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# ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

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# DYNAMICAL SYMMETRY BREAKING IN QCD AND QUARK MASSES

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## ABSTRACT

Using an effective potential approach for composite operators, I study dynamical symmetry breaking in QCD-like gauge theories. The analysis is extended to massive quarks in QCD with three flavors and the masses of the pseudoscalar octet mesons and their decay constants are calculated. Renormalization group corrections are taken into account. The effective potential depends on the standard parameters of QCD: $\Lambda_{QCD}$, $m_u$, $m_d$, $m_s$ and on a further mass scale $\mu$ which discriminates between the infrared and the ultraviolet regimes. A good fit for the meson masses (agreement within 3%) and for the decay constants is obtained for the following values of the quark masses at 1 $GeV$: $m_u = 5.8\ MeV$, $m_d = 8.4\ MeV$ and $m_s = 118\ MeV$. These values essentially agree with the values obtained by quite different methods except for certain sum rules estimates of $m_s$ which give larger values.

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## 1. INTRODUCTION

There are now various reasons to believe that $SU(3)_c$ gauge theory of quarks and gluons (quantum chromodynamics (QCD)) is the best candidate theory of hadron physics even though many essential properties that it is presumed to have such as confinement, dynamical mass generation and chiral symmetry breaking, are still poorly understood.

Because of the complexity of the strong interaction phenomena which this theory describes, it is impossible to solve it exactly and some sort of approximation is nedeed.

In fact, in the high momentum (or short-distance) regime, quarks and gluons can be treated as weakly interacting particles in perturbative QCD, but, at the other end of the scale is low-energy hadronic physics. Here the interacting units are not individual quarks and gluons but hadrons. So far we cannot solve QCD in this domain.

In this work we will focus on the problem of the realization of the chiral symmetry in QCD and on the causes and consequences of its spontaneous and explicit breaking.

From the vast amount of experimental informations available, some highly fruitful ideas had been developed long before QCD was invented. We are referring to what is known as "current algebra".

The mass matrix $\mathbf{m}$ in the QCD Lagrangian density is a phenomenological quantity whose origin is unknown. If QCD leads to quark confinement, as it is assumed, then the mass parameters are not observable quantities. However they can be determined in terms of observable hadronic masses through "current algebra" methods.

In this framework the quark mass problem shows up only implicitly in the symmetry properties of the Lagrangian that is in the commutation rules involving the currents and the energy momentum tensor.

The approximate unitary symmetry of the strong interactions implies that the masses of the light quarks are almost equal $m_u \simeq m_d \simeq m_s$ and the breaking of this global flavor symmetry was identified as an octet term very early [1].

Since isospin conservation is a much better symmetry than the whole flavor $SU(3)$, the relation $m_u \simeq m_d$ should hold to a higher degree of accuracy than $m_d \simeq m_s$. Also, the mass parameters for quarks $c$, $b$, $t$ should all be much larger than those of $u$, $d$, $s$ because we see no trace of flavor $SU(4)$ or higher symmetries in the hadronic spectrum.

If we put $m_u = m_d$ then isospin will become exactly conserved. In this limit the neutron and the proton will have the same mass and so will the pions $\pi^+$, $\pi^-$ and $\pi^0$.

But this does not explain the smallness of pion mass $(M_\pi/M_p = 0.14)$ which makes the pion very special among the hadrons.

To understand this Nambu [2] suggested that there is a limit, which is an idealized theoretical construct, in which the pion is a massless Goldstone boson associated with spontaneous symmetry breaking.

To translate these ideas in QCD let us consider the chiral limit $m_u = m_d = 0$.

Since $m_u$ and $m_d$, as we will show later, are small compared to the nucleon mass, this limit gives a reasonably good description of ordinary hadronic physics in which strangeness and higher flavors do not play a direct role.

This unperturbed system is invariant under the global symmetry group $SU(2)_V \otimes SU(2)_A \otimes U(1)_V \otimes U(1)_A$ where the subscripts V and A stand respectively for "vector" and "axial-vector".

How are these symmetries manifested in nature?

The $U(1)_V$ is manifested directly as barion number conservation whereas the $U(1)_A$ is broken by the Adler, Bell and Jackiw axial anomaly. The $SU(2)_V$ invariance leads to the isospin conservation and in fact the hadron fall into easily recognizable isospin multiplets.

On the other hand a direct manifestation of $SU(2)_A$ would require that each isospin multiplet is accompanied by a mirror multiplet of the same mass but with opposite parity. No hint of this can be found in the hadronic spectrum.

Assuming that the real world is well approximated by the chiral limit and that the flavor and chiral invariance are valid to a similar order of approximation, one must conclude that these symmetries are not realized in the same manner.

In general, given a set of generators $L^j$ of symmetry transformations of the Lagrangian density $\mathcal{L}$, we have two possibilities:

$$L^j|0\rangle = 0 \tag{1.1}$$

which is called Wigner-Weyl symmetry, or

$$L^j|0\rangle \neq 0 \tag{1.2}$$

called Nambu-Goldstone symmetry.

Two theorems are specially relevant with respect to these questions.

The first, due to Coleman [3], asserts that "the invariance of the vacuum is the invariance of the world" or, in more straightward terms, the physical states (including bound states) are invariant under the transformations of a group of Wigner-Weyl symmetries.

In this way it is strongly suggested that the $SU(2)_V$ is an approximated Wigner-Weyl symmetry and that the chiral $SU(2)_L \otimes SU(2)_R$ contains Nambu-Goldstone type generators.

The second relevant theorem is due to Goldstone [4]. It states that for each generator that fails to annihilate the vacuum there must exist a massless boson with the quantum numbers of that generator.

Therefore we may explain the smallness of the $\pi$ masses because in the limit $m_u, m_d \to 0$ we would have $M_\pi \to 0$.

The mass of the physical pions then comes from the explicit chiral symmetry breaking parameters $m_u$ and $m_d$. Since $m_u$ and $m_d$ are small, the pions are *almost* massless and the axial-vector currents to which they couple are *almost* conserved.

Let the pion states be denoted by $|\pi_j\rangle$ where $j = 1, 2, 3$ is the isospin index. The axial-vector currents $J^k_{\mu 5}$ $k = 1, 2, 3$ can annihilate states of the same quantum numbers as the pions and hence connect the pion states to the vacuum. By Lorentz invariance and isospin consevation we can write

$$\langle 0|J^k_{\mu 5}(x)|\pi_j(p)\rangle = i\ p_\mu\ \delta_{kj} f_\pi e^{-ipx} \tag{1.3}$$

where $p_\mu$ is the pion 4-momentum anf $f_\pi$ is the pion decay constant. Then, taking the divergence

$$\langle 0|\partial^\mu J^k_{\mu 5}(x)|\pi_j(p)\rangle = \delta_{kj} f_\pi\ M_\pi^2 e^{-ipx} \tag{1.4}$$

which expresses the property of the pions of being the Goldstone bosons in the chiral limit since $\partial^\mu J_{\mu 5} = 0$ implies $M_\pi = 0$.

One can defines

$$\phi_k(x) \equiv \frac{1}{M_\pi^2 f_\pi} \partial^\mu J^k_{\mu 5}(x) \tag{1.5}$$

then

$$\langle 0|\phi_k(x)|\pi_j(p)\rangle = \delta_{kj} e^{-ipx} \tag{1.6}$$

The content of PCAC is a rule for using $\phi_k(x)$ as a pion field operator in the chiral limit.

Let us spend some more words on the Nambu-Goldstone realization of the chiral $G = SU(2)_L \otimes SU(2)_R$ symmetry.

The six generators of this group $Q^i$ and $Q^i_5$ (i=1,2,3) are such that

$$Q^i|0\rangle = 0 \tag{1.7}$$

$$Q^i_5|0\rangle \neq 0 \tag{1.8}$$

Hence the generators to which the isospin vector currents are associated do annihilate the vacuum and generate a subgroup H of G, the stability group of the vacuum, (here $H = SU(2)_{L+R} = SU(2)_V$), while the generators $Q^i_5$ lie in the quotient space

$$Q^i_5 \in Lie\ G/Lie\ H \tag{1.9}$$

This phenomenon is commonly referred as the spontaneous symmetry breaking of the symmetry G down to the symmetry H (SSB) [5].

This means that, even though the Lagrangian is chiral invariant, this symmetry is not reflected algebrically in the S-matrix elements since it is not a symmetry of the ground state. The symmetry is not really broken, of course the axial-vector currents are exactly conserved and the pions remain massless through all perturbative orders of the Lagrangian.

Furthermore, since there is strong evidence for approximate unitary $SU(3)$ symmetry in hadronic physics, one might entertain the idea of an extended chiral-symmetric limit corresponding to $m_u \simeq m_d \simeq m_s = 0$.

By the same reasoning as before, one concludes that this extended chiral symmetry is spontaneously broken and is manifested through the existence of Goldstone particles identifiable with the eight lightest pseudoscalar mesons $(\pi, \mathrm{K}, \eta)$.

In this limit the global symmetry group is $SU(3)_V \otimes SU(3)_A \otimes U(1)_V \otimes U(1)_A$ with the $SU(3)_A$ realized in the Goldstone mode and the $SU(3)_V$ realized directly in the "eightfold way".

Let us remark that the eightfold way is an approximate symmetry of the strong interactions not because the quarks $u$, $d$, and $s$ have similar masses, but because their mass differences are small in comparison to hadron masses.

The explicit chiral symmetry violation raises the masses of the pseudoscalar mesons to finite values while the $SU(3)_V$ violation leads to mass splitting in the pion octet as well as those in all other flavor multiplets [6].

This arises from the perturbation Lagrangian density

$$-(m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s) \tag{1.10}$$

whose effects are usually treated by "chiral perturbation theory" [7] which is a combination of "current algebra" and "extended PCAC".

The question is now the following: what is the dynamical reason why $SU(3)_A$ manifests itself in the Goldstone mode?

In the non-renormalizable pre-QCD model of Nambu and Jona-Lasinio [8] the cause of spontaneous symmetry breakdown is a direct nucleon-nucleon attraction.

The scheme is motivated by the observation of an interesting analogy between the properties of Dirac particles and the quasi-particle excitations that appear in the theory of superconductivity of Bardeen, Cooper and Schrieffer (BCS) [9].

The characteristic feature of the BCS theory is that it produces an energy gap between the ground state and the excited states of a superconductor.

This gap is due to the fact that the attractive phonon-mediated interaction between electrons, produce correlated pairs of electrons with opposite momenta and spin near the Fermi surface.

As the energy gap in a superconductor is created by an effective electron-electron attraction, one assumes that the mass of a Dirac particle is also due to some interaction between massless bare fermions.

In ref. [8] a simplified non-renormalized model of a chirally invariant four fermion interaction is considered.

The implications are that the nucleon mass is generated by some primary interaction between originally massless fermions and that the same interaction is also responsible for the formation of pseudoscalar zero-mass bound states of fermion-antifermion pairs which may be regarded as idealized pions.

The presence in the physical spectrum of massless particles is a manifestation of spontaneous symmetry breaking.

However, here the Goldstone bosons are composite particles and the Goldstone mechanism turns out to be secondary in the sense that it is not related to the fundamental Lagrangian but it is related to the effective Lagrangian of hadron interaction.

Hence, in the contest of a non-renormalizable direct fermion-antifermion interaction, chiral symmetry is spontaneously broken by the dynamics of the strong interactions. This phenomenon is called dynamical symmetry breaking (DSB).

Just as the effective electron-electron attraction in superconductivity arises from the more fundamental electron-phonon interaction, the Nambu-Jona-Lasinio model represents an effective low-energy description of the strong quark-gluon gauge interaction.

So the further problem will be to explain how the color forces really lead to the dynamical breakdown of chiral symmetry. In such a way the problem will be reduced to the dynamical realization of a linear $\sigma$-model [10].

As it is known, in the unstable (symmetric) phase of the $\sigma$-model there are $n^2$ scalar and $n^2$ pseudoscalar tachyons assigned to the $(n, n^*) \oplus (n^*, n)$ representation of the chiral $SU(n)_L \otimes SU(n)_R$ group. In addition there are $n$ left and $n$ right massless fermions assigned to the $(n,0)$ and $(0,n)$ representations respectively.

The occurence of tachyons indicates that the vacuum of the normal phase with massless fermions is unstable.

Under the vacuum rearrangement (phase transition), the symmetry is lowered to $SU(n)_V \otimes U(1)_V$. Then $n^2 - 1$ pseudoscalar tachyons are transformed into massless Goldstone bosons and $n^2$ scalars and one pseudoscalar (associated with the $U(1)_A$ broken by the axial anomaly) are transformed into massive bosons. Fermions acquire mass due to the Yukawa-type interaction with scalar bosons.

In the framework of QCD, hadrons are represented by bound states of quarks and antiquarks. So one must determine the forces which can lead to such tightly bound states as tachyons and Goldstone bosons in the unstable and stable phases respectively.

One expects that the binding of the fermions, coming from the strong action of the color forces at distances of the size of Goldstone bosons, results in the appearence of condensates breaking spontaneously chiral symmetry.

The crude but basic idea is the following. Consider a bound state of a pair of massless quark and antiquark. Because of the uncertainty principle, the energy of the ground state in a fully relativistic formulation will be given by

$$E^2 \simeq p^2 - g^2/r^2 \simeq p^2(1-g^2) \qquad (1.11)$$

where $p$ and $r$ denote the relative momentum and coordinate respectively and $g$ is the gauge coupling constant.

When $g$ exceeds order one, there will be a tachyon bound state indicating the instability of the vacuum. In order to cure this instability, the vacuum rearranges itself and gives mass to quarks.

The existence of a critical value for the coupling is essential for the mechanism of dynamical mass generation.

The gauge coupling in quantum chromodynamics is asymptotically free and becomes strong at large distances. So there will be a scale at which the ground state of the theory has an indefinite number of massless fermion pairs which can be created by the strong coupling.

Since we still expect the bound state to be invariant under Lorentz and color $SU(3)_c$ transformations, it will only contain pairs with vanishing total momentum, angular momentum and color charge but with a net chiral charge.

More generally the situation is that the vacuum $|\Omega\rangle$ will have the property that an operator which destroys a fermion pair has a non zero vacuum expectation value

$$\langle\Omega|\overline{\Psi}_{Li}\Psi_{Rj}|\Omega\rangle = \langle\Omega|\overline{\Psi}_{Ri}\Psi_{Lj}|\Omega\rangle = v\delta_{ij} \tag{1.12}$$

corresponding to equal condensation of pairs for each flavor.

So, in the limit in which the global $SU(n)_V$ is an unbroken symmetry, we will have the same dynamical mass generation for each quark flavor.

These arguments summarize the basic points of physics that we wish to discuss in detail in this work. It remains to carry out this analysis more completely and precisely.

Our first priority is to learn how to do a more quantitative computation of chiral symmetry breaking. Basically we need to know how to test whether the energy of the vacuum is lowered when some fermion bilinear acquires a nonzero vacuum expectation value.

The standard method to describe symmetry breaking in field theories when the quantity acquiring a vacuum expectation value is a scalar field $\phi$ is to evaluate the effective potential [11].

The main property of the effective potential is that it turns out to be equal to the energy of the vacuum under the constraint that the vacuum expectation value of $\phi$ has some definite value $\phi_c$. So one needs only to minimize this functional with respect to $\phi_c$ in order to determine the vacuum value of $\phi$ and the various phases of the theory.

A series expansion for the effective potential was derived by Jackiw [12]. Each order of the series corresponds to an infinite set of Feynman diagrams with a fixed numbers of loops.

This functional evaluation of the effective potential results very useful since it is important to be able to study the higher-order multiloop graphs, if not explicitly, at least in general terms.

In fact there exist phenomena which cannot be easily seen in perturbative series. A clear example is given by the formation of bound states which can never be observed in a finite order of a loop expansion. Necessarily they require at least an infinite subset of all orders ( remember that in the case of a chirally symmetric theory the invariance of the Lagrangian guarantees that the mass term in the fermion propagator will never appear in any order of perturbation theory).

So, what we need is an approximation scheme that preserves some of the non-linear features of field theory which, presumably, leads to these co-operative and coherent effects.

With the effective potential series as introduced by Jackiw, it is possible to sum large classes of ordinary perturbation-series diagrams so it represents a formalism especially appropriate for the study of DSB.

Actually in this case one expects that the breaking of the theory is due to the formation of bound states (condensates) playing the role of the previous elementary scalar fields $\phi$. So one needs an appropriate generalization of the effective potential for composite operators.

This was introduced by Cornwall, Jackiw and Tomboulis (CJT) [13]. The idea is to introduce inside the generating functional of the Green functions of the theory, a bilocal source $J(x,y)$ coupled to the composite operator we are interested in and then to Legendre-transform to a generalized effective action.

The functional $\Gamma$ they obtain in this way for a scalar theory, depends not only on the possible expectation value of scalar field $\phi_c(x)$, but also on $G(x,y)$, a possible expectation value of $\mathrm{T}(\phi(\mathrm{x})\phi(\mathrm{y}))$ and it represents the generating functional in $\phi_c$ of the two particle irreducible Green functions expressed in terms of the propagator $G$. (The conventional effective action is merely $\Gamma(\phi_c, G)$ at $J(x,y)=0$).

Physical solutions require

$$\frac{\delta\Gamma(\phi_c, G)}{\delta\phi_c}=0 \tag{1.13}$$

$$\frac{\delta\Gamma(\phi_c, G)}{\delta G}=0 \tag{1.14}$$

Eq. (1.13) reproduces the equations of motion of the theory while eq. (1.14) is nothing but the Schwinger-Dyson equation for the full propagator $G$.

We see that this formalism is especially appropriate for the study of dynamical symmetry violation which is characterized by the fact that even though (1.13) has only the symmetric solution $\phi_c(x) = 0$, symmetry breaking solutions exist for (1.14).

In ref. [13] a formal series expansion is derived for the generalized effective action consisting on a systematic resummation of graphs with a fixed number of loops.

So one has to evaluate $\Gamma_{CJT}$ up to a certain loop approximation and derive the stationary conditions of eqs. (1.13) and (1.14) corresponding to vanishing sources.

A non-symmetric solution of eq. (1.14) for the composite operator $G$ is a signal for dynamical symmetry breaking.

For example, in the case of spontaneous chiral symmetry breaking ($\chi$SB), all this procedure is equivalent to turn on some external field ( analogous to a magnetic field orienting a potentially ferromagnetic system) coupled to the bilinear $(\overline{\psi}\psi)$, construct the ordered vacuum in the presence of this field and then see if the order of this vacuum survives when we turn off this field. The vacuum expectation value of the composite operator $\langle\overline{\psi}\psi\rangle$ is the order parameter characterizing the phase transition.

But, as it has been pointed out in refs. [14] and [15], the CJT functional has an intrinsic defect: it is not bounded from below. In particular, in the case of a $SU(N)$ fermion gauge theory, a detailed numerical study [16] shows that all chiral symmetry breaking stationary points are saddle points.

This is a very unpleasant property if one wants to perturbe the vacuum of the theory to find its excitations.

To cure this instability problem of the CJT formulation R. Casalbuoni, D. Dominici, R. Gatto and I, have introduced a modification of the CJT effective action which corresponds to a different choice of the source-dependent term [18], [22].

Our functional has the same stationary points as the CJT one but does not suffer from the problem of unboundness from below and also has the property that the solutions of the Schwinger-Dyson equation for the propagator correspond to real minima.

The main difference in the two cases resides in the choice of the dynamical variable. In the case of fermionic gauge theories the CJT effective action is a functional of the full fermion propagator $S$ while our effective action results completely espressed in terms of the fermion proper self-energy $\Sigma$.

We have applied this formalism to the study of the dynamical breaking of the chiral symmetry in QCD-like gauge theories.

Our main hypothesis is that the relevant contribution for the $\chi$SB phenomenon comes from relatively short-distance effects as also suggested by computer simulations [17]. This justifies our calculation for the effective action which are performed in the two-loop approximation.

Our strategy consists in introducing a parameter $\mu$ as an infrared cutoff. We have assumed the self-energy of the fermions as a constant at energies lower than $\mu$ whereas, for greater energies, we have used the behaviour given by the Operator Product Expansion (OPE).

The parametrization of the fermion self-energy is in terms of the renormalized fermionic condensates. The condensates are indeed our variational parameters to be determined by looking at the minimum of the generalized effective potential.

In our first works on this subject [18], [19] and [20], we have discussed the so called "rigid case" in which the logarithmic corrections coming from the renormalization group analysis are neglected.

In this approximation it is possible to derive analytically the complete expression for the effective potential at two fermion loops.

The result is that in the case of massless fermions, the theory has two phases: the chirally symmetric phase and the broken phase into the diagonal flavor subgroup when the gauge coupling constant exceeds a critical value.

In the case of massive quarks, it is necessary to renormalize the composite operator wave function. This leads to a condition which is equivalent to the Adler-Dashen requirement in the limit of vanishing quark masses and also ensures the absence of spontaneous breaking of parity.

In this framework, since the lowest energy vacuum corresponds to a local minimum of our effective potential, it is possible to calculate the masses of the pseudoscalar mesons. In fact they are simply related to the second derivatives of the effective potential which, in our model, turn out to be positive definite (remember that our stationary points are minima).

In ref. [20], in collaboration with A. Barducci, we find that, in QCD with three flavors, a good fit for the pseudoscalar meson masses (singlet sector excluded) can be obtained with the quark mass ratios

$$m_d/m_u = 1.94 \qquad m_s/m_d = 21.7 \qquad (m_s - \hat{m})/(m_d - m_u) = 43.14$$

$$(\hat{m} = (m_u + m_d)/2)$$

to be compared, for example, with the corresponding values $1.76 \pm 0.13$, $19.6 \pm 1.6$, $43.5 \pm 2.2$, given in [21].

The further step consists in generalizing all these results to the case in which one considers the gauge coupling constant and the fermion self-energy corrected by the renormalization group analysis in the leading logarithmic approximation.

In the case of massless quarks we have discussed the different ansatzes with or without logarithmic corrections and we have found that the result of spontaneous symmetry breaking of chiral symmetry for large couplings remains stable [22] and [23]. In particular we find that, when the leading-log expressions are used both for $g$ and for $\Sigma$, then $\chi$SB does occur in QCD with three flavors for

$$\alpha_s = \frac{g^2(\mu)}{4\pi} \gtrsim 0.73\pi \qquad (1.15)$$

with an $U(3)_V$ residual symmetry.

The analysis of the massive case has been carried out in ref. [24] and represents the central part of this work.

We evaluate the effective potential as a functional of the proper fermion self-energy $\Sigma$ in QCD with three flavors in the general case in which both spontaneous and explicit breakdown of the chiral symmetry are present.

The calculations are in the two-loop approximation, use is made of the Landau gauge and of the renormalization group improved expression for the gauge coupling constant.

From the analysis of the asymptotical equations of the Green functions we deduce the form of the test function to adopt for $\Sigma$.

We assume a constant behaviour in the infrared region of momentum and a decreasing as $1/p^2$ $(logs)$ for $p > \mu$ which is completely consistent with the OPE analysis.

By substituting in the effective action we find an expression which is ultraviolet finite. This fact is connected with the use of renormalization improved expressions for $\Sigma$ and $g$ which completely regularize the theory in the two-loop approximation we are considering.

However a finite part of the effective potential remains to be fixed with a suitable normalization condition. The natural choice for it comes from the expression of the effective potential for small masses and, in this limit, it is equivalent to the Adler-Dashen requirement.

Our method consists now in making a convenient ansatz for $\Sigma$ in terms of a set of parameters related to the fermionic condensates and then in minimizing the effective potential with respect to these parameters.

We find that, also in the massive case, the effective potential evaluated at the minimum factorizes out in the sum of separate contributions, one for each flavor.

We determine in this way the values of the condensates for the quarks $u$, $d$ and $s$ at the minimum. They depend on the parameters of our model: the renormalization invariant mass $\Lambda_{QCD}$, the three quark masses $m_u$, $m_d$ and $m_s$ and the further scale $\mu$ we have introduced in order to separate the infrared from the ultraviolet region of momenta.

Our task is to determine these parameters from the experimental data.

We are in the position of calculating the masses of the pseudoscalar mesons which represent the pseudo-Goldstone bosons of our derivation.

Also we can derive an expression for the decay coupling constants for the pseudoscalar meson octet by following refs. [25] and [26].

The necessary ingredients are the normalization of our pseudoscalar dynamical variables and the vertex function of the pseudoscalar composite fields. Then the decay constants are evaluated from the coupling of the meson fields $|\pi_j\rangle$ to the axial-vector currents $J^k_{\mu 5}(x)$ $(j, k = 1, ..8)$. (The mixing in the 3-8 sector has been explicitly taken into account).

We have in this way a system of coupled equations and we determine the parameters of our model with an iterative procedure.

We obtain a very good fit for the meson masses (agreement within 3%) and for the decay coupling constants for the following values of the quark masses at 1 GeV

$$\begin{aligned} m_u(1) &= 5.8\ MeV \\ m_d(1) &= 8.4\ MeV \\ m_s(1) &= 118\ MeV \end{aligned} \tag{1.16}$$

These values agree with the valeus obtained by quite different methods, except for certain sum rule estimates of $m_s(1)$ which give larger values.

In sect.2 we review some fundamental aspects of alternative forms of the effective potential for composite operators and we look at the nature of their stationary points. Then we derive our modified version for the effective action and discuss its properties.

In sect.3 we evaluate the effective action for a QCD-like gauge theory of massive fermions as a functional of $\Sigma$ which, at the phisical point, turns out to be the proper fermion self-energy.

In sect.4 we derive the ultraviolet behaviour of the fermion self-energy in the general case in which both spontaneous and explicit breakdown of the chiral symmetry are present. We also make some comments in favour of the use of the so called "regular solution" for the self-energy.

In sect.5 the variational ansatz for $\Sigma$ is discussed.

In sect.6 we analize the cancellation of the ultraviolet divergences in $\Gamma$ and we normalize our functional.

In sect.7, in order to better understand the pattern of the dynamical breakdown of the chiral symmetry, we analize the properties of the massless effective potential.

Sect.8 is devoted to the comparison with other studies of the dynamical symmetry breaking phenomenon.

In sect.9 we discuss the general massive case in the case of QCD with three flavors.

Then we calculate the masses and the decay coupling constants of the octet pseudoscalar mesons in terms of the parameters of our model in sects.10 and 11 respectively.

The numerical results are given in sec.12 while in sec.13 our discussion focuses on the comparison of the values we get for the quark masses with the values obtained by different methods.

Conclusions and further developments are in sec.14.

## 2. EFFECTIVE ACTION FOR COMPOSITE OPERATORS, REVIEW OF ALTERNATIVE FORMULATIONS

The main tool in the following analysis is a modified version of the effective action for composite operators introduced by Cornwall, Jackiw and Tomboulis (CJT) [13].

The physical interest in the study of the effective action and, more particularly in the study of the effective potential, is the fact that the minima of this functional determine the physical vacuum of the theory.

Of course this is particularly relevant when one expects a non trivial vacuum as in the case of spontaneously broken symmetries.

When the breaking of the theory is due to the formation of bound-states (condensates), as in the case of strongly interacting fermionic theories, the technique of the effective action for composite operators turns out to be very useful. In fact it consists of a systematic resummation of graphs which is capable of describing non-perturbative phenomena in a sequence of approximations.

The more direct way to construct an effective action describing the interactions between the elementary degrees of freedom of the theory and the collective modes (composite fields), is to introduce an auxiliary field in the generating functional and to develop a loop expansion in it.

This approach leads to the so called "collective variables" or "auxiliary field" (AF) method (see for example [27]).

Unfortunately the (AF) technique suffers of severe limitations because it can be usefully applied only in the case of quartic interactions.

A more general formalism to study dynamical symmetry breaking was introduced by Domokos and Suranyi [28], and rediscussed some years later by Cornwall, Jackiw and Tomboulis [13].

In this method, one introduces a "classical " bilocal field which turns out to be the one-particle propagator of the theory and defines a generalized effective action such that its variations with respect to the usual "classical" field and to the bilocal field reproduce the equations of motion of the theory and generate the Schwinger-Dyson (SD) equation for the propagator (gap equation).

We believe that this is the most efficient way to our disposal to discuss DSB.

We will review the AF and the CJT methods for a fermion gauge theory showing that, in the lowest approximation, these formulations are equivalent in the sense that the stationary points in the two cases simply give the same dynamics.

We will then introduce a modified version of the CJT functional having the same local extrema as the CJT one but a different asymptotic behaviour turning out to be bounded from below [29],[18] and [22].

The main advantage of the new form of the action is that the solutions of the gap equation correspond to real minima. This can be seen from the explicit expression and from the general analysis performed by various authors [30] and [31].

This stability property is essential if we want to do something more than just find the extrema of the effective potential and in particular if we want to perturbe the vacuum to find its excitations.

Let us recapitulate here the salient points of the CJT formalism in the case of a fermion gauge field theory in its euclidean formulation.

This technique consists in introducing a bilocal source $J(x,y)$ coupled to the operator $\overline{\psi}(x)\psi(y)$ in the generating functional of the theory $Z_{CJT}\ [J]$

$$Z_{CJT}\ [J] = e^{-W_{CJT}\ [J]} = \\ = \frac{1}{N}\int \mathcal{D}\psi\ \mathcal{D}\overline{\psi}\ \mathcal{D}A_\mu\ e^{-[I(\psi,\overline{\psi},A_\mu)\ +\ \overline{\psi}J\psi]} \tag{2.1}$$

where $N$ is a normalization constant (it will be omitted from now on), and $I(\psi,\overline{\psi},A_\mu)$ is the classical euclidean action for the gauge theory evaluated in the Landau gauge (see sect. 3). Clearly it also contains ghost terms but, for sake of simplicity, we have written down only the dependence on the fermion and gauge fields. Use is made of the shorthand notation

$$\overline{\psi}J\psi = \int d^4x\ d^4y\ \overline{\psi}_\alpha(x)J_{\alpha\beta}(x,y)\psi_\beta(y) \tag{2.2}$$

where the index $\alpha$ is a collective index for spinor and flavor variables. Also we have not explicitly introduced in $Z_{CJT}\ [J]$ the usual linear sources coupled to $\psi$ and $\overline{\psi}$ because we are not interested in their effects.

Let us define the "classical" bilocal field $S$

$$\frac{\delta W_{CJT}}{\delta J} = -S \tag{2.3}$$

and introduce the effective action $\Gamma_{CJT}$ as the Legendre transform of the generating functional of the connected Green functions $W_{CJT} = -\log Z_{CJT}$

$$\Gamma_{CJT}\,[S] = W_{CJT} - \frac{\delta W_{CJT}}{\delta J}\,J \tag{2.4}$$

It follows that

$$\frac{\delta \Gamma_{CJT}}{\delta S} = J \tag{2.5}$$

The Legendre variable $S$ conjugate to $J$ will be the CJT dynamical variable.

For physical processes ($J = 0$), $S$ has to satisfy the stationary condition for the effective action $\delta\Gamma_{CJT}/\delta S = 0$.

We will show that this is nothing but the Schwinger-Dyson equation for the fermion propagator and so $S$ coincides with the exact fermion propagator when the source $J$ is turned off.

A remark is now in order. If one is interested only in translationally invariant solutions of the SD equation, one can take the composite field $S$ to be a function of the space-time difference $(x - y)$.

In this way an overall factor of space-time volume factorizes out and the effective potential for composite operators $V_{CJT}$ may be defined

$$V_{CJT}\,[S]\,\Omega = \Gamma_{CJT}\,[S]\Big|_{tran.inv.} \qquad \Omega = \int d^4x \tag{2.6}$$

which, in the CJT formulation, is equivalent to consider the generating functional for zero-momentum two-particle irreducible Green functions, expressed in terms of the propagator $S$.

$\Gamma_{CJT}$ (and equivalently $V_{CJT}$) can be expressed in the euclidean space as the following formal series

$$\Gamma_{CJT}\,(S) = -\mathrm{Tr}\log S^{-1} - \mathrm{Tr}(S_0^{-1}S) - \Gamma_2(S) \tag{2.7}$$

where $S_0$ is the free fermion propagator and $\Gamma_2(S)$ is the sum of all the two-particle irreducible vacuum diagrams of the theory evaluated with fermionic propagator equal to $S$.

By inserting (2.7) in (2.5) one gets the following expression for $J$

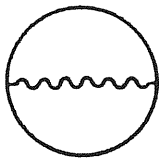
$$J = \frac{\delta \Gamma_{CJT}}{\delta S} = S^{-1} - S_0^{-1} - \frac{\delta \Gamma_2}{\delta S} \tag{2.8}$$

From eq. (2.8) the SD equation follows by setting $J = 0$. It is clear that

$$\Sigma = -\frac{\delta \Gamma_2}{\delta S} \tag{2.9}$$

represents the fermion self-energy when the source is off.

We will evaluate $\Gamma_2$ at the lowest order, that is at the two-loops level corresponding to a single gluon exchange.

$\Gamma_2 =$  (2.10)

This approximation will be improved by taking into account renormalization group effects which amounts to use the running coupling constant at the vertices (see sect. 3).

The important fact is that $\Gamma_2$ in this approximation is a quadratic expression in $S$, in fact $(\delta^2\Gamma_2/\delta S^2)$ is nothing but the gluon propagator plus possible corrections not involving explicitly fermions.

For this reason it follows (the trace operation is understood)

$$\Gamma_2 = \frac{1}{2}\, S\, \frac{\delta^2 \Gamma_2}{\delta S^2}\, S = \frac{1}{2}\, S\, \frac{\delta \Gamma_2}{\delta S} \tag{2.11}$$

or

$$\frac{\delta^2 \Gamma_2}{\delta S^2}\, S = \frac{\delta \Gamma_2}{\delta S} \tag{2.12}$$

It is easy to see that the approximation we are using for $\Gamma_2$ corresponds to integrate formally in the generating functional $Z$ the gluon fields and then to expand up to the fourth order in the fermionic fields.

In this way one obtains an effective four-fermion interaction and, instead of using $\Gamma_{CJT}$, one can introduce a collective variable, the auxiliary field $\Phi$ related to the bilocal fermion-antifermion composite field.

The strenght of the effective four-fermion interaction, within the specified approximation, is given by $(\delta^2\Gamma_2/\delta S^2)$.

One performs the standard trick of the functional identity

$$\int D\Phi\ e^{-\frac{1}{2}(\Phi - \psi\overline{\psi})D(\Phi - \psi\overline{\psi})} = const. \tag{2.13}$$

where $D$ is an arbitrary operator. In order to eliminate the quadrifermionic term, we will choose $D = \delta^2\Gamma_2/\delta S^2$.

Then one defines a new functional $Z_{AF}$ depending on a bilocal source $J(x,y)$ which is now coupled to the auxiliary field $\Phi$

$$\begin{aligned} Z_{AF}\ [J] &= e^{-W_{AF}\ [J]} = \\ &= \int D\psi\ D\overline{\psi}\ D\Phi\ e^{-[I_{eff}(\psi,\overline{\psi}) + \frac{1}{2}(\Phi - \psi\overline{\psi})\frac{\delta^2\Gamma_2}{\delta S^2}(\Phi - \psi\overline{\psi}) - J\Phi]} \end{aligned} \tag{2.14}$$

where $I_{eff}(\psi,\overline{\psi})$ results from the integration over the gauge fields and from the expansion up to the forth order in the fermionic fields.

From the equations of motions for the auxiliary field $\Phi$ one gets

$$\Phi_{\alpha\beta}(x,y) = \psi_\alpha(x)\overline{\psi}_\beta(y) \tag{2.15}$$

In this sense $\Phi$ can be identified with the operator $\psi\overline{\psi}$.

In order to build up an effective action relatively to auxiliary composite field, let us integrate on the fermion fields

$$Z_{AF}\ [J] = \int D\Phi\ e^{-[-\mathrm{Tr}\log(S_0^{-1} + \frac{\delta^2\Gamma_2}{\delta S^2}\Phi) + \frac{1}{2}\Phi\frac{\delta^2\Gamma_2}{\delta S^2}\Phi - J\Phi]} \tag{2.16}$$

If we do a stationary phase approximation, i.e. tree approximation in the $\Phi$ field, we get

$$W_{AF}\ [J] = -\mathrm{Tr}\log(S_0^{-1} + \frac{\delta^2\Gamma_2}{\delta S^2}\Phi_0) + \frac{1}{2}\Phi_0\frac{\delta^2\Gamma_2}{\delta S^2}\Phi_0 - J\Phi_0 \tag{2.17}$$

where $\Phi_0$ is the solution of the classical equation of motion.

One can Legendre transform $W_{AF}\ [J]$ in the usual way by defining

$$\frac{\delta W_{AF}}{\delta J} = -\Phi_c \tag{2.18}$$

and

$$\Gamma_{AF}\ [\Phi_c] = W_{AF} - \frac{\delta W_{AF}}{\delta J}\ J \tag{2.19}$$

from which it follows

$$\frac{\delta \Gamma_{AF}}{\delta \Phi_c} = \ J \tag{2.20}$$

In the lowest order approximation we can choose $\Phi_c = \Phi_0$.

By substituting eq. (2.17) in eq. (2.19) we obtain the auxiliary field effective action in the tree approximation for the composite field $\Phi_c$ [30]

$$\Gamma_{AF}\ [\Phi_c] = -\mathrm{Tr}\log(S_0^{-1} + \frac{\delta^2\Gamma_2}{\delta S^2}\Phi_c) + \frac{1}{2}\Phi_c\frac{\delta^2\Gamma_2}{\delta S^2}\Phi_c \tag{2.21}$$

Then from eq. (2.20) one gets:

$$J = \left(-(S_0^{-1} + \frac{\delta^2\Gamma_2}{\delta S^2}\Phi_c)^{-1} + \Phi_c\right)\frac{\delta^2\Gamma_2}{\delta S^2} \tag{2.22}$$

which gives $\Phi_c$ as a functional of J.

Let us look at eqs. (2.8) and (2.22). They represent the stationary conditions for $\Gamma_{CJT}$ and $\Gamma_{AF}$ respectively.

Switching the source $J$ off and using eq. (2.12) in (2.8) they respectively read

$$S^{-1} = S_0^{-1} + \frac{\delta^2\Gamma_2}{\delta S^2}S \tag{2.23}$$

$$\Phi_c^{-1} = S_0^{-1} + \frac{\delta^2\Gamma_2}{\delta S^2}\Phi_c \tag{2.24}$$

This means that the CJT and AF formulations are equivalent in the lowest approximation (which is the tree approximation for the auxiliary field and the lowest non-trivial (two-loops) order in the CJT formalism) in the sense that $S$ and $\Phi_c$ satisfy the same gap equation at the phisical point and so describe the same dynamics.

However, the functional form of $\Gamma_{CJT}$ and $\Gamma_{AF}$ are different and the effective actions for these two cases do differ outside the stationary points.

We can easily show that the difference between $W_{CJT}$ and $W_{AF}$ is a quadratic term in $J(x,y)$ essentially due to the fact that the source in the generating functional is coupled to $\overline{\psi}(x)\psi(y)$ in the first case and to $\Phi(x,y)$ in the second one.

If we start from eq. (2.1) and use the same functional trick (2.13), we obtain, after integrating over the fermion fields,

$$e^{-W_{CJT}} = \int \mathcal{D}\Phi \; e^{-[-\mathrm{Tr}\log(S_0^{-1} + \frac{\delta^2\Gamma_2}{\delta S^2}\Phi + J) + \frac{1}{2}\Phi\frac{\delta^2\Gamma_2}{\delta S^2}\Phi]} \tag{2.25}$$

If we use the invariance of the volume element under translations, we can change the integration variable

$$\Phi \rightarrow \Phi + (\frac{\delta^2\Gamma_2}{\delta S^2})^{-1} \; J \tag{2.26}$$

and get

$$e^{-W_{CJT}} =$$

$$= \int \mathcal{D}\Phi \; e^{-[-\mathrm{Tr}\log(S_0^{-1} + \frac{\delta^2\Gamma_2}{\delta S^2}\Phi) + \frac{1}{2}\Phi\frac{\delta^2\Gamma_2}{\delta S^2}\Phi + \frac{1}{2}J(\frac{\delta^2\Gamma_2}{\delta S^2})^{-1}J - J\Phi]} \tag{2.27}$$

that is

$$W_{CJT} = W_{AF} + \frac{1}{2}J \left(\frac{\delta^2\Gamma_2}{\delta S^2}\right)^{-1} J \tag{2.28}$$

So, the whole effect of introducing the auxiliary field $\Phi$ and coupling a source to it, is to add a term with a quadratic $J$ dependence to $W_{CJT}$.

We will see that this term is responsible for changing the stability properties of the effective potentials in the two formulations.

In fact, let us introduce in the standard way the "auxiliary field" effective potential

$$V_{AF}\;[\Phi_c]\;\Omega = \Gamma_{AF}\;[\Phi_c]\Big|_{tran.inv.} \qquad \Omega = \int d^4x \tag{2.29}$$

It is clear that, in the AF formalism, the second derivative of $V_{AF}\;[\Phi_c]$ can be interpreted as the mass of the $\Phi_c$ field. So, its positivity is a necessary condition for the validity of the composite field loop expansion.

We will show that the auxiliary field effective potential does have a local minimum corresponding to the lowest energy vacuum of the theory.

On the other hand, it happens that, even in the free field case, $V_{CJT}$ turns out to be unbounded from below (see [14] and [15]) and, in general, that the absence of a lower bound and the related saddle point behaviour for the solutions of the gap equation of $V_{CJT}$, is an intrinsic defect of the CJT formulation.

Haymaker, Matzuki and Cooper ([16] and [30]), have shown that, in the case of a $SU(N)$ fermion gauge theory, under certain physical conditions imposed on the solution of the gap equation, the lowest energy stationary point is a saddle point for $V_{CJT}$ while it is a local minimum for $V_{AF}$.

To determine if a solution of the gap equation is at a local minimum, a saddle point or a maximum, we need to solve an eigenvalue equation for the curvature operator.

Let us consider the case of a massless fermion gauge theory and let us parametrize the fermion propagator in the Landau gauge as following

$$S^{-1}(p) = i\hat{p} - \Sigma(p) \qquad \Sigma(p) = -\frac{\delta \Gamma_2}{\delta S(p)} \tag{2.30}$$

Since we are looking for a solution of the gap equation which is a singlet under the internal and Lorentz variables, the self-energy can be set

$$\Sigma_{\alpha\beta}\Big|_{extr} = \overline{\Sigma}\ \delta_{\alpha\beta} \tag{2.31}$$

Also the following assumptions are made ([16], [30])

i) the solution to the gap equation is spherically symmetric in four momentum space,

ii) $\Gamma_2$ gives a negative contribution to the effective potential in full space,

iii) the self-energy function $\Sigma$ has a constant behaviour as $p \to 0$ and approachs to zero as $p^{-a}$, $a > 0$ for $p \to \infty$.

(Notice that all of them will be satisfied in the model we will develop in the next sections).

Let us substitute (2.30) in eq. (2.7) with $\Gamma_2$ evaluated at the two-loop order.

Then, performing the angular integration in euclidean momentum space and taking the traces over $\gamma$-matrices we obtain, after a constant renormalization, the following expression for the CJT effective potential

$$
\begin{aligned}
V_{CJT} &= \frac{Nn}{2\pi^2}\Bigg[\int dp\ p^3\ \frac{\Sigma^2(p^2)}{p^2+\Sigma^2(p^2)} - \\
&- \frac{1}{2}\int dp\ p^3\ \log(1+\frac{\Sigma^2(p^2)}{p^2}) - \\
&- \frac{1}{2}\int dp\ p^3 \int dq\ q^3\ \frac{\Sigma(p^2)}{p^2+\Sigma^2(p^2)} D(p,q)\ \frac{\Sigma(q^2)}{q^2+\Sigma^2(q^2)}\Bigg]
\end{aligned} \tag{2.32}
$$

where $N$ is the dimensionality of the gauge group, $n$ is the number of flavors and the gauge coupling constant is included in the definition of the kernel $D(p,q)$.

Due to the above assumptions the self-ernergy $\Sigma$ is a function of the square of the momentum, $D(p,q)$ is positive definite and it is symmetric in $p$ and $q$ even if it includes the running coupling constant.

From eq. (2.32) we derive the gap equation for $\Sigma$ and the expression for the curvature operator

$$
\frac{\delta V_{CJT}}{\delta \Sigma(p^2)} = A(p)\left[\Sigma(p^2) - \int dq\ q^3 D(p,q)\ \frac{\Sigma(q^2)}{q^2+\Sigma^2(q^2)}\right] = 0 \tag{2.33}
$$

$$
\frac{\delta^2 V_{CJT}}{\delta \Sigma(p^2)\delta\Sigma(q^2)} = A(p)\delta(p-q) - A(p)D(p,q)A(q) \tag{2.34}
$$

where we have omitted the overall constant factor $(Nn/2\pi^2)$ and

$$
A(p) = p^3\frac{p^2-\Sigma^2(p^2)}{(p^2+\Sigma^2(p^2))^2} \tag{2.35}
$$

In order to compare the curvatures of CJT and AF formulations, let us derive an expression for the AF effective potential for a massless fermion gauge theory when the same assumptions are made.

Here

$$
\Sigma = -\frac{\delta^2\Gamma_2}{\delta S^2}\Phi_c \tag{2.36}
$$

Substituting in eq. (2.21) we get

$$
V_{AF} = -\mathrm{Tr}\log\ (i\hat{p} - \Sigma(p^2)) + \frac{1}{2}\mathrm{Tr}\ (\Sigma\ \left(\frac{\delta^2\Gamma_2}{\delta S^2}\right)^{-1}\ \Sigma) \tag{2.37}
$$

Let us remark that in the case we are considering

$$\frac{\delta^2\Gamma_2}{\delta S_{\beta\alpha}(p)\delta S_{\delta\gamma}(q)} = \delta_{\alpha\gamma}\delta_{\beta\delta}D(p,q) \tag{2.38}$$

Then, performing as before the angular integration in euclidean momentum space and taking traces over $\gamma$-matrices we obtain

$$\begin{aligned} V_{AF} = \frac{Nn}{2\pi^2}\Big[ & -\frac{1}{2}\int dp\ p^3\ \log(1+\frac{\Sigma^2(p^2)}{p^2})+ \\ & +\frac{1}{2}\int dp\ p^3 \int dq\ q^3\ \Sigma(p^2)\ D^{-1}(p,q)\ \Sigma(q^2)\Big] \end{aligned} \tag{2.39}$$

where the inverse of $D(p,q)$ is defined as

$$\int dr\ r^3\ D^{-1}(p,r)\ D(r,q) = \frac{1}{p^3}\ \delta(p-q) \tag{2.40}$$

From equation (2.39) we derive the gap equation for $\Sigma$ and the expression for the curvature operator in the AF formulation

$$\frac{\delta V_{AF}}{\delta\Sigma(p^2)} = -p^3\frac{\Sigma(p^2)}{p^2+\Sigma^2(p^2)} + p^3\int dq\ q^3\ D^{-1}(p,q)\ \Sigma(q^2) = 0 \tag{2.41}$$

$$\frac{\delta^2 V_{AF}}{\delta\Sigma(p^2)\delta\Sigma(q^2)} = -A(p)\delta(p-q) + p^3D^{-1}(p,q)q^3 \tag{2.42}$$

(we are again omitting the $(Nn/2\pi^2)$ factor).

It is easy to prove (by using (2.40)) that eq. (2.41) gives us the same solution as eq. (2.33) showing once again that the CJT and the AF formulations lead to the same stationary points.

Let us now expand the effective potential about a solution $\overline{\Sigma}$ of the gap equation

$$V(\overline{\Sigma}+\delta\Sigma) = V(\overline{\Sigma}) + \frac{1}{2}\int dp\ dq\ \delta\Sigma(p^2)\frac{\delta^2 V}{\delta\Sigma(p^2)\delta\Sigma(q^2)}\delta\Sigma(q^2) + \cdots \tag{2.43}$$

It is possible to show that in the CJT case, the second term in (2.43) can be negative or positive by choosing appropriate variations $\delta\Sigma$ i.e. the solution is a saddle point and also that the same solution of the gap equation which is at a saddle point of the CJT effective potential, is at a local minimum of the AF effective potential [30].

Let us consider the following eigenvalue equation

$$\int dq\, [\delta(p-q) - D(p,q)A(q)]\ \phi_n(q) = \lambda_n \phi_n(p) \tag{2.44}$$

with the $\phi_n$ satisfying the orthogonality relation

$$\int dp\ dq\ \phi_m(p)A(p)D(p,q)A(q)\phi_n(q) = \delta_{mn} \tag{2.45}$$

Let us expand the variation $\delta\Sigma(p)$ in the $\phi_n$ basis

$$\delta\Sigma(p) = \sum_n c_n \phi_n(p) \tag{2.46}$$

Then the expectation value of the CJT curvature given in eq. (2.34) has the following expression

$$\begin{aligned}\int dp\ dq\ \sum_{m,n} c_m\phi_m(p)A(p)[\delta(p-q) - D(p,q)A(q)]c_n\phi_n(q) = \\ = \int dp\ dq\ \sum_{m,n} c_m c_n \lambda_n \phi_m(p)A(p)\phi_n(q)\end{aligned} \tag{2.47}$$

From eq. (2.44) one can derive

$$\phi_n(p) = \frac{1}{1-\lambda_n}\int dq\ D(p,q)A(q)\phi_n(q) \tag{2.48}$$

Then, substituting in (2.47) and using the orthogonality relation (2.45) we get

$$\int dp\ dq\ \delta\Sigma(p^2)\frac{\delta^2 V_{CJT}}{\delta\Sigma(p^2)\delta\Sigma(q^2)}\delta\Sigma(q^2) = \sum_n c_n^2 \frac{\lambda_n}{1-\lambda_n} \tag{2.49}$$

The eigenvalue equation (2.44) has been numerically studied in ref. [16]. The result is that the eigenvalues corresponding to the ultraviolet region are positive and less than 1, while those corresponding to the infrared region are larger than 1, which makes the value of $(\lambda_n/(1-\lambda_n))$ negative.

This clearly means a saddle point solution for the CJT effective potential.

Let us now examine the expectation value of the curvature (2.42) for the AF effective potential.

By using eqs. (2.44) and (2.45) we get

$$\begin{aligned}\int dp\, dq\, \sum_{m,n} c_m\phi_m(p)[p^3 D^{-1}(p,q)q^3 - A(p)\delta(p-q)]c_n\phi_n(q) = \\ = \int dp\, dq\, dr\, \sum_{m,n} c_m c_n \phi_m(p)[p^3 D^{-1}(p,q) r^3 \delta(q-r) - \\ - A(q)p^3 D^{-1}(p,r)D(r,q)r^3]\phi_n(q)\end{aligned} \tag{2.50}$$

where we have also used eq. (2.40).

Then, from the eigenvalue equation (2.44)

$$\begin{aligned}\int dp\, dq\, \delta\Sigma(p^2)\frac{\delta^2 V_{AF}}{\delta\Sigma(p^2)\delta\Sigma(q^2)}\delta\Sigma(q^2) = \\ = \sum_{m,n} c_n c_m \lambda_n \int dp\, dr\, \phi_m(p) p^3 D^{-1}(p,r) r^3 \phi_n(r)\end{aligned} \tag{2.51}$$

Multiplying both sides of (2.48) by $\int dp\, p^3 D^{-1}(p,t)$, we get

$$\int dp\, p^3 D^{-1}(p,t)\phi_n(p) = \frac{1}{1-\lambda_n}\frac{1}{t^3}A(t)\phi_n(t) \tag{2.52}$$

where use has been made of eq. (2.40).

Let us substitute the result (2.52) in eq. (2.51)

$$\begin{aligned}\int dp\, dq\, \delta\Sigma(p^2)\frac{\delta^2 V_{AF}}{\delta\Sigma(p^2)\delta\Sigma(q^2)}\delta\Sigma(q^2) = \\ = \sum_{m,n} c_n c_m \frac{\lambda_n}{1-\lambda_n}\int dp\, \phi_m(p)A(p)\phi_n(p) = \\ = \sum_{m,n} c_n c_m \frac{\lambda_n}{(1-\lambda_n)^2}\int dp\, dq\, \phi_m(p)A(p)D(p,q)A(q)\phi_n(q) = \\ = \sum_n c_n^2 \frac{\lambda_n}{(1-\lambda_n)^2}\end{aligned} \tag{2.53}$$

where we have used again eqs. (2.48) and (2.45).

Since $\lambda_n$ is positive, as showed by the numerical study of ref. [16], the expectation value of the curvature of the AF effective potential is positive definite.

This fact ensures that the solution to the gap equation for the AF effective potential is at a local minimum even though, the same solution is at a saddle point for the CJT effective potential.

Taking into account these properties of the CJT and the AF effective potentials, we have introduced a further functional which is a modified version of the CJT one not suffering from the problem of unboundness from below.

In particular our effective action will be as general as the CJT one (not being restricted to the case of four-linear interactions), will have the same stationary points as $\Gamma_{CJT}$ and $\Gamma_{AF}$ but it will have the same functional form and so the same asymptotic behaviour as the AF one.

In this way it will be clear that the instability due to the presence of saddle points is an artifact of the particular choice of the effective potential and that it disappears when one chooses an alternative but equivalent form.

We have shown that $W_{AF}$ can be obtained by adding a source dependent term to $W_{CJT}$.

Let us change the definition of the source $J(x,y)$

$$J \equiv \frac{\delta^2 \Gamma_2}{\delta S^2} L \tag{2.54}$$

Then eq. (2.28) can be rewritten as

$$W_{AF} = W_{CJT} - \Gamma_2(S+L) + \Gamma_2(S) + \frac{\delta \Gamma_2}{\delta S} L \tag{2.55}$$

where we have used the property of $\Gamma_2(S)$ of being a quadratic functional of $S$ (eqs. (2.11) and (2.12)).

Let us consider the explicit expression for $W_{CJT}$ in terms of $\Gamma_{CJT}$

$$\begin{aligned} W_{CJT} &= \Gamma_{CJT} - JS = \Gamma_{CJT} - \frac{\delta \Gamma_2}{\delta S} L = \\ &= -\mathrm{Tr} \log\, (S^{-1}) - \mathrm{Tr}\, (S_0^{-1} S) - \Gamma_2(S) - \frac{\delta \Gamma_2}{\delta S} L \end{aligned} \tag{2.56}$$

where we have used the formal series representation (2.7) for $\Gamma_{CJT}$.

Substituting eq. (2.56) in (2.57) one gets

$$W_{AF}\, [L] = -\mathrm{Tr} \log\, (S^{-1}) - \mathrm{Tr}\, (S_0^{-1} S) - \Gamma_2(S+L) \tag{2.57}$$

and

$$\frac{\delta W_{AF}}{\delta L} = \left( S^{-1} - S_0^{-1} - \left.\frac{\delta \Gamma_2}{\delta S}\right|_{S+L} \right) \frac{\delta S}{\delta L} - \left.\frac{\delta \Gamma_2}{\delta S}\right|_{S+L} \tag{2.58}$$

Let us insert (2.54) into the gap equation (2.8)

$$\frac{\delta^2 \Gamma_2}{\delta S^2} L = S_0^{-1} - S^{-1} - \frac{\delta \Gamma_2}{\delta S} \tag{2.59}$$

Then, remembering that $\Gamma_2$ is a quadratic functional, that is

$$(S + L) \frac{\delta^2 \Gamma_2}{\delta S^2} = \left.\frac{\delta \Gamma_2}{\delta S}\right|_{S+L} \tag{2.60}$$

if follows

$$S_0^{-1} = S^{-1} + \left.\frac{\delta \Gamma_2}{\delta S}\right|_{S+L} \tag{2.61}$$

So, by substituting (2.61) in (2.58), we obtain

$$\frac{\delta W_{AF}}{\delta L} = -\left.\frac{\delta \Gamma_2}{\delta S}\right|_{S+L} = \Sigma|_{S+L} \equiv \tilde{\Sigma} \tag{2.62}$$

which means that the variable conjugate to the source $L$ in the AF formalism turns out to be $\tilde{\Sigma}$.

Finally, let us perform the Legendre transform of $W_{AF}\,[L]$ with respect to $L$ in order to get the "auxiliary field" effective action as a functional of $\tilde{\Sigma} = -(\delta \Gamma_2 / \delta S)|_{S+L}$

$$\begin{aligned} \Gamma_{AF}\,[\tilde{\Sigma}] &= W_{AF} - \frac{\delta W_{AF}}{\delta L} L = \\ &= -\mathrm{Tr}\log \left( S_0^{-1} + \left.\frac{\delta \Gamma_2}{\delta S}\right|_{S+L} \right) + \\ &+ \mathrm{Tr} \left( \left.\frac{\delta \Gamma_2}{\delta S}\right|_{S+L} (S + L) \right) - \Gamma_2(S + L) \end{aligned} \tag{2.63}$$

where again eq. (2.61) has been used.

The different functional form of $\Gamma_{CJT}$ and $\Gamma_{AF}$ is due to the use of different sources, $J$ and $L$ respectively. This means that the two effective actions describe the dynamics of the different composite fields $S$ and $\tilde{\Sigma}$ to which the sources are linearly attached.

A third alternative for the source term gives our result.

Since we want $\Sigma$ to be our dynamical variable (and not $\tilde{\Sigma}$), it is quite natural to define a new action simply redefining $S$ as $(S+L)$ in (2.63)

$$\Gamma\ [\Sigma] = -\mathrm{Tr}\log\ \left(S_0^{-1} + \frac{\delta\Gamma_2}{\delta S}\right) + \mathrm{Tr}\ \left(\frac{\delta\Gamma_2}{\delta S}\ S\right) - \Gamma_2(S) \tag{2.64}$$

This is the effective action which has been used in [18], [19], [20], [22], [23], [24], [25], [26] and [29].

It is clear from the derivation that $\Gamma_{AF}[\tilde{\Sigma}]$ and $\Gamma[\Sigma]$ have the same functional form. This means that the second derivatives of the two effective potentials with respect to their respective variables, evaluated with sources turned off, are equal ($\tilde{\Sigma} = \Sigma$ at the physical point).

This makes us sure that our potential has local minima as stationary points.

There is a general proof of this property [31] and also our analytical and numerical calculations fully confirm the validity of the statement.

Let us now derive the relation between our functional $\Gamma$ and $\Gamma_{CJT}$.

By using eq. (2.8) in (2.7) one obtains

$$\begin{aligned}\Gamma_{CJT}\ [S] &= -\mathrm{Tr}\log\ \left(S_0^{-1} + \frac{\delta\Gamma_2}{\delta S} + J\right) + \mathrm{Tr}\ \left(\frac{\delta\Gamma_2}{\delta S} + J\right)S - \Gamma_2(S) = \\ &= -\mathrm{Tr}\log\ \left(S_0^{-1} + \frac{\delta\Gamma_2}{\delta S}\right) + \mathrm{Tr}\log\ (1 + \left(S_0^{-1} + \frac{\delta\Gamma_2}{\delta S}\right)^{-1} J) + \\ &\quad + \mathrm{Tr}\ \left(\frac{\delta\Gamma_2}{\delta S}\ S\right) + \mathrm{Tr}\ (JS) - \Gamma_2(S)\end{aligned} \tag{2.65}$$

Then, if we use again (2.8), we may write

$$\begin{aligned}\mathrm{Tr}\log\ (1 + \left(S_0^{-1} + \frac{\delta\Gamma_2}{\delta S}\right)^{-1} J) &= \mathrm{Tr}\log\ (1 + (S^{-1} - J)^{-1}J) = \\ &= \mathrm{Tr}\log\ (1 - SJ)^{-1}\end{aligned} \tag{2.66}$$

Let us now insert (2.66) in (2.65) and compare with (2.64)

$$\Gamma_{CJT} = \Gamma + \mathrm{Tr}\log\ (1 - SJ) + \mathrm{Tr}\ (JS) \tag{2.67}$$

From eq. (2.67) the following relations follow

$$\Gamma_{CJT}\big|_{J=0} = \Gamma\big|_{J=0} \tag{2.68}$$

$$\left.\frac{\delta\Gamma_{CJT}}{\delta S}\right|_{J=0} = \left.\frac{\delta\Gamma}{\delta S}\right|_{J=0} \tag{2.69}$$

However

$$\left.\frac{\delta^2\Gamma_{CJT}}{\delta^2 S}\right|_{J=0} \neq \left.\frac{\delta^2\Gamma}{\delta^2 S}\right|_{J=0} \tag{2.70}$$

Furthermore, let us perform the functional derivative of $\Gamma$ as given in (2.64) with respect to $\Sigma$ ($\Sigma = -\delta\Gamma_2/\delta S$)

$$\frac{\delta\Gamma}{\delta\Sigma} = (S_0^{-1} - \Sigma)^{-1} - S - \Sigma\frac{\delta S}{\delta\Sigma} - \frac{\delta\Gamma_2}{\delta S}\frac{\delta S}{\delta\Sigma} \tag{2.71}$$

Then, the stationary condition $\delta\Gamma/\delta\Sigma = 0$ leads to the right SD equation

$$S^{-1} = S_0^{-1} + \Sigma \tag{2.72}$$

Another good reason to use $\Gamma$ instead of $\Gamma_{CJT}$, is related to eq. (2.62)

$$\begin{aligned} \frac{\delta W_{AF}}{\delta L} = -\left.\frac{\delta\Gamma_2}{\delta S}\right|_{S+L} = \frac{\delta W_{AF}}{\delta J}\frac{\delta J}{\delta L} = \\ = -\Phi_c\frac{\delta^2\Gamma_2}{\delta S^2} = \Sigma|_{S+L} \end{aligned} \tag{2.73}$$

which shows the simple relation between the self-energy and the vacuum expectation value of the composite field $\Phi$ ($\Phi_c = \langle\Phi\rangle$).

It follows that a series expansion of the effective action in $\Sigma = -\delta\Gamma_2/\delta S$ gives essentially the 1PI Green functions relative to the field $\Phi$, while a series expansion in $S$ as in the CJT case, does not have so a direct physical meaning.

In other words $\Sigma$ describes the physical excitations of the theory around the vacuum and an analogous situation does not hold in the CJT formulation because due to the fact that $(\delta^2 V_{CJT}/\delta\Sigma^2)$ is not positive definite.

## 3. EVALUATION OF THE EFFECTIVE ACTION IN QCD-LIKE GAUGE THEORIES

Let us evaluate the effective action for an $SU(N)$ QCD-like gauge theory whithin our modification of CJT functional formalism.

The mechanism of the spontaneous chiral symmetry breaking in non-abelian massless gauge theories has been studied in refs. [18], [22] and [23].

Here this mechanism is extended to the realistic situation when both spontaneous and explicit breakdown of the global chiral symmetry take place.

The calculations are for $\theta = 0$ ( $\theta$ is the parameter connected with the axial anomaly).

The classical euclidean lagrangian density of the strong interaction of the fermions $\Psi$, mediated by a set of vector gluons $\mathbf{A}_\mu$ which are the gauge bosons of the symmetry group $SU(N)$ is

$$\begin{aligned}\mathcal{L} = \overline{\Psi}\mathrm{S}_0^{-1}\Psi - i\ g\ \overline{\Psi}\hat{\mathbf{A}}\Psi + gauge\ terms \\ +ghost\ terms + gauge\ fixing\end{aligned} \tag{3.1}$$

where $\Psi$ are $n$ multiplets of $SU(N)$ each of them assigned to the fundamental representation of the gauge group and $\mathrm{S}_0$ is the free fermion propagator which, in a theory renormalizated at the point $p^2 = \mu^2$, has the following expression

$$\mathrm{S}_0(p) = [Z_\Psi(\mu,\Lambda)(i\hat{p} - \mathrm{m}_0(\Lambda))]^{-1} \tag{3.2}$$

Here $\Lambda$ is an ultraviolet cutoff, $Z_\Psi(\mu,\Lambda)$ is the renormalization constant for the fermion propagator and

$$\mathrm{m}_0(\Lambda) = \mathrm{m}(\mu) - \delta\mathrm{m}(\mu,\Lambda) \tag{3.3}$$

where $\mathrm{m}(\mu)$ is the $n \times n$ renormalized mass matrix which is responsible for the explicit breakdown of the chiral symmetry and $\delta\mathrm{m}(\mu,\Lambda)$ is the mass counterterm.

In fact we can write the lagrangian in eq. (3.1) as a sum of two contributions:

$$\mathcal{L} = \mathcal{L}_0 - Z_\Psi(\mu,\Lambda)\ \overline{\Psi}\mathrm{m}_0(\Lambda)\Psi \tag{3.4}$$

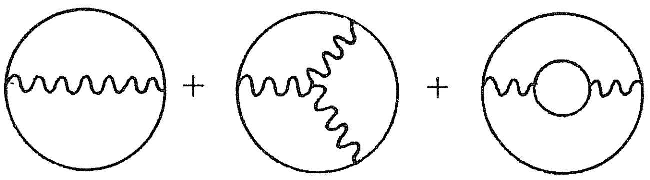
where $\mathcal{L}_0$ is invariant under the transformations of the flavour group $U(n)_L \otimes U(n)_R$ (more precisely it is invariant under the global chiral $SU(n)_L \otimes SU(n)_R$ and the $U(1)_{L+R}$ groups since the divergence of the singlet axial-vector current connected with the $U(1)_{L-R}$ group is non-zero even in the chiral limit due to the axial anomaly).

The expression to be evaluated is (see eq. (2.64))

$$\Gamma\,[\Sigma] = -\mathrm{Tr}\log\left(\mathrm{S}_0^{-1} + \frac{\delta\Gamma_2}{\delta \mathrm{S}}\right) + \mathrm{Tr}\left(\frac{\delta\Gamma_2}{\delta \mathrm{S}}\mathrm{S}\right) - \Gamma_2(\mathrm{S}) \tag{3.5}$$

with

$$\Sigma = -\frac{\delta\Gamma_2}{\delta \mathrm{S}} \tag{3.6}$$

$\Gamma_2 =$  $+ \cdots$ (3.7)

In eq. (3.5) S is the full fermion propagator which, at the physical point, satisfies the gap equation

$$\mathrm{S}^{-1}(p) = \mathrm{S}_0^{-1}(p) - \Sigma(p) \tag{3.8}$$

with $\Sigma$ equal to the fermion self-energy function.

We will show that, in the chiral limit, the theory possesses two phases: the chiral phase and the broken phase into the diagonal subgroup and, in particular, that spontaneous symmetry breaking occurs when the coupling constant $g$ exceeds some critical value.

This spontaneous symmetry breaking is accompained by $n^2-1$ composite Goldstone bosons which are associated to each unbroken generator of the coset space $SU(n)_L \otimes SU(n)_R / SU(n)_{L+R}$.

Actually the lagrangian in eq. (3.1) is not chirally invariant because of the quark mass term.

However, as it will follow from our analysis, there is a phase of the theory in which one has dynamical generation of the fermionic mass due to the formation of quark-antiquark condensates.

For sake of simplicity, we will keep on calling this phenomenon spontaneous chiral symmetry breaking ($\chi$SB) even if, clearly, this term is not appropriate. In the case we are analyzing the particle spectrum contains pseudo-Goldstone bosons which have acquired a mass induced by the explicit chiral symmetry breaking.

Following some suggestions from lattice calculations, we assume that the main contribution to the effective action for the spontaneous chiral symmetry breaking phenomenon comes from short-distance effects.

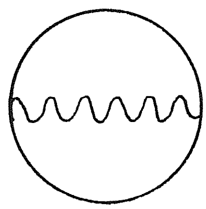
In fact, computer simulations in lattice gauge theories suggest that the range of forces responsible for $\chi$SB is relatively short and independent of confinement [17].

This suggestion is consistent with the idea that the pion is a thightly bound quark-antiquark state characterized by a size which is less than the distance at which the confinement forces dominate.

For this reason we will introduce an infrared cutoff for the confinement region and we will focus our attention on the short-distance dynamics.

In this range it is sensible to perform a loop expansion of the effective action.

In fact for large momenta, in virtue of the asymptotic freedom of the gauge theory, one can neglect the multiloop contributions and evaluate $\Gamma_2$ to the lowest order

$\Gamma_2 =$  (3.9)

In this graph one has to decide the form of the vertex and of the gauge field propagator.

The renormalization group analysis and the asymptotic freedom allow us to use the free expression for the vertex and gauge field propagator and to improve this approximation with the running coupling constant (see the next section).

But, as far as the vertex is concerned, the situation is more subtle because, in principle, one can run in some difficulties in order to satisfy the Ward identities. Let us examine this point.

We can express the inverse of the full fermion propagator in the following general form:

$$\mathrm{S}^{-1}(p) = i\mathrm{Z}(p^2)\ \hat{p} - \Sigma'(p^2) \tag{3.10}$$

Then the Ward identity for the vertex function reads

$$\begin{aligned}(q_1 - q_2)^\mu \Gamma_\mu &= \mathrm{S}^{-1}(q_1) - \mathrm{S}^{-1}(q_2) = \\ &= i\mathrm{Z}(q_1^2)\ \hat{q}_1 - i\mathrm{Z}(q_2^2)\ \hat{q}_2 - \Sigma'(q_1^2) + \Sigma'(q_2^2)\end{aligned} \tag{3.11}$$

This equation can be satisfied by taking

$$\begin{aligned}\Gamma_\mu = i\gamma_\mu + \frac{(q_1 - q_2)_\mu}{(q_1 - q_2)^2} \Big[ & i(\mathrm{Z}(q_1^2) - 1)\ \hat{q}_1 - i(\mathrm{Z}(q_2^2) - 1)\ \hat{q}_2 - \\ & - \Sigma'(q_1^2) + \Sigma'(q_2^2) \Big]\end{aligned} \tag{3.12}$$

However, in the evaluation of $\Gamma_2$, $\Gamma_\mu$ is always saturated with the gauge field propagator.

Therefore, if we adopt the Landau gauge, the gauge field propagator is transverse and we can safely use the free expression for the vertex. In other gauges the corrections to the free vertex are needed in order to satisfy the Ward identity (3.11).

This is the reason why all our calculations will be performed in the Landau gauge.

Also, as we shall see, there will be some other simplifications in this gauge. For example the wave function renormalization constant $Z_\Psi(\mu, \Lambda)$ in the Landau gauge is equal to one in the approximation we are considering.

It remains the problem of the gauge invariance.

By itself, the phenomenon of the spontaneous chiral symmetry breaking is gauge invariant since the chiral group currents are singlets with respect to the gauge group.

As stated before we will construct the effective action as a functional of $\Sigma$ and we will describe $\chi$SB with the help of this function which, at the physical point, represents the fermion dynamical mass.

In the next section we will derive a relation between the scalar part of the fermion self-energy and the condensate $\langle\overline{\Psi}\Psi\rangle_\mu$.

So if one chooses the vacuum expectation value $\langle\overline{\Psi}\Psi\rangle_\mu$, which is a gauge invariant quantity, as the order parameter, it is reasonable that all the results one finds in such a picture are gauge invariant.

The expression for $\Gamma_2$ we obtain is then the following

$$\Gamma_2 = -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\,(i\,g(p,q))^2 \\ \mathrm{Tr}\,[\mathrm{S}(p)T^a\gamma^\mu \mathrm{S}(q)T^a\gamma^\nu]\;D_{\mu\nu}(p-q)\int d^4x \tag{3.13}$$

where $T^a$ $a = 1,..,N^2-1$ are the hermitean generators of the gauge group in the fundamental representation, $g(p,q)$ is the running coupling constant and

$$D_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{3.14}$$

In the region of momenta larger than the renormalization group invariant mass of the theory $M_0$ (in QCD $M_0 = \Lambda_{QCD}$), we will assume, in the leading logarithmic approximation, the following form for the function $g^2(p,q)$ [32]

$$g^2(p,q) = \Theta(p^2-q^2)\;g^2(p) + \Theta(q^2-p^2)\;g^2(q) \tag{3.15}$$

However, we know that the running coupling constant $g^2(p)$ becomes singular for $p^2 = M_0^2$, a singularity due to the use of perturbation theory in a region where the coupling becomes strong.

Unfortunately in eq. (3.13) one has to integrate upon all the range of momenta and consequently one has to make an ansatz for the coupling constant in the infrared region.

Since the attitude we take here is that the spontaneous chiral symmetry breaking is dominated by short distance effects, we will substitute the infrared behaviour of $g^2(p)$ with a constant by introducing a mass scale $\mu$ characterizing the separation between the large distance and the small distance regions.

On the other hand, for values of $p^2 > \mu^2$ we will assume the standard renormalization group expression for $g^2(p)$ which provides the effective cutoff of interaction at small distances.

So, the expression we will use for the running coupling constant in the leading log approximation is the following

$$g^2(p) = 2\,b \left[ \Theta(\mu^2 - p^2)\, \frac{1}{\log(\mu^2/M_0^2)} + \Theta(p^2 - \mu^2)\, \frac{1}{\log(p^2/M_0^2)} \right] \qquad (3.16)$$

with $b = 24\pi^2/(11N - 2n)$.

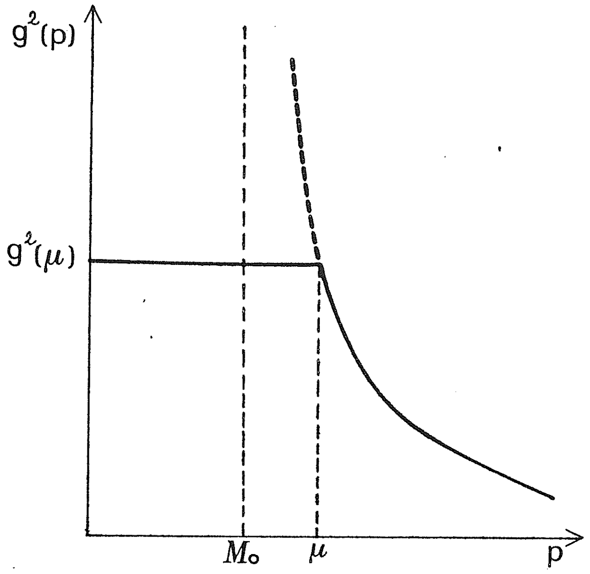



*FIG. 1. Running coupling constant in QCD-like gauge theories.*

In this way the expression (3.13) for $\Gamma_2$ is not merely the ladder approximation consisting of a single gauge field exchange but, with the insertion of the running coupling constant, it takes automatically into account the vertex perturbative corrections at least in the leading logarithmic approximation.

A further observation is in order. We have choosen the scale $\mu$ separating the infrared and the ultraviolet region to be coincident with the point at which we renormalize the theory.

As it will be clear in the next section, this fact leads to some simplifications, for example in the relation holding between the value of the minimum of the effective potential and the corresponding value of the fermion-antifermion condensate.

However the constant $\mu$ is a free parameter of our model.

In order to evaluate $\Gamma_2$, let us parametrize the fermion propagator in the following way

$$[\mathrm{S}(p)]_{Aa}^{Bb} = \delta_A^B \mathrm{S}(p)_a^b = \delta_A^B\, [i\mathrm{A}(p^2)_a^b\, \hat{p} + \mathrm{B}(p^2)_a^b + i\gamma_5 \mathrm{C}(p^2)_a^b] \qquad (3.17)$$

with $A, B = 1, .., N \quad a, b = 1, .., n$.

Notice that, from the assumption of a fermion propagator S which is function only of the space time difference, the translational invariance of the effective action follows and, as a consequence, the space-time volume $\Omega = \int d^4x$ factorizes out in $\Gamma_2$ (see eq. (3.13)).

Let us substitute the parametrization (3.17) in (3.13) and evaluate the trace over the color and the spinor indices

$$\Gamma_2 = 6NC_2\Omega \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{g^2(p,q)}{(p-q)^2} \text{tr } [\text{B}(p^2)\text{B}(q^2) + \text{C}(p^2)\text{C}(q^2)] - \\ - 2NC_2\Omega \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{g^2(p,q)}{(p-q)^2} \text{tr } [\text{A}(p^2)\text{A}(q^2)] \; E(p,q) \tag{3.18}$$

where $C_2 = (N^2-1)/2N = \sum_a T^a T^a \quad a = 1, .., N^2-1$ is the quadratic Casimir of the fermion representation, the trace is over the flavor indices and

$$E(p,q) = 1 - \frac{1}{2}\left[(p^2+q^2) + \frac{(p^2-q^2)^2}{(p-q)^2}\right] \frac{1}{(p-q)^2} \tag{3.19}$$

The expression for $g^2(p,q)$ we use, does not depend on the angle between $p$ and $q$ (see eqs. (3.15) and (3.16)). Therefore one can perform the angular integration in (3.18) by the help of the following formulae

$$\int d\Omega \; \frac{1}{(p-q)^2} = \frac{2\pi^2}{pq} \; e^{-|\log q/p|} \tag{3.20}$$

$$\int d\Omega \; \frac{1}{(p-q)^4} = \frac{2\pi^2}{pq} \; \frac{e^{-|\log q/p|}}{|p^2-q^2|} \tag{3.21}$$

The result is

$$\int d\Omega \; E(p,q) = 0 \tag{3.22}$$

and so there is no contribution in $\Gamma_2$ from the matrix A defined in (3.17).

This is obviously due to the non-renormalization of the wave function in the Landau gauge at this order.

We are so left with a dependence of $\Gamma_2$ only on the matrices B and C.

Remember that, as pointed out in the previous section, our task is to express $\Gamma_2$ as a functional of $\Sigma = -\delta\Gamma_2/\delta\text{S}$ which, when the Schwinger-Dyson equation is satisfied, is nothing but the fermion self-energy.

In order to do that, let us separate the scalar from the pseudoscalar contribution by defining

$$\Sigma(p^2) = \Sigma_s(p^2) + i\gamma_5 \Sigma_p(p^2) \tag{3.23}$$

Then, performing the functional derivative of $\Gamma_2$ given in (3.13) with respect to $\mathrm{S}(q^2)$ and using the parametrization (3.17) we obtain

$$\Sigma_s(q^2) = -3C_2 \int \frac{d^4p}{(2\pi)^4} \, \mathrm{B}(p^2) \, \frac{g^2(p,q)}{(p-q)^2} \tag{3.24}$$

$$\Sigma_p(q^2) = 3C_2 \int \frac{d^4p}{(2\pi)^4} \, \mathrm{C}(p^2) \, \frac{g^2(p,q)}{(p-q)^2} \tag{3.25}$$

Here $\Sigma_s$ and $\Sigma_p$ are matrices in the flavor space.

After inserting (3.15), let us perform the angular integration in (3.24)

$$\Sigma_s(q^2) = -\frac{3C_2}{16\pi^2}\left[\frac{g^2(q)}{q^2}\int_0^{q^2} dp^2 \, p^2 \, \mathrm{B}(p^2) + \right. \\ \left. + \int_{q^2}^{\infty} dp^2 \, \mathrm{B}(p^2) \, g^2(p)\right] \tag{3.26}$$

We can invert the relation between $\Sigma_s$ and $\mathbf{B}$ by applying an appropriate differential operator to both sides of (3.26).

In particular, let us differentiate with respect to $q^2$

$$\frac{d}{dq^2}\,\Sigma_s(q^2) = -\frac{3C_2}{16\pi^2}\frac{d}{dq^2}\left(\frac{g^2(q)}{q^2}\right)\int_0^{q^2} dp^2 \, p^2 \, \mathrm{B}(p^2) \tag{3.27}$$

that is

$$-\frac{16\pi^2}{3C_2}\;\frac{1}{\frac{d}{dq^2}\left(\frac{g^2(q)}{q^2}\right)}\;\frac{d}{dq^2}\,\Sigma_s(q^2) = \int_0^{q^2} dp^2 \, p^2 \, \mathrm{B}(p^2) \tag{3.28}$$

and, differentiating once again we obtain

$$\mathrm{B}(q^2) = -\frac{16\pi^2}{3C_2}\;\frac{1}{q^2}\;\frac{d}{dq^2}\left[\frac{1}{\frac{d}{dq^2}\left(\frac{g^2(q)}{q^2}\right)}\frac{d}{dq^2}\Sigma_s(q^2)\right] \tag{3.29}$$

and analogously

$$C(q^2) = \frac{16\pi^2}{3C_2}\,\frac{1}{q^2}\,\frac{d}{dq^2}\left[\frac{1}{\frac{d}{dq^2}\left(\frac{g^2(q)}{q^2}\right)}\frac{d}{dq^2}\Sigma_p(q^2)\right] \tag{3.30}$$

Let us remark that in deriving eqs. (3.29) and (3.30) it is crucial to assume (3.15). The expressions we find are however valid for any choice of $g^2(p)$.

An important property of $\Gamma_2$ in the two loop approximation is to be a quadratic functional of S.

This means that we can reexpress $\Gamma_2$ by using the Euler theorem for homogeneous functionals (see eqs. (2.11) and (2.12))

$$\Gamma_2 = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta\Gamma_2}{\delta \mathrm{S}}\mathrm{S}\right) = -\frac{1}{2}\mathrm{Tr}(\Sigma \mathrm{S}) \tag{3.31}$$

By substituting all these results in (3.18) and performing the angular integration we obtain

$$\begin{aligned}\Gamma_2 = \frac{2N}{3C_2}\Omega\int dp^2\ \mathrm{tr}\Bigg[&\Sigma_s(p^2)\,\frac{d}{dp^2}\left(\frac{1}{\frac{d}{dp^2}\left(\frac{g^2(p)}{p^2}\right)}\frac{d}{dp^2}\Sigma_s(p^2)\right)+\\ &+\Sigma_p(p^2)\,\frac{d}{dp^2}\left(\frac{1}{\frac{d}{dp^2}\left(\frac{g^2(p)}{p^2}\right)}\frac{d}{dp^2}\Sigma_p(p^2)\right)\Bigg]\end{aligned} \tag{3.32}$$

This is the final form for $\Gamma_2$ which results completely expressed in terms of $\Sigma_s$ and $\Sigma_p$.

Let us now evaluate the logarithmic contribution in $\Gamma$ (see eq. (3.5))

$$\begin{aligned}\mathrm{Tr}\ \log\left(\mathrm{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta \mathrm{S}}\right) &= \mathrm{Tr}\ \log\left(i\hat{p}-\mathrm{m}_0(\Lambda)-\Sigma_s(p^2)-i\gamma_5\Sigma_p(p^2)\right) =\\ &=N\Omega\int\frac{d^4p}{(2\pi)^4}\ \log\mathrm{Det}\ \left(i\hat{p}-\mathrm{m}_0(\Lambda)-\Sigma_s(p^2)-i\gamma_5\Sigma_p(p^2)\right)\equiv\\ &\equiv N\Omega\int\frac{d^4p}{(2\pi)^4}\ \log\mathrm{Det}\ \left(i\hat{p}-\mathrm{M}\right)\end{aligned} \tag{3.33}$$

where we have defined

$$\mathrm{M} = (\mathrm{m}_0(\Lambda) + \Sigma_s(p^2)) + i\gamma_5\Sigma_p(p^2) \tag{3.34}$$

The following relations hold

$$\mathrm{M}\mathrm{M}^\dagger = [\mathrm{m}_0(\Lambda) + \Sigma_s(p^2)]^2 + \Sigma_p^2(p^2) \tag{3.35}$$

$$\mathrm{M}\hat{p} = \hat{p}\mathrm{M}^\dagger \tag{3.36}$$

$$\begin{aligned} \hat{p}\gamma_5\mathrm{M} &= \hat{p}\gamma_5[(\mathrm{m}_0 + \Sigma_s) + i\gamma_5\Sigma_p] = \\ &= [(\mathrm{m}_0 + \Sigma_s) - i\gamma_5\Sigma_p]\ \hat{p}\gamma_5 = \\ &= \mathrm{M}^\dagger\hat{p}\gamma_5 \end{aligned} \tag{3.37}$$

$$\begin{aligned} \mathrm{Det}\ (i\hat{p} - \mathrm{M}) &= \mathrm{Det}\left(\hat{p}\gamma_5\ \frac{(i\hat{p} - \mathrm{M})}{p^2}\gamma_5\hat{p}\right) = \\ &= \mathrm{Det}\left(\frac{-i\hat{p} - \mathrm{M}^\dagger}{p^2}\ p^2\right) = \\ &= \mathrm{Det}\ \big( - i\hat{p} - \mathrm{M}^\dagger\big) = \\ &= \mathrm{Det}\ \big(i\hat{p} - \mathrm{M}\big)^\dagger \end{aligned} \tag{3.38}$$

from which

$$\begin{aligned} \big|\mathrm{Det}\ \big(i\hat{p} - \mathrm{M}\big)\big|^2 &= \mathrm{Det}\ \big(i\hat{p} - \mathrm{M}\big)\ \mathrm{Det}\ \big(i\hat{p} - \mathrm{M}\big)^\dagger = \\ &= \mathrm{Det}\ \big(p^2 + \mathrm{M}\mathrm{M}^\dagger\big) = \\ &= \det\big[(p^2 + \mathrm{M}\mathrm{M}^\dagger)\big]^4 \end{aligned} \tag{3.39}$$

where the last determinant is only on the flavor space.

Our result is then

$$\mathrm{Tr}\ \log\left(\mathrm{S}_0^{-1} + \frac{\delta\Gamma_2}{\delta\mathrm{S}}\right) = \frac{N\Omega}{8\pi^2}\int dp^2\ p^2\ \log\ \det\ (p^2 + \mathrm{M}\mathrm{M}^\dagger) \tag{3.40}$$

Let us now write down the final form of the effective action $\Gamma$ as a functional of $\Sigma$.

Observing that in the two loop approximation

$$\mathrm{Tr}\left(\frac{\delta\Gamma_2}{\delta\mathrm{S}}\mathrm{S}\right) - \Gamma_2 = \Gamma_2 \tag{3.41}$$

we get

$$\Gamma\,[\Sigma] = \Omega \Bigg[ -\frac{N}{8\pi^2} \int dp^2\; p^2\; \log\; \det\; \Big( p^2 \delta_{ab} + $$
$$+ [\mathrm{m}_0(\Lambda) + \Sigma_s(p^2)]^2_{ab} + [\Sigma_p^2(p^2)]_{ab} \Big) +$$
$$+ \frac{2N}{3C_2} \int dp^2\; \mathrm{tr} \Bigg[ \Sigma_s(p^2)\; \frac{d}{dp^2} \left( \frac{1}{\frac{d}{dp^2}\left(\frac{g^2(p)}{p^2}\right)} \frac{d}{dp^2} \Sigma_s(p^2) \right) +$$
$$+ \Sigma_p(p^2)\; \frac{d}{dp^2} \left( \frac{1}{\frac{d}{dp^2}\left(\frac{g^2(p)}{p^2}\right)} \frac{d}{dp^2} \Sigma_p(p^2) \right) \Bigg] \Bigg] \tag{3.42}$$

with $\Sigma(p^2) = \Sigma_s(p^2) + i\gamma_5 \Sigma_p(p^2)$.

As expected, the volume element $\Omega$ factorizes out, and we can define the effective potential

$$\Gamma = \Omega\; V$$

Our method will consist now in making a convenient ansatz for $\Sigma(p^2)$ in terms of a set of parameters related to the fermionic condensates and then in evaluating these parameters by minimizing the effective potential with respect to them.

In order to get a hint for the ansatz to be made, in the next section we will analize the ultraviolet asymptotics of the fermion self-energy function for our $SU(N)$ gauge theory.

## 4. THE ULTRAVIOLET ASYMPOTICS OF THE FERMION SELF-ENERGY FUNCTION

We will obtain some restrictions on the mechanism of spontaneous chiral symmetry breaking directly from the equations of the theory [33].

In particular we will consider the Ward identities relating the unrenormalized proper axial-vector vertex function with the fermion bare propagator in the massive case.

These quantities depend on an ultraviolet cutoff $\Lambda$.

Only after having introduced the renormalized functions and going over the deep euclidean region of momenta, we will perform the limit $\Lambda \to \infty$.

For sake of simplicity, we will consider the case of an $SU(N)$ gauge theory of $n$ fermions having the same bare mass, that is, we will restrict to a bare mass matrix which is proportional to the identity in the flavor space ($\mathrm{m}_0(\Lambda) = m_0(\Lambda)\ \mathbf{1}$).

The Ward identities read

$$\begin{aligned} ip^\mu \Gamma^{(0)i}_{5\mu}(q_2, q_1, \Lambda) = & -2im_0(\Lambda)\Gamma^{(0)i}_5(q_2, q_1, \Lambda) + \\ & + \gamma_5 \frac{\lambda_i}{2} S^{(0)-1}(q_1, \Lambda) + S^{(0)-1}(q_2, \Lambda)\frac{\lambda_i}{2}\gamma_5 \end{aligned} \tag{4.1}$$

where $p = q_2 - q_1$, $\Gamma^{(0)i}_{5\mu}$ are the bare vertices of the colorless axial vector currents $J^i_{\mu 5} = \overline{\Psi}\gamma_\mu\gamma_5\frac{\lambda_i}{2}\Psi$, $\Gamma^{(0)i}_5$ are the bare vertices of the colorless pseudoscalar densities $J^i_5 = \overline{\Psi}\gamma_5\frac{\lambda_i}{2}\Psi$, $\lambda_i$ are the matrices of the fundamental representation of the $SU(n)$ algebra normalized to $\mathrm{tr}(\lambda_i\lambda_j) = 2\delta_{ij} \quad i,j = 1,..,n^2-1$ and $S^{(0)-1}$ is the inverse bare fermion propagator.

The vertices $\Gamma^{(0)i}_{5\mu}$ and $\Gamma^{(0)i}_5$ satisfy the equations of the Bethe-Salpeter type

$$\begin{aligned} \Gamma^{(0)i}_{5\mu}(q_2, q_1, \Lambda)_{\alpha\beta} = & \frac{\lambda_i}{2}(\gamma_\mu\gamma_5)_{\alpha\beta} + \int^\Lambda \frac{d^4k}{(2\pi)^4} K^{(0)}(q_2, q_1, k, \Lambda)_{\alpha\beta\alpha'\beta'} \\ & [S^{(0)}(k+p, \Lambda)\Gamma^{(0)i}_{5\nu}(k+p, k, \Lambda)S^{(0)}(k, \Lambda)]_{\alpha'\beta'} \end{aligned} \tag{4.2}$$

$$\begin{aligned} \Gamma^{(0)i}_5(q_2, q_1, \Lambda)_{\alpha\beta} = & \frac{\lambda_i}{2}(i\gamma_5)_{\alpha\beta} + \int^\Lambda \frac{d^4k}{(2\pi)^4} K^{(0)}(q_2, q_1, k, \Lambda)_{\alpha\beta\alpha'\beta'} \\ & [S^{(0)}(k+p, \Lambda)\Gamma^{(0)i}_5(k+p, k, \Lambda)S^{(0)}(k, \Lambda)]_{\alpha'\beta'} \end{aligned} \tag{4.3}$$

where $K$ is the fermion-antifermion scattering kernel.

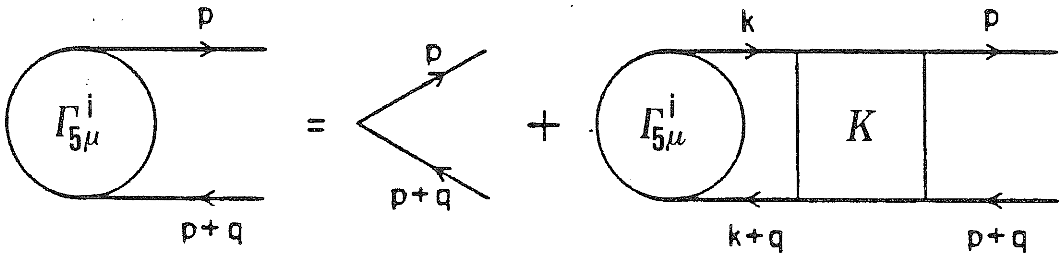



*FIG. 2. Graphical representation of the integral equation for the axial-vector vertex function.*

By substituting (4.2) and (4.3) in (4.1) we get

$$\begin{aligned} i\hat{p}\gamma_5 + \int^\Lambda \frac{d^4k}{(2\pi)^4} K^{(0)}(q_2, q_1, k, \Lambda)\ [S^{(0)}(k+p,\Lambda)\gamma_5 + \gamma_5 S^{(0)}(k,\Lambda)] = \\ = 2m_0(\Lambda)\gamma_5 + \gamma_5 S^{(0)-1}(q_1,\Lambda) + S^{(0)-1}(q_2,\Lambda)\gamma_5 \end{aligned} \tag{4.4}$$

Now, by taking the limit $p \to 0$ and defining $q = (q_1+q_2)/2$ we get

$$\begin{aligned} \gamma_5 S^{(0)-1}(q,\Lambda) + S^{(0)-1}(q,\Lambda)\gamma_5 = -2m_0(\Lambda)\gamma_5 + \\ + \int^\Lambda \frac{d^4k}{(2\pi)^4} K^{(0)}(q,k,\Lambda)\ [S^{(0)}(k,\Lambda)\gamma_5 + \gamma_5 S^{(0)}(k,\Lambda)] \end{aligned} \tag{4.5}$$

Let us pass on the renormalized functions

$$\begin{aligned} S(k) &= Z_\Psi^{-1}(\mu,\Lambda) S^{(0)}(k,\Lambda) \\ K(q,k) &= Z_\Psi^2(\mu,\Lambda) K^{(0)}(q,k,\Lambda) \end{aligned} \tag{4.6}$$

where $Z_\Psi(\mu,\Lambda)$ is the usual renormalization constant for the fermion propagator and $\mu$ is the renormalization point which, as before, is chosen to be coincident with our mass scale $\mu$.

One then obtain the equation

$$\begin{aligned} \gamma_5 S^{-1}(q) + S^{-1}(q)\gamma_5 = -2Z_\Psi(\mu,\Lambda) m_0(\Lambda)\gamma_5 + \\ + \int^\Lambda \frac{d^4k}{(2\pi)^4} K(q,k)\ [S(k)\gamma_5 + \gamma_5 S(k)] \end{aligned} \tag{4.7}$$

Let us parametrize

$$S^{-1}(k) = iZ(k^2)\ \hat{k} - \Sigma'(k^2) \tag{4.8}$$

Substituting in (4.7) we get

$$\begin{aligned} \gamma_5 \Sigma'(q^2) = & Z_\Psi(\mu, \Lambda) m_0(\Lambda) \gamma_5 + \\ & + \int^\Lambda \frac{d^4k}{(2\pi)^4}\ K(q,k)\ [S(k)\gamma_5\Sigma'(k^2)S(k)] \end{aligned} \tag{4.9}$$

As usual in considering the ultraviolet asymptotics, let us go over to the deep euclidean region of momenta in eq. (4.9).

Since the ultraviolet asymptotics of $Z(k^2)$ and $K(q,k)$ are insensitive to the mass term, they should not be changed when spontaneous chiral symmetry breaking is taken into account.

Therefore in the leading logarithmic approximation, one can take for them the expressions following from the renormalization group analysis [34].

Assuming the validity of the usual arguments that the main contribution to the integral on the right-hand side of eq. (4.9) in the limit $q^2 \to \infty$ comes from the region $k^2, (q-k)^2 >> M_0^2$ ($M_0$ is the free dimensional parameter of the theory), we will use the following expressions

$$\begin{aligned} K(q,k)_{\alpha\beta\alpha'\beta'} &= C_2\ (ig(q,k))^2 (\gamma^\mu)_{\beta\beta'} (\gamma^\nu)_{\alpha\alpha'} D_{\mu\nu}(q-k) \\ Z(k^2) = 1 & \qquad\qquad S(k) = -\frac{i}{\hat{k}} \end{aligned} \tag{4.10}$$

where

$$C_2 = (N^2-1)/2N$$

$$D_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

and $g(q,k)$ is the running coupling constant previously introduced in eq. (3.15).

Let us spend some words about these choices. According to the renormalization group analysis and thanks to the asymptotic freedom of the gauge interaction, it is meaningfull to approximate the kernel $K$ in the large momentum region with the lowest perturbative order but inserting the running coupling constant which takes automatically into account of the vertex perturbative corrections at least in the leading logarithmic approximation. Remember that we have used the same arguments in the calculation of $\Gamma_2$ (see eq. (3.13)).

In this way the expression (4.10) for the kernel is not merely the asymptotic limit of the ladder graph, but, with the insertion of the running coupling constant, faithfully represents the complete (relevant) kernel.

Also, since we are working in the Landau gauge, there is no wave function renormalization at this order ($Z_\Psi(\mu, \Lambda) = 1$). This means that the anomaluos dimension $\gamma_\Psi$ is equal to zero in this approximation and so there are no logarithmic corrections to the lowest perturbative order of the proper four-fermion scattering amplitude.

Substituting eq. (4.10) we find that (4.9) in the deep euclidean region takes the following form

$$\Sigma'(q^2) = m_0(\Lambda) + 3C_2 \int^{\Lambda} \frac{d^4k}{(2\pi)^4}\, g^2(q,k) \frac{\Sigma'(k^2)}{k^2(q-k)^2} \tag{4.11}$$

In the region $k^2, (q-k)^2 >> M_0^2$ we are considering, we will assume for the function $g^2(q,k)$ the behaviour expected from eqs. (3.15) and (3.16) in the large momentum range [32].

Substituting in (4.11) we get

$$\begin{aligned}\Sigma'(q^2) = m_0(\Lambda) + 3C_2\, g^2(q) \int_0^{q^2} \frac{d^4k}{(2\pi)^4} \frac{\Sigma'(k^2)}{k^2(q-k)^2} + \\ + 3C_2 \int_{q^2}^{\Lambda^2} \frac{d^4k}{(2\pi)^4}\, g^2(k) \frac{\Sigma'(k^2)}{k^2(q-k)^2}\end{aligned} \tag{4.12}$$

According to our assumption of the ultraviolet dominance, we will use a regularized expression for $\Sigma'(k^2)$ in the $k \to 0$ limit (see next section) and so we will not have infrared divergences in (4.12).

Let us now integrate over the angles and obtain the final result

$$\begin{aligned}\Sigma'(q^2) = m_0(\Lambda) + \frac{3C_2}{16\pi^2} \Bigg[ \frac{g^2(q)}{q^2} \int_0^{q^2} dk^2\, \Sigma'(k^2) + \\ + \int_{q^2}^{\Lambda^2} dk^2\, g^2(k) \frac{\Sigma'(k^2)}{k^2} \Bigg]\end{aligned} \tag{4.13}$$

showing that the chiral symmetry violating part of the fermion self- energy satisfies an homogeneous integral equation.

For the determination of the ultraviolet asympotics of the dynamical mass function $\Sigma'(q^2)$, one has to solve (4.13) with a finite $\Lambda$ and only after to perform the $\Lambda \to \infty$ limit.

Eq. (4.13) can be transformed into a differential equation.

Let us differentiate with respect to $q^2$

$$\frac{d}{dq^2}\Sigma'(q^2) = \frac{3C_2}{16\pi^2}\frac{d}{dq^2}\left(\frac{g^2(q)}{q^2}\right)\int_0^{q^2} dk^2\ \Sigma'(k^2) \tag{4.14}$$

so

$$\frac{1}{\frac{d}{dq^2}\left(\frac{g^2(q)}{q^2}\right)}\frac{d}{dq^2}\Sigma'(q^2) = \frac{3C_2}{16\pi^2}\int_0^{q^2} dk^2\ \Sigma'(k^2) \tag{4.15}$$

By differentiating once again, we find out that the solutions of (4.13) satisfy the second order differential equation

$$\frac{16\pi^2}{3C_2}\frac{d}{dq^2}\left[\frac{1}{\frac{d}{dq^2}\left(\frac{g^2(q)}{q^2}\right)}\frac{d}{dq^2}\Sigma'(q^2)\right] = \Sigma'(q^2) \tag{4.16}$$

and the boundary condition

$$\begin{aligned}&\left[q^2\ \frac{d}{dq^2}\Sigma'(q^2)+\right.\\ &\left.+\left(1-\frac{q^2}{g^2(q)}\ \frac{d}{dq^2}\ g^2(q)\right)\left(\Sigma'(q^2)-m_0(\Lambda)\right)\right]_{q^2=\Lambda^2} = 0\end{aligned} \tag{4.17}$$

The general solution of eq. (4.16) takes the form

$$\Sigma'(q^2) = a_1\ \Sigma_1(q^2) + a_2\ \Sigma_2(q^2) \tag{4.18}$$

with

$$\Sigma_1(q^2)_{q^2\to\infty} \sim \left(\log\frac{q^2}{M_0^2}\right)^{-d} \tag{4.19}$$

$$\Sigma_2(q^2)_{q^2\to\infty} \sim \frac{1}{q^2}\left(\log\frac{q^2}{M_0^2}\right)^{d-1} \tag{4.20}$$

and

$$d = \frac{3C_2 b}{8\pi^2}$$

In the literature these two solutions are commonly referred as the irregular and the regular solution respectively.

By substituting (4.18) in the boundary condition (4.17) and retaining only the leading contributions for large values of $\Lambda$ we get

$$a_1 = -a_2\, d\, \frac{1}{\Lambda^2} \left(\log \frac{\Lambda^2}{M_0^2}\right)^{2d-2} + m_0(\Lambda) \left(\log \frac{\Lambda^2}{M_0^2}\right)^{d} \tag{4.21}$$

Now we can remove the cutoff $\Lambda$.

Remembering that, in the leading logarithmic approximation, the relation between the bare mass and the mass renormalized at the point $\mu$ reads

$$\begin{aligned} m(\mu) &= m_0(\Lambda)\; Z_m^{-1}(\mu, \Lambda) \\ Z_m(\mu, \Lambda) &= \left(\frac{\log(\mu^2/M_0^2)}{\log(\Lambda^2/M_0^2)}\right)^{d} \end{aligned} \tag{4.22}$$

we obtain

$$a_1 = m(\mu) \left(\log \frac{\mu^2}{M_0^2}\right)^{d} \tag{4.23}$$

Then the result is that the constant $a_1$ is proportional to the explicit chiral symmetry breaking parameter and so it vanishes in the chiral limit.

In this way we have found that the asymptotic behaviour of the irregular solution $\Sigma_1(q^2)$ exactly corresponds to the result of a straightforward renormalization group analysis in the case of a bare fermion mass different from zero.

Hence we expect that the solution which actually represents chiral symmetry realized in the Goldstone mode has the softer asymptotic behaviour of $\Sigma_2(q^2)$.

It is possible to express the constant $a_2$ through the phenomenological parameter $\langle\overline{\Psi}\Psi\rangle_\mu$ (this is a shorthand notation to indicate

$$\lim_{x\to 0} \sum_{\alpha, A} \langle 0|\overline{\Psi}_{a\alpha A}(0)\Psi_a^{\alpha A}(x)|0\rangle_\mu$$

where $\alpha$ and $A$ are the spinor and color indices respectively and no summation is performed on the flavor index $a$).

Indeed, in the theory with cutoff we have

$$\begin{aligned}\langle\overline{\Psi}\Psi\rangle_\Lambda &= -\lim_{x\to 0}\langle 0|\mathrm{T}\ \Psi(x)\overline{\Psi}(0)|0\rangle_\Lambda = \\ &= -\lim_{k^2\to\infty} \mathrm{Tr}\int^\Lambda \frac{d^4k}{(2\pi)^4} S^{(\Lambda)}(k) = \\ &= \lim_{k^2\to\infty} a_2 4N \int^\Lambda \frac{d^4k}{(2\pi)^4}\frac{\Sigma_2(k^2)}{k^2} = \\ &= a_2\frac{2N}{3C_2 b}\left(\log\frac{\Lambda^2}{M_0^2}\right)^d\end{aligned} \tag{4.24}$$

where we have only considered the $\Sigma_2$ contribution to the dynamical mass because the explicit symmetry breaking term does not contribute due to the definition of the T product.

Also, in deriving (4.24), we have used $\Sigma_2^{(\Lambda)} = \Sigma_2$ since in the Landau gauge there is no wave function renormalization at this order.

From the relation between the bare and the renormalized condensate

$$\langle\overline{\Psi}\Psi\rangle_\Lambda = \left(\frac{\log(\Lambda^2/M_0^2)}{\log(\mu^2/M_0^2)}\right)^d \langle\overline{\Psi}\Psi\rangle_\mu \tag{4.25}$$

we finally determine $a_2$

$$a_2 = \frac{3C_2 b}{2N}\left(\log\frac{\mu^2}{M_0^2}\right)^{-d}\langle\overline{\Psi}\Psi\rangle_\mu \tag{4.26}$$

(notice that both the constants $a_1$ and $a_2$ are renormalization group invariant, i.e. independent on $\mu$).

Summing up, the ultraviolet asymptotics of the fermion mass function in the case in which both spontaneous and explicit chiral symmetry breaking take place is given by

$$\begin{aligned}\Sigma'(q^2)_{q^2\to\infty} &\sim m(\mu)\left(\frac{\log(q^2/M_0^2)}{\log(\mu^2/M_0^2)}\right)^{-d} + \\ &+ \frac{3C_2}{4N}\langle\overline{\Psi}\Psi\rangle_\mu \frac{g^2(q)}{q^2}\left(\frac{\log(q^2/M_0^2)}{\log(\mu^2/M_0^2)}\right)^d\end{aligned} \tag{4.27}$$

Let use notice that, in the case in which one wants to take into account also of the pseudoscalar contribution in $\Sigma'$, the previous considerations are still true and one obtains an extra term in (4.27) proportional to the pseudoscalar condensate.

So, separating the scalar from the pseudoscalar contribution in $\Sigma'$ we get

$$\begin{aligned} \Sigma'_s(q^2)_{q^2\to\infty} \sim m(\mu)\left(\frac{\log(q^2/M_0^2)}{\log(\mu^2/M_0^2)}\right)^{-d} + \\ + \frac{3C_2}{4N}\langle\overline{\Psi}\Psi\rangle_\mu \frac{g^2(q)}{q^2}\left(\frac{\log(q^2/M_0^2)}{\log(\mu^2/M_0^2)}\right)^{d} \end{aligned} \tag{4.28}$$

$$\Sigma'_p(q^2)_{q^2\to\infty} \sim \frac{3C_2}{4N}\langle\overline{\Psi}i\gamma_5\Psi\rangle_\mu \frac{g^2(q)}{q^2}\left(\frac{\log(q^2/M_0^2)}{\log(\mu^2/M_0^2)}\right)^{d} \tag{4.29}$$

These results are consistent with those obtained by means of the Wilson Operator Product Expansion (OPE) analysis [35] (for the explicit comparison see [23]).

In the OPE evaluation of $\Sigma'(q^2)$ the factors $g^2(q)$ and $(\log(q^2/M_0^2))^d$ are due to the renormalization group improvement of the Wilson coefficients and in particular $(\log(q^2/M_0^2))^d$ comes from the anomalous dimension of $(\overline{\Psi}\Psi)$ or equivalently of $(\overline{\Psi}i\gamma_5\Psi)$.

It is surprising that, although the first papers concerning the ultraviolet asymptotics of the fermion self-energy in QCD, appeared in the middle of seventies ([34] and [35]), until now there is no common opinion about the form of this asymptotics.

For example, recent studies of the spontaneous chiral symmetry breaking of massless QCD in the framework of variational approach, some authors [36], [37] and [38] agree with us by using the regular solution $\Sigma_2(q^2)$ as given in (4.20), while other authors [39], [40] and [41] use the irregular solution $\Sigma_1(q^2)$ given in (4.19).

Also, in a recent paper [42], K. Stam affirms that the ultraviolet asymptotics of the dynamical quark mass has the irregular form (4.19) while the regular solution is only an artefact of the Hartree-Fock approximation.

In a later work [43], V.A. Miransky has criticized the approach used by Stam and we agree with him in stating that only the regular solution can ensure the conservation of the anomaly-free axial vector currents $J^i_{\mu 5} = \overline{\Psi}\gamma_\mu\gamma_5\frac{\lambda_i}{2}\Psi$ in massless QCD (this condition is of course necessary to guarantee the spontaneous character of chiral $SU(n)_L \otimes SU(n)_R$ symmetry breaking).

In fact, let us specialize the previous analysis of the ultraviolet asymptotics of the fermion self-energy function, to the chiral case.

In the theory with the ultraviolet cutoff $\Lambda$ and the bare mass $m_0(\Lambda)$, the axial vector currents satisfy the equation

$$\partial^\mu J^i_{\mu 5} = 2m_0(\Lambda)\left(\overline{\Psi}\gamma_5 \frac{\lambda_i}{2}\Psi\right)_\Lambda \tag{4.30}$$

Due to the property of asymptotic freedom, the dependence on $\Lambda$ of the composite operator $(\overline{\Psi}\gamma_5\frac{\lambda_i}{2}\Psi)_\Lambda$ is well known (see eq. (4.25))

$$\left(\overline{\Psi}\gamma_5 \frac{\lambda_i}{2}\Psi\right)_\Lambda = Z_m^{-1}(\mu,\Lambda)\left(\overline{\Psi}\gamma_5 \frac{\lambda_i}{2}\Psi\right)_\mu \tag{4.31}$$

with $Z_m$ given in (4.22).

So the condition ensuring the conservation of the axial-vector currents is a rapid decrease of the bare mass $m_0(\Lambda)$ as $\Lambda \to \infty$

$$\lim_{\Lambda\to\infty} m_0(\Lambda) Z_m^{-1}(\mu,\Lambda) = 0 \tag{4.32}$$

The condition (4.32) is necessary and sufficient to determine uniquely the asymptotics of the fermion self-energy.

In fact, substituting it in the boundary condition (4.17) and using (4.18), we find that the coefficient $a_1$ is equal to zero in the $\Lambda \to \infty$ limit and therefore the only regular solution $\Sigma_2(q^2)$ does contribute in (4.18).

Let us also remark that, in a paper subsequent to [42], L.J. Reinders and K. Stam [44] discuss the dynamical quark mass function in the frame of the Operator Product Expansion.

They find that, asymptotically, the regular solution is consistent with the OPE, while the result of the analytic continuation to lower values of $p^2$, leads to a freezing of the quark self- energy at its treshold value reached for $p^2 = m^2_{dyn}$ where $m_{dyn}$ is a sort of constituent quark mass.

As we will see in the next section, the ansatz for the dynamical quark mass function we will use, completely agrees with this result.

## 5. ANSATZ FOR THE FERMION SELF-ENERGY

When the bilocal source $J(x,y)$ is off, that is at the physical point, the Schwinger-Dyson equation for the fermion propagator must hold

$$\mathrm{S}^{-1}(p) = \mathrm{S}_0^{-1}(p) + \frac{\delta\Gamma_2}{\delta \mathrm{S}} = i\hat{p} - \mathrm{m}_0(\Lambda) - \Sigma(p^2) \tag{5.1}$$

with the self-energy function $\Sigma$ evaluated at the minimum of the effective potential.

In the previous section we have parametrized $\mathrm{S}^{-1}$ in the following way

$$\mathrm{S}^{-1}(p) = i\hat{p} - \Sigma'(p^2) \tag{5.2}$$

and we have derived an expression for $\Sigma'(p^2)$ in the range of large momenta (remember that, in the approximation we are considering, $\mathrm{Z}(p^2) = 1$ at this order in the Landau gauge).

From the comparison of eqs. (5.1) and (5.2) we can deduce the asymptotic behaviour of the function $\Sigma(p^2)$ at the physical point.

The main assumption will be that, also outside the extremum of the effective potential, the function $\Sigma(p^2)$ has the asymptotic behaviour suggested by the Schwinger-Dyson equation.

In order to have a hint on the form of the test function for the fermion self-energy in the infrared region of momenta, let us consider the gap equations for the scalar and the pseudoscalar parts of $\Sigma'$.

Let us start with the relation

$$\begin{aligned} \mathrm{S}(p) &= i\mathrm{A}(p^2)\hat{p} + \mathrm{B}(p^2) + i\gamma_5 \mathrm{C}(p^2) = \\ &= (i\hat{p} - \Sigma'_s(p^2) - i\gamma_5\Sigma'_p(p^2))^{-1} \end{aligned} \tag{5.3}$$

From eq. (5.3) we get (after diagonalization in flavor space)

$$\begin{aligned} \mathrm{A}(p^2) &= -\frac{1}{p^2 + {\Sigma'_s}^2(p^2) + {\Sigma'_p}^2(p^2)} \\ \mathrm{B}(p^2) &= -\frac{\Sigma'_s(p^2)}{p^2 + {\Sigma'_s}^2(p^2) + {\Sigma'_p}^2(p^2)} \\ \mathrm{C}(p^2) &= \frac{\Sigma'_p(p^2)}{p^2 + {\Sigma'_s}^2(p^2) + {\Sigma'_p}^2(p^2)} \end{aligned} \tag{5.4}$$

Taking into account eqs. (3.24) and (3.25) and using (5.4) we get the non-linear SD equations for $\Sigma'_s$ and $\Sigma'_p$

$$\Sigma'_s(q^2) = \mathrm{m}_0(\Lambda) + 3C_2 \int \frac{d^4p}{(2\pi)^4} \frac{\Sigma'_s(p^2)}{p^2 + \Sigma'^2_s(p^2) + \Sigma'^2_p(p^2)} \frac{g^2(p,q)}{(p-q)^2} \quad (5.5)$$

$$\Sigma'_p(q^2) = 3C_2 \int \frac{d^4p}{(2\pi)^4} \frac{\Sigma'_p(p^2)}{p^2 + \Sigma'^2_s(p^2) + \Sigma'^2_p(p^2)} \frac{g^2(p,q)}{(p-q)^2} \quad (5.6)$$

Let us examine the integrals on the right hand side of eqs. (5.5) and (5.6) in the ultraviolet region of $p^2$ by inserting the asymptotic behaviours of $\Sigma'_s$ and $\Sigma'_p$.

As far as the pseudoscalar part is concerned, the integral representation (5.6) is clearly ultraviolet convergent because

$$\Sigma'_p(p^2)_{p^2\to\infty} \sim \frac{1}{p^2}(logs)$$

Also for the scalar part there are no problems due to the asymptotic behaviour of the mass term in $\Sigma'_s$.

We have in fact

$$\begin{aligned}
&\mathrm{m}_0(\Lambda) + 3C_2 \int \frac{d^4p}{(2\pi)^4} \frac{\Sigma'_s(p^2)}{p^2 + \Sigma'^2_s(p^2) + \Sigma'^2_p(p^2)} \frac{g^2(p,q)}{(p-q)^2}\Bigg|_{p^2\to\infty} \sim \\
&\sim \mathrm{m}_0(\Lambda) + \frac{3C_2}{16\pi^2}\int_0^{\Lambda^2} dp^2\, \Sigma'_s(p^2)\, \frac{g^2(p)}{p^2}\Bigg|_{p^2\to\infty} \sim \\
&\sim \mathrm{m}_0(\Lambda) + \frac{3C_2}{16\pi^2}\int_0^{\Lambda^2} \frac{dp^2}{p^2}\, \mathrm{m}(\mu) \left(\frac{\log(p^2/M_0^2)}{\log(\mu^2/M_0^2)}\right)^{-d} \frac{2b}{\log(p^2/M_0^2)} = \\
&= \mathrm{m}_0(\Lambda) - \mathrm{m}(\mu)\left(\frac{\log(\Lambda^2/M_0^2)}{\log(\mu^2/M_0^2)}\right)^{-d} = 0
\end{aligned} \quad (5.7)$$

where we have used eq. (4.22) relating the bare mass to the renormalized one in the leading logarithmic approximation.

In other words, the asymptotic behaviours (4.28) and (4.29) give us a perfectly UV regularized theory in the leading log approximation.

On the contrary, if one ignores the logarithmic corrections coming from the renormalization group analysis, the integral representation (5.5) is ultraviolet divergent.

In fact, in this case, the asymptotic behaviour of $\Sigma'(p^2)$ is simply

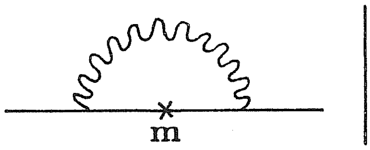
$$\Sigma'(p^2)_{p^2\to\infty} \sim \mathrm{m}(\mu) + \frac{3C_2g^2}{4N}\left(\langle\overline{\Psi}\Psi\rangle_\mu + i\gamma_5\langle\overline{\Psi}i\gamma_5\Psi\rangle_\mu\right)\frac{1}{p^2} \tag{5.8}$$

By substituting in eqs. (5.5) and (5.6) one finds an ultraviolet divergence for the scalar part, which is proportional to the mass parameter

$$\Sigma'_s\big|_{div} = \mathrm{m}(\mu)\ \frac{3C_2g^2}{16\pi^2}\ \log\frac{\Lambda^2}{\mu^2} \tag{5.9}$$

This is exactly the divergence corresponding to the diagram

$$\delta\mathrm{m}(\mu,\Lambda) \quad = $$




$_{div}$ (5.10)

which must be subtracted in order to regularize the theory.

It is clear that in our formulation, the renormalization group prediction for the fermion self-energy takes automatically into account this minimal subtraction procedure.

Let us now examine the behaviour of the integrals of eqs. (5.5) and (5.6) in the infrared region of momenta.

To this end, let us perform the $q^2 \to 0$ limit and let us suppose that

$$\Sigma'(q^2)_{q^2\to 0} \sim (q^2)^{-\alpha} \qquad \alpha > 0 \tag{5.11}$$

It is easy to see that, with this assumption, one has a finite contribution at the lower limit of integration while, on the left hand side of eqs. (5.5) and (5.6), one has an infrared divergence.

It follows that, for $q^2 \to 0$, $\Sigma'(q^2)$ must go to a finite constant.

As stated before, we will assume that these results are valid also outside the extremum. Thus the test function for $\Sigma(q^2)$ will be assumed to have a constant value in the IR region of momenta. This value can be identified with the fermion dynamical mass.

Therefore, using our preferred renormalization point $\mu$, we will make the following ansatz for $\Sigma(q^2)$

$$\Sigma(q^2) = \mathrm{m}(\mu)f_1(p^2) - \mathrm{m}_0(\Lambda) + (\mathrm{s} + i\gamma_5\mathrm{p})f_2(p^2) \tag{5.12}$$

with

$$f_1(p^2) = \Theta(\mu^2 - p^2) + \Theta(p^2 - \mu^2) f(p^2)^{-d} \tag{5.13}$$

$$f_2(p^2) = \mu \left[ \Theta(\mu^2 - p^2) + \Theta(p^2 - \mu^2) \frac{\mu^2}{p^2} f(p^2)^{d-1} \right] \tag{5.14}$$

$$f(p^2) = \frac{\log(p^2/M_0^2)}{\log(\mu^2/M_0^2)} \tag{5.15}$$

We will use the parameter-dependent test function (5.12) for the fermion self-energy in our effective potential formalism to investigate the stability of the theory.

The fields $\mathbf{s}_{ab}$ and $\mathbf{p}_{ab}$ $\quad a, b = 1, .., n$ which here are constant fields because we are only interested in the evaluation of the effective potential, will be our variational parameters. The minimum of the effective potential will determine the values of these parameters corresponding to the optimal form of the test function for $\Sigma(p^2)$.

The matrices $\mathbf{s}$ and $\mathbf{p}$ evaluated at the extremum of the effective potential, let us call them $\langle \mathrm{s} \rangle$ and $\langle \mathrm{p} \rangle$, can be related to the fermionic condensates.

This can be seen directly by the comparison of our ansatz with the OPE expansion which coincides with the expressions given in eqs. (4.28) and (4.29).

We obtain in this way

$$\langle \mathbf{s}_{ab} \rangle = \frac{3C_2}{4N} \frac{g^2(\mu)}{\mu^3} \langle \overline{\Psi}_a \Psi_b \rangle_\mu \tag{5.16}$$

$$\langle \mathbf{p}_{ab} \rangle = \frac{3C_2}{4N} \frac{g^2(\mu)}{\mu^3} \langle \overline{\Psi}_a i\gamma_5 \Psi_b \rangle_\mu \tag{5.17}$$

$$a, b = 1, .., n.$$

Let us notice that, if we didn't have chosen the scale $\mu$ separating the ultraviolet and the infrared region of momenta, to be coincident with the renormalization point of the theory, and we had renormalized at $p^2 = \bar{\mu}^2$, then an extra factor $(\log(\mu^2/M_0^2)/\log(\bar{\mu}^2/M_0^2))^d$ would have been present in eqs. (5.16) and (5.17).

An obvious question is now in order: how much will our following results depend on the particular choice of the ansatz for the fermion self-energy?

As a check, in the case in which one neglects the logarithmic corrections coming from the renormalization group analysis (let us call this case "the rigid case"), we have considered the following smooth test function for $\Sigma(p^2)$

$$\Sigma(p^2) = \mu \ (\mathrm{s} + i\gamma_5 \mathrm{p}) \ \frac{\mu^2}{p^2 + \mu^2} \tag{5.18}$$

Explicit calculations have shown that the qualitative picture of dynamical chiral symmetry breaking does not change and also quantitatively the results in the two cases do not differ very much. This is a confirmation of the ultraviolet dominance for the $\chi$SB phenomenon.

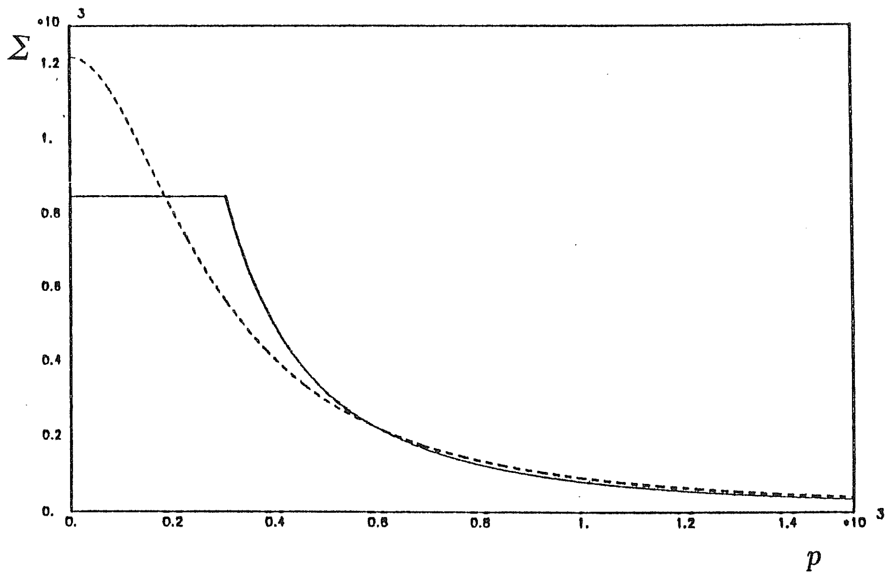



*FIG. 3. Self-energy functions for the quark u in QCD with three flavors (rigid case). The curves shown correspond to the ansatz (5.19) with* $\mu = 306$ *MeV and* $\langle \chi_u \rangle = -2.76$ ([25]) *(solid line), and to the ansatz (5.18) with* $\mu = 282$ *MeV and* $\langle \chi_u \rangle = -4.31$ *(dashed line).*

In the rigid case, eq. (5.12) simply becomes ([18], [19] and [20])

$$\Sigma(p^2) = \mu \ (\mathrm{s} + i\gamma_5 \mathrm{p}) \ \left[ \Theta(\mu^2 - p^2) + \Theta(p^2 - \mu^2) \frac{\mu^2}{p^2} \right] \tag{5.19}$$

In Fig. 3 the shape of the two different test functions (5.19) and (5.18) for the quark self-energy evaluated at the minimum of the effective potential is shown.

In particular the curves refer to the quark $u$ and specializing (5.16) to QCD ($N = 3$, $C_2 = 4/3$) we have introduced

$$\langle \chi_u \rangle = \langle s_{11} \rangle = \frac{g^2}{3\mu^3} \langle \bar{u}u \rangle_\mu \tag{5.20}$$

The numerical fit for the octet meson masses and decay constants (see next sections), gives higher values for the quark condensates in the case in which the ansatz smooth (5.18) is used.

Actually in Fig. 3 we have plotted $-\Sigma_u$ since the scalar condensate has a negative value (the pseudoscalar is vanishing).

## 6. CANCELLATION OF THE ULTRAVIOLET DIVERGENCES IN THE EFFECTIVE ACTION AND NORMALIZATION CONDITION

Let us substitute our ansatz (5.12) for $\Sigma(p^2)$ in the expression (3.41) for the effective action and let us start with analyzing $\Gamma_2$.

By performing an integration by parts in (3.32) we obtain

$$\Gamma_2 = \frac{2N}{3C_2}\,\Omega\Bigg(\frac{1}{2}\left[\frac{1}{\frac{d}{dp^2}\left(\frac{g^2(p)}{p^2}\right)}\,\mathrm{tr}\left(\frac{d}{dp^2}(\Sigma_s^2(p^2)+\Sigma_p^2(p^2))\right)\right]_0^{\Lambda^2} - \\ -\int_0^{\Lambda^2} dp^2\,\frac{1}{\frac{d}{dp^2}\left(\frac{g^2(p)}{p^2}\right)}\left[\mathrm{tr}\left(\frac{d}{dp^2}\Sigma_s(p^2)\right)^2+\mathrm{tr}\left(\frac{d}{dp^2}\Sigma_p(p^2)\right)^2\right]\Bigg) \tag{6.1}$$

with

$$\Sigma_s(p^2) = \mathrm{m}(\mu)f_1(p^2) - \mathrm{m}_0(\Lambda) + \mathrm{s}f_2(p^2) \tag{6.2}$$

$$\Sigma_p(p^2) = \mathrm{p}f_2(p^2) \tag{6.3}$$

and $f_1(p^2)$, $f_2(p^2)$ given in eqs. (5.13) and (5.14).

The surface term in (6.1) gives a vanishing contribution.

In fact, for $p^2 = 0$ we get obviously zero. Let us calculate explicitly the contribution for $p^2 = \Lambda^2$.

Introducing

$$f_\Lambda = f(p^2)\big|_{\Lambda^2} = \frac{1}{2a}\log\frac{\Lambda^2}{M_0^2} = 1 + \frac{1}{2a}\log\frac{\Lambda^2}{\mu^2} \tag{6.4}$$

$$a = \log\frac{\mu}{M_0}$$

with $f(p^2)$ defined in (5.15) we have

$$\frac{1}{\frac{d}{dp^2}\left(\frac{g^2(p)}{p^2}\right)}\,\mathrm{tr}\left[\frac{d}{dp^2}(\Sigma_s^2(p^2)+\Sigma_p^2(p^2))\right]\Bigg|_{p^2=\Lambda^2} = \tag{6.5}$$

$$
\begin{aligned}
&= \mathrm{tr}\left(\mathrm{m}^2(\mu)\right)\left(\frac{2ad}{b}\ \Lambda^2\ \frac{f_\Lambda^{-2d+1}}{1+2af_\Lambda}\right)+ \\
&+\mathrm{tr}\left(\mathrm{s}^2+\mathrm{p}^2\right)\left(\frac{4a}{b}\ \frac{\mu^6}{\Lambda^2}\ \frac{f_\Lambda^{2d}}{1+2af_\Lambda}\right)- \\
&-\mathrm{tr}\left(\mathrm{m}_0(\Lambda)\cdot\mathrm{s}\right)\left(\frac{4a^2}{b}\ \mu^3\ \frac{f_\Lambda^{d+1}}{1+2af_\Lambda}\right)- \\
&-\mathrm{tr}\left(\mathrm{m}(\mu)\cdot\mathrm{m}_0(\Lambda)\right)\left(\frac{2ad}{b}\ \Lambda^2\ \frac{f_\Lambda^{-d+1}}{1+2af_\Lambda}\right)+ \\
&+\mathrm{tr}\left(\mathrm{m}(\mu)\cdot\mathrm{s}\right)\left(\frac{4a^2}{b}\ \mu^3\ \frac{f_\Lambda}{1+2af_\Lambda}\right)
\end{aligned}
$$

Remembering that, in the leading log approximation

$$
\mathrm{m}_0(\Lambda)=\mathrm{m}(\mu)Z_m(\mu,\Lambda)=\mathrm{m}(\mu)f_\Lambda^{-d} \tag{6.6}
$$

with $Z_m(\mu,\Lambda)$ defined in (4.22), and performing the $\Lambda\to\infty$ limit we get

$$
\begin{aligned}
&\frac{1}{\frac{d}{dp^2}\left(\frac{g^2(p)}{p^2}\right)}\ \mathrm{tr}\left[\frac{d}{dp^2}(\Sigma_s^2(p^2)+\Sigma_p^2(p^2))\right]\Bigg|_{p^2=\Lambda^2\to\infty}\sim \\
&\sim\mathrm{tr}\left(\mathrm{m}^2(\mu)\right)\left(\frac{2ad}{b}\ \Lambda^2\ \frac{f_\Lambda^{-2d+1}}{1+2af_\Lambda}\right)- \\
&-\mathrm{tr}\left(\mathrm{m}(\mu)\cdot\mathrm{s}\right)\left(\frac{4a^2}{b}\ \mu^3\ \frac{f_\Lambda}{1+2af_\Lambda}\right)- \\
&-\mathrm{tr}\left(\mathrm{m}^2(\mu)\right)\left(\frac{2ad}{b}\ \Lambda^2\ \frac{f_\Lambda^{-2d+1}}{1+2af_\Lambda}\right)+ \\
&+\mathrm{tr}\left(\mathrm{m}(\mu)\cdot\mathrm{s}\right)\left(\frac{4a^2}{b}\ \mu^3\ \frac{f_\Lambda}{1+2af_\Lambda}\right)=0
\end{aligned} \tag{6.7}
$$

showing that no contribution arises from the surface term of $\Gamma_2$ in virtue of the mass renormalization.

We are so left with

$$
\Gamma_2=-\frac{2N}{3C_2}\ \Omega\int_0^{\Lambda^2}dp^2\ \frac{1}{\frac{d}{dp^2}\left(\frac{g^2(p)}{p^2}\right)}\ \mathrm{tr}\left[\frac{d}{dp^2}\left(\Sigma_s^2(p^2)+\Sigma_p^2(p^2)\right)\right] \tag{6.8}
$$

For large values of momenta we have

$$
\begin{aligned}
\operatorname{tr}\left[\frac{d}{dp^2}\left(\Sigma_s^2(p^2)+\Sigma_p^2(p^2)\right)\right]_{p^2\to\infty} &\sim \\
&\sim \operatorname{tr}\left(\mathrm{m}^2(\mu)\right)\left(\frac{d^2}{4a^2}\,\frac{1}{p^4}\,f(p^2)^{-2d-2}\right)+ \\
&+2\operatorname{tr}\left(\mathrm{m}(\mu)\cdot\mathrm{s}\right)\left(-\frac{d}{4a^2}\,\left(d-1-2af(p^2)\right)\frac{\mu^3}{p^6}\,f(p^2)^{-3}\right)+ \\
&+\operatorname{tr}\left(\mathrm{s}^2+\mathrm{p}^2\right)\left(\frac{1}{4a^2}\,\left(d-1-2af(p^2)\right)^2\frac{\mu^6}{p^8}\,f(p^2)^{2d-4}\right)
\end{aligned}
\tag{6.9}
$$

so

$$
\Gamma_2\Big|_{UV} = -\frac{2N}{3C_2}\,\Omega\,\left[\operatorname{tr}\left(\mathrm{m}^2(\mu)\right)\mathcal{A}+2\operatorname{tr}\left(\mathrm{m}(\mu)\cdot\mathrm{s}\right)\mathcal{B}+\operatorname{tr}\left(\mathrm{s}^2+\mathrm{p}^2\right)\mathcal{C}\right] \tag{6.10}
$$

with

$$
\begin{aligned}
\mathcal{A} &= -\frac{d^2}{2b}\int^{\Lambda^2} dp^2\,\frac{f(p^2)^{-2d}}{1+2af(p^2)} \\
\mathcal{B} &= \frac{d}{2b}\,\mu^3\int^{\Lambda^2}\frac{dp^2}{p^2}\,\left(d-1-2af(p^2)\right)\,\frac{f(p^2)^{-1}}{1+2af(p^2)} \\
\mathcal{C} &= -\frac{1}{2b}\,\mu^6\int^{\Lambda^2}\frac{dp^2}{p^4}\,\left(d-1-2af(p^2)\right)^2\,\frac{f(p^2)^{2d-2}}{1+2af(p^2)}
\end{aligned}
\tag{6.11}
$$

Notice that the first term in (6.10) does not depend on the fields s and p and so it can be neglected since it only represents an additive constant.

Furthermore, the integral in $\mathcal{C}$ is convergent for large values of momenta.

It remains to analize the ultraviolet divergences arising from the term proportional to $\operatorname{tr}\left(\mathrm{m}(\mu)\cdot\mathrm{s}\right)$.

Let us perform a change of variable in $\mathcal{B}$

$$
t=\log\frac{p^2}{\mu^2} \tag{6.12}
$$

then

$$
\begin{aligned}
\mathcal{B} &= \frac{d}{2b}\,\mu^3\int^{\log\Lambda^2/\mu^2} dt\,\left(d-1-2a-t\right)\,\frac{2a}{(2a+t)(1+2a+t)} = \\
&= \frac{d}{b}\,a\,\mu^3\left[(d-1)\log\left(2a+\log\frac{\Lambda^2}{\mu^2}\right)-d\,\log\left(1+2a+\log\frac{\Lambda^2}{\mu^2}\right)\right]
\end{aligned}
\tag{6.13}
$$

Taking the $\Lambda \to \infty$ limit, we finally find a divergence of the form

$$\mathcal{B}\Big|_{div} = \lim_{\Lambda\to\infty} -\frac{d}{b}\ a\ \mu^3 \log\log\frac{\Lambda^2}{\mu^2} \tag{6.14}$$

which, substituted in eq. (6.10) gives

$$\Gamma_2\Big|_{div} = \lim_{\Lambda\to\infty} \frac{N\Omega}{2\pi^2}\ a\ \mu^3\ \mathrm{tr}\big(\mathbf{m}(\mu)\cdot\mathbf{s}\big)\ \log\log\frac{\Lambda^2}{\mu^2} \tag{6.15}$$

Let us now examine the ultraviolet divergences in the logarithmic term of $\Gamma$. Let us call it $\Gamma_{log}$.

$$\Gamma_{log} = -\frac{N\Omega}{8\pi^2}\int^{\Lambda^2} dp^2\ p^2\ \log\ \det\ \left(p^2\delta_{ab} + [\mathbf{m}_0(\Lambda) + \Sigma_s(p^2)]^2_{ab} + [\Sigma^2_p(p^2)]_{ab}\right) \tag{6.16}$$

Substituting our ansatz for the large momenta behaviour of $\Sigma_s$ and $\Sigma_p$ and neglecting an infinite constant we obtain

$$\begin{aligned}\Gamma_{log} \sim -\frac{N\Omega}{8\pi^2}\int^{\Lambda^2} dp^2\ p^2\ \log\ \det\ \Bigg(\delta_{ab} &+ [\mathbf{m}^2(\mu)]_{ab}\frac{1}{p^2}f_1^2(p^2) + \\ &+ 2\ [\mathbf{m}(\mu)\cdot\mathbf{s}]_{ab}\frac{1}{p^2}f_1(p^2)f_2(p^2) + \\ &+ [\mathbf{s}^2 + \mathbf{p}^2]_{ab}\frac{1}{p^2}f_2^2(p^2)\Bigg)\end{aligned} \tag{6.17}$$

In the $p^2 \to \infty$ limit, we can use in (6.17) the relation

$$\det(1 + \mathbf{A}) = 1 + \mathrm{tr}\mathbf{A} \tag{6.18}$$

which holds for any infinitesimal matrix $\mathbf{A}$.

Also we can expand the logarithm and obtain

$$\begin{aligned}\Gamma_{log} \sim -\frac{N\Omega}{8\pi^2}\int^{\Lambda^2} dp^2\ \Big(&\mathrm{tr}\big(\mathbf{m}^2(\mu)\big)f_1^2(p^2) + \\ &+ 2\mathrm{tr}\big(\mathbf{m}(\mu)\cdot\mathbf{s}\big)f_1(p^2)f_2(p^2) + \\ &+ \mathrm{tr}\big(\mathbf{s}^2 + \mathbf{p}^2\big)f_2^2(p^2)\Big)\end{aligned} \tag{6.19}$$

In (6.19) the first term does not depend on the fields $\mathbf{s}$ and $\mathbf{p}$ and can be neglected.

Substituting the expressions (5.13) and (5.14) for $f_1(p^2)$ and $f_2(p^2)$, one finds that an ultraviolet divergence arises only from the term proportional to $\mathrm{tr}\big(\mathrm{m}(\mu)\cdot \mathrm{s}\big)$.

The explicit calculation gives

$$\begin{aligned}\Gamma_{log}\Big|_{div} &= \lim_{\Lambda\to\infty}\left[-\frac{N\Omega}{8\pi^2}\mu^3\int^{\Lambda^2}\frac{dp^2}{p^2}\,f(p^2)^{-1}\right]\;2\mathrm{tr}\big(\mathrm{m}(\mu)\cdot \mathrm{s}\big)\sim\\ &\sim-\lim_{\Lambda\to\infty}\frac{N\Omega}{2\pi^2}\;a\;\mu^3\;\mathrm{tr}\big(\mathrm{m}(\mu)\cdot \mathrm{s}\big)\;\log\log\frac{\Lambda^2}{\mu^2}\end{aligned} \tag{6.20}$$

which exactly cancels the divergent part of $\Gamma_2$ (eq. (6.15)).

In this way we have a completely UV regularized expression for $\Gamma$.

This is due to fact that we are taking into account the renormalization group effects in the leading log approximation. As a consequence in fact, we don't obtain the usual logarithmic mass divergence from $\Gamma_{log}$ but it remains only a divergent term which goes to infinity like $\log\log(\Lambda^2/\mu^2)$.

In other words, the insertion of the running mass $\mathrm{m}(\mu)f_1(p^2)$ in the scalar part of the fermion self-energy, regularizes the theory at one loop order, at least in the leading log approximation, while the residual divergence is cancelled by the two loop contribution of $\Gamma_2$.

Remember that, also in the discussion of the Schwinger-Dyson equation (see sect. 5), the asymptotic behaviour for large momentum of the mass term in $\Sigma'_s$ is responsible for the regularization of the theory in the ultraviolet range.

Let us remark that, in the case in which one neglects the renormalization group effects, the situation of the ultraviolet divergences changes.

As observed in the previous section, in the rigid case a logarithmic divergence appears in the gap equation for $\Sigma'_s$ and, in order to cancel it, one has to use a subtraction procedure by inserting a mass counterterm.

Let us analize the ultraviolet divergences in $\Gamma$ in this simplified rigid case.

The ansatz for $\Sigma$ is given in eq. (5.19).

Inserting it in $\Gamma_2$ we get a finite expression. But, let us examine the logarithmic term of $\Gamma$ in the range of large momenta.

Subtracting, as usual, an infinit constant which is independent on the fields, we get

$$\Gamma_{log} \sim -\frac{N\Omega}{8\pi^2}\int^{\Lambda^2} dp^2\; p^2\;\log\;\det\left(\delta_{ab} + [\mathrm{m}^2(\mu)]_{ab}\frac{1}{p^2} + 2\;[\mathrm{m}(\mu)\cdot \mathrm{s}]_{ab}\frac{\mu^3}{p^4} + \right.$$
$$\left. + [\mathrm{s}^2 + \mathrm{p}^2]_{ab}\frac{\mu^6}{p^6}\right) \tag{6.21}$$

Using the relation (6.18) and expanding the logarithm in (6.22) one finds an ultraviolet divergence only from the term proportional to $\mathrm{tr}(\mathrm{m}(\mu)\cdot \mathrm{s})$

$$\Gamma_{log}\Big|_{div} = \lim_{\Lambda\to\infty} -\frac{N\Omega}{4\pi^2}\mu^3\;\log\frac{\Lambda^2}{\mu^2}\;\mathrm{tr}\big(\mathrm{m}(\mu)\cdot \mathrm{s}\big) \tag{6.22}$$

which is a logarithmic divergence.

Recalling the relation between s evaluated at the extremum of the effective potential and the scalar fermion condensate (5.16), we can write the divergent term in the form

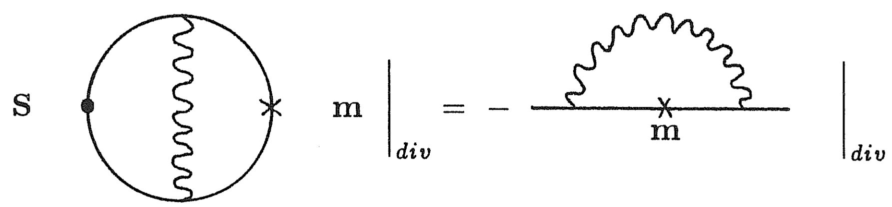
$$\begin{aligned} -\frac{3g^2C_2}{16\pi^2}\Omega\;\log\frac{\Lambda^2}{\mu^2}\;\mathrm{tr}\big(\mathrm{m}(\mu)\cdot \mathrm{S}\big) &= \\ = -\Omega\;\mathrm{tr}\big(\delta\mathrm{m}(\mu,\Lambda)\cdot \mathrm{S}\big)& \end{aligned} \tag{6.23}$$

with

$$\mathrm{S}_{ab} = \overline{\Psi}_a\Psi_b$$

$$\delta\mathrm{m}(\mu,\Lambda) = \mathrm{m}(\mu)\;\frac{3g^2C_2}{16\pi^2}\;\log\frac{\Lambda^2}{\mu^2} \tag{6.24}$$

Eq. (6.24) represents exactly the divergence of the composite operator S which is equal to minus the usual mass divergence evaluated at the one loop order. Diagrammatically



We see that, in the rigid case, it is necessary to renormalize the potential by adding a wave function renormalization term for the composite operator S,

$$\mathrm{tr}(\delta Z_{\mathrm{S}}(\mu,\Lambda)\cdot \mathrm{S})$$

such that

$$\delta Z_{\mathrm{S}}(\mu,\Lambda)\Big|_{div} = \Omega\ \delta \mathrm{m}(\mu,\Lambda)\Big|_{div} \tag{6.25}$$

The finite part of $\delta Z_{\mathrm{S}}$ has to be determined by a suitable normalization condition which will be discussed in a while.

Let us come back to the general case.

We have proved that the ansatz we have made for the self-energy asymptotic behaviour, garantees the cancellation of the mass divergences.

What we need now is a normalization condition in order to fix the finite part of the effective potential.

The natural choice comes from the expression of the effective potential for small masses [20]

$$\lim_{m_a\to 0}\frac{1}{m_a}\frac{\partial V}{\partial \chi_a}\Big|_{extr} = \frac{4N\mu^3}{3C_2 g^2(\mu)} \qquad a=1,..,n \tag{6.26}$$

with

$$\langle \chi_a\rangle = \langle \mathrm{s}_{aa}\rangle = \frac{3C_2 g^2(\mu)}{4N\mu^3}\langle \overline{\Psi}_a \Psi_a\rangle_\mu$$

In eq. (6.26) $m_a$ are the $n$ eigenvalues of the fermion mass matrix and the partial derivative is made with respect to the explicit dependence of $V$ on $\chi_a$ (explicit symmetry breaking part) and is evaluated at the minimum of the potential.

To understand this choice, let us derive an expression for the effective action in the limit of small masses.

We can think to add the mass term by the following replacement in the bilocal source

$$\mathbf{J}(x,y) \to \mathbf{J}(x,y) + \mathrm{m}\ \delta^4(x-y) \tag{6.27}$$

Then the generating functional of the connected Green functions in the presence of a small mass term can be written

$$W[\mathrm{m},\mathbf{J}]_{\mathrm{m}\to 0} \sim W[0,\mathbf{J}] + \int d^4x\ \mathrm{tr}\left[\mathrm{m}\ \frac{\delta W[0,\mathbf{J}]}{\delta \mathbf{J}(x,x)}\right] \tag{6.28}$$

where the trace over the spinor and the color indices is understood.

We can now calculate the effective action at its axtremum $\mathbf{J} = 0$ obtaining

$$\begin{aligned}\Gamma(\mathrm{m})\big|_{\mathrm{J}=0} = W(\mathrm{m},0)_{\mathrm{m}\to 0} \sim W(0,0) - \mathrm{tr}\left[\mathrm{m}\int d^4x\ \mathrm{S}(x,x)\right]_{\mathrm{J}=0} \sim \\ \sim \Gamma(0)\big|_{\mathrm{J}=0} + \Omega\ \mathrm{tr}\ (\mathrm{m}\ \langle\overline{\Psi}\Psi\rangle)\end{aligned} \qquad (6.29)$$

in fact $\mathrm{S}(x,x)$ is nothing but minus the scalar condensate.

Therefore we have to require

$$\lim_{\mathrm{m}\to 0}\frac{1}{\Omega}\ \mathrm{tr}\left(\mathrm{m}^{-1}\ \frac{\delta\Gamma}{\delta\langle\overline{\Psi}\Psi\rangle}\right)\bigg|_{extr} = 1 \qquad (6.30)$$

or equivalently, by using the relation between the scalar condensate and the scalar field s (5.16)

$$\lim_{\mathrm{m}\to 0}\mathrm{tr}\left(\mathrm{m}^{-1}\ \frac{\partial V(\mathrm{s}^2+\mathrm{p}^2,\mathrm{s})}{\partial \mathrm{s}}\right)\bigg|_{extr} = \frac{4N\mu^3}{3C_2 g^2(\mu)} \qquad (6.31)$$

As we will see, at the minimum of the effective potential we will have

$$\langle \mathrm{p}_{ab}\rangle = 0 \quad \forall\ a,b \qquad (6.32)$$

$$\langle \mathrm{s}_{ab}\rangle = 0 \quad for \quad a \neq b \qquad (6.33)$$

and consequentely a factorization of the effective potential itself in the sum on $n$ pieces, one for each flavor.

Hence, if we choose a diagonal mass matrix, as we will do, the normalization conditions to be imposed will be those of eq. (6.26).

At this point, having verified that the expression (3.41) for the effective action results IR and UV finite once inserted the ansatz (5.12) for the self-energy and having introduced the suitable normalization condition for $\Gamma$ at the physical point, we are in order to give an expression of the effective potential $V$ ($\Gamma = \Omega\ V$) as a function of the variational parameters of the theory s and p

$$\begin{aligned}V = \frac{N\mu^4}{4\pi^2}\bigg[c\ A_1\ \mathrm{tr}\ (\mathrm{s}^2+\mathrm{p}^2) + (A_2+\delta z_f)\ \mathrm{tr}\ (\mathrm{m}(\mu)\cdot\mathrm{s}) - \\ -\frac{1}{2}\int_0^\infty dy\ y\ \log\det\Big(y\delta_{ab} + \mathrm{x}_{ab}^2 + \mathrm{z}_{ab}^2\Big)\bigg]\end{aligned} \qquad (6.34)$$

where

$$
\begin{aligned}
c &= \frac{8\pi^2}{3C_2 g^2(\mu)} \\
A_1 &= 1 + \frac{d^2}{2a}\int_0^1 du \, \frac{F(u)^{2d-2}}{1+2aF(u)} \\
A_2 &= \frac{2a}{\mu}\left[\int_0^1 \frac{du}{u}\,\frac{1}{1+2aF(u)} - (d-1)\log\frac{1+2a}{2a}\right] \\
F(u) &= 1 - \frac{1}{2a}\log u \\
\mathrm{x} &= \ \mathrm{m}\,\frac{f_1(y)}{\mu} + \mathrm{s}\,\frac{f_2(y)}{\mu} \\
\mathrm{z} &= \ \mathrm{p}\,\frac{f_2(y)}{\mu} \\
f_1(y) &= \Theta(1-y) + \Theta(y-1)\left(1+\frac{1}{2a}\log y\right)^{-d} \\
f_2(y) &= \mu\left[\Theta(1-y) + \Theta(y-1)\frac{1}{y}\left(1+\frac{1}{2a}\log y\right)^{d-1}\right] \\
a &= \log\frac{\mu}{M_0}
\end{aligned}
\tag{6.35}
$$

and we have introduced the finite counterterm $\delta z_f$ to be determined in order to satisfy the normalization condition (6.26).

## 7. MASSLESS EFFECTIVE POTENTIAL

If the mass parameter in the classical Lagrangian density (3.1) is equal to zero, one recovers the invariance of the theory under the chiral group.

Let us consider the logarithmic contribution in $\Gamma$ in the massless case

$$\Gamma_{log} = -\mathrm{Tr}\ \log\left(\mathbf{S}_0^{-1} + \frac{\delta\Gamma_2}{\delta\mathbf{S}}\right) = -\mathrm{Tr}\ \log\left(i\hat{p} - \Sigma_s(p^2) - i\gamma_5\Sigma_p(p^2)\right) =$$
$$= -N\Omega \int \frac{d^4p}{(2\pi)^4} \log\ \mathrm{Det}\left(i\hat{p} - (\mathbf{s} + i\gamma_5\mathbf{p})f_2(p^2)\right) \tag{7.1}$$

where we have used eqs. (6.2) and (6.3) in the massless case and $f_2(p^2)$ is given in (5.14).

The matrix

$$\mathbf{A} = \mathbf{s} + i\gamma_5\mathbf{p} \tag{7.2}$$

considered as an $n \times n$ complex matrix, can be diagonalized by a chiral rotation using two unitary matrices $\mathbf{U}$ and $\mathbf{V}$

$$\mathbf{A}_d = \mathbf{U}\mathbf{A}^\dagger\mathbf{V} \tag{7.3}$$

Since this transformation leaves the $\Gamma_2$ contribution in the effective action invariant, we can simply insert $\mathbf{A}_d$ in the evaluation of the determinant

$$\log\ \mathrm{Det}\left(i\hat{p} - \mathbf{A}_d f_2(p^2)\right) =$$
$$= \log\ \mathrm{Det}\left(i\hat{p} - \left[\left(s_0 + \sum_{i=1}^{n-1} s_i h_i\right) + i\gamma_5\left(p_0 + \sum_{i=1}^{n-1} p_i h_i\right)\right] f_2(p^2)\right) \tag{7.4}$$

where we have expanded $\mathbf{A}_d$ in terms of the generators of the Cartan subalgebra of $U(n)$, $\mathbf{1}$ and $h_i$ $\ i = 1,..,n-1$ normalized as $\mathrm{tr}(h_i h_j) = \delta_{ij}$.

Substituting (7.4) in (7.1), one obtains

$$\Gamma_{log} = - N\Omega \int \frac{d^4p}{(2\pi)^4} \log\ \mathrm{Det}\left(\left[i\hat{p} - (\chi_1 + i\gamma_5\pi_1)f_2(p^2)\right]\cdot\right.$$
$$\left.\cdot\left[i\hat{p} - (\chi_2 + i\gamma_5\pi_2)f_2(p^2)\right]\cdots\left[i\hat{p} - (\chi_n + i\gamma_5\pi_n)f_2(p^2)\right]\right) = \tag{7.5}$$
$$= - N\Omega \sum_{a=1}^{n} \int \frac{d^4p}{(2\pi)^4} \log\ \mathrm{Det}\left[i\hat{p} - (\chi_a + i\gamma_5\pi_a)f_2(p^2)\right]$$

where

$$
\begin{aligned}
\chi_a &= s_{aa} = s_0 + \sum_{i=1}^{n-1} s_i (h_i)_{aa} \\
\pi_a &= p_{aa} = p_0 + \sum_{i=1}^{n-1} p_i (h_i)_{aa} \\
a &= 1, .., n
\end{aligned}
\tag{7.6}
$$

that is $\chi_a$ and $\pi_a$ are the eigenvalues of the matrices s and p respectively.

On the other hand

$$
\mathrm{tr}(\mathbf{s}^2 + \mathbf{p}^2) = \sum_{a=1}^{n} (\chi_a^2 + \pi_a^2) \tag{7.7}
$$

Therefore, evaluating as before the determinant over the $\gamma$-matrices in $\Gamma_{log}$ and performing the integration over the angular variables we obtain the following expression for the effective potential in the massless case which we will indicate as $V^{(0)}$ (we omit an additive infinite constant)

$$
\begin{aligned}
V^{(0)} &= \frac{N\mu^4}{4\pi^2} \sum_{a=1}^{n} V_1^{(0)}(\chi_a, \pi_a) \\
V_1^{(0)}(\chi, \pi) &= \Bigg( c \left[ 1 + \frac{d^2}{2a} \int_0^1 du \, \frac{F(u)^{2d-2}}{1 + 2aF(u)} \right] (\chi^2 + \pi^2) - \\
&\quad - \frac{1}{2} \int_0^1 dy \; y \; \log \left( 1 + \frac{\chi^2 + \pi^2}{y} \right) - \\
&\quad - \frac{1}{2} \int_0^1 \frac{du}{u^3} \; \log \left( 1 + (\chi^2 + \pi^2) \; u^3 \; F(u)^{2d-2} \right) \Bigg)
\end{aligned}
\tag{7.8}
$$

where $c$, $F(u)$ and $a$ are quantities defined in (6.36).

As expected $V^{(0)}$ is a completely finite quantity both in the ultraviolet and in the infrared regime.

The theory is in fact regularized in the infrared by the assumed constant behaviour of the self-energy in the $p \rightarrow 0$ limit, whereas the convergence in the ultaviolet follows in the massless case from the physical meaning of $\chi$ and $\pi$.

In fact, from the relations (5.16), (5.17) and (7.6) it follows that $\chi$ and $\pi$ have operator dimension equal to three.

However, due to the chiral invariance, linear terms in $\chi$ and $\pi$ are forbidden so that the effective potential $V^{(0)}$ must start at least with $(\chi^2+\pi^2)$ in an expansion in the composite fields.

Therefore, due to the absence of operators of dimension lower or equal to 4, no ultraviolet divergences are expected in $V^{(0)}$.

In the simple case in which one neglects the logarithmic corrections due to the renormalization group analysis, it is possible to perform all the integrations explicitly and to find an analytic expression of the effective potential as a function of $\phi^2 = \chi^2 + \pi^2$.

This case, which we have called the rigid case, corresponds to a fixed coupling constant $g^2(p) = g^2(\mu) = g^2$ and to

$$\left(\frac{\log(p^2/M_0^2)}{\log(\mu^2/M_0^2)}\right)^d = 1$$

which is equivalent to take equal to zero the anomalous dimension of the composite operator $(\overline{\Psi}\Psi)$.

Then, calling $\overline{V}$ the effective potential in the rigid case, we get

$$\overline{V}^{(0)} = \frac{N\mu^4}{4\pi^2} \sum_{a=1}^{n} \overline{V}_1^{(0)}(\phi_a^2)$$

$$\begin{aligned}\overline{V}_1^{(0)}(\phi^2) = {} & (c-\frac{1}{4})\phi^2 + \frac{1}{4}\phi^4 \log\frac{1+\phi^2}{\phi^2} + \\ & + \frac{1}{8}\phi^{4/3}\log\frac{1-\phi^{2/3}+\phi^{4/3}}{(1+\phi^{2/3})^2} - \\ & - \frac{1}{4}\sqrt{3}\phi^{4/3}\left(\frac{\pi}{2} - \arctan\frac{2-\phi^{2/3}}{\sqrt{3}\phi^{2/3}}\right)\end{aligned} \tag{7.9}$$

It is interesting to notice that the simplicity of this case is mainly due to the fact that, ignoring the *logs*, one recovers the exact scale invariance.

In fact the scale $\mu$ appears in the effective potential only as an overall factor and the minima of $\overline{V}^{(0)}$, which determines the phase structure of the theory, depend only on $c$ which is inversely proportional to coupling constant $(c = (8\pi^2)/(3C_2 g^2))$.

From eq. (7.9) we get easily the behaviour of $\overline{V}_1^{(0)}(\phi^2)$ for $\phi^2 \to 0$

$$\overline{V}_1^{(0)}(\phi^2)_{\phi^2\to 0} \sim (c-1)\phi^2 \tag{7.10}$$

showing that the theory is unstable at $\phi = 0$ when $c < 1$ and spontaneous symmetry breaking occurs.

Thus, for

$$\alpha_s > \frac{2\pi}{3C_2} \tag{7.11}$$

$(\alpha_s = g^2/(4\pi))$, we have a local minimum of the effective potential at $\phi^2 \neq 0$ which corresponds to a non zero vacuum expectation value (VEV) for $\phi^2$.

For example, in QCD with triplet fermions, the critical point is for $\alpha_s = \pi/2$.

The graphical representation of the potential $\overline{V}_1^{(0)}$ of eq. (7.9) as a function of the two variables $c$ and $\phi$ is given in Fig. 4.

The development of the separate minima in $\phi$ from a unique minimum is clearly exhibited.

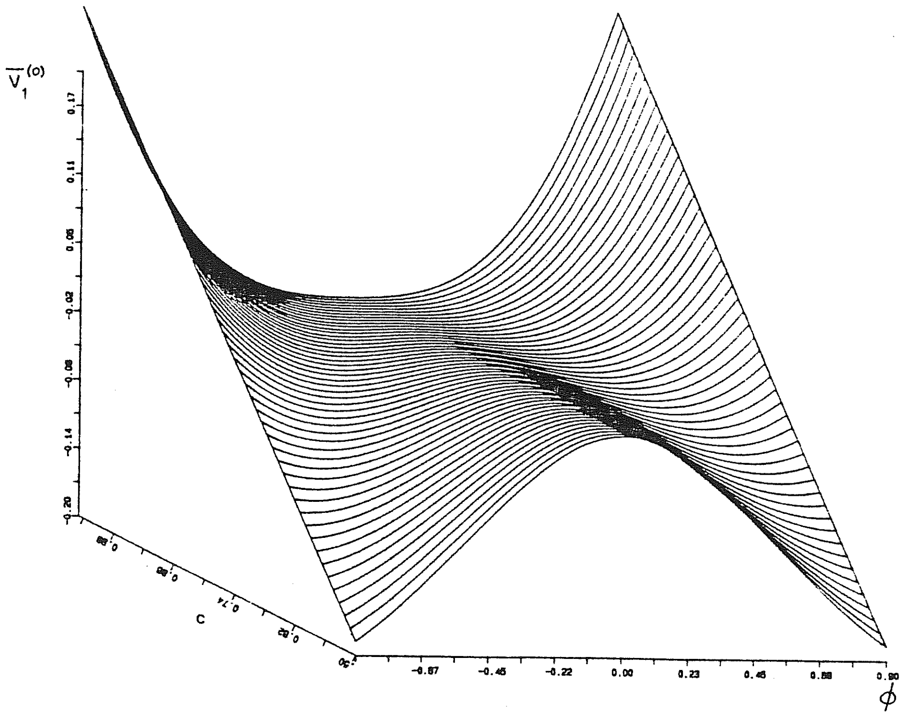



*FIG. 4. The effective potential $\overline{V}_1^{(0)}$ as a function of the field $\phi$ and of the parameter c (c is inversely proportional to $\alpha_s$). By decreasing the value of c one goes continously from a shape having a single minimum in $\phi$ (the highest section in the figure) to a shape with two distinct degenerate minima. The bifurcation trajectory of the minima corresponds to the appearence of chiral symmetry breaking.*

In Fig. 5 we have the potential $\overline{V}_1^{(0)}$ as a function of $\chi$ and $\pi$ with its typical shape of degenerate minima lying on a circle.

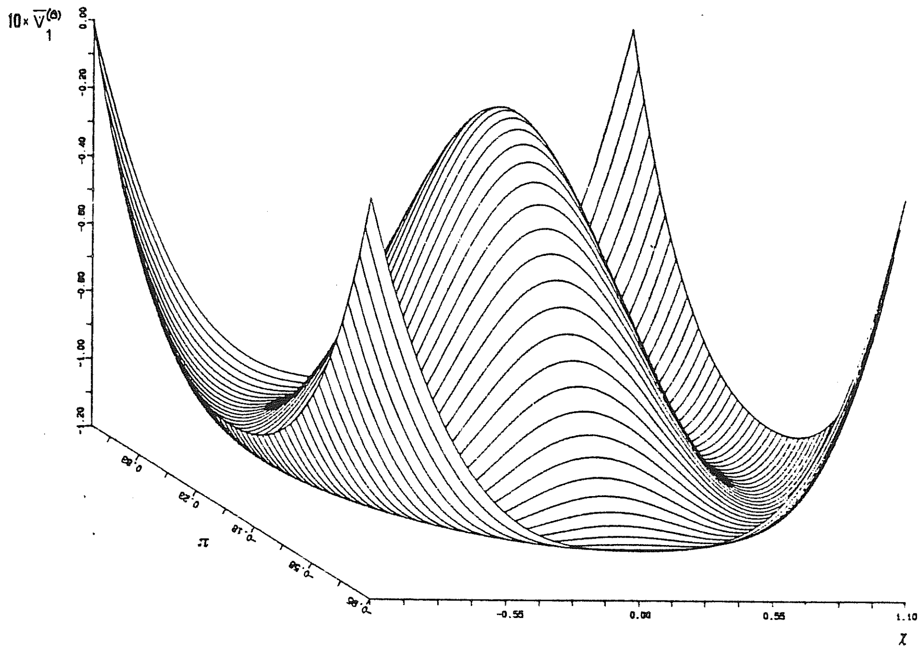


*FIG. 5. The effective potential* $\overline{V}_1^{(0)}$ *on the* $\pi$ *-* $\chi$ *plane. The figure is for* $c = 0.6$.

Without loss of generality, we can suppose that the VEV of $\chi_a$ is different from zero and the VEV of $\pi_a$ is equal to zero (there is no spontaneous breaking of the parity).

Since the effective potential has the factorized form given in eq. (7.8), every $\chi_a$ has the same VEV, $\langle \chi_a \rangle = u$.

From the expression of $\chi_a$ in eq. (7.6), we then find

$$\begin{aligned} &\langle s_0 \rangle = u \\ &\langle s_i \rangle = 0 \qquad i = 1, .., n-1 \end{aligned} \tag{7.12}$$

In this way we have proved that, in the framework of our simplified model, the quark-antiquark-gluon interaction provides the Goldstone realization of the chiral symmetry due to the spontaneous breaking of $SU(n)_L \otimes SU(n)_R$ to $SU(n)_{L+R}$.

The fields $p_i \quad i = 1, .., n-1$ are the composite massless Goldstone bosons of the chiral symmetry breaking ( $p_0$ is not massless due to the axial anomaly) and eqs. (7.6) and (5.17) give the relations between the $p_i$ and the pseudoscalar condensates.

From eq. (7.9) we can easily extract the asymptotic behaviour of $\overline{V}_1^{(0)}$ for large values of the field $\phi^2$

$$\overline{V}_1^{(0)}(\phi^2)_{\phi^2\to\infty} \sim \; c\phi^2 \tag{7.13}$$

showing that the effective potential is bounded from below (obviously for $c > 0$).

Let us also observe that, the $\phi \to 0$ limit

$$\overline{V}_1^{(0)}(\phi^2)_{\phi^2\to 0} \sim \; (c-1)\phi^2 + \frac{3}{16}\phi^4 - \frac{1}{4}\phi^4 \log \, \phi^2 \tag{7.14}$$

differs from a $\sigma$-model potential [10] because of the last logarithmic term.

Such a logarithmic singularity comes from the vanishing quark mass otherwise the $\phi \to 0$ limit of the effective potential would be analytic.

(Notice that in ref. [18], eq. (42), there is a misprinting in the coefficient of $\phi^4$).

It is important to remark that higher fermionic loops do not change these results.

In fact, the instability for $c < 1$ is exhibited by the quadratic term in $\phi$ while, a higher order contribution to $\Gamma_2$, would contain at least one more fermionic propagator leading to a power of $\phi$ greater than two.

As far as the asymptotic behaviour is concerned, every fermionic propagator gives a contribution which goes to zero for large fields.

So, the characteristic of our effective potential to be bounded from below, remains true to higher loop orders.

One can also study numerically the minima of the potential of eq. (7.9) and determine their behaviour with $c$.

Due to the connection between $\langle\chi\rangle$ and the scalar condensate renormalized at $\mu$, we get the way of varying of $\langle\overline{\Psi}\Psi\rangle_\mu$ with the coupling constant. This is illustrated in Fig. 6 where

$$\left(\frac{1}{N}\,\frac{\langle\overline{\Psi}\Psi\rangle_\mu}{\mu^3}\right)^{\frac{1}{3}} = \left(\frac{c\,\langle\chi(c)\rangle}{2\pi^2}\right)^{\frac{1}{3}}$$

is plotted as a function of $c$ showing explicitly the phase transition at $c = 1$.

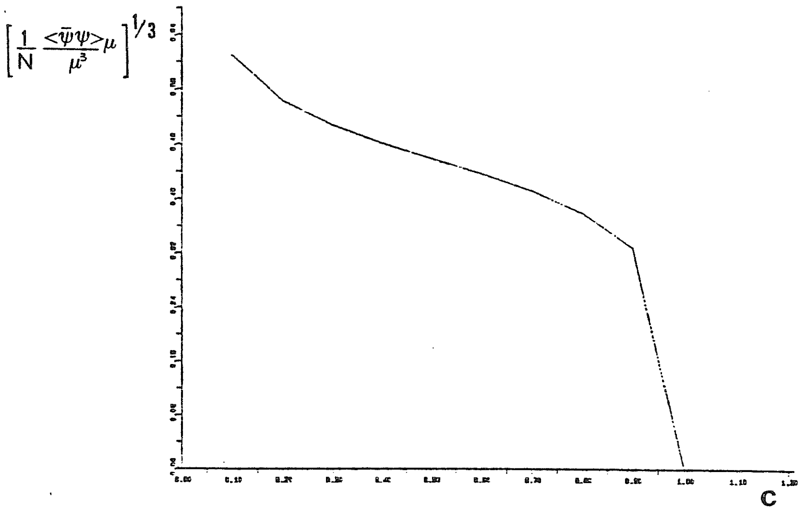



*FIG. 6. Way of varying of the scalar condensate $\langle\overline{\Psi}\Psi\rangle_\mu$ with $c = 2\pi/(3C_2\alpha_s)$. The phase transition is for $c = 1$.*

Let us go back to the complete case with logarithmic corrections taken into account.

First of all we will show that, also in this case the effective potential $V^{(0)}$ of eq. (7.8) is bounded from below.

Starting from eq. (7.8), we can easily derive the following expression

$$V^{(0)}(\phi^2) - \overline{V}^{(0)}(\phi^2) = \frac{N\mu^4}{4\pi^2}\left(\frac{cd^2}{2a}\int_0^1 du\ \frac{F(u)^{2d-2}}{1+2aF(u)}\ \phi^2 - \right.$$
$$\left. -\frac{1}{2}\int_0^1 \frac{du}{u^3}\ \log\frac{1+\phi^2\ u^3\ F(u)^{2d-2}}{1+\phi^2\ u^3}\right) \tag{7.15}$$

In eq. (7.15) the first term is positive definite because $F(u) > 0$ for $0 < u \leq 1$.

As far as the second term is concerned, one has to consider the quantity

$$F(u)^{d-1} = \left(1 - \frac{1}{2a}\log u\right)^{d-1}$$

The expression (7.15) is positive definite when $F(u)^{d-1} \leq 1$. Because $u \leq 1$, it follows that $F(u)^{d-1} \leq 1$ according to $(d-1) \leq 0$.

Remember that

$$d = \frac{3bC_2}{8\pi^2} = \frac{9C_2}{11N - 2n}$$

so, in QCD with color triplet fermions ($N = 3$, $C_2 = 4/3$), one has $d = 12/(33 - 2n)$, and $d \leq 1$ for $n \leq 21/2$. This means that, if we have less than six families, $V^{(0)}(\phi^2) - \overline{V}^{(0)}(\phi^2) > 0$ for any value of $\phi^2$.

We have derived in this way a rigorous lower bound for $V^{(0)}(\phi^2)$. This bound clearly shows that also $V^{(0)}(\phi^2)$ is bounded from below.

The numerical analysis of $V^{(0)}(\phi^2)$ is straightforward showing the occurence of the spontaneous chiral symmetry breaking phenomenon $SU(n)_L \otimes SU(n)_R \rightarrow SU(n)_{L+R}$.

In this general case, the effective potential is a function of the ratio of the two scales $\mu$ and $M_0$ and it appears more natural to parametrize the results with $\mu/M_0$ rather than in terms of the variable $c$.

One finds that, in the case of QCD with three flavors ($M_0 = \Lambda_{QCD}$), the theory undergoes chiral symmetry breaking for $\mu/\Lambda_{QCD} \lesssim 1.355$ which corresponds to

$$\alpha_s = \frac{g^2(\mu)}{4\pi} \gtrsim 0.73\pi \tag{7.16}$$

The behaviour of the effective potential near the critical point is showed in Fig. 7.

Remember that, in the previous simplified discussion in which we neglected the logarithmic corrections, we found $(\alpha_s)_{crit} = \pi/2$.

The somewhat higher value of $(\alpha_s)_{crit}$ in this case with respect to the case without *logs*, is qualitatively understandable since, in the latter case the coupling constant does not depend on the momentum whereas in the former it goes down to zero for large $p$. In some loose sense, in the rigid case we are averaging the running coupling constant all over the range of momentum.

We can definitly conclude that the overall qualitative picture remains unchanged when we take the renormalization group corrections into account.

Furthermore, the quantitative differences in the two cases are not dramatic (13% for the critical value of $\mu/\Lambda_{QCD}$).

Let us also remark that, as we will see in the massive case, the two potentials $V$ and $\overline{V}$ must be compared for different values of $\mu/\Lambda_{QCD}$ (this quantity must be recalculated in each case from the experimental data).

In this way we will find that the observable quantities like the values of the quark masses and condensates, vary very little from the rigid to the general case ($\sim$ 10%).

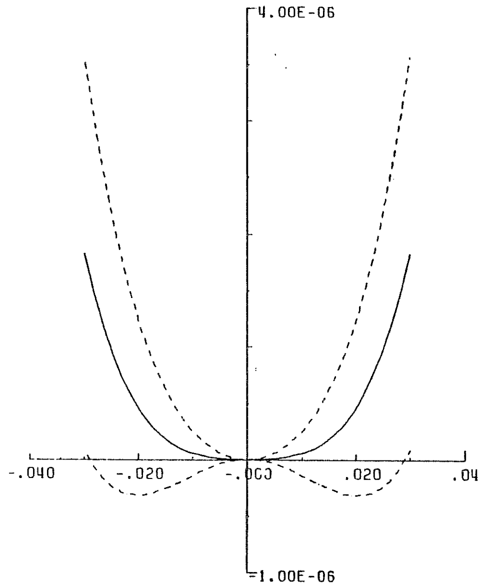



*FIG. 7. Behaviour of* $V^{(0)}(\phi^2)$ *near the critical point in the case of QCD with three flavors. The curves shown correspond to the values* $\mu/\Lambda_{QCD} = 1.354$ *(dash-dotted line), 1.355 (solid line), 1.357 (dashed line).*

## 8. COMPARISON WITH OTHER STUDIES OF THE DYNAMICAL CHIRAL SYMMETRY BREAKING PHENOMENON

The method of the effective action for composite operators, as we have seen, proves to be a convenient tool in studying field theories with dynamical symmetry breaking.

The application of the direct variational method to the problem of DSB in QCD, has allowed us to take into account the non linear aspects of the problem and to justify the results obtained first in the framework of the linearized approximation (see for example [45], [46] and [47]).

In fact a way to investigate the dynamical mechanism of the spontaneous breakdown of chiral invariance in massless gauge theories is based on the exact solution of the linearized Bethe-Salpeter (BS) equations for the pseudoscalar Goldstone bosons.

The main hypothesis of this approach is that the mechanism of the condensates formation responsible for the spontaneous symmetry breaking, comes from the strong gauge forces acting at distances of the order of the size of the Goldstone bosons and the crucial point is the assumption that these distances are smaller than those at which the confinement forces dominate.

In the model considered in ref. [45] the dynamics of condensation is described by BS equations for the fermion-antifermion tightly bound states in which the parameters of infrared and ultraviolet cutoffs are introduced in order to pick out the momentum range which is responsible for binding.

The infrared cutoff is identified with the confinement scale.

Since the result of the analysis gives a critical value $(g)_{crit}$ for the coupling constant at which chiral symmetry breaking occurs and, since in non abelian gauge theories the domain of strong coupling $(g(p) \gtrsim (g)_{crit})$ is the region of small momenta, they identify the ultraviolet cutoff with the value of $p$ at which $g(p) \sim (g)_{crit}$.

The kernels of the BS equations, are taken in the ladder approximation with values of the coupling constant $g$ and of the fermion dynamical mass $m$ equal to the values of the running coupling constant $g(p)$ and of the fermion mass function $\Sigma(p^2)/Z(p^2)$ respectively, averaged in the appropriate momentum range ($S^{-1}(p) = iZ(p^2)\hat{p} - \Sigma(p^2)$).

With these approximations, it happens that the BS equations have the solutions for the thightly bound states with $M^2 \leq 0$ if

$$\alpha = \frac{g^2}{4\pi}\frac{N^2-1}{2N} \qquad \text{for} \quad SU(N)$$

exceeds its critical value

$$(\alpha)_{crit} = \frac{\pi}{3} \tag{8.1}$$

In QCD this means

$$(\alpha_s)_{crit} = (\frac{g^2}{4\pi})_{crit} = \frac{\pi}{4} \tag{8.2}$$

This model has been applied to the investigation of the spectrum of the pseudoscalar mesons in QCD with $n$ flavors [46].

The main results obtained are the following.

In the symmetric unstable phase, there exist $n^2$ pseudoscalar tachyons while, in the phase in which the vacuum rearrangement results in spontaneous breakdown of the chiral $SU(n)_L \otimes SU(n)_R$ symmetry, a fermion mass appears.

The spectrum of the fermion dynamical masses can be determined by requiring that tachyons disappear in the stable phase and, instead of them, the $(n^2-1)$-plet of pseudoscalar Goldstone bosons appears (the singlet under $SU(n)_{L+R}$ acquiring mass through the Adler-Bell-Jackiw anomaly of the $U(1)_{L-R}$ current).

Also, in the case in which there is spontaneous and explicit breakdown of chiral symmetry, the mass formula for the pseudoscalar mesons is derived in this model and, once compared with the mass formula of the current algebra, it provides a dynamical realization of the PCAC hypothesis.

From the Ward identities for the axial currents, it follows that such a way of determining the dynamical fermion mass, is equivalent to looking for non trivial solutions for the linearized Schwinger-Dyson equation for the self-energy $\Sigma(p^2)$ in the ladder approximation.

An improved form of the Scwinger-Dyson equation for the quark propagator in QCD, has been numerically studied by K. Higashijima [36].

He also assumes that the short range force rather than the confining force is responsible for chiral symmetry breaking in QCD and so he approximates the kernel of the SD equation by the one-gluon exchange but improving with the running coupling constant.

In order to tame the infrared singularities of

$$\lambda(t) = \frac{3C_2 g^2(t)}{4\pi^2} \qquad t = \log \frac{p}{\Lambda_{QCD}}$$

he defines a non confining QCD-like model by

$$\lambda(t) = \begin{cases} A/t_c, & \text{if } t \leq t_c; \\ A/t, & \text{if } t > t_c. \end{cases} \tag{8.3}$$

with

$$A = \frac{3C_2}{11 - 2n/3}$$

(which exactly corresponds to our choice for the running coupling constant).

The numerical result is that, in the case of triplet quarks and $n = 3$, for $t_c \lesssim 0.88$ the constituent quark mass remains no vanishing in the chiral limit, that is, the chiral symmetry is spontaneously broken.

Since the parameter $t_c$ corresponds to our $\log(\mu/\Lambda_{QCD})$, the critical point results for

$$\left(\frac{\mu}{\Lambda_{QCD}}\right)_{crit} \simeq 2.4 \tag{8.4}$$

which corresponds to a broken phase for

$$\alpha_s = \frac{g^2(\mu)}{4\pi} \gtrsim \frac{\pi}{4} \tag{8.5}$$

Let us notice that this is completely consistent with the critical value (8.2) of the coupling constant of refs. [45], [46] and [47] in which the linearized SD equation is considered.

This is not surprising since the dynamical mass function goes to zero as a power for large values of momenta and so its contribution at the denominator of SD equation is quite inessential in the range of momenta at which chiral symmetry breaking occurs.

In fact, let us write down the non-linear BS equation for the fermion self-energy (see eq. (5.5)) in the massless case

$$\Sigma(q^2) = 3C_2 \int \frac{d^4p}{(2\pi)^4} \frac{\Sigma(p^2)}{p^2 + \Sigma^2(p^2)} \frac{g^2(p,q)}{(p-q)^2} \tag{8.6}$$

(we are only considering the scalar part of the self-energy and $Z(p^2) = 1$ (Landau gauge)).

In the large range of momenta, it is sensible to neglect the self-energy contribution at the denominator on the right hand side of eq. (8.6) due to the asymptotical behaviour of $\Sigma$ itself. In this way one just recovers the linearized form of the Schwinger- Dyson equation.

As far as the variational methods are concerned, let us mention the result obtained by M.E. Peskin [15] with a simple stability analysis of the CJT effective action.

In order to compute whether chiral symmetry breaking can be induced by one-gluon exchange in a $SU(N)$ gauge theory of massless fermions (lowest order approximation), he works in the following simplified framework: he considers a fixed coupling constant $g$ and expands $\Gamma_{CJT}$ into quadratic order in the fermion self-energy $\Sigma$.

Since he expects that for $g^2$ sufficiently small, the vacuum is chirally symmetric, he restricts his attention to study the stability of the symmetric vacuum.

He finds that the kinematical terms in $\Gamma_{CJT}$ stabilize the chirally symmetric state $\Sigma = 0$, and so the interactions must counteract this effect.

The explicit calculation of $\Gamma_2$ truncated up to the second order in $\Sigma$ shows that, a criterion for an instability is

$$\frac{3C_2 g^2}{4\pi^2} > 1 \tag{8.7}$$

which, for quarks in the fundamental of $SU(3)$ gives again

$$\alpha_s > \frac{\pi}{4}.$$

We notice that this result represents only a criterion for the vacuum instability and it depends on the crude approximations done.

In general, other computations based on the CJT effective action formalism, give a higher value for the critical coupling constant.

For example in ref. [38], an analysis completely equivalent to ours is performed but using the CJT functional.

Their numerical analysis for QCD with three flavors gives for the coupling constant the critical value

$$(\alpha_s)_{crit} \simeq \frac{\pi}{2} \tag{8.8}$$

Some similar results have been obtained by P. Castorina and S.Y. Pi [37].

They use the original CJT variational principle and reach the same conclusions on the chiral symmetry breaking as [38] and we find. We, however, disagree with some statements of these authors.

First of all, their potential is not bounded from below since we have shown in sect. 2 that all the stationary points corresponding to chiral symmetry breaking solutions, are actually saddle points.

So their statement of boundness from below of the $V_{CJT}$ effective potential comes simply from the fact that they have not analized the behaviour of the potential in the appropriate range of parameters.

This has also been proved directly by V.P.Gusynin and Yu. Sitenko in ref. [38] in which the computer calculations show the monotone decreasing of $V_{CJT}(\chi)$ with increasing $\chi$.

The unboundness from below of this function is clearly preserved if any finite numbers of loops is taken into account in evaluating the potential since the contribute of the multiloop diagrams, as we have already observed, vanishes in the limit of large dynamical mass.

We don't agree with another statement in ref. [37] about the fact that the logarithmic behaviour of the coupling constant and the fermion self-energy is crucial for the stability of the dynamical symmetry breaking solutions.

As it is shown in our previous analysis (sect. 7), this is not true since the case with the logarithmic corrections taken into account, simply confirms the results obtained in the rigid case.

We think that the differences that P. Castorina and S.Y. Pi find in the two cases, only concern the lack of finite symmetry breaking minimum in their calculations with non logarithmic ansatz.

As a final remark, it is interesting to notice that the variational methods with the specific ansatzes for the fermion self-energy give the higher value for the critical coupling constant as compared to the methods based on the exact solution of the linearized equation or the numerical solution of the non-linear SD equation. This is a general feature of the variational calculations and clearly indicates that the true form of the self-energy is more complicated than that of the type (5.12).

In particular, the constant behaviour in the infrared region ($p < \mu$), is a rather crude approximation.

# 9. MASSIVE EFFECTIVE POTENTIAL IN QCD WITH THREE FLAVORS

We can now discuss the general case of massive fermions and examine the particular predictions of the formalism for QCD.

We will consider the case of three flavors $u$, $d$, $s$ by introducing a diagonal mass matrix for the three quarks

$$\mathrm{m} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \tag{9.1}$$

Let us calculate explicitly the determinant in (6.35) by using the standard relation holding for $3 \times 3$ matrices

$$\det\mathbf{A} = \frac{1}{3}\left[(\mathrm{tr}\mathbf{A}^3) - \frac{3}{2}(\mathrm{tr}\mathbf{A})(\mathrm{tr}\mathbf{A}^2) + \frac{1}{2}(\mathrm{tr}\mathbf{A})^3\right] \tag{9.2}$$

We get in this way [24]

$$V = \frac{N\mu^4}{4\pi^2}\left[c\ A_1\ \mathrm{tr}\ (\mathrm{s}^2 + \mathrm{p}^2) + (A_2 + \delta z_f)\ \mathrm{tr}\ (\mathrm{m}(\mu)\cdot \mathrm{s}) - \right. \\ \left. -\frac{1}{2}\int_0^\infty dy\ y\ \log\ A_3\right] \tag{9.3}$$

where

$$\mathrm{s}_a^b = \left(s_\alpha \frac{\lambda_\alpha}{\sqrt{2}}\right)_a^b \tag{9.4}$$

$$\mathrm{p}_a^b = \left(p_\alpha \frac{\lambda_\alpha}{\sqrt{2}}\right)_a^b \tag{9.5}$$

$$\lambda_i \ = \ Gell - Mann\ matrices; \quad \lambda_0 = \sqrt{\frac{2}{3}}; \quad i = 1,...,8; \quad \alpha = 0,...,8.$$

and

$$\begin{aligned} A_3 &= y^3 + 3y^2E + y(3E^2 - C^2) + E^3 - EC^2 + \frac{2}{3}D \\ C^2 &= C_kC_k\ , \qquad D = d_{klm}C_kC_lC_m \\ C_k &= \sqrt{\frac{2}{3}}(x_0x_k + z_0z_k) + \frac{1}{2}d_{ijk}(x_ix_j + z_iz_j) + f_{ijk}x_iz_j \\ E &= \frac{1}{3}\mathrm{tr}(\mathrm{x}^2 + \mathrm{z}^2) \qquad i,j,k,l,m, = 1,..,8 \end{aligned} \tag{9.6}$$

The quantities $c$, $A_1$, $A_2$, $\mathrm{x}$ and $\mathrm{z}$ are the same we have previously introduced in (6.36) and $\delta z_f$ is the finite counterterm which will be determined in a while by imposing the normalization condition (6.26).

It is possible to show that, also in the massive case, the effective potential (9.3) has a local minimum for vanishing charged fields.

We will prove this statement for $n$ flavors in the symmetric case in which $\mathrm{m}_a^b = m\ \delta_a^b$ [19].

It is convenient to expand the fields $\mathrm{s}_{ab}$ (and $\mathrm{p}_{ab}$) in terms of the Cartan basis of $U(n)$

$$\mathrm{s} = s_0 + \sum_{i=1}^{n-1} s_i h_i + \sum_{\alpha=1}^{(n-1)n/2} (s_\alpha e_\alpha + s_{-\alpha} e_{-\alpha}) \tag{9.7}$$

and analogously for $\mathrm{p}$.

We want to show that the potential $V$ has a minimum on the surface defined by

$$\begin{aligned} s_i &= s_\alpha = s_{-\alpha} = 0 \\ p_i &= p_\alpha = p_{-\alpha} = 0 \end{aligned} \tag{9.8}$$

Let us consider the potential $V$ at the points defined by

$$\begin{aligned} s_\alpha &= s_{-\alpha} = 0 \\ p_\alpha &= p_{-\alpha} = 0 \end{aligned} \tag{9.9}$$

We can prove that, on this surface, $V$ decomposes, as in the massless case, in the sum of $n$ contributions, one for each flavor

$$V = \frac{N\mu^4}{4\pi^2} \sum_{a=1}^{n} V_1(\chi_a, \pi_a, m) \tag{9.10}$$

with $\chi_a$ and $\pi_a$ defined in (7.6).

This factorization can be easily checked in the case of QCD with three flavors, by considering all the components of the fields equal to zero except for those in the 0, 3 and 8 directions in eq. (9.3).

Due to the structure of (9.10), the minimum of $V$ will be at the point $\langle \chi_a \rangle = v$ and $\langle \pi_a \rangle = w$ independent on $a$ (remember that we are considering the $U(n)$ symmetric case). So, using eq. (7.6) this means

$$\begin{aligned} \langle s_0 \rangle &= v \\ \langle p_0 \rangle &= w \\ \langle s_i \rangle = \langle p_i \rangle &= 0 \qquad i = 1, .., n-1 \end{aligned} \tag{9.11}$$

This is a minimum on the surface defined by (9.9).

It remains to be shown that this point is a minimum also along the charged field direction.

Due to the $SU(n)$ invariance of the potential, the first derivatives of $V$ will have the form

$$\begin{aligned} \frac{\partial V}{\partial s_i} &= A\ s_i + B\ p_i \\ \frac{\partial V}{\partial p_i} &= B\ s_i + C\ p_i \end{aligned} \tag{9.12}$$

$$\begin{aligned} \frac{\partial V}{\partial s_\alpha} &= A\ s_{-\alpha} + B\ p_{-\alpha} \\ \frac{\partial V}{\partial p_\alpha} &= B\ s_{-\alpha} + C\ p_{-\alpha} \end{aligned} \tag{9.13}$$

where $A$, $B$ and $C$ are $SU(n)$ invariants.

At the point defined by eq. (9.11) and (9.9), eq. (9.12) implies

$$AC - B^2 \geq 0, \qquad A \geq 0 \tag{9.14}$$

and we see from (9.13) that the eigenvalues of the second derivatives of the potential are positive also along the charged directions.

Some observations are now in order.

On the basis of the previous considerations alone, one cannot exclude the possibility that $V$ has some other isolated minima. However, one can notice that, in the zero mass case, the potential $V$ does not show any other isolated minimum and therefore, for small $m$, the minimum we found is certainly the absolute one.

This result is in agreement with the one found by C. Wafa and E. Witten [48]. In fact they have shown that in vector gauge theories with $\theta = 0$ ( $\theta$ is the parameter connected with the axial anomaly), the global vector symmetries are not spontaneously broken and that, in the case of a symmetric mass matrix, all the condensates are equal (the theory breaks down to $SU(n)_V$).

Let us also remark that the previous arguments apply rigorously only to the symmetric case but, for small values of the fermionic masses, we do expect the same conclusions in virtue of the continuity of the effective potential function.

We will then assume that, also for a general diagonal mass matrix for the quarks, the matrix of the condensate is diagonal at the phisical point.

This leads again to a factorization at the minimum of the effective potential in a sum of indipendent terms.

The result for QCD with $n = 3$ follows by considering the charged fields equal to zero in (9.3). Then, using the definition (7.6) for $\chi_a$ and $\pi_a$ with $a = u, d, s$ and $(\lambda_0/\sqrt{2},\ \lambda_3/\sqrt{2},\ \lambda_8/\sqrt{2})$ as basis of the Cartan subalgebra of $U(3)$, we get

$$V = \frac{3\mu^4}{4\pi^2} \sum_{a=u,d,s} V_1(\chi_a, \pi_a, \alpha_a)$$

$$\begin{aligned} V_1(\chi,\pi,\alpha) = {} & c\ A_1\ (\chi^2 + \pi^2) + \mu\ \alpha\ \chi(A_2 + \delta z_f) - \\ & -\frac{1}{2}\int_0^1 dy\ y\ \log\left(1 + \frac{2\alpha\chi + (\chi^2+\pi^2)}{y+\alpha^2}\right) - \\ & -\frac{1}{2}\int_0^1 \frac{du}{u^3}\ \log\left(1 + \frac{2\alpha\chi u^2 F(u)^{-1} + (\chi^2+\pi^2)u^3 F(u)^{2d-2}}{1+\alpha^2 u F(u)^{-2d}}\right) \end{aligned} \tag{9.15}$$

with $\alpha_a = m_a/\mu$ and $d = 4/9$.

We can now determine $\delta z_f$ by imposing the normalization condition (6.27). The result is the following

$$\begin{aligned} A_2 + \delta z_f = \frac{1}{\mu}\Bigg[ & 2c + \int_0^1 du\ \frac{u}{u + \langle\chi^0\rangle^2} + \\ & + \int_0^1 \frac{du}{u} \frac{F(u)^{-1}}{1 + \langle\chi^0\rangle^2 u^3 F(u)^{2d-2}}\Bigg] \end{aligned} \tag{9.16}$$

where $\langle\chi^0\rangle$ is the value of the field $\chi$ at the minimum of the potential in the massless case.

By substituting (9.16) in (9.15) we get the final form of the effective potential (with charged fields equal to zero) for QCD with three massive flavors

$$V_1(\chi,\pi,\alpha) = c\left[1+\frac{d^2}{2a}\int_0^1 du\,\frac{F(u)^{2d-2}}{1+2aF(u)}\right](\chi^2+\pi^2)+$$
$$+\alpha\chi\left[2c+\int_0^1 du\,\frac{u}{u+\langle\chi^0\rangle^2}+\right.$$
$$\left.+\int_0^1\frac{du}{u}\frac{F(u)^{-1}}{1+\langle\chi^0\rangle^2u^3F(u)^{2d-2}}\right]-$$
$$-\frac{1}{2}\int_0^1 dy\;y\;\log\left(1+\frac{2\alpha\chi+(\chi^2+\pi^2)}{y+\alpha^2}\right)-$$
$$-\frac{1}{2}\int_0^1\frac{du}{u^3}\log\left(1+\frac{2\alpha\chi u^2F(u)^{-1}+(\chi^2+\pi^2)u^3F(u)^{2d-2}}{1+\alpha^2uF(u)^{-2d}}\right) \tag{9.17}$$

where

$$F(u)=1-\frac{1}{2a}\log u \qquad a=\log\frac{\mu}{\Lambda_{QCD}}$$

Remember that the following relations hold

$$\begin{aligned}
\langle\chi_u\rangle=\langle s_{uu}\rangle &= \frac{1}{\sqrt{3}}\langle s_0\rangle+\frac{1}{\sqrt{2}}\langle s_3\rangle+\frac{1}{\sqrt{6}}\langle s_8\rangle=\frac{g^2(\mu)}{3\mu^3}\langle\bar{u}u\rangle_\mu\\
\langle\chi_d\rangle=\langle s_{dd}\rangle &= \frac{1}{\sqrt{3}}\langle s_0\rangle-\frac{1}{\sqrt{2}}\langle s_3\rangle+\frac{1}{\sqrt{6}}\langle s_8\rangle=\frac{g^2(\mu)}{3\mu^3}\langle\bar{d}d\rangle_\mu\\
\langle\chi_s\rangle=\langle s_{ss}\rangle &= \frac{1}{\sqrt{3}}\langle s_0\rangle-\frac{\sqrt{2}}{\sqrt{3}}\langle s_8\rangle=\frac{g^2(\mu)}{3\mu^3}\langle\bar{s}s\rangle_\mu
\end{aligned} \tag{9.18}$$

and analogously for $\langle\pi_a\rangle$ related to $\langle\overline{\Psi}_a i\gamma_5\Psi_a\rangle_\mu$.

So we are to determine the values of the quark condensates from the stationarity points of $V_1(\chi_a,\pi_a,\alpha_a)$ given in eq. (9.17).

Let us consider the simplified case of a symmetric mass matrix $\mathrm{m}_a^b = m\,\delta_a^b$ $(\alpha=m/\mu)$ and let us neglect the logarithmic corrections (rigid case) [19].

From the previous discussion it follows that the potential around the minimum defined in (9.11), has the following form ($n$ flavors)

$$\overline{V}=\frac{N\mu^4}{4\pi^2}\,n\,\overline{V}_1(s_0,p_0,\alpha)$$

$$\overline{V}_1(s_0,p_0,\alpha)=c(s_0^2+p_0^2)+\alpha s_0\log\frac{\Lambda^2}{\mu^2}-$$
$$-\frac{1}{2}\int_0^{\Lambda^2/\mu^2}dy\;y\;\log\left(1+\frac{2\alpha s_0f(y)+(s_0^2+p_0^2)f(y)^2}{y+\alpha^2}\right) \tag{9.19}$$

where

$$f(y) = \mu \left[\Theta(1-y) + \Theta(y-1)\frac{1}{y}\right]$$

and we have regularized the expected ultraviolet logarithmic divergence, linear in $s_0$, with the cutoff $\Lambda$.

The integral in (9.19) can be evaluated exactly but the expression one finds is not very useful because it depends on the roots of a cubic equation.

However, one can derive some general features of $\overline{V}_1$ simply by its integral representation: i) the function $\overline{V}_1$ has stationary points on the line $p_0 = 0$, ii) the hessian matrix of $\overline{V}_1$ is positive definite on the stationary points outside the line $p_0 = 0$, iii) in the asymptotic limit $s_0, p_0 \to \infty$, $\overline{V}_1$ goes like $c(s_0^2 + p_0^2)$.

The qualitative picture we get from these considerations is that two possibilities arise:

1) $\overline{V}_1$ has only one minimum on the line $p_0 = 0$

2) $\overline{V}_1$ has two minima at the points $(\langle s_0\rangle, \langle p_0\rangle)$ and $(\langle s_0\rangle, -\langle p_0\rangle)$.

These two possibilities correspond to two different phases of the theory. In particular, the second one, corresponds to a phase where P and CP are broken spontaneously by a non vanishing vacuum expectation value of a pseudoscalar fermion pair.

In order to better understand the effect of $\langle p_0\rangle \neq 0$, let us perform a chiral rotation on the fermionic fields

$$\Psi \to e^{i\gamma_5 \frac{\varphi}{2}}\,\Psi \tag{9.20}$$

corresponding to a rotation on the plane $(\langle s_0\rangle, \langle p_0\rangle)$:

$$\begin{aligned} \langle s_0\rangle' &= \langle s_0\rangle \cos\varphi + \langle p_0\rangle \sin\varphi \\ \langle p_0\rangle' &= -\langle s_0\rangle \sin\varphi + \langle p_0\rangle \cos\varphi \end{aligned} \tag{9.21}$$

By choosing

$$\tan\varphi = \frac{\langle p_0\rangle}{\langle s_0\rangle}$$

we rotate the minimum along the line $p_0 = 0$, but the effect on the mass term in the Lagrangian is

$$m\overline{\Psi}\Psi \to m \frac{\langle s_0 \rangle}{(\langle s_0 \rangle^2 + \langle p_0 \rangle^2)^{1/2}} \overline{\Psi}\Psi + m \frac{\langle p_0 \rangle}{(\langle s_0 \rangle^2 + \langle p_0 \rangle^2)^{1/2}} \overline{\Psi} i\gamma_5 \Psi \tag{9.22}$$

that is the Lagrangian acquires a P-violating contribution.

Phisically, the spontaneous violation of the parity we obtain, is due to the degeneracy $p_0 \to -p_0$ of the potential and to the fact that, under a parity transformation $p_0 \to -p_0$. When the theory develops a non zero VEV for $p_0$, we have to choose between two degenerate vacua and break spontaneously P and consequently CP.

The numerical analysis of function (9.19) completely confirms the previous considerations. In particular we have found that the CP-violating phase is present and the $(c, \alpha)$ phase diagram in Fig. 8 shows the boundary of this region.

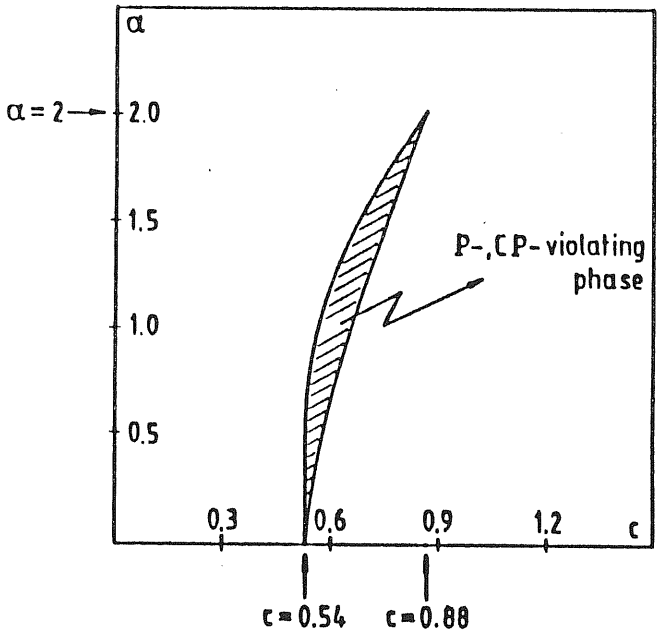
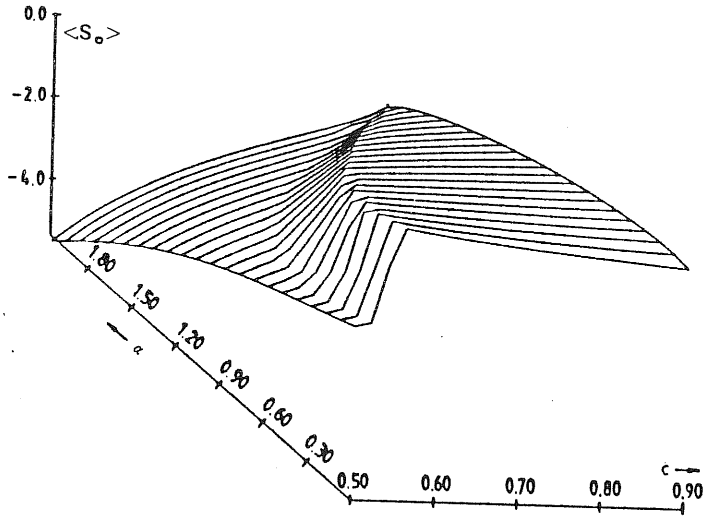



*FIG. 8. The phase diagram for SU(N) color gauge theory with n massive fermions. The vertical axis is $\alpha = m/\mu$, ratio of the fermion mass m and the momentum scale $\mu$. The horizontal axis is $c = 2\pi/(3C_2\alpha_s)$ where $\alpha_s = g^2/(4\pi)$ and g is the color gauge coupling. In the shadowed region P and CP are spontaneously violated by vacuum expectation values. These results are obtained with $\theta = 0$.*

In Fig. 9 we have reported the values of the minima $\langle s_0 \rangle$ and $\langle p_0 \rangle$ as functions of $c$ and $\alpha$.

*(a) VEV of the scalar* $s_0$




*(b) VEV of the pseudoscalar* $p_0$

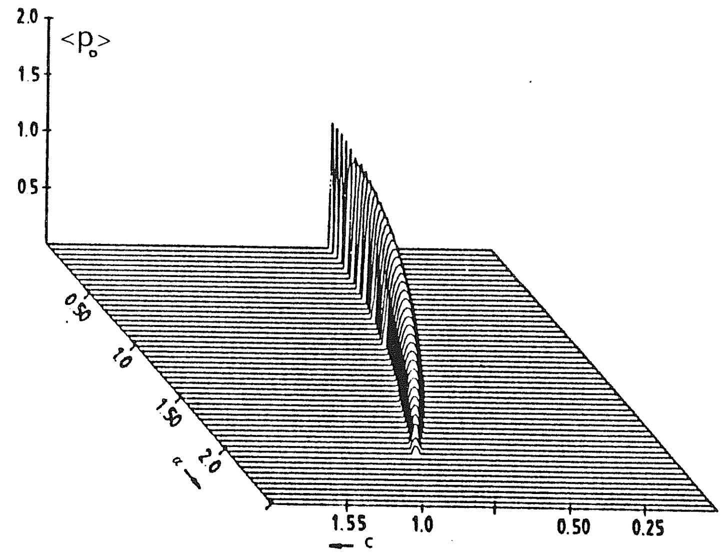



*FIG. 9. The vacuum expectation values of (a)* $s_0$ *and (b)* $p_0$ *as functions of* $\alpha$ *and c. The region where* $\langle p_0 \rangle \neq 0$ *[in (b)] is the P- and CP-violating phase. This region corresponds to the step rises of* $\langle s_0 \rangle$ *[see (a)] at constant* $\alpha$ *and increasing c. In (a), for convenience of visualization, the horizontal c-axis runs from right to left.*

Let us now expand eq. (9.19) for small $\alpha$. We get

$$\overline{V}_1(s_0, p_0, \alpha)_{\alpha \to 0} \to \overline{V}_1(s_0, p_0, 0) + \alpha s_0 \; N(s_0, p_0) \tag{9.23}$$

where

$$N(s_0, p_0) = 1 - (s_0^2 + p_0^2) \log \left( \frac{1 + s_0^2 + p_0^2}{s_0^2 + p_0^2} \right) + \frac{1}{3} \log(1 + s_0^2 + p_0^2) \tag{9.24}$$

A numerical study shows that the function $N(s_0, p_0)$ changes its sign for $(s_0^2 + p_0^2)^{1/2} = 1.110320$, therefore, the absolute minimum of $\overline{V}_1$ goes from negative to positive values of $s_0$.

Due to the continuity of $\overline{V}_1$, this may happen only if the minimum moves from one point to another "outside" the line $p_0 = 0$.

From the analysis of the minima at $m = 0$ one finds that the value $(s_0^2 + p_0^2)^{1/2} = 1.110320$ corresponds to $c = 0.54$ and this explains the starting point of the CP-violating region in Fig. 8.

One can ask where QCD-like theories lie on the phase diagram of Fig. 8.

To give an answer to this question, remember that in eq. (9.19) we have regularized the potential by a minimal subtraction procedure but we have not yet imposed the normalization condition of eq. (6.26).

As we will see in detail later on, this requirement will constrain the theory to satisfy the Adler-Dashen condition for small quark masses [20], [25].

In the case we are analyzing here, that is without the logarithmic corrections, (6.26) is a non trivial condition since it determines the value of the coupling constant $g$ (and so the value of $c$).

In fact, from the expansion for small masses (9.23) we get

$$N(s_0, p_0)\Big|_{extr} = N(\langle s_0(c) \rangle) = 2c \tag{9.25}$$

which becomes, in this case, only an equation in $c$ since, as we have noticed in sect. 7, the minima of the effective potential in the massless case depend only on $c$ (remember that $\overline{V}^{(0)}$ depends on $\mu$ only for an overall scale).

One can show that eq. (9.25) has one and only one solution which can be obtained numerically. The result is $c = 0.32$.

In Fig. 10 the curves $y = N(\langle s_0(c)\rangle)$ and $y = 2c$ are plotted as functions of $c$. Their intersection represent the solution of eq. (9.25).

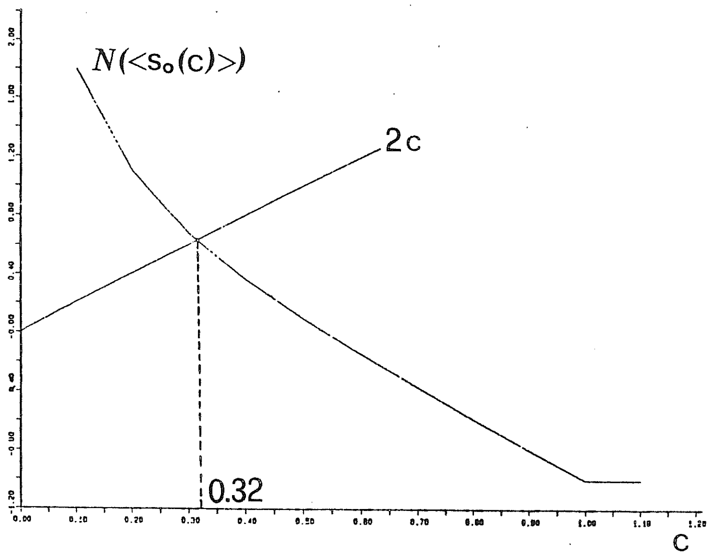



*FIG. 10. Graphical solution of eq. (9.25). It shows that, for QCD-like gauge theories analyzed in the rigid case, the requirement of the Adler-Dashen condition in the small masses limit, leads to* $c = 0.32$ *or equivalently* $\alpha_s = (1/C_2)\ 2.08\ \pi$.

In this way $g^2$ is fixed while $\mu$ is left undetermined.

This is nothing but the dimensional trasmutation (S. Coleman and E. Weinberg [11]) and the theory is now completely identified by the scale parameter $\mu$ (besides the fermion masses).

Let us observe that this result does not remain true when one takes the renormalization group corrections into account since the theory in this case depends explicitly on the infrared mass scale $\mu$.

The conclusion is then the following: no spontaneous parity violation appears in QCD-like gauge theories in the approximation in which one neglects the *logs*.

This can be seen from the phase diagram in Fig. 8 which shows that the line $c = 0.32$ never meets the P- and CP-violating region.

It thus appears that the normalization condition (6.26), or equivalently the imposition of the Adler-Dashen requirement, constraints the theory in a region of the phase diagram in $c$ and $\alpha$ where P and CP conservation are guaranteed.

This result is consistent with a general result found by C. Wafa and E.Witten [49].

We can now ask what will happen in the general case.

We expect that all the previous conclusions remain true also in the case with *logs* since, as previously proved, the logarithmic corrections don't change the picture.

Furthermore, we have a check on this assumption.

In fact, as we will see in the next sections, we have performed our calculations of the masses of the pseudo-Goldstone bosons for vanishing values of the pseudoscalar condensates at the minimum of the effective potential.

What we find is a positive value for the masses, as it must be, and, equivalently, a positive value for the second derivatives of the effective potential with respect to the pseudoscalar fields showing that the points we are considering are true minima of the effective potential itself.

The final conclusion is that we can put $\pi = 0$ in eq. (9.17) and simply minimize $V_1(\chi, 0, \alpha) \equiv V_1(\chi, \alpha)$ with respect to $\chi$.

In Fig. 11 we have plotted $V_1(\chi, \alpha)$ as given in eq. (9.17) as a function of $\chi$ in the massless case and for a value of the quark mass equal to 5.8 $MeV$. We see that in the massive case the degeneracy is removed and we have a minimum for a negative value of $\chi$.

In Fig. 12 we can see how the shape of the effective potential changes for an increasing value of the quark mass (118 $MeV$).

The values 5.8 and 118 $MeV$ correspond to the masses, renormalized at 1 $GeV$, of the quark $u$ and $s$ respectively, as we will obtain in sect. 12 from the fit of the octet meson masses.

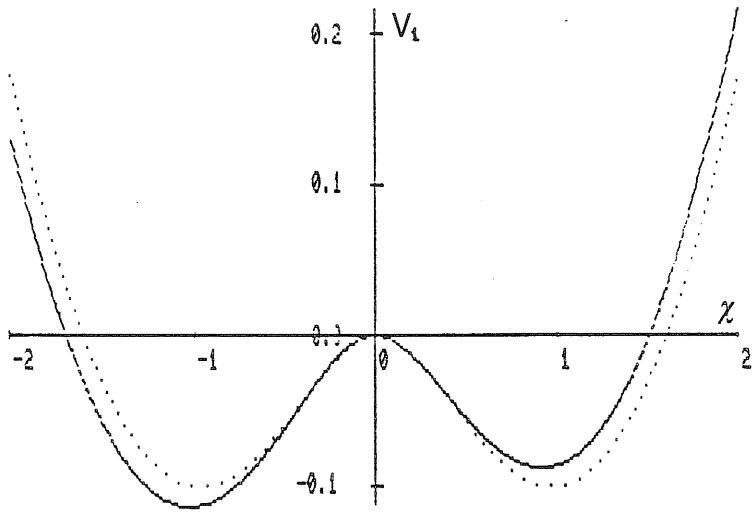



*FIG. 11. Effective potential $V_1$ for QCD with three flavors as a function of $\chi$ for $m = 0$ (dashed line) and for $m = 5.8$ MeV (solid line). The curves are for a value of $\mu/\Lambda_{QCD} = 1.11$.*

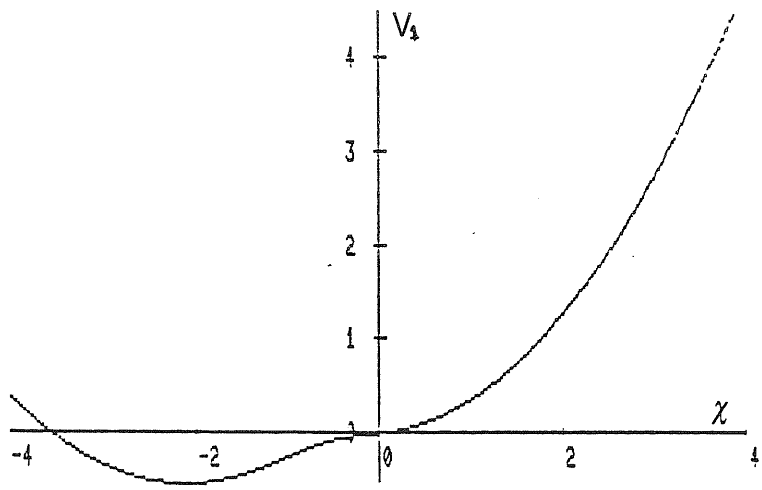



*FIG. 12. Effective potential $V_1$ for QCD with three flavors as a function of $\chi$ for $m = 118$ MeV and $\mu/\Lambda_{QCD} = 1.11$.*

The boundness from below of our effective potential is evident in these graphs.

## 10. PSEUDOSCALAR MESON MASSES

As we have seen in the previous sections, if the coupling constant in the infrared region, exceeds a critical value, then the dynamical breakdown of the chiral symmetry in the massless case occurs.

According to the Goldstone theorem, we expect $n^2 - 1$ massless Goldstone bosons relative to the breaking of $SU(n)_L \otimes SU(n)_R$ to $SU(n)_{L+R}$ (as stated before we are neglecting the $U(1)_A$ problem).

These particles are represented by the composite fields $\mathrm{p}_{ab}$ $\quad a, b = 1, .., n$ whose vacuum expectation values are related to the pseudoscalar condensates $\langle \overline{\Psi}_a i\gamma_5 \Psi_b \rangle_\mu$ by eq. (5.17).

When a mass matrix $\mathrm{m}$ for the quarks is allowed, the Goldstone bosons acquire mass and, in the case of QCD with three flavors, they are represented by the octet of the pseudoscalar mesons.

What we are going to do, is to calculate the masses of the pseudoscalar mesons of the octect in the framework of our model.

In order to better visualize the main points of our program, let us concentrate, as an illustration, on the case of a single flavor in which the potential $V$ is a function of the scalar field $\chi$, the pseudoscalar field $\pi$ and the quark mass $m$.

First of all, one has to properly normalize the field $\pi$ with respect to the canonical pseudo-Goldstone field $\phi_\pi$

$$\phi_\pi = b_\pi \ \pi \tag{10.1}$$

The constant $b_\pi$ can be obtained in terms of the pion decay constant $f_\pi$. Performing a chiral rotation in (10.1) and taking the vacuum expectation values we get

$$b_\pi = -\frac{1}{\sqrt{2}} \frac{f_\pi}{\langle \chi \rangle} \tag{10.2}$$

where, as usual, $\langle \chi \rangle$ stands for the value of $\chi$ at the minimum of $V$.

To compute the mass of the pseudoscalar meson (the pseudo-Goldstone boson described by the field $\phi_\pi$), one has to take the second derivative of the effective potential with respect to the field $\pi$, evaluate it at the minimum and opportunely normalize it

$$M_\pi^2 = \left.\frac{d^2V}{d\phi_\pi^2}\right|_{extr} = \left.\frac{1}{b_\pi^2}\frac{d^2V}{d\pi^2}\right|_{extr} = 2\left.\frac{\langle\chi\rangle^2}{f_\pi^2}\frac{d^2V}{d\pi^2}\right|_{extr} \tag{10.3}$$

Since $\langle\pi\rangle = 0$, the following relation holds

$$\left.\frac{d^2V}{d\pi^2}\right|_{extr} = -\left.\frac{1}{\langle\chi\rangle}\frac{\partial V}{\partial\chi}\right|_{extr} \tag{10.4}$$

where the derivative of $V$ on the right hand side means the derivative with respect to the explicit dependence on $\chi$ ($\partial V/\partial\chi$ is proportional to the explicit chiral symmetry breaking term).

Eq. (10.4) represents the Goldstone theorem. Substituting it in eq. (10.3) we get

$$M_\pi^2 = -\frac{2}{f_\pi^2}\langle\chi\rangle\left.\frac{\partial V}{\partial\chi}\right|_{extr} = -\frac{2}{f_\pi^2}\langle\overline{\psi}\psi\rangle_\mu\left.\frac{g^2(\mu)}{3\mu^3}\frac{\partial V}{\partial\chi}\right|_{extr} \tag{10.5}$$

where we have used the relation between the scalar field at the minimum and the scalr condensate.

Taking into account the normalization condition (6.26), eq. (10.5) reproduces, as expected, the Adler-Dashen formula in the small mass limit

$$\left.M_\pi^2\right|_{m\to 0} \sim -2m\,\frac{1}{f_\pi^2}\langle\overline{\psi}\psi\rangle_\mu \tag{10.6}$$

Let us now apply this procedure to the general case of QCD with three flavors in order to obtain an expression for the masses of the pseudoscalar octet mesons [24]. Remember that the effective potential we have calculated depends on the standard parameters of QCD: $\Lambda_{QCD}$, $m_u$, $m_d$, $m_s$ and the further mass scale $\mu$.

So we will proceed in the following way.

We will derive a convenient generalization of eq. (10.5) which will allow us to compute directly the pseudo-Goldstone masses in terms of $\Lambda_{QCD}$, $m_u$, $m_d$, $m_s$ and $\mu$ and of the decay coupling constants $f_{ij}\quad i,j = 1,..,8$.

Then we will derive within our formalism an expression for $f_{ij}$ and, finally, we will determine the parameters of our theory from the experimental data. Having a system of coupled equations, this determination will be performed numerically.

First of all we need to normalize our dynamical variables $p_i \quad i = 1,..,8$ (remember that we are neglecting the $U(1)_A$ problem) defined in (9.5).

Let us call $\overline{p}_i$ the fields which diagonalize the matrix of the second derivatives of the effective potential given in eq. (9.3), then

$$\overline{p}_i = \sum_{j=1}^{8} p_j \; \mathrm{a}_{ji} \tag{10.7}$$

where a is an orthogonal $n \times n$ matrix which is non diagonal only in the 3-8 sector.

The coefficients $b_i$ which relate the fields $\overline{p}_i$ to the physical fields $\phi_i$

$$\phi_i = b_i \; \overline{p}_i = b_i \sum_{j=1}^{8} p_j \; \mathrm{a}_{ji} \tag{10.8}$$

can be determined through standard current algebra arguments.

In fact, let us remind the standard relations in pion physics generalized to the case of three quarks flavors

$$i f_{ij} = \langle 0 | \left[ Q_5^i, \phi_j(0) \right] | 0 \rangle \qquad i,j = 1,..,8 \tag{10.9}$$

where $\phi_j$ are the canonical pseudo-Goldstone boson fields related to the $p_j$ by eq. (10.8),

$$Q_5^i = \int d^3\vec{x} \; \Psi^\dagger(x) \gamma_5 \frac{\lambda_i}{2} \Psi(x) \tag{10.10}$$

are the axial charges and $f_{ij}$ are the decay coupling constants for the meson octet and they are defined by

$$\langle 0 | J^i_{\mu 5}(x) | \pi_j(p) \rangle = i \; p_\mu \; f_{ij} e^{-ipx} \quad i,j = 1,..,8 \tag{10.11}$$

Here $J^i_{\mu 5}$ are the axial vector currents and the $|\pi_j(p)\rangle$ are the meson octet states with four-momentum $p_\mu$ which satisfy (see (1.6))

$$\langle 0 | \phi_i(x) | \pi_j(p) \rangle = \delta_{ij} \; e^{-ipx} \tag{10.12}$$

Combining eq. (9.5) with (5.17) we find that the fields $p_i$ are related to the pseudoscalar fermion condensates by the relation

$$p_i = F \left( \overline{\Psi} i \gamma_5 \frac{\lambda_i}{\sqrt{2}} \Psi \right) \tag{10.13}$$

where

$$F = \frac{g^2(\mu)}{3\mu^3} \tag{10.14}$$

is the usual dimensional normalization factor.

Substituting (10.13) in (10.8) and then in (10.9) we find

$$f_{ij} = -\frac{\sqrt{2}}{3}\ \mathrm{a}_{ij}\ b_j\ F\ \mathrm{tr}\langle\overline{\Psi}\Psi\rangle_\mu - \sum_{k,l=1}^{8} \mathrm{a}_{kj}\ b_j\ d_{ikl}\ F\ \mathrm{tr}\langle\overline{\Psi}\frac{\lambda_l}{\sqrt{2}}\Psi\rangle_\mu \tag{10.15}$$

So, for the sector with charge or strangeness different from zero ($\mathrm{a}_{ij} = \delta_{ij}$), one gets (remember $\langle\chi_a\rangle = F\ \langle\overline{\Psi}_a\Psi_a\rangle_\mu\ \ a = u, d, s$)

$$\begin{aligned} b_1 = b_2 &= -\frac{\sqrt{2} f_{\pi^\pm}}{\langle\chi_u\rangle + \langle\chi_d\rangle} = b_{\pi^\pm} \\ b_4 = b_5 &= -\frac{\sqrt{2} f_{K^\pm}}{\langle\chi_u\rangle + \langle\chi_s\rangle} = b_{K^\pm} \\ b_6 = b_7 &= -\frac{\sqrt{2} f_{K^0,\overline{K^0}}}{\langle\chi_d\rangle + \langle\chi_s\rangle} = b_{K^0,\overline{K^0}} \end{aligned} \tag{10.16}$$

where, according to the usual conventions, we have set

$$\begin{aligned} f_{11} &= f_{22} = f_{\pi^\pm} \\ f_{44} &= f_{55} = f_{K^\pm} \\ f_{66} &= f_{77} = f_{K^0,\overline{K^0}} \end{aligned} \tag{10.17}$$

For the sector with $Q = S = 0$, one obtains

$$f_{\mu\nu} = \sum_{\rho=3,8} \mathcal{A}_{\mu\rho}\ \mathrm{a}_{\rho\nu}\ b_\nu; \qquad \mu,\ \nu = 3, 8 \tag{10.18}$$

with

$$\begin{aligned} \mathcal{A}_{33} &= -\frac{1}{\sqrt{2}}\ (\langle\chi_u\rangle + \langle\chi_d\rangle) \\ \mathcal{A}_{38} = \mathcal{A}_{83} &= -\frac{1}{\sqrt{6}}\ (\langle\chi_u\rangle - \langle\chi_d\rangle) \\ \mathcal{A}_{88} &= -\frac{1}{3\sqrt{2}}\ (\langle\chi_u\rangle + \langle\chi_d\rangle + 4\langle\chi_s\rangle) \end{aligned} \tag{10.19}$$

Summarizing, the coefficients $b_i$ can be expressed in terms of the decay constants of the pseudoscalar mesons and of the values of the fields $\chi_a$ ($a =$

$u, d, s$), at the minimum of the effective potential, but, due to the mixing in the 3-8 sector, one has also to take into account the matrix a which transforms the fields $p_i$ in the mass eigenstates (notice that in the $SU(2)$-symmetric case $(m_u = m_d)$, $\mathrm{a}_{ij} = \delta_{ij}$).

Let us now derive an expression for the masses of the pseudoscalar octet mesons by multiplying the second derivatives of the effective potential with respect to the pseudoscalar fields by the appropriate factors relating the physical fields $\phi_i$ to our variables $p_i$ (clearly we do not compute the $\eta'$ mass since we have not considered the effects of the $U(1)_A$ anomaly).

The mass matrix is given by

$$\begin{aligned} M_{ij}^2 = M_i^2 \delta_{ij} &= \left. \frac{d^2 V}{d\phi_i d\phi_j} \right|_{extr} = \\ &= \frac{1}{b_i b_j} \sum_{k,l=1}^{8} \mathrm{a}_{ki} \mathrm{a}_{lj} \left. \frac{d^2 V}{dp_k dp_l} \right|_{extr} \equiv \frac{1}{b_i b_j} \sum_{k,l=1}^{8} \mathrm{a}_{ki} \mathrm{a}_{lj} V_{kl} \\ & i, j = 1, ..., 8 \end{aligned} \tag{10.20}$$

A direct calculation of the second derivatives evaluated at the extremum of the effective potential, leads to the following result

$$V_{11} = V_{22} = \frac{3\mu^4}{4\pi^2} \left[ 2cA_1 - \frac{1}{\mu^2} \int_0^\infty dy\, y \left( \frac{x_u}{x_u + x_d} \frac{1}{y + x_u^2} + \right. \right.$$
$$\left. \left. + \frac{x_d}{x_u + x_d} \frac{1}{y + x_d^2} \right) f_2(y)^2 \right]$$

$$V_{33} = \frac{3\mu^4}{4\pi^2} \left[ 2cA_1 - \frac{1}{2\mu^2} \int_0^\infty dy\, y \left( \frac{1}{y + x_u^2} + \frac{1}{y + x_d^2} \right) f_2(y)^2 \right]$$

$$V_{44} = V_{55} = \frac{3\mu^4}{4\pi^2} \left[ 2cA_1 - \frac{1}{\mu^2} \int_0^\infty dy\, y \left( \frac{x_u}{x_u + x_s} \frac{1}{y + x_u^2} + \right. \right.$$
$$\left. \left. + \frac{x_s}{x_u + x_s} \frac{1}{y + x_s^2} \right) f_2(y)^2 \right]$$

$$V_{66} = V_{77} = \frac{3\mu^4}{4\pi^2} \left[ 2cA_1 - \frac{1}{\mu^2} \int_0^\infty dy\, y \left( \frac{x_d}{x_d + x_s} \frac{1}{y + x_d^2} + \right. \right.$$
$$\left. \left. + \frac{x_s}{x_d + x_s} \frac{1}{y + x_s^2} \right) f_2(y)^2 \right]$$

$$V_{88}=\frac{3\mu^4}{4\pi^2}\Bigg[2cA_1-\frac{1}{\mu^2}\int_0^\infty dy\ y\left(\frac{1}{6}\frac{1}{y+x_u^2}+\right.$$

$$\left.+\frac{1}{6}\frac{1}{y+x_d^2}+\frac{2}{3}\frac{1}{y+x_s^2}\right)f_2(y)^2\Bigg]$$

$$V_{38}=V_{83}=\frac{3\mu^4}{4\pi^2}\left[\frac{1}{2\sqrt{3}}\frac{1}{\mu^2}\int_0^\infty dy\ y\left(\frac{1}{y+x_d^2}-\frac{1}{y+x_u^2}\right)f_2(y)^2\right] \tag{10.21}$$

where $c$, $A_1$, $f_2(y)$ are defined in eq. (6.36) and $x_a$ $(a=u,d,s)$ are the eigenvalues, evaluated at the extremum of $V$, of the matrix x defined in (6.36).

We want now to derive a more physically transparent expression for the $V_{kl}$.

As stated before, when one sets the charged fields equal to zero, the effective potential factorizes in the sum of three contributions, one for each flavor, as showed in eq. (9.15).

For the further developments, it is convenient to rewrite eq. (9.15) in the following form

$$V=\frac{3\mu^4}{4\pi^2}\sum_{a=u,d,s}V_1(\phi_a^2,\chi_a,m_a)$$

$$V_1(\phi^2,\chi,m)=c\ A_1\ \phi^2+m\chi\ (A_2+\delta z_f)-$$

$$-\frac{1}{2}\int_0^\infty dy\ y\ \log\left(y+\frac{m^2}{\mu^2}f_1^2(y)+\phi^2\frac{f_2^2(y)}{\mu^2}+2\chi\frac{m}{\mu}\frac{f_1(y)}{\mu}f_2(y)\right) \tag{10.22}$$

where $\phi^2=\chi^2+\pi^2$, $A_2+\delta z_f$ is given in (9.16) and $c$, $A_1$, $f_1(y)$, $f_2(y)$ are given in (6.36).

The extremum condition is

$$\frac{dV_1}{d\chi}=2\ \chi\ \frac{\partial V_1}{\partial\phi^2}+\frac{\partial V_1}{\partial\chi}=0 \tag{10.23}$$

that is

$$\left.\frac{\partial V_1}{\partial\phi^2}\right|_{extr}=-\frac{1}{2\langle\chi\rangle}\left.\frac{\partial V_1}{\partial\chi}\right|_{extr}=c\ A_1-$$

$$-\frac{1}{2\mu^2}\int_0^\infty dy\ y\ \frac{f_2^2(y)}{y+(mf_1(y)/\mu+\langle\chi\rangle f_2(y)/\mu)^2} \tag{10.24}$$

So, for the various flavors $a = u, d, s$ one gets

$$-\frac{1}{\langle\chi_a\rangle}\frac{\partial V}{\partial \chi_a}\bigg|_{extr} = \frac{3\mu^4}{4\pi^2}\left[2\ c\ A_1 - \frac{1}{\mu^2}\int_0^\infty dy\ y\ \frac{f_2^2(y)}{y + x_a^2}\right] \tag{10.25}$$

Let us substitute eq. (10.25) in (10.21).

For the 3-8 sector we get simply

$$\begin{aligned}
V_{33} &= -\frac{1}{2\langle\chi_u\rangle}\frac{\partial V}{\partial \chi_u}\bigg|_{extr} - \frac{1}{2\langle\chi_d\rangle}\frac{\partial V}{\partial \chi_d}\bigg|_{extr} \\
V_{88} &= -\frac{1}{6\langle\chi_u\rangle}\frac{\partial V}{\partial \chi_u}\bigg|_{extr} - \frac{1}{6\langle\chi_d\rangle}\frac{\partial V}{\partial \chi_d}\bigg|_{extr} - \frac{2}{3\langle\chi_s\rangle}\frac{\partial V}{\partial \chi_s}\bigg|_{extr} \\
V_{38} = V_{83} &= \frac{1}{2\sqrt{3}}\frac{1}{\langle\chi_d\rangle}\frac{\partial V}{\partial \chi_d}\bigg|_{extr} - \frac{1}{2\sqrt{3}}\frac{1}{\langle\chi_u\rangle}\frac{\partial V}{\partial \chi_u}\bigg|_{extr}
\end{aligned} \tag{10.26}$$

Let us now examine $V_{11}$ as given in eq (10.21). We can rewrite it in the following form

$$\begin{aligned}
V_{11} = \frac{3\mu^4}{4\pi^2}\Bigg[2cA_1 - \frac{1}{\mu^2}\int_0^\infty dy\ y\ &\left(\frac{1}{y + x_u^2} - \frac{x_d}{x_u + x_d}\frac{1}{y + x_u^2} +\right. \\
&\left. + \frac{x_d}{x_u + x_d}\frac{1}{y + x_d^2}\right) f_2^2(y)\Bigg] = \\
= -\frac{1}{\langle\chi_u\rangle}\frac{\partial V}{\partial \chi_u}\bigg|_{extr} &- \frac{3\mu^2}{4\pi^2}\int_0^\infty dy\ y\ \frac{x_d(x_u - x_d)}{(y + x_u^2)(y + x_d^2)}\ f_2^2(y)
\end{aligned} \tag{10.27}$$

or, analogously, we can obtain

$$V_{11} = -\frac{1}{\langle\chi_d\rangle}\frac{\partial V}{\partial \chi_d}\bigg|_{extr} - \frac{3\mu^2}{4\pi^2}\int_0^\infty dy\ y\ \frac{x_u(x_d - x_u)}{(y + x_u^2)(y + x_d^2)}\ f_2^2(y) \tag{10.28}$$

So we can write an expression $ud$ symmetric for $V_{11}$ by summing eqs. (10.27) and (10.28) and dividing by 2.

The same arguments apply to $V_{44}$ and $V_{66}$ and the result is

$$\begin{aligned}
V_{11} = V_{22} &= -\frac{1}{2\langle\chi_u\rangle}\frac{\partial V}{\partial \chi_u}\bigg|_{extr} - \frac{1}{2\langle\chi_d\rangle}\frac{\partial V}{\partial \chi_d}\bigg|_{extr} + V_{ud} \\
V_{44} = V_{55} &= -\frac{1}{2\langle\chi_u\rangle}\frac{\partial V}{\partial \chi_u}\bigg|_{extr} - \frac{1}{2\langle\chi_s\rangle}\frac{\partial V}{\partial \chi_s}\bigg|_{extr} + V_{us} \\
V_{66} = V_{77} &= -\frac{1}{2\langle\chi_d\rangle}\frac{\partial V}{\partial \chi_d}\bigg|_{extr} - \frac{1}{2\langle\chi_s\rangle}\frac{\partial V}{\partial \chi_s}\bigg|_{extr} + V_{ds}
\end{aligned} \tag{10.29}$$

We recall that here the partial derivatives are respect to the explicit dependence on $\chi_a$ and

$$V_{ab} = \frac{3\mu^2}{8\pi^2} \int_0^\infty dy\; y\; \frac{(x_a - x_b)^2}{(y + x_a^2)(y + x_b^2)} f_2^2(y) \tag{10.30}$$

$$a = u, d, s,$$

Inserting (10.29) in (10.20) we obtain a generalization of the Goldstone theorem as expressed in (10.4) to the case of three flavors.

In fact in eq. (10.29) the dependence on the explicit symmetry breaking part of the effective potential, has been isolated. This means that, in the $\mathbf{m} \to 0$ limit, the $V_{kl}$ are all naively equal to zero independently on the values of the fields $\chi_a$ at the minimum.

This is not clear if one uses directly eq. (10.21) for evaluating $V_{kl}$ with obvious problems from a computational point of view.

As an example, let us consider the $SU(2)$-symmetric case given by $m_u = m_d = m$.

Clearly in this case $\langle\chi_u\rangle = \langle\chi_d\rangle = \langle\chi\rangle$ and $\mathbf{a}_{ij} = \delta_{ij}$.

Then, using the first of eq. (10.16), we obtain for the pion

$$M_\pi^2 = -\frac{1}{b_\pi^2} \frac{1}{\langle\chi\rangle} \frac{\partial V}{\partial \chi}\bigg|_{extr} = -\frac{2}{f_\pi^2} \langle\chi\rangle \frac{\partial V}{\partial \chi}\bigg|_{extr} \tag{10.31}$$

and, with the normalization condition (6.26), we get for $m \to 0$

$$M_\pi^2\bigg|_{m \to 0} \sim -2m \frac{1}{f_\pi^2} \langle\overline{\psi}\psi\rangle_\mu \tag{10.32}$$

where $\langle\overline{\psi}\psi\rangle_\mu = \langle\overline{u}u\rangle_\mu = \langle\overline{d}d\rangle_\mu$.

Eq. (10.32) is the standard Adler-Dashen relation.

As already observed, there is a mixing between the components along 3 and 8 directions (see eq. (10.26)).

This is of course expected and we have to diagonalize the matrix of the second derivatives of the effective potential in order to get the masses of the physical $\pi^0$ and $\eta_8$. (Notice, however, that the $\eta_8$ is not the true physical particle but the result of undoing the mixing between the members of the octect and of the singlet of $SU(3)$ giving rise to the physical $\eta$ and $\eta'$).

The explicit calculation, performed up to the order

$$\left(\frac{m_u - m_d}{m_s}\right)^2$$

gives for the 3-8 sector

$$\begin{pmatrix} a_{33} & a_{38} \\ a_{83} & a_{88} \end{pmatrix} = \frac{1}{N} \begin{pmatrix} 1 & -\dfrac{V_{38}}{V_{33} - V_{88}} \\ \dfrac{V_{38}}{V_{33} - V_{88}} & 1 \end{pmatrix} \tag{10.33}$$

with

$$N = \sqrt{1 + \frac{V_{38}^2}{(V_{33} - V_{88})^2}} \; .$$

## 11. PSEUDOSCALAR MESON DECAY CONSTANTS

In our context there are at least two possibilities to evaluate the decay constants of the pseudoscalar mesons: i) to evaluate the couplings of the fields $\phi_i$ to the axial currents $J^j_{5\mu}$, ii) to evaluate the residue at the pole of the meson propagator.

In the first case we need to know the meson-quark-antiquark vertex, while in the second case the knowledge of the bound state wave function of the pseudoscalar fields is required.

These quantities can be determined directly from the effective action if we are able to extend our previous calculations from constant fields s and p to arbitrary functions of the space-time.

In fact from the effective potential one can extract amplitudes for composite operators of vanishing four-momentum but, to derive amplitudes of non zero momentum, one has to allow for space-time dependence in the composite fields.

Our effective action $\Gamma$ (see (2.65)) consists of two terms: the $\Gamma_2$ term, which in our approximation is a quadratic expression in the composite fields (so the relation $\mathrm{Tr}(\mathrm{S}\ \delta\Gamma_2/\delta\mathrm{S}) - \Gamma_2 = \Gamma_2$, holds), and the logarithmic term which can be interpreted as the sum of all the graphs with a fermionic loop.

Therefore, in order to determine the meson-quark-antiquark vertex, we only need to generalize the logarithmic term of $\Gamma$ to the case of local fields.

Let us recall the structure of the self-energy for constant fields s and p (eq. (5.12))

$$-\frac{\delta\Gamma_2}{\delta\mathrm{S}(p)} = \Sigma(p^2) = \mathrm{m}(\mu) f_1(p^2) - \mathrm{m}_0(\Lambda) + (\mathrm{s} + i\gamma_5\mathrm{p}) f_2(p^2) \qquad (11.1)$$

with $f_1(p^2)$ and $f_2(p^2)$ given in (5.13) and (5.14) respectively.

For sake of simplicity, from now on we will omit the square in the arguments of $\Sigma$, $f_1$ and $f_2$.

The operator we need to generalize is

$$\mathrm{S}_0^{-1} + \frac{\delta\Gamma_2}{\delta\mathrm{S}} \qquad \text{with} \qquad \mathrm{S}_0^{-1}(p) = i\hat{p} - \mathrm{m}_0(\Lambda)$$

in the Landau gauge.

We will use a Weyl symmetrization prescription [25],[26] that is

$$\begin{aligned} \langle x|\mathrm{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta \mathrm{S}}|y\rangle = \langle x|i\hat{p}-\mathrm{m}(\mu)f_1(p)- \\ -\frac{1}{2}\Big[\mathrm{s(x)}+i\gamma_5\mathrm{p(x)},f_2(\mathrm{p})\Big]_+|y\rangle \end{aligned} \tag{11.2}$$

where x and p have the canonical definition

$$\begin{aligned} \mathrm{x}_\mu|x\rangle &= x_\mu|x\rangle \\ \mathrm{p}_\mu|p\rangle &= p_\mu|p\rangle \\ \big[\mathrm{x}_\mu,\mathrm{p}_\nu\big] &= -i\ g_{\mu\nu} \end{aligned} \tag{11.3}$$

Notice that this prescription maintains the charge-conjugation property of the fermion propagator

$$C\left[\left(\mathrm{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta\mathrm{S}}\right)(x,y)\right]C^{-1} = \left(\mathrm{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta\mathrm{S}}\right)^{\mathrm{T}}(y,x) \tag{11.4}$$

We shall also see that the Weyl symmetrization prescription leads to a meson-quark-antiquark vertex which is consistent with the axial Ward-identity.

In order to evaluate Tr log $(\mathrm{S}_0^{-1}+\delta\Gamma_2/\delta\mathrm{S})$, we traslate the composite fields s and p with respect to their value at the minimum of the effective potential.

Let us define

$$\langle \mathrm{s}\rangle = \begin{pmatrix} \langle\chi_u\rangle & 0 & 0 \\ 0 & \langle\chi_d\rangle & 0 \\ 0 & 0 & \langle\chi_s\rangle \end{pmatrix} \tag{11.5}$$

and introduce the operators

$$\begin{aligned} \overline{\mathrm{S}}^{-1}(p) &= i\hat{p}-\mathrm{m}(\mu)f_1(p)-\langle \mathrm{s}\rangle f_2(p) \\ \Phi(x) &= \mathrm{s}(x)+i\gamma_5\mathrm{p}(x)-\langle\mathrm{s}\rangle \\ \mathbf{R} &= \frac{1}{2}\Big[\Phi(\mathrm{x}),f_2(\mathrm{p})\Big]_+ \end{aligned} \tag{11.6}$$

where $\overline{\mathrm{S}}$ can be interpreted as the quark propagator with mass $\langle\mathrm{s}\rangle f_2(p)$ dynamically generated.

Substituting in the logarithmic term of $\Gamma$ we obtain

$$\begin{aligned}\mathrm{Tr}\ \log\left(\mathrm{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta\mathrm{S}}\right) &= \mathrm{Tr}\ \log\ \overline{\mathrm{S}}^{-1}+\mathrm{Tr}\ \log\ (1-\overline{\mathrm{S}}\mathrm{R}) = \\ &= \mathrm{Tr}\ \log\ \overline{\mathrm{S}}^{-1}-\sum_{n=1}^{\infty}\frac{1}{n}\ \mathrm{Tr}\ (\overline{\mathrm{S}}\mathrm{R})^n\end{aligned} \tag{11.7}$$

The last trace can be calculated inserting intermediate eigenstates of $\mathrm{x}_\mu$

$$\begin{aligned}\mathrm{Tr}\ (\overline{\mathrm{S}}\mathrm{R})^n = \int\prod_{i=1}^{2n}d^4x_i\ \mathrm{tr}\ \big[&\langle x_1|\overline{\mathrm{S}}|x_2\rangle\langle x_2|\mathrm{R}|x_3\rangle\cdot \\ &\cdots\langle x_{2n-1}|\overline{\mathrm{S}}|x_{2n}\rangle\langle x_{2n}|\mathrm{R}|x_1\rangle\big]\end{aligned} \tag{11.8}$$

From the definition of the operator $\mathbf{R}$, introducing

$$f_2(x-y)\equiv\langle x|f_2(\mathrm{p})|y\rangle \tag{11.9}$$

we get

$$\begin{aligned}\langle x_{2i}|\mathrm{R}|x_{2i+1}\rangle &= \frac{1}{2}\big(\Phi(x_{2i})f_2(x_{2i}-x_{2i+1})+f_2(x_{2i}-x_{2i+1})\Phi(x_{2i+1})\big) = \\ &= \frac{1}{2}\int d^4z_i\ \left[\delta^4(z_i-x_{2i})+\delta^4(z_i-x_{2i+1})\right]\ f_2(x_{2i}-x_{2i+1})\ \Phi(z_i) = \\ &= \int d^4z_i\ V(x_{2i},x_{2i+1};z_i)\ \Phi(z_i)\end{aligned} \tag{11.10}$$

where we have defined

$$V(x,y;z)\equiv\frac{1}{2}\left[\delta^4(z-x)+\delta^4(z-y)\right]\ f_2(x-y) \tag{11.11}$$

So, using

$$\overline{\mathrm{S}}(x-y)\equiv\langle x|\overline{\mathrm{S}}(\mathrm{p})|y\rangle \tag{11.12}$$

we finally obtain

$$\begin{aligned}\mathrm{Tr}\ (\overline{\mathrm{S}}\mathrm{R})^n = \int\prod_{i=1}^{2n}d^4x_i\prod_{j=1}^{n}d^4z_j\ &\overline{\mathrm{S}}(x_1-x_2)\ V(x_2,x_3;z_1)\ \Phi(z_1)\cdot \\ &\cdot\overline{\mathrm{S}}(x_3-x_4)\ V(x_4,x_5;z_2)\ \Phi(z_2)\cdot \\ &\cdots\ \overline{\mathrm{S}}(x_{2n-1}-x_{2n})\ V(x_{2n},x_1;z_n)\ \Phi(z_n)\end{aligned} \tag{11.13}$$

From eq. (11.13) it is clear that the operator Tr $(\overline{\mathbf{S}}\mathbf{R})^n$ corresponds to $n$ bound state fields $\mathbf{\Phi}$ emerging from a fermionic loop calculated with fermion propagator $\overline{\mathbf{S}}$ as illustrated in Fig. 13.

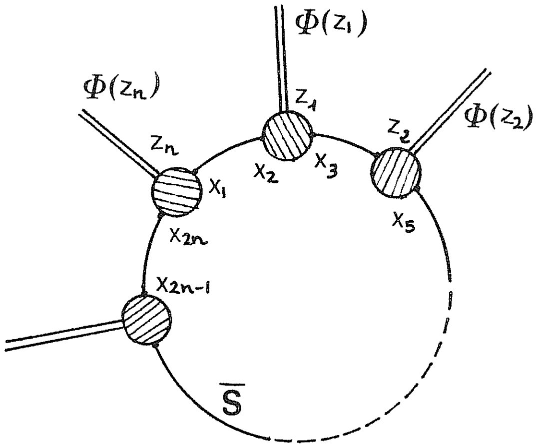



*FIG. 13. Graphical representation of the operator* Tr $(\overline{\mathbf{S}}\mathbf{R})^n$ *given in eq. (11.13).*

The vertex $V(x,y;z)$ has the following expression in the momentum space

$$V(x,y;z) = \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\, V(p,q)\; e^{-ipx + iqy + i(p-q)z}$$
$$V(p,q) = \frac{1}{2}\left[f_2(p) + f_2(q)\right] \tag{11.14}$$

where the momenta are as in Fig. 14.

Actually we are interested in the pseudoscalar bound state vertices. We can read their expressions directly from eqs. (11.13) and (11.14)

$$\mathbf{V}_5^j(p,q) = i\gamma_5 \frac{\lambda^j}{2\sqrt{2}}\left[f_2(p) + f_2(q)\right] \tag{11.15}$$

This is the effective coupling $g_{p\bar{q}q}$.

To get the effective coupling $g_{\phi\bar{q}q}$ for physical mesons, we must properly normalize the fields $p_i$ by using eq. (10.8).

The result is

$$
\begin{aligned}
G_5^i(p,q) &= \frac{1}{b_i}\sum_{j=1}^{8} V_5^j(p,q)\ a_{ji} = \\
&= i\gamma_5 \frac{1}{b_i}\sum_{j=1}^{8} \frac{\lambda^j}{2\sqrt{2}}\ a_{ji}\ \left[f_2(p) + f_2(q)\right]
\end{aligned}
\tag{11.16}
$$

The $G_5^i$ are the physical pseudoscalar meson-quark-antiquark vertices.

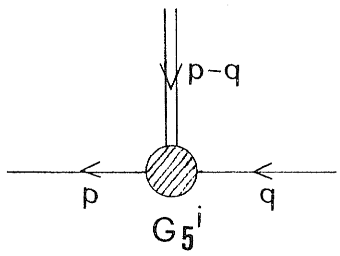



*FIG. 14. The pseudoscalar bound-state vertices with two fermion lines of momenta p and q whereas $p-q$ is a bound state line.*

We will now prove that the expression (11.16) is exactly what one needs in order to satisfy the axial vector Ward identities relating the proper axial-vector vertex functions $\Gamma^i_{5\mu}$ and the fermion propagator S [50].

Let us consider the case of massless quarks. Then the axial Ward identities have the following form (see eq. (4.1))

$$
iq^\mu \Gamma^i_{5\mu}(p+q,p) = \gamma_5 \frac{\lambda^i}{2} S^{-1}(p) + S^{-1}(p+q)\gamma_5 \frac{\lambda^i}{2} \tag{11.17}
$$

(remember that in the massless case the fermion propagator is proportional to the unit matrix in the flavor space since the condensates in the self- energy have all the same value for each flavor).

Let us substitute the expression for the fermion propagator as given by the Schwinger-Dyson equation in the massless case

$$
S^{-1}(p) = i\hat{p} - \overline{\Sigma}(p) \tag{11.18}
$$

where $\overline{\Sigma}$ is the self-energy evaluated at the minimum of the effective potential. We get

$$iq^{\mu}\Gamma^{i}_{5\mu}(p+q,p) = i\hat{q}\gamma_5\frac{\lambda^i}{2} - \gamma_5\frac{\lambda^i}{2}\left[\overline{\Sigma}(p+q)+\overline{\Sigma}(p)\right] \qquad (11.19)$$

From eq. (11.19) it follows that one can have a non zero dynamical quark mass if and only if $\Gamma^{i}_{5\mu}$ has a pseudoscalar pole at $q^2 = 0$ (Goldstone pole) whith residue proportional to the pion decay constant $f_\pi$.

Then we can write

$$\Gamma^{i}_{5\mu}(p+q,p) = \frac{\lambda^i}{2}\gamma_\mu\gamma_5 - f_\pi \mathrm{G}^{i}_{5}(p+q,p)\ \frac{q_\mu}{q^2} + \tilde{\Gamma}^{i}_{5\mu}(p+q,p) \qquad (11.20)$$

Here $\mathrm{G}^{i}_{5}(p+q,p)$ represents the proper pseudoscalar meson-quark- antiquark vertex function and $\tilde{\Gamma}^{i}_{5\mu}(p+q,p)$ is a term regular at $q^2 = 0$ which can be ignored in the approximation we are considering since it is of order $g^2$.

Finally, the comparison of eq. (11.19) with eq. (11.20), gives the following expression for pion-quark-antiquark vertex function

$$\mathrm{G}^{i}_{5}(p+q,p) = -i\gamma_5\frac{\lambda^i}{2}\ \frac{1}{f_\pi}\left[\overline{\Sigma}(p+q)+\overline{\Sigma}(p)\right] \qquad (11.21)$$

which is in complete agreement with (11.16). In fact, in the approximation of massless quarks we have

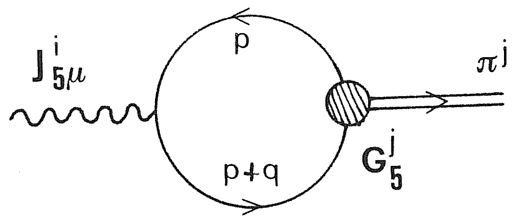
$$\begin{aligned} \mathrm{a}_{ij} &= \delta_{ij} \\ b_1 = b_2 &= -\frac{1}{\sqrt{2}}\ \frac{f_\pi}{\langle\chi\rangle} \\ \overline{\Sigma}(p) &= \langle\chi\rangle f_2(p) \end{aligned} \qquad (11.22)$$

It is worth to stress that the result (11.16) crucially depends on the symmetrization prescription used.

In practice, what we have shown here is that our effective action reproduces, at one loop level, the results of the Dynamical Perturbation Theory (DPT) introduced some time ago by H. Pagels and S. Stokar [50].

At this point, having derived the meson-quark-antiquark vertex function, it is possible to obtain an expression for the meson decay constants $f_{ij}$ by directly evaluating the coupling of the fields $\phi_j$ to the axial-currents $J^{i}_{5\mu}$.

From the diagram




one gets (remember that we are working in the Euclidean space)

$$\langle 0|J^i_{5\mu}(0)|\pi_j(q)\rangle = iq_\mu f_{ij} =$$
$$= 3\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[i\gamma_\mu\gamma_5\frac{\lambda_i}{2}\mathrm{S}(p+q)\mathrm{G}^j_5(p+q,p)\mathrm{S}(p)\right] \tag{11.23}$$

Substituting the expressions for the quark propagator and for the meson-quark-antiquark vertex, we get

$$\begin{aligned} iq_\mu f_{ij} &= \frac{3}{\sqrt{2}b_j}\sum_{k=1}^{8}\mathrm{a}_{kj}\int\frac{d^4p}{(2\pi)^4}\,[f_1(p+q)+f_1(p)] \\ &\mathrm{Tr}\left(i\gamma_\mu\gamma_5\frac{\lambda_i}{2}\left[i(\hat{p}+\hat{q})-\overline{\Sigma}(p+q)\right]^{-1}i\gamma_5\frac{\lambda_k}{2}\left[i\hat{p}-\overline{\Sigma}(p)\right]^{-1}\right)\end{aligned} \tag{11.24}$$

where

$$\overline{\Sigma}(p) = \begin{pmatrix} \overline{\Sigma}_u(p) & 0 & 0 \\ 0 & \overline{\Sigma}_d(p) & 0 \\ 0 & 0 & \overline{\Sigma}_s(p)\end{pmatrix} \tag{11.25}$$

with $\overline{\Sigma}_a(p) = m_a f_1(p) + \langle\chi_a\rangle f_2(p) \qquad a = u, d, s.$

Let us evaluate the traces over the spinor and the flavor indices in (11.24)

$$\begin{aligned} q_\mu f_{ij} = \frac{3}{\sqrt{2}b_j}\sum_{k=1}^{8}\mathrm{a}_{kj}\sum_{a,b=u,d,s}\mathrm{c}^{ik}_{ab}\int\frac{d^4p}{(2\pi)^4}\,[f_2(p+q)+f_2(p)]\cdot \\ \cdot\frac{p_\mu\left(\overline{\Sigma}_a(p+q)-\overline{\Sigma}_b(p)\right)-q_\mu\overline{\Sigma}_b(p)}{\left((p+q)^2+\overline{\Sigma}^2_a(p+q)\right)\left(p^2+\overline{\Sigma}^2_b(p)\right)}\end{aligned} \tag{11.26}$$

where $\mathbf{c}^{ik}$ are $3\times 3$ matrices of the following form

$$\mathbf{c}^{11} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} = \mathbf{c}^{22} \qquad \mathbf{c}^{44} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0\end{pmatrix} = \mathbf{c}^{55}$$

$$\mathbf{c}^{66} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{c}^{77} \tag{11.27}$$

$$\mathbf{c}^{33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{c}^{88} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\mathbf{c}^{38} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{c}^{83}$$

In order to obtain an expression for $f_{ij}$, we will differentiate with respect to $q_\nu$ in eq. (11.26) and, according to the soft pion limit, we will take $q = 0$. In this way we get

$$f_{ij} = \frac{1}{b_j} \sum_{k=1}^{8} \mathbf{M}_{ik} \, \mathbf{a}_{kj} \tag{11.28}$$

with

$$\begin{aligned} g_{\mu\nu} \, \mathbf{M}_{ik} = \frac{3}{\sqrt{2}} \sum_{a,b=u,d,s} \mathbf{c}_{ab}^{ik} \int \frac{d^4p}{(2\pi)^4} \\ \Bigg[ \frac{\partial f_2(p)}{\partial p_\nu} \, p_\mu \, \frac{\overline{\Sigma}_a(p) - \overline{\Sigma}_b(p)}{\left(p^2 + \overline{\Sigma}_a^2(p)\right)\left(p^2 + \overline{\Sigma}_b^2(p)\right)} + \\ + 2 \, f_2(p) \, \frac{p_\mu \, \dfrac{\partial \overline{\Sigma}_a(p)}{\partial p_\nu} - g_{\mu\nu} \, \overline{\Sigma}_b(p)}{\left(p^2 + \overline{\Sigma}_a^2(p)\right)\left(p^2 + \overline{\Sigma}_b^2(p)\right)} - \\ - 2 \, f_2(p) \, \frac{\left(2 \, p_\mu p_\nu + \dfrac{\partial \overline{\Sigma}_a^2(p)}{\partial p_\nu} \, p_\mu\right)\left(\overline{\Sigma}_a(p) - \overline{\Sigma}_b(p)\right)}{\left(p^2 + \overline{\Sigma}_a^2(p)\right)^2 \left(p^2 + \overline{\Sigma}_b^2(p)\right)} \Bigg] \end{aligned} \tag{11.29}$$

For the sector with charge and strangeness different from zero, inserting (10.16) in (11.28), one finds

$$\begin{aligned} f_{\pi^\pm}^2 &= -\frac{\langle \chi_u \rangle + \langle \chi_d \rangle}{\sqrt{2}} \, \mathbf{M}_{11} \\ f_{K^\pm}^2 &= -\frac{\langle \chi_u \rangle + \langle \chi_s \rangle}{\sqrt{2}} \, \mathbf{M}_{44} \\ f_{K^0,\overline{K^0}}^2 &= -\frac{\langle \chi_d \rangle + \langle \chi_s \rangle}{\sqrt{2}} \, \mathbf{M}_{66} \end{aligned} \tag{11.30}$$

(clearly from eq. (11.29) we get $\mathbf{M}_{11} = \mathbf{M}_{22}$, $\mathbf{M}_{44} = \mathbf{M}_{55}$ and $\mathbf{M}_{66} = \mathbf{M}_{77}$).

As an illustration, let us derive a representation fo $f_\pi^2$ in the case of massless quarks. Then

$$g_{\mu\nu}\mathbf{M}_{11} = 6\sqrt{2}\int \frac{d^4p}{(2\pi)^4}\, f_2(p)\, \frac{p_\mu \dfrac{\partial\overline{\Sigma}(p)}{\partial p_\nu} - g_{\mu\nu}\,\overline{\Sigma}(p)}{\left(p^2+\overline{\Sigma}^2(p)\right)^2} \tag{11.31}$$

So, using the relations (11.22) and performing tha angular integration we obtain

$$f_\pi^2 = \frac{3}{(2\pi)^2}\int_0^\infty dp^2\, p^2\, \frac{\overline{\Sigma}^2(p) - \frac{1}{2}\,p^2\,\overline{\Sigma}(p)\,\dfrac{\partial\overline{\Sigma}(p)}{\partial p^2}}{\left(p^2+\overline{\Sigma}^2(p)\right)^2} \tag{11.32}$$

which is the expression given by H. Pagels and S. Stokar [50]. We recall that Euclidean variables have been used.

On the other hand, for the sector with $Q = S = 0$ one obtains

$$f_{\mu\nu} = \frac{1}{b_\nu}\sum_{\rho=3,8}\mathbf{M}_{\mu\rho}\mathbf{a}_{\rho\nu} \tag{11.33}$$

which, toghether with eq. (10.18) gives the following results

$$f_{\mu\nu}f_{\mu\nu} = \sum_{\rho,\tau=3,8}\mathbf{M}_{\mu\rho}\mathbf{a}_{\rho\tau}\mathcal{A}_{\mu\tau}\mathbf{a}_{\tau\nu} \tag{11.34}$$

$$\begin{aligned} b_3^2 &= \frac{\sum_{\rho=3,8}\mathbf{M}_{3\rho}\mathbf{a}_{\rho 3}}{\sum_{\rho=3,8}\mathcal{A}_{3\rho}\mathbf{a}_{\rho 3}} \\ b_8^2 &= \frac{\sum_{\rho=3,8}\mathbf{M}_{8\rho}\mathbf{a}_{\rho 8}}{\sum_{\rho=3,8}\mathcal{A}_{8\rho}\mathbf{a}_{\rho 8}} \end{aligned} \tag{11.35}$$

Eq. (11.34), in which there is no summation on the indices $\mu$ and $\nu$, gives the square of the decay constants in the 3-8 sector.

Here the matrices $\mathbf{M}$ and $\mathcal{A}$ depend only on the parameters of our model: $\mu$, $\Lambda_{QCD}$, $m_u$, $m_d$ and $m_s$. Therefore the $f_{ij}$'s are completely determined as functions of these quantities.

On the other hand, with eq. (11.35) we can calculate the coefficients $b_3$ and $b_8$ which enter in the expressions of $M^2_{\pi^0}$ and $M_{\eta_8}$ (see eq. (10.20)).

As an example let us consider the $SU(2)$ symmetric case ($m_u = m_d$).

Then $\mathrm{a}_{ij} = \delta_{ij}$ and also $\mathcal{A}_{38} = \mathcal{A}_{83} = 0$; $\quad \mathrm{M}_{38} = \mathrm{M}_{83} = 0$ so

$$
\begin{aligned}
f_{33}^2 = f_{\pi^0}^2 = & \mathrm{M}_{33}\ \mathcal{A}_{33} = -\frac{\langle\chi_u\rangle + \langle\chi_d\rangle}{\sqrt{2}}\ \mathrm{M}_{33} \\
f_{88}^2 = f_{\eta_8}^2 = & \mathrm{M}_{88}\ \mathcal{A}_{88} = -\frac{\langle\chi_u\rangle + \langle\chi_d\rangle + 4\langle\chi_s\rangle}{3\sqrt{2}}\ \mathrm{M}_{88}
\end{aligned}
\tag{11.36}
$$

$$
b_3^2 = \frac{\mathrm{M}_{33}}{\mathcal{A}_{33}} \qquad\qquad b_8^2 = \frac{\mathrm{M}_{88}}{\mathcal{A}_{88}} \tag{11.37}
$$

Using (11.36) in (11.37) we get

$$
\begin{aligned}
b_3^2 &= \frac{f_{\pi^0}^2}{\mathcal{A}_{33}^2} = \frac{2 f_{\pi^0}^2}{(\langle\chi_u\rangle + \langle\chi_d\rangle)^2} \\
b_8^2 &= \frac{f_{\eta_8}^2}{\mathcal{A}_{88}^2} = \frac{18 f_{\eta_8}^2}{(\langle\chi_u\rangle + \langle\chi_d\rangle + 4\langle\chi_s\rangle)^2}
\end{aligned}
\tag{11.38}
$$

Eq. (10.20) for the 3-8 sector in the symmetric case reads

$$
M_3^2 = \frac{1}{b_3^2}\, V_{33} \qquad\qquad M_8^2 = \frac{1}{b_8^2}\, V_{88} \tag{11.39}
$$

which means that, as expected, that there is no mixing.

So we can identify

$$
M_3 = M_{\pi^0} \qquad\qquad M_8 = M_{\eta_8} \tag{11.40}
$$

and the following relations hold

$$
\begin{aligned}
M_{\pi^0}^2 = & \frac{1}{2\ f_{\pi^0}^2}\Big(\langle\chi_u\rangle + \langle\chi_d\rangle\Big)^2 \frac{d^2V}{dp_3 dp_3} \\
M_{\eta_8}^2 = & \frac{1}{18\ f_{\eta_8}^2}\Big(\langle\chi_u\rangle + \langle\chi_d\rangle + 4\langle\chi_s\rangle\Big)^2 \frac{d^2V}{dp_8 dp_8}
\end{aligned}
\tag{11.41}
$$

As it follows from the previous considerations, in the general case one has to correct eq. (11.41) with terms of order

$$
\left(\frac{m_u - m_d}{m_s}\right)^2 .
$$

## 12. NUMERICAL RESULTS

The expressions we have found in the previous sections for the decay coupling constants and for the pseudoscalar meson masses, are functions of the parameters of our model: $\mu$, $\Lambda_{QCD}$, $m_u$, $m_d$ and $m_s$.

The problem now is to determine these parameters from the experimental data.

We have a system of coupled equations given by (10.20) and (11.28), so this determination can be done only in an approximate way.

In order to realize this program, we start from the $SU(2)$ sector with $m_u = m_d = 0$.

Then the representation for $f_\pi^2$ (see eq. (11.32)), gives $f_\pi/\Lambda_{QCD}$ as a function of $c$

$$\frac{f_\pi}{\Lambda_{QCD}} = F(c) \tag{12.1}$$

(remember that in QCD with three flavors

$$c = \frac{2\pi^2}{g^2(\mu)} \qquad \text{or equivalently} \qquad \frac{\mu}{\Lambda_{QCD}} = e^{4/9\ c}\ ).$$

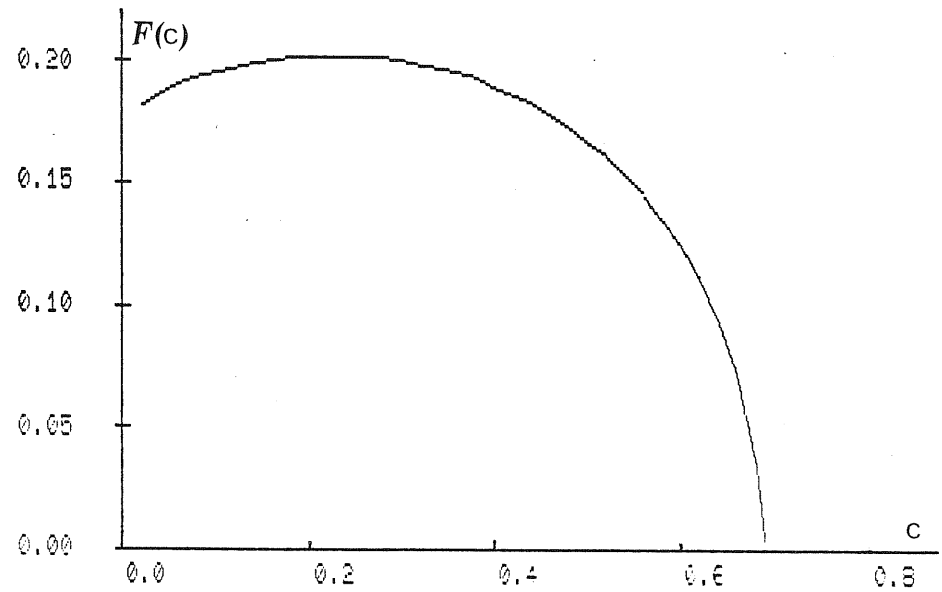



*FIG. 15. Graphical representation of $f_\pi/\Lambda_{QCD}$ as a function of c in the massless case. The curve reaches a maximum value $F_{max}$ for* $c = 0.23$.

The numerical analysis of eq. (12.1), shows that the function $F(c)$ has a maximum, $F_{max}$, when $c = 0.23$. This means that in our model $\Lambda_{QCD} \geq f_\pi / F_{max}$.

So, in order to reproduce the experimental value $f_\pi = 93\ MeV$ with the possible smallest value of $\Lambda_{QCD}$, we have to fix $c = 0.23$ corresponding to the maximum for $F$ (see Fig. 15).

In this way, having fixed the value of $c$, we can determine $\mu$ (or $\Lambda_{QCD}$) from the experimental value of $f_\pi$.

Furthermore, from the value of the minimum of the effective potential for massless quarks, one can extract the values of the condensates $\langle \overline{u}u^0 \rangle_\mu = \langle \overline{d}d^0 \rangle_\mu$ (the superscript 0 means $m = 0$).

Using these values in the Adler-Dashen relation (10.32), we obtain a first approximation for the quark mass $m_u = m_d$ given the experimental value of the pion mass.

Substituting these first results in eq. (11.28) with $i, j, k = 1$ we redetermine $\mu$ given $f_\pi = 93\ MeV$.

Then we iterate this procedure.

From the minimum of the effective potential we obtain the values of the condensates for the quarks $u$ and $d$ and then, from the second derivative of the effective potential evaluated at the minimum, we extract the values of the quark masses. Schematically

$$\begin{array}{lcl} f_\pi = 93\ MeV & \longrightarrow & \mu \\ \dfrac{dV}{d\chi} = 0 & \longrightarrow & \langle \overline{u}u \rangle_\mu = \langle \overline{d}d \rangle_\mu \\ M_\pi^2 = (139\ MeV)^2 & \longrightarrow & \hat{m}(\mu) = \dfrac{m_u(\mu) + m_d(\mu)}{2} \end{array}$$

where we have explicitily indicated the dependence on the renormalization point $\mu$ of the quark masses and condensates.

The iteration converges very fastly and the results we obtain are

$$\begin{aligned} \mu &= 497\ MeV \\ \Lambda_{QCD} &= 449\ MeV \\ \hat{m}(\mu) &= 18\ MeV \end{aligned} \tag{12.2}$$

With these values, it is easy to determine $m_s$, for instance given $M_{K^\pm} = 494\ MeV$. The result is

$$m_s(\mu) = 294\ MeV \tag{12.3}$$

Finally, assuminig that the elettromagnetic mass difference between $K^\pm$ and $K^0, \overline{K^0}$ is of order of 1.5 $MeV$ [21], we can calculate the difference $m_u(\mu) - m_d(\mu)$ and, combining with the previous result (12.3), we find

$$\begin{aligned} m_u(\mu) &= 14.5\ MeV \\ m_d(\mu) &= 21\ MeV \end{aligned} \tag{12.4}$$

We are now ready to calculate the masses and the decay constants for the octet mesons.

The values we get [24] are the following to be compared with the experimental ones.

| | *evaluated* | *experimental* |
|---|---|---|
| $M_{\pi^\pm}$ | 139 $MeV$ | 139.6 $MeV$ |
| $M_{\pi^0}$ | 138.7 $MeV$ | 135 $MeV$ |
| $M_{K^\pm}$ | 492 $MeV$ | 494 $MeV$ |
| $M_{K^0,\overline{K^0}}$ | 498 $MeV$ | 498 $MeV$ |
| $M_{\eta_8}$ | 546 $MeV$ | (566) $MeV$ |
| $f_\pi$ | 93 $MeV$ | 93 $MeV$ |
| $f_K$ | 105 $MeV$ | $f_K/f_\pi = 1.17$ |
| $f_\eta$ | 111 $MeV$ | $f_\eta/f_\pi = 1.3$ |

The fit for the meson masses is very good (agreement within 3%) and the ratios $f_K/f_\pi = 1.13$ and $f_\eta/f_\pi = 1.19$ are in rather good agreement with the experimental results [51] and with the current algebra calculations [52].

Some observations are now in order.

It is known that the main contribution to the $\pi^\pm - \pi^0$ mass difference is electromagnetic. The mass splitting we find in the framework of our model, comes only from the explicit $SU(2)$ breaking due to $m_u \neq m_d$ and therefore, it has to be compared with the current algebra predictions [21], [53].

For the fit we have reported, the mass difference is $(M_{\pi^\pm} - M_{\pi^0}) = 0.3\ MeV$ of which $0.11\ MeV$ come from the $\pi^0 - \eta_8$ mixing.

Here $M_{\eta_8}$ is the mass of the eigthth component of the octet.

However, $\eta_8$ mixes with the singlet $\eta_0$ because of the $SU(3)$ breaking and the physical states are given by

$$\begin{aligned} \eta &= \eta_8 \ \cos\theta - \eta_0 \ \sin\theta \\ \eta' &= \eta_8 \ \sin\theta + \eta_0 \ \cos\theta \end{aligned} \tag{12.5}$$

where $\theta$ must be determined in order to diagonalize the mass squared matrix

$$\begin{pmatrix} M^2_{\eta_0} & M^2_{08} \\ M^2_{80} & M^2_{\eta_8} \end{pmatrix} \tag{12.6}$$

Since we are ignoring the mixing with the $SU(3)$ singlet, the output of our model is $M_{\eta_8}$ and it has not to be compared with the experimental value of $M_\eta$, but with the prediction of the modified Gell-Mann-Okubo mass formula which yields to $M_{\eta_8} = 566\ MeV$ [53].

Let us also puntualize that in the determination of $f_{\pi^0}$ and $f_{\eta_8}$, we have neglected the mixing terms since their contributions is almost irrelevant. Our explicit calculation gives in fact $f_{38}, f_{83} \simeq 1.5\ MeV$. (See also [52] where the determination of the off-diagnal matrix elements of the meson decay constants is performed in the context of chiral perturbation theory).

Having determined the values of the quark masses, one can extract the corresponding numerical values of the quark condensates from the minima of the effective potential.

Let us remember that the numbers we get for the masses and for the condensates must be interpreted as the values at the renormalization point $\mu = 497\ MeV$.

In order to compare our results with the values obtained by quite different methods, quoted in literature, let us perform a rescaling at $1\ GeV$.

Recalling that the current quark masses scale as:

$$m(\overline{\mu}) = m(\mu) \ \left( \frac{\log \overline{\mu}/\Lambda_{QCD}}{\log \mu/\Lambda_{QCD}} \right)^{-d} \tag{12.7}$$

with $d = 4/9$, and that $m \langle\overline{\psi}\psi\rangle$ is a renormalization group invariant quantity

$$m(\overline{\mu}) \langle\overline{\psi}\psi\rangle_{\overline{\mu}} = m(\mu) \langle\overline{\psi}\psi\rangle_{\mu} \tag{12.8}$$

we obtain

$$\begin{aligned} m_u(1) &= 5.8\ MeV & \langle\bar{u}u\rangle_1 &= (-223)^3\ MeV^3 \\ m_d(1) &= 8.4\ MeV & \langle\bar{d}d\rangle_1 &= (-225)^3\ MeV^3 \\ m_s(1) &= 118\ MeV & \langle\bar{s}s\rangle_1 &= (-284)^3\ MeV^3 \end{aligned} \tag{12.9}$$

First of all, let us compare these results with those of ref. [25] where the same calculations are performed but in the approximation in which the logarithmic corrections coming from the renormalization group analysis are neglected (rigid case).

| | *rigid case* | *logs* |
|---|---|---|
| $\mu$ | $306\ MeV$ | $497\ MeV$ |
| $\Lambda_{QCD}$ | $265\ MeV$ | $449\ MeV$ |
| $m_u(1)$ | $6.5\ MeV$ | $5.8\ MeV$ |
| $m_d(1)$ | $9.5\ MeV$ | $8.4\ MeV$ |
| $m_s(1)$ | $122\ MeV$ | $118\ MeV$ |
| $\langle\overline{u}u\rangle_1$ | $-(218)^3\ MeV^3$ | $-(223)^3\ MeV^3$ |
| $\langle\overline{d}d\rangle_1$ | $-(220)^3\ MeV^3$ | $-(225)^3\ MeV^3$ |
| $\langle\overline{s}s\rangle_1$ | $-(284)^3\ MeV^3$ | $-(284)^3\ MeV^3$ |

We see that, even if the values for $\mu$ and $\Lambda_{QCD}$ the two cases are quite different, the variations for the quark masses and condensates are small. This shows once again that the whole picture is essentially insensitive to the corrections due to the renormalization group.

The same numerical analysis in the rigid case has been also performed by using the smooth ansatz (5.18) for the fermion self-energy just to have an indication of the dependence of our results on the particular test function for $\Sigma$.

As anticipated in sect. 5, the differences we find are not dramatic. In particular we obtain a good fit for $\mu = 282\ MeV$, $\Lambda_{QCD} = 220\ MeV$, $m_u(1) = 3.6\ MeV$, $m_d(1) = 5\ MeV$ and $m_s(1) = 82\ MeV$ to which correspond a little bit higher values for the quark condensates ($\langle \overline{u}u \rangle_1 = -(262)^3\ MeV^3$, $\langle \overline{d}d \rangle_1 = -(264)^3\ MeV^3$, $\langle \overline{s}s \rangle_1 = -(341)^3\ MeV^3$).

Other quantities which can be introduced are the invariant quark masses and condensates given by

$$\overline{m} = m(\mu)\ \ (\log \mu / \Lambda_{QCD})^d \tag{12.10}$$

$$\overline{\langle \overline{\psi}\psi \rangle} = \langle \overline{\psi}\psi \rangle_\mu\ \ (\log \mu / \Lambda_{QCD})^{-d} \tag{12.11}$$

Then, with our values of $\Lambda_{QCD} = 449\ MeV$ we obtain

$$\begin{aligned} \overline{m}_u &= 5.25\ MeV & \overline{\langle \overline{u}u \rangle} &= (-261)^3\ MeV^3 \\ \overline{m}_d &= 7.6\ MeV & \overline{\langle \overline{d}d \rangle} &= (-232)^3\ MeV^3 \\ \overline{m}_s &= 106.4\ MeV & \overline{\langle \overline{s}s \rangle} &= (-293)^3\ MeV^3. \end{aligned}$$

## 13. COMPARISON WITH CURRENT ALGEBRA AND SUM RULES PREDICTIONS FOR THE QUARK MASSES

The earliest informations about quark masses derived from current algebra although, in this framework, the quark mass problem showed up only implicitly in the symmetry properties of the Hamiltonian i.e. in the commutation rules involving the currents and the energy-momentum tensor.

We know that for massless quarks, QCD with three flavors possesses a global symmetry $SU(3)_L \otimes SU(3)_R \otimes U(1)_{L+R}$ (the $U(1)_{L-R}$ is broken by the axial anomaly).

The $G = SU(3)_L \otimes SU(3)_R$ chiral symmetry is spanned by the generators which are the charges associated to the Noëther axial vector and vector currents

$$\begin{aligned} J^i_{5\mu}(x) &= \overline{\Psi}\gamma_\mu\gamma_5\frac{\lambda^i}{2}\Psi \\ J^i_\mu(x) &= \overline{\Psi}i\gamma_\mu\frac{\lambda^i}{2}\Psi \\ i &= 1,..,8 \end{aligned} \tag{13.1}$$

The symmetry framework inherited from the success of the current algebra and PCAC is the one where the axial charges do not annihilate the vacuum i.e. the chiral symmetry is realized á la Nambu-Goldstone [2], [4].

In this scheme, the chiral flavor group is broken spontaneously by the quark vacuum condensates down to a subgroup $H = SU(3)_{L+R}$ with respect to which the vacuum condensates are symmetric (remember that we are dealing with the massless case so $\langle\overline{u}u\rangle = \langle\overline{d}d\rangle = \langle\overline{s}s\rangle$).

This spontaneous breaking is accompained by 8 Goldstone bosons which are associated to each unbroken generator of the coset space $G/H$.

On the other hand the vector charges annihilate the vacuum and the corresponding symmetry is realized á la Wigner-Weyl so that particles are classified in irreducible representations of $SU(3)_{L+R}$.

As it is clear from our previous analysis, the massless Goldstone bosons acquire a mass induced by an explicit breaking of the $SU(3)_L \otimes SU(3)_R$ global symmetry due to the quark mass terms in the QCD Lagrangian.

In this way the divergence of the axial-vector currents are non zero and they are associated to the meson decay constants.

Also, since the spectrum of the pseudoscalar mesons in the octet does not show a degeneracy in their masses, an explicit breaking of the $SU(3)_{L+R}$ is expected and its strenght is measured by $m_s - (m_u + m_d)/2$.

There are convincing estimates of the quark mass ratios from the comparison of various current algebra Ward identities at zero momentum transfer with physical parameters like the masses and the decay constants of hadrons.

The success of the current algebra predictions is mainly due to the fact that the ratio of the quark masses is defined unambiguosly as it is scale independent.

For example, a standard way to extract informations about the quark mass ratios is based on the chiral expansion [21].

In fact, since the masses of the light quarks $u$, $d$, $s$ turn out to be small in comparison to their typical kinetic energy, the deviations from chiral symmetry may be studied by treating the quark mass term in the Hamiltonian as a perturbation, with massless QCD as the unperturbed system.

The chiral symmetry implies a set of Ward identities which link the various Green functions and therefore interrelate the expansion coefficients.

In this way, by expanding the mass of the bound states in powers of the quark masses, one obtains the first order mass formulae. For the pseudoscalar meson octet they read

$$\begin{aligned} M^2_{\pi^\pm} &= (m_u + m_d)\ B + \mathcal{O}(m_q \log m_q) \\ M^2_{K^\pm} &= (m_u + m_s)\ B + \mathcal{O}(m_q \log m_q) \\ M^2_{K^0,\overline{K^0}} &= (m_d + m_s)\ B + \mathcal{O}(m_q \log m_q) \end{aligned} \tag{13.2}$$

with the same constant

$$B = -\frac{2}{f_\pi^2} \langle \overline{u} u \rangle$$

(in fact, in the chiral limit, $f_\pi = f_K$ and $\langle \overline{u}u \rangle = \langle \overline{d}d \rangle = \langle \overline{s}s \rangle$), showing that the Gell-Mann-Okubo additive rule [54] is well satisfied by the pseudoscalar octet.

So, if one ignores the higher order corrections, eq. (13.2) gives the quark mass ratios in terms of the masses of the pseudo Goldstone bosons.

The study of higher order terms in the quark mass expansion does not reduce to the study of a simple Taylor series since, in the chiral limit around which one is expanding, the theory contains massless physical particles (Goldstone bosons) which generate infrared singularities [55].

To obtain a reliable approximation scheme, it is necessary to reorder the expansion and summing up the leading infrared singularities that occur to all orders in the quark mass.

Armed with the estimates of the higher order terms in the quark mass expansion, one can analyze the quark mass ratios $m_u : m_d : m_s$ given the experimental values of the meson and barion masses.

For example in ref. [21] the ratio $R = (m_s - \hat{m})/(m_d - m_u)$ with $\hat{m} = (m_u + m_d)/2$, is determined on the basis of five independent manifestations of isospin breaking ($K^+ - K^0$, $p - n$, $\Sigma^+ - \Sigma^0$, $\Xi^0 - \Xi^-$ and $\rho - \omega$ mixing).

Treating the values they obtain in this way as independent, they find

$$R = 43.5 \pm 2.2 \quad (42.65) \tag{13.3}$$

where we have reported in parenthesis the value we get in the framework of our model.

From the observed masses of $\pi$, $K$ and $\eta$, one can extract the value of the ratio $m_s/\hat{m}$ and, from the analysis of the corresponding higher order terms in the quark mass expansion, one gets [21]

$$m_s/\hat{m} = 25.0 \pm 2.5 \quad (16.62) \tag{13.4}$$

The above results for $R$ and $m_s/\hat{m}$ imply the following values for the related quark mass ratios [21]

$$\begin{aligned} m_d/m_u &= 1.76 \pm 0.13 \quad (1.45) \\ m_s/m_d &= 19.6 \pm 1.6 \quad (14.05) \\ m_s/m_u &= 34.5 \pm 5.1 \quad (20.34) \end{aligned} \tag{13.5}$$

Let us notice that the discrepancies between the chiral perturbation theory results and ours (written in parenthesis), are essentially due to the fact that in our calculations $f_\pi \neq f_K \neq f_\eta$ and also the values of the quark condensates are different for each flavor.

While it is possible to calculate the quark mass ratios on the basis of current algebra symmetries alone, the absolute value of the light quark masses is not known very accurately even if there has been substantial progress in its determination during the last few years.

The framework which has been used is a variety of QCD sum rules.

The method of QCD sum rules is based on the duality between the QCD expression of the hadronic Green functions and their spectral representation wich can be derived from the analytical properties of such Green functions.

Formally, the kind of objects one is dealing with are two-point functions like

$$\Pi(q^2) = i \int d^4x \; \langle 0 | \mathrm{T} \; \mathcal{O}(x) \mathcal{O}^{\dagger}(0) | 0 \rangle \; e^{iqx} \qquad (13.6)$$

where $\mathcal{O}(x)$ denotes a composite operator of quark and gluon fields with specified quantum numbers.

Ever since the advent of QCD, there has been a lot of effort to find ways of relating the behaviour of two-point functions like that in eq. (13.6) in deep euclidean region where the theory makes firm predictions, to the properties of the resonances of multihadron states which the operator $\mathcal{O}(x)$ can extract from the vacuum.

To this end one considers the dispersion relation

$$\Pi(q^2) = \int_0^{\infty} ds \; \frac{1}{s - q^2 - i\epsilon} \; \frac{1}{\pi} \; \mathrm{Im} \; \Pi(s) + \mathit{subtractions} \qquad (13.7)$$

which follows from the analyticity properties of two-point functions like (13.6).

The relation (13.7) can be regarded as a duality relation in the sense that the weighted average of the hadronic spectral function $(1/\pi)$ Im $\Pi(s)$ in the right hand side (which, up to kinematical factors, is a total cross-section), for sufficientely large space-like $q^2$ values, must match $\Pi(q^2)$ in the left hand side which, up to subtractions, is a calculable quantity in QCD.

Various forms of duality sum rules which follow more or less directly from eq. (13.7) have been proposed in literature.

In all of them the left hand side is evaluated by theory and compared to phenomenological input in the right hand side.

Following the original ideas developed by Shifman, Vainshtein and Zakharov [56], there has been a lot of effort to improve on a purely QCD evaluation of the left hand side of equations like (13.7).

These authors have proposed to use the Wilson Operator Product Expansion of the time ordered product in eq. (13.6) to parametrize non-perturbative effects due to the confining nature of the QCD vacuum which, at short distances, appear as power corrections to asymptotic freedom behaviour.

In order to extract informations about light quark masses, the appropriate two point functions are those involving the divergence of the axial-vector currents

$$\psi_5(q^2) = i \int d^4x \; \langle 0|\mathrm{T} \; \partial^\mu J_{5\mu}(x) \partial^\nu J^\dagger_{5\nu}(0)|0\rangle \; e^{iqx} \tag{13.8}$$

and the corresponding two point functions associated to the divergence of the vector currents

$$\psi(q^2) = i \int d^4x \; \langle 0|\mathrm{T} \; \partial^\mu J_\mu(x) \partial^\nu J_\nu(0)|0\rangle \; e^{iqx} \tag{13.9}$$

The reason to consider these particular two-point functions is that in QCD the operators $\partial^\mu J_{5\mu}$ and $\partial^\mu J_\mu$, which are renormalization group invariant operators, are proportional to the sums and to the differences of quark masses respectively.

For example, there are two familiar Ward identities which relate $\psi_5(0)$ and $\psi(0)$ to products of quark masses and vacuum expectation values of quark-antiquark fields

$$\begin{aligned} \psi_5(0) &= -\,(m_u + m_d)\; \langle \bar{u}u + \bar{d}d \rangle \\ \psi(0) &= -\,(m_u - m_d)\; \langle \bar{u}u - \bar{d}d \rangle \end{aligned} \tag{13.10}$$

The variety of QCD sum rules found in literature, is due mainly to the freedom one has a priori to exploit optimally the informations both on the hadronic side and on the QCD side of these equations.

Although some consensus on the values of the light quark masses has by now been reached among various group of authors, there are still inconsistent results and the errors remain rather large.

For example, Gasser and Leutwyler [21] show that the sum rules they consider for the divergence of the axial current, are consistent with

$$\hat{m}(1) = 7 \pm 2\ MeV, \qquad m_s(1) = 180 \pm 50\ MeV \tag{13.11}$$

With the results for the ratios $m_u : m_d : m_s$ given in eq. (13.5), the estimate $\hat{m}(1) = 7 \pm 2\ MeV$ amounts to the following results

$$\begin{aligned} m_u(1) &= 5.1 \pm 1.5\ MeV \quad (5.8) \\ m_d(1) &= 8.9 \pm 2.6\ MeV \quad (8.4) \\ m_s(1) &= 175 \pm 55\ MeV \quad (118) \end{aligned} \tag{13.12}$$

These absolute values are not known very accurately. However, within the large errors, we see that our estimates (in parenthesis) for the $u$ and $d$ quark masses at the same renormalization point compare very well while the situation for the strange quark is much more critical.

Also, if one uses the value $\hat{m}(1) = 7 \pm 2\ MeV$ in the Adler-Dashen relation, one can fix the order parameter $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle$ from the experimental values of $M_\pi$ and $f_\pi$. The result is

$$\langle \bar{u}u \rangle_1 = (-225 \pm 25)^3\ MeV^3 \quad (-(223)^3) \tag{13.13}$$

which agrees very well with the value we find.

In a recent paper, S. Narison [57] discusses in detail various determinations of chiral symmetry breaking parameters from the light-meson systems by using the SVZ-Laplace transform QCD sum rules and also calculates a weighted average of various estimates coming from different methods.

Concerning the light quark $u$, $d$ and $s$ masses he gets (for $100 < \Lambda_{QCD} < 150\ MeV$)

$$\begin{aligned} m_u(1) &= 5.1 \pm 0.9\ MeV \quad (5.8) \\ m_d(1) &= 9.0 \pm 1.6\ MeV \quad (8.4) \\ m_s(1) &= 148.4 \pm 15.3\ MeV \quad (118) \end{aligned} \tag{13.14}$$

As far as the estimates of the ratios of the quark vacuum condensates are concerned, the results he gets are

$$\begin{aligned} \langle \bar{d}d \rangle / \langle \bar{u}u \rangle &= 1 - (1 \pm 0.3)\ 10^{-2} \quad (1 + 3\ 10^{-3}) \\ \langle \bar{s}s \rangle / \langle \bar{u}u \rangle &= 0.6 \pm 0.2 \quad (1.08) \end{aligned} \tag{13.15}$$

and, as usual, we have reported in parenthesis the values we get from eq. (12.9).

Also C.A. Dominguez and E. De Rafael [58] have recently presented an improved determination of the light quark masses in QCD by combining the information provided by the effective chiral Lagrangian of QCD at long distances (see for example [21],[52]), and the QCD behaviour at short distances within the combined framework of Gaussian sum rules and finite energy sum rules.

The main result of their work is the determination of the sum of the running $u$ and $d$ masses at 1 $GeV$

$$m_u(1) + m_d(1) = 15.5 \pm 2.0 \; MeV \quad (14.2) \tag{13.16}$$

Then, they determine the strange quark mass from the combination of the result (13.16) with the current algebra determination of the ratio $m_s/\hat{m}$ (for example, the value in ref. [21] is given in eq. (13.4)) and they get

$$m_s(1) = 199 \pm 33 \; MeV \quad (118) \tag{13.17}$$

Also, the best determination in their framework of the down and up quark mass difference, follows from the combination of the current algebra determination of the ratio $(m_d - m_u)/(m_d + m_u)$ with eq. (13.16). They obtain in this way the individual values

$$\begin{aligned} m_u(1) &= 5.6 \pm 1.1 \; MeV \quad (5.8) \\ m_d(1) &= 9.9 \pm 1.1 \; MeV \quad (8.4) \end{aligned} \tag{13.18}$$

Let us also mention the results obtained by L.J. Reinders and H.R. Rubinstein [59] about the determination of the mass and the condensate of the strange quark.

Their strategy consists in taking advantage of the constraints coming from heavy quark physics and then in analyzing the light quark meson channels with strange quarks.

They find that the compatibility of all sum rules they calculate, requires a very narrow window for $m_s\langle\bar{s}s\rangle$.

As a consequence they establish that

$$m_s \langle \bar{s}s \rangle = -(210 \pm 5)^4 \ MeV^4 \quad (-(228)^4) \tag{13.19}$$

and

$$m_s(1) = 110 \pm 10 \ MeV \quad (118) \tag{13.20}$$

They also suggest the following value for the condensate

$$\langle \bar{s}s \rangle_1 = (0.8 \pm 0.1) \ \langle \bar{u}u \rangle_1 \tag{13.21}$$

which, however, does not agree with their previous results. In fact, taking litterally the values given in eqs. (13.19) and (13.20) one would get

$$\langle \bar{s}s \rangle_1 = -(260 \pm 20)^3 \ MeV^3 \quad (-(284)^3) \tag{13.22}$$

which is in very good agreement with our result.

Summing up, we can say that, as far as the value of $(m_u(1) + m_d(1))$ is concerned, one finds in the literature values ranging between 10 and 19 $MeV$ [60] which agree very well with our results.

However, the situation for the mass of the strange quark is much more confused. One finds evaluations varying from 100 up to 230 $MeV$.

At the same time, there is a general tendency to indicate that the value of the condensate $\langle \bar{q}q \rangle$ decreases for heavier quarks.

Notice that, in our approach, the condensate increases with the mass simply because the extremum of the effective potential moves further away from the origin as the symmetry breaking increases.

This is clearly shown in Fig. 16 where we have a plot of the the cubic root of the quark-antiquark condensate as a function of the quark mass.

However, as it is shown in Fig. 16 ($b$), the condensate varies very slowly for large values of the current quark mass. For example, it varies of $\sim$ $(0.3 \ GeV)^3$ when passing from a value of 20 $GeV$ to a value of 40 $GeV$ for the mass.

This means that, for heavy quarks, the effects of the condensate become neglibible as also expected from the asymptotic behaviour of the quark self-energy.

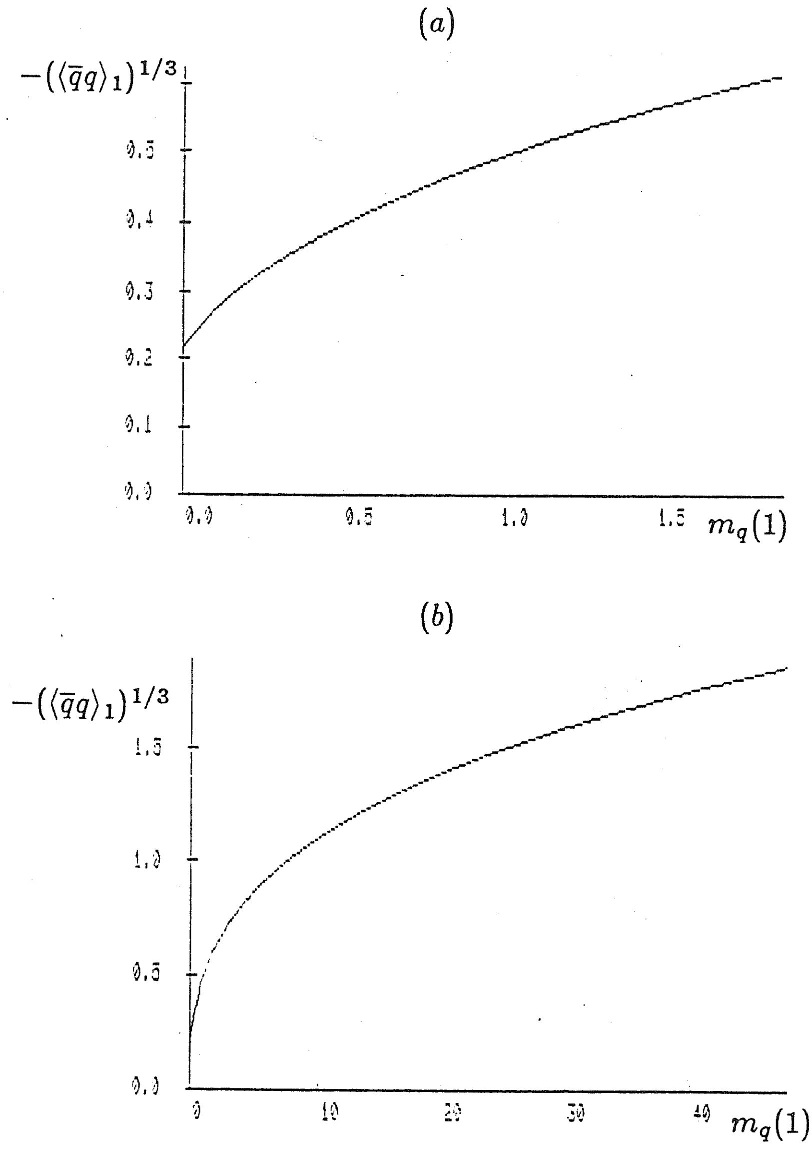



*FIG. 16. Way of varying of* $-(\langle\overline{q}q\rangle_1)^{1/3}$ *with* $m_q(1)$. *The values are in GeV. In (a) the current quark mass ranges from* 0 *to* 1.5 *GeV while in (b) it reaches the value of* 40 *GeV.*

As an example of this phenomenon let us compute the "constituent quark mass".

D. Politzer [35] has proposed a definition which, in the euclidean region reads

$$m_{const} = \overline{\Sigma}'(p_E^2 = -4m_{const}^2) \tag{13.23}$$

where $\overline{\Sigma}'(p^2) = m_0(\Lambda) + \overline{\Sigma}(p^2)$ is evaluated at the extremum of the effective potential.

Using our previous results, we get the following form for the Politzer's equation

$$
\begin{aligned}
m_{const} = m(\mu) & \left(1 + \frac{1}{2a} \log \frac{4m_{const}^2}{\mu^2}\right)^{-4/9} + \\
& + \mu\chi \frac{\mu^2}{4m_{const}^2} \left(1 + \frac{1}{2a} \log \frac{4m_{const}^2}{\mu^2}\right)^{-5/9}
\end{aligned}
\tag{13.24}
$$

with $a = \log(\mu/\Lambda_{QCD})$.

A numerical study of eq. (13.24) with the values of $\mu$, $\Lambda_{QCD}$, $m_u$, $m_d$ and $m_s$ given in (12.2), (12.3) and (12.4) and the values of the condensates renormalized at $\mu$ obtained by rescaling in (12.9), gives essentially the same constituent mass for all the three light quarks

$$
\begin{aligned}
(m_u)_{const} &= 275 \ MeV \\
(m_d)_{const} &= 276 \ MeV \\
(m_s)_{const} &= 288 \ MeV
\end{aligned}
\tag{13.25}
$$

Here the constituent mass is largely dominated by the condensate scale.

In the case of heavy quarks the situation is quite different.

If, for instance, we look at the charm quark, using the value $m_c(2m_c) = 1.01 \ GeV$ for the charm mass, we find

$$
(m_c)_{const} = 0.97 \ GeV \tag{13.26}
$$

Hence, although the large value of the charm condensate ($\langle \bar{c}c \rangle_{2m_c} \sim -(600)^3 \ MeV^3$), the constituent mass differs from the current mass of only 4%.

This means that, already for the charm quark, the effects of the condensate are negligible for a quantity like the constituent mass. Obviously this phenomenon will be stronger for much heavier quarks.

## 14. CONCLUSIONS AND FURTHER DEVELOPMENTS

We have analyzed dynamical mass generation in QCD-like fermion gauge theories with running gauge coupling constant and fermion self-energy corrected by the renormalization group analysis in the leading logarithmic approximation.

Use has been made of a variational method based upon an effective potential for composite operators which is a modified version of the one introduced by Cornwall, Jackiw and Tomboulis.

This formalism deals with a non local order parameter for which a non local source $J(x, y)$ is introduced.

Since this effective potential corresponds to the vacuum energy only when the source function vanishes, there is an ambiguity of adding an arbitrary polinomial of the source function itself satisfying some suitable conditions.

In particular our choice corresponds to a functional which has the same local extrema as the CJT one but has the convenient property of boundness from below. $V_{CJT}$ does not enjoy this property and this instability is reflected in the saddle point character of its stationary points.

We have shown that the different choices of the source term in the two cases correspond to different choices of the dynamical variable. Our effective potential results completely expressed in terms of the fermion self-energy $\Sigma$ and our variational method consists in making use of a parameter dependent test function for $\Sigma$ to investigate the stability of the theory.

The parametrization of the self-energy is in terms of constant fields related to the fermionic condensates once evaluated at the minimum of the effective potential.

Following some suggestions from lattice calculations we have assumed that the main contribution for the spontaneous chiral symmetry breaking phenomenon comes from short distances effects and is independent on confinement.

For this reason we have introduced a parameter $\mu$ as an infrared cutoff and we have focused our attention on the short-distance dynamics.

In this range it is sensible to perform a loop expansion of the effective action and to retain only the lowest non trivial contribution.

So, we have calculated the effective action at the two loop order and, according to the renormalization group analysis, we have improved this approximation with the insertion of the running coupling constant at the vertices (the calculations are performed in the Landau gauge).

As far as the momentum dependence of the fermion self-energy test function is concerned, we have assumed a constant behaviour in the infrared region ($p < \mu$) and a fall down like $1/p^2$ ($logs$) for $p > \mu$ as suggested by the Operator Product Expansion analysis.

In the case of massless fermions we find that the theory possesses two phases: the chirally symmetric phase and the broken phase to the diagonal flavor subgroup, depending on the value of the coupling constant renormalized at the point $\mu$ (which is chosen to be coincident with the mass scale which discriminates the IR from the UV region of momenta).

For example, in the case of QCD with three flavors, the numerical calculations show that the color gauge dynamics spontaneously breaks the chiral symmetry down to $SU(3)_{L+R}$ by giving equal vacuum expectation value to the scalar quark-antiquark pair $\langle \overline{u}u \rangle = \langle \overline{d}d \rangle = \langle \overline{s}s \rangle$, for $\alpha_s = g^2(\mu)/(4\pi) \gtrsim 0.73\pi$.

This value, which is in agreement with the one obtained by other variational methods with the specific ansatzes for the fermion self-energy, is higher than the value that people find with methods based on the exact solution of the linearized Schwinger-Dyson equation or the numerical solution of the non- linear one. We think that this is due to the fact that the true form of the fermion self-energy is more complicated than the one used. In particular the constant behaviour of $\Sigma$ in the infrared region is a rather crude approximation and in the definition of the critical value of $\alpha_s$ are involved also the low momentum components of the theory.

The central part of this work is represented by the extension of the effective potential formalism to the realistic situation when both spontaneous and explicit breakdown of the global chiral symmetry take place.

Because we believe that chiral symmetry breaking in QCD is realized in a dynamical way, we have examined the particular predictions of our formalism for QCD. In this way it has been possible to compare the theoretical results with the experimental ones and to have a check, in a quantitative way, of the quality of the approach used.

We have considered the case of three flavors $u$, $d$ and $s$ and the result is a minimum of the effective potential corresponding to a vanishing value of all the pseudoscalar condensates (no spontaneous P and CP violation) and of the scalar charged condensates (the global vector symmetries are not spontaneously broken).

We have determined the values of the quark condensates $\langle \overline{u}u \rangle$, $\langle \overline{d}d \rangle$ and $\langle \overline{s}s \rangle$ from the stationary points of the effective potential. They depend on the parameters of our model: the renormalization invariant mass $\Lambda_{QCD}$, the quark masses $m_u$, $m_d$ and $m_s$ and the further scale $\mu$.

In order to determine these parameters from the experimental data, we have derived an explicit expression for the masses and for the decay constants of the pseudoscalar octet mesons which represent the pseudo-Goldstone bosons of the chiral symmetry breaking.

The equations relating the meson masses to the second derivatives of the effective potential with respect to the pseudoscalar fields, can be recast in a form which only depends on the explicit symmetry breaking part of the effective potential and is particularly useful from a computational point of view. Also, when the normalization condition for the effective potential in the small quark mass limit is imposed, these equations represent a generalization to the case of three flavors of the Adler-Dashen relation.

Allowing a general space-time dependence of the variational parameters, have calculated the effective action at one loop order by using a Weyl symmetrization prescription in order to solve the quantum mechanical ordering problem.

In this way it has been possible to estract, directly from our functional, the expression for the meson-quark-antiquark vertices.

This is a necessary ingredient for the calculation of the pseudoscalar meson decay constants together with the normalization factors relating our dynamical variables to the canonical pseudo-Goldstone fields.

The espressions for the masses and the decay costants of the pseudoscalar mesons represent a system of coupled equations so, the determination of the parameters of our model has been carried out in an approximate way.

The experimental inputs are

$$f_\pi = 93\ MeV$$
$$M_{\pi^\pm} = 139\ MeV$$
$$M_{K^\pm} = 494\ MeV$$
$$(M_{K^+} - M_{K^0})^\gamma = 1.5\ MeV$$

and we get a very good fit for the octet meson masses (agreement within 3%) with the following choices:

$$\mu = 497\ MeV$$
$$\Lambda_{QCD} = 449\ MeV$$
$$m_u(1) = 5.8\ MeV$$
$$m_d(1) = 8.4\ MeV$$
$$m_s(1) = 118\ MeV$$

where $\mu/\Lambda_{QCD}$ has been fixed in order to obtain the maximum value for $\Lambda_{QCD}$ in the massless case given $f_\pi$ and the quark masses are renormalized at 1 $GeV$. (In our calculations the mixing in the 3-8 sector has been taken into account while we have not faced the $U(1)_A$ problem.)

A comparison of our results with current algebra and sum rules predictions shows that our estimates for the $u$ and $d$ quark masses and condensates agree very well while the situation for the strange quark is much more critical.

However the indications given in the literature for $m_s(1)$ are confused. One finds evaluations varying from 100 $MeV$ up to 230 $MeV$.

So, within the large errors, we can conclude that the variational approach we have used for studying dynamical mass generation has led to quantitative results which are essentially in agreement with those obtained by quite different methods.

This proves the validity of the method and encourages us to use it for the study of the DSB phenomenon in other different contexts.

What we have in mind is the application of this variational formalism to technicolor-type models [61].

Let us spend some words about this argument.

Although the $SU(2) \otimes U(1)$ gauge theory of electroweak interactions has met with considerable experimental success, the symmetry breaking mechanism that generates the masses for the $W^{\pm}$, the $Z^0$ and the fermions remains yet to be understood.

The original version of the electroweak theory makes use of fundamental scalars to break the gauge symmetry.

A serious problem one encounters in this way is the so called "naturalness" problem connected with the fact that, unless some particular symmetry arises (like supersymmetry), there is nothing to prevent the Higgs mass to become as large as the typical cutoff of the theory.

Another problem with fundamental scalars is that the Yukawa couplings to fermions are quite arbitrary. They must be adjusted by hand to fit the various fermion masses.

To circumvent these problems, during the last few years there have been many attempts to find alternative formulations of the standard model avoiding elementary Higgs fields. Various schemes have been proposed and, in all of them (except the case of supersymmetric theories), the Higgs fields are composite ones.

Then, no naturalness problem arises, but the question of spontaneous symmetry breaking becomes a non-perturbative issue and one has to look for appropriate tools in order to be able to deal properly with it.

An example is represented by theories in which the electroweak symmetry breaking is driven by the condensation of a fermion bilinear due to a strong vector gauge force called technicolor (TC).

In these theories the technifermions T must exhibit a global chiral symmetry at least as large as $SU(2)_L \otimes SU(2)_R$ and its spontaneous breakdown to $SU(2)_{L+R}$ produces the three requisite Goldstone bosons to trigger the Higgs mechanism.

This phenomenon can be studied by using the same tecnique as for the $\chi$SB in $QCD$ provided that the confinement scale and also the chiral symmetry breaking scale are of the order of several hundred $GeV$. In this way this mechanism provides an adequate origin for the masses of the $W^{\pm}$ and $Z^0$ vector bosons.

The problem with technicolor theories is that they do not easily generate the masses of quarks and leptons.

In fact additional interactions must be introduced playing the role of the Yukawa couplings in the conventional scalar field Higgs theory.

The scale M associated with these new interactions must be larger than the technicolor scale and, at momenta small compared to M, the new interactions will take the form of the effective four-fermion couplings with strenght of order $1/M^2$. Fermion masses are then generated by the condensation of technifermions and they are proportional to $\langle \overline{T}T \rangle / M^2$.

The natural expectation based on the experience with QCD is that $\langle \overline{T}T \rangle$ is of the order of $\Lambda_{TC}^3$ ($\Lambda_{TC}$ is the confinement scale of the technicolor theory). So, the scale M must be much larger than $\Lambda_{TC}$ in order to explain the mass of any known fermions.

But, in addition to the four-fermion interactions involving two ordinary fermions and two technifermions, there will be others involving four ordinary fermions and also four technifermions. These too are expected to be of strenght $1/M^2$.

The problem with the four ordinary fermion interactions is that they typically contain flavor-changing neutral currents.

The standard electroweak model with the GIM mechanism has succesfully predicted the rates of flavor-changing neutral current processes and technicolor models must do the same in order to become viable alternative to it.

It is possible to show that in order to avoid a variety of experimental constraints, the scale M must be at least of the order of 300 $TeV$. But, with M of this order, the typical size of a fermion mass will be no more than 1 $MeV$, much less than many of the quark and lepton masses.

The problem of flavor-changing neutral currents has long been a fatal desease to the TC theories [62] and nowadays many people believe that TC is dead.

There is also another problem that is likely to plague any such theory. The full global symmetry of the technifermions will typically be larger than $SU(2)_L \otimes SU(2)_R$. If so, there will be more Goldstone bosons than can be absorbed by the $W^{\pm}$ and $Z^0$.

Some of these will remain massless until the effects of the four technifermion interactions, that explicitly break some of the relevant chiral symmetry of the technisector, are included.

The pseudo masses generated in this way, can be computed by chiral perturbation theory and they are proportional to $\langle \overline{\mathrm{T}}\mathrm{T} \rangle$.

The existence of these light pseudo-Goldstone bosons is a phenomenological embarassment and there is the necessity of a mechanism capable to raise the pseudo masses above current experimental bounds. For example with a two-order-of-magnitude enhancement of the technifermion condensate, these pseudo masses can be pushed well out of the range so far excluded by experiment.

Recently, T. Appelquist and L.C.R. Wijewardhana [63] have proposed a modification of the TC dynamycs leading to a higher value of the fermion condensate but leaving the Goldstone bosons decay constant $F$, which determines the $W$ and $Z$ masses, essentially unaltered.

In their model the asymptotic freedom of the TC theory is maintained but it is assumed that the large number of fermions expected in a realistic TC theory substantially slows down the running of the coupling.

They show that the slow running modifies the ultraviolet behaviour of the theory. In particular they find that there is a critical coupling $\alpha_c$ that the running coupling $\alpha(p)$ must exceed before chiral condensation can set in and that the solution of the Schwinger-Dyson equation for constant $\alpha > \alpha_c$ has a $1/p$ power behaviour multiplied by an oscillatory function.

They use the mass scale M as an UV cutoff. Then, the relatively slow fall of the technifermion self-energy allows a higher value for the cutoff M than naively expected for a given value of the fermion mass.

This raising of the cutoff leads to a suppression of the flavor-changing neutral currents and also raises the pseudo masses above current accelerator bounds. They find that fermion masses of the order of 100 $MeV$ and pseudo masses of 100 $GeV$ can be obtained for M$\sim$ 300 $TeV$.

On the other hand, $W^{\pm}$ and $Z^0$ masses remain essentially unaltered since the condensate $\langle\overline{\mathrm{T}}\mathrm{T}\rangle_{\mathrm{M}}$ is clearly much more sensitive to the high momentum behaviour of the self-energy than is the decay constant $F$.

Due to the interesting properties of this model, we can think of applying all our variational formalism to the case of chiral symmetry breaking in asymptotically free theories with slowly running couplings.

The ansatz for the self-energy to be used in this case is clearly different from the one used in QCD. In fact the test function will start from a constant value for momenta $p < \Lambda_{TC}$, will fall slowly like $1/p\ \Phi(logs)$ for a significant range $p > \Lambda_{TC}$ and then will take the asymptotic form $1/p^2\ (logs)$.

All these topics are now under study and represent some of the further developments of this work.

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