x \in \Omega$ (see Remark 2.7 of Chapter I). It is clear that $\text{graph } \Lambda \in \mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbf{R}^k)$.

By the Aumann-von Neumann Selection Theorem 1.2 there exists a $\widehat{\mathcal{B}}(\Omega)$ -measurable selection ψ of the multivalued function Λ and, by the definition of $\Lambda(x)$, the function ψ satisfies (3.2). \square

Theorem 3.3. *Let $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ with $u(x) \in \text{co}\{u_1(x), \dots, u_k(x)\}$ for a.e. $x \in \Omega$. If g is the function given by Lemma 3.1 and $\psi: \Omega \rightarrow \Sigma_k$ is a $\widehat{\mathcal{B}}(\Omega)$ -measurable function such that (3.2) holds, then $g(\cdot, \psi(\cdot))$ is μ -measurable and*

$$(3.3) \quad G(u, A) = \int_A g(x, \psi(x)) d\mu + \nu_0(A)$$

for every $A \in \mathcal{A}(\Omega)$.

Let us explicitly notice that if u is as above, then $G(u, \Omega) < +\infty$ by Proposition 2.9. The proof of Theorem 3.3 heavily relies on the following approximation lemma, which essentially reduces the problem to the case of a constant ψ .

Lemma 3.4. *Let u and ψ be as in Theorem 3.3. Let $\lambda \in \text{ri}\Sigma_k$ with $d(\lambda, \partial\Sigma_k) = \eta > 0$, let $0 < \varepsilon < \eta$ and $B \in \mathcal{B}(\Omega)$ such that $|\psi(x) - \lambda| \leq \varepsilon$ for p -q.e. $x \in B$. Then,*

$$|G(u, B) - G(u_\lambda, B)| \leq \frac{\varepsilon}{\eta} \sum_{i=1}^k G(u_i, B).$$

Proof. Let us define $v = u + t(u_\lambda - u)$ on Ω , with $t = 1 + \eta/\varepsilon$. It turns out that $v(x) \in \text{co}\{u_1(x), \dots, u_k(x)\}$ for p -q.e. $x \in B$ and

$$u_\lambda = \frac{1}{t}v + (1 - \frac{1}{t})u \quad \text{on } \Omega.$$

In order to get from v a function which belongs to $\text{co}\{u_1, \dots, u_k\}$ a.e. on Ω , we consider the projection $w = P_k(v; u_1, \dots, u_k)$ as defined in Corollary 2.4 of Chapter I. Set

$$z = \frac{1}{t}w + (1 - \frac{1}{t})u \quad \text{on } \Omega.$$

By Lemma 2.2 of Chapter I, the function w , and hence z , belongs to $W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$. Moreover, since $w = P_k(v; u_1, \dots, u_k) = v$ p -q.e. on B (see Corollary 1.4), we have $z = u_\lambda$ p -q.e. on B . By the convexity of G and Proposition 2.9

$$\begin{aligned} G(z, B) &\leq \frac{1}{t} G(w, B) + (1 - \frac{1}{t}) G(u, B) \\ &\leq \frac{\varepsilon}{\eta} \sum_{i=1}^k G(u_i, B) + G(u, B). \end{aligned}$$

In view of the locality property of G on Borel sets (Proposition 2.6) we have $G(z, B) = G(u_\lambda, B)$; thus

$$G(u_\lambda, B) \leq G(u, B) + \frac{\varepsilon}{\eta} \sum_{i=1}^k G(u_i, B).$$

The inequality

$$G(u, B) \leq G(u_\lambda, B) + \frac{\varepsilon}{\eta} \sum_{i=1}^k G(u_i, B)$$

can be obtained analogously defining now $v = u_\lambda + t(u - u_\lambda)$ with $t = \eta/\varepsilon > 1$. \square

Proof of Theorem 3.3. Let us fix $A \in \mathcal{A}(\Omega)$.

Step 1. Assume that $\psi(x) \in \text{ri}\Sigma_k$ for every $x \in \Omega$.

Given $\eta > 0$, let us define $\Sigma_{k,\eta} = \{\lambda \in \Sigma_k : d(\lambda, \partial\Sigma_k) \geq \eta\}$ and $A_\eta = \psi^{-1}(\Sigma_{k,\eta}) \cap A$. For every $\varepsilon \in]0, \eta[$ we can fix a finite partition $(B_j)_{j \in J}$ of $\Sigma_{k,\eta}$ by means of Borel sets having diameter less than ε , and a family $(\lambda_j)_{j \in J}$ of elements of $\Sigma_{k,\eta}$ such that $\lambda_j \in B_j$ for every $j \in J$. Let us define $E_j = \psi^{-1}(B_j) \cap A$ for every $j \in J$. Since ψ is $\widehat{B}(\Omega)$ -measurable, the sets A_η and E_j are in $\widehat{B}(\Omega)$. According to the convention made in Section 1, for every $z \in W^{1,p}(\Omega, \mathbf{R}^m)$ the completion of the measures μ , ν_0 and $G(z, \cdot)$ will be still denoted by μ , ν_0 and $G(z, \cdot)$. By Lemma 3.4, for every $j \in J$

$$|G(u, E_j) - G(u_{\lambda_j}, E_j)| \leq \frac{\varepsilon}{\eta} \sum_{i=1}^k G(u_i, E_j).$$

This and Lemma 3.1 imply

$$\begin{aligned} G(u, A_\eta) - \nu_0(A_\eta) &= \sum_{j \in J} [G(u, E_j) - \nu_0(E_j)] \\ &\leq \sum_{j \in J} [G(u_{\lambda_j}, E_j) - \nu_0(E_j) + \frac{\varepsilon}{\eta} \sum_{i=1}^k G(u_i, E_j)] \\ &= \sum_{j \in J} \left[\int_{E_j} g(x, \lambda_j) d\mu + \frac{\varepsilon}{\eta} \sum_{i=1}^k G(u_i, E_j) \right]. \end{aligned}$$

Since $g(x, \cdot)$ is convex and bounded by 1 in Σ_k , it is Lipschitz continuous in $\Sigma_{k,\eta}$ with constant $1/\eta$; thus

$$\begin{aligned} G(u, A_\eta) - \nu_0(A_\eta) &\leq \sum_{j \in J} \left[\int_{E_j} g(x, \psi(x)) d\mu + \frac{1}{\eta} \int_{E_j} |\psi(x) - \lambda_j| d\mu + \frac{\varepsilon}{\eta} \sum_{i=1}^k G(u_i, E_j) \right] \\ &\leq \int_{A_\eta} g(x, \psi(x)) d\mu + \frac{\varepsilon}{\eta} [\mu(A_\eta) + \sum_{i=1}^k G(u_i, A_\eta)]. \end{aligned}$$

(Note that $g(\cdot, \psi(\cdot))$ is μ -measurable since g is Borel measurable and ψ is $\widehat{\mathcal{B}}(\Omega)$ -measurable). Since ε is arbitrary, we get

$$G(u, A_\eta) - \nu_0(A_\eta) \leq \int_{A_\eta} g(x, \psi(x)) d\mu.$$

Now, taking into account that $\psi(x) \in \text{ri}\Sigma_k$ for every $x \in \Omega$ and letting $\eta \rightarrow 0^+$, we obtain

$$G(u, A) - \nu_0(A) \leq \int_A g(x, \psi(x)) d\mu.$$

The reverse inequality can be obtained in a completely analogous way.

Step 2. We consider now the general case $\psi: \Omega \rightarrow \Sigma_k$.

Let $b_0 = \frac{1}{k}(e_1 + \dots + e_k)$ be the barycenter of Σ_k (e_1, \dots, e_k are the elements of the standard basis of \mathbf{R}^k). For every $0 \leq \sigma \leq 1$ define $\psi_\sigma = b_0 + \sigma(\psi - b_0)$ and $u_\sigma = \sum_{i=1}^k \psi_\sigma^i u_i = u_0 + \sigma(u - u_0)$, where $u_0 = \frac{1}{k}(u_1 + \dots + u_k)$. If $0 \leq \sigma < 1$ then $\psi_\sigma(x) \in \text{ri}\Sigma_k$ for every $x \in \Omega$ (see [39], Theorem 6.1); therefore, by Step 1 we have

$$G(u_\sigma, A) = \int_A g(x, \psi_\sigma(x)) d\mu + \nu_0(A).$$

Observe now that the lower semicontinuity and the convexity of G imply that

$$\lim_{\sigma \rightarrow 1^-} G(u_\sigma, A) = G(u, A).$$

Moreover, the continuity of $g(x, \cdot)$ and the dominated convergence theorem yield

$$\lim_{\sigma \rightarrow 1^-} \int_A g(x, \psi_\sigma(x)) d\mu = \int_A g(x, \psi(x)) d\mu.$$

This concludes the proof. □

We point out that the values $u(x)$ of the function u enter the integral representation of Theorem 3.3 through the parameters $\psi^i(x)$ for which $u(x) = \sum_{i=1}^k \psi^i(x)u_i(x)$. When looking for an integrand depending directly on the values of u , the main difficulty we meet is that the expression of $u(x)$ as a convex combination of $u_1(x), \dots, u_k(x)$ is not necessarily unique. This problem is essentially overcome by the following lemma.

Lemma 3.5. *Let $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ and $u(x) \in \text{co}\{u_1(x), \dots, u_k(x)\}$ for a.e. $x \in \Omega$. Let g be the function defined in Lemma 3.1 and*

$$N = \{x \in \Omega : \exists \lambda, \lambda' \in \Sigma_k \text{ } u(x) = u_\lambda(x) = u_{\lambda'}(x) \text{ and } g(x, \lambda) \neq g(x, \lambda')\}$$

(N is defined up to sets of zero p -capacity). Then, $N \in \widehat{\mathcal{B}}(\Omega)$ and $\mu(N) = 0$.

Proof. As in the proof of Lemma 3.2 we can fix quasi continuous, Borel measurable representatives of u, u_1, \dots, u_k ; the set N is now well defined all over Ω . Consider the multivalued map Γ from Ω into $\Sigma_k \times \Sigma_k$ defined by

$$\Gamma(x) = \{(\lambda, \lambda') \in \Sigma_k \times \Sigma_k : u(x) = u_\lambda(x) = u_{\lambda'}(x) \text{ and } g(x, \lambda) \neq g(x, \lambda')\}.$$

Arguing as in the proof of Lemma 3.2 and taking into account that g is Borel measurable on $\Omega \times \Sigma_k$, we obtain that $\text{graph}\Gamma \in \mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbf{R}^k) \otimes \mathcal{B}(\mathbf{R}^k)$. By the Projection Theorem (Theorem 1.1) we get $N \in \widehat{\mathcal{B}}(\Omega)$. The Aumann-von Neumann Selection Theorem (Theorem 1.2) implies the existence of two $\widehat{\mathcal{B}}(\Omega)$ -measurable functions $\sigma_1, \sigma_2: \Omega \rightarrow \Sigma_k$ such that $(\sigma_1|_N, \sigma_2|_N)$ is a selection of Γ on N . Define for $j = 1, 2$

$$\psi_j = \begin{cases} \psi, & \text{on } \Omega \setminus N, \\ \sigma_j, & \text{on } N, \end{cases}$$

where $\psi: \Omega \rightarrow \Sigma_k$ is the $\widehat{\mathcal{B}}(\Omega)$ -measurable function given in Lemma 3.2. Then ψ_1 and ψ_2 are $\widehat{\mathcal{B}}(\Omega)$ -measurable functions such that

$$(3.4) \quad u(x) = \sum_{i=1}^k \psi_1^i(x)u_i(x) = \sum_{i=1}^k \psi_2^i(x)u_i(x) \quad \text{for } p\text{-q.e. } x \in \Omega,$$

$$(3.5) \quad g(x, \psi_1(x)) \neq g(x, \psi_2(x)) \quad \text{for every } x \in N.$$

By Theorem 3.3, (3.4) implies that

$$\int_A g(x, \psi_1(x)) d\mu = \int_A g(x, \psi_2(x)) d\mu$$

for every $A \in \mathcal{A}(\Omega)$. Hence,

$$g(x, \psi_1(x)) = g(x, \psi_2(x)) \quad \text{for } \mu\text{-a.e. } x \in \Omega.$$

Together with (3.5) this yields that $\mu(N) = 0$. □

For future convenience we single out a technical remark about measurability of functions.

Remark 3.6. (i) Let T be a Borel subset of \mathbf{R}^m . Given $g: \Omega \times \mathbf{R}^m \times T \rightarrow [0, +\infty]$, let us define $f: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ by setting $f(x, \xi) = \inf_{t \in T} g(x, \xi, t)$. If $g(x, \cdot, t)$ is continuous for every $(x, t) \in \Omega \times T$, uniformly with respect to $t \in T$, then $f(x, \cdot)$ is continuous for every $x \in \Omega$. Assume, in addition, that $g(\cdot, \xi, \cdot)$ is $\widehat{\mathcal{B}}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$ -measurable for every $\xi \in \mathbf{R}^m$. Then $f(\cdot, \xi)$ is $\widehat{\mathcal{B}}(\Omega)$ -measurable for every $\xi \in \mathbf{R}^m$, hence f is $\widehat{\mathcal{B}}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$ -measurable.

Indeed, given $\xi \in \mathbf{R}^m$ and $s \in \mathbf{R}$, the set

$$E_s = \{x \in \Omega : f(x, \xi) < s\} = \{x \in \Omega : \exists t \in T \ g(x, \xi, t) < s\}$$

is the projection on Ω of the set $\{(x, t) \in \Omega \times T : g(x, \xi, t) < s\}$. Since $(x, t) \mapsto g(x, \xi, t)$ is $\widehat{\mathcal{B}}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$ -measurable, by the Projection Theorem (Theorem 1.1) we get $E_s \in \widehat{\mathcal{B}}(\Omega)$.

(ii) Let $f: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ be a $\widehat{\mathcal{B}}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$ -measurable function. Then for every positive finite Borel measure μ on Ω there exists a set $N \in \mathcal{B}(\Omega)$ with $\mu(N) = 0$ such that $f|_{(\Omega \setminus N) \times \mathbf{R}^m}$ is a Borel function.

Indeed, for every $E \in \mathcal{B}_\mu(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$ there exists $N \in \mathcal{B}(\Omega)$ with $\mu(N) = 0$ such that $E \setminus (N \times \mathbf{R}^m) \in \mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$.

Finally, we are able to prove the main result of this section, i.e., the integral representation on the pointwise convex combinations of a finite number of fixed functions.

Theorem 3.7. Let $G \in \mathcal{G}_p$, $k \in \mathbf{N}$, $u_1, \dots, u_k \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$, and assume $G(u_i, \Omega) < +\infty$ for every $i = 1, \dots, k$. Let ν_0 be the positive finite Borel measure introduced in Proposition 2.7. Then, there exist a positive finite Borel measure μ on Ω , absolutely continuous with respect to the p -capacity, and a function $f: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ with the following properties:

- (i) for every $x \in \Omega$ the function $f(x, \cdot)$ is convex and lower semicontinuous on \mathbf{R}^m ;
- (ii) f is $\widehat{\mathcal{B}}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$ -measurable;
- (iii) for every $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that $u(x) \in \text{co}\{u_1(x), \dots, u_k(x)\}$ for a.e. $x \in \Omega$, the function $f(\cdot, u(\cdot))$ is μ -measurable on Ω and $G(u, A) = \int_A f(x, u(x)) d\mu + \nu_0(A)$ for every $A \in \mathcal{A}(\Omega)$. Moreover, the restriction of $f(x, \cdot)$ to $\text{co}\{u_1(x), \dots, u_k(x)\}$ is continuous for μ -a.e. $x \in \Omega$.

Proof. Let us fix quasi continuous Borel measurable representatives of u_1, \dots, u_k . For every $\lambda \in \Sigma_k$ and $x \in \Omega$ we set $u_\lambda(x) = \sum_{i=1}^k \lambda^i u_i(x)$ and $C_k(x) = \text{co}\{u_1(x), \dots, u_k(x)\}$. Let us define the function $f: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ as

$$f(x, \xi) = \begin{cases} \inf_{\substack{\lambda \in \Sigma_k \\ u_\lambda(x) = \xi}} g(x, \lambda), & \text{if } \xi \in C_k(x), \\ +\infty, & \text{otherwise,} \end{cases}$$

where g is the function introduced in Lemma 3.1.

Let us prove (i). Fix $x \in \Omega$; the multivalued function from $C_k(x)$ to Σ_k defined by $\xi \mapsto \{\lambda \in \Sigma_k : u_\lambda(x) = \xi\}$ has closed graph and compact range; hence it is upper semicontinuous. Moreover, by the continuity of $g(x, \cdot)$, the function $S \mapsto \inf\{g(x, \lambda) : \lambda \in S\}$, defined on the compact subsets of Σ_k , is continuous with respect to the Hausdorff metric and decreasing with respect to inclusion. Therefore, we can deduce the lower semicontinuity of $f(x, \cdot)$ on $C_k(x)$. This immediately implies the lower semicontinuity of $f(x, \cdot)$ on \mathbf{R}^m , while the convexity of $f(x, \cdot)$ can be easily verified directly. Hence, (i) holds true.

Let us prove (ii). For every $x \in \Omega$ let us consider the Moreau-Yosida transforms of $f(x, \cdot)$, defined by

$$f_s(x, \xi) = \inf_{\eta \in C_k(x)} [f(x, \eta) + s|\xi - \eta|] \quad (s \in \mathbf{N})$$

for every $\xi \in \mathbf{R}^m$. Since $f(x, \cdot)$ is lower semicontinuous on \mathbf{R}^m , for every $x \in \Omega$ and $\xi \in \mathbf{R}^m$ we have

$$(3.6) \quad f(x, \xi) = \sup_{s \in \mathbf{N}} f_s(x, \xi).$$

Let us prove that for every $\xi \in \mathbf{R}^m$, $f_s(\cdot, \xi)$ is $\widehat{\mathcal{B}}(\Omega)$ -measurable. Note that

$$\begin{aligned} f_s(x, \xi) &= \inf_{\eta \in C_k(x)} \inf_{\substack{\lambda \in \Sigma_k \\ u_\lambda(x) = \eta}} [g(x, \lambda) + s|\xi - \eta|] \\ &= \inf_{\lambda \in \Sigma_k} [g(x, \lambda) + s|\xi - u_\lambda(x)|]. \end{aligned}$$

Remark 3.6 shows that f_s is $\widehat{\mathcal{B}}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$ -measurable; by (3.6) the same is true for f .

Let us now turn to the proof of (iii). Fix $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that $u(x) \in \text{co}\{u_1(x), \dots, u_k(x)\}$ for a.e. $x \in \Omega$ and choose a quasi continuous representative

of u . Since such a representative is $\mathcal{B}_\mu(\Omega)$ -measurable (recall that μ is absolutely continuous with respect to the p -capacity), $f(\cdot, u(\cdot))$ is μ -measurable on Ω . By Lemma 3.2 and Theorem 3.3, for every $A \in \mathcal{A}(\Omega)$ we have

$$G(u, A) = \int_A g(x, \psi(x)) d\mu + \nu_0(A),$$

where $\psi: \Omega \rightarrow \Sigma_k$ is a $\widehat{\mathcal{B}}(\Omega)$ -measurable function such that (3.2) holds. By the definition of f we have

$$f(x, u(x)) = \inf_{\substack{\lambda \in \Sigma_k \\ u_\lambda(x) = u(x)}} g(x, \lambda) \quad \text{for } p\text{-q.e. } x \in \Omega.$$

Let N be the set given in Lemma 3.5. Then

$$f(x, u(x)) = g(x, \psi(x)) \quad \text{for } p\text{-q.e. } x \in \Omega \setminus N.$$

This proves (iii) since $\mu(N) = 0$. □

The following proposition shows that, given the measures μ and ν , the function f obtained in the integral representation theorem is essentially unique.

Proposition 3.8. *Let G, u_1, \dots, u_k be as in Theorem 3.7. Let μ and ν be two positive finite Borel measures on Ω , with μ absolutely continuous with respect to the p -capacity. Assume that two functions $f_1, f_2: \Omega \times \mathbb{R}^m \rightarrow [0, +\infty]$ satisfy conditions (i)–(iii) of Theorem 3.7 with ν_0 replaced by ν . Then $f_1(x, \xi) = f_2(x, \xi)$ for μ -a.e. $x \in \Omega$ and for every $\xi \in \text{co}\{u_1(x), \dots, u_k(x)\}$.*

Proof. From the finiteness of $G(u_i, \Omega)$ for $i = 1, \dots, k$ and from property (iii), we deduce that $f_1(\cdot, u_i(\cdot)) < +\infty$, $f_2(\cdot, u_i(\cdot)) < +\infty$ μ -a.e. on Ω . The convexity of $f_1(x, \cdot)$ and $f_2(x, \cdot)$ then guarantees that $f_1(x, \cdot)$ and $f_2(x, \cdot)$ are finite on $\text{co}\{u_1(x), \dots, u_k(x)\}$ for μ -a.e. $x \in \Omega$. By Theorem 10.1 in [39], it follows that $f_1(x, \cdot)$ and $f_2(x, \cdot)$ restricted to $\text{ri co}\{u_1(x), \dots, u_k(x)\}$ are continuous for μ -a.e. $x \in \Omega$. By (iii), for every $\lambda \in \Sigma_k \cap \mathbb{Q}^k$ we have

$$\int_A f_1(x, u_\lambda(x)) d\mu = \int_A f_2(x, u_\lambda(x)) d\mu$$

for every $A \in \mathcal{A}(\Omega)$; hence, there exists a set $N \in \mathcal{B}(\Omega)$, $\mu(N) = 0$ such that $f_1(x, u_\lambda(x)) = f_2(x, u_\lambda(x))$ for every $x \in \Omega \setminus N$ and $\lambda \in \Sigma_k \cap \mathbb{Q}^k$. Since the functions $f_1(x, \cdot)$ and $f_2(x, \cdot)$ restricted to $\text{ri co}\{u_1(x), \dots, u_k(x)\}$ are continuous, we have $f_1(x, \xi) = f_2(x, \xi)$ for every $x \in \Omega \setminus N$ and $\xi \in \text{ri co}\{u_1(x), \dots, u_k(x)\}$. By the continuity along line segments ([39], Corollary 7.5.1) $f_1(x, \xi) = f_2(x, \xi)$ for every $\xi \in \text{co}\{u_1(x), \dots, u_k(x)\}$. □

4. Auxiliary lemmas

We collect here some results we shall use in the next section.

Lemma 4.1. *Let X be a separable metric space and let $F: X \rightarrow \mathbb{R}$ be lower semicontinuous. Then there exists a countable subset D of X with the following property: for every $x \in X$ there exists a sequence (x_h) in D converging to x and such that $(F(x_h))$ converges to $F(x)$.*

Proof. It is enough to take a countable dense subset E of the epigraph of F and consider as D the projection of E onto X . \square

Lemma 4.2. *Let $d \in \mathbb{N}$ and X be a subset of \mathbb{R}^d . Let H be a Lipschitz multivalued function from X to \mathbb{R}^m with non-empty, compact and convex values. Then, there exists a sequence (h_j) of Lipschitz functions from X to \mathbb{R}^m such that*

$$H(x) = \text{cl}\{h_j(x) : j \in \mathbb{N}\}$$

for every $x \in X$, where cl denotes the closure in \mathbb{R}^m .

Proof. Let (ξ_j) be a dense sequence in \mathbb{R}^m ; for every $x \in X$, define $h_j(x) = P(\xi_j, H(x)) \in H(x)$, where P is the projection map given in Theorem 2.3 of Chapter I. Since P and H are both Lipschitzian, so is h_j .

Given $\xi \in H(x)$ and $\varepsilon > 0$ there exists $\xi_j \in \mathbb{R}^m$ such that $|\xi - \xi_j| < \varepsilon$. If L denotes a Lipschitz constant for P , then

$$|\xi - h_j(x)| = |P(\xi, H(x)) - P(\xi_j, H(x))| \leq L|\xi - \xi_j| < L\varepsilon;$$

we conclude that $\xi \in \text{cl}\{h_j(x) : j \in \mathbb{N}\}$. \square

We shall now state a result due to G. Bouchitté and M. Valadier concerning the commutativity property for the operations of integration and infimum. To this aim we need the following notion of C^1 -convexity which is essentially the notion of C^1 -stability introduced in [10].

Definition 4.3. Given a positive Radon measure λ on Ω and a set \mathcal{H} of λ -measurable functions from Ω into \mathbf{R}^m , we say that \mathcal{H} is C^1 -convex if for every finite family $(u_i)_{i \in I}$ of elements of \mathcal{H} and for every family $(\alpha_i)_{i \in I}$ of non-negative functions of $C^1(\Omega) \cap W^{1,\infty}(\Omega)$ such that $\sum_i \alpha_i = 1$ in Ω , we have that $\sum_i \alpha_i u_i$ belongs to \mathcal{H} .

Let λ be a positive Radon measure on Ω and \mathcal{H} be a family of λ -measurable functions from Ω to \mathbf{R}^m . Then, there exists a closed valued λ -measurable multifunction $\Gamma: \Omega \rightarrow \mathbf{R}^m$ (i.e., such that $\Gamma^{-1}(C) = \{x \in \Omega : \Gamma(x) \cap C \neq \emptyset\}$ is λ -measurable for every closed subset C of \mathbf{R}^m) with the following properties (see [41], Proposition 14):

- (i) for every $w \in \mathcal{H}$ we have $w(x) \in \Gamma(x)$ for λ -a.e. $x \in \Omega$;
- (ii) if $\Phi: \Omega \rightarrow \mathbf{R}^m$ is a closed valued λ -measurable multifunction such that for every $w \in \mathcal{H}$, $w(x) \in \Phi(x)$ for λ -a.e. $x \in \Omega$, then $\Gamma(x) \subseteq \Phi(x)$ for λ -a.e. $x \in \Omega$.

This multifunction Γ is unique up to λ -equivalence and will be denoted by $\lambda\text{-ess sup}_{w \in \mathcal{H}} \{w(\cdot)\}$.

The next theorem is taken from [10], Theorem 1.

Theorem 4.4. Let λ be a positive Radon measure on Ω and let \mathcal{H} be a C^1 -convex family of λ -measurable functions from Ω into \mathbf{R}^m . Let $f: \Omega \times \mathbf{R}^m \rightarrow]-\infty, +\infty]$ be a $\mathcal{B}_\lambda(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$ -measurable function such that $f(x, \cdot)$ is convex on \mathbf{R}^m for λ -a.e. $x \in \Omega$. Suppose that $f(\cdot, u(\cdot)) \in L^1(\Omega, \lambda)$ for every $u \in \mathcal{H}$ and let $\Gamma(x) = \lambda\text{-ess sup}_{u \in \mathcal{H}} \{u(x)\}$. Then

$$\inf_{u \in \mathcal{H}} \int_{\Omega} f(x, u(x)) d\lambda = \int_{\Omega} \inf_{z \in \Gamma(x)} f(x, z) d\lambda.$$

We point out that in the next section it will be crucial the use of the following technical result, proven in Chapter I, Lemma 4.2:

Lemma 4.5. Let (w_k) be a sequence of functions in $W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ converging in $L^\infty(\Omega, \mathbf{R}^m)$ to a function $w \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$. Then there exists a sequence (v_k) in $W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that $v_k(x) \in \text{co}\{w_1(x), \dots, w_k(x)\}$ for a.e. $x \in \Omega$ and (v_k) converges to w strongly in $W^{1,p}(\Omega, \mathbf{R}^m)$.

Lemma 4.6. *Let λ be a positive Borel measure on Ω . Let (γ_h) and γ be non-negative functions in $L^1(\Omega, \lambda)$ satisfying*

$$(4.1) \quad \gamma(x) \leq \liminf_{h \rightarrow \infty} \gamma_h(x) \quad \text{for } \lambda\text{-a.e. } x \in \Omega,$$

$$(4.2) \quad \int_{\Omega} \gamma d\lambda \geq \limsup_{h \rightarrow \infty} \int_{\Omega} \gamma_h d\lambda.$$

Then, (γ_h) converges to γ strongly in $L^1(\Omega, \lambda)$.

Proof. Let us note that, by the Fatou Lemma, (4.2) ensures that

$$(4.3) \quad \int_{\Omega} \gamma d\lambda = \lim_{h \rightarrow \infty} \int_{\Omega} \gamma_h d\lambda.$$

In view of (4.1) we have

$$\gamma \leq \liminf_{h \rightarrow \infty} (\gamma_h \wedge \gamma) \leq \limsup_{h \rightarrow \infty} (\gamma_h \wedge \gamma) \leq \gamma \quad \text{on } \Omega.$$

Thus, the dominated convergence theorem guarantees that $(\gamma_h \wedge \gamma)$ converges to γ in $L^1(\Omega, \lambda)$, and, in particular

$$(4.4) \quad \int_{\Omega} \gamma_h \wedge \gamma d\lambda \rightarrow \int_{\Omega} \gamma d\lambda.$$

By noticing that $\gamma_h + \gamma = (\gamma_h \wedge \gamma) + (\gamma_h \vee \gamma)$, (4.3) and (4.4) permit to conclude that

$$\int_{\Omega} \gamma_h \vee \gamma d\lambda \rightarrow \int_{\Omega} \gamma d\lambda;$$

hence $(\gamma_h \vee \gamma)$ converges to γ in $L^1(\Omega, \lambda)$, being $\gamma_h \vee \gamma \geq \gamma$. Now, the conclusion can be obtained by using again the relation $\gamma_h = (\gamma_h \wedge \gamma) + (\gamma_h \vee \gamma) - \gamma$ on Ω . \square

5. Integral representation on $W^{1,p}(\Omega, \mathbb{R}^m) \cap L^\infty(\Omega, \mathbb{R}^m)$

The main result of this section is the integral representation of the functionals of the class \mathcal{G}_p on the bounded functions of $W^{1,p}(\Omega, \mathbb{R}^m)$ (Theorem 5.4).

Given $G \in \mathcal{G}_p$, let us introduce the least closed valued multifunction having the elements of $\text{dom} G(\cdot, \Omega)$ among its selections.

Proposition 5.1. *Let $G \in \mathcal{G}_p$ and let A be an open subset of Ω with $\text{dom}G(\cdot, A) \neq \emptyset$. Then there exists a closed valued multifunction K_A from A to \mathbf{R}^m , unique up to sets of p -capacity zero, such that*

- (i) *for every $u \in \text{dom}G(\cdot, A)$ we have $u(x) \in K_A(x)$ for p -q.e. $x \in A$;*
- (ii) *if H is a closed valued multifunction from A to \mathbf{R}^m such that for every $u \in \text{dom}G(\cdot, A)$ we have $u(x) \in H(x)$ for p -q.e. $x \in A$, then $K_A(x) \subseteq H(x)$ for p -q.e. $x \in A$.*

Moreover, K_A satisfies the following properties:

- (iii) *K_A is quasi lower semicontinuous and $K_A(x)$ is convex for p -q.e. $x \in A$;*
- (iv) *if (u_k) is a countable dense subset of $\text{dom}G(\cdot, A)$, then*

$$K_A(x) = \text{cl}\{u_k(x) : k \in \mathbf{N}\} = \text{cl}\left(\bigcup_{k=1}^{\infty} C_k(x)\right) \quad \text{for } p\text{-q.e. } x \in A,$$

where $C_k(x) = \text{co}\{u_1(x), \dots, u_k(x)\}$.

Proof. The same argument applied in Proposition 3.3 in Chapter I works now for the subset $\{u|_A : u \in \text{dom}G(\cdot, A)\}$ of $W^{1,p}(A, \mathbf{R}^m)$. \square

Remark 5.2. Let A and A' be open subsets of Ω , with $\text{dom}G(\cdot, A) \neq \emptyset$ and $\text{dom}G(\cdot, A') \neq \emptyset$. If K_A and $K_{A'}$ are the multifunctions given by the previous proposition, then $K_A = K_{A'}$ p -q.e. on $A \cap A'$.

It is enough to give the proof in the case $A' \subseteq A$. Since $\text{dom}G(\cdot, A) \subseteq \text{dom}G(\cdot, A')$, the inclusion $K_A(x) \subseteq K_{A'}(x)$ for p -q.e. $x \in A'$ follows immediately from property (i) satisfied by $K_{A'}$ and property (ii) applied to K_A and

$$H(x) = \begin{cases} K_{A'}(x), & \text{if } x \in A', \\ \mathbf{R}^m, & \text{if } x \in A \setminus A'. \end{cases}$$

To get the opposite inclusion let us choose $u_0 \in \text{dom}G(\cdot, A)$. Fix now $u \in \text{dom}G(\cdot, A')$ and $A'' \in \mathcal{A}(\Omega)$ with $A'' \subset\subset A'$. If φ is a function in $C_0^1(A')$, with $\varphi = 1$ on A'' and $0 \leq \varphi \leq 1$, by the C^1 -convexity and the locality property of G on open sets, we have

$$\begin{aligned} G(\varphi u + (1 - \varphi)u_0, A) &\leq G(\varphi u + (1 - \varphi)u_0, A') + G(\varphi u + (1 - \varphi)u_0, A \setminus \text{supp}\varphi) \\ &\leq G(u, A') + G(u_0, A') + G(u_0, A \setminus \text{supp}\varphi) < +\infty. \end{aligned}$$

Therefore, $\varphi u + (1 - \varphi)u_0 \in \text{dom}G(\cdot, A)$ so that $u(x) \in K_A(x)$ for p -q.e. $x \in A''$. By the arbitrariness of A'' we deduce that $u(x) \in K_A(x)$ for p -q.e. $x \in A'$. By applying property (ii) we conclude that $K_{A'}(x) \subseteq K_A(x)$ for p -q.e. $x \in A'$.

Lemma 5.3. *Let $s > 0$ and let T_s be the orthogonal projection onto the ball $\overline{B}_s(0)$ defined in Lemma 2.5. Then for every $u, v \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$*

$$G(u + T_s \circ (v - u), A) \leq G(u, A) + G(v, A).$$

Proof. It is enough to consider the case $A \subset\subset \Omega$. Let $\varphi \in C_0^1(\Omega)$, $\varphi = 1$ on A , $0 \leq \varphi \leq 1$. By Lemma 3.6 in Chapter I there exists a sequence (ψ_h) of functions in $C^\infty(\mathbf{R}^n)$ such that $0 \leq \psi_h \leq 1$ and $(\psi_h \varphi(v - u))$ converges to $T_s \circ [\varphi(v - u)]$ weakly in $W^{1,p}(\Omega, \mathbf{R}^m)$ as h goes to ∞ . Since $G(\cdot, A)$ is weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbf{R}^m)$ (recall that $G(\cdot, A)$ is convex) we get

$$\begin{aligned} G(u + T_s \circ [\varphi(v - u)], A) &\leq \liminf_{h \rightarrow \infty} G(u + \psi_h \varphi(v - u), A) \\ &\leq G(u, A) + G(v, A), \end{aligned}$$

where in the last estimate we have used the C^1 -convexity of G . Now the conclusion can be obtained by applying the locality property of G on open sets. \square

Theorem 5.4. *Let $G \in \mathcal{G}_p$ with $\text{dom}G(\cdot, \Omega) \cap L^\infty(\Omega, \mathbf{R}^m) \neq \emptyset$. Let ν_0 be the positive finite Borel measure introduced in Proposition 2.7 and let $K = K_\Omega$ be the closed valued multifunction from Ω to \mathbf{R}^m given by Proposition 5.1 for $A = \Omega$. Then, there exist a positive finite Borel measure μ on Ω , absolutely continuous with respect to the p -capacity, and a Borel function $f: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ with the following properties:*

- (i) *for every $x \in \Omega$, the function $f(x, \cdot)$ is convex and lower semicontinuous on \mathbf{R}^m ;*
- (ii) *for every $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ and for every $A \in \mathcal{A}(\Omega)$*

$$G(u, A) = \begin{cases} \int_A f(x, u(x)) d\mu + \nu_0(A), & \text{if } u(x) \in K(x) \text{ for } p\text{-q.e. } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

Proof. Step 1. Let (u_i) be a sequence of functions in $\text{dom}G(\cdot, \Omega) \cap L^\infty(\Omega, \mathbf{R}^m)$ which will be specified in Step 2. We construct now the measure μ and the integrand f (see (5.3)) satisfying (i) and we prove that for every $A \in \mathcal{A}(\Omega)$ and for every $k \in \mathbf{N}$ we have

$$(5.1) \quad G(u, A) = \int_A f(x, u(x)) d\mu + \nu_0(A)$$

whenever $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ and $u(x) \in \text{co}\{u_1(x), \dots, u_k(x)\}$ for a.e. $x \in A$.

Fix quasi continuous Borel measurable representatives of (u_i) . For every $x \in \Omega$ define $C_k(x) = \text{co}\{u_1(x), \dots, u_k(x)\}$. By Theorem 3.7, for every $k \in \mathbf{N}$ there exist a positive finite Borel measure μ_k on Ω , absolutely continuous with respect to the p -capacity, and a function $f_k: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ such that

- (a) for every $x \in \Omega$ the function $f_k(x, \cdot)$ is convex and lower semicontinuous on \mathbf{R}^m ; moreover, the restriction of $f_k(x, \cdot)$ to $C_k(x)$ is continuous for μ_k -a.e. $x \in \Omega$;
- (b) f_k is $\widehat{\mathcal{B}}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$ -measurable;
- (c) for every $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that $u(x) \in C_k(x)$ for a.e. $x \in \Omega$ the function $f_k(\cdot, u(\cdot))$ is μ -measurable on Ω and for every $A \in \mathcal{A}(\Omega)$

$$(5.2) \quad G(u, A) = \int_A f_k(x, u(x)) d\mu_k + \nu_0(A).$$

By a standard cut-off argument we obtain that (5.2) still holds if $u(x) \in C_k(x)$ for a.e. $x \in A$.

Let μ be a positive finite Borel measure on Ω absolutely continuous with respect to the p -capacity and such that $\mu_k \ll \mu$ for every $k \in \mathbf{N}$ (for instance, take $\mu(B) = \sum_{k=1}^{\infty} 2^{-k} \frac{\mu_k(B)}{\mu_k(\Omega)}$ for every $B \in \mathcal{B}(\Omega)$). Define

$$g_k(x, \xi) = f_k(x, \xi) \frac{d\mu_k}{d\mu}(x),$$

where $d\mu_k/d\mu$ is a fixed (μ -measurable) representative of the Radon-Nikodym derivative of μ_k with respect to μ . By Proposition 3.8, there exists a set $N \in \mathcal{B}(\Omega)$ with $\mu(N) = 0$ such that $g_k(x, \xi) = g_{k+1}(x, \xi)$ for every $k \in \mathbf{N}$, $x \in \Omega \setminus N$ and $\xi \in C_k(x)$. Hence, we can define $g: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ as

$$g(x, \xi) = \begin{cases} g_k(x, \xi), & \text{if } x \in \Omega \setminus N \text{ and } \xi \in C_k(x) \text{ for some } k \in \mathbf{N}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $C_k(x) = \{\sum_{i=1}^k \lambda^i u_i(x) : \lambda \in \Sigma_k\}$, by Theorem III.9 and Proposition III.13 in [14], the graph of C_k belongs to $\mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$. Recalling the definition of g_k it follows that g is $\widehat{\mathcal{B}}(\Omega) \otimes \mathcal{B}(\mathbf{R}^m)$ -measurable. An easy check gives the convexity of $g(x, \cdot)$ on \mathbf{R}^m for every $x \in \Omega$.

Now, for every $x \in \Omega$ let us set $h(x, \cdot) = \text{sc}^-g(x, \cdot)$, where $\text{sc}^-g(x, \cdot)$ denotes the lower semicontinuous envelope of $g(x, \cdot)$. It turns out that

$$h(x, \xi) = \sup_{s \in \mathbb{N}} g_s(x, \xi),$$

where $g_s(x, \xi) = \inf_{\eta \in \mathbb{R}^m} [g(x, \eta) + s|\xi - \eta|]$. By Remark 3.6, for every $s \in \mathbb{N}$ there exists a set $Z_s \in \mathcal{B}(\Omega)$, with $\mu(Z_s) = 0$, such that $g_s|_{(\Omega \setminus Z_s) \times \mathbb{R}^m}$ is a Borel function. Set $Z = \bigcup_{s \in \mathbb{N}} Z_s$; then $\mu(Z) = 0$ and $h|_{(\Omega \setminus Z) \times \mathbb{R}^m}$ is Borel measurable. Now we are in a position to define the function f as

$$(5.3) \quad f(x, \cdot) = \begin{cases} h(x, \cdot) = \text{sc}^-g(x, \cdot) & \text{if } x \in \Omega \setminus Z \\ 0 & \text{if } x \in Z. \end{cases}$$

Then, f is a Borel function on $\Omega \times \mathbb{R}^m$ and satisfies (i) (see [39], Theorem 7.4).

Let us prove that for μ -a.e. $x \in \Omega$

$$(5.4) \quad f(x, \cdot) = g_k(x, \cdot) \quad \text{on } C_k(x).$$

Let us fix $x \in \Omega \setminus (N \cup Z)$ and $k \in \mathbb{N}$. Let $H(x)$ be the affine hull of $\bigcup_{k=1}^{\infty} C_k(x)$. As the sequence $(C_k(x))$ is increasing, there exists $h \geq k$ such that the interior of $C_h(x)$ relative to $H(x)$ is non-empty. Since $g(x, \cdot) = g_h(x, \cdot)$ on $C_h(x)$, and the restriction of $g_h(x, \cdot)$ to $C_h(x)$ is continuous, we have $f(x, \cdot) = g_h(x, \cdot)$ on the interior of $C_h(x)$ relative to $H(x)$. As $C_h(x)$ is a polytope, the restriction of $f(x, \cdot)$ to $C_h(x)$ is continuous (see [39], Theorem 10.2), hence $f(x, \cdot) = g_h(x, \cdot)$ on $C_h(x)$. Since $C_k(x) \subseteq C_h(x)$ and $g_k(x, \cdot) = g_h(x, \cdot)$ on $C_k(x)$, we conclude that (5.4) is satisfied.

Let us now prove (5.1). Fix $k \in \mathbb{N}$, $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(\Omega, \mathbb{R}^m) \cap L^\infty(\Omega, \mathbb{R}^m)$ with $u(x) \in C_k(x)$ for a.e. $x \in A$. For every $0 < \sigma < 1$ let us define $u_\sigma = \sigma u + (1 - \sigma)u_0$, with $u_0 = \frac{1}{k} \sum_{i=1}^k u_i$. Then $u_\sigma(x) \in \text{ri} C_k(x)$ for p -q.e. $x \in A$. Therefore, by (5.4) and the integral representation formula (5.2) satisfied by f_k , we get

$$G(u_\sigma, A) = \int_A f(x, u_\sigma(x)) d\mu + \nu_0(A).$$

Since every lower semicontinuous proper convex function is continuous along line segments (see [39], Corollary 7.5.1) it turns out that

$$\begin{aligned} \lim_{\sigma \rightarrow 1^-} G(u_\sigma, A) &= G(u, A) \\ \lim_{\sigma \rightarrow 1^-} \int_A f(x, u_\sigma(x)) d\mu &= \int_A f(x, u(x)) d\mu \quad \text{for every } x \in \Omega. \end{aligned}$$

We thus obtain (5.1).

Step 2. We choose now a suitable sequence (u_i) , dense in $\text{dom}G(\cdot, \Omega) \cap L^\infty(\Omega, \mathbf{R}^m)$, to which Step 1 will be applied.

Let \mathcal{D} be a countable base for the open subsets of Ω , closed under finite unions. For every $A \in \mathcal{D}$ we can apply Lemma 4.1 to $G(\cdot, A)$ on $\text{dom}G(\cdot, \Omega) \cap L^\infty(\Omega, \mathbf{R}^m)$ (with the $W^{1,p}(\Omega, \mathbf{R}^m)$ topology); this yields the existence of a set $\mathcal{G}_A \subseteq \text{dom}G(\cdot, \Omega) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that for every $u \in \text{dom}G(\cdot, \Omega) \cap L^\infty(\Omega, \mathbf{R}^m)$ there exists a sequence (u_h) in \mathcal{G}_A satisfying

$$\begin{aligned} u_h &\rightarrow u && \text{strongly in } W^{1,p}(\Omega, \mathbf{R}^m), \\ G(u_h, A) &\rightarrow G(u, A) && \text{in } \mathbf{R}. \end{aligned}$$

Let (u_i) be an enumeration of $\bigcup_{A \in \mathcal{D}} \mathcal{G}_A$; starting from (u_i) we then construct by means of Step 1 a Borel function $f: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ satisfying (i) and (5.1).

Step 3. Let us prove that for every $u \in \text{dom}G(\cdot, \Omega) \cap L^\infty(\Omega, \mathbf{R}^m)$ and for every $A \in \mathcal{A}(\Omega)$

$$(5.5) \quad G(u, A) \geq \int_A f(x, u(x)) d\mu + \nu_0(A).$$

Fix $u \in \text{dom}G(\cdot, \Omega) \cap L^\infty(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{D}$. By Step 2 it is possible to extract a sequence (u_{i_h}) from $\{u_i : i \in \mathbf{N}\}$ such that

$$\begin{aligned} u_{i_h} &\rightarrow u && p\text{-q.e. in } \Omega \quad (\text{hence } \mu\text{-a.e.}), \\ G(u_{i_h}, A) &\rightarrow G(u, A) && \text{in } \mathbf{R}. \end{aligned}$$

Therefore, by (5.1)

$$G(u, A) = \lim_{h \rightarrow \infty} \int_A f(x, u_{i_h}(x)) d\mu + \nu_0(A);$$

by the Fatou Lemma and the lower semicontinuity of $f(x, \cdot)$ we get (5.5) for every $A \in \mathcal{D}$. The result for an arbitrary $A \in \mathcal{A}(\Omega)$ can be obtained by approximation, since each $A \in \mathcal{A}(\Omega)$ is the union of an increasing sequence of elements of \mathcal{D} (recall that \mathcal{D} is closed under finite unions).

Step 4. It is now easy to prove that for every $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$ the inequality (5.5) holds.

Given $A \in \mathcal{A}(\Omega)$ and $u \in \text{dom}G(\cdot, A) \cap L^\infty(\Omega, \mathbf{R}^m)$, let $U \subset\subset A$ and $\varphi \in C_0^1(A)$ with $\varphi = 1$ on U and $0 \leq \varphi \leq 1$. Set $u_\varphi = \varphi u + (1 - \varphi)w$, where w belongs to $\text{dom}G(\cdot, \Omega) \cap L^\infty(\Omega, \mathbf{R}^m)$, which is non-empty by assumption. By the convexity and the locality property of G we have $G(u_\varphi, \Omega) < +\infty$. Therefore, Step 3 applied to u_φ yields

$$G(u_\varphi, U) \geq \int_U f(x, u_\varphi(x)) d\mu + \nu_0(U);$$

since $\varphi = 1$ on U we get

$$G(u, U) \geq \int_U f(x, u(x)) d\mu + \nu_0(U).$$

As $U \subset\subset A$ is arbitrary, the conclusion is easily achieved.

Step 5. Let $K = K_\Omega$ be the closed valued multifunction from Ω to \mathbf{R}^m given by Proposition 5.1 for $A = \Omega$. The aim is now to prove that for every $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ such that $u(x) \in K(x)$ for p -q.e. $x \in A$, we have

$$(5.6) \quad G(u, A) \leq \int_A f(x, u(x)) d\mu + \nu_0(A).$$

Recall that for every $k \in \mathbf{N}$ and $x \in \Omega$ we have $C_k(x) = \text{co}\{u_1(x), \dots, u_k(x)\}$, where (u_i) is the sequence given in Step 2. By Lemmas 2.5 and 5.3, $\text{dom}G(\cdot, \Omega) \cap L^\infty(\Omega, \mathbf{R}^m)$, which is non-empty by assumption, is dense in $\text{dom}G(\cdot, \Omega)$. Hence, (u_i) is dense in $\text{dom}G(\cdot, \Omega)$ and, by Proposition 5.1, $K(x) = \text{cl}(\bigcup_{k=1}^\infty C_k(x))$ for p -q.e. $x \in \Omega$.

Fix $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$ such that $u(x) \in K(x)$ for p -q.e. $x \in A$. Clearly, we can assume that the right-hand side in (5.6) is finite. Moreover, we can consider open sets $A \subset\subset \Omega$ with smooth boundary, so that there exists an extension operator $W^{1,p}(A) \rightarrow W^{1,p}(\Omega)$.

In a first moment we work with the additional assumption that $u(x)$ is in the closure of $\bigcup_{k=1}^\infty C_k(x)$ “uniformly” for $x \in A$; more precisely, given a sequence (r_h) of positive numbers decreasing to 0, we require that for every $h \in \mathbf{N}$ there exists $n_h \in \mathbf{N}$ such that

$$(5.7) \quad B_{r_h/2}(u(x)) \cap C_k(x) \neq \emptyset \quad \text{for } p\text{-q.e. } x \in A$$

for every $k \geq n_h$.

To achieve (5.6) we look for a sequence (v_h) of functions in $W^{1,p}(A, \mathbf{R}^m) \cap L^\infty(A, \mathbf{R}^m)$ and a strictly increasing sequence (k_h) of positive integers such that $v_h \in C_{k_h}(x)$ for p -q.e. $x \in A$, (v_h) converges to u in $W^{1,p}(A, \mathbf{R}^m)$ and

$$(5.8) \quad \limsup_{h \rightarrow \infty} \int_A f(x, v_h(x)) d\mu \leq \int_A f(x, u(x)) d\mu.$$

Indeed, as ∂A is smooth, we can assume that (v_h) is a sequence in $W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ converging in $W^{1,p}(\Omega, \mathbf{R}^m)$ to a function v such that $v = u$ a.e. on A . By the lower semicontinuity of $G(\cdot, A)$ and the integral representation (5.1) obtained in Step 1, we can then conclude

$$\begin{aligned} G(u, A) &= G(v, A) \leq \liminf_{h \rightarrow \infty} G(v_h, A) \\ &\leq \limsup_{h \rightarrow \infty} \left(\int_A f(x, v_h(x)) d\mu + \nu_0(A) \right) \leq \int_A f(x, u(x)) d\mu + \nu_0(A). \end{aligned}$$

Let us first construct a sequence (w_h) of functions in $W^{1,p}(A, \mathbf{R}^m) \cap L^\infty(A, \mathbf{R}^m)$ and a strictly increasing sequence (k_h) of positive integers, with the following properties:

$$(5.9) \quad \begin{aligned} w_h(x) &\in C_{k_h}(x) && \text{for } p\text{-q.e. } x \in A \\ w_h &\rightarrow u && \text{uniformly on } A \\ \limsup_{h \rightarrow \infty} \int_A f(x, w_h(x)) d\mu &\leq \int_A f(x, u(x)) d\mu. \end{aligned}$$

To this aim let us prove that for every $h \in \mathbf{N}$ there exists $k_h \in \mathbf{N}$ such that

$$(5.10) \quad \inf_{w \in \mathcal{H}_k^h} \int_A f(x, w(x)) d\mu < \int_A f(x, u(x)) d\mu + r_h,$$

for every $k \geq k_h$, where

$$\mathcal{H}_k^h = \{w \in W^{1,p}(A, \mathbf{R}^m) \cap L^\infty(A, \mathbf{R}^m) : w(x) \in \overline{B}_{r_h}(u(x)) \cap C_k(x) \text{ for } p\text{-q.e. } x \in A\}.$$

Let us fix $h \in \mathbf{N}$ and let n_h be as in (5.7). For every fixed $k \geq n_h$ we want to apply Theorem 4.4 to the set \mathcal{H}_k^h . For this purpose let us verify that

$$(5.11) \quad \mu\text{-ess sup}_{w \in \mathcal{H}_k^h} \{w(x)\} = \overline{B}_{r_h}(u(x)) \cap C_k(x) \quad \text{for } p\text{-q.e. } x \in A.$$

Let $\Xi = \{(\xi, \xi_1, \dots, \xi_k) \in (\mathbf{R}^m)^{k+1} : d(\xi, \text{co}\{\xi_1, \dots, \xi_k\}) \leq r_h/2\}$ and let H be the multivalued function from Ξ to \mathbf{R}^m defined by

$$H(\xi, \xi_1, \dots, \xi_k) = \overline{B}_{r_h}(\xi) \cap \text{co}\{\xi_1, \dots, \xi_k\}.$$

By Theorem 1 in [33], H is lipschitzian. Hence, we can apply Lemma 4.2 to H obtaining a sequence (h_j) of Lipschitz functions from Ξ to \mathbf{R}^m such that

$$H(\xi, \xi_1, \dots, \xi_k) = \text{cl} \left(\bigcup_{j=1}^{\infty} \{h_j(\xi, \xi_1, \dots, \xi_k)\} \right).$$

Since $B_{r_h/2}(u(x)) \cap C_k(x) \neq \emptyset$ for p -q.e. $x \in A$, we can define $z_j = h_j(u, u_1, \dots, u_k)$ p -q.e. on A for every $j \in \mathbf{N}$. By Lemma 2.2 of Chapter I, $z_j \in W^{1,p}(A, \mathbf{R}^m) \cap L^\infty(A, \mathbf{R}^m)$. Thus $z_j \in \mathcal{H}_k^h$, and

$$\overline{B}_{r_h}(u(x)) \cap C_k(x) = H(u(x), u_1(x), \dots, u_k(x)) = \text{cl} \left(\bigcup_{j=1}^{\infty} \{z_j(x)\} \right)$$

for p -q.e. $x \in A$. Hence, (5.11) holds.

Moreover, since every $w \in \mathcal{H}_k^h$ is a convex combination of u_1, \dots, u_k , we have

$$\int_A f(x, w(x)) d\mu \leq \int_A \sum_{i=1}^k f(x, u_i(x)) d\mu < +\infty.$$

We can now apply Theorem 4.4; by (5.11), for every $k \geq n_h$ we have

$$(5.12) \quad \inf_{w \in \mathcal{H}_k^h} \int_A f(x, w(x)) d\mu = \int_A \inf_{\xi \in C_k^h(x)} f(x, \xi) d\mu,$$

where $C_k^h(x) = \overline{B}_{r_h}(u(x)) \cap C_k(x)$. Since $u(x) \in K(x)$ for p -q.e. $x \in A$, in view of the continuity property along line segments for a proper, lower semicontinuous convex function, for p -q.e. $x \in A$ we can approximate $u(x)$ by a sequence $(\xi_k(x))$ in $\text{ri}K(x)$ such that

$$f(x, u(x)) = \lim_{k \rightarrow \infty} f(x, \xi_k(x)).$$

As $\text{ri}K(x) \subseteq \bigcup_{k=1}^{\infty} C_k(x)$ (see [39], Theorem 6.3), we can suppose that $\xi_k(x) \in C_k(x)$ for every $k \in \mathbf{N}$. Thus, for every $h \in \mathbf{N}$ we have

$$(5.13) \quad \inf_{k \in \mathbf{N}} \inf_{\xi \in C_k^h(x)} f(x, \xi) \leq f(x, u(x)) \quad \text{for } p\text{-q.e. } x \in A.$$

Moreover, for every $k \geq n_h$ the set $C_k^h(x)$ is non-empty for p -q.e. $x \in A$. Therefore, the convexity of f ensures that

$$\int_A \inf_{\xi \in C_k^h(x)} f(x, \xi) d\mu \leq \int_A \sum_{i=1}^k f(x, u_i(x)) d\mu < +\infty;$$

by (5.13) and by the monotone convergence theorem it follows that

$$\inf_{k \in \mathbb{N}} \int_A \inf_{\xi \in C_k^h(x)} f(x, \xi) d\mu \leq \int_A f(x, u(x)) d\mu.$$

This inequality, together with (5.12), proves (5.10).

Let (k_h) be the sequence given in (5.10) which we can assume to be strictly increasing. For every $h \in \mathbb{N}$, by (5.10) there exists a function $w_h \in W^{1,p}(A, \mathbb{R}^m) \cap L^\infty(A, \mathbb{R}^m)$ such that

$$\begin{aligned} w_h(x) &\in \overline{B}_{r_h}(u(x)) \cap C_{k_h}(x) \quad \text{for } p\text{-q.e. } x \in A, \\ \int_A f(x, w_h(x)) d\mu &\leq \int_A f(x, u(x)) d\mu + r_h. \end{aligned}$$

It is easy to verify that (w_h) satisfies the properties in (5.9).

Let us set now $\gamma_h = f(\cdot, w_h(\cdot))$ and $\gamma = f(\cdot, u(\cdot))$. We claim that

$$(5.14) \quad \gamma_h \rightarrow \gamma \quad \text{strongly in } L^1(A, \mu).$$

Indeed, as (w_h) converges to u p -q.e. on A , by the lower semicontinuity of $f(x, \cdot)$ we get $\gamma(x) \leq \liminf_{h \rightarrow \infty} \gamma_h(x)$. By (5.9) and Lemma 4.6, it follows that (γ_h) converges to γ in the strong topology of $L^1(A, \mu)$.

In view of (5.14) it is not restrictive to assume that for every $h \in \mathbb{N}$

$$(5.15) \quad \int_A |\gamma_h - \gamma| d\mu \leq \frac{1}{2^h}.$$

At this point let us apply Lemma 4.5 to the sequence $(w_j)_{j \geq h}$ for every $h \in \mathbb{N}$. We obtain a sequence $(v_{h,j})_{j \geq h}$ of functions in $W^{1,p}(A, \mathbb{R}^m) \cap L^\infty(A, \mathbb{R}^m)$ such that

$$\begin{aligned} v_{h,j}(x) &\in \text{co}\{w_h(x), w_{h+1}(x), \dots, w_j(x)\} \quad \text{for } p\text{-q.e. } x \in A, \\ v_{h,j} &\rightarrow u \quad \text{strongly in } W^{1,p}(A, \mathbb{R}^m) \text{ as } j \rightarrow \infty. \end{aligned}$$

By a standard argument we can find a strictly increasing sequence (j_h) of positive integers such that (v_{h,j_h}) converges to u in $W^{1,p}(A, \mathbb{R}^m)$. Define $v_h = v_{h,j_h}$ for every $h \in \mathbb{N}$. Then $v_h \in W^{1,p}(A, \mathbb{R}^m) \cap L^\infty(A, \mathbb{R}^m)$ and

$$\begin{aligned} v_h(x) &\in \text{co}\{w_h(x), \dots, w_{j_h}(x)\} \quad \text{for } p\text{-q.e. } x \in A, \\ v_h &\rightarrow u \quad \text{strongly in } W^{1,p}(A, \mathbb{R}^m). \end{aligned}$$

In particular, a suitable sequence (k_h) exists such that $v_h(x) \in C_{k_h}(x)$ for p -q.e. $x \in A$. Now we only need to verify that (5.8) holds for the sequence (v_h) just obtained. By Lemma 3.2 we can write $v_h(x) = \sum_{i=h}^{k_h} \psi_h^i(x) w_i(x)$ for p -q.e. $x \in A$, where $\psi_h: A \rightarrow \Sigma_{k_h-h+1}$ are μ -measurable. Let us now make use of the convexity of f , together with (5.15):

$$\begin{aligned} \int_A f(x, v_h(x)) d\mu &\leq \sum_{i=h}^{k_h} \int_A \psi_h^i(x) |\gamma_i(x) - \gamma(x)| d\mu + \int_A \gamma(x) d\mu \\ &\leq \sum_{i=h}^{k_h} \frac{1}{2^i} + \int_A \gamma(x) d\mu \leq \int_A \gamma(x) d\mu + \frac{1}{2^h}. \end{aligned}$$

This implies

$$\limsup_{h \rightarrow \infty} \int_A f(x, v_h(x)) d\mu \leq \int_A f(x, u(x)) d\mu.$$

Finally, let us remove the additional assumption (5.7). Fix $U \subset\subset A$ and a sequence (r_h) of positive real numbers decreasing to 0. For every $\varepsilon > 0$ there exists an open set $A_\varepsilon \subseteq \Omega$, with $\text{cap}_p(A_\varepsilon, \Omega) < \varepsilon$, such that $u_i|_{\Omega \setminus A_\varepsilon}$ and $u|_{\Omega \setminus A_\varepsilon}$ are continuous for every $i \in \mathbb{N}$. In particular, the multifunction C_k is continuous on $\Omega \setminus A_\varepsilon$ with respect to the Hausdorff metric. By Lemma 4.3 in Chapter I for every $h \in \mathbb{N}$ there exists $n_h^\varepsilon \in \mathbb{N}$ such that $B_{r_h/2}(u(x)) \cap C_k(x) \neq \emptyset$ for every $k \geq n_h^\varepsilon$ and for every $x \in U \setminus A_\varepsilon$. Let z_ε be the capacitary potential of A_ε and $u_\varepsilon = (1 - z_\varepsilon)u + z_\varepsilon u_1$, where u_1 is the first term of the sequence (u_i) . Then one can easily check that (u_ε) converges to u in $W^{1,p}(\Omega, \mathbb{R}^m)$, that $u_\varepsilon \in K(x)$ for p -q.e. $x \in U$, and that for every $h \in \mathbb{N}$ there exists $n_h^\varepsilon \in \mathbb{N}$ such that $B_{r_h/2}(u_\varepsilon(x)) \cap C_k(x) \neq \emptyset$ for p -q.e. $x \in U$ and for every $k \geq n_h^\varepsilon$. Therefore we can apply the previous result for u_ε and U in place of u and A ; this gives

$$G(u_\varepsilon, U) \leq \int_U f(x, u_\varepsilon(x)) d\mu + \nu_0(U).$$

Since

$$\int_U f(x, u_\varepsilon(x)) d\mu \leq \int_A [(1 - z_\varepsilon(x))f(x, u(x)) + z_\varepsilon(x)f(x, u_1(x))] d\mu$$

and $\int_A f(x, u(x)) d\mu < +\infty$ by assumption, the lower semicontinuity of G and the dominated convergence theorem imply

$$G(u, U) \leq \liminf_{\varepsilon \rightarrow 0^+} G(u_\varepsilon, U) \leq \int_A f(x, u(x)) d\mu + \nu_0(A).$$

Taking the supremum for $U \subset\subset A$ we get (5.6).

Step 6. In view of Step 4 and Step 5 we get

$$G(u, A) = \int_A f(x, u(x)) d\mu + \nu_0(A)$$

for every $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ with $u(x) \in K(x)$ for p -q.e. $x \in A$. Property (ii) now follows by taking into account that if $u \in \text{dom}G(\cdot, A)$ then $u(x) \in K(x)$ for p -q.e. $x \in A$ by Remark 5.2. \square

6. Integral representation on $W^{1,p}(\Omega, \mathbf{R}^m)$

We now eliminate (Theorem 6.1) the restrictive condition $u \in L^\infty(\Omega, \mathbf{R}^m)$ considered in the previous section. Furthermore, Proposition 6.3 will allow us to treat in a unified way both cases of the representation formula established in Theorem 5.4(ii). Thus, we achieve (Theorem 6.5) the conclusive integral representation theorem, which is the main result of the chapter.

Given $G \in \mathcal{G}_p$ let us define

$$(6.1) \quad \bar{\nu}(B) = \inf\{G(u, B) : u \in W^{1,p}(\Omega, \mathbf{R}^m)\},$$

for every $B \in \mathcal{B}(\Omega)$. It is easily seen that the proof of Proposition 2.7 still works for the set function $\bar{\nu}$ on every $\Omega' \in \mathcal{A}(\Omega)$ with $\text{dom}G(\cdot, \Omega') \neq \emptyset$. Therefore, on such sets, $\bar{\nu}$ is a positive finite Borel measure.

Theorem 6.1. *Let $G \in \mathcal{G}_p$ and assume $\text{dom}G(\cdot, \Omega) \neq \emptyset$. Then the conclusions of Theorem 5.4 still hold with u (in item (ii)) ranging all over $W^{1,p}(\Omega, \mathbf{R}^m)$ and ν_0 replaced by the measure $\bar{\nu}$ defined in (6.1).*

Proof. For every $v \in W^{1,p}(\Omega, \mathbf{R}^m)$ and for every $B \in \mathcal{B}(\Omega)$ let us define

$$\begin{aligned} X_v &= \{u \in W^{1,p}(\Omega, \mathbf{R}^m) : u - v \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)\}, \\ \nu_v(B) &= \inf\{G(u, B) : u \in X_v\}. \end{aligned}$$

By a suitable application of Theorem 5.4, it turns out that for every $v \in \text{dom}G(\cdot, \Omega)$ there exist a positive finite Borel measure μ_v on Ω , absolutely continuous with respect to the p -capacity, and a Borel function $f_v: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ such that

- (i) for every $x \in \Omega$ the function $f_v(x, \cdot)$ is convex and lower semicontinuous on \mathbf{R}^m ;
- (ii) for every $u \in X_v$ and for every $A \in \mathcal{A}(\Omega)$

$$(6.2) \quad G(u, A) = \begin{cases} \int_A f_v(x, u(x)) d\mu_v + \nu_v(A), & \text{if } u(x) \in K(x) \text{ for } p\text{-q.e. } x \in A, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $K = K_\Omega$ is the closed valued multifunction from Ω to \mathbf{R}^m given by Proposition 5.1 for $A = \Omega$.

Step 1. Let us show first that for every $v \in \text{dom}G(\cdot, \Omega)$, $u \in W^{1,p}(\Omega, \mathbf{R}^m)$, and $A \in \mathcal{A}(\Omega)$ we have

$$(6.3) \quad G(u, A) < +\infty \quad \text{if and only if} \quad \begin{cases} u(x) \in K(x) & \text{for } p\text{-q.e. } x \in A, \\ \int_A f_v(x, u(x)) d\mu_v + \nu_v(A) < +\infty. \end{cases}$$

By the definition of K and Remark 5.2, if $G(u, A) < +\infty$ then $u(x) \in K(x)$ for p -q.e. $x \in A$. Hence, let us assume that $u(x) \in K(x)$ for p -q.e. $x \in A$ and prove that $G(u, A) < +\infty$ if and only if $\int_A f_v(x, u(x)) d\mu_v + \nu_v(A) < +\infty$.

For every $k \in \mathbf{N}$, let $T_k: \mathbf{R}^m \rightarrow \mathbf{R}^m$ be the orthogonal projection onto the ball $\overline{B}_k(0)$; by Lemma 2.5, for every $w \in W^{1,p}(\Omega, \mathbf{R}^m)$ the function $T_k \circ w$ belongs to $W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$, and the sequence $(T_k \circ w)$ converges to w in the strong topology of $W^{1,p}(\Omega, \mathbf{R}^m)$ as k tends to ∞ .

For every $k \in \mathbf{N}$ let us set $u_k = v + T_k \circ (u - v)$. By (6.2) we have

$$(6.4) \quad G(u_k, A) = \int_A f_v(x, u_k(x)) d\mu_v + \nu_v(A).$$

Assume now $G(u, A) < +\infty$. By (6.4) and Lemma 5.3 we have

$$\int_A f_v(x, u_k(x)) d\mu_v + \nu_v(A) \leq G(u, A) + G(v, A).$$

Since, up to a subsequence, (u_k) converges to u p -q.e. on Ω , the Fatou Lemma and the lower semicontinuity of $f_v(x, \cdot)$ ensure that

$$\int_A f_v(x, u(x)) d\mu_v + \nu_v(A) \leq G(v, A) + G(u, A) < +\infty.$$

Conversely, assume $\int_A f_v(x, u(x)) d\mu_v + \nu_v(A) < +\infty$. For every $k \in \mathbb{N}$, by (6.2), (6.4), and by the convexity of $f_v(x, \cdot)$ we get

$$(6.5) \quad \begin{aligned} G(u_k, A) &\leq \int_A f_v(x, u(x)) d\mu_v + \int_A f_v(x, v(x)) d\mu_v + \nu_v(A) = \\ &= \int_A f_v(x, u(x)) d\mu_v + G(v, A). \end{aligned}$$

Hence, by the lower semicontinuity of $G(\cdot, A)$ we conclude that $G(u, A) < +\infty$.

Step 2. Let us fix $A \in \mathcal{A}(\Omega)$ and $u, v \in \text{dom} G(\cdot, \Omega)$. We claim that

$$(6.6) \quad G(u, A) = \int_A f_v(x, u(x)) d\mu_v + \nu_v(A).$$

Let us show first that for every $w \in X_u$ with $w(x) \in \text{co}\{u(x), v(x)\}$ for p -q.e. $x \in \Omega$, we have

$$(6.7) \quad G(w, A) \leq \int_A f_v(x, w(x)) d\mu_v + \nu_v(A).$$

Let us fix $w \in X_u$ with $w(x) \in \text{co}\{u(x), v(x)\}$ for p -q.e. $x \in \Omega$, and let $u_k = v + T_k \circ (w - v)$ for $k \in \mathbb{N}$. By the lower semicontinuity of $G(\cdot, A)$ and by (6.2)

$$(6.8) \quad G(w, A) \leq \liminf_{k \rightarrow \infty} G(u_k, A) = \liminf_{k \rightarrow \infty} \int_A f_v(x, u_k(x)) d\mu_v + \nu_v(A).$$

Note that (u_k) converges to w p -q.e. on Ω . Since $u_k(x)$ is on the segment with endpoints $u(x)$ and $v(x)$, by the convexity of f_v it turns out that $f_v(x, u_k(x)) \leq f_v(x, v(x)) + f_v(x, u(x))$ for p -q.e. $x \in \Omega$. From (6.3) we have $\int_A f_v(x, v(x)) d\mu_v < +\infty$ and $\int_A f_v(x, u(x)) d\mu_v < +\infty$. Hence, by the continuity property of $f_v(x, \cdot)$ along line segments ([39], Corollary 7.5.1) and the dominated convergence theorem

$$(6.9) \quad \lim_{k \rightarrow \infty} \int_A f_v(x, u_k(x)) d\mu_v = \int_A f_v(x, w(x)) d\mu_v.$$

In view of (6.8), this implies (6.7).

From (6.7) with $w = u$ we obtain

$$G(u, A) \leq \int_A f_v(x, u(x)) d\mu_v + \nu_v(A).$$

Let us now prove the opposite inequality.

If in (6.7) we apply (6.2) to represent $G(w, A)$ we obtain

$$\int_A f_u(x, w(x)) d\mu_u + \nu_u(A) \leq \int_A f_v(x, w(x)) d\mu_v + \nu_v(A).$$

By exchanging now the roles of u and v we obtain that for every $w \in X_v$ with $w(x) \in \text{co}\{u(x), v(x)\}$ for p -q.e. $x \in \Omega$

$$\int_A f_v(x, w(x)) d\mu_v + \nu_v(A) \leq \int_A f_u(x, w(x)) d\mu_u + \nu_u(A).$$

Now, if we take $w = v + T_k \circ (u - v)$ and argue as for (6.9), by the Fatou Lemma we get

$$\begin{aligned} \int_A f_v(x, u(x)) d\mu_v + \nu_v(A) &\leq \liminf_{k \rightarrow \infty} \int_A f_u(x, v(x) + T_k(u(x) - v(x))) d\mu_u + \nu_u(A) \\ &= \int_A f_u(x, u(x)) d\mu_u + \nu_u(A) = G(u, A). \end{aligned}$$

Step 3. For every $v \in \text{dom}G(\cdot, \Omega)$, $A \in \mathcal{A}(\Omega)$, and $u \in \text{dom}G(\cdot, A)$ it turns out that

$$(6.10) \quad G(u, A) = \int_A f_v(x, u(x)) d\mu_v + \nu_v(A).$$

This follows by applying the same argument used in Step 4 of Theorem 5.4.

Step 4. Since $\text{dom}G(\cdot, \Omega) \neq \emptyset$, there exists a function v for which (6.10) holds for every $A \in \mathcal{A}(\Omega)$ and $u \in \text{dom}G(\cdot, A)$. Finally, we obtain that for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$

$$(6.11) \quad G(u, A) = \begin{cases} \int_A f_v(x, u(x)) d\mu_v + \nu_v(A), & \text{if } u(x) \in K(x) \text{ for } p\text{-q.e. } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

Indeed, if $u(x) \in K(x)$ for p -q.e. $x \in A$ and $u \notin \text{dom}G(\cdot, A)$, then by (6.3) we have $\int_A f_v(x, u(x)) d\mu_v + \nu_v(A) = +\infty$.

So far we have proved the integral representation by means of any of the measures ν_v with $v \in \text{dom}G(\cdot, \Omega)$. We claim that for every $v \in \text{dom}G(\cdot, \Omega)$ and $B \in \mathcal{B}(\Omega)$

$$(6.12) \quad \nu_v(B) = \inf\{G(u, B) : u \in W^{1,p}(\Omega, \mathbf{R}^m)\}.$$

Let $B \in \mathcal{B}(\Omega)$ and $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ with $G(u, B) < +\infty$. In view of the definition of G on Borel sets, by (6.11) we have

$$G(u, B) = \int_B f_v(x, u(x)) d\mu_v + \nu_v(B) \geq \nu_v(B);$$

hence, $\inf\{G(u, B) : u \in W^{1,p}(\Omega, \mathbf{R}^m), G(u, B) < +\infty\} \geq \nu_v(B)$. By the definition of $\nu_v(B)$, this implies (6.12). \square

The following proposition shows that, given the measures μ and ν , the function f obtained in the integral representation theorem is essentially unique.

Proposition 6.2. *Let $G \in \mathcal{G}_p$ with $\text{dom}G(\cdot, \Omega) \neq \emptyset$ and let $K = K_\Omega$ be the closed valued multifunction from Ω to \mathbf{R}^m given by Proposition 5.1 for $A = \Omega$. Let μ and ν be two positive finite Borel measures on Ω , with μ absolutely continuous with respect to the p -capacity, and let $f_1, f_2: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ be two Borel functions such that $f_1(x, \cdot)$ and $f_2(x, \cdot)$ are convex and lower semicontinuous on \mathbf{R}^m for μ -a.e. $x \in \Omega$. Assume that for every $A \in \mathcal{A}(\Omega)$ and for every $u \in \text{dom}G(\cdot, A)$ we have $G(u, A) = \int_A f_i(x, u(x)) d\mu + \nu(A)$ for $i = 1, 2$. Then $f_1(x, \xi) = f_2(x, \xi)$ for μ -a.e. $x \in \Omega$ and for every $\xi \in K(x)$.*

Proof. By a translation we can easily reduce the problem to the case $G(0, \Omega) < +\infty$. Let (u_i) and $C_k(x)$ be as in the proof of Theorem 5.4. By Proposition 2.9 we have

$$G(u, A) = \int_A f_1(x, u(x)) d\mu + \nu(A) = \int_A f_2(x, u(x)) d\mu + \nu(A)$$

for every $A \in \mathcal{A}(\Omega)$ and for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ with $u(x) \in C_k(x)$ for p -q.e. $x \in \Omega$. By Proposition 3.8 it turns out that $f_1(x, \xi) = f_2(x, \xi)$ for μ -a.e. $x \in \Omega$ and for every $\xi \in C_k(x)$. Hence the equality holds for every $\xi \in \text{ri}K(x) \subseteq \bigcup_{k=1}^\infty C_k(x)$, and, therefore, for every $\xi \in K(x)$ by the continuity along line segments (see [39], Corollary 7.5.1). \square

Proposition 6.3. *Let $K(x)$ be a closed and convex valued multifunction from Ω to \mathbf{R}^m for which there exists a sequence (u_k) of functions in $W^{1,p}(\Omega, \mathbf{R}^m)$ such that $K(x) = \text{cl}\{u_k(x) : k \in \mathbf{N}\}$ for p -q.e. $x \in \Omega$. Then, there exists a positive finite Borel measure ρ on Ω , absolutely continuous with respect to the p -capacity, such that for every $A \in \mathcal{A}(\Omega)$ and for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ the following conditions are equivalent:*

- (i) $u(x) \in K(x)$ for p -q.e. $x \in A$,
- (ii) $u(x) \in K(x)$ for ρ -a.e. $x \in A$.

Proof. It is not restrictive to assume that $0 \in K(x)$ for p -q.e. $x \in \Omega$. Moreover, we can suppose that $u_k \in W^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$ for every $k \in \mathbf{N}$. Indeed, if $T_h \circ u_k$, with $h \in \mathbf{N}$, denotes the truncation introduced in Lemma 2.5, it turns out that $K(x) = \text{cl}\{(T_h \circ u_k)(x) : h, k \in \mathbf{N}\}$, since $K(x)$ is closed and convex and $(T_h \circ u_k)_h$ converges to u_k strongly in $W^{1,p}(\Omega, \mathbf{R}^m)$, as h tends to ∞ .

Let us note that, by a standard cut-off argument, it is enough to consider the case $A = \Omega$. Moreover, (i) clearly implies (ii) as ρ is absolutely continuous with respect to the p -capacity.

Step 1. Here we prove that (ii) implies (i) for $u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$ under the additional assumption that $\partial\Omega$ is smooth.

Let us define the convex sets

$$\begin{aligned}\mathcal{K} &= \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K(x) \text{ for } p\text{-q.e. } x \in \Omega\}, \\ \mathcal{K}_k &= \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : u(x) \in K(x) + \tfrac{1}{k}B_1(0) \text{ for } p\text{-q.e. } x \in \Omega\},\end{aligned}$$

for every $k \in \mathbf{N}$. Since $W_0^{1,p}(\Omega, \mathbf{R}^m)$ is separable, the set \mathcal{K}_k is the intersection of a countable family of closed half-spaces of $W_0^{1,p}(\Omega, \mathbf{R}^m)$. Hence, there exists a sequence $(\mu_{k,h})_h$ in $W^{-1,p'}(\Omega, \mathbf{R}^m)$, with $p' = \frac{p}{p-1}$, and a sequence $(a_{k,h})_h$ in \mathbf{R} such that

$$\mathcal{K}_k = \bigcap_{h \in \mathbf{N}} \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : \langle \mu_{k,h}, u \rangle \geq a_{k,h}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,p'}(\Omega, \mathbf{R}^m)$ and $W_0^{1,p}(\Omega, \mathbf{R}^m)$.

Denote by $\mathcal{M}(\Omega, \mathbf{R}^m)$ the space of all \mathbf{R}^m -valued Radon measures on Ω with bounded total variation. We say that an element $T \in W^{-1,p'}(\Omega, \mathbf{R}^m)$ belongs to $\mathcal{M}(\Omega, \mathbf{R}^m)$ if there exists $\mu \in \mathcal{M}(\Omega, \mathbf{R}^m)$ such that

$$\langle T, \varphi \rangle = \int_{\Omega} \varphi d\mu$$

for every $\varphi \in C_0^\infty(\Omega, \mathbf{R}^m)$. In this case T and μ will be identified.

Let us prove that $\mu_{k,h} \in \mathcal{M}(\Omega, \mathbf{R}^m)$ for every $h, k \in \mathbf{N}$. Fix $\varphi \in C_0^\infty(\Omega, \mathbf{R}^m)$ with $\|\varphi\|_\infty \leq 1$. Since $u_1 + \frac{1}{k}\varphi$ and $u_1 - \frac{1}{k}\varphi$ belong to \mathcal{K}_k , we have

$$-k(\langle \mu_{k,h}, u_1 \rangle - a_{k,h}) \leq \langle \mu_{k,h}, \varphi \rangle \leq k(\langle \mu_{k,h}, u_1 \rangle - a_{k,h}).$$

Therefore, there exists $C_{k,h} > 0$ such that $|\langle \mu_{k,h}, \varphi \rangle| \leq C_{k,h} \|\varphi\|_\infty$ for every $\varphi \in C_0^\infty(\Omega, \mathbf{R}^m)$. Hence $\mu_{k,h}|_{C_0^\infty(\Omega, \mathbf{R}^m)}$ can be uniquely extended to a continuous linear functional on the space of continuous functions on Ω vanishing on $\partial\Omega$. We conclude by the Riesz representation theorem.

Since $\mathcal{K} = \bigcap_{k \in \mathbf{N}} \mathcal{K}_k$, we can assert that there exists a sequence (μ_h) in $W^{-1,p'}(\Omega, \mathbf{R}^m) \cap \mathcal{M}(\Omega, \mathbf{R}^m)$ and a sequence (a_h) in \mathbf{R} such that

$$(6.13) \quad \mathcal{K} = \bigcap_{h \in \mathbf{N}} \{u \in W_0^{1,p}(\Omega, \mathbf{R}^m) : \langle \mu_h, u \rangle \geq a_h\}.$$

Moreover, as $\mu_h \in W^{-1,p'}(\Omega, \mathbf{R}^m) \cap \mathcal{M}(\Omega, \mathbf{R}^m)$, by [30] and [11], Lemma 2 we have that $|\mu_h|$ is absolutely continuous with respect to the p -capacity. For every $B \in \mathcal{B}(\Omega)$ define

$$\rho(B) = \sum_{h=1}^{\infty} 2^{-h} \frac{|\mu_h|(B)}{|\mu_h|(\Omega)}$$

(clearly, we can assume that $|\mu_h|(\Omega) > 0$ for every $h \in \mathbf{N}$). Then ρ is a positive finite Borel measure on Ω absolutely continuous with respect to the p -capacity. Let g_h be the Radon-Nikodym derivative of μ_h with respect to ρ . Then $g_h \in L^1(\Omega, \rho)$. By Corollary 6 in [11], for every $u \in W_0^{1,p}(\Omega, \mathbf{R}^m) \cap L^\infty(\Omega, \mathbf{R}^m)$, we have $u \cdot g_h \in L^1(\Omega, \rho)$ and

$$(6.14) \quad \langle \mu_h, u \rangle = \int_{\Omega} u \cdot g_h \, d\rho.$$

Let us now prove that (ii) implies (i). Fix $u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$ with $u(x) \in K(x)$ for ρ -a.e. $x \in \Omega$. Assume first that u belongs to $L^\infty(\Omega, \mathbf{R}^m)$.

For every $h \in \mathbf{N}$ and $v \in \mathcal{K} \cap L^\infty(\Omega, \mathbf{R}^m)$, by (6.14) we have

$$a_h \leq \langle \mu_h, v \rangle = \int_{\Omega} v \cdot g_h \, d\rho,$$

hence

$$(6.15) \quad a_h \leq \inf_{v \in \mathcal{K} \cap L^\infty} \int_{\Omega} v \cdot g_h \, d\rho.$$

In view of the fact that the functions u_k are in $L^\infty(\Omega, \mathbf{R}^m)$, it turns out that $K(x) = \rho$ -ess sup $\{v(x)\}$ for ρ -a.e. $x \in \Omega$; then Theorem 4.4 yields

$$(6.16) \quad \inf_{v \in \mathcal{K} \cap L^\infty} \int_{\Omega} v \cdot g_h \, d\rho = \int_{\Omega} \inf_{\xi \in K(x)} \xi \cdot g_h(x) \, d\rho.$$

By the assumption $u(x) \in K(x)$ for ρ -a.e. $x \in \Omega$, by (6.15) and (6.16), it follows that

$$a_h \leq \int_{\Omega} u(x) \cdot g_h(x) \, d\rho = \langle \mu_h, u \rangle$$

for every $h \in \mathbf{N}$. By (6.13) this proves that $u \in \mathcal{K}$, i.e., $u(x) \in K(x)$ for p -q.e. $x \in \Omega$.

Consider now a general $u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$; let us note that, since $K(x)$ is convex and $0 \in K(x)$ for p -q.e. $x \in \Omega$, the condition $u(x) \in K(x)$ for ρ -a.e. $x \in \Omega$ implies

that $(T_h \circ u)(x) \in K(x)$ for ρ -a.e. $x \in \Omega$ and for every $h \in \mathbf{N}$. The previous step and the p -q.e. convergence of $(T_h \circ u)$ to u allow us to conclude as $K(x)$ is closed.

Step 2. Let us now prove that (ii) implies (i) for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ without assuming the smoothness of the boundary of Ω .

Let (Ω_h) be a sequence of open subsets of Ω with $\Omega_h \subset\subset \Omega_{h+1}$, $\bigcup_h \Omega_h = \Omega$, and $\partial\Omega_h$ smooth. Let φ_h be a $C_0^1(\Omega_h)$ function with $\varphi_h = 1$ on Ω_{h-1} and $0 \leq \varphi_h \leq 1$. Define

$$K_h(x) = \text{cl}\{\varphi_h(x)u_k(x) : k \in \mathbf{N}\}$$

for p -q.e. $x \in \Omega_h$. By Step 1 there exists a positive finite Borel measure ρ_h on Ω_h , absolutely continuous with respect to the p -capacity, and such that for every $u \in W_0^{1,p}(\Omega, \mathbf{R}^m)$ the condition $u(x) \in K_h(x)$ for p -q.e. $x \in \Omega_h$ is equivalent to the condition $u(x) \in K_h(x)$ for ρ_h -a.e. $x \in \Omega_h$. We can consider ρ_h as a measure on Ω by setting $\rho_h(B) = \rho_h(B \cap \Omega_h)$ for $B \in \mathcal{B}(\Omega)$. Let us define

$$\rho(B) = \sum_{h=1}^{\infty} 2^{-h} \frac{\rho_h(B)}{\rho_h(\Omega)}$$

for every $B \in \mathcal{B}(\Omega)$. Then ρ is a positive finite Borel measure on Ω which is absolutely continuous with respect to the p -capacity.

Let us fix $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ with $u(x) \in K(x)$ for ρ -a.e. $x \in \Omega$. Then, $\varphi_h(x)u(x) \in K_h(x)$ for ρ_h -a.e. $x \in \Omega_h$, so that $\varphi_h(x)u(x) \in K_h(x)$ for p -q.e. $x \in \Omega_h$. Since $\varphi_h = 1$ on Ω_{h-1} , we obtain that $u(x) \in K(x)$ for p -q.e. $x \in \Omega_{h-1}$. As h is arbitrary, we conclude that $u(x) \in K(x)$ for p -q.e. $x \in \Omega$. \square

Lemma 6.4. *Let $G \in \mathcal{G}_p$ and define Ω_0 to be the union of all $A \in \mathcal{A}(\Omega)$ such that $\text{dom}G(\cdot, A) \neq \emptyset$. Then $\text{dom}G(\cdot, A) \neq \emptyset$ for every $A \in \mathcal{A}(\Omega)$ with $A \subset\subset \Omega_0$.*

Proof. By induction we can reduce ourselves to prove that, whenever A_1, A_2 are open subsets of Ω with $\text{dom}G(\cdot, A_1) \neq \emptyset$ and $\text{dom}G(\cdot, A_2) \neq \emptyset$, then $\text{dom}G(\cdot, A) \neq \emptyset$ for every open set $A \subset\subset A_1 \cup A_2$.

Let A_1, A_2 and A be as above, and let $A' \subset\subset A_1$ with $A \subset\subset A' \cup A_2$. We shall show that $\text{dom}G(\cdot, A' \cup A_2) \neq \emptyset$, which clearly implies $\text{dom}G(\cdot, A) \neq \emptyset$. Consider a function $\varphi \in C_0^1(A_1)$ with $\varphi = 1$ on A' and $0 \leq \varphi \leq 1$. By assumption we can find $u \in \text{dom}G(\cdot, A_1)$ and $v \in \text{dom}G(\cdot, A_2)$; define $w = \varphi u + (1 - \varphi)v$. Then, by the usual properties of the class \mathcal{G}_p it is easy to see that $G(w, A' \cup A_2) < +\infty$. \square

Theorem 6.5. *Let Ω be an open subset of \mathbf{R}^n (not necessarily bounded) and let $G \in \mathcal{G}_p$. Then, there exist a positive finite Borel measure μ on Ω , absolutely continuous with respect to the p -capacity, a positive Borel measure ν on Ω , and a Borel function $f: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ with the following properties:*

- (i) *for every $x \in \Omega$ the function $f(x, \cdot)$ is convex and lower semicontinuous on \mathbf{R}^m ;*
- (ii) *for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and for every $A \in \mathcal{A}(\Omega)$*

$$(6.17) \quad G(u, A) = \int_A f(x, u(x)) d\mu + \nu(A).$$

Proof. Let $\bar{\nu}$ be the function on $\mathcal{B}(\Omega)$ defined in (6.1). Let us set

$$\nu(A) = \sup\{\bar{\nu}(A') : A' \in \mathcal{A}(\Omega), A' \subset\subset A\}$$

for every $A \in \mathcal{A}(\Omega)$. Clearly ν is increasing with respect to the inclusion, and $\nu(\emptyset) = 0$. Moreover, for every $A \in \mathcal{A}(\Omega)$

$$(6.18) \quad \nu(A) < +\infty \Rightarrow A \subseteq \Omega_0.$$

Clearly ν is inner regular on $\mathcal{A}(\Omega)$; therefore, by Proposition 5.5 and Theorem 5.6 in [26], to prove that ν can be extended to a Borel measure on Ω it suffices to show that ν is subadditive and superadditive on $\mathcal{A}(\Omega)$.

Let $A_1, A_2 \in \mathcal{A}(\Omega)$, and note that

$$(6.19) \quad \nu(A_1 \cup A_2) = \sup\{\bar{\nu}(A'_1 \cup A'_2) : A'_i \in \mathcal{A}(\Omega), A'_i \subset\subset A_i \ (i = 1, 2)\}.$$

If $\nu(A_1)$ and $\nu(A_2)$ are finite, then $A_1, A_2 \subseteq \Omega_0$ by (6.18). Since, by Lemma 6.4, $\bar{\nu}$ is a measure on every $\Omega' \subset\subset \Omega_0$, from (6.19) it follows that $\nu(A_1 \cup A_2) \leq \nu(A_1) + \nu(A_2)$. In a similar way we get superadditivity.

This allows us to conclude that the set function $\nu: \mathcal{B}(\Omega) \rightarrow [0, +\infty]$ defined by

$$\nu(B) = \inf\{\nu(A) : A \in \mathcal{A}(\Omega), B \subseteq A\}$$

is a Borel measure on Ω and that $\nu(B) = \bar{\nu}(B)$ for every $B \in \mathcal{B}(\Omega)$ with $B \subset\subset \Omega_0$.

Let us construct now μ and f . Let (Ω_h) be a sequence of open subsets of Ω_0 with a smooth boundary such that $\Omega_h \subset\subset \Omega_{h+1}$ and $\Omega_0 = \bigcup_h \Omega_h$. In particular, $\text{dom}G(\cdot, \Omega_h) \neq \emptyset$ by Lemma 6.4. Since there exists an extension operator from

$W^{1,p}(\Omega_h, \mathbf{R}^m)$ to $W^{1,p}(\Omega, \mathbf{R}^m)$, it is possible to apply Theorem 6.1 to each Ω_h using $W^{1,p}(\Omega, \mathbf{R}^m)$ instead of $W^{1,p}(\Omega_h, \mathbf{R}^m)$. Let K_{Ω_h} be the multifunction defined in Proposition 5.1 for $A = \Omega_h$; then there exist a positive finite Borel measure μ_h on Ω_h , absolutely continuous with respect to the p -capacity, and a Borel function $f_h: \Omega_h \times \mathbf{R}^m \rightarrow [0, +\infty]$ such that

- (a) for every $x \in \Omega_h$ the function $f_h(x, \cdot)$ is convex and lower semicontinuous on \mathbf{R}^m ;
- (b) for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega_h)$

$$G(u, A) = \begin{cases} \int_A f_h(x, u(x)) d\mu_h + \overline{\nu}(A), & \text{if } u(x) \in K_{\Omega_h}(x) \text{ for } p\text{-q.e. } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

Moreover, we have a uniqueness property for the integrand as stated in Proposition 6.2.

By Proposition 6.3, for every $h \in \mathbf{N}$ there exists a positive finite Borel measure ρ_h on Ω_h , absolutely continuous with respect to the p -capacity, such that for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ the condition $u(x) \in K_{\Omega_h}(x)$ for p -q.e. $x \in \Omega_h$ is equivalent to the condition $u(x) \in K_{\Omega_h}(x)$ for ρ_h -a.e. $x \in \Omega_h$. Let us consider μ_h and ρ_h as measures on Ω by setting $\mu_h(B) = \mu_h(B \cap \Omega_h)$ and $\rho_h(B) = \rho_h(B \cap \Omega_h)$ for every $B \in \mathcal{B}(\Omega)$. Define

$$\mu(B) = \sum_{h=1}^{\infty} 2^{-h} \frac{(\mu_h + \rho_h)(B)}{(\mu_h + \rho_h)(\Omega)},$$

$$g_h(x, \xi) = \begin{cases} f_h(x, \xi) \frac{d\mu_h}{d\mu}(x), & \text{if } x \in \Omega_h \text{ and } \xi \in K_{\Omega_h}(x), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $d\mu_h/d\mu$ is a fixed Borel representative of the Radon-Nikodym derivative. Then μ is a positive finite Borel measure on Ω , absolutely continuous with respect to the p -capacity. Since for every $h \in \mathbf{N}$ there is a sequence (v_i) in $W^{1,p}(\Omega, \mathbf{R}^m)$ such that $K_{\Omega_h}(x) = \text{cl}\{v_i(x) : i \in \mathbf{N}\}$, by Theorem III.9 and Proposition III.13 in [14], the graph of K_{Ω_h} belongs to $\mathcal{B}(\Omega_h) \times \mathcal{B}(\mathbf{R}^m)$. Therefore $g_h: \Omega_h \times \mathbf{R}^m \rightarrow [0, +\infty]$ is a Borel function, and for every $x \in \Omega_h$, the function $g_h(x, \cdot)$ is convex and lower semicontinuous on \mathbf{R}^m .

By recalling that $K_{\Omega_h} = K_{\Omega_{h+1}}$ p -q.e. on Ω_h (see Remark 5.2), and by using the uniqueness property of the integrand mentioned above, we easily obtain that $g_h(x, \cdot) = g_{h+1}(x, \cdot)$ for μ -a.e. $x \in \Omega_h$.

Therefore, there exists a Borel function $f: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ satisfying (i) and such that for every $h \in \mathbf{N}$

$$f(x, \cdot) = g_h(x, \cdot) \quad \text{on } \mathbf{R}^m \text{ for } \mu\text{-a.e. } x \in \Omega_h.$$

Let us now prove (ii). Fix $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$. If $A \setminus \Omega_0 \neq \emptyset$, then $\nu(A) = +\infty$ by (6.18). On the other hand, by the definition of Ω_0 we have $G(u, A) = +\infty$ for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$. Therefore

$$G(u, A) = \int_A f(x, u(x)) d\mu + \nu(A).$$

Let now $A \subseteq \Omega_0$ and $A' \subset\subset A$. Then $A' \subseteq \Omega_h$ for a suitable $h \in \mathbf{N}$. In view of the properties of the measure ρ_h , from the definition of g_h and f we easily obtain

$$G(u, A') = \int_{A'} g_h(x, u(x)) d\mu + \bar{\nu}(A') = \int_{A'} f(x, u(x)) d\mu + \bar{\nu}(A').$$

Therefore (ii) follows from the definition of ν taking the supremum for $A' \subset\subset A$. \square

Remark 6.6. Let $G \in \mathcal{G}_p$ and Ω_0 be as in Lemma 6.4. By Proposition 5.1 and Remark 5.2, there exists a closed valued multifunction K from Ω_0 to \mathbf{R}^m , unique up to sets of the p -capacity zero, such that

$$(6.20) \quad K(x) = K_A(x) \quad \text{for } p\text{-q.e. } x \in A$$

whenever $A \in \mathcal{A}(\Omega)$ and $\text{dom}G(\cdot, A) \neq \emptyset$. Moreover, $K(x)$ is non-empty and convex for p -q.e. $x \in \Omega_0$.

It is clear that the function f constructed in the proof of Theorem 6.5 satisfies the additional condition $f(x, \xi) = +\infty$ for μ -a.e. $x \in \Omega_0$ and for every $\xi \notin K(x)$. This is not necessarily true for every function f which satisfies conditions (i) and (ii) of Theorem 6.5. Let us consider, for instance, the functional

$$G(u, A) = \begin{cases} 0, & \text{if } u = 0 \text{ a.e. on } A, \\ +\infty, & \text{otherwise,} \end{cases}$$

in the case $n = m = 1$ and $\Omega = \mathbf{R}$. Then, clearly, $K(x) = \{0\}$ for p -q.e. $x \in \mathbf{R}$ and

$$(6.21) \quad G(u, A) = \int_A f(x, u(x)) dx,$$

with

$$f(x, \xi) = \begin{cases} 0, & \text{if } \xi = 0, \\ +\infty, & \text{if } \xi \neq 0. \end{cases}$$

But (6.21) holds also with $f(x, \xi) = a(x)|\xi|^2$, where $a: \mathbf{R} \rightarrow [0, +\infty[$ is any finite valued Borel function such that $\int_A a(x) dx = +\infty$ for every open subset A of \mathbf{R} (see [31], Section 43, Exercise 7).

Remark 6.7. If $\text{dom}G(\cdot, \Omega) = \emptyset$ and ν is not necessarily finite, the uniqueness result of Proposition 6.2 still holds, with an obvious localization of the proof, in the weaker form:

$$f_1(x, \xi) = f_2(x, \xi) \quad \text{for } \mu\text{-a.e. } x \in \Omega_0 \text{ and for every } \xi \in K(x),$$

where Ω_0 is defined in Lemma 6.4 and $K(x)$ is now defined by (6.20).

7. Quadratic functionals

In this section we show how certain algebraic properties of the functional G are inherited by the integrand which appears in the representation of G according to Theorem 6.5. We recall that a cone in a vector space X (with vertex at 0) is a set K such that $tx \in K$ for every $t > 0$ and for every $x \in K$.

Definition 7.1. Let X be a real vector space and let $p \in \mathbf{R}$. We say that a function $f: X \rightarrow [0, +\infty]$ is:

- (i) *positively homogeneous of degree p* on a cone K if $G(tx) = t^p G(x)$ for every $t > 0$ and for every $x \in K$;
- (ii) a (non-negative) *quadratic form* (with extended real values) on X if there exist a linear subspace Y of X and a symmetric bilinear form $B: Y \times Y \rightarrow \mathbf{R}$ such that

$$G(x) = \begin{cases} B(x, x), & \text{if } x \in Y, \\ +\infty, & \text{if } x \in X \setminus Y. \end{cases}$$

We shall refer to Y as the *domain* of G .

Remark 7.2. In the previous definition it is not restrictive to assume that B is defined over all of $X \times X$. Indeed, let Z be an algebraic complement of Y in X and denote by $P: X \rightarrow Y$ the canonical projection on Y associated to the pair (Y, Z) . Then, it is enough to consider the extension $(x, y) \mapsto B(Px, Py)$ defined for every $(x, y) \in X \times X$. As a consequence, if X is finite dimensional and $\dim X = m$, then there exists an $m \times m$ symmetric matrix (a_{ij}) such that $G(x) = \sum_{i,j=1}^m a_{ij} x^i x^j$ for every $x \in Y$, where x^1, \dots, x^m denote the components of x with respect to a fixed basis of X .

Theorem 7.3. *Let $G \in \mathcal{G}_p$ and $p \in \mathbf{R}$. Let f , μ , ν be as in Theorem 6.5 and let Ω_0 and K be as in Remark 6.6. Assume that $f(x, \xi) = +\infty$ for μ -a.e. $x \in \Omega_0$ and for every $\xi \notin K(x)$. Then the following properties hold:*

- (i) *if $G(\cdot, A)$ is positively homogeneous of degree p on $W^{1,p}(\Omega, \mathbf{R}^m)$ for every $A \in \mathcal{A}(\Omega)$, then $K(x)$ is a closed convex cone for p -q.e. $x \in \Omega_0$, and $f(x, \cdot)$ is positively homogeneous of degree p on $K(x)$ for μ -a.e. $x \in \Omega_0$; if, in addition, $p \neq 0$, then $\nu(B) = 0$ for every $B \in \mathcal{B}(\Omega_0)$;*
- (ii) *if $G(\cdot, A)$ is a quadratic form on $W^{1,p}(\Omega, \mathbf{R}^m)$ for every $A \in \mathcal{A}(\Omega)$, then $\nu = 0$, $K(x)$ is a linear subspace of \mathbf{R}^m for p -q.e. $x \in \Omega$, and for μ -a.e. $x \in \Omega$ the function $f(x, \cdot)$ is a quadratic form on \mathbf{R}^m with domain $K(x)$.*

Proof. Proof of (i). For every $A \in \mathcal{A}(\Omega)$ the positive homogeneity of degree p implies that $tu \in \text{dom}G(\cdot, A)$ whenever $t > 0$ and $u \in \text{dom}G(\cdot, A)$. Recalling the definition and properties of K_A given in Proposition 5.1, it is easy to see that, if $\text{dom}G(\cdot, A) \neq \emptyset$, then there exists a set $N \subseteq \Omega$ with $\text{cap}_p(N) = 0$ and such that for every $x \in A \setminus N$, $q \in \mathbf{Q}^+ \setminus \{0\}$ and $\xi \in K_A(x)$ we have $q\xi \in K_A(x)$. Since $K_A(x)$ is closed, it follows that $K_A(x)$ is a cone for every $x \in A \setminus N$. The convexity of $K_A(x)$ is proved in Proposition 5.1(iii). By the definition (6.20) of $K(x)$ and by the definition of Ω_0 we conclude that $K(x)$ is a closed convex cone for p -q.e. $x \in \Omega_0$.

Let us now prove that $f(x, \cdot)$ is positively homogeneous of degree p on $K(x)$ for μ -a.e. $x \in \Omega_0$. Let us first consider the case $p = 0$. If $A \in \mathcal{A}(\Omega)$ and $u \in \text{dom}G(\cdot, A)$, then the function $t \mapsto G(tu, A)$ from $[0, 1]$ into $[0, +\infty]$ is convex and lower semicontinuous; moreover, $G(tu, A) = G(u, A) < +\infty$ for every $t > 0$. Therefore, $G(0, A) = \lim_{t \rightarrow 0^+} G(tu, A) = G(u, A)$ for every $A \in \mathcal{A}(\Omega)$ and $u \in \text{dom}G(\cdot, A)$. This shows that $0 \in K(x)$ for p -q.e. $x \in \Omega_0$. By the uniqueness of the integrand stated in Remark 6.7, we conclude that $f(x, \xi) = f(x, 0)$ for μ -a.e. $x \in \Omega_0$ and for every $\xi \in K(x)$.

Assume now $p \neq 0$. Let $A \in \mathcal{A}(\Omega)$ and $w \in \text{dom}G(\cdot, A)$. Since $\nu(A) \leq G(tw, A) = t^p G(w, A)$ for every $t > 0$, taking the limit as $t \rightarrow 0^+$ or $t \rightarrow +\infty$ according to whether $p > 0$ or $p < 0$, we get $\nu(A) = 0$. In view of the positive homogeneity of $G(\cdot, A)$ and (6.17) we have

$$G(u, A) = \int_A \frac{1}{t^p} f(x, tu(x)) d\mu$$

for every $A \in \mathcal{A}(\Omega)$, $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $t > 0$. Now, by the uniqueness of the

integrand (Remark 6.7), we have $f(x, \xi) = (1/t^p)f(x, t\xi)$ for μ -a.e. $x \in \Omega_0$ and for every $\xi \in K(x)$.

Proof of (ii). Assume that $G(\cdot, A)$ is a quadratic form for every $A \in \mathcal{A}(\Omega)$. Then $G(0, A) = 0$ for every $A \in \mathcal{A}(\Omega)$, hence $\Omega_0 = \Omega$, $\nu(B) = 0$ for every $B \in \mathcal{B}(\Omega)$, and $f(x, 0) = 0$ for μ -a.e. $x \in \Omega$. Directly from Definition 7.1 it follows that $\text{dom}G(\cdot, A)$ is a linear space; in particular, $u + v$ and $-u$ belong to $\text{dom}G(\cdot, A)$ whenever $u, v \in \text{dom}G(\cdot, A)$. As in the first part of (i), it can be shown that for p -q.e. $x \in \Omega$, $\xi + \eta$ and $-\xi$ belong to $K(x)$ if $\xi, \eta \in K(x)$. Since $K(x)$ is a cone (part (i)), this guarantees that $K(x)$ is a linear subspace of \mathbf{R}^m for p -q.e. $x \in \Omega$.

If X is a (real) vector space, it is well known (Fréchet-Von Neumann-Jordan Theorem, see, for instance, [42]) that a function $G: X \rightarrow [0, +\infty]$ is a quadratic form if and only if $G(0) = 0$, G is positively homogeneous of degree 2, and satisfies the following “parallelogram identity”:

$$G(\xi + \eta) + G(\xi - \eta) = 2G(\xi) + 2G(\eta)$$

for every $\xi, \eta \in X$. Since $f(x, 0) = 0$ for μ -a.e. $x \in \Omega$ and $f(x, t\xi) = t^2 f(x, \xi)$ for μ -a.e. $x \in \Omega$ and for every $t > 0$, $\xi \in K(x)$ (see part (i)), to complete the proof of (ii) it remains only to show that $f(x, \cdot)$ satisfies the parallelogram identity on $K(x)$ for μ -a.e. $x \in \Omega$. Define the functional $H: [W^{1,p}(\Omega, \mathbf{R}^m)]^2 \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ as

$$H(u, v, A) = G(u + v, A) + G(u - v, A) = 2G(u, A) + 2G(v, A).$$

Since $\nu = 0$, from (6.17) we obtain

$$(7.1) \quad \begin{aligned} H(u, v, A) &= \int_A 2[f(x, u(x)) + f(x, v(x))] d\mu = \\ &\int_A [f(x, u(x) + v(x)) + f(x, u(x) - v(x))] d\mu \end{aligned}$$

for every $A \in \mathcal{A}(\Omega)$. Since $[W^{1,p}(\Omega, \mathbf{R}^m)]^2$ can be identified with $W^{1,p}(\Omega, \mathbf{R}^{2m})$, we can apply Remark 6.7 to the functional H , with the set $K(x) \times K(x)$ playing the role of $K(x)$ for p -q.e. $x \in \Omega$. Therefore, (7.1) gives that

$$2(f(x, \xi) + f(x, \eta)) = f(x, \xi + \eta) + f(x, \xi - \eta)$$

for μ -a.e. $x \in \Omega_0$ and for every $(\xi, \eta) \in K(x) \times K(x)$. □

Corollary 7.4. *Let $G \in \mathcal{G}_p$. Assume that $G(\cdot, A)$ is a quadratic form on $W^{1,p}(\Omega, \mathbf{R}^m)$ for every $A \in \mathcal{A}(\Omega)$. Then there exist:*

- (i) *a positive finite Borel measure μ on Ω , absolutely continuous with respect to the p -capacity,*
- (ii) *a symmetric $m \times m$ matrix (a_{ij}) of Borel functions from Ω into \mathbf{R} such that $\sum_{i,j=1}^m a_{ij}(x)\xi^i\xi^j \geq 0$ for p -q.e. $x \in \Omega$ and for every $\xi \in \mathbf{R}^m$,*
- (iii) *for every $x \in \Omega$ a linear subspace $V(x)$ of \mathbf{R}^m ,*

with the following properties: for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$

- (a) *if $G(u, A) < +\infty$, then $u(x) \in V(x)$ for p -q.e. $x \in A$;*
- (b) *if $u(x) \in V(x)$ for p -q.e. $x \in A$, then $G(u, A) = \int_A \sum_{i,j=1}^m a_{ij}(x)u^i(x)u^j(x) d\mu$.*

Proof. The conclusion follows from Theorems 6.5 and 7.3(ii), and from Remark 7.2. \square

Chapter III

LIMITS OF MINIMUM PROBLEMS WITH CONVEX OBSTACLES FOR VECTOR VALUED FUNCTIONS

Introduction

It is well known that a relaxation phenomenon may asymptotically occur for a sequence of minimum problems of the calculus of variations with varying obstacles. In this chapter a similar analysis is carried out for functionals defined on vector valued Sobolev functions, drawing general properties of the limit behaviour.

Given an open subset Ω of \mathbf{R}^n , let us consider the following problem:

$$(0.1) \quad \min \left\{ \int_{\Omega} W(x, Du(x)) dx : u \in H_0^1(\Omega, \mathbf{R}^m), u(x) \in K(x) \text{ for q.e. } x \in A \right\},$$

where: $m \geq 1$, $W(x, \eta)$ is quadratic in η and non-negative, K is a multifunction from Ω to the closed convex subsets of \mathbf{R}^m , $A \in \mathcal{A}(\Omega)$ (the family of the open subsets of Ω), and q.e. means quasi everywhere with respect to the usual capacity. Our aim is to discuss some general properties of the asymptotic behaviour of a sequence of problems of the form (0.1) relative to an arbitrary sequence of multifunctions (K_h) and under suitable assumptions on W . Hence, we shall first be interested in establishing a convergence result (compactness), and secondly in analysing the general features of the limit. This study will be developed in the context of Γ -convergence.

Some remarks are now in order. Let us notice that the multifunction K in (0.1) will be subject to no regularity condition other than the closedness and convexity. With a view to considering also constraints on “thin” sets (such as lines in \mathbf{R}^2 or surfaces in \mathbf{R}^3), the condition $u(x) \in K(x)$ is required to hold up to sets of null capacity. In the scalar case (i.e., $m = 1$) problem (0.1) takes the form

$$\min \left\{ \int_{\Omega} W(x, Du(x)) dx : u \in H_0^1(\Omega, \mathbf{R}^m), \varphi(x) \leq u(x) \leq \psi(x) \text{ for q.e. } x \in A \right\},$$

where φ and ψ are arbitrary functions from Ω to $[-\infty, +\infty]$. Thus, the asymptotic analysis of a sequence of minimum problems with unilateral ($\varphi = -\infty$ or $\psi = +\infty$) or

bilateral obstacles falls within this framework. Accordingly, we shall recover, as particular cases, some of the previous well-known results on the subject: see [13], [25], [22], [18], [17], [7], [8], [5] and [23].

As usual in this setting, problem (0.1) is identified by means of the functional $\int_{\Omega} W(x, Du) dx + G(u, A)$, with

$$(0.2) \quad G(u, A) = \begin{cases} 0, & \text{if } u(x) \in K(x) \text{ for q.e. } x \in A, \\ +\infty, & \text{otherwise,} \end{cases}.$$

($u \in H^1(\Omega, \mathbf{R}^m)$, $A \in \mathcal{A}(\Omega)$). Actually, we shall consider sequences in an abstract class \mathcal{G}_2 which contains in particular the obstacle constraint functionals. $G: H^1(\Omega, \mathbf{R}^m) \times \mathcal{A}(\Omega) \rightarrow [0 + \infty]$ belongs to \mathcal{G}_2 if: (i) for every $A \in \mathcal{A}(\Omega)$ the function $G(\cdot, A)$ is lower semicontinuous on $H^1(\Omega, \mathbf{R}^m)$; (ii) for every $u \in H^1(\Omega, \mathbf{R}^m)$ the set function $G(u, \cdot)$ is (the trace of) a Borel measure on Ω ; (iii) $G(u, A) = G(v, A)$ whenever $u, v \in H^1(\Omega, \mathbf{R}^m)$, $A \in \mathcal{A}(\Omega)$, and $u|_A = v|_A$; (iv) for every $A \in \mathcal{A}(\Omega)$ the function $G(\cdot, A)$ is convex on $H^1(\Omega, \mathbf{R}^m)$ and, if $u, v \in H^1(\Omega, \mathbf{R}^m)$ then $G(w, A) \leq G(u, A) + G(v, A)$ whenever w is a convex combination of u and v with smooth coefficients. These conditions are singled out since they are stable under Γ -convergence and strong enough to allow a suitable representation for the functionals of the class. Indeed, in Section 2 we recall the fundamental integral representation results for \mathcal{G}_2 , based on Chapter II. In Theorem 4.1 (compactness) we prove that every sequence in \mathcal{G}_2 admits, up to a subsequence, a limit functional which still belongs to \mathcal{G}_2 . Both the results are then applied to a sequence of obstacle problems, yielding as a limit a functional G of the form

$$(0.3) \quad G(u, A) = \int_A g(x, u(x)) d\mu + \nu(A) \quad (u \in H^1(\Omega, \mathbf{R}^m), A \in \mathcal{A}(\Omega)),$$

where μ and ν are positive Borel measures on Ω and $g: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$ is a Borel function, convex and lower semicontinuous in the second variable.

Section 5 is devoted to Dirichlet problems in perforated domains, which correspond to multifunctions K_h taking only the values $\{0\}$ and \mathbf{R}^m . Consequently, the representation (0.3) becomes

$$G(u, A) = \int_A u^T M u a d\rho \quad (u \in H^1(\Omega, \mathbf{R}^m), A \in \mathcal{A}(\Omega)),$$

where M is a symmetric $m \times m$ matrix of real Borel functions on Ω with $\xi^T M \xi \geq 0$ for every $\xi \in \mathbf{R}^m$, $a: \Omega \rightarrow [0, +\infty]$ is a Borel function and ρ is a positive Radon measure with

$\rho \in H^{-1}(\Omega)$. Bounds for the ratio between the maximum and the minimum eigenvalue of M are then obtained.

Examples of concrete situations which lead to problems of the class considered above may be found, for instance, in linearized elasticity. Let us denote by $e(u)$ the linearized strain tensor $\frac{1}{2}((Du)^T + Du)$ and set

$$W(Du) = \frac{\lambda}{2} |\operatorname{div} u|^2 + \mu |e(u)|^2, \quad (\lambda, \mu > 0)$$

(u is the displacement with respect to a reference configuration Ω). $\int_{\Omega} W(Du) dx$ represents the strain energy obtained by linearization of the constitutive equation of a homogeneous, isotropic, elastic material whose reference configuration is a natural state (i.e., with null residual stress). (If one prefers, W can be seen as the approximation up to the second order with respect to $e(u)$ of the stored energy function of a homogeneous, isotropic, hyperelastic material whose reference configuration is a natural state; see, e.g., [16] for further details).

A kind of application which well fits the framework of obstacle problems outlined above, is the inclusion of a very rigid material into an elastic body (think, e.g., of a body fixed by “nails” at a rigid support). We can model this situation by adding to the energy functional $\int_{\Omega} W(Du) dx$ a term of the form (0.2), with $K(x)$ taking only the values $\{0\}$ and \mathbf{R}^3 . As the inclusions increase in number and decrease in thickness, an asymptotic analysis is needed.

1. Notation and preliminaries

Throughout this chapter p is a fixed real number, $1 < p < +\infty$, m, n are two fixed positive integers, and Ω is an open subset of \mathbf{R}^n , possibly unbounded. We recall that we denote by $\mathcal{A}(\Omega)$ the family of the open subsets of Ω and by $\mathcal{B}(\Omega)$ the family of its Borel subsets. The elements of \mathbf{R}^m will be usually considered as column vectors.

As usual, we shall set $H^1(\Omega, \mathbf{R}^m) = W^{1,2}(\Omega, \mathbf{R}^m)$ and $H_0^1(\Omega, \mathbf{R}^m) = W_0^{1,2}(\Omega, \mathbf{R}^m)$ (\mathbf{R}^m will be dropped if $m = 1$).

We recall that a subset A of Ω is said to be *p-quasi open* (resp. *p-quasi closed*) if for every $\varepsilon > 0$ there exists an open set A_ε with $\operatorname{cap}_p(A_\varepsilon, \Omega) < \varepsilon$ such that $A \cup A_\varepsilon$ (resp. $A \setminus A_\varepsilon$) is an open set (resp. closed set).

Dealing with p -capacities, we shall drop the prefix p in the case $p = 2$.

For simplicity, we indicate by $\mathcal{M}_p(\Omega)$ the class of all positive Borel measures μ on Ω such that $\mu(B) = 0$ whenever $B \in \mathcal{B}(\Omega)$ and $\text{cap}_p(B, \Omega) = 0$ (absolute continuity with respect to capacity).

We say that a Radon measure μ on Ω (that is, a Borel measure on Ω , finite on compact sets) belongs to $W^{-1,p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, if there exists a constant $c > 0$ such that $\int_{\Omega} \varphi d\mu \leq c \|\varphi\|_{W^{1,p}(\Omega)}$ for every $\varphi \in C_0^\infty(\Omega)$. It is well known that if μ is a positive Radon measure on Ω which belongs to $W^{-1,p'}(\Omega)$, then μ belongs to $\mathcal{M}_p(\Omega)$.

Γ -convergence. (For more details on this subject see [20]) Let (X, d) be an arbitrary metric space. Let (F_h) be a sequence of functions from X to $\overline{\mathbf{R}}$, and let F be a function from X to $\overline{\mathbf{R}}$. We set

$$F'(x) = \Gamma\text{-}\liminf_{h \rightarrow \infty} F_h(x) = \inf \{ \liminf_{h \rightarrow \infty} F_h(x_h) : (x_h) \text{ converging to } x \text{ in } (X, d) \},$$

$$F''(x) = \Gamma\text{-}\limsup_{h \rightarrow \infty} F_h(x) = \inf \{ \limsup_{h \rightarrow \infty} F_h(x_h) : (x_h) \text{ converging to } x \text{ in } (X, d) \}.$$

It turns out that these infima are actually minima.

Definition 1.1. We say that (F_h) Γ -converges to F if $F = F' = F''$ on X .

It turns out that (F_h) Γ -converges to F if and only if the following conditions are satisfied:

- (a) for every $x \in X$ and for every sequence (x_h) converging to x in (X, d) , $F(x) \leq \liminf_{h \rightarrow \infty} F_h(x_h)$;
- (b) for every $x \in X$ there exists a sequence (x_h) converging to x in (X, d) such that $F(x) = \lim_{h \rightarrow \infty} F_h(x_h)$.

A functional $F: X \times \mathcal{A}(\Omega) \rightarrow \overline{\mathbf{R}}$ is said to be *lower semicontinuous* (on X) if $F(\cdot, A)$ is lower semicontinuous on X for every $A \in \mathcal{A}(\Omega)$. F is said to be *increasing* (on $\mathcal{A}(\Omega)$) if the set function $F(x, \cdot)$ is increasing on $\mathcal{A}(\Omega)$ for every $x \in X$. The *inner regular envelope* of F is the functional $F_-: X \times \mathcal{A}(\Omega) \rightarrow \overline{\mathbf{R}}$ defined by

$$F_-(x, A) = \sup \{ F(x, B) : B \in \mathcal{A}(\Omega), B \subset\subset A \}.$$

F is said to be *inner regular* on $\mathcal{A}(\Omega)$ if $F_- = F$.

Let now (F_h) be a sequence of increasing functionals on $X \times \mathcal{A}(\Omega)$, and let $F', F'': X \times \mathcal{A}(\Omega) \rightarrow \overline{\mathbf{R}}$ be the functionals defined by

$$F'(\cdot, A) = \Gamma\text{-}\liminf_{h \rightarrow \infty} F_h(\cdot, A), \quad F''(\cdot, A) = \Gamma\text{-}\limsup_{h \rightarrow \infty} F_h(\cdot, A).$$

The functionals F' and F'' are lower semicontinuous and increasing, but in general not inner regular.

Definition 1.2. We say that (F_h) $\bar{\Gamma}$ -converges to F in (X, d) if $F = (F')_- = (F'')_-$ on $X \times \mathcal{A}(\Omega)$.

Remark 1.3. The $\bar{\Gamma}$ -limit F turns out to be lower semicontinuous, increasing and inner regular.

We say that a subset \mathcal{R} of $\mathcal{A}(\Omega)$ is *rich* in $\mathcal{A}(\Omega)$ if, for every family $(A_t)_{t \in \mathbf{R}}$ in $\mathcal{A}(\Omega)$, with $A_s \subset \subset A_t$ whenever $s, t \in \mathbf{R}$, $s < t$, the set $\{t \in \mathbf{R} : A_t \notin \mathcal{R}\}$ is at most countable.

Proposition 1.4. Let (F_h) be as above and let $F: X \times \mathcal{A}(\Omega) \rightarrow \bar{\mathbf{R}}$ be a lower semicontinuous, increasing and inner regular functional. If the topology of X has a countable base, then the following conditions are equivalent:

- (a) (F_h) $\bar{\Gamma}$ -converges to F ;
- (b) there exists a rich set \mathcal{R} in $\mathcal{A}(\Omega)$ such that $(F_h(\cdot, A))$ Γ -converges to $F(\cdot, A)$ in X for every $A \in \mathcal{R}$.

If A is an open subset of \mathbf{R}^n , we shall use the symbol Γ_A to denote the Γ -convergence in the metric space $W^{1,p}(A, \mathbf{R}^m)$ endowed with the metric of $L^p(A, \mathbf{R}^m)$. A similar notation will be adopted for the $\bar{\Gamma}$ -convergence.

2. Further results about the class \mathcal{G}_p

In this section we consider a subclass of \mathcal{G}_2 which is relevant for the study of Dirichlet problems on perforated domains. We recall that the most general form of the functionals of \mathcal{G}_p is given in Theorem 6.5 of Chapter II, i.e.

Theorem 2.1. Let $G \in \mathcal{G}_p$. Then, there exist a finite measure $\mu \in \mathcal{M}_p(\Omega)$, a positive Borel measure ν on Ω , and a Borel function $g: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$, with the following properties:

- (i) for every $x \in \Omega$ the function $g(x, \cdot)$ is convex and lower semicontinuous on \mathbf{R}^m ;
- (ii) for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and for every $A \in \mathcal{A}(\Omega)$

$$G(u, A) = \int_A g(x, u(x)) d\mu + \nu(A).$$

Remark 2.2. Let us notice that Theorem 2.2 in [19] guarantees that given a positive Radon measure $\mu \in \mathcal{M}_p(\Omega)$, there exist a positive Radon measure $\rho \in W^{-1,p'}(\Omega)$ and a non-negative Borel function $b: \Omega \rightarrow [0, +\infty[$ such that $\mu(B) = \int_B b d\rho$ for every Borel subset B of Ω . Therefore, in Theorem 2.1 and in the following one we are allowed to replace the finite measure μ with the product $b\rho$.

We recall that if X is a real vector space, we say that a function $F: X \rightarrow [0, +\infty]$ is a (non-negative) *quadratic form* (with extended real values) on X if there exist a linear subspace Y of X and a symmetric bilinear form $B: Y \times Y \rightarrow \mathbf{R}$ such that

$$F(x) = \begin{cases} B(x, x), & \text{if } x \in Y, \\ +\infty, & \text{if } x \in X \setminus Y. \end{cases}$$

A functional $G \in \mathcal{G}_p$ will be said to be *quadratic* if $G(\cdot, A)$ is a quadratic form on $W^{1,p}(\Omega, \mathbf{R}^m)$ for every $A \in \mathcal{A}(\Omega)$. In particular $\text{dom}G(\cdot, \Omega) \neq \emptyset$ since $G(0, \Omega) = 0$. By Theorem 7.3 and Proposition 5.1 in Chapter II we can associate to G a multifunction V_G from Ω to \mathbf{R}^m , unique up to sets of p -capacity zero, such that

- (i) $V_G(x)$ is a linear subspace of \mathbf{R}^m ;
- (ii) for every $u \in \text{dom}G(\cdot, \Omega)$ we have $u(x) \in V_G(x)$ for p -q.e. $x \in \Omega$;
- (iii) if H is a closed valued multifunction from Ω to \mathbf{R}^m such that for every $u \in \text{dom}G(\cdot, \Omega)$ we have $u(x) \in H(x)$ for p -q.e. $x \in \Omega$, then $V_G(x) \subseteq H(x)$ for p -q.e. $x \in \Omega$.

Moreover, V_G enjoys the following property:

- (iv) if (u_k) is a countable dense subset of $\text{dom}G(\cdot, \Omega)$ then $V_G(x) = \text{cl}\{u_k(x) : k \in \mathbf{N}\}$ for p -q.e. $x \in \Omega$.

From Corollary 7.4 in Chapter II we obtain:

Theorem 2.3. Let G be a quadratic functional of \mathcal{G}_p . Then there exist a finite measure $\mu \in \mathcal{M}_p(\Omega)$ and a symmetric $m \times m$ matrix M of Borel functions from Ω to \mathbf{R} with $\xi^T M(x) \xi \geq 0$ for p -q.e. $x \in \Omega$ and for every $\xi \in \mathbf{R}^m$, with the following properties:

for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$

- (i) if $G(u, A) < +\infty$, then $u(x) \in V_G(x)$ for p -q.e. $x \in A$;
- (ii) if $u(x) \in V_G(x)$ for p -q.e. $x \in A$, then

$$G(u, A) = \int_A u^T M u d\mu.$$

The next proposition contains a significant case in which we can recover, for a quadratic functional, an integral representation valid on the whole $W^{1,p}(\Omega, \mathbf{R}^m)$, though maintaining a quadratic form in the integrand function and a Radon measure with finite energy. In particular it applies to the scalar case ($m = 1$).

Proposition 2.4. *Let G be a quadratic functional of \mathcal{G}_p . Assume that V_G may take only the values $\{0\}$ and \mathbf{R}^m . Then there exist*

- (i) *a positive Radon measure $\nu \in W^{-1,p'}(\Omega)$;*
- (ii) *a Borel function $a: \Omega \rightarrow [0, +\infty]$;*
- (iii) *a symmetric $m \times m$ matrix M of Borel functions from Ω to \mathbf{R} with $\xi^T M(x) \xi \geq 0$ for p -q.e. $x \in \Omega$ and for every $\xi \in \mathbf{R}^m$,*

with the property that

$$G(u, A) = \int_A u^T M u a d\nu$$

for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$.

Moreover, denoting by E the set $\{x \in \Omega : V_G(x) = \{0\}\}$ (which is defined up to sets of p -capacity zero), we can find M and a such that M equals the identity matrix p -q.e. on E and a is finite valued p -q.e. on $\Omega \setminus E$.

Proof. Let μ and P be the measure and the matrix in the representation of G according to Theorem 2.3.

Let E be a representative of $\{x \in \Omega : V_G(x) = \{0\}\}$. From property (iv) above satisfied by V_G , it follows that E is p -quasi closed. Moreover, we may assume that E is a Borel subset of Ω . Denote by ∞_E the Borel measure on \mathbf{R}^n defined as follows: $\infty_E(B) = 0$ if $\text{cap}_p(B \cap E) = 0$ and $\infty_E(B) = +\infty$ otherwise. Let us show that there exist a Radon measure $\rho \in W^{-1,p'}(\mathbf{R}^n)$ and a Borel function $b: \mathbf{R}^n \rightarrow [0, +\infty]$ such that

$$(2.1) \quad \int_A v^2 d\infty_E = \int_A v^2 b d\rho$$

for every $A \in \mathcal{A}(\mathbf{R}^n)$ and $v \in W^{1,p}(A)$. By Theorem 5.7 in [19] there exist a positive Radon measure $\rho \in W^{-1,p'}(\mathbf{R}^n)$ and a Borel function $f: \mathbf{R}^n \times \mathbf{R} \rightarrow [0, +\infty]$, with $f(x, \cdot)$ increasing and lower semicontinuous on \mathbf{R} for every $x \in \mathbf{R}^n$, such that

$$\int_A (v^+(x))^2 d\infty_E = \int_A f(x, v(x)) d\rho$$

for every $A \in \mathcal{A}(\mathbf{R}^n)$ and $v \in W^{1,p}(A)$. We can now apply the argument of Lemma 2.4 in [21], thus obtaining the existence of a Borel function $b: \mathbf{R}^n \rightarrow [0, +\infty]$ such that

$$\int_A f(x, v(x)) d\rho = \int_A (v^+(x))^2 b(x) d\rho$$

for every $A \in \mathcal{A}(\mathbf{R}^n)$ and $v \in W^{1,p}(A)$. This easily yields (2.1).

Let us notice that, since E is p -quasi closed, by Lemma 1.5 in [19] there exists an increasing sequence (v_h) in $W^{1,p}(\mathbf{R}^n)$ converging p -q.e. to the characteristic function of $\Omega \setminus E$. We can apply (2.1) to the sequence (v_h) . By the monotone convergence theorem we conclude that

$$(2.2) \quad \int_{\Omega \setminus E} b d\rho = 0.$$

Let us now define on Ω a matrix-valued Borel function M by setting $M = I$ (identity matrix) on E , and $M = P$ on $\Omega \setminus E$. Moreover, let us set $\sigma = \mu + b\rho$. We claim that

$$(2.3) \quad G(u, A) = \int_A u^T M u d\sigma$$

for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$.

Assume first that $u = 0$ p -q.e. on $E \cap A$. We have

$$\int_A u^T M u d\sigma = \int_A u^T P u d\mu + \int_A u^T M u b d\rho.$$

In view of the definition of μ and P , the first term on the right-hand side equals $G(u, A)$. The second one is zero by (2.2). Hence we get (2.3) in the case $u = 0$ p -q.e. on $E \cap A$.

Assume now that $\text{cap}_p(\{x \in \Omega : u(x) \neq 0\} \cap E \cap A) > 0$. By Theorem 2.3, $G(u, A) = +\infty$. On the other hand

$$\int_A u^T M u d\sigma \geq \int_{A \cap E} |u|^2 b d\rho = \int_A |u|^2 b d\rho = \int_A |u|^2 d\infty_E = +\infty.$$

Up to now we have proved (2.3). According to Remark 2.2 we can replace μ by $b'\rho'$, where $b': \Omega \rightarrow [0, +\infty[$ is a Borel function and ρ' is a Radon measure in $W^{-1,p'}(\Omega)$. Let $\nu = \rho' + \rho$; then ν is a Radon measure on Ω which belongs to $W^{-1,p'}(\Omega)$. Define $a: \Omega \rightarrow [0, +\infty]$ as a Borel representative of the function $b' \frac{d\rho'}{d\nu} + b \frac{d\rho}{d\nu}$. In view of (2.2) we can assume that a is finite on $\Omega \setminus E$. Then (2.3) yields the desired representation of G . \square

3. Some convergence results in the class \mathcal{G}_p

Let $W: \Omega \times \mathbf{R}^{mn} \rightarrow \mathbf{R}$ be a function such that

- (a) $W(\cdot, \eta)$ is Borel measurable on Ω for every $\eta \in \mathbf{R}^{mn}$ and $W(x, \cdot)$ is convex on \mathbf{R}^{mn} for a.e. $x \in \Omega$;
- (b) there exists a function $a \in L^1(\Omega)$ and a constant $b > 0$ such that

$$0 \leq W(x, \eta) \leq a(x) + b|\eta|^p \quad \text{for a.e. } x \in \Omega \text{ and for every } \eta \in \mathbf{R}^{mn};$$

- (c) for every $A \in \mathcal{A}(\Omega)$ there exists a constant $c(A) > 0$ such that

$$\int_A |Du|^p dx \leq c(A) \left(\int_A W(x, Du) dx + \int_A |u|^p dx \right)$$

for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$.

Let us define

$$(3.1) \quad F(u, B) = \int_B W(x, Du(x)) dx$$

for every $A \in \mathcal{A}(\Omega)$, for every $u \in W^{1,p}(A, \mathbf{R}^m)$ and $B \in \mathcal{B}(A)$. By the Carathéodory Continuity Theorem, $F(\cdot, A)$ is continuous on $W^{1,p}(A, \mathbf{R}^m)$ for every $A \in \mathcal{A}(\Omega)$. Since $F(\cdot, A)$ is convex, it is also lower semicontinuous in the weak topology of $W^{1,p}(A, \mathbf{R}^m)$.

Proposition 3.1. *Let (G_h) be a sequence of functionals of \mathcal{G}_p . Assume that there exists a functional $G: W^{1,p}(\Omega, \mathbf{R}^m) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ such that the sequence $(F + G_h)$ $\bar{\Gamma}_\Omega$ -converges to $F + G$. Then G satisfies properties (i), (ii) and (iii) of Definition 2.1 of Chapter II.*

We need the following lemma.

Lemma 3.2. *Let A, A' and B be open subsets of Ω , with $A' \subset\subset A$; let $u, v \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $\varphi \in C_0^1(A)$ with $\varphi = 1$ on A' and $0 \leq \varphi \leq 1$. Then for every $0 < \varepsilon < 1/2$ we have*

$$F(\varphi u + (1-\varphi)v, A' \cup B) \leq F(u, A) + F(v, B) + C(\varepsilon + \varepsilon^{1-p} \|D\varphi\|_{L^\infty(A)}^p \|u-v\|_{L^p(U \cap B, \mathbf{R}^m)}^p),$$

where $U = \{x \in \Omega : D\varphi(x) \neq 0\}$ and C is a positive constant depending only on the function a and the constant b in the foregoing condition (b) and monotonically on $\max(\|u\|_{W^{1,p}(A, \mathbf{R}^m)}, \|v\|_{W^{1,p}(B, \mathbf{R}^m)})$.

Proof. Define $w = \varphi u + (1 - \varphi)v$. By the convexity of $W(x, \cdot)$ we have

$$\begin{aligned} F(w, A' \cup B) &\leq (1 - \varepsilon) \int_{A' \cup B} [\varphi W(x, \frac{Du}{1 - \varepsilon}) + (1 - \varphi) W(x, \frac{Dv}{1 - \varepsilon})] dx + \\ &\quad + \varepsilon \int_{A' \cup B} W(x, \frac{1}{\varepsilon}(u - v) \otimes D\varphi(x)) dx. \end{aligned}$$

In view of the boundedness condition (b), it follows that

$$\begin{aligned} (3.2) \quad F(w, A' \cup B) &\leq F(\frac{u}{1 - \varepsilon}, A) + F(\frac{v}{1 - \varepsilon}, B) + \\ &\quad + \varepsilon \|a\|_{L^1(A' \cup B)} + \varepsilon^{1-p} b \|D\varphi\|_{L^\infty(A)}^p \|u - v\|_{L^p(U \cap B, \mathbf{R}^m)}^p. \end{aligned}$$

Let us apply the well-known Lipschitz estimate for finite valued convex functions to the functions $t \mapsto F(tu, A)$ and $t \mapsto F(tv, B)$ on \mathbf{R} . Again by the boundedness condition (b), it follows that for every $0 < \varepsilon < 1/2$

$$F(\frac{u}{1 - \varepsilon}, A) \leq F(u, A) + C'\varepsilon, \quad F(\frac{v}{1 - \varepsilon}, B) \leq F(v, B) + C'\varepsilon,$$

where C' is a constant with the same properties stated for C . Together with (3.2) this concludes the proof. \square

Proof of Proposition 3.1.

(i) By Remark 1.3, for every $A \in \mathcal{A}(\Omega)$ the functional $(F + G)(\cdot, A)$ is lower semicontinuous in $L^p(\Omega, \mathbf{R}^m)$; moreover, $F(\cdot, A)$ is continuous in $W^{1,p}(\Omega, \mathbf{R}^m)$. It follows that $G(\cdot, A)$ is lower semicontinuous on $W^{1,p}(\Omega, \mathbf{R}^m)$.

(ii) Let us show that $(F + G)(u, \cdot)$, and hence $G(u, \cdot)$, is the trace of a Borel measure on Ω for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$.

For every $A \in \mathcal{A}(\Omega)$ define

$$\begin{aligned} H'(\cdot, A) &= \Gamma_\Omega\text{-}\liminf_{h \rightarrow \infty} [F(\cdot, A) + G_h(\cdot, A)], \\ H''(\cdot, A) &= \Gamma_\Omega\text{-}\limsup_{h \rightarrow \infty} [F(\cdot, A) + G_h(\cdot, A)]. \end{aligned}$$

By following the proof of Theorem 18.5 and Proposition 18.4 in [20], it can be easily seen that it is enough to show that for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ the set function $H''(u, \cdot)$ is weakly subadditive, i.e.,

$$(3.3) \quad H''(u, A' \cup B) \leq H''(u, A) + H''(u, B)$$

whenever $A', A, B \in \mathcal{A}(\Omega)$ with $A' \subset \subset A$.

Let us fix $A', A, B \in \mathcal{A}(\Omega)$, $A' \subset \subset A$ and $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ such that $H''(u, A)$, $H''(u, B) < +\infty$. Let (u_h) and (v_h) be two sequences in $W^{1,p}(\Omega, \mathbf{R}^m)$ converging to u in $L^p(\Omega, \mathbf{R}^m)$ and such that

$$(3.4) \quad \begin{aligned} H''(u, A) &= \limsup_{h \rightarrow \infty} [F(u_h, A) + G_h(u_h, A)], \\ H''(u, B) &= \limsup_{h \rightarrow \infty} [F(v_h, B) + G_h(v_h, B)]. \end{aligned}$$

Let $\varphi \in C_0^1(A)$ with $\varphi = 1$ on a neighbourhood of A' and $0 \leq \varphi \leq 1$. Define $w_h = \varphi u_h + (1 - \varphi)v_h$. Then (w_h) converges to u in $L^p(\Omega, \mathbf{R}^m)$. Thus $H''(u, A' \cup B) \leq \limsup_{h \rightarrow \infty} [F(w_h, A' \cup B) + G_h(w_h, A' \cup B)]$. Let us estimate the right hand side. For every $h \in \mathbb{N}$ we have

$$\begin{aligned} G_h(w_h, A' \cup B) &\leq G_h(u_h, \overline{A'}) + G_h(v_h, B \setminus A) + G_h(\varphi u_h + (1 - \varphi)v_h, (A \cap B) \setminus \overline{A'}) \\ &\leq G_h(u_h, \overline{A'}) + G_h(v_h, B \setminus A) + G_h(u_h, (A \cap B) \setminus \overline{A'}) + G_h(v_h, (A \cap B) \setminus \overline{A'}); \end{aligned}$$

hence

$$(3.5) \quad G_h(w_h, A' \cup B) \leq G_h(u_h, A) + G_h(v_h, B).$$

Let us fix $0 < \varepsilon < 1/2$. Notice that the finiteness of $H''(u, A)$ and $H''(u, B)$, and the coerciveness condition (c) guarantee that $(\|u_h\|_{W^{1,p}(A, \mathbf{R}^m)})$ and $(\|v_h\|_{W^{1,p}(B, \mathbf{R}^m)})$ are bounded. By Lemma 3.2

$$(3.6) \quad F(w_h, A' \cup B) \leq F(u_h, A) + F(v_h, B) + C(\varepsilon + \varepsilon^{1-p} \|D\varphi\|_{L^\infty(A)}^p \|u_h - v_h\|_{L^p(\Omega, \mathbf{R}^m)}^p),$$

where C can be chosen independently of h . From (3.5) and (3.6) we get

$$\begin{aligned} H''(u, A' \cup B) &\leq \limsup_{h \rightarrow \infty} [F(u_h, A) + G_h(u_h, A)] + \\ &\quad \limsup_{h \rightarrow \infty} [F(v_h, B) + G_h(v_h, B)] + 2\varepsilon C. \end{aligned}$$

By (3.4) and the arbitrariness of ε we obtain (3.3).

(iii) Let $A \in \mathcal{A}(\Omega)$ and $u, v \in W^{1,p}(\Omega, \mathbf{R}^m)$ with $u = v$ a.e. on A . We shall show that for every $A_1, A_2 \in \mathcal{A}(\Omega)$ with $A_1 \subset \subset A_2 \subset \subset A$ we have

$$H'(u, A_1) \leq H'(v, A_2).$$

Indeed, since $F + G$ is the inner regular envelope of H' , from this inequality it follows that $(F + G)(u, A) \leq (F + G)(v, A)$, and we conclude by exchanging the roles of u and v .

Let (u_h) be a sequence converging in $L^p(\Omega, \mathbf{R}^m)$ to u , and such that

$$H'(v, A_2) = \liminf_{h \rightarrow \infty} [F(v_h, A_2) + G_h(v_h, A_2)].$$

Let $\varphi \in C_0^1(A_2)$ with $\varphi = 1$ on A_1 and $0 \leq \varphi \leq 1$. Define $w_h = \varphi v_h + (1 - \varphi)u$ for every $h \in \mathbf{N}$. Then (w_h) converges to u in $L^p(\Omega, \mathbf{R}^m)$. Hence, by the local property of F and G_h

$$\begin{aligned} H'(u, A_1) &\leq \liminf_{h \rightarrow \infty} [F(w_h, A_1) + G_h(w_h, A_1)] \leq \\ &\liminf_{h \rightarrow \infty} [F(v_h, A_2) + G_h(v_h, A_2)] = H'(v, A_2). \end{aligned}$$

□

Let $\varphi \in W^{1,p}(\Omega, \mathbf{R}^m)$. For every $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(A, \mathbf{R}^m)$ we set

$$F_\varphi(u, A) = \begin{cases} F(u, A) & \text{if } u - \varphi \in W_0^{1,p}(A, \mathbf{R}^m), \\ +\infty & \text{otherwise in } W^{1,p}(A, \mathbf{R}^m). \end{cases}$$

Proposition 3.3. *Let (G_h) and G be as in Proposition 3.1. Then*

- (i) $F(\cdot, \Omega) + G_h$ $\bar{\Gamma}_\Omega$ -converges to $F(\cdot, \Omega) + G$;
- (ii) for every $\varphi \in W^{1,p}(\Omega, \mathbf{R}^m)$

$$F_\varphi(\cdot, \Omega) + G_h \quad \bar{\Gamma}_\Omega\text{-converges to} \quad F_\varphi(\cdot, \Omega) + G;$$

more precisely, if $A \in \mathcal{A}(\Omega)$ with $A \subset\subset \Omega$ is such that $(F(\cdot, \Omega) + G_h(\cdot, A))$ Γ_Ω -converges to $F(\cdot, \Omega) + G(\cdot, A)$, then $(F_\varphi(\cdot, \Omega) + G_h(\cdot, A))$ Γ_Ω -converges to $F_\varphi(\cdot, \Omega) + G(\cdot, A)$.

Proof. (i) Let G' and G'' be defined as follows

$$\begin{aligned} F(\cdot, \omega) + G'(\cdot, \omega, A) &= \Gamma_\Omega\text{-}\liminf_{h \rightarrow \infty} [F(\cdot, \omega) + G_h(\cdot, A)], \\ F(\cdot, \omega) + G''(\cdot, \omega, A) &= \Gamma_\Omega\text{-}\limsup_{h \rightarrow \infty} [F(\cdot, \omega) + G_h(\cdot, A)], \end{aligned}$$

for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $\omega, A \in \mathcal{A}(\Omega)$ with $A \subseteq \omega$. Let us first prove that for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$

$$(3.7) \quad G'(u, \omega, A) \leq G'(u, \Omega, A) \quad \text{if } \omega, A \in \mathcal{A}(\Omega) \text{ with } A \subseteq \omega;$$

$$(3.8) \quad G''(u, \Omega, A) \leq G''(u, \omega, A) \quad \text{if } \omega, A \in \mathcal{A}(\Omega) \text{ with } A \subset\subset \omega.$$

Let us fix u , ω and A as in (3.7). By definition of G' there exists a sequence (u_h) converging to u in $L^p(\Omega, \mathbf{R}^m)$ such that

$$F(u, \Omega) + G'(u, \Omega, A) = \liminf_{h \rightarrow \infty} [F(u_h, \Omega) + G_h(u_h, A)].$$

We can assume that $G'(u, \Omega, A) < +\infty$ and that the lower limit is a limit. It follows, by the coerciveness of F that (u_h) is bounded in $W^{1,p}(\Omega, \mathbf{R}^m)$, hence converging to u weakly in $W^{1,p}(\Omega, \mathbf{R}^m)$. Since the functional $F(\cdot, \Omega \setminus \omega)$ is lower semicontinuous in the weak topology of $W^{1,p}(\Omega, \mathbf{R}^m)$, we have

$$\begin{aligned} F(u, \Omega) + G'(u, \omega, A) &= F(u, \omega) + G'(u, \omega, A) + F(u, \Omega \setminus \omega) \\ &\leq \liminf_{h \rightarrow \infty} [F(u_h, \omega) + G_h(u_h, A)] + \liminf_{h \rightarrow \infty} F(u_h, \Omega \setminus \omega) \\ &\leq \liminf_{h \rightarrow \infty} [F(u_h, \Omega) + G_h(u_h, A)] = F(u, \Omega) + G'(u, \Omega, A). \end{aligned}$$

This yields (3.7).

Let now ω and A be as in (3.8) and assume $G''(u, \omega, A) < +\infty$. By definition of G'' there exists a sequence (u_h) converging to u in $L^p(\Omega, \mathbf{R}^m)$ such that

$$F(u, \omega) + G''(u, \omega, A) = \limsup_{h \rightarrow \infty} [F(u_h, \omega) + G_h(u_h, A)].$$

Let $0 < \varepsilon < 1/2$ and K be a compact set with $A \subseteq K \subseteq \omega$ and $F(u, \omega \setminus K) < \varepsilon$. Consider a function $\varphi \in C_0^1(\omega)$ with $\varphi = 1$ in a neighbourhood of K and $0 \leq \varphi \leq 1$. Let $w_h = \varphi u_h + (1 - \varphi)u$; since, because of the coerciveness of F , (u_h) is bounded in $W^{1,p}(\omega, \mathbf{R}^m)$, by Lemma 3.2 there exists a constant $C > 0$ depending only on the function a and the constant b in condition (b), such that

$$F(w_h, \Omega) \leq F(u_h, \omega) + F(u, \Omega \setminus K) + C(\varepsilon + \varepsilon^{1-p} \|D\varphi\|_{L^\infty(\omega)}^p \|u_h - u\|_{L^p(\Omega, \mathbf{R}^m)}^p).$$

Therefore

$$\begin{aligned} F(u, \Omega) + G''(u, \Omega, A) &\leq \limsup_{h \rightarrow \infty} [F(u_h, \omega) + G_h(u_h, A)] + F(u, \Omega \setminus K) + C\varepsilon \\ &\leq F(u, \omega) + G''(u, \omega, A) + F(u, \Omega \setminus \omega) + \varepsilon + C\varepsilon \\ &= F(u, \Omega) + G''(u, \omega, A) + \varepsilon + C\varepsilon. \end{aligned}$$

Since ε is arbitrary we obtain (3.8).

We are now in a position to conclude the proof of (i). By assumption and Proposition 1.4 there exists a rich set \mathcal{R}_1 in $\mathcal{A}(\Omega)$ such that $G'(u, A, A) = G''(u, A, A) = G(u, A)$ for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{R}_1$. Therefore, by (3.7) and (3.8), for every $A \in \mathcal{R}_1$ we have

$$\begin{aligned} \sup_{A' \subset \subset A} G(u, A') &\leq \sup_{A' \subset \subset A} G'(u, \Omega, A') \leq \sup_{A' \subset \subset A} G''(u, \Omega, A') \\ &\leq \sup_{A' \subset \subset A} G'''(u, A, A') \leq G'''(u, A, A) = G(u, A). \end{aligned}$$

Since $G(u, \cdot)$ is a measure, it follows that for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{R}_1$

$$G(u, A) = \sup_{A' \subset \subset A} G'(u, \Omega, A') = \sup_{A' \subset \subset A} G''(u, \Omega, A').$$

On the other hand, since $G'(u, \Omega, A)$ and $G''(u, \Omega, A)$ are increasing with respect to A and lower semicontinuous with respect to u on $W^{1,p}(\Omega, \mathbf{R}^m)$, by Proposition 15.15 in [20], there exists a rich subset \mathcal{R}_2 of $\mathcal{A}(\Omega)$ such that

$$G'(u, \Omega, A) = \sup_{A' \subset \subset A} G'(u, \Omega, A') \quad G''(u, \Omega, A) = \sup_{A' \subset \subset A} G''(u, \Omega, A'),$$

for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{R}_2$. We conclude that $G'(u, \Omega, A) = G''(u, \Omega, A) = G(u, A)$ for every $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ and for every A in the rich subset $\mathcal{R}_1 \cap \mathcal{R}_2$ of $\mathcal{A}(\Omega)$. Taking Proposition 1.4 into account, this concludes the proof of (i).

(ii) The proof of (ii) can be easily obtained from the proof of Theorem 4.3 in [21], making use of Lemma 3.2, as above, in place of the “J-property”. \square

Remark 3.4. Let $G: W^{1,p}(\Omega, \mathbf{R}^m) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ be an increasing local functional. We extend the definition of G by setting, for every $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(A, \mathbf{R}^m)$,

$$\tilde{G}(u, A) = \sup_{\substack{B \in \mathcal{A}(\Omega) \\ B \subset \subset A}} G(\varphi_B u, B),$$

where φ_B denotes a $C_0^1(A)$ function with $\varphi_B = 1$ on B . It is easy to see that \tilde{G} is increasing, local and inner regular. Moreover, if $u \in W^{1,p}(A, \mathbf{R}^m)$ admits an extension \tilde{u} to $W^{1,p}(\Omega, \mathbf{R}^m)$ (in particular if A has smooth boundary), then $\tilde{G}(u, A) = G_-(\tilde{u}, A)$.

If $G \in \mathcal{G}_p$, the extension \tilde{G} is nothing but the natural extension allowed by the integral representation of Theorem 2.3. In such cases we shall use the same symbol G .

Let now (G_h) and G be as in Proposition 3.1 and $\omega \in \mathcal{A}(\Omega)$. If A is an open subset of Ω such that $A \subset\subset \omega$ and $(F(\cdot, A) + G_h(\cdot, A))$ Γ_Ω -converges to $F(\cdot, A) + G(\cdot, A)$, then $(F(\cdot, A) + G_h(\cdot, A))$ Γ_ω -converges to $F(\cdot, A) + \tilde{G}(\cdot, A)$. In particular

$$F + G_h \quad \bar{\Gamma}_\omega\text{-converges to} \quad F + \tilde{G}.$$

Indeed, if ψ denotes a function in $C_0^1(\omega)$, with $\psi = 1$ on A , then for every $u \in W^{1,p}(\omega, \mathbf{R}^m)$ $\tilde{G}(u, A) = G(\psi u, A)$ and the Γ_Ω -lower (resp. upper) limit on $(\psi u, A)$ coincides with the Γ_ω -lower (resp. upper) limit on (u, A) .

The relevance of the next proposition lies in the fact that it provides a convergence result for *all* the open subsets of Ω .

Proposition 3.5. *Let (G_h) and G be as in Proposition 3.1 and assume that $G \in \mathcal{G}_p$. Let $\varphi \in W^{1,p}(\Omega, \mathbf{R}^m)$ and assume that $G_h(\varphi, \Omega) = 0$ for every $h \in \mathbf{N}$. Then*

$$(3.9) \quad F_\varphi(\cdot, A) + G_h(\cdot, A) \quad \Gamma_A\text{-converges to} \quad F_\varphi(\cdot, A) + G(\cdot, A)$$

for every $A \in \mathcal{A}(\Omega)$.

Proof. For every $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(A, \mathbf{R}^m)$ let us define

$$\begin{aligned} H'_\varphi(\cdot, A) &= \Gamma_A\text{-}\liminf_{h \rightarrow \infty} [F_\varphi(\cdot, A) + G_h(\cdot, A)], \\ H''_\varphi(\cdot, A) &= \Gamma_A\text{-}\limsup_{h \rightarrow \infty} [F_\varphi(\cdot, A) + G_h(\cdot, A)]. \end{aligned}$$

To prove (3.9) we have to show that

$$H''_\varphi(u, A) \leq F_\varphi(u, A) + G(u, A) \leq H'_\varphi(u, A)$$

for every $A \in \mathcal{A}(\Omega)$ and for every $u \in W^{1,p}(A, \mathbf{R}^m)$.

(i) Let us prove that

$$(3.10) \quad F_\varphi(u, A) + G(u, A) \leq H'_\varphi(u, A).$$

Let us fix $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(A, \mathbf{R}^m)$. Let (u_h) be a sequence in $W^{1,p}(A, \mathbf{R}^m)$ converging to u in $L^p(A, \mathbf{R}^m)$. Let us show that

$$(3.11) \quad F_\varphi(u, A) + G(u, A) \leq \liminf_{h \rightarrow \infty} [F_\varphi(u_h, A) + G_h(u_h, A)].$$

It is not restrictive to suppose that the right hand side of this inequality is finite and the lower limit is a limit. Hence, we can assume that $u_h - \varphi \in W_0^{1,p}(A, \mathbf{R}^m)$. By the coerciveness of F , we obtain that (u_h) converges to u weakly in $W^{1,p}(A, \mathbf{R}^m)$. Thus $u - \varphi \in W_0^{1,p}(A, \mathbf{R}^m)$.

Let \mathcal{R} be a rich subset of $\mathcal{A}(\Omega)$ such that $(F(\cdot, B) + G_h(\cdot, B))$ Γ_Ω -converges to $F(\cdot, B) + G(\cdot, B)$ for every $B \in \mathcal{R}$. Therefore, for every $B \in \mathcal{R}$ with $B \subseteq A$ we have

$$\begin{aligned} F(u, B) + G(u, B) &\leq \liminf_{h \rightarrow \infty} [F(u_h, B) + G_h(u_h, B)] \leq \\ &\leq \liminf_{h \rightarrow \infty} [F(u_h, A) + G_h(u_h, A)]. \end{aligned}$$

By taking the supremum over all such B , we get (3.11); hence, (3.10).

(ii) Let us now prove that for every $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(A, \mathbf{R}^m)$

$$(3.12) \quad H''_\varphi(u, A) \leq F_\varphi(u, A) + G(u, A).$$

Fix $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(A, \mathbf{R}^m)$ such that the right hand side of (3.12) is finite. Hence, $u - \varphi \in W_0^{1,p}(A, \mathbf{R}^m)$ and $F_\varphi(u, A) = F(u, A)$.

Let us first consider the additional assumption:

(A) there exists a function $u_0 \in W_0^{1,p}(A, \mathbf{R}^m)$, with $u_0 = 0$ outside a compact subset of A , such that $u = u_0 + \varphi$;

Let \mathcal{R} be a rich subset of $\mathcal{A}(A)$ such that

$$(3.13) \quad F(\cdot, B) + G_h(\cdot, B) \quad \Gamma_A\text{-converges to} \quad F(\cdot, B) + G(\cdot, B)$$

for every $B \in \mathcal{R}$.

Let $\varepsilon > 0$ and K be a compact subset of A with $u_0 = 0$ on $A \setminus K$ and $F(u, A \setminus K) < \varepsilon$. Let us fix $B \in \mathcal{R}$ with $K \subseteq B \subset\subset A$. In view of (3.13) there exists a sequence (v_h) in $W^{1,p}(A, \mathbf{R}^m)$ converging to u in $L^p(A, \mathbf{R}^m)$ and such that

$$F(u, B) + G(u, B) \geq \limsup_{h \rightarrow \infty} [F(v_h, B) + G_h(v_h, B)].$$

By the coerciveness of F we can assume that (v_h) is bounded on $W^{1,p}(B, \mathbf{R}^m)$.

Let $\psi \in C_0^1(B)$ with $\psi = 1$ on a neighbourhood U of K and $0 \leq \psi \leq 1$. Define $u_h = \psi v_h + (1 - \psi)u$. Then (u_h) converges to u in $L^p(A, \mathbf{R}^m)$. By Lemma 3.2 there exists a constant $C > 0$ such that for every $\varepsilon > 0$ and $h \in \mathbf{N}$

$$F(u_h, A) \leq F(v_h, B) + F(u, A \setminus K) + C(\varepsilon + \varepsilon^{1-p} \|D\psi\|_{L^\infty(B)}^p \|v_h - u\|_{L^p(A, \mathbf{R}^m)}^p),$$

Since $u_h = v_h$ p -q.e. on U , and $G_h(u, B \setminus K) = G_h(\varphi, B \setminus K) = 0$, we have

$$\begin{aligned} G_h(u_h, A) &= G_h(u_h, B) = G_h(u_h, K) + G_h(u_h, B \setminus K) \leq \\ &\leq G_h(u_h, K) + G_h(v_h, B \setminus K) + G_h(u, B \setminus K) = G_h(v_h, B). \end{aligned}$$

It follows that

$$\limsup_{h \rightarrow \infty} [F(u_h, A) + G_h(u_h, A)] \leq F(u, A) + G(u, A) + (C + 1)\varepsilon.$$

Thus, by the arbitrariness of ε , we get (3.12) under the assumption (A).

Let us now drop condition (A), first in the case $u - \varphi \in L^\infty(A, \mathbf{R}^m)$. It is easy to see that there exists a sequence (z_h) of non-negative functions in $W_0^{1,p}(A)$ with compact support in A converging to $|u - \varphi|$ in $W^{1,p}(A)$ and such that $z_h \uparrow |u - \varphi|$ p -q.e. on A . Therefore, for every $\varepsilon > 0$ the sequence

$$(3.14) \quad \left(\frac{z_h}{|u - \varphi| + \varepsilon} \right) (u - \varphi) \quad \text{converges to} \quad u_\varepsilon = \frac{|u - \varphi|}{|u - \varphi| + \varepsilon} (u - \varphi) \quad \text{in } W^{1,p}(A, \mathbf{R}^m).$$

as $h \rightarrow \infty$. Let us show that

$$(3.15) \quad u_\varepsilon \rightarrow u - \varphi \quad \text{in } W^{1,p}(A, \mathbf{R}^m) \quad \text{as } \varepsilon \rightarrow 0.$$

Equivalently, we prove that $\left(\frac{\varepsilon}{|u - \varphi| + \varepsilon} (u - \varphi) \right)$ converges to 0 in $W^{1,p}(A, \mathbf{R}^m)$. Clearly, we have the convergence in $L^p(A, \mathbf{R}^m)$. Define $S_\varepsilon = \frac{\varepsilon}{|\xi| + \varepsilon} \xi$ for every $\xi \in \mathbf{R}^m$. It turns out that for every $\sigma > 0$, S_ε is lipschitzian on $\{\xi \in \mathbf{R}^m : |\xi| \geq \sigma\}$ with Lipschitz constant bounded by $(\sqrt{n} + 1) \frac{\varepsilon}{\sigma + \varepsilon}$; in particular, S_ε is lipschitzian on \mathbf{R}^m with Lipschitz constant bounded by $\sqrt{n} + 1$. Therefore, for every $\sigma > 0$ and $\varepsilon > 0$ with $\sigma > \varepsilon$, we have

$$\begin{aligned} \int_A |Du_\varepsilon|^p dx &= \int_{A \cap \{|u - \varphi| < \sigma\}} |D(S_\varepsilon \circ (u - \varphi))|^p dx + \int_{A \cap \{|u - \varphi| \geq \sigma\}} |D(S_\varepsilon \circ (u - \varphi))|^p dx \leq \\ &\leq (\sqrt{n} + 1)^p \int_{A \cap \{|u - \varphi| < \sigma\}} |D(u - \varphi)|^p dx + (\sqrt{n} + 1)^p \left(\frac{\varepsilon}{\sigma} \right)^p \int_A |D(u - \varphi)|^p dx. \end{aligned}$$

We conclude by taking first the limit as $\varepsilon \rightarrow 0$ and then the limit for $\sigma \rightarrow 0$.

From (3.14) and (3.15), by a diagonalization argument it follows that there exists a sequence (ψ_h) in $W_0^{1,p}(A, \mathbf{R}^m)$ with compact support in A and $0 \leq \psi_h \leq 1$, such that $(\psi_h(u - \varphi))$ converges to $(u - \varphi)$ strongly in $W^{1,p}(A, \mathbf{R}^m)$.

By applying (3.12) for the functions $u_h = \varphi + \psi_h(u - \varphi) = \psi_h u + (1 - \psi_h)\varphi$, we get

$$F_\varphi(u_h, A) + G(u_h, A) \geq H_\varphi''(u_h, A).$$

The lower semicontinuity of $H''_\varphi(\cdot, A)$ yields

$$(3.16) \quad H''_\varphi(u, A) \leq \liminf_{h \rightarrow \infty} [F_\varphi(u_h, A) + G(u_h, A)].$$

Clearly, $\lim_{h \rightarrow \infty} F_\varphi(u_h, A) = F_\varphi(u, A)$. By Theorem 2.4 there exists a measure $\mu \in \mathcal{M}_p(\Omega)$, a positive Borel measure ν on Ω and a Borel function $g: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$, convex and lower semicontinuous in the second variable, such that

$$G(w, A) = \int_A g(x, w(x)) d\mu + \nu(A)$$

for every $w \in W^{1,p}(\Omega, \mathbf{R}^m)$. The condition $G_h(\varphi, \Omega) = 0$ for every $h \in \mathbf{N}$ implies that $G(\varphi, \Omega) = 0$; hence $g(x, \varphi(x)) = 0$ for μ -a.e. $x \in \Omega$. By the continuity along line segments for lower semicontinuous proper convex functions (see [39], Corollary 7.5.1), from $G(u_h, A) = \int_A g(x, u_h(x)) d\mu + \nu(A)$ it follows that $\lim_{h \rightarrow \infty} G(u_h, A) = G(u, A)$. In view of (3.16) we obtain (3.12) in the case $u - \varphi \in L^\infty(A, \mathbf{R}^m)$.

Let us now deal with the general case. By Lemma 2.5 in Chapter II there exists a sequence (ζ_h) of functions such that $0 \leq \zeta_h \leq 1$, $\zeta_h(u - \varphi) \in W_0^{1,p}(A, \mathbf{R}^m) \cap L^\infty(A, \mathbf{R}^m)$ and $(\zeta_h(u - \varphi))$ converges to $(u - \varphi)$ strongly in $W^{1,p}(A, \mathbf{R}^m)$. Finally, it is enough to repeat the previous argument with u_h replaced by $\varphi + \zeta_h(u - \varphi)$, since $\zeta_h(u - \varphi) \in L^\infty(A, \mathbf{R}^m)$. \square

4. A compactness theorem

Let us now consider the particular case when $W(x, \cdot)$ is a quadratic form. More precisely, we assume that $W: \Omega \times \mathbf{R}^m \rightarrow \mathbf{R}$ is a Borel function such that for a.e. $x \in \Omega$ and for every $\eta \in \mathbf{R}^{mn}$

$$(4.1) \quad W(x, \eta) = \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^m a_{\alpha\beta}^{ij}(x) \eta_i^\alpha \eta_j^\beta,$$

where $a_{\alpha\beta}^{ij} \in L^\infty(\Omega)$, $a_{\alpha\beta}^{ij} = a_{\beta\alpha}^{ji}$, and the coerciveness condition (c) of Section 3 holds.

Let us notice that condition (c) easily implies that $W(x, \eta) \geq 0$ for a.e. $x \in \Omega$ and for every $\eta \in \mathbf{R}^{mn}$. Therefore, denoting by Λ a bound for the L^∞ -norm of $a_{\alpha\beta}^{ij}$, we have

$$(4.2) \quad 0 \leq W(x, \eta) \leq \Lambda |\eta|^2 \quad \text{for a.e. } x \in \Omega \text{ and for every } \eta \in \mathbf{R}^{mn}.$$

As W is non-negative, $W(x, \cdot)$ is convex. It follows that also conditions (a) and (b) of Section 3 are satisfied.

As already mentioned in the introduction, the usual energy density of linearized elasticity, i.e., $W(x, \eta) = \frac{\lambda}{2} |\text{tr} \frac{1}{2}(\eta^T + \eta)|^2 + \mu |\frac{1}{2}(\eta^T + \eta)|^2$, is of type (4.1) on every bounded open subset Ω of \mathbf{R}^3 . Indeed (see for instance [34], Theorem 3.4), the following Korn's inequality holds: for every $u \in H^1(\Omega, \mathbf{R}^3)$

$$\|u\|_{H^1(\Omega, \mathbf{R}^3)}^2 \leq c(\Omega) \left(\int_{\Omega} \frac{1}{2} |(Du)^T + Du|^2 dx + \int_{\Omega} |u|^2 dx \right).$$

Theorem 4.1 (compactness). *Let (G_h) be a sequence of functionals in \mathcal{G}_2 . Then there exists a subsequence $(G_{h_k})_k$ of (G_h) and a functional G in \mathcal{G}_2 such that $(F + G_{h_k})_k$ $\bar{\Gamma}_{\Omega}$ -converges to $F + G$.*

Proof. By Theorem 16.9 in [20] there exists a subsequence $(G_{h_k})_k$ of (G_h) such that $(F + G_{h_k})$ is $\bar{\Gamma}_{\Omega}$ -convergent. The limit functional can be written as $F + G$, where $G: H^1(\Omega, \mathbf{R}^m) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$. Proposition 3.1 guarantees that G satisfies properties (i), (ii) and (iii) in the definition of the class \mathcal{G}_2 . Therefore, only (iv) remains to be proved.

Take A , u , v and φ as in (iv) of Definition 2.1 in Chapter II and assume $G(u, A) + G(v, A) < +\infty$. Let (u_k) and (v_k) be sequences in $H^1(\Omega, \mathbf{R}^m)$ converging to u and v , respectively, in $L^2(\Omega, \mathbf{R}^m)$ and such that

$$\begin{aligned} F(u, A) + G(u, A) &= \limsup_{k \rightarrow \infty} [F(u_k, A) + G_{h_k}(u_k, A)] \\ F(v, A) + G(v, A) &= \limsup_{k \rightarrow \infty} [F(v_k, A) + G_{h_k}(v_k, A)]. \end{aligned}$$

By the coerciveness of F it is not restrictive to suppose that (u_k) and (v_k) converge weakly in $H^1(\Omega, \mathbf{R}^m)$ to u and v , respectively. Then, denoting by (σ_k) an infinitesimal sequence, which can change from line to line, we have

$$\begin{aligned} F(\varphi u_k + (1 - \varphi)v_k, A) - F(\varphi u + (1 - \varphi)v, A) &\leq F(\varphi(u_k - u) + (1 - \varphi)(v_k - v), A) + \sigma_k \\ &\leq \int_A W(x, \varphi D(u_k - u) + (1 - \varphi)D(v_k - v)) dx + \sigma_k \leq F(u_k - u, A) + F(v_k - v, A) + \sigma_k \\ &\leq F(u_k, A) - F(u, A) + F(v_k, A) - F(v, A) + \sigma_k. \end{aligned}$$

Therefore, using the C^1 -convexity of G_{h_k}

$$\begin{aligned} G(\varphi u + (1 - \varphi)v, A) &\leq \limsup_{k \rightarrow \infty} [F(u_k, A) + G_{h_k}(u_k, A) - F(u, A)] \\ &\quad + \limsup_{k \rightarrow \infty} [F(v_k, A) + G_{h_k}(v_k, A) - F(v, A)]. \end{aligned}$$

This implies that $G(\varphi u + (1 - \varphi)v, A) \leq G(u, A) + G(v, A)$. The convexity of $G(\cdot, A)$ can be proved analogously. \square

Remark 4.2. By applying Theorem 11.10 in [20] it is easy to see that if each G_h is quadratic, then so is the limit functional G .

We can now combine Theorems 4.1 and 2.4, Proposition 3.5 and the properties of Γ -convergence to obtain convergence results for minimum problems with obstacles. As an example we have:

Corollary 4.3. *Let $\varphi \in H^1(\Omega, \mathbf{R}^m)$. Let (K_h) be a sequence of multifunctions from Ω to \mathbf{R}^m with closed convex values and such that $\varphi(x) \in K_h(x)$ for every $h \in \mathbf{N}$ and for q.e. $x \in \Omega$. Then there exist a subsequence $(K_{h_k})_k$ of (K_h) , a finite measure $\mu \in \mathcal{M}_2(\Omega)$, a positive Borel measure ν on Ω , and a Borel function $g: \Omega \times \mathbf{R}^m \rightarrow [0, +\infty]$, convex and lower semicontinuous in the second variable, such that for every $A \in \mathcal{A}(\Omega)$, the values*

$$(\mathcal{P}_k) \quad \min \left\{ \int_A W(x, Du(x)) dx : u - \varphi \in H_0^1(A, \mathbf{R}^m), u(x) \in K_{h_k}(x) \text{ for q.e. } x \in A \right\}$$

converge, as $k \rightarrow \infty$, to

$$(\mathcal{P}) \quad \min \left\{ \int_A W(x, Du(x)) dx + \int_A g(x, u(x)) d\mu + \nu(A) : u - \varphi \in H_0^1(A, \mathbf{R}^m) \right\}.$$

Moreover, if M_k and M denote the set of minimum points of problems \mathcal{P}_k and \mathcal{P} respectively, then for every neighbourhood U of M in $\varphi + H_0^1(A, \mathbf{R}^m)$ with respect to the topology of $L^2(A, \mathbf{R}^m)$ there exists $l \in \mathbf{N}$ such that $M_k \subseteq U$ for every $k \geq l$.

5. Dirichlet problems in perforated domains

As in the previous section we shall assume that the energy functional F introduced in (3.1) is quadratic, i.e., the integrand function W is of the form (4.1).

Given a quadratic functional $G \in \mathcal{G}_2$, recall that V_G , defined in Section 1, is a linear-space-valued map on Ω which gives pointwisely the “greatest” set of admissible directions allowed to the functions by the finiteness of G . Hence, the condition $G < +\infty$ cannot select some particular directions when V_G takes only the values $\{0\}$ and \mathbf{R}^m . Here we shall take into account such a special class of \mathcal{G}_2 , which enjoys, as we shall see, nice properties of “closure” with respect to Γ -convergence under suitable assumptions on the energy functional F . In particular, this class contains the functionals which arise dealing with Dirichlet problems on perforated domains, i.e.,

$$G_E(u, A) = \begin{cases} 0, & \text{if } u = 0 \text{ q.e. on } A \cap E, \\ +\infty, & \text{otherwise,} \end{cases}$$

when E is a subset of Ω .

If G is a quadratic functional of \mathcal{G}_2 , we shall say that a pair (M, μ) represents G if M is a symmetric $m \times m$ matrix of Borel functions from Ω to \mathbf{R} with $\xi^T M(x) \xi \geq 0$ for q.e. $x \in \Omega$ and for every $\xi \in \mathbf{R}^m$, μ is a measure in $\mathcal{M}_2(\Omega)$ and

$$G(u, A) = \int_A u^T M u d\mu$$

for every $u \in H^1(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$.

Definition 5.1. For every $\gamma > 0$ let $\mathcal{G}_2(\gamma)$ be the class of all the functional $G \in \mathcal{G}_2$ such that

- (i) G is quadratic;
- (ii) $V_G(x) \in \{\{0\}, \mathbf{R}^m\}$ for q.e. $x \in \Omega$;
- (iii) there exists a pair (M, μ) which represents G and such that the ratio between the maximum and the minimum eigenvalue of $M(x)$ is bounded by γ for μ -a.e. $x \in \Omega$.

Theorem 5.2. *Let $\gamma > 0$ and (G_h) be a sequence in $\mathcal{G}_2(\gamma)$. Assume that there exists $G \in \mathcal{G}_2$ such that $(F + G_h)$ $\bar{\Gamma}_\Omega$ -converges to $F + G$. Then G is quadratic and $V_G(x) \in \{\{0\}, \mathbf{R}^m\}$ for q.e. $x \in \Omega$.*

Proof. G is quadratic by Remark 4.2.

For every $h \in \mathbf{N}$ let (M_h, μ_h) be a pair which represents G_h , and let $\alpha_h(x)$ and $\beta_h(x)$ the maximum and the minimum eigenvalue of $M_h(x)$. Let \mathcal{R} be a rich subset of $\mathcal{A}(\Omega)$ such that $(F(\cdot, A) + G_h(\cdot, A))$ Γ_Ω -converges to $F(\cdot, A) + G(\cdot, A)$ for every $A \in \mathcal{R}$. Let us show that for every $A \in \mathcal{R}$

$$(5.1) \quad Ru \in \text{dom}G(\cdot, A) \quad \text{for every } u \in \text{dom}G(\cdot, A) \text{ and } R \in \mathbf{O}^+,$$

where \mathbf{O}^+ denotes the set of positive definite orthogonal $m \times m$ matrices. Indeed, if $A \in \mathcal{R}$ and $u \in \text{dom}G(\cdot, A)$, then there exists a sequence (u_h) converging to u in $L^2(\Omega, \mathbf{R}^m)$ such that

$$(5.2) \quad F(u, A) + G(u, A) \geq \limsup_{h \rightarrow \infty} [F(u_h, A) + G_h(u_h, A)].$$

In view of (4.2) we have

$$\begin{aligned} F(Ru, A) + G(Ru, A) &\leq \liminf_{h \rightarrow \infty} [F(Ru_h, A) + G_h(Ru_h, A)] \\ &\leq \Lambda \limsup_{h \rightarrow \infty} \int_A |Du_h|^2 dx + \limsup_{h \rightarrow \infty} G_h(Ru_h, A). \end{aligned}$$

By (5.2) and the coerciveness of F , the sequence $(\|Du_h\|_{L^2(A, \mathbf{R}^m)})$ is bounded. Hence, to prove (5.1) we have only to show the boundedness of $(G_h(Ru_h, A))$. It turns out that

$$\begin{aligned} G_h(Ru_h, A) &= \int_A u_h^T R^T M_h R u_h d\mu_h \leq \int_A \alpha_h |u_h|^2 d\mu_h \leq \gamma \int_A \beta_h |u_h|^2 d\mu_h \\ &\leq \gamma \int_A u_h^T M_h u_h d\mu_h = \gamma G_h(u_h, A); \end{aligned}$$

therefore, by (5.2).

$$\limsup_{h \rightarrow \infty} G_h(Ru_h, A) \leq \gamma [F(u, A) + G(u, A)] < +\infty.$$

Thus (5.1) is proved.

Let us fix $A \in \mathcal{R}$ and let (u_k) be a dense sequence in $\text{dom}G(\cdot, A)$ (which is non-empty since $G(0, \Omega) = 0$). By Proposition 5.1 and Remark 6.6 in Chapter II

$$(5.3) \quad V_G(x) = \text{cl}\{u_k(x) : k \in \mathbb{N}\} \quad \text{for q.e. } x \in A.$$

In view of (5.1) we obtain that $G(Ru_k, A) < +\infty$ for every $k \in \mathbb{N}$ and $R \in \mathbf{O}^+$. By the definition of V_G , it follows that $Ru_k(x) \in V_G(x)$ for q.e. $x \in A$, for every $k \in \mathbb{N}$ and R in a fixed basis of \mathbf{O}^+ . By linearity

$$(5.4) \quad Ru_k(x) \in V_G(x), \quad \text{for q.e. } x \in A, \text{ for every } k \in \mathbb{N} \text{ and } R \in \mathbf{O}^+.$$

Let $E = \{x \in A : u_k(x) = 0 \text{ for every } k \in \mathbb{N}\}$; E is defined up to sets of capacity zero. By (5.3), $V_G(x) = \{0\}$ for q.e. $x \in E$. Moreover, by taking (5.4) into account, $V_G(x) = \mathbb{R}^m$ for q.e. $x \in \Omega \setminus E$. \square

Remark 5.3. The previous theorem fails if we drop condition (iii) in Definition 5.1. For instance, consider the sequence $(F + G_h)$ with $F(u, A) = \int_A |Du|^2 dx$ and $G_h(u, A) = \int_A (u^1)^2 dx + \int_A h(u^2)^2 dx$ ($A \in \mathcal{A}(\Omega)$, $u = (u^1, u^2) \in H^1(\Omega, \mathbb{R}^2)$).

Together with Proposition 2.4 the previous theorem yields a result about the limit of Dirichlet problems in perforated domains:

Corollary 5.4. *Let (E_h) be a sequence of subsets of Ω . Assume that there exists a functional $G \in \mathcal{G}_2$ such that the sequence $(F + G_{E_h})$ $\bar{\Gamma}_\Omega$ -converges to $F + G$. Then G is quadratic, V_G is the null space $\{0\}$ or the whole \mathbb{R}^m for q.e. $x \in \Omega$ and, for every $u \in H^1(\Omega, \mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$, the functional G can be written in the form*

$$(5.5) \quad G(u, A) = \int_A u^T M u a \, d\nu,$$

where ν is a positive Radon measure on Ω with $\nu \in H^{-1}(\Omega)$, $a: \Omega \rightarrow [0, +\infty]$ is a Borel function and M is a symmetric $m \times m$ matrix of Borel functions from Ω to \mathbb{R} with $\xi^T M(x) \xi \geq 0$ for q.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^m$.

Before giving the compactness result for the class $\mathcal{G}_2(\gamma)$, let us consider the special case when the functional $F(\cdot, A)$ is invariant with respect to rotations of \mathbb{R}^m ; more precisely, we require that

$$(5.6) \quad F(Ru, A) = F(u, A) \quad \text{for every } u \in H^1(\Omega, \mathbb{R}^m), A \in \mathcal{A}(\Omega) \text{ and } R \in \mathbf{O}^+.$$

It is not difficult to verify that condition (5.6) is equivalent to each of the following ones:

(5.7) for every $i, j = 1, \dots, n$, the matrix $(a_{\alpha\beta}^{ij}(x))_{\alpha\beta}$ is a multiple of the identity for a.e. $x \in \Omega$;

(5.8) for every $x \in \Omega$ there exists a symmetric $n \times n$ matrix $A(x)$ such that

$$W(x, \eta) = \sum_{\alpha=1}^m w(x, \eta^\alpha) \quad \text{for every } \eta = (\eta_i^\alpha)_{\substack{\alpha=1, \dots, m \\ i=1, \dots, n}} \in \mathbf{R}^{mn},$$

where $w(x, \zeta) = \zeta^T A(x) \zeta$ ($\zeta \in \mathbf{R}^n$).

In the case these conditions are satisfied, we shall set, for every $A \in \mathcal{A}(\Omega)$ and $v \in H^1(A)$,

$$\widehat{F}(v, A) = \int_A w(x, Dv(x)) dx,$$

where w is given in (5.8).

As (5.8) shows, we are essentially in a scalar case. Indeed, we have the following result.

Lemma 5.5. *Assume that F satisfies condition (5.6). Let (μ_h) be a sequence in $\mathcal{M}_2(\Omega)$ and*

$$\widehat{G}_h(v, A) = \int_A v^2 d\mu_h, \quad G_h(u, A) = \int_A |u|^2 d\mu_h$$

for every $A \in \mathcal{A}(\Omega)$, $v \in H^1(A)$ and $u \in H^1(A, \mathbf{R}^m)$. Suppose that there exists a functional $\widehat{G}: H^1(\Omega) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ such that

$$\widehat{F} + \widehat{G}_h \quad \overline{\Gamma}_\Omega\text{-converges to} \quad \widehat{F} + \widehat{G}.$$

Then

(i) *there exist a Borel function $a: \Omega \rightarrow [0, +\infty]$ and a Radon measure $\nu \in H^{-1}(\Omega)$ such that $\widehat{G}(v, A) = \int_A v^2 a d\nu$ for every $v \in H^1(\Omega)$ and $A \in \mathcal{A}(\Omega)$;*

(ii) *$F + G_h \quad \overline{\Gamma}_\Omega\text{-converges to} \quad F + G$,*

where $G(u, A) = \int_A |u|^2 a d\nu$ for every $u \in H^1(\Omega, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$.

Proof. Item (i) follows immediately from Theorem 4.1 and Proposition 2.4 applied for $m = 1$, while (ii) is a simple check. \square

By Theorem 5.2, a sequence (G_h) in $\mathcal{G}_2(\gamma)$ gives rise to a limit functional which still satisfies properties (i) and (ii) in Definition 5.1. To obtain also a bound on the ratio of the eigenvalues, we have to strengthen the coerciveness of F .

Theorem 5.6. *Assume that there exists a constant $\alpha > 0$ such that*

$$(5.9) \quad F(u, A) \geq \alpha \int_A |Du(x)|^2 dx$$

for every $u \in H_0^1(A, \mathbf{R}^m)$ and $A \in \mathcal{A}(\Omega)$. Let $\gamma > 0$ and let (G_h) be a sequence in $\mathcal{G}_2(\gamma)$. Assume that there exists a functional $G \in \mathcal{G}_2$ such that $(F + G_h)$ $\bar{\Gamma}_\Omega$ -converges to $F + G$. Then $G \in \mathcal{G}_2(\gamma')$, with $\gamma' = \max(\frac{\Lambda}{\alpha}, \gamma)$.

Proof. For every $h \in \mathbf{N}$ let (M_h, μ_h) be a pair which represents G_h , and let $\alpha_h(x)$ and $\beta_h(x)$ be the maximum and the minimum eigenvalue of $M_h(x)$.

Since G is quadratic, by Proposition 2.4 there exists a pair $(M, a\nu)$ representing G , with $a: \Omega \rightarrow [0, +\infty]$ Borel function and ν Radon measure in $H^{-1}(\Omega)$. Moreover, denoting by E a representative of the set $\{x \in \Omega : V_G(x) = \{0\}\}$, we can assume that M is the identity matrix on E on $\Omega \setminus E$. We claim that the ratio between the maximum and the minimum eigenvalue of $M(x)$ is bounded by γ' for $a\nu$ -a.e. $x \in \Omega$, where $\gamma' = \max(\frac{\Lambda}{\alpha}, \gamma)$.

Let ω be a fixed bounded open subset of Ω . Clearly, it is enough to show that the estimate holds $a\nu$ -a.e. on ω .

By (5.9) and (4.2) we have

$$\alpha \int_\omega |Du|^2 dx + \int_A \beta_h |u|^2 d\mu_h \leq F(u, \omega) + G_h(u, A) \leq \Lambda \int_\omega |Du|^2 dx + \int_A \alpha_h |u|^2 d\mu_h$$

for every $A \in \mathcal{A}(\omega)$ and for every $u \in H^1(\omega, \mathbf{R}^m)$ with constant value on the boundary of ω ; therefore, for the same u and A ,

$$(5.10) \quad \alpha \int_\omega |Du|^2 dx + G_h^1(u, A) \leq F(u, \omega) + G_h(u, A) \leq \gamma' (\alpha \int_\omega |Du|^2 dx + G_h^1(u, A))$$

where $\gamma' = \max(\frac{\Lambda}{\alpha}, \gamma)$, and

$$G_h^1(u, A) = \int_A \beta_h |u|^2 d\mu_h.$$

By Remark 3.4 and (ii) of Proposition 3.3, we have

$$(5.11) \quad F_\varphi(\cdot, \omega) + G_h \quad \bar{\Gamma}_\omega\text{-converges to} \quad F_\varphi(\cdot, \omega) + G$$

for every $\varphi \in H^1(\omega, \mathbf{R}^m)$.

Define $D(u, A) = \alpha \int_A |Du|^2 dx$ ($A \in \mathcal{A}(\Omega)$, $u \in H^1(A, \mathbf{R}^m)$). By Theorem 4.1 and Lemma 5.5, there exist a subsequence $(G_{\sigma(h)}^1)$ of (G_h^1) , a Borel function $b: \Omega \rightarrow [0, +\infty]$ and a Radon measure $\rho \in H^{-1}(\Omega)$, such that $(D + G_{\sigma(h)}^1)$ $\bar{\Gamma}_\Omega$ -converges to $D + G^1$, where

$$G^1(u, A) = \int_A |u|^2 b d\rho, \quad (u \in H^1(\Omega, \mathbf{R}^m), A \in \mathcal{A}(\Omega)).$$

In view of Remark 3.4 and (ii) of Proposition 3.3, we have

$$(5.12) \quad D_\varphi(\cdot, \omega) + G_{\sigma(h)}^1 \quad \bar{\Gamma}_\omega\text{-converges to} \quad D_\varphi(\cdot, \omega) + G^1,$$

where $D_\varphi(\cdot, \omega) = D(\cdot, \omega)$ on $\varphi + H_0^1(\omega, \mathbf{R}^m)$ and $D_\varphi(\cdot, \omega) = +\infty$ otherwise on $H^1(\omega, \mathbf{R}^m)$.

Let us now fix $\xi \in \mathbf{R}^m$ with $|\xi| = 1$. In view of (5.10), (5.11) and (5.12) with $\varphi = \xi$, we easily get that

$$(5.13) \quad \begin{aligned} \alpha \int_\omega |Du|^2 dx + \int_A |u|^2 b d\rho &\leq F(u, \omega) + \int_A u^T M u a d\nu \\ &\leq \gamma' (\alpha \int_\omega |Du|^2 dx + \int_A |u|^2 b d\rho) \end{aligned}$$

for every $u \in H^1(\omega, \mathbf{R}^m)$ with $u - \xi \in H_0^1(\omega, \mathbf{R}^m)$ and for every $A \in \mathcal{A}(\omega)$.

We claim that

$$(5.14) \quad \int_V b d\rho \leq \int_V \xi^T M \xi a d\nu \leq \gamma' \int_V b d\rho$$

for every quasi open subset V of ω .

Let us fix $V \subseteq \omega$, V quasi open. For every $h \in \mathbf{N}$ there exists a set $Z_h \subseteq \omega$ such that $V \cup Z_h$ is open and $\text{cap}(Z_h, \omega) < 1/h$. Denote by w_h the capacitary potential of Z_h with respect to ω ; then $w_h \in H_0^1(\omega)$, $w_h = 1$ q.e. on Z_h and (w_h) converges to zero strongly in $H^1(\omega)$.

Let us now show the first inequality in (5.14). Take $u = (1 - w_h)\xi$ and $A = V \cup Z_h$ in (5.13). Since

$$\lim_{h \rightarrow \infty} \int_\omega |D((1 - w_h)\xi)|^2 dx = 0, \quad \lim_{h \rightarrow \infty} F((1 - w_h)\xi, \omega) = 0$$

we have

$$\liminf_{h \rightarrow \infty} \int_V (1 - w_h)^2 b d\rho \leq \liminf_{h \rightarrow \infty} \int_{V \cup Z_h} (1 - w_h)^2 \xi^T M \xi a d\nu \leq \int_V \xi^T M \xi a d\nu.$$

Therefore, by the Fatou Lemma,

$$\int_V b \, d\rho \leq \int_V \xi^T M \xi a \, d\nu.$$

The second inequality in (5.14) can be proved analogously.

Let u be a fixed function of $\text{dom} G^1(\cdot, \omega)$ and $\varepsilon > 0$. Set $V = \{x \in \omega : |u(x)| > \varepsilon\}$. Since V is quasi open, from (5.14) it follows that for every $A \in \mathcal{A}(\omega)$

$$\int_{V \cap A} b \, d\rho \leq \int_{V \cap A} \xi^T M \xi a \, d\nu \leq \gamma' \int_{V \cap A} b \, d\rho.$$

Let us now notice that $\int_V b \, d\rho < +\infty$, since

$$+\infty > G^1(u, \omega) = \int_\omega |u|^2 b \, d\rho \geq \varepsilon^2 \int_V b \, d\rho.$$

Hence, there exists a subset $N = N(\xi, u, \varepsilon)$ of ω such that $(\nu + \rho)(N) = 0$ and

$$(5.15) \quad b \frac{d\rho}{d(\nu + \rho)} \leq \xi^T M \xi a \frac{d\nu}{d(\nu + \rho)} \leq \gamma' b \frac{d\rho}{d(\nu + \rho)}$$

on $V \setminus N$, where $d\rho/d(\nu + \rho)$ and $d\nu/d(\nu + \rho)$ are fixed Borel representatives of the Radon-Nykodim derivatives.

Let E_1 be a representative of the set $\{x \in \omega : V_{G_1}(x) = \{0\}\}$. Letting ξ , u and ε above vary in countable dense subsets of $\{\xi \in \mathbf{R}^m : |\xi| = 1\}$, $\text{dom} G^1(\cdot, \omega)$ and \mathbf{R}^+ respectively, we obtain that (5.15) holds $(\nu + \rho)$ -a.e. on $\omega \setminus E_1$ for every $\xi \in \mathbf{R}^m$ with $|\xi| = 1$.

Let us notice that $E_1 = E$ up to sets of capacity zero. Indeed, by a standard cut-off argument, from (5.13) it follows that $\text{dom} G(\cdot, A) = \text{dom} G^1(\cdot, A)$ for every $A \in \mathcal{A}(\omega)$ with $A \subset\subset \omega$. In view of the definition of V_G (see Section 2) and Remark 5.2 in Chapter II, $V_G = V_{G_1}$ q.e. on A for every $A \subset\subset \omega$. It turns out that E_1 coincides with E up to sets of capacity zero.

Therefore, (5.15) holds $(\nu + \rho)$ -a.e. on $\omega \setminus E$. Since $0 < \frac{d\nu}{d(\nu + \rho)} < +\infty$ ν -a.e. on Ω , and a is finite on $\Omega \setminus E$, (5.15) implies that

$$b \frac{d\rho}{d(\nu + \rho)} \left(a \frac{d\nu}{d(\nu + \rho)} \right)^{-1} \leq \xi^T M \xi \leq \gamma' b \frac{d\rho}{d(\nu + \rho)} \left(a \frac{d\nu}{d(\nu + \rho)} \right)^{-1}$$

$a\nu$ -a.e. on $\omega \setminus E$ for every $\xi \in \mathbf{R}^m$ with $|\xi| = 1$. This proves the stated bound on the eigenvalues of M on $\omega \setminus E$. Since M is the identity matrix on E , the proof of the theorem is accomplished. \square

As an application of the previous theorem we have:

Corollary 5.7. *Let E_h and G be as in Corollary 5.4.*

(i) *If there exists $\lambda > 0$ such that*

$$W(x, \eta) \geq \lambda |\eta|^2 \quad \text{for a.e. } x \in \Omega \text{ and for every } \eta \in \mathbf{R}^{mn},$$

then $G \in \mathcal{G}_2(\Lambda/\lambda)$.

(ii) *If $n = m = 3$ and there exists $\kappa > 0$ such that*

$$W(x, \eta) \geq \kappa \left| \frac{1}{2}(\eta^T + \eta) \right|^2 \quad \text{for a.e. } x \in \Omega \text{ and for every } \eta \in \mathbf{R}^3,$$

then $G \in \mathcal{G}_2(2\Lambda/\kappa)$.

Proof. (i) follows immediately from Theorem 5.6. As to (ii), it is enough to notice that, by Korn's inequality (see, for instance, [34] Theorem 3.1)

$$\frac{1}{2} \int_A |Du|^2 dx \leq \int_A \left| \frac{1}{2}((Du)^T + Du) \right|^2 dx$$

for every $u \in H_0^1(A, \mathbf{R}^m)$. □

References

- [1] ALBERTI G.: Integral representation of local functionals. Preprint Scuola Normale, Pisa. 1990.
- [2] ALBERTI G., BUTTAZZO G.: Integral representation of functionals defined on Sobolev spaces. *Composite Media and Homogenization Theory (Trieste, 1990)*, 1-12, Birkhäuser, Boston, 1991.
- [3] AMBROSIO L., DAL MASO G.: On the relaxation in $BV(\Omega; \mathbb{R}^m)$ of quasi-convex integrals. *J. Funct. Anal.*, to appear.
- [4] AMBROSIO L., PALLARA D.: Integral representation of relaxed functionals on $BV(\mathbb{R}^n; \mathbb{R}^k)$ and polyhedral approximation. Preprint Scuola Normale, Pisa, submitted to *Indiana Univ. Math. J.*
- [5] ATTOUCH H. : Variational Convergence for Functions and Operators. Pitman, London. 1984.
- [6] ATTOUCH H. , PICARD C. : Problèmes variationnels et théorie du potentiel non linéaire. *Ann. Fac. Sci. Toulouse.* 1 (1979), 89-136.
- [7] ATTOUCH H. , PICARD C. : Asymptotic analysis of variational problems with constraints of obstacle type. *Publ. Math. d'Orsay* (1982), 82-107.
- [8] ATTOUCH H. , PICARD C. : Variational inequalities with varying obstacles: the general form of the limit problem. *J. Funct. Anal.* 50 (1983), 329-386.
- [9] BOUCHITTÉ G., DAL MASO G.: Integral representation and relaxation of convex local functionals on $BV(\Omega)$. Preprint SISSA, Trieste, 1991.
- [10] BOUCHITTÉ G. , VALADIER M. : Integral representation of convex functionals on a space of measures. *J. Funct. Anal.* 80 (1988), 398-420.
- [11] BREZIS H., BROWDER F.E.: Some properties of higher order Sobolev spaces. *J. Math. Pures Appl.* (9) 61 (1982), 245-259.
- [12] BUTTAZZO G.: Semicontinuity, Relaxation and Integral Representation Problems in the Calculus of Variations. Pitman Res. Notes in Math. 207, Longman, Harlow, 1989.
- [13] CARBONE L. , COLOMBINI F. : On convergence of functionals with unilateral constraints. *J. Math. Pures Appl.* (9) 59 (1980), 465-500.
- [14] CASTAING C., VALADIER M.: Convex Analysis and Measurable Multifunctions. Lecture Notes in Math. 580, Springer-Verlag, Berlin, 1977.
- [15] CHOQUET G. : Lectures on Analysis. Vol. I. Mathematics Lecture Note Series, Benjamin, Inc., 1969.
- [16] CIARLET P.G. : Mathematical Elasticity. Volume I: Three-dimensional Elasticity. North-Holland (1988).
- [17] CIORANESCU D. , MURAT F. : Un terme étrange venu d'ailleurs, I, II. *Nonlinear Partial Differential Equations and Their Applications. Collège de France Seminar. Vol. II*, 98-133, *Res. Notes in Math.*, 60, Pitman, London, 1982, *Vol. III*, 154-178, *Res. Notes in Math.*, 70, Pitman, London, 1983.
- [18] DAL MASO G. : Asymptotic behaviour of minimum problems with bilateral obstacles. *Ann. Mat. Pura Appl.* (4) 129 (1981), 327-366.

- [19] DAL MASO G. : On the integral representation of certain local functionals. *Ricerche Mat.* **32** (1983), 85-113.
- [20] DAL MASO G. : An Introduction to Γ -convergence. Birkhäuser, Boston, to appear.
- [21] DAL MASO G. , DEFRANCESCHI A. : Limits of nonlinear Dirichlet problems in varying domains. *Manuscripta Math.* **61** (1988), 251-278.
- [22] DAL MASO G. , LONGO P. : Γ -limits of obstacles. *Ann. Mat. Pura Appl. (4)* **128** (1980), 1-50.
- [23] DAL MASO G. , MOSCO U. : Wiener's criterion and Γ -convergence. *Appl. Math. Optim.* **63** (1987), 15-63.
- [24] DAL MASO G. , PADERNI G. : Integral representation of some convex local functionals. *Ricerche Mat.* **36** (1987), 197-214.
- [25] DE GIORGI E. , DAL MASO G. , LONGO P. : Γ -limiti di ostacoli. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **68** (1980), 481-487.
- [26] DE GIORGI E. , LETTA G. : Une notion générale de convergence faible pour des fonctions croissantes d'ensemble. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **4** (1977), 61-99.
- [27] FEDERER H. , ZIEMER W. P. : The Lebesgue set of a function whose distribution derivatives are p th power summable. *Indiana Univ. Math. J.* **22** (1982), 139-158.
- [28] FONSECA I. , RYBKA P. : Relaxation of multiple integrals in the space $BV(\Omega; \mathbb{R}^p)$. Research Report, Carnegie Mellon University, Pittsburgh, 1991.
- [29] GILBARG D. , TRUDINGER N. S. : Elliptic partial differential equations of second order. Springer-Verlag, New York, 1983.
- [30] GRUN-REHOMME M. : Caractérisation du sous-différentiel d'intégrandes convexes dans les espaces de Sobolev. *J. Math. Pures Appl.* **56** (1977), 149-156.
- [31] HALMOS P.R. : Measure Theory. Van Nostrand, Princeton, 1950.
- [32] HIAI F. , UMEGAKI H. : Integrals, conditional expectations, and martingales of multivalued functions. *J. Multivariate Anal.* **7** (1977), 149-182.
- [33] LOJASIEWICZ S. , Jr. : Parametrizations of convex sets. *J. Approx. Theory*, to appear.
- [34] NECAS J. , HLAVACEK I. : Mathematical Theory of Elastic-Plastic Bodies: an Introduction. Elsevier, Amsterdam, 1981.
- [35] MORREY C. B., Jr. : Multiple integrals in the calculus of variations. Springer-Verlag, Berlin, 1966.
- [36] MOSCO U. : Convergence of convex sets and of solutions of variational inequalities. *Adv. in Math.* **3** (1969), 510-585.
- [37] ORNELAS A. : Parametrization of Carathéodory multifunctions. *Rend. Sem. Mat. Univ. Padova* **83** (1990), 33-44.
- [38] PICARD C. Problème biharmonique avec obstacles variables. Thèse, Univ. Paris-Sud, 1984.
- [39] ROCKAFELLAR R. T. : Convex Analysis. Princeton Univ. Press, 1970.
- [40] SAINT PIERRE J. : Point de Steiner et sections lipschitziennes. Sem. d'Analyse convexe. Montpellier, 1985.

- [41] VALADIER M.: Multi-applications mesurables à valeurs convexes compactes. *J. Math. Pures Appl.* **50** (1971), 265-297.
- [42] YOSIDA K.: Functional Analysis. Springer-Verlag, Berlin, 1965.
- [43] ZIEMER W. P. : Weakly differentiable Functions. Springer-Verlag, Berlin, 1989.

