CONFORMAL TODA FIELD THEORIES

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Contents

Introduction

The role of Lie algebras in determining the properties of the Toda field theories in two dimensions has been known for a long time. This is made visible by the classification of the Toda field equations in terms of the generalized Cartan matrices and by those analysis relating symmetries of such models with corresponding symmetries of the associated Dynkin diagrams [24]. Moreover, it has been shown that the problem of constructing solutions of the Toda field theories can be posed as a factorization problem in the underlying Lie algebra. Particularly, the Gauss decomposition of the associated simple Lie group constitutes the clue in order to produce explicit solutions of the Toda equations [23]. Subsequently, the possibility to regard Toda field theories as hamiltonian reductions of Wess-Zumino-Novikov-Witten (WZNW) models – this being in itself a vast field of research – emphasizes the importance of such algebraic and geometrical structures [18].

However, as a consequence of the recent results in Conformal Field Theory and in the study of Quantum Groups, and, on the other hand, with the appearance of new promising directions of research, such as Matrix Models and two-dimensional Topological Gravity, the situation seems to be somehow upset. Due to their rich structure, which connects them with all these fields, Toda field theories in two dimensions have become a favorite research topic in theoretical physics.

Going into more details, Toda field theories based on finite dimensional Lie algebras underlie many remarkable conformal field theory models – notably all sort of minimal models –; indeed they possess both the necessary conformal structure, which manifests itself for example in the chiral splitting, and the integrable structure, which is connected with their hidden quantum group symmetry. This last represent the peculiar feature of the "quantized" version of all that algebraic machinery whose importance we are here mentioning. Both structures are simultaneously expressed in an elegant form by the exchange algebra.

Another reason of interest is the connection between the sl_2 Toda field theory, i.e. the Liouville theory, and string theory and 2-dimensional gravity, together with the evoked possibility that sl_n Toda field theory might bear a relation to 2-dimensional gravity coupled to conformal matter, in much the same way as the latter combination appears in matrix models. The present state of affairs does not even allow us to exclude that there might be a direct connection between matrix models and Liouville or Toda theories formulated on the lattice.

Still linked to their algebraic and geometric structure is the W-algebra symmetry, which characterizes the Toda field theories. In fact such theories are at the origin of the present interest in W-algebras. The geometrical meaning of W-algebras is still rather obscure. However, if we remember that the Liouville equation is well-known to be the basis of the uniformization theory of Riemann surfaces, it is reasonable to expect geometry to play a deep role in Toda field theories and, viceversa, that the latter might lead to significant geometrical developments.

All this sounds pretty appealing to all those who have followed the most recent developments in theoretical physics. On the other hand, even though the research in this field has been intensive, many questions in Toda field theories are still unanswered. Among the latter we quote in particular the problem of constructing conformal blocks in W-algebra minimal models and the geometrical meaning of W-algebras.

To this aim, the following brief summary of the principal features which characterize those general structures, underlying Toda field theories, could be of some utility, also in consideration of the developments to be analyzed in the next Chapters. Obviously an exhaustive treatment of the subject is a task that we shall not pursue in the present Introduction and, for every further deepening, we refer to the concerning literature (see refs.[4, 14, 13, 26, 6]).

After the present Introduction, the content of this Thesis is organized in three Chapters.

In Chapter 1, we recall the main features which characterize the classical Toda field theories associated to finite dimensional Lie algebras $\mathfrak{g}$. Particular emphasis is devoted to the conformal properties of the model and to the past attempts for separating the chiralities in the Toda phase space. In this context, we first consider the Leznov-Saveliev analysis, based on the Gauss decomposition of the Lie group corresponding to $\mathfrak{g}$, and, then, the reconstructive approach proposed in ref.[7].

The unanswered question concerning the splitting the chiralities, which concludes Chapter 1, constitutes the matter of study of Chapter 2, where the Liouville model is addressed. Here, the problem is faced starting from a more thorough analysis of the transformation undergone by the symplectic structure, when we try to single out the chiral and antichiral content of the theory. On these grounds, we furnish the solution to the problem, so to completely implement the splitting program, by a more general interpretation of the Leznov-Saveliev decomposition. This allows us to determine the explicit realization of the Drinfel'd-Sokolov linear systems introduced in ref.[7] and, therefore, a global parametrization of the phase space of the Liouville model in terms of free bosonic oscillator modes.

The scheme of the Drinfel'd-Sokolov linear systems, proposed in ref.[7] and realized in Chapter 2 for the sl_2 case, is exploited in Chapter 3 to formulate a lattice version of Toda field theories, following ref.[8]. By referring to this prescription, we study, on one hand, the classical conformal properties of the theory at the lattice level and, on the other hand, the quantization of sl_n Toda field theories. As for the first topic, we find the discretize counterpart of the W_3 algebra. Moreover, our quantization program leads to the exchange algebra of the sl_n Bloch wave basis. The related exchange matrix has been already achieved in a completely different context, which, however, does not provide a correct interpretation in a Quantum Group framework.

Geometrical structure of integrable systems in two dimensions: the zero-curvature representation

As it is well known, a necessary and sufficient condition to establish the integrability of a dynamical system with an infinite number of degrees of freedom does not exist. However, in (1+1) dimensions, many interesting evolution equations which admit exact solutions, such as the Korteweg de Vries (KdV) equation, the non-linear Schrödinger equation, the sine-Gordon equation, exhibit a so-called zero-curvature representation. This means that they can be regarded as the compatibility conditions of a linear system or, in a more geometrical language, as the zero-curvature condition of a suitable flat connection on the (1+1) dimensional space-time. Therefore, one can inquire whether such a structure, with such peculiar features, might constitute the criterion for characterizing at least a class of integrable systems. We will return to this question in the following, when we will face the problem from a more general point of view. Let us now briefly recall the main properties of the zero-curvature representation, referring to its simple geometrical interpretation, as the name suggests.

To fix the notation, consider a non-relativistic evolution equation

$$u_t = f[u] \,, \tag{I.1}$$

where with u_t we denote the time derivative of the wave function $u = u(x,t)$ and $f[u]$ is a non-linear function of u and its space derivatives. In a similar fashion, we will indicate with u_x the derivative with respect to the spatial variable. We say that the eq.(I.1) admits a *zero-curvature representation* if it can be recast in the form

$$\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} = 0 \,, \tag{I.2}$$

or, in components,

$$\mathbf{F}_{xt} = [\mathbf{D}_x, \mathbf{D}_t] = 0 \,. \tag{I.3}$$

Here $\mathbf{F}$ and $\mathbf{D}$ are the curvature and the covariant derivative, respectively, relative to the connection $\mathbf{A} = \mathbf{A}[u]$. The condition (I.2) is equivalent to state that the parallel transport along a curve joining $\mathbf{x}_0 = (x_0, t_0)$ to $\mathbf{x} = (x,t)$ depends in fact only on the initial and final points $\mathbf{x}_0$ and $\mathbf{x}$. Therefore, let v denotes an element of a vector bundle associated with the principal fibre bundle on which A is defined and let ρ be the representation[1] of the structural group on the fibre of such a vector bundle. Then the condition (I.2) corresponds to the relation

$$v(\mathbf{x}) = \rho(\Psi(\mathbf{x};\mathbf{x}_0))\, v(\mathbf{x}_0) \,, \qquad \Psi(\mathbf{x};\mathbf{x}_0) = \overleftarrow{\mathrm{P}}\, e^{-\int_\gamma \mathbf{A}} \,, \tag{I.4}$$

where γ in an arbitrary path from $\mathbf{x}_0$ to $\mathbf{x}$, or to its differential counterpart

$$\begin{aligned} \mathbf{D}_x v &= (\partial_x + \mathbf{A}_x) v = 0 \\ \mathbf{D}_t v &= (\partial_t + \mathbf{A}_t) v = 0 \end{aligned} \,. \tag{I.5}$$

The expression of Ψ in eq.(I.4), where $\overleftarrow{\mathrm{P}}$ denotes the path-ordering ($\overrightarrow{\mathrm{P}}$ will indicate the reverse path-ordering), is to remember the relation

$$\mathbf{A} = -d\Psi\, \Psi^{-1} \,, \tag{I.6}$$

[1]Hereafter the indication of the representation will be understood unless this will cause any ambiguity

a further equivalent way to formulate the zero-curvature condition.

Eq.(I.5) is the *linear system* associated with the integrable equation (I.1) and the eq.(I.2) can be viewed as compatibility condition of such an overdetermined system.

Clearly a similar construction is by no means unique. Indeed we can always perform a *gauge transformation*

$$\mathbf{A} \to {}^{g}\mathbf{A} = g^{-1}\mathbf{A}g + g^{-1}dg \tag{I.7}$$

without affecting the zero-curvature condition (I.2). Therefore, with the above geometrical interpretation, we have mapped the phase space of the integrable dynamical system into an orbit of gauge equivalent connections. We will show later that this identification constitutes the crucial starting point of a different algebraic approach.

To give a concrete content to the above definitions, consider a classical example, the already recalled KdV equation

$$u_t = u_{xxx} + 6uu_x \, . \tag{I.8}$$

The associated flat connection is

$$\mathbf{A}_x^{\mathrm{KdV}} = \begin{pmatrix} 0 & -1 \\ \lambda^2 + u & 0 \end{pmatrix} \qquad \mathbf{A}_t^{\mathrm{KdV}} = \begin{pmatrix} u_x & 4\lambda - 2u \\ p[u] & -u_x \end{pmatrix}$$

with

$$p[u] = 2u^2 + u_{xx} - 2\lambda^2 u - 4\lambda^4 \, .$$

Here λ is an arbitrary constant, usually called the spectral parameter of the theory, which caused a degeneracy of the zero curvature representation, common feature of many integrable systems which exhibit a similar structure.

We can compute

$$\begin{aligned} [\mathbf{A}_x^{\mathrm{KdV}}, \mathbf{A}_t^{\mathrm{KdV}}] &= \begin{pmatrix} -u_{xx} & 2u_x \\ 2(\lambda^2 + u)u_x & u_{xx} \end{pmatrix} \\ \partial_x \mathbf{A}_t^{\mathrm{KdV}} &= \begin{pmatrix} u_{xx} & -2u_x \\ 2(2u - \lambda^2)u_x + u_{xxx} & -u_{xx} \end{pmatrix} \\ \partial_t \mathbf{A}_x^{\mathrm{KdV}} &= \begin{pmatrix} 0 & 0 \\ u_t & 0 \end{pmatrix} \end{aligned}$$

which clearly shows how imposing $\mathbf{F}_{xt}^{\mathrm{KdV}} = 0$ yields back eq.(I.8).

When the base space (that is the space-time of the evolution equation) is the xt-plane, the principal fibre bundle supporting the flat connection $\mathbf{A}$ is trivial and, therefore, the above geometrical description could seem superfluous and it reduces to a nice exercise. Nevertheless interesting features arise when periodic space boundary condition are chosen. This is to say that the phase space of the evolution equation is the space of smooth functions on the cylinder $Cyl = \{0 \leq x < 2\pi, -\infty < t < +\infty\}$. In this case the conserved quantities acquire a simple geometrical interpretation. Indeed, since the space-time is no longer simply connected, it is possible to construct a non trivial holonomy of the flat connection $\mathbf{A}$,

$$T_{\mathbf{x}_0} = \overleftarrow{\mathrm{P}}\, e^{-\oint_{\mathcal{C}} \mathbf{A}} \tag{I.9}$$

associated to the loop[2] $\mathcal{C}$ at $\mathbf{x}_0$ around the cylinder, once the base point $\mathbf{x}_0 = (x_0, t_0)$ is fixed.

If we consider $T_{\mathbf{x}_0}$ as a function of the time component of the base point, $T_{x_0}(t)$, it easy to verify that

$$T_{x_0}(t) \;=\; S(t)\, T_{x_0}(t_0)\, S^{-1}(t)\,. \tag{I.10}$$

Indeed, it is sufficient to notice as, in order to go from (x_0, t_0) to (x_0, t), we can either draw firstly a loop at fixed time t_0 and then evolve to (x_0, t), keeping constant the space component x_0, or the contrary, drawing this time a loop at the time t. Due to the flatness of the connection, the two path are equivalent and therefore we have

$$T_{x_0}(t)\left(\overleftarrow{\mathrm{P}}\, e^{-\int_{x_0} \mathbf{A}_t}\right) \;=\; \left(\overrightarrow{\mathrm{P}}\, e^{-\int_{x_0} \mathbf{A}_t}\right) T_{x_0}(t_0) \tag{I.11}$$

With the identification

$$S(t) = \overleftarrow{\mathrm{P}}\, e^{-\int_{x_0} \mathbf{A}_t} \qquad S^{-1}(t) = \overrightarrow{\mathrm{P}}\, e^{-\int_{x_0} \mathbf{A}_t}\,,$$

the evolution law (I.10) is recovered.

The relevant consequence is that the trace of the holonomy (I.10) is conserved in time. If we take into account the dependence on a spectral parameter, then

$$\mathfrak{T}(\lambda) \;=\; \mathrm{tr}\, T_{\mathbf{x}_0}(\lambda)$$

can be viewed as the generating function of the conserved quantities of the dynamical system. Hereafter we refer to the holonomy (I.9) as the *monodromy matrix*, it determining the monodromy behaviour of the solutions of the linear system (I.5).

[2] More precisely, being $\mathbf{A}$ a flat connection, the holonomy is associated to the corresponding homotopy class of $\mathcal{C}$

Algebraic structure of integrable systems in two dimensions: the Lax pair realization

The integrability of the KdV equation is better known as an application of the classical Inverse Scattering Method. From this point of view, the crucial step is to determine a *Lax pair* of linear differential operators, such that the flow of the dynamical system can be written as

$$\partial_t L = [L, M] \,. \tag{I.12}$$

Then, the conserved quantities are characterized as those functions of L which are invariant under the adjoint action of M. In particular, the spectrum of L is left unchanged by the flow (I.12). This seems to suggest that the zero-curvature representation could be interpreted in a similar way. Actually, this immediately follows once we set

$$\begin{aligned} L &= \partial_x + \mathbf{A}_x \\ M &= \mathbf{A}_t \,, \end{aligned}$$

which leads immediately to recast the zero-curvature condition (I.2) in the form (I.12).

Moreover, to assimilate the two different formalisms, we could apply a usual procedure in the framework of ordinary differential equations which replaces the n-order linear differential operators L with a first order $n \times n$ differential operator $\tilde{L} = d + \mathbf{A}$. The operator $\tilde{L}$ is equivalent to the original L in the sense that their Kernels are in one-to-one correspondence: if $\tilde{v} \in Ker\, \tilde{L}$, its last component belongs to $Ker\, L$ and conversely, from an element in $Ker\, L$ we can construct one in $Ker\, \tilde{L}$. Then the matrix $\mathbf{A}$ may be regarded as belonging to the Lie algebra $sl(n)$.

A deeper analysis shows that flows as those produced by Lax pairs naturally arise in two different general frameworks, that of the pseudo-differential operators and that of the Baxter-Lie algebras.

In the former case we can associate to every linear n-order ordinary differential operator,

$$L = d^n + \sum_{j=1}^{n} a_j d^{n-j} \,,$$

an infinite hierarchy of flows defined by means of a commutation relation

$$\delta_k L = [M_k, L] \,, \tag{I.13}$$

where δ_k is the corresponding infinitesimal variation. Actually, requiring that eq.(I.13) is a well-posed definition, which is equivalent to impose that the commutator in eq.(I.13) closes in the space generated by L (*i.e.* that it is of order less or equal to $n-1$), one determines the form of the differential operators M_k. It turns out to be[3]

$$M_k = (L^{\frac{k}{n}})_+ ,$$

where $(L^{\frac{k}{n}})_+$ is the ordinary differential part of the pseudo-differential operator $L^{\frac{k}{n}} = D^k$, obtained by defining D as such unique pseudo-differential operator that satisfies the identity $D^n = L$.

The relevant upshots of the previous construction, in consideration of which this appears as a generalization of the classical Inverse Scattering Method, can be summarized in the following points:

i) as immediate consequences of the commutator form of the flow equations, these flows are isospectral and commute, moreover

ii) the traces $I_k = \mathrm{tr}(L^{\frac{k}{n}}) = \int \mathrm{res}\, L^{\frac{k}{n}}$ form an infinite set of conserved quantities in involution;

iii) this hierarchy of integrable flows has also an interesting hamiltonian interpretation: indeed there exist two Hamiltonian structures reproducing the flows $\delta_k L$ as

$$\delta_k L = \{I_k, L\}^{(1)} = \{I_{k-1}, L\}^{(2)} \tag{I.14}$$

and

iv) the second hamiltonian structure provides a realization of the classical (Poisson) version of the Virasoro algebra and of its extentions, the W-algebras. Indeed, this is the upshot when we compute the Poisson relations among the coefficients of the differential operator L, referring to the bracket $\{\cdot,\cdot\}^{(2)}$.

Baxter-Lie algebras represent a completely different context. In technical language, these are the Lie bialgebras corresponding to coboundary Poisson-Lie group. This means that we are considering on the Lie algebra $\mathfrak{g}$ the structure induced by defining on the related Lie group G the Poisson bracket

$$\{g \overset{\otimes}{,} g\} := -[r, g\otimes g] , \tag{I.15}$$

[3]More precisely one should take into account also the flow generated by functions, but, since it causes only a shift in the function on which L acts, this is used to bring L in the canonical form with no d^{n-1} term.

which is referred to as the *Sklyanin bracket.* The element $g \in G$ is here identified with the coordinate functions and

$$\{g \overset{\otimes}{,} g\}_{ij,kl} := \{g_{ik}, g_{jl}\} .$$

Moreover $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a solution of the modified classical Yang-Baxter equation, which were also antisymmetric with respect to the invariant scalar product. Denoting C the Casimir element in $\mathfrak{g} \otimes \mathfrak{g}$, this is equivalent to say that $r^{\pm} = r \pm C$ are solutions of the better known *classical Yang-Baxter equation*

$$[r_{12}, r_{23}] + [r_{12}, r_{13}] + [r_{13}, r_{23}] = 0 . \tag{I.16}$$

Here the indices $1, 2, 3$ label the three copies of $\mathfrak{g}$ in the tensor product $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ on which this equation has to be considered. It is well-known that the condition (I.16) can be achieved as the "classical" limit of the Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} ,$$

when R is identified with the exponentiation of the classical r-matrix, $R = e^{-i\hbar r}$, and the limit $\hbar \rightarrow 0$ is taken. Differently, in the context we are here considering, the requirement (I.16) for the classical r-matrix follows imposing the Jacobi identity[4] for the Sklyanin bracket (I.15). Furthermore, we can verify that the group G equipped with (I.15) acquires the structure of a (coboundary) Poisson-Lie group, the definition of the Sklyanin bracket being compatible with the group multiplication of G.

The induced structure on the Lie algebra level is obtained by differentiating the Sklyanin bracket at the identity of G, and interpreting the result as a Lie bracket in the dual space $\mathfrak{g}^*$. The pair $(\mathfrak{g}, \mathfrak{g}^*)$ is the tangent Lie bialgebra of G. If there is in $\mathfrak{g}$ an invariant scalar product $\langle \cdot, \cdot \rangle$, a coboundary Poisson-Lie group structure in G corresponds to the definition in the Lie algebra of a further Lie bracket

$$[X, Y]_R := [R(X), Y] + [X, R(Y)] , \qquad X, Y \in \mathfrak{g} . \tag{I.17}$$

Here $R : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the endomorphism associated to $r = \sum r^{\mu\nu} X_\mu \otimes X_\nu$ through the invariant scalar product, $R(X) = \sum r^{\mu\nu} \langle X_\nu, X \rangle X_\mu$. Again the Jacobi identity for the Lie bracket (I.17) is guaranteed by the properties of the r-matrix. This is equivalent to say that the Lie algebra of a coboundary Poisson-Lie group G is a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}_R)$, where $\mathfrak{g}^*$ has been identified with $\mathfrak{g}$

[4] Actually the condition (I.16) can be relaxed, requiring only the *Ad*-invariance of the *lhs*. However, every coboundary Poisson-Lie group can be always brought again to the particular case in which $r^{\pm}$ satisfy the classical Yang-Baxter equation.

equipped with the Lie bracket (I.17). This particular structure is known as *Baxter-Lie algebra* and, reversing the above derivation, one can show that given $(\mathfrak{g}, \mathfrak{g}_R)$ the corresponding Lie group G is a coboundary Poisson-Lie group.

A further important result is that the two Lie brackets in $\mathfrak{g}$ allow us to define in the Lie algebra an interesting double Poisson structure by introducing the corresponding *Kirillov brackets*

$$\{\varphi, \psi\}(L) := L([d\varphi(L), d\psi(L)]) \tag{I.18}$$

$$\{\varphi, \psi\}_R(L) := L([d\varphi(L), d\psi(L)]_R) . \tag{I.19}$$

Here $L \in \mathfrak{g}^*$ and $\varphi, \psi \in C^\infty(g^*)$, while the differentials $d\varphi(L), d\psi(L) \in \mathfrak{g}$. In the above notation we have emphasized the distinction between $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$ just for clarity; however, here as in the following, it should be remembered that $\mathfrak{g}$ and $\mathfrak{g}^*$ can be identified by the invariant scalar product.

The connection with the previous general consideration on flows referable to a Lax form, which is the main motivation of the above definitions, turns out to be a fairly immediate consequence. Indeed, from the analysis of hamiltonian flows on the symplectic leaves of such Poisson structures, we can establish the following two results, in which we find the prescription to construct integrable systems:

i) the Ad^*-invariant function on $\mathfrak{g}^*$ are in involution with respect to the Poisson bracket (I.19);

ii) the equation of motion on $\mathfrak{g}^*$ defined by the Ad^*-invariant Hamiltonian h with respect to the bracket (I.19) has the generalized Lax form

$$\frac{dL}{dt} = ad^*M \cdot L = [L, M] , \tag{I.20}$$

where $M = R(dh)$, $L \in \mathfrak{g}^*$ and the last equality is allowed by the identification of the Lie algebra with its dual space.

The geometrical meaning of such statements is fairly simple. Indeed, in the present case we have two systems of orbits in $\mathfrak{g}^*$, generated by the adjoint actions of G, that is Ad^* and Ad^*_R. The point *ii)* says that hamiltonian equations of motion, with Hamiltonian being an Ad^*-invariant function h respect both. Therefore, eq.(I.20) means that the velocity vector is always tangent to the intersection of Ad^*-orbit and Ad^*_R-orbit of $L \in \mathfrak{g}^*$. If these intersections coincide with the higher dimensional torus of the action-angle variables for our hamiltonian systems (which is always the case when finite dimensional simple Lie algebras are considered), then this implies complete integrability.

An interesting example is that involving infinite dimensional Lie algebras. This allows us to include in the previous formalism the case of the integrable systems in two dimensions, reproducing the zero-curvature representation in the form (I.20). Let us consider a Lie algebra $\mathfrak{g}$ (which could be also an infinite dimensional algebra, as a loop algebra or a Kac-Moody algebra) with an invariant scalar product $\langle\cdot,\cdot\rangle$. Then denote $\tilde{\mathfrak{g}} = C^\infty(S^1,\mathfrak{g})$ and $\tilde{G} = C^\infty(S^1,G)$ the corresponding group. Moreover suppose that $\mathfrak{g}$ is a Baxter-Lie algebra and, therefore, that there exists $R \in End(\mathfrak{g})$ with the properties already recalled. We can consider in $\tilde{\mathfrak{g}}$ the pointwise bracket

$$[\tilde{X},\tilde{Y}]_\sim(x) := [\tilde{X}(x),\tilde{Y}(x)]$$

and, in a similar fashion, we can extend the definition of the operator R to $\tilde{\mathfrak{g}}$ by setting

$$\tilde{R}(\tilde{X})(x) := R(\tilde{X}(x)).$$

The Lie algebra $\tilde{\mathfrak{g}}$ turns out to be so provided with the structure of a Baxter-Lie algebra. Since our aim is to reproduce the scheme of the zero-curvature representation in terms of Ad^* orbit, it is necessary to consider a central extension of $\tilde{\mathfrak{g}}$. Hence, let $\hat{\mathfrak{g}}$ be the central extension of $\tilde{\mathfrak{g}}$ defined by the Maurer-Cartan 2-cocycle

$$\omega(\tilde{X},\tilde{Y}) = (d\tilde{X},\tilde{Y})_\sim = \int_{S^1} dx\,(\partial_x\tilde{X}(x),\tilde{Y}(x)), \tag{I.21}$$

with an obvious meaning of the notation. We can consider as well the corresponding central extension of $\tilde{\mathfrak{g}}_R$ when the 2-cocycle (I.21) is deformed in the usual way

$$\omega_{\tilde{R}}(\tilde{X},\tilde{Y}) = \omega(\tilde{R}(\tilde{X}),\tilde{Y}) + \omega(\tilde{X},\tilde{R}(\tilde{Y})).$$

Nevertheless, since R, and then $\tilde{R}$, are, by definition, antisymmetric with respect to the invariant scalar product, it is easy to verify that $\omega_{\tilde{R}} = 0$[5]. Therefore, after the usual identification $\hat{\mathfrak{g}}^* \simeq \tilde{g} \oplus \mathbb{C}$, made possible by the definition of the invariant scalar product $\langle(\tilde{L},e),(\tilde{X},\lambda)_\wedge\rangle := (\tilde{L},\tilde{X}) + e\mu$, we can consider the coadjoint actions of $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_{\tilde{R}}$ on $\hat{\mathfrak{g}}^*$, which are given by the

[5]As we will see later, this is the origin of an important property of the related Poisson structure, the so-called *ultralocality property*, which will become prerequisite for a lot of further developments.

expressions

$$ad^*_\wedge \tilde{X} \cdot (\tilde{L}, e) \;:=\; ([\tilde{L}, \tilde{X}]_\sim + e\partial_x X, 0) \tag{I.22}$$

$$ad^*_{\hat{R}} \tilde{X} \cdot (\tilde{L}, e) \;:=\; ([\tilde{L}, \tilde{X}]_{\hat{R}}, 0)\,, \tag{I.23}$$

respectively. Here the endomorphism R is extended by setting

$$\hat{R}(\hat{X}) = \hat{R}(\tilde{X}, \lambda) = \tilde{R}(\tilde{X})\,.$$

The conclusions of points *i)* and *ii)* can be now immediately generalized to such an infinite dimensional case, when the coadjoint actions Ad^* and Ad^*_R are replaced by the integrated versions of eqs.(I.22,I.23), respectively. The equation of motion (I.20) on $\hat{\mathfrak{g}}^*$ acquires the form

$$\partial_t \tilde{L} \;=\; \partial_x \tilde{M} + [\tilde{L}, \tilde{M}]\,, \tag{I.24}$$

which is that of a zero-curvature condition. Here $\tilde{M} = \tilde{R}(dh)$, h being an $Ad^*_\wedge$-invariant function on $\hat{\mathfrak{g}}^*$.

Remark 1 Eq.(I.22) integrates into the coadjoint action of $\tilde{G}$

$$Ad^*g \cdot (\tilde{L}, e) \;=\; (g^{-1}\tilde{L}g + eg^{-1}\partial_x g, e)\,, \qquad g \in \tilde{G} \tag{I.25}$$

which is nothing else that the gauge transformation (I.7).

Remark 2 The $Ad^*_\wedge$-invariant functions on $\hat{\mathfrak{g}}^*$ have the form $\tilde{L} \mapsto \varphi(T(\tilde{L}))$, where $T(\tilde{L})$ is the monodromy matrix of the linear system with connection $\tilde{L}$ and $\varphi \in C^\infty(G)$ is any Ad^*-invariant function on G. Indeed we can show that $(\tilde{L}, e)$ and $(\tilde{L}', e')$ are in the same coadjoint orbit of $\tilde{G}$ in $\hat{\mathfrak{g}}^*$ if and only if $e = e'$ and the monodromy matrices $T(\tilde{L})$ and $T(\tilde{L}')$ are conjugate in G.

Remark 3 The derivation of the Poisson properties of the monodromy matrix leads to a fairly relevant upshot. Indeed, one finds

$$\{T(\tilde{L}) \overset{\otimes}{,} T(\tilde{L})\} \;=\; -[r, T(\tilde{L}) \otimes T(\tilde{L})]\,,$$

which coincides with the definition (I.15). Moreover one can show that the monodromy function $\tilde{L} \mapsto T(\tilde{L})$ is a Poisson map between $\mathfrak{g}^*_{\hat{R}}$ and G equipped with the Sklyanin bracket, that is it establishes an exact correspondence between the two Poisson structures.

$\sim * \sim$

Toda field theories occupy a very peculiar place in both the geometrical framework underlying the zero-curvature approach and in the two, apparently disconnected, algebraic contexts described above, that of the pseudo-differential operator algebra and that of Baxter-Lie algebras. A more detailed discussion of the properties of the zero-curvature representation of Toda field theories will be given in the next Chapter. Let us now spend few words about the second aspect.

The starting observation is that such theories realize the two algebraic structures in a complementary way. As we will see more explicitely in the next Chapter, Toda field theories produce the natural symplectic structure induced on an orbit of $\mathfrak{g}^*_{\tilde{R}}$ when they are considered in their zero-curvature representation. This is equivalent to state that the space component of the flat connection fulfills the Poisson bracket relation

$$\{\mathbf{A}_x(y) \overset{\otimes}{,} \mathbf{A}_x(y')\} \;=\; [r, \mathbf{A}_x(y) \otimes 1 + 1 \otimes \mathbf{A}_x(y)]\delta(y-y')\,. \tag{I.26}$$

By adopting the identification and the convention already specified for eq.(I.15), the above relation coincides with the Kirillov bracket corresponding to the definition (I.23). Notice that the vanishing of the $\tilde{R}$-deformed 2-cocycle $\omega_{\tilde{R}}$ (already recalled the ultralocality property) guarantees here that terms proportional to any derivative of the δ-function do not appear in the Poisson bracket (I.26). Moreover, due to the ultralocality property, it is easy to achieve from eq.(I.26) the Poisson bracket of the monodromy matrix which is related to the flat connection $\mathbf{A}$. The result is again the Sklyanin bracket relation, which confirms what stated in the remark 3 and, on the other hand, constitutes an alternative way to show that Toda field theories realize the scheme of Baxter-Lie algebras.

The peculiar feature, that establishes the link between Toda field theories and the results pointed out in discussing the pseudo-differential operator algebra approach, is their conformal invariance. Indeed, this makes possible to construct two completely decoupled sectors of the theory, corresponding to the the opposite chiralities; one sector depending only on the right moving light cone variable $x_+ = x + t$, while the other depending on the the left moving one, $x_- = x - t$. In a general fashion, in each of these sectors one can associate to the (anti)chiral connection a differential operator

$$\begin{aligned}\mathbf{A}(x_+) &\mapsto \partial_+^n + \sum_{i=2}^{n} a_i(x_+)\partial_+^{n-i} \\ \bar{\mathbf{A}}(x_-) &\mapsto \partial_-^n + \sum_{i=2}^{n} \bar{a}_i(x_-)\partial_-^{n-i} .\end{aligned}$$

A direct computation shows that the Poisson relations among the coefficients a_i, and, in a parallel way, $\bar{a}_i$, constitute a realization of the classical W-algebras, in the same way as the Poisson bracket of the second hamiltonian structure in eq.(I.14), reproducing the flows $\delta_k L$ of eq.(I.13), does. In other words, the coefficients a_i ($\bar{a}_i$) turn out to be the chiral (antichiral) components of the generators of the extended conformal invariance underlying the theory. In particular, a_2 and $\bar{a}_2$ coincide with the chiral and antichiral components of the improved energy-momentum tensor.

Chapter 1

An outlook of classical conformal invariant Toda field theories

This Chapter contains a summary of the main properties characterizing the Toda field theories associated to finite dimensional Lie algebras. We will devote particular care to outline those feature, which will constitute the basis of the arguments analyzed in Chapter 2, as their conformal invariance and the splitting of the chiralities. Moreover, we will recall the approach related to the Drinfel'd-Sokolov linear systems, as it is implemented in ref.[7], which allow us to reconstruct a real periodic local solution of the Toda equations in terms of free modes of definite chirality. Also in this case, we refer to Chapter 2 for a discussion about the extent of these results and their interpretation.

1.1 The model

Two-dimensional Toda field theories describe an interaction of exponential form among r massless scalar fields ϕ_i $(i = 1, \ldots, r)$ on a $(1+1)$ dimensional space-time. Their Lagrangian is often given in the form (see §3.3)

$$\mathcal{L} = \tfrac{1}{2}\frac{a_{ij}}{|\alpha_j|^2}\partial_\mu\phi_i\partial^\mu\phi_j - V(\phi)\,, \tag{1.1}$$

where the repeated indices are understood to be summed over and the potential is

$$V(\phi) = \sum_{i=1}^{r} m_i^2 e^{(a\cdot\phi)_i}\,, \qquad \text{with } (a\cdot\phi)_i = \sum_{j=1}^{r} a_{ij}\phi_j\,. \tag{1.2}$$

Here a_{ij} is the Cartan matrix[1], $a_{ij} = 2\frac{\langle\alpha_i,\alpha_j\rangle}{|\alpha_i|^2}$, and α_i $(i = 1,\ldots,r)$ is the simple root system of a certain finite dimensional Lie algebra. These characterization of the coefficients in the Lagrangian (1.1) determines the integrability of the models. The corresponding equations of motion read

$$\partial_\mu\partial^\mu\phi_k + m_k^2|\alpha_k|^2\, e^{(a\cdot\phi)_i} = 0\,. \tag{1.3}$$

Such equations exhibit an important symmetry. Let us consider the light-cone preserving reparametrizations of the space-time,

$$\begin{cases} x_+ = f(x'_+) \\ x_- = g(x'_-) \end{cases}, \tag{1.4}$$

where f and g are regular functions. As it is well-known, this means that we are performing a conformal transformation of a 2-dimensional minkowskian space-time. Moreover, let the transformation laws for the fields ϕ_k be

$$\phi'_k(x'_+,x'_-) = \phi_k(f(x'_+),g(x'_-)) + \rho_k \ln[f'(x'_+)g'(x'_-)]\,, \tag{1.5}$$

with ρ_k such that $a_{ik}\rho_k = 1$, $\forall i$ (the existence of such a vector is guaranteed by the properties of Cartan matrices). Accordingly, since the metric changes by a factor $f'g'$, the ϕ'_k's constitute still a solution of the equations of motion (1.3). This is to say that the model is invariant under the conformal transformation (1.4) and (1.5).

For mere reasons of convenience in view of algebraic manipulations, we prefer to consider the Lagrangians of Toda field theories in the "mass-eigenvector" fields φ_i $(i = 1,\ldots,r)$. In this way, they become

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\varphi\cdot\partial^\mu\varphi - \sum_{\alpha\ simple} m_\alpha^2 e^{\alpha\cdot\varphi}\,. \tag{1.6}$$

In this basis, the equations of motion in the light-cone variables $x_\pm = x \pm t$ read

$$\partial_+\partial_-\varphi_k = \frac{1}{4}\sum_{\alpha\ simple} m_\alpha^2\alpha_k\, e^{\alpha\cdot\varphi} \tag{1.7}$$

[1]Although this case will not be consider in the present Thesis, we would remember that conformal invariant Toda field theories can be construct also with generalized Cartan matrices, which turn out to be associated to Kac-Moody algebras [5].

where α_k are the component of the root vector [remember that $\partial_\pm = 1/2(\partial_x \pm \partial_t)$ and therefore $4\partial_+\partial_- = -\Box$].

We want to remark once again that, in the case of Toda field theories, integrability and conformal invariance of the model arise at the same time, when the coefficients of the Lagrangian acquire a meaning in a Lie algebra context and, thus, the link between such an algebraic framework and Toda field theories is established. This relation, together with the role played by coboundary Poisson-Lie group and Baxter-Lie algebras, is made evident by the zero-curvature formulation of the theory.

Assigned the Cartan matrix a_{ij}, or, equivalently, the simple root system $\{\alpha\}$, we can identify the associated simple Lie algebra $\mathfrak{g}$ of rank r. Let $\mathfrak{g}$ be equipped with an invariant scalar product $\langle\cdot,\cdot\rangle$. Once a Cartan subalgebra (CSA) is chosen, we can fix in it an orthonormal basis $\{H_i\}_{i=1,\ldots,r}$. Then, consider the corresponding Cartan-Weyl basis, whose Lie brackets are

$$[H_i, E_{\pm\alpha}] = \pm\alpha(H_i)E_{\pm\alpha}\,,$$

$$[E_\alpha, E_{-\alpha}] = h_\alpha\,,$$

with h_α the element in the CSA dual to the root α,

$$h_\alpha = \langle E_\alpha, E_{-\alpha}\rangle \sum_k \alpha_k H_k\,.$$

In the above equation, it is $\alpha_k = \alpha(H_k)$.

The equations of motion of the Toda field theory, associated to the Lie algebra $\mathfrak{g}$, can now be introduced as the zero-curvature condition of the flat connection

$$\mathbf{A}_+ = \partial_+\mathbf{\Phi} + e^{ad\mathbf{\Phi}}\mathcal{E}_+ \qquad \mathbf{A}_- = -\partial_-\mathbf{\Phi} + e^{ad\mathbf{\Phi}}\mathcal{E}_-\,, \tag{1.8}$$

the field $\mathbf{\Phi}$ taking values in the CSA, $\mathbf{\Phi} : M^2 \to \mathcal{H}$, with M^2 the two-dimensional space-time, and

$$\mathcal{E}_\pm = \sum_{\alpha_i\ simple} E_{\pm\alpha_i}\,. \tag{1.9}$$

By imposing the zero curvature condition for $\mathbf{A}$, we achieve the equation

$$\partial_+\partial_-\mathbf{\Phi} = \frac{1}{2}\sum_{\alpha_i\ simple} e^{2\alpha_i(\mathbf{\Phi})}h_\alpha\,. \tag{1.10}$$

Notice that, in order to recover the equation of motion of the Toda field theory, it is sufficient to substitute $\mathbf{\Phi} = \frac{1}{2}\varphi_i H_i$. This yields eq.(1.7) with $m_\alpha = 2$, $\forall\alpha$.

As for the hamiltonian formalism, this is deduced in the usual way by the lagrangian one, with the introduction the canonical Poisson brackets between the scalar fields φ_i and their conjugate momenta, π_i. Such relations can be cast in the compact form[2]

$$\{\mathbf{\Pi_\Phi}(x,t) \overset{\otimes}{,} \mathbf{\Phi}(y,t)\} = \delta(x-y) \cdot t_0 , \tag{1.11}$$

with

$$t_0 = \sum_i H_i \otimes H_i \tag{1.12}$$

and $\mathbf{\Pi_\Phi} = \partial_t \mathbf{\Phi}$.

From the zero-curvature formulation standpoint, the equal time canonical symplectic structure, given in the above equation, translates into a fundamental relation for the space component of the flat connection in eq.(1.8), $\mathbf{A}_x = \mathbf{A}_+ + \mathbf{A}_-$. Indeed, by a direct computation, we find

$$\{\mathbf{A}_x(y) \overset{\otimes}{,} \mathbf{A}_x(y')\} = [r, \mathbf{A}_x(y) \otimes 1 + 1 \otimes \mathbf{A}_x(y)]\delta(y-y') , \tag{1.13}$$

where $r \in \mathfrak{g} \otimes \mathfrak{g}$ is defined up to a term proportional to the Casimir element $C \in \mathfrak{g} \otimes \mathfrak{g}$,

$$r = \sum_{\alpha\ positive} \frac{(E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha)}{\langle E_\alpha, E_{-\alpha} \rangle} + \lambda C ,$$

$$C = t_0 + \sum_{\alpha\ positive} \frac{(E_\alpha \otimes E_{-\alpha} + E_{-\alpha} \otimes E_\alpha)}{\langle E_\alpha, E_{-\alpha} \rangle} .$$

When $\lambda = \pm 1$, then $r(\lambda = \pm 1) = r^\pm$ are solutions of the classical Yang-Baxter equation recalled in eq.(I.16). The extent of this result has been already discussed in the Introduction.

1.2 Conformal properties

The importance of the conformal invariance, pointed out in the previous Section, has been already suggested in the Introduction, where we hinted at the

[2] Often in the following the time dependence will be dropped: then, it shall be understood that the time is set to $t = 0$.

related structure. Here we will go into some details. To this end, an indispensable premise is to construct a chiral split recasting of the phase space. We can identify this, either with the space of connection gauge equivalent to (1.8), in agreement with the general framework of the zero-curvature representation, or, following the approach related to Baxter-Lie algebras, with the coadjoint orbit $\tilde{G}$ in $\hat{\mathfrak{g}}^*$.

The analysis by Leznov and Saveliev [23] represents the first attempt to face the problem. Only in order to fix the notation, let us consider the case of $\mathfrak{g} = sl(n, \mathbb{R})$ Toda field theories: the following result can be easily generalized to any simple Lie algebra. Denote with $\mathfrak{g} = \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$ its root eigenspace decomposition, where we have chosen the Cartan subalgebra (CSA) $\mathcal{H}$ and $\mathcal{N}_+$ ($\mathcal{N}_-$) is the direct sum of positive (negative) root eigenspaces. In the case $\mathfrak{g} = sl(n, \mathbb{R})$ we can identify the CSA as the subalgebra of $n \times n$ traceless diagonal matrices, while $\mathcal{N}_\pm$ coincide with $n \times n$ upper and lower triangular matrices with zero diagonal entries, respectively.

On the other hand, the transport matrix Ψ, which corresponds to the connection $\mathbf{A}$ through the relation (I.6),

$$\mathbf{A}_\pm = -\partial_\pm \Psi \, \Psi^{-1} , \tag{1.14}$$

can be formally represented by the path-ordered exponential

$$\Psi(\mathrm{x}) = \overleftarrow{\mathrm{P}} \exp\left[- \int^x (\mathbf{A}_+ + \mathbf{A}_-) \right] ,$$

where we refer to an unspecified base-point.

As an element of the group $G = SL(n, \mathbb{R})$ of $n \times n$ unimodular matrices with real entries, Ψ can be represented by the Gauss decomposition[3] in two different ways. Indeed, we can factorize Ψ in the two products

$$\Psi = \begin{cases} \exp(H^{(-)}) n_- m_+ \\ \exp(H^{(+)}) n_+ m_- \end{cases} , \tag{1.15}$$

where $H^{(\pm)}$ take values in the CSA and $m_\pm$, $n_\pm$ in $\exp \mathcal{N}_\pm$, respectively, and, therefore, are upper (the $+$ ones) and lower (the $-$ ones) triangular matrix with one on the diagonal. The comparison with eq.(1.14) shows that m_+ and m_- depend only on x_+ and x_-, respectively, and that $H^{(+)}$ ($H^{(-)}$) equals $-\mathbf{\Phi}$

[3]To make this statement rigorous, we should restrict the argument only to elements in the big cell of G, the open subset obtained by exponentiating the Lie algebra. Actually this difficulty can be easily overcome.

(Φ), modulo chiral (antichiral) terms. Such considerations lead to reorder the decompositions (1.15) in the form

$$\Psi = \begin{cases} \exp(\Phi)V \\ \exp(-\Phi)\bar{V} \end{cases}, \tag{1.16}$$

where V and $\bar{V}$ are very close to the chiral objects we are seeking, their first and last row being, respectively, chiral and antichiral. This follows from the above analysis, but, on the other hand, it turns out to be immediately evident, once the linear systems fulfilled by V and $\bar{V}$ are determined. Indeed, by a direct computation, we obtain

$$\begin{cases} (\partial_+ + \mathbf{A}^{\mathrm{v}}_+)V = 0 \\ (\partial_- + \mathbf{A}^{\mathrm{v}}_-)V = 0 \end{cases}, \qquad \text{with} \qquad \begin{aligned} \mathbf{A}^{\mathrm{v}}_+ &= 2\partial_+\Phi + \mathcal{E}_+ \\ \mathbf{A}^{\mathrm{v}}_- &= e^{-2ad_\Phi}\mathcal{E}_- \end{aligned} \tag{1.17}$$

and, parallelly, for the $\bar{V}$

$$\begin{cases} (\partial_+ + \mathbf{A}^{\bar{\mathrm{v}}}_+)\bar{V} = 0 \\ (\partial_- + \mathbf{A}^{\bar{\mathrm{v}}}_-)\bar{V} = 0 \end{cases}, \qquad \text{with} \qquad \begin{aligned} \mathbf{A}^{\bar{\mathrm{v}}}_+ &= e^{2ad_\Phi}\mathcal{E}_+ \\ \mathbf{A}^{\bar{\mathrm{v}}}_- &= -2\partial_-\Phi + \mathcal{E}_- . \end{aligned} \tag{1.18}$$

Remark 4 Denote $\rho^{(r)}$ a matrix representation of $sl(n, \mathbb{F})$ with highest weight $\lambda^{(r)}_{max}$ and lowest weight $\lambda^{(r)}_{min}$. The conclusion of the Leznov-Saveliev analysis can be reformulated by referring to this particular representation. This means that we can define for each $\rho^{(r)}$ the two chiral vectors

$$\xi^{(r)} = \langle \lambda^{(r)}_{max}|V , \qquad \partial_-\xi^{(r)} = 0 , \tag{1.19}$$

$$\bar{\xi}^{(r)} = \langle \lambda^{(r)}_{min}|\bar{V} , \qquad \partial_+\bar{\xi}^{(r)} = 0 . \tag{1.20}$$

Such non-local fields summarize the entire dynamical content of the theory. In particular, since, by definition, the identity

$$e^{-2\Phi} = V\bar{V}^{-1}$$

holds, we can reconstruct the solution of the Toda equations from the knowledge of the vector ξ in the fundamental representations. Indeed, the above equation allows us to determine the relations

$$e^{-2\Lambda_i(\Phi)} = \langle \Lambda_i|V\bar{V}^{-1}|\Lambda_i\rangle = \xi^{(i)} \cdot {}^{T}\bar{\xi}^{(i)} . \tag{1.21}$$

Here, Λ_i $(i = 1,\dots,n)$ are the fundamental highest weight of $sl(n,\mathbb{F})$, $\xi^{(i)}$ and $\bar{\xi}^{(i)}$ the corresponding chiral and antichiral vector, respectively (in the following, we will drop every label concerning the representation, when the defining one is considered, or, otherwise, when it is not essential in the context). The suffix τ denotes

$$^{\tau}\bar{\xi} = w^{-1} \cdot {}^{t}\bar{\xi} \qquad\qquad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where "t" represents the transpose operation. As a by-product, eq.(1.21) tells us that the ξ^i's and the $\bar{\xi}^i$'s are conformal primary fields of weight $\Delta_i = \langle \Lambda_i, \rho \rangle$. This follows from the conformal transformation laws of the fields φ_i [see eq.(1.5)].

Remark 5 Notice that $\mathbf{A}^{V}$ and $\mathbf{A}^{\bar{V}}$ can be derived by gauge transforming the flat connection (1.8) with $g = e^{\mathbf{\Phi}}$ and $g = e^{-\mathbf{\Phi}}$, respectively. However, this analysis is not sufficient for the phase space to be considered completely split into a right-moving and a left-moving sector. Indeed, neither the $(-)$ component of $\mathbf{A}^{V}$ nor the $(+)$ component of $\mathbf{A}^{\bar{V}}$ are vanishing and, on the other hand, V and $\bar{V}$ do not come out to be decoupled from the point of view of the symplectic structure [see eqs.(2.20), (2.21) and (2.22)], even if we are restricted to their entries of definite chirality. In this connection, from eq.(1.13), or, more precisely, from its integrate version concerning the transport matrix Ψ, one achieves the Poisson brackets

$$\begin{aligned}
\{\xi(x) \overset{\otimes}{,} \xi(y)\} &= \xi(x) \otimes \xi(y) \cdot [\vartheta(x-y)\, r^{+} + \vartheta(y-x)\, r^{-}]\,, \\
\{\bar{\xi}(x) \overset{\otimes}{,} \bar{\xi}(y)\} &= \bar{\xi}(x) \otimes \bar{\xi}(y) \cdot [\vartheta(x-y)\, r^{-} + \vartheta(y-x)\, r^{+}]\,, \\
\{\xi(x) \overset{\otimes}{,} \bar{\xi}(y)\} &= -\xi(x) \otimes \bar{\xi}(y) \cdot r^{-}\,, \\
\{\bar{\xi}(x) \overset{\otimes}{,} \xi(y)\} &= -\bar{\xi}(x) \otimes \xi(y) \cdot r^{+}\,,
\end{aligned} \tag{1.22}$$

which we will refer to as the *exchange algebra* in the ξ basis (see, for example, ref.[2] to find an explicit derivation).

As will be clarified in the next Chapter, the limit of the above analysis lies in how one faces the group factorization problem. Eq.(1.16) represents the simplest, albeit incomplete, answer to the question: we have not taken any care of the transformation which the symplectic structure undergoes and this causes inconsistencies in what follows.

Nevertheless, also such a partial result leads to some remarkable consequences.

If we limit ourself to the right-moving sector, the vector $\xi^{(r)}$ turns out to be a basis of solutions of the ordinary differential equation of order $dim(\rho^{(r)})$, associated to the $+$ component in the linear system (1.17) (the $-$ component reduces to the condition on $\xi^{(r)}$ to be chiral). In this regard, since higher dimensional representations do not give us any further information, let us consider the defining representation, in which ξ fulfills the n order ordinary differential equation

$$\left(\partial_+^n + \sum_{i=2}^{n} u_i\, \partial_+^{n-i}\right) \xi \;=\; 0\,. \tag{1.23}$$

By a direct derivation based on the expression of the connection as quoted in eq.(1.8), we obtain the coefficients u_i as functions of the field φ_i. In particular, u_2 comes out to coincide with the chiral component of the traceless energy-momentum tensor. On the other hand, knowing the ξ's, we can reconstruct the differential equation itself in the wronskian form

$$\det \begin{pmatrix} \xi & \xi_1 & \cdots & \xi_n \\ \xi' & \xi_1' & \cdots & \xi_n' \\ \cdots & \cdots & \cdots & \cdots \\ \xi^{(n)} & \xi_1^{(n)} & \cdots & \xi_n^{(n)} \end{pmatrix} = 0\,, \tag{1.24}$$

where $\xi = (\xi_1, \ldots, \xi_n)$. This means that it is possible to compute the Poisson algebra of the coefficients u_i, starting from the exchange algebra of the ξ's. The upshot is that the u_i's are the generators of the extended conformal algebra (the W-algebra) underlying the theory. Therefore, the canonical simplectic structure (1.11) translates here into the second hamiltonian structure of the generalized KdV equation. This confirms, already at this stage of our analysis, what the final considerations of the Introduction has announced in advance.

1.3 Drinfel'd-Sokolov linear systems: a reconstructive approach

Although the conformal fields ξ and $\bar{\xi}$ do not completely fulfill our chiral splitting program, two points of the above discussion seem to suggest the solution. Firstly, the realization of the W-algebra generators in terms of the ξ's (as for the chiral sector) gives a fundamental meaning to the exchange algebra of this chiral vector, while a specular argument holds for $\bar{\xi}$. Furthermore, and this is the second aspect, eq.(1.21) contains the recipe to work out a solution of the Toda equation of motion, once we are able to solve the linear systems (1.17) and (1.18). When presented in this form, all the matter sounds a bit tautological, since ξ and $\bar{\xi}$ contain explicitly the Toda fields $\mathbf{\Phi}$ in their expressions. Therefore, we need to know in advance the solutions of the Toda field equations in order for the above reconstruction scheme to work. It is evident that, confining ourselves to eq.(1.21), we do not get any explicit representation of the space of classical solutions.

Nevertheless, these considerations clearly show how to proceed: the idea is to produce a realization of both the identity (1.21) and the exchange algebra (1.22), as long as its two chiral halves are taken into account separately, in terms of the modes of right-moving and left-moving (and, therefore, free) fields. This has been developed in ref.[7]. The basic observation, from which we move, is that the higher order differential equations [as, for instance, that shows in eq.(1.23)], fulfilled by ξ and $\bar{\xi}$, respectively, can be associated to the linear system

$$\begin{cases} \partial_+ Q_+ - (\mathbf{p} - \mathcal{E}_+)Q_+ = 0 \\ \partial_- Q_+ = 0 \end{cases} , \tag{1.25}$$

for the chiral sector, and to its antichiral parallel

$$\begin{cases} \partial_+ Q_- = 0 \\ \partial_- Q_- + Q_-(\bar{\mathbf{p}} - \mathcal{E}_-) = 0 \end{cases} . \tag{1.26}$$

Here $\mathbf{p}$ and $\bar{\mathbf{p}}$ are chiral and antichiral fields, respectively, which take values in the CSA. Since our aim is to reproduce the original symplectic structure in each chiral sector, we impose that the connection in eqs.(1.25) and (1.26) have the same Poisson brackets exhibited by $\mathbf{A}^{\nabla}$ and $\bar{\mathbf{A}}^{\bar{\nabla}}$. This means that

$$\{\mathbf{p}(x) \overset{\otimes}{,} \mathbf{p}(y)\} = -(\partial_x - \partial_y)\delta(x-y)\, t_0 \,, \tag{1.27}$$

$$\{\mathbf{p}(x) \overset{\otimes}{,} \bar{\mathbf{p}}(y)\} = 0 \,, \tag{1.28}$$

$$\{\bar{\mathbf{p}}(x) \overset{\otimes}{,} \bar{\mathbf{p}}(y)\} = (\partial_x - \partial_y)\delta(x-y)\, t_0 \,. \tag{1.29}$$

The fields $\mathbf{p}$ and $\bar{\mathbf{p}}$ are completely decoupled, also in respect with their Poisson relations. Therefore, they are good candidates for parametrizing the phase space of the Toda field theory, once the splitting of the chiralities were implemented.

In order to recover a solution of the Toda field equation as formally given by eq.(1.21), but which now involves the new variables introduced by this reconstruction procedure, we define the basis σ and $\bar{\sigma}$,

$$\begin{aligned} \sigma^{(r)}(x) &= \langle \lambda^{(r)} | Q_+(x) , \\ \bar{\sigma}^{(r)}(x) &= Q_-(x) | \lambda^{(r)} \rangle . \end{aligned} \tag{1.30}$$

Obviously, this basis fulfills (up to a τ-transposition in the antichiral half) the exchange algebra (1.22), except that σ and $\bar{\sigma}$ Poisson commute. Thus, we can reconstruct a solution of the equation (1.10) through the identification

$$e^{-2\Lambda_i(\Phi)(x_+,x_-)} = \sigma^i(x_+) M \bar{\sigma}^i(x_-) , \tag{1.31}$$

where M is a constant matrix to be determined. Some words have to be spent on the meaning of this factor. In order to solve the Toda field equation, observe that, if the solution $Q_+(x)$ and $Q_-(x)$ of the linear systems (1.25) and (1.26) are normalized by the conditions

$$Q_+(0) = 1 \qquad\qquad Q_-(0) = 1 , \tag{1.32}$$

then, the projection of M on the fundamental highest weight vectors plays the role of initial condition for the Toda fields Φ. On the other hand, the reconstruction procedure implemented by eq.(1.31) fulfills the canonical symplectic structure (1.11) only if we endow M with a dynamical meaning. In particular, this is necessary for the locality property

$$\{ \Phi(x,t) \overset{\otimes}{,} \Phi(y,t) \} = 0 \tag{1.33}$$

to hold.

In agreement with ref.[7], we will refer to the eqs.(1.25) and (1.26) as the *Drinfel'd-Sokolov linear system* and to the above reconstruction scheme as the *Drinfel'd-Sokolov prescription.*

More interesting features arise if we consider boundary conditions which are periodic in the space variable, that is if we solve the Toda field equations on a cylindrical space-time, $M^2 = \{x, t : \ 0 < x \leq 2\pi\}$. In this case $\mathrm{p}(x)$ and $\bar{\mathrm{p}}(x)$ are periodic and, therefore, can be expanded in the Fourier series

$$\mathrm{p}(x) = \sum_n \mathrm{p}_n e^{inx} , \qquad\qquad \bar{\mathrm{p}}(x) = \sum_n \bar{\mathrm{p}}_n e^{inx} .$$

The modes $\mathbf{p}_n$ and $\bar{\mathbf{p}}_n$ realize the algebra of free bosonic oscillators,

$$\{\mathbf{p}_n \overset{\otimes}{,} \mathbf{p}_m\} = -\frac{in}{\pi}\delta_{n+m}\, t_0\,, \tag{1.34}$$

$$\{\bar{\mathbf{p}}_n \overset{\otimes}{,} \bar{\mathbf{p}}_m\} = \frac{in}{\pi}\delta_{n+m}\, t_0\,. \tag{1.35}$$

On a cylindrical space-time, we have to take into account a further feature, which characterizes the field content of the theory, that is, the monodromy behaviour. In the case of the fields σ and $\bar{\sigma}$, this is ruled by the left and right monodromy matrices

$$S = Q_+(2\pi), \qquad \bar{S} = Q_-(2\pi)\,,$$

so that they are shifted by S and $\bar{S}$, respectively, after each cycle. Therefore, in order to make the solution (1.31) periodic in x, we should impose

$$S \cdot M \cdot \bar{S} = M\,. \tag{1.36}$$

The solution of this equation is subordinate to the possibility for the monodromy matrices to be diagonalized. Since S and $\bar{S}$ are both triangular matrices (upper and lower triangular, respectively), they are always conjugate to diagonal matrices[4]. Therefore, let us set

$$\begin{aligned} S &= g\kappa g^{-1}\,, & \kappa &= e^{2\pi \mathbf{p}_0}\,, \\ \bar{S} &= \bar{g}^{-1}\bar{\kappa}\bar{g}\,, & \bar{\kappa} &= e^{-2\pi\bar{\mathbf{p}}_0}\,. \end{aligned} \tag{1.37}$$

Accordingly, the condition in eq.(1.36) will be satisfied if

$$M = gD\bar{g}$$

$$\kappa\bar{\kappa} = 1\,,$$

where $D \in \exp(\mathcal{H})$, $\mathcal{H}$ being the CSA. The second condition simply means that $\mathbf{p}_0 = \bar{\mathbf{p}}_0$, which is a natural identification from a physical point of view. Nevertheless, as we will see in the following, the possibility to keep $\mathbf{p}_0$ and $\bar{\mathbf{p}}_0$ distinguished often simplify the treatment. Finally, the locality condition (1.33) determines the expression of D in terms of the field variables of the reconstructed phase space, so that

$$D = \Theta\bar{\Theta}, \qquad \text{with} \left\{ \begin{array}{l} \Theta = e^{\mathbf{q}-\mathbf{k}} \\ \bar{\Theta} = e^{\bar{\mathbf{q}}+\bar{\mathbf{k}}} \end{array} \right., \tag{1.38}$$

[4]This is strictly true when a discrete set of unacceptable values of the zero modes is ruled out, so to exclude the case of parabolic monodromy matrices.

where

$$\mathbf{k} = \sum_{n\neq 0} \frac{i\mathbf{p}_n}{n}, \qquad \bar{\mathbf{k}} = \sum_{n\neq 0} \frac{i\bar{\mathbf{p}}_n}{n}.$$

In eq.(1.38) we have introduced $\mathbf{q}$ and $\bar{\mathbf{q}}$, the conjugate variables of the zero modes $\mathbf{p}_0$ and $\bar{\mathbf{p}}_0$,

$$\begin{aligned} \{\mathbf{q} \overset{\otimes}{,} \mathbf{p}_0\} &= \tfrac{1}{2\pi} t_0 , \\ \{\bar{\mathbf{q}} \overset{\otimes}{,} \bar{\mathbf{p}}_0\} &= \tfrac{1}{2\pi} t_0 , \end{aligned} \tag{1.39}$$

respectively. Notice that, only adding the above relations, the Poisson algebra (1.35) turns out to be non-degenerate.

From another point of view, the above procedure leads to the explicit form of the *Bloch wave basis* of solutions of the Drinfel'd-Sokolov linear system. Indeed, let us consider the new fields

$$\begin{aligned} \psi^{(r)}(x) &= \sigma^{(r)}(x) g \Theta , \\ \bar{\psi}^{(r)}(x) &= \bar{\Theta}\bar{g}\bar{\sigma}^{(r)}(x) . \end{aligned} \tag{1.40}$$

It is easy to verify that the fields ψ and $\bar{\psi}$ have diagonal monodromy κ and $\bar{\kappa}$, respectively. This good monodromy behaviour has its counterpart in a fairly complicate Poisson relations. Indeed, the $(\psi,\bar{\psi})$ basis obeys the exchange algebra

$$\begin{aligned} \{\psi(x) \overset{\otimes}{,} \psi(y)\} = &-\psi(x)\otimes\psi(y)\cdot\Big[\epsilon(x-y)(r^+ - r^-)+ \\ &-\coth(\pi ad_{1\,\mathbf{p}_0})\cdot(r^+ - t_0) - \coth(\pi ad_{2\,\mathbf{p}_0})\cdot(r^- + t_0)\Big] \end{aligned} \tag{1.41}$$

$$\begin{aligned} \{\bar{\psi}(x) \overset{\otimes}{,} \bar{\psi}(y)\} = &\Big[\epsilon(x-y)(r^+ - r^-) + \coth(\pi ad_{1\,\bar{\mathbf{p}}_0})\cdot(r^- + t_0)+ \\ &+\coth(\pi ad_{2\,\bar{\mathbf{p}}_0})\cdot(r^+ - t_0)\Big]\cdot\bar{\psi}(x)\otimes\bar{\psi}(y) \end{aligned} \tag{1.42}$$

while, only as long as the $\mathbf{p}_0$ and $\bar{\mathbf{p}}_0$ are considered as independent modes, chiral and antichiral fields are decoupled. In the above equations, the labels $1,2$ indicate which group element in the tensor product the adjoint action applies to.

The solution of the Toda field equation (1.10), associated to the above Drinfel'd-Sokolov prescription, turns out to be expressed in terms of the Bloch wave basis as

$$e^{-2\Lambda_i(\Phi)(x_+,x_-)} = \psi^i(x_+)\cdot\bar{\psi}^i(x_-) . \tag{1.43}$$

It can be immediately shown that the above relation produces both periodic and local solutions, provided we reduce the phase space by imposing $\mathbf{p}_0 = \bar{\mathbf{p}}_0$.

This approach constitutes certainly a powerful tool in the analysis of further developments, as could be periodic lattice discretization and quantization of the model, and the just remembered expression of the Bloch wave basis of the theory could be quoted among the interesting results produced through the Drinfel'd-Sokolov prescription. On the other hand a problem is left still open: the realization of such construction in terms of the original field content of Toda field theories and, consequently, the completion of the program dating back to Leznov and Saveliev works. The answer to these questions is matter of the next Chapter.

Chapter 2

Splitting the chiralities

Chapter 1 has left open a number of fundamental questions. Firstly, the separation of the phase space into two sectors of definite chirality is still an incomplete program (the Drinfel'd-Sokolov prescription is only a reconstructive procedure). As already pointed out, the conformal fields ξ and $\bar{\xi}$, right- and left-moving, respectively, are coupled by their exchange algebra and, moreover, it can be shown that $\mathbf{A}^{\vee}$ can not be brought into a chiral connection by a well-defined gauge transformation, and similarly for $\mathbf{A}^{\bar{\vee}}$ (see §2.1, below). This means that the Drinfel'd-Sokolov linear systems can not be immediately recovered.

From a different standpoint, we have proposed through the Drinfel'd-Sokolov linear systems a representation of the solutions of the Toda equations in terms of the independent free fields $\mathbf{p}$ and $\bar{\mathbf{p}}$. We can still wonder to what extent can we represent Toda field theories by means of free bosonic oscillators. In this sense the analysis which we will implement in this Chapter represents the completion of ref.[7]. The unanswered questions concern the characterization of the subset in the space of solutions, which can be reconstructed by the recipe synthesized in §1.3, and, eventually, whether it is possible to parametrize the whole space in a similar fashion. A further problem to be faced is represented by the relation between the interacting fields of each Toda Lagrangian and the free modes, and, in particular, whether and how this mapping transforms the symplectic structure. The aim we want to achieve is the proof of an exact correspondence between, on one hand, the canonical Poisson brackets in the Toda field theory, which we can translate in the Kirillov brackets for the space component of the flat connection $\mathbf{A}$ in eq.(1.8), and, on the other hand, the Poisson brackets for the bosonic oscillators, which practically represent the chiral and antichiral connections in the Drinfel'd-Sokolov linear system framework.

Although the questions which we are wondering are fundamental by themselves, in order to clarify the structure underlying Toda field theories, they could seem somehow redundant, since they are certainly just two different sides of the same problem. Nevertheless, taking into account both these points of view leads to a better comprehension of their possible solutions, as we will verify soon.

In this Chapter we address the above two problems for the sl_2 Toda field theory, the Liouville model. We will consider the theory defined in a cylindrical space-time, where space is represented by a circle and time by a straight line. Therefore, periodic space boundary conditions are assumed. We want to emphasize that the main remarks, which will allow us to achieve our aim, are not peculiar features of the simple Toda field theory under consideration, but could be seen in the more general context of sl_n Toda field theories. The Liouville model represents a meaningful example which greatly helps in avoiding those technical difficulties which instead could masquerade the real nature of the problem.

The following treatment is organized in four Sections. The first two Sections represent an introductory discussion which, however, constitute an essential premise to focalize the problem. Most of the features, which are there analyzed, are crucial to realize our aim and their comprehension was quite sketchy up to now. Our upshot is the construction of a *one-to-one correspondence* between solutions of Liouville equation and the appropriate *free bosonic oscillator fields* which allow for the chiral splitting of the physical phase space and, at the same time, preserve the canonical symplectic structure of the Liouville field theory. This just represents the realization of the Drinfel'd-Sokolov linear systems. A detailed presentation and discussion of our results is contained in §2.3, while their peculiar features are summarized and clarified in §2.4, in view of further developments (see ref.[1]).

2.1 The $(\xi,\bar{\xi})$ basis

Our first task is to identify the origin of the coupling problem between ξ and $\bar{\xi}$. In order to avoid unessential technical difficulties, we restrict by now to consider the simplest example of 2-dimensional Toda field theory, the Liouville model. Most part of the peculiar features, which are related to the chiral splitting program, arises already in this particular context. In the general treatment summarized in §1.1, the Liouville field theory is described by the Lagrangians (1.1) and (1.6), when only one scalar field is involved. Referring to its zero-curvature representation, the Liouville field equation is associated to the flat connection (1.8), where the related Lie algebra is $sl(2,\mathbb{R})$.

Therefore, let us introduce some notation relevant to this case. We choose in $sl(2,\mathbb{R})$ the Chevalley basis of generators, which is constituted by the 2×2 traceless matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.1}$$

whose Lie brackets are

$$[H, E_\pm] = \pm 2E_\pm\,, \qquad\qquad [E_+, E_-] = H\,.$$

The associated flat connection reads

$$\mathbf{A}_+ = \begin{pmatrix} \frac{1}{2}\partial_+\varphi & e^\varphi \\ 0 & -\frac{1}{2}\partial_+\varphi \end{pmatrix}, \qquad \mathbf{A}_- = \begin{pmatrix} -\frac{1}{2}\partial_-\varphi & 0 \\ e^\varphi & \frac{1}{2}\partial_-\varphi \end{pmatrix}, \tag{2.2}$$

and corresponds to the *Liouville equation*

$$\partial_+\partial_-\varphi = e^{2\varphi}\,. \tag{2.3}$$

The general solution of this equation is well-known from the original works of Poincaré and is expressed by the classical formula

$$e^{2\varphi} = \frac{u'v'}{(1-uv)^2}\,, \tag{2.4}$$

where

$$u = u(x_+)\,, \qquad v = v(x_-)\,.$$

As in the general case, if we denote with π the conjugate momentum of φ, *i.e.* $\pi = \partial_t\varphi$, the canonical symplectic structure

$$\{\pi(x), \varphi(x')\}_{e.t.} = 4\,\delta(x-x')$$

leads to the already mentioned Kirillov bracket

$$\{\mathbf{A}_x(y), \mathbf{A}_x(y')\}_{e.t.} = [r, \mathbf{A}_x(y)\otimes 1 + 1 \otimes \mathbf{A}_x(y)]\delta(y-y')\,. \tag{2.5}$$

According to the general treatment, $r \in \mathfrak{g} \otimes \mathfrak{g}$ can be identified with either of the classical r-matrices

$$r^{\pm} = \pm\frac{1}{2}(H \otimes H + 4E_{\pm} \otimes E_{\mp}) .$$

In this particular case, let us quote, once again, the integrated version of eq.(2.5),

$$\{\Psi(x), \Psi(x)\}_{e.t.} = -[r, \Psi \otimes \Psi] , \tag{2.6}$$

which summarizes the symplectic structure on the group of regular maps with values in $SL(2, \mathbb{R})$, in the same way as eq.(2.5) does for the corresponding Lie algebra. The subscript *e.t.* in the above Poisson relations is to remember that they are equal time equations. Hereafter it will be understood and we choose to set the time to zero. Running again through the Leznov-Saveliev analysis, we find the $\xi = (\xi_{11}, \xi_{12})$, $\bar{\xi} = (\bar{\xi}_{21}, \bar{\xi}_{22})$ basis in the Liouville case as matrix elements of

$$V = \begin{pmatrix} \xi_{11} & \xi_{12} \\ * & * \end{pmatrix} , \qquad \bar{V} = \begin{pmatrix} * & * \\ \bar{\xi}_{21} & \bar{\xi}_{22} \end{pmatrix} .$$

Furthermore, the flat connection $\mathbf{A}$ quoted in eq.(2.2) is brought into the gauge transformed connections

$$\mathbf{A}_{+}^{V} = \begin{pmatrix} \partial_{+}\varphi & 1 \\ 0 & -\partial_{+}\varphi \end{pmatrix} , \qquad \mathbf{A}_{-}^{V} = \begin{pmatrix} 0 & 0 \\ e^{2\varphi} & 0 \end{pmatrix} , \tag{2.7}$$

$$\mathbf{A}_{+}^{\bar{V}} = \begin{pmatrix} 0 & e^{2\varphi} \\ 0 & 0 \end{pmatrix} , \qquad \mathbf{A}_{-}^{\bar{V}} = \begin{pmatrix} -\partial_{-}\varphi & 0 \\ 1 & \partial_{-}\varphi \end{pmatrix} . \tag{2.8}$$

Because of the particular choice of boundary conditions, we can classify the field bases in the phase space of the theory according to their monodromy behaviour. In the case of the $(\xi, \bar{\xi})$ basis, this is ruled by the monodromy matrix

$$T = \Psi(2\pi) =: \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} , \tag{2.9}$$

so that, after a cycle, this basis transforms as

$$\begin{aligned} \xi &\to \xi \cdot T \qquad \text{or} \quad \begin{cases} \xi_{11} \to \alpha\xi_{11} + \gamma\xi_{12} \\ \xi_{12} \to \beta\xi_{11} + \delta\xi_{12} \end{cases} , \\ \bar{\xi} &\to \bar{\xi} \cdot T \qquad \text{or} \quad \begin{cases} \bar{\xi}_{21} \to \alpha\bar{\xi}_{21} + \gamma\bar{\xi}_{22} \\ \bar{\xi}_{22} \to \beta\bar{\xi}_{21} + \delta\bar{\xi}_{22} \end{cases} . \end{aligned} \tag{2.10}$$

Here and hereafter, we identify the base point of the holonomy T as the origin of our frame. Notice that the relations (2.10) do not combine the elements of the $(\xi,\bar{\xi})$ basis with those entries of V and $\bar{V}$ which have not definite chirality. This means that, at least as far as the behaviour described in eq.(2.10) is concerned, the monodromy shift respects the chirality structure of the phase space. This is not a completely expected result, since the identification of ξ and $\bar{\xi}$ as field of definite chirality occurs on account of the solution of a group factorization problem, which has no apparent link with the geometrical structure of the zero-curvature representation.

Finally, we have to take into account the monodromy matrix T in the description of the involved symplectic structure. Therefore, the collection of Poisson brackets has to be completed by adding the relations

$$
\begin{aligned}
\{\xi(x) \overset{\otimes}{,} T\} &= \xi(x) \otimes T \cdot r^- \,, \\
\{\bar{\xi}(x) \overset{\otimes}{,} T\} &= \bar{\xi}(x) \otimes T \cdot r^+ \,,
\end{aligned}
\tag{2.11}
$$

to the $(\xi,\bar{\xi})$ exchange algebra quoted in eq.(1.22).

Let us now turn to consider the chirality problem for the $(\xi,\bar{\xi})$ basis, as already formulated in the introduction of this Chapter. We synthesize it with the following question: *can we gauge transform* $\mathbf{A}^{\vee}$ $(\mathbf{A}^{\bar{\vee}})$ *into a chiral (antichiral) connection* $\mathbf{A}^{(+)} = \mathbf{A}_+^{(+)}\, dx_+$ $(\mathbf{A}^{(-)} = \mathbf{A}_-^{(-)}\, dx_-)$*?*

If we limit ourself to the right-moving sector, the solution of this problem lies in the possibility to determine a group element $g \in C^\infty(S^1, Sl_2)$, which induces the gauge transformation[1]

$$
\begin{aligned}
V &\to g^{-1} V \,, \\
\mathbf{A}_\pm^{\vee} &\to g^{-1} \mathbf{A}_\pm^{\vee} g + g^{-1} \partial_\pm g \,,
\end{aligned}
\tag{2.12}
$$

such that ${}^g\mathbf{A}_-^{\vee}$ vanishes. It is sufficient to consider g in the lower unipotent subgroup

$$
g = \begin{pmatrix} 1 & 0 \\ a_+ & 1 \end{pmatrix} .
$$

Imposing the condition we aim for, we find

$$
{}^g\mathbf{A}_-^{\vee} = 0 \qquad \Rightarrow \qquad \partial_- a_+ + e^{2\varphi} = 0
$$

[1]The following considerations have to be contextualized in the Baxter-Lie algebra framework. In this sense we shall regard the gauge transformation, we are going to construct, as an infinitesimal translation, tangent to a coadjoint orbit in $\hat{\mathfrak{g}}^*$. Further geometrical analyses are not, by now, among our purposes.

and, in order for a_+ to solve the above equation, we have to require

$$a_+ \;=\; -\partial_+\varphi + (\text{chiral terms})\,. \tag{2.13}$$

In consequence of this gauge transformation, the $(+)$ component of the connection $\mathbf{A}^{\vee}$ is mapped into

$$\mathbf{A}_+^{(+)} \;:=\; {}^g\mathbf{A}_+^{\vee} = \begin{pmatrix} -p & 1 \\ \partial_+ a_+ - 2(\partial_+\varphi)a_+ - a_+^2 & p \end{pmatrix}, \tag{2.14}$$

which, compared with eq.(2.13) and substituting a_+, becomes

$$\mathbf{A}_+^{(+)} \;=\; \begin{pmatrix} -p & 1 \\ [\mathcal{T} - \partial_+ p - p^2] & p \end{pmatrix}. \tag{2.15}$$

Here p denotes the chiral terms in the expression (2.13) and $\mathcal{T}$ is the chiral component of the energy-momentum tensor of φ,

$$\mathcal{T}\,(=\mathcal{T}_{++}) \;=\; -\partial_+^2\varphi + (\partial_+\varphi)^2\,.$$

This could seem the answer to the initial question, which, in spite of our premise, would turn out to be affirmative. Nevertheless, the above result is still unsatisfactory, and this for two reasons. The connection $\mathbf{A}_+^{(+)}$ is still acting on ${}^gV$, which has no definite chirality. Moreover, looking further from a physical point of view, if our aim is to identify the chiral variable p in eq.(2.15), with the free field of the Drinfel'd-Sokolov linear system, we expect that the chiral component of the energy-momentum tensor can be expressed in terms of p. The effect of this condition is to equate the lower-diagonal entry of $\mathbf{A}_+^{(+)}$ to zero. Concerning the first argument, consider the explicit form of ${}^gV$,

$${}^gV \;=\; \begin{pmatrix} \xi_{11} & \xi_{12} \\ -a_+\xi_{11} + \xi_{21} & -a_+\xi_{12} + \xi_{22} \end{pmatrix}.$$

It is evident that, if we require ${}^gV$ to be upper triangular, a chiral object is singled out. This represents a further condition on a_+, which determines its chiral part to be

$$p \;=\; -\partial_+\varphi - \xi_{21}\xi_{11}^{-1}\,. \tag{2.16}$$

It is easy to verify by a direct computation that $\partial_- p = 0$. Now, the remark which rules out the just constructed gauge transformation, follows from the monodromy behaviour shown in eq.(2.10). Actually, we shall consider its completion in the monodromy relation

$$V \;\to\; V\cdot T\,. \tag{2.17}$$

In account of this equation, the element of the group, which induces such a gauge transformation, is not univalent, *i.e.* $g \notin C^\infty(S^1, Sl_2)$. The above prescription turns out to be hindered even more seriously when we observe that, even before g, the bosonic field p is not periodic, in disagreement with the expectation based on the Drinfel'd-Sokolov approach.

By allowing for a parallel treatment of the antichiral sector, we would bring $\mathbf{A}^{\bar{V}}$ in the form

$$\mathbf{A}_+^{(+)} := {}^g\mathbf{A}_+^V = \begin{pmatrix} -\bar{p} & 0 \\ 1 & \bar{p} \end{pmatrix}, \tag{2.18}$$

where

$$\bar{p} = \partial_- \varphi + \bar{\xi}_{12}\bar{\xi}_{22}^{-1}. \tag{2.19}$$

Thus, as it is expected, we find again the same difficulties met in the chiral case [the monodromy shift for $\bar{V}$ is implemented exactly as in eq.(2.17)].

2.2 The symplectic structure

Before coming to a general interpretation of what arose in the previous Section for the particular example represented by the Liouville model, we would point out a further aspect of the problem. We have already noticed that the exchange algebra in the $(\xi, \bar{\xi})$ basis, as quoted in eq.(1.22), presents still coupling terms between the right- and left-moving sectors. We can wonder how far the problem goes back and, in particular, which are the relations with the hill-definiteness of the gauge transformation that we have just discussed. To answer this questions, we should analyze the transformation which the symplectic structure undergoes when the connections $\mathbf{A}^V$ and $\mathbf{A}^{\bar{V}}$ in eqs.(2.7) and (2.8) are brought into a form of definite chirality by the above prescription. Therefore, the upshot of this Section will be the computation of the Poisson relations involving the momentum p, introduced in eqs.(2.13) and (2.15), and its antichiral counterpart, $\bar{p}$. Before coming to this result, some general considerations, which concern the symplectic structure, are in order.

For the particular geometry of the space-time to be properly taken into account, the notation introduced in eq.(1.22) needs to be more carefully defined. The $(\xi, \bar{\xi})$ exchange algebra, there quoted, is explicitly derived in ref.[2], where he computation is described in detail. The crucial feature is represented by the circumstances which makes the fundamental Poisson bracket

(2.5) independent on the derivatives of the δ-function. This is the often recalled ultralocality property. This allows us to obtain the exchange algebra of the complete V and $\bar{V}$ matrices, which consists in the collection of Poisson relations

$$
\begin{aligned}
\{V(x) \overset{\otimes}{,} V(y)\} = {} & \vartheta(x-y) V(x) \otimes V(y) \cdot \\
& \cdot \Big[r^+ - 2\, V^{-1}(y) \otimes V^{-1}(y) \cdot E_+ \otimes E_- \cdot V(y) \otimes V(y) \Big] + \\
+ {} & \vartheta(y-x) V(x) \otimes V(y) \cdot \\
& \cdot \Big(r^- + 2\, V^{-1}(x) \otimes V^{-1}(x) \cdot E_- \otimes E_+ \cdot V(x) \otimes V(x) \Big] \, ,
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
\{V(x) \overset{\otimes}{,} \bar{V}(y)\} = {} & \vartheta(x-y) V(x) \otimes \bar{V}(y) \cdot \\
& \cdot \Big[r^- + 2\, \bar{V}(y) \otimes \bar{V}(y) \cdot E_- \otimes E_+ \cdot \bar{V}^{-1}(y) \otimes \bar{V}^{-1}(y) \Big] + \\
+ {} & \vartheta(y-x) V(x) \otimes \bar{V}(y) \cdot \\
& \cdot \Big[r^- + 2\, V^{-1}(x) \otimes V^{-1}(x) \cdot E_- \otimes E_+ \cdot V(x) \otimes V(x) \Big] \, ,
\end{aligned} \tag{2.21}
$$

$$
\begin{aligned}
\{\bar{V}(x) \overset{\otimes}{,} \bar{V}(y)\} = {} & \vartheta(x-y) \bar{V}(x) \otimes \bar{V}(y) \cdot \\
& \cdot \Big[r^- + 2\, \bar{V}(y) \otimes \bar{V}(y) \cdot E_- \otimes E_+ \cdot \bar{V}^{-1}(y) \otimes \bar{V}^{-1}(y) \Big] + \\
+ {} & \vartheta(y-x) \bar{V}(x) \otimes \bar{V}(y) \cdot \\
& \cdot \Big[r^+ - 2\, \bar{V}(x) \otimes \bar{V}(x) \cdot E_+ \otimes E_- \cdot \bar{V}^{-1}(x) \otimes \bar{V}^{-1}(x) \Big] \, .
\end{aligned} \tag{2.22}
$$

In the above relations, the ambiguous feature concerns the meaning of the function ϑ. The fact that V and $\bar{V}$ are defined on the covering space $\mathbb{R}$ can deceive us. Indeed, the function ϑ has to be regarded as a form-factor on S^1, rather than as a Heavyside θ-function[2]. More precisely, if we consider an equal time slice S^1 of our cylindrical space-time and we fix on it an origin and an orientation, then $\vartheta(x-y)$ distinguishes between the two ordering, $x > y$ ($\vartheta = 1$) and $x < y$ ($\vartheta = 0$). In this sense we say that such a function keeps trace of the geometrical structure of the problem and of the nature of V and $\bar{V}$, related by definition to the parallel transport. From a more technical standpoint, we can easily figure out the reason why we should interpret the function ϑ as a form-factor by remembering the representation of the transport matrix in terms of path-ordered exponential. If we perform the calculation of the Poisson brackets just expanding Ψ in terms Chen integrals, the meaning of the notation in eqs.(2.20), (2.21) and (2.22) follows immediately.

[2] Since here we are considering a circle at equal time, ϑ cannot be regarded as the primitive of the δ-function, which cannot be integrate on S^1.

The $(\xi, \bar{\xi})$ exchange algebra can be easily derived from the above equations by properly projecting them on the highest and lowest weight vectors. To complete the description of the symplectic structure, we should consider those Poisson relations which involves the monodromy matrix T, as we noticed in §2.1 [see eq.(2.11)]. However, no different features arise in this further computation.

The second remark concerns a meaningful expression to which p, and analogously $\bar{p}$, can be reduced. Through the x-dependence of ξ and $\bar{\xi}$, which is encoded in the linear systems fulfilled by V and $\bar{V}$, eq.(2.16) can be recast in the form

$$p = \partial_x \log \xi_{11} \tag{2.23}$$

and, in a similar fashion, eq.(2.19) becomes

$$\bar{p} = -\partial_x \log \bar{\xi}_{22} \,. \tag{2.24}$$

These formulas can be interpreted as the "bosonization" rules for the $-1/2$ spin fields ξ and $\bar{\xi}$. A similar result for the Liouville model is already known in the literature (see ref.[19]), but limited to the case of open string boundary conditions, which imply a (crucial) reality requirement.

In order to realize our purpose, we reverse the sense of these bosonization prescriptions and exploit them to derive the Poisson brackets of p and $\bar{p}$ from the $(\xi, \bar{\xi})$ exchange algebra. The result is

$$\begin{aligned} \{p(x), p(y)\} &= -2\delta'(x-y)\,, \\ \{\bar{p}(x), \bar{p}(y)\} &= 2\delta'(x-y)\,, \\ \{p(x), \bar{p}(y)\} &= -4\xi_{11}(x)^{-2}\bar{\xi}_{22}^{-2}(y)\,. \end{aligned} \tag{2.25}$$

Remark 1 In the computation of the Poisson brackets (2.25) we have to pay care to the derivative with respect to x, which appears in the bosonization formulas (2.23) and (2.24). This operation could cause the ultralocality property to partially fails. Therefore, a naive calculation could bring to an incorrect result and the explicit development of path-ordered expressions could become necessary.

The above equations clearly show where the coupling problem between the fields of definite chirality, ξ and $\bar{\xi}$ arises, and its close connection with the difficulties met in the previous Section, about the univaluedness of the "would-be" bosonic fields p and $\bar{p}$.

2.3 From the factorization problem to the free field representation

Reconsidering what we have done up to now, the relevance of the monodromy behaviour (2.10) becomes apparent with great evidence. In particular, the only possibility to obtain a well-defined gauge transformation from the treatment of §2.1 lies in the solution of the diagonalization problem for T. Indeed, only in the case of diagonal monodromy, p and $\bar{p}$, once expressed as in eqs.(2.16) and (2.19), turn out to be univalent. This should not be surprising as we come back to consider eq.(2.14): there, it clearly appears that the problem of recovering the Drinfel'd-Sokolov linear system from the connection $\mathbf{A}^V$ reduces to the study of the solution of the ordinary differential equation which arises when we equate to zero the lower-diagonal entries in $\mathbf{A}_+^{(+)}$. This is an equation of Riccati type and its monodromy matrix coincides with T.

By referring to §2.3.1 for a detailed discussion about the diagonalization of the monodromy matrix, here we intend to show how the Leznov-Saveliev analysis can be more carefully interpreted, in the light of what already observed. This leads to the desired solution of the chirality problem and, hence, to the free field parametrization of the physical phase space. Therefore, rather than on the properties of the connection $\mathbf{A}$, we direct our attention on the definition of V and $\bar{V}$. The first remark is that these objects are not univocally defined. Indeed, if we require the chirality behaviour of the first row of V and the last one of $\bar{V}$ to be preserved, a great arbitrariness is still left and the class of equivalent choices is determined by the invariance

$$\begin{cases} V \mapsto n_- V g_0 \\ \bar{V} \mapsto n_+ \bar{V} g_0 \end{cases} . \tag{2.26}$$

Here $n_\pm : S^1 \to \exp \mathcal{N}_\pm$ and g_0 belongs to the group G, which coincides with $SL(2, \mathbb{R})$ in the case here analyzed. Notice that, while the left multiplication by $n_\pm$ can be interpreted as a gauge transformation (see §2.1), the right action of G shifts only the initial condition, without affecting the dynamics. On the other hand, the right multiplication by g_0 implies a change by conjugation of the monodromy matrix, *i.e.* $T \mapsto {}^{g_0}T = g_0^{-1} T g_0$. A further remarkable feature of the symmetry (2.26) is that it does not act on the solution of the Toda field equation, which can be reconstructed according to the formula

$$e^{-\varphi} = \langle \Lambda | (n_- V g_0)(n_+ \bar{V} g_0)^{-1} | \Lambda \rangle =$$

$$= \langle \Lambda | V \bar{V}^{-1} | \Lambda \rangle .$$

[see eq.(1.21) and remember that $\varphi = 2\Lambda(\Phi)$]. Finally, notice that this symmetry is a general feature of the Leznov-Saveliev analysis and concerns not only the simple case of the Liouville model, but every Toda field theory associated to any simple Lie algebra.

In consideration of the induced transformation undergone by the monodromy matrix, we exploit such an invariance property to impose the condition of diagonal monodromy. Running through the prescription of §2.1, now we obtain

$$V_+ := g^{-1} V g_0 = \begin{pmatrix} \xi_{11}^{g_0} & \xi_{12}^{g_0} \\ 0 & 1/\xi_{11}^{g_0} \end{pmatrix} \quad \Rightarrow \quad a_+ = \frac{\xi_{21}^{g_0}}{\xi_{11}^{g_0}} \tag{2.27}$$

and, for the antichiral sector,

$$V_- := \bar{g}^{-1} \bar{V} g_0 = \begin{pmatrix} 1/\bar{\xi}_{11}^{g_0} & 0 \\ \bar{\xi}_{21}^{g_0} & \bar{\xi}_{22}^{g_0} \end{pmatrix} \quad \Rightarrow \quad a_- = \frac{\xi_{12}^{g_0}}{\xi_{22}^{g_0}} . \tag{2.28}$$

These equations clearly show that a_+ is univalent when the monodromy matrix is upper triangular [$\gamma = 0$ in eq.(2.9)], while for the univaluedness of a_- we have to require T to be lower triangular ($\beta = 0$). Therefore, imposing that $D = g_0^{-1} T g_0$ is a diagonal matrix represent a necessary and sufficient condition for both the gauge transformations g and $\bar{g}$ to be well-defined (where this has the meaning emphasized in §2.1).

2.3.1 Diagonalization of the monodromy matrix

In order to implement the above prescription, we have to prove that the condition of diagonal monodromy can be required. The possibility to conjugate the monodromy matrix T to a diagonal matrix obviously lies in the properties of the flat connection (2.2). As already observed, the monodromy matrix can be represented as the path ordered exponential

$$T = \overleftarrow{\mathrm{P}}\, e^{-\oint_C \mathbf{A}} , \tag{2.29}$$

where C is a closed path emanating from a given base point (the origin of our frame) and homotopic to an equal time loop. The fundamental remark consists in the simple observation that different choices of the path C correspond to different factorizations of T in the group $SL(2, \mathbb{R})$. This turns out to be a crucial point in order to make the "algebraic" properties of the connection readable at the group level of the monodromy matrix. For instance, let us identify $\mathcal{C}$ with an equal time path. Then the expression (2.29) becomes

$$T = \overleftarrow{\mathrm{P}}\, e^{-\int_0^{2\pi} dx\, \mathbf{A}_x}\,,$$

with

$$\mathbf{A}_x = \mathbf{A}_+ + \mathbf{A}_- = \partial_t \Phi + e^{\varphi}(E_+ + E_-)\,.$$

Although H and $(E_+ + E_-)$ span the vector space of symmetric matrices in $sl(2,\mathbb{R})$, they do not constitute a closed subalgebra. Therefore, the properties of $\mathbf{A}_x$ do not survive the path-ordered exponentiation. Actually, the particular form of $\mathbf{A}$ seems to suggest the choice of a loop moving along the light-cone variable directions, on which the parallel transport is ruled by the connection components $\mathbf{A}_+$ and $\mathbf{A}_-$, given in eq.(2.2). Indeed, they take values in the Borel subalgebras of $sl(2,\mathbb{R})$ and, therefore, the corresponding monodromy factors belong to the associated Borel subgroup. The simplest possibility to realize this kind of path is represented by the curve

$$\mathcal{C} \,:\, (x,t) = \begin{cases} (\tau,\tau) & \text{for } \tau \in [0,\pi] \\ (\tau, 2\pi - \tau) & \text{for } \tau \in [\pi, 2\pi] \end{cases}, \tag{2.30}$$

which, in light-cone coordinates[3] is parametrized as

$$\mathcal{C} \,:\, (x_+,x_-) = \begin{cases} (2\tau, 0) & \text{for } \tau \in [0,\pi] \\ (2\pi, 2\tau - 2\pi) & \text{for } \tau \in [\pi, 2\pi] \end{cases}. \tag{2.31}$$

Remark 2 A reason which could make necessary a more elaborated choice of the path is represented by the possibility that a singular behaviour of the flat connection arises. Indeed, when we assume the equivalence of integration paths which are homotopic, at the same time we understand some regularity hypothesis, concerning the connection $\mathbf{A}$ and, therefore, the solution of the Liouville equation: this ensure us that any singularity could hinder the path deformation.

On the path (2.31), the monodromy matrix factorizes in two terms corresponding to the right- and left-moving components,

$$T = \Psi_-(2\pi)\Psi_+(2\pi)\,,$$

[3]Due to our definition of the light-cone variables, there is a scale mismatch between the (x_+, x_-) frame and the (x,t) one.

with

$$\Psi_+(x) = \overleftarrow{\mathrm{P}} \exp\left[-\int_0^x dx_+\, \mathbf{A}_+(x_+, x_- = 0)\right],$$

$$\Psi_-(x) = \overleftarrow{\mathrm{P}} \exp\left[-\int_0^x dx_-\, \mathbf{A}_-(x_+ = 2\pi, x_-)\right].$$

In order to obtain the explicit expression of the monodromy matrix, we have to solve the linear systems fulfilled by Ψ_+ and Ψ_-. This amounts to find the independent solutions of the chiral and antichiral halves of the linear system which is associated to the the Liouville model,

$$(\partial_+ + \mathbf{A}_+)\Psi_+ = 0$$

$$(\partial_- + \mathbf{A}_-)\Psi_- = 0$$

with the boundary conditions $\Psi_+(0) = 1$ and $\Psi_-(0) = \Psi_+(2\pi)$. Actually, we do not need the complete form of the solution, which is a rather complicate non-local function of φ. A well-known classification of the element of $SL(2,\mathbb{R})$ guarantees that if $\mathrm{tr}\, T > 2$, then the monodromy matrix is of hyperbolic type, that is, its eigenvalues are real and positive. This statement can be easily verified for a 2×2 unimodular matrix. However, this classification has a meaningful interpretation in terms of the Iwasawa decomposition of the group, which allows us to make a generalization to higher order unimodular matrices.

If we reduce the aim of our computation to the proof of the above requirement, we can work out two crucial results.

If we introduce the notation

$$\Psi_+(2\pi) = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{11}^{-1} \end{pmatrix}, \qquad \Psi_-(2\pi) = \begin{pmatrix} v_{22}^{-1} & 0 \\ v_{21} & v_{22} \end{pmatrix},$$

then, for a general, regular periodic solution of the Liouville equation, we have: φ,

i) $\Psi_+(2\pi)$ and $\Psi_-(2\pi)$ are hyperbolic matrices and,

ii) due to the positivity of the off-diagonal elements of $\mathbf{A}_+$ and $\mathbf{A}_-$ [see eq.(2.2)], the corresponding off-diagonal entries of $\Psi_+(2\pi)$ and $\Psi_-(2\pi)$, *i.e.* u_{11} and v_{22}, are negative.

Therefore, it is now matter of matrix multiplication to prove that

$$\mathrm{tr}\, T = v_{22}^{-1}u_{11} + v_{22}u_{11}^{-1} + v_{21}u_{12} > 2\cosh(v_{22}^{-1}u_{11}) > 2,$$

q.e.d.

Finally, let us formulate an apparently trivial observation about the the matrix $g_0 \in SL(2,\mathbb{F})$ which conjugates the monodromy matrix into a diagonal form. Indeed, it is a well-known algebraic result that g_0 cannot be univocally determined. If we consider the canonical form

$$g_0 = R \cdot S$$

with

$$R = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

we are still left with two possibilities, corresponding to the two orderings of the eigenvalues on the diagonal of the matrix $D = g_0^{-1} T g_0$. By imposing that, at a first step, R brings T into an upper triangular form, we obtain

$$D = \begin{pmatrix} \alpha + \beta a & 0 \\ 0 & \delta - \beta a \end{pmatrix},$$

where a fulfills the equation

$$\beta\, a^2 + (\alpha - \delta)\, a - \gamma = 0 . \qquad (2.32)$$

In the above equation $\alpha, \beta, \gamma, \delta$ are the entries of the monodromy matrix, as given by eq.(2.9). The two choices for D and, therefore, for g_0, are related to the two solutions $a_{1,2}$ of the quadratic equation. Obviously we have $\alpha + \beta a_{1,2} = \delta - \beta a_{2,1}$. This simple observation is the reason for a degeneracy in the free field parametrization of the Liouville phase space.

2.3.2 The Drinfel'd-Sokolov linear systems

After the solution of the diagonalization problem, our analysis has reached the result to bring the connections $\mathbf{A}^{\nabla}$ and $\mathbf{A}^{\bar{\nabla}}$ in the Drinfel'd-Sokolov type form

$$\mathbf{A}^{(+)} = \begin{pmatrix} -p & 1 \\ 0 & p \end{pmatrix}, \qquad \mathbf{A}^{(+)} = \begin{pmatrix} -\bar{p} & 0 \\ 1 & \bar{p} \end{pmatrix},$$

respectively, where p and $\bar{p}$ are free periodic fields. They can be represented through the bosonization formulas

$$\begin{aligned} p &= \partial_x \log \xi_{11}^{g_0} = \\ &= \partial_x \log\left(\xi_{11} + a\xi_{12}\right), \end{aligned} \qquad (2.33)$$

$$\begin{aligned}\bar{p} &= -\partial_x \log \bar{\xi}^{g_0}_{22} = \\ &= -\partial_x \log \left((1+ab)\, \bar{\xi}_{22} + b\bar{\xi}_{21} \right) .\end{aligned} \tag{2.34}$$

This means that we formally recover a realization of the Drinfel'd-Sokolov linear systems in terms of the interacting field φ. In order to give a complete meaning to this statement, we need to verify that the original canonical symplectic structure for φ and its conjugate field is transformed in the symplectic structure which characterizes the Drinfel'd-Sokolov construction. In particular, the field p and $\bar{p}$ must fulfill the bosonic Poisson relations (1.27) and (1.29), and must also decouple as in eq.(1.28). This can be verified by following the same steps as outlined in §2.2. Indeed, we can as well exploit the bosonization formulas (2.33) and (2.34), taking into account the new Poisson algebra which is fulfilled by ξ^{g_0} and $\bar{\xi}^{g_0}$. Notice that, in the diagonalization procedure, the $(\xi, \bar{\xi})$ basis undergoes an $SL(2,\mathbb{F})$ transformation, whose coefficients have a dynamics value, *i.e.* they do not Poisson commute, as they are functions of the monodromy matrix elements. This observation makes the Poisson brackets involving ξ^{g_0} and $\bar{\xi}^{g_0}$ not very appealing. If we look at the counterpart of this Bloch wave basis in the Drinfel'd-Sokolov prescription, we can figure out that the unreadable form of this Poisson algebra originates from the zero modes problem. Indeed, while the Drinfel'd-Sokolov construction keeps distinct the chiral zero modes from the antichiral ones and identifies them only when the solution of the Toda field is recovered, we have never enlarged the original phase space of the theory with our approach and, in consequence of this, p_0 and $\bar{p}_0$ naturally coincides (see the next Subsection). In spite of the appearance of this rather obscure region, the Poisson structure realized by p and $\bar{p}$ in eqs.(2.33) and (2.34) turns out to be the correct one,

$$\begin{aligned}\{p(x) \overset{\otimes}{,} p(y)\} &= -2\delta'(x-y)\,, \\ \{p(x) \overset{\otimes}{,} \bar{p}(y)\} &= 0\,, \\ \{\bar{p}(x) \overset{\otimes}{,} \bar{p}(y)\} &= 2\delta'(x-y)\,.\end{aligned}$$

Remark 3 We can observe that the role played up to now by each component of the $(\xi, \bar{\xi})$ basis is rather unequal. This is outlined by the bosonization formulas (2.23) and (2.24), as well as (2.33) and (2.34), where only the (1,1) component (respectively, the (2,2) one) appears. Actually, it is possible to construct a completely parallel mapping of the zero-curvature representation into the Drinfel'd-Sokolov scheme whose upshot are the momenta

$$\tilde{p} = \partial_x \log \xi^{g_0}_{12}\,,$$

$$\bar{\tilde{p}} = -\partial_x \log \bar{\xi}_{21}^{g_0} .$$

This exactly represents the degeneracy to which we hinted at the end of our monodromy diagonalization program. Indeed, let D and $\tilde{D}$ be the two diagonal forms to which the monodromy matrix is conjugated through $g_0 = g_0|_{a=a_1}$ and $\tilde{g}_0 = g_0|_{a=a_2}$, respectively. They are related by the equation

$$\tilde{D} = w \cdot D \cdot w^{-1} ,$$

where the matrix w,

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

implements the Weyl transformation in the group $SL(2, \mathbb{R})$. Therefore, the existence of two equivalent possibilities for the diagonalization prescription reflects into the invariance

$$\begin{aligned} e^{-\varphi} &= \langle \Lambda | (V\tilde{g}_0)(\bar{V}\tilde{g}_0)^{-1} | \Lambda \rangle = \\ &= \langle \Lambda | (\tilde{V} g_0)(\bar{\tilde{V}} g_0)^{-1} | \Lambda \rangle , \end{aligned}$$

with

$$\tilde{V} = V w = \begin{pmatrix} \xi_{12} & -\xi_{11} \\ * & * \end{pmatrix} ,$$

$$\bar{\tilde{V}} = \bar{V} w = \begin{pmatrix} * & * \\ \bar{\xi}_{21} & -\bar{\xi}_{22} \end{pmatrix} .$$

The two copies of free fields which parametrize the Liouville phase space appear completely equivalent, at this level of the discussion. However, we can make a distinction if the regularity of the momenta p $(\tilde{p})$ and $\bar{p}$ $(\bar{\tilde{p}})$ is involved.

Remark 4 The bosonization formulas (2.33) and (2.34) allows us to recognize the existing connection between the zero modes of p and of $\bar{p}$ and the monodromy matrix eigenvalues. Indeed

$$\begin{aligned} p_0 &= \int_0^{2\pi} dx \, (\partial_x \log \xi_{11}^{g_0}) = \\ &= \log(\alpha + \beta a) , \end{aligned}$$

$$\begin{aligned} \bar{p}_0 &= \int_0^{2\pi} dx \, (-\partial_x \log \bar{\xi}_{22}^{g_0}) = \\ &= -\log(\delta - \beta a) . \end{aligned}$$

According to our expectation, these equations show that the chiral zero mode coincides with the antichiral one.

2.3.3 Zero modes and conjugate variables

Let us recall the main aspects of the problem we are treating in the present Chapter. Our aim is to represent the phase space of the Liouville model by a global free mode parametrization. Since the model under consideration exhibits a conformal symmetry, we know that this free field representation allows for the separation of the chiralities. Furthermore our guess is that such a program can be realized by a one-to-one correspondence between the space of solutions of the Liouville equation and the related Drinfel'd-Sokolov reconstructive scheme. In §2.3.2 we have shown how, starting from a solution of the Liouville equation, it is possible to single out the expressions of Drinfel'd-Sokolov type momenta, p and $\bar{p}$. This permits to define Q_+ and Q_- as in §1.3, but do not guarantee that the solution recovered by the by the Drinfel'd-Sokolov prescription is neither the original one nor univocally defined. Facing this problem amounts to fixing a suitable conditions, which allows us to identify any element of the phase space which is chosen as initial data and the reconstructed one. Its solution implies the realization of the variable q, conjugate variable of the zero mode, in the Liouville phase space. The natural requirement that we are led to impose is

$$\langle \Lambda | V \bar{V}^{-1} | \Lambda \rangle = \langle \Lambda | Q_+ g_s \rho \bar{g}_s Q_- | \Lambda \rangle \,. \tag{2.35}$$

Indeed, the first term of the above equation is just a tautological rewriting of the given solution φ, while the second one represent the main ingredient in the Drinfel'd-Sokolov recipe. If we come back to the definition of V_+ and V_- in eqs.(2.27) and (2.28), which are the products of the gauge transformation g, the condition (2.35) turns out to be equivalent to the identification

$$V_+ V_-^{-1} = Q_+ g_s \rho \bar{g}_s Q_- \,.$$

If we evaluate this equation at the origin ($x_+ = 0, x_- = 0$), the result is

$$V_+(0) V_-^{-1}(0) = g_s \rho \bar{g}_s \,.$$

If we remember the meaning of the notation, explained in §1.3 (there g_s and $\bar{g}_s$ are g and $\bar{g}$, respectively, while ρ coincides with D), clearly the *rhs* of the above equation is nothing but the Gauss decomposition of the matrix at the left. Solving this factorization problem, we obtain

$$\rho = \begin{pmatrix} (1+ab)\, e^{-\varphi_0} & 0 \\ 0 & (1+ab)^{-1}\, e^{\varphi_0} \end{pmatrix} ,$$

where $\varphi_0 = \varphi(x_+ = 0, x_- = 0)$ and a, b are the function of the monodromy matrix elements which parametrize the entries of the diagonalizing g_0. As in the Drinfel'd-Sokolov construction ρ contains q, the conjugate variable of $p_0 = \log(\alpha + \beta\, a)$, we should verify that

$$\{p_0, \rho\} = \rho\,.$$

The calculation is long but uneventful, when some subtleties are properly taken into account. Indeed, the result follows from direct computation, when the problems announced in Remark 1 are carefully considered.

2.4 Conclusions

In summary, the upshot of the previous Section consists in the realization of the Drinfel'd-Sokolov linear systems

$$\begin{aligned} \partial_+ Q_+ - (\mathbf{p} - \mathcal{E}_+)Q_+ &= 0 \\ \partial_- Q_- + Q_-(\bar{\mathbf{p}} - \mathcal{E}_-) &= 0 \end{aligned} \,,$$

where $\mathbf{p} = pH$ and $\bar{\mathbf{p}} = \bar{p}H$ are a chiral and an antichiral momentum, respectively. This is made possible by the identifications

$$p = \partial_x \log \xi_{11}^{g_0}\,, \tag{2.36}$$

$$\bar{p} = -\partial_x \log \bar{\xi}_{22}^{g_0}\,, \tag{2.37}$$

or, equivalently,

$$\tilde{p} = \partial_x \log \xi_{12}^{g_0}\,, \tag{2.38}$$

$$\bar{\tilde{p}} = -\partial_x \log \bar{\xi}_{21}^{g_0}\,, \tag{2.39}$$

where $g_0 \in G$ is determined by imposing that g_0 sets by conjugation the monodromy matrix into a diagonal form. The meaning of this result can be understood in the framework of the conformal invariance, which characterizes the model. Indeed, in consideration of such a symmetry, we are led to search a representation of the Poisson algebra of observables as the direct sum of two sectors, which correspond to the opposite chiralities. This program is implemented by p and $\bar{p}$ in eq.(2.37) [or, equivalently, by $\tilde{p}$ and $\bar{\tilde{p}}$ in eq.(2.39)]. According to a slightly different point of view, we can interpret the momenta

p and $\bar{p}$ [or, equivalently, $\tilde{p}$ and $\bar{\tilde{p}}$] as the generators of the free bosonic modes, which realize the Poisson algebra

$$\{p_n \overset{\otimes}{,} p_m\} = -\frac{in}{\pi}\delta_{n+m} ,$$

$$\{\bar{p}_n \overset{\otimes}{,} \bar{p}_m\} = \frac{in}{\pi}\delta_{n+m} .$$

Their definition allows us to globally parametrize the space of solution of the Liouville equation.

A still open problem concerns the formulation a definitive recipe to properly take into account the symplectic structure transformation under the affine action of g_0, in order to give a meaningful interpretation of the $(\xi^{g_0}, \bar{\xi}^{g_0})$ exchange algebra. This could represents the starting point for a generalization, which is not yet a crude extension of the previous results, to Toda field theories.

Chapter 3

Quantum Toda field theories

This Chapter mainly deals with the quantization of Toda field theories based on a finite dimensional Lie algebra in a periodic space lattice [10].

The reason for the interest in Toda field theories is twofold: on the one hand these theories, which we recall are characterized by a W-algebra symmetry, underlie a vast set of conformal field theories, in particular the W-Minimal Models [17] [20]; on the other hand they define the so-called W-Gravities, which are generalizations of the Liouville theory that might relate to the most recent results from Matrix Models and Topological Gravity.

While the quantum sl_2 Toda theory, i.e. the Liouville theory, has been properly analyzed in the literature, the attempts to do the same for a general Toda field theory are fewer and much less complete.

Motivated by the renewed interest in Toda theories, in this Chapter we want to analyze in detail the quantization of the sl_p, and in particular the sl_3, Toda field theory. As it is common feature of most quantum theory , sl_p Toda field theories require to be renormalized. The use of a lattice cutoff (as suggested in refs.[15]) reveals to be a natural choice. It essentially preserves the continuum symmetries of the theory and, in particular, we can point out a lattice conformal symmetry, as it is done in ref.[16] in the case of the WZNW model. Another possibility to regularize the theory would be the normal ordering of vertex operators, employed in refs.[19, 11, 21]. However, the scheme of the Drinfel'd-Sokolov construction, on which we base our recipe for the discretization, does not naturally match with this operation. One can easily figure out the reasons of such an incongruence when we remember the role there played by non-local fields, representable as path-ordered exponential (trace of the holonomy problem in the original zero-curvature approach – see Chapter 1 and the references there quoted –). The problem lies in the incompatibility of the normal ordering prescription and the path-ordered structures. The lattice regularization overcomes also this problem.

On the other hand, the lattice approach should not be regarded only as a mere, albeit powerful, tool to properly quantize the theory. We have already emphasized the manifestation of a discretize version of the Conformal symmetry, which comes into play as an invariance under a block spin type renormalization group. On this regard, in §3.1.1 we will proposed an interesting quadratic algebra, which plays the role of the lattice W_3 algebra. The study of a possible relation between discretized Toda field theories and Matrix Models, suggested by the most recent literature, could be a further reason of interest in this analysis. Therefore, we will start this Chapter with illustrating and pointing out the main features of the Toda field theories on the lattice.

Once the discretized version of Toda field theories is formulated, the scheme of ref.[7], relative to the continuum case, will be implemented in this context, til to the computation, on the lattice, of the exchange algebra in the Bloch wave basis. This allows us to recover the result found in ref.[20], from a completely independent point of view. Moreover, the problem to link the exchange matrix in the Bloch wave basis (to be compared with the exchange matrix of the vertex operators, dressed in a Coulomb gas fashion, in refs.[20, 11, 21]) to the quantum R-matrix in a Quantum Group framework, turns out to be completely overcome.

3.1 Classical theory on the lattice

The formulation of Toda field theories on a (periodic) space lattice with N sites runs parallel to the construction of Chapter 1. We recall in the following the recipe found in ref.[8]. This treatment is addressed to Toda field theories associated to every simple Lie algebra. Moreover we will understand the representation labels where they are not explicitly required by the context. They can be supplied in an obvious way. We limit ourselves to one chirality, since the other chirality has completely parallel formulas. Moreover we set $t = 0$.

The discretized version of eq.(1.25) should consist in the parallel transport on a lattice spacing Δ

$$Q_n = L_n Q_{n-1} , \tag{3.1}$$

where one is led to identify

$$\begin{aligned} \mathrm{L}_n &= \exp\left\{-\int_{x_{n-1}}^{x_n} \mathrm{A}\right\} = \\ &= \exp\left\{-\Delta \mathrm{A}_n + O(\Delta^2)\right\} = \\ &= \exp\left\{\Delta(\mathrm{p}_n - \mathcal{E}_+) + O(\Delta^2)\right\} . \end{aligned}$$

However, in order to give a precise meaning to L_n and consequently to Q_n we have to fix a criterion to solve the usual ambiguity inherent in the arbitrariness of the $O(\Delta^2)$ terms, typical of any discretization procedure. Since the exchange algebra relations were the peculiar feature of the theory in the continuum, we assume it as a requirement to fix the lattice deformation. Therefore, Q_n still satisfies the Poisson bracket

$$\{Q_n \overset{\otimes}{,} Q_m\} =$$

$$= \begin{cases} Q_n \otimes Q_m \left[-r + Q_m^{-1} \otimes Q_m^{-1}(r - t_0) Q_m \otimes Q_m\right] , & \text{if } n > m \\ \left[r, Q_n \otimes Q_n\right] , & \text{if } n = 0 \\ Q_n \otimes Q_m \left[-r + Q_n^{-1} \otimes Q_n^{-1}(r + t_0) Q_n \otimes Q_n\right] , & \text{if } n \le m . \end{cases} \tag{3.2}$$

Here, as in the continuum counterpart, r is defined up to Casimir terms and, therefore, can be replaced by the classical r-matrices of sl_p,

$$r^+ = t_0 + 2 \sum_{\alpha \; positive} \frac{E_\alpha \otimes E_{-\alpha}}{\langle E_\alpha, E_{-\alpha}\rangle}$$

$$r^- = -t_0 - 2 \sum_{\alpha \; positive} \frac{E_{-\alpha} \otimes E_\alpha}{\langle E_{-\alpha}, E_\alpha\rangle} .$$

Thus, through the relation $L_n = Q_n Q_{n-1}^{-1}$, we find the Poisson bracket algebra of the discrete transport matrix L_n

$$\begin{aligned} \{L_n \overset{\otimes}{,} L_m\} &= \delta_{n,m}[r, L_n \otimes L_m] + \\ &- \delta_{n,m+1} L_n \otimes 1 \cdot t_0 \cdot 1 \otimes L_m + \\ &+ \delta_{n,m-1} 1 \otimes L_m \cdot t_0 \cdot L_n \otimes 1 . \end{aligned} \tag{3.3}$$

In order to recover the original equation in the continuum, it is sufficient to assume that

$$L_n = 1 + \Delta(\mathrm{p}_n - \mathcal{E}_+) + O(\Delta^2).$$

Indeed, taking the limit $\Delta \to 0$ we obtain eq.(1.27).

Remark 1 Eq.(3.3) is a non-ultralocal generalization of the usual ultralocal formula

$$\{L_n \stackrel{\otimes}{,} L_m\} = \delta_{n,m}[r, L_n \otimes L_m],$$

which are at the basis of the Hamiltonian approach of the Inverse Scattering Method.

Remark 2 Jocobi identity is satisfied due to the Yang-Baxter equation on r and the fact that

$$[r_{12}, H \otimes 1 + 1 \otimes H] = 0, \qquad \forall H \in CSA.$$

Till now we have not imposed any boundary condition. The periodicity of the lattice leads to the identification

$$L_{n+N} = L_n \tag{3.4}$$

and, hence, to define the monodromy matrix S by means of

$$Q_{N+n} = Q_n S.$$

Here $S = \prod_{i=1}^{N} L_i$ and, therefore, it satisfies the Poisson brackets

$$\begin{aligned} \{Q_n \stackrel{\otimes}{,} S\} = {} & Q_n \otimes S \left(-r + Q_n^{-1} \otimes Q_n^{-1} \cdot (r + t_0) \cdot Q_n \otimes Q_n \right. \\ & \left. -1 \otimes S^{-1} \cdot t_0 \cdot 1 \otimes S\right), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \{S \stackrel{\otimes}{,} S\} = {} & S \otimes S \left(-r + S^{-1} \otimes S^{-1} \cdot r \cdot S \otimes S + \right. \\ & \left. + S^{-1} \otimes 1 \cdot t_0 \cdot S \otimes 1 - 1 \otimes S^{-1} \cdot t_0 \cdot S \otimes 1\right). \end{aligned} \tag{3.6}$$

In the computation of the above relations, we have to remember that, due to the periodic boundary condition (3.4) and the non-ultralocality of the Poisson brackets (3.3), it is $\{L_1 \stackrel{\otimes}{,} L_N\} \neq 0$.

Since eqs.(3.2),(3.5),(3.6) coincide exactly with the exchange algebra of ref.[7] in a discretize version, as we intend to obtain with our construction, all the consequent steps to the definition of the Bloch wave basis can now be reproduced. In particular, we need to introduce the lattice analog of the matrix ρ, which must satisfy the Poisson brackets

$$\{Q_n \overset{\otimes}{,} \rho\} = -\alpha Q_n \otimes \rho \cdot t_0 , \tag{3.7}$$

$$\{S \overset{\otimes}{,} \rho\} = -\alpha S \otimes \rho \cdot t_0 - \beta t_0 \cdot S \otimes \rho , \tag{3.8}$$

$$\{\rho \overset{\otimes}{,} \rho\} = 0 . \tag{3.9}$$

In the above expression, α and β are two arbitrary constants such that $\alpha+\beta = 1$. Since the final results expressed in the Bloch basis do not depend on the value of these constants, hereafter we will choose

$$\alpha = 0 .$$

Finally, given any highest weight vector $|\lambda^{(r)}_{max}\rangle$, we can construct the discrete σ basis

$$\sigma^{(r)}_n = \langle \lambda^{(r)}_{max}| Q_n$$

and, then, the ψ basis

$$\psi^{(r)}_n = \sigma^{(r)}_n g \rho$$

so to verify both periodicity and locality of $\psi^{(r)}_n \bar{\psi}^{(r)}_n$.

3.1.1 Conformal algebra and its extensions on the lattice

As an interesting application of the above formalism, we study how the conformal properties of the theory are deformed on the lattice. A fundamental remark concerns the Poisson bracket in eq.(3.3). As can be easily verified by a direct computation, it is still satisfied if we substitute L_n with

$$L'_n = L_n L_{n+1} . \tag{3.10}$$

This feature presents a general meaningful interpretation. Let us consider the increasing map $D : \mathbb{Z} \to \mathbb{Z}$, such that $D(n+N') = D(n) + N \quad (N' \leq N)$. The map

$$Q_n \mapsto Q'_n = Q_{D(n)} , \tag{3.11}$$

Q'_n being defined on a periodic lattice which identifies the sites 0 and N', leaves the Poisson structure invariant (*i.e.*, it is a Poisson map). Therefore, although the discretization procedure violates the conformal invariance, we can regard this Kadanoff type transformation as the lattice version of the conformal one (see ref.[16]). Furthermore, as shown in refs.[15], we can define the lattice equivalent of the generators of the (extended) conformal algebra, having conserved on the lattice the structure of the exchange algebra of the σ basis. If we consider, in order to fix the notation, the Lie algebra sl_p, such a definition can be given by means of the discretize version of the p order ordinary differential

$$\det \begin{pmatrix} \sigma & \sigma^1 & \cdots & \sigma^p \\ \sigma' & (\sigma^1)' & \cdots & (\sigma^p)' \\ \cdots & \cdots & \cdots & \cdots \\ \sigma^{(p)} & (\sigma^1)^{(p)} & \cdots & (\sigma^p)^{(p)} \end{pmatrix} = 0 \,,$$

i.e. the finite difference equation

$$\det \begin{pmatrix} \sigma_n & \sigma^1 & \cdots & \sigma^p \\ \sigma_{n+1} & \sigma^1_{n+1} & \cdots & \sigma^p_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n+p} & \sigma^1_{n+p} & \cdots & \sigma^p_{n+p} \end{pmatrix} = 0 \,. \tag{3.12}$$

Here σ^i $(i = 1, \ldots, p)$ are the elements of the σ basis in the defining representation.

Thus, the generators of the (extended) conformal algebra, that in the continuum are given in terms of σ_i by a wronskian type expression, can be naturally identified on the lattice with the coefficients of eq.(3.12).

The case of sl_2 has been treated in refs.[15]. In these references, the cubic algebra

$$\begin{aligned} \{S_n, S_m\} \;=\; -S_n S_m[(4 \;-\; & S_n - S_m)(\delta_{n,m-1} - \delta_{n,m+1}) + \\ + \; & S_{n-1}\delta_{n,m-2} - S_{m-1}\delta_{n,m-2}] \end{aligned} \tag{3.13}$$

has been found. In the above equation, we introduced the quantity

$$S_n = 4\,\frac{W^{(1)}_{n+1} W^{(1)}_{n-1}}{W^{(2)}_n W^{(2)}_{n-1}} \,, \tag{3.14}$$

where $W^{(p)}_n = \sigma^1_n \sigma^2_{n+p} - \sigma^2_n \sigma^1_{n+p}$. The Virasoro algebra is recovered from eq.(3.13) in the limit $S_n \to 1 + \Delta^2 u(x)$. Some difference can be outlined in the higher rank cases. This becomes evident already for sl_3, computing the Poisson brackets of the lattice generators[1]

$$W_n^{(i)} = \epsilon^{ijkl} \sigma^1_{n+j} \sigma^2_{n+k} \sigma^3_{n+l} \,, \qquad i,j,k,l = 0,1,2,3\,, \tag{3.15}$$

where ϵ is the completely antisymmetric symbol. In the **3** representation we obtain the quadratic algebra

$$\begin{aligned}
\left\{W_n^{(1)}, W_m^{(1)}\right\} &= -\tfrac{1}{3} W_n^{(1)} W_m^{(1)} (\delta_{n\,m-3} - \delta_{n\,m-1} + \delta_{n\,m+1} - \delta_{n\,m+3}) + \\
&\quad + W_n^{(2)} W_{n+2}^{(3)} \delta_{n\,m-1} - W_{n+1}^{(3)} W_{n-1}^{(2)} \delta_{n\,m+1} \\
\left\{W_n^{(1)}, W_m^{(2)}\right\} &= -\tfrac{1}{3} W_n^{(1)} W_m^{(2)} (\delta_{n\,m-3} + \delta_{n\,m-2} - 2\delta_{n\,m-1} + \\
&\quad + \delta_{n\,m} - \delta_{n\,m+1} + \delta_{n\,m+2} - \delta_{n\,m+3}) + \\
&\quad + W_n^{(3)} W_{n+2}^{(3)} \delta_{n\,m-1} - W_{n+1}^{(3)} W_{n-2}^{(3)} \delta_{n\,m+2} \\
\left\{W_n^{(1)}, W_m^{(3)}\right\} &= -\tfrac{1}{3} W_n^{(1)} W_m^{(3)} (\delta_{n\,m-3} + \delta_{n\,m-2} - \delta_{n\,m-1} - \delta_{n\,m+2}) \\
\left\{W_n^{(2)}, W_m^{(2)}\right\} &= -\tfrac{1}{3} W_n^{(2)} W_m^{(2)} (\delta_{n\,m-3} - \delta_{n\,m-1} + \delta_{n\,m+1} - \delta_{n\,m+3}) + \\
&\quad - W_n^{(3)} W_{n+1}^{(1)} \delta_{n\,m-1} + W_n^{(1)} W_{n-1}^{(3)} \delta_{n\,m+1} \\
\left\{W_n^{(2)}, W_m^{(3)}\right\} &= -\tfrac{1}{3} W_n^{(2)} W_m^{(3)} (\delta_{n\,m-3} + \delta_{n\,m} - \delta_{n\,m+1} - \delta_{n\,m+2}) \\
\left\{W_n^{(3)}, W_m^{(3)}\right\} &= -\tfrac{1}{3} W_n^{(3)} W_m^{(3)} (\delta_{n\,m-2} + \delta_{n\,m-1} - \delta_{n\,m+1} - \delta_{n\,m+2})
\end{aligned} \tag{3.16}$$

As revealed by its Poisson algebra and in analogy to what seen in the sl_2 Toda field theory, $W_n^{(3)}$ represents the opposite chirality. However, differently from what found in that case, here it does not seem to be possible to construct rational combinations of the $W_n^{(i)}$, which play the role of discrete generators of the extended Virasoro (W_3) algebra and which are completely decoupled from

[1] We remark that if we construct analogous W_n's in the ψ_n basis, the above algebra does not change. Indeed, as it happens in the continuum case, this is an intrinsic result, concerning the coefficients of the finite difference equation (3.12) by themselves, that is, it does not depend on the particular basis of solutions, which have been chosen to represent them.

the antichiral modes, like the S_n of refs.[15]. This feature indicates that the fundamental algebra is the quadratic one, as that quoted in eq.(3.16).

3.2 Quantum theory on the lattice

The next step is the choice of a quantization map which suitably deforms the classical Poisson relations. In the Introduction we have pointed out the deep link between the algebraic structure underlying the Toda models and their integrability properties; we have also hinted at the corresponding "quantized" structure, the quasitriangular Hopf algebras whose quantum R-matrix gives back, in the "classical" limit ($q = e^{-i\hbar} \to 1$, q being the deformation parameter of the quantum algebra), the original classical r-matrix. Therefore, we are naturally led to a quantization prescription which follows such an algebraic path. This is the basic idea which is implemented in [8], whose results will be summarized in the subsection below.

3.2.1 General formulas

We introduce the notation

$$X_1 = X \otimes I, \qquad X_2 = I \otimes X,$$

for any matrix X. Moreover we will use for the quantum operators the same notation as for their classical counterparts.

According to the above considerations, the quantized version of eq.(3.3 is

$$R_{12} L_{1n} L_{2n} = L_{2n} L_{1n} R_{12}, \tag{3.17}$$

$$L_{1n} L_{2,n+1} = L_{2,n+1} A_{12} L_{1n}, \tag{3.18}$$

where

$$A_{12} = e^{i\hbar t_0},$$

$$R_{12} = 1 - i\hbar r_{12} + O(\hbar^2).$$

R_{12} must satisfy the quantum Yang-Baxter equation. Hereafter we denote with $R^{\pm}_{12}$ the two solutions of the Yang-Baxter equation, whose classical limits are $r^{\pm}_{12}$, respectively. Since we have $R^{-}_{12} = (R^{+}_{12})^{-1}$, we can use indifferently one of these exchange matrices in eq.(3.17).

The crucial property of the commutation relations (3.17) and (3.18) is represented by their integrability. However, one needs previously to define the quantum operator Q_n, whose expression is not just a trivial generalization of its classical analogue. Indeed, we have to slightly modify its expression in terms of L_n with the introduction of the matrix

$$B = e^{\frac{i\hbar}{2}\sum_i H_i^2} .$$

so that

$$Q_n = L_n B L_{n-1} B \dots B L_1 , \tag{3.19}$$

Then, the quantum Q_n satisfies the closed algebra

$$R_{12} Q_{1n} Q_{2n} = Q_{2n} Q_{1n} R_{12} , \tag{3.20}$$

$$Q_{1n} Q^{-1}_{1m} A_{21} R_{12} Q_{1m} Q_{2m} = Q_{2m} Q_{1n} R_{12} , \qquad n > m , \tag{3.21}$$

with $n, m < N$.

In a periodic lattice with N sites we introduce the quantum monodromy matrix via

$$Q_{n+N} = Q_n B S .$$

Therefore, the quantum monodromy matrix S fulfills the commutation relations

$$R_{12} S_1 A_{12} S_2 = S_2 A_{12} S_1 R_{12} , \tag{3.22}$$

$$A_{12} R_{12} Q_{1n} Q_{2n} = Q_{1n} S_1^{-1} Q_{2n} A_{12} S_1 R_{12} . \tag{3.23}$$

Finally, the quantum matrix operator $\rho \in \exp(\mathcal{H})$ has the following properties:

$$\rho_1 \rho_2 = \rho_2 \rho_1 , \tag{3.24}$$

$$A_{12} S_1 \rho_2 A_{12} = \rho_2 S_1 , \tag{3.25}$$

$$Q_{1n} \rho_2 = \rho_2 Q_{1n} , \tag{3.26}$$

where we adopt the same choice as in eqs.(3.7), (3.8 and (3.9).

Let us define now the quantum σ basis

$$\sigma_n^{(r)} = \langle \lambda_{max}^{(r)} | Q_n . \tag{3.27}$$

From eq.(3.21) we obtain the exchange algebra

$$\begin{aligned} \sigma_{1n}^{(r)} \sigma_{2m}^{(r')} &= \sigma_{2m}^{(r')} \sigma_{1n}^{(r)} (R_{12}^{+})^{(r,r')}, \qquad & n > m \\ \sigma_{1n}^{(r)} \sigma_{2m}^{(r')} &= \sigma_{2m}^{(r')} \sigma_{1n}^{(r)} (R_{12}^{-})^{(r,r')}, \qquad & n < m . \end{aligned} \tag{3.28}$$

From eq.(3.20) we can also obtain the exchange algebra for $n = m$

$$q^{\lambda_{max}^{(r)} \cdot \lambda_{max}^{(r')}} \sigma_{1n}^{(r)} \sigma_{2n}^{(r')} = \sigma_{2n}^{(r')} \sigma_{1n}^{(r)} (R_{12}^{+})^{(r,r')} \tag{3.29}$$

3.2.2 Quantum sl_3 Toda field theory on the lattice

The quantum exchange algebra

The purpose of this subsection is to use the general formulas of the previous subsection to define the Block wave basis $\psi_n^{(r)}$ and find the relevant exchange algebra in the sl_3 case.

We start with the quantum R-matrix to be inserted in the above formulas. From the universal R-matrix for $U_q(sl_3)$ as given in refs.[22, 25] (see Appendix A), we obtain

$$(R_{12}^{+})_{ij,kl}^{(\mathbf{3},\mathbf{3})} = (R_{12}^{+})_{ij,kl}^{(\mathbf{3}^*,\mathbf{3}^*)} = \begin{cases} q^{\frac{2}{3}} & i=j=k=l \\ q^{-\frac{1}{3}} & i=k\neq j=l \\ q^{-\frac{1}{3}} x & i=l<j=k \\ 0 & \text{otherwise} \end{cases} \tag{3.30}$$

and

$$(R_{12}^{+})_{ij,kl}^{(\mathbf{3},\mathbf{3}^*)} = (R_{12}^{+})_{ij,kl}^{(\mathbf{3}^*,\mathbf{3})} = \begin{cases} q^{\frac{1}{3}} & i=j, \ \ k=l, \ \ i+j\neq 3 \\ q^{-\frac{2}{3}} & i=k, \ \ j=l, \ \ i+j=3 \\ q^{-\frac{2}{3}} x & i=l=1, \ j=3, \ k=2 \ \text{ and } \ i=j=2, \ k=3, \ l=1 \\ -q^{-\frac{2}{3}} x & i=l=1, \ j=k=3 \\ 0 & \text{otherwise} \end{cases} \tag{3.31}$$

where $q = e^{-i\hbar}$, while, here and hereafter, we denote $x = q - q^{-1}$. Using these equations we can easily derive the exchange algebra in the σ basis, which will not be written down explicitly here.

As a first step in the direction of the ψ basis, we have to diagonalize the upper triangular monodromy matrices S and S^* in the two fundamental representations

$$S = \begin{pmatrix} A & D & F \\ 0 & B & E \\ 0 & 0 & C \end{pmatrix}, \qquad S^* = \begin{pmatrix} A^* & D^* & F^* \\ 0 & B^* & E^* \\ 0 & 0 & C^* \end{pmatrix}.$$

Eqs.(3.22) and (3.23) allow us to compute the commutation relations of the entries of S and S^* among themselves and with the component of the σ_n's. One finds that the diagonal elements $A\ B\ C$ and $A^*\ B^*\ C^*$ commute with everything (except ρ, see below) while, for example,

$$DF = q^{-1}\,FD\,, \qquad EF = q\,FE\,, \qquad ED = q^{-1}DE + x\,BF \tag{3.32}$$

and

$$\begin{aligned} D\sigma_n^1 &= q^{-1}\,\sigma_n^1 D + x\,\sigma_n^2 A & \qquad D\sigma_n^2 &= q\,\sigma_n^2 D \\ E\sigma_n^1 &= \sigma_n^1 & E\sigma_n^2 &= q^{-1}\,\sigma_n^2 E + x\,\sigma_n^3 B \\ F\sigma_n^1 &= q^{-1}\,\sigma_n^1 F + x\,\sigma_n^3 A & F\sigma_n^2 &= \sigma_n^2 F + x\,\sigma_n^3 D \end{aligned}$$

$$\begin{aligned} D\sigma_n^3 &= \sigma_n^3 D \\ E\sigma_n^3 &= q\,\sigma_n^3 E \\ F\sigma_n^3 &= q\,\sigma_n^3 F\,. \end{aligned} \tag{3.33}$$

We will not write down here the remaining relations, except for

$$AC^* = A^*C = BB^*\,. \tag{3.34}$$

This relation explains why we can express the exchange algebras in terms of zero modes A, B, C only (see Appendix B).

Next we diagonalize S and S^* with upper triangular matrices g and g^*, respectively, which have unity entries on the main diagonals. That means that

$$S = g\kappa g^{-1}, \qquad S^* = g^*\kappa^*(g^*)^{-1}\,,$$

κ and κ^* being diagonal matrices, whose main diagonals coincide with the main diagonal of S and S^*, i.e. A, B, C and A^*, B^*, C^*, respectively. It is immediate to compute the commutators of the entries of g and g^* with all the operators introduced so far.

Next we introduce the matrix ρ and the analogous ρ^*.

In a similar way, for the conjugate variables to the zero modes

$$\rho = \begin{pmatrix} \rho_1 & & \\ & \rho_2 & \\ & & \rho_3 \end{pmatrix}, \qquad \rho^* = \begin{pmatrix} \rho_1^* & & \\ & \rho_2^* & \\ & & \rho_3^* \end{pmatrix}. \tag{3.35}$$

Due to eqs.(3.24) and (3.26) and (3.25), ρ_i and ρ_i^* $(i = 1, 2, 3)$ commute among themselves and with all the operators we introduced so far, except for the elements of S and S^*. For these we have

$$S_{ij}\rho_k = \rho_k S_{ij} q^{\frac{6\delta_{i,k}-2}{3}} \qquad S_{ij}\rho_k^* = \rho_k^* S_{ij} q^{\frac{2-6\delta_{i+k,4}}{3}} \tag{3.36}$$

and two more equations which can be obtained by "starring" these two (with the understanding that this formal * operation is involutive).

After introducing the complete set of operators of the theory, we can draw some immediate useful conclusions. First, ABC and $A^*B^*C^*$ belong to the center of the theory. Second, $\rho_2\rho_2^*$, $\rho_1\rho_3^*$ and $\rho_3\rho_1^*$ also commute with everything else. Therefore, we can and will henceforth impose

$$ABC = 1 = A^*B^*C^* \tag{3.37}$$

and

$$\rho_2\rho_2^* = \rho_1\rho_3^* = \rho_3\rho_1^*. \tag{3.38}$$

This will allow us to simplify many formulas. In particular, eq.(3.37) allows us to parametrize the zero modes as follows:

$$\begin{aligned} A &= q^{\frac{N-1}{3}} e^{2\pi(\frac{1}{\sqrt{2}}p_0^1 + \frac{1}{\sqrt{6}}p_0^2)}, & A^* &= q^{\frac{N-1}{3}} e^{2\pi(\sqrt{\frac{2}{3}}p_0^2)}, \\ B &= q^{\frac{N-1}{3}} e^{2\pi(-\frac{1}{\sqrt{2}}p_0^1 + \frac{1}{\sqrt{6}}p_0^2)}, & B^* &= q^{\frac{N-1}{3}} e^{2\pi(\frac{1}{\sqrt{2}}p_0^1 - \frac{1}{\sqrt{6}}p_0^2)}, \\ C &= q^{\frac{N-1}{3}} e^{2\pi(-\sqrt{\frac{2}{3}}p_0^2)}, & C^* &= q^{\frac{N-1}{3}} e^{2\pi(-\frac{1}{\sqrt{2}}p_0^1 - \frac{1}{\sqrt{6}}p_0^2)}, \end{aligned} \tag{3.39}$$

in agreement with the conventions chosen for the Cartan subalgebra (CSA) basis and with our quantization procedure.

What remains to be done is to define the quantum Block wave basis

$$\psi_n = \sigma_n g \rho, \qquad \psi_n^* = \sigma_n^* g^* \rho^* \tag{3.40}$$

and compute its exchange algebra. The calculation is long but uneventful and the result has the form

$$\psi_{1n}\psi_{2m} = \psi_{2m}\psi_{1n}(R_{12}^{\pm}(\mathbf{p}_0))^{(\mathbf{3},\mathbf{3})}, \qquad \begin{cases} + & n > m \\ - & n < m \end{cases} \tag{3.41}$$

$$\psi_{1n}\psi_{2m}^* = \psi_{2m}^*\psi_{1n}(R_{12}^{\pm}(\mathbf{p}_0))^{(\mathbf{3},\mathbf{3}^*)}, \qquad \begin{cases} + & n > m \\ - & n < m \end{cases}, \tag{3.42}$$

where the argument $\mathbf{p}_0$ is to remember the dependence on the zero modes. The entries of the (zero mode dependent) quantum R-matrix[2] in the Bloch wave basis are written down explicitly in Appendix B.

This completes our proof about the relation between the quantum R-matrix of Jimbo and Rosso [22, 25] and the quantum R-matrix in the Bloch wave basis. Such a relation cannot be envisaged as an (operator-valued) similarity transformation since, for example in eq.(3.41), $g_1\rho_1$ does not commute with ψ_{2m}. We can only say that the relation is specified by the operator-valued change of basis (3.40).

In order to discuss periodicity and locality, one has to repeat everything for the discretization of the antichiral half, in order to calculate the exchange algebra of

$$\bar{\psi}_n = \bar{\rho}\bar{g}\bar{\sigma}_n, \qquad \bar{\psi}_n^* = \bar{\rho}^*\bar{g}^*\bar{\sigma}_n^*. \tag{3.43}$$

The result of this calculation can also be found in Appendix B.

Periodicity and locality.

In analogy with the continuum case, we define

$$e^{-\varphi_n} = \psi_n\bar{\psi}_n, \qquad e^{-\varphi_n^*} = \psi_n^*\bar{\psi}_n^*.$$

We find

$$e^{-\varphi_{n+N}} = q^{\frac{4}{3}}\left(A\bar{A}\psi_n^1\bar{\psi}_n^1 + B\bar{B}\psi_n^2\bar{\psi}_n^2 + C\bar{C}\psi_n^3\bar{\psi}_n^3\right)$$

$$e^{-\varphi_{n+N}^*} = q^{\frac{4}{3}}\left(A^*\bar{A}^*\psi_n^{*1}\bar{\psi}_n^{*1} + B^*\bar{B}^*\psi_n^{*2}\bar{\psi}_n^{*2} + C^*\bar{C}^*\psi_n^{*3}\bar{\psi}_n^{*3}\right).$$

[2]Here we enlarge the notion of R-matrix. Indeed the exchange matrices of eqs.(3.41) and (3.42) are not solutions of the YBE's, but of a modified version of them.

Since $A\bar{A}, B\bar{B}, C\bar{C}, A^*\bar{A}^*, B^*\bar{B}^*, C^*\bar{C}^*$ commute with all the operators of the theory, we can project out of the full Hilbert space $\mathcal{H}$ the subspace $\mathcal{H}_0$, where

$$A\bar{A} = B\bar{B} = C\bar{C} = A^*\bar{A}^* = B^*\bar{B}^* = C^*\bar{C}^* = q^{-\frac{4}{3}}. \tag{3.44}$$

In $\mathcal{H}_0$ both $e^{-\varphi_n}$ and $e^{-\varphi_n^*}$ are periodic.

To prove locality, we compute

$$\begin{aligned}
[e^{-\varphi_n}, e^{-\varphi_m}] = & \; x\,\frac{B\bar{B} - A\bar{A}}{(\bar{B}-\bar{A})(B-A)}\left(\psi_n^1\psi_m^2\bar{\psi}_n^2\bar{\psi}_m^1 - \psi_m^1\psi_n^2\bar{\psi}_m^2\bar{\psi}_n^1\right) + \\
& + x\,\frac{C\bar{C} - A\bar{A}}{(\bar{C}-\bar{A})(C-A)}\left(\psi_n^1\psi_m^3\bar{\psi}_n^3\bar{\psi}_m^1 - \psi_m^1\psi_n^3\bar{\psi}_m^3\bar{\psi}_n^1\right) + \\
& + x\,\frac{C\bar{C} - B\bar{B}}{(\bar{C}-\bar{B})(C-B)}\left(\psi_n^2\psi_m^3\bar{\psi}_n^3\bar{\psi}_m^2 - \psi_m^2\psi_n^3\bar{\psi}_m^3\bar{\psi}_n^2\right)
\end{aligned}$$

Therefore, the commutator vanishes in the subspace $\mathcal{H}_0$. The same conclusion holds if we consider $[e^{-\varphi_n^*}, e^{-\varphi_m^*}]$. Next let us consider $[e^{-\varphi_n}, e^{-\varphi_m^*}]$. This commutator is more complicated than the previous ones. However it can be proven to be a combination of terms, each of which factorizes out either $B\bar{B} - A\bar{A}$ or $C\bar{C} - A\bar{A}$ or $C\bar{C} - B\bar{B}$. Therefore, we can conclude again that, in $\mathcal{H}_0$,

$$[e^{-\varphi_n}, e^{-\varphi_m^*}] = 0$$

This completes the derivation of our result as far as the sl_3 Toda field theory is concerned. It is perhaps useful spending a few words to give the reader the coordinates of this result in the prospect of evaluating correlation functions. The lattice analogues of the conformal blocks are given by expressions like

$$\langle\, \theta_\infty | \psi_{n_1}\psi_{n_2}\cdots\psi_{n_k} | \theta_0 \,\rangle$$

and

$$\langle\, \bar{\theta}_\infty | \bar{\psi}_{n_1}\bar{\psi}_{n_2}\cdots\bar{\psi}_{n_k} | \bar{\theta}_0 \,\rangle$$

(we consider here only the **3** representation) where the θ states tend to the corresponding conformal vacua in the continuum limit. Putting together the two halves, we compute

$$\left\langle e^{-\varphi(x_1)} e^{-\varphi(x_2)} \ldots e^{-\varphi(x_k)} \right\rangle, \tag{3.45}$$

where $e^{-\varphi(x)}$ is the continuum limit of $e^{-\varphi_n}$. Single-valuedness and locality of eq.(3.45) is then guaranteed by the condition (3.44).

3.3 Comparison with previous results.

In the previous Section we computed the exchange algebra for the sl_3 Toda field theory in a periodic lattice. Since this algebra does not depend on the lattice spacing, we can immediately translate it into a continuous language by the simple replacements

$$\psi_n^i \to \psi^i(x)\,, \qquad \psi_n^{*i} \to \psi^{*i}(x)\,, \qquad \text{etc.}, \qquad \theta(n-m) \to \theta(x-y)\,.$$

In this way we can compare our results with those of ref.[20] (see also ref.[11] and, for the specific case of sl_3, ref.[21]). There, following a different approach, the sl_{l+1} exchange algebra in the Bloch wave basis for the defining representation was calculated to be

$$\begin{aligned}
\phi_j(\sigma)\phi_j(\sigma') &= e^{-i\hbar\epsilon\, l/l+1}\,\phi_j(\sigma')\phi_j(\sigma)\,,\\
\phi_j(\sigma)\phi_k(\sigma') &= e^{i\hbar\epsilon/l+1}\frac{sin(\hbar(\varpi_{jk}+1)}{sin(\hbar\varpi_{jk})}\,\phi_k(\sigma')\phi_j(\sigma)+ \\
&\quad + e^{i\hbar\epsilon/l+1}\frac{sin(\hbar)e^{-i\hbar\varpi_{jk}}}{sin(\hbar\varpi_{ij})}\,\phi_j(\sigma')\phi_k(\sigma)\,,
\end{aligned} \tag{3.46}$$

where $\epsilon = \text{sign}(\sigma-\sigma')$ and

$$\varpi_{jk} = (\lambda_k - \lambda_j)\cdot\vec{\varpi} \qquad \lambda_i = \text{weights of the defining representation}$$

$$\vec{\varpi} = -\frac{i}{2}\sqrt{\frac{2\pi}{\hbar}}\,\Lambda_0 \quad , \quad \Lambda_0 = \sum_{i=1}^{n}\Lambda_i\tilde{p}_0^i$$

$$\Lambda_i = \text{fundamental weights}\,.$$

Here, the zero modes $\tilde{p}_0^i$ $(i = 1,\ldots,l)$ correspond to the weight space basis, which consists of the simple root system $\{\alpha_i\}_{i=1,\ldots,l}$ (that is to say, to the basis $\{h_i\}_{i=1,\ldots,l}$ of the CSA, $h_i = e_{ii} - e_{i+1\,i+1}$). Considering as an example the case of sl_3, the $\tilde{p}_0^{1,2}$ are related to the zero modes introduced in eqs.(3.39) by the linear transformation

$$\tilde{p}_0^1 = \sqrt{\frac{4\pi}{\hbar}}(\sqrt{2}p_0^1)\,, \qquad \tilde{p}_0^2 = \sqrt{\frac{4\pi}{\hbar}}(-\frac{1}{\sqrt{2}}p_0^1 + \sqrt{\frac{3}{2}}p_0^2)\,. \tag{3.47}$$

Taking into account such a rotation, together with the usual identification $q = e^{-i\hbar}$, we obtain the relations

$$\begin{aligned}
(q-q^{-1})\frac{A}{B-A} &= -\frac{sin(\hbar)}{sin(\hbar\varpi_{12})}e^{-i\hbar\varpi_{12}}\,, & (q-q^{-1})\frac{B}{B-A} &= \frac{sin(\hbar)}{sin(\hbar\varpi_{21})}e^{-i\hbar\varpi_{21}}\,,\\
(q-q^{-1})\frac{B}{C-B} &= -\frac{sin(\hbar)}{sin(\hbar\varpi_{23})}e^{-i\hbar\varpi_{23}}\,, & (q-q^{-1})\frac{C}{C-B} &= \frac{sin(\hbar)}{sin(\hbar\varpi_{32})}e^{-i\hbar\varpi_{32}}\,,\\
(q-q^{-1})\frac{A}{C-A} &= -\frac{sin(\hbar)}{sin(\hbar\varpi_{13})}e^{-i\hbar\varpi_{13}}\,, & (q-q^{-1})\frac{C}{C-A} &= \frac{sin(\hbar)}{sin(\hbar\varpi_{31})}e^{-i\hbar\varpi_{31}}\,.
\end{aligned}$$

This allows us to identify the off-diagonal coefficients of the operator algebra (3.46) with the corresponding elements of the zero mode dependent R-matrices of eq.(3.41) (see Appendix B, §B.1.). As for the diagonal entries, it is important to remark that they coincide only up to a change in the normalization of the Bloch wave basis. Indeed, in order to reproduce the operator algebra of §B.1, the vertex fields ϕ_i, $i = 1, 2, 3$, should be multiplied by the factors

$$\begin{aligned} c_1(\tilde{p}_0) &= [sin(\hbar\varpi_{13})]^a\,[sin(\hbar\varpi_{12})]^{1-a} \;, \\ c_2(\tilde{p}_0) &= [sin(\hbar(\varpi_{12}+1))]^a\,[sin(\hbar\varpi_{23})]^{1-a} \;, \\ c_3(\tilde{p}_0) &= [sin(\hbar(\varpi_{23}+1)]^a\,[sin(\hbar(\varpi_{13}+1))]^{1-a} \;, \end{aligned} \tag{3.48}$$

respectively, where a is an arbitrary parameter. Since the c_i's depend only on the zero modes, this operation does not modify the monodromy behaviour of the fields, which still constitute a Bloch wave basis. After this transformation, while the off-diagonal elements remain unchanged, the diagonal ones take the expressions[3]

$$R^{21}_{12}(\tilde{p}_0) \;\rightarrow\; \tfrac{sin(\hbar\varpi_{12})}{sin[\hbar(\varpi_{12}+1)]}\,R^{21}_{12}(\tilde{p}_0) \;=\; e^{i\hbar\epsilon/3} \;=\; q^{-\frac{1}{3}} \;,$$

$$R^{32}_{23}(\tilde{p}_0) \;\rightarrow\; \tfrac{sin(\hbar\varpi_{23})}{sin[\hbar(\varpi_{23}+1)]}\,R^{32}_{23}(\tilde{p}_0) \;=\; e^{i\hbar\epsilon/3} \;=\; q^{-\frac{1}{3}} \;,$$

$$R^{31}_{13}(\tilde{p}_0) \;\rightarrow\; \tfrac{sin(\hbar\varpi_{13})}{sin[\hbar(\varpi_{13}+1)]}\,R^{31}_{13}(\tilde{p}_0) \;=\; e^{i\hbar\epsilon/3} \;=\; q^{-\frac{1}{3}} \;,$$

$$\begin{aligned} R^{12}_{21}(\tilde{p}_0) \;&\rightarrow\; \tfrac{sin[\hbar(\varphi_{12}+1)]}{sin(\hbar\varpi_{12})}\,R^{12}_{21}(\tilde{p}_0) \;= \\ &= e^{i\hbar\epsilon/3}\left[cos^2(\hbar) - sin^2(\hbar)\tfrac{cos^2(\hbar\varpi_{12})}{sin^2(\hbar\varpi_{12})}\right] \;=\; q^{-\frac{1}{3}}\left[1-(q-q^{-1})^2\tfrac{AB}{(B-A)^2}\right] \;, \end{aligned}$$

$$\begin{aligned} R^{23}_{32}(\tilde{p}_0) \;&\rightarrow\; \tfrac{sin[\hbar(\varphi_{23}+1)]}{sin(\hbar\varpi_{23})}\,R^{23}_{32}(\tilde{p}_0) \;= \\ &= e^{i\hbar\epsilon/3}\left[cos^2(\hbar) - sin^2(\hbar)\tfrac{cos^2(\hbar\varpi_{23})}{sin^2(\hbar\varpi_{23})}\right] \;=\; q^{-\frac{1}{3}}\left[1-(q-q^{-1})^2\tfrac{BC}{(C-B)^2}\right] \;, \end{aligned}$$

$$\begin{aligned} R^{13}_{31}(\tilde{p}_0) \;&\rightarrow\; \tfrac{sin[\hbar(\varphi_{13}+1)]}{sin(\hbar\varpi_{13})}\,R^{13}_{31}(\tilde{p}_0) \;= \\ &= e^{i\hbar\epsilon/3}\left[cos^2(\hbar) - sin^2(\hbar)\tfrac{cos^2(\hbar\varpi_{13})}{sin^2(\hbar\varpi_{13})}\right] \;=\; q^{-\frac{1}{3}}\left[1-(q-q^{-1})^2\tfrac{AC}{(C-A)^2}\right] \;, \end{aligned}$$

while the entries R^{ii}_{ii} do not transform.

[3] Following ref.[21], we use the notation $\phi_i\phi_j = \sum_{kl} R^{kl}_{ij}\phi_k\phi_l$ for the R-matrix elements.

The same can be repeated when, instead of the representations **3** and **3**, we have $\mathbf{3}^*$ and $\mathbf{3}^*$ or **3** and $\mathbf{3}^*$.

In conclusion, we have shown that the R-matrices in the Bloch wave basis of [20, 11, 21] are the same as the ones we exhibit in Appendix B, except for the renormalization pointed out above. We should however bear in mind that only by virtue of such a change of normalization can the locality property of the previous subsection be fulfilled. Furthermore, we remark again that our derivation answers the question opened in such references, about the relation between the R-matrix of Drinfel'd, Jimbo and Rosso and the R-matrix, that appears in the exchange algebra of the Bloch wave basis.

3.4 The sl_p case

It is easy to generalize the above results to the sl_p case, at least as far as the defining representation is concerned.

The quantum R-matrix in the defining representation of sl_p is

$$R_{ij,kl} = \begin{cases} q^{\frac{p-1}{p}} & i=j=k=l \\ q^{-\frac{1}{p}} & i=k,\ j=l,\ i\neq j \\ q^{-\frac{1}{p}}x & i=l<j=k \\ 0 & \text{otherwise}, \end{cases}$$

where $x = q - q^{-1}$, as above (see Appendix A). Let us denote by A_i and $\bar{A}_i$ $(i = 1, ..., p)$ the diagonal elements S_{ii} and $\bar{S}_{ii}$ of the monodromy matrices S and $\bar{S}$, respectively. The exchange algebra in the Bloch wave basis is:

for $n > m$

$$\begin{aligned} \psi^i_n\psi^i_m &= q^{\frac{n-1}{n}}\psi^i_m\psi^i_n\,, \\ \psi^i_n\psi^j_m &= q^{-\frac{1}{n}}\left[\psi^j_m\psi^i_n + x\tfrac{A_i}{A_j-A_i}\psi^i_m\psi^j_n\right]\,, && i<j \\ \psi^i_n\psi^j_m &= q^{-\frac{1}{n}}\left[\left(1-x^2\tfrac{A_iA_j}{(A_i-A_j)^2}\right)\psi^j_m\psi^i_n + x\tfrac{A_i}{A_i-A_j}\psi^i_m\psi^j_n\right]\,; && i>j \end{aligned} \tag{3.49}$$

for $n < m$

$$
\begin{aligned}
\psi_n^i \psi_m^i &= q^{-\frac{n-1}{n}} \psi_m^i \psi_n^i \,, \\
\psi_n^i \psi_m^j &= q^{\frac{1}{n}} \left[\psi_m^j \psi_n^i + x \tfrac{A_j}{A_j - A_i} \psi_m^i \psi_n^j \right] \,, && i < j \\
\psi_n^i \psi_m^j &= q^{\frac{1}{n}} \left[\left(1 - x^2 \tfrac{A_i A_j}{(A_i - A_j)^2}\right) \psi_m^j \psi_n^i + x \tfrac{A_j}{A_i - A_j} \psi_m^i \psi_n^j \right] \,, && i > j \,.
\end{aligned}
\tag{3.50}
$$

Likewise we can write down the exchange algebra for $\bar{\psi}_n$ and construct $\psi_n \bar{\psi}_n$. Periodicity of these objects is guaranteed in the subspace $\mathcal{H}_0$ of the total Hilbert space $\mathcal{H}$, where the conditions

$$
A_i \bar{A}_i = q^{\frac{n+1}{n}} \tag{3.51}
$$

are satisfied. As for locality, we find

$$
[\psi_n \bar{\psi}_m, \psi_n \bar{\psi}_m] = x \sum_{i<j} \frac{A_j \bar{A}_j - A_i \bar{A}_i}{(A_j - A_i)(\bar{A}_j - \bar{A}_I)} \left(\psi_n^i \psi_m^j \bar{\psi}_n^j \bar{\psi}_m^i - \psi_m^i \psi_n^j \bar{\psi}_m^j \bar{\psi}_n^i \right) \,. \tag{3.52}
$$

Hence, also in this general case, the condition (3.51) guarantees locality as well.

Appendix A

The $U_q(sl_n)$ R-matrix in the fundamental representations

A.1 Generalities

The quantum group[1] $U_q(sl_n)$ (the quantum universal enveloping algebra of sl_n) is the algebra on $\mathbb{C}$ generated by e_i, f_i, k_i, k_i^{-1}, with $i = 1, \ldots, n-1$, modulo the relations

$$
\begin{aligned}
k_i k_i^{-1} &= k_i^{-1} k_i = 1\,, & k_i k_j &= k_j k_i\,,\\
k_i e_j k_i^{-1} &= q_i^{a_{ij}/2} e_j\,, & k_i f_j k_i^{-1} &= q_i^{-a_{ij}/2} f_j\,,\\
& [e_i, f_j] = \delta_{ij}\, \frac{k_i^2 - k_i^{-2}}{q_i - q_i^{-1}}\,, &&
\end{aligned}
\tag{A.1}
$$

and the Chevalley relations. Here $q_i = q^{\langle \alpha_i, \alpha_i \rangle /2}$, $\langle \cdot, \cdot \rangle$ being the invariant inner product on the root system $\oplus \mathbb{C}\alpha_i$, with $(\alpha_i, \alpha_i) \in \mathbb{Z}$, q is a free parameter and $a_{ij} = 2\langle a_i, a_j \rangle / \langle a_i, a_i \rangle$ is the Cartan matrix of sl_n.

Remark. These relations are satisfied by the defining representation of sl_n when one identifies

$$
k_i^{\pm} = q^{\pm h_i/2}\,, \qquad e_i = \hat{e}_{i,i+1}\,, \qquad f_i = \hat{e}_{i+1,i}\,. \tag{A.2}
$$

Here we have denoted $(\hat{e}_{ij})_{lm} = \delta_{il}\delta_{jm}$ and $h_i = [\hat{e}_{i\,i+1}, \hat{e}_{i+1\,i}] = \hat{e}_{i\,i} - \hat{e}_{i+1\,i+1}$. Indeed the Chevalley relations, which are homogeneous cubic equations in e_i

[1] About notations and definitions, we refers to [25]

or f_i, are immediately satisfied, while[2]

$$\begin{aligned} k_i^2 - k_i^{-2} &= (q\,\hat{e}_{i,i} + q^{-1}\,\hat{e}_{i+1,i+1}) - (q^{-1}\hat{e}_{i,i} + q\hat{e}_{i+1,i+1}) \\ &= (q - q^{-1})\,h_i \\ &= (q_i - q_i^{-1})\,h_i\,. \end{aligned}$$

Similarly we can show that the fundamental representations of $U_q(sl_n)$ coincide with those of $U(sl_n)$ through the identification (A.2).

Therefore, from the expression of the Universal R-matrix of $U_q(sl_n)$, we can derive those corresponding to the fundamental representations

$$R_{\mathrm{fun}} = q^{t_0}\left[1 + q^{-1}(q-q^{-1})\sum_{\alpha>0}(-q^{-2})^{l(\alpha)-1}x_\alpha \otimes y_\alpha\right]. \tag{A.3}$$

Here we have introduced the following notations:

$$t_0 = \sum_i H_i \otimes H_i\,, \qquad \{H_i\}_{i=1,\dots,n-1} = \text{orthonormal basis of the CSA}\,;$$

$$l(\alpha) = \text{level of the root } \alpha\,;$$

$x,\, y$:

$$x_{\alpha_i} = \hat{e}_{i\,i+1}\,q^{-h_i/2}\,, \qquad y_{\alpha_i} = \hat{e}_{i\,i+1}\,q^{h_i/2}\,, \qquad (\alpha_1 \text{ simple})\,,$$

$$x_\alpha = ad_{x_{\alpha_i}}(x_{\alpha'})\,, \qquad y_\alpha = \tilde{ad}_{y_{\alpha_i}}(y_{\alpha'})\,, \qquad \begin{cases} \alpha = \epsilon_i - \epsilon_j \quad i < j-1 \\ \alpha' = \epsilon_{i+1} - \epsilon_j\,, \end{cases}$$

being $\{\epsilon_i\}_{i=1,\dots,n-1}$ the dual basis of $\{h_i\}_{i=1,\dots,n-1}$, while ad and $\tilde{ad}$ are the adjoint representations

$$\begin{aligned} ad &= (L \otimes R)(id \otimes S)\Delta \\ \tilde{ad} &= (L \otimes R)(id \otimes \tilde{S})\tilde{\Delta}\,, \end{aligned}$$

whit Δ, $\tilde{\Delta} = \tau \circ \Delta$ ($\tau(x \otimes y = y \otimes x)$ the two coproducts and S, $\tilde{S}$ the corresponding antipodes.

[2]As concern the last identification, remember that, if we consider the bilinear form $(\cdot,\cdot)_{\mathrm{fun}} : sl_n \otimes n \to \mathbb{F} \,:\, (X,Y) \mapsto (X,Y)_{\mathrm{fun}} = \mathrm{tr}(XY)$, then $\langle \alpha_i, \alpha_i \rangle = 2$, $\langle \alpha_i, \alpha_j \rangle = -1$.

A.2 Examples

The R-matrix in the defining representation

As for the orthonormal basis of the Cartan subalgebra, introduced to define t_0, we have

$$H_i = \frac{1}{\sqrt{k(k+1)}} \sum_{k=1}^{i} k h_k\,, \qquad i = 1, ..., n-1\,. \tag{A.4}$$

Moreover, in the defining representation of sl_n we find

$$x_{\alpha_i} = \hat{e}_{i,i+1}\left(q^{-1/2}\,\hat{e}_{i,i} + q^{1/2}\,\hat{e}_{i+1,i+1}\right) = q^{1/2}\,\hat{e}_{i,i+1}\,,$$
$$y_{\alpha_i} = \hat{e}_{i+1,i}\left(q^{1/2}\,\hat{e}_{i,i} + q^{-1/2}\,\hat{e}_{i+1,i+1}\right) = q^{1/2}\,\hat{e}_{i+1,i}\,,$$

with α_i, $i = 1, \ldots, n-1$, simple. Then the action of the adjoint representations give us

$$\begin{aligned} x_{\alpha_i+\alpha_{i+1}} &= (L\otimes R)(x_{\alpha_i}\otimes -q^{-h_i}\otimes q^{h_i}\,x_{\alpha_i})(x_{\alpha_{i+1}}) = \\ &= x_{\alpha_i}x_{\alpha_{i+1}} - q^{-h_i}x_{\alpha_{i+1}}q^{h_i}x_{\alpha_i} = \\ &= q\,\hat{e}_{i,i+1}\hat{e}_{i+1,i+2} - q\,\hat{e}_{i+1,i+2}\hat{e}_{i,i+1} = q\,\hat{e}_{i,i+2}\,, \\ x_{\alpha_{i-1}+\alpha_i+\alpha_{i+1}} &= ad_{x_{alpha_i-1}}(x_{\alpha_i+\alpha_{i+1}}) = q^{3/2}\,\hat{e}_{i-1,i+2}\,, \\ &\ \ \cdots \end{aligned}$$

so that generally we obtain

$$x_\alpha = q^{1/2\,l(\alpha)}\,\hat{e}_{k,l}\,, \qquad \alpha = \epsilon_k - \epsilon_l \quad (k<l)\,,$$

where the level $l(\alpha)$ is defined as the difference $l-k$. Analogously, we achieve the result

$$y_\alpha = q^{3/2\,l(\alpha)-1}\,\hat{e}_{l,k}\,, \qquad \alpha = \epsilon_k - \epsilon_l \quad (k<l)\,.$$

Therefore, in terms of the "classical" generators $\hat{e}_{l,k}$, the R-matrix of the quantum group $U_q(sl_n)$ in the defining representation has the expression

$$R_{\text{fun}} = q^{t_0}\left[1 + (q-q^{-1})\sum_{k<l}\hat{e}_{k,l}\otimes\hat{e}_{l,k}\right]\,. \tag{A.5}$$

The $U_q(sl_3)$ R-matrix in the fundamental representations

The Lie algebra sl_3 has two fundamental representations, which we will denote as $\mathbf{3}$, the defining representation, and $\mathbf{3}^*$, its conjugate representation.

$\mathbf{3}$. The generators are represented by

$$\begin{aligned} H_1^{\mathbf{3}} &= \frac{1}{\sqrt{2}} h_1 , \qquad & H_2^{\mathbf{3}} &= \frac{1}{\sqrt{6}}(h_1 + 2h_2) , \\ E_{\alpha_1}^{\mathbf{3}} = \hat{e}_{12} , \qquad E_{\alpha_2}^{\mathbf{3}} &= \hat{e}_{23} , \qquad & E_{\alpha_1+\alpha_2}^{\mathbf{3}} &= \hat{e}_{13} . \end{aligned} \tag{A.6}$$

$\mathbf{3}^*$. The generators are represented by

$$\begin{aligned} H_1^{\mathbf{3}^*} &= \frac{1}{\sqrt{2}} h_2 , \qquad & H_2^{\mathbf{3}^*} &= \frac{1}{\sqrt{6}}(2h_1 + h_2) , \\ E_{\alpha_1}^{\mathbf{3}^*} = \hat{e}_{23} , \qquad E_{\alpha_2}^{\mathbf{3}^*} &= \hat{e}_{12} , \qquad & E_{\alpha_1+\alpha_2}^{\mathbf{3}^*} &= -\hat{e}_{13} . \end{aligned} \tag{A.7}$$

Therefore,

$$R_{12}^{\mathbf{3}\otimes\mathbf{3}} = q^{-1/3} \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix} \tag{A.8}$$

$$R_{12}^{\mathbf{3}\otimes\mathbf{3}^*} = q^{-2/3} \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & x & 0 & -q^{-1}x & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix} \tag{A.9}$$

where $x = (q - q^{-1})$, while the subscript $_{12}$ is to remember that we have used the matrix tensor product $(A \otimes B)_{ij,kl} = A_{ij}B_{kl}$. The R-matrix in the $\mathbf{3}^* \otimes \mathbf{3}^*$ representation coincides with that of eq.(A.8). This is a general feature: the R-matrices corresponding to two conjugate representations coincide.

Appendix B

The exchange algebra in the sl_3 case

B.1 The ψ ψ exchange algebra

In the defining representation we find ($x = q - q^{-1}$):

i) case $n > m$

$$\psi_n^1\psi_m^1 = q^{\frac{2}{3}}\,\psi_m^1\psi_n^1$$

$$\psi_n^1\psi_m^2 = q^{-\frac{1}{3}}\,\psi_m^2\psi_n^1 - q^{-\frac{1}{3}}x\frac{A}{B-A}\,\psi_m^1\psi_n^2$$

$$\psi_n^1\psi_m^3 = q^{-\frac{1}{3}}\,\psi_m^3\psi_n^1 - q^{-\frac{1}{3}}x\frac{A}{C-A}\,\psi_m^1\psi_n^3$$

$$\psi_n^2\psi_m^1 = q^{-\frac{1}{3}}\left[1 - x^2\frac{AB}{(B-A)^2}\right]\psi_m^1\psi_n^2 + q^{-\frac{1}{3}}x\frac{B}{B-A}\,\psi_m^2\psi_n^1$$

$$\psi_n^2\psi_m^2 = q^{\frac{2}{3}}\,\psi_m^2\psi_n^2$$

$$\psi_n^2\psi_m^3 = q^{-\frac{1}{3}}\,\psi_m^3\psi_n^2 - q^{-\frac{1}{3}}x\frac{B}{C-B}\,\psi_m^2\psi_n^3$$

$$\psi_n^3\psi_m^1 = q^{-\frac{1}{3}}\left[1 - x^2\frac{AC}{(C-A)^2}\right]\psi_m^1\psi_n^3 + q^{-\frac{1}{3}}x\frac{C}{C-A}\,\psi_m^3\psi_n^1$$

$$\psi_n^3\psi_m^2 = q^{-\frac{1}{3}}\left[1 - x^2\frac{BC}{(C-B)^2}\right]\psi_m^2\psi_n^3 + q^{-\frac{1}{3}}x\frac{C}{C-B}\psi_m^3\psi_n^2$$

$$\psi_n^3\psi_m^3 = q^{\frac{2}{3}}\psi_m^3\psi_n^3$$

ii) case $n = m$

$$\psi_n^1\psi_n^2 = \frac{B-A}{qB-q^{-1}A}\psi_n^2\psi_n^1$$

$$\psi_n^2\psi_n^3 = \frac{C-B}{qC-q^{-1}B}\psi_n^3\psi_n^2$$

$$\psi_n^1\psi_n^3 = \frac{C-A}{qC-q^{-1}A}\psi_n^3\psi_n^1$$

iii) case $n < m$

$$\psi_n^1\psi_m^1 = q^{-\frac{2}{3}}\psi_m^1\psi_n^1$$

$$\psi_n^1\psi_m^2 = q^{\frac{1}{3}}\psi_m^2\psi_n^1 - q^{\frac{1}{3}}x\frac{B}{B-A}\psi_m^1\psi_n^2$$

$$\psi_n^1\psi_m^3 = q^{\frac{1}{3}}\psi_m^3\psi_n^1 - q^{\frac{1}{3}}x\frac{C}{C-A}\psi_m^1\psi_n^3$$

$$\psi_n^2\psi_m^1 = q^{\frac{1}{3}}\left[1 - x^2\frac{AB}{(B-A)^2}\right]\psi_m^1\psi_n^2 + q^{\frac{1}{3}}x\frac{A}{B-A}\psi_m^2\psi_n^1$$

$$\psi_n^2\psi_m^2 = q^{-\frac{2}{3}}\psi_m^2\psi_n^2$$

$$\psi_n^2\psi_m^3 = q^{\frac{1}{3}}\psi_m^3\psi_n^2 - q^{\frac{1}{3}}x\frac{C}{C-B}\psi_m^2\psi_n^3$$

$$\psi_n^3\psi_m^1 = q^{\frac{1}{3}}\left[1 - x^2\frac{AC}{(C-A)^2}\right]\psi_m^1\psi_n^3 + q^{\frac{1}{3}}x\frac{A}{C-A}\psi_m^3\psi_n^1$$

$$\psi_n^3\psi_m^2 = q^{\frac{1}{3}}\left[1 - x^2\frac{BC}{(C-B)^2}\right]\psi_m^2\psi_n^3 + q^{\frac{1}{3}}x\frac{B}{C-B}\psi_m^3\psi_n^2$$

$$\psi_n^3\psi_m^3 = q^{-\frac{2}{3}}\psi_m^3\psi_n^3$$

The exchange algebra $\psi^*\ \psi^*$ can be obtained from this by simply starring it, *i.e.* by replacing everywhere ψ, A, B, C by ψ^*, A^*, B^*, C^*, respectively.

The antichiral algebra $\bar\psi\ \bar\psi$ can be obtained from the above following the recipe: in order to get the exchange relation of $\bar\psi^i_n\ \bar\psi^j_m$, take the exchange relation of $\psi^j_m\ \psi^i_n$ and bar everything including A, B and C. For example, for $n > m$, we get

$$\bar\psi^1_n\bar\psi^2_m = q^{\frac{1}{3}}\left[1 - x^2\frac{\bar A\bar B}{(\bar B-\bar A)^2}\right]\bar\psi^2_m\bar\psi^1_n + q^{\frac{1}{3}}x\frac{\bar A}{\bar B-\bar A}\bar\psi^1_m\bar\psi^2_n\,.$$

Of course the algebra $\bar\psi^*\ \bar\psi^*$ is obtained by simply "starring" the algebra $\bar\psi\ \bar\psi$.

B.2 The $\psi\ \psi^*$ exchange algebra

This algebra is given by:

i) case $n > m$

$$\psi^1_n\psi^{*1}_m = q^{\frac{1}{3}}\,\psi^{*1}_m\psi^1_n$$

$$\psi^1_n\psi^{*2}_m = q^{\frac{1}{3}}\,\psi^{*2}_m\psi^1_n$$

$$\begin{aligned}\psi^1_n\psi^{*3}_m &= q^{-\frac{2}{3}}\,\psi^{*3}_m\psi^1_n - q^{-\frac{2}{3}}x\tfrac{A}{B-A}\,\psi^{*2}_m\psi^2_n + \\ &\quad + q^{-\frac{2}{3}}x\tfrac{A}{C-A}\,\tfrac{qC-q^{-1}B}{C-B}\,\psi^{*1}_m\psi^3_n\end{aligned}$$

$$\psi^2_n\psi^{*1}_m = q^{\frac{1}{3}}\psi^{*1}_m\psi^2_n$$

$$\begin{aligned}\psi^2_n\psi^{*2}_m &= q^{-\frac{2}{3}}x\tfrac{B}{B-A}\,\psi^{*3}_m\psi^1_n + q^{-\frac{2}{3}}\left[1 - x^2\tfrac{AB}{(B-A)^2}\right]\psi^{*2}_m\psi^2_n + \\ &\quad - q^{-\frac{2}{3}}x\tfrac{B}{C-B}\,\tfrac{q^{-1}B-qA}{B-A}\,\tfrac{qC-q^{-1}A}{C-A}\,\psi^{*1}_m\psi^3_n\end{aligned}$$

$$\psi^2_n\psi^{*3}_m = q^{\frac{1}{3}}\,\psi^{*3}_m\psi^2_n$$

$$\begin{aligned}\psi^3_n\psi^{*1}_m &= -q^{-\frac{2}{3}}x\tfrac{C}{C-A}\,\tfrac{q^{-1}C-qB}{C-B}\,\psi^{*3}_m\psi^1_n + \\ &\quad + q^{-\frac{2}{3}}x\tfrac{C}{C-B}\,\tfrac{qB-q^{-1}A}{B-A}\,\tfrac{q^{-1}C-qA}{C-A}\,\psi^{*2}_m\psi^2_n + \\ &\quad + q^{-\frac{2}{3}}\left[1 - x^2\tfrac{AC}{(C-A)^2}\right]\left[1 - x^2\tfrac{BC}{(C-B)^2}\right]\psi^{*1}_m\psi^3_n\end{aligned}$$

$$\psi_n^3 \psi_m^{*2} = q^{\frac{1}{3}} \psi_m^{*2} \psi_n^3$$

$$\psi_n^3 \psi_m^{*3} = q^{\frac{1}{3}} \psi_m^{*3} \psi_n^3$$

iii) case $n < m$

$$\psi_n^1 \psi_m^{*1} = q^{-\frac{1}{3}} \psi_m^{*1} \psi_n^1$$

$$\psi_n^1 \psi_m^{*2} = q^{-\frac{1}{3}} \psi_m^{*2} \psi_n^1$$

$$\begin{aligned} \psi_n^1 \psi_m^{*3} &= q^{\frac{2}{3}} \psi_m^{*3} \psi_n^1 - q^{\frac{2}{3}} x \frac{B}{B-A} \psi_m^{*2} \psi_n^2 + \\ &\quad + q^{\frac{2}{3}} x \frac{C}{C-A} \frac{qC - q^{-1}B}{C-B} \psi_m^{*1} \psi_n^3 \end{aligned}$$

$$\psi_n^2 \psi_m^{*1} = q^{-\frac{1}{3}} \psi_m^{*1} \psi_n^2$$

$$\begin{aligned} \psi_n^2 \psi_m^{*2} &= q^{\frac{2}{3}} x \frac{A}{B-A} \psi_m^{*3} \psi_n^1 + q^{\frac{2}{3}} \left[1 - x^2 \frac{AB}{(B-A)^2}\right] \psi_m^{*2} \psi_n^2 + \\ &\quad - q^{\frac{2}{3}} x \frac{C}{C-B} \frac{q^{-1}B - qA}{B-A} \frac{qC - q^{-1}A}{C-A} \psi_m^{*1} \psi_n^3 \end{aligned}$$

$$\psi_n^2 \psi_m^{*3} = q^{-\frac{1}{3}} \psi_m^{*3} \psi_n^2$$

$$\begin{aligned} \psi_n^3 \psi_m^{*1} &= -q^{\frac{2}{3}} x \frac{A}{C-A} \frac{q^{-1}C - qB}{C-B} \psi_m^{*3} \psi_n^1 + \\ &\quad + q^{\frac{2}{3}} x \frac{B}{C-B} \frac{qB - q^{-1}A}{B-A} \frac{q^{-1}C - qA}{C-A} \psi_m^{*2} \psi_n^2 + \\ &\quad + q^{\frac{2}{3}} \left[1 - x^2 \frac{AC}{(C-A)^2}\right] \left[1 - x^2 \frac{CB}{(C-B)^2}\right] \psi_m^{*1} \psi_n^3 \end{aligned}$$

$$\psi_n^3 \psi_m^{*2} = q^{-\frac{1}{3}} \psi_m^{*2} \psi_n^3$$

$$\psi_n^3 \psi_m^{*3} = q^{-\frac{1}{3}} \psi_m^{*3} \psi_n^3$$

The exchange algebra for $n = m$ is the same as for $n > m$, with the *rhs* multiplied by $q^{-\frac{1}{3}}$.

By "starring" this algebra we obtain the algebra $\psi^* \ \psi$.

The corresponding antichiral algebra $\bar{\psi} \ \bar{\psi}^*$ can be obtained according to the recipe: to get $\bar{\psi}_n^i\ \bar{\psi}_m^{*j}$, take $\psi_m^{*j}\ \psi_n^i$ and bar everything including A, B and C.

From this, one can write down the algebra $\bar{\psi}^*\ \bar{\psi}$.

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