RENORMALIZATION AND CRITICAL PHENOMENA IN THE PARISI AND WU STOCHASTIC QUANTIZATION SCHEME.

Ph. D. Thesis of

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PART I.

INTRODUCTION.

Since the first attempts of giving a mathematical description to the Physics of the Elementary Particles, the physicists had to face the problem of quantizing a classical field, as a consequence of the indetermination principle and of the locality dictated by the theory of the Relativity.

The first quantization scheme devised (the Canonical Quantization [1]) is based on a generalization of the ordinary Quantum Mechanics to systems with infinite degrees of freedom, and the fields are represented by operators satisfying Canonical Commutation Relations [2].

The second scheme of quantization proposed, relies on the Feynman's integral over the paths [3]. In this approach one deals only with classical fields and the transition amplitudes are obtained computing a sum over all the field configurations, each one weighed by the exponential of i times the Classical Action. The Path Integral (P.I.) formalism is a very useful tool for both the perturbative and non-perturbative insights of the

Quantum Field Theory [4][5]. Moreover, after a Wick rotation, a remarkable link between Euclidean Quantum Field Theory and Statistical Physics is apparent: the former can be considered as described by the partition function of a statistical system at the equilibrium, $\hbar$ being the temperature while the Action represents the energy.

In this perspective, a third method of quantization has been proposed: the Stochastic Quantization [6]. Here the field has to be considered as a stochastic process satisfying a stochastic differential equation [7]. The random system described by this field evolves in a new, fictitious time direction, and, prepared at the initial time in some configuration, it reaches an equilibrium situation in the infinite time limit. For a large class of diffusion equations, it is possible to show the equivalence of this procedure with the P.I. approach [8], provided that a suitable choice of the drift force has been done.

The existence of a new quantization scheme is not only interesting by itself, but can help for a deeper understanding of the properties of the Quantum Systems and it can provide new computational tools. There are some advantages in the Stochastic Quantization Scheme (S.Q.S.) with respect to the P.I. formalism. For example it partially avoids the intriguing definition of the sum

over the configurations, needed in the P.I.. It provides new, probably non-perturbative, regularization schemes [9][10][11]. It is very easy the derivation of the rules for perturbative expansions [12]. There is a very interesting relation between the S.Q. and supersymmetric models [13] and the Nicolai mapping [14]. Gauge fields can be quantized without the need of a gauge fixing [6]. New numerical computations can be based on this method [15] in alternative with respect to the usual Monte Carlo simulations.

On the other hand there are , of course, some disadvantages with respect to the P.I.. For instance, the stochastic graphs arising in the perturbative expansions are more numerous and much more difficult to evaluate. There are some ambiguities in the mathematical formalism of the stochastic processes (Ito's calculus versus Stratonovitch one [7]), strictly related to the ambiguities in the definition of the measure for the P.I. [16]. Some features of the P.I. having relevant physical interpretations like the vacuum-vacuum transition amplitude and the effective potential are not here easy to be identified. This list could be much longer, but I

trust in the fact that a lot of unclear features of the Stochastic Quantization will be clarified in a future. In this class of open problems falls the problem of the

anomalies, of the gauge invariance of the renormalized stochastic theory, of the relevance of topological non trivial configurations like solitons and instantons, of the possibility that the stochastic regularization preserves supersymmetry.

In this thesis, I will focus my attention on the problem of the renormalization in the stochastic quantization scheme.

In Part II. I give a mathematical introduction to the stochastic processes (II.1) and to the diffusion equations (II.2). The main purpouse of this introduction is to make clear and natural the idea underlying the stochastic quantization. The stochastic quantization is described in detail in the Sect.III.3 for both the perturbative and the non perturbative aspects. In Sect.III.4 a formal discussion of the emergence of the Ward identities in the stochastic quantization scheme is reported.

In Part III. the problem of the renormalization is analyzed. In particular I discuss the consistence of the stochastic regularization (III.1) and the possibility of renormalizing the stochastic theory also in presence of some peculiar features of the stochastic regularization (III.2). To this end we make a non trivial use of a hidden BRS symmerty. In Sect.III.3 a first example of renormalization procedure is given. Moreover the

background field method for the stochastic quantization scheme is introduced. In Sect. III.4 a detailed analysis of the renormalization group equations and the interplay between stochastic quantization and the theory of (dynamical) critical phenomena is reported.

In Part IV. I show the possibility of using the new methods produced by the stochastic quantization to get new numerical values for the critical exponents of $\lambda\phi^4$ in d=3. In Sect.IV.1 I remind the main features of the ε-expansion and set the problem. In Sect.IV.2 I explain the idea on which the computations are based, while in Sect.IV.3 the computations and the results are reported.

PART II

II 1. BRIEF REVIEW OF THE STOCHASTIC PROCESSES.

I am going here to give some informations concerning the theory of the random processes needed for a comlpete understanding of the ideas related to the Stochastic Quantization. For this reason, I will not follow a sophisticated mathematical approach, but rather the approach of refs. [7][17], that is to extract from the mathematical theory only the relevant points for the physical purpouses and to use a rather simplified language.

A random variable X is defined as a function from the set of the results of some experiment to some set I on the real axis, characterized by a probability distribution P(x), such that:

$$\text{a)} \quad P(x) \geq 0 \qquad \forall \; x \in I$$

$$\text{b)} \quad \int_I P(x)\, dx = 1 \qquad (II.1.1)$$

The probability of finding the random variable X in the interval (x, x+dx) is given by

$$p(x, x+dx) = P(x)\, dx \qquad (II.1.2)$$

The knowledge of P(x) allows the computation of the moments, i.e. the expectation values of x^k :

$$m_k = E(x^k) = \int_I x^k P(x)\, dx \qquad (II.1.3)$$

This gives the set of previsions we are able to do about the random variable.

It is straightforward to generalize this to the case of a n-dimensional random variable $\underline{X} = (X_1, X_2, \ldots , X_n)$: we introduce the joint probability distribution $P(x_1, x_2, \ldots , x_n)$, non-negative and normalized on the volume

$$V = \prod_{i=1, \ldots, n} I_i$$

A stochastic process is an ∞ - dimensional random variable: X=X(t) and t is a continuous real index, often called the time. Let X_i the random variables at time $t=t_i$ and x_i the corresponding values. A complete description of the stochastic process is given if we know the joint probability distribution $P^{(n)}(x_1 t_1 ; x_2 t_2 ; \ldots ; x_n t_n)$ for any n and, for semplicity, we consider $t_1 \geq t_2 \geq \ldots \geq t_n$.

$$P^{(n)}(x_1t_1 ; x_2t_2 ;...; x_nt_n) dx_1...dx_n \qquad (II.1.4)$$

gives the probability that X(t) takes the value in the interval $(x_1 ; x_1+dx_1)$ at time t_1, $(x_2 ; x_2+dx_2)$ at time t_2 and so on. Of course $P^{(n)}$ must satisfy:

$$P^{(n)}(1...n)\geq 0 \quad \text{positivity}$$

$$P^{(n)}(1...n) \quad \text{symmetric}$$

$$\int dx_{n+1} ... dx_{n+k} P^{(n+k)}(1...n ; n+1 ... n+k)=P^{(n)}(1 ...n)$$

compatibility (II.1.5)

We can define the conditional probability densities:

$$R^{(n|m)}(1...n|n+1...n+m)=P^{(n+m)}(1...n+m)/P^{(m)}(n+1...n+m) \qquad (II.1.6)$$

giving the probability of having X(t) in $(x_i ; x_i+dx_i)$ at t_i for $1\leq i\leq n$ if X(t) was in the interval written above for every $n+1\leq i\leq n+m$. It immediately follows from the definition that:

$$\int dx_1...dx_n R^{(n|m)}(1...n|n+1...n+m)=1 \qquad (II.1.7)$$

$$\int dx_k \, R^{(n|m)}(1...k...n|n+1...n+m)=R^{(n-1|m)}(1...\mathbb{k}...n|...n+m) \qquad (II.1.8)$$

$$\int dx_k \, R^{(n|m)}(1...n|n+1...k...n+m)R^{(1|m-1)}(k|n+1...\mathbb{k}...n+m)$$

$$=R(n|m-1)(1...n|n+1...\mathbb{k}...n+m) \qquad (II.1.9)$$

where we have indicated with $\mathbb{k}$ the lack of the k-th variable.

It is also possible to define a measure for the process:

$$d\mu[x(t)]=\mu[x(t)] \; d[x(t)] \qquad (II.1.10)$$

giving the probability of having a trajectory in a "tube" around x(t). If C is the set of the trajectories passing through $(x_i ; t_i)$ i=1...n, we have:

$$\int_C d\mu[x(t)]=1$$

$$\int_{C(1...n)} d\mu[x(t)]=P^{(n)}(1...n) \qquad (II.1.11)$$

We will be mainly interested in the so called Markov processes and in understanding what happens when the Markov property does not hold any more. This property

can be expressed by the fact that:

$$R^{(n|m)}(1...n|n+1...n+m)=R^{(n|1)}(1...n|n+1) \qquad (II.1.12)$$

for $t_1>t_2>...>t_n>t_{n+1}>...>t_{n+m}$. As a consequence of this property, we get

$$R^{(n|1)}(1...n|n+1)=R(1|2)R(2|3)...R(n|n+1) \qquad (II.1.13)$$

where we have written $R^{(1|1)}$, the transition probability, dropping the indices. A Markov process is completely specified once we know the probability density P(x; t) and the transition probability R(xt|x't'), since any joint probability can be obtained:

$$P^{(n)}(1...n)=R(1|2)R(2|3)...R(n-1|n)P(n) \qquad (II.1.14)$$

and we can compute the probability distribution when we know it at an earlier time:

$$P(xt)=\int dy\ R(xt|y\tau)P(y\ \tau) \quad \text{for } t>\tau \qquad (II.1.15)$$

If the stochastic process is continuous,

$$\lim_{t\to\tau} R(xt|y\tau) = \delta(x-y) \qquad \text{(II.1.16)}$$

Applying the Markov property on eq.(II.1.9) for n=1, m=2, we get the so called Chapman -Kolmogorov Equation:

$$R(xt|y\tau)=\int dz\ R(xt|zt')R(zt'|y\tau) \qquad \text{(II.1.17)}$$

This is the fundamental equation at which the transition probability of a Markov process must obey. A Markov process with R depending only from the difference of the times is a time homogeneous process, if moreover P(x, t+t')=P(x,t), the process is stationary.

As a useful example to our purpouses we remind the "random walk" : a discrete random variable $X_t=n$ moving in a discretized time variable $t= s\tau$ with $n\in Z$ and $s\in Z^+$.If the probability of increasing n is p for one time step and the probability of decreasing is q, the Markov process is specified by

$$P(n;0)=\delta_{n;0}$$

$$R(n\ ,\ s+1|\ m,\ s\ )=p\delta_{n,m+1}+q\delta_{n,m-1} \qquad \text{(II.1.18)}$$

with p+q=1. The process is time homogeneous but not stationary, and:

$$P(n,s)=s!\{\ [(n+s)/2]!\ [(s-n)/2]!\ \}^{-1}\ q^{(s-n)/2}\ p^{(n+s)/2} \qquad (II.1.19)$$

The continuous limit of this process is known as Wiener process. With the initial condition $P(x;0)=\delta(x)$, the process is specified by:

$$P(x;t)=(2\pi\sigma t)^{-1/2}\ \exp\{\ -x^2/(2\sigma t)\}$$

$$R(x_2t_2|x_1t_1)=$$

$$=(2\pi\sigma[t_2-t_1])^{-1/2}\exp\{-(x_2-x_1)^2/(2\sigma[t_2-t_1])\} \qquad (II.1.20)$$

The joint probability is simply:

$$P^{(n)}(x_1t_1...x_nt_n)=\prod_{i=1...n}(2\pi\sigma[t_i-t_{i-1}])^{-1/2}$$

$$\exp\{-(x_i-x_{i-1})^2/(2\sigma[t_i-t_{i-1}])\} \qquad (II.1.21)$$

with $t_0=x_0=0$. The process is time homogeneous and Gaussian with covariance matrix

$$S_{ij} = < (x(t_i)-E[x(t_i)])\ (x(t_j)-E[x(t_j)]) >= \sigma(t_i-t_{i-1})\ \delta_{ij}$$

(II.1.22)

The Wiener process is of central importance in the theory of the stochastic processes since we can define a measure for the continuous trajectories: writing $t_i-t_{i-1}=\tau$, and keeping $T=N\tau$ fixed when $N\to\infty$, we have

$$d\mu[x(t)]=\lim\ N\to\infty\ P(x_1 t_1 \ldots x_N t_N)=$$

$$=1/N\ \ [dx(t)]\ \exp\{-\int_0^T dt/2\sigma\ \ \dot{x}(t)^2\ \} \qquad \text{(II.1.23)}$$

It is easy to obtain the expectation values:

$$E\{x(t)\}=0$$

$$E\{x(t)x(t')\}=\sigma\ \min\ (t,t') \qquad \text{(II.1.24)}$$

It is interesting to notice that the measure for the process

$$\eta(t)=dx(t)/dt \qquad \text{(II.1.25)}$$

is a Gaussian measure with a δ-function for kernel. $\eta(t)$ is a Gaussian, δ-correlated (or white) stochastic

Markovian process:

$$E\{\eta(t)\}=0$$

$$E\{\eta(t)\eta(t')\}=\sigma\,\delta(t-t') \qquad \text{(II.1.26)}$$

The Gaussian white noise $\eta(t)$ and the corresponding measure will be the building blocks of the stochastic quantization. However, from this summary about stochastic processes, it is clear that the Gaussian noise is not a mathematically well defined object. In fact it is, formally, the derivative of a Wiener process, regardless to the fact that actually a Wiener process is a continuous, but nowhere differentiable mapping. In this sense we must keep in mind that, although we will always deal with η, if we meet some troubles or want to give rigorous proofs, we must go to the well defined Wiener process.

II.2. FOKKER-PLANCK AND LANGEVIN EQUATIONS.

For a Markov process, because of the Chapman-Kolmogorov equation (eq.(II.1.17)), we can relate the probabilities at different times, once we give the initial conditions:

$$\lim_{t\to to} P(x,t) = P(x,t_o)$$

$$\lim_{t\to to} R(x,t;\, x_o,t_o) = \delta(x-x_o) \qquad \text{(II.2.1)}$$

The transition probability can be written in a short notation:

$$R(x,t|\, y,\tau) = \langle x|R\,(t|\tau)|y\rangle \qquad \text{(II.2.2)}$$

and eq.(II.1.17), for t=t'+dt' becomes:

$$R\,(t'+dt'\,|\tau) = R\,(t'+dt'\,|t')\;R\,(t'\,|\tau) \qquad \text{(II.2.3)}$$

Defining:

$$R\,(t'+dt'\,|t') = \mathbb{1} + \Gamma\,(t')dt' \qquad \text{(II.2.4)}$$

we get the Master Equation:

$$\frac{\partial}{\partial t} R\,(t|\tau)=\Gamma(t)R\,(t|\tau) \qquad \text{(II.2.5)}$$

The function $\Gamma(x,y;\,t)= \langle x|\,\Gamma(t)|y\rangle$, describes the short time properties of the Markov process. From eq.(II.1.7) we see that:

$$\int dx\;\Gamma\;(x,y;\,t)=0 \qquad \text{(II.2.6)}$$

From eq.(II.2.4), we see that for $x\neq y$ Γ is nothing but the transition probability per unit time of going from y to x: w(x,y;t). Imposing eq.(II.2.6), we get

$$\Gamma(x,y;t)= w(x,y;t) - \int dx'\; w(x',y;t)\;\delta(x-y) \qquad \text{(II.2.7)}$$

Using now eq.(II.1.15), we can write the Master equation in the Gain-Loss form:

$$\frac{\partial P(x,t)}{\partial t} = \int dy\;[\; w(x,y;t)P(y,t) - w(y,x;t)P(x,t)] \qquad \text{(II.2.8)}$$

From the definition of homogeneous process, it is easy to get that, in that case, Γ and w are independent from t. Note that if the process is stationary, i.e. $\partial_t P=0$, from eq.(II.2.8), we get the detailed balance relation:

$$w(x,y)P(y)=w(y,x)P(x) \qquad \text{(II.2.9)}$$

If the stochastic process is such that only the first two moments

$$a_n(x)=\int dx' \ (x'-x)^n \ w(x',x) \qquad \text{(II.2.10)}$$

are different from zero, the process is called a diffusion process and the Master equation reduces to the Fokker-Planck (F.P.) equation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[a_1(x)P(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[a_2(x)P(x,t)\right] \qquad \text{(II.2.11)}$$

or, alternatively, on the transition probabilities, we have the forward Fokker-Planck:

$$\partial_t \ R(x,t|x_0,t_0)=\left[-\partial_x \ a_1(x) + \partial_x{}^2 \ a_2(x)/2 \ \right] \ R(x,t|xo,to) \qquad \text{(II.2.12)}$$

and the backward F.P. :

$$\partial_{t_0} R(x,t|x_0,t_0) = [+ a_1(x) \partial_x + a_2(x)/2 \partial_x^2] R(x,t|x_0,t_0) \qquad (II.2.13)$$

Another way of studying the time evolution of a stochastic process is through an equation directly written for the stochastic process itself. The most important of these stochastic equations is the Langevin equation [18], introduced to describe the Brownian motion [19]:

$$\frac{dx(t)}{dt} = K[x(t)] + \eta(t) \qquad (II.2.14)$$

where K[x] is the drift force and η a Gaussian noise. This equation is equivalent to a F.P. equation. In fact, the average of an arbitrary function of x can be written as:

$$\langle f[x(t)] \rangle = \int d\eta \exp\{ -\int \eta^2/2\sigma \} f[x(t)] =$$

$$= \int dx\, f[x]\, P(x,t) \qquad (II.2.15)$$

Taking the t derivative, we get:

$$\int dx\ f(x)\ \partial_t\ P(x,t) = \int d\eta\ \exp\{ -\int \eta^2/2\sigma \}\ \partial_x f\ \{K[x]+\eta\} \qquad (II.2.16)$$

Performing an integration by parts, we obtain:

$$\int d\eta\ \exp\{-\int\eta^2/2\sigma \}\ \frac{df}{dx}\ \eta =$$

$$= \sigma \int d\eta\ \exp\{-\int\eta^2/2\sigma \}\ \frac{d^2f}{dx^2}\ \frac{dx(t)}{d\eta(t)} =$$

$$= \frac{\sigma}{2} \int d\eta\ \exp\{-\int\eta^2/2\sigma \}\ \frac{d^2f}{dx^2} \qquad (II.2.17)$$

Eq.(II.2.16) becomes:

$$\int dx\ f[x]\ \partial_t\ P(x,t) = < \partial_x f\ K[x] + \sigma/2\ \partial_x{}^2 f >=$$

$$= \int dx\ f[x]\ \{ - \partial_x\ K[x] + \sigma/2\ \partial_x{}^2\ ]\ P(x,t) \qquad (II.2.18)$$

giving a F.P. equation for P(x,t) with:

$$a_1(x)= K[x] \qquad a_2(x)=\sigma/2 \qquad (II.2.19)$$

It has to be noted that, when the process becomes

stationary, the F.P. equation reduces to:

$$\partial_X \{ K[x] - \sigma/2 \, \partial_X \} \, P(x)=0 \qquad (II.2.20)$$

A solution of this equation is:

$$P_{eq}(x) = \exp\{ 2/\sigma \int^X K[x'] \, dx' \} \qquad (II.2.21)$$

if this P_{eq} is normalizable. More in general, the F.P. equation can be set in a Schrodinger like equation by the following definition for ψ:

$$P(x,t)= \exp\{ \int K/\sigma \} \, \psi(x;t) \qquad (II.2.22)$$

The equation for ψ is:

$$\partial_t \psi = -[- \sqrt{\sigma/2} \, \partial_X - K/\sqrt{2\sigma} \,][- \sqrt{\sigma/2} \, \partial_X + K/\sqrt{2\sigma} \,] \, \psi \qquad (II.2.23)$$

Expanding in eigenfunctions of the " Hamiltonian ", we see that $\psi(t)$ converges (if in the spectrum a mass gap exists) to the eigenfunction with lowest eigenvalue,

given by:

$$\psi = \exp \{ \int^x K/\sigma \} \qquad \text{(II.2.24)}$$

In any case we can avoid the language of the F.P. equation and work directly on the solutions of the Langevin equation (II.2.14), or better, on its differential, mathematically meaningful, version :

$$dx(t) = K[x(t)]\, dt + dW(t) \qquad \text{(II.2.25)}$$

where dW(t) is a Wiener process. If one tries to write the solution of this equation in the form:

$$x(t)=x(t_o) + \int_{to}^{t} K[x(t')]\, dt' + \int_{to}^{t} dW \qquad \text{(II.2.26)}$$

has to face with a fundamental problem concerning stochastic differential equations: the usual Riemann-Stieljtes integral is not well defined on stochastic processes. In fact, for an integral of the kind $\int_{to}^{t} G(W)\, dW$, the partial sums

$$S_n = \Sigma_{i=1\ldots n}\, G(\tau_i)\, [W(t_i)-W(t_{i-1})] \qquad \text{(II.2.27)}$$

with $t_0 \leq t_1 \leq \ldots \leq t_n = t$ and $\tau_i \in (t_{i-1} ; t_i)$, are not defined and depend on the choice of the intermediate points τ_i, as a consequence of the unbound variation of the Wiener process. To give a meaning to S_n we must give a prescription for the choice of τ_i:

$$\tau_i = t_{i-1} (1-a) + a t_i \qquad \text{(II.2.28)}$$

The choice a=0 defines the "Ito calculus": the sums are evaluated using the value of the Wiener process at the beginning of the time intervals. This is a natural choice, intuitive from the physical point of view, and advantagous from the mathematical point of view (rigorous theorems can be proved in this framework). Neverthless in this calculus the rules of the usual calculus don't hold any more. A useful rule to remember is that , when expanding at first order, we must take on the same footing dt and $(dW)^2 = dt$, giving some unexpected terms in changing the variables.

The choice a=1/2 on the contrary, allows the use of the standard rules (Stratonovich calculus). The white noise is considered here as the limit of some smoother, symmetric distribution. Moreover, when we write the

corresponding Fokker-Planck, we meet the usual midpoint prescription for the path integral.

It is interesting to see how some ambiguities of the path integral emerge in the stochastic scheme. In any case, we will never meet any explicit difficulty related to these ambiguities, and we'll be allowed to use the rules of the standard calculus directly on η.

II.3 PARISI AND WU QUANTIZATION OF A SCALAR FIELD.

The Green functions of a quantum theory:

$$G^{(n)}(x_1,...,x_n) = <T\phi(x_1)...\phi(x_n)> \qquad (II.3.1)$$

can be obtained in the path integral approach by taking functional derivatives, with respect to an external source J(x), of the generating functional:

$$Z(J)= \int [d\phi] \exp\{ i/\hbar\, S[\phi] + \int d^dx\, J(x)\phi(x) \} \qquad (II.3.2)$$

where $S[\phi]$ is the action and $[d\phi]$ the measure of the functional integration. After a Wick rotation, Z(J) looks like the partition function of a statistical system at the equilibrium with probability distribution:

$$P_{eq}(\phi) = \exp\{ -S_E[\phi]/\hbar \} \qquad (II.3.3)$$

where $S_E[\phi]$ is the Euclidean action. To be defined we

choose:

$$S_E[\phi]= \int d^dx \{ 1/2\ \partial^\mu\phi\partial_\mu\phi+1/2m^2\phi^2+V(\phi) \} \qquad (II.3.4)$$

The basic idea of the stochastic quantization is to find the stochastic process having $P_{eq}(\phi)$ of eq.(II.3.3) as equilibrium distribution. Comparing eq.(II.3.3) with eq.(II.2.21), we see that, if

$$2/\sigma \int^x K(x')dx' = -S[\phi] \qquad (II.3.5)$$

(we drop the subscript E and put $\hbar$=1), the stochastic process satisfying the Langevin equation:

$$\frac{\partial\phi(x;t)}{\partial t} = -\frac{\sigma}{2}\frac{\delta S[\phi]}{\delta\phi(x;t)} + \eta(x;t) \qquad (II.3.6)$$

has $P_{eq}(\phi)$ as equilibrium distribution. Here $\phi(x;t)$ is a field in d Euclidean dimensions and evolves in a fictitious (or stochastic) time t . $\eta(x;t)$ is a Gaussian white noise:

$$<\eta(x;t)>=0 \qquad <\eta(x;t)\eta(y;t')>=2\delta^d(x-y)\delta(t-t')$$

$$<\eta(x_1;t_1)...\eta(x_{2n};t_{2n})>=\Sigma_{\text{possible pairs}}\Pi_{\text{pairs}}<\eta(x_i;t_i)\eta(x_j;t_j)> \qquad (II.3.7)$$

where we have set $\sigma=2$ and the last equation gives a Wick decomposition for the Gaussian noise. If ϕ_η is a solution of eq.(II.3.6), we have that the Green functions are obtained in the t infinite limit of the average of n fields ϕ_η computed at the same time:

$$G^{(n)}(x_1...x_n)=\lim_{t\to\infty}<\phi_\eta(x_1;t)...\phi_\eta(x_n;t)> \qquad (II.3.8)$$

Here the averages have to be understood in the sense of the probability distribution of η:

$$P(\eta) = \exp\{ -1/4 \int d^dx\, dt\, \eta(x;t)\eta(x;t) \} \qquad (II.3.9)$$

It has to be remarked that eq.(II.3.6) is not the only choice, and some kernel can be introduced:

$$\frac{\partial\phi}{\partial t} = - \int K(x-y) \frac{\delta S[\phi]}{\delta\phi(y;t)} d^dy + \eta(x;t)$$

$$P'(\eta)=e^{-1/4\int d^dx\, d^dy\, dt\, \eta(x;t)\, K^{-1}(x-y)\, \eta(y;t)} \qquad (II.3.10)$$

since ϕ has the same equilibrium distribution $P_{eq}(\phi)=\exp\{-S[\phi]\}$.

In the following we will consider

$$V(\phi) = \lambda/4!\ \phi^4 \qquad \text{(II.3.11)}$$

and give a perturbative solution as a power series in λ. First, we define a stochastic propagator by:

$$(\partial_t-\partial^\mu\partial_\mu+m^2)G(x-y;t-t')=\delta^d(x-y)\delta(t-t') \qquad \text{(II.3.12)}$$

Explicitly:

$$G(x-y;t-t')=\int d^dp/(2\pi)^d\theta(t-t')\exp\{ip(x-y)-(p^2+m^2)(t-t')\} \qquad \text{(II.3.13)}$$

where we have taken only the forward propagation since we will be interested in the t infinite limit. The formal solution of eq.(II.3.6) with initial condition $\phi(x;0)=\phi_0(x)$ is:

$$\phi(x;t)=\int d^d y dt G(x-y;t-\tau)\,[\,\eta(y;\tau) - \lambda/3!\,\phi^3(y;\tau)\,] \; + \\ + \int G(x-y;t)\phi_0(y)d^d y \qquad \text{(II.3.14)}$$

We can choose $\phi_0(x)=0$ since, in the perturbative framework at least [8], the equilibrium situation is independent from the choice of the initial configuration. Eq.(II.3.14) is suitable for an expansion in λ: going to the Fourier trasform:

$$\phi(k;t)=\int d^d x\ \phi(x;t)\ \exp\{-ikx\ \} \qquad \text{(II.3.15)}$$

we have, at first order:

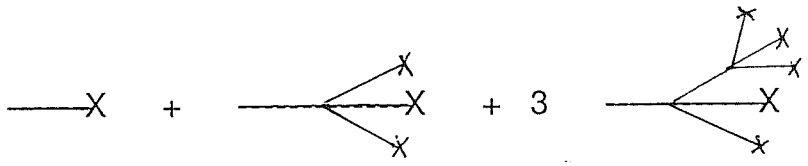
$$\phi(k;t)=\int d\tau\ G_k(t-\tau)\eta(k;\tau) - \lambda/3!\int dt \prod_{i=1\ldots3} d^d q_i/(2\pi)^d \\ G_k(t-\tau)\ G_{qi}(t-t_i)\eta(q_i;t_i)dt_i\ \delta^d(k-\Sigma_{i=1}{}^3 q_i)(2\pi)^d \; + \\ + O(\lambda^2) \qquad \text{(II.3.16)}$$

A useful graphical representation can be given using the following rules:

a line t ____k____ $\tau = G_k(t-\tau)$ is a stochastic propagator,

a cross X = $\eta(q;t)$ is a noise. When a line ends on a cross, an integration over dt is understood.

Eq.(II.3.16) can be represented graphically by:

$\phi(k;t)=$  $+ O(\lambda^3)$

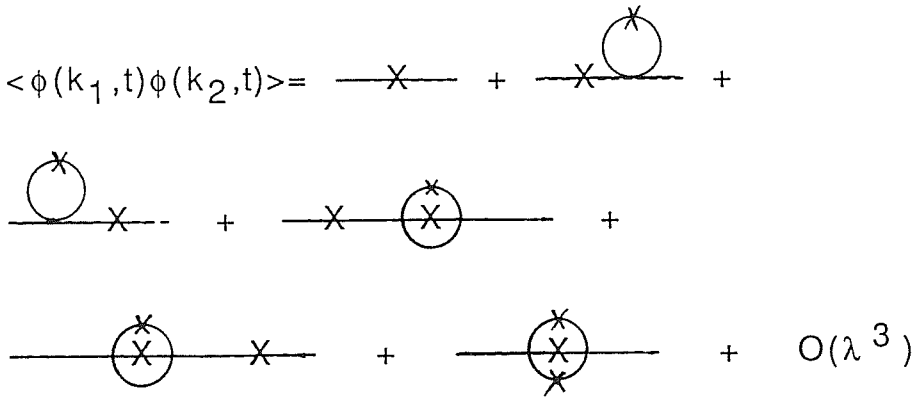
(II.3.17)

where every vertex carries a factor $-\lambda/3!\delta^d(k-\Sigma_i q_i)(2\pi)^d$ and an integration over $\Pi_i d^d q_i/(2\pi)^d\, d\tau$ is understood.

A finite time Green function is now obtained taking averages of products of ϕ's. The average is:

$$<\eta(k_1,t)\eta(k_2,t')>=2(2\pi)^d\delta(k_1+k_2)\delta(t-t') \qquad \text{(II.3.18)}$$

now represented by a cross in the middle of a line. For example:

$<\phi(k_1,t)\phi(k_2,t)>=$  $+ O(\lambda^3)$

(II.3.19)

The combinatorial factor can be directly computed counting the multiplicity of the tree diagrams and the number of contractions coming from the Wick decomposition, but a useful rule can be given: each vertex carries a $-\lambda$ (without 1/3!), then one has to multiply for the times of independent choices for the external legs, and divide for the number of topologically equivalent internal lines.

The first graph of eq.(II.3.19) (the tree level) gives:

$$<\phi(k_1,t)\phi(k_2,t)>=<G_{k1}(t-t_1)G_{k2}(t-t_2)\eta(k_1,t_1)\eta(k_2,t_2)>=$$

$$=\int dt\ \exp\{-(k_1{}^2+m^2)(t-\tau)-(k_2{}^2+m^2)(t-\tau)\}\ \theta(t-\tau)$$

$$2(2\pi)^d\delta^d(k_1+k_2)=$$

$$= (2\pi)^d\delta^d(k_1+k_2)\ \ [1-\exp\{-2t(k_1{}^2+m^2)\}]\ /(k_1{}^2+m^2) \qquad \text{(II.3.20)}$$

It is now easy to see that for $t\to\infty$ this reduces to the usual Euclidean propagator. In the same way it is possible to show that the sum of the graphs of the same shape in the loop expansion (eq.(II.3.19)), converges to the corresponding graph of the usual Q.F.T. [8][20].

The perturbative expansion of the n-point average

$<\phi(x_1,T)...\phi(x_n,T)>$ produces connected as well as disconnected pieces; moreover, the connected pieces can be classified according to the 1PI graphs, in the same way we usually do in the path integral. A n-point 1PI graph has the form $\Gamma^{(n)}(x_1,T_1;...;x_k,T_k;x_{k+1},t_{k+1};...;x_n,t_n)$ where the T_i are uncostrained time variables, while the tj are constrained to be smaller with respect to some Ti becouse of the θ–functions in the stochastic propagator. To avoid this distinction, bothering the computations, it is convenient to define a Laplace transformed 1PI function [21]:

$$\Gamma^{(n)}(x_1,T_1;...;x_k,T_k;x_{k+1},s_{k+1};...;x_n,s_n)=$$

$$=\int\Pi_j ds_j \Gamma^{(n)}(x_1,T_1;...;x_k,T_k;x_{k+1},t_{k+1};...;x_n,t_n)\exp\{-s_j t_j\} \qquad (II.3.21)$$

These 1PI stochastic graphs are the building blocks of the perturbative expansion and we will use them to compute the renormalization constants.

It is interesting to remind that we can also give a path integral formulation of the stochastic theory, which means that we go from the Langevin approach to the

Fokker-Planck one [22]. In this way we obtain a local action $S_{FP}[\phi]$ which allows us to calculate the stochastic averages by conventional path integral methods and the perturbative expansion can be obtained in the standard path integral framework. To this end, we can write the generating average:

$$Z(J)=<\exp\{\phi J\}>=\int[d\eta]\ \exp\{-1/4\int(\eta^2+\phi_\eta J)\} \qquad (II.3.22)$$

Here J is an external source, and all the Green functions can be obtained taking functional derivatives of Z(J) with respect to J, and setting J to zero. ϕ_η is the solution of the Langevin equation (II.3.6). Eq.(II.3.22) can be rewritten as:

$$Z(J)=\int[d\eta][d\phi]P(\phi;0)\delta(\phi-\phi_\eta)\ \exp\{-1/4\int\eta^2+\phi J\} \qquad (II.3.23)$$

$P(\phi;0)$ is the initial probability distribution. For instance, in the case of eq.(II.3.14), we have:

$$P(\phi;0)=\prod_x \delta\{\phi(x;0)-\phi_0(x)\} \qquad (II.3.24)$$

Writing the argument of the δ-function explicitly, we get:

$$Z(J)=\int[d\eta][d\phi]P(\phi;0)\ \det|\ \delta\eta/\delta\phi\ |\ \exp\{-1/4\int\eta^2+\phi J\}$$
$$\delta(\partial_t\phi+\delta S/\delta\phi-\eta)$$

(II.3.25)

The integration over η is trivial and we are left with the computation of the determinant:

$$\det|\delta\eta/\delta\phi| = \exp\{\ Tr\ ln\ [\ \partial_t + \delta^2S/\delta\phi(t)\delta\phi(t')\ ]\ \delta(t-t')\ \}$$

(II.3.26)

For a system with a discrete set of degrees of freedom (d=0 or a lattice regularized system), eq.(II.3.26) is not only formal, but we can compute it, using the fact that , apart for a constant:

$$\det|\delta\eta/\delta\phi|= \exp\{\ Tr\ ln\ [\ \delta(t-t') + \theta(t-t')\ \delta^2S/\delta\phi(t)\delta\phi(t')\ ]\}$$

(II.3.27)

The θ-function appearing here is the inverse of the operator ∂_t, in the forward propagation as required by the causality of the stochastic process. Expanding the

logarythm, all the terms vanish because of the trace, but the first one. Since we follow the midpoint prescription, $\theta(0)=1/2$, and we get:

$$\det|\delta\eta/\delta\phi| = \exp\{1/2\int dt\ tr_x\ \delta^2 S/\delta\phi^2\} \qquad (II.3.28)$$

Eq.(II.3.25) becomes:

$$Z(J)=\int[d\phi]P(\phi,0)\ \exp\{\int d^dx dt\ [\ -1/4\ (\partial_t\phi^2 + (\delta S/\delta\phi)^2 + 2\partial_t\phi\ \delta S/\delta\phi\ ) + 1/2\ \delta^d(0)\ \delta^2 S/\delta\phi^2\ ]\} \qquad (II.3.29)$$

The term $\partial_t\phi\ \delta S/\delta\phi$ is actually a total derivative and contributes only for boundary terms. In a lattice regularization the last term in the bracket acts like a counterterm to cancel some divergences of the perturbation theory, while in a dimensional regularization it can be disregarded. Rescaling $t\rightarrow 2t$, we can easyly identify the expression in the bracket with the forward Fokker-Planck action:

$$S_{FP}= \int[\ 1/2\ \partial_t\phi^2 + U(\phi)]$$

$$U(\phi) = 1/8\ (\delta S/\delta\phi)^2 - 1/4\ \delta^2 S/\delta\phi^2 \qquad (II.3.30)$$

If we had chosen the backward Green function for ∂_t, we would get now the backward Fokker-Planck.

This formulation of the stochastic quantization ha been useful in discussing some supersymmetric properties [23] and to provide a dynamics to study statistical systems in a finite geometry [24].

II.4. SYMMETRIES AND WARD IDENTITIES.

Before the discussion of the problem of the regularization and renormalization, we want to show how some properties of the system, relevant from a physical point of view, can be described in the stochastic quantization approach as well as in the path integral [25].

In particular we know that if a system has some symmetry at the classical level, this symmetry is very relevant for the structure of the quantized system and will manifest itself at the quantum level through a set of identities among the Green functions (the Ward identities). In the path integral approach the Ward identities come from the invariance of the generating functional Z(J); we want now to show how these identities appear in the stochastic quantization scheme. To this end we must discuss more in general than ref[9] how the stochastic quantization procedure has to be used in presence of internal (global or local) symmetries and how these symmetries will manifest themselves.

Let the action S(U) be invariant under the transformations of a group G, where U_α is the set of fields defining S. To be general enough we can think that

U_α represents different fields transforming linearly with some representation of the group G (for example $U_\alpha=(\phi_i;\ \phi_i^*,\ \psi_j,\ \Psi_j,\ A_{\mu a})$). The transformation of U_α under $g\in G$ is:

$$U_\alpha \rightarrow U_\alpha{}^g = (\exp\{ i\varepsilon^a I^a \})_{\alpha\beta}\ U_\beta \qquad \text{(II.4.1)}$$

where I^a are the generators of G in some (in general reducible) representation and ε^a are the infinitesimal parameters. If for some value of α U_α is a gauge field, we should add to the right hand side of eq.(II.4.1) in the gauge sector also a piece proportional to $\partial_\mu \varepsilon^a$, but if ε does not depend on t (the fictitious time of the stochastic evolution), this piece will not affect the following discussion, since $\partial_t A_{\mu a}$ only transforms linearly with the adjoint representation of G and no inhomogeneous term appears. In any case:

$$\delta U_\alpha{}^g / \delta U_\beta = (\exp \{ i\varepsilon^a I^a)_{\alpha\beta} \qquad \text{(II.4.2)}$$

is the representative of the group element. Let us

write the G-invariant quadratic form as :

$$Q = U_\alpha \omega^{\alpha\beta} U_\beta \qquad (II.4.3)$$

where ω is such that

$$Q^g = U_\alpha{}^g \omega^{\alpha\beta} U_\beta{}^g = U_\alpha{}^g (\omega^{\alpha\beta} U_\beta)^{g-1} = Q \qquad (II.4.4)$$

so the characterizing property of ω, in matrix form, is:

$$\omega I + I^T \omega = 0 \qquad (II.4.5)$$

In general ω connects different representations in order to get the singlets Q. Some explicit examples will be given later.

We are now ready to introduce the Langevin equation for U_α in a covariant form [26]:

$$\partial_t U_\alpha = -\omega_{\alpha\beta}\ \delta S[U]/\delta U_\beta + \eta_\alpha \qquad (II.4.6)$$

with $\omega_{\alpha\beta}=(\omega^{\alpha\beta})^{-1}$ and the noise-noise correlation is

given by:

$$< \eta_\alpha(x_1, t_1)\, \eta_\beta(x_2, t_2) > = 2\omega_{\alpha\beta}\, \delta(x_1-x_2)\, \delta(t_1-t_2) \qquad (II.4.7)$$

The probability density for η is

$$P(\eta)= \exp\{ -1/4\, \eta_\alpha \omega^{\alpha\beta} \eta_\beta \} \qquad (II.4.8)$$

manifestly invariant under G-transformations

$$P(\eta)=P(\eta^g) \qquad (II.4.9)$$

Since the quantum Green functions are obtained in the equal time $t\to\infty$ limit of averages over η:

$$G^{(n)}{}_{\alpha_1 \ldots \alpha_n}(x_1 \ldots x_n)=\lim_{t\to\infty} <U_{\alpha 1}(x_1,t) \ldots U_{\alpha n}(x_n,t)>_\eta \qquad (II.4.10)$$

we must in general study the behaviour under G-transformations of functionals $F[U(\eta)]$ where $U(\eta)$ is the solution of eq.(II.4.6)

$$<F[U(\eta)]>_\eta = \int [d\eta] P(\eta) F[U(\eta)] = <F[U(\eta^g)]>_{\eta^g} = <F[U(\eta^g)]>_\eta \qquad (II.4.11)$$

the last equality being a consequence of the invariance of $P(\eta)$, (assuming naively the invariance of the measure in η (i.e. presence of no anomaly)); $U(\eta^g)$ is the solution of the following equation:

$$\partial_t U_\alpha(\eta^g) = -\omega_{\alpha\beta}\delta S[U]/\delta U_\beta + \eta_\alpha{}^g \qquad \text{(II.4.12)}$$

Since S is G-invariant, we have:

$$\delta S[U]/\delta U = \delta S[U^{g-1}]/\delta U^{g-1}\ \delta U^{g-1}/\delta U = (\delta S[U^{g-1}]/\delta U^{g-1})^{g-1} \qquad \text{(II.4.13)}$$

where eq.(II.4.2) has been used; because of the properties of ω (i.e. the covariance of the Langevin equation) we get:

$$\omega_{\alpha\beta}\ \delta S[U]/\delta U_\beta = (\ \omega_{\alpha\beta}\ \delta S[U^{g-1}]/\delta U^{g-1}\ )^g \qquad \text{(II.4.14)}$$

We now insert this identity in eq.(II.4.12) and apply on the left the element $g^{-1} \in G$:

$$\partial_t U_\alpha{}^{g-1}(\eta^g) = -\omega_{\alpha\beta}\ \delta S[U^{g-1}]/\delta U_\beta{}^{g-1} + \eta_\alpha \qquad \text{(II.4.15)}$$

By inspection we see that this is nothing else that the starting Langevin equation (II.4.6); so, if the solution is unique, we get

$$U_\alpha(\eta^g)=U_\alpha{}^g(\eta) \qquad \text{(II.4.16)}$$

i.e. the solution with the G-rotated source is the G-transformed of the original solution. Let us now write the average over η of eq.(II.4.11) as a functional integral over the U fields using the Fokker-Planck action for the stochastic quantization , introduced by Gozzi [22]:

$$<F[U(\eta)]>_\eta = \int[dU]\ \exp\{-S_{FP}(U)\}\ F(U) \qquad \text{(II.4.17)}$$

obtained once the η have been integrated away taking into account the form of the Langevin equation. From eq.(II.4.11) and using eq.(II.4.16) we get:

$$<F[U^g(\eta)]>_\eta=<F[U(\eta)]>_\eta \qquad \text{(II.4.18)}$$

so that:

$$\int[dU]e^{-S_{FP}[U]}F[U] = \int[dU^g]e^{-S_{FP}[U^g]}F[U^g]=$$

$$=\int[dU^g]e^{-S_{FP}[U^g]}F[U] = \int[dU]e^{-S_{FP}[U^g]}F[U] \qquad (II.4.19)$$

using the invariance of the measure; we get

$$S_{FP}[U^g] = S_{FP}[U] \qquad (II.4.20)$$

(as can also be checked directly knowing the explicit form of S_{FP}: it is built using the matrix ω). So, if the action S[U] defining the path integral quantization is G-invariant, the action S_{FP} defining the stochastic quantization is invariant too. This has an immediate consequence on the Ward identities; let us consider the following generating functional of the Green functions:

$$Z(J_\alpha(x,t);\lambda)=$$

$$=\int[dU]\exp\{-\int[L_{FP}(t)\lambda+(1-\lambda)\delta(t)S+U_\alpha J_\alpha(\lambda+(1-\lambda)\delta(t))]dt\} \qquad (II.4.21)$$

where J_α is a source transforming with g^{-1}. As discussed in ref.[27] we can prepare the system at $t=-\infty$ so that at t=0 it is already at the equilibrium ; for $\lambda=0$ we have the usual path integral generating functional,

while for $\lambda=1$ we generate the stochastic Green functions at the equilibrium. However we have proved that the whole interpolating action is G-invariant for every value of λ and then the Ward identities:

$$\delta Z(J;\lambda)/\delta J_\alpha \, \delta J_\alpha = 0 \qquad \text{(II.4.22)}$$

hold for every λ; of course the same equations are valid for $W=\ln Z$ and its legendre transform Γ. This is useful since Γ is much more difficult to compute in the stochastic case than in the usual Quantum Field Theory. Moreover it is clear that for $\lambda=0$ (the purely stochastic case) the Fokker Planck action is still invariant (if a suitable choice of the initial conditions has been done), even if the system had been prepared at t=0 so that it reaches the equilibrium only in the $t\to\infty$ limit. This means that, although the structure of the Green functions is different in the two quantization schemes (being equal only in the $t\to\infty$ limit),the relations among the Green functions are fixed only by the symmerty structure of the system and don't depend on the quantization procedure (path integral or stochastic equations). Gauge models should be excluded in this case since they do have a dependance on the initial conditions [28] and a further

investigation is required. To make this formal discussion more explicit we breafly give two examples.

Suppose G to be SU(N) and U_α the reducible representation $(\psi, \underline{\psi})$; in this case eq.(II.4.4) gives ω to be a 2nx2n matrix with vanishing diagonal blocks. Moreover, if we require chiral invariance and impose the fictitious time to have dimensions p^{-2} (in order to ensure the convergence of the factors $\exp\{-p^2 t\}$ in the stochastic propagators), we get

$$\omega = \left(\begin{array}{c|c} 0 & i\partial\delta_{ij} \\ \hline i\partial\delta_{ij} & 0 \end{array}\right) \qquad \text{(II.4.23)}$$

which is the usual choice for the fermion case [9].

Let now G be O(N) and U_α a N dimensional real vector ϕ_i; since the generators of O(N) are antisymmstric, eq.(II.4.5) gives correctly $\omega_{ij}=\delta_{ij}$. We can take $S=\int 1/2[\partial^\mu\phi_i\partial_\mu\phi_i+m^2\phi^2]+V(\phi^2)$ and verify explicitly eq.(II.4.16); the formal solution of the Langevin equation for the rotated source is:

$$\phi_i(\eta^R) = \phi_i(\eta)+\varepsilon_a\delta_a\phi_i(\eta) = G\cdot\{\eta^R-\delta V(\phi^2(\eta^R))/\delta\phi^2\ 2\phi_i(\eta^R)\} \qquad \text{(II.4.24)}$$

where $\eta^R \equiv \eta_k + i\varepsilon^a I^a{}_{ki}\eta_i$; G is the usual stochastic propagator. We expand the right hand side in terms of ε^a and solve for the unknown $\delta_a\phi_i(\eta)$; the solution is easiliy found to be $\delta_a\phi_i = iI^a{}_{ij}\phi_j(\eta)$ as dictated by eq.(II.4.16). It is now trivial to check that the usual Ward identities [29]:

$$<\phi_i>=0 \qquad <\phi_i(p)\phi_j(q)>=\delta_{ij}P(p,q)$$

$$<\phi_i\phi_j\phi_k>=0 \qquad <\phi_i\phi_j\phi_k\phi_l>\equiv(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{kj}) \tag{II.4.25}$$

are satisfied at any time t because of the correlation functions of the noise, the Gaussian properties of the noises, and the form of the solution (eq.(II.4.24)); P(p,q) is of course different from the usual propagator:

$$P(p,q)=(1-\exp\{-2p^2t\})(2\pi)^4\delta^{(4)}(p+q)/p^2+O(g) \tag{II.4.26}$$

being equal only in the $t\to\infty$ limit.

In conclusion, we have shown, in a very general framework, some consequences of the symmetries in the stochastic quantization scheme once one stresses on the covariance properties of the Langevin equation, and in particular we have proved that the old (i.e. derived in the path integral approach) Ward identities still hold.

PART III. RENORMALIZATION

III.1. REGULARIZATION SCHEMES.

As a trivial consequence of the fact that in the $T\to\infty$ limit the stochastic perturbative expansion reproduces the usual perturbative expansion, we must face with the problem of the divergences of the stochastic graphs. As a simple example, we show how a divergence arises in the loop obtained gluing togheter the end points of a free propagator:

(x,T)  $= \int dy\ \delta(x-y)\langle\phi(x,T)\phi(y,T)\rangle =$

$$= \int d^d p/(2\pi)^d \int_0^T dt_1\, dt_2\, 2\delta(t_1-t_2)$$

$$\exp\{-(p^2+m^2)(2T-t_1-t_2)\} \qquad \text{(III.1.1)}$$

The integration over p is Gaussian and can be easily performed, provided $(2T-t_1-t_2)\geq 0$. The divergence arises when the integration over p is no more damped by Gaussian factors,i.e. for $t_1=t_2\to T$. Of course we could regularize the theory by dimensional regularization, but we also discover different possibilities. Since a loop, by

construction, must contain at least one cross,i.e. a contraction of two noises, we can modify either the space or the fictitious time part of the correlation function for η, introducing suitable kernels in place of the δ-functions appearing in eq.(II.3.7).For example we could take:

$$<\eta(p,t)\eta(q,t')>=2(2\pi)^d\left(\Lambda^2/(\Lambda^2+q^2)\right)^n\delta^d(p+q)\delta(t-t') \qquad (III.1.2)$$

In this way any crossed propagator will have additional momenta in the denominator and , for sufficiently high n, all the ultraviolet divergences will be regularized. Letting $\Lambda\to\infty$ we recover the unregularized theory. This choice however does not respect the gauge invariance if we are dealing with gauge fields instead then scalar fields. On the other hand, we can introduce a kernel for the fictitious time correlation:

$$<\eta(x,t)\eta(y,t')>= 2\delta^d(x-y)\ \alpha(t-t') \qquad (III.1.3)$$

If $\alpha(t-t')$ has a zero of sufficiently high order when $t\to t'$, the ultraviolet divergences are regularized. It is

easy to see how this works by inspection of the simple example given in eq.(III.1.1.) for $\alpha(t_1-t_2)$ in place of $\delta(t_1-t_2)$. An apparent advantage of this choice, is that now the symmetries of the physical space are untouched. A first proposal for α is the family of kernels [9]:

$$\alpha(t)= 1/2n!\ \Lambda^2\ (\Lambda^2|t|)^n\ \exp\{-\Lambda^2|t|\} \qquad \text{(III.1.4)}$$

A more convenient, from a practical point of view, is [10]:

$$\alpha_\sigma(t-t') = \sigma/2\ |t-t'|^{\sigma-1} \qquad \text{(III.1.5)}$$

in which the regulator is the real parameter σ, and

$$\lim_{\sigma\to 0}\ \alpha_\sigma(t-t') = \delta(t-t') \qquad \text{(III.1.6)}$$

The non locality in the fictitious time means that the stochastic process is no more Markovian. A Markovian process can be obtained in the limit of eq.(III.1.6). The main motivation for this choice rely on the renormalization of the theory and in fact we can consider

two related perspectives. First, one can introduce $\alpha_\sigma(t-t')$ as a device for regularizing the theory. By the choice of eq.(III.1.5) one obtains that the divergences appear as poles in σ and one can devise a minimal subtraction scheme in which the poles are reabsorbed into the renormalization constants, while keeping the dimensionality d fixed, for instance d=4 in the $\lambda\phi^4$ theory. The advantage of keeping the dimensionality fixed should be relevant for instance in the case of supersymmetric theories [30]. Second, one can consider a theory defined through the Langevin equation for $\sigma\neq 0$. This should give a non-Markovian theory where the average $<\phi...\phi>_\eta$ is not equal to the corresponding physical Green functions of a Euclidean field theory. One can nevertheless imagine to renormalize that theory, which would be renormalizable for values of the dimensionality d related to σ, and also introduce the corresponding renormalization group equations, and study the critical phenomena. This appears to be convenient because of the observation that for d fixed , say d=3 for the typical critical phenomena one is interested in, there exists a particular value $\sigma^*=\sigma$ for which the non-Markovian theory is renormalizable and asimptotically free in the infrared, providing a

systematic method for a subsequent extrapolation to the physical point $\sigma=0$, in the same spirit of the ε-expansion algorithm for computing critical exponents, here $(\sigma-\sigma^*)$ playing the role of the expansion parameter [31].

Numerous questions arise when considering the renormalization pattern of the non-Markovian theory. We will report here an analysis of the features of the theory based on the Langevin equation and on eq.(III.1.5), including the Markovian and non-Markovian cases [32].

We discuss a proper definition of the non-Markovian case and, by means of perturbation theory, the convergence of the $T\to\infty$ limit for the general non-Markovian case, for which the Fokker-Planck equation cannot be derived.

In order to define the measure $[d\rho(\eta)]$ for the Gaussian process, we look for the "momentum representation", i.e. we describe $\eta(t)$ by means of its Fourier transform

$$\eta(\nu)=1/\sqrt{2\pi}\int dt\ \eta(t)\ \exp\{-i\nu t\} \qquad \text{(III.1.7)}$$

In this representation the kernel α^{-1} of the bilinear form which defines the Gaussian distribution is diagonal

and we get

$$[d\rho(\eta)]=N\ \exp\{-1/4\int dv\ |\eta\ (v)|^2\alpha^{-1}(v)\ \}\ [d\eta\ ] \qquad (III.1.8)$$

The requirement is that the kernel $\alpha_\sigma^{-1}(v)$ is positive definite. As we want to use it for regularizing the theory, we consider a family of kernels α_σ, labelled by the real variable σ, such that, for $\sigma\rightarrow 0$, we get back $\delta(t-t')$ and that the ultraviolet divergences are expressed as poles in σ. A useful form is the following:

$$\alpha_\sigma(v)= \cos\ (\sigma\pi/2)\ \Gamma(\sigma+1)|v|^{-\sigma} \qquad (III.1.9)$$

which is positive for $\sigma\in\ I_n$, where I_n is the interval $I_n=(-1+4n;\ 1+4n)$ for $n\geq 0$.

By Fourier transforming back to the variable t we get α_σ as given in eq.(III.1.5), which was verified [10] to provide the desired analyticity properties, i.e. it shares common features with the analytic [34] and dimensional regularizations. Notice that an analytic continuation in σ, if needed, is implicit in the above formula for the Fourier transformation. Analytic continuation is also required [9]

to provide the desired regularization of ultraviolet divergences, because one must reach the region where $\alpha_\sigma(t-t')$ has zeros of sufficiently large order for $t \to t'$ and so we must take $\sigma \in I_n$ with n sufficiently large. After the divergences have been cancelled, an analytic continuation from I_n to the interval of interest (usually I_o) through unphysical regions will be needed.

We still want to prove the convergence for infinite fictitious time of the regularized graphs of the perturbative expansion. The proof of the convergence of the perturbative expansion of the usual (Markovian) stochastic quantization [8] is based on the Fokker-Planck equation, but in the non-Markovian case we cannot use it. The main reason is that [7] , as discussed in Section II.1, the Fokker-Planck equation is a consequence of the characterizing property of the Markovian systems for the conditional probabilities (eq.(II.2.12)). In order to discuss the non-Markovian case, let us consider the graphs of the perturbation theory. According to the discussion of Sect. II.3, each graph appearing in the perturbative evaluation of eq.(II.3.8), is obtained by contracting togheter some tree diagrams according to eq.(III.1.3). Hence, any graph can be decomposed into the tree diagrams which have generated it. Each vertex and

each cross in the tree diagrams involves an integration over the corresponding time variables.Therefore, a tree diagram with V vertices gives, for a $\lambda\phi^n$ theory, V time variables t_i and (n-2)V+1 variables τ_α from the vertices and the crosses respectively. The variables τ_α and τ_α' appear in $\alpha_\sigma(\tau_\alpha-\tau_\alpha')$, and also in the propagators. All these time variables have T, the external time at which the Green functions are computed, as an upper bound.

A line starting at t and ending at t' (our convention is t>t') in the tree diagram is a stochastic propagator G, eq.(II.3.13). The presence of the θ-functions induces an orientation and chronological ordering in the sets of the t_i's and τ_α's.

Moreover, every oriented line G ends either on a vertex or on a cross, and not more than one G can end at the same point, because of the topological properties of the tree diagrams. We have a one to one correspondence between the integration variables and the set of G's by associating to each time variable the stochastic propagator having that time variable as smaller time. In this way we have associated to each time variable a difference of times (the argument of the associated G,

see eq.(II.3.13)). Let's perform the change of integration variables from the t_i's and τ_α's to the associated differences z_i's and y_α's. These are suitable integration variables since they are positive definite because of the θ-functions and the exponentials in eq.(II.3.13) are always damping factors. So the integration is performed over damping exponentials times the functions α_σ which have as arguments sums and differences of z's and y's. The integration interval for each variable runs from 0 to a time value t_{sup} bounded by T,i.e. t_{sup} = [T - (a sum of z's)] and this is the only place where T appears. When we take the $T\to\infty$ limit, the upper integration value goes to infinity and the integration is convergent due to the exponential damping,at least for $m\neq 0$ (since we consider the case when, like in eq.(III.1.5), $\alpha_\sigma(z-y)$ behaves as a power of (z-y)). In the case m=0, which is interesting for the theory of critical phenomena, the problem of the limit $T\to\infty$ is the same as the problem of the infrared divergences [35]. Indeed, T provides an infrared regulator and we can always put m=0 before $T\to\infty$ (this could be useful for discussing the renormalization of non-linear

σ models for which a symmetric infrared regularization is desirable). On this point we will come again later (see Sect. III.4).

In conclusion we remark that the stochastic quantization allows new regularization schemes by means of non-local kernals in the space [36] or in the time part of the noise correlations. It is important to note that the symmetric structure of the theory is preserved (apparently also for chiral and supersymmetric theories) and the regularization scheme is probably non-perturbative [37]. We have shown that not only the Markovian case is consistent, but it is also possible to define a meaningful non-Markovian theory and obtain, in this way, new informations about the related physical system.

III.2. RENORMALIZATION PROPERTIES.

Once the theory has been regularized, we must discuss the renormalizability of the perturbative expansion. In the following we will use the regularization produced by eq.(III.1.3) with the choice for α_σ given in eq.(III.1.5). Moreover we will let d, the dimensionality of the Euclidean space, unspecified, in order to discuss at the same time the renormalization in the dimensional and in the stochastic regularization [31].

The first, naive, consideration is that a theory is renormalizable if the coupling constant is dimensionless. From the Langevin equation and the noise correlation we can read the canonical dimensions (in length):

$$[\phi] = x^{\sigma+1-d/2} \qquad [t] = x^2$$

$$[\eta] = x^{\sigma-1-d/2} \qquad [\lambda] = x^{-2\sigma-4+d} \qquad \text{(III.2.1)}$$

where a $\lambda\phi^4$ coupling has been considered. The requirement of a dimensionless coupling constant furnishes the following fundamental relation between σ

and d:

$$2\sigma = d-4 \qquad \text{(III.2.2)}$$

This naive argument can be implemented by a power counting analysis of the perturbative expansion. In fact, a generic graph with L loops, E_0 uncrossed external legs and E_c crossed ones, has a superficial degree of divergence:

$$D/2= L\ d/2\ -M(\sigma-1)-N \qquad \text{(III.2.3)}$$

where M is the number of crossed internal lines and N the number of internal time variables over which we integrate. Using the topological properties of the graphs for the $\lambda\phi^4$ theory:

$$2M=2V+E_0-E_c$$

$$V=L-1+(E_0+E_c)/2$$

$$N=2M+V-1 \qquad \text{(III.2.4)}$$

we can rewrite the superficial degree of divergence

as:

$$D/2=L(d/2-2-\sigma)+3+\sigma-(3/2+\sigma)E_0-E_c/2 \qquad (III.2.5)$$

We see that, if eq.(III.2.2) holds, D is independent of the order of the perturbative expansion and only a finite number of divergent graphs occours.

In particular, when $E_0=1$ $E_c=1$ we find a quadratic divergence and when $E_0=1$ $E_c=3$ a logarythmic one. For $E_0=2$ $E_c=0$ a logarythmic divergence occours only if $\sigma=0$, $d=4$, according to the fact that only in this case such a graph (and the related counterterm) becomes local in the fictitious time. In this way we have not only verified the naive argument given at the beginning, but also identified the graphs to be computed in order to renormalize the theory.

A much more complete proof of the renormalizability can be based upon a hidden BRS invariance typical of the stochastic quantization [23][32]. It is in fact possible to give a different functional formulation to the stochastic quantization with respect to what we did in Sect.II.3.

The Langevin equation can be considered as a constraint linking ϕ to η :

$$\partial_t\phi + \delta S/\delta\phi = L(\phi) = \eta \qquad \text{(III.2.6)}$$

and η can have some, not specified, probability distribution $[d\rho(\eta)]$. The generating functional can be written as:

$$Z(J)=\int[d\rho(\eta)]\,[d\phi]\,\delta(\phi-\phi_\eta)\,e^{\phi J} =$$

$$=\int[d\phi]\,[d\rho(\eta)]\,\delta(L(\phi)-\eta)\,\det|\delta L(\phi)/\delta\phi|\,e^{\phi J} \qquad \text{(III.2.7)}$$

Now we write the determinant using anticommuting (ghost) variables, and introduce an auxiliary field for the δ-function:

$$Z(J)=\int[d\phi]\,[dc]\,[d\underline{c}]\,[d\omega]\,e^{-A+J\phi} \qquad \text{(III.2.8)}$$

with

$$A=-W(\omega) + L(\phi)\omega - \underline{c}\,\delta L/\delta\phi\; c \qquad \text{(III.2.9)}$$

Here $W(\omega)$ is the generating functional of the noise

correlations:

$$e^{W(\omega)} = \int [d\rho(\eta)]\, e^{\omega\eta} \qquad \text{(III.2.10)}$$

The action A is very similar to the gauge fixing part and the Fadeev-Popov part of the effective action of a gauge theory [4]. In particular we observe the following BRS invariance:

$$\delta\phi = \chi c \qquad \delta c = 0 \qquad \delta\underline{c} = \chi\omega \qquad \delta\omega = 0 \qquad \text{(III.2.11)}$$

In fact:

$$\delta A = \delta L/\delta\phi\ \chi c\omega - \chi\omega\ \delta L/\delta\phi\ c - \underline{c}\ \delta^2 L/\delta\phi^2\ cc = 0 \qquad \text{(III.2.12)}$$

The last term vanishes in virtue of the Grassmanian character of c and, if we have more scalar fields ϕ_a, in virtue of the symmetry of the second derivative of L. It is interesting to write A in the superspace, by introducing a superfield:

$$\Phi(x,t;\theta,\underline{\theta}) = \phi(x,t) + \underline{\theta}c + \underline{c}\theta + \underline{\theta}\theta\omega \qquad \text{(III.2.13)}$$

The BRS invariance is nothing but a translation $\underline{\theta} \rightarrow \underline{\theta} + \chi$

in the superspace. In the case of the Langevin equation there exists also an additional symmetry corresponding to the translation:

$$t \to t - \underline{\theta}\chi \qquad \theta \to \theta + \chi \tag{III.2.14}$$

The covariant derivatives corresponding to these two transformations are

$$D = -i\partial_\theta \qquad \underline{D} = i(\theta\partial_t - \partial_{\underline{\theta}}) \tag{III.2.15}$$

giving a simple realization of supersymmetry:

$$\{ D; \underline{D} \} = \partial_t \qquad \{ D ; D \} = \{ \underline{D} ; \underline{D} \} = 0 \tag{III.2.16}$$

The action can be written as:

$$A = \int dt d\theta d\underline{\theta} \{ D\Phi \underline{D}\Phi - S(\Phi) \} \tag{III.2.17}$$

This supersymmetry is tightly related to the time reversal properties of the system [22].

Since our regularization, based on eq.(III.1.5), defines a particular measure, it does not break the symmetry in eq.(III.2.11), and then the renormalized action A_R

satisfies the Ward identity induced by eq.(III.2.11):

$$\delta A_R/\delta\phi\ c + \omega\ \delta A_R/\delta\underline{c} = 0 \qquad \text{(III.2.18)}$$

If one is able to prove, by power counting and fermion number conservation, that A_R is at most quadratic in the ghost fields, i.e. A_R has the form:

$$A_R = -\underline{c}M(\phi;\omega)c + \Sigma(\phi;\omega) \qquad \text{(III.2.19)}$$

then the Ward identity eq.(III.2.18) implies:

$$M(\phi;\omega) = \delta L(\phi;\omega)/\delta\phi$$

$$\Sigma(\phi;\omega) = \omega L_R(\phi;\omega) - W_R(\omega) \qquad \text{(III.2.20)}$$

Comparing eqs.(III.2.20) and (III.2.19) with eq.(III.2.9), we see that the theory is renormalizable if the dimensional analysis, applied to $L_R(\phi;\omega)$, does not allow a dependance on ω, i.e. L_R is a function of ϕ only, and moreover it is polynomial in ϕ and its derivatives of the same form as it appears in the Langevin equation. The

renormalization scheme can be based on a renormalized Langevin equation:

$$Z_t\partial_t\phi=-Z_\phi(\partial_\mu\partial^\mu+m^2+\delta m^2)\phi+Z_V\delta V/\delta\phi+Z_\eta\eta \qquad \text{(III.2.21)}$$

when the dimensional analysis fixes $W_R(\omega)$ to be at most quadratic in ω.

We will consider mainly the theory $\lambda\phi^4$ for d>2, since it is relevant for the application to the critical phenomena , both in the Markovian and non-Markovian case. As an additional example we also discuss the theory $g\phi^3$. In this latter case $[g]=x^{-\sigma+d/2-3}$. We will be interested in the case for which the coupling constant is dimensionless, and therefore we require eq.(III.2.2) for the case $\lambda\phi^4$, and

$$2\sigma+6=d \qquad \text{(III.2.22)}$$

for the case $g\phi^3$. Following the discussion of Sect.III.1, we also require $\sigma\in I_0$ and therefore restrict

$$2<d<6 \qquad \text{(III.2.23)}$$

for the case $\lambda\phi^4$, and

$$4<d<8 \qquad \text{(III.2.24)}$$

for the case $g\phi^3$.

Since a four ghost term has dimensions:

$$[\int \underline{c}\, c\, \underline{c}\, c\, dt\, d^dx\,] = x^{-d+2} \qquad \text{(III.2.25)}$$

it is not ultraviolet divergent for $d>2$, and therefore eq.(III.2.19) holds. Let us discuss the form of L_R: it has critical dimensions $x^{\sigma-d/2-1}$. The non-interacting part of it, i.e. the part containing only ϕ and only linearly, will have the form:

$$Z_t\partial_t\phi+Z_\phi(-\partial_\mu\partial^\mu)\phi+(m^2+\delta m^2)\phi \qquad \text{(III.2.26)}$$

For the case of the $\lambda\phi^4$ theory, the interaction part of L_R will be of the form : $Z_V\lambda_R\phi^3/3!$. Notice that terms of the form ϕ^2 or $\omega\phi$ are forbidden by the symmetry $\phi\rightarrow-\phi$, $\omega\rightarrow-\omega$. If there is an additional $g\phi^3$ interaction, besides $\lambda\phi^4$, still keeping eq.(III.2.2), one sees that a term $\omega\phi$

would appear multiplied by one power of λ at least, making it not divergent.

For the case of the $g\phi^3$ theory, the interaction part of L_R will be of the form: $Z_V g_R \phi^2/2$, except for the particular case d=4, corresponding to a non-Markovian situation with σ=-1, due to eq.(III.2.22), where a term $\omega\phi$ can appear.

Let's repeat the same dimensional analysis on $W_R(\omega)$. In a minimal subtraction scheme, a potentially divergent term of the form $\int\omega^k$ dtd^{d}x appearing in $W_R(\omega)$ can give a logarythmic divergence (i.e. the relevant one) only if its dimensionality is a positive even integer. It turns out that, in the case of the $\lambda\phi^4$ and $g\phi^3$ theories, a logarythmically divergent term, quadratic in ω, occours for d=4 and d=6 respectively, i.e. in the Markovian case (σ=0). The function $W(\omega)$ generating the noise-noise correlations remains quadratic under renormalization and the renormalization constant Z_η, appearing in:

$$W(\omega)=Z_\eta^2 \int dtdt'd^dx\ \omega(x;t)\alpha_\sigma(t-t')\ \omega(x;t') \qquad \text{(III.2.27)}$$

is actually required only in the Markovian case, being

equal to one in the non-Markovian cases. We conclude that $\lambda\phi^4$ is renormalizable in general when eqs.(III.2.2) and (III.2.23) hold and $g\phi^3$ is renormalizable when eqs.(III.2.22) and (III.2.24) hold.

In both the cases $L_R(\phi)$ can be set in the form:

$$L_R(\phi)= Z_t\partial_t\phi + \delta S_R/\delta\phi \qquad \text{(III.2.28)}$$

where S_R is the renormalized version of the action S. Notice that at σ=-1 (the lower extreme value of I_o), corresponding to d=2 for $\lambda\phi^4$ and d=4 for $g\phi^3$, the non-Markovian case is not renormalizable. In fact, for $\lambda\phi^4$, four ghost terms and terms containing higher powers of ω and ϕ can appear, while, for $g\phi^3$, non quadratic and ϕ dependent noise-noise correlations would be required.

Let's remind an important feature of the Markovian case. The supersymmetry of eq.(III.2.14), implies an additional Ward identity [23]:

$$Z_t=Z_\eta^{\,2} \qquad \text{(III.2.29)}$$

On the contrary, if the process is non-Markovian, it is not possible to write A in the supersymmetric form because now $W(\omega)$ is no more local in time, while only local terms can appear in the supersymmetric action. In this case, eq.(III.2.29) does not work any more: a non trivial Z_t will be , in general, required, while, as previously seen, $Z_\eta=1$.

III.3. THE BACKGROUND FIELD METHOD.

Let us rewrite the Langevin equation in the presence of an external deterministic source J(x;t):

$$\partial_t\phi(x;t)= -F_o[\phi(x;t)] + \sqrt{\varepsilon}\,\eta(x;t) - J(x;t) \qquad \text{(III.3.1)}$$

where

$$Fo[\phi(x;t)]=\delta S[\phi]/\delta\phi(x;t) \qquad \text{(III.3.2)}$$

and ε is a small parameter playing the role of $\hbar$. As an example we will consider the action S of a $g\phi^3$ theory where, for generality, a linear term in ϕ has been added:

$$S=\int d^dx\{1/2\ \phi(-\partial_\mu\partial^\mu+m^2)\phi + g/3!\ \phi^3 + k\phi\ \} \qquad \text{(III.3.3)}$$

We want to compute directly the "effective force" for the stochastic process in analogy with the definition of effective action in the usual Quantum Field Theory.

In the path integral approach, one defines the effective action $\Gamma[\tilde{\phi}]$ by means of a Legendre

transform of the functional generating the connected Green functions. The argument of Γ , $\tilde{\phi}(x)$, is the expectation value (in the sense of the path integral) of the quantum field $\phi(x)$ in the presence of an external source J(x). The effective action generalizes the classic action, in the sense that the equation of motion for $\tilde{\phi}$ is equal to the "quantum equation of motion":

$$< \delta S[\phi]/\delta\phi >= \delta\Gamma[\tilde{\phi}]/\delta\tilde{\phi} = -J \qquad \text{(III.3.4)}$$

We define the effective force for the stochastic system extending the validity of eq.(III.3.4) to finite time:

$$<F_o[\phi(x;t)]>_\eta = F^{eff}[\tilde{\phi}(x;t)]$$

$$\tilde{\phi}(x;t) = <\phi(x;t)>_\eta \qquad \text{(III.3.5)}$$

where now the averages must be understood over the noise measure eq.(III.1.8). Taking the average of eq.(III.3.1) we get an equation for F^{eff}:

$$\partial_t\tilde{\phi} = -F^{eff}[\tilde{\phi}]-J \qquad \text{(III.3.6)}$$

During the calculation of $F^{eff}[\tilde{\phi}]$ we will meet divergences and hence, as we have seen in Sect.III.2, counterterms must be introduced:

$$Z_t \partial_t \tilde{\phi} = -F_R{}^{eff}[\tilde{\phi}] - J \qquad \text{(III.3.7)}$$

where $F_R{}^{eff}$ means the renormalized effactive force, i.e. in $F_R{}^{eff}$ the renormalization constants Z_ϕ, δm^2, Z_V, δk appear. The computation of eq.(III.3.7) can be done using a background field method [46]. We define a background field $\Phi(x;t)$ as the solution of eq.(III.3.1) when ε has been set equal to zero:

$$\partial_t \Phi(x;t) = -F_0[\Phi(x;t)] - J \qquad \text{(III.3.8)}$$

This deterministic equation (a heat equation) corresponds to the tree level approximation of.eq.(III.3.7). Let's shift

$$\phi(x;t) = \Phi(x;t) + \xi(x;t) \qquad \text{(III.3.9)}$$

and

$$\tilde{\phi}(x;t)=\Phi(x;t)+\tilde{\xi}(x;t) \qquad \text{(III.3.10)}$$

where $\tilde{\xi}$ is the mean value of the stochastic field ξ. The crucial observation in the background field method is that [46] the effective action Γ, expressed as a functional of Φ, can be computed summing only vacuum graphs (i.e. graphs with no external ξ leg) in the background Φ. Here we will see that the same feature occours for the effective force $F_R^{eff}(\Phi)$. Of course, since the force is the derivative of the action, now we will have vacuum diagrams with one external amputated leg, at the point (x;t) corresponding to $\delta/\delta\Phi(x;t)$, represented by a dot in fig.1

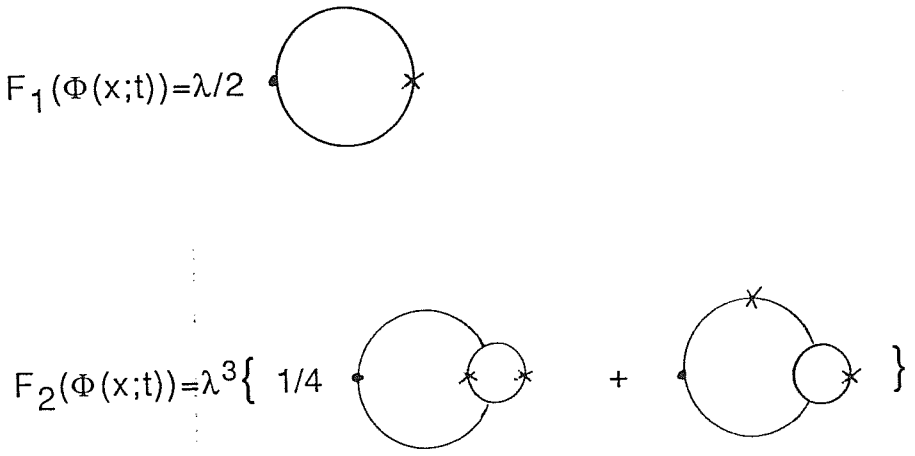


Fig.1.Diagrams for the effective force.

Let's compute directly $F_R^{eff}(\Phi)$. From eqs.(III.3.1) and (III.3.9) we get:

$$\partial_t \xi = -\{ F'_o(\Phi)\, \xi + 1/2\, F''_o\, \xi^2 + \dots \} + \sqrt{\varepsilon}\, \eta \qquad \text{(III.3.11)}$$

This equation can be formally solved defining a stochastic propagator in the backgroud $\Phi(x;t)$ by:

$$[\, \partial_t + F'_o(\Phi(x;t))]\; G(x-y;t-\tau;\, \Phi) = \delta^d(x-y)\delta(t-t) \qquad \text{(III.3.12)}$$

Eq.(III.3.11) is solved by:

$$\xi(x;t) = \int d^d y d\tau\; G(x-y;t-\tau;\, \Phi)\; \{ -1/2F''_o(\Phi)\xi^2(y;\tau) + -1/3!F'''_o\xi^3 + \dots + \sqrt{\varepsilon}\, \eta(y;t)\, \} \qquad \text{(III.3.13)}$$

By taking the average we get:

$$\tilde{\xi} = \langle \xi \rangle = \Sigma_1^{\infty} \varepsilon^n \xi_n = G \bullet \{ -1/2F''_o \langle \xi^2 \rangle - 1/3!F'''_o \langle \xi^3 \rangle + \dots \} \qquad \text{(III.3.14)}$$

Remembering the usual perturbative expansion in the

stochastic quantization scheme, we note that the expansion in ε is nothing but a loop expansion. Defining $\xi_0=\Phi$, we can solve, order by order, the equation for F_R^{eff}:

$$\partial_t \Sigma_0^{\infty} \varepsilon^N \xi_N = -\Sigma_0^{\infty} \varepsilon^N F_N[\Phi] - J \qquad \text{(III.3.15)}$$

We produce in this way a set of algebraic equations defining F_N in terms of the preceding F_i's; the general form is:

$$J\delta_{N;0} + \sum_{k=0}^{N} \sum_{m1\ldots mk=1}^{\Sigma mi \le N} 1/k!\; F^{(k)}_{N-(m1+\ldots+mk)}\; \xi_{m1}\ldots\xi_{mk}$$

$$= -\partial_t \xi_N \qquad \text{(III.3.16)}$$

where $F_i^{(k)}$ is the k-th derivative with respect to $\tilde{\phi}$ computed at $\tilde{\phi}=\Phi$ and convolutions in $(x_i;\, t_i)$ are understood. For the reader's convenience, we show here the first equations:

$N=0 \qquad J+F_0[\Phi]=-\partial_t\Phi$

$N=1 \qquad F_1+F_0^{(1)}\xi_1=-\partial_t\xi_1$

$$N=2 \qquad F_2+F_1^{(1)}\xi_1+F_0^{(1)}\xi_2+1/2F_0^{(2)}(\xi_1)^2=-\partial_t\xi_2 \tag{III.3.17}$$

We consider the example of eq.(III.3.18). In this case, $F_0^{(2)}=g$ while higher derivatives vanish, and we can, as usual, represent eq.(III.3.13) as an expansion in tree diagrams.

Eq.(III.3.14) is represented by the sum of stochastic graphs of Fig.2 . The terms in brackets of fig.2 are ξ_1 and ξ_2 respectively.

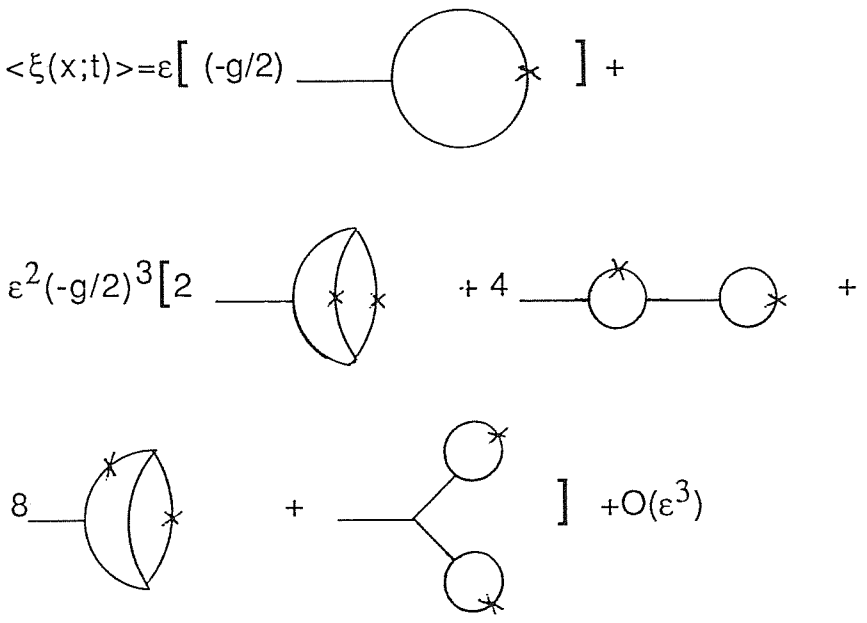


Fig.2 Graphical representation of the average.

Eq.(III.3.17) for N=1 gives the first correction to the force (see Fig.1a):

$$F_1[\Phi(x;t)]=g/2 \int d^d y d\tau_1 d\tau_2 \; G(x-y;\, t-\tau_1;\, \Phi)\; G(x-y;\, t-\tau_2;\, \Phi)$$

$$2\,\alpha_\sigma(\tau_1-\tau_2) \qquad \text{(III.3.18)}$$

Notice that the external stochastic propagator of ξ_1 has been removed because of eq.(III.3.12).

We then expand $F_1(\Phi(x))$ in the background field and compute the divergent part of the graphs with zero, one and two background fields as external legs. These divergences will be absorbed by the renormalization constants δk for zero external leg, Z_ϕ,Z_t, δm^2 for one and Z_v for two.

The second order correction can be obtained from eq.(III.3.17) for N=2. It is easy to see that the various terms, appearing in the equation for F_2, arrange themselves in such a way that F_2 is expressed only by means of 1PI graphs with no external leg, as shown in fig.1b.

A final remark: following the discussion of Sect.III.2,

one should also care about the renormalization of the noise correlations. But, it must be noticed that the only place where the renormalization constant Z_η appears is the crossed line (for zero background):

$$t \text{ —}\times\text{— } t' = Z_\eta^2/Z_t Z_\phi [p^2+m^2+\delta m^2]$$

$$\{\exp[-Z_\phi(p^2+m^2+\delta m^2)|t-t'|/Z_t]+$$

$$-\exp[-Z_\phi(p^2+m^2+\delta m^2)(t+t')/Z_t]\}$$

(III.3.20)

In the Markovian case, the Ward identity discussed in Sect.III.2 (eq.(III.2.29)) holds and we have not to care about Z_η. On the other hand, for the non-Markovian case, we have seen in Sect. III.2 that $Z_\eta=1$. We then conclude that the renormalization scheme based on the effective force, as previously discussed, is sufficient, and the explicit introduction of a renormalization constant Z_η is not required. Further, by using eq.(III.3.20) and applying the topological arguments about the structure of the stochastic graphs (Sect.III.1) to the graphs defining the correction F_n to the effective force, it is possible to show that:

$$F_n(t;Z_t) = (Z_t^{\sigma-1} Z_\eta^{\,2})^n \, F_n(t/Z_t;1) \qquad \text{(III.3.21)}$$

where we have indicated only the dependance of F_n from the external time t and from Z_t.

In the Markovian case $\sigma=0$ eq.(III.3.21) shows that the equilibrium ($t\to\infty$) does not depend on Z_t because of eq.(III.2.29) so that the only wave function renormalization is given by Z_ϕ (the usual renormalization constant Z of the Quantum Field Theory). On the contrary, in the non-Markovian case $Z_\eta=1$ and the equilibrium wave function renormalization needs both Z_t and Z_ϕ.

III.4.RENORMALIZATION GROUP EQUATIONS.

In the prevoius chapters it has been shown how a renormalization procedure can be based directly on the Langevin equation and that this procedure is consistent not only when the stochastic noise is Markovian, but also in the more general case of non Markovian noises.

To absorb the divergences of the perturbative expansion, some renormalization constants have to be introduced in the Langevin equation. For all of them but two, it is easy the identification with the renormalization constants of the usual quantum field theory. The remaning two constants, called Z_t and Z_η in the following, are tightly connected with the stochastic character of the process. It is our aim to clarify their role. In particular, while Z_η turns out to be irrelevant as will be shown, the constant Z_t is related to the evolution in the stochastic time and is relevant in discussing how the system converges towards the equilibrium.

This is not only interesting from the point of view of the Quantum Field Theory, but also in the framework of the theory of the critical phenomena. In fact , the Langevin equation has been extensively used to provide a dynamics to the critical systems[38], in order to compute the relaxation time towards the equilibrium [39] and the scaling properties of the Green functions out of the equilibrium. In some sense we have an extension to the dynamics of the well known connection between the path integral formulation of Euclidean Quantum Field Theory and equilibrium Statistical Physics. Recently, in this perspective, making use of some results obtained studying the stochastic quantization, the critical dynamics of statistical systems in a finite geometry has been discussed and the relaxation time in a finite volume has been computed [24]

Moreover, another relation between the stochastic quantization in the massless case and the theory of the critical phenomena should be considered: in the former we are interested in the infinite fictitious time behaviour and this turns out to be similar to the approach to the critical temperature for a statistical system. In fact, the fictitious time provides an infrared regulator: if, on the Langevin equation:

$$\partial_T \varphi(x;T) = -\frac{\delta S[\varphi]}{\delta \varphi(x;T)} + \eta(x;T) \qquad \text{(III.4.1)}$$

(where S is the classical Euclidean action and η the random noise) we perform a Laplace transform with respect to T, the left hand side will act as an extra mass term, vanishing as $T \to \infty$.

Alternatively, taking for instance the action:

$$S[\varphi] = \int d^d x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\varphi^2 m^2 + \frac{\lambda}{4!}\varphi^4 \right] \qquad \text{(III.4.2)}$$

we can compute the stochastic, equal time, 2-point Green function and from this we can get the correlation lenght [29] as a function of m^2 and T. At the tree level this reads:

$$\xi^2(m^2;T) = \frac{1}{m^2} - \frac{2Te^{-2Tm^2}}{1-e^{-2Tm^2}} \qquad \text{(III.4.3)}$$

In the $T\to\infty$ limit, we recover the usual divergence $\xi\sim(m^2)^{-\nu}$ of the quantum theory near the critical temperature m=0, with the tree level value for the critical exponent $\nu=\frac{1}{2}$. On the other hand, the stochastic quantization of the massless theory exhibits, at finite time, $\xi\sim T^{\nu_T}$. Thus, the theory is infrared regulated and it becomes critical only in the T infinite limit, when the correlation lenght diverges with the critical exponent ν_T, whose tree level value is $\nu_T=1/2$. The value of ν_T is changed by the renormalization procedure, as happens for ν. We will see that this change is controlled by the renormalization constant Z_t.

In the following we will extensively use the standard techniques of the Renormalization Group (R.G.) equations and of the Callan-Symanzik method (see ref.[29][40]).

As already happened in the connection between path integral Quantum Field Theory and Statistical Physics, it is worthwhile to be noticed that, the study of the Stochastic Quantization can renew the interest in dynamical critical phenomena. On the other hand, techniques developed in the framework of the dynamical statistical systems can be used in the Stochastic Quantization scheme.

The renormalized Langevin equation:

$$Z_t\partial_T\phi_R(x;T) = -[Z_\phi(-\Box+m^2+\delta m^2)\phi_R + Z_V\lambda_R/3!\mu^\omega\phi^3{}_R] + \eta_R \qquad \text{(III.4.4)}$$

with

$$\eta_R(x;T) = Z\eta\,\eta_B(x;T) \qquad \text{(III.4.5)}$$

Equations (III.4.1) , (III.4.2) and the 2-noise correlation

$$\langle\eta(x;T)\eta(y;T')\rangle = 2\delta(x-y)\delta(T-T')\alpha_\sigma(T-T') \qquad \text{(III.4.6)}$$

(α_σ as given in eq.(III.1.5)) have to be thought of as the bare version of the theory.

The coupling constant of the regularized theory has dimensions μ^ω, μ being a mass scale and

$$\omega = 2\sigma+4-d \qquad \text{(III.4.7)}$$

while the renormalized coupling constant λ_R is dimensionless. The renormalization point

occours for $\omega^*=2\sigma^*+4-d^*=0$. If the renormalization is done keeping $d^*\equiv 4$ and $\sigma^*\equiv 0$, we are studying what we call in the following the Markovian case. Otherwise, we are dealing with a non-Markovian case.

Up to now the renormalization constants have been, in principle, arbitrary, dimensionless functions of the cut-off Λ, of the coupling constant λ_R, and of the other dimensional parameters T, μ, m^2. Their form is fixed by the normalization conditions. For example, in a normalization at zero momentum (used in the framework of the Callan-Symanzik equation), they will depend on λ_R and $\Lambda^2 T$.

For our purposes, it is better to choose a minimal subtraction scheme where, by dimensional analysis, we see that:

$$Z_i = Z_i(\lambda_R;\frac{\Lambda}{\mu}) \tag{III.4.8}$$

and the renormalization constants don't depend on the fictitious time since all the finite parts in the counterterms have been set to zero.

We now define:

$$T = T_R = \frac{Z_t}{Z_\varphi} T_B \tag{III.4.9}$$

and observe that, using eq. (III.4.6) , we have:

$$\langle \eta_R(x;T_R)\eta_R(y;T_R)\rangle = Z_\eta^2 \left(\frac{Z_t}{Z_\varphi}\right)^{\sigma-1} \langle \eta_B(x;T_B)\eta_B(y;T_B)\rangle \tag{III.4.10}$$

Eq. (III.4.4) becomes:

$$Z_t Z_\eta^{-1} \left(\frac{Z_t}{Z_\varphi}\right)^{-(1+\sigma)/2} \partial_{T_B}\varphi_R(x;T_R) =$$

$$= -Z_\varphi Z_\eta^{-1} \left(\frac{Z_t}{Z_\varphi}\right)^{(1-\sigma)/2} (-\Box + m^2 + \delta m^2)\varphi_R(x;T_R) +$$

$$-Z_V \frac{\lambda_R}{3!}\mu^\omega Z_\eta^{-1} \left(\frac{Z_t}{Z_\varphi}\right)^{(1-\sigma)/2} \varphi_R{}^3(x;T_R) + \eta_B(x;T_B) \tag{III.4.11}$$

This equation goes into the bare one, if we define:

$$Z^{1/2} = Z_\eta^{-1} Z_\varphi^{(1+\sigma)/2} Z_t^{(1-\sigma)/2} \tag{III.4.12}$$

and

$$Z^{1/2}\varphi_R(x;T_R) = \varphi_B(x;T_B) \tag{III.4.13}$$

As seen before, in the non-Markovian case ($\sigma^* \neq 0$) $Z_\eta \equiv 1$, and so it is not necessary to introduce it. Moreover, in the Markovian case ($\sigma^* = 0$), eq.(III.2.29 holds

so that $Z^{1/2} = Z_\varphi^{1/2}$, Z_η disappears from (III.4.11), and Z_t enters only through the definition of T_R. We then conclude that Z_η never plays any role. On the contrary, Z_t is relevant in the approach to the equilibrium for the Markovian case, but disappears once the equilibrium has been reached (the usual Quantum Field Theory does not know anything about Z_t). In the non-Markovian case instead, the effect of Z_t has to be taken into account also at the equilibrium.

We define the connected, finite time, N point Green functions as the connected part of the average $\langle \varphi(p_1;T)...\varphi(p_N;T) \rangle$. The relation between bare and renormalized Green functions is:

$$G_B^{(N)}(p_i;\lambda_B;T_B;\Lambda) = Z^{N/2} G_R^{(N)}(p_i;\lambda_R;T_R;\mu) \tag{III.4.14}$$

By dimensional analysis, we also get:

$$G_R^{(N)}(\varrho p_i;\lambda_R;\varrho^{-2}T_R;\varrho\mu) = \varrho^D G_R^{(N)}(p_i;\lambda_R;T_R;\mu) \tag{III.4.15}$$

with

$$D = d(1-N)+N(\frac{d}{2}-1-\sigma) = d(1-N)+N \tag{III.4.16}$$

the last equality being valid at the renormalization point $\omega^*=0$. On equations (4.14) and (4.15) we can use the renormalization group techniques to understand the role played by Z_t and, more in general, the behaviour during the approach to the equilibrium.

As we have seen, since in the massless case T_R acts as an infrared cut-off, the stochastic massless theory at finite time is well defined. To write the renormalization group equations, we take the derivative $\mu\frac{d}{d\mu}$ of (III.4.14) keeping the bare quantities fixed and using the renormalization constants in the minimal subtraction scheme as given in (III.4.8).

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}+(\gamma_t-\gamma_\varphi)T_R\frac{\partial}{\partial T_R}+\frac{N}{2}\gamma\right]G^{(N)}(p_i;\lambda_R;T_R;\mu)=0 \qquad \text{(III.4.16)}$$

where, as usual:

$$\beta(\lambda_R)=\mu\frac{\partial\lambda_R}{\partial\mu}\Big|_B \qquad \gamma=\mu\frac{d}{d\mu}\ln Z\Big|_B$$

$$\gamma_i=\mu\frac{\partial}{\partial\mu}\ln Z_t\Big|_B \qquad \gamma_\varphi=\mu\frac{\partial}{\partial\mu}\ln Z_\varphi\Big|_B \qquad \text{(III.4.17)}$$

Using the dimensional properties given by (III.4.15), we obtain the final form of the R.G. equation:

$$\left[-s\frac{\partial}{\partial s}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}+[\gamma_t-\gamma_\varphi+2]T_R\frac{\partial}{\partial T_R}+[D+\frac{N}{2}\gamma]\right]G^{(N)}(sp_i;\lambda_R;T_R;\mu)=0$$

(III.4.18)

The solution is:

$$G^{(N)}(sp_i;\lambda_R;T_R;\mu)=s^D e^{\frac{N}{2}\int_1^s\frac{ds'}{s'}\gamma(\lambda(s'))}\,G^{(N)}(p_i;\lambda(s);T(s);\mu) \qquad \text{(III.4.19)}$$

with the running parameters $\lambda(s)$ and T(s) defined by:

$$s\frac{d\lambda(s)}{ds}=\beta(\lambda(s))$$

$$\lambda(s=1)=\lambda_R \qquad \text{(III.4.20)}$$

and

$$s\frac{dT(s)}{ds}=[\gamma_t(\lambda(s))-\gamma_\varphi(\lambda(s))+2]T(s)$$

$$T(s=1)=T_R \qquad \text{(III.4.21)}$$

We know that for $\omega>0$ (i.e. below the critical dimension $d_c=4$, while keeping $\sigma^*=0$) the theory is superrenormalizable from the ultraviolet point of view: if λ_R is of order ω and sufficiently close to the origin, $\lambda(s)$ is driven towards the ultraviolet fixed point $\lambda^*_{u.v.}=0$, so that γ_t and γ_φ are small corrections to the factor 2 in the bracket of (III.4.21). This means that T(s) goes to infinity as $s\to\infty$. From (III.4.19), we see that the high momentum behaviour of the finite T_R Green function (the l.h.s.) furnishes an approximation of the behaviour of the quantum Green function, i.e. of the Green function of the critical theory. It has to be remarked that this approximation works outside of the critical region, i.e. when the theory behaves like a free one. On the contrary, if $\omega=0$, this approximation does not work for $\lambda\varphi^4$ since $\lambda=0$ suddenly becomes an infrared fixed point. However, it can become a useful approximation in discussing the gauge theories and their numerical simulations since, in that case, the origin is an ultraviolet fixed point.

On the other hand, since we are dealing with a massless theory, we are actually interested in studying the critical region, i.e. small momenta and large T_R where, as shown in (III.4.3), the correlation length diverges. Moreover, we will see that is this the region in which the T_R infinite limit has to be discussed.

To this end, we write (III.4.16) at $\lambda_R=\lambda^*$, an infrared fixed point of the β-function:

$$\left[\mu\frac{\partial}{\partial\mu}+(\gamma_t^*-\gamma_\varphi^*)T_R\frac{\partial}{\partial T_R}+\frac{N}{2}\gamma^*\right]G^{(N)}{}_R(p_i;\lambda^*;T_R;\mu)=0 \qquad \text{(III.4.22)}$$

where a star quantity means that it has been computed at the fixed point. We want to stress that we are authorized in doing so since, as discussed in ref.[40], the behaviour in the critical region has to be studied by the renormalized theory in which $\lambda_R=\lambda^*$. The solution is:

$$G^{(N)}{}_R(p_i;T_R;\mu) = \mu^{-\frac{N}{2}\gamma^*}\Phi(p_i;\mu^{-(\gamma_t^*-\gamma_\varphi^*)}T_R) \qquad \text{(III.4.23)}$$

with Φ some function of the momenta and of $T(\mu)$ only. Now we use the canonical dimension of G:

$$G_R^{(N)}(p_i;T_R;\mu) = \varrho^D G_R^{(N)}(\varrho^{-1}p_i;\varrho^2T_R;\varrho^{-1}\mu) \qquad \text{(III.4.24)}$$

and get

$$G_R^{(N)}(p_i;T_R;\mu) = \mu^{-\frac{N}{2}\gamma^*}\varrho^{\frac{N}{2}\gamma^*+D}\Phi(\varrho^{-1}p_i;(\varrho^{-1}\mu)^{-(\gamma_t^*-\gamma_\varphi^*)}\varrho^2T_R) \qquad \text{(III.4.25)}$$

Since ϱ is arbitrary, we can choose:

$$(\varrho^{-1}\mu)^{-(\gamma_t^*-\gamma_\varphi^*)}\varrho^2T_R = 1 \qquad \text{(III.4.26)}$$

whose solution is

$$\varrho = \mu(T_R\mu^2)^{\frac{-1}{2+\gamma_t^*-\gamma_\varphi^*}} \qquad \text{(III.4.27)}$$

Eq.(4.25)) becomes:

$$G_R^{(N)}(p_i;T_R;\mu) = \mu^D (T_R\mu^2)^{-\frac{D+N\gamma^*/2}{2+\gamma_t^*-\gamma_\varphi^*}} \Phi\left[\mu^{-1}(T_R\mu^2)^{\frac{1}{2+\gamma_t^*-\gamma_\varphi^*}} p_i\right] \tag{III.4.28}$$

The Green function depends on the momenta only through the combination $\xi_T p_i$, with

$$\xi_T = \mu^{-1}(T_R\mu^2)^{\frac{1}{2+\gamma_t^*-\gamma_\varphi^*}} \tag{III.4.29}$$

This gives the time dependent correlation length diverging with T_R. In analogy with the critical exponent ν defined for the theory above the critical temperature, we can define

$$\nu_T^{-1}=2+\gamma_t^*-\gamma_\varphi^* \tag{III.4.30}$$

giving the scaling of ξ_T when $T_R\to\infty$.

Eq.(4.28)) can be rewritten as:

$$G_R^{(N)}(p_i;T_R;\mu) = \xi_T^{-D}(\mu\xi_T)^{-\frac{N}{2}\gamma^*} \Phi(\xi_T p_i) \tag{III.4.31}$$

In this way, we see that the problem of the T_R infinite limit, for the massless theory, reduces to the problem of studying the system in the critical region, i.e. when $p_i\xi_T\to\infty$. The critical theory (the quantum, T infinite, massless theory) exists at the renormalization point $\omega=0$ (in the dimensional regularization this means that it exists at the critical dimension $d_c=4$). For $\omega>0$ (below the critical dimension) the critical theory acquires a meaning only through the ω-expansion (the ϵ-expansion in the Markovian case) around the critical value $\omega^*=0$ ($\epsilon^*=d^*-4=0$).

In particular, the Green functions of the critical theory (denoted with $G_Q^{(N)}$) satisfy R.G. equations similar to eq.(III.4.22):

$$\left(\mu\frac{\partial}{\partial\mu}+\frac{N}{2}\gamma^*\right) G_Q^{(N)}(p_i;\mu) = 0 \tag{III.4.32}$$

For N=2, using the dimensional analysis, we have the well known behaviour:

$$G_Q^{(2)}(p;\mu) = \mu^{-\gamma^*} p^{(\gamma^*-2)} C \tag{III.4.33}$$

where C is some constant. The analogous equation for the T_R finite theory, can be obtained from (III.4.25) for N=2 and $\varrho = p$:

$$G_R^{(2)}(p;T_R;\mu) = \mu^{-\gamma^*} p^{(\gamma^*-2)} \Phi^{(2)}(p\,\xi_T) \tag{III.4.34}$$

As the T_R infinite limit is taken, ξ_T diverges and (III.4.33) implies that:

$$\lim_{y\to\infty} \Phi^{(2)}(y) = C \tag{III.4.35}$$

so that the stochastic theory converges to the critical one.

Since the difference between (III.4.22) and (III.4.32) is given by a term proportional to the time derivative of the Green function, it is interesting to see how the time derivatives go to zero. To this end, we write a R.G. equation for $G'^{(N)}_R=\partial_{T_R} G^{(N)}_R$ and study how it behaves as a function of T_R (as in the Callan-Symanzik approach). The scaling property of $G'^{(N)}_R$ now is:

$$G'^{(N)}_B(p_i;\lambda_B;T_B;\Lambda) = Z^{\frac{N}{2}}\frac{Z_t}{Z_\varphi}G'^{(N)}_R(p_i;\lambda_R;T_R;\mu) \tag{III.4.36}$$

The R.G. equation has the form:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}+(\gamma_t-\gamma_\varphi)T_R\frac{\partial}{\partial T_R}+\frac{N}{2}\gamma+\gamma_t-\gamma_\varphi\right]G'^{(N)}_R(p_i;\lambda_R;T_R;\mu) = 0 \tag{III.4.37}$$

At the fixed point λ^*, following step by step the discussion done for $G^{(N)}_R$ (from (III.4.22) to (III.4.28)), we arrive to the conclusion that:

$$G'^{(N)}_R(p_i;T_R;\mu) = \mu^{D+2}\,(T_R\mu^2)^{-\frac{D+2+\frac{N}{2}\gamma^*+\gamma_t^*-\gamma_\varphi^*}{2+\gamma_t^*-\gamma_\varphi^*}}F^{(N)}(p_i\xi_T) \tag{III.4.38}$$

For semplicity we take N=2 and discuss the Markovian case only. Since D= -2 and , in the Markovian case, $\gamma^*=\gamma_\varphi^*$, we get

$$G'^{(2)}_R(p;T_R;\mu) = (T_R\mu^2)^{-\frac{\gamma_t^*}{2+\gamma_t^*-\gamma_\varphi^*}}F^{(2)}(p\,\xi_T) \tag{III.4.39}$$

Alternatively, this equation can be written in the same form of eq.(III.4.34):

$$G'^{(2)}_R(p;T_R;\mu) = \mu^{-\gamma_t^*}p^{\gamma_t^*}H^{(2)}(p\,\xi_T) \tag{III.4.40}$$

It is interesting to observ that the anomalous scaling is completely determinated by the anomalous dimension of Z_t. Moreover, since we know that the convergence of the stochastic theory is controlled, in general, by exponential factors of the kind $exp(-T(p^2+m^2))$, here we see that $G'^{(2)}_R$ goes to zero, as a consequence of (III.4.35), in virtue of the fact that the exponential damping factors contained in the function $H^{(2)}$ can have only the form $\exp(-p^2\xi_T^2)$, as dictated by (III.4.40). When T_R goes to infinity, ξ_T diverges and the exponential goes to zero, since in the critical region $p\,\xi_T>>1$.

Once the massless theory has been solved, there exists a general procedure to extend the theory to the massive case [29][40]. The Green functions of the massive theory can be expanded in terms of the Green functions of the massless theory with zero momentum mass insertions. In doing so, the R.G. equations for the massive theory have, as ingredients, the renormalization constants of the critical theory. With this techinque it is strightforward to extend (III.4.16) to the massive case:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}+(\gamma_t-\gamma_\varphi)T_R\frac{\partial}{\partial T_R}+\right.$$

$$\left.+\gamma_{\varphi^2}\, m_R^2\frac{\partial}{\partial m_R^2}+\frac{N}{2}\gamma\right] G_R^{(N)}(p_i;\lambda_R;T_R;m_R^2;\mu) = 0 \qquad \text{(III.4.41)}$$

where

$$\gamma_{\varphi^2} = -\mu\frac{\partial}{\partial\mu}\ln Z_{\varphi^2}\Big|_B \qquad \text{(III.4.42)}$$

is the anomalous dimension related to the mass insertion for the massless theory. It is possible to write the solution:

$$G(sp_i;\lambda_R;T_R;m_R^2;\mu) = e^{\int_1^s\frac{ds'}{s'}[D+\frac{N}{2}\gamma]}\, G(p_i;\lambda(s);T(s);m^2(s);\mu)$$

with the running mass defined by:

$$s\frac{dm^2(s)}{ds} = (\gamma_{\varphi^2}-2)m^2(s)$$

$$m^2(s=1) = m_R^2$$

However , it is more convenient to change our perspective to the Callan-Symanzik point of view again, in order to understand what happens when we now vary m_R^2. Following the discussion for m=0 , we fix $\lambda_R = \lambda^*$ in eq.(III.4.41):

$$\left[\mu\frac{\partial}{\partial\mu}+(\gamma_t^*-\gamma_\varphi^*)T_R\frac{\partial}{\partial T_R}+\gamma^*_{\varphi^2}\, m_R^2\frac{\partial}{\partial m_R^2}+\frac{N}{2}\gamma^*\right] G_R^{(N)}(p_i;T_R;m_R^2;\mu) = 0 \qquad \text{(III.4.43)}$$

The solution of this equation can be written in the form:

$$G_R^{(N)}(p_i;T_R;m_R^2;\mu) = \mu^{-\frac{N}{2}\gamma^*}\chi(p_i;T_R\mu^{-(\gamma_t^*-\gamma_\varphi^*)};\, m_R^2\mu^{-\gamma^*_{\varphi^2}}) \qquad \text{(III.4.44)}$$

Using the dimensional property

$$G(p_i;T_R;m_R^2;\mu) = \varrho^D G(\varrho^{-1}p_i;\varrho^2 T_R;\varrho^{-2}m_R^2;\varrho^{-1}\mu) \tag{III.4.45}$$

we get the result:

$$G_R^{(N)}(p_i;T_R;m_R^2;\mu) = \varrho^D(\varrho^{-1}\mu)^{-\frac{N}{2}\gamma^*}\chi\left(\varrho^{-1}p_i;\varrho^2 T_R(\varrho^{-1}\mu)^{-(\gamma_t^*-\gamma_\varphi^*)};\varrho^{-2}m_R^2(\varrho^{-1}\mu)^{-\gamma^*_{\varphi^2}}\right)$$

(III.4.46)

We now set

$$\varrho = \mu\left(\frac{m_R^2}{\mu^2}\right)^{(2-\gamma^*_{\varphi^2})^{-1}} \tag{III.4.47}$$

so that the last argument in the function χ is equal to 1 . This gives the usual form to the correlation length above the critical temperature:

$$\xi = \varrho^{-1} = \mu^{-1}\left(\frac{m_R^2}{\mu^2}\right)^{-\nu} \tag{III.4.48}$$

with

$$\nu = \frac{1}{2-\gamma^*_{\varphi^2}} \tag{III.4.49}$$

Eq.(4.46)now becomes:

$$G_R^{(N)}(p_i;T_R;m_R^2;\mu) = \mu^D\left(\frac{m_R}{\mu}\right)^{(2D-N\gamma^*)(2-\gamma^*_{\varphi^2})^{-1}}\chi(\xi p_i;\mu^2 T_R(\frac{m_R}{\mu})^{4(2-\gamma^*_{\varphi^2})^{-1}(1+\frac{1}{2}(\gamma_t^*-\gamma_\varphi^*)})$$

(III.4.50)

From this equation we see that, the exponential damping $\exp -T_R m_R^2$, controlling the convergence towards the equilibrium, now reads $\exp -T_R/T_o$, with T_o, the relaxation time [38], given by:

$$T_o = \mu^{-2}\left(\frac{m_R}{\mu}\right)^{-4(2-\gamma^*_{\varphi^2})^{-1}(1+\frac{\gamma_t^*-\gamma_\varphi^*}{2})} \tag{III.4.51}$$

The tree level result $T_o \sim m_R^{-2}$, predicted by the Landau- Khalatnikov theory [39], now is changed and T_o scales with the critical exponent z:

$$T_o \sim (m^2)^{-\nu z} \sim \xi^z \tag{III.4.52}$$

From (III.4.51) and (III.4.49), we can identify z:

$$z = 2+\gamma_t^*-\gamma_\varphi^* \qquad \text{(III.4.53)}$$

This result was first obtained in the framework of the R.G. equations by De Dominicis, Brezin and Zinn-Justin [41].

We will use the numerical values of next Section, obtained in the renormalization of the stochastic theory, to compute z at least at the first order and we will compare our results with the ε-expansion values [41][42].

In conclusion, we have shown that the T infinite limit of the stochastic quantization in the massless case can be studied with a method similar to that used for a quantum theory near the critical temperature. In particular, the exponent ν_T has been computed .

On the other hand, when the theory is massive, we have been able to compute the relaxation time and the related critical exponent.

In this way we achived various results: we clarified the role of the extra renormalization constants, proved the convergence of the stochastic theory even in the massless case, obtained the possibility of giving new numerical values to a dynamical critical exponent (for which, up to now, the available information is rather poor and contradictory).

IV. COMPUTATION OF THE CRITICAL EXPONENTS.

IV.1. THE ε-EXPANSION.

In the following we want to give a brief review of the problem of producing theoretical previsions on the behaviour of a statistical system near the critical temperature [29][40]. From the point of view of the field theory, this implies the study of the scaling behaviour of a system described by a scalar field (or a set of scalar fields if an additional rotational symmetry is concerned), with a 4-point self interaction. The mass term is related to the deviation from the critical temperature, since, as seen in Sect.III.4, when $m \to 0$ the correlation length diverges, and we are in presence of a phase transition. Since it is well known that the mean field approximation works well only at dimensionality $d>4$ ($d=4$ being the critical dimension), one is forced to look for some perturbative expansion in the coupling constant λ of the interaction term $\lambda\phi^4$, in order to get some information about the behaviour at the physical dimension $d=3$. However, this is not a straightforward

application of the standard tecniques of the perturbation theory. In fact, although from the ultraviolet point of view $\lambda\phi^4$ in d=3 is a super-renormalizable theory, when $m\to 0$ severe infrared divergences occour and the theory is no longer renormalizable in the infrared region. But we are interested in this region in statistical physics, since we want to compute the long scale properties. The non renormalizability of the theory is simply expressed by the fact that λ acquires dimensions μ^{ε}, with μ a mass scale and $\varepsilon=4-d$:

$$\lambda=\lambda_0\mu^{\varepsilon} \qquad \text{(IV.1.1)}$$

λ_0 is a dimensionless coupling constant. The first consequence is that we haven't any more a small parameter to expand the Green functions: a loop expansion is actually an expansion in λ ,and λ is small provided $\mu/\Lambda<<1$, where Λ is a typical mass scale. But since we are interested in the long distances, Λ is typically sent to zero and λ cannot be considered any

more a small perturbative parameter. Moreover, we are unable to fix the renormalization conditions: for example, consider the 2-point Green function at the tree level $\Gamma^{(2)}(p^2)=p^2 \to 0$ for $p^2 \to 0$. After a naive loop expansion has been performed, it is not still possible to require that $\Gamma^{(2)}(p^2;\lambda) \to 0$ for $p^2 \to 0$, since $\Gamma^{(2)}(p^2;\lambda)=\Sigma_L \mu^{\varepsilon L} \lambda_o{}^L p^{2-\varepsilon L} C_L$ and, for positive ε there will always exist a L such that $2-\varepsilon L<0$. The way out to this situation consists in a double expansion in loops and in ε, considering ε as a small parameter , i.e. the dimensionality of the system to be close to 4. With this technique the previous loop expansion becomes:

$$\Gamma^{(2)}(p^2;\lambda)=\Sigma_k \Sigma_L (\varepsilon L)^k/k!\ \lambda_o{}^L (\ln \mu/p^2)^k\ p^2\ C_L \Big|_{\varepsilon=0} \qquad \text{(IV.1.2)}$$

and no logarythmic term will overwhelm the p^2 factor. Of course now every quantity has to be computed at $\varepsilon=0$ where the theory is renormalizable and we can use the standard methods. Moreover, ε is now the small parameter we were looking for: after the ß-function has been obtained and its zeros computed in terms of ε, any

quantity computed at the fixed point will be expressed as a power series in terms of the small parameter ε. Among the fixed points of the ß-function we must verify that an infrared attractive fixed point exists and that this fixed point is indeed of order ε: the long scale behaviour of the system will be described by the critical exponents computed at this fixed point, the perturbative expansion being justified by the smallness of ε. After we have obtained the relevant quantities in terms of powers of ε, an extrapolation to the point $\varepsilon=1$ is needed in order to reach again the physical situation.

The first information we would like to get, is about the range of the correlations. In fact, the correlation between points separated by the distance r, goes as:

$$G(r) \cong \exp\{ -m^2 r\}\, r^{-d+2+\eta} \qquad \text{(IV.1.3)}$$

When m=0 (critical temperature) the behaviour of the correlations is dictated by a power law in which the 2-d is the classical (tree or mean field approximation) exponent, while η is the quantum correction. It is clear that in the field theory language, η is the anomalous

dimension of the 2-point Green function, and can be computed from the wave function renormalization constant Z (eq.(III.4.12)):

$$\eta = \mu \frac{\partial \ln Z}{\partial \mu}\Big|_B \qquad \text{(IV.1.4)}$$

Another interesting quantity is related to the susceptibility in the critical region:

$$\chi \cong |T-T_c|^{-\gamma} \qquad \text{(IV.1.5)}$$

The critical exponent γ can be computed in terms of η and $\gamma_\phi 2$, the anomalous dimension of the mass insertion:

$$\gamma = \{1+\gamma_\phi 2/(2-\eta)\}^{-1} \qquad \text{(IV.1.6)}$$

As we have seen in Sect.III.4, another interesting critical exponent to be computed is z, given in eq.(III.4.53).

In the following table we report the values for these

exponents for the first orders in ε [29][41] and from the high temperature expansion [43].

γ	1.167	1.24	1.250±0.003
η	0.019	0.037	0.041±0.01
z-2	0.013	0.006	--
	O(ε)	O(ε^2)	H.T.E.

Table 1. The values of the critical exponents in the ε-expansion and in the High Temperature Expansion.

IV.2.THE STOCHASTIC QUANTIZATION APPROACH

In this chapter we will put togheter everything we've learn about the stochastic quantization, the stochastic regularization, and the renormalization to devise a new method for computing the critical exponents.

First, in Sect. III.1, we have seen that with the stochastic quantization it is possible to introduce a new regularization scheme, that this scheme is consistent if the parameter σ of the kernel α_σ is contained in the interval $I_0=(-1;1)$ and that the theory converges for large stochastic times.

Second, in Sect. III.2, we have proved that $\lambda\phi^4$ is also renormalizable provided the fundamental relation between σ and the dimension d of the space is satisfied:

$$2\sigma+4-d=0 \qquad \text{(IV.2.1)}$$

Moreover we have proved that the renormalization can be done directly at the level of the Langevin equation by introducing the renormalization constants Z_t, Z_ϕ, Z_V, Z_η (see eq.(III.2.2)).

Third, in Sect. III.4, we have seen that it is possible to set m=0 before taking the $T\to\infty$ limit and that the theory converges to the critical one in this limit. At the same time, in a minimal subtraction scheme, the renormalization constants do not depend on T and so we can get all the relevant information about the critical theory.

Using all these facts, we can propose the following program to compute the critical exponents. We can set d=3, the physical dimension, from the very beginning, and, according to eq.(IV.2.1), $\sigma^*=-1/2$. The relevant 1PI Green functions have been found in Sect.III.2: the 2-point Green function $\Gamma^{(2)}$ and the 4-point $\Gamma^{(4)}$, and the mass insertion $\Gamma^{(A;2)}$. We can compute the divergent parts of these 1PI Green functions as poles in

$$\sigma-\sigma^*= \omega/2 \qquad \text{(IV.2.2)}$$

where ω is a new parameter (it plays the same role as ε in the ε-expansion). The divergences of $\Gamma^{(2)}$ will be absorbed by Z_t and Z_ϕ , while the divergences of $\Gamma^{(4)}$ by

Z_V. If, to define the mass insertion we add to the Langevin equation the term $2Z_A A_R \phi_R$, the mass insertion renormalization constant , taking into account also the wave function renormalization, is:

$$Z_{\phi^2} = Z_A Z_t^{1-\sigma} Z_\phi^{\sigma} \qquad \text{(IV.2.3)}$$

and the anomalous dimension:

$$\gamma_{\phi^2} = \mu\, \partial_\mu \ln Z_{\phi^2} \Big|_B \qquad \text{(IV.2.4)}$$

Once we know the renormalization constants, we can , first of all, compute the ß-function:

$$\beta(\lambda) = \mu \partial_\mu \lambda_R \Big|_B$$

$$\lambda_B = \mu^{\omega} Z_V Z_t^{\sigma-1} Z_\phi^{-(\sigma+2)} \lambda_R \qquad \text{(IV.2.5)}$$

The knowledge of the ß-function allows us to identify its zeros and compute the value $\lambda^* = \lambda^*(\omega)$ of the infrared attractive fixed point. From Z_ϕ and Z_t we can compute the wave function renormalization (see eq.(III.4.12)). Its

anomalous dimension at the fixed point gives the critical exponent η. From γ_{ϕ^2} computed at the fixed point and using eq.(IV.1.6) we can get γ. As an additional result we can get the critical exponent z using eq.(III.4.53).

IV.3.COMPUTATIONS AND RESULTS.

At the first order we have to compute the divergent part of the graphs of fig.3a and 3b for $\Gamma^{(2)}$ and $\Gamma^{(4)}$. The graph 3b) can be computed analytically and we obtain:

$$Z_v=1+\lambda 3R/\omega \qquad \text{(IV.3.1)}$$

since Z_t and Z_ϕ have at least contributions of order λ^2. Here $R=(2\sqrt{\pi})^{-3}$. In a similar way we get the mass insertion renormalization constant:

$$Z_\phi{}^2=1+\lambda R/\omega \qquad \text{(IV.3.2)}$$

a) b)

Fig.3a) graph for the wave function renormalization
Fig.3b) graph for the vertex renormalization.

From the graph of fig.3a, we can get Z_t and Z_ϕ. To this end we have to perform some integral numerically. We get:

$$Z_t=1-\lambda^2/2\ R_t R^2/\omega$$

$$Z_\phi=1-\lambda^2/2\ R_\phi R^2/\omega \qquad \text{(IV.3.3)}$$

with $R_\phi=0.196\pm0.003$ and $R_t=0.264\pm0.004$.

The ß-function is:

$$ß=-\omega\lambda+\lambda^2 3R \qquad \text{(IV.3.4)}$$

The fixed point occours at

$$\lambda^*=\omega/3R \qquad \text{(IV.3.5)}$$

As aspected, it is infrared attractive and of order ω. The numerical values for γ, η and z are:

$$\gamma=1.167$$

$$\eta=0.055\pm0.001$$

$$z-2=0.0075\pm0.0008 \qquad \text{(IV.3.6)}$$

The value of γ is the same as for the first order in the ε-expansion. The value of η is higher with respect to that obtained with the ε-expansion, while z-2 is lower. It is now interesting to go to the second order approximation and look if we can get closer to the high temperature expansion values.

The second order graphs contributing to the ß-function can be constructed starting from the expansion

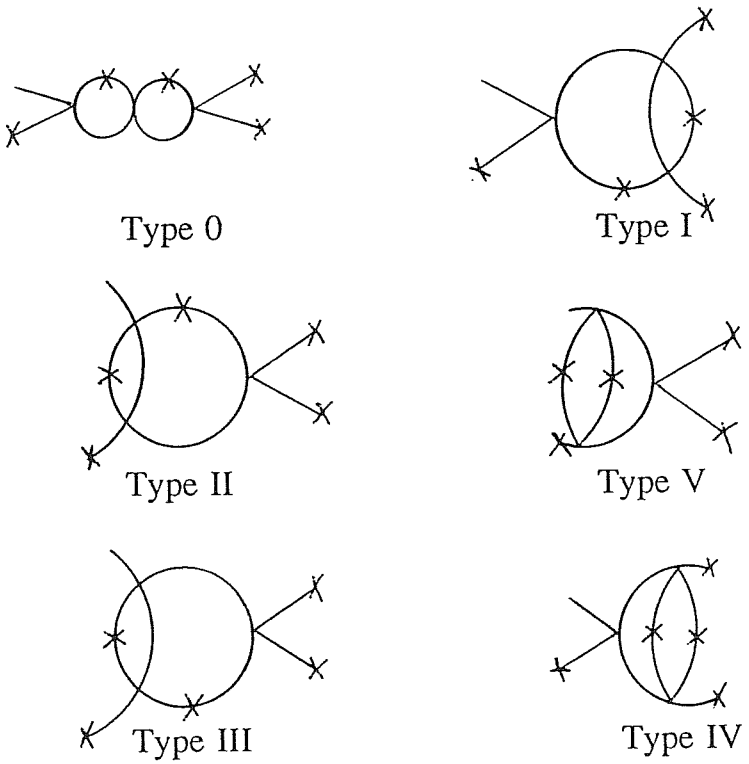


Fig.4 The two loops graphs contributing to $\Gamma^{(4)}$.

in tree diagrams of the field ϕ , and then contracting pairs of random sources. We need essentially to compute the divergent part of $\Gamma^{(4)}$.

By using the notation explained earlier, we get at order λ^3 , the graphs of fig. 4 (where the combinatorial factors have been explicitly shown). The IV and V types have a subloop with a cross on each line of the loop; we know that this kind of subloop is not divergent and so these graphs have just a divergence coming from the larger loop : a simple pole divergence. On the other hand , the other graphs have also divergent subloops and so we aspect them to show a double pole plus a simple pole structure. For example the type 0 is essentially contributing just with a double pole: its diverging part is the square of that of the simple loop we studied for the ß - function at order 1 loop (fig.3b), as can be easily seen:

$$\text{Divergent part of type 0} = (R/ \omega) \qquad \text{(IV.3.7)}$$

This gives rise to the problem of the cancellation of the double poles: in fact in the ß- function the double pole can be , at best, multiplied by ω and so there must be a cancellation , in order to get a finite ß- function.

Let us compute the ß - function at order λ^3 . We write:

$$\lambda_B = \mu^{\omega}\ Z_v\ Z_s\ \lambda_R \qquad Z_s = Z_t^{\sigma-1} Z_\phi^{-(\sigma+2)} \tag{IV.3.8}$$

and

$$Z_v = 1 + \lambda\ 3R/\omega + \lambda^2 \delta_2 Z_v \qquad Z_s = 1 + \lambda^2 \delta_2 Z_s$$

with

$$\delta_2 Z_s = (\sigma-1)\ \delta_2 Z_t - (\sigma+2)\ \delta_2 Z_\phi \tag{IV.3.9}$$

where $\delta_1 Z_v = 3R/\omega$, $\delta_2 Z_t$ and $\delta_2 Z_\phi$ are given by eq.(IV.3.1), (IV.3.2), (IV.3.3).

$$\beta = \mu\ \partial_\mu (\mu^{-\omega}\ Z_v^{-1}\ Z_s^{-1}\ \lambda_B) =$$

$$= -\ \omega\lambda + 3R\lambda^2 + 2\lambda^3\omega\ \delta_2 Z_s - 2\lambda^3\omega 9R^2/\omega^2 + 2\omega\lambda^3 \delta_2 Z_v$$

$$+ O(\lambda^4) \tag{IV.3.10}$$

Since $\delta_2 Z_s$ has only single poles, in order to get a finite ß- function , $\delta_2 Z_v$ must exibit a double pole part with the form:

$$\delta_2 Z_v = 9\, R^2/\omega^2 + \text{single pole part} \qquad \text{(IV.3.11)}$$

This is not only important from a theoretical point of view, but it provides also a check of the combinatorial factors we have computed. The result of our computation is that for each graph of fig.4, the double pole is equal to $(R/\omega)^2$ times the combinatorial factors indicated for each graph, times 1/n for n loops, i.e. 1/2 for this case, times the number of diverging subloops of the graph. In the same way we have obtained the cancellation of the double poles for the mass insertion and for the wave function renormalization. Indeed we get

$$\gamma_{\phi^2} = -\lambda R - 2\omega\lambda^2\, \delta_2 Z_{\phi^2} + \omega\lambda^2 (R/\omega)^2 + 3\,\lambda^2 R^2/\omega + O(\lambda^3) \qquad \text{(IV.3.12)}$$

and the result is finite since

$$\delta_2 Z_{\phi^2} = 2\,(R/\omega)^2 + \text{single pole part} \qquad \text{(IV.3.13)}$$

For the wave function we write

$$Z = 1 + \lambda^2\, r/\omega + \lambda^3\, \delta_3 Z \qquad r = -R^2(3R_t + R_\phi)/4 \qquad \text{(IV.3.14)}$$

and then

$$\eta = -2\,\omega\lambda^2\, r/\omega + 6R\,\lambda^3\, r/\omega - 3\,\omega\lambda^3\delta_3 Z + O(\lambda^4) \qquad \text{(IV.3.15)}$$

The result is finite since

$$\delta_3 Z = 2\,Rr/\omega^2 + \text{single pole part} \qquad \text{(IV.3.16)}$$

To evaluate the single pole part of the various diagrams of fig.4, we have to extract the pole in $1/\omega$ and compute numerically the residue, once the one loop counterterm has been taken into account.

We computed the residues S_i of ω^{-1} by using the program VEGAS (a Monte Carlo adaptive multidimensional integration routine [44]) on a GOULD 32/97 Computer. We got the following results, corresponding to the various graphs of fig.4:

Type I $\quad S_I = R^2\,(\,0.00 \pm 0.08\,)/4$

Type II $\quad S_{II} = R^2\,(\,0.00 \pm 0.04\,)/4$

Type III $\quad S_{III} = R^2\,(\,2.07 \pm 0.03\,)/4$

Type IV $\quad S_{IV} = R^2\,(\,7.38 \pm 0.1)/4$

Type V $\quad S_V = R^2\,(1.03 \pm 0.01)/4 \qquad \text{(IV.3.17)}$

Taking into account all the combinatorial factors the sum of the two loops contributions is:

$$-3!\ \lambda^3 U/4\omega = -3!\ \lambda^3/4\omega(S_I + 1/2S_{II} + 1/2S_{III} + 1/4S_{IV} + 1/4S_V)$$

$$U = 3.2 \pm 0.1 \qquad \text{(IV.3.18)}$$

Defining

$$\lambda R = k \qquad \text{(IV.3.19)}$$

we find, from eq.(IV.3.10), that the ß - function is:

$$ß(k)/k = -\omega + 3k - a k^2 + O(k^3)$$

where $$a = -3/2\ (R_t + R_\phi - 2U) = 9.0 \pm 0.4 \qquad \text{(IV.3.20)}$$

The fixed point now is

$$k^* = \omega/3 + a/27\ \omega^2 + O(\omega^3) \qquad \text{(IV.3.21)}$$

For the mass insertion we find that the two loops

contributions involve the previously studied graphs of type 0,II, III, V of fig.4. The second order term in Z_A is :

$$\delta_2 Z_A = 2(R/ \omega)^2 - U_A R^2/4\omega$$

$$\text{with } U_A = S_{II} + S_{III} + 1/2\, S_V = 2.62 \pm 0.07 \qquad \text{(IV.3.22)}$$

Expanding in powers of λ eq.(IV.2.3), we get $\delta_2 Z_\phi{}^2$ from $\delta_2 Z_A$:

$$\delta_2 Z_\phi{}^2 = 2(R/ \omega)^2 - R^2(U_A + 3R_t - R_\phi)/4\omega \qquad \text{(IV.3.23)}$$

Finally we get :

$$\gamma_\phi{}^2 = -R\lambda + \lambda^2 R^2/2\, (U_A + 3R_t - R_\phi) = -k + k^2\, c \qquad \text{(IV.3.24)}$$

where $c = 1.61 \pm 0.04$ and k is defined in eq.(IV.3.19).

We then compute $\gamma_\phi{}^{2*}$ at the fixed point :

$$\gamma_\phi{}^{2*} = - \omega/3 + \omega^2(3\, c - a)/27 + O(\omega^3) \qquad \text{(IV.3.25)}$$

The critical exponent γ is now obtained expanding eq.(IV.1.6) in powers of ω :

$$\gamma = 1 + \omega/6 + \omega^2/4\ (\ 2a - 6c + 3\ )/27 + O(\omega^3) \qquad \text{(IV.3.26)}$$

Extrapolating to $\omega=1$, this gives

$$\gamma = 1.27 \pm 0.01 \qquad \text{(IV.3.27)}$$

This is our result; it is really a good one, because we remember that from the ε– expansion we get $\gamma_\varepsilon = 1.24$ and the value of the high temperature expansion is $\gamma_T = 1.250 \pm 0.003$. [43].

Let us now compute the second order correction to the η critical exponent. There are three graphs contributing at this order (see fig.5) to the wave function renormalization.

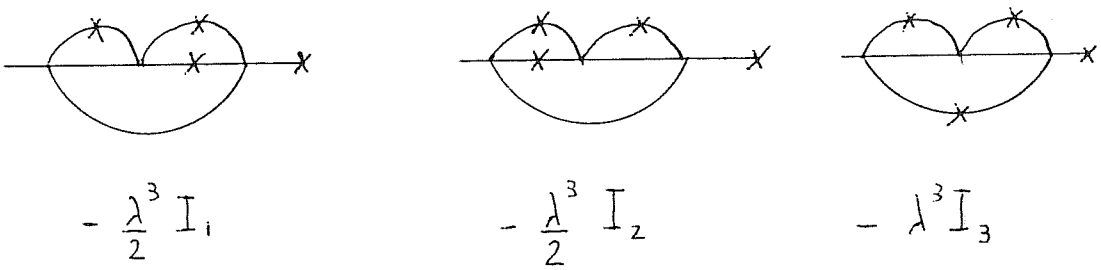


Fig.5 The three loops graphs contributing to the wave function renormalization.

The inverse of the complete propagator is

$$\Gamma^{(2)} = G^{-1} - \lambda^2/2\, W + \lambda^3\{ I_3 + 1/2\, I_2 + 1/2\, I_1 + \text{- counterterms}\} \qquad \text{(IV.3.28)}$$

where W is the first graph of fig.3a (the two loops graph).

From this we can compute the renormalization constants:

$$Z_t = 1 + \lambda^2/2\ G_t/\omega + \lambda^3\{ RG_t/\omega^2 - T/\omega \} + O(\lambda^4)$$

$$Z_\phi = 1 + \lambda^2/2\ G_\phi/\omega + \lambda^3\{ RG_\phi/\omega^2 - \phi/\omega \} + O(\lambda^4) \qquad \text{(IV.3.29)}$$

where, we have written $G_t = -R_t R^2$ and $G_\phi = -R_\phi R^2$; R, R_t , R_f have been defined previously , and T and ϕ are the single pole residues of the graphs of Fig. 5. We can now compute η in terms of these quantities:

$$\eta = \mu\partial_\mu \ln Z \qquad \text{for} \quad Z = Z_t^{(3-\omega)/2}\, Z_\phi^{(1+\omega)/2} \qquad \text{(IV.3.30)}$$

The result, at the fixed point k* is:

$$\eta = -(3/2\ G_t + 1/2 G_\phi)\omega^2/(9R^2) + \{ -(3G_t + G_\phi)\ a/(81R^2) +$$

$$+ (G_t - G_\phi)/(18R^2) + (3T + \phi)/(18R^3)\}\omega^3 \quad \text{(IV.3.31)}$$

where the quantity 'a' has been defined in eq.(IV.3.20).

At order ω^2 we get the old result; at order ω^3 of course we need the numerical values of T and ϕ :

$$\eta = 0.055\omega^2 + [\ 0.106 + (\ 3T + \phi\ )/(18R^3)] \quad \text{(IV.3.32)}$$

Calling S_{it} and $S_{i\phi}$ the residues of the single poles of the s and p2 derivatives of the graphs I_i (see fig. 5), we have:

$$T = S_{3t} + 1/2\ S_{2t} + 1/2\ S_{1t}$$

$$\phi = S_{3\phi} + 1/2\ S_{2\phi} + 1/2\ S_{1\phi} \quad \text{(IV.3.33)}$$

S_{it} and $S_{i\phi}$ have been computed numerically using the program VEGAS, like previously. All these numbers are given as a result of 7 dimensional integrals; the integrands are rather singular functions , due to the presence of a very delicate mechanism of cancellation of

logarythmic singularities inside them; this has not allowed a very accurate determination of these numbers due to the limits in CPU time and in the precision of the Computer. Moreover, an inspection of the equation defining η (eq.(IV.3.32)), shows that , in order to get a result compatible with the experimental value, an almost complete cancellation should occur between the number 0.106 coming from the two loops corrections to the value of k* and from the exponents in eq.(IV.3.30), and the term $(3T+\phi)$. One of our main results is that this indeed happens. However this cancellation makes also the statistical error on the final result rather large (since errors must be added in absolute values). Our estimates are, for the quantities defined previously:

$$S_{1\phi} = (-0.41\pm0.03)R^3 \qquad S_{1t} = (0.87\pm0.12)R^3$$

$$S_{2\phi} = (0.25\pm0.04)R^3 \qquad S_{2t} = (-0.37\pm0.07)R^3$$

$$S_{3\phi} = (-0.27\pm0.05)R^3 \qquad S_{3t} = (-0.85\pm0.12)R^3 \qquad \text{(IV.3.34)}$$

and therefore

$$T = (-0.6\pm0.2)R^3 \qquad \phi = (-0.35\pm0.08)R^3 \qquad \text{(IV.3.35)}$$

We get from that:

$$\eta = 0.055\ \omega^2 + \omega^3\ (-0.013 \pm 0.03) \qquad \text{(IV.3.36)}$$

Extrapolating to $\omega=1$ we get the result

$$\eta = 0.042 \qquad \text{(IV.3.37)}$$

with a large statistical error ($\pm$ 0.03).

To get a small error we have to increase the accuracy in the numerical integration at least of a factor 10, which is, for now, out of the computer possibilities. However it is already a good result to obtain η in the range of values estimated by means of the (resummed) ε-expansion and of the high temperature expansion [40][43]. We can notice that the critical exponent η seems , in every method, to be much more difficult to compute, giving a large error, with respect to the exponent γ. The experimental value for η is also rather uncertain: $\eta_{exp} = 0.045 \pm 0.015$ [45].

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